A Study of Generalized Weighted Composition Operators on Weighted Hardy Spaces

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Declaration

I declare that this thesis entitled **'A Study of Generalized Weighted Composition Operators on Weighted Hardy Spaces'** has been prepared by me under the guidance of Dr. Sunil Kumar Sharma, Assistant Professor of Department of Mathematics, M. P. Govt. Degree College, Amb, Dist. Una, Himachal Pradesh. No part of this thesis has formed the basis for the award of any degree or fellowship previously.

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CERTIFICATE BY ADVISOR

I hereby affirm as under that:

1. The thesis presented by Rohit Gandhi is worthy of consideration for the award of the degree of Doctor of Philosophy.

2. He/she has pursued the prescribed course of research.

3. The work is original contribution of the candidate.

4. The candidate has incorporated all the suggestions made by the Department Doctoral Board during Pre-Submission Seminar held on 06-April-2017.

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Abstract

The research work presented in this thesis deals with the study of "**Generalized Weighted Composition Operators on Weighted Hardy Spaces**".

The study of composition operators in the past has been done by many mathematicians on different function spaces. The properties like boundedness, compactness of the operators were discussed and used by many researchers. Examples of composition operators are the shift operators. Also composition operators can be viewed as the generalization of translation map or rotation map on the unit disk in the complex plane $\mathbb{C}$. The idea of composition operators is a motivation for the existence of other operators like multiplication operators, weighted composition operators. The operator-theoretic properties like boundedness, compactness, Hermitian, Isometric, Unitary weighted composition operators are discussed in the present thesis. A weighted composition operator $W_{\theta,\phi}$ is the product of multiplication operator M_θ and composition operator C_ϕ that is $W_{\theta,\phi} = M_\theta C_\phi$. Multiplication operators have its origin in diagonal matrices. For example, on a finite dimensional space an operator is a multiplication operator iff its matrix is a diagonal matrix. Multiplication operators plays an important role in the theory of operators via spectral theory for normal operators which states that every normal operator is unitarily equivalent to a multiplication operator. Multiplication operators, Composition operators, Weighted composition operators having the subject matter of study over the last a few decades.

As this theory of operators advances, the generalized composition operators, generalized multiplication operators come into existence and are studied on function spaces having elements analytic on the open unit disk (see[56],[57]). Recently, the study of generalized weighted composition operators have been initiated by many researchers. As it contains all the above mentioned operators, if taken as a particular case. For example, no(zero) derivative makes generalized weighted composition operator as weighted composition operator.

During the last decade, we can see the study of generalized weighted composition operator on different function spaces. They have been studied from $F(p, q, s)$ space to Bloch type space, Bloch type space to Bergman space, Area Nevalinna space to Bloch type space, H^∞ to Logarthim Bloch space, Zygmund space to Bloch-Orlicz type space, Weighted Bergman space (see[76],[75],[90],[91], [92], [3])in which properties like compactness and boundedness of the operator have been discussed.

The aim of our research is to study the properties of generalized weighted composition operators on weighted Hardy spaces. The following objectives have been set in order to fulfill this aim:

1. To characterize the adjoint of generalized weighted composition operators on weighted Hardy spaces.
2. To characterize the boundedness, compactness of generalized weighted composition operators on weighted Hardy spaces.
3. To characterize the conditions for which the generalized weighted composition operator becomes Fredhlom on weighted Hardy spaces.
4. Compute the spectra of generalized composition operator, generalized multiplication operator and generalized weighted composition operator.
5. To investigate Hermitian, Normal, Quasinormal generalized weighted composition operators.

In order to achieve the objectives as mentioned above, the various techniques used by many mathematicians have been applied. Mainly, the approach in our research work is strongly influenced by the books and papers of Cowen and MacCuler[10][12], Shapiro[50], Gunatillake[17][18][19], Sharma and Komal[56][57][58][59][60].

The thesis is composed of four chapters. The Chapter I contains introductory material to be used in subsequent chapters. In the second chapter we study adjoint of generalized composition operator, generalized multiplication operator and generalized weighted composition operator using evaluation kernel on weighted Hardy space. The adjoint of these operators are also characterized on n^{th} derivative of evaluation kernel. The norm estimate of generalized composition operators as well as of generalized weighted composition operators are also discussed. We know that in general the differential operator $D : C^1[a,b] \to C^1[a,b]$, defined by $Df = f'$ the derivative of f, where $C^1[a,b]$ is the Banach space of continuously differentiable functions is not a bounded operator. However, In the case of weighted Hardy space there do exist differential bounded operators. Anti-Differential operators on weighted Hardy spaces are also consider in this chapter.

In Chapter III we make a study of bounded and compact generalized weighted composition operators on weighted Hardy spaces. In this chapter we explore the conditions on the inducing maps which makes generalized weighted composition operators to be bounded and compact. The theory is illustrated with the help of nice examples. It is shown that a generalized weighted composition operator on weighted Hardy space is never isometric, However Hermitian and Fredhlom generalized weighted composition operator exist and they are also discussed in the chapter III.

The eigen value problem in the operator theory is a difficult problem. In the Chapter IV we discussed the eigen value problem for multiplication operators and generalized multiplication operators on weighted Hardy spaces. In the end of this chapter we obtain spectra of generalized composition operators on the weighted Hardy spaces.

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(Rohit Gandhi)

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Notations

$\in$	=	belongs to
$\cup$	=	union
$\cap$	=	intersection
$\mathbb{N}$	=	set of all positive integers
N_0	=	set of all positive integers including zero
R_+	=	set of all non-negative real numbers
$\mathbb{C}$	=	set of complex numbers
$\mathbb{R}$	=	set of real numbers
$\mathbb{D}$	=	open unit disk in the complex plane
$\bar{\mathbb{D}}$	=	closed unit disk in the complex plane
Ω	=	region in the open unit disk $\mathbb{D}$
$A \subset B$	=	A is a subset of B
$\forall$	=	for all
$\exists$	=	there exists or there exist
$\Rightarrow$	=	implies
$\iff$	=	if and only if
$\emptyset$	=	empty set
$\|\|x\|\|$	-	norm of the element $x \in R^n$
$\|x\|$	=	modulus of x
T^*	=	adjoint of an operator T
H	=	Hilbert space
$B(H)$	=	set of all bounded linear operators
$C[a,b]$	=	space of all continuous functions on closed interval $[a,b]$
$C^n[a,b]$	=	space of all n times continuously differentiable functions on closed interval $[a,b]$
$dimX$	=	dimension of X
$D(T)$	=	domain of T
$R(T)$	=	range of T
$kerT$	=	null space of T or kernel of T

$M^{\perp}$	=	set orthogonal to M
$M^{\perp\perp}$	=	set orthogonal to $M^{\perp}$
D	=	Differential operator
D_a	=	Anti-differential operator
$\inf A$	=	infimum(or greatest lower bound)of A
$\sup A$	=	supremum (or least upper bound)of A
$\sigma(T)$	=	spectrum of T

Dedicated to
Almighty God and my entire family.

Chapter 1

Introduction and Preliminaries

1.1 Literature Review

Let X be a non empty set and F be a field. Let also $V(X)$ be a linear space consisting of a functions $f : X \to F$. If $\phi : X \to X$ is a self map on X. Then a composition operator $C_\phi : V(X) \to V(X)$ induced by ϕ is defined by $C_\phi f = fo\phi$ for every $f \in V(X)$. Initially it can be seen that C_ϕ is a linear map and can be viewed as the generalization of translation map on real line or rotation map on the unit circle. If $\phi(t) = t$ for each t, (the identity function) then $C_\phi = I$, the identity operator. Further, if $\theta : X \to F$ is a function, then a multiplication operator $M_\theta : V(X) \to V(X)$ induced by θ is defined by the equation $M_\theta f = \theta.f$ for every $f \in V(X)$. The class of multiplication operators and the class of composition operators are combined to yield another important class of operators called weighted composition operators. Thus a weighted composition operator is of the type $M_\theta C_\phi$ or $C_\phi M_\theta$ and we usually denote it by $W_{\theta,\phi}$. Weighted composition operator $W_{\theta,\phi} : V(X) \to V(X)$ induced by θ and ϕ is defined by $W_{\theta,\phi} f = \theta.fo\phi$ for every $f \in V(X)$.

Composition Operators had made their appearance in many research areas. The first appearance these operators had made in 1871 in a paper of Schroder[48] where it was asked to find a function f and α such that $(fog)(z) = \alpha f(z)$ for every given analytic self-map g and for every z in the appropriate domain. Koeings[24] solved the Schroder's equation in case of unit disk in $\mathbb{C}$. The operators were used in Littlewood subordination theory[33]. B.O. Koopman used composition operators in studying statistical mechanics. Banach[4] hmself used these operators to study

the isometeries of Banach space of continuous functions. Neuman and Halmos[41] used these operators in a study of ergodic transformation.
Systematic study of composition operators was started by Nordgren[42] in 1968, in which he discussed the boundedness and norm structure of C_ϕ. Further properties of C_ϕ were related to the existence of fixed points of ϕ. He also had described the spectrum of C_ϕ in the case when ϕ is a linear fractional transformation of the unit disk onto itself. Perhaps Schwartz[49] wrote his Ph.D thesis "Composition operators on H^p spaces" in 1969. Ridge[44] also wrote his Ph.D thesis "Composition operators" in 1969 followed by Singh[63] who completed his Ph.D on composition operators in 1972 under the supervision of Prof. Eric. A. Nordgen.
Composition operators are studied mainly on three types of function spaces.

(A) The underlying spaces are measures spaces and the inducing maps are measurable functions.

(B) The underlying spaces are space of continuous functions and the inducing maps are continuous functions.

(C) The underlying spaces are taken to be regions in $\mathbb{C}$ or $\mathbb{C}^n$ and the inducing maps are holomorphic functions.

Although composition operators have been studied on many spaces, the majority of the literature is available on spaces whose functions are analytic on some set or in which the norm structure is closely connected to the analytic structure.
The systematic study initiated in the 1970s has been continued and extended in several directions during the last decade. Worth mention are some names such as Attle[2], Bourdon[5], Cowen[10], Cowen and MacCuler[14]-[16], Cowen and Gallardo-Gutirrez[11], Kamowitz[23], Komal[25]-[29], Kumar[31], MacCluer[36]-[37], Manhas[38]-[39], Roan[45], Shapiro[50]-[52], Sharma[55], Singh[63], Singh and Manhas[66]-[67], Singh and Kumar[65], Singh and Komal[64], Somasundaram[68], Yousefi[78]-[85], Zorboska[94]-[96] who explored the properties of the composition operators on different function spaces. Arora, Datt and Verma[1], Bourdon and Narayan[6], Bourdon and Shang[7], Cowen, Gunatillake and Ko[12], Contreras and Hernandez-Diaz[8, 9], Gunatillake[17]-[19], Manhas[39], Ohno and Takagi[43], Shields[61], Takagi[71], Ueki, Sei-Ichino and Luo[72], Yuan, Zhou and Tianjin[87] studied weighted composition operators on function spaces in which the functions under study are analytic. Sharma and Komal[56]-[59] introduced the study

of generalized composition operators and generalized multiplication operators on weighted Hardy spaces. Stevic[70], A. K. Sharma[54] also studied generalized composition operators on weighted Bergman spaces. Zhu[90, 91], Hu, Qing and Zhu[21] in 2009 had discussed the compactness and boundedness of Generalized weighted composition operators on weighted Bergman spaces. Yang[75, 76] studied generalized weighted composition operators on $F(p, q, s)$ space to the Bloch-type space.

1.2 Preliminaries

In this section we have discussed the results which are useful for studying the topic.

A complex vector space H is called an *inner product space* if to each ordered pair of vectors x and y in H is associated a complex number $\langle x, y\rangle$, called *inner product* of x and y, such that the following rules hold:

1. $\langle y, x\rangle = \overline{\langle x, y\rangle}$
2. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
3. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ if $x, y \in H$,$\alpha \in \mathbb{C}$
4. $\langle x, x, \rangle \geq 0$
5. $\langle x, x, \rangle = 0$ only if $x = 0$

For fixed y, $\langle x, y\rangle$ is therefore a linear function of x. For fixed x, it is a conjugate-linear function of y. Such functions of two variables are sometimes called sesquilinear.

If $\langle x, y\rangle = 0$, x is said to be *orthogonal* to y, and the notation $x \perp y$ is sometimes used. Since $\langle x, y\rangle = 0$ implies $\langle y, x\rangle = 0$, the relation $\perp$ is symmetric. If $A \subset H$ and $B \subset H$, the notation AB means that $x \perp y$ whenever $x \in A$ and $y \in B$.

Every inner product space can be a normed by defining

$$||x|| = \langle x, x\rangle^{\frac{1}{2}}$$

Let $\{\beta_n\}$ be a sequence of positive real numbers with $\beta_0 = 1$. Consider the function $f(z)$ which is analytic on the open unit disk $\mathbb{D}$, then it can be written as $f(z) = \sum_{n=0}^{\infty} f_n z^n$ where $\{f_n\}$ is a sequence of complex numbers, we say that $f \in H^2(\beta)$ iff $\sum_{n=0}^{\infty} |f_n|^2 \beta_n^2 < \infty$. $H^2(\beta)$ is a Banach space with respect to the norm $||f||^2 = \sum_{n=0}^{\infty} |f_n|^2 \beta_n^2$ and a Hilbert space with respect to the inner product $\langle f, g \rangle = \sum_{n=0}^{\infty} f_n \bar{g}_n \beta_n^2$, where $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $g(z) = \sum_{n=0}^{\infty} g_n z^n$ are elements of $H^2(\beta)$. The space $H^2(\beta)$ is known as weighted Hardy space. In other words weighted Hardy space $H^2(\beta)$ is a set containing the analytic functions on the unit disk, where the monomials $\{1, z, z^2, ...\}$ constitute a complete orthogonal set of non-zero vectors in $H^2(\beta)$. For simplicity, let $e_k(z) = z^k$ and $\hat{e_k}(z) = \frac{z^k}{\beta_k}$, clearly $\{\hat{e_k} : k \in N_0\}$ is an orthonormal basis for $H^2(\beta)$ where $N_0 = \mathbb{N} \cup \{0\}$. Each weighted Hardy space is characterized by a weight sequence $(\{\beta_n\})$ defined for each non-negative integer j by $\beta_j = ||z^j||$. If $\lim_{n\to\infty} \beta_n^{1/n} = 1$ or $\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1$, the space $H^2(\beta)$ contains analytic functions in the unit disk.

Some well known special cases of this types of spaces are, The **classical Hardy space** denoted by H^2 for which $\beta_n = 1$ for all $n \in N_0$. This space has a norm defined by

$$||f||^2 = \lim_{r\to 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \quad for \quad f \in H^2$$

The **Bergman space** represented by $A^2(\mathbb{D})$, (where $\mathbb{D}$ represents unit disk in the complex plane) when $\beta_n = \frac{1}{\sqrt{n+1}}$ for every $n \in N_0$. The norm defined on this space is

$$||f||^2 = \int_{\mathbb{D}} |f(x+iy)|^2 \frac{dxdy}{\pi} \quad for \quad f \in A^2(\mathbb{D})$$

and the **Dirichlet space** when $\beta_n = \sqrt{n+1}$ for every $n \in N_0$. The space is embedded with the norm as

$$||f||^2 = \int_{\mathbb{D}} |f'(x+iy)|^2 \frac{dxdy}{\pi} \quad for \quad f \quad in \quad Dirichlet \quad Space$$

Further by taking $\beta_n = n!$ the space is known as Fischer space.

The various properties of the weighted Hardy space $H^2(\beta)$ depends on weights

$\{\beta_n\}$. These weights can be visualized as a generating functions and many properties of $H^2(\beta)$ can be characterized.
The generating function for the weighted Hardy space $H^2(\beta)$ is the function

$$k(z) = \sum_{n=0}^{\infty} \frac{z^n}{\beta_n^2}$$

It can be seen that the generating function is analytic on the unit disk, with the help of generating function we can find the value of function $f \in H^2(\beta)$ at any point in the open unit disk $\mathbb{D}$. This leads to origin of Evaluation function $K_w(z)$ known as point evaluation function or evaluation kernel, as for every $w \in \mathbb{D}$ and for all $f \in H^2(\beta)$, we have $f(w) = \langle f, K_w \rangle$, where $K_w(z) = k(\bar{w}z)$.
Moreover, it can be easily seen that $||K_w|| = k(|w|^2)$.
For the Hardy space $H^2(\mathbb{D})$, evaluation at w in the disk is given by $f(w) = \langle f, K_w \rangle$ where

$$K_w(z) = \frac{1}{1 - \bar{w}z} \quad \textit{and} \quad ||K_w|| = \frac{1}{\sqrt{1 - |w|^2}}$$

The Bergman space which is denoted by $A^2(\mathbb{D})$, evaluation at w in the disk is given by $f(w) = \langle f, K_w \rangle$ where

$$K_w(z) = \frac{1}{(1 - \bar{w}z)^2} \quad \textit{and} \quad ||K_w|| = \frac{1}{1 - |w|^2}$$

In the Dirichlet space, evaluation at w in the disk is given by $f(w) = \langle f, K_w \rangle$ where

$$K_w(z) = \frac{1}{\bar{w}z} \log\left(\frac{1}{1 - \bar{w}z}\right) \quad \textit{and} \quad ||K_w||^2 = \frac{1}{|w|^2} \log\left(\frac{1}{1 - |w|^2}\right)$$

.

Clearly $||K_w||$ is an increasing function of $|w|$ and $\langle f, K_w \rangle = f(w)$ for every $w \in \mathbb{D}$.
For any positive integer n, the n^{th} derivative evaluation kernel at a, $K_a^{[n]}$ is the function in $H^2(\beta)$ so that

$$\langle f, K_a^{[n]} \rangle = f^{(n)}(a)$$

for f in $H^2(\beta)$. Using K_w the adjoint of composition operator C_ϕ can be characterized as follow

$$\langle f, C_\phi^* K_w \rangle = \langle C_\phi f, K_w \rangle = f(\phi(w)) = \langle f, K_{\phi(w)} \rangle$$

which shows that $C_\phi^* K_w = K_{\phi(w)}$.

Fixed Points

If ϕ is an analytic map of the open unit disk $\mathbb{D}$ to itself and b is a point of the closed unit disk(denote it by $\bar{\mathbb{D}}$), we will call b a fixed point of ϕ if $\lim_{r \to 1} \phi(rb) = b$. Again, let $\phi : \mathbb{D} \to \mathbb{D}$ is an analytic map from open unit disk to itself which is different from the identity map. Then

1. The function ϕ can have at most one fixed point inside the open unit disk.
2. If ϕ has a fixed point "a" inside the open unit disk then $|\phi^{'}(a)| \leq 1$.
3. If ϕ has no fixed points inside the open unit disk then it will have at least one fixed point on the unit circle and for only one of these points, say "a", $|\phi^{'}(a)| \leq 1$. The absolute value of the derivative at other fixed points on the unit circle are either greater that 1 or they do not exist at all. Therefore it is clear that ϕ has exactly one fixed point "a" on the closed unit disk where $|\phi^{'}(a)| \leq 1$. This point is known as the **Denjoy- Wolff** point of ϕ.

Let H be a Hilbert space. A transformation $T : H \to H$ is called a linear transformation if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all $\alpha, \beta \in F(field)$ and $x, y \in H$.

In addition, if there exist $M > 0$ such that

$$||Tx|| \leq M||x||$$

for every $x \in H$, we say that T is a bounded linear transformation or a bounded linear operator.

The set of all bounded linear operators from H into itself is denoted by $B(H)$. If $T \in B(H)$, then by Riesz Representation theorem there exist $S \in B(H)$ such that

$$\langle Tx, y \rangle = \langle x, Sy \rangle$$

for all $x, y \in H$. The operator S is called adjoint of T and we often denote it by T^*. Further, T is called

1. Self Adjoint or Hermitian if
$$T^* = T$$

2. Unitary if
$$T^*T = I = TT^*$$

3. Normal if
$$T^*T = TT^*$$

4. Quasi Normal if
$$T^*TT = TT^*T$$

5. Hyponormal if
$$TT^* \leq T^*T$$

6. Isometry if
$$T^*T = I$$

7. Compact if $\overline{T(E)}$ is compact for every bounded subset E of H.

8. Fredholm if $kerT$ and $kerT^*$ are finite dimensional and range of T is closed.

Let Ω be a region in open unit disk $\mathbb{D}$ in the complex plane C.

Definition 1.1. **Generalized composition operator:** Let $\phi : \Omega \to \Omega$ be an analytic mapping. Then a generalized composition operator $C_\phi^d : H^2(\beta) \to H^2(\beta)$ is defined by $C_\phi^d f = f' o\phi$, where f' is the derivative of $f \in H^2(\beta)$.

It can be verified that C_ϕ^d is a linear operator. For if a and b in $\mathbb{C}$ and $f, g \in H^2(\beta)$.

$$\begin{aligned}(C_\phi^d(af + bg))(z) &= (af + bg)'(\phi(z)) \\ &= af'(\phi(z)) + bg'(\phi(z)) \\ &= (aC_\phi^d f + bC_\phi^d g)(z)\end{aligned}$$

Definition 1.2. **Generalized multiplication operator:** Let $\theta : \Omega \to \mathbb{C}$ be a holomorphic function. Then generalized multiplication operator $M_\theta^d : H^2(\beta) \to H^2(\beta)$ is defined by $M_\theta^d = \theta.f'$, where f' represents the derivative of function f .

Definition 1.3. **Generalized weighted composition operator:** Let $\theta : \Omega \to \mathbb{C}$ and $\phi : \Omega \to \Omega$ be the analytic maps. The generalized weighted composition operator on $H^2(\beta)$ is denoted by $W^d_{\theta,\phi}$ and is defined by $W^d_{\theta,\phi}f = \theta.f'o\phi$, where f' is the derivative of f which further belongs to the weigthed Hardy space $H^2(\beta)$. The generalized weighted composition operator include many well known operator. For, if we take $d = 0$(no derivative) then $W^d_{\theta,\phi} = W_{\theta,\phi}$ which is a weighted composition operators studied by Gunatillake[17][18][19] . For $\theta(z) = 1$ and $d = 0$ generalized weighted composition operator becomes composition operator. Further for $\theta(z) = \phi'(z)$ we we have $W^d_{\theta,\phi} = DC_\phi$ differentiation of composition operator which was studied by Stevic [69]. If $\phi(z) = z$ for every $z \in \mathbb{D}$, then $W^d_{\theta,\phi} = M^d_\theta$ which is a generalized multiplication operator. The generalized composition operator and generalized multiplication operator on weighted Hardy space were studied by Sharma and Komal[56]-[57].

Definition 1.4. **Differential operator:** Let f be a mapping in $H^2(\beta)$ into itself. Then the differential operator D on $H^2(\beta)$ is defined by

$$D(\sum_{n=0}^{\infty} f_n z^n) = \sum_{n=0}^{\infty} n f_n z^{n-1}$$

Definition 1.5. **Anti-differential operator:** For any function f in $H^2(\beta)$. The anti-differential operator denoted by D_a is defined as

$$D_a(\sum_{n=0}^{\infty} f_n z^n) = \sum_{n=0}^{\infty} \frac{f_n z^{n+1}}{n+1}$$

Definition 1.6. The complex number λ is an eigenvalue of the bounded operator T if $Tf = \lambda f$ for some nonzero f; the vector f is then said to be an eigenvector of T. The set of all eigenvalues of T is called the point spectrum of T and is denoted by $\Pi_0(T)$.

Definition 1.7. If T is a bounded linear operator on a Hilbert space H, the spectrum of T, denoted by $\sigma(T)$, is the set of all complex numbers λ such that $T - \lambda I$ is not invertible, where I is the identity operator on H.

Definition 1.8. For a bounded operator T on Hilbert space H, a closed subspace M is called a non-trivial **invariant subspace** of T if $M \neq 0$ and $M \neq H$ and

$$x \in M \implies Tx \in M$$

The properties related to the spectra given below are very elementary and very well known.
Let T be a bounded linear operator.

- If $||I - T|| < 1$, then T is invertible.
- The spectrum of T is a nonempty compact subset of $\mathbb{C}$.
- If T is an invertible operator, then
$$\sigma(T^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma(A)\}.$$
- If T^* denoted the adjoint of T, then
$$\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(A)\}.$$
- If T is an operator on a finite dimensional space, then $\sigma(T) = \Pi_0(T)$. Further for the operators on infinite-dimensional spaces, $\Pi_0(T)$ may be the empty set.

Throughout the thesis, the symbol B(H) denote the Banach algebra of all bounded linear operator on H into itself and N_o denote the set $\{0, 1, 2, 3, ...\}$.

The present thesis is a study of Generalized weighted composition operators acting on weighted Hardy spaces. The thesis is composed of four chapters. The introductory material is presented in the chapter I. This chapter also contain the historical background of the composition operators. The definition of all operators which we have used in the thesis are defined in the chapter I. The chapter II is a study of Generalized composition operators acting in weighted Hardy spaces. This chapter consists of four section. In the first section we compute the adjoint of generalized composition operators using evaluation kernel on weighted Hardy spaces. This section is concluded with an example. In the next section we estimate the norm of generalized multiplication operators by using evaluation kernel i.e. $|\theta(w)| \leq ||M_\theta^d|| \frac{||K_w||}{|K_w^{[1]}||}$ for each w belongs to the open unit disk $\mathbb{D}$ subset of $\mathbb{C}$. We also prove that if $M_\theta^d \in B(H^2(\beta))$, where $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$, then $\frac{|\theta(w)|}{\beta_1 \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}} \leq ||M_\theta^d||$.

The adjoint of generalized weighted composition operators by using evaluation kernel on $H^2(\beta)$ are studied. Further in the first theorem of the third section we characterized the adjoint of generalized weighted composition operators by using evaluation kernel on $H^2(\beta)$. Norm of generalized weighted composition operator is also estimated in this section. In the end of this section we give corollary that if $W^d_{\theta,\phi}$ is a bounded operator on the Hardy space $H^2(\mathbb{D})$ such that $|\phi(w)| < 1$, then θ is uniformly bounded on the open unit disk. In the last section of this chapter we have studied the Anti-Differential operator on weighted Hardy space. We first characterize the condition of boundedness of Anti differential operator on weighted Hardy space. In general the differential operator $D : C^1[a, b] \to C^1[a, b]$, where $C^1[a, b]$ is the Banach space of continuously differential functions is not a bounded operator. But we have shown that, in case of weighted Hardy space differential operator can be bounded. Adjoint of anti- differential operator is already characterized in the paper of Sharma and Komal[59]. The Hermitian, normal, quasinormal and hyponormal Anti-Differential operators on weighted Hardy spaces are considered in this section.

The main purpose of chapter III is the study of generalized weighted composition operators on weighted Hardy spaces. This chapter is divided into four sections. In the first section we first characterize bounded generalized weighted composition operators and then we characterize compact generalized weighted composition operators on $H^2(\beta)$. In the next section we have shown that the only Hermitian weighted composition operator is the zero operator, which is not true in the earlier known Hermitian weighted composition operator on other function spaces. It is also shown that generalized weighted composition operator is not isometric on $H^2(\beta)$. A necessary and sufficient condition for the generalized weighted composition operators to be Fredhlom are investigated in the third section. The theorem is concluded with a suitable examples. In the end if this section we have shown that generalized weighted composition operator has a non-trivial invariant subspace. In the fourth section of this chapter we have shown that generalized weighted composition operator on $H^2(\beta)$ commutes iff $\theta = \theta o\phi$.

The fourth chapter is divided into two sections. In the first section we discussed the spectra of generalized weighted composition operators on weighted Hardy spaces. It is shown by Gunatillake[17] that the spectrum of weighted composition operator $W_{\theta,\phi}$ on $H^2(\beta)$ is contained in the set $\{0, \theta(a), \theta(a)\phi'(a), \theta(a)(\phi'(a))^2, ...\}$. Cowen,Gunatillake and Ko[12] studied Hermitian weighted composition operators on weighted Hardy spaces in which they characterized the adjoint of weighted composition operator and also discussed the eigen values of the operator. In this chapter, we will see how the spectrum of weighted composition operator $W_{\theta,\phi}$ is found under the assumption that $W_{\theta,\phi}$ is compact, with ϕ having fixed point inside the open unit disk, we have discussed the results which can help us to characterize the eigen values of generalized composition operator C_ϕ^d, generalized multiplication operator M_θ^d on $H^2(\beta)$. We have also characterize the adjoint of generalized composition operator, generalized multiplication operator and generalized weighted composition operator on the derivative of evaluation kernel on $H^2(\beta)$. The second section of this chapter is devoted to find the spectra of Multiplication operator, generalized multiplication operator and generalized composition operators on weighted Hardy space. We have shown that $\Pi_0(M_\theta) \subseteq \{\theta(0)\}$ is an eigen value of M_θ iff $\theta(n) = 0$ for every $n \geq 1$. The reverse inclusion hold if θ is a constant function. In the next result we will see that spectra of generalized multiplication operator is $n\alpha$ i.e. $\Pi_0(M_\theta^d) = n\alpha \quad for \quad k = 1 \quad and \quad \emptyset(emptyset) \quad for \quad k \geq 2$ where $\theta(z) = \alpha z^k$, α is any number and $n \in N$. In the end of this chapter we have characterize the condition to find the eigen vlaue λ of generalized composition operator on $H^2(\beta)$. We have shown that λ is an eigen value of C_ϕ^d if $|\lambda| < \limsup \left(\frac{n!}{\beta_n}\right)^{\frac{1}{n}}$, where $\phi(z) = z$ for every z in open unit disk and λ is an eigen value of C_ϕ^d if $|\lambda| < \limsup \left(\frac{n! a^{\frac{n(n+1)}{2}}}{\beta_n}\right)^{\frac{1}{n}}$, where $\phi(z) = az$ for every z in open unit disk and a be any real number.

In the end of the thesis, a bibliography has been given which by no means is an exhaustive one but lists only those research papers and books which have been referred to in the main text of the thesis.

Chapter 2

Generalized Composition Operators on Weighted Hardy Spaces [1]

2.1 Adjoint of a generalized composition operator using evaluation kernel on a weighted Hardy spaces

The adjoint of the composition operators on different function spaces were found by Cowen and MacCuler. As adjoint of operator helps to characterize the other properties of the operator like Hermitian, isometry, normality etc. So in this chapter we establish the adjoint of generalized composition operator, generalized multiplication operator and generalized weighted composition operator using evaluation kernel on weighted Hardy space. Anti differential operators on weighted Hardy spaces are also studied in the last section of this chapter.
For sake of convenience we give here the Theorem[2.16] of Cowen and MacCuler[15] in the form of lemma given below

Lemma 2.1. *Let $f \in H^2(\beta)$ and $K_w(z)$ be an evaluation kernel. Then*

$$\langle f, K_w^{[1]} \rangle = f^{'}(w) \tag{2.1}$$

[1]Results of this chapter is published in International Journal of Mathematical Analysis,Vol. 9-2015, No. 14, 655-660.

where $K_w^{[1]}$ *is the first derivative evaluation kernel at* w*.*

Proof. Given that $f \in H^2(\beta)$,
So

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.2}$$

and

$$K_w(z) = \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{\beta_n^2} \tag{2.3}$$

Now

$$K_w^{[1]}(z) = \sum_{n=1}^{\infty} \frac{n z^n \overline{w}^{n-1}}{\beta_n^2} \tag{2.4}$$

Now if we take

$$\begin{aligned}
\langle f, K_w^{[1]} \rangle &= \langle \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} \frac{(n+1) z^{n+1} \overline{w}^n}{\beta_{n+1}^2} \rangle \\
&= \sum_{n=1}^{\infty} n f_n w^{n-1} \\
&= f^{'}(w)
\end{aligned}$$

Therefore we can say that

$$\langle f, K_w^{[1]} \rangle = f^{'}(w) \tag{2.5}$$

Hence the result. □

Theorem 2.2. *Let* $C_\phi^d \in B(H^2(\beta))$*. Then* $C_\phi^{d*} K_w = K_{\phi(w)}^{[1]}$*, where* C_ϕ^{d*} *is the adjoint of* C_ϕ^d*.*

Proof. Let $f \in H^2(\beta)$. Now by using the property of point evaluation kernel and Lemma(2.1), we have

$$\begin{aligned}
\langle f, C_\phi^{d*} K_w \rangle &= \langle C_\phi^d f, K_w \rangle \\
&= \langle f^{'} o \phi, K_w \rangle \\
&= f^{'}(\phi(w)) \\
&= \langle f, K_{\phi(w)}^{[1]} \rangle
\end{aligned}$$

Therefore, we can say that $\quad C_\phi^{d*} K_w = K_{\phi(w)}^{[1]}$ □

Example 2.1. *Let* $\Omega = \{z \in \mathbb{C} : |z| < e^{-1}\}$ *and* $\phi(z) = z^2$ *for all* $z \in \Omega$. *Then* $\phi : \Omega \to \Omega$ *is an analytic map. For every* $n \in \mathbb{N} \cup \{0\}$ *define* $\beta_n = e^{-n}$. *Then* $H^2(\beta) \neq 0$ *as* $e_1 \in H^2(\beta)$. *We first show that* $C_\phi^d : H^2(\beta) \to H^2(\beta)$ *is a bounded operator. Take* $f(z) = \sum_{n=0}^{\infty} f_n z^n$ *in* $H^2(\beta)$, *then*

$$||f||^2 = \sum_{n=0}^{\infty} |f_n|^2 \beta_n^2 < \infty$$

Now

$$\begin{aligned} C_\phi^d f &= \sum_{n=1}^{\infty} n f_n (\phi(z))^{n-1} \\ &= \sum_{n=1}^{\infty} n f_n z^{2n-2} \end{aligned}$$

Consider

$$\begin{aligned} ||C_\phi^d f||^2 &= \sum_{n=0}^{\infty} (n+1)^2 |f_{n+1}|^2 \beta_{2n}^2 \\ &= \sum_{n=0}^{\infty} ((n+1)\frac{\beta_{2n}}{\beta_{n+1}})^2 |f_{n+1}|^2 \beta_{n+1}^2 \end{aligned} \tag{2.6}$$

But

$$\begin{aligned} (n+1)\frac{\beta_{2n}}{\beta_{n+1}} &= \frac{(n+1)e^{n+1}}{e^{2n}} \\ &= \frac{(n+1)}{e^{n-1}} \\ &\leq e \quad \textit{for every} \quad n = 0, 1, 2, 3, ... \end{aligned} \tag{2.7}$$

Therefore from (2.6)

$$\begin{aligned} ||C_\phi^d f||^2 &\leq e^2 \sum_{n=0}^{\infty} |f_{n+1}|^2 \beta_{n+1}^2 \\ &\leq e^2 ||f||^2 \end{aligned} \tag{2.8}$$

or $\quad ||C_\phi^d f|| \leq e||f|| \quad$ *for every* $\quad f \in H^2(\beta)$

Hence C_ϕ^d *is a bounded operator.*

Consider

$$\begin{aligned}\langle f, C_\phi^{d^*} K_w\rangle &= \langle C_\phi^d f, K_w\rangle \\ &= f^{'}(\phi(w)) = f^{'}(w^2) \\ &= \sum_{n=0}^{\infty}(n+1)f_{n+1}w^{2n}\end{aligned}$$

On the other side $\langle f, K_{\phi(w)}^{[1]}\rangle = \langle f, K_{w^2}^{[1]}\rangle = \sum_{n=0}^{\infty}(n+1)f_{n+1}w^{2n}$. *which shows that*

$$C_\phi^{d^*} K_w = K_{\phi(w)}^{[1]}$$

Theorem 2.3. *Let f be any function on $H^2(\beta)$. Let $C_\phi^{d^*}$ denote the adjoint of C_ϕ^d, then*

$$C_\phi^{d^*} K_a^{[n]} = \sum_{j=0}^{n} \overline{\alpha_j(a)} K_{\phi(a)}^{[j]} + \overline{(\phi'(a))^n} K_{\phi(a)}^{[n+1]}$$

where C_ϕ^d is a bounded operator in $H^2(\beta)$, "a" is a point inside the open unit disk and $K_a^{[n]}$ is the n^{th} derivative evaluation kernel at a.

Proof. For f be any function on $H^2(\beta)$, Consider

$$\begin{aligned}\langle f, C_\phi^{d^*} K_a^{[n]}\rangle &= \langle C_\phi^d f, K_a^{[n]}\rangle \\ &= \langle f^{'} o\phi, K_a^{[n]}\rangle \\ &= (f^{'} o\phi)^{(n)}(a)\end{aligned} \tag{2.9}$$

By evaluating the R.H.S of above equation and using the algebraic properties of inner product, we have

$$\langle f, C_\phi^{d^*} K_a^{[n]}\rangle = \langle f, \sum_{j=0}^{n} \overline{\alpha_j(a)} K_{\phi(a)}^{[j]} + \overline{(\phi'(a))^n} K_a^{[n+1]}\rangle \tag{2.10}$$

Since f is arbitrary, Hence

$$C_\phi^{d^*} K_a^{[n]} = \sum_{j=0}^{n} \overline{\alpha_j(a)} K_{\phi(a)}^{[j]} + \overline{(\phi'(a))^n} K_{\phi(a)}^{[n+1]}$$

.

□

2.2 Adjoint and norm estimate of a generalized multiplication operator using evaluation kernel

Theorem 2.4. *Let $M_\theta^d \in B(H^2(\beta))$. Then $M_\theta^{d^*} K_w = \bar{\theta}(w) K_w^{[1]}$, where $M_\theta^{d^*}$ is the adjoint of M_θ^d.*

Proof. Let $f \in H^2(\beta)$,
We have

$$\begin{aligned}\langle f, M_\theta^{d^*} K_w \rangle &= \langle M_\theta^d f, K_w \rangle \\ &= \langle \theta f', K_w \rangle \\ &= \theta(w) f'(w) \\ &= \theta(w) \langle f, K_w^{[1]} \rangle\end{aligned}$$

This proves that $\quad M_\theta^{d^*} K_w = \bar{\theta}(w) K_w^{[1]}$. □

Example 2.2. *Let M_θ^d be a bounded operator. For $\theta(z) = z$, Consider*

$$\begin{aligned}\langle f, M_\theta^{d^*} K_w \rangle &= \langle M_\theta^d f, K_w \rangle \\ &= \langle \theta f', K_w \rangle \\ &= w f'(w) = \sum_{n=0}^{\infty} n f_n w^n\end{aligned}$$

Now it can be seen that $\langle f, \bar{\theta}(w) K_w^{[1]} \rangle = \sum_{n=0}^{\infty} n f_n w^n$.

Theorem 2.5. *Let $\mathbb{D}$ be the open unit disk in complex plane $\mathbb{C}$. If $M_\theta^d \in B(H^2(\beta))$, then $|\theta(w)| \leq ||M_\theta^d|| \frac{||K_w||}{||K_w^{[1]}||}$ for each $w \in \mathbb{D}$.*

Proof. Let $f_w = \frac{K_w}{||K_w||}$, then $||f_w|| = 1$
Since M_θ^d is bounded, therefore

$$||M_\theta^{d^*} f_w|| \leq ||M_\theta^d|| \tag{2.11}$$

$$||M_\theta^{d^*} \frac{K_w}{||K_w||}|| \leq ||M_\theta^d|| \tag{2.12}$$

$$||M_\theta^{d^*} K_w|| \leq ||M_\theta^d|| ||K_w|| \tag{2.13}$$

By using theorem (2.4), we have

$$||\bar{\theta}(w) K_w^{[1]}|| \leq ||M_\theta^d|| ||K_w|| \tag{2.14}$$

Hence the result $\quad |\theta(w)| \leq ||M_\theta^d|| \frac{||K_w||}{||K_w^{[1]}||}$ □

Theorem 2.6. *Suppose that* $M_\theta^d \in B(H^2(\beta))$*, where* $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$*. Then* $\frac{|\theta(w)|}{\beta_1 \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}} \leq$ $||M_\theta^d||$.

Proof. By theorem (2.5), we have

$$|\theta(w)| \leq ||M_\theta^d|| \frac{||K_w||}{||K_w^{[1]}||} \tag{2.15}$$

As we know that

$$||K_w||^2 = \sum_{n=0}^{\infty} \frac{|w|^{2n}}{\beta_n^2}$$

Therefore for any $|w| < 1$, it is easy to see that $||K_w|| < \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}$

Also $\quad ||K_w^{[1]}|| \geq \frac{1}{\beta_1}$ implies $\quad \frac{1}{||K_w^{[1]}||} \leq \beta_1$

Therefore from inequality (2.15) we have $|\theta(w)| \leq ||M_\theta^d|| \beta_1 \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}$

This proves that $\quad \frac{|\theta(w)|}{\beta_1 \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}} \leq ||M_\theta^d||$ □

Theorem 2.7. *Let* f *be any function on* $H^2(\beta)$*. Let* $M_\theta^{d^*}$ *denote the adjoint of* M_θ^d*, then*

$$M_\theta^{d^*} K_a^{[n]} = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \overline{\theta^{(n-r)}(a)} K_a^{[r+1]}$$

where M_θ^d *is a bounded operator in* $H^2(\beta)$*, "a" is a point inside the open unit disk and* $K_a^{[r+1]}$ *is the* $(r+1)^{th}$ *derivative evaluation kernel at* a*.*

Proof. Let f be any function in $H^2(\beta)$. Then consider

$$\langle f, M_\theta^{d^*} K_a^{[n]} \rangle = \langle M_\theta^d f, K_a^{[n]} \rangle = \langle \theta f', K_a^{[n]} \rangle \tag{2.16}$$

$$= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \theta^{(n-r)}(a) f^{(r+1)}(a) \tag{2.17}$$

$$= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \theta^{(n-r)}(a) \langle f, K_a^{[r+1]} \rangle \tag{2.18}$$

Therefore we have

$$\langle f, M_\theta^{d^*} K_a^{[n]} \rangle = \langle f, \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \overline{\theta^{(n-r)}(a)} K_a^{[r+1]} \rangle \tag{2.19}$$

Since f is arbitrary, Hence the result. □

2.3 Adjoint of a generalized weighted composition operator using evaluation kernel

Theorem 2.8. *Suppose $W_{\theta,\phi}^d \in B(H^2(\beta))$. Then $W_{\theta,\phi}^{d^*} K_w = \bar{\theta}(w) K_{\phi(w)}^{[1]}$, where $W_{\theta,\phi}^{d^*}$ is the adjoint of $W_{\theta,\phi}^d$.*

Proof. Let $f \in H^2(\beta)$. Then

$$\begin{aligned}
\langle f, W_{\theta,\phi}^{d^*} K_w \rangle &= \langle W_{\theta,\phi}^d f, K_w \rangle \\
&= \langle \theta . f^{'} o\phi, K_w \rangle \\
&= \theta(w) f^{'}(\phi(w)) \\
&= \theta(w) \langle f, K_{\phi(w)}^{[1]} \rangle \quad \textit{by using lemma } (2.1) \\
&= \langle f, \theta(\bar{w}) K_{\phi(w)}^{[1]} \rangle
\end{aligned}$$

therefore, we have

$$W_{\theta,\phi}^{d^*} K_w = \bar{\theta}(w) K_{\phi(w)}^{[1]}$$

Hence the result. □

Example 2.3. *Let $\Omega = \{z \in \mathbb{C} : |z| < e^{-1}\}$. For every $n \in \mathbb{N} \cup \{0\}$, let $\beta_n = e^{-n}$. Let $\phi : \Omega \to \Omega$ be defined by $\phi(z) = z^2$. Let $\theta : \Omega \to C$ be defined by $\theta(z) = z^2$. We first show that $W_{\theta,\phi}^d$ is a bounded operator.*

Take $f(z) = \sum_{n=0}^{\infty} f_n z^n$ *in* $H^2(\beta)$, *then* $||f||^2 = \sum_{n=0}^{\infty} |f_n|^2 \beta_n^2$

Now

$$\begin{aligned}(W^d_{\theta,\phi}f)(z) &= \theta(z)f^{'}(\phi(z))\\ &= z^2\sum_{n=1}^{\infty} nf_n z^{2n-2}\\ &= \sum_{n=1}^{\infty} nf_n z^{2n}\end{aligned}$$

Therefore

$$\begin{aligned}||W^d_{\theta,\phi}f||^2 &= \sum_{n=0}^{\infty} n^2|f_n|^2\beta_{2n}^2\\ &= \sum_{n=0}^{\infty}\left(n\frac{\beta_{2n}}{\beta_n}\right)^2|f_n|^2\beta_n^2\end{aligned} \tag{2.20}$$

But

$$\frac{n\beta_{2n}}{\beta_n} = \frac{ne^n}{e^{2n}} = \frac{n}{e^n} \leq 1 \tag{2.21}$$

Hence form equation (2.20) $||W^d_{\theta,\phi}f||^2 \leq \sum_{n=0}^{\infty}|f_n|^2\beta_n^2 \leq ||f||^2$

This proves that $W^d_{\theta,\phi}f$ *is a bounded operator.*

For the adjoint of $W^d_{\theta,\phi}$*, consider*

$$\begin{aligned}\langle f, W^{d^*}_{\theta,\phi}K_w\rangle &= \langle W^d_{\theta,\phi}f, K_w\rangle\\ &= \langle\theta.f^{'}o\phi, K_w\rangle\\ &= w^2f^{'}(\phi(w)) = w^2f^{'}(w^2)\\ &= \sum_{n=1}^{\infty} nf_n w^{2n}\end{aligned}$$

Similarly it can be seen that

$$\langle f, \bar{\theta}(w)K^{[1]}_{\phi(w)}\rangle = \sum_{n=1}^{\infty} nf_n w^{2n}$$

Theorem 2.9. *If* $W^d_{\theta,\phi} \in B(H^2(\beta))$*, Then* $|\theta(w)| \leq ||W^d_{\theta,\phi}||\frac{||K_w||}{||K^{[1]}_{\phi(w)}||}$ *for each* w *in the open unit disk.*

Proof. Let $f_w = \frac{K_w}{||K_w||}$

Then $||f_w|| = 1$

Since $W^d_{\theta,\phi}$ is bounded, therefore

$$||W^{d*}_{\theta,\phi} f_w|| \leq ||W^{d*}_{\theta,\phi}||||f_w|| = ||W^d_{\theta,\phi}|| \tag{2.22}$$

This implies

$$||W^{d*}_{\theta,\phi} \frac{K_w}{||K_w||}|| \leq ||W^d_{\theta,\phi}|| \tag{2.23}$$

$$||W^{d*}_{\theta,\phi} K_w|| \leq ||W^d_{\theta,\phi}||||K_w|| \tag{2.24}$$

By using theorem (2.8), we have

$$||\bar{\theta}(w) K^{[1]}_{\phi(w)}|| \leq ||W^d_{\theta,\phi}||||K_w|| \tag{2.25}$$

So we have

$$|\theta(w)| \leq ||W^d_{\theta,\phi}|| \frac{||K_w||}{||K^{[1]}_{\phi(w)}||} \tag{2.26}$$

□

Theorem 2.10. *Suppose that generalized weighted composition operator* $W^d_{\theta,\phi} \in B(H^2(\beta))$, *where* $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$. *Then the norm* $||W^d_{\theta,\phi}||$ *is bounded below by* $\frac{|\theta(w)|}{\beta_1 \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}}$.

Proof. By the theorem (2.9) we have

$$|\theta(w)| \leq ||W^d_{\theta,\phi}|| \frac{||K_w||}{||K^{[1]}_{\phi(w)}||} \tag{2.27}$$

Also we know that

$$||K_w||^2 = \sum_{n=0}^{\infty} \frac{|w|^{2n}}{\beta_n^2} \tag{2.28}$$

and for any $|w| < 1$ it is easy to see that $||K_w|| < \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}$

Clearly

$$||K^{[1]}_{\phi(w)}|| \geq \frac{1}{\beta_1} \tag{2.29}$$

Therefore from equation (2.27) we have

$$|\theta(w)| \leq ||W^d_{\theta,\phi}||\beta_1\sqrt{\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}} \tag{2.30}$$

Hence

$$\frac{|\theta(w)|}{\beta_1\sqrt{\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}}} \leq ||W^d_{\theta,\phi}|| \tag{2.31}$$

□

Corollary 2.11. *Let $W^d_{\theta,\phi}$ be a bounded operator on the Hardy space $H^2(\mathbb{D})$ where ϕ is an analytic map from the unit disk into itself such that $|\phi(w)| < 1$, then θ is uniformly bounded on the open unit disk.*

Proof. Let $w \in \mathbb{D}$ and for $H^2(\mathbb{D})$, we take $\beta(n) = 1$
So by the result given in theorem (2.9) we have

$$|\theta(w)| \leq ||W^d_{\theta,\phi}||\frac{||K_w||}{||K^{[1]}_{\phi(w)}||} \tag{2.32}$$

Therefore

$$|\theta(w)| \leq ||W^d_{\theta,\phi}||\frac{1}{\sqrt{1-|w|^2}||K^{[1]}_{\phi(w)}||} \tag{2.33}$$

Hence θ is uniformly bounded on the open unit disk. □

Theorem 2.12. *Let f be any function on $H^2(\beta)$. Let ${W^d_{\theta,\phi}}^*$ denote the adjoint of $W^d_{\theta,\phi}$, then*

$${W^d_{\theta,\phi}}^* K^{[n]}_a = \sum_{j=0}^{n}\overline{\alpha_j(a)}K^{[j]}_{\phi(a)} + \overline{\theta(a)(\phi'(a))^n}K^{[n+1]}_{\phi(a)}$$

where $W^d_{\theta,\phi}$ is a bounded operator in $H^2(\beta)$, "a" is a point inside the open unit disk and $K^{[n+1]}_a$ is the $(n+1)^{th}$ derivative evaluation kernel at a.

Proof. Let f be any function in $H^2(\beta)$, then

$$\langle f, {W^d_{\theta,\phi}}^* K^{[n]}_a\rangle = \langle W^d_{\theta,\phi}f, K^{[n]}_a\rangle \tag{2.34}$$

$$= \langle \theta f' o\phi, K^{[n]}_a\rangle \tag{2.35}$$

Now differentiate n times the function $\theta f^{'} o\phi$ and evaluate it at a. This gives us,

$$\langle f, {W^{d}_{\theta,\phi}}^{*} K^{[n]}_{a}\rangle = \sum_{j=0}^{n} \alpha_j(a) f^{(j)}(\phi(a)) + \theta(a) f^{(n+1)}(\phi(a))(\phi^{'}(a))^{n} \tag{2.36}$$

Now (2.36) can be written as

$$\langle f, {W^{d}_{\theta,\phi}}^{*} K^{[n]}_{a}\rangle = \sum_{j=0}^{n} \alpha_j(a) \langle f, K^{[j]}_{\phi(a)}\rangle + \theta(a)(\phi^{'}(a))^{n} \langle f, K^{[n+1]}_{\phi(a)}\rangle \tag{2.37}$$

Now by using the algebraic properties of the inner product in the equation (2.37) we get,

$$\langle f, {W^{d}_{\theta,\phi}}^{*} K^{[n]}_{a}\rangle = \langle f, \sum_{j=0}^{n} \overline{\alpha_j(a)} K^{[j]}_{\phi(a)} + \overline{\theta(a)(\phi^{'}(a))^{n}} K^{[n+1]}_{\phi(a)}\rangle \tag{2.38}$$

Since f is arbitrary, which shows that

$${W^{d}_{\theta,\phi}}^{*} K^{[n]}_{a} = \sum_{j=0}^{n} \overline{\alpha_j(a)} K^{[j]}_{\phi(a)} + \overline{\theta(a)(\phi^{'}(a))^{n}} K^{[n+1]}_{\phi(a)}$$

□

Theorem 2.13. *Let f be any function on $H^2(\beta)$. Let $W^{*}_{\theta,\phi}$ denote the adjoint of $W_{\theta,\phi}$, then*

$$W^{*}_{\theta,\phi} K^{[n]}_{a} = \sum_{j=0}^{n-1} \overline{\alpha_j(a)} K^{[j]}_{a} + \overline{\theta(a)(\phi^{'}(a))^{n}} K^{[n]}_{a}$$

where "a" is a Denjoy- Wolff point of the composition map ϕ is inside the open unit disk and $K^{[n]}_{a}$ is the n^{th} derivative evaluation kernel at a.

Proof. Let $f \in H^2(\beta)$, Then

$$\langle f, W^{*}_{\theta,\phi} K^{[n]}_{a}\rangle = \langle W_{\theta,\phi} f, K^{[n]}_{a}\rangle \tag{2.39}$$

$$= \langle \theta f o\phi, K^{[n]}_{a}\rangle \tag{2.40}$$

Now differentiate n times the function "$\theta f o\phi$" and evaluate it at a. This gives us,

$$\langle f, W^{*}_{\theta,\phi} K^{[n]}_{a}\rangle = \sum_{j=0}^{n-1} \alpha_j(a) f^{(j)}(a) + \theta(a) f^{(n)}(a)(\phi^{'}(a))^{n} \tag{2.41}$$

Now (2.41) can be written as

$$\langle f, W_{\theta,\phi}^{*} K_a^{[n]}\rangle = \sum_{j=0}^{n-1} \alpha_j(a)\langle f, K_a^{[j]}\rangle + \theta(a)(\phi^{'}(a))^n \langle f, K_a^{[n]}\rangle \tag{2.42}$$

By using the properties of the inner product on the right hand side of (2.42) we get,

$$\langle f, W_{\theta,\phi}^{*} K_a^{[n]}\rangle = \langle f, \sum_{j=0}^{n-1} \overline{\alpha_j(a)} K_a^{[j]} + \overline{\theta(a)(\phi'(a))^n} K_a^{[n]}\rangle \tag{2.43}$$

Since f is arbitrary, which shows that

$$W_{\theta,\phi}^{*} K_a^{[n]} = \sum_{j=0}^{n-1} \overline{\alpha_j(a)} K_a^{[j]} + \overline{\theta(a)(\phi'(a))^n} K_a^{[n]}$$

. □

Note: For if $\theta(z) = 1$, then $W_{\theta,\phi} = C_\phi$. In this case we have

$$C_\phi^{*} K_a^{[n]} = \sum_{j=0}^{n-1} \overline{\alpha_j(a)} K_a^{[j]} + \overline{(\phi'(a))^n} K_a^{[n]} \tag{2.44}$$

Theorem 2.14. *Suppose that the inducing map ϕ satisfies $\phi(a) = 0$ for some $a \in \mathbb{D}$. If $W_{\theta,\phi}^d$ is unitary, then*

$$\theta = c\frac{K_a \beta_1}{||K_a||} \quad \textit{where} \quad |c| = 1 \tag{2.45}$$

Proof. Suppose that $W_{\theta,\phi}^d$ is unitary, then

$$W_{\theta,\phi}^d W_{\theta,\phi}^{d*} K_a = K_a$$

By using theorem [2.8], we have

$$W_{\theta,\phi}^d \left(\bar{\theta}(a) K_{\phi(a)}^{[1]}\right) = K_a$$

$$W_{\theta,\phi}^d \left(\bar{\theta}(a) K_0^{[1]}\right) = K_a$$

$$W_{\theta,\phi}^d \left(\bar{\theta}(a) \frac{z}{\beta_1^2}\right) = K_a$$

$$\frac{\bar{\theta}(a)}{\beta_1^2} W_{\theta,\phi}^d z = K_a$$

$$\frac{\bar{\theta}(a)}{\beta_1^2}\theta(z) = K_a$$

Therefore

$$\theta(z) = \frac{K_a\beta_1^2}{\bar{\theta}(a)} \tag{2.46}$$

Now for $z = a$

$$\theta(\bar{a})\theta(a) = K_a(a)\beta_1^2$$
$$|\theta(a)|^2 = ||K_a||^2\beta_1^2$$

Hence we have

$$\theta(a) = ||K_a||\beta_1 \tag{2.47}$$

Using equation [2.46] and [2.47], we have

$$\theta = c\frac{K_a\beta_1}{||K_a||}$$

□

2.4 Differential and Anti-Differential operators on weighted Hardy spaces

Theorem 2.15. *Let $f \in H^2(\beta)$. Then the anti-differential operator $D_a : H^2(\beta) \to H^2(\beta)$ is bounded iff the sequence $\frac{\beta_{n+1}}{(n+1)\beta_n}$ where $n \in N_0$ is a bounded sequence.*

Proof. It is given that f is an element of $H^2(\beta)$.Therefore $f(z) = \sum_{n=0}^{\infty} f_n z^n$.
Also let us assume that there exist $M > 0$ such that $\frac{\beta_{n+1}}{(n+1)\beta_n} \leq M$.
Then

$$\begin{aligned}||D_af||^2 &= \sum_{n=0}^{\infty}|\frac{f_n}{n+1}|^2\beta_{n+1}^2 \\ &= \sum_{n=0}^{\infty}\frac{|f_n|^2}{(n+1)^2}\frac{\beta_{n+1}^2}{\beta_n^2}\beta_n^2 \\ &\leq M^2\sum_{n=0}^{\infty}|f_n|^2\beta_n^2 \\ &= M^2||f||^2\end{aligned}$$

hence we have

$$||D_a f|| \leq M||f||$$

Conversely let us assume that D_a is bounded, then there exist a real number $M > 0$ such that

$$||D_a \hat{e_n}|| \leq M \quad where \quad \hat{e_n} = \frac{z^n}{\beta_n}$$

which implies that

$$\frac{\beta_{n+1}}{(n+1)\beta_n}||\hat{e}_{n+1}|| \leq M$$

Hence the result. □

We know that in general the differential operator $D : C^1[a, b] \to C^1[a, b]$, where $C^1[a, b]$ is the Banach space of continuously differential functions is not a bounded operator. However, in case of weighted Hardy space it can be bounded as shown in the following theorem.

Theorem 2.16. *Let $f \in H^2(\beta)$. Then the differential operator $D : H^2(\beta) \to H^2(\beta)$ is bounded iff the sequence $\frac{(n+1)\beta_n}{\beta_{n+1}}$ where $n \in N_0$ is a bounded sequence.*

Proof. Let us suppose that the condition is true i.e. $\frac{(n+1)\beta_n}{\beta_{n+1}}$ is a bounded sequence, then there exist $M > 0$ such that $\frac{(n+1)\beta_n}{\beta_{n+1}} \leq M$

Now for any $f \in H^2(\beta)$, we have

$$\begin{aligned}||Df||^2 &= \sum_{n=0}^{\infty} |(n+1)f_{n+1}|^2 \beta_n^2 \\ &= \sum_{n=0}^{\infty} (n+1)^2 |f_n|^2 \frac{\beta_n^2}{\beta_{n+1}^2} \beta_{n+1}^2 \\ &\leq M^2 \sum_{n=0}^{\infty} |f_{n+1}|^2 \beta_{n+1}^2 \\ &= M^2 ||f||^2\end{aligned}$$

Conversely, let us suppose that the differential operator D is bounded, therefore for any $f \in H^2(\beta)$ we have

$$||D\hat{e}_n|| \leq M \qquad for \;\; some \;\; real \;\; no. \quad M > 0$$

which shows that

$$\frac{(n+1)\beta_n}{\beta_{n+1}} \leq M$$

Hence $\{\frac{(n+1)\beta_n}{\beta_{n+1}}\}$ is a bounded sequence. ☐

Theorem 2.17. *Let $f \in H^2(\beta)$. Then*

$$D_a^* f = \sum_{n=0}^{\infty} \frac{f_{n+1}}{(n+1)} \left(\frac{\beta_{n+1}}{\beta_n}\right)^2 z^n$$

where D_a^* is the adjoint of D_a.

Proof. For any $n \in N_0$, Consider

$$\begin{aligned} \langle D_a^* e_{n+1}, f\rangle &= \langle e_{n+1}, D_a f\rangle \\ &= \frac{f_n}{n+1}\beta_{n+1}^2 \\ &= \frac{f_n}{n+1}\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 \beta_n^2 \\ &= \frac{1}{n+1}\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 \langle e_n, f\rangle \quad \forall \quad f \in H^2(\beta) \end{aligned}$$

Therefore

$$D_a^* e_{n+1} = \frac{1}{n+1}\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 e_n \quad and \quad D_a^* e_0 = 0 \tag{2.48}$$

Now for $\quad f = \sum_{n=0}^{\infty} f_n e_n$

$$\begin{aligned} D_a^* f &= \sum_{n=0}^{\infty} f_n D_a^* e_n \\ &= f_0 D_a^* e_0 + \sum_{n=0}^{\infty} f_{n+1} D_a^* e_{n+1} \\ &= \sum_{n=0}^{\infty} \frac{f_{n+1}}{n+1}\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 e_n \qquad by \ using \ equation \ (2.48) \\ &= \sum_{n=0}^{\infty} \frac{f_{n+1}}{n+1}\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 z^n \end{aligned}$$

This completes the proof. ☐

Theorem 2.18. *Let $D_a \in B(H^2(\beta))$. Then D_a is Hermitian if and only if $\left(\frac{\beta_n}{n\beta_{n-1}}\right) \quad \forall \quad n \in \mathbb{N}$ is a constant sequence.*

Proof. Suppose that D_a is Hermitian. Then for $f = e_n$, we have

$$||D_a^* e_n|| = ||D_a e_n|| \quad \forall \quad n \in \mathbb{N}$$

$$||\frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1}|| = ||\frac{e_{n+1}}{n+1}|| \quad by \quad using \quad equation \quad (2.48)$$

$$\frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 \beta_{n-1} = \frac{\beta_{n+1}}{n+1}$$

$$\frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right) = \frac{\beta_{n+1}}{(n+1)\beta_n} \tag{2.49}$$

which shows that $\left(\frac{\beta_n}{n\beta_{n-1}}\right) \forall \quad n \in \mathbb{N}$ is a constant sequence

Conversely, let us suppose that $\left(\frac{\beta_n}{n\beta_{n-1}}\right) \quad \forall \quad n \in \mathbb{N}$ is a constant sequence

Then for $f = \sum_{n=1}^{\infty} f_n e_n$, consider

$$\begin{aligned}
||D_a^* f||^2 &= \langle D_a^* f, D_a^* f\rangle \\
&= \langle \sum_{n=1}^{\infty} \frac{f_n}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1}, \sum_{n=1}^{\infty} \frac{f_n}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1}\rangle \\
&= \sum_{n=1}^{\infty} |f_n|^2 \frac{1}{n^2}\left(\frac{\beta_n}{\beta_{n-1}}\right)^4 \beta_{n-1}^2 \\
&= \sum_{n=1}^{\infty} |f_n|^2 \left(\frac{\beta_{n+1}}{(n+1)\beta_n}\right)^2 \beta_n^2 \quad using \quad equation \quad [2.49] \\
&= \sum_{n=1}^{\infty} |f_n|^2 \left(\frac{\beta_{n+1}}{(n+1)}\right)^2 \\
&= ||D_a f||^2
\end{aligned}$$

Hence D_a is a Hermitian. □

Theorem 2.19. *Let $D_a \in B(H^2(\beta))$. Then D_a is Normal if and only if $\left(\frac{\beta_n}{n\beta_{n-1}}\right) \quad \forall \quad n \in \mathbb{N}$ is a constant sequence.*

Proof. Let us suppose first that D_a is a normal operator, then for $f = e_n$ consider

$$\begin{aligned} D_a D_a^* e_n &= D_a \frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1} \\ &= \frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 \frac{e_n}{n} \\ &= \frac{1}{n^2}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_n \\ &= \left(\frac{\beta_n}{n\beta_{n-1}}\right)^2 e_n \end{aligned} \tag{2.50}$$

$$\begin{aligned} D_a^* D_a e_n &= D_a^*\left(\frac{e_{n+1}}{n+1}\right) \\ &= \frac{1}{n+1}\left(\frac{1}{n+1}\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 e_n\right) \\ &= \left(\frac{\beta_{n+1}}{(n+1)\beta_n}\right)^2 e_n \end{aligned} \tag{2.51}$$

Hence form equations (2.50) and (2.51) we have,

$$\left(\frac{\beta_{n+1}}{(n+1)\beta_n}\right) = \left(\frac{\beta_n}{n\beta_{n-1}}\right)$$

which shows that $\left(\frac{\beta_n}{n\beta_{n-1}}\right)$ is a constant sequence.
Conversely suppose that $\left(\frac{\beta_n}{n\beta_{n-1}}\right)$ is a constant sequence, let $\frac{\beta_n}{n\beta_{n-1}} = c \quad \forall \quad n \ in \ \mathbb{N}$
Then

$$\begin{aligned} D_a D_a^* e_n &= D_a\left(\frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1}\right) \\ &= \frac{1}{n^2}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_n \\ &= c^2 e_n \end{aligned} \tag{2.52}$$

$$\begin{aligned} D_a^* D_a e_n &= D_a^*\left(\frac{e_{n+1}}{n+1}\right) \\ &= \frac{1}{(n+1)^2}\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 e_n \\ &= c^2 e_n \end{aligned} \tag{2.53}$$

Form equations (2.52) and (2.53), we can see that

$$D_a D_a^* e_n = D_a^* D_a e_n$$

or

$$D_a D_a^* = D_a^* D_a$$

which proves that D_a is normal. □

Theorem 2.20. *Let $D_a \in B(H^2(\beta))$. Then D_a^* is quasinormal if and only if $\left(\frac{\beta_n}{n\beta_{n-1}}\right) \quad \forall \quad n \in \mathbb{N}$ is a constant sequence.*

Proof. Suppose that D_a^* is quasi normal, then for $f = e_{n+1}$ we have

$$D_a D_a^* D_a^* e_{n+1} = D_a^* D_a D_a^* e_{n+1}$$

Consider

$$\begin{aligned} D_a D_a^* D_a^* e_{n+1} &= D_a D_a^* \left[\frac{1}{n+1} \left(\frac{\beta_{n+1}}{\beta_n} \right)^2 e_n \right] \\ &= D_a \frac{1}{n+1} \left(\frac{\beta_{n+1}}{\beta_n} \right)^2 \left[\frac{1}{n} \left(\frac{\beta_n}{\beta_{n-1}} \right)^2 e_{n-1} \right] \\ &= \frac{1}{n(n+1)} \left(\frac{\beta_{n+1}}{\beta_n} \right)^2 \left(\frac{\beta_n}{\beta_{n-1}} \right)^2 \frac{e_n}{n} \\ &= \frac{1}{n^2(n+1)} \left(\frac{\beta_{n+1}}{\beta_n} \right)^2 \left(\frac{\beta_n}{\beta_{n-1}} \right)^2 e_n \end{aligned} \tag{2.54}$$

Again let us consider

$$\begin{aligned} D_a^* D_a D_a^* e_{n+1} &= D_a^* D_a \left[\frac{1}{n+1} \left(\frac{\beta_{n+1}}{\beta_n} \right)^2 e_n \right] \\ &= D_a^* \left[\frac{1}{n+1} \left(\frac{\beta_{n+1}}{\beta_n} \right)^2 \frac{e_{n+1}}{n+1} \right] \\ &= \frac{1}{(n+1)^2} \left(\frac{\beta_{n+1}}{\beta_n} \right)^2 \frac{1}{(n+1)} \left(\frac{\beta_{n+1}}{\beta_n} \right)^2 e_n \\ &= \frac{1}{(n+1)^3} \left(\frac{\beta_{n+1}}{\beta_n} \right)^4 e_n \end{aligned} \tag{2.55}$$

From equation (2.54) and (2.55), we have

$$\frac{1}{n^2(n+1)}\left(\frac{\beta_{n+1}}{\beta_n}\right)^2\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_n = \frac{1}{(n+1)^3}\left(\frac{\beta_{n+1}}{\beta_n}\right)^4 e_n$$

which implies that $\frac{1}{n}\frac{\beta_n}{\beta_{n-1}} = \frac{1}{n+1}\frac{\beta_{n+1}}{\beta_n}$ which shows that $\left(\frac{\beta_n}{n\beta_{n-1}}\right)$ is a constant sequence.

For converse we suppose that $\left(\frac{\beta_n}{n\beta_{n-1}}\right)$ is a constant sequence $for \;\; all \;\; n \in \mathbb{N}$, For this, let $\left(\frac{\beta_n}{n\beta_{n-1}}\right) = c$

Then

$$\begin{aligned} D_a D_a^* D_a^* e_n &= D_a D_a^* \left[\frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1}\right] \\ &= D_a \frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 \left[\frac{1}{n-1}\left(\frac{\beta_{n-1}}{\beta_{n-2}}\right)^2 e_{n-2}\right] \\ &= \frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 \frac{1}{n-1}\left(\frac{\beta_{n-1}}{\beta_{n-2}}\right)^2 \frac{e_{n-1}}{n-1} \\ &= nc^4 e_{n-1} \end{aligned} \tag{2.56}$$

and

$$\begin{aligned} D_a^* D_a D_a^* e_n &= D_a^* D_a \left[\frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1}\right] \\ &= D_a^* \left[\frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 \frac{e_n}{n}\right] \\ &= \frac{1}{n^2}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 \frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1} \\ &= nc^4 e_{n-1} \end{aligned} \tag{2.57}$$

Hence from equation (2.56) and (2.57) we see that D_a^* is an quasinormal operator. □

Theorem 2.21. *Let $D_a \in B(H^2(\beta))$. Then D_a^* is hyponormal if and only if $\left(\frac{\beta_n}{n\beta_{n-1}}\right) \;\; \forall \;\; n \in \mathbb{N}$ is a decreasing sequence.*

Proof. Let us suppose that D_a^* be a hyponormal, Then

$$||D_a^* e_n|| \geq ||D_a e_n||$$

$$||\frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_n|| \geq ||\frac{e_{n+1}}{n+1}||$$

$$\frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 \beta_{n-1} \geq \frac{1}{n+1}\beta_{n+1}$$

$$\frac{1}{n}\frac{\beta_n^2}{\beta_{n-1}} \geq \frac{\beta_{n+1}}{n+1}$$

$$\frac{\beta_n}{n\beta_{n-1}} \geq \frac{\beta_{n+1}}{(n+1)\beta_n} \quad \forall \quad n \in \mathbb{N}$$

which shows that $\left(\frac{\beta_n}{n\beta_{n-1}}\right)$ is a decreasing sequence.
Conversely, let us assume that $\left(\frac{\beta_n}{n\beta_{n-1}}\right)$ is a decreasing sequence $\quad \forall \quad n \in \mathbb{N}$. we will show that D_a^* is a hyponormal
Consider

$$\begin{aligned} D_a^* f &= \sum_{n=1}^{\infty} f_n D_a^* e_n \\ &= \sum_{n=1}^{\infty} f_n \frac{1}{n}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 e_{n-1} \end{aligned} \tag{2.58}$$

and

$$D_a f = \sum_{n=0}^{\infty} f_n \frac{e_{n+1}}{n+1} \tag{2.59}$$

Then

$$\begin{aligned} ||D_a^* f||^2 &= \sum_{n=1}^{\infty} |f_n|^2 \frac{1}{n^2}\left(\frac{\beta_n}{\beta_{n-1}}\right)^4 \beta_{n-1}^2 \\ &= \sum_{n=1}^{\infty} |f_n|^2 \left(\frac{1}{n^2}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2 \beta_n^2\right) \end{aligned} \tag{2.60}$$

Since $\left(\frac{\beta_n}{n\beta_{n-1}}\right)$ is a decreasing sequence $\quad\forall \quad n \in \mathbb{N}$, therefore

$$\begin{aligned}\sum_{n=1}^{\infty}|f_n|^2\left(\frac{1}{n^2}\left(\frac{\beta_n}{\beta_{n-1}}\right)^2\beta_n^2\right) &\geq \sum_{n=0}^{\infty}|f_n|^2\left(\frac{\beta_{n+1}}{(n+1)\beta_n}\right)^2\beta_n^2\\ &= \sum_{n=0}^{\infty}|f_n|^2\frac{\beta_{n+1}^2}{(n+1)^2}\\ &= ||\sum_{n=0}^{\infty} f_n D_a e_n||^2\\ &= ||\sum_{n=0}^{\infty} D_a(f_n e_n)||^2\\ &= ||D_a f||^2\end{aligned} \tag{2.61}$$

which implies that $||D_a^* f|| \geq ||D_a f||$. Hence D_a^* is a Hyponormal operator. $\square$

Chapter 3

Generalized Weighted Composition Operators on Weighted Hardy Spaces[2]

3.1 Bounded and Compact generalized weighted composition operators on weighted Hardy spaces

Boundedness of a generalized composition operators on the weighted Hardy space is characterized in Sharma and Komal[56]. In this chapter we will see that under what conditions on inducing maps generalized weighted composition operators becomes bounded and compact. Further, by using the examples the concept is also discussed. It has been found that generalized weighted composition operator on weighted Hardy space is not isometric. Hermitian and Fredholm generalized weighted composition operator is also discussed in this chapter.

Theorem 3.1. *Let $\theta : \mathbb{D} \to \mathbb{C}$ and $\phi : \mathbb{D} \to \mathbb{D}$ be two mappings such that $\{\theta.\phi^n : n \in N_0\}$ is a orthogonal family. Then $W^d_{\theta,\phi} : H^2(\beta) \to H^2(\beta)$ is bounded if and only if $\exists M > 0$ such that $||\theta.\phi^{n-1}|| \leq \frac{M}{n}\beta_n$ for all $n \in \mathbb{N}$, where $\phi^n(z) = (\phi(z))^n$.*

[2]Results of this chapter is communicated in Scopus indexed Journal.

Proof. We first assume that $W^d_{\theta,\phi}$ is a bounded operator,
then $\exists\ M > 0$ such that

$$||W^d_{\theta,\phi}\hat{e}_n|| \leq M \quad \forall\ n \in \mathbb{N}, \qquad \text{where} \qquad \hat{e_n} = \frac{e_n}{\beta_n}$$

the above inequality becomes

$$n||W_{\theta,\phi}\frac{e_{n-1}}{\beta_n}|| \leq M \quad \forall\ \ n \in \mathbb{N}$$

equivalently above can be written as

$$\frac{n}{\beta_n}||\theta.\phi^{n-1}|| \leq M \quad \forall\ \ n \in \mathbb{N}$$

Conversely we assume that the condition of the theorem is satisfied.
Take $\quad f \in H^2(\beta)$,
then for $\quad f = \sum_{n=0}^{\infty} f_n\hat{e}_n$
we have $\quad ||f||^2 = \sum_{n=0}^{\infty} |f_n|^2$
consider

$$\begin{aligned} ||W^d_{\theta,\phi}f||^2 &= ||\sum_{n=1}^{\infty} \frac{nf_n}{\beta_n} W_{\theta,\phi}e_{n-1}||^2 \\ &= \sum_{n=1}^{\infty} \frac{n^2|f_n|^2}{\beta_n^2}||\theta.\phi^{n-1}||^2 \\ &\leq M^2||f||^2 \end{aligned}$$

This proves that $W^d_{\theta,\phi}$ is a bounded operator. □

Theorem 3.2. *Let $W^d_{\theta,\phi} \in B(H^2(\beta))$. Suppose $\{\theta.\phi^{n-1} : n \in \mathbb{N}\}$ is an orthogonal family. Then $W^d_{\theta,\phi}$ is a compact operator if and only if $\frac{n||\theta.\phi^{n-1}||}{\beta_n} \to 0$ as $n \to \infty$.*

Proof. Suppose $W^d_{\theta,\phi}$ is a compact operator,
then $\quad \hat{e}_n \to 0 \quad$ weakly, $\quad$ where $\quad \hat{e_n} = \frac{e_n}{\beta_n}$
Therefore

$$||W^d_{\theta,\phi}\hat{e}_n|| \to 0 \quad as\ n \to \infty$$

or that

$$\frac{n}{\beta_n}||\theta.\phi^{n-1}|| \to 0 \quad as\ n \to \infty$$

Conversely suppose that condition is true.
Then for given $\epsilon > 0$ there exist $m \in \mathbb{N}$ such that

$$\frac{n||\theta.\phi^{n-1}||}{\beta_n} < \epsilon \ \ \forall \ n \geq m$$

Let

$$f = \sum_{n=0}^{\infty} f_n \hat{e}_n$$

For $m \in \mathbb{N}$, define

$$A_m f = \sum_{n=1}^{m} \frac{n f_n \theta.\phi^{n-1}}{\beta_n}$$

Clearly A_m is a compact operator.
Consider

$$\begin{aligned}
||W^d_{\theta,\phi} f - A_m f||^2 &= ||\sum_{n=1}^{\infty} \frac{n f_n}{\beta_n}(\theta.\phi^{n-1}) - \sum_{n=1}^{m} \frac{n f_n}{\beta_n}(\theta.\phi^{n-1})||^2 \\
&= \sum_{n=m+1}^{\infty} n^2 |f_n|^2 \frac{||\theta.\phi^{n-1}||^2}{\beta_n^2} \\
&\leq \epsilon^2 \sum_{n=m+1}^{\infty} |f_n|^2 \\
&\leq \epsilon^2 ||f||^2
\end{aligned}$$

This is true for every $f \in H^2(\beta)$, so on taking supremum over all $f(\neq 0) \in H^2(\beta)$, we obtain

$$||W^d_{\theta,\phi} - A_m|| < \epsilon$$

Hence $W^d_{\theta,\phi}$ is a compact operator being the limit of compact operators. □

Example 3.1. *Let $\theta : \mathbb{D} \to \mathbb{C}$ and $\phi : \mathbb{D} \to \mathbb{D}$ be defined by $\theta(z) = z$ and $\phi(z) = \frac{z}{2}$ for every $z \in \mathbb{D}$.*
Let $\beta_n = n$ and $H^2(\beta)$ is a Hilbert space of analytic function on the unit disc. We have

$$\begin{aligned}
||\theta.\phi^{n-1}|| &= ||e_1.(\frac{1}{2}e_1)^{n-1}|| \\
&= ||\frac{1}{2^{n-1}} e_1^n|| \\
&= \frac{n}{2^{n-1}}
\end{aligned}$$

Now $\frac{n}{\beta_n}||\theta.\phi^{n-1}|| = ||\theta.\phi^{n-1}|| = \frac{n}{2^{n-1}} \quad \to 0 \text{ as } \quad n \to \infty$
Therefore $W^d_{\theta,\phi} : H^2(\beta) \to H^2(\beta)$ *is a compact operator.*

3.2 Hermitian and Isometric generalized weighted composition operators on weighted Hardy spaces

In this section we obtain an interesting result that the only Hermitian weighted composition operator is the zero operator, which is not true in the earlier known Hermitian weighted composition operator on other function spaces. It is shown that generalized weighted composition operator is not isometric.

Theorem 3.3. *Let* $W^d_{\theta,\phi} \in B(H^2(\beta))$*, also let* $\theta(1) = 0$ *and* $\phi(0) = 0$*. Then* $W^d_{\theta,\phi}$ *is Hermitian operator if and only if* $W^d_{\theta,\phi} = 0$*.*

Proof. We first suppose that $W^d_{\theta,\phi}$ is an Hermitian operator.
Then

$$\langle W^{d*}_{\theta,\phi} e_1, e_0\rangle = \langle W^d_{\theta,\phi} e_1, e_0\rangle$$

which implies

$$\langle e_1, W^d_{\theta,\phi} e_0\rangle = \langle \theta.e_0 o\phi, e_0\rangle$$
$$\langle e_1, 0\rangle = \langle \theta, e_0\rangle$$
$$0 = \theta(0)$$

hence

$$\theta(0) = 0$$

Again, let us consider

$$\langle W^{d*}_{\theta,\phi} e_2, e_1\rangle = \langle W^d_{\theta,\phi} e_2, e_1\rangle$$

which implies that

$$\begin{aligned}
\langle e_2, W^d_{\theta,\phi} e_1\rangle &= \langle \theta.2e_1 o\phi, e_1\rangle \\
\langle e_2, \theta.e_0 o\phi\rangle &= 2\langle \theta.e_1 o\phi, e_1\rangle \\
\langle e_2, \theta\rangle &= 2\langle \theta.\phi, e_1\rangle \\
&= 2\theta(1)\phi(0) \\
\overline{\theta(2)}\beta_2^2 &= 0 \\
\theta(2) &= 0
\end{aligned}$$

Next consider

$$\begin{aligned}
\langle W^{d*}_{\theta,\phi} e_n, e_1\rangle &= \langle W^d_{\theta,\phi} e_n, e_1\rangle \\
\langle e_n, W^d_{\theta,\phi} e_1\rangle &= \langle n\theta.\phi^{n-1}, e_1\rangle \\
\langle e_n, \theta.e_0 o\phi\rangle &= \langle n\theta.\phi^{n-1}, e_1\rangle \quad for\ n \geq 3 \\
\langle e_n, \theta\rangle &= \langle n\theta.\phi^{n-1}, e_1\rangle \quad for\ n \geq 3 \\
\overline{\theta(n)}\beta_n^2 &= 0 \quad for\ n \geq 3
\end{aligned}$$

Therefore

$$\overline{\theta(n)} = 0$$

or

$$\theta(n) = 0 \;\; \forall\, n \in \mathbb{N}$$

Thus $W^d_{\theta,\phi} = 0$. The proof of the converse part is trivial. □

Theorem 3.4. *Let $W^d_{\theta,\phi} \in B(H^2(\beta))$. Then $W^d_{\theta,\phi}$ is not an isometry.*

Proof. If possible, suppose $W^d_{\theta,\phi}$ is an isometry. Then

$$||W^d_{\theta,\phi} f|| = ||f|| \text{ for every } f \in H^2(\beta)$$

Taking $f = e_0$, we get

$$||W^d_{\theta,\phi} e_0|| = 0$$

and

$$||e_0|| = \beta_0$$

which implies that $\beta_0 = 0$. Which is not possible.
Hence $W^d_{\theta,\phi}$ is never an isometry. □

3.3 Fredholm generalized weighted composition operators on weighted Hardy spaces

A necessary and sufficient condition for a generalized weighted composition operator to be Fredholm is investigated in this section.

Theorem 3.5. *Let $\theta : \mathbb{D} \to \mathbb{C}$ and $\phi : \mathbb{D} \to \mathbb{D}$ be two mappings such that $\{\theta.\phi^{n-1} : n \in \mathbb{N}\}$ is an orthogonal family. Then $W^d_{\theta,\phi}$ has closed range if and only if there exists $\epsilon > 0$ such that*

$$\frac{n}{\beta_n}||\theta.\phi^{n-1}|| \geq \epsilon \quad \forall \quad n \in \mathbb{N}$$

Proof. We first assume that $W^d_{\theta,\phi}$ has closed range. Then $W^d_{\theta,\phi}$ is bounded away from zero on $(kerW^d_{\theta,\phi})^{\perp}$. Therefore there exists $\epsilon > 0$ such that

$$\begin{aligned} ||W^d_{\theta,\phi}e_n|| \geq \epsilon||e_n|| \quad &\forall \quad n \in \mathbb{N} \\ ||n \ \ \theta.\phi^{n-1}|| \geq \epsilon\beta_n \quad &\forall \quad n \in \mathbb{N} \end{aligned}$$

which implies that

$$\frac{n}{\beta_n}||\theta.\phi^{n-1}|| \geq \epsilon \quad \forall \quad n \in \mathbb{N}$$

Conversely for every $f \in H^2(\beta)$, we have

$$\begin{aligned} ||W^d_{\theta,\phi}f||^2 &= ||\sum_{n=0}^{\infty} f_n W^d_{\theta,\phi}e_n||^2 \\ &= \sum_{n=0}^{\infty} n^2|f_n|^2||\theta.\phi^{n-1}||^2 \\ &\geq \epsilon^2\sum_{n=0}^{\infty}|f_n|^2\beta_n^2 \\ &= \epsilon^2||f||^2 \text{ for every } f \in (ker\ W^d_{\theta,\phi})^{\perp} \end{aligned}$$

Thus $W^d_{\theta,\phi}$ is bounded away from zero on $(kerW^d_{\theta,\phi})^{\perp}$. Hence $W^d_{\theta,\phi}$ has closed range. ☐

Example 3.2. *Let $\theta = e_1$ and $\phi : \mathbb{D} \to \mathbb{D}$ be define by $\phi(z) = z \quad \forall \ z \in \mathbb{D}$*
Then

$$\frac{n}{\beta_n}||\theta.\phi^{n-1}|| = \frac{n}{\beta_n}||e_n|| = n \geq 1 \quad \forall \ n \in \mathbb{N}$$

Hence $W^d_{\theta,\phi}$ has closed range.

Theorem 3.6. *Let $\theta : \mathbb{D} \to \mathbb{C}$ and $\phi : \mathbb{D} \to \mathbb{D}$ be two mappings such that $\{\theta.\phi^{n-1} : n \in N\}$ is a basis for $H^2(\beta)$. Then $W^d_{\theta,\phi}$ is Fredholm if and only if there exists $\epsilon > 0$ such that*

$$\frac{n}{\beta_n}||\theta.\phi^{n-1}|| \geq \epsilon \quad \textit{for every} \quad n \in N_0$$

Proof. Suppose that the condition is true. Then in view of the theorem (3.5) $W^d_{\theta,\phi}$ has closed range. Also $kerW^d_{\theta,\phi}$ is finite dimensional. We next show that $kerW^{d^*}_{\theta,\phi}$ is zero dimensional. Let $g \in kerW^{d^*}_{\theta,\phi}$.
Then

$$W^{d^*}_{\theta,\phi} g = 0$$

Therefore for $n \in \mathbb{N}$ we have

$$\begin{aligned} 0 &= \langle W^{d^*}_{\theta,\phi} g, e_n \rangle \\ &= n\langle g, \theta.\phi^{n-1} \rangle \quad \forall \ n \geq 1 \end{aligned}$$

implies that

$$g = 0$$

Thus

$$kerW^{d^*}_{\theta,\phi} = \{0\}$$

This proves that $W^d_{\theta,\phi}$ is Fredholm. The converse is easy to prove. ☐

Example 3.3. *Let $\theta : \mathbb{D} \to \mathbb{C}$ be defined by $\theta(z) = 1$ for all $z \in \mathbb{D}$, $\phi : \mathbb{D} \to \mathbb{D}$ be defined by $\phi(z) = z$, let $\beta_n = n!$.*
Then

$$\theta.\phi^{n-1} = e_{n-1} \quad for \quad n \geq 1$$

and

$$\frac{n}{\beta_n}||e_{n-1}|| = \frac{n\beta_{n-1}}{\beta_n}$$
$$= \frac{n(n-1)!}{n!} = 1 \quad \forall\ n \in \mathbb{N}$$

Suppose $f \in kerW^d_{\theta,\phi}$
Then

$$W^d_{\theta,\phi}f = 0$$

or

$$kerW^d_{\theta,\phi} = span\{e_0\}$$

and

$$kerW^{d^*}_{\theta,\phi} = \{0\}$$

Also $W^d_{\theta,\phi}$ *has close range in view of theorem [3.6]. Hence* $W^d_{\theta,\phi}$ *is Fredholm.*

Example 3.4. *Let* $\theta : \mathbb{D} \to \mathbb{C}$ *be defined by* $\theta(z) = z^2$ *and* $\phi : \mathbb{D} \to \mathbb{D}$ *be defined by* $\phi(z) = z^2$ *and* $\beta_n = e^{-n}$. *Then* $W^d_{\theta,\phi}$ *is bounded.*
But

$$kerW^{d^*}_{\theta,\phi} = span(\{e_{2n-1} : n \in \mathbb{N}\} \cup \{e_0\})$$

This proves that $kerW^{d^*}_{\theta,\phi}$ *is infinite dimensional. Hence* $W^d_{\theta,\phi}$ *is not Fredholm.*

Theorem 3.7. *Let* $W^d_{\theta,\phi} \in B(H^2(\beta))$. *Suppose* $\{\theta.\phi^{n-1}\}_{n=1}^{\infty}$ *is an orthogonal family. Then* $W^d_{\theta,\phi}$ *has non-trivial invariant subspace.*

Proof. Let M = span$\{e_0\}$. Then $M \subset kerW^d_{\theta,\phi}$
Next if $f \in kerW^d_{\theta,\phi}$, then

$$W^d_{\theta,\phi}f = 0$$

This implies

$$||W^d_{\theta,\phi}f||^2 = 0$$
$$\sum_{n=1}^{\infty} n^2|f_n|^2||\theta.\phi^{n-1}||^2 = 0$$

This implies that

$$f_n = 0 \ \ \forall \ \ n \in \mathbb{N}$$

Hence $f = \alpha e_0$ so that $f \in M$. Thus ker $W^d_{\theta,\phi} = M$, which is invariant under $W^d_{\theta,\phi}$. □

In the last part of this section we have discussed some results on generalized weighted composition operators which can be helpful further in the study of new properties of the operator on weighted Hardy spaces.

Theorem 3.8. *Let θ be analytic on the unit disk and $\phi : \mathbb{D} \to \mathbb{D}$ be analytic map. Let a be a real number and C_a be the composition operator given by $(C_a f)(z) = f(e^{ia}z)$ for every $f \in H^2(\beta)$. The operator C_a is unitary on $H^2(\beta)$ and if $W^d_{\theta,\phi}$ is bounded then*

$$C_a^* W^d_{\theta,\phi} C_a = W^d_{\bar{\theta},\bar{\phi}}$$

where $\bar{\theta}(z) = \theta(e^{-ia}z)$ *and* $\bar{\phi}(z) = e^{ia}\phi(e^{-ia}z)$

Proof. Consider

$$\begin{aligned}
(C_a^* W^d_{\theta,\phi} C_a)(z) &= C_a^* W^d_{\theta,\phi} f(e^{ia}z) \\
&= C_a^*(\theta(z) f^{'}(e^{ia}\phi(z)) \\
&= \theta(e^{-ia}z) f^{'}(e^{ia}\phi(e^{-ia}z)) \\
&= \bar{\theta}(z) f^{'}(\bar{\phi}(z)) \\
&= (W^d_{\bar{\theta},\bar{\phi}} f)(z)
\end{aligned}$$

Hence the result. □

Theorem 3.9. *Let $W^d_{\theta,\phi} \in B(H^2(\beta))$. Then $W^d_{\theta,\phi} = W^d_{\phi,\theta}$ iff $\theta = \theta o \phi$.*

Proof. Let $f \in H^2(\beta)$ such that $W^d_{\theta,\phi} f = W^d_{\phi,\theta} f$
For $f = e_n$,we have $W^d_{\theta,\phi} e_n = W^d_{\phi,\theta} e_n$ for every $n \in \mathbb{N}$ particularly take $n = 1$
then we will get

$$\begin{aligned}
W^d_{\theta,\phi} e_1 &= W^d_{\phi,\theta} e_1 \\
\theta.(e_0 o \phi) &= (\theta.e_0) o \phi \\
\theta.e_0 &= \theta o \phi \\
\theta &= \theta o \phi
\end{aligned}$$

Conversely, let us suppose that $\theta = \theta o \phi$ Then

$$\begin{aligned} W^d_{\theta,\phi} f &= \theta.(f^{'} o \phi) \\ &= (\theta o \phi).(f^{'} o \phi) \\ &= (\theta.f^{'}) o \phi \\ &= W^d_{\phi,\theta} f \end{aligned}$$

Hence proved. □

Chapter 4

Spectra of Generalized Composition Operators and Generalized Multiplication Operators on Weighted Hardy Spaces[3]

It was shown by Gunatillake[17] that the spectrum of weighted composition operator $W_{\theta,\phi}$ on weighted Hardy space $H^2(\beta)$ is contained in the set

$$\{0, \theta(a), \theta(a)\phi^{'}(a), \theta(a)(\phi^{'}(a))^2, ...\}$$

. Cowen, Gunatillake and Ko[12] studied Hermitian weighted composition operators on weighted Hardy spaces in which they characterized the adjoint of weighted composition operator and also discussed the eigen value of the operator. In this chapter, we will see how the spectrum of weighted composition operator $W_{\theta,\phi}$ is found under the assumption that $W_{\theta,\phi}$ is compact, with ϕ having fixed point inside the open unit disk. Using the idea, we have discussed the results which can help us to characterize the eigen values of generalized composition operator C_ϕ^d, generalized multiplication operator M_θ^d on weighted Hardy space $H^2(\beta)$.

[3]Results of this paper has accepted in the AIP conference proceedings.

4.1 Spectra of generalized composition operators

In this section we have discussed the eigen value problem for generalized composition operator on weighted Hardy space.

Theorem 4.1 (Gunatillake). *Suppose $W_{\theta,\phi}$ is a compact operator on the weighted Hardy space $H^2(\beta)$. If the Denjoy-Wolff point "a" of the composition map ϕ is inside the open unit disk then the set*

$$\{0, \theta(a), \theta(a)\phi^{'}(a), \theta(a)(\phi^{'}(a))^2, \theta(a)(\phi^{'}(a))^3, ...\}$$

contains the spectrum.

Proof. It is well known that a compact operator on an infinite dimensional space is not invertible hence 0 is in the spectrum.
Let $\lambda \neq 0$ be an eigen value for $W_{\theta,\phi}$. Therefore there exist a function call it $f(z) \neq 0$ holomorphic on unit disk such that

$$W_{\theta,\phi} f(z) = \lambda f(z)$$

$$\theta(z) f(\phi(z)) = \lambda f(z) \tag{4.1}$$

Let f have a zero of order n at a, i.e. $f^n(a) \neq 0$ and $f(a) = f^{'}(a) = ... = f^{n-1}(a) = 0$. If $n = 0$, then at $z = a$ from equation(4.1) we have $\lambda = \theta(a)$. For $n \geq 0$, differentiate the equation(4.1) n times, then

$$\sum_{i=0}^{n-1} \alpha_i(z) f^{(i)}(\phi(z)) + \theta(z) f^{(n)}(\phi(z))(\phi^{'}(z))^n = \lambda f^{(n)}(z) \tag{4.2}$$

where $f^{(n)}$ stands for the n^{th} derivative of f and ${\alpha_j}^s$ are functions which consists of various products of derivative of θ and ϕ. Now let us take $z = a$, Since f has a zero of order of n at a, so the equation(4.2) becomes

$$\sum_{i=0}^{n-1} \alpha_i(a) f^{(i)}(\phi(a)) + \theta(a) f^{(n)}(\phi(a))(\phi^{'}(a))^n = \lambda f^{(n)}(a) \tag{4.3}$$

using that a is a fixed point, finally we have $\lambda = \theta(a)(\phi'(a))^n$. So the above computations shows that only possible eigenvalues are of the form $\theta(a)(\phi'(a))^n$.

□

Theorem 4.2. *Let $C_\phi^d \in B(H^2(\beta))$. Let λ is an eigen value of C_ϕ^d if $|\lambda| < \limsup \left(\frac{n!}{\beta_n}\right)^{\frac{1}{n}}$, where $\phi(z) = z$ for every z in open unit disk.*

Proof. Since λ is an eigen value of C_ϕ^d therefore there exist $f \neq 0$ such that

$$(C_\phi^d f)(z) = \lambda f(z) \tag{4.4}$$

Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$. Then $C_\phi^d f(z) = \sum_{n=0}^{\infty}(n+1)f_{n+1}z^n$ Using equation (4.4) we get

$$f(z) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f_o z^n$$

Consider

$$||f||^2 = \sum_{n=0}^{\infty} |f_n|^2 \beta_n^2 = \sum_{n=0}^{\infty} |\lambda^n/n!|^2 |f_o|^2 \beta_n^2$$

Therefore the series in the above equation converges if

$$\limsup \left(|\lambda^n/n!|^2 \beta_n^2\right)^{\frac{1}{n}} < 1$$

which implies that

$$|\lambda| < \limsup (n!/\beta_n)^{\frac{1}{n}}$$

Hence the result. □

Theorem 4.3. *Let $C_\phi^d \in B(H^2(\beta))$. Let λ is an eigen value of C_ϕ^d if $|\lambda| < \limsup \left(\frac{n! a^{\frac{n(n+1)}{2}}}{\beta_n}\right)^{\frac{1}{n}}$, where $\phi(z) = az$ for every z in open unit disk and a be any real number.*

Proof. Since λ is an eigen value of C_ϕ^d therefore there exist $f \neq 0$ such that

$$(C_\phi^d f)(z) = \lambda f(z) \tag{4.5}$$

Let $f(z)=\sum_{n=0}^{\infty} f_n z^n$.

Then $C_\phi^d f(z)=\sum_{n=0}^{\infty}(n+1)f_{n+1}a^n z^n$

Using equation (4.5) we get

$$f(z)=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!a^{\frac{n(n+1)}{2}}}f_o z^n$$

Consider

$$||f||^2=\sum_{n=0}^{\infty}|f_n|^2\beta_n^2=\sum_{n=0}^{\infty}|\lambda^n/n!a^{\frac{n(n+1)}{2}}|^2|f_o|^2\beta_n^2$$

Therefore the series in the above equation converges if

$$\limsup\left(|\lambda^n/n!a^{\frac{n(n+1)}{2}}|^2\beta_n^2\right)^{\frac{1}{n}}<1$$

which implies that

$$|\lambda|<\limsup\left(\frac{n!a^{\frac{n(n+1)}{2}}}{\beta_n}\right)^{\frac{1}{n}}$$

Hence the result. □

4.2 Spectra of generalized multiplication operators

In this section we will find the spectra of multiplication operators, generalized multiplication operators on weighted Hardy spaces.

Theorem 4.4. *Let M_θ be a multiplication operator in $H^2(\beta)$. Then $\Pi_0(M_\theta)\subseteq\{\theta(0)\}$ is an eigen value of M_θ iff $\theta(n)=0$ for every $n\geq 1$, where $\Pi_0(M_\theta)$ denotes the set of eigen values of M_θ. The reverse inclusion hold if θ is a constant function.*

Proof. Suppose λ is an eigen value of M_θ, Then there exist $f\in H^2(\beta)$, such that

$$M_\theta f=\lambda f$$

which implies that

$$\theta f = \lambda f$$

But

$$(\theta f)(n) = \sum_{k=0}^{n} \theta(n-k) f(k) \tag{4.6}$$

$$\sum_{k=0}^{n} \theta(n-k) f(k) = \lambda f(n) \tag{4.7}$$

For $n = 0$, we have

$$\theta_0 f_0 = \lambda f_0$$

Then

$$\lambda = \theta_0 \qquad if \qquad f_0 \neq 0$$

In case, $f_0 = 0$, then we take $n = 1$, so that

$$\theta_0 f_1 + \theta_1 f_0 = \lambda f_1$$

This implies that $\theta_0 f_1 = \lambda f_1$ which further implies that $\theta_0 = \lambda$ provided $f_1 \neq 0$. Further if $f_1 = 0$, then we look for $n = 2$ in which case again we get $\theta_0 = \lambda$ provided $f_2 \neq 0$. But $f \neq 0$, so there exist some $n_0 \in \mathbb{N}$ such that $f_{n_0} \neq 0$. Repeating the above proof for $n = n_0$, we get

$$\theta_0 = \lambda$$

Thus

$$\Pi_0(M_\theta) \subseteq \{\theta(0)\}$$

For the reverse inclusion suppose $\theta_n = 0 \quad \forall \quad n \geq 1$.
Take $f = e_0$
Then

$$M_\theta f = M_\theta e_0 = M_{\theta_0} e_0 = \theta_0 e_0$$

therefore θ_0 is an eigen value of M_θ. □

Theorem 4.5. *Let $M_\theta^d \in B(H^2(\beta))$ and $\theta(z) = \alpha z^k \quad \forall \quad z \in C$, where α is any number, then*

$$\Pi_0(M_\theta^d) = \begin{cases} n\alpha & \text{for } k = 1 \\ \emptyset & \text{for } k \geq 2 \end{cases}$$

where $\quad n \in \mathbb{N}$.

Proof. Suppose $\theta(z) = \alpha z^k$, For $k = 1$ we have $\theta = \alpha z$ (also $\theta_n = 0 \quad \forall \quad n \neq k$)
Again for $k = 1$, first take $\lambda = n\alpha$ we show that $M_\theta^d f = \lambda f$ for some $f \in H^2(\beta)$
Take $f = e_n (= z^n)$, Then

$$M_\theta^d f = \theta f^{'} = \theta e_n^{'} = \alpha e_1 n e_{n-1} = \alpha n e_n = \lambda f$$

Hence $\lambda = n\alpha$ is an eigen value
Therefore

$$\{n\alpha : n \in \mathbb{N}\} \subseteq \Pi_0(M_\theta^d)$$

Conversely suppose that $\lambda \in \Pi_0(M_\theta^d)$
Then there exist $f \in H^2(\beta)$ such that

$$M_\theta^d = \lambda f$$

which implies

$$\theta f^{'} = \lambda f$$

In other words

$$\alpha z\{f_1 + 2f_2 z^2 + ...\} = \lambda\{f_0 + f_1 z + f_2 z^2 + ...\}$$

This implies that $f_0 = 0, \quad \alpha f_1 = \lambda f_1$, so we have $\lambda = \alpha \quad if \quad f_1 \neq 0$. If $f_1 = 0$, we have $2\alpha f_1 = \lambda f_2 \quad or \quad \lambda = 2\alpha \quad provided \quad f_2 \neq 0$
Since $f \neq 0$, so $f(n) \neq 0 \quad for \quad some \quad n \in \mathbb{N}$
Comparing coefficients of z^{n-1}, we have $\alpha n f_n = \lambda f_n$ which implies that

$$\lambda = \alpha n$$

Hence

$$\lambda \in \{\alpha n : n \in \mathbb{N}\}$$

Hence we can say that

$$\Pi_0(M_\theta^d) = \{\alpha n : n \in \mathbb{N}\}$$

For $k = 2$, we have $\theta(z) = \alpha z^2$
Now

$$(M_\theta^d f)(z) = (\theta f^{'})(z) = \lambda z$$

therefore

$$\alpha z^2\{f_1 + 2f_2 z^2 + ...\} = \lambda\{f_0 + f_1(z) + f_2(z)^2 + ...\}$$

Equating coefficients of like powers of z on both sides, we get

$$f_0 = 0, \quad f_1 = 0, \quad f_2 = 0, \quad and \quad so \quad on$$

Therefore there exist no non zero f such that $M_\theta^d f = \lambda f$
Hence

$$\Pi_0(M_\theta^d) = \emptyset(emptyset)$$

□

Conclusion

The properties like Boundedness, Compactness, Isometry, Hermitian, Ajoint of Generalized weighted composition operators on weighted Hardy spaces have been obtained in this thesis. Since evaluation kernel gives the value of function at any point in the open unit disk, so using the techniques of C. C. Cowen and G. Gunatillake, We compute the adjoint of generalized composition operator, generalized multiplication operator and generalized weighted composition operator on weighted Hardy space have been characterized.

After introducing some important definitions and fundamental results for the composition operators on different function spaces in Chapter I, the adjoint and norm estimate of generalized weighted composition operators has been found in the Chapter II. In general Differential operators on function spaces is not bounded but in this chapter Differential and Anti-Differential operators boundedness on weighted Hardy space have been discussed and found that in weighted Hardy space we can make differential operator bounded. Properties of operator like Normal, Hyponormal, Quasinormal are also discussed in the last part of Chapter II.

By assuming that product of inducing maps $\theta : \mathbb{D} \to \mathbb{C}$ and $\phi : \mathbb{C} \to \mathbb{C}$ constituting an orthogonal family the compactness and boundedness of generalized weighted composition operator on weighted Hardy space have been discussed in Chapter III. Fredhlom and Closed range for the operator has been given in this chapter.

In Chapter IV, spectra of generalized composition operator, generalized multiplication operator and generalized weighted composition operator have been discussed.

The thesis has a big scope of study for the researchers working in the area of operator theory. In this thesis, We have initiated a study of generalized weighted composition operators on weighted Hardy spaces. A lot more is yet to be explored. Applications of weighted composition operators to other branches of Mathematical sciences like Dynamical systems etc. are yet to be explored.

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List of Research Papers Published/Accepted/Communicated

(i) Rohit Gandhi, Sunil Kumar Sharma and B. S. Komal, *Adjoint of Generalized weighted composition operators using Evaluation Kernel on weighted Hardy space*, **International Journal of Mathematical Analysis**, **Vol. 9-2015, No. 14, 655-660**. Online link: $http://dx.doi.org/10.12988/ijma.2015.0512$.

(ii) Rohit Gandhi, Sunil Kumar Sharma and B. S. Komal, *Generalized composition operators and Evaluation kernel on weighted Hardy space*, **Book Algebra, Geometry, Analysis and their Applications outcome of International conference on Algebra, Geometry, Analysis and their Applications(ICAGAA-14)**, **2016, 59-65**.

(iii) Rohit Gandhi, Sunil Kumar Sharma and B. S. Komal, *Spectra of Generalized composition operators on weighted Hardy space*, accepted in **AIP conference proceeding(RAFAS conference conducted by Lovely Professional University, Phagwara, Punjab, India from November 25-26,2016)**.

(iv) Rohit Gandhi, Sunil Kumar Sharma and B. S. Komal, *Generalized weighted composition operators on weighted Hardy spaces*, communicated in **Indian Journal of Mathematics (IJM), Al.M.S.), 2017**.

Research Papers Presented in National/International Conferences

(i) Attended and presented a paper entitled *Spectra of Generalized composition operators on weighted Hardy space* in in TCPDE-16 held in Panjab University, India from December 05-10, 2016.

(ii) Attended and presented a paper entitled *"Spectra of Generalized composition operators on weighted Hardy space"* in "International conference on Recent Advances in Fundamental and Applied Sciences(RAFAS-16)" held in Lovely Professional University,Phagwara, Punjab, India from November 25-26,2016.

(iii) Attended and presented a paper entitled *Generalized composition operators and Evaluation kernel on weighted Hardy spaces*in "National conference on Role of Mathematics and Computer science in advancement of Physics(RMCSAP-16)" held in Government Degree College, Kathua(J and K), India from February 26-27,2016.

(iv) Attended and presented a paper entitled *"Generalized composition operators and Evaluation kernel on weighted Hardy spaces* in "International conference on Algebra, Geometry, Analysis and their Applications(ICAGAA-14)" held in Jamia Millia Islamia, New Delhi, India from Nov 27-29,2014.

(v) Attended and presented a paper entitled *Generalized weighted composition operator on weighted Hardy space* in "International conference on advances in pure and applied mathematics(ICAPAM-2014)" held on March 7-9, 2014 organized by Department of Mathematics , JLN Government college, Haripur(Manali), Himachal Pradesh, India

(vi) Attended and Presented a Paper entitled *Normal and Hermitian operators on weighted Hardy spaces* in "National conference on advances in mathematics and its application (AMA-2013)" held on June 25-27, 2013 organised by the Department of Mathematics, NIT Hamirpur , Himachal Pradesh.

(vii) Attended and presented a paper entitled *Hermitian operators on weighted Hardy spaces* in "Bhartiya Vigyan Sammelan and Expo" held on October 11-14, 2012 organised by the Lovely Professional University, Phagwara, Punjab.

Participation in Workshops/Seminars

(i) Attended *"National Workshop on Latex"* held at Chitkara University, Rajpura, Punjab from July 8-10, 2013.

(ii) Attended in workshop on "*Mathematical Modeling and Computational Techniques*" on September 27-28, 2013 organized by the University Institute of Engineering and Technology, Panjab University, Chandigarh.