STUDY OF WEAK AND STRONG FORMS OF COMPACTNESS OF TOPOLOGICAL SPACES

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We referred to [2], [3], [4], [5], [6], and [7] for understanding of compactness and its related concepts in topology.

1. Introduction

Topology emerged as a step towards abstraction of well known properties of the Real or Euclidean Line ($\mathbb{R}$) to develop new classes of spaces. In the beginning, the burning question was which property(s) of $\mathbb{R}$ would prove useful in this quest of defining new classes of spaces that would pave the way for useful applications and interesting problems. The answer was compactness (which was refereed to as bicompactness in the early years). Topologists spent the next years formulating the fertile definition of compactness in topological spaces. Two important theorems of real analysis; Bolzano-Weierstrass Theorem and Borel-Lebesgue Lemma became the primary sources of putting together the modern definition of compactness in topological spaces.

2. Preliminaries

A **topological space** is a pair (X, τ) which consists of an arbitrary set X and a collection τ of subsets of X which satisfies the three axioms:

(1) The empty set ϕ and X belong to the collection τ.
(2) All arbitrary union of elements of τ belong to the collection τ.
(3) All finite intersection of elements of τ belong to the collection τ.

This collection τ is called a **topology** for X and each set in the collection τ is called an **open set**. The pair $\mathbb{R}^n$ together with the standard topology is an example of a topological space that is studied exclusively in real analysis. We define a subset of X to be a **closed set** if its complement is an open set of X. Hence, given a topological space X, the following holds:

(1) ϕ and X are closed in X.
(2) All arbitrary intersection of closed sets in X are also closed in X.
(3) All finite unions of closed sets in X are also closed in X.

We may redefine a topology on a set X by starting with a collection of closed sets that satisfy the three properties as stated above and then defining an open set as the complement of a closed set.There are many possibilities for a topology on an arbitrary set M. Therefore we compare the sizes of these topologies. If $\tau \subset \tau^{'}$ then we say $\tau^{'}$ is finer than τ or τ is coarser than $\tau^{'}$. Therefore, indiscrete topology is the coarsest topology we know and the discrete topology is the finest topology we know.

Another important concept of an open set is the **neighbourhood**. We say that a neighbourhood of a point of X is an open set of X to which the point belongs. A neighbourhood of a subset Y of X is an open set of X that contains Y. A point p of the set X is called an **accumulation point** or **limit point** of a subset Y of X when every neighbourhood of p intersects Y at some point of Y distinct from p. A specific kind of the accumulation point is the ω**-accumulation point** for which every open set containing p must contain infinitely many points of Y. A **limit point of a sequence** $x_{n n \in \mathbb{Z}^+}$ of points x_n of a space X is defined such that every open set containing p contains all but finite number of points of the sequence. The sequence is then said to **converge** to the point p. A weaker condition on p is that every open set containing p contains infinitely many points of the sequence. Then p is called the **accumulation point of the sequence**. An accumulation point p of Y is called a **complete accumulation point** if for each neighbourhood U of p, $|U \cap Y| = |Y|$. The set of all accumulation points of a subset Y of X is called the **derived set** of Y, denoted usually by Y'. The set Y with its derived set Y' is called its **closure**, denoted by $\bar{Y}$ or $Cl(Y)$. Equivalently, the closure $\bar{Y}$ is the intersection of all the closed sets (and therefore the smallest closed set) which contain the set Y. Related to the closure of a set is the **interior** of a set A of space X, which is defined as the largest open set contained in A. Therefore, the interior of A is the union of all the open sets contained in A and we denote it by $Int(A)$. A subset Y of X is called a **dense set** if $\bar{Y} = X$. That is every point of X is an accumulation point of Y. Two subsets A, B of a space X are **separated** sets if neither of the two sets contain the accumulation points of the other or $A \cap \bar{B} = \bar{A} \cap B = \phi$.

Metric Topology. For a nonempty set X, the function $d\colon X \times X \to \mathbb{R}$ is called a **metric on X** if d satisfies the following three properties.

(1) $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq d(x, y) + d(y, z)$,

for all $x, y, z \in X$. We call the pair (X, d) a **metric space with metric d**. The third property mentioned above is known as the triangle inequality of the metric. We refer to the metric as the distance function too. We will now define the topology of a metric space. In a metric space (X, d), a subset U of X is an open set if for every $x \in U$ there is an $\epsilon > 0$ such that the ball $B(x, \epsilon) = \{y \in X | d(x, y) > \epsilon\}$ is contained in U. The set of all such open sets of X is called the **topology induced by metric** or the topology of a metric space. It is only appropriate that we discuss the importance of the triangular inequality property of the metric here. It guarantees that around each point y such that $d(x, y) > \epsilon$, there is a smaller $B(y, \delta)$ ball entirely contained in the ball $B(x, \epsilon)$ around M. Therefore, open ball is really open in X. Now if X is a topological space, X is said to be **metrizable** if there exists a metric d on the set X that induces the topology of X as explained previously.

Subbasis and Basis. We can generate a topology for a space X from a collection of subsets of X. A collection of subsets $\mathscr{S}$ of X is called a **subbasis** if we construct the collection of open sets (or topology on X) by taking unions of finite intersections of elements of the subbasis along with ϕ and X. If the union of elements of $\mathscr{S}$ is X and if every point belonging to the intersection of any two elements of $\mathscr{S}$ also belongs to an element of $\mathscr{S}$ contained in the intersection, then that collection $\mathscr{S}$ is called a **basis**. Hence, all the open sets of X can be taken as unions of elements of the basis. A **local basis** at a point p of X is a collection of neighbourhoods of p such that every open set that contains p contains at least one element of the collection.

Subspace Topology. There is a natural way to define a subspace of a given topological space(X, τ). For a subset Y of X, we define the **subspace topology** τ_Y as the collection of intersections of every open sets of X with Y. Then (Y, τ_Y) is the required subspace of a space X. Now consider a space X (by which, we mean a topological space from here on) with a particular property. We then say that it is a **hereditary** property if every subspace of the space X has the property. If only all the closed subspaces (these are closed subsets of X together with the subspace topology) has the property, then it is called a **weakly hereditary** property.

Functions. Functions (or maps defined from one space to another) are great tools for studying properties of spaces and making new spaces from existing ones. We define a function f from space X to space Y to be **continuous** if the inverse image of all open sets of Y is an open set in X. This is equivalent to proving that the inverse image of all closed sets should be closed or for each subset A of X, $f(\bar{A}) \subset \overline{f(A)}$. A function is an **open map** when the image of every open set is open and is a **closed map** f is when the image of every closed set is closed. A function f from space X to space Y is a **homeomorphism** when it is bijective and both f and f^{-1} are continuous. We then say X is homeomorphic (or **topologically equivalent**) to Y or $X \cong Y$. A property is said to be a **topological invariant** if whenever one space has the given property, any other space homeomorphic to it also has the same property. A **Urysohn function** for disjoint subsets A, B of X, is a continuous function $f\colon X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

3. Separation Axioms

Topological spaces without any additional restrictions (or axioms) imposed on the topology behave rather too wildly and pondering over invariance of properties or any applications of these spaces becomes a very complicated and uninteresting mental activity. Here we discuss more axioms a topology may satisfy so that we create the necessary grounds for discussion of various forms of compactness that exist in topological spaces.

These axioms, if imposed on a topology of a space, specify the extent to which distinct points or closed sets of the space may be separated by open sets. They are listed below. Let X be a topological space.

$\mathbf{T_0}$ axiom: For all distinct points $x, y \in X$, there exists a neighbourhood of one of the two pints that does not contain the other. If X satisfies the T_0 axiom, then X is a **Kolmogorov** Space.

$\mathbf{T_1}$ axiom: For all distinct points $x, y \in X$, there exist a neighbourhood of each point that does not contain the other. If X satisfies the T_1 axiom, then X is called a **Fréchet** Space.

$\mathbf{T_2}$ axiom: For all distinct points $x, y \in X$, there exist disjoint neighbourhoods of both points. If X satisfies the T_2 axiom, then X is a **Hausdorff** Space.

$\mathbf{T_3}$ axiom: For all points $x \in X$ and each closed sets A of X that does not contain x, there exists disjoint neighbourhoods of x and A.

$\mathbf{T_4}$ axiom: For every disjoint closed sets A, B of X, there exists disjoint neighbourhoods of A and B.

$\mathbf{T_5}$ axiom: For all separated sets in X, there exists disjoint neighbourhoods of A and B.

Along with the above commonly used axioms we list two more useful axioms as follows.

$\mathbf{T_{2\frac{1}{2}}}$ axiom: For all distinct points $x, y \in X$, there exist neighbourhoods of both points such that the intersection of the closures of the neighbourhoods is empty. If X satisfies the T_2 axiom, then X is a **completely Hausdorff** Space.

$\mathbf{T_{3\frac{1}{2}}}$ axiom: For all points $x \in X$ and each closed sets A of X that does not contain x, there is a Urysohn function for A and x.

We observe that in T_0 space, no two distinct points in such a space can be limit points of each other. In T_1 space, all singleton sets are closed sets, and all points in T_2 space are intersection of their closed neighbourhoods. In T_3 space,each open set contains a closed neighbourhood around each of its points or each closed set is the intersection of all of its closed neighbourhoods. A space that satisfies T_4 axiom is characterized by the fact that every open set in the space contains a closed neighbourhood of each closed set contained in it and every subset of T_5 space which contains an open set, also contains its closure. It must be appreciated that each of these axioms stand independently of the three axioms of a topology define on a set. But they are not really independent of each other. For instance, observe that T_2 implies T_1 which in turn implies T_0. However, the converse is not true. Now T_3 and T_4 do not imply or implied by any other axiom in the list. Also note that, though T_5 imply T_4, it does not imply any of the other axioms. In addition, a completely Hausdorff space is a Hausdorff space and every $T_{3\frac{1}{2}}$ is T_3 though not necessarily T_2.
Now we can employ the separation axioms to define stronger properties. We define a space which is both T_0 and T_3 (hence, T_2 as well) to be a **regular** space. Hence every regular space is completely Hausdorff. A space which is T_0 and $T_{3\frac{1}{2}}$ is called **completely regular**. Thus completely regular spaces are regular. A space which is both T_1 and T_4 (consequently, T_3 too) as a **normal** space. Hence, normal spaces are completley regular. A space which is both T_1 and T_5 is a **completely normal space**.

4. Compactness

Before we discuss the fundamental concept in point-set topology, we will first define a cover.

Covers. By a **cover** $\mathscr{A}$ of a space X, we mean a collection $\{A_\alpha | \alpha \in J\}$ of subsets A_α of X such that the union of its elements is X. We say that the collection covers X. An **open cover** is a cover which consists of only open sets A_α of X. By a **subcover** of the cover $\mathscr{A}$ of X, we mean a subfamily $\{A_\alpha | \alpha \in K\}$,$K \subset J$, that is a cover of X. A cover V_β of a space X is a **refinement** of a cover U_α if for each V_β there exists a U_α such that $V_\beta \subset U_\alpha$. A cover is **point finite** if each point belongs to only finitely many sets in the cover, and it is **locally finite** if each point has some neighbourhood which intersects only finitely many members of the cover.

Compactness. A topology τ defined for a set X determines the number of open sets available in the space (X, τ). A space X will satisfy any of the separation axioms if the topology has enough open sets to provide disjoint neighbourhoods for distinct points and disjoint subsets of X. Now, compactness, is defined in the literature as,

"X is ***compact*** *if every open cover of X contains a finite subcover"*

or equivalently, X is compact if every collection of closed subsets whose intersection is empty contains a finite subcollection whose intersection is empty. Further using the definition of refinement, we can say that a space X is compact if every open cover $\mathscr{A}$ of X has a finite open refinement $\mathscr{B}$ that covers X. This definition is equivalent to the earlier one because given such a refinement $\mathscr{B}$, we can find for each element of $\mathscr{B}$ an element of $\mathscr{A}$ containing it. Hence, we will obtain a finite subcollection of $\mathscr{A}$ that covers X. We can observe that compactness puts a limit to the number of open sets in a topology. Therefore, separation axioms and compactness are two different notions that aids us in understanding the abstract structure and behaviour of each topological space. Moreover, in compact spaces, generalization from local to global properties is possible. If X is a compact space such that open sets of X have a property P that X may not have, but also such that if U and V have it, then so does their union. Then if X has this property locally (which means that every point of X has a neighbourhood with property P), X itself has the property too by assuming the property is inductively transferred to finite unions. So if a function f is locally bounded, then f is bounded. or, if $\mathscr{A}$ is a locally finite cover, then $\mathscr{A}$ is finite. And if A is a locally finite subset of X, then A is finite. Conversely, this means that if A is an infinite subset of X, then there exists a point of X all of whose neighbourhoods contain infinitely many points of A. This point is an ω-accumulation point of A. Now from the definition of compactness, we can conclude that every closed subspace of a compact space is compact. As a result, compactness is a weakly hereditary. Moreover, in Hausdorff spaces, every compact subspace is closed. With further careful analysis, we see that every closed interval in $\mathbb{R}$ is compact. In addition, the image of a compact space under any continuous map is compact.

Now we can generalize the notion of compactness stated above in two ways. First, by weakening the requirement that subcovers must be finite or only requiring that countable open covers have finite subcovers instead of all open covers. Second, by the use of various types of refinements of a cover.

Weak Forms of Compactness. A **Lindelöf** space is a topological space where every open cover has a countable subcover. A **countably compact** space is one where every countable open cover has a finite subcover or equivalently, every sequence has an accumulation point (in the space itself). A space is **sequentially compact** if every sequence has a convergent subsequence and **limit point compact** if every infinite subset of the space has a limit point. Hence, compactness implies all of these weak forms and sequential compactness implies countable compactness, which in turn imply limit point compactness. However, the converses are not true. But, in a T_1 space, limit point compactness is equivalent to countable compactness. Also every countably compact Lindelöf space is compact and every limit point compact space such that each infinite subset has a complete accumulation point is compact. A space is **locally compact**, if each of its points is contained in a compact subspace. Again, every compact space is locally compact but the converse is not true. In Hausdorff spaces, there exists a stronger form of local compactness because in such spaces, compact sets are closed. This definition makes the concept of locally compact a more intuitively 'local' property. A space X is **strong local compact** if at each point x of X, given a neighbourhood U containing x, there exists a neighbourhood of x whose closure is contained in U.

Countability Axioms and Separability. Compactness indirectly impose a limitation on the number of open sets in a topology. But countability axioms restricts the number of basis elements, thereby directly limiting the number of open sets. We define a space to be **separable** if it has a dense subset. A **second-countable** (or completely separable, or perfectly separable) space is the one that has a countable basis. A **first-countable** space has a local countable basis at each of its points. We can see easily that most topological spaces we know are first-countable and every second-countable space is first-countable. In second-countable spaces, compactness is equivalent to countable compactness. By similar argument, we see that in first-countable spaces, countable compactness is equivalent to sequential compactness.

Paracompactness. This is another way of generalizing compactness. Several compactness properties which have both local and global aspects rely on the concept of a refinement of a cover. A space X is **paracompact** if every open cover has a locally finite open refinement that covers X. Every compact space is paracompact. Paracompactness, like compactness, is also a weakly hereditary property. A space X is **metacompact** if every open cover has a point finite open refinement. Every paracompact Hausdorff space is normal and every regular Lindelöf space is paracompact. Also every metrizable space is paracompact.

5. Recent Developments in Weak and Strong Forms of Compactness

Here we discuss briefly discuss the concept of generalized preopen compactness and its relation to other known types of compactness along with some new separation axioms as presented in [1]. One of the main reasons for constructing such a concept is to look at some modified forms of continuity, separation axioms etc.

Definitions. A subset A of a space X, is **preopen** if for every open set U that contains A, $A \subset Int(Cl(A))$ and **preclosed** if $Cl(Int(A)) \subset A$. Space X is **strongly compact** if every preopen cover has a finite subcover. A space that contains two disjoint dense subsets are called **resolvable** and the space is **strongly irresolvable** if every open subspace of

X is irresolvable (which means we cannot represent it as a disjoint union of two dense subsets). X is said to be **quasi-H-closed** or QHC if every open cover of X has a finite subfamily, the closures of whose members cover X.

Generalized Preopen Sets. A is a **generalized preopen** (or gpo-) set if for each preclosed subset U of X that contains A, $Cl(A) \subset U$. Every open set is a gpo- set. The complement of a gpo- set is **generalized preclosed** (or gpc-). Defined equivalently, a subset A of a space X is a gpc- set if and only if for each preopen set U that is contained in A, $U \subset Int(A)$. For two gpo- compact sets A and B, $A \cap B$ is not generally gpo- but arbitrary unions of gpo- sets are gpo- sets. A space is **gpo- irresolvable** if every preopen set is a gpo- set.

Generalized Preopen Compactness. A space X is **gpo- compact** if every gpo- cover (which is a cover consisting of gpo- set) of X contains a finite subcover. Every gpo-compact space is a compact space. If a space is both gpo- irresolvable and gpo- compact, then it is strongly compact. A space X is **gpo- regular** if for every gpo- cover A_α of X and for every $x \in A_\alpha(x) \in A$, there exists a preopen set U_α such that $x \in U_\alpha \subset A_\alpha$. Then we see that if X is gpo- regular and a strongly compact space, then it is gpo- compact.

New Separation Axioms. Some new separation axioms were also introduced in [1].

- X is called a $\mathrm{T_{p0}}$ space if for each pair of distinct points $x, y \in X$, there is either a gpo- set containing x but not y or a gpo- set containing y but not x.
- X is called a $\mathrm{T_{p1}}$ space if for each pair of distinct points $x, y \in X$, there exist a gpo- set containing x but not y and a gpo- set containing y but not x.
- X is called a $\mathrm{T_{p2}}$ space if for each pair of distinct points $x, y \in X$, there exist disjoint gpo- sets U and V such that $x \in U$ and $y \in V$.
- X is called a weak regular space if for each closed subset A and each point x of X not belonging to A, there exist gpo- sets U and V such that $x \in U$, $A \subset V$ and $U \cap V = \phi$.
- X is called a weak normal space if for each pair of disjoint closed subsets A and B of X there exist gpo- sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$.

A weak regular $\mathrm{T_{p1}}$ space is called a $\mathrm{T_{p3}}$ space. A weak normal $\mathrm{T_{p1}}$ space is called a $\mathrm{T_{p4}}$ space.

6. Future Plan

We will extend our study of weak and strong forms of compactness for Dissertation II by exploring further the following.

1. Connections between classical compactness and gpo- compactness and study of spaces that satisfy the new axioms introduced in [1].
2. Product of gpo- compact spaces.
3. Other possible ways of generalizing open sets and construction of new separation axioms.
4. Existence of any suitable application of generalized preopen compactness, hopefully in topology itself.

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