STUDY ON WEAK AND STRONG FORM OF CONNECTEDNESS OF TOPOLOGICAL SPACES

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1. Introduction

The concept of topological space comes the analysis of real line along with Euclidean space. A detailed investigation of continuous functions initiates this concept. It is a space where neither shape matters nor angles if holes remain preserved. Under this space, we can stretch, crumple and bend any topological space, but we cannot tear or glue it. There is a very famous example which says that a coffee cup and a donut is same in terms of topology. Due to having so many unique properties, there are some complications in handling this space, due to which we generally try to convert topological space to a space in which we are feasible with i.e. Metrizable Space. Hence, to do the same we need to have homomorphism between topological space and metrizable space. But it is not always possible to find a function for the same purpose. So, we need to know what minimum conditions we can impose on any set to obtain our aim. One of the way of finding these minimum conditions is connectedness. Further, we have an obstacle which says that we can not always approach Connectedness directly. Then we need to approach to its weaker and stronger forms i.e. local connectedness. This discussion justifies our main motive behind the topic i.e. Study on Weak and Strong form of Connectedness of Topological Spaces. Under the same discussion, we have cut-point space, a wonderful tool to impose required conditions. In this report we are going to discuss some important terms and results on cut point space. These results will help us to reach our main motive of discussion.

2. Notations, Definitions and Preliminaries

A topological space is a set with some topology defined on it. It is represented by **t-space**. Any collection of subsets of X is called as topology τ on X if it satisfy following three conditions:

- ϕ and X lie in τ.
- Random unions of members of τ lies in τ.
- Finite intersections of members of τ lies in τ.

Members of τ are known as **open sets**, where o-set represents open set. Point of a given set is actually an element of that set and it is denoted by **pt**. We can compare any two topologies defined on set X if atleast one of them contains the other one. If X is any finite set, then it is easy to induce topology τ on it; but, if X is infinite, then we require a subcollection of τ which can generate each and every o-set of X. This type of subcollection is known as the **Basis** for topology τ on set X. Any subcollection of power set(collection of all subsets) of X is known as a Basis for a topology on X if it satisfy the following:

- For each element of the set X, we have atleast one member of subcollection which contains that element.
- If there is an existence of an non-empty intersection between two members of the subcollection, then for every element x of X lying in the intersection, there is an existence of atleast one member of the subcollection, consisting of elements of that intersection only, contains x.

There are various ways to induce topology on a set. One way is **order topology**. It can only be applied to those sets which has an order relation. Order topology has o-sets in terms of open intervals. Basis of an order topology defined on X is collection of all open intervals (c, d), where c and d lies in X.
If there is an existence of smallest element of the set, then basis should also contain all intervals of form $[c_0, d)$. Similarly, if there is an existence of largest element of the set, then basis should contain all intervals of form $(c, d_0]$, where c_0 represents the smallest and d_0 represents the largest element of the set. There are some named intervals like (a, ∞) or $(-\infty, a)$ or $[a, \infty)$ or $(-\infty, b]$.
We named these intervals as **rays** determined by a; (a, ∞) and $(-\infty, a)$ represents open rays whereas $[a, \infty)$ and $(-\infty, b]$ represents closed rays. We can also induce new topology from given topologies on the same set with the aid of **product topology** i.e. second way of inducing topology from existing ones. If (X, τ_X) and (Y, τ_Y) are two spaces, then the basis for product topology defined on $X \times Y$ is a collection of all sets of form $U \times V$ where $U \in \tau_X$ and $V \in \tau_Y$. There is another way to relate basis of $X \times Y$ with basis of X and Y as follows: product topology is a collection of all subsets of form $B \times C$ where B lies in basis of X and C lies in basis of Y.
We can also relate topology on $X \times Y$ and τ_X(or τ_Y) with the help of **projection map**; a map f from $X \times Y$ to X such that $f(x, y) = x$. Third way of inducing topology is **subspace topology**. If (X, τ_X) be a space and Y be its subset, then subspace topology defined in Y is a collection of all those subsets of X which can be scripted as intersection of an o-set of X with Y.
We can form a relation between the basis of subspace topology on Y and basis of topology on X as follows: basis of subspace topology on Y is a collection of all those subsets of Y which can be scripted as the intersection of the basis element (of X) with Y. Some of those subsets which are not open can be covered in terms of **closed sets**; subset of X having complement (w.r.t. X) open in X, where w.r.t. represents with respect to. Closed set is represented by c-set.
Topology on a space (X, τ) can also be defined in terms of c-sets as follows:

- ϕ and X have compliments (w.r.t. X) open in X.
- Random intersections of c-sets of τ has compliment (w.r.t. X) open in X.
- Finite unions of c-sets of τ has compliment (w.r.t. X) open in X.

We can obtain a c-set or an o-set around any particular subset A of a space in terms of interior and closure of a set A. The union of all those o-sets contained by A is known as **interior** of A and intersection of all those c-sets which contains A is known as **closure** of A. For a subset A of a space X, $\boldsymbol{A^-}$ and $\boldsymbol{A^o}$ represents closure and interior of A respectively. If $A \subset Y \subset X$, then $\boldsymbol{cl_Y A}$ describes the closure of A in Y.
Closure is a hereditary property and we can justify this property as follows: if X be a

space and Y be subset of X with subspace topology, then subset A of Y will have closure (in Y) if it can be written in form of intersection of closure of A (in X) with Y. There is another relation between any given subset and o-sets (or c-sets) of X in terms of **limit pts**. A subset A of a space X has a limit pt x if every o-set of X containing x, also intersects A in some other pt than x. Limit pt of A may or may not be present in A. If each limit pt of A belong to A, then that A becomes closed in X.

Limit pts help us in finding the smallest c-set near any subset of X; but, the smallest possible set containing its limit pts is closed singleton set. It is not necessary to have all singletons of X as closed singletons. But suppose we are having all singletons of X as closed singletons, then we need to introduce a new types of space, Hausdorff space; a space in which every finite pt set is closed. Hausdorff Space is represented as **H-space.** H-space is a space in which each one pt can be separated from another pt by the aid of o-sets. We can also define **H-space** as a space in which for every two distinct pt x and y, we have two distinct nonempty o-sets U and V such that $x \in U$ and $y \in V$.

We can understand one space with the aid of another space by **continuous functions**. If (X, τ_X) and (Y, τ_Y) be two spaces, then function $f : Y \longrightarrow X$ is continuous if for each $V \in \tau_Y$, the set $f^{-1}(V) \in \tau_X$. Stronger form of continuous function is **homeomorphism** which says that if f is a bijection map and both the function f and the inverse function f^{-1} are continuous, then f is homeomorphism from X to Y. Under continuous function, we have an intermediate theorem (well known theorem in real analysis) mentioned below.

"*If X be a C.T. Space and Y be a completely ordered set with the order topology such that $f : Y \longrightarrow X$ is continuous and $c, d \in X$ is such that $f(c)$ is less than $f(d)$, then there is an existence of $r = f(a) \in Y$ such that r lies between $f(c)$ and $f(d)$, where $a \in X$.*"

Continuity of f is not the only highlight of this theorem, properties of the space Y also playing a vital role here.

3. Connectedness

Connectedness of the space X plays a vital role in intermediate value theorem. Connectedness is defined as follows: A space X is said to be connected if there is no existence of any pair U, V of disjoint nonempty open subsets of X having union equal to X. As we have defined connectedness in terms of o-sets of X, so it is obviously a topological property. We can also verify that space X is connected if and only if there is absence of such proper nonempty subset A of X which gives $A \in \tau_X$ and $X - A \in \tau$. There are two weaker forms of connectedness; **path connected** and **locally connected**.

Before going to path connectedness, we shall discuss what actually path means. Given pts x and y of the space X, a path in X from x to y is a continuous map $f : [a, b] \longrightarrow X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be path connected if every pair of pts of X can be joined by a path in X. Now, Suppose that we need arbitrary small neighborhood for each pt of X to be connected, then condition required for obtaining the same is that X should be local connectedness. A space X is said to be **locally connected at** x if for every neighborhood U of x, there is an existence of a connected neighborhood V of x lying in U. If X is locally connected at its each and every pt, then it is known as **locally connected**. Similarly, X is said to be **locally path connected at** x if for every neighborhood U of x, there is an existence of path connected neighborhood V of x lying in U. If X is locally path connected at its each and every pt, then it is known as **locally path connected**.

By $Y = \boldsymbol{C} \mid \boldsymbol{D}$, we means two nonempty subsets C and D of t-space Y such that $(Y - \{x\}) = C \cup D$ and $\overline{B} \cup A = \overline{A} \cup B = \phi$, where x is known as cut pt of Y. A space X is called $\boldsymbol{T_{1/2}}$ if for every element x of X we have either $\{x\} \in \tau$ or $X - \{x\} \in \tau$. A space X is called $\boldsymbol{H(i)}$ if it is compact with a condition that closure of members of finite subcollection of every open covering of X cover X. $\boldsymbol{ctX}$ represents the set of all cut pts of a space X. $\boldsymbol{B_x^*}$ is used for the set $B_x \cup \{x\}$.
For a connected subset Y of $X - \{x\}$, $B_x(Y)$ represents the separating subset of $X - \{x\}$ containing Y, $\boldsymbol{B_x^* Y}$ is used for the set $B_x(Y) \cup \{x\}$. For a pt $b \in (X - \{x\})$, $\boldsymbol{B_x(b)}$ represents the separating subset of $X - \{x\}$ containing b. $\boldsymbol{B_x^*(b)}$ is used for the set $B_x(b) \cup \{x\}$. $\boldsymbol{S(c, d)}$ represents the set of all separating pts between c and d. After joining the pts c and d with $S(c, d)$, the obtained set is represented by $\boldsymbol{S[c, d]}$. A space X is called as **space with end pts**, if there is an existence of c and d in X such that $X = S[c, d]$.
By the definition of a cut pt x of a space X, there is an existence of separation $A \mid B$ of $X - \{x\}$ which need not to be unique. For obtaining unique separation, we need to impose some conditions on the separating sets of $X - \{x\}$. **Strong cut pt** is introduced for the same purpose. As by name we can conclude that strong cut pt is stronger than that of a cut pt. A pt x of a space X is called as strong cut pt if the separating sets of $X - \{x\}$ are connected.
Connected ordered topological space (C.O.T.S.) is defined as a connected space with a condition that for three pt subset Y of X, there is an existence of a pt y in Y such that Y meets two connected components of $X - \{y\}$. Connected Topological Space is represented by **C.T. space** and $H(i)$ Connected Topological Space is represented by **H.C.T. space**. If X be a **H(i) two pt cut pt space**, then the dismissal of any two pt disconnected set makes the space disconnected.

4. Cut Point Space

Now, we are restricting our discussion to only those C.T. spaces in which we have at least one pt which can make the space disconnected after its dismissal. This type of pt is known as **cut pt**. Recalling definition of cut pt and cut pt space as follows: if X be a nonempty C.T. space, then x is said to be cut pt of X if there is an existence of $A, B \subset X$ such that $A \cap \overline{B} = B \cap \overline{A} = \phi$ and $A \cup B = X/\{x\}$. If every $x \in X$ is a cut pt of X, then X is known as **cut pt space**.
B. Honari and Y. Bahrampour [2] provides a generalized background of cut pt space. Discussion initiates from set of real numbers by a result which says that union of n straight lines in $\mathbb{R}$ is a cut pt space if and only if either all of them are concurrent or precisely $n - 1$ of them are collateral. If there is an existence of subset A, B of X such that $(X - \{x\}) = A \mid B$, then we can show that x is a cut pt of X by showing intersection of A with closure of B is empty. Similar argument we can use for showing intersection of B with closure of A equal to ϕ. Following example will follow parallel proof:

"*If* $X_1 = \{f(x, y) \in \mathbb{R}^2 : x \leq 0 \;\&\; |y| = 1\}$ *and* $X_2 = \{f(x, y) \in \mathbb{R}^2 : x > 0 \;\&\; y = sin(1/x)\}$ *are two spaces, then their union is a cut pt space.*"

Before going further we should know about some of the important terms i.e. **closed pts** or **open pts**. Closed pt is an element of X which lies in a singleton c-set in X. Similarly, open pt is an element of X which lies in a singleton o-set in X. On Closed and Open pts, we have a result mentioned below:

"*If* X *be a connected topological space and* x *be a cut pt of* X *such that* $(X - \{x\}) = A \mid B$*, then* x *is either an open pt or closed pt.*"

Also, we can add that if x is an open pt, then A and B are closed in X and if x is a closed pt, then A and B are open in X. We can prove it by using following result: if $(X - \{x\})$ is not connected, then there is an existence of proper subset of $(X - \{x\})$ which is both open and closed in $(X - \{x\})$. After then, we have to show that if X is a C.T. space and x be its cut pt such that $(X - \{x\}) = A \,|\, B$, then $A \cup \{x\}$ is connected. For obtaining the same, we have to assume that $A \cup \{x\}$ is not connected and afterwards we can show show that X is not connected which will be a contradiction.
Till now discussion mostly held was for one cut pt of X, but suppose if X has two cut pts, then we have a result as follows:

"*If x and y be two cut pts of C.T. space X such that $(X - \{x\}) = A \,|\, B$ and $(X - \{y\}) = C \,|\, D$ and $x \in C$ and $y \in A$, then $D \subset A$ and $B \subset C$.*"

As we can see that $x \in C$; so, it will not contain in D and connectedness of $D \cap \{y\}$ which indicates that $D \cap \{y\} \subset (X - \{x\})$. As $y \in A$ so connectedness of A indicates that $D \cap \{y\} \subset A$ which further gives $D \subset A$. Similar argument we can follow for $C \subset B$. Now, discussion comes related to some important result on the cardinality of cut pt space. In terms of closed pts, we have a result as follows:

"*If X is a cut pt space, then the cardinality of collection of closed pts of X is infinite.*"

Its easy to prove it by mathematical induction. As we can see that for $n = 1$ result is trivially true. At $n = k$, we have to assume that there are k closed pts of X. After then we have to show that there is an existence of $(k + 1)^{th}$ closed pt which is different from rest k closed pts. For this we need to use following result.

"*If X be a C.T. space and x be a cut pt of it such that $(X - \{x\}) = A \,|\, B$ and $A \in ctX$, then A contains at least one closed pt*"

If we assume that all pts of A are open pts, then it will lead to a contradiction that there is an existence of a pt which cannot disconnect X. Hence we can see that there is a relation between cut pt space and compactness which further indicates that if X be a cut pt space, then X is non-compact. In one of the previous results, we discussed about that subset A of cut pt space X which has a cut pt; but, suppose if there is absence of such proper subset A of cut pt space X which has a cut pt means there does not exist proper cut pt subspace of X, then X is known as **irreducible cut pt space**. As by name we can see that X cannot be reduced to a space with all its pts as cut pts. Related to the same, we have a result mentioned follows:

"*If X is an irreducible cut pt space, then for every $x \in X$, $(X - \{x\})$ has exactly two components.*"

We can prove it by using the fact that if X is a cut pt space, $x \in X$ and $(X - \{x\}) = A \,|\, B$ such that A is not connected, then $A \cup \{x\}$ is a cut pt space. We can relate irreducible cut pt space and connectedness as follows: if X be an irreducible cut pt space, $x \in X$ and $(X - \{x\}) = A \,|\, B$, then there are exactly two pts $y \in A$ and $z \in B$ such that $\{x, y\}$ and $\{x, z\}$ are connected. At last we have a main result which says that a t-space X is an irreducible cut pt space if and only if X is homeomorphic to the Khalimsky line". It can be proven by mathematical induction i.e. we find a subset Y of X that is homeomorphic to the Khalimsky line and then irreducibility of X concludes $X = Y$.

5. H.C.T. Spaces and Cut Points

After B. Honari and Y. Bahrampour [2], Devender Kumar Kamboj and Vinod Kumar [3] presented a wonderful work on *H.C.T.* spaces and cut pts. Under their work many important results were obtained, out of which some are mentioned below:

"*Cut pt space is always non* $H(i)$."

It can be obtained by using the result which says that every *H.C.T.* Space has at least two non-cut pts. Further we have a result that if $x \in \text{ct}X$, then B_x contains a non-cut pt of X. After showing that B_x^* is both connected and $H(i)$, we can reach our result. We have a result on non cut pt which says:

"*y is a non-cut pt of X if it is a non-cut pt of B_x^* in B_x^* and $y \neq x$*"

where X is a C.T. space and $x \in \text{ct}X$. As $B_x^* - \{y\}$ and C_x^* are connected which implies our result (As $X\{y\} = (B_x^* - \{y\}) \cup C_x^*$ is connected). By using the above result we can obtain another result which says:

"*If X be a H.C.T. Space and $x \in ctX$, then B_x contains a non-cut pt of X.*"

As we know that there is non existence of proper connected subset of X containing $X - \text{ct}X$. So, $X - \text{ct}X \neq \phi$ and one of the component of $X - \{x\}$ contains a non-cut pt of X, implies our result. We have a result on two pt cut pt space as follows:

"*If $ctX \neq \phi$, then there is a two pt disconnected subspace $\{c, d\}$ of X such that $X - \{c, d\}$ is connected.*"

From the previous results, we can obtain that B_x^* and C_x^* have at least two non-cut pts in B_x^* and C_x^* respectively which indicates that $B_x^* - \{c\}$ and $C_x^* - \{d\}$ are connected. Hence disconnectedness of $\{c, d\}$ proves our result.

"*Let X has exactly two non-cut pts, say, c and d, then we have $X = S[c, d]$; i.e. X is a space with end pts*"

As we can see that, each one of B_x and C_x contains a non-cut pt of X. As $X - \text{ct}X = \{c, d\}$, so these non-cut pts should be c and d which further indicates that $c \in B_x$ and $d \in C_x$ or conversely; hence implies the required result. A result on proper subsets of $H(i)$ space is mentioned below:

"*If Y is a $H(i)$ space such that $c, d \in (Y - ct\,Y)$ and each $\{c\}$ and $\{d\}$ are either open or closed in Y such that $Y - \{c, d\}$ is disconnected, then there is an existence of a proper $H(i)$ connected subset of X which contain both c and d.* "

If X be a $H(i)$ two pt cut pt space, then from above result, we can obtain the following:

"*For any two open or closed pts of X, there is an existence of a proper $H(i)$ connected subset of X containing those pts*"

This can be shown by using the fact that if X is a *H.C.T.* space such that the dismissal of any two pt disconnected space makes the space disconnected, then $\text{ct}X = \phi$. We have a result on strong cut pt as follows:

"*If X be a H.C.T. space with exactly two non-cut pts, then every cut pt of X is a strong cut pt*"

It can be obtained by using the fact that if X is a C.T. space and $x \in \text{ct}X$ such that $c \in B_x$ and $B_x - \{c\} \subset S(c, d)$, then B_x is connected. At last, we have a result on ordered space as follows:

"*If H.C.T. space X have exactly two non-cut pts, then it is C.O.T.S.*"

For proving the same, it is sufficient to prove that for every three pts of X, one pt is a separating pt between the other two.

6. Conclusion

In terms of applications of cut pts along with the fact that the many C.T. spaces (i.e. the Khalimsky line) having cut pts are not $T1$; the supposition of separation axioms is prevented to the greatest extent. In $H(i)$ spaces, no separation axioms are supposed. That's we have discussed concepts related to $H(i)$ in this report; as we can obtain better results under the same.
In our final report, our focus will be to obtain the following relations:

- Relation between Local Connectedness and Cut pt Space.
- Relation between Cut pt Space and Metrizable Space.

There is a wonderful contribution by E. D. Shirley [6] in finding the relation between semi local connectedness and cut pts in terms of metric continua; so, we will also try to achieve our first goal with some remarkable results in the same approach. Andreas W.M. Dress along with Katharina T. Huber, Jacobus Koolen and Vincent Moulton[1] presented a commendable work on cut pts in metric spaces. So, we might be taking help from the same to achieve useful results under our second goal.

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