Lovely Professional University

Master of Science

Common Fixed Point in Metric Spaces and G-Metric Space

A project submitted in fulfilment of the requirements for the degree of Master of Science

in the

Department of Mathematics
School of Chemical Engineering & Physical Sciences
Lovely Faculty of Technology and Sciences

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April 2017

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Date: April 2017

"Thanks to my solid academic training, today I can write hundreds of words on virtually any topic without possessing a shred of information, which is how I got a good job in journalism."

Dave Barry

LOVELY PROFESSIONAL UNIVERSITY
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Abstract

Master of Science

Common Fixed Point in Metric Spaces and G-Metric Space

by NIHARIKA SHARMA

In this project, we prove some fixed point theorem for a weakly compatible pair os maps satisfying generalized integral type contraction alongwith E.A. property and CLR property in G-metric space also we proved a common fixed point theorem for six maps satisfying generalized (ψ, ϕ)-integral type contraction in G-metric spaces.

2010 Mathematics Subject Classification: 47H10, 54H25.

Key words and phrases: Weakly compatible maps, weak contraction, generalized weak contraction, E.A. property, (CLR_f) property, R-weakly commuting mapping of type (A_g), R-weakly commuting mapping of type (A_f), R-weakly commuting mapping of type (P).

Acknowledgements

It is great pleasure for me to present the project report on "**Common Fixed Point in Metric Space and G-Metric Space**". Every work accomplished is a pleasure, a sense of fulfillment. However many people always motivate, criticize and appreciate a work with their objective ideas and opinions. I would like to use this opportunity to thank all, who have directly or indirectly helped us to complete this capstone-1.

Firstly, I would like to thank Dr. Manoj Kumar. I never would've completed my capstone-1 without the help of Dr. Manoj Kumar.

Next, I would like to thank all people who gave their valuable time and feedback to this project. I would also like to thank my collage for supporting us with resources which beyond any doubt have helped me.

At last I would prefer to specify my deep feeling to our head of department Mr. Kulwinder Singh and head of school Dr. Ramesh Chand Thakur and for providing all necessary facilities and inspiring.

Niharika Sharma
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hat (it) **S**tands **F**or

This work is dedicated to my parents because without their blessings and constant support it was not possible to complete my work. . . .

Chapter 1

Introduction

This chapter elaborates the basic definitions, results and notations, which are required in the subsequent chapters. It begins with the study of core material of thesis along with a resume of previously known results.

In 1922, the Polish mathematician, Banach proved a common fixed point theorem, which ensures the existence and uniqueness of a fixed point under appropriate conditions. This result of Banach is known as Banach's fixed point theorem or Banach contraction principle, which states that " Let (X, d) be a complete metric space. If T satisfies

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

for each $x, y \in X$, where $0 < k < 1$, then T has a unique fixed point in X". This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. This principle is basic tool in fixed point theory.

Many authors extended, generalized and improved Banach fixed point theorem in different ways. For the last quarter of the 20^{th} century, there has been a considerable interest in the study of common fixed point of pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were obtained in this direction by various authors. The generalization of Banach's fixed point theorem by Jungck [9] gave a new direction to the "Fixed point theory Literature". This theorem has had many applications, but suffers from the drawback that the definition requires that T be continuous throughout X. There then follows a flood of papers involving contractive definition that do not require the continuity of T. This result was further generalized and extended in various ways by many authors. On the other hand, Sessa [22] coined the notion of weak commutativity and proved common fixed point theorem for a pair of mappings.

1.1 Background of fixed point theory

Fixed point theory itself is a beautiful mixture of analysis, topology and geometry. Over since last 50 years, fixed point theory has been revealed itself as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in diverse fields such as in biology, chemistry, economics, engineering, game theory and physics. The point at which the curve y = f (x) and the line y = x intersects gives the solution of the curve, and the point of intersection is the fixed point of the curve. The usefulness of the concrete applications has increased enormously due to the development of accurate techniques for computing fixed points. Fixed point theory is rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include numerical methods like Newton-Raphson method, establishing Picards Existence theorem, existence of solution of integral equations and a system of linear equations.

Significance of Fixed Points

Fixed points are the points which remain invariant under a map/transformation. Fixed points tell us which parts of the space are pinned in plane, not moved, by the transformation. The fixed points of a transformation restrict the motion of the space under some restrictions. We note that fixed point problems and root finding problems f(x) = 0 are equivalent. Now, the question arise what type of problems have the fixed point. The fixed point problems can be elaborated in the following manner:

(i) What functions/maps have a fixed point?

(ii) How do we determine the fixed point?

(iii) Is the fixed point unique?

Next, we state a result which gives us the guarantee of existence of fixed points. Suppose g is continuous self map on [a, b]. Then, we have the following conclusions:

If the range of the mapping y = g(x) satisfies $y\epsilon[a,b]$ for all $x\epsilon[a,b]$, then g has a fixed point in [a, b]. Suppose that g(x) is defined over (a, b) and that a positive constant k ¡ 1 exists with $|g(x)| \leq k$ for all $x\epsilon(a,b)$, then g has a unique fixed point p in [a, b].

Now, suppose that (X, d) be a complete metric space and $T : X \to X$ be a map. The mapping T satisfies a Lipschitz condition with constant $\alpha \leq 0$ such that d $(Tx, Ty) \leq d(x, y)$, for all x, y in X. For different values of α, we have the following cases:

(a) T is called a contraction mapping if $\alpha < 1$;

(b) T is called non-expansive if $\alpha \leq 1$;

(c) T is called contractive if $\alpha = 1$.

It is clear that contraction contractive non-expansive Lipschitz. However, converse may not true in either case as:

(i) The identity map $I : X \to X$, where X is a metric space, is non-expansive but not contractive.

(ii) Let$X = [0, \infty)$ be a complete metric space equipped with the metric of absolute value. Define, $f : X \to X$ given by $f(x) = x + 1/x$. Then f is contractive map, while f is not a contraction.

There are two important fixed point theorems: one is Brouwers, and the other Banachs fixed point theorem. Brouwers fixed point theorem is existential by its nature. Brouwer (1912): Every continuous self map on the closed unit ball $C = x : \|x\| \leq 1$ in Rn has a fixed point.

The elegant Banachs fixed point theorem solves:
(a) the problem on the existence of a unique solution to an equation,

(b) gives a practical method to obtain approximate solutions and

(c) gives an estimate of such solutions.

The applications of the Banachs fixed theorem and its generalizations are very important in diverse disciplines of mathematics, statistics, engineering and economics. In 1922, Banach [8] proved a fixed-point theorem and called it Banach Fixed Point Theorem/Banach Contraction Principle which is considered as the mile stone in fixed point theory. This theorem states that if T is self mapping of a complete metric space (X, d) and there exists a number $h\epsilon[0, 1)$, such that for all x, $y\epsilon X$,

$$d(Tx, Ty) \leq hd(x, y)$$

then T has a unique fixed point, i.e., every contraction map on a complete metric space has a fixed point. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. This theorem was generalized and improved in different ways by various authors. This principle has had many applications but it suffers from one drawback - the definition requires that T be continuous throughout X.

Definition 1.1. Let X be a nonempty set and $T : X \to X$ be a mapping. A solution of an equation $Tx = x$ is called a fixed point of T.

Example 1.1. . *Examples of fixed point [i- iv] are as follows:*

(i) A translation mapping has no fixed point, that is, $Tx = x + 3$ *for all* $x \epsilon R$.

(ii) A mapping $T : R \to R$ *defined by* $Tx = \frac{x}{p} - (p - 1)$, *where p is a positive integer, then* $x = p$ *is the unique fixed point.*

(iii) A mapping $T : R \to R$ *defined by* $Tx = x^2$ *has two fixed points 0 and 1.*

(iv) A mapping $T : R \to R$ *defined by* $Tx = x$, *has infinitely many fixed points, i.e, every point of domain is a fixed point of* T.

Therefore, from the above examples one can conclude that a mapping may have a unique fixed point, it may have more than one or even infinitely many fixed points and it may not have any fixed point. Theorems dealing with the existence and construction of a solution to an operator equation Tx = x form the part of fixed point theory.

We note that every contraction mapping is continuous and uniformly continuous but converse need not be true. The first answer of this question was given by Kannan [55] in 1968, who proved a fixed point theorem for operators which dont have to be continuous.

Kannan (1968): If T is self mapping of a complete metric space X satisfying

(1.2)

$$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)]$$

for all x, y in X and $0 < k < 1/2$, then T has unique fixed point in X. We note that a map T is not continuous even though T has a fixed point. However, in every case, the maps involved are continuous at the fixed point. Therefore, Kannan type and their generalizations have been considered as an important class of mappings in fixed point theory. Following Kannan, Chatterjea [15] proved a fixed point theorem for operator which satisfies the condition: there exists $c\epsilon[0, 1/2)$ such that for all $x, y\epsilon X$,

(1.3)

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$$

Rhoades [92] had shown that these three conditions (1.1), (1.2) and (1.3) are independent. Zamrfescu [119] combined the conditions (1.1), (1.2) and (1.3) as follows:

there exist the real numbers a, b and c satisfying $0 \leq a < 1, 0 \leq b < 1/2$, and $0 \leq c < 1/2$; such that for each x, $y\epsilon X$ at least one of the following is true:

$$(z1) d(Tx, Ty) \leq ad(x, y)$$

$$(z2) d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$$

$$(z3) d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$$

In 1983, Rus [94] gave another generalization of Banach contraction principle replacing the condition (1.1) with the next condition as follows:

there is a comparison function $\varphi : \mathbb{R}+ \rightarrow \mathbb{R}+$such that

(1.4)
$$d(Tx,Ty) \leq \varphi(d(x,y)) forall x, y \epsilon X.$$

A generalization of Kannan Theorem was made by Bianchini [9], who replace the condition (1.2) with:

there is $a \in [0, 1)$ such that for all x, $y \in X$. (1.5)

$$d(Tx,Ty) \leq \text{a max } \{d(x,Tx), d(y,Ty)\}.$$

In these condititions the operator T has a unique fixed point.

Chapter 2

Review of the Literature

2.1 VARIOUS TYPES OF SPACES

We emphasis our research mainly on the following spaces:

Metric spaces
G-metric spaces

2.2 Metric Spaces

In 1906, Maurice Frechet (1878-1973), a French mathematician, introduced the notion of metric space, which is derived from the word metor (measure). Further, he pioneered the study of such spaces and their applications to different areas of mathematics. Though, the definition presently in use is given by the German mathematician, Felix Hausdroff (1868-1942) in 1914.

Let X be an arbitrary set. Let $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:
(i) $d(x, y) \geq 0$; $d(x, y) = 0$ iff $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$
for all $x, y, z \in X$. The set X together with the metric d, i.e., (X, d) is called a metric space.
Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be
(i) convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$ as $n \to \infty$ or $lim_{n\to\infty} x_n = x$.

(ii) Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number $n(\varepsilon)$ such that for all $n > m > n(\varepsilon), d(x_n, x_m) < \varepsilon$.
(iii) complete if every Cauchy sequence is convergent in X.

2.3 G-metric spaces

In 2003, Mustafa and Sims [8] have shown that most of the results concerning Dhages D-metric space are invalid. Therefore, they introduced an improved version of the generalized metric space structure, and called it as G-metric spaces.

In 2006, Mustafa and Sims [8] introduced the concept of G-metric space as follows:
Let X be a nonempty set, and let $G : XXX \to \mathbb{R}+$ be a function satisfying the following:

(G1) G(x, y, z) = 0 if x = y = z,
(G2) $0 < G(x, x, y) forallx, yinXwithz \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z) forallx, y, zinXwithz \neq y$,
(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = . . . (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all x, y, z, a in X (rectangle inequality).
Then the function G is called a generalized metric or, more specifically, a G-metric on X and the pair (X, G) is called a G-metric space.

Let (X, G) be a G-metric space. Then for $x_0 \epsilon X, r > 0$, the G-ball with center x_0 and radius r is

$$BG(x_0, r) = y \in X; G(x_0, y, y) < r$$

Let (X, G) be a G-metric space. Then a sequence x_n is

G-convergent to x if $\lim_{m,n \to \infty} G(x, x_n, x_m) = 0$; i.e., for any $\epsilon > 0$, there exists $N \epsilon N$(set of natural numbers) such that $G(x, x_n, x_m) < \epsilon$, for all n, m N. We call x as the limit of the sequence and write x_n x or $\lim_{n \to \infty} x_n = x$.

said to be G-Cauchy if for each $\epsilon > 0$, there is N N such that $G(x_n, x_m, x_1) < \epsilon$, for all n, m, l N that is if $G(x_n, x_m, x1) < 0$ as n, m, l .

said to be G-complete(or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

A G - metric space (X, G) is called a symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all x, $y \epsilon X$.

Now, we state some basic results which are useful for our results in G-metric spaces.

Let (X, G) be a G-metric space. Then, for any x, y, z, $a \epsilon X$ it follows that:

(i) if $G(x, y, z) = 0$, then $x = y = z$,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(y, x, x)$,

(vi) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq (G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

Various types of mappings

Now, we discuss various types of mappings in metric spaces which are basic tools for our further study of different spaces. The different types of mappings used in different chapter are akin to metric spaces and these mappings can also be defined on the same line in other spaces.

First, we discuss various types of mappings in metric spaces. In metric fixed point theory, Banach gives classical theorem known as Banach Contraction Principle, which is the beginning of this theory. This principle gives:

(i) existence and uniqueness of fixed points.

(ii) methods for obtaining approximative fixed points.

The contraction principle has had many applications which are scattered throughout almost all the branches of mathematics.

Various types of mappings in metric spaces

In 1974, Pfeffer [85] showed that an involution r of a circle S has a fixed point if and only if there exist a free involution of S which commutes with r. This observation leads to interdependence between commutativity and existence of fixed points.

In 1976, Jungck [47] proved a common fixed point theorem for commuting maps, generalizing the Banachs fixed point theorem.

In 1979, Das and Naik [23] generalized Jungcks result. Since then various generalizations of commuting mappings have been proposed and studied by several authors. On the other hand, in 1982, Sessa [103] defined the notion of weak commutativity as follows: Two self-mappings f and g of a metric space (X, d) are said to be weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all x in X.

In 1984, Khan et. al. [13] addressed a new category of fixed point problems with the help of a control function and called it altering distance function.

A function $\Psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:
(i) $\Psi(0) = 0$,
(ii) Ψ is continuous and monotonically non-decreasing.

Khan et. al. proved the following fixed point theorem using altering distance function as follows: Let (X, d) be a complete metric space. Let Ψ be an altering distance function and $f : X \to X$ be a self-mapping which satisfies the following inequality:

(2.1)

$$\Psi(d(fx, fy))c\Psi(d(x, y))$$

for all $x, y \in X$ and for some $0 < c < 1$. Then f has a unique fixed point. Altering distance has been used in metric fixed point theory in a number of papers. Some of the works utilizing the concept of altering distance function are noted in [3, 16, 20, 21].

In 2000 and 2005, Chaudhary et al. ([6] and [7]) extend the notion of altering distance to two variables and three variables.

In 1986, Jungck [48] coined the notion of compatible mappings and proved common fixed point theorems related to these maps.

Two self-mappings f and g of a metric space (X, d) are said to be compatible if $\lim_{n\to\infty} d(fgy_n, gfx_n = 0$, whenever x_n is a sequence in Xsuch that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fx_n = t$ for some t in X. This concept has been useful for obtaining fixed point theorems for compatible mappings satisfying contractive conditions and assuming continuity of atleast one of the mappings. It has been known from the paper of Kannan [12] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point. This paper was a genesis for a multitude of fixed point papers over the next two decades.

The evolution of weak commutativity and compatibility give new direction to fixed point theory and researchers start to relax the condition of commutativity and compatibility to improve common fixed point theorems. Consequently, the recent literature of metric fixed point theory has witnessed the evolution of several weak conditions of commutativity such as: Compatible mappings of type (A) ([49]), Compatible mappings of type (B), Compatible mappings of type (P), Compatible mappings of type (C), Biased maps ([50]) and several others whose lucid survey and illustration are available in Murthy [71].

In 1994, Pant [17] introduced the notion of R-weakly commuting mappings in metric spaces, firstly to widen the scope of the study of common fixed point theorems from the class of compatible to the wider class of R-weakly commuting mappings. Secondly, maps are not necessarily continuous at the fixed point.

A pair of self-mappings (f, g) of a metric space (X, d) is said to be R-weakly commuting if there exists some such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all x in X.

In 1996, Jungck [11] introduced the concept of weakly compatible maps as follows:

Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

In 2002, Aamri and Moutawakil introduced property (E.A.) The property (E.A.) buys containment of range without any continuity requirement besides relaxing the commutativity requirement at the point of coincidence. In general, a pair enjoying property (E.A.) need not follow the pattern of containment of range of one map into the range of other. Moreover, the range space. We also note that the property (E.A.) needs not to satisfy E.A property.

Definition 2.1. Two self mappings f and g of a matric space (X, d) are said to satisfy E.A. property if there exists a sequence x_n in X such that
$lim_{n \to \infty} fx_n = lim_{n \to \infty} gx_n = t$ for some $t \in X$.

In 2011, sintunavarat et. al introduced the notion of (CLR_f) property as follow:

Definition 2.2. Two self mappings f and g of a matric space (X, d) are said to satisfy (CLR_f) property if there exists a sequence x_n in X such that
$lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = fx$ for some $tinX$.

Example 2.1. *Let* $X = [0, 1]$ *be endowed with the Euclidean metric* $d(x, y) = |x - y|$ *and let* $fx = x$ *and* $gx = x^2$ *for each* $x \in X$.
Consider the sequence $x_n = 1/n$ *so that* $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = 0 = f(0)$. *Hance the pair* (f, g) *satisfy the* (CLR_f) *property.*

2.4 Objective of the study

The objective of our study is to meet the following
To design a frame work to survey the fixed point in G-metric space is study by other researcher using
Weakly compatible mappings
E.A. property
(CLR) property

Chapter 3

Main Result

In this chapter first of all we prove a fixed point theorem for a single map. Then we prove some common fixed theorem for a pairs of self maps satisfying weakly compatible property along with E.A property and CLR property. Also we prove common fixed theorem for six self maps satisfying generalized(ψ, ϕ)-integral type contraction in G-metric space.

3.1 Common Fixed Point Theorems for Generalized $\psi_{\int\varphi}$-Weakly Contractive Mappings in G-Metric Space

Definition 3.1. Let (X, G) be a G-metric space and $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a Lebesgue integrable mapping. A mapping $T : X \longrightarrow X$ is said to be $\psi_{\int\varphi}$ - weakly contractive if for all x, y, z in X
$\psi(\int_0^{G(Tx,Ty,Tz)} \varphi(t)dt) \leq \psi(\int_0^{G(x,y,z)} \varphi(t)dt) - \phi(\int_0^{G(x,y,Tz)} \varphi(t)dt)$
where$\phi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous and non-decreasing function such that $\phi(t) = 0 = \psi(t)$ if and only if t=0.

Theorem 3.2. *Let (X, G) be a complete G-metric space and* $T : X \rightarrow X is \psi_{\int\varphi}$ *- weakly contractive mapping, where* $\varphi : [0, \infty) \rightarrow [0, \infty)$ *is a Lebesgue integrable mapping which is summable, non-negative and such that*
$\int_0^{\varepsilon} \varphi(t)dt > 0, for each \varepsilon > 0.$
then T has a unique fixed point.

Proof. Let $x_0 \epsilon X$ be an arbitrary point and choose a sequence x_n in X such that $x_n = Tx_{n-1} for all n > 0.$

for (2.1),we have

$$\psi(\int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt) = \psi(\int_0^{G(Tx_n,Tx_{n-1},Tx_{n-1})}\varphi(t)dt)$$

$$\leq \psi(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt) - \phi(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt)$$

$$\leq \psi(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt)$$

using monotone property of ψ-function, we have

$$\int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt \leq \int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt$$

let $y_n = \int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt, then 0 \leq y_n \leq y_{n-1} for all n > 0$. It follows thats the sequence y_n is monotone decreasing and lower bounded, so, there exists $r \geq 0$, such that

$$\lim_{n\to\infty}\int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt = \lim_{n\to\infty} y_n = r.$$

Then (by the lower semi-continuity of ϕ)

$$\phi(r) \leq \lim_{n\to\infty}\inf \phi(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt)$$

let, if possible, $r > 0$.

taking upper limit as $n \to \infty$ on either side of (1.3), we get

$$\psi(r) \leq \psi(r) - \lim_{n\to\infty}\inf \phi(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt)$$

$$\leq (r) - \phi(r), a contradiction.$$

thus, r=0, i.e., $\lim_{n\to\infty}\int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt = \lim_{n\to\infty} y_n = 0$

therefore, we have

$\lim_{n\to\infty} G(x_{n+1}, x_n, x_n) = 0$

now, we prove that x_n is a G- Cauchy sequence. let if possible, x_n is not a G-Cauchy sequence. Then, there exists, an $\varepsilon > 0$ and subsequences $x_{m(k)} and x_{n(k)} of x_n$ with $n(k) > m(k) > k$ such that

$G(x_{n(k)-1}, x_{m(k)}, x_{m(k)}) \geq \varepsilon$

Let m(k) be the least positive integer exceeding n(k) satisfying and such that

$G(x_{n(k)-1}, x_{m(k)}, x_{m(k)}) < \varepsilon$, for every integer k.

then, we have

$\leq G(x_{n(k)}, x_{m(k)}, x_{m(k)})$

$\leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)}, x_{m(k)})$

$< \varepsilon + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1})$.
now

$0 < \delta = \int_0^{\varepsilon} \varphi(t)dt \leq \int_0^{G(x_{n(k)}, x_{m(k)}, x_{m(k)})} \varphi(t)dt \leq \int_0^{\varepsilon + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1})} \varphi(t)dt.$

Letting $k \to \infty$ and using, we get

$\lim_{k\to\infty} \int_0^{G(x_{n(k)}, x_{m(k)}, x_{m(k)})} \varphi(t)dt = \delta$

By the triangular inequality,

$G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{m(k)-1}, x_{m}$

$G(x_{n(k)-1}, x_{m(k)}, x_{m(k)-1}) \leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{m(k)}, x_{m(t)}, x_{m(k)-1})$

therefore, we have

$\int_0^{G(x_{n(k)}, x_{m(k)}, x_{m(k)})} \varphi(t)dt \leq \int_0^{G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)})} \varphi(t)d$

$\int_0^{G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1})} \varphi(t)dt \leq \int_0^{G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1})} \varphi(t)dt$

Letting $k \to \infty$ in the above two inequalities, we get

$\lim_{k\to\infty} \int_0^{G(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1})} \varphi(t)dt = \delta$

Taking $x = x_{n(k)-1}, y = x_{m(k)-1}, z = x_{m(k)-1}$, we get

$\psi(\int_0^{G(Tx_{n(k)-1},Tx_{m(k)-1},Tx_{m(k)-1})} \varphi(t)dt) = \psi(\int_0^{G(x_{n(k)},x_{m(k)},x_{m(k)})} \varphi(t)dt)$

$\leq \psi(\int_0^{G(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1})} \varphi(t)dt) - \phi(\int_0^{G(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1})} \varphi(t)dt)$

Letting $k \to \infty$, and property of ψ, ϕ, we get

$$\psi(\delta) \leq \psi(\delta) - \phi(\delta)$$

a contradiction, since $\delta > 0$.

thus x_n is G-Cauchy sequence. since X is a complete metric space, there exists u in X such that
$\lim_{n\to\infty} x_n = u$

Taking $x = x_{n-1}, y = u, z = u$, we get

$$\psi(\int_0^{G(Tx_{n-1},Tu,Tu)} \varphi(t)dt) = \psi(\int_0^{G(x_n,Tu,Tu)} \varphi(t)dt)$$

$$\leq (\int_0^{G(x_{n-1},u,u)} \varphi(t)dt) - \phi(\int_0^{G(x_{n-1},u,u)} \varphi(t)dt)$$

Letting $n \to \infty$ and property of ψ, ϕ, we get

$\psi(\int_0^{G(u,Tu,Tu)} \varphi(t)dt) \leq \psi(0) - \phi(0) = 0$

Which implies that

$\int_0^{G(u,Tu,Tu)} \varphi(t)dt = 0$

Thus $G(u, Tu, Tu) = 0$, that is, u=Tu
Now, we prove that u is the unique fixed point of T. let us suppose that v be another common fixed point of T, i.e., $Tv = v$.
Putting x=u, y=v, z=v, we get

$$\psi(\int_0^{G(Tu,Tv,Tv)} \varphi(t)dt) = \psi(\int_0^{G(u,v,v)} \varphi(t)dt)$$

$$\leq (\int_0^{G(u,v,v)} \varphi(t)dt) - \phi(\int_0^{G(u,v,v)} \varphi(t)dt)$$

From here, we get

$\phi(\int_0^{G(u,v,v)} \varphi(t)dt) = 0$

which implies that, G(u, v, v)=0, that is, u=v.

Hence u is the unique fixed point of T.

□

Weakly compatible mapping and E.A. property:

Theorem 3.3. *Let (X, G) be a G-metric space and let f and g be weakly compatible self maps of X satisfying and the followings:*
f and g satisfy the E.A. property,
fX is closed subset of X.
Then f and g have a unique fixed point.

Proof. Since f and g satisfy the E.A. property, there exists a sequence x_n in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = x_0 for some x_0 \epsilon X$
since fX is closed subset of X, therefore, we have

$\lim_{n\to\infty} fx_n = fz for some z \epsilon X$

Now, we claim that fz = gz
From, we have

$\psi(\int_0^{G(gx_n,gz,gz)} \varphi(t)dt) \leq \psi(\int_0^{G(fx_n,fz,fz)} \varphi(t)dt) - \phi(\int_0^{G(fx_n,fz,fz)} \varphi(t)dt)$

from and property of ψ, ϕ, we have

$\psi(\int_0^{G(fz,gz,gz)} \varphi(t)dt) \leq \psi(0) - \phi(0) = 0$, implies that, $\int_0^{G(fz,gz,gz)} \varphi(t)dt = 0$

Thus, we have G(fz, gz, gz)=0, implies that, fz=gz.

Now, we show that gz is the common fixed point of f and g.
suppose that, $gz \neq fz$.
Since f and g are weakly compatible, gfz=fgz and therefore, ffz=ggz.
From, we have

$\psi(\int_0^{G(gz,ggz,ggz)} \varphi(t)dt) \leq \psi(\int_0^{G(fz,fgz,fgz)} \varphi(t)dt) - \phi(\int_0^{G(fz,fgz,fgz)} \varphi(t)dt)$

$= \psi(\int_0^{G(gz,ggz,ggz)} \varphi(t)dt) - \phi(\int_0^{G(gz,ggz,ggz)} \varphi(t)dt)$

$< \psi(\int_0^{G(gz,ggz,ggz)} \varphi(t)dt)$, a contradiction

Thus, ggz=gz.
Hence gz is the common fixed point of f and g.
Finally, we show that the fixed point is unique.
Let u and v be two common fixed point of f and g such that $u \neq v$.
From, we have

$\psi(\int_0^{G(u,v,v)}\varphi(t)dt) = \psi(\int_0^{G(gu,gv,gv)}\varphi(t)dt)$

$\leq \psi(\int_0^{G(fu,fv,fv)}\varphi(t)dt) - \phi(\int_0^{G(fu,fv,fv)}\varphi(t)dt)$

$\doteq \psi(\int_0^{G(u,v,v)}\varphi(t)dt) - \phi(\int_0^{G(u,v,v)}\varphi(t)dt)$

$< \psi(\int_0^{G(u,v,v)}\varphi(t)dt)$, a contradiction.

therefor, u=v, and hence the uniqueness follows.

□

Weakly compatible mapping and (CLR) property:

Theorem 3.4. *Let (X, G) be a G-metric space and let f and g be weakly compatible self maps os X satisfying and the following;*
f and g satisfy (CLR_f) *property.*
Then f and g have a unique common fixed point.

Proof. since f and g satisfy the (CLR_f) property, there exists a sequence x_n in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some $x \epsilon X$

From, we have

$\psi(\int_0^{G(gx_n,gx,gx)}\varphi(t)dt) \leq \psi(\int_0^{G(fx_n,fx,fx)}\varphi(t)dt) - \phi(\int_0^{G(fx_n,fx,fx)}\varphi(t)dt)$

Letting $n \to \infty$ and using conditions of ψ, ϕ, we get

$\psi(\int_0^{G(fx,gx,gx)}\varphi(t)dt) \leq \psi(\int_0^{G(fx,fx,fx)}\varphi(t)dt) - \phi(\int_0^{G(fx,fx,fx)}\varphi(t)dt)$

$\doteq \psi(0) - \phi(0) = 0$, implies that, $fw = fgx = gfx = gw$

Thus, G(fx, gx, gx)=0, that is fx=gx.
let w=fx=gx

Since f and g are weakly compatible, fgx = gfx, implies that, fw = fgx = gfx = gw.
Now, we claim that tw = w.
Let, if possible, $tw \neq w$
From, we have

$$\psi(\int_0^{G(gw,w,w)} \varphi(t)dt) \doteq \psi(\int_0^{G(gw,gx,gx)} \varphi(t)dt)$$

$$\leq \psi(\int_0^{G(fw,fx,fx)} \varphi(t)dt) - \phi(\int_0^{G(fw,fx,fx)} \varphi(t)dt)$$

$$\doteq \psi(\int_0^{G(gw,w,w)} \varphi(t)dt) - \phi(\int_0^{G(gw,w,w)} \varphi(t)dt)$$

$$< \psi(\int_0^{G(gw,w,w)} \varphi(t)dt), \text{ a contradiction}$$

Hence fw = w = gw.
Hence, w is the common fixed point of f and g.
Finally, we show that the fixed point is unique.
Let v be another common fixed point of f and g suh that fv = v = gv.
Now, we claim that w = v.
Let, if possible, $w \neq v$
From, we have

$$\psi(\int_0^{G(w,v,v)} \varphi(t)dt) = \psi(\int_0^{G(gw,gv,gv)} \varphi(t)dt)$$

$$\leq \psi(\int_0^{G(fw,fv,fv)} \varphi(t)dt) - \phi(\int_0^{G(fw,fv,fv)} \varphi(t)dt)$$

$$\doteq \psi(\int_0^{G(w,v,v)} \varphi(t)dt) - \phi(\int_0^{G(w,v,v)} \varphi(t)dt)$$

$$< \psi(\int_0^{G(w,v,v)} \varphi(t)dt), \text{ a contradiction.}$$

Terefore, w = v, and hence the uniqueness follows.

□

3.2 Common fixed point theorem for six mapping satisfying generalized contractive condition of integral type in G-metric spaces

Definition 3.5. Let (X, G) be a G-metric space, and let x_n be a sequence of point of X. Then the two axioms of Wilson in G-metric space are as follows:
$(w.3)$ Given x_n, x and y in X, $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_n, y) = 0$ imply $x = y$.
$(w.4)$ Given x_n, y_n and x in X, $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_n, y_n) = 0$ imply that

$$\lim_{n\to\infty} G(y_n, y_n, x) = 0$$

.

Definition 3.6. Let (X, G) be a G-metric space and $A, B, S, T, g, h : X \to X$. The pairs $(A, S), (B, T) and (g, h)$ satisfy the common (E.A)property if there exist three sequences $x_n, y_n, and z_n$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gz_n = \lim_{n\to\infty} hz_n = t \in X$$

Theorem 3.7. *Let (X, G) be a $G-$metric space which satisfies $(W.3)$. Let A, B, S, T, h and g be six self-mappings of X such that*
(i)$A(X) \subset T(X), B(X) \subset S(X)$ and $h(X) \subset G(X)$

$$(ii)\psi(\int_0^{G(Ax,By,hz)} Q(t)dt) \leq \psi(\int_0^{GL(x,y,z)+(1-a)M(x,y,z)} Q(t)dt) - \phi(\int_0^{GL(x,y,z)+(1-a)M(x,y,z)} Q(t)dt) \tag{3.1}$$

for all $x, y \epsilon X$,$\psi : R_+ \longrightarrow R_+$ such that $0 < \psi(t) < t$ and $\psi : R_+ \longrightarrow R_+$ is a lebesgue integrable mapping which is summable , non negative and such that

$$\int_0^t \psi(t) > 0 for all \varepsilon > 0 \tag{3.2}$$

where

$$L(x, y, z) = max\{G(gx, Ty, Sz), G(gx, By, hz), G(By, Ty, Sz)\}$$
$$M(x, y, z) = max\{G^2(gx, Ty, Sz), G(gx, By, hz), G(By, Ty, Sz), G(gx, Ty, Sz), G(gx, By, hz), G^2(By, Ty, Sz)\}^{1/2}$$

and $0 \leq a \leq 1$, *suppose that* $(A, g)(B, T) and (h, S)$ *satisfy the common (E.A) property and are weakly compatible. If one of the subspace AX, gX, BX, TX, SX and hX of X is complete, then A, g, B, T, S and h have a unique common fixed point in X.*

Proof. Since $(A, g)(B, T) and (h, S)$ satisfy the common (E.A) property, there exist three sequence $\{x_n\}, \{y_n\} and \{z_n\}$ in X such that

$$\lim_{n\to\infty} G(Ax_n, z, z) = \lim_{n\to\infty} G(By_n, z, z)$$

$$\lim_{n\to\infty} G(Ty_n, z, z) = \lim_{n\to\infty} G(gx_n, z, z)$$

$$= \lim_{n\to\infty} G(hz_n, z, z) = 0$$

for some $z \epsilon X$ Now suppose that gX is a complete subspace of X. Then z = gu for some $u \epsilon X$ Consequently, we have

$$\lim_{n\to\infty} G(Ax_n, gu, gu)$$

$$= \lim_{n\to\infty} G(Sz_n, gu, gu) = \lim_{n\to\infty} G(By_n, gu, gu) = \lim_{n\to\infty} G(Ty, gu, gu) \tag{3.3}$$

$$= \lim_{n\to\infty} G(gx_n, gu, gu) = \lim_{n\to\infty} G(hz_n, gu, gu) = 0 \tag{3.4}$$

If $Au \neq z$ and using (2.1) then we get

$$\psi(\int_0^{G(Au, By_n, hz_n)} \varphi(t)dt) \leq \psi(\int_0^{GL(u,y_n,z_n)+(1-a)M(u,y_n,z_n)} \varphi(t)dt) - \phi \int_0^{GL(u,y_n,z_n)+(1-a)M(u,y_n,z_n)} \varphi(t)dt \tag{3.5}$$

where

$$L(u, v, z_n) = max G(gu, Ty_n, Sz_n), G(gu, By_n, hz_n), G(By_n, Ty_n, Sz_n)$$

,

$$M(u, y_n, z_n) = [max G^2(gu, Ty_n, Sz_n), G(gu, By_n, hz_n), G(By_n, Ty_n, Sz_n), G(gu, Ty_n, Sz_n), G(gu, By_n, h$$

Taking $n \to \infty$, we get $L(u, y_n, z_n) = 0$ and M(x, y, z)=0 respectively Now (2.5) becomes

$$\lim_{n\to\infty} \int_0^{G(Au, By_n, gz_n)} \varphi(t)dt = 0$$

and (2.2) implies that $\lim_{n\to\infty} G(Au, bY_n, gz_n) = 0$ By (W.3), we have z = Au = gu. Since $AX \subset TX$, there exists $v \epsilon X such that z = Az = Tv$

if $Bv \neq z$ using (2.1) we have,

$$\int_0^{G(z,Bv,gz_n)} \varphi(t)dt \leq \psi(\int_0^{aL(u,v,z_n)+(1-a)M(u,v,z_n)} \varphi(t)dt) - \phi \int_0^{aL(u,v,z_n)+(1-a)M(u,v,z_n)} \varphi(t)dt \tag{3.6}$$

where $L(u,v,z_n) = maxG(gu,Tv,Sz_n), G(g_n,Bv,hz_n), G(Bv,Tv,Sz_n)$,
$M(u,v,z_n) = [maxG^2(gu,Tv,Sz_n), G(gu,Bv,Sz_n), G(gu,Tv,Sz_n), G(gu,Bv,hz_n), G^2(Bv,Tv,Sz_n)]^{1/2}$
form which we get

$$L(u,v,z_n) = G(z,Bv,z)$$

and

$$M(u,v,z_n) = [G^2(z,Bv,z)]^{1/2}$$

respectively, Now (2.6) becomes

$$\int_0^{G(z,Bv,z)} \varphi(t)dt \leq \psi(\int_0^{aG(z,Bv,z)+(1-a)G(z,Bv,z)} \varphi(t)dt)$$

$$= \psi(\int_0^{G(z,Bv,z)} \varphi(t)dt) < \int_0^{G(z,Bz,z)} \varphi(t)dt$$

which is a contradiction. Hence,

$$\int_0^{G(z,Bz,z)} \varphi(t)dt = 0$$

and (2.2) implies that $\lim_{n\to\infty} G(z,Bz,z) = 0$
i.e, $z = Bz = Tv$
Since (A,g) is weakly compatible, so Agu=gAu Whenever Au=gu which implies

$$Az = gz \tag{3.7}$$

let us show that z is a common fixed point of A, g, B, T, S and h.
if $z \neq Az$, again using (2.1), we get

$$\int_0^{G(Az,z,z)} \varphi(t)dt \leq \psi(\int_0^{aL(z,v,z_n)+(1-a)M(z,v,z_n)} \varphi(t)dt) - \phi(\int_0^{aL(z,v,z_n)+(1-a)M(z,v,z_n)} \varphi(t)dt \tag{3.8}$$

$$\leq \psi(\int_0^{aL(z,v,z_n)+(1-a)M(z,v,z_n)} \varphi(t)dt)$$

where

$$L(z,v,z_n) = maxG(gz,Tv,Sz_n), G(gz,Bv,hz_n), G(Bv,Tv,Sz_n)$$

$M(z,v,z_n) = [maxG^2(gz,Tv,Sz_n),G(gz,Bz,hz_n),G(Bz,Tv,Sz_n),G(gz,Bv,hz_n),G^2(Bz,Tv,Sz_n)]^{1/2}$
form which we get $L(z,v,z_n) = G(Az,z,z)$ and
$M(u,v,z_n) = [G^2(Az,z,z)]^{1/2}$
respectively. Now (2.8) becomes

$$\int_0^{G(Az,z,z)} \varphi(t)dt \leq \psi(\int_0^{aG(Az,z,z)+(1-a)G(Az,z,z)} \varphi(t)dt)$$
$$= \psi(\int_0^{G(Az,z,z)} \varphi(t)dt) < \int_0^{G(Az,z,z)} \varphi(t)dt$$

which is a contradiction. Therefore,

$$\int_0^{G(Az,z,z)} \varphi(t)dt = 0$$

and (2.2) implies that G(Az, z, z)=0 i.e., z=Az=gz
similarly, the weak compatibility of B and T implies BTv=TBv, i.e., Bz=Tz. if $z \neq Bz$ by using (2.1)and (2.2), a similar calculation to the above yields z=Bz=Tz. Thus, z is a common fixed point of A, g, B, T, S and h.
When TX is assumed to be a complete of X, then the proof is similar. on the other hand, the cases in which AX, or BX and SX or hX is a complete subspace of X are similar to the cases in which TX or gX is complete, respectively, by(2.1).
For the uniqueness of the common fixed piont z, let $w \neq z$ be another common fixed point of A, g, B, T, S and h. then, using (2.1), we obtain

$$\int_0^{G(w,z,z)} \varphi(t)dt \leq \psi(\int_0^{aL(w,z,z)+(1-a)M(w,z,z)} \varphi(t)dt) - \phi(\int_0^{aL(w,z,z)+(1-a)M(w,z,z)} \varphi(t)dt)$$
$$= \psi(\int_0^{G(w,z,z)} \varphi(t)dt) < \int_0^{G(w,z,z)} \varphi(t)dt$$

which is a contradiction. Therefore $\int_0^{G(w,z,z)} \varphi(t)dt = 0 and (2.2) implies that z = w.$
For $\psi(t) = 1$ in theorem 2.1, we obtain the following corollary:

□

Corollary 3.8. *Let (X, G) be a G-metric space which satisfies $(w.3)$*
Let A, B, S, T, h and g be six self mapping of X such that
(i) $A(x) \subset T(x), B(x) \subset S(x)$ and $h(x) \subset g(x)$
(ii) $G(Ax, By, hz) \leq \Psi(aL(x,y,z) + (1-a)M(x,y,z) - \phi(aL(x,y,z) + (1-a)M(x,y,z)$
for all $x, y, \epsilon X$ where
$L(x,y,z) = maxG(gx,Ty,Sz)G(gx,By,hz),G(By,Ty,Sz)$
$M(x,y,z) = [maxG^2(gx,Ty,Sz),G(gx,By,hz),G(By,Ty,Sz),G(gx,Ty,Sz),G(gx,By,hz),G^2(By,T$
and $0 \leq a \leq 1$. suppose that $(A,g),(B,T)$ and (h,S) satisfy the common (E.A) property and are weakly compatible.

If one of the subspace AX, gX, BX, TX, SX *and* hX *of* X *is complete, then* A, g, B, T, S *and* h *have a unique common fixed point in* X*.*

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