Fixed Point Theorems in Metric Spaces and b-Metric Spaces

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ABSTRACT

In this report, first of all, we shall study about metric spaces and b-metric spaces. Secondly, we prove certain fixed point theorems in these spaces.

1. INTRODUCTION

1.1 Background of fixed point theory

Fixed point theory itself is a beautiful mixture of analysis, topology and geometry. Over since last 50 years, fixed point theory has been revealed itself as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in diverse fields such as in biology, chemistry, economics, engineering, game theory and physics. The point at which the curve y = f (x) and the line y = x intersects gives the solution of the curve, and the point of intersection is the fixed point of the curve. The usefulness of the concrete applications has increased enormously due to the development of accurate techniques for computing fixed points.

Fixed point theory is rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include numerical methods like Newton-Raphson method, establishing Picard's Existence theorem, existence of solution of integral equations and a system of linear equations.

1.2 Significance of Fixed Points

Fixed points are the points which remain invariant under a map/transformation. Fixed points tell us which parts of the space are pinned in plane, not moved, by the transformation. The fixed points of a transformation restrict the motion of the space under some restrictions.

We note that fixed point problems and root finding problems $f(x) = 0$ are equivalent.

Now, the question arise what type of problems have the fixed point. The fixed point problems can be elaborated in the following manner:

(i) What functions/maps have a fixed point ?

(ii) How do we determine the fixed point ?

(iii) Is the fixed point unique ?

Next, we state a result which gives us the guarantee of existence of fixed points.

Suppose g is continuous self map on [a, b]. Then, we have the following conclusions:

(i) If the range of the mapping $y = g(x)$ satisfies $y \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in [a, b].

(ii) Suppose that $g'(x)$ is defined over (a, b).

(iii) a positive constant $k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$, then g has a unique fixed point p in [a, b].

Now, suppose that (X, d) be a complete metric space and $T : X \rightarrow X$ be a map. The mapping T satisfies a Lipschitz condition with constant $\alpha \geq 0$ such that $d(Tx, Ty) \leq \alpha\, d(x, y)$, for all x, y in X. For different values of α, we have the following cases:

(a) T is called a contraction mapping if $\alpha < 1$;

(b) T is called non-expansive if $\alpha \leq 1$;

(c) T is called contractive if $\alpha = 1$.

It is clear that contraction ⇒ contractive ⇒ non-expansive⇒ Lipschitz. However, converse may not true in either case as:

(i) The identity map I: X →X, where X is a metric space, is non-expansive but not contractive.

(ii) Let X = [0, ∞) be a complete metric space equipped with the metric of absolute value. Define, f : X →X given by f(x) = x + 1/x. Then f is contractive map, while f is not a contraction.

Definition 1.1 Let X be a nonempty set and T : X → X be a mapping. A solution of an equation Tx = x is called a fixed point of T.

Example 1.1 Examples of fixed point are as follows:

(i) A translation mapping has no fixed point, that is,

Tx = x + 3 for all x ∈R.

(ii) A mapping T : R → R defined by Tx = $\frac{x}{p}$ (p−1), where p is a positive integer, then x = −p is the unique fixed point.

(iii) A mapping T : R → R defined by Tx = x^2 has two fixed points 0 and 1.

(iv) A mapping T : R → R defined by Tx = x, has infinitely many fixed points, i.e, every point of domain is a fixed point of T.

Therefore, from the above examples one can conclude that a mapping may have a unique fixed point, it may have more than one or even infinitely many fixed points and it may not have any fixed point. Theorems dealing with the existence and construction of a solution to an operator equation Tx = x form the part of fixed point theory.

1.3 Metric Spaces

In 1906, Maurice Frechet (1878-1973), a French mathematician, introduced the notion of metric space, which is derived from the word metor (measure). Further, he pioneered the study of such spaces and their applications to different areas of mathematics. Though, the definition presently in use is given by the German mathematician, Felix Hausdroff (1868-1942) in 1914.

Let X be an arbitrary set. Let d : X × X → $\mathbb{R}^+$ satisfies the following conditions:

(i) d(x, y) ≥ 0; d(x, y) = 0 iff x = y,

(ii) d(x, y) = d(y, x),

(iii) d(x, z) ≤ d(x, y) + d(y, z)

for all x, y, z ∈ X. The set X together with the metric d, i.e., (X, d) is called a metric space.

Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be

(i) convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $\{x_n\} \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

(ii) Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number $n(\varepsilon)$ such that for all $n > m > n(\varepsilon)$, $d(x_n, x_m) \prec \varepsilon$.

(iii) complete if every Cauchy sequence is convergent in X.

1.4 b-Metric Spaces

The concept of b−metric spaces was introduced by Bakht [7] in 1989, who used it to prove a generalization of the Banach contraction principle in spaces endowed with such kind of metrics. Since then, this notion has been used by many authors to obtain various fixed point theorems.

Definition 1.2 Let X be a non-empty set and $k \geq 1$ a given real number. A function $d : X \times X \to \mathbf{R}^+$ is a b−metric iff for each x, y, z ∈ X, following conditions are satisfied:

(b1) $d(x, y) = 0$ iff $x = y$,

(b2) $d(x, y) = d(y, x)$,

(b3) $d(x, z) \leq k\,[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b−metric space.

It should be noted that the class of b−metric spaces is effectively larger than that of metric spaces. Indeed, a b-metric is a metric if and only if $k = 1$.

Example 1.2 Let (X, d) be a metric space and $\rho(x, y) = d(x, y)^p$ where $p>1$ is a real number. We show that ρ is a b−metric with $k = 2^{p-1}$. Obviously, conditions (b1) and (b2) of definition 1.1 are satisfied. If $1 < p < \infty$, then convexity of the function $f(x) = x^p$ $(x > 0)$ implies that $\left(\frac{a+b}{2}\right)^p \leq \frac{a^p}{2} + \frac{b^p}{2}$ that is, $(a + b)^p \leq 2^{p-1}(a^p+b^p)$ holds.

Thus for each x, y, z ∈ X, we have

$$\rho(x, y) = (d(x, y))^p \leq (d(x, z) + d(z, y))^p$$

$$\leq 2^{p-1}\left((d(x, z))^p + (d(z, y))^p\right) = 2^{p-1}(\rho(x, z) + \rho(z, y))$$

So condition (b3) of definition 1.1 holds and ρ is a b−metric. Note that (X, ρ) is not necessarily a metric space.

For example, if X = R be the set of real numbers and $d(x, y) = |x - y|$ a usual metric, then $\rho(x, y) = (x - y)^2$ is a b−metric on R with $k = 2$, but not a metric on R, as the triangle inequality for a metric does not hold.

Before stating our results, we present some definitions and propositions in a b−metric space.

Definition 1.3 Let (X, d) be a b−metric space. Then a sequence $\{x_n\}$ in X is called:

(a) convergent if and only if there exists x ∈ X such that $d(x_n, x) \to 0$, as $n \to +\infty$. In this case, we write $\lim_{n\to\infty} x_n = x$

(b) Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to +\infty$.

Proposition 1.1 In a b−metric space (X, d) the following assertions hold:

(i) a convergent sequence has a unique limit.

(ii) each convergent sequence is Cauchy,

(iii) in general, a b-metric is not continuous.

Definition 1.4 Let (X, d) be a b−metric space. If Y is an on empty subset of X, then the closure $\bar{Y}$ of Y is the set of limits of all convergent sequences of points in Y, i.e., $\bar{Y} = \{x \in X\}$ there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} x_n = x$

Definition 1.5 Let (X, d) be a b−metric space. Then a subset Y ⊂X is called closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have x ∈ Y (i.e.$\bar{Y} = Y$).

Definition 1.6 The b−metric space (X, d) is complete if every Cauchy sequence in X converges.

In general a b−metric function d for k>1 is not jointly continuous in all of its two variables. Following is an example of a b−metric which is not continuous.

Example 1.3 Let $X = \mathbb{N} \cup \{\infty\}$ and $D : X \times X \to R$ defined by:

$$D(m,n) = \begin{cases} 0 & if\ m = n \\ \left|\frac{1}{m} - \frac{1}{n}\right| & if\ m,n\ are\ even\ or\ mn = \infty \\ 5 & if\ m,n\ are\ odd\ and\ m \neq n \\ 2 & otherwise \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$,we have

$$D(m,p) \leq 3(D(m,n) + D(n,p))$$

Thus, (X, D) is a b- metric space with k = 3. If $x_n = 2n$, for each $n \in \mathrm{N}$, the

$$D(2n,\infty) = \frac{1}{2n} \to 0\ as\ n \to \infty$$

i.e. , $x_n \to \infty$,but $D(x_{2n},1) = 2 \to D(\infty,1), as\, n \to \infty$
as b-metric is not continuous in general , so we need the following simple Lemma about the b-convergent sequences.

Lemma 1.1 Let (X, d) be a b- metric space with $k \geq 1$.Suppose that $\{x_n\} and \{y_n\}$ are b-convergent to x and y, respectively. Then we have,

$$\frac{1}{k^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq k^2 d(x,y)$$

In particular, if x = y, then we have, $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover for each $z \in X$ we have,

$$\frac{1}{k} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq kd(x,z)$$

Lemma 1.2 Let (X, d) is a b-metric space. If there exists two sequences $\{x_n\} and \{y_n\}$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$, whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$ then $\lim_{n\to\infty} y_n = t.$

Proof: By a triangle inequality in b-metric space, we have

$$d(y_n, t) \leq k(d(y_n, x_n) + d(x_n, t))$$

Now by taking the upper limit when $n \to \infty$ in the above inequality we get,

$$\limsup_{n\to\infty} d(y_n, t) \leq k(\limsup_{n\to\infty} d(x_n, y_n) + \limsup_{n\to\infty} d(x_n, t) = 0$$

Definition 1.7 Let (X, d) be a b-metric space. A pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\to\infty} d(f\, gx_n, g\, fx_n) = 0$, whenever $\{x_n\}$is a sequence in X such that :

$$\lim_{n\to\infty} f\, x_n = \lim_{n\to\infty} g\, x_n = t \; for\, some, \, t \in X.$$

2. Literature Review:

There are two important fixed point theorems: one is Brouwer's(1912), and the other Banach's(1922) fixed point theorem. Brouwer's fixed point theorem is existential by its nature.

The elegant Banach's fixed point theorem solves:-

a) the problem on the existence of a unique solution to an equation,
b) gives a practical method to obtain approximate solutions and
c) gives an estimate of such solutions.

The applications of the Banach's fixed theorem and its generalizations are very important in diverse disciplines of mathematics, statistics, engineering and economics.

In 1922, Banach [5] proved a fixed-point theorem and called it Banach Fixed Point Theorem/Banach Contraction Principle which is considered as the mile stone in fixed point theory. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. This theorem was generalized and improved in different ways by various authors. This principle has had many applications but it suffers from one drawback - the definition requires that T be continuous throughout X.

Therefore, from the above examples one can conclude that a mapping may have a unique fixed point, it may have more than one or even infinitely many fixed points and it may not have any fixed point.

We note that every contraction mapping is continuous and uniformly continuous but converse need not be true.

We note that a map T is not continuous even though T has a fixed point. However, in every case, the maps involved are continuous at the fixed point. Therefore, Kannan type and their generalizations have been considered as an important class of mappings in fixed point theory.

In 1996, Jungck [4] introduced the concept of weakly compatible maps as follows

Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

In 2002, Aamri and Moutawakil [6]introduced property (E.A.) as follows:-

Two self-mappings f and g of a metric space (X, d) are said to satisfy property (E. A.) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

The property (E.A.) buys containment of ranges without any continuity requirement besides relaxing the commutativity requirement at the points of coincidence. In general, a pair enjoying property (E.A.) need not follow the pattern of containment of range of one map into the range of other. Moreover, the completeness requirement of the space is weakened to a natural condition of completeness of the range space. We also note that the property (E.A.) needs not to satisfy the compatible property.

In 2011, Sintunavarat et al. [7] introduced the notion of (CLR_f) property as follows:-

Two self-mappings f and g of a metric space (X, d) are said to satisfy (CLR_f) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some x in X.

3. APPLICATIONS OF FIXED POINT THEORY

3.1 In Analysis

An important theoretical application of Banach fixed point theorem is the proof of:-

1.The existence of solution of differential equations.

2. The uniqueness of solutions of differential equations.

3.2 Application to image compression

The best way to store an image in memory is to store the color of each pixel. There are two problems with this method:-

- It requires an enormous quantity of memory.
- If we try to enlarge the image, for instance for using it in a large poster, then the pixels will become larger squares and we will be missing information on how to fill the details in these squares.

3.3 The Page Rank algorithm

Google's success as a search engine comes from its algorithm: the PageRank algorithm. In this algorithm, one computes a fixed point of a linear operator on $\mathbb{R}^n$ which is a contraction, and this fixed point (which is a vector) yields the ordering of the pages. In practice, the fixed point (which is an eigenvector of the eigenvalue 1) is calculated approximately as $\boldsymbol{P_n}$ for n sufficiently large.

3.4 Using Games to Find Fixed Points

It is surprisingly easy to use the existence of equilibrium in two person games to prove Kakutani's fixed point theorem in full generality. The key idea has a simple description. Fix a nonempty compact convex $X \subset R^{\mathbf{d}}$, and let $F : X \to X$ be a (not necessarily convex valued or upper semicontinuous) correspondence with compact values. We can define a two person game with strategy sets S = T = X by setting

$$u(s,t) = -min\|s-x\|^2 \quad \text{for all } x \in F(t)$$

$$v(s,t) = \begin{cases} 0 & s \neq t \\ 1 & s = t \end{cases}$$

If (s, t) is a Nash equilibrium, then s ∈ F (t) and t= s, so s = t is a fixed point. Conversely, if x is a fixed point, then (x, x) is a Nash equilibrium.

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