Some Fixed Point Theorems in b-Metric Spaces

A PROJECT REPORT SUBMITTED TO

LOVELY PROFESSIONAL UNIVERSITY,PUNJAB

For partial fulfillment of the requirements of the degree of

M.Sc.(H)Mathematics

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(REGISTRATION NUMBER:11605779)

UNDER THE SUPERVISIONOF

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1. **Abstract**

In this report we shall study about b-metric spaces and prove some fixed point theorems in these spaces.

2. **Introduction**

In 1993, Czerwik introduced the notion of b-metric spaces and observed a characterization of Banach contraction principle in the context of complete b-metric space.
The Fixed Point Theory is one of the most powerful and productive tools from the nonlinear analysis and it can be considered the kernel of the nonlinear analysis. The best known result from the Fixed Point Theory is Banach's Contraction Principle, which can be considered the beginning of this theory.
Following Petrusel and Rus, we say that S is a Picard operator if S has a unique fixed point $u*$ and $\lim_{n\longrightarrow\infty} S^n y = u*$ for all $y \in Y$ and is weakly Picard operator if the sequence $(S^n y)_{n\in\mathbb{N}}$ converges, for all $y \in Y$ and the limit is a fixed point of S. One of the ways to generalize and extend the Banach's Principle Contraction is to substitute the condition with a weaker condition or independent one. Because any contraction is a continous operator, it is natural to ask:Are there contraction conditions which do not imply the continuity of the operator?
The first answer of this question was given by R.Kannan in 1968 who proved a fixed point theorem for operators which don't have to be continous and replace the condition with: there exists $m \in [0, \frac{1}{2})$ such that
Following Kannan, Chatterjea proved a fixed point theorem for operators which satisfies the condition: there exists $n \in [0, \frac{1}{2})$ such that
It's well know, see Rhoades, that these conditions are independent.Using these conditions, L.Ciric , S.Reich and I.A. Rus proved a fixed point theorem using a very general condition: there is a nonnegative numbers h, i, j with $h + i + j < 1$ such that
By combining these conditions in an inspired manner, T.Zamrfescu considered the following class of operators: there is $h \in [0, 1)$ and $i, j \in [0, \frac{1}{2})$ such that for any $x, y \in Y$ atleast one of the following holds:
(z1)$\sigma(Sx, Sy) \leq h\sigma(x, y)$;
(z2)$\sigma(Sx, Sy) \leq i[\sigma(x, Sx) + \sigma(y, Sy)]$;
(z3)$\sigma(Sx, Sy) \leq j[\sigma(x, Sy)+\sigma(y, Sx)]$. Using the comparison function in 1983, I.A. Rus gave another generalizatin of Banach Contraction Principle replacing the first condition with the condition: there is a comparison function. $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that
$\sigma(Sx, Sy) \leq \alpha(\sigma(x, y)), \forall x, y \in Y$.

3. **The background of metrical fixed point theory**

The fixed point theory is concerned with finding conditions on the structure that the set Y must be endowed as well as on the properties of the operator $S : Y \longrightarrow Y$, in order to obtain results on:
(1) the existence and uniqueness of fixed points;
(2) the data dependence of fixed points;
(3) the construction of fixed points.
The ambient spaces Y involved in fixed point theory cover a variety of spaces:

lattice, metric space, normed linear space, generalized metric space,uniform space, linear topological space etc., while the conditions imposed on the operator S are generally metrical or compactness type conditions.
In order to prove several convergence theorems, we shall use various elementary results concerning recurrent inequalities, as the following lemmas: Let h_n, i_n be sequences of non negative numbers and a constant l,$0 \leq l < 1$, so that
$h_{n+1} \leq lh_n + i_n, \quad n \geq 0$
(1) If $\lim_{n \longrightarrow \infty} i_n = 0$, then $\lim_{n \longrightarrow \infty} h_n = 0$.
(2) If $\sum i_n < \infty$, then $\sum h_n < \infty$. Let y_n be a sequence of nonnegative real numbers. Then,
$\lim_{n \longrightarrow \infty} y_n = 0 \Longleftrightarrow \lim_{n \longrightarrow \infty} \sum k^{n-i} y_i = 0, \quad k \in [0, 1)$

Definition 3.1. *Let (Y, σ) be a metric space and $A, B : Y \longrightarrow Y$ be two mappings. We say that A and B are commuting if*
$ABy = BAy, \forall y \forall \in Y.$
As a generalization of this notion, Sessa defined A and B to be weakly commuting if
$\sigma(ABy, BAy) \leq \sigma(Ay, By), \quad \forall y \in Y.$

Definition 3.2. *Let (Y, σ) be a metric space and $A, B : Y \longrightarrow Y$ be two mappings. We say that A and B are compatible, as a generalization of weakly commuting, if*
$\lim_{n \longrightarrow \infty} \sigma(ABy_n, BAy_n) = 0,$
whenever y_n is a sequence in Y such that
$\lim_{n \longrightarrow \infty} Ay_n = \lim_{n \longrightarrow \infty} By_n = b, \quad b \in Y.$

Definition 3.3. *A and B mappings satisfy (E.A.) property if there exists a sequence $\{y_n\} \in Y$ such that*
$\lim_{n \longrightarrow \infty} Ay_n = \lim_{n \longrightarrow \infty} By_n = b, \quad forsome \quad b \in Y.$

4. Review of Literature

In 1906, M.Frechet generalized the notion of distance and extended it to arbitrary sets which includes the real line $\mathbb{R}$ as a particular case.

Definition 4.1. ***(Metric Spaces)*** *Let X be any non-empty set. A metric for X is a function d with domain $X \times X$ and range $[0, \infty[$ such that*

(i) $d(x, y) \geq 0 \quad \forall \quad x, y \in X$ *(Non-negative property)*
(ii) $d(x, y) = 0 \iff x = y$
(iii) $d(x, y) = d(y, x) \quad \forall \quad x, y \in X$ *(Symmetry)*
(iv) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall \quad x, y, z \in X$ *(Triangle Inequality)*

The set X with the metric d is called a metric space and it is denoted by (X,d). When there is no confusion about the metric d on X, we shall say that X is a metric space in place of saying that (X, d) is metric space.

Definition 4.2. ***(Pseudo-metric space)*** *A mapping d of $X \times X$ into $[0, \infty[$ is called a pseudometric for X iff d satisfies the axioms (i),(iii),(iv) of the above definition of metric space on X and the axiom $(ii)' d(x, x) = 0 \quad \forall \quad x \in X$ Clearly every metric space is a pseudometric but every pseudometric need not be metric space.*

Ex.1 : Let R be the set of real numbers. Then the function $d : R \times R \to R$ defined by $d(x, y) = |x - y|$, $\forall x, y \in R$ is a metric on R.The metric d is known as the **usual metric space** on R.

Ex.2 : Let X be a non-empty set and define a mapping $d : X \times X \to R$ as follows $d(x, y) = \binom{0, when x=y}{1, when x \neq y}$, $\forall 0 x, y \in X$. The metric d is known as **discrete metric space** on R.

Ex.3 : Let R^2 be the set of all ordered pairs of real numbers and let $d : R^2 \times R^2 \to R$ be defined by $d(x, y) = \{(x_1 - y_1)^2 + (x_2 - y_2)^2\}^{\frac{1}{2}}$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The metric d is known as **euclidean plane** on R^2.

Definition 4.3. ***b-metric spaces*** *Let Y be a nonempty set and $q \geq 1$ be a given real number. A function $\sigma : Y \times Y \longrightarrow [0, \infty)$ is a b-metric if, for all $x, y, z \in Y$, the following conditions are satisfied:*

(i) $\sigma(x, y) = 0$ *if and only if* $x = y$,
(ii) $\sigma(x, y) = \sigma(y, x)$,
(iii) $\sigma(x, z) \leq q[\sigma(x, y) + \sigma(y, z)]$.

The pair (Y, σ) is called a b-metric space.
A metric space is evidently a b-metric space but a b-metric space need not be a metric space.

Example 4.1. *Let $Y = 1, 2, 3$ and $d(1, 3) = d(4, 3) = s \geq 4, d(3, 2) = d(2, 1) = d(2, 3) = d(3, 2) = 5$, and $d(1, 1) = d(2, 2)d(3, 3) = 0$, Then $d(p, q) \leq s\underline{2}[d(p, r) + d(r, q)]$ for all $p, q, r \in Y$. If $s > 4$, then the ordinary triangle inequality does not hold.*

Definition 4.4. *Let $\{y_n\}$ be a sequence in a b-metric space (Y, σ).*
a. $\{y_n\}$ is called a b-convergent if and only if there is $y \in Y$ such that $\sigma(y_n, y) \longrightarrow 0$ as $n \longrightarrow \infty$.
b. $\{y_n\}$ is a b-Cauchy sequence if and only if $\sigma(y_n, y_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$.
c. A b-metric space is said to be complete if and only if each b-cauchy sequence in this space is b-convergent.

Definition 4.5. ***Convergent sequence***
A sequence $< x_n >$ in a metric space (X, d) is said to converge to $x \in X$, if given $\epsilon > 0$,we can find a positive integer m (depending on ϵ) such that $d(x_n, x) < \epsilon$, whenever $n \geq m$.
Equivalently, the sequence $< x_n >$ is said to converge to ,$x \in X$ if for given $\epsilon > 0$ there exists a positive integer m (depending on ϵ) such that$x_n \in S(x, \epsilon)$ for all $n \geq m$.
If $< x_n >$ converges to x, we say that x is a limit of the sequence and we write

$$\lim_{x \to \infty} x_n = n \quad or \quad x_n \to x \quad as \quad n \to \infty \quad or \quad d(x_n, x) \to 0 \quad as \quad n \to \infty \tag{4.1}$$

Theorem 4.2. *Limit of a sequence , if it exists is unique.*

Definition 4.6. ***Cauchy sequence*** *A sequence $< x_n >$ in a metric space (X, d) is said to be Cauchy sequence iff for each $\epsilon > 0$ there exists a positive integer p such that*

$$d(x_n, x_m) < \epsilon \quad \forall \quad n, m \geq p. \tag{4.2}$$

Theorem 4.3. *Every convergent sequence is a Cauchy sequence but the converse is not necessarily true .*

Definition 4.7. ***Complete metric space***
A metric space is said to be complete if every Cauchy sequence in X converges.
A metric space which is not complete is called incomplete.

Definition 4.8. ***Contraction Mapping Principle***
(Fixed point of mapping:) *Let X be a non-empty set and let $T : X \to X$ be a mapping on X itself. A point $x \in X$ is called a fixed point of T if $T(x) = x$.*
(Contraction mapping:) *A mapping T from a metric space (X, d) to itself is said to be a contraction mapping if for some real number α such that $0 < \alpha < 1$,*

$$d(T_x, T_y) \leq \alpha d(x, y) \quad \forall \quad x, y \in X \tag{4.3}$$

Theorem 4.4. ***Banach's Fixed Point Theorem***
Every contraction mapping S on a complete metric space (Y, σ) has a unique fixed point.
OR
Let (Y, σ) be a complete metric space and S a contraction mapping of Y into itself.Then there exist a unique fixed point of S, say u, and $\lim_{n \longrightarrow \infty} S^n y = u$ for each $y \in Y$.

Proof. Let $y_0 \in Y$ be arbitrary.Put $y_1 = Sy_0, y_2 = Sy_1 = S^2 y_0, ..., y_n = Sy_{n-1} = S^n y_0$. We will show that y_n is a cauchy sequence. We have

$$\sigma(y_k, y_{k+1}) = \sigma(Sy_{k-1}, Sy_k) \leq \lambda \Sigma \sigma(Sy_{k-2}, Sy_{k-1}) \leq ... \leq \lambda^k \sigma(y_0, y_1). \tag{4.4}$$

Using the triangle inequality, for any n and p we get

$$\sigma(y_n, y_{n+p}) \leq \Sigma(y_k, y_{k+1}) \leq \Sigma \lambda^k \sigma(y_0, y_1) \leq \frac{\lambda^n}{1 - \lambda} \sigma(y_0, y_1). \tag{4.5}$$

Since for an arbitrary $\epsilon > 0$ there exists n_0 such that

$$\frac{\lambda^{n_0}}{1 - \lambda} \sigma(y_0, y_1) < \epsilon, \tag{4.6}$$

we have for each $n \geq n_0$ and all $p \geq 1$,

$$\sigma(y_n, y_{n+p}) < \epsilon \tag{4.7}$$

Therefore y_n is a cauchy sequence. Since Y is complete , it converges to some point in X, say u.

$$\lim_{n \longrightarrow \infty} y_n = u \tag{4.8}$$

Since S is continuous, we have

$$Su = S(\lim_{n \longrightarrow \infty} y_n) = \lim_{n \longrightarrow \infty} (Sy_n) = \lim_{n \longrightarrow \infty} y_{n+1} = u, \tag{4.9}$$

i.e., u is a fixed point of S. Suppose now that there exist $u_1 \in Y$, $u_1 \neq u$, such that $Su_1 = u_1$. Then we have

$$\sigma(u_1, u) = \sigma(Su_1, S_u) \leq \lambda \sigma(u_1, u) < \sigma(u_1, u) \tag{4.10}$$

a contradiction. Therefore u is a unique fixed point of S. □

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