Lovely Professional University

M.Sc.(Hons.) Mathematics

Some Fixed Point Problems in Various Spaces

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1. Introduction

Fixed point theory is a beautiful mixture of Analysis, Topology and Geometry. In a wide range of mathematical problems the existence of a solution is equivalent to the existence of a fixed point for a suitable map. In Mathematics, a fixed point theorem is a result saying that a function F will have at least one fixed point (a point x for which $F(x) = x$) under some defined transformation on F. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences. Fixed point results provide conditions under which maps have solutions. Over the last 50 years or so, the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. Many nonlinear problems can be described in unitary manner by the following scheme. For a given object T, find another object x satisfying two conditions:

- The object y belongs to a given class Y of objects;
- The object y is in a certain relation R to the object T.

An object y satisfying these conditions will be called a solution of the given problem. This problem can be described by $\{y \in Y : yRT\}$. In turn, any problem of the above form can be written equivalently as a fixed point problem $y = Ty$ where $T : E \to E$ is a corresponding operator defined on a non-empty space E that allows us to use constructive fixed point tool for obtaining the desired solution. The theory of fixed points is of interest in itself as it provides ways to check the existence of solution to a set of equations. In particular, fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory and physics. Constructive fixed point theorem (e.g. Banach Fixed Point Theorem) is used in medical science when some relevant biological or chemical process is modeled by equations. Non-constructive fixed point theorems like Brouwers, Schauders, Lefscetz, Knaster-Tarski will provide criteria for the existence of a fixed point. They will contribute to a qualitative understanding of the models but can also serve as a basis for decisions about where to look for solutions of equations. As to real world applications, there are famous examples like the existence of market equilibrium in economics. Fixed Point Theorem is a natural ingredient in the support of analysis and solutions methods for variational inequality and nonlinear optimization problems. The theory of fixed point theorems is a powerful tool to determine uniqueness of solutions to dynamical systems and is widely used in theoretical and applied analysis. So the attraction of Fixed Point Theory to a large number of mathematicians is understandable. The theory is also helpful, in existence theory, for the solutions of differential equations and integral equations.

2. Literature Review

The origin of fixed point theory lies in the method of successive approximations to establish the existence and uniqueness of solutions of differential equations. This method is associated with the work of celebrated mathematicians Cauchy, Liovilli, Lipschitz, Peano, Fredhlom and Picard. The actual theoretic approach of fixed point was originated from the work of Picard. However, it was the Polish mathematician Banach, who underlined the idea into an abstract framework which is also suitable for application other than the differential equations. From the historical point of view, the major classical result in the fixed point theory is due to Brouwer, however the three turning points in fixed point theorem for nonlinear analysis are: Brouwer's fixed theorem [1], Banach fixed point theorem [2] and Browders fixed point theorem [7].

Banach [2] was one of the founders of functional analysis who introduced and proved an important result on fixed points in metric spaces called Banach contraction principle. This principle is used to establish the existence of solution for differential and an integral equation. The important property of a contraction mapping is its behaviour in a complete metric space. In this mapping the distance between the images of any two points is always less than the distance between the points. The more generalised concept of contractive mappings was given by Edelstein [3]. Another generalization of Banach contraction theorem was given by Boyd and Wong [8] assuming an upper semi-continuous auxiliary function. But, according to him assumption of semi continuity on auxiliary function can be dropped if the space X is metrically convex. Rakotch [4] also modified Banach contraction theorem by replacing the Lipschitz constant by a non-negative monotonic decreasing function, whose value lies in the interval [0, 1). Later, many generalization of the Banach contraction principles has given by many mathematician in various space [5], [6] and [9]-[17].

Recently Azam et al. [18] obtained the more general metric space, which is well known as complex valued metric spaces and gave sufficient conditions for the existence of common fixed points satisfying contractive mappings. S. Bhatt et al. [19] without using the notion of continuity proved a common fixed point theorem for weakly compatible maps in complex valued metric spaces. F. Rouzkard and M. Imdad [20] proved some common fixed point theorems satisfying rational type contraction mapping in the framework of complex valued metric space. C. Klin-eam and C. Suanoom [25] proved certain common fixed-point theorems for two single-valued mappings satisfy certain metric inequalities.

W. Sintunavarat and P. Kumam [22] gave the more generalized contraction mappings than Azam et al. [18] and also as an application of the proved result a solution to the integral equations of Urysohns' type has been obtained. K. Sitthikul and S. Saejung [23] owning the concept introduced by Azam et al. [18] proved several fixed point theorems for mappings satisfying certain point-dependent contractive conditions in complex valued metric spaces. R. Tiwari and D.P. Shukla [24] generalized the results of S. Bhatt et al. [19] and obtained common fixed point theorem for six maps in complex valued metric spaces for commuting and weakly compatible pair of maps.

Azam et al.[26] established the existence of common fixed points for multi-valued mappings in complex valued metric space. Y.R. Sharma [27] generalized the results of R. Tiwari [24] and obtained a common fixed point theorem for six maps in complex valued metric space having commuting, weakly compatible pair of maps and satisfying different type of inequality. J. Ahmad et al. [29] using the notion of multivalued contractive mappings proved common fixed point theorems without using the condition of continuity. M.A. Kutbi et al. [30] introduced and studied the notion of common coupled fixed points for a pair of mappings in complex valued metric space and demonstrate the existence and uniqueness of the common coupled fixed points in a complete complex-valued metric space in view of diverse contractive conditions. Also, mappings are well supported by nontrivial examples [30]. Khan et al. [32] owning the concept of complex valued metric spaces introduced by Azam et al. [18] obtained sufficient conditions for the existence of common fixed point for a pair of mapping satisfying generalized rational type contraction.

From the above literature survey, it is clear that, concept of Banach's contraction principle was the foundation stone of fixed point theory and many researchers are using this principle as a base in one or the other way to solve number of fixed point problems. Even after so many years, the scope of this principle has not been reduced. Yet there are a number of open problems in the arena of fixed point theory in various spaces, which can be solved using Banach's contraction principle. At present, many researchers are trying to generalize Banach's contraction principle in various spaces and finding the scope of applicability of fixed point theory in pure mathematics, applied mathematics as well as in other sciences.

3. Problem Formulation

Most of the results in fixed point theory are available in Hilbert and normed linear spaces. Consideration of fixed point problems in more general spaces viz. convex metric spaces, metric linear spaces and metric space is quite challenging. Since the results available in these more general spaces do not constitute a unified theory, we will make an attempt in this direction. Keeping in view, it has been planned to formulate the various problems concerning the topic.

(1) **Research Problems:** The emphasis of the present research will be
- On computation and nature of fixed point in various spaces.
- On the existence and uniqueness of fixed points in various spaces.

(2) **Research Methodology:** The Research methodology to be adopted while fulfilling the objectives will consist of the following phases:
  (a) **Formulation of the Problem:** Problem will be formulated after consulting the research articles and books.
  (b) **Solution or Proof of the Problems:** The problem defined in the step (a) will be proved or solved by applying the relevant methodology and technique.

The solved problems will be published in the reputed journals in the form of research papers.

4. Objective

The objective of the present study is concerned with finding the conditions in order to obtain the result

- On existence of fixed points;
- On uniqueness of fixed points;
- The construction of fixed points.

in various spaces. At last, we will make an attempt to find the application of the obtained results.

5. Methodology

In the proposed research work, by consulting research articles from journals, books and internet, an attempt will be made to find the relevant methodology/technique to solve the framed problems or theorems in various spaces. The Proposed study is planned into the three phases.
Phase I. This phase will be utilized to collect the existing literature on the subject in the form of research articles from journals, books, etc.
Phase II. In this phase formulation of the various problems concerning the topic will be done. An attempt will be made in this phase to prove or solve the formulated problems based upon the existing theories and methodologies decided in the phase I.
Phase III. The proposed work will be presented in the form of thesis consisting of an introduction to available literature. Further, studies will be extended to establish the results regarding fixed point theorems various spaces

6. Progress of Work

As discussed in the literature review many researchers have proved fixed point problems in various spaces using different approaches. We have generalized the contraction mappings and proved fixed point problems using the existing approaches in the framework of complex valued metric space. Those findings in brief are given below:

(1) **Some results on common fixed points for rational type contraction mappings in complex valued metric spaces** : In this paper using the concept of Azam et

al. [18], we have proved few theorems for a pair of mappings satisfying rational type contraction conditions in the framework of complex valued metric spaces. The obtained results generalize and extend some of the existing problems in the arena of fixed point theory.

(2) **Some common fixed point problems using the concept of C-Cauchy sequence and C-complete in complex valued metric spaces** : In, 2013 W. Sintunavarat et al. [31] introduced the concept of C-Cauchy sequence and C-complete in complex valued metric spaces proved the existence and uniqueness of common fixed point theorems in the framework of C-complete complex valued metric spaces. Using the concept of C-Cauchy sequence and C-complete in complex valued metric spaces we have obtained few results for more generalized rational type contraction mappings.

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