Lovely Professional University

Master of Science

Existence of Fixed Point Theorems in Metric Spaces and Multiplicative Metric Spaces

A project submitted in fulfilment of the requirements for the degree of Master of Science

in the

Department of Mathematics
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Declaration of Authorship

I, Navpreet, declare that this project titled, "Existence of Fixed Point Theorems in Metric Spaces and Multiplicative Metric Spaces" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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Certificate

This is to certify that NAVPREET has completed Project titled "Existence of Fixed Point Theorems in Metric Spaces and Multiplicative Metric Spaces" under my guidance and supervision. To the best of my knowledge, the present work is the result of his/her original investigation and study. No part of the project has ever been submitted for any other degree at any University.
The project is fit for the submission and the partial fulfilment of the conditions for the award of Master of Science in Mathematics.

Signed:

Supervisor: DR. MANOJ KUMAR

Date: April 2017

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Abstract

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Existence of Fixed Point Theorems in Metric Spaces and Multiplicative Metric Spaces

by NAVPREET

In this project, we proved a common fixed point theorem for two pairs of weakly compatible maps satisfying a general contractive condition in complete metric space. Also, we proved some common fixed point theorems for weakly C-contractive mappings in multiplicative metric spaces.

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Contents

Dedicated to my Parents

Chapter 1

Introduction and Preliminaries

This chapter elaborates the basic definitions, results and notations, which are required in the subsequent chapters. It begins with the study of core material of thesis along with the previously known results.

1.1 Background of Fixed Point Theory

Fixed point theory is a beautiful mixture of analysis, topology and geometry. Over since last 50 years, fixed point theory has been revealed itself as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in divers fields such as in biology, chemistry, economics, engineering, game theory and physics. The point at which the curve $y = f(x)$ and the line $y = x$ intersects gives the solution of the curve, and the point of intersection is the fixed point of the curve. The usefulness of the concrete applications has increased enormously due to the development of accurate techniques for computing fixed points.
Fixed point theory is a rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include numerical methods like Newton-Raphson method, establishing Picard Existence Theorem, existence of solution of integral equations and a system of linear equations.

1.1.1 Significance of fixed points

Fixed points are points which remain invariant under a map/transformation. Fixed points tells us which part of the space are pinned in plane, not moved, by the transformation. The fixed points of a transformation restrict the motion of the space under some restrictions.

We note that fixed points problems and root finding problems $f(x) = 0$ are equivalent. Now the question arises what types of problems have the fixed points. The fixed points problem can be elaborated in following manner:

1. what functions/maps have fixed points?
2. How do we determine the fixed points?
3. Is the fixed point unique?

Next we state a result which gives us the gurantee of existence of fixed points. Suppose g is continuous self map on $[a, b]$. Then we have the following conclusions:

If the range of the mapping $y = g(x)$ satisfies $y \in [a, b]$ for all $x \in [a, b]$, then g has a fixed points in $[a, b]$.

Suppose that $g'(x)$ is defined over (a, b) and that a positive constant $k < 1$ exist with $|g'(x)| \leq k$ for all $x \in (a, b)$,then g has unique fixed point p in [a,b].

Now suppose that (X, d) be a complete metric space and $T : X \to X$ be a map. The mapping T satisfies a Lipschitz condition with constant $\alpha \geq 0$ such that $d(Tx, Ty) \leq \alpha(x, y)$, for all $x, y \in X$. For different values of α, we have the following cases:

1. T is called contraction mapping if $\alpha < 1$
2. T is called non-expensive if $\alpha \leq 1$
3. T is called contractive if $\alpha = 1$

It is clear that contraction$\Rightarrow$ contractive$\Rightarrow$ non-expensive$\Rightarrow$ Lipschitz. However, converse may not true in either case as:

The identity map $I : X \to X$ is a metric space, is non-expansive but not contractive.

Let $X = [0, \infty)$ be a complete metric space equipped with the metric of absolute value. Define, $f : X \to X$ given by $f(x) = x + \frac{1}{x}$. Then f is contractive map, while f is not contraction.

There are two fixed points theorem: one is Brouwers, and the other Banach fixed point theorem. Brouwers fixed point theorem is extential by its nature.

Brouwer(1912): Every continuous self map on the closed unit ball $C = x : |x| \leq 1$ in $\mathbb{R}^n$ has a fixed point.

The elegant Banach fixed point theorem solves:
the problem on the existence of a unique solution to an equation,
gives a practical method to obtain approximate solutions and
gives an estimate of such solutions.
The applications of Banach fixed point theorem and its generalization are very important in diverse disciplines of mathematics, statistics, engineering and economics.

In 1922, Banach [8] proved a fixed point theorem and called it Banach Fixed point theorem/Banach Contraction Principle which is considered as the mile stone on fixed point theory. This theorem states that if T is self mapping of a complete metric space (X, d) and there exist a number $h \in [0, 1)$, such that for all $x, y \in X$,

$$d(Tx, Ty) \leq hd(x, y) \tag{1.1}$$

Then T be continuous throughout X.

This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. This theorem was generalized and improved in different ways by various authors. This principle has had many applications but it suffers from one drawback-the definition requires that T be continuous throughout X.

Definition 1.1. Let X be a non empty set and $T : X \to X$ be a mapping. A solution of an equation $Tx = x$ is called fixed point of T.

Example 1.1. *Examples of fixed points are following:*

(i) *A translation mapping has no fixed points,that is $Tx = x + 3$ for all $x \in R$.*

(ii) *A mapping $T : R \to R$ defined by $Tx = \frac{x}{p} - (p - 1)$ where p is positive integer, then $x = -p$ is the unique fixed point.*

(iii) *A mapping $T : R \to R$ defined by $Tx = x^2$ has two fixed points 0 and 1.*

(iv) *A mapping $T : R \to R$ defined by $Tx = x$, has infinitely many points, i.e,every point of domain is a fixed point of T.*

Therefore, from the above examples one can conclude that a mapping may have unique fixed point, it may have more than one or even infinitely many fixed points and it may have no fixed point. Therefore dealing with the existence and construction of a solution to an operator equation $Tx = x$ from the part of fixed point theory.

We note that every mapping is continuous and uniformly continuous but converse need not be true. The first answer of this question is given by Kannan in 1968, who proved a fixed point theorem for operators which do not have to be continuous.

Kannan (1968): If T is a self mapping and a complete metric space X satisfying

$$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)] \tag{1.2}$$

for all $x, y \in X$ and $0 \leq k < \frac{1}{2}$, then T has unique fixed point in X. We note that a map T is not continuous even though T has fixed point. However, in every case, the maps involved are continuous at the fixed. Therefore, Kannan type and their generalizations have been considered as an important class of mapping in fixed point theory. Following Kannan, Chatterjea proved a fixed point theorem for operator which satisfies the condition: there exist $c \in [0, \frac{1}{2})$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \tag{1.3}$$

Rhoades had shown that these conditions (1.1), (1.2) , (1.3) are independent. Zamrfescu combined the conditions (1.1), (1.2) , (1.3) as follows:
there exist a number a, b and c satisfying $0 \leq b < 1$, $0 \leq a < \frac{1}{2}$, and $0 \leq c < \frac{1}{2}$; such that for each $x, y \in X$ at least one of the following is true:
(z_1) $d(Tx, Ty) \leq ad(x, y)$;
(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;
(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.
In 1983, Rus gave another generalization of Banach contraction principle replacing the condition (1.1) with the next condition as follows:
there is a comparison function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) \tag{1.4}$$

for all $x, y \in X$.
A generalization of Kannan Theorem was made by Bianchini, who replace the condition (1.2) with:
there is $a \in [0, 1)$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq a.max\{(d(x, Tx), d(y, Ty)\} \tag{1.5}$$

In these conditions the operator T has a unique fixed point.

1.2 Various Types Of Spaces

We emphasis our research mainly on metric spaces and multiplicative metric spaces.

1.2.1 Metric Spaces

In 1906, Maurice Frechet (1878-1973), a French mathematician, introduced the notation of metric space, which derived from the word metor (measure). Further, he pioneered the study of such spaces and their applications to different areas of mathematics. Thought, the definition presently in use is given by the German mathematician. Felix Hausdroff(1868-1942) in 1914.

Definition 1.2. Let X be an arbitrary set. Let $d : X \times X \to \mathbb{R}_+$ satisfies the following conditions :

(i) $d(x, y) \geq 0$, $d(x, y) = 0$ iff $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$

for all $x, y, z \in X$. The set X together with the metric d, i.e, (X, d) is called metric space.

Definition 1.3. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be

(i) Convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$ as $n \to \infty$ or $lim_{n\to\infty} x_n = x$.

(ii) Cauchy sequence if and only if for each $\epsilon > 0$ there exists a natural number $n(\epsilon)$ such that for all $n > m > n(\epsilon)$, $d(x_n, x_m) < \epsilon$.

(iii) Complete if every cauchy sequence is convergent in X.

The study of common fixed point of mappings satisfying contractive condition have been a very active field of research during recent years. The most general of common fixed point theorems pertaining to four mappings A, B, S and T of a metric space (X, d) uses either a Banach-type contractive condition of the form

$$d(Ax, By) \leq km(x, y)\ (0 \leq k < 1), \text{where}$$

$$m(x,y) = max\{d(Ax,By), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)+d(Ty,Ax)}{2}\},$$

or a Meir-Keeler-type (ϵ, δ)-contractive condition, that is, given $\epsilon > 0$, there exist a $\delta > 0$ such that

$$\epsilon \leq m(x,y) < \epsilon + \delta \Rightarrow d(Ax,By) < \epsilon,$$

or a φ-contractive condition of the form

$$d(Ax,By) \leq \varphi(m(x,y)),$$

involving a contractive gauge function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) < t$ for each $t > 0$. Note that Banach-type contractive condition is a special case of both conditions Meir-Keeler-type (ϵ, δ)-contractive and φ-contractive. A φ-contractive condition does not guarantee the existence of a fixed point unless some additional condition is assumed. Moreover, a φ-contractive condition, in general, does not imply the Meir-Keeler-type (ϵ, δ)-contractive conditions.
In this , we will prove a common fixed point theorem for four weakly compatible self-maps satisfying a general contractive condition and also prove common fixed point theorems for weakly compatible maps along with E.A. and (CLR) properties.

1.2.2 Multiplicative Metric Spaces

Let X be a non-empty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x,y) \geq 1$, $d(x,y) = 1$ iff $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii $d(x,y) \leq d(x,z).d(z,y)$ (multiplicative triangular inequality)

for all $x, y, z \in X$. The set X together with the multiplicative metric d, i.e, (X,d) is called Multiplicative Metric Space.

Definition 1.4. Let (X,d) be multiplicative metric space. Then a sequence $\{x_n\}$ is said to be a multiplicative convergent to x for every multiplicative open ball

$$B_\epsilon(x) = \{y \in X | d(x,y) < \epsilon, \epsilon > 1\},$$

there exist a natural number N such that $n \geq N$ then $x_n \in B_\epsilon(x)$ that is, $d(x_n, x) \to 1$ as $n \to \infty$.

Definition 1.5. Let (X, d) be multiplicative metric space. Then a sequence $\{x_n\}$ is said to be cauchy sequence if for all $\epsilon > 1$ there exist $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$, $\forall$ $m, n > N$ that is $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

A multiplicative metric space is said to be complete if every multiplicative cauchy sequence in it is a multiplicative convergent sequence.

Chapter 2

Review of the Literature

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of all real numbers, the set of all positive real numbers and the set of all natural numbers,respectively.

It is well known that the set of positive real number $\mathbb{R}_+$ is not a complete according to the usual metric. To overcome this problem, in 2008, Bashirov[2] introduced the concept of multiplicative metric space as follows:

Definition 2.1. Let X be a non-empty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x, y) \geq 1$, $d(x, y) = 1$ iff $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z).d(z, y)$ (multiplicative triangular inequality)

for all $x, y, z \in X$. The set X together with the multiplicative metric d, i.e, (X, d) is called Multiplicative Metric Space.

Example 2.1. *Let $\mathbb{R}^n_+$ be the collection of all n-tuples of positive real numbers. Let $d^*(u, v) : \mathbb{R}^n_+ \to \mathbb{R}$ be defined as follows:*

$$d^*(u, v) = |\frac{u_1}{v_1}|^* \cdot |\frac{u_2}{v_2}|^* \cdots |\frac{u_n}{v_n}|^*,$$

where $u = (u_1, u2, \cdots u_n)$, $v = (v_1, v2, \cdots v_n) \in \mathbb{R}^n_+$ and $|\cdot|^* : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by:

$$|k|^* = \begin{cases} k, & \text{if } k \geq 1; \\ \\ \frac{1}{k}, & \text{if } k < 1, \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric space are satisfied and $(\mathbb{R}^n_+, d)$ *is a multiplicative metric space.*

Example 2.2. *Let* $d : \mathbb{R} \times \mathbb{R} \to [1, \infty)$ *be defined as* $d(x, y) = a^{|x-y|}$, *where* $x, y \in \mathbb{R}$ *and* $a > 1$. *Then* d *is a multiplicative metric and* $(\mathbb{R}, d)$ *is a multiplicative metric space.*

Remark 2.2. We note that Example 2.1 is valid for positive real numbers and Example 2.2 is valid for all real numbers.

Example 2.3. *Let* (X, d) *be a metric space. Define mapping* d_a *on* X *by*

$$d_a(x, y) = a^{d(x,y)} = \begin{cases} 1, & \text{if } x = y; \\ \\ a, & \text{if } x \neq y, \end{cases}$$

where $x, y \in X$ *and* $a > 1$. *Then* d_a *is called a discrete multiplicative metric and* (X, d_a) *is known as the discrete multiplicative metric space.*

Example 2.4. *Let* $X = C^*[a, b]$ *be the collection of all real-valued multiplicative continuous function on* $[a, b] \subset \mathbb{R}_+$, *then* (X, d) *is a mutiplicative continuous metric space with* d *defined by*
$d(f, g) = sup_{x\in[a,b]}|\frac{f(x)}{g(x)}|$ *for arbitrary f,g* $\in X$.

Remark 2.3. Neither every metric is multiplicative metric nor every multiplicative metric is metric. The mapping d^* defined above is multiplicative metric but not but not metric as it does not triangular inequality,
Consider $d^*(\frac{1}{3}, \frac{1}{2}) + d^*(\frac{1}{2}, 3) = \frac{3}{2} + 6 = 7.5 < 9 = d^*(\frac{1}{2}, 3)$.
On the other hand the usual metric on $\mathbb{R}$ is not multiplicative metric as it does not satisfy multiplicative triangular inequality,since
$d(2, 3).d(3, 6) = 3 < 4 = d(2, 6)$

Definition 2.4. Let (X, d) be multiplicative metric space. Then a sequence $\{x_n\}$ is said to be a multiplicative convergent to x for every multiplicative open ball

$$B_\epsilon(x) = \{y \in X|d(x, y) < \epsilon, \epsilon > 1\},$$

there exist a natural number N such that $n \geq N$ then $x_n \in B_\epsilon(x)$ that is, $d(x_n, x) \to 1$ as $n \to \infty$.

Definition 2.5. Let (X, d) be multiplicative metric space. Then a sequence $\{x_n\}$ is said to be cauchy sequence if for all $\epsilon > 1$ there exist $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$, $\forall$ $m, n > N$ that is $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

A multiplicative metric space is said to be complete if every multiplicative cauchy sequence in it is a multiplicative convergent sequence.

Remark 2.6. The set of positive real number $\mathbb{R}_+$ is not complete according to usual metric.
Let $X = \mathbb{R}_+$ and the sequence $\{x_n\} = \frac{1}{n}$.
It is obvious $\{x_n\}$ is a cauchy sequence in X and X is not a complete metric space, Since $0 \in \mathbb{R}^+$.In case of multiplicative metric space, we take a sequence $\{x_n\} = a^{\frac{1}{n}}$, where $a > 1$. Then $\{x_n\}$ is a cauchy sequence, since for $n \geq m$,
$d(x_n, x_m) = |\frac{x_n}{x_m}| = |a^{\frac{1}{n}-\frac{1}{m}}| \leq a^{\frac{1}{n}-\frac{1}{m}} < a^{\frac{1}{m}} < \epsilon$ if $m > \frac{loga}{log\epsilon}$,
where

$$|a| = \begin{cases} a, & \text{if } a \geq 1; \\ \\ \frac{1}{a}, & \text{if } a < 1, \end{cases}$$

Also $x_n \to 1$ as $n \to \infty$ and $1 \in \mathbb{R}_+$. Hence (X, d) is complete multiplicative metric space.
In 2012, Ozavsar and Cevikel [11] gave the concept of multiplicative contraction mappings and proved some fixed point theorem of such mappings in a multiplicative metric space.

Definition 2.7. Let f be mapping of a multiplicative metric space (X, d) into self. Then f is said to be multiplicative contraction if there exist a real constant $\lambda \in [0, 1)$ such that
$d(fx, fy) \leq d^{\lambda}(x, y)$ for all $x, y \in X$.

Gu et al.[6] introduced the notation of commutative and weak contraction mappings in a complete multiplicative metric space and proved some fixed point theorems for these mappings.

Chapter 3

Common Fixed Point Theorems In Metric Space

3.1 Objective of the study

Our objective was to meet the following:
To design a framework to survey the study of fixed point in metric spaces and multiplicative metric spaces as studied by other researchers using
(i) weakly compatible property,
(ii) E.A. property,
(iii) CLR property.

Some Common Fixed Point Theorems for Four Self-Mappings satisfying a general contractive condition

Definition 3.1. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Definition 3.2. Two self-mapping f and g of metric space (X, d) are said to be satisfy **E.A.** property if there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = t$ for some t in X.

Definition 3.3. Two self-mapping f and g of metric space (X, d) are said to be satisfy (CLR_f) property if there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = fx$ for some x in X.

Theorem 3.4. *Let A, B, S and T be self maps of metric space (X, d) satisfying the followings:*

(i) $SX \subseteq BX$, $TX \subseteq AX$,

(ii) *for all x, y in X, there exists a right continuous functions $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(0) = 0 = \phi(0)$ and $\psi(s) < s$, $\phi(s) < s$ for $s > 0$ such that*

$$\psi(d(Sx, Ty)) \leq \psi(m(x, y)) - \phi(m(x, y)), \text{ where}$$

$$m(x, y) = max\{d(Ax, By), d(Sx, Ax), d(Ty, By), \tfrac{1}{2}(d(Sx, By) + d(Ty, Ax))\}.$$

If AX, BX, SX or TX is complete subspace of X, then the pair (A, S) or (B, T) have a coincidence point.
Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point of X. From (i), we conclude that $\{y_n\}$ in X as follows:

$$y_{2n+1} = Sx_{2n} = Bx_{2n+1}, y_{2n+2} = Tx_{2n+1} = Ax_{2n+2}, \text{ for all } n = 0, 1, 2, ... \tag{3.1}$$

Define $d_n = d(y_n, y_{n+1})$. Suppose that $d_{2n} = 0$ for some n. Then $y_{2n} = y_{2n+1}$, that is, $Tx_{2n-1} = Ax_{2n} = Sx_{2n} = Bx_{2n+1}$. A and S have coincidence point.
Similarly, if $d_{2n+1} = 0$, then B and T have coincidence point. Assume that $d_n \neq 0$ for each n. From (ii), we have

$$\psi(d(Sx_{2n}, Tx2n + 1)) \leq \psi(m(x_{2n}, x_{2n+1})) - \phi(m(x_{2n}, x_{2n+1})), \text{ where} \tag{3.2}$$

$$\begin{aligned}(m(x_{2n}, x_{2n+1})) &= max\{d(Ax_{2n}, Bx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}),\\ &\frac{1}{2}(d(Sx_{2n}, Bx2n + 1) + d(Tx_{2n+1}, Ax_{2n}))\}\\ &= max\{d_{2n}, d_{2n+1}\}.\end{aligned} \tag{3.3}$$

Thus from (3.2) we have

$$\psi(d(Sx_{2n}, Tx2n + 1)) \leq \psi(max\{d_{2n}, d_{2n+1}\} - \phi(max\{d_{2n}, d_{2n+1}\}). \tag{3.4}$$

Now, if $d_{2n+1} \geq d_{2n}$, for some n, then from (3.4), we have

$$\begin{aligned}\psi(d_{2n+1}) &\leq \psi(d_{2n+1}) - \phi(d_{2n+1})\\ &< \psi(d_{2n+1}) \text{ a contradiction.}\end{aligned} \tag{3.5}$$

Thus, from $d_{2n} > d_{2n+1}$ for all n, and so, from (3.4), we have

$$\psi(d_{2n+1}) \leq \psi(d_{2n}) - \phi(d_{2n}), \text{ for all } n \in \mathbb{N} \tag{3.6}$$

Similarly

$$\psi(d_{2n}) \leq \psi(d_{2n-1}) - \phi(d_{2n-1})$$
$$\psi(d_{2n-1}) \leq \psi(d_{2n-2}) - \phi(d_{2n-2})$$

In general, we have for all n= 1,2,3,...,

$$\begin{aligned} \psi(d_n) &\leq \psi(d_{n-1}) - \phi(d_{n-1}) \\ &< \psi(d_{n-1}). \end{aligned} \tag{3.7}$$

Hence the sequence $\{\psi(d_n)\}$ is monotonically decreasing and bounded below. Thus, there exists, $r \geq 0$, such that

$$lim_{n\to\infty}\psi(d_n) = r. \tag{3.8}$$

From (3.7), we deduce that

$$0 \leq \phi(d_{n-1}) \leq \psi(d_{n-1}) - \psi(d_n).$$

Letting limit as $n \to \infty$ and using (3.8), we get

$$lim_{n\to\infty}\phi(d_{n-1}) = 0 \Rightarrow lim_{n\to\infty}d_{n-1} = lim_{n\to\infty\infty}d(y_{n-1}, y_n) = 0, or \tag{3.9}$$

$$lim_{n\to\infty}d_n = lim_{n\to\infty}d(y_n, y_{n+1}) = 0. \tag{3.10}$$

Now, we show that $\{y_n\}$ is a cauchy sequence. For this, it is sufficient to show that $\{y_n\}$ is a cauchy sequence. Let, if possible $\{y_n\}$ is not a cauchy sequence. Then there exist an $\epsilon > 0$ such that for each even integer $2k$ there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$d(y_{2n(k)}, y_{2m(k)}) \geq \epsilon. \tag{3.11}$$

For every even integer $2k$, suppose that $2m(k)$ be the least positive integer exceeding $2n(k)$ satisfying (3.11) such that

$$d(y_{2n(k)}, y_{2m(k)-2}) < \epsilon. \tag{3.12}$$

From (3.11), we have

$$\begin{aligned}\epsilon &\leq d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).\end{aligned}$$

Using (3.10) and (3.12) in the above inequality, we get

$$lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \quad (3.13)$$

Also, by the triangular inequality,

$$\begin{aligned}|d(y_{2n(k)}, y_{2m(k)-1}) + d(y_{2n(k)}, y_{2m(k)})| &\leq d_{2m(k)-1}, \\ |d(y_{2n(k)+1}, y_{2m(k)-1}) + d(y_{2n(k)}, y_{2m(k)})| &\leq d_{2m(k)-1} + d_{2m(k)}. \quad (3.14)\end{aligned}$$

Using (3.10), we get

$$lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) = \epsilon \quad (3.15)$$

From (ii), we have

$$\psi(d(Sx_{2n(k)}, Tx_{2m(k)-1})) \leq \psi(m(x_{2n(k)}, x_{2m(k)-1})) - \phi(m(x_{2n(k)}, x_{2m(k)-1})), \quad (3.16)$$

where

$$\begin{aligned}m(x_{2n(k)}, x_{2m(k)-1}) = max\{&d(Ax_{2n(k)}, Bx_{2m(k)-1}), d(Sx_{2n(k)}, Ax_{2n(k)}), \\ &d(Tx_{2m(k)-1}, Bx_{2m(k)-1}), \\ &\frac{d(Sx_{2n(k)}, Bx_{2m(k)-1}) + d(Tx_{2n(k)}, Ax_{2m(k)-1})}{2}\} \\ = max\{&d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)}, y_{2n(k)+1}), d(y_{2m(k)-1}, y_{2m(k)}), \\ &\frac{d(y_{2n(k)+1}, y_{2m(k)-1}) + d(y_{2n(k)}, y_{2m(k)-1})}{2}\}.\end{aligned}$$

Letting limit as $k \to \infty$ and using (3.15), we get $\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$, a contradiction, since $\epsilon > 0$. Thus,$\{y_{2n}\}$ is a cauchy sequence and so $\{y_n\}$ is a cauchy sequence.
Now, suppose that $A(X)$ is a complete. Note that $\{y_{2n}\}$ is contained in $A(X)$ and has a limit in $A(x)$, say u, that is, $lim_{n\to\infty} y_{2n} = u$. Let $v \in A^{-1}u$. Then $Av = u$. Now, we shall prove that $Sv = u$. Let, if Possible, $Sv \neq u$, that is, $d(Sv, u) = p > 0$. Putting $x = v$ and $y = x_{2n-1}$ in (ii), we have

$$\psi(d(Sv, Tx_{2n-1})) \leq \psi(m(v, x_{2n-1})) - \phi(m(v, x_{2n-1})).$$

Letting limit as $n \to \infty$, we have

$$lim_{n\to\infty}\psi(d(Sv, Tx_{2n-1})) \leq lim_{n\to\infty}\psi(m(v, x_{2n-1})) - lim_{n\to\infty}\phi(m(v, x_{2n-1})), \text{ where} \tag{3.17}$$

$$\begin{aligned} lim_{n\to\infty}m(v, x_{2n-1}) &= lim_{n\to\infty}[max\{d(u, y_{2n-1}), d(Sv, u), d(y_{2n}, y_{2n-1}), \\ &\frac{d(Sv, y_{2n-1}) + d(y_{2n}, u)}{2}\}] \\ &= max\{d(u, u), d(Sv, u), d(u, u), \frac{d(Sv, u) + d(u, u)}{2}\} \\ &= d(Sv, u) = p. \end{aligned}$$

Thus, from (3.17), we have

$$\psi(d(Sv, u)) \leq \psi(p) - \phi(p), \text{ that is },$$

$$\psi(p) \leq \psi(p) - \phi(p), \text{ a contradiction, since } p > 0.$$

Thus, $Sv = u = Av$.
Hence u is the coincidence point of the pair (A, S).
Since $SX \subseteq BX$, $Sv = u$, implies that, $u \in BX$.
Let $w \in B^{-1}u$. Then $Bw = u$. By using the same arguments as above, one can easily verify that, $Tw = u = Bw$, that is, u is the coincidence point of the pair (B, T).
The same result holds, if we assume that BX is complete instead of AX.
Now, if TX is complete, then by (i), $u \in TX \subseteq AX$.
Similarly, If SX is complete, then $u \in SX \subseteq BX$.
Now, since the pairs (A, S) and (B, T) are weakly compatible, so

$$u = Sv = Av = Tw = Bw, \text{ then}$$

$$Au = ASv = SAv = Su. \tag{3.18}$$

$$Bu = BTw = TBw = Tu.$$

Now, we claim that $Tu = u$.

Let, if possible, $Tu \neq u$. From (ii), we have

$$\begin{aligned}
\psi(s(u,Tu)) &= \psi(d(Sv,Tu)) \\
&\leq \psi(m(v,u)) - \phi(m(v,u)), \text{ where} \\
m(v,u) &= max\{d(Av,Bu), d(Sv,Av), d(Tu,Bu), \frac{d(Sv,Bu)+d(Tu,Av)}{2}\} \\
&= max\{d(u,Tu), d(u,u), 0, \frac{d(u,Tu)+d(Tu,u)}{2}\} \\
&= d(u,Tu).
\end{aligned}$$

Thus, we thus

$$\begin{aligned}
\psi(d(u,Tu)) &\leq \psi(d(u,Tu)) - \phi(d(u,Tu)) \\
&< \psi(d(u,Tu)), \text{ contradiction}
\end{aligned}$$

So, $Tu = u$.

Similarly, $Su = u$.

Thus, we get $Au = Su = Bu = Tu = u$.

Hence u is the common fixed point of A, B, S and T.

Now, we claim that $u = z$.

Let, if possible, $u \neq z$.

$$\begin{aligned}
\psi(d(u,z)) &= \psi(d(Su,Tz)) \\
&\leq \psi(m(u,z)) - \phi(m(u,z)) \\
&= \psi(d(u,z)) - \phi(d(u,z)),
\end{aligned}$$

Since$m(u,z) = d(u,z) < \psi(d(u,z)))$, a contradiction.

Thus, $u = z$, and the uniqueness follows. □

Chapter 4

Common Fixed Point Theorems In Multiplicative Metric Space

Main Result

In this chapter, we prove some common fixed point theorems for weakly C-contractive mapping in Multiplicative Metric Space.

Definition 4.1. A mapping $T : X \to X$, where (X, d) is a multiplicative metric space is said to be a C-contractive if there exist $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq [d(x, Ty).d(y, Tx)]^{\alpha}.$$

Definition 4.2. A mapping $T : X \to X$, where (X, d) is a multiplicative metric space is said to be a weakly C-contractive if for all $x, y \in X$,

$$d(Tx, Ty) \leq [(d(x, Ty).d(y, Tx)]^{\alpha} - \varphi(d(x, Ty), d(y, Tx)),$$

where $\varphi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 1$.

Definition 4.3. Let T and S be two self mappings of a multiplicative metric space (X, d). T and S are said to be weakly compatible if for all $x \in X$ the equality $Tx = Sx \Rightarrow TSx = STx$.

Theorem 4.4. *Let (X, d) be a complete multiplicative metric space and let E be a non-empty closed subset of X. Let $T, S : E \to E$ such that,*

$$d(Tx, Sy) \leq [(d(Rx, Sy).d(Ry, Tx)]^{\frac{1}{2}} - \varphi(d(Rx, Sy), d(Ry, Tx)), \tag{4.1}$$

for every pair $(x,y) \in X \times X$,where $\varphi : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\varphi(x,y) = 0$ if and only if $x = y = 1$ and $R : E \to X$ satisfying the following hypothesis:

(i) *$TE \subseteq RE$ and $SE \subseteq RE$.*

(ii) *The pairs (T,R) and (S,R) are weakly compatible.*

In addition, assume that $R(E)$ is a closed subset of X. Then, T and R and S have a unique common fixed points.

Proof. Let $x_0 \in E$ be arbitrary. Using 4.1 there exist two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $y_0 = Tx_0 = Rx_1$, $y_1 = Sx_1 = Rx_2$, $y_2 = Tx_2 = Rx_3...$, $y_{2n} = Tx_{2n} = Rx_{2n+1}$, $y_{2n+1} = Tx_{2n+1} = Rx_{2n+2}$,...
We complete the proof in three steps.
Step I. We will prove that $lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.
Let $n = 2k$.Using condition (4.1), we obtain that

$$\begin{aligned} d(y_{2k+1}, y_{2k}) &= d(Tx_{2k}, Sx_{2k+1}) \\ &\leq [(d(Rx_{2k}, Sx_{2k+1}).d(Rx_{2k+1}, Tx_{2k})]^{\frac{1}{2}} - \varphi(d(Rx_{2k}, Sx_{2k+1}), d(Rx_{2k+1}, Tx_{2k})) \\ &= [d(y_{2k-1}, y_{2k+1}).d(y_{2k}, y_{2k})]^{\frac{1}{2}} - \varphi(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k})) \\ &\leq [d(y_{2k-1}, y_{2k+1})]^{\frac{1}{2}} \\ &\leq [d(y_{2k-1}, y_{2k}).d(y_{2k}, y_{2k+1})]^{\frac{1}{2}} \qquad (4.2) \end{aligned}$$

Hence,$d(y_{2k+1}, y_{2k}) \leq d(y_{2k}, y_{2k+1})$.
If $n = 2k+1$, similarly we can prove that

$$d(y_{2k+2}, y_{2k+1}) \leq d(y_{2k+1}, y_{2k}),$$

Thus $d(y_{n+1}, y_n)$ is a decreasing sequence of non-negative real numbers and hence it is convergent.
Assume that, $lim_{n\to\infty} d(y_{n+1}, y_n) = r$ from the above argument we have,

$$\begin{aligned} d(y_{n+1}, y_n) &\leq (d(y_{n-1}, y_{n+1})^{\frac{1}{2}} \\ &\leq [d(y_{n-1}, y_n).d(y_n, y_{n+1})]^{\frac{1}{2}} \qquad (4.3) \end{aligned}$$

If $n \to \infty$, we have

$$r \leq lim_{n\to\infty} (d(y_{n-1}, y_{n+1})^{\frac{1}{2}} \leq r.$$

Therefore,$lim_{n\to\infty}d(y_{n-1},y_{n+1}) = r^2$. We have proved in (4.2)

$$\begin{aligned} d(y_{2k+1},y_{2k}) &= d(Tx_{2k},Sx_{2k+1}) \\ &\leq [(d(y_{2k-1},y_{2k+1}).d(y_{2k},y_{2k})]^{\frac{1}{2}} - \varphi(d(y_{2k-1},y_{2k+1}),d(y_{2k},y_{2k})) \end{aligned} \quad (4.4)$$

Now, if $k \to \infty$ and using the continuity of φ we obtain

$$r \leq r - \varphi(r^2, 1),$$

and consequently, $\varphi(r^2, 1) = 0$. This gives us that

$$r = lim_{n\to\infty}d(y_{n+1}, y_n) = 1 \quad (4.5)$$

by our assumption about φ.

StepII. $\{y_n\}$ is cauchy sequence,

Since $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$,

It is sufficient to show that the subsequence $\{y_{2n}\}$ is a cauchy sequence.

Suppose that $\{y_{2n}\}$ is not a cauchy sequence. Then there exist $\epsilon > 0$ for which we can find subsequence $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ such that $n(k)$ is the least index for which $n(k) > m(k) > k$ and $d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon$.

This means that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \epsilon. \quad (4.6)$$

From triangular inequality

$$\epsilon \leq d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2n(k)-2}).d(y_{2n(k)-2}, y_{2n(k)-1}).d(y_{2n(k)-1}, y_{2n(k)}) \quad (4.7)$$

Letting $k \to \infty$ and using (4.5) we conclude that

$$\begin{aligned} \epsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \\ &\leq \epsilon.1.1 = \epsilon \end{aligned}$$

Therefore

$$d(y_{2m(k)}, y_{2n(k)}) = \epsilon \quad (4.8)$$

Moreover, we have

$$|d(y_{2m(k)}, y_{2n(k)+1}).d(y_{2m(k)}, y_{2n(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1}) \tag{4.9}$$

and

$$|d(y_{2n(k)}, y_{2m(k)-1}).d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1}) \tag{4.10}$$

and

$$|d(y_{2n(k)}, y_{2m(k)-2}).d(y_{2n(k)}, y_{2m(k)-1})| \leq d(y_{2m(k)-2}, y_{2m(k)-1}) \tag{4.11}$$

using (4.5),(4.8),(4.9),(4.10) and (4.11) we get

$$\begin{aligned} lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) &= lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)-1}) \\ &= lim_{k\to\infty} d(y_{2m(k)-2}, y_{2n(k)}) \\ &= \epsilon \end{aligned} \tag{4.12}$$

Now, from (4.1) we have

$$\begin{aligned} d(y_{2m(k)-1}, y_{2n(k)}) &= d(Tx_{2n(k)}, Sx_{2m(k)-1}) \\ &\leq [d(Rx_{2n(k)}, Sx_{2m(k)-1}).d(Rx_{2m(k)-1}, Tx_{2n(k)})]^{\frac{1}{2}} \\ &- \varphi(d(Rx_{2n(k)}, Sx_{2m(k)-1}), d(Rx_{2m(k)-1}, Tx_{2n(k)})) \\ &= [d(y_{2n(k)-1}, y_{2m(k)-1}).d(y_{2m(k)-2}, y_{2n(k)})]^{\frac{1}{2}} \\ &- \varphi(d(y_{2n(k)-1}, y_{2m(k)-1}), d(y_{2m(k)-2}, y_{2n(k)})) \\ &\leq [d(y_{2m(k)-1}, y_{2m(k)}).d(y_{2m(k)}, y_{2m(k)+1})]^{\frac{1}{2}} \end{aligned} \tag{4.13}$$

If $k \to \infty$ in the above inequality, from (4.12) and the continuity of φ, we have

$$\epsilon \leq [(\epsilon.\epsilon)]^{\frac{1}{2}} - \varphi(\epsilon, \epsilon)$$

$$\epsilon \leq \epsilon - \varphi(\epsilon, \epsilon)$$

$$\varphi(\epsilon, \epsilon) \leq 0$$

$$\varphi(\epsilon, \epsilon) = 0$$

and from the last inequality $\varphi(\epsilon, \epsilon) = 0$. By our assumption about φ, we have $\epsilon = 1$ which is contradiction (because $\epsilon < 1$).

Step III. T, S and R have a common fixed point.

Since (X, d) is complete and $\{y_n\}$ is cauchy, there exist $z \in X$ such that $lim_{y_n\to\infty} = z$.

Since E is closed and $\{y_n\}$ is contained a sequence in E, we have $z \in E$. By assumption $R(E)$ is closed, so there exist $u \in E$ such that $z = Ru$.
For all $n \in N$,

$$\begin{aligned} d(Tu, y_{2n+1}) &= d(Tu, Sx_{2n+1}) \\ &\leq [d(Ru, Sx_{2n+1}).d(Rx_{2n+1}, Tu)]^{\frac{1}{2}} \\ &- \varphi(d(Ru, Sx_{2n+1}), d(Rx_{2n+1}, Tu)) \\ &= [d(z, y_{2n+1}).d(y_{2n}, Tu)]^{\frac{1}{2}} - \varphi(d(z, y_{2n+1}), d(y_{2n}, Tu) \end{aligned} \tag{4.14}$$

If $n \to \infty$,

$$d(Tu, z) \leq [(d(z, z).d(z, Tu)]^{\frac{1}{2}} - \varphi(d(z, z), d(z, Tu))$$

and hence

$$\varphi(1, d(z, Tu)) \leq -[d(Tu, z)]^{\frac{1}{2}} \leq 0,$$

$$\varphi(1, d(z, Tu)) = 0$$

Therefore $d(z, Tu) = 1$. Therefore $Tu = z$.
Similarly $Su = z$. So $Tu = Su = Ru = z$. Since the pairs (R, T) and (R, S) are weakly compatible, we have $Tz = Sz = Rz$. Now we can have

$$\begin{aligned} d(Tz, y_{2n+1}) &= d(Tz, Sx_{2n+1}) \\ &\leq [d(Rz, Sx_{2n+1}).d(Rx_{2n+1}, Tz)]^{\frac{1}{2}} \\ &- \varphi(d(Rz, Sx_{2n+1}), d(Rx_{2n+1}, Tz)) \\ &= [d(Rz, y_{2n+1}).d(y_{2n}, Tz)]^{\frac{1}{2}} \\ &- \varphi(d(Rz, y_{2n+1}), d(y_{2n}, Tu) \end{aligned} \tag{4.15}$$

If $n \to \infty$, since $Tu = Su = Rz$, we obtain

$$d(Tz, z) = [d(Tz, z) + d(z, Tz))]^{\frac{1}{2}} - \varphi(d(Tz, z), d(z, Tz))$$

. Hence, $\varphi(d(Tz, z), d(z, Tz)) = 1$ and so $d(Tz, z) = 1$. Therefore $Tz = z$ and from $Tz = Sz = Rz = z$.
Uniqueness of common fixed point follows from (4.1).
Let u and z be two common fixed points of R, S and T.

$$\therefore Ru = Tu = Su = u \text{ and}$$

$$Rz = Tz = Sz = z$$

from (4.1)

$$\begin{aligned}
d(Tu,Sz) &\leq [(d(Ru,Sz).d(Rz,Tu)]^{\frac{1}{2}} - \varphi(d(Ru,Sz),d(Rz,Tu)) \\
d(Tu,z) &\leq [(d(u,z).d(z,u)]^{\frac{1}{2}} - \varphi(d(u,z),d(z,u)) \\
d(u,z) &\leq (u,z) - \varphi(d(u,z),d(z,u)) \\
\varphi(d(u,z),d(z,u)) &\leq d(u,z) - d(u,z) \\
\varphi(d(u,z),d(z,u)) &\leq 0 \\
\varphi(d(u,z),d(z,u)) &= 0 \\
d(z,u) &= 0 \\
u &= z
\end{aligned}$$

□

Chapter 5

Fixed Point Results for (ϵ, δ)-Uniformly Locally Contractive mappings in Multiplicative Metric Space

Definition 5.1. Let (M, d) be a multiplicative metric space and let $f : M \to M$ be a mapping of M into itself.f is said to be (ϵ, λ)-uniformly locally contractive if

$$1 < d(x, y) < \epsilon \Rightarrow d(fx, fy) < d(x, y)^{\alpha} \quad (5.1)$$

where $\epsilon > 1$ and $\alpha \in [0, 1)$.

Definition 5.2. A multiplicative metric space M is said to be ϵ-chainable if for every $a, b \in M$ there is a finite set of points $a = x_0, x_1, ..., x_m = b$ (m may depends on both a and b) such that $d(x_{i-1}, x_i) < \epsilon(i = 1, 2, ..., m)$

Theorem 5.3. *Let (M, d) be a complete multiplicative metric ϵ-chainable space and let $f : M \to M$ be an (ϵ, λ)-uniformly locally contractive self mapping on M. Then there exist a unique point $u \in M$ such that $fu = u$.*

Proof. Let $x \in M$. Consider the ϵ -chain $x = x_0, x_1, ..., x_m = fx$. Since $d(x_{i-1,i}, x_i) < \epsilon$, then by (5.1) we have

$$d(fx_{i-1}, fx_i) \leq d(x_{i-1}, x_i)^{\alpha} < \epsilon^{\alpha} < \epsilon.$$

Thus by (5.1) we get

$$d(f(fx_{i-1}), f(fx_i)) \leq d(fx_{i-1}, fx_i)^{\alpha} < \epsilon^{\alpha^2},$$

Continuing this process, we get

$$d(f^n x_{i-1}, f^n x_i) \leq d(f^{n-1} x_{i-1}, f^{n-1} x_i)^{\alpha} \leq ... \leq d(x_{i-1}, x_i)^{\alpha^n} < \epsilon^{\alpha^n}.$$

Therefore by triangular inequality we get

$$d(f^n x, f^{n+1} x) \leq \Pi_{i=1}^{m} d(f^n x_{i-1}, f^n x_i) < \epsilon^{m\alpha^n}.$$

Hence, the sequence $\{f^n x\}$ is multiplicative cauchy.
Indeed, if p and q, $(p < q)$ are any positive integer, then

$$\begin{aligned} d(f^p x, f^q x) &\leq \Pi_{i=p}^{q-1} d(f^i x, f^{i+1} x) \\ &\leq \Pi_{i=p}^{q-1} \epsilon^{m\alpha^i} \\ &< \Pi_{i=p}^{\infty} \epsilon^{m\alpha^i} \\ &= \epsilon^{m\alpha^p} . \epsilon^{m\alpha^{p+1}} ... \\ &= \epsilon^{m\alpha^p} (1 + \epsilon + \epsilon^2 + ...) \\ &= \frac{\epsilon^{m\alpha^p}}{\epsilon - 1} \to 1 \text{as } q > p \to \infty \end{aligned}$$

The completeness of M guarantees the existence of some point $u \in M$ such that $lim_{n\to\infty} f^n x = u$. Since any (ϵ, α)-uniformly locally contractive mapping is continuous, it follows that

$$fu = f(lim_{n\to\infty} f^n x) = lim_{n\to\infty} f^{n+1} x = lim_{n\to\infty} f^n x = u.$$

Hence, u is a fixed point of f. Now, we show that u is the unique fixed point of f. On the contrary, let us suppose that $v \neq u$ such that $fv = v$ and let $u = x_0, x_1, ..., x_k = v$ be an ϵ-chain. Then we have

$$\begin{aligned} 1 < d(u, v) &= d(f^n u, f^n v) \\ &\leq \Pi_{i=1}^{k} d(f^n x_{i-1}, f^n x_i) \\ &\leq \epsilon^{k\alpha^n} \to 1 \text{ as } n \to \infty \text{ a contradiction.} \end{aligned}$$

Hence $v = u$ □

Definition 5.4. Let (M, d) be a multiplicative metric space. A self map f on M is said to be a weakly multiplicative contractive if $\exists$ a function $\alpha : (1, \infty) \to [0, 1)$ with

$sup\{\alpha(c) : 1 < a \leq c \leq b\} < 1$ and such that

$$d(fx, fy) = d(x, y)^{\alpha(d(x,y))}. \tag{5.2}$$

Theorem 5.5. *Let (M, d) be a complete multiplicative metric space. Let $f : M \to M$ be a weakly multiplicative mapping. Then f has a unique fixed point.*

Proof. Let $x \in M$ be arbitrary. Consider the sequence $\{f_n x\}$. If $d(f^n x, f^{n+1}x) = 1$ for some n, then $ff^n x = f^n x$ i.e; fx is a fixed point of f and so the conclusion of theorem follows. Suppose now that $d(f^n x, f^{n+1}x) > 1$ for all $n \in N$. Then, as $\alpha(c) < c$ for $c > 1$, from (5.2) we have that f is multiplicative contractive. So we get

$$\begin{aligned} d(f^n x, f^{n+1}x) &= d(ff^{n-1}x, ff^n x) \\ &\leq d(f^{n-1}x, f^n x)^{\alpha(d(f^{n-1}x, f^n x))} \\ &< d(f^{n-1}x, f^n x). \end{aligned}$$

Thus $\{d(f^n x, f^{n+1}x)\}$ is a monotone decreasing sequence of reals and so it converges. Let

$$lim_{n\to\infty} d(f^n x, f^{n+1}x) = r$$

Then $r < d(fx, f^{n+1}x) \leq d(x, fx)$ for all $n \geq 0$. We show that $r = 1$. Suppose the contrary that $r > 1$ and set $\alpha = sup\{\alpha(c) : 1 < r \leq c \leq d(x, fx)\}$ Then $\alpha(d(f^n x, f^{n+1}x)) \leq \alpha$ for all $n \geq 0$ and so we have

$$1 < r < d(f^n x, f^{n+1}x) \leq d(f^n x, f^{n-1}x)^{\alpha} \leq ..... \leq d(x, fx)^{\alpha^n} \to 1 \text{as n} \to \infty$$

a contradiction. Therefore, $r = 1$.

Now, we show that $\{f^n x\}$ is a multiplicative cauchy sequence. Let $\epsilon > 1$ and set $1 < \alpha(\epsilon) = sup\{\alpha(c) : \frac{\epsilon}{2} \leq c \leq \epsilon\}$. Since $lim_{n\to\infty} d(f^n x, f^{n+1}x) = 1$ and $\alpha(\epsilon) - 1 > 0$, there exist $n_0 \in N$ such that

$$d(f^n x, f^{n+1}x) < \epsilon^{\frac{\alpha(\epsilon)-1}{2}} \text{ for all } n \geq n_0. \tag{5.3}$$

Let $n \geq n_0$ be any fixed positive integer. We shall show, by induction, that

$$d(f^n x, f^m x) < \epsilon \text{ for all } m > n \geq n_0. \tag{5.4}$$

for $m = n+1$, (5.4) follows from (5.3). Assume now that (5.4) holds for some $m \geq n+1$. If $M(d(f^n x, f^m x)) \geq \frac{\epsilon}{2}$, then from (5.2) we have

$$d(ff^n x, ff^m x) \leq d(f^n x, f^m x)^{\alpha(\epsilon)} < \epsilon^{\alpha(\epsilon)}.$$

Thus, by multiplicative triangular inequality and (5.3) we get

$$\begin{aligned} d(f^n x, f^{m+1} x) &\leq d(f^n x, ff^n x).d(ff^n x, ff^m x) \\ &< \epsilon^{\frac{\alpha(\epsilon)-1}{2}}.\epsilon^{\alpha(\epsilon)} < \epsilon. \end{aligned}$$

If $d(f^n x, f^m x) < \frac{\epsilon}{2}$, then by the multiplicative triangular inequality and (2.3) we get

$$\begin{aligned} d(f^n x, f^{m+1} x) &\leq d(f^n x, f^m x).d(f^m x, f^{m+1} x) \\ &< \frac{\epsilon}{2}.\epsilon^{\frac{\alpha(\epsilon)-1}{2}} < \epsilon. \end{aligned}$$

Therefore, $d(f^n x, f^{m+1} x) < \epsilon$, which completes the induction from (5.4). We conclude that $\{f^n x\}$ is a multiplicative cauchy sequence. The multiplicative completeness of f guarantees that the existence of some point $u \in M$ such that $lim_{n\to\infty} f^n x = u$. By continuity of f, it follows :

$$fu = f(lim_{n\to\infty} f^n x = u) = lim_{n\to\infty} ff^n x = u$$

Multiplicative contractivity of f implies that the uniqueness of fixed point. □

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