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# Fixed Point Theorems in Metric Space and G-Metric Space

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# FIXED POINT THEOREM IN METRIC SPACE AND G-METRIC SPACE

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Abstract. The purpose of this report is to study about metric space and investigate the properties related to the metric space.

## 1. Introduction

Historically metric space developed as an outgrowth of analysis rather than of geometry. However metric space resembles geometry very closely in its spirit and indeed can be considered as an abstract form of geometry. It will therefore be instructive to see how the definition of metric space can be approached in a very natural way starting from geometry. The journey from euclidean geometry to metric spaces can be conveniently broken into two parts. If we take a close look at euclidean spaces we see that their entire structure is based upon two basic concepts, that of distances between two points. By generalization of euclidean spaces the concept of metric space developed. First replace the euclidean space by an abstract set $X$ and the euclidean distance function by an abstract distance function. This will be a real valued function $d$ on the product $X \times X$; if $x$, $y \in X$ then $d(x, y)$ would be the distance between $x$ and $y$. To give reasonable generalization of euclidean spaces one must considered some properties of $d$ as axioms. These are positivity, symmetry and the triangle inequality properties. The function $d$ is called a metric on the set $X$. A set $X$ together with metric $d$ on $X$ is called a metric space.

A metric space can be be thought of as very basic space having a geometry, with only few axioms. Metric spaces are generalization of the real line, in which some of the theorems that hold for $\mathbb{R}$ remain valid. Some of the main results in the real analysis are

(1) Cauchy sequences convergence
(2) For continuous functions $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$,
(3) Continuous real functions are bounded on intervals of type $[a, b]$ and satisfy the intermediate value theorem.

At first sight, it is difficult to generalize these theorems, say to sequences and continuous functions of several variable such as $f(x, y)$. This is the aim of this abstract course: to show how these theorems apply in a much more general setting than $\mathbb{R}$. The fundamental ingredient that is needed is that of a distance or metric. This is not enough, however, to get the best results distance need to be of a nice type, called a complete metric.

In what follows that the metric space $X$ will denote an abstract set with a distance function defined on it which satisfying some axioms.

## 2. Metric Space

In this section we studied the most important notions of metric spaces are the generalization of concepts of real line.

**Definition 2.1** (Metric Space)**.** Let $X$ be a non-empty set whose elements we shall call points, is said to be a metric space if with any two points $p$ and $q$ of $X$ there is associated a real number $d(p, q)$, called the distance from $p$ to $q$, where $d$ is called a metric. A metric on $X$ is a function

$$d : X \times X \to \mathbb{R}$$

having the following properties:

(1) $d(p, q) \geq 0$ for all $p$, $q \in X$; equality holds if and only if $x = y$.
(2) $d(p, q) = d(p, q)$ for all $p, q \in X$.
(3) $d(p, q) \leq d(p, r) + d(q, r)$ for all $p, q, r \in X$.

Therefor the ordered pair $(X, d)$ is known as the metric space on set $X$ with metric $d$.

**Example 2.1.** *let us look at the plane $\mathbb{R}^2$. For $p = (p_1, p_2)$ and $q = (q_1, q_2)$, two points in $\mathbb{R}^2$, we introduced the Euclidean distance formula $d(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$.*

*The ordered pair* $(\mathbb{R}^2, d)$ *is a metric space because* $d(p,q) \geq 0$ *for all* $p, q \in \mathbb{R}^2$ *and second condition of metric space is also trivial. Now for any* $r = (r_1, r_2) \in \mathbb{R}^2$ *check the triangle inequality* $d(p,q) \leq d(p,r) + d(r,q)$*. The points* $p$*,* $q$*, and* $r$ *forms a triangle in the plane* $\mathbb{R}^2$*, since the sum of the lengths of any two sides must be greater than or equal to the remaining side. Thus the triangle inequality holds for* $d$*.*

For each $x \in X$ and $\epsilon > 0$, define $B(x,\epsilon) = \{p \in X \,|\, d(x,p) < \epsilon\}$. The set $B(x,\epsilon)$ is called the open ball of radius $\epsilon$ centered at $x$.

For each $x \in X$ and $\epsilon > 0$, define $B[x,\epsilon] = \{p \in X \,|\, d(x,p) \leq \epsilon\}$. The set $B[x,\epsilon]$ is called the closed ball of radius $\epsilon$ centered at $x$.

Consider the usual metric space $(\mathbb{R}, d)$, the open ball $B(x,\epsilon)$ is the open interval $(x - \epsilon, x + \epsilon)$ while the closed ball $B[x,\epsilon]$ is the closed interval $[x - \epsilon, x + \epsilon]$.

**Lemma 2.2.** *Let* $y$ *be in* $X$ *and assume* $r > 0$*. Then for every* $x \in B(y,r)$ *there exists an* $\epsilon > 0$ *such that* $B(x,\epsilon) \subset B(y,r)$*.*

**Example 2.2.** *Given a set* $X$ *define*

$$d(x,y) = 1 \qquad \text{if } x \neq y,$$
$$d(x,y) = 0 \qquad \text{if } x = y.$$

*From the definition of* $d(x,y)$ *we can consider that* $d(x,y) \geq 0$ *and* $d(x,y) = d(y,x)$ *for all* $x$*,* $y \in X$*. For any* $z \in X$ *let us check the triangular inequality*

$$d(x,y) \leq d(x,z) + d(y,z).$$

*if* $x$*,* $y$ *and* $z$ *are equal than* $d(x,y)$*,* $d(x,z)$ *and* $d(z,y)$ *are zero.*

**Definition 2.3** (Open set)**.** In a metric space, a subset is said to be a open set if it is a neighbourhood of each of its points.

For the usual metric space $(\mathbb{R}, d)$ the open interval is an open set. Some properties related to open sets are given below [4]

(1) Every open ball is an open set but converse is not true.
(2) A subset of metric space is open if it is a union of open balls.
(3) In any metric space $(X,d)$ the set $X$ and the empty set are open sets.
(4) The union of arbitrary collection of open sets is open.
(5) The intersection of finite number of open sets is open.

Two metrics $d$ and $d'$ on the same set $X$ are said to be equivalent, if every open set in metric space $(X,d)$ is open in metric space $(X,d')$ and vice versa.

Let $(X,d)$ be a metric space and let $A$ and $B$ any two non-empty subsets of $X$. The distance between these two sets is defined as $d(A,B) = inf\{d(x,y) \,|\, x \in A, y \in B\}$.

**Definition 2.4** (Diameter of a Set)**.** Let $(X,d)$ be a metric space and let $A$ be any non-empty subset of $X$. Then the the diameter of $A$ is defined as the maximum distance between any two points of $A$.

The diameter of empty set is considered as minus infinity.

## 3. Cauchy Sequence

Let $(X,d)$ be a metric space. A function whose domain is the set of natural numbers and range a subset of $X$ is called a sequence. When $X = \mathbb{R}$, then the sequence is known as the real sequence.

**Definition 3.1** (Convergent Sequence)**.** A sequence $(x_n)$ in a metric space $(X, d)$ is said to converges to $x \in X$, if given $\epsilon > 0$, we can find a positive integer $m$ depending on $\epsilon$ such that $d(x_n, x) < \epsilon$ whenever $n \geq m$.

A limit of a sequence, if it exists is unique.

**Definition 3.2** (Cauchy sequence)**.** A sequence of real numbers $\{a_n\}$ is a Cauchy sequence provided for every $\epsilon > 0$ there exists a natural number $N$ so that for $n, m \geq N$ implies $|a_n - a_m| \leq \epsilon$.

Every convergent sequence is a cauchy sequence. For example the sequence $\{\frac{1}{n}\}$ converges to 0. It is a cauchy sequence.

Now we define the cauchy sequence in metric space $(X, d)$. We say that a sequence $x_1, x_2, \dots$ in a metric space $X$ is a cauchy sequence if for all $\epsilon > 0$, we can find a natural number $N$, such that $d(x_n, x_m) \leq \epsilon$ where $n, m > N$.

For checking any sequence is cauchy sequence first try to check it is a convergent sequence or not because every convergent sequence is a cauchy sequence. Suppose a sequence $\{x_n\}$ converges to $x$, let $\epsilon > 0$ then there exists a some number $N$ such that $d(x_n, x) < \frac{\epsilon}{2}$, it follows that for $n, m > N$

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon$$

so that sequence $\{x_n\}$ is a cauchy sequence.

**Example 3.1.** *Consider space $C[0, 1]$ of continuous function on closed interval $[0, 1]$ with metric d, than the sequence*

$$f_n(x) = \begin{cases} 0 & \textit{if } x \leq \frac{1}{2}; \\ n(x - \frac{1}{2}) & \textit{if} \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n}; \\ 1 & \textit{if } \frac{1}{2} + \frac{1}{n} \leq x. \end{cases}$$

*Then the sequence $\{f_n\}$ is a cauchy sequence.*

If a cauchy sequence has a subsequence that converges to $x$, then so be the sequence converges to $x$. Since every convergent sequence is a cauchy sequence but the converse is not true.

**Example 3.2.** *consider the sequence $\{x_n\}$ where $x_n = \frac{\sqrt{2}}{n}$ for each $n \in \mathbb{N}$. Note that each $x_n$ is an irrational numbers. But the sequence is converges to zero which is a rational number. Thus $\{x_n\}$ converges in $\mathbb{R}$. It does not converges in set of irrational numbers.*

**Example 3.3.** *Let $x_1 \in \mathbb{N}$ and let $x_n$ be sequence defined by $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for each $n \in \mathbb{N}$, than the sequence $x_n$ converges to a irrational number $\sqrt{2}$. So the sequence $x_n$ converges in $\mathbb{R}$ but not in $\mathbb{Q}$.*

Example 3.2 and 3.3 demonstrate that each of the set of irrational numbers and set of rational numbers are not well behaved metric space. Because there exists some sequence in each space that do not converges to element in space. The sequence are well behaved in the sense that they do converge in $\mathbb{R}$. The following section we formalize the idea of well behaved sequence in metric space.

## 4. Complete Metric Space

**Definition 4.1.** A metric space $(X, d)$ is complete if every cauchy sequence in $X$ converges to a point in $X$.

The real interval $(0, 1)$ with usual metric is not a complete space; the sequence $x_n = \frac{1}{n}$ is cauchy but does not converge to element of $(0, 1)$. $\mathbb{R}$ is a complete metric space, since every sequence of real numbers is converges to a real number.

Completion axioms states that every cauchy sequence of real number converges.

**Example 4.1.** *Consider the closed unit interval* $[0, 1]$*. It is a complete metric space under absolute value metric.*

**Example 4.2.** *Consider the sequence* $\{f_n\}$ *where it is defined as*

$$f_n(x) = \begin{cases} nx & \text{if } x \leq \frac{1}{n}; \\ 1 & \text{if } x \geq \frac{1}{n}. \end{cases}$$

*on unit interval* $[0, 1]$ *in* $\mathbb{R}$*. Since the sequence does not converge in* $C[0, 1]$*. Therefore the space* $C[0, 1]$ *would not be complete.*

With euclidean metric $(d_2)$ the space $\mathbb{R}^K$ is complete. Consider $a = (a_1, a_2, \ldots, a_K), b = (b_1, b_2, \ldots, b_K)$ are elements of $\mathbb{R}^k$. Therefore for $1 \leq i \leq K$ than

$$d_2(a, b) = \sqrt{(a_1 - b_1)^2 + \cdots + (a_k - b_k)^2} \geq |a_i - b_i|$$

. Now let $\{x_n\}$ be a cauchy sequence in $\mathbb{R}^k$ where $x_n = (x_n^{(1)}, \ldots, x_n^{(K)})$, let $\epsilon > 0$ and let $N$ be such that $d_2(x_n, x_m) < \epsilon$ for all $n, m > N$, for each $1 \leq i \leq k$ we have

$$|x_n^{(i)} - x_m^i| \leq d_2(x_n, x_m) < \epsilon$$

for all $n, m > N$. This mean the sequence $(x_n^i)$ of the $i$th coordinate is also cauchy and therefore it converges. Thus $\mathbb{R}^k$ is complete. [3]

Some properties of complete metric spaces are

(1) Completeness is preserved under isometries.
(2) Complete metric space is not a hereditary property, but the closed subspace of complete metric space is closed. It is necessary and sufficient condition for any closed subspace of complete metric space to be a complete metric space.

**Definition 4.2** (Contraction mapping)**.** A mapping $f$ from metric space $(X, d)$ to itself is said to be a contraction mapping if for some real number $\alpha$ such that $0 < \alpha < 1$,

$$d(f_x, f_y) \leq \alpha d(x, y).\text{all} x, y \in X$$

Any mapping is contraction in the sense that the distance between the images of any two points is less than that between their pre-images. If $f$ is a contraction on $X$ then it is continuous on $X$. Every contraction mapping is uniform continuous but converse is not true.

Now we give a theorem related to contraction mapping on complete metric space.

**Theorem 4.3.** *Every contraction mapping* $f$ *on a complete metric space has a unique fixed point.*

**Definition 4.4** (Compact Metric Space)**.** A metric space is said to be compact if every open cover of metric space has a finite open subcover.

The real line $\mathbb{R}$ is not compact, but its closed and bounded subset in particular $[0, 1]$ is a compact metric space. Now the properties of the compact metric space are given below

(1) Compactness on a metric is not a hereditary property. The closed subset of compact set is compact.
(2) A continuous image of compact metric space is compact.
(3) Compactness is preserved under isometries.

If every infinite subset of metric space has a limit point, then we said that the given metric space satisfying the Bolzano-Weirstass property. If every sequence in metric space has convergent subsequence, then the given metric space is sequentially compact metric space. The necessary and sufficient condition for any metric space to be sequentially compact if it satisfing the Bolzano-Weirstass property.

**Definition 4.5** (Locally Compact Spaces)**.** A metric space is said to be locally compact if and only if every point in metric space has at least one neighbourhood whose closure is compact.

Every compact metric space is a locally compact, but converse is not true. For example discrete space is a locally compact space but it is not a compact space.

## 5. A new approach to generalized Metric Spaces

In this section we investigate a new approach to generalized metric space which was introduced by the Zead Mustafa and Brailey Sims which is an alternative robust generalization of metric spaces. Namely, that of a G-metric space, where the G-metric satisfies the axioms:

(1) $G(x, y, z) = 0$ if $x = y = z$,
(2) $0 < G(x, z, y); whenever x \neq y$,
(3) $G(x, x, y) \leq G(x, y, z)$ whenever $z \neq y$,
(4) $G$ is a symmetric function of its three variable, and
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$

During the sixties, 2-metric spaces were introduced by Gahler

**Definition 5.1** (2-metric Spaces)**.** Let $X$ be a nonempty set, and let $\mathbb{R}$ denote the real numbers. A function $d : X \times X \times X \to \mathbb{R}^+$ satisfying the following properties:

(1) For distinct points $x, y \in X$, there is $z \in X$, such that $d(x, y, z) \neq 0$
(2) $d(x, y, z) = 0$ if two of the triple $x, y, z \in X$ are equal.
(3) $d(x, y, z) = d(x, z, y) = \ldots$ (Symmetry in all three variables)
(4) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ for all $x, y, z, a \in X$,

is called a 2-metric, on $X$. The set $X$ equipped with such a 2-metric is called a 2-metric space.

It is clear that taking $d(x, y, z)$ to be the area of the triangle with vertices at $x, y$ and $z$ in $\mathbb{R}^2$ provides an example of a 2-metric Gahler claimed that a 2-metric is a generalization of the usual notation of a metric, but different authors proved that there is no relation between these two functions. The contraction mapping theorem in metric spaces and in 2-metric spaces are unrelated.

This consideration led Bapure Dhage to introduce a new class of generalized metrics called D-metrics. In subsequent series of papers Dhage attempted to develop topological structures in such spaces. He claimed that D-metrics provide a generalization of ordinary metric functions and went on to present several fixed point results. A D-metric need not be a continuous function of its variables. All of these consideration leads us to seek a more appropriate notion of generalized metric space. [1]

**Definition 5.2** (G-metric)**.** Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying:

(1) $G(x, y, z) = 0$ if $x = y = z$
(2) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$,
(3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
(4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (Symmetry in all three variables), and
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, ()rectangle inequality.

Than the function $G$ is called a generalized metric or, more specifically G-metric on $X$, the pair $(X, G)$ is called a G-metric space. [2]

When $G(x, y, z)$ is the perimeter of the triangle than these are trivially holds for $x, y, z \in \mathbb{R}^2$. A G-metric space is symmetric if $G(x, y, y) = G(x, x, y)$, for all $x, y \in X$. If $G$ is derives from an underlying metric via $E_s$ or $E_m$ than the following G-metric space is symmetric.

**Example 5.1.** *Let $X = \{a, b\}$, Let $G(a, a, a) = G(b, b, b) = 0$ , $G(a, a, b) = 1$ and $G(a, b, b) = 2$ and extend $G$ to all of $X \times X \times X$ by symmetry in the variables. Then it is a G-metric, but it is not symmetric.*

Let $(X, G)$ be a $G$ metric space than the following properties are easily derived from the axioms of the G-metric space.

(1) If $G(x, y, z) = 0$ then $x = y = z$,
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(3) $G(x, y, y) \leq 2G(y, x, x)$,
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(6) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$,
(7) $|G(x, y, z) - G(x, y, a)| \leq max\{G(a, z, z), G(z, a, a)\}$
(8) $|G(x, y, z) - G(x, y, a)| \leq G(x, a, z)$,
(9) $|G(x, y, z) - G(y, z, z)| \leq max\{G(x, z, z), G(z, x, x)\}$,
(10) $|G(x, y, y) - G(y, x, x)| \leq max\{G(y, x, x), G(x, y, y)\}$

**Proposition 5.3.** *Let $(X, G)$ be a G-metric space and let $k > 0$, then $G_1$ and $G_2$ are also G-metrics on $X = \bigcup_{i=1}^{n} A_i$, where,*

(1) $G_1(x, y, z) = min\{k, G(x, y, z)\}$, *and*
(2) $G_2(x, y, z) = \frac{G(x,y,z)}{k+G(x,y,z)}$.
(3)

$$G_3(x, y, z) = \begin{cases} G(x, y, z), & \textit{if for some } i \textit{ we have } x, y, z \in A_i, \\ k + G(x, y, z), & \textit{otherwise}, \end{cases}$$

is also a G-metric.

**Proposition 5.4.** *Let $(X, G)$ be a G-metric space, then the following are equivalent*

(1) *$(X, G)$ is symmetric.*
(2) *$G(x, y, y) \leq G(x, y, a)$ for all $x, y, a \in X$.*
(3) *$G(x, y, z) \leq G(x, y, a) + G(x, y, b)$ for all $x, y, z, a, b \in X$.*

## 6. Learning Outcomes

Write here Learning Outcomes of the report.

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