Basic Mathematics I

DMTH201

LOVELY PROFESSIONAL UNIVERSITY

BASIC MATHEMATICS – I

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SYLLABUS

Basic Mathematics – I

Objectives: This course is designed to provide an introduction to the fundamental concepts in Mathematics. After the completion of this course prospective students will be able to apply the concepts of basic Mathematics in the professional course.

S. No.	Description
1	Trigonometric Functions of Sum and Difference of Two Angles
2	Allied Angles. Transformation ormulae , Inverse Trigonometric Functions
3	Matrix, Types of Matrices, Matrix Operations, Addition, Substraction, Multiplication of Matrices, Transpose of Matrix, Symmetric and Skew Symmetric Matrix
4	Adjoint of Matrix, Inverse of a Matrix using Elementary operation and Determinants Method
5	Minors and co-factors , Determinant , Solution of system of equations, Inverse of Matrix using determinants
6	Distance between two points, Slope of a line, Various forms of the equation of a line,
7	Distance of a Point from a Line, Circle.
8	Functions, Different types of functions, Limits and Continuity, Rules and Standard Procedures
9	Differentiability, Derivatives of Exponential, Logarithmic and Parametric Functions, Logarithmic Differentiation,
10	Rate of Change, Tangents and Normal, Maxima and Minima.

CONTENTS

Unit 1: Trigonometric Functions-I

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define trigonometeric functions of a real number
- Draw the graphs of trigonometeric functions
- Interpret the graphs of trigonometeric functions

Introduction

Trigonometry is a branch of mathematics that deals with triangles, circles, oscillations and waves; it is totally crucial to much of geometry and physics. You will often hear it described as if it was all about triangles, but it is a lot more exciting than that. For one thing, it works with all angles, not just triangles. For another, it describes the behaviour of waves and resonance, which are at the root of how matter works at the most basic level. They are behind how sound and light move, and there are reasons to suspect they are involved in our perception of beauty and other facets of how our minds work, so trigonometry turns out to be fundamental to pretty much everything.

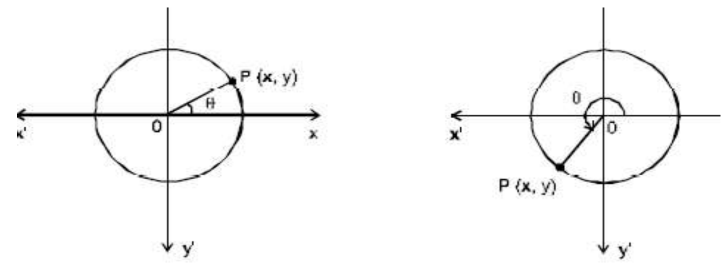
Notes

Any time you want to figure out anything to do with angles, or turning, or swinging, there's trigonometry involved.

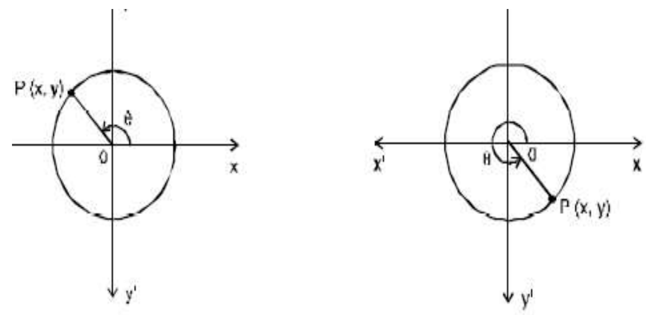
1.1 Trigonometric Coordinates

As you have already studied the trigonometric ratios of acute angles as the ratio of the sides of a right angled triangle. You have also studied the trigonometric identities and application of trigonometric ratios in solving the problems related to heights and distances.

While considering, a unit circle you must have noticed that for every real number between 0 and 2p, there exists a ordered pair of numbers x and y. This ordered pair (x, y) represents the Coordinates of the point P.



(a) (b)



(c) (d)

If we consider $\theta = 0$ on the unit circle, we will have a point whose coordinates are (1, 0).

If $\theta = \pi/2$, then the corresponding point on the unit circle will have its coordinates (0, 1).

In the above figures you can easily observe that no matter what the position of the point, corresponding to every real number q we have a unique set of coordinates (x, y). The values of x and y will be negative or positive depending on the quadrant in which we are considering the point.

Considering a point P (on the unit circle) and the corresponding coordinates (x, y), we define trigonometric functions as:

Sin $\theta = y$, cos $\theta = x$

Tan $\theta = y/x$ (for $x \neq 0$), cot $\theta = x/y$ (for $y \neq 0$)

Sec $\theta = 1/x$ (for $x \neq 0$), cosec $\theta = 1/y$ (for $y \neq 0$)

Notes

Now let the point P move from its original position in anti–clockwise direction. For various positions of this point in the four quadrants, various real numbers q will be generated. We summarise, the above discussion as follows. For values of q in the:

I quadrant, both x and y are positive.

II quadrant, x will be negative and y will be positive.

III quadrant, x as well as y will be negative.

IV quadrant, x will be positive and y will be negative.

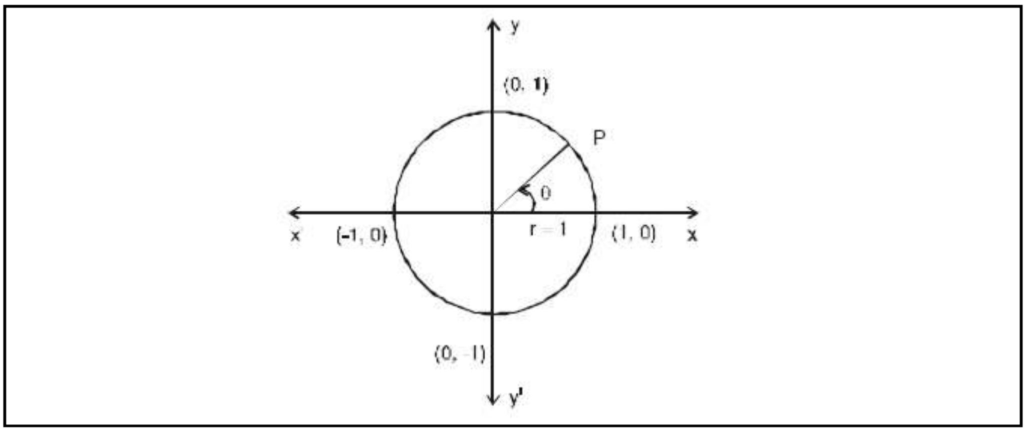
or

I quadrant	II quadrant	III quadrant	IV quadrant.
All positive	sin positive	tan positive	cos positive
Cosec positive	cot positive	sec positive	

Where what is positive can be remembered by:

	All	sin	tan	cos
Quadrant	I	II	III	IV

If (x, y) are the coordinates of a point P on a unit circle and q , the real number generated by the position of the point, then $\sin\theta = y$ and $\cos\theta = x$. This means the coordinates of the point P can also be written as ($\cos\theta$, $\sin\theta$) From Figure you can easily see that the values of x will be between –1 and +1 as P moves on the unit circle. Same will be true for y also. Thus, for all P on the unit circle.



$-1 < x > 1$ and $-1 < y > 1$

Thereby, we conclude that for all real numbers θ

$-1 < \cos\theta > 1$ and $-1 < \sin\theta > 1$

In other words, sin and cos θ can not be numerically greater than 1

Similarly, $\sec\theta = 1/\cos$ $(\theta \neq n\pi/2)$

Notes

Task

1. Find the most general value of θ satisfying :

(i) $\sin\theta = \frac{\sqrt{3}}{2}$ (ii) $\text{cosec}\,\theta = \sqrt{2}$

(iii) $\sin\theta = -\frac{\sqrt{3}}{2}$ (iv) $\sin\theta = -\frac{1}{\sqrt{2}}$

2. Find the most general value of θ satisfying :

(i) $\cos\theta = -\frac{1}{\sqrt{2}}$ (ii) $\sec\theta = -\frac{2}{\sqrt{3}}$

(iii) $\cos\theta = \frac{\sqrt{3}}{2}$ (iv) $\sec\theta = -\sqrt{2}$

3. Find the most general value of θ satisfying :

(i) $\tan\theta = -1$ (ii) $\tan\theta = \sqrt{3}$

(iii) $\cot\theta = -1$

1.2 Trigonometric Function

The trigonometric ratios – sine, cosine, and tangent – are based on properties of right triangles. The function values depend on the measure of the angle. The functions are outlined below.

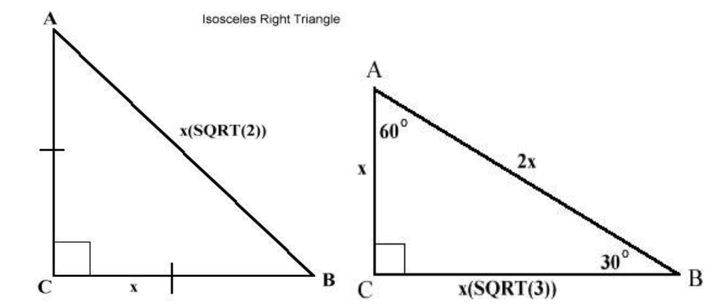
sine x = (side opposite x)/hypotenuse

cosine x = (side adjacent x)/hypotenuse

tangent x = (side opposite x)/(side adjacent x)

In the figure, sin A = a/c, cosine A = b/c, and tangent A = a/b.

There are two special triangles you need to know, 45 – 45 – 90 and 30 – 60 – 90 triangles. They are depicted in the figures below.



The figures show how to find the side lengths of those types of triangles. Besides knowing how to find the length of any given side of the special triangles, you need to know their trig. ratio values (they are always the same, no matter the size of the triangle because the trig. ratios depend on the measure of the angle). A table of these values is given below:

	0°	30°	45°	60°	90°
Sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

Notes

1.2.1 Reciprocal Ratio

The reciprocal ratios are trigonometric ratios, too. They are outlined below:

cotangent x = 1/tan x = (adjacent side)/(opposite side)

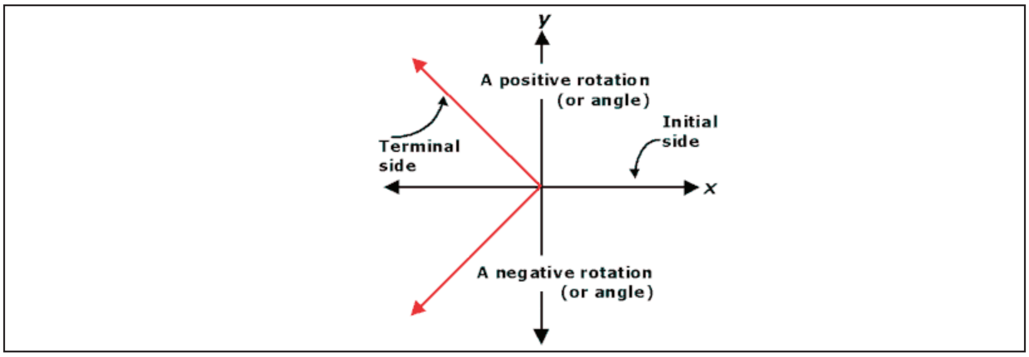
secant x = 1/cos x = (hypotenuse)/(adjacent side)

cosecant x = 1/sin x = (hypotenuse)/(opposite side)

1.2.2 Rotations of Angles

Angles are also called *rotations* because they can be formed by rotating a ray around the origin on the coordinate plane. The *initial side* is the x-axis and the ray that has been rotated to form an angle is the terminal side.

Example:



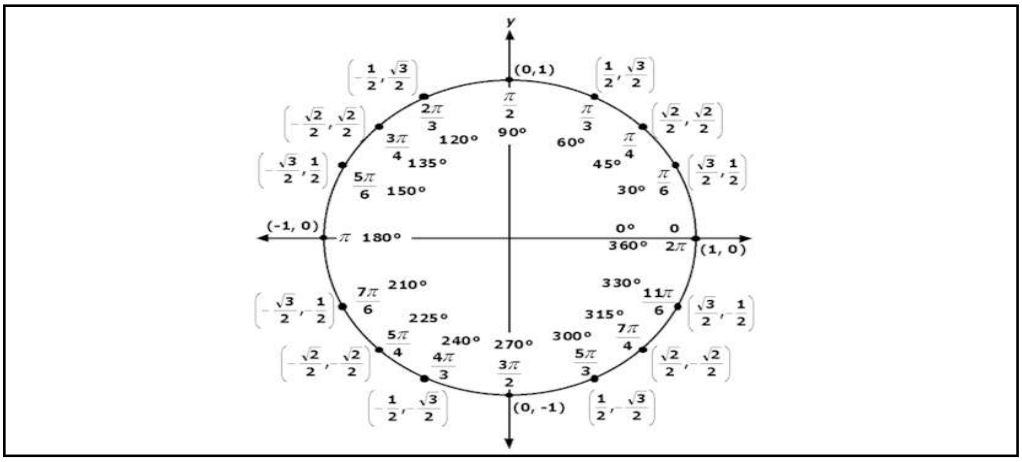
Reference angles are useful when dealing with rotations that end in the second, third, or fourth quadrants.

Radians

Up until now, you have probably only measured angles using degrees. Another useful measure, based on the unit circle, is called radians.

The figure shows measures in degrees and radians on the unit circle that you should probably memorize, as they are commonly used measures.

Notes



Sometimes, it will be necessary to convert from radians to degrees or vice versa. To convert from degrees to radians, multiply by ((π)/180°). To convert from radians to degrees, multiply by (180°/(π)).

1.3 Sines and Cosines Defined

Sine and cosine are periodic functions of period 360°, that is, of period 2π. That's because sines and cosines are defined in terms of angles, and you can add multiples of 360°, or 2π, and it doesn't change the angle.

Properties of Sines & Cosines following from this definition

There are numerous properties that we can easily derive from this definition. Some of them simplify identities that we have seen already for acute angles.

Thus,

$$\sin(t + 360°) = \sin t, \text{ and}$$
$$\cos(t + 360°) = \cos t.$$

Many of the current applications of trigonometry follow from the uses of trig to calculus, especially those applications which deal straight with trigonometric functions. So, we should use radian measure when thinking of trig in terms of trig functions. In radian measure that last pair of equations read as:

$$\sin(t + 2\pi) = \sin t, \text{ and}$$
$$\cos(t + 2\pi) = \cos t.$$

Sine and cosine are complementary:

$$\cos t = \sin(\pi/2 - t)$$
$$\sin t = \cos(\pi/2 - t)$$

We've seen this before, but now we have it for any angle t. It's true because when you reflect the plane across the diagonal line y = x, an angle is exchanged for its complement.

The Pythagorean identity for sines and cosines follows directly from the definition. Since the point B lies on the unit circle, its coordinates x and y satisfy the equation $x^2 + y^2 = 1$. But the coordinates are the cosine and sine, so we conclude

$$\sin^2 t + \cos^2 t = 1.$$

We're now ready to look at sine and cosine as functions.

Sine is an odd function, and cosine is an even function. You may not have come across these adjectives "odd" and "even" when applied to functions, but it's important to know them. A function f is said to be an odd function if for any number x, f(–x) = –f(x). A function f is said to be an even function if for any number x, f(–x) = f(x). Most functions are neither odd nor even functions, but it's important to notice when a function is odd or even. Any polynomial with only odd degree terms is an odd function, for example, $f(x) = x^5 + 8x^3 - 2x$. (Note that all the powers of x are odd numbers.) Similarly, any polynomial with only even degree terms is an even function. For example, $f(x) = x^4 - 3x^2 - 5$. (The constant 5 is $5x^0$, and 0 is an even number.)

Sine is an odd function, and cosine is even

$$\sin -t = -\sin t,$$

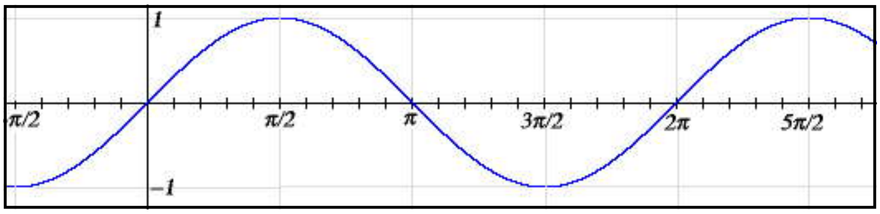
$$\text{and } \cos -t = \cos t.$$

These facts follow from the symmetry of the unit circle across the x-axis. The angle $-t$ is the same angle as t except it's on the other side of the x-axis. Flipping a point (x,y) to the other side of the x-axis makes it into $(x,-y)$, so the y-coordinate is negated, that is, the sine is negated, but the x-coordinate remains the same, that is, the cosine is unchanged.

An obvious property of sines and cosines is that their values lie between –1 and 1. Every point on the unit circle is 1 unit from the origin, so the coordinates of any point are within 1 of 0 as well.

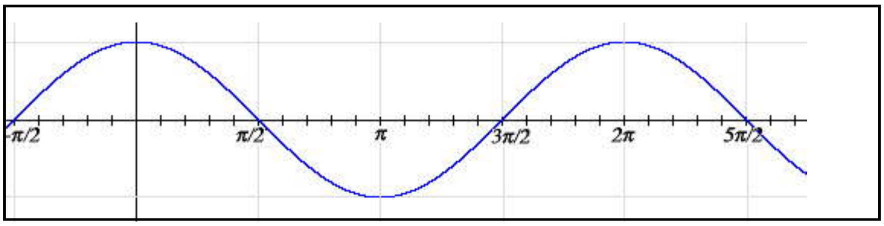
1.3.1 The Graphs of the Sine and Cosine Functions

Let's continue to use t as a variable angle. A good way to understand a function is to look at its graph. Let's start with the graph of sin t. Take the horizontal axis to be the t-axis (rather than the x-axis as usual), take the vertical axis to be the y-axis, and graph the equation y = sin t. It looks like this.



First, note that it is periodic of period 2π. Geometrically, that means that if you take the curve and slide it 2π either left or right, then the curve falls back on itself. Second, note that the graph is within one unit of the t-axis. Not much else is obvious, except where it increases and decreases. For instance, sin t grows from 0 to $\pi/2$ since the y-coordinate of the point B increases as the angle AOB increases from 0 to $\pi/2$.

Next, let's look at the graph of cosine. Again, take the horizontal axis to be the t-axis, but now take the vertical axis to be the x-axis, and graph the equation x = cos t.

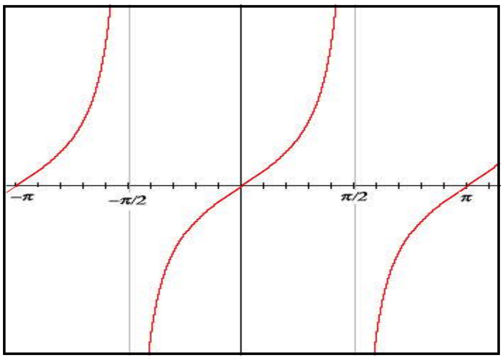


Notes

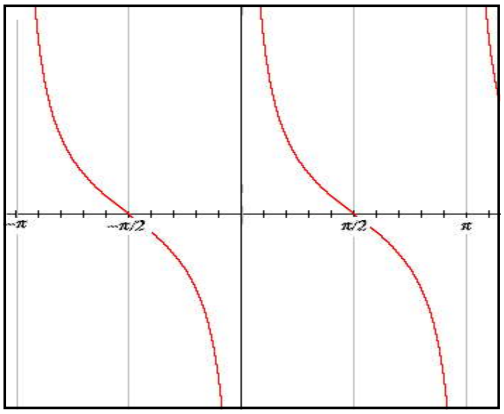
Note that it looks just like the graph of sin t except it's translated to the left by $\pi/2$. That's because of the identity cos t = sin ($\pi/2 + t$). Although we haven't come across this identity before, it easily follows from ones that we have seen: cos t = cos $-t$ = sin ($\pi/2 - (-t)$) = sin ($\pi/2 + t$).

1.3.2 The Graphs of the Tangent and Cotangent Functions

The graph of the tangent function has a vertical asymptote at x =$\pi/2$. This is because the tangent approaches infinity as t approaches $\pi/2$. (Actually, it approaches minus infinity as t approaches $\pi/2$ from the right as you can see on the graph.



You can also see that tangent has period π; there are also vertical asymptotes every π unit to the left and right. Algebraically, this periodicity is uttered by tan ($t + \pi$) = tan t.

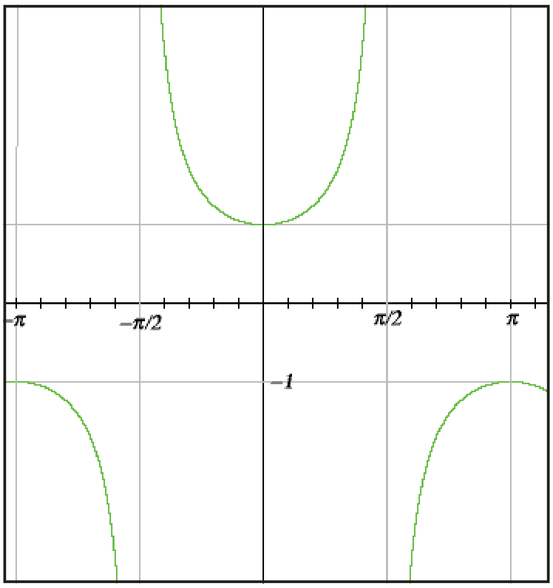


This similarity is simply because the cotangent of t is the tangent of the complementary angle $\pi - t$.

1.3.3 The Graphs of the Secant and Cosecant Functions

The secant is the reciprocal of the cosine, and as the cosine only takes values between –1 and 1, therefore the secant only takes values above 1 or below –1, as shown in the graph. Also secant has a period of 2π.

Notes



As you would expect by now, the graph of the cosecant looks much like the graph of the secant.

1.3.4 Domain and Range of Trigonometric Functions

From the definition of sine and cosine functions, we observe that they are defined for all real numbers. Further, we observe that for each real number x, $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$

Thus, domain of $y = \sin x$ and $y = \cos x$ is the set of all real numbers and range is the interval [–1, 1], i.e., $-1 \leq y \leq 1$. Range and domains of trigonometric function are given in a table shown below:

	I quadrant	**II quadrant**	**III quadrant**	**IV quadrant**
Sine	increases from 0 to 1	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0
Cosine	decreases from 1 to 0	decreases from 0 to – 1	increases from –1 to 0	increases from 0 to 1
Tan	increases from 0 to ∞	increases from –∞ to 0	increases from 0 to ∞	increases from –∞ to 0
Cot	decreases from ∞ to 0	decreases from 0 to –∞	decreases from ∞ to 0	decreases from 0 to –∞
Sec	increases from 1 to ∞	increases from –∞ to –1	decreases from –1 to –∞	decreases from ∞ to 1
Cosec	decreases from ∞ to 1	increases from 1 to ∞	increases from –∞ to –1	decreases from –1 to –∞

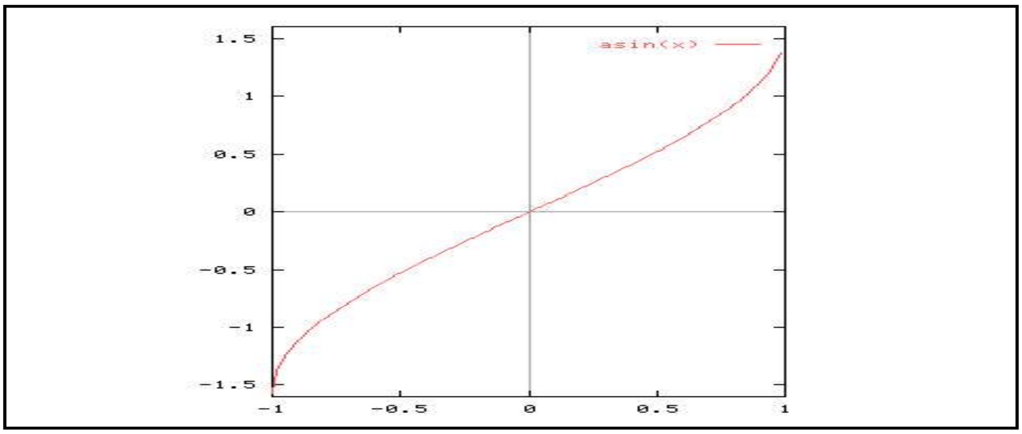
Task Find the cot and cosec values of triangle where, equals to 30, 60 and 45 degree.

Notes

1.4 Inverse Trigonometric Functions

1.4.1 The Arcsine Function

Till now there was the restriction on the domain of the sine function to [-π/2, π/2]. Now this restriction is invertible because each image value in [–1, 1] corresponds to exactly one original value in [–π/2, π/2]. The inverse function of that restricted sine function is called the arcsine function. We write arcsin(x) or asin(x). The graph y = arcsin(x) is the mirror image of the restricted sine graph with respect to the line y = x. The domain is [–1, 1] and the range is [-π/2 , π/2].

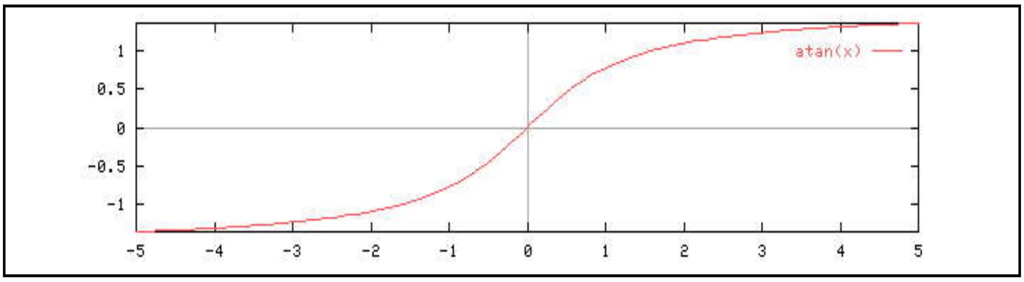


1.4.2 The Arccos Function

Like sine there was restriction on the domain of the cosine function to [0 ,π]. But now this restriction is invertible because each image value in [–1, 1] corresponds to exactly one original value in [0,π]. The inverse function of that restricted cosine function is called the arccosine function. We write arccos(x) or acos(x). The graph y = arccos(x) is the mirror image of the restricted cosine graph with respect to the line y = x. The domain is [–1, 1] and the range is [0,π].

1.4.3 The Arctan Function

We restrict the domain of the tangent function to [–π /2, π /2]. The inverse function of that restricted tangent function is called the arctangent function. We write arctan(x) or atan(x). The graph y = arctan(x) is the mirror image of the restricted tangent graph with respect to the line y = x. The domain is R and the range is [–π /2, π /2].



1.4.4 The Arccot Function

We restrict the domain of the cotangent function to [0, π]. The inverse function of that restricted cotangent function is called the arccotangent function. We write arccot(x) or acot(x). The graph y

= arccot(x) is the mirror image of the restricted cotangent graph with respect to the line y = x. The domain is R and the range is [0, π].

Transformations

As with the trigonometric functions, the related functions can be created using simple transformations.

y = 2.arcsin(x – 1) comes about by moving the graph of arcsin(x) one unit to the right, and then by multiplying all the images by two. The domain is [0, 2] and the range is [–π, π].

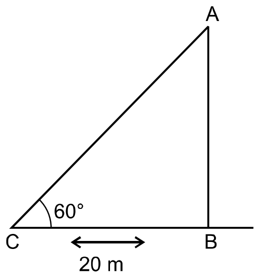
Example 1: A stairs stands vertically on the ground. From a point on the ground, which is 20 m away from1 the foot of the tower, the angle of elevation of the top of the stairs is found to be 60°. Find the height of the stairs.

Solution: First let us draw a simple diagram to represent the problem. Here AB represents the stairs, CB is the distance of the point from the stairs and ∠ ACB is the angle of elevation. Now we have to find the height 9 stairs that is AB. Also, ACB is a triangle, right-angled at B.

Now, $\tan 60° = \dfrac{AB}{BC}$

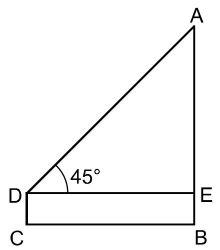
i.e., $\sqrt{3} = \dfrac{AB}{15}$

i.e., $AB = 15\sqrt{3}$



Hence the height of stair is $15\sqrt{3}$ m.

Example 2: A scientist 1.5 m tall is 28.5 m away from a satellite. The angle of elevation of the top of the satellite from satellite eyes is 45°. What is the height of the scientist?



Solution: Here, AB is the satellite, CD the observer and ∠ ADE the angle of elevation. In this case, ADE is a triangle, right-angled at E and we are required to find the height of the chimney.

We have $AB = AE + BE = AE + 1.5$

and $DE = CB = 28.5$ m

Notes

To solve this, we choose a trigonometric ratio, which involves both AE and DE. Let us choose the tangent of the angle of elevation.

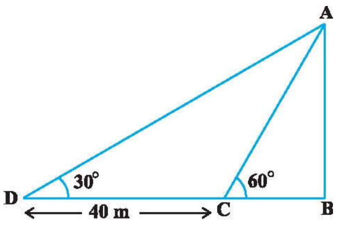
Now, $\tan 45° = \dfrac{AE}{DE}$

i.e., $1 = \dfrac{AE}{28.5}$

Therefore, $AE = 10\sqrt{3}$

So the height of the satellite (AB) = (28.5 + 1.5) m = 30 m.

Example 3: The shadow of a building standing on a level ground is found to be 40 m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the building.



Solution: Let AB is the building and BC is the length of the shadow when the Sun's altitude is 60°, i.e., the angle of elevation of the top of the building from the tip of the shadow is 60° and DB is the length of the shadow, when the angle of elevation is 30°.

Now, let AB be h m and BC be x m. According to the question, DB is 40 m longer than BC.

So, $DB = (40 + x)$ m

Now, we have two right triangles ABC and ABD.

In Δ ABC, $\tan 60° = \dfrac{AB}{BC}$

or, $\sqrt{3} = \dfrac{h}{x}$ (1)

In Δ ABC, $\tan 30° = \dfrac{AB}{BD}$

i.e., $\dfrac{1}{\sqrt{3}} = \dfrac{h}{x = 40}$ (2)

From (1), we have $h = x\sqrt{3}$

Putting this value in (2), we get $(x\sqrt{3})\sqrt{3} = x + 40$, i.e., $3x = x + 40$

i.e., $x = 20$

Som $h = 20\sqrt{3}$ [From (1)]

Therefore, the height of the building is $20\sqrt{3}$ m.

Notes

1.5 Summary

Inverse of a trigonometric function exists if we restrict the domain of it.

(i) $\sin^{-1} x = y$ if $\sin y = x$ where $-1 \le x \le 1,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii) $\cos^{-1} x = y$ if $\cos y = x$ where $-1 \le x \le 1,\ 0 \le y \le \pi$

(iii) $\tan^{-1} x = y$ if $\tan y = x$ where $x \in R,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$

(iv) $\cot^{-1} x = y$ if $\cot y = x$ where $x \in R,\ 0 < y < \pi$

(v) $\sec^{-1} x = y$ if $\sec y = x$ where $x \ge 1,\ 0 \le y < \frac{\pi}{2}$ or $x \le -1, \frac{\pi}{2} < y \le \pi$

(vi) $\text{cosec}^{-1} x = y$ if $\text{cosec}\, y = x$ where $x \ge 1, 0 < y \le \frac{\pi}{2}$

or $x \le -1, -\frac{\pi}{2} \le y < 0$

Graphs of inverse trigonometric functions can be represented in the given intervals by interchanging the axes as in case of $y = \sin x$, etc.

1.6 Self Assessment

Multiple Choice Questions

1. The Principal value of

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{5}$

2. $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$ equals to

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{2\pi}{2}$ (d) $\frac{2\pi}{4}$

3. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(a) π (b) $\frac{-\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

4. If $\sin^{-1} x = y$ then

(a) $o \le y \le p$ (b) $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ (c) $o < y < p$ (d) $\frac{\pi}{2} < y < \frac{\pi}{2}$

5. $\cos^{-1}\left(\cos^{-1}\frac{7\pi}{6}\right)$ is equal to

(a) $\frac{7\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

6. $\tan^{-1}\sqrt{3} - \cot^{-1}\left(-\sqrt{3}\right)$ is equal to

(a) π (b) $\frac{-\pi}{2}$ (c) o (d) $2\sqrt{3}$

Fill in the blanks:

7. Inverse trigonometric function is also called as ....

8. The value of an inverse trigonometric functions which lies in its principal value branch is called as .... of that inverse trigonometric functions.

Notes

9. For suitable value of domain is equal to ………………
10. For suitable value of domain cot-1 (-x) is equal to ....

1.7 Review Questions

1. Prove each of the following:

(a) $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{8}{17}\right)=\sin^{-1}\left(\frac{77}{85}\right)$

(b) $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{1}{9}\right)=\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

(c) $\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{27}{11}\right)$

2. Prove each of the following:

(a) $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{23}{11}\right)$

(b) $\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}$

(c) $\tan^{-1}\left(\frac{1}{8}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{1}{3}\right)$

3. (a) Prove that 2 $\sin^{-1}$ x = $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

(b) Prove that 2 $\cos^{-1}$ x = $\cos^{-1}\left(2x^2-1\right)$

(c) Prove that $\cos^{-1}$ x = $2\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)=2\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

4. Prove the following:

(a) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}$

(b) $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=\frac{\pi}{4}-x$

(c) $\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)=0$

5. Solve each of the following:

(a) $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

(b) $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\mathrm{cosec}\,x)$

(c) $\cos^{-1} x + \sin^{-1}\left(\frac{1}{2}x\right) = \frac{\pi}{6}$

(d) $\cot^{-1} x - \cot^{-1}(x+2) = \frac{\pi}{12}, x > 0$

7. Solve that

$$\tan^{-1}(1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right)$$

8. Find out solution of given equation

$$\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

9. Show that

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x, \frac{1}{\sqrt{2}} \le x \le 1$$

10. Verify that given equation is relevant or not ?

$$\sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1} x$$

Answers: Self Assessment

1.	(c)	2.	(a)
3.	(b)	4.	(b)
5.	(b)	6.	(b)
7.	Arc functions	8.	Principal value
9.	$-\mathrm{cosec}^{-1} x$	10.	$\pi - \cot^{-1} x$

1.8 Further Readings

Books

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

Ncert Mathematics books class XI

Ncert Mathematics books class XII

Smith and Minton. *Calculus Early Trancendental,* Third Edition. McGraw Hill.2008

Online links

http://www.suitcaseofdreams.net/Trigonometric_Functions.htm

http://library.thinkquest.org/20991/alg2/trigi.html

http://www.intmath.com/trigonometric-functions/5-signs-of-trigonometric-functions.php

Notes

Unit 2: Trigonometric Functions-II

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss computing trignometeric functions
- Explain transformations of products into sums and inverse
- Discuss trignometric functions of multiples examples

Introduction

In last unit you have studied about the trigonometric functions. The inverse trigonometric functions play an important role in calculus for they serve to define many integrals. The concepts of inverse trigonometric functions is also used in science and engineering. The inverse trigonometric functions are the inverse functions of the trigonometric functions, written $\cos^{-1}$, $\cot^{-1}$, $\csc^{-1}$, $\sec^{-1}$, $\sin^{-1}$ and $\tan^{-1}$. The inverse trig functions are similar to any other inverse functions. In this unit we will study about computing different tignomatric functions and transformation of products into sums and inverse.

2.1 Computing Trigonometric Functions

Ptolemy (100–178) produced one of the earliest tables for trigonometry in his work, the Almagest, and he incorporated the mathematics needed to develop that table. It was a table of chords for every arc from 1/2° through 180° in intervals of 1/2°. Also he explained how to exclaim between the given angles. let's look at how to create tables for sines and cosines using his methods. First, based on the Pythagorean theorem and similar triangles, the sines and cosines of certain angles can be computed directly. In particular, you can directly find the sines and cosines for the angles 30°, 45°, and 60° as described in the section on cosines. Ptolemy knew two other angles that could be constructed, namely 36° and 72°. These angles were constructed by Euclid in Proposition IV.10 of his Elements. Like Ptolemy, we can use that construction to compute the trig functions for those angles. At this point we could compute the trig functions for the angles 30, 36, 45, 60, and 72 degrees, and, of course we know the values for 0 and 90 degrees, too.

Keeping in mind the sine of an angle, the cosine of the complementary angle

$$\cos t = \sin (90° - t) \qquad \sin t = \cos (90° - t)$$

So you have the trig functions for 18 and 54 degrees, too.

Use of the half–angle formulas for sines and cosines to compute the values for half of an angle if you know the values for the angle. If it is an angle between 0° and 90°, then

$$\sin t/2 = ((1 - \cos t) / 2) \qquad \cos t/2 = ((1 + \cos t) / 2)$$

Using these, from the values for 18, 30, and 54 degrees, you can find the values for 27, 15, and 9 degrees, and, therefore, their complements 63, 75, and 81 degrees.

With the help of the sum and difference formulas,

$$\sin (s + t) = \sin s \cos t + \cos s \sin t$$

$$\sin (s - t) = \sin s \cos t - \cos s \sin t$$

$$\cos (s - t) = \cos s \cos t + \sin s \sin t$$

$$\cos (s + t) = \cos s \cos t - \sin s \sin t$$

you can find the sine and cosine for 3° (from 30° and 27°) and then fill in the tables for sine and cosine for angles from 0° though 90° in increments of 3°.

Again, using half–angle formulas, you could produce a table with increments of 1.5° (that is, 1° 30′), then 0.75° (which is 45′), or even of 0.375° (which is 22′ 30″).

2.1.1 Addition and Subtraction of Trigonometric Functions

Earlier we have learnt about circular measure of angles, trigonometric functions, values of trigonometric functions of specific numbers and of allied numbers.

You may now be interested to know whether with the given values of trigonometric functions of any two numbers A and B, it is possible to find trigonometric functions of sums or differences.

You will see how trigonometric functions of sum or difference of numbers are connected with those of individual numbers. This will help you, for instance, to find the value of trigonometric functions of $\pi/12$ can be expressed as $\pi/4 - \pi/6$

$5\pi/12$ can be expressed as $\pi/4 + \pi/6$

How can we express $7\pi/12$ in the form of addition or subtraction?

In this section we propose to study such type of trigonometric functions.

Notes

Addition formulae

For any two numbers A and B,

cos (A + B) = cos A cosB – sinA sinB

In given figure trace out

$\angle$SOP = A

$\angle$POQ = B

$\angle$SOR = –B

Where points P, Q, R, S lie on the unit circle.

Coordinates of P, Q, R, S will be (cos A, sin A),

[cos (A + B), sin (A + B)],

[cos (–B), sin (–B)], and (1, 0).

From the given figure, we have

side OP = side OQ

$\angle$POR = $\angle$QOS (each angle = $\angle$B +$\angle$ QOR)

side OR = side OS

ΔPOR $\cong$ ΔQOS (by SAS)

$\therefore$ PR = QS

PR = $\sqrt{}$(cos A – cos B)^2 + (sin A – sin(–B)^2

QS = $\sqrt{}$(cos A + B – 1)^2 + (sin A + B – 0)^2

Since PR^2 = QS^2

$\therefore$ cos^2 A + cos^2 B – 2cos A cos B + sin^2 A + sin^2 B + 2sin A sin B

= cos^2(A + B) + 1 – 2cos(A + B) + sin^2(A + B)

$\Rightarrow$ 1 + 1 – 2(cos A cos B – sin A sin B) = 1 + 1 – 2cos(A + B)

$\Rightarrow$ cosA cosB – sinA sinB = cos (A + B) (I)

For any two numbers A and B, cos (A– B) = cos A cos B + sin A sin B

Proof: Replace B by – B in (I)

cos(A – B) = cos A cos B + sin A sin B

$\therefore$ cos (–B) = cos B and sin(–B) = –sin B

For any two numbers A and B

sin(A + B) = sin A cos B + cos A sin B

Proof: We know that cos (π/2 – A) = sin A

sin (π/2 – A) = cos A

sin(A + B) = cos[π/2 – (A + B)

= cos[(π/2 – A) + B]

= cos(π/2 – A) cos B + Sin (π/2 – A) or $\quad \sin(A + B) = \sin A \cos B + \cos A \sin B$ .....(II)

For any two numbers A and B

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Proof: Replacing B by – B in (2), we have

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

or $\quad \sin(A - B) = \sin A \cos B - \cos A \sin B$

Example: Find the value of each of the following:

(i) $\sin 5\pi/12$ (ii) $\cos\pi/12$ (iii) $\cos 7\pi/12$

Solution

(a) (i) $\sin 5\pi/12 = \sin(\pi/4 + \pi/6) = \sin \pi/4.\cos \pi/6 + \cos\pi/4.\sin\pi/6$

$= 1/\sqrt{2}.\sqrt{3}/2 + 1/\sqrt{2}.1/2$

$\therefore \quad \sin 5\pi/12 = \sqrt{3} + 1/\sqrt{2}.\ 1/2 = \sqrt{3} + 1/2\sqrt{2}$

(ii) $\cos \pi/12 = \cos(\pi/4 - \pi/6)$

$= \cos\pi/4 . \cos\pi/6 + \sin\pi/4 + \sin\pi/6$

$= 1/\sqrt{2}.\sqrt{3}/2 + 1/\sqrt{2}.1/2 = \sqrt{3} + 1/2\sqrt{2}$

$\therefore \quad \cos \pi/12 = \sqrt{3} + 1/2/\sqrt{2}$

Observe that $\sin 5\pi/12 = \cos\pi/12$

(iii) $\cos 7\pi/12 = \cos(\pi/3 + \pi/4)$

$= \cos\pi/3 . \cos\pi/4 - \sin\pi/3 . \sin\pi/4$

$= 1/2.\ 1/\sqrt{2} - \sqrt{3}/2.\ 1/\sqrt{2} = 1 - \sqrt{3}\ /2\sqrt{2}$

$\therefore \quad \cos 7\pi/12 = 1 - \sqrt{3}/2\sqrt{2}$

2.1.2 Transformation of Products into Sums and Inverse

Transformation of Products into Sums or Differences

We know that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

By adding and subtracting the first two formulae, we get respectively

$$2\sin A \cos B = \sin(A + B) + \sin(A - B) \quad \ldots(1)$$

and

$$2\cos A \sin B = \sin(A + B) - \sin(A - B) \quad \ldots(2)$$

Notes

Similarly, by adding and subtracting the other two formulae, we get

$$2\cos A \cos B = \cos(A + B) + \cos(A - B) \qquad \ldots(3)$$

and

$$2\sin A \sin B = \cos(A - B) - \cos(A + B) \qquad \ldots(4)$$

We can also reference these as

$$2\sin A \cos B = \sin(\text{sum}) + \sin(\text{difference})$$

$$2\cos A \sin B = \sin(\text{sum}) - \sin(\text{difference})$$

$$2\cos A \cos B = \cos(\text{sum}) + \cos(\text{difference})$$

$$2\sin A \sin B = \cos(\text{difference}) - \cos(\text{sum})$$

Transformation of sums or differences into products

In the above results put

$$A + B = C$$

$$A - B = D$$

Then A= C + D/2 and B = C – D/2 and (1), (2), (3) and (4) becomes

$$\sin C + \sin D = 2\sin C + D/2 \cos C - D/2$$

$$\sin C - \sin D = 2\cos C + D/2 \sin C - D/2$$

$$\cos C + \cos D = 2\cos C + D/2 \cos C - D/2$$

$$\cos C - \cos D = 2\sin C + D/2 \sin C - D/2$$

Further applications of addition and subtraction formulae

We shall prove that

(i) $\sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B$ or $\cos^2 B - \sin^2 A$

Proof: (i) $\sin(A + B) \sin(A - B)$

$$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$$

$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$

$$= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$$

$$= \sin^2 A - \sin^2 B$$

(ii) $\cos(A + B) \cos(A - B)$

$$= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$$

$$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$$

$$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B$$

$$= \cos^2 A - \sin^2 B$$

$$= (1 - \sin^2 A) - (1 - \cos^2 B)$$

$$= \cos^2 B - \sin^2 A$$

Example: Express the following products as a sum or difference

(i) $2\sin 3\theta \cos 2\theta$

(ii) $\cos 6\theta \cos \theta$

Solution:

(i) $(2\sin 3\theta \cos 2\theta = \sin(3\theta + 2\theta) + \sin(3\theta - 2\theta)$

$= \sin 5\theta + \sin\theta$

(ii) $\cos 6\theta \cos\theta = 1/2\,(2\cos 6\theta \cos\theta)$

$= 1/2\,[\cos(6\theta + \theta) + \cos(6\theta - \theta)]$

$= 1/2(\cos 7\theta + \cos 5\theta)$

2.1.3 Trigonometric Functions of Multiples of Angles

(a) To express sin 2A in terms of sin A, cos A and tan A.

We know that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

By putting B = A, we get

$$\sin 2A = \sin A \cos A + \cos A \sin A$$

$$= 2\sin A \cos A$$

$\therefore$ sin 2A can also be written as

$$\sin 2A = \frac{2\sin A \cos A}{\cos^2 A + \sin^2 A} \qquad (\because\ 1 = \cos^2 A + \sin^2 A)$$

Dividing numerator and denominator by $\cos^2 A$, we get

$$\sin 2A = \frac{2\left(\dfrac{\sin A \cos A}{\cos^2 A}\right)}{\dfrac{\cos^2 A}{\cos^2 A} + \dfrac{\sin^2 A}{\cos^2 A}} = \frac{2\tan A}{1 + \tan^2 A}$$

(b) To express cos 2A in terms of sin A, cos A and tan A.

We know that

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Putting B = A, we have

$$\cos 2A = \cos A \cos A - \sin A \sin A$$

or $\cos 2A = \cos^2 A - \sin^2 A$

Also $\cos 2A = \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

i.e, $\cos 2A = 2\cos^2 A - 1 \quad \Rightarrow \quad \cos^2 A = \dfrac{1 + \cos 2A}{2}$

Notes

Also cos 2A = $\cos^2$ A – $\sin^2$ A

= 1 – $\sin^2$ A – $\sin^2$ A

i.e., cos 2A = 1 – 2$\sin^2$ A $\Rightarrow$ $\sin^2 A = \frac{1-\cos 2A}{2}$

$$\therefore \quad \cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

Dividing the numerator and denominator of R.H.S. by $\cos^2$ A, we have

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

(c) To express tan 2A in terms of tan A.

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2\tan A}{1-\tan^2 A}$$

Thus we have derived the following formulae :

$$\sin 2A = 2\sin A \cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A \ \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\cos^2 A = \frac{1+\cos 2A}{2}, \ \sin^2 A = \frac{1-\cos 2A}{2}$$

Example: If $A = \frac{\pi}{6}$, verify the following:

(i) 2tan A = 2sin 2A 2sin A cos A = $\frac{2\tan A}{1+\tan^2 A}$

(ii) cos 2A cos A sin 2A 2cos A 1 1 2sin A = $\frac{1-\tan^2 A}{1+\tan^2 A}$

Solution

(i)

$$\sin 2A = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$2\sin A\cos A = 2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\times\frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\frac{2\tan A}{1+\tan^2 A} = \frac{2\tan\frac{\pi}{6}}{1+\tan^2\frac{\pi}{6}} = \frac{\left(2\times\frac{1}{\sqrt{3}}\right)}{1+\frac{1}{3}} = \frac{2}{\sqrt{3}}\times\frac{3}{4} = \frac{\sqrt{3}}{2}$$

Thus, it is verified that **Notes**

$$\sin 2A = 2\sin A \cos A = \frac{2\tan A}{1+\tan^2 A}$$

(ii) $$\cos 2A = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\cos^2 A - \sin^2 A = \cos 2\frac{\pi}{6} - \sin 2\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$$2\cos^2 A - 1 = 2\cos^2\frac{\pi}{6} - 1 = 2\times\left(\frac{\sqrt{3}}{2}\right)^2 - 1$$

$$= 2\times\frac{3}{4} - 1 = \frac{1}{2}$$

$$1 - 2\sin^2 A = 1 - 2\sin\frac{\pi}{6} = 1 - 2\times\left(\frac{1}{2}\right)^2 = 1 - 2\times\frac{1}{4} = \frac{1}{2}$$

$$\frac{1-\tan^2 A}{1+\tan^2 A} = \frac{1-\tan^2\frac{\pi}{6}}{1+\tan^2\frac{\pi}{6}} = \frac{1-\left(\frac{1}{\sqrt{3}}\right)^2}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{2}{3}\times\frac{3}{4} = \frac{1}{2}$$

Thus, it is verified that

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

2.1.4 Trigonometric Function of 3A in terms of A

(a) sin 3A in terms of sin A

Substituting 2A for B in the formula

$$\sin (A + B) = \sin A \cos B + \cos A \sin B, \text{ we get}$$

$$\sin(A + 2A) = \sin A \cos 2A + \cos A \sin 2A$$

$$= \sin A(1 - 2\sin^2 A) + (\cos A \times 2\sin A \cos A)$$

$$= \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A)$$

$$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$$

$$\therefore \quad \sin 3A = 3\sin A - 4\sin^3 A \qquad ....(1)$$

(b) cos 3A in terms of cos A

Substituting 2A for B in the formula

$$\cos(A + B) = \cos A \cos B - \sin A \sin B, \text{ we get}$$

$$\cos(A + 2A) = \cos A \cos 2A - \sin A \sin 2A$$

$$= \cos A(2\cos^2 A - 1) - (\sin A) \times 2\sin A \cos A$$

Notes

$$= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A)$$

$$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$$

$$\therefore \quad \cos^3 A = 4\cos^3 A - 3\cos A \qquad ....(2)$$

(c) tan 3A in terms of tan A

Putting B = 2A in the formula

$$\tan(A + B) = \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \text{ we get}$$

$$\tan(A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1 - \tan^2 A}}{1 - \tan A \times \dfrac{2\tan A}{1 - \tan^2 A}}$$

$$= \frac{\dfrac{\tan A - \tan^3 A + 2\tan A}{1 - \tan^2 A}}{\dfrac{1 - \tan^2 A - 2\tan^2 A}{1 - \tan^2 A}}$$

$$\therefore \quad = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \qquad ...(3)$$

(d) Formulae for $\sin^3 A$ and $\cos^3 A$

Q $\quad \sin 3A = 3\sin A - 4\sin^3 A$

$\therefore \quad 4\sin^3 A = 3\sin A - \sin 3A$

or $\quad \sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

Thus, we have derived the following formulae:

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

$$\cos^3 A = \frac{3\cos A + \cos 3A}{4}$$

Example 4: If $A = \dfrac{\pi}{4}$, verify that

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Notes

Solution

(i) $$\sin 3A = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

$$3\sin A - 4\sin^3 A = 3\sin\frac{\pi}{4} - 4\sin^3\frac{\pi}{4}$$

$$= 3\times\frac{1}{\sqrt{2}} - 4\times\left(\frac{1}{\sqrt{2}}\right)^3$$

$$= \frac{3}{\sqrt{2}} - \frac{4}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Thus, it is verified that $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $$\cos 3A = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$4\cos^3 A - 3\cos A = 4\times\left(\frac{1}{\sqrt{2}}\right)^3 - 3\times\frac{1}{\sqrt{2}}$$

$$= \frac{4}{2\sqrt{2}} - \frac{3}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

Thus, it is verified that $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) $$\tan 3A = \tan\frac{3\pi}{4} = -1,$$

$$\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \frac{3\times 1 - 1^3}{1 - 3\times 1^2} = \frac{2}{-2} = -1$$

Thus, it is verified that $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Task

1. If $A = \frac{\pi}{3}$, verify that

   (a) $\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$ (b) $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$

   (c) $\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$

2. Find the values of $\sin\frac{\pi}{12}$ and $\sin\frac{\pi}{24}$.

Contd...

Notes

3. Determine the values of

(a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$

(c) $\tan\frac{\pi}{8}$

2.2 Trigonometric Functions and Submultplication of Angles

$\frac{A}{2},\frac{A}{3},\frac{A}{4}$ are called submultiples of A.

It has been proved that

$$\sin^2 A = \frac{1-\cos 2A}{2}, \cos 2A = \frac{1+\cos 2A}{2}, \tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$$

Replacing A by $\frac{A}{2}$, we easily get the following formulae for the sub–multiple $\frac{A}{2}$:

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}, \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \text{ and } \tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

We will choose either the positive or the negative sign depending on whether corresponding value of the function is positive or negative for the value of $\frac{A}{2}$. This will be clear from the following examples:

Examples 5: Find the values of $\cos\frac{\pi}{12}$ and $\cos\frac{\pi}{24}$.

Solution: We use the formulae $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$ and take the positive sign, because $\cos\frac{\pi}{12}$ and $\cos\frac{\pi}{24}$ are both positive.

$$\cos\frac{\pi}{12} = \pm\sqrt{\frac{1+\cos\frac{\pi}{6}}{2}}$$

$$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{2\times 2}}$$

Notes

$$= \sqrt{\frac{4+2\sqrt{3}}{8}}$$

$$= \sqrt{\frac{\left(\sqrt{3}+1\right)^2}{8}}$$

$$\left(\because 4+2\sqrt{3} = 1+3+2\sqrt{3} = \left(1+\sqrt{3}\right)^2\right)$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos\frac{\pi}{24} = \sqrt{\frac{1+\cos\frac{\pi}{12}}{2}}$$

$$= \sqrt{\frac{\left(1+\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{2}}$$

$$= \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}}$$

$$= \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}}$$

2.3 Some Important Trigonometric Equations

You are well–known with the equations like simple linear equations, quadratic equations in algebra.

You must have also learnt how to solve this type of equations.

Thus,

(i) $x - 3 = 0$ gives one value of x as a solution.

(ii) $x^2 - 9 = 0$ gives two values of x.

You must have noticed, the number of values depends upon the degree of the equation.

Now we require thinking as to what will happen in case x's and y's are replaced by trigonometric functions.

Thus solution of the equation $\sin\theta - 1 = 0$, will give

$\sin\theta = 1$ and $\pi/2$, $\pi/5$, $9\pi/2$, …………

Obviously, the solution of simple equations with only finite number of values does connot essentially hold good in case of trigonometric equations.

So, we will try to find the ways of finding solutions of such equations.

Notes

To find the general solution of the equation sin θ = 0

It is given that $\sin\theta = 0$

But we know that $\sin 0, \sin\pi, \sin 2\pi, ..., \sin n\pi$ are equal to 0

$\therefore \quad \theta = n\pi, n \in N$

But we know that $\sin(-\theta) = -\sin\theta = 0$

$\therefore \quad \sin(-\pi), \sin(-2\pi), \sin(-3\pi), ..., \sin(-n\pi) = 0$

$\therefore \quad \theta = n\pi, n \in I.$

Thus, the general solution of equations of the type $\sin\theta = 0$ is given by $\theta = n\pi$ where n is an integer.

To find the general solution of the equation cos θ = 0

It is given that $\cos\theta = 0$

But in practice we know that $\cos \pi/2 = 0$. Therefore, the first value of θ is

$$\theta = \pi/2 \qquad ...(1)$$

We know that $\cos(\pi + \theta) = -\cos\theta$ or $\cos(\pi + \pi/2) = -\cos\pi/2 = 0$.

or $\cos 3\pi/2 = 0$

In the same way, it can be found that 9

$\cos 5\pi/2, \cos 7\pi/2, \cos 9\pi/2, ..., \pi \cos(2n + 1)\pi/2$ are all zero

$\therefore \quad (2n + 1)\pi/2, n \in N$

But we know that $\cos(-\theta) = \cos\theta$

$\therefore \quad \cos(-\pi/2) = \cos(-3\pi/2) = \cos(-5\pi/2) = \cos\{-(2n-1)\pi/2\} = 0$

$\theta = (2n + 1)\pi/2, n \in I$

Therefore, $\theta = (2n + 1)\pi/2$ is the solution of equations $\cos\theta = 0$ for all numbers whose cosine is 0.

To find a general solution of the equation tan θ = 0

It is given that $\tan\theta = 0$

or $\sin\theta/\cos\theta = 0$ or $\sin\theta = 0$

i.e. $\theta = n\pi, n \in I.$

We have consider above the general solution of trigonometric equations, where the right hand is zero. In the following equation, we take up some cases where right hand side is non–zero.

To find the general solution of the equation sinθ = sin α

It is given that $\sin\theta = \sin\alpha$

$\Rightarrow \quad \sin\theta - \sin\alpha = 0$

or $\quad 2\cos(\theta + \alpha/2)\sin(\theta - \alpha/2) = 0$

$\therefore \quad$ Either $\cos(\theta + \alpha/2) = 0$ or $\sin(\theta - \alpha/2) = 0$

Notes

$\Rightarrow \quad \frac{\theta+\alpha}{2} = (2p+1)\frac{\pi}{2} \quad$ or $\quad \frac{\theta-\alpha}{2} = q\pi, p, q \in I$

$\Rightarrow \quad \theta = (2p + 1)\pi - \alpha \quad$ or $\quad \theta = 2\pi + \alpha \qquad \ldots(1)$

From (1), we get

$\theta = n\pi + (-1)^n \alpha, n \in I$ as the general solution of the equation $\sin \theta = \sin \alpha$

To find the general solution of the equation cos θ = cos α

It is given that, $\cos \theta = \cos \alpha$

$\Rightarrow \quad \cos \theta - \cos \alpha = 0$

$\Rightarrow \quad -2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$

$\therefore$ Either, $\sin\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow \quad \frac{\theta+\alpha}{2} = p\pi \quad$ or $\quad \frac{\theta-\alpha}{2} = q\pi, p, q \in I$

$\Rightarrow \quad \theta = 2p\pi - \alpha \quad$ or $\quad \theta = 2p\pi + \alpha \qquad ....(1)$

From (1), we have

$\theta = 2n\pi \pm \alpha, n \in I$ as the general solution of the equation $\cos \theta = \cos \alpha$

To find the general solution of the equation tan θ= tan α

It is given that, $\tan \theta = \tan \alpha$

$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha} = 0$

$\Rightarrow \quad \sin \theta \cos \alpha - \sin \alpha \cos \theta = 0$

$\Rightarrow \quad \sin(\theta - \alpha) = 0$

$\Rightarrow \quad \theta - \alpha = n\pi, n \in I$

$\Rightarrow \quad \theta = n\pi + \alpha \ n \in I$

Similarly, for $\operatorname{cosec} \theta = \operatorname{cosec} \alpha$, the general solution is

$\theta = n\pi + (-1)^n \alpha$

and, for $\sec \theta = \sec \alpha$, the general solution is

$\theta = 2n\pi \pm \alpha$

and for $\cot \theta = \cot \alpha$

$\theta = n\pi + \alpha$ is its general solution

Notes

If $\sin^2\theta = \sin^2\alpha$, then

$$\frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2}$$

$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha$, n I

$\Rightarrow \quad \theta = n\pi \pm \alpha$

Similarly, if $\cos^2\theta = \cos^2\alpha$, then

$$\alpha = n\pi \pm \alpha, n \in I$$

Again, if $\tan^2\theta = \tan^2\alpha$, then

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$

$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha$

$\Rightarrow \quad \theta = n\pi \pm \alpha$, $n \in I$ is the general solution.

Example 6: Find the general solution of the following equation:

(a) $\cot\theta = -\sqrt{3}$

(b) $4\sin^2\theta = 1$

Solution

(a) $\sin\theta = -\sqrt{3}$

$$\tan\theta = -\frac{1}{\sqrt{3}} = -\tan\frac{\pi}{6} = \tan\left(\pi - \frac{\pi}{6}\right) = \tan\frac{5\pi}{6}$$

$\therefore \quad \theta = n\pi + \frac{5\pi}{6}, n \in I$

(b) $4\sin^2 q = 1 \Rightarrow \quad \sin^2\theta + \frac{1}{4} = \left(\frac{1}{2}\right)^2 = \sin^2\frac{\pi}{6}$

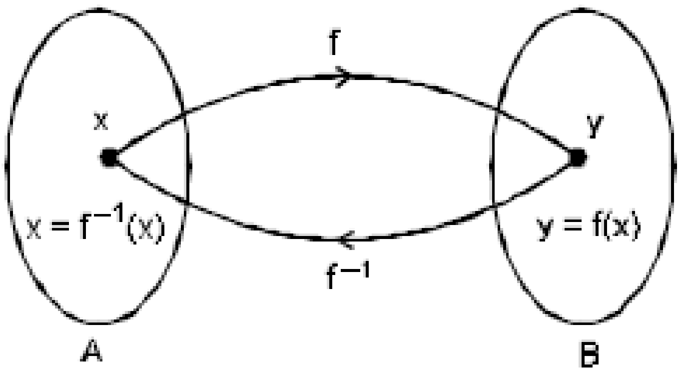
$\Rightarrow \quad \sin\theta = \sin\left(\pm\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{6}, n \in I$

2.4 Inverse of a Trigonometric Function

Notes

In the previous lesson, you have studied the definition of a function and different kinds of functions. We have defined inverse function.



Let us briefly recall:

Let f be a one–one onto function from A to B.

Let y be an arbitary element of B. Then, f being onto, $\exists$ an element $x \in A$ such that $f(x) = y$. Also, f being one–one, then x must be unique. Thus for each $y \in B$, $\exists$ a unique element $x \in A$ such that $f(x) = y$. So we may define a function, denoted by f^{-1} as $f^{-1} : B \to A$

$\therefore$ $$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

The above function f^{-1} is called the inverse of f. A function is invertiable if and only if f is one–one onto.

It this case the domain of f^{-1} is the range of f and the range of f^{-1} is the domain f.

Let us take another example.

We define a function: f: Car $\to$ Registration No.

If we write, g : Registration No. $\to$ Car, we see that the domain of f is range of g and the range of f is domain of g.

So, we say g is an **inverse function** of f, i.e., $g = f^{-1}$.

In this lesson, we will learn more about inverse trigonometric function, its domain and range, and simplify expressions involving inverse trigonometric functions.

2.4.1 Possibility of Inverse of Every Function

Take two ordered pairs of a function (x_1, y) and (x_2, y)

If we invert them, we will get (y, x_1) and (y, x_2)

This is not a function because the first member of the two ordered pairs is the same.

Now let us take another function:

$$\left(\sin\frac{\pi}{2}, 1\right), \left(\sin\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{and} \left(\sin\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

Notes

Writing the inverse, we have

$$\left(1, \sin\frac{\pi}{2}\right), \left(\frac{1}{\sqrt{2}}, \sin\frac{\pi}{4}\right) \text{ and } \left(\frac{\sqrt{3}}{2}, \sin\frac{\pi}{3}\right)$$

which is a function.

Let us consider some examples from daily life.

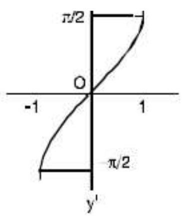
f : Student → Score in Mathematics

Do you think f^{-1} will exist?

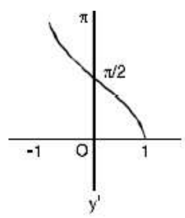
It may or may not be because the moment two students have the same score f^{-1} will cease to be a function. Because the first element in two or more ordered pairs will be the same. So we conclude that

every function is not invertible.

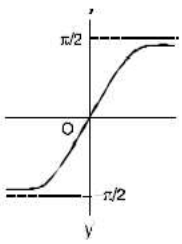
2.4.2 Graphical Representation of Inverse of Trigonometric Function



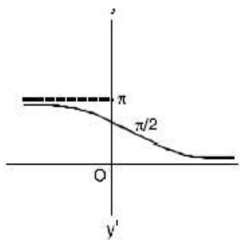
$y = \sin^{-1} x$



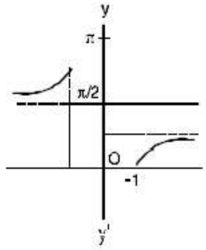
$y = \cos^{-1} x$



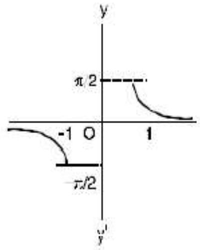
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \text{cosec}^{-1} x$

2.5 Property of Inverse of Trigonometric Function

Property 1: $\sin^{-1}(\sin\theta) = \theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Solution: Let $\sin\theta = x$

$\Rightarrow \quad \theta = \sin^{-1} x$

$= \sin^{-1}(\sin\theta) = \theta$

Also $\quad \sin(\sin^{-1} x) = x$

Similary, we can prove that

(i) $\quad \cos^{-1}(\cos\theta) = \theta,\ 0 \le \theta \le \pi$

(ii) $\quad \tan^{-1}(\tan\theta) = \theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Property 2

(i) $\quad \text{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

(ii) $\quad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\quad \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Solution

(i) Let $\quad \text{cosec}^{-1} x = \theta$

$\Rightarrow \quad x = \text{cosec}\,\theta$

$\Rightarrow \quad \frac{1}{x} = \sin\theta$

$\therefore \quad \theta = \sin^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad \text{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

(ii) Let $\quad \cot^{-1} x = \theta$

$\Rightarrow \quad x = \cot\theta$

$\Rightarrow \quad \frac{1}{x} = \tan\theta$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$

Notes

$\therefore \quad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\quad \sec^{-1} x = \theta$

$\Rightarrow \quad x = \sec\theta$

$\therefore \quad \frac{1}{x} = \cos\theta \quad$ or $\quad \theta = \cos^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

Property 3

(i) $\sin^{-1}(-x) = -\sin^{-1} x$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Solution

(i) Let $\sin^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \sin\theta \quad$ or $\quad x = -\sin\theta = \sin(-\theta)$

$\therefore \quad -q = \sin^{-1} x \quad$ or $\quad q = -\sin^{-1} x$

or $\quad \sin^{-1}(-x) = -\sin^{-1} x$

(ii) Let $\tan^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \tan\theta \quad$ or $\quad x = -\tan\theta = \tan(-\theta)$

$\therefore \quad q = -\tan^{-1} x \quad$ or $\quad \tan^{-1}(-x) = -\tan^{-1} x$

(iii) Let $\cos^{-1}(-x) = \theta$

$\Rightarrow \quad -x = \cos\theta \quad$ or $\quad x = -\cos\theta = \cos(\pi - \theta)$

$\therefore \quad \cos^{-1} x = \pi - \theta$

$\therefore \quad \cos^{-1}(-x) = \pi - \cos^{-1} x$

Property 4

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Solution

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Let $\sin^{-1} x = \theta$ $\Rightarrow$ $x = \sin\theta = \left(\frac{\pi}{2} - \theta\right)$

or $\cos^{-1} x = \left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow$ $\theta + \cos^{-1} x = \frac{\pi}{2}$ or $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(ii) Let $\cot^{-1} x = \theta$ $\Rightarrow$ $x = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

$\therefore$ $\tan^{-1} x = \frac{\pi}{2} - \theta$ or $\theta + \tan^{-1} x = \frac{\pi}{2}$

or $\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}$

(iii) Let $\operatorname{cosec}^{-1} x = \theta$

$\Rightarrow$ ...$x = \operatorname{cosec}\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

$\therefore$ $\sec^{-1} x = \frac{\pi}{2} - \theta$ or $\theta + \sec^{-1} x = \frac{\pi}{2}$

$\Rightarrow$ $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Property 5

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Solution

(i) Let $\tan^{-1} x = \theta$, $\tan^{-1} y = \phi$ $\Rightarrow$ $x = \tan\theta$, $y = \tan\phi$

We have to prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

By substituting that above values on L.H.S. and R.H.S., we have

L.H.S. = $\theta + \phi$ and R.H.S. = $\tan^{-1}\left[\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right]$

Notes

$$= \tan^{-1}\left[\tan(\theta+\phi)\right] = q + f = \text{L.H.S.}$$

$\therefore$ The result holds.

Similarly (ii) can be proved.

Property 6: $2\tan^{-1}x = \sin^{-1}\left[\frac{2x}{1+x^2}\right] = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right] = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$

(i) (ii) (iii) (iv)

Let x = tan θ

Substituting in (i), (ii), (iii), and (iv) we get

$$2\tan^{-1}x = 2\tan^{-1}(\tan\theta) = 2\theta \quad \ldots(i)$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta)$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta \quad \ldots(ii)$$

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)$$

$$= \cos^{-1}(\cos^2\theta - \sin^2\theta)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta \quad \ldots(iii)$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) = 2\theta \quad \ldots(iv)$$

From (i), (ii), (iii) and (iv), we get

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Property 7

(i) $$\sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$= \sec^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right]$$

Notes

$$= \cot^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$$

$$= \text{cosec}^{-1}\left[\frac{1}{x}\right]$$

(ii) $\cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$

$$= \text{cosec}^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right]$$

$$= \cot^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$= \sec^{-1}\left[\frac{1}{x}\right]$$

Proof: Let $\sin^{-1} x = \theta \Rightarrow \sin\theta = x$

(i) $\cos\theta = \sqrt{1-x^2}$, $\tan\theta = \frac{x}{\sqrt{1-x^2}}$, $\sec\theta = \frac{1}{\sqrt{1-x^2}}$, $\cot\theta = \frac{\sqrt{1-x^2}}{x}$

and $\text{cosec}\,\theta = \frac{1}{x}$

$\therefore$ $\sin^{-1} x = \theta = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \text{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) Let cos–1 x = θ $\Rightarrow$ $x = \cos\theta$

$\therefore$ $\sin\theta = \sqrt{1-x^2}$, $\tan\theta = \frac{\sqrt{1-x^2}}{x}$, $\sec\theta = \frac{1}{x}$, $\cot\theta = \frac{x}{\sqrt{1-x^2}}$

Notes

and $\quad \operatorname{cosec} \theta = \dfrac{1}{\sqrt{1-x^2}}$

$$\cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \sec^{-1}\left(\frac{1}{x}\right)$$

Property 8

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

(ii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left[xy\sqrt{1-x^2}\sqrt{1-y^2}\right]$

(iii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(iv) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right]$

Proof

(i) Let $x = \sin \theta$, $y = \sin \phi$, then

L.H.S. = $\theta + \phi$

R.H.S. = $\sin^{-1} (\sin \theta \cos \phi + \cos \theta \sin \phi)$

$$= \sin^{-1}\left[\sin(\theta+\phi)\right] = \theta + \phi$$

$\therefore$ L.H.S. = R.H.S

(ii) Let $x = \cos \theta$ and $y = \cos \phi$

L.H.S. = $\theta + \phi$

R.H.S. = $\cos^{-1} (\cos \theta \cos \phi - \sin \theta \sin \phi)$

$$= \cos^{-1}\left[\cos(\theta+\phi)\right] = \theta + \phi$$

$\therefore$ L.H.S. = R.H.S

(iii) Let $x = \sin \theta$, $y = \sin \phi$

L.H.S. = $\theta - \phi$

R.H.S = $\sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

= $\sin^{-1}\left[\sin\theta\sqrt{1-\sin^2\phi} - \sin\phi\sqrt{1-\sin^2\theta}\right]$

= $\sin^{-1}$ [sin θ cos φ – cos θ sin φ]

= $\sin^{-1}\left[\sin(\theta-\phi)\right] = \theta - \phi$

∴ L.H.S. = R.H.S.

(iv) Let x = cos θ, y = cos φ

∴ L.H.S. = θ – φ

R.H.S. = $\cos^{-1}$ [cos θ cos φ + sin θ sin φ]

= $\cos^{-1}\left[\cos(\theta-\phi)\right] = \theta - \phi$

∴ L.H.S. = R.H.S.

2.6 Derivatives of Exponential Functions

Let y = e^x be an exponential function. …(i)

∴ y + δy = $e^{(x + \delta x)}$ (Corresponding small increments) …(ii)

From (i) and (ii), we have

∴ δy = $e^{x + \delta x} - e^x$

Dividing both sides by δx and taking the limit as δx → 0

∴ $$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} e^x \frac{\left[e^{\delta x} - 1\right]}{\delta x}$$

⇒ $$\frac{dy}{dx} = e^x . 1 = e^x$$

Thus, we have $\frac{d}{dx}\left(e^x\right) = e^x$.

Working rule: $\frac{d}{dx}\left(e^x\right) = e^x . \frac{d}{dx}(x) = e^x$

Next, let y = $e^{ax + b}$

Then y + δy = $ea^{(x + \delta x)} + b$

[δx and δy are corresponding small increments]

∴ δy = $e^{a(x + \delta x) + b} - e^{ax - b}$

= $e^{ax + b}\,[a^{edx} - 1]$

Notes

$$\therefore \quad \frac{\delta y}{\delta y} = e^{ax+b}\frac{\left[e^{a\delta x}-1\right]}{\delta x}$$

$$= a\,e^{ax+b}\frac{e^{a\delta x}-1}{a\delta x} \qquad \text{[Multiply and divide by a]}$$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta y} = a\cdot e^{ax+b}\cdot\lim_{\delta x\to 0}\frac{e^{a\delta x}-1}{a\,\delta x}$$

or

$$\frac{dy}{dx} = a\cdot e^{ax+b}\cdot 1 \qquad \left[\lim_{x\to 0}\frac{e^{x}-1}{x}=1\right]$$

$$= ae^{ax+b}$$

2.6.1 Derivatives of Logarithmic Functions

We first consider logarithmic function

Let $\quad y = \log x \qquad \ldots$(i)

$\therefore \quad y + \delta y = \log(x + \delta x) \qquad \ldots$(ii)

(δx and δy are corresponding small increments in x and y)

From (i) and (ii), we get

$$\delta y = \log(x + \delta x) - \log x$$

$$= \log\frac{x+\delta x}{x}$$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\log\left[1+\frac{\delta x}{x}\right]$$

$$= \frac{1}{x}\cdot\frac{1}{\delta x}\log\left[1+\frac{\delta x}{x}\right] \qquad \text{[Multiply and divide by x]}$$

$$= \frac{1}{x}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

Taking limits of both sides, as $\delta x \to 0$, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{1}{x}\lim_{\delta x\to 0}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

Notes

$$\frac{dy}{dx} = \frac{1}{x}\cdot\log\left\{\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right\}$$

$$= \frac{1}{x}\log e \qquad \left[\because \lim_{dx\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} = e\right]$$

$$= \frac{1}{x}$$

Thus, $$\frac{d}{dx}(\log x) = \frac{1}{x}$$

Next, we consider logarithmic function

$$y = \log(ax + b) \qquad \ldots(i)$$

$\therefore$ $$y + \delta y = \log[a(x + \delta x) + b] \qquad \ldots(ii)$$

[δx and δy are corresponding small increments]

From (i) and (ii), we get

$$\delta y = \log[a(x + \delta x) + b] - \log(ax + b)$$

$$= \log\frac{a(x+\delta x)+b}{ax+b}$$

$$= \log\frac{(ax+b)+a\delta x}{ax+b}$$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right]$$

$$= \frac{a}{ax+b}\cdot\frac{ax+b}{a\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right]$$

$$\left[\text{Multiply and divide by } \frac{a}{ax+b}\right]$$

$$= \frac{a}{ax+b}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

Talking limits on both sides as $dx \to 0$

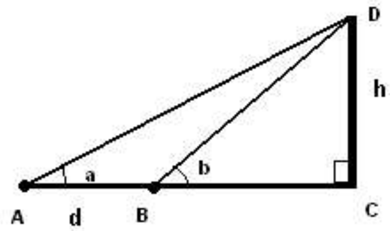
$\therefore$ $$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{a}{ax+b}\lim_{\delta x\to 0}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

Notes

or $$\frac{dy}{dx} = \frac{a}{ax+b}\log e \qquad \left[\because \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e\right]$$

or, $$\frac{dy}{dx} = \frac{a}{ax+b}$$

Example 1: From point A, an observer notes that the angle of elevation of the top of a tower (C,D) is a (degrees) and from point B the angle of elevation is b (degrees). Points A, B and C (the bottom of the tower) are collinear. The distance between A and B is d. Find the height h of the tower in terms of d and angles a and b.



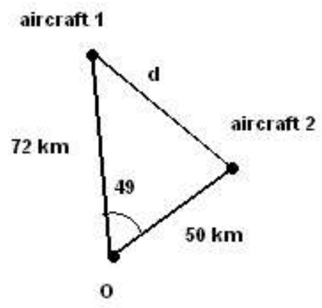
Solution:

1. Let x be the distance between points B and C, hence in the right triangle ACD we have tan(a) = h / (d + x)
2. and in the right triangle BCD we have tan(b) = h / x
3. Solve the above for x x = h / tan(b)
4. Solve tan(a) = h / (d + x) for h h = (d + x) tan(a)
5. Substitute x in above by h / tan(b) h = (d + h / tan(b)) tan(a)
6. Solve the above for h to obtain. h = d tan(a) tan(b) / [tan(b) - tan(a)]

Problem 1: An aircraft tracking station determines the distance from a common point O to each aircraft and the angle between the aicrafts. If angle O between the two aircrafts is equal to 49^o and the distances from point O to the two aircrafts are 50 km and 72 km, find distance d between the two aircrafts.(round answers to 1 decimal place).

Solution to Problem 1:

1. A diagram to the above problem is shown below



Notes

2. The cosine law may be used as follows:

   $d^2 = 72^2 + 50^2 - 2\,(72)(50)\cos(49^{\circ})$

3. Solve for d and use calculator.

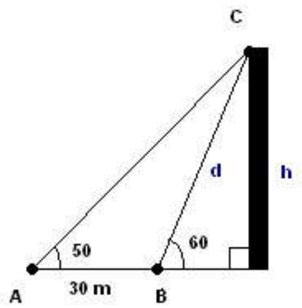
   $d = \text{SQRT}\,[72^2 + 50^2 - 2\,(72)(50)\cos(49^{\circ})]$

   (approximately) = 54.4 km

Task

1. A triangle has sides equal to 4 m, 11 m and 8 m. Find its angles (round answers to 1 decimal place).
2. A ship leaves port at 1 pm traveling north at the speed of 30 miles/hour. At 3 pm, the ship adjusts its course 20 degrees eastward. How far is the ship from the port at 4pm? (round to the nearest unit).

Problem 2: The angle of elevation to the top C of a building from two points A and B on level ground are 50 degrees and 60 degrees respectively. The distance between points A and B is 30 meters. Points A, B and C are in the same vertical plane. Find the height h of the building (round your answer to the nearest unit).



Solution to Problem 2:

1. We consider triangle ABC. Angle B internal to triangle ABC is equal to

   $B = 180^{\circ} - 60^{\circ} = 120^{\circ}$

2. In the same triangle, angle C is given by.

   $C = 180^{\circ} - (50^{\circ} + 120^{\circ}) = 10^{\circ}$

3. Use sine law to find d.

   $d / \sin(50) = 30 / \sin(10)$

4. Solve for d.

   $d = 30 * \sin(50) / \sin(10)$

5. We now consider the right triangle.

   $\sin(60) = h / d$

6. Solve for h.

   $h = d * \sin(60)$

7. Substitute d by the expression found above.

   $h = 30 * \sin(50) * \sin(60) / \sin(10)$

Notes

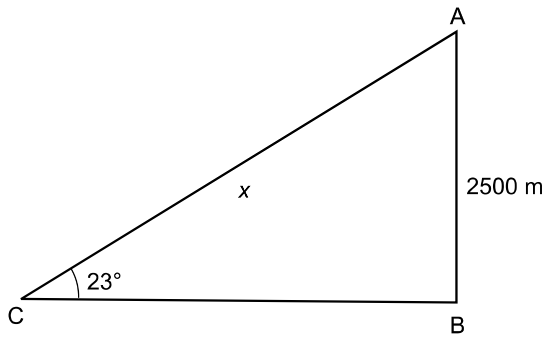
8. Use calculator to approximate h.

   h = (approximately) 115 meters.

The angle of elevation of an aeroplane is 23°. If the aeroplane's altitude is 2500 m, how far away is it?

Answer: A represents aeroplane

Let the distance be x. Then .



You can walk across the Bridge and take a photo of the House from about the same height as top of the highest sail. This photo was taken from a point about 500 m horizontally from the House and we observe the waterline below the highest sail as having an angle of depression of 8°. How high above sea level is the highest sail of the House?

This is a simple tan ratio problem.

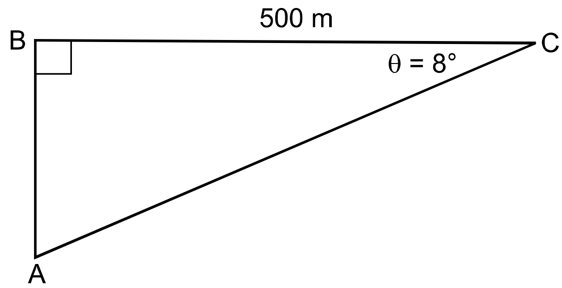
tan 8° = h/500

So

h = 500 tan 8° = 70.27 m.

So the height of the tallest point is around 70 m.

[The actual height is 67.4 m.]



2.7 Summary

(i) In this unit you have studied about trigonometric functions. There are six trigonometric functions, these are sine, cosine, tan, cot, sec and cosec.

(ii) By Pythagorean Theorem.

   $\sin^2 a + \cos^2 a = 1$

(iii) $1 + \tan^2 x = \sec^2 x$

(iv) $1 + \cot^2 x = \text{cosec}^2 x$

(v) Sum of cosine of two angles

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

(vi) Difference of cosine of two angles

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

(vii) Sum of sines of two angles

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

(viii) Difference of sines of two angles

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

2.8 Self Assessment

1. The exact value of cos 75°

(a) $\frac{\sqrt{6}-\sqrt{2}}{4}$

(b) 1

(c) $1/\sqrt{2}$

(d) $\sqrt{2}$

2. Value of $3\tan^2(B/2) - 1 = 0$

(a) $1/\sqrt{3}$

(b) $\sqrt{1}/3$

(c) $1/2$

(d) $2/\sqrt{3}$

3. Vale of sin 2A

(a) $1/2$

(b) $2/\sqrt{3}$

(c) $\sqrt{2}$

(d) $1/\sqrt{2}$

4. Sin 120° – cos 150° is equals to

(a) $\sqrt{3}$

(b) $\sqrt{2}$

(c) $\sqrt{2}/3$

(d) $3/\sqrt{2}$

Notes

5. Tan θ is equals to
   (a) sin θ/cos θ
   (b) cos/sin θ
   (c) $\sin^2 \theta/\cos \theta$
   (d) $\cos^2 \theta/\sin \theta$
6. Value of tan 75 is
   (a) $(\sqrt{3}+1)/(\sqrt{3}-1)$
   (b) $(\sqrt{3}-1)/(\sqrt{3}+1)$
   (c) $(\sqrt{2}/3+1)/(2/\sqrt{3}-1)$
   (d) $(1/3+1)/(1/3+\sqrt{2})$
7. Value of cot 45 is
   (a) 1/2
   (b) $\sqrt{2}$
   (c) $\sqrt{3}$
   (d) 1
8. Value of cot 45° – cos 30°
   (a) $\sqrt{3}$
   (b) $-\sqrt{3}$
   (c) $2\sqrt{3}$
   (d) $-2\sqrt{3}$
9. cos (a) + cos (b) is equals to
   (a) $2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
   (b) $-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$
   (c) $2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
   (d) $2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$
10. Value of sin (a) – sin (b) is
   (a) $2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

Notes

(b) $-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

(c) $2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

(d) $2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

2.9 Review Questions

1. What's the exact value of *tan* 15° or *tan* (π/12)?
2. Find all solutions for $\sin^2 (E/2) - \cos^2 (E/2) = 1$.
3. Solve $\frac{\cot(x-y)}{\cot y - \cot x} = \cot x \cot y + 1$
4. Prove that :

   $$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}$$
5. Prove that : $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$
6. Verify that $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$
7. Prove that $\sin^2 x + 2 \sin 4x + \sin 6x = 4 \cos^2 x \sin 4x$
8. Solve the given equation : $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$
9. Prove the cosine sum of two angles
10. Prove the tan sum of two angles

Answers: Self Assessment

1.	(a)	2.	(b)
3.	(a)	4.	(a)
5.	(a)	6.	(a)
7.	(d)	8.	(b)
9.	(a)	10.	(d)

2.10 Further Readings

Books

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

NCERT Mathematics books class XI

NCERT Mathematics books class XII

Smith and Minton. *Calculus Early Trancendental,* Third Edition. McGraw Hill.2008

Notes

Online links

http://www.suitcaseofdreams.net/Trigonometric_Functions.htm

http://library.thinkquest.org/20991/alg2/trigi.html

http://www.intmath.com/trigonometric–functions/5–signs–of–trigonometric–functions.php

Unit 3: Matrix

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Explain the meaning of matrix.
- Discuss different types of matrices.
- Describe the matrix operation such as addition, subtraction and multiplication.
- Understand the transpose of matrix.
- Explain the symmetric and skew symmetric matrix.

Introduction

In earlier units you have studied about the trigonometric functions of sum and difference of two angles and inverse trigonometric functions.

A matrix was first introduced to solve systems of linear equations. In 1750, G. Cramer gave a rule called Cramer's rule to solve the simultaneous equations. Sir Arthur Cayley introduced the theory of matrices. If all the equations of a system or model are linear, then matrix algebra provides an efficient method of their solution than the traditional method of elimination of variables. Just like ordinary algebra, matrix algebra has operations like addition and subtraction. In this unit you will generalize matrix algebra and different types of matrices.

3.1 Matrix

A matrix is an array of numbers arranged in certain number of rows and columns. If there are $m \times n$ numbers ($i = 1$ to m and $j = 1$ to n), we can write a matrix with m rows and n columns as

Notes

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A matrix having m rows and n columns is called a matrix of order $m \times n$. The individual entries of the array, are termed as the elements of matrix A.

A matrix can be indicated by enclosing an array of numbers by parentheses [ ] or ( ).

Matrices are usually denoted by capital letters A, B, C, ... etc., while small letters like a, b, c, ... etc. are used to denote the elements of a matrix.

In order to locate an element of a matrix one has to specify the row and column to which it belongs. For example, lies in ith row and jth column of A.

Example: $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 7 \\ -1 & 4 & 10 \end{bmatrix}$

$$C = \begin{bmatrix} -3 & 5 \\ 0 & 6 \\ 7 & -12 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 10 & 7 \\ 11 & -2 & 8 \end{bmatrix}$$

Notes

1. We shal follow the notations namely A= $[a_{ij}]$ m × n to indicat that A is a matrix of order m × n.
2. We shall consider only those matrices whose elements are real numbers or functions telling real value.

Order (Type of a Matrix)

If a matrix has m rows and n columns then the matrix is said to be of order $m \times n$.

In the above examples, A is of order 2×2, B is of order 2×3, C is of order 3×2, D is of order 3×3.

Example.

In an examination of Economics, 25 students from college A, 28 Students from college B and 35 students from college C appeared. The number of students passing the examination were 14, 18, 20 and those obtaining distinction were 7, 10 and 15 respectively. Express the above information in matrix form.

Solution

We assume that each column represents the information about a college. Similarly, let first row represent total number of students appeared, second row represent the number of students passed and third row represent the number of students who obtained distinction. The required matrix can be written as

$$\begin{array}{r c} & \text{College} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{r} \text{Appeared} \\ \text{Passed} \\ \text{Distinction} \end{array} & \begin{bmatrix} 25 & 28 & 35 \\ 14 & 18 & 20 \\ 7 & 10 & 15 \end{bmatrix} \end{array}$$

3.2 Equality of Matrices

Notes

Two matrices A and B are said to be equal if they are of the same order and the corresponding elements of A and B are equal:

Example: (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 8 & -7 & 4 \end{bmatrix}_{2 \times 3}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 8 & -7 & 4 \end{bmatrix}_{2 \times 3}$

The orders are same and the corresponding elements are equal.

$\therefore\ A = B.$

(2) If $A = \begin{bmatrix} x & 2 \\ 0 & y \end{bmatrix}_{2 \times 2}$ and $B = \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix}$ then

$$A = B \longleftrightarrow x = -1 \text{ and } y = 4.$$

3.3 Types of Matrices

1. ***Rectangular matrix:*** A matrix of order $m \times n$ is called a rectangular matrix.

   Example: $A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 7 & 0 \end{bmatrix}_{2 \times 3}$ is a rectangular matrix.

2. ***Square matrix:*** A matrix in which the number of rows is equal to the number of columns (i.e., $m \times m$ matrix) is called square matrix.

   Example: $A = \begin{bmatrix} -1 & 4 \\ 7 & 0 \end{bmatrix}_{2 \times 2}$ $B = \begin{bmatrix} 2 & 6 & -11 \\ 5 & 0 & 8 \\ 7 & -4 & 1 \end{bmatrix}_{3 \times 3}$

3. ***Diagonal matrix:*** A square matrix in which all the elements except the principal diagonal elements are zero, is called a diagonal matrix.

   Example: $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ are diagonal matrices.

Notes If A = $[a_{ij}]$ is a sq matrix of order n, then elements entries a_{11}, a_{22} … a_{nn} are said to constitute diagonal of the matrix A. Thus of A = $\begin{bmatrix} 1 & -3 & 1 \\ 2 & 4 & -1 \\ 3 & 5 & 6 \end{bmatrix}$, Them elements of the diagonal of A are 1, 4, 6.

4. ***Scalar matrix:*** A diagonal matrix in which all the principal diagonal elements are equal, is called a scalar matrix.

Notes

Example: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

5. ***Unit matrix (or Identity matrix):*** A scalar matrix in which all the principal diagonal elements are equal to 1 is called a unit matrix.

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

A unit (or identity) matrix is denoted by I.

6. ***Null matrix (or zero matrix):*** A matrix in which all the elements are 0 is called a null matrix. A null (or zero) matrix is denoted by 0.

Example: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times2}, C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

7. ***Row matrix:*** A matrix having only one row and any number of columns (i.e., $1\times n$ matrix) is called a row matrix.

Example: $\begin{bmatrix} 1 & -3 & 0 \end{bmatrix}_{1\times3}$

8. ***Column matrix:*** A matrix having only one column and any number of rows (i.e., $m\times1$ matrix) is called a column matrix.

Example: $\begin{bmatrix} 2 \\ -7 \\ 0 \end{bmatrix}_{3\times1}$

Example.

Cars and Jeeps are produced in two manufacturing units, M_1 and M_2 of a company. It is known that the unit M_1 manufactures 10 cars and 5 Jeeps per day and unit M_2 manufactures 8 cars and 9 Jeeps per day. Write the above information in a matrix form. Multiply this by 2 and explain its meaning.

Solution:

Let A denote the required matrix. Let first row of A denote the output of M_1 and second denote the output of M_2. Further, let first column of A represent the number of cars and second the number of Jeeps.

Thus, the matrix

Notes

$$A = \begin{array}{c} \\ M_1 \\ M_2 \end{array} \begin{array}{c} \text{Cars} \quad \text{Jeeps} \\ \begin{bmatrix} 10 & 5 \\ 8 & 9 \end{bmatrix} \end{array}$$

Further, $2A = \begin{bmatrix} 20 & 10 \\ 16 & 18 \end{bmatrix}$. This matrix gives the number of cars and Jeeps produced by each unit of the company in two days.

3.4 Operation of Matrices

1. ***Addition of matrices:*** Addition of two matrices A and B is defined if and only if they are of the same order.

Notes If A and B are not of same order, then A + B is not defined. e.g. $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, then A +B is not defined. We may observe that addition of matrices is an example of binary operation or the set of matrices of same order.

If A and B are matrices of the same order then their sum $A + B$ is obtained by adding the corresponding elements of A and B.

Example: $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & 10 \end{bmatrix}_{2\times 3}, B = \begin{bmatrix} 0 & -1 & 7 \\ 5 & 8 & 15 \end{bmatrix}_{2\times 3}$

then $A + B = \begin{bmatrix} 2+0 & -1+(-1) & 0+7 \\ 4+5 & 7+8 & 10+15 \end{bmatrix}_{2\times 3}$

$$= \begin{bmatrix} 2 & -2 & 7 \\ 9 & 15 & 25 \end{bmatrix}_{2\times 3}$$

$$B + A = \begin{bmatrix} 0+2 & (-1)+(-1) & 7+0 \\ 5+4 & 8+7 & 15+10 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 2 & -2 & 7 \\ 9 & 15 & 25 \end{bmatrix}_{2\times 3}$$

$\therefore\ A + B = B + A.$

2. ***Subtraction of matrices:*** Subtraction of two matrices A and B is defined if and only if they are of the same order.

If A and B are matrices of the same order then their difference $A - B$ is obtained by subtracting the elements of B by the corresponding elements of A.

Notes

Example: If $A = \begin{bmatrix} 2 & 0 \\ 5 & -1 \\ 4 & 7 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} -1 & 8 \\ 9 & 0 \\ 7 & -3 \end{bmatrix}_{3\times 2}$

then $A - B = \begin{bmatrix} 2-(-1) & 0-8 \\ 5-9 & -1-0 \\ 4-7 & 7-(-3) \end{bmatrix}_{3\times 2}$

$$= \begin{bmatrix} 3 & -8 \\ -4 & -1 \\ -3 & 10 \end{bmatrix}_{3\times 2}$$

$$B - A = \begin{bmatrix} -1-2 & 8-0 \\ 9-5 & 0-(-1) \\ 7-4 & -3-7 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} -3 & 8 \\ 4 & 1 \\ 3 & -10 \end{bmatrix}_{3\times 2}$$

$\therefore A - B \neq B - A.$

3. ***Scalar multiplication:*** If A is a matrix of order $m \times n$ and k is a scalar, then the matrix kA is obtained by multiplying all the elements of A by k.

Example: If $A = \begin{bmatrix} 2 & -5 & 4 \\ -7 & 3 & 10 \end{bmatrix}_{2\times 3}$

then $2A = \begin{bmatrix} 4 & -10 & 8 \\ -14 & 6 & 20 \end{bmatrix}_{2\times 3}$

and $\frac{1}{2}A = \begin{bmatrix} 1 & \frac{-5}{2} & 2 \\ -\frac{7}{2} & \frac{3}{2} & 5 \end{bmatrix}_{2\times 3}$

4. ***Multiplication of matrices:*** Multiplication of matrices is defined if and only if the number of columns of the first matrix is equal to the number of rows of the second matrix. i.e., if A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$ then only AB is defined and AB will be a matrix of order $m \times p.$ The mode of multiplication is always row $\times$ column.

Notes Multiplication of diagonal matrices of same order will be commutative.

Notes

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}_{2 \times 3}$ and $B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}_{3\times 2}$

then $AB = \begin{bmatrix} a_1 & \overrightarrow{b_1} & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \downarrow$

$$= \begin{bmatrix} a_1x_1 + b_1x_2 + c_1x_3 & a_1y_1 + b_1y_2 + c_1y_3 \\ a_2x_1 + b_2x_2 + c_2x_3 & a_2y_1 + b_2y_2 + c_2y_3 \end{bmatrix}_{2 \times 2}$$

Also $BA = \begin{bmatrix} x_1 & \overrightarrow{y_1} \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \downarrow$

$$= \begin{bmatrix} x_1a_1 + y_1a_2 & x_1b_1 + y_1b_2 & x_1c_1 + y_1c_2 \\ x_2a_1 + y_2a_2 & x_2b_1 + y_2b_2 & x_2c_1 + y_2c_2 \\ x_3a_1 + y_3a_2 & x_3b_1 + y_3b_2 & x_3c_1 + y_3c_2 \end{bmatrix}_{3 \times 3}$$

From these, we observe that $AB \neq BA.$ Hence in general $AB \neq BA.$

Task If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$ find the value of a, b, c, x, y and z.

Example.

A firm produces chairs, tables and cupboards, each requiring three types of raw materials increase length timber, nails and varnish. You are given below, the units of different raw materials required for producing one unit of each product:

Product	Timber (c.ft.)	Nails (dozens)	Varnish (litres)
Chair	0.7	2	1
Table	1	4	1.5
Cupboard	3.2	6	2

If the firm produces 300 units of each product, find the quantity of each raw material using matrix algebra.

Solution:

Let $A = \begin{bmatrix} 0.7 & 1 & 3.2 \\ 2 & 4 & 6 \\ 1 & 1.5 & 2 \end{bmatrix}$

Where each row gives the requirement of a raw materials (timber, nails or varnish) to produce one unit of each product.

Notes

Let, $$B = \begin{bmatrix} 300 \\ 300 \\ 300 \end{bmatrix}$$

Then, $$AB = \begin{bmatrix} 0.7 & 1 & 3.2 \\ 2 & 4 & 6 \\ 1 & 1.5 & 2 \end{bmatrix} \begin{bmatrix} 300 \\ 300 \\ 300 \end{bmatrix} = \begin{bmatrix} 1470 \\ 3600 \\ 1350 \end{bmatrix}$$

Thus the requirement is: 1470 c.ft. of timber, 3600 dozens of nails and 350 litres of varnish.

Example

In a certain city, there are 50 colleges and 400 schools. Each school and college have 18 Peons, 5 Clerks and 1 Cashier. Each college in addition has one Section Officer and one Librarian. The monthly salary of each of them is as follows:

Peon: ₹ 1,200; Clerk: ₹ 2,000; Cashier: ₹ 2,400; Section Officer: ₹ 2,800 and Librarian: ₹ 3,600. Using the matrix notation, find (i) total number of posts of each kind in schools and colleges taken together, (ii) the total monthly salary bill of all the schools and colleges taken together.

Solution:

The number of posts of each kind in a school or a college can be written as a column vector.

(i) The total number of posts of each kind, in schools and colleges taken together, can be written as column matrix P, as shown below, where the first and second column vectors give the number of posts of each kind in a school and a college respectively.

$$P = 400\begin{bmatrix} 18 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 50\begin{bmatrix} 18 \\ 5 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7200 + 900 \\ 2000 + 250 \\ 400 + 50 \\ 0 + 50 \\ 0 + 50 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 8100 \\ 2250 \\ 450 \\ 50 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \text{Peons} \\ \text{Clerks} \\ \text{Cashiers} \\ \text{S.Officers} \\ \text{Librarians} \end{array}$$

(ii) Salaries for different posts can be written as row matrix S, as shown below:

$$S = [1200 \quad 2000 \quad 2400 \quad 2800 \quad 3600]$$

$$\text{Total salary bill} = SP = [1200 \quad 2000 \quad 2400 \quad 2800 \quad 3600]\begin{bmatrix} 8100 \\ 2250 \\ 450 \\ 50 \\ 50 \end{bmatrix}$$

$$= 1200 \times 8100 + 2000 \times 2250 + 2400 \times 450 + 2800 \times 50 + 3600 \times 50$$

$$= ₹\ 1{,}56{,}20{,}000.$$

Example 15.

A firm produces three products A, B and C, which it sells in two markets. Annual sales in units are given below:

	Units Sold		
Market	*A*	*B*	*C*
I	8000	4000	16000
II	7000	18000	9000

If the prices per unit of A, B and C are ₹ 2.50, ₹ 1.25 and ₹ 1.50 and the costs per unit are ₹ 1.70, ₹ 1.20 and ₹ 0.80 respectively, find total profit in each market by using matrix algebra.

Solution:

Let Q be the matrix of the quantities sold.

$$Q = \begin{bmatrix} 8000 & 4000 & 16000 \\ 7000 & 18000 & 9000 \end{bmatrix}$$

We can also write $P = \begin{bmatrix} 2.50 \\ 1.25 \\ 1.50 \end{bmatrix}$ and $C = \begin{bmatrix} 1.70 \\ 1.20 \\ 0.80 \end{bmatrix}$, as the matrices of prices and costs respectively.

Note: P and C can also be written as row matrices.

The respective total revenue and cost matrices are

$$TR = QP = \begin{bmatrix} 8000 & 4000 & 16000 \\ 7000 & 18000 & 9000 \end{bmatrix}\begin{bmatrix} 2.50 \\ 1.25 \\ 1.50 \end{bmatrix} = \begin{bmatrix} 49000 \\ 53500 \end{bmatrix},$$

and $$TC = QC = \begin{bmatrix} 8000 & 4000 & 16000 \\ 7000 & 18000 & 9000 \end{bmatrix}\begin{bmatrix} 1.70 \\ 1.20 \\ 0.80 \end{bmatrix} = \begin{bmatrix} 31200 \\ 40700 \end{bmatrix}$$

$$\text{The profit matrix} = TR - TC = \begin{bmatrix} 49000 \\ 53500 \end{bmatrix} - \begin{bmatrix} 31200 \\ 40700 \end{bmatrix} = \begin{bmatrix} 17800 \\ 12800 \end{bmatrix}$$

Hence the profits from market I and II are ₹ 17,800 and ₹ 12,800 respectively.

Alternatively, the profit matrix can be written as = $Q [P - C]$.

Example

A firm produces three products P_1, P_2 and P_3 requiring the mix-up of three materials M_1, M_2 and M_3. The per unit requirement of each product for each material (in units) is as follows:

$$A = \begin{array}{c c} & \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} & \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{bmatrix} \end{array}$$

Using matrix notations, find:

(i) The total requirement of each material if the firm produces 100 units of each product.

(ii) The per unit cost of production of each product if the per unit cost of materials M_1, M_2 and M_3 are ₹ 5, ₹ 10 and ₹ 5 respectively.

(iii) The total cost of production if the firm produces 200 units of each product.

Solution:

Let $B = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}\begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array}$ and $C = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}\begin{array}{c} M_1 \\ M_2 \\ M_3 \end{array}$ denote the output vector and the cost of material vector respectively.

Notes

(i) The total requirement of each material is given by

$$A\text{¢}B = \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \end{array} \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 2 & 4 & 2 \\ 3 & 2 & 4 \\ 1 & 5 & 2 \end{bmatrix} \end{array} \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} = \begin{bmatrix} 800 \\ 900 \\ 800 \end{bmatrix}$$

(ii) The per unit cost production of each product is given by

$$AC = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{bmatrix} \end{array} \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 45 \\ 65 \\ 60 \end{bmatrix}$$

(iii) The total cost of production of 200 units of each product

$$\begin{bmatrix} 200 & 200 & 200 \end{bmatrix} \begin{bmatrix} 45 \\ 65 \\ 60 \end{bmatrix} = 9000 + 13000 + 12000 = 34000.$$

Tasks

A manufacturer produces three products A, B and C, which are sold in Delhi and Calcutta. The annual sales of these products are given below:

	Product		
	A	B	C
Delhi	5000	7500	15000
Calcutta	9000	12000	8700

If the sale price of the products A, B and C per unit be ₹ 2, 3 and 4 respectively, calculate total revenue from each centre by using matrices.

3.5 Transpose of a Matrix

If A is a matrix of order $m \times n$, then the matrix obtained by interchanging the rows and columns is called the transpose of A and is denoted by A' or $A^{T.}$ A' will be a matrix of order $n \times m$.

Example: If $A = \begin{bmatrix} 1 & 3 & -5 \\ -4 & 7 & 8 \end{bmatrix}_{2\times 3}$ then $A' = \begin{bmatrix} 1 & -4 \\ 3 & 7 \\ -5 & 8 \end{bmatrix}_{3 \times 2}$

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix}$, find $A+B$, $A-B$, $2A+3B$, $2A-3B$, $5A+B$, $A-7B$

Solution:

$$A+B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+0 \\ 3+4 & 4+7 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 7 & 11 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1+1 & 2-0 \\ 3-4 & 4-7 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & -3 \end{bmatrix}$$

$$2A+3B=2\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+3\begin{bmatrix}-1 & 0\\ 4 & 7\end{bmatrix}=\begin{bmatrix}2-3 & 4-0\\ 6+12 & 8+21\end{bmatrix}=\begin{bmatrix}-1 & 4\\ 18 & 29\end{bmatrix}$$

$$2A-3B=2\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}-3\begin{bmatrix}-1 & 0\\ 4 & 7\end{bmatrix}=\begin{bmatrix}2+3 & 4-0\\ 6-12 & 8-21\end{bmatrix}=\begin{bmatrix}5 & 4\\ -6 & -13\end{bmatrix}$$

$$5A+B=5\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 4 & 7\end{bmatrix}=\begin{bmatrix}5-1 & 10+0\\ 15+4 & 20+7\end{bmatrix}=\begin{bmatrix}4 & 10\\ 19 & 27\end{bmatrix}$$

$$A-7B=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}-7\begin{bmatrix}-1 & 0\\ 4 & 7\end{bmatrix}=\begin{bmatrix}1+7 & 2-0\\ 3-28 & 4-49\end{bmatrix}=\begin{bmatrix}8 & 2\\ -25 & -45\end{bmatrix}$$

Example: If $A=\begin{bmatrix}-1 & 5\\ 0 & 6\end{bmatrix}, B=\begin{bmatrix}2 & 1\\ 3 & -8\end{bmatrix}$

Verify $(A+B)'=A'+B'$ and $(A-B)'=A'-B'$.

Solution:

$$A=\begin{bmatrix}-1 & 5\\ 0 & 6\end{bmatrix}, B=\begin{bmatrix}2 & 1\\ 3 & -8\end{bmatrix}$$

$$\therefore A'=\begin{bmatrix}-1 & 0\\ 5 & 6\end{bmatrix}, B'=\begin{bmatrix}2 & 3\\ 1 & -8\end{bmatrix}$$

$$A+B=\begin{bmatrix}-1 & 5\\ 0 & 6\end{bmatrix}+\begin{bmatrix}2 & 1\\ 3 & -8\end{bmatrix}=\begin{bmatrix}-1+2 & 5+1\\ 0+3 & 6-8\end{bmatrix}\begin{bmatrix}1 & 6\\ 3 & -2\end{bmatrix}$$

$$\therefore (A+B)'=\begin{bmatrix}1 & 3\\ 6 & -2\end{bmatrix} \qquad ...(1)$$

$$A'+B'=\begin{bmatrix}-1 & 0\\ 5 & 6\end{bmatrix}+\begin{bmatrix}2 & 3\\ 1 & -8\end{bmatrix}=\begin{bmatrix}-1+2 & 0+3\\ 5+1 & 6-8\end{bmatrix}$$

$$\Rightarrow A'+B'=\begin{bmatrix}1 & 3\\ 6 & -2\end{bmatrix} \qquad ...(2)$$

From (1) and (2) ; $(A+B)'=A'+B'$

$$A-B=\begin{bmatrix}-1 & 5\\ 0 & 6\end{bmatrix}-\begin{bmatrix}2 & 1\\ 3 & -8\end{bmatrix}=\begin{bmatrix}-1-2 & 5-1\\ 0-3 & 6+8\end{bmatrix}=\begin{bmatrix}-3 & 4\\ -3 & 14\end{bmatrix}$$

$$\therefore (A-B)'=\begin{bmatrix}-3 & -3\\ 4 & 14\end{bmatrix} \qquad ...(3)$$

$$A'-B'=\begin{bmatrix}-1 & 0\\ 5 & 6\end{bmatrix}-\begin{bmatrix}2 & 3\\ 1 & -8\end{bmatrix}=\begin{bmatrix}-1-2 & 0-3\\ 5-1 & 6+8\end{bmatrix}$$

Notes

$$\Rightarrow A' - B' = \begin{bmatrix} -3 & -3 \\ 4 & 14 \end{bmatrix} \quad ...(4)$$

From (3) and (4) $(A - B)' = A' - B'$.

Example: Find the matrices A and B given that $2A + B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ and $3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

Solution:

$$2A + B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} \quad ...(1)$$

$$3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix} \quad ...(2)$$

Multiply (1) by (2)

$$\therefore (1) \times 2 \Rightarrow 4A + 2B = \begin{bmatrix} 4 & 6 & 2 \\ 2 & 8 & 0 \end{bmatrix}$$

$$(2) \Rightarrow 3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}$$

Subtracting, we get $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix}$

Substituting this in (1), we get

$$2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix} + B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 10 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0 & 3-0 & 1-2 \\ 1-0 & 4-10 & 0+10 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -6 & 10 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -6 & 10 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ find AB and BA.

Solution:

$$AB = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+(-2)(1) & 2(1)+(-2)1 \\ -2(1)+2(1) & (-2)1+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & 2-2 \\ -2+2 & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(2)+1(-2) & 1(-2)+1(2) \\ 1(2)+1(-2) & 1(-2)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & -2+2 \\ 2-2 & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example: Simplify: $\begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-1)+(-1)0 & 1(0)+(-1)(-8) \\ (-2)(-1)+(-3)0 & (-2)(0)+(-3)(-8) \end{bmatrix}$$

$$= \begin{bmatrix} -1+0 & 0+8 \\ 2+0 & 0+24 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 2 & 24 \end{bmatrix}$$

Example: Find x and y if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2(4)+3(-1) \\ 4(4)+5(-1) \end{bmatrix} = \begin{bmatrix} 8-3 \\ 16-5 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Equating the corresponding elements, we get

$x = 5, y = 11.$

Notes

Example: If $A = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$, prove that $A^2 - 4I = 0$ where I is the unit matrix of second order.

Solution:

$$A^2 = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 0+4 & 0+0 \\ 0+0 & 4+0 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$-4I = -4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\therefore A^2 - 4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 & 0+0 \\ 0+0 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\therefore A^2 - 4I = 0.$$

Example: If $A = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$, prove that $A^2 - 8A + 13I = 0.$

Solution:

$$A^2 = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 9+2 & 3+5 \\ 6+10 & 2+25 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 16 & 27 \end{bmatrix}$$

$$-8A = -8\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -24 & -8 \\ -16 & -40 \end{bmatrix}$$

$$13I = 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$$

Adding, $A^2 - 8A + 13I = \begin{bmatrix} 11 & 8 \\ 16 & 27 \end{bmatrix} + \begin{bmatrix} -24 & -8 \\ -16 & -40 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$

$$= \begin{bmatrix} 11-24+13 & 8-8+0 \\ 16-16+0 & 27-40+13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore A^2 - 8A + 13I = 0.$$

Notes

Example: If $A = \begin{bmatrix} 5 & 2 & -1 \\ 0 & 7 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 1 & -1 \end{bmatrix}$

Verify that $(AB)' = B'A'$.

Solution:

$$A' = \begin{bmatrix} 5 & 0 \\ 2 & 7 \\ -1 & 1 \end{bmatrix}, B' = \begin{bmatrix} -3 & 4 & 1 \\ 1 & 7 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 2 & -1 \\ 0 & 7 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5(-3)+2(4)+(-1)1 & 5(1)+2(7)+(-1)(-1) \\ 0(-3)+7(4)+1(1) & 0(1)+7(7)+1(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -15+8-1 & 5+14+1 \\ 0+28+1 & 0+49-1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -8 & 20 \\ 29 & 48 \end{bmatrix}$$

$$\therefore (AB)' = \begin{bmatrix} -8 & 29 \\ 20 & 48 \end{bmatrix} \qquad ...(1)$$

$$B'A' = \begin{bmatrix} -3 & 4 & 1 \\ 1 & 7 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 2 & 7 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-3)5+4(2)+1(-1) & (-3)0+4(7)+1(1) \\ 1(5)+7(2)+(-1)(1) & 1(0)+7(7)+(-1)1 \end{bmatrix}$$

$$= \begin{bmatrix} -15+8-1 & 0+28+1 \\ 5+14+1 & 0+49-1 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} -8 & 29 \\ 20 & 48 \end{bmatrix} \qquad ...(2)$$

From (1) and (2), $(AB)' = B'A'$

Notes

Example: If $A = \begin{bmatrix} 2 & -4 \\ 4 & 1 \end{bmatrix}$, find $(A')^2 - A' + I$.

Solution:

$$A = \begin{bmatrix} 2 & -4 \\ 4 & 1 \end{bmatrix}, \therefore A' = \begin{bmatrix} 2 & 4 \\ -4 & 1 \end{bmatrix}$$

$$(A')^2 = A'A' = \begin{bmatrix} 2 & 4 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-16 & 8+4 \\ -8-4 & -16+1 \end{bmatrix}$$

$$\therefore (A')^2 = \begin{bmatrix} -12 & 12 \\ -12 & -15 \end{bmatrix}$$

$$-A' = \begin{bmatrix} -2 & -4 \\ 4 & -1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore (A')^2 - A' + I = \begin{bmatrix} -12-2+1 & 12-4+0 \\ -12+4+0 & -15-1+1 \end{bmatrix} = \begin{bmatrix} -13 & 8 \\ -8 & -15 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$, find A^3.

Solution:

$$A^2 = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9+4 & 12-4 \\ 3-1 & 4+1 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 2 & 5 \end{bmatrix}$$

$$\therefore\ A^3 = A^2.A = \begin{bmatrix} 13 & 8 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 39+8 & 52-8 \\ 6+5 & 8-5 \end{bmatrix}$$

$$\therefore A^3 = \begin{bmatrix} 47 & 44 \\ 11 & 3 \end{bmatrix}$$

Notes

Example: Find the product of $A = [x \quad y \quad z]$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Solution:

A is a matrix of order 1×3, B is of 3×3 and C is of 3×1.

$\therefore ABC$ will be a matrix of order 1×1.

$$AB = [x \quad y \quad z]_{1 \times 3} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3 \times 3}$$

$$= \left[ax + hy + gz \quad hx + by + fz \quad gx + fy + cz \right]_{1 \times 3}$$

$$(AB)C = \left[ax + hy + gz \quad hx + by + fz \quad gx + fy + cz \right] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left[(ax + hy + gz)\, x + (hx + by + fz)y + (gx + fy + cz)z \right]$$

$$= [ax^2 + hxy + gzx + hxy + by^2 + fyz + gzx + fyz + cz^2]$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx]_{1 \times 1}$$

Example: If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$.

Solution:

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$-4A = \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix}$$

Notes

$$-5I = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\text{Adding} \Rightarrow A^2 - 4A - 5I = \begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\therefore\ A^2 - 4A - 5I = 0.$$

Example: If $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 7 & -5 \\ 0 & 18 & 10 \end{bmatrix}, B = \begin{bmatrix} -2 & 4 & 8 \\ 0 & 6 & 3 \\ 5 & 7 & 11 \end{bmatrix}$

Verify that $2(A+B) = 2A + 2B.$

Solution:

$$A + B = \begin{bmatrix} 1-2 & -2+4 & 3+8 \\ 4+0 & 7+6 & -5+3 \\ 0+5 & 8+7 & 10+11 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 11 \\ 4 & 13 & -2 \\ 5 & 15 & 21 \end{bmatrix}$$

$$2(A+B) = 2\begin{bmatrix} -1 & 2 & 11 \\ 4 & 13 & -2 \\ 5 & 15 & 21 \end{bmatrix}$$

$$2(A+B) = \begin{bmatrix} -2 & 4 & 22 \\ 8 & 26 & -4 \\ 10 & 30 & 42 \end{bmatrix} \quad ...(1)$$

$$2A = 2\begin{bmatrix} 1 & -2 & 3 \\ 4 & 7 & -5 \\ 0 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 6 \\ 8 & 14 & -10 \\ 0 & 16 & 20 \end{bmatrix}$$

$$2B = 2\begin{bmatrix} -2 & 4 & 8 \\ 0 & 6 & 3 \\ 5 & 7 & 11 \end{bmatrix} = \begin{bmatrix} -4 & 8 & 16 \\ 0 & 12 & 6 \\ 10 & 14 & 22 \end{bmatrix}$$

$$\therefore\ 2A+2B=\begin{bmatrix}2 & -4 & 6\\ 8 & 14 & -10\\ 0 & 16 & 20\end{bmatrix}+\begin{bmatrix}-4 & 8 & 16\\ 0 & 12 & 6\\ 10 & 14 & 22\end{bmatrix}$$

$$=\begin{bmatrix}2-4 & -4+8 & 6+16\\ 8+0 & 14+12 & -10+6\\ 0+10 & 16+14 & 20+22\end{bmatrix}$$

$$2A+2B=\begin{bmatrix}-2 & 4 & 22\\ 8 & 26 & -4\\ 10 & 30 & 42\end{bmatrix} \qquad ...(2)$$

From (1) and (2), $2(A+B)=2A+2B.$

Example: If $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, B=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, prove that $A^2+B^2-2I=0.$

Solution:

$$A^2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0+1 & 0+0\\ 0+0 & 1+0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$B^2=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}1+0 & 0+0\\ 0+0 & 0+1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$-2I=-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}$$

Adding, $A^2+B^2-2I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}$

$$=\begin{bmatrix}1+1-2 & 0+0+0\\ 0+0+0 & 1+1-2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$\therefore\ A^2+B^2-2I=0.$

Example: Solve for x, y, z given that $\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2 & 3 & 4\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}\begin{bmatrix}4\\ 5\\ 6\end{bmatrix}$

Solution:

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2 & 3 & 4\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}\begin{bmatrix}4\\ 5\\ 6\end{bmatrix}$$

Notes

$$= \begin{bmatrix} 8+15+24 \\ 16+25+36 \\ 28+40+54 \end{bmatrix} = \begin{bmatrix} 47 \\ 77 \\ 122 \end{bmatrix}$$

$\therefore x = 47, y = 77, z = 122.$

Example: Evaluate: $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} - 3\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} - 3\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \end{bmatrix} - \begin{bmatrix} 27 & 24 & 21 \\ 18 & 15 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-27 & 3+2-24 & 5+4-21 \\ 2+6-18 & 4+8-15 & 6+10-12 \end{bmatrix} = \begin{bmatrix} -26 & -19 & -12 \\ -10 & -3 & 4 \end{bmatrix}$$

Example: Find the matrix X such that $A + 2X = B$ given that $A = \begin{bmatrix} 5 & -1 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -5 \\ 4 & 9 \end{bmatrix}$

Solution:

$A + 2X = B$

$\therefore 2X = B - A$

$\therefore X = \frac{1}{2}(B - A)$

$$= \frac{1}{2}\left\{\begin{bmatrix} 2 & -5 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} 5 & -1 \\ 4 & 7 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ 0 & 2 \end{bmatrix}$$

i.e., $X = \begin{bmatrix} -\frac{3}{2} & -2 \\ 0 & 1 \end{bmatrix}$

3.6 Summary

- In this unit we have studied the concepts of matrices their importance in solving real world problems of business. While a matrix is an array of numbers arranged into certain number of rows and columns.
- A matrix is an ordered rectangular array of numbers or functions.A matrix having m rows and n columns is called a matrix of order $m \times n$.

- $[a_{ij}]m \times 1$ is a column matrix and $[a_{ij}]1 \times n$ is a row matrix.
- An $m \times n$ matrix is a square matrix if $m = n$.
- $A = [a_{ij}]m \times m$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.
- $A = [a_{ij}]n \times n$ is a scalar matrix if $a_{ij} = 0$, when $i \neq j$, $a_{ij} = k$, (k is some constant), when $i = j$.
- $A = [a_{ij}]n \times n$ is an identity matrix, if $a_{ij} = 1$, when $i = j$, $a_{ij} = 0$, when $i \neq j$.
- A zero matrix has all its elements as zero.
- $A = [a_{ij}] = [b_{ij}] = B$ if (i) A and B are of same order, (ii) $a_{ij} = b_{ij}$ for all possible values of i and j.

3.7 Keywords

Column Matrix: A matrix having only one column.

Matrix: An array of numbers arranged in certain numbers of rows and columns.

Rectangular Matrix: A matrix consisting of m rows and n columns.

Row Matrix: A matrix having only one row.

Square Matrix: If the number of rows of a matrix is equal to its number of columns, the matrix is said to be a square matrix.

3.8 Self Assessment

Fill in the blanks:

1. A matrix having m rows and n columns is called a matrix of order.................
2. A matrix consisting of m rows and n columns, where $m \neq n$, is called a.................
3. If the number of rows of a matrix is equal to its number of columns, the matrix is said to be a ....
4. A matrix having only one row is called a.................
5. A diagonal matrix in which all the diagonal elements are equal to 1 is called.................

Multiple choice Questions:

6. $A = \left[a_{ij}\right]m \times n$ is a Square Matrix, it

   (a) m < n

   (b) m > n

   (c) m = n

   (d) None of these

7. Which of the given values of x and y make the following pair of matrices equal to

   $\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$

   (a) $x = \frac{-1}{3}, y = 7$

   (b) Not possible to find

Notes

(c) $y = 7, x = \frac{-2}{3}$

(d) $x = \frac{-1}{3}, y = \frac{-2}{3}.$

8. If A + B are symmetric matrices of same order, then AB - BA is a

(a) Skew symmetric matrix

(b) Symmetric matrix

(c) Zero matrix

(d) Identify matrix

9. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, Then $A + A' = I$, If value of α is

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{3}$

(c) π

(d) $\frac{3\pi}{2}$

3.9 Review Questions

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ then show that $A^3 - 23A - 40I = 0$

2. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ then complete (A + B) and (B – C)

Verify that A + (B – C) = (A + B) – C.

3. Simplify $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

4. Show that

$$\begin{bmatrix} 6 & 7 & 9 \\ 2 & 4 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 6 & 7 & 9 \\ 9 & 2 & 3 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 & 1 \\ 6 & 7 & 9 \\ 9 & 2 & 3 \end{bmatrix} \begin{bmatrix} 6 & 7 & 9 \\ 2 & 4 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

Notes

5. Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Answers: Self Assessment

1.	m × n	2.	Rectangular Matrix
3.	Equality of Matrix	4.	Row Matrix
5.	Unit Matrix	6.	(c)
7.	(b)	8.	(a)
9.	(b)		

3.10 Further Readings

Books

D C Sanchethi and V K Kapoor, *Business Mathematics*

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005

Sivayya and Sathya Rao, *An Introduction to Business Mathematics*

Online links

http://www.suitcaseofdreams.net/Trigonometric_Functions.htm

http://library.thinkquest.org/20991/alg2/trigi.html

http://www.intmath.com/trigonometric–functions/5–signs–of–trigonometric–functions.php

Notes

Unit 4: Determinants

CONTENTS

Objectives

After studying this unit, you will be able to:

- Explain the meaning of determinants
- Discuss matrix representation of a linear equation system
- Describe inverse of a matrix using determinants
- Understand solution of equations

Introduction

A determinant was first introduced to solve systems of linear equations. In 1750, G. Cramer gave a rule called Cramer's rule to solve the simultaneous equations. In the previous unit, we have studied about matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices.

In this unit, we shall study determinants up to order three only with real entries. Also, we will study minors, cofactors and applications of determinants in finding the area of a triangle, adjoint and inverse of a square matrix, consistency and inconsistency of system of linear equations and solution of linear equations in two or three variables using inverse of a matrix.

4.1 Determinant of a Square Matrix

Notes

To every square matrix A, a real number is associated. This real number is called its determinant and is denoted by $\Delta(A)$.

Example: If $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ then its determinant is denoted by $\Delta(A) = \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix}$.

The value of this determinant is determined as $\Delta(A) = (-1)4 - (3\times 2) = -4 - 6 = -10$.

In general if $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ then its value is $\Delta = a_1b_2 - a_2b_1$.

Similarly if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then its value is $\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$\Rightarrow \Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$.

$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$ is called a 2nd order determinant.

> *Notes* For matrix A, |A| is read as determinant of A not modules of A only square matrices have determinants.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

is called a 3rd order determinant.

The rows are represented by R_1, R_2, R_3, the columns are represented by C_1, C_2, C_3.

4.2 Minor of an Element of a Square Matrix

The minor of an element of a square matrix A is defined to be the determinant obtained by deleting the row and column in which the element is present.

Example: $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ then

minor of $a_1 = b_2$

minor of $b_1 = a_2$

minor of $a_2 = b_1$

minor of $b_2 = a_1$

Notes

$$\text{If } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\text{Minor of } a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2$$

$$\text{Minor of } b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = a_2c_3 - a_3c_2$$

$$\text{Minor of } c_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$$

$$\text{Minor of } a_2 = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = b_1c_3 - b_3c_1$$

$$\text{Minor of } b_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$$

Notes Minor of an element of a determinant of order n(n ≥ 2) is a determinant of order n – 1.

$$\text{Minor of } c_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = a_1b_3 - a_3b_1$$

$$\text{Minor of } a_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1$$

$$\text{Minor of } b_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$

$$\text{Minor of } c_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

4.3 Cofactor of an Element of a Square Matrix

The cofactor of an element of a square matrix is defined to be $(-1)^{i+j} \times$ (minor of the element) where *i* and *j* are the number of row and column in which the element is present.

Here $(-1)^{i+j}$ will be equal to 1 if $i + j$ is even and will be equal to -1 if $i + j$ is odd.

$$\text{If } A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

Cofactor of $a_1 = (-1)^{1+1}(b_2) = +b_2$

Cofactor of $b_1 = (-1)^{1+2}(a_2) = -a_2$

Cofactor of $a_2 = (-1)^{2+1}(b_1) = -b_1$

Cofactor of $b_2 = (-1)^{2+2}(a_1) = +a_1$

The signs of the cofactors are $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$$\text{If } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Cofactor of $a_1 = (-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = +(b_2c_3 - b_3c_2)$

Cofactor of $b_1 = (-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -(a_2c_3 - a_3c_2)$

Notes If element of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

Cofactor of $c_1 = (-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = +(a_2b_3 - a_3b_2)$

Cofactor of $a_2 = (-1)^{2+1}\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = -(b_1c_3 - b_3c_1)$

Cofactor of $b_2 = (-1)^{2+2}\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = +\ (a_1c_3 - a_3c_1)$

Cofactor of $c_2 = (-1)^{2+3}\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = -\ (a_1b_3 - a_3b_1)$

Cofactor of $a_3 = (-1)^{3+1}\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = +\ (b_1c_2 - b_2c_1)$

Cofactor of $b_3 = (-1)^{3+2}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = + -(a_1c_2 - a_2c_1)$

Notes

$$\text{Cofactor of } c_3 = (-1)^{3+3}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = +(a_1b_2 - a_2b_1)$$

The signs of cofactors are $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$.

The cofactors of $a_1, b_1, c_1 \quad a_2, b_2, c_2, a_3, b_3, c_3$ are denoted by capitals A_1, B_1, C_1 $A_2, B_2, C_2, A_3, B_3, C_3$ respectively.

4.4 Adjoint of a Square Matrix

The adjoint of a square matrix A is the transpose of the matrix of the cofactors of the elements of A and is denoted by Adj. A.

If $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, then

Cofactor of $a_1 = +\,b_2$ I column

Cofactor of $b_1 = -a_2$

Cofactor of $a_2 = -b_1$ II column

Cofactor of $b_2 = +\,a_1$

$$\therefore \qquad Adj.\ A = \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}$$

> *Notes* To find the adjoint of a 2nd order square matrix, interchange the elements of the principal diagonal and change the signs of the elements of the other diagonal.

Example: If $A = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}$, then $Adj.\ A = \begin{bmatrix} 7 & -3 \\ 1 & 2 \end{bmatrix}$ ...(1)

This can be calculated and verified

Cofactor of $2 = +(7) = 7$ I column

Cofactor of $3 = -(-1) = 1$

Cofactor of $-1 = -(3) = -3$ II column

Cofactor of $7 = +(2) = 2$

$Adj.\ A = \begin{bmatrix} 7 & -3 \\ 1 & 2 \end{bmatrix}$ which is the same as (1)

Notes

$$\text{If } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \text{ then}$$

Cofactor of $a_1 = +(b_2c_3 - b_3c_2) = A_1$

Cofactor of $b_1 = -(a_2c_3 - a_3c_2) = B_1$ I column

Cofactor of $c_1 = +(a_2b_3 - a_3b_2) = C_1$

Cofactor of $a_2 = -(b_1c_3 - b_3c_1) = A_2$

Cofactor of $b_2 = +(a_1c_3 - a_3c_1) = B_2$ II column

Cofactor of $c_2 = -(a_1b_3 - a_3b_1) = C_2$

Cofactor of $a_3 = +(b_1c_2 - b_2c_1) = A_3$

Cofactor of $b_3 = -(a_1c_2 - a_2c_1) = B_3$ III column

Cofactor of $c_3 = +(a_1b_2 - a_2b_1) = C_3$

$$\therefore \quad Adj.\ A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

Notes $A(Adj.\ A) = (Adj\ A)\ A = |A|\ I$ where I is the identity matrix of the same order as that of A.

4.5 Singular and Non-singular Matrices

A square matrix A is said to be singular if $|A| = 0$ and is said to be non-singular if $|A| \neq 0$.

Example: $\Delta = \begin{vmatrix} -2 & -1 \\ 14 & 7 \end{vmatrix} = (-2)7 - (14)(-1) = -14 + 14 = 0.$

$\therefore \Delta$ is singular.

$$\Delta = \begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} = 7 - 12 = -5 \neq 0.$$

$\therefore \Delta$ is non-singular.

4.6 Inverse of a Square Matrix

Inverse of a square matrix is defined if and only if it is non-singular. The inverse of a non-singular square matrix A is denoted by A^{-1}.

Notes

A^{-1} is determined by using the formula

$$A^{-1} = \frac{Adj.A}{|A|} \text{ where } |A| \neq 0.$$

> Notes If A is inverse of B, then B is also inverse of A.

4.7 Solution of a System of Linear Simultaneous Equations (Cramer's Rule)

1. To solve the simultaneous equations in two variables:

$$a_1x + b_1y = c_1 \quad ...(1)$$

$$a_2x + b_2y = c_2 \quad ...(2)$$

Find $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0.$

Replace a_1, a_2 by c_1, c_2 to get

$$\Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1$$

Replace b_1, b_2 by c_1, c_2 to get

$$\Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$

Then by Cramer's Rule

$$x = \frac{\Delta_1}{\Delta} \text{ and } y = \frac{\Delta_2}{\Delta}.$$

2. To solve the simultaneous equations in three variables

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Find $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$

Replace a_1, a_2, a_3 by d_1, d_2, d_3

Notes

$$\therefore \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Replace b_1, b_2, b_3 by d_1, d_2, d_3

$$\therefore \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

Replace c_1, c_2, c_3 by d_1, d_2, d_3

$$\therefore \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then by Cramer's Rule,

$$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta},\ z = \frac{\Delta_3}{\Delta}.$$

Tasks Evaluate the following determinants

1. $\begin{vmatrix} -5 & 3 \\ 2 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 4 & 5 & -1 \end{vmatrix}$

Example: If $\begin{vmatrix} 3 & 4 & x \\ 2 & 1 & 3 \\ -5 & -1 & 2 \end{vmatrix} = -40$, find x.

Solution:

$$\begin{vmatrix} 3 & 4 & x \\ 2 & 1 & 3 \\ -5 & -1 & 2 \end{vmatrix} = -40$$

$$\Rightarrow 3(2+3) - 4(4+15) + x(-2+5) = -40$$

$$\Rightarrow 15 - 76 + 3x = -40$$

$$\Rightarrow 3x = -40 + 76 - 15$$

$$\Rightarrow 3x = 21$$

$$\Rightarrow x = 7$$

Notes

Example: Find the value of x: $\begin{vmatrix} 1 & 4 & 5 \\ 2 & x & 0 \\ 3 & 5 & 8 \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 1 & 4 & 5 \\ 2 & x & 0 \\ 3 & 5 & 8 \end{vmatrix} = 0$$

$$\Rightarrow 1(8x - 0) - 4(16 - 0) + 5(10 - 3x) = 0$$

$$\Rightarrow 8x - 64 + 50 - 15x = 0$$

$$\Rightarrow -7x - 14 = 0$$

$$\Rightarrow 7x = -14$$

$$\Rightarrow x = -2$$

Example: Find the value of x if $\begin{bmatrix} 2 & -1 & x \\ 0 & 1 & 5 \\ 1 & 3 & -1 \end{bmatrix}$ is singular.

Solution:

$$\begin{bmatrix} 2 & -1 & x \\ 0 & 1 & 5 \\ 1 & 3 & -1 \end{bmatrix} \text{ is singular } \Rightarrow \begin{vmatrix} 2 & -1 & x \\ 0 & 1 & 5 \\ 1 & 3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow 2(-1 - 15) + 1(0 - 5) + x(0 - 1) = 0$$

$$\Rightarrow -32 - 5 - x = 0$$

$$\Rightarrow x = -37$$

Example: Find x if $\begin{vmatrix} x & 2 & 2 \\ 2 & x & 2 \\ 2 & 2 & x \end{vmatrix} = 0.$

Solution:

$$\begin{vmatrix} x & 2 & 2 \\ 2 & x & 2 \\ 2 & 2 & x \end{vmatrix} = 0$$

$$\Rightarrow x(0 - 4) - 2(2x - 4) + 2(4 - 0) = 0$$

$$\Rightarrow -4x - 4x + 8 + 8 = 0$$

$$\Rightarrow -8x + 16 = 0 \therefore x = 2$$

Example: If $\begin{bmatrix} x & 2 & x+2 \\ 3 & 5 & 8 \\ x+1 & 7-x & 12 \end{bmatrix}$ is singular, find the value of x.

Solution:

The given matrix is singular $\Rightarrow$ its determinant = 0

i.e., $\begin{vmatrix} x & 2 & x+2 \\ 3 & 5 & 8 \\ x+1 & 7-x & 12 \end{vmatrix} = 0$

$$\Rightarrow x(60-56+8x)-2(36-8x-8)+(x+2)(21-3x-5x-5)=0$$

$$\Rightarrow 4x+8x^2-56+16x+16x+32-8x^2-16x=0$$

$$\Rightarrow 20x-24=0$$

$$\Rightarrow x=\frac{24}{20}=\frac{6}{5}.$$

Example: Evaluate the following determinants

1. $\begin{vmatrix} 20 & 21 \\ 22 & 23 \end{vmatrix}$ 2. $\begin{vmatrix} 40 & 41 & 42 \\ 41 & 42 & 43 \\ 42 & 43 & 44 \end{vmatrix}$ 3. $\begin{vmatrix} 77 & 78 & 79 \\ 75 & 74 & 73 \\ 76 & 75 & 74 \end{vmatrix}$

4. $\begin{vmatrix} 4200 & 4201 \\ 4202 & 4203 \end{vmatrix}$ 5. $\begin{vmatrix} 12 & 0 & 0 \\ 4 & 3 & 0 \\ 2 & 2 & -3 \end{vmatrix}$

Solution:

1. Let $\Delta = \begin{vmatrix} 20 & 21 \\ 22 & 23 \end{vmatrix}$

$R_1 - R_2$

$= \begin{vmatrix} -2 & -2 \\ 22 & 23 \end{vmatrix}$

$C_1 - C_2$

$= \begin{vmatrix} 0 & -2 \\ -1 & 23 \end{vmatrix}$

Expand

$= 0(23)-(-1)(-2) = -2$

Notes

2. Let $\Delta = \begin{vmatrix} 40 & 41 & 42 \\ 41 & 42 & 43 \\ 42 & 43 & 44 \end{vmatrix}$

$R_1 - R_2$ and $R_2 - R_3$

$= \begin{vmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ 42 & 43 & 44 \end{vmatrix}$

$R_1 - R_2$

$= \begin{vmatrix} 0 & 0 & 0 \\ -1 & -1 & -1 \\ 42 & 43 & 44 \end{vmatrix}$

Expand

$0(-44+43) - 0(-44+42) + 0(-43+42)$

$= 0 - 0 + 0 = 0$

3. Let $\Delta = \begin{vmatrix} 77 & 78 & 79 \\ 75 & 74 & 73 \\ 76 & 75 & 74 \end{vmatrix}$

$R_1 - R_2$ and $R_2 - R_3$

$= \begin{vmatrix} 2 & 4 & 6 \\ -1 & -1 & -1 \\ 76 & 75 & 74 \end{vmatrix}$

$C_1 - C_2$ and $C_2 - C_3$

$= \begin{vmatrix} -2 & -2 & 6 \\ 0 & 0 & -1 \\ 1 & 1 & 74 \end{vmatrix}$

Expand

$= (-2)\,(0+1) + 2(0+1) + 6(0-0)$

$= -2 + 2 + 0 = 0$

4. Let $\Delta = \begin{vmatrix} 4200 & 4201 \\ 4202 & 4203 \end{vmatrix}$

$R_1 - R_2$

Notes

$$= \begin{vmatrix} -2 & -2 \\ 4202 & 4203 \end{vmatrix}$$

$C_1 - C_2$

$$= \begin{vmatrix} 0 & -2 \\ -1 & 4203 \end{vmatrix}$$

Expand

$= 0(4203) - (-1)(-2)$

$= -2$

5. Let $\Delta = \begin{vmatrix} 12 & 0 & 0 \\ 4 & 3 & 0 \\ 2 & 2 & -3 \end{vmatrix}$

Expand

$= 12(-9-0) - 0(-12-0) + 0(8-6)$

$= -108 - 0 + 0 = -108$

4.8 Examples on Adjoint and Inverse

Example: Find the adjoint of the following matrices

1. $\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix}$ 2. $\begin{bmatrix} -1 & 5 \\ 2 & 8 \end{bmatrix}$ 3. $\begin{bmatrix} 3 & 0 \\ 5 & 9 \end{bmatrix}$

4. $\begin{bmatrix} 2 & -1 & 5 \\ 4 & 0 & 1 \\ 9 & -3 & 6 \end{bmatrix}$ 5. $\begin{bmatrix} 1 & -5 & 6 \\ 0 & 7 & 11 \\ 5 & -2 & 4 \end{bmatrix}$ 6. $\begin{bmatrix} 9 & -1 & 2 \\ 0 & 8 & 1 \\ -5 & 1 & 7 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 7 & -3 \\ 4 & 10 & 1 \\ -3 & 5 & 8 \end{bmatrix}$

Solution:

1. $A = \begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix}$

Cofactor of 2 = + (7) = 7

Cofactor of –1 = – (4) = – 4 I column

Cofactor of 4 = – (– 1) = 1

Notes

Cofactor of 7 = + (2) = 2 II column

$$\therefore Adj\ A = \begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix}$$

2. $A = \begin{bmatrix} -1 & 5 \\ 2 & 8 \end{bmatrix}$

Cofactor of –1 = + (8) = 8

Cofactor of 5 = – (2) = – 2 I column

Cofactor of 2 = – (5) = – 5

Cofactor of 8 = + (– 1) = – 1 II column

$$\therefore Adj\ A = \begin{bmatrix} 8 & -5 \\ -2 & -1 \end{bmatrix}$$

3. $A = \begin{bmatrix} 3 & 0 \\ 5 & 9 \end{bmatrix}$

Cofactor of 3 = + (9) = 9

Cofactor of 0 = – (5) = – 5 I column

Cofactor of 5 = – (0) = 0

Cofactor of 9 = + (3) = 3 II column

$$\therefore Adj\ A = \begin{bmatrix} 9 & 0 \\ -5 & 3 \end{bmatrix}$$

4. $A = \begin{bmatrix} 2 & -1 & 5 \\ 4 & 0 & 1 \\ 9 & -3 & 6 \end{bmatrix}$

Cofactor of 2 = + (0 + 3) = 3

Cofactor of – 1 = – (24 – 9) = – 15 I column

Cofactor of 5 = + (– 12 – 0) = – 12

Cofactor of 4 = – (– 6 + 15) = – 9

Cofactor of 0 = + (12 – 45) = – 33 II column

Cofactor of 1 = – (– 6 + 9) = – 3

Cofactor of 9 = + (– 1 – 0) = – 1

Cofactor of – 3 = – (2 – 20) = 18 III column

Cofactor of 6 = + (0 + 4) = 4

$$\therefore Adj\ A = \begin{bmatrix} 3 & -9 & -1 \\ -15 & -33 & 18 \\ -12 & -3 & 4 \end{bmatrix}$$

Notes

5. $A = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 7 & 11 \\ 5 & -2 & 4 \end{bmatrix}$

Cofactor of 1 = + (28 + 22) = 50

Cofactor of – 5 = – (0 – 55) = 55 I column

Cofactor of 6 = + (0 – 35) = – 35

Cofactor of 0 = – (– 20 + 12) = 8

Cofactor of 7 = + (4 – 30) = – 26 II column

Cofactor of 11 = – (– 2 + 25) = – 23

Cofactor of 5 = + (– 55 – 42) = – 97

Cofactor of – 2 = – (11 – 0) = – 11 III column

Cofactor of 4 = + (7 – 0) = 7

$$\therefore Adj\ A = \begin{bmatrix} 50 & 8 & -97 \\ 55 & -26 & -11 \\ -35 & -23 & 7 \end{bmatrix}$$

6. $A = \begin{bmatrix} 9 & -1 & 2 \\ 0 & 8 & 1 \\ -5 & 1 & 7 \end{bmatrix}$

Cofactor of 9 = + (56 – 1) = 55

Cofactor of – 1 = – (0 + 5) = – 5 I column

Cofactor of 2 = + (0 + 40) = 40

Cofactor of 0 = – (– 7 – 2) = 9

Cofactor of 8 = + (63 – 10) = 53 II column

Cofactor of 1 = – (9 – 5) = – 4

Cofactor of – 5 = + (– 1 – 16) = – 17

Cofactor of 1 = – (9 – 0) = – 9 III column

Cofactor of 7 = + (72 – 0) = 72

$$\therefore Adj\ A = \begin{bmatrix} 55 & 9 & -17 \\ -5 & 53 & -9 \\ 40 & -4 & 72 \end{bmatrix}$$

7. $A = \begin{bmatrix} 2 & 7 & -3 \\ 4 & 10 & 1 \\ -3 & 5 & 8 \end{bmatrix}$

Notes

Cofactor of 2 = + (80 – 5) = 75

Cofactor of 7 = – (32 + 3) = – 35 I column

Cofactor of – 3 = + (20 + 30) = 50

Cofactor of 4 = – (56 + 15) = – 71

Cofactor of 10 = + (16 – 9) = 7 II column

Cofactor of 1 = – (10 + 21) = – 31

Cofactor of – 3 = + (7 + 30) = 37

Cofactor of 5 = – (2 + 12) = – 14 III column

Cofactor of 8 = + (20 – 28) = – 8

$$\therefore Adj\ A = \begin{bmatrix} 75 & -71 & 37 \\ -35 & 7 & -14 \\ 50 & -31 & -8 \end{bmatrix}$$

Example: Find the inverses of the following matrices provided they exist:

1. $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ 2. $\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$ 5. $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ 6. $\begin{bmatrix} -1 & -2 & 0 \\ 3 & 1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$

7. $\begin{bmatrix} 0 & -2 & 4 \\ 1 & 7 & 3 \\ 2 & 5 & -4 \end{bmatrix}$ 8. $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \\ 3 & 4 & -5 \end{bmatrix}$

Solution:

1. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 0 + 2 = 2 \neq 0$$

Cofactor of 1 = + (0) = 0

Cofactor of – 1 = – (2) = – 2 I column

Cofactor of 2 = – (– 1) = 1

Cofactor of 0 = + (1) = 1 II column

$$\therefore \text{Adj}\ A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

Notes

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}}{2} = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

2. Let $A = \begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} -2 & 0 \\ 4 & 1 \end{vmatrix} = -2 - 0 = -2 \neq 0$$

Cofactor of –2 = + (1) = 1

Cofactor of 0 = – (4) = – 4 I column

Cofactor of 4 = – (0) = 0

Cofactor of 1 = + (– 2) = – 2 II column

$$\therefore \text{Adj}\ A = \begin{bmatrix} 1 & 0 \\ -4 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{\begin{bmatrix} 1 & 0 \\ -4 & -2 \end{bmatrix}}{-2} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 2 & 1 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = 4 + 3 = 7 \neq 0$$

Cofactor of 1 = + (4) = 4

Cofactor of – 1 = – (3) = – 3 I column

Cofactor of 3 = – (– 1) = 1

Cofactor of 4 = + (1) = 1 II column

$$\therefore \text{Adj}\ A = \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{\begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix}}{7} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{1}{7} \end{bmatrix}$$

Notes

4. Let $A = \begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & -2 \\ 3 & 7 \end{vmatrix} = 35 + 6 = 41$

Cofactor of 5 = + (7) = 7

Cofactor of – 2 = – (3) = – 3 I column

Cofactor of 3 = – (–2) = 2

Cofactor of 7 = + (5) = 5 II column

$\therefore$ Adj $A = \begin{bmatrix} 7 & 2 \\ -3 & 5 \end{bmatrix}$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{\begin{bmatrix} 7 & 2 \\ -3 & 5 \end{bmatrix}}{41} = \begin{bmatrix} \frac{7}{41} & \frac{2}{41} \\ -\frac{3}{41} & \frac{5}{41} \end{bmatrix}$$

5. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= 1(-2-1) - 2(1-1) - 1(-1-2)$

$= -3 - 0 + 3 = 0$

Since $|A| = 0,$ the inverse does not exist.

6. $\begin{bmatrix} -1 & -2 & 0 \\ 3 & 1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$

$|A| = \begin{vmatrix} -1 & -2 & 0 \\ 3 & 1 & 5 \\ 4 & 7 & -1 \end{vmatrix}$

$= -1(-1-35) + 2(-3-20) + 0(21-4)$

$= 36 - 46 = -10 \neq 0$

Cofactor of – 1 = + (– 1 – 35) = – 36

Cofactor of – 2 = – (– 3 – 20) = 23 I column

Notes

Cofactor of 0 = + (21 – 4) = 17

Cofactor of 3 = – (2 – 0) = – 2

Cofactor of 1 = + (1 – 0) = 1 II column

Cofactor of 5 = – (– 7 + 8) = – 1

Cofactor of 4 = + (– 10 – 0) = – 10

Cofactor of 7 = – (– 5 – 0) = 5 III column

Cofactor of – 1 = + (– 1 + 6) = 5

$$\therefore \text{Adj } A = \begin{bmatrix} -36 & -2 & -10 \\ 23 & 1 & 5 \\ 17 & -1 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{-10}\begin{bmatrix} -36 & -2 & -10 \\ 23 & 1 & 5 \\ 17 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{18}{5} & \frac{1}{5} & 1 \\ \frac{-23}{10} & -\frac{1}{10} & \frac{-5}{10} \\ -\frac{17}{10} & \frac{1}{10} & -\frac{1}{2} \end{bmatrix}$$

7. Let $A = \begin{bmatrix} 0 & -2 & 4 \\ 1 & 7 & 3 \\ 2 & 5 & -4 \end{bmatrix}$

$$|A| = 0(-28-15) + 2(-4-6) + 4(5-14)$$

$$= 0 - 20 - 36 = -56 \neq 0$$

Cofactor of 0 + (– 28 – 15) = – 43

Cofactor of – 2 = – (– 4 – 6) = 10 I column

Cofactor of 4 = + (5 – 14) = – 9

Cofactor of 1 = – (8 – 20) = 12

Cofactor of 7 = + (0 – 8) = – 8 II column

Cofactor of 3 = – (0 + 4) = – 4

Cofactor of 2 = + (– 6 – 28) = – 34

Cofactor of 5 = – (0 – 4) = 4 III column

Cofactor of – 4 = + (0 + 2) = 2

Notes

$$\therefore \text{Adj } A = \begin{bmatrix} -43 & 12 & -34 \\ 10 & -8 & 4 \\ -9 & -4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{-56} = \begin{bmatrix} -43 & 12 & -34 \\ 10 & -8 & 4 \\ -9 & -4 & 2 \end{bmatrix}$$

$$\text{i.e., } A^{-1} = \begin{bmatrix} \frac{43}{56} & \frac{-3}{14} & \frac{17}{28} \\ \frac{-5}{28} & \frac{1}{7} & -\frac{1}{14} \\ \frac{9}{56} & \frac{1}{14} & -\frac{1}{28} \end{bmatrix}$$

8. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \\ 3 & 4 & -5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \\ 3 & 4 & -5 \end{vmatrix}$$

$$= 2(-10-0)+1(-5-0)+1(4-6)$$

$$= -20-5-2 = -27 \neq 0$$

Cofactor of 2 = + (– 10 – 0) = – 10

Cofactor of – 1 = – (– 5 – 0) = 5 I column

Cofactor of 1 = + (4 – 6) = – 2

Cofactor of 1 = – (5 – 4) = – 1

Cofactor of 2 = + (–10 – 3) = – 13 II column

Cofactor of 0 = – (8 + 3) = – 11

Cofactor of 3 = + (0 – 2) = – 2

Cofactor of 4 = – (0 – 1) = 1 III column

Cofactor of – 5 = + (4 + 1) = 5

$$\therefore \text{Adj } A = \begin{bmatrix} -10 & -1 & -2 \\ 5 & -13 & 1 \\ -2 & -11 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{-27} = \begin{bmatrix} -10 & -1 & -2 \\ 5 & -13 & 1 \\ -2 & -11 & 5 \end{bmatrix}$$

Notes

Example: For the following matrices find A^{-1} and verify that (i) A (Adj A) = (Adj A) A = $|A|$ I and (ii) $AA^{-1} = A^{-1}A = I$

1. $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

Solution:

1. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = 2 + 2 = 4 \neq 0$$

Cofactor of 1 = + (2) = 2

Cofactor of – 1 = – (2) = –2 I column

Cofactor of 2 = – (– 1) = 1

Cofactor of 2 = + (1) = 1 II column

$$\therefore \text{ Adj } A = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{-1}{2} & \frac{1}{4} \end{bmatrix}$$

$$A(\text{Adj } A) = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 1-1 \\ 4-4 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\Rightarrow A\ (\text{A}dj\ \text{A}) = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 4I = |\text{A}|\ I\ (\because\ |A| = 4)$$

Similarly it can be verified that $(Adj\ A)\,A = |A|\ I.$

$$\text{Now, } AA^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2+2 & 1-1 \\ 4-4 & 2+2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$\therefore AA^{-1} = I.$$

Similarly, it can be verified that $A^{-1}A = I.$

Notes

2. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

$|A| = 1(1-0) + 1(2-0) + 1(4-3) = 4 \neq 0.$

Cofactor of 1 = + (1 – 0) = 1

Cofactor of – 1 = – (2 – 0) = – 2 I column

Cofactor of 1 = + (4 – 3) = 1

Cofactor of 2 = – (– 1 –2) = 3

Cofactor of 1 = + (1 – 3) = – 2 II column

Cofactor of 0 = – (2 + 3) = – 5

Cofactor of 3 = + (0 – 1) = – 1

Cofactor of 2 = – (0 – 2) = 2 III column

Cofactor of 1 = + (1 + 2) = 3

$$\text{Adj } A = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -2 & 2 \\ 1 & -5 & 3 \end{bmatrix}$$

$$\therefore A(Adj\ A) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ -2 & -2 & 2 \\ 1 & -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+1 & 3+2-5 & -1-2+3 \\ 2-2+0 & 6-2+0 & -2+2+0 \\ 3-4+1 & 9-4-5 & -3+4+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= 4I = |A|\, I \ (\because |A| = 4)$

Similarly, we can verify that $(Adj\ A)A = |A|\, I$.

$\therefore A\,(\text{Adj } A) = (\text{Adj } A)\, A = |A|\, I$

Notes

Now, $A^{-1} = \dfrac{Adj\ A}{|A|}$

$$= \frac{1}{4}\begin{bmatrix} 1 & 3 & -1 \\ -2 & -2 & 2 \\ 1 & -5 & 3 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 1 & 3 & -1 \\ -2 & -2 & 2 \\ 1 & -5 & 3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore AA^{-1} = I$

Similarly it can be verified that $A^{-1}A = I.$

$\therefore AA^{-1} = A^{-1}A = I.$

4.9 Examples on Simultaneous Equations (Cramer's Rule)

Example: Using Cramer's Rule, solve the following equations:

1. $3x + 4y = 7,\ 4x - 3y = 5$
2. $3x + 3y = 12,\ 2x + 4y = 12$
3. $6x + 4y = 10,\ x + 7y = 8$
4. $x + 6y = -16,\ -2x + 3y = -13$
5. $2x + y = 4,\ 3x + 4y = 11$
6. $x + y + z = 11,\ 2x - 6y - z = 0,\ 3x + 4y + 2z = 0$
7. $x + 3y - z = 4,\ x - 2z = -5,\ 3x + y = 5$
8. $z + 2x + 1 = 0,\ -y + z + 2 = 0,\ x + 2y = 5$

Solution:

1. $3x + 4y = 7$

   $4x - 3y = 5$

   $$\Delta = \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} = -9 - 16 = -25$$

   $$\Delta_1 = \begin{vmatrix} 7 & 4 \\ 5 & -3 \end{vmatrix} = -21 - 20 = -41$$

Notes

$$\Delta_2 = \begin{vmatrix} 3 & 7 \\ 4 & 5 \end{vmatrix} = 15 - 28 = -13$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-41}{-25} = \frac{41}{25}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-13}{-25} = \frac{13}{25}$$

$$\therefore\ x = \frac{41}{25},\ y = \frac{13}{25}$$

2. $3x + 3y = 12$

$$2x + 4y = 12$$

$$\Delta = \begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix} = 12 - 6 = 6$$

$$\Delta_1 = \begin{vmatrix} 12 & 3 \\ 12 & 4 \end{vmatrix} = 48 - 36 = 12$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 \\ 2 & 12 \end{vmatrix} = 36 - 24 = 12$$

$$x = \frac{\Delta_1}{\Delta} = \frac{12}{6} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{12}{6} = 2$$

$$\therefore\ x = 2,\ y = 2$$

3. $6x + 4y = 10$

$$x + 7y = 8$$

$$\Delta = \begin{vmatrix} 6 & 4 \\ 1 & 7 \end{vmatrix} = 42 - 4 = 38$$

$$\Delta_1 = \begin{vmatrix} 10 & 4 \\ 8 & 7 \end{vmatrix} = 70 - 32 = 38$$

$$\Delta_2 = \begin{vmatrix} 6 & 10 \\ 1 & 8 \end{vmatrix} = 48 - 10 = 38$$

$$\therefore\ x = \frac{\Delta_1}{\Delta} = \frac{38}{38} = 1$$

Notes

$$y = \frac{\Delta_2}{\Delta} = \frac{38}{38} = 1$$

$\therefore x = 1, y = 1$

4. $x + 6y = -16$

$-2x + 3y = -13$

$$\Delta = \begin{vmatrix} 1 & 6 \\ -2 & 3 \end{vmatrix} = 3 + 12 = 15$$

$$\Delta_1 = \begin{vmatrix} -16 & 6 \\ -13 & 3 \end{vmatrix} = -48 + 78 = 30$$

$$\Delta_2 = \begin{vmatrix} 1 & -16 \\ -2 & -13 \end{vmatrix} = -13 - 32 = -45$$

$$x = \frac{\Delta_1}{\Delta} = \frac{30}{15} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-45}{15} = -3$$

5. $2x + y = 4$

$3x + 4y = 11$

$$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5$$

$$\Delta_1 = \begin{vmatrix} 4 & 1 \\ 11 & 4 \end{vmatrix} = 16 - 11 = 5$$

$$\Delta_2 = \begin{vmatrix} 2 & 4 \\ 3 & 11 \end{vmatrix} = 22 - 12 = 10$$

$$\therefore\ x = \frac{\Delta_1}{\Delta} = \frac{5}{5} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{10}{5} = 2$$

$\therefore x = 1, y = 2$

6. $x + y + z = 11$

$2x - 6y - z = 0$

$3x + 4y + 2z = 0$

Notes

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{vmatrix}$$

$$= 1(-12+4) - 1\,(4+3) + 1(8+18)$$

$$= -8 - 7 + 26 = 11$$

$$\Delta_1 = \begin{vmatrix} 11 & 1 & 1 \\ 0 & -6 & -1 \\ 0 & 4 & 2 \end{vmatrix}$$

$$= 11(-12+4) - 0 + 0$$

$$= -88$$

$$\Delta_2 = \begin{vmatrix} 1 & 11 & 1 \\ 2 & 0 & -1 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= 1(0-0) - 11(4+3) + 1(0-0)$$

$$= -77$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 11 \\ 2 & -6 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

$$= 1(0-0) - 1(0-0) + 11(8+18)$$

$$= 286$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-88}{11} = -8$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-77}{11} = -7$$

$$z = \frac{\Delta_3}{\Delta} = \frac{286}{11} = 26$$

7. $x + 3y - z = 4$

$$x - 2z = -5$$

$$3x + y = 5$$

$$\Delta = \begin{vmatrix} 1 & 3 & -1 \\ 1 & 0 & -2 \\ 3 & 1 & 0 \end{vmatrix}$$

$= 1(0+2) - 3(0+6) - 1(1-0)$

$= 2 - 18 - 1 = -17$

$$\Delta_1 = \begin{vmatrix} 4 & 3 & -1 \\ -5 & 0 & -2 \\ 5 & 1 & 0 \end{vmatrix}$$

$= 4(0+2) - 3(0+10) - 1(-5-0)$

$= 8 - 30 + 5 = -17$

$$\Delta_2 = \begin{vmatrix} 1 & 4 & -1 \\ 1 & -5 & -2 \\ 3 & 5 & 0 \end{vmatrix}$$

$= 1(0+10) - 4(0+6) - 1(5+15)$

$= 10 - 24 - 20 = -34$

$$\Delta_3 = \begin{vmatrix} 1 & 3 & 4 \\ 1 & 0 & -5 \\ 3 & 1 & 5 \end{vmatrix}$$

$= 1(0+5) - 3(5+15) + 4(1-0)$

$= 5 - 60 + 4 = -51$

$$\therefore\ x = \frac{\Delta_1}{\Delta} = \frac{-17}{-17} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-34}{-17} = 2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-51}{-17} = 3$$

$\therefore\ x = 1,\ y = 2,\ z = 3$

8. $z + 2x + 1 = 0 \Rightarrow 2x + 0y + z = -1$

$-y + z + 2 = 0 \Rightarrow 0x - y + z = -2$

$x + 2y = 5 \Rightarrow x + 2y + 0z = 5$

$$\Delta = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix}$$

$= 2(0-2) - 0(0-1) + 1(0+1)$

Notes

$$= -4 + 1 = -3$$

$$\Delta_1 = \begin{vmatrix} -1 & 0 & 1 \\ -2 & -1 & 1 \\ 5 & 2 & 0 \end{vmatrix}$$

$$= (-1)(0-2) - 0(0-5) + 1(-4+5)$$

$$= 2 - 0 + 1 = 3$$

$$\Delta_2 = \begin{vmatrix} 2 & -1 & 1 \\ 0 & -2 & 1 \\ 1 & 5 & 0 \end{vmatrix}$$

$$= 2(0-5) + 1(0-1) + 1(0+2)$$

$$= -10 - 1 + 2 = -9$$

$$\Delta_3 = \begin{vmatrix} 2 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 2 & 5 \end{vmatrix}$$

$$= 2(-5+4) - 0(0+2) - 1(0+1)$$

$$= -2 - 1 = -3$$

$$\therefore\ x = \frac{\Delta_1}{\Delta} = \frac{3}{-3} = -1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-9}{-3} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-3}{-3} = 1$$

Example

Show that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1^2 + a_2^2 & a_1b_1 + a_2b_2 \\ a_1b_1 + a_2b_2 & a_1^2 + b_2^2 \end{vmatrix}$

Solution:

Let $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$\therefore$ $D^2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1^2 + a_2^2 & a_1b_1 + a_2b_2 \\ a_1b_1 + a_2b_2 & b_1^2 + b_2^2 \end{vmatrix}.$

Notes

Example

Prove that $\begin{vmatrix} a^2-bc & b^2-ca & c^2-ab \\ c^2-ab & a^2-bc & b^2-ca \\ b^2-ca & c^2-ab & a^2-bc \end{vmatrix} = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2$

Solution:

Let A, B and C be the cofactors of a, b and c respectively in $\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$. We note that the determinant on the L.H.S. of the given equation is a determinant of cofactors.

Let $$\Delta_1 = \begin{vmatrix} a^2-bc & b^2-ca & c^2-ab \\ c^2-ab & a^2-bc & b^2-ca \\ b^2-ca & c^2-ab & a^2-bc \end{vmatrix} = \begin{vmatrix} A & B & C \\ C & A & B \\ B & C & A \end{vmatrix}$$

Then $$\Delta_1\Delta = \begin{vmatrix} A & B & C \\ C & A & B \\ B & C & A \end{vmatrix}\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} aA+bB+cC & cA+aB+bC & bA+cB+aC \\ aC+bA+cB & cC+aA+bB & bC+cA+aB \\ aB+bC+cA & cB+aC+bA & bB+cC+aA \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix}$$

Thus, $\Delta_1\Delta = \Delta^3$ or $\Delta_1 = \Delta^2$. Hence the result.

Note: The solution of the above example is based on property (7) of determinants.

Example

Prove that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$.

Solution:

We can write $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

Notes

Further,
$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix}$$

$$= abc - a^3 - b^3 + abc + abc - c^3 = -(a^3 + b^3 + c^2 - 3abc)$$

$$\therefore \qquad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (a^3 + b^3 + c^3 - 3abc)^2.$$

Example

A transport company uses 3 types of trucks T_1, T_2 and T_3 to transport 3 types of vehicles V_1, V_2 and V_3. The capacity of each truck in terms of 3 types of vehicles is given below:

	V_1	V_2	V_3
T_1	1	3	2
T_2	2	2	3
T_3	3	2	2

Using matrix method, find:

(i) The number of trucks of each type required to transport 85, 105 and 110 vehicles of V_1, V_2 and V_3 types respectively.

(ii) Find the number of vehicles of each type which can be transported, if the company has 10, 20 and 30 trucks of each type respectively.

Solution:

(i) Let x_1, x_2 and x_3 be the number of trucks of type T_1, T_2 and T_3, respectively. Then we can write

$$x_1 + 2x_2 + 3x_3 = 85$$
$$3x_1 + 2x_2 + 2x_3 = 105$$
$$2x_1 + 3x_2 + 2x_3 = 110$$

Denoting the coefficient matrix by A, we have $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}$

Further, $|A| = 4 + 8 + 27 - 12 - 6 - 12 = 9$

The cofactors of the elements of A are:

$C_{11} = -2$, $C_{12} = -2$, $C_{13} = 5$, $C_{21} = 5$, $C_{22} = -4$, $C_{23} = 1$ $C_{31} = -2$, $C_{32} = 7$, $C_{33} = -4$

Thus, $$A^{-1} = \frac{1}{9}\begin{bmatrix} -2 & 5 & -2 \\ -2 & -4 & 7 \\ 5 & 1 & -4 \end{bmatrix}$$

Notes

and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -2 & 5 & -2 \\ -2 & -4 & 7 \\ 5 & 1 & -4 \end{bmatrix}\begin{bmatrix} 85 \\ 105 \\ 110 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 135 \\ 180 \\ 90 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \\ 10 \end{bmatrix}$

Hence $x_1 =$ 15, $x_2 =$ and $x_3 = 10$.

(ii) The number of vehicles of each type, that can be transported, are given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 140 \\ 130 \\ 140 \end{bmatrix}\begin{matrix} V_1 \\ V_2 \\ V_3 \end{matrix}$$

4.10 Economic Applications

Two-Commodity Market Equilibrium

Let the demand and supply equations of the two commodities, 1 and 2, be as given below:

$$Q_1^d = a_1 + b_1P_1 + c_1P_2$$
$$Q_1^s = e_1 + f_1P_1 + g_1P_2$$
$$Q_2^d = a_2 + b_2P_1 + c_2P_2$$
$$Q_2^s = e_2 + f_2P_1 + g_2P_2$$

The two-commodity market will be in equilibrium if $Q_1^d = Q_1^s$ and $Q_2^d = Q_2^s$.

The first condition implies that

$$a_1 + b_1P_1 + c_1P_2 = e_1 + f_1P_1 + g_1P_2$$

or $(b_1 - f_1)P_1 + (c_1 - g_1)P_2 = -(a_1 - e_1)$ ...(1)

Similarly, the second equilibrium condition implies that

$$a_2 + b_2P_1 + c_2P_2 = e_2 + f_2P_1 + g_2P_2$$

or $(b_2 - f_2)P_1 + (c_2 - g_2)P_2 = -(a_2 - e_2)$ ... (2)

Let us assume, for convenience, that

$$a_i = b_i - f_i,\ b_i = c_i - g_i, \text{ and } g = -(a_i - e_i)(i = 1, 2)$$

Thus, equations (1) and (2) can be written as

$$\begin{matrix} \alpha_1P_1 + \beta_1P_2 = \gamma_1 \\ \alpha_2P_1 + \beta_2P_2 = \gamma_2 \end{matrix} \text{ or } \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

This is a system of two equations in two unknowns P_1 and P_2, which can be solved either by matrix inversion method or by Cramer's Rule. We shall, however, solve this by matrix inversion method.

Let $A = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$, $\therefore\ |A| = a_1b_2 - a_2b_1$

Notes

Also $C = \begin{bmatrix} \beta_2 & -\alpha_2 \\ -\beta_1 & \alpha_1 \end{bmatrix}$ $\therefore A^{-1} = \dfrac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\begin{bmatrix} \beta_2 & -\beta_1 \\ -\alpha_2 & \alpha_1 \end{bmatrix}$

Further, $$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\begin{bmatrix} \beta_2 & -\beta_1 \\ -\alpha_2 & \alpha_1 \end{bmatrix}\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

Thus equilibrium prices are $P_1 = \dfrac{\gamma_1\beta_2 - \gamma_2\beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1}$ and $P_2 = \dfrac{\gamma_2\alpha_1 - \gamma_1\alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1}$.

On substituting these prices either in demand or in supply equation, we can obtain the equilibrium quantities of the two commodities.

The two-commodity model can be easily generalised to the case of n-commodities. It will consist of n equations in n prices as shown below:

$$\begin{aligned} a_{11}P_1 + a_{12}P_2 + \ldots.. + a_{1n}P_n &= g_1 \\ a_{21}P_1 + a_{22}P_2 + \ldots.. + a_{2n}P_n &= g_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}P_1 + a_{n2}P_2 + \ldots.. + a_{nn}P_n &= g_n \end{aligned}$$

National Income Model

The simplest form of the Keynesian model of national-income determination is given by the following system of equations:

$Y = C + I_0$

$C = a + bY \qquad (a > 0, 0 < b < 1)$

We note here that Y (the level of national income) and C (the level of national consumption) are endogenous variables. The above equations must be rearranged so that all the endogenous variables appear only on the L.H.S of the equations.

Thus, we have

$$\begin{aligned} Y - C &= I_0 \\ -bY + C &= a \end{aligned}$$

Using matrix notation, the above equations can be written as

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 \\ a \end{bmatrix}$$

Applying Cramer's rule, we get

$$Y = \frac{\begin{vmatrix} I_0 & -1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{I_0 + a}{I - b} \text{ and } C = \frac{\begin{vmatrix} 1 & I_0 \\ -b & a \end{vmatrix}}{1 - b} = \frac{a + bI_0}{1 - b}.$$

Notes

Example

For the following market conditions, find the equilibrium quantities and prices by using matrix inverse method.

$$Q_1^D = 45 - 2P_1 + 3P_2 - 7P_3, \qquad Q_1^S = -5 + 4P_1$$

$$Q_2^D = 16 + 2P_1 - P_2 + 3P_3, \qquad Q_2^S = -19 + 5P_2$$

$$Q_3^D = 30 - P_1 + 2P_2 - 8P_3, \qquad Q_3^S = -6 + 2P_3$$

Solution:

Using the equilibrium condition $Q_1^D = Q_1^S$, we can write

$$45 - 2P_1 + 3P_2 - 7P_3 = -5 + 4P_1$$

or $$-6P_1 + 3P_2 - 7P_3 = -50$$

or $$6P_1 - 3P_2 + 7P_3 = 50 \qquad ...(1)$$

Similarly $$Q_2^D = Q_2^S$$

$$16 + 2P_1 - P_2 + 3P_3 = -19 + 5P_2$$

or $$2P_1 - 6P_2 + 3P_3 = -35 \qquad ...(2)$$

and $$Q_3^D = Q_3^S$$

$$30 - P_1 + 2P_2 - 8P_3 = -6 + 2P_3$$

or $$-P_1 + 2P_2 - 10P_3 = -36$$

or $$P_1 - 2P_2 + 10P_3 = 36 \qquad ...(3)$$

The system of equations given by (1), (2) and (3) can be written as the matrix equation

$$\begin{bmatrix} 6 & -3 & 7 \\ 2 & -6 & 3 \\ 1 & -2 & 10 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 50 \\ -35 \\ 36 \end{bmatrix}. \text{ Let } A = \begin{bmatrix} 6 & -3 & 7 \\ 2 & -6 & 3 \\ 1 & -2 & 10 \end{bmatrix}$$

$\therefore$ $$|A| = -360 - 9 - 28 + 42 + 36 + 60 = -259.$$

Since $|A| \neq 0$, the solution is unique. Writing the matrix of cofactors as

$$C = \begin{bmatrix} -54 & -17 & 2 \\ 16 & 53 & 9 \\ 33 & -4 & -30 \end{bmatrix} \quad \therefore A^{-1} = -\frac{1}{259}\begin{bmatrix} -54 & 16 & 33 \\ -17 & 53 & -4 \\ 2 & 9 & -30 \end{bmatrix}$$

Thus, $$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = -\frac{1}{259}\begin{bmatrix} -54 & 16 & 33 \\ -17 & 53 & -4 \\ 2 & 9 & -30 \end{bmatrix}\begin{bmatrix} 50 \\ -35 \\ 36 \end{bmatrix}$$

Hence $$P_1 = -\frac{-50\times 54 - 35\times 16 + 36\times 33}{259} = 8$$

$$P_2 = -\frac{-50\times 17 - 35\times 53 - 36\times 4}{259} = 11$$

Notes

$$P_3 = -\frac{50\times 2 - 35\times 9 - 36\times 30}{259} = 5$$

Further,

$$Q_1 = -5 + 4 \times 8 = 27 \text{ (using supply equation)}$$

$$Q_2 = -19 + 5 \times 11 = 36$$

$$Q_3 = -6 + 2 \times 5 = 4.$$

Example

A manufacturer produces two types of products X and Y. Each product is first processed in machine M_1 and then sent to another machine M_2 for finishing. Each unit of X requires 20 minutes time on machine M_1 and 10 minutes time on M_2, whereas each unit of Y requires 10 minutes time on machine M_1 and 20 minutes time on M_2. The total time available on each machine is 600 minutes and is fully utilized in the production of X and Y. Calculate the number of units of two types of products produced by constructing a matrix equation of the form AX = B and then solve it by matrix inversion method.

Solution:

Let x and y denote the number of units produced of X and Y respectively. Time taken on M_1 by the production of x units of X and y units of Y is $20x + 10y$ and this should be equal to 600 minutes.

Thus we have $$20x + 10y = 600 \qquad ...(1)$$

Similarly, we can write an equation representing the time taken on machine M_2. This equation is given by

$$10x + 20y = 600 \qquad ...(2)$$

Writing equation (1) and (2) in matrix form $\begin{bmatrix} 20 & 10 \\ 10 & 20 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 600 \\ 600 \end{bmatrix}$

or $AX = B$, where $A = \begin{bmatrix} 20 & 10 \\ 10 & 20 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 600 \\ 600 \end{bmatrix}$

Now $|A|$ = 400 – 100 = 300 ¹ 0. Thus, the system has a unique solution.

We write $C = \begin{bmatrix} 20 & -10 \\ -10 & 20 \end{bmatrix}$ $\therefore A^{-1} = \frac{1}{300}\begin{bmatrix} 20 & -10 \\ -10 & 20 \end{bmatrix}$

Thus the solution is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{300}\begin{bmatrix} 20 & -10 \\ -10 & 20 \end{bmatrix}\begin{bmatrix} 600 \\ 600 \end{bmatrix} = \frac{1}{300}\begin{bmatrix} 12,000 - 6,000 \\ -6,000 + 12,000 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

From the above, we can write x = 20 and y = 20.

Example

The prices, in rupees per unit, of the three commodities X, Y and Z are x, y and z respectively. A purchases 4 units of Z and sells 3 units of X and 5 units of Y. B purchases 3 units of Y and sells 2 units of X and 1 unit of Z. C purchases 1 unit of X and sells 4 units of Y and 6 units of Z. In the process A, B and C earn ₹ 6000, 5,000 and 13,000 respectively. Using matrices, find the prices of the three commodities (note that selling the unit is positive earning and buying the units is negative earning.)

Notes

Solutionl

The given information can be written as the following set of equations:

$$\begin{aligned} 3x+5y-4z &= 6,000 \\ 2x-3y+z &= 5,000 \\ -x+4y+6z &= 13,000 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6,000 \\ 5,000 \\ 13,000 \end{bmatrix}$$

Let A be the matrix of the coefficients

$\therefore$ $|A| = -54 - 5 - 32 + 12 - 12 - 60 = -151 \neq 0.$

Writing the matrix of cofactors, we get

$$C = \begin{bmatrix} -22 & -13 & 5 \\ -46 & 14 & -17 \\ -7 & -11 & -19 \end{bmatrix} \quad \therefore A^{-1} = -\frac{1}{151}\begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}$$

Thus, $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{151}\begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}\begin{bmatrix} 6,000 \\ 5,000 \\ 13,000 \end{bmatrix}$$

$$x = \frac{-22\times 6,000 - 46\times 5,000 - 7\times 13,000}{-151} = 3,000$$

$$y = \frac{-13\times 6,000 + 14\times 5,000 - 11\times 13,000}{-151} = 1,000$$

$$z = \frac{5\times 6,000 - 17\times 5,000 - 19\times 13,000}{-151} = 2,000.$$

Example

A company has two productions departments, P_1 and P_2, and three service departments, S_1, S_2 and S_3. The direct cost allocated to each of the departments and the percentage of total cost of each service department apportioned to various departments are given below:

Deptt.	Direct Cost (Rs.)	Percentage Allocation of Total Cost		
		S_1	S_2	S_3
P_1	60,000	40	35	25
P_2	74,000	20	40	20
S_1	6,000	0	20	35
S_2	8,000	15	0	20
S_3	68,500	25	5	0

Notes

Determine the total cost (allocated and apportioned) for each production department by using matrix algebra.

Solution:

First of all, we find the total cost of each service department S_1, S_2 and S_3. Let C_1, C_2 and C_3 denote the total cost of the service departments S_1, S_2 and S_3 respectively. Therefore, we can write

$$C_1 = 6,000 + 0 \times C_1 + 0.20C_2 + 0.35C_3$$

or $$C_1 - 0.20C_2 - 0.35C_3 = 6,000 \quad ...(1)$$

Similarly, $$C_2 = 8,000 + 0.15C_1 + 0 \times C_2 + 0.20C_3$$

or $$-0.15C_1 + C_2 - 0.20C_3 = 8,000 \quad ...(2)$$

and $$C_3 = 68,500 + 0.25C_1 + 0.05C_2 + 0 \times C_3$$

or $$-0.25C_1 - 0.05C_2 + C_3 = 68,500 \quad ...(3)$$

From (1), (2) and (3), we get $\begin{bmatrix} 1 & -0.20 & -0.35 \\ -0.15 & 1 & -0.20 \\ -0.25 & -0.05 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 6,000 \\ 8,000 \\ 68,500 \end{bmatrix}$

Let $$A = \begin{bmatrix} 1 & -0.20 & -0.35 \\ -0.15 & 1 & -0.20 \\ -0.25 & -0.05 & 1 \end{bmatrix}$$

$\therefore$ $$|A| = 1 - 0.01 - 0.002625 - 0.0875 - 0.01 - 0.03 = 0.86 \text{ (approx.)}$$

Also $$|A_1| = \begin{vmatrix} 6000 & -0.20 & -0.35 \\ 8000 & 1 & -0.20 \\ 68500 & -0.50 & 1 \end{vmatrix} = 34395$$

$$|A_2| = \begin{vmatrix} 1 & 6000 & -0.35 \\ -0.15 & 8000 & -0.20 \\ -0.25 & 68500 & 1 \end{vmatrix} = 25796.25$$

$$|A_3| = \begin{vmatrix} 1 & -0.20 & 6000 \\ -0.15 & 1 & 8000 \\ -0.25 & -0.05 & 68500 \end{vmatrix} = 68790$$

Thus, using Cramer's rule, we have

$$C_1 = \frac{34395}{0.86} = 39994.19,\ C_2 = \frac{26796.25}{0.86} = 29995.64,\ C_3 = \frac{68790}{0.86} = 79988.37$$

The total cost of the two Production Departments, denoted by P_1 and P_2, is given by

Notes

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 60000 \\ 74000 \end{bmatrix} + \begin{bmatrix} 0.40 & 0.35 & 0.25 \\ 0.20 & 0.40 & 0.20 \end{bmatrix} \begin{bmatrix} 39994.19 \\ 29995.64 \\ 79988.37 \end{bmatrix} = \begin{bmatrix} 106493.24 \\ 109994.77 \end{bmatrix}$$

Thus, $P_1 =$ ₹ 1,06,493.24 and $P_2 =$ ₹ 1,09,994.77.

Example

An amount of ₹ 4,000 is distributed into three investments at the rate of 7%, 8% and 9% per annum respectively. The total annual income is ₹ 317.50 and the annual income from the first investment is ₹ 5 more than the income from the second. Find the amount of each investment.

Solution:

Let x_1, x_2 and x_3 denote the amount of first, second and third investments respectively. We can write

$$x_1 + x_2 + x_3 = 4{,}000 \quad .... (1)$$

Also $$0.07x_1 + 0.08x_2 + 0.09x_3 = 317.50$$

or $$7x_1 + 8x_2 + 9x_3 = 31{,}750 \quad .... (2)$$

Further, $$0.07x_1 - 0.08x_2 = 5$$

or $$7x_1 - 8x_2 = 500 \quad .... (3)$$

Writing the above equations in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 7 & 8 & 9 \\ 7 & -8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4000 \\ 31750 \\ 500 \end{bmatrix} \text{ or } AX = D$$

Here $$|A| = 23$$

Also $$|A_1| = \begin{vmatrix} 4000 & 1 & 1 \\ 31750 & 8 & 9 \\ 500 & -8 & 0 \end{vmatrix} = 34500,$$

$$|A_2| = \begin{vmatrix} 1 & 4000 & 1 \\ 7 & 31750 & 9 \\ 7 & 500 & 0 \end{vmatrix} = 28750$$

and $$|A_3| = \begin{vmatrix} 1 & 1 & 4000 \\ 7 & 8 & 31750 \\ 7 & -8 & 500 \end{vmatrix} = 28750$$

$\therefore$ $$x_1 = \frac{34500}{23} = ₹\ 1{,}500$$

and $$x_2 = x_3 = \frac{28750}{23} = ₹\ 1{,}250.$$

Notes

Example

To control a crop disease, it is necessary to use 8 units of chemical A, 14 units of chemical B and 13 units of chemical C. One barrel of spray P contains 1 unit of A, 2 units of B and 3 units of C. One Barrel of spray Q contains 2 units of A, 3 units of B and 2 units of C. One barrel of spray R contains 1 unit of A, 2 units of B and 2 units of C. Find how many barrels of each spray be used to just meet the requirement?

Solution:

Let x barrels of spray P, y barrels of spray Q and z barrels of Spray R be used to just meet the requirement.

The above information can be written as the following matrix equation.

$$x + 2y + z = 8$$
$$2x + 3y + 2z = 14$$
$$3x + 2y + 2z = 13$$

Let $$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix}$$

$$|A| = 6 + 12 + 4 - 9 - 4 - 8 = 1$$

$$A^{-1} = \text{Adj } A = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix}$$

$\therefore$ $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

i.e. $x = 1, y = 2, z = 3$ barrels of spray P, Q and R respectively.

Example

An amount of ₹ 65,000 is invested in three investments at the rate of 6%, 8% and 9% per annum, respectively. The total annual income is ₹ 4,800. The income from the third investment is ₹ 600 more than the income from second investment. Using matrix algebra, determine the amount of each investment.

Solution:

Let x, y and z be the amount invested in the three investments. Thus, we can write

$$x + y + z = 65000$$
$$0.06x + 0.08y + 0.09z = 4800 \quad \text{or} \quad 6x + 8y + 9z = 4{,}80{,}000$$
$$-0.08y + 0.09z = 600 \quad \text{or} \quad 8y - 9z = -60{,}000$$

We can write $$A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 8 & 9 \\ 0 & 8 & -9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 65,000 \\ 4,80,000 \\ -60,000 \end{bmatrix}$$

$$|A| = -72 + 48 - 72 + 54 = -42 \neq 0.$$

Also $$A^{-1} = -\frac{1}{42}\begin{bmatrix} -144 & 17 & 1 \\ 54 & -9 & -3 \\ 48 & -8 & 2 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{42}\begin{bmatrix} -144 & 17 & 1 \\ 54 & -9 & -3 \\ 48 & -8 & 2 \end{bmatrix}\begin{bmatrix} 65,000 \\ 4,80,000 \\ -60,000 \end{bmatrix} = \begin{bmatrix} 30,000 \\ 15,000 \\ 20,000 \end{bmatrix}$$

Hence $x = 30,000$, $y = 15,000$ and $z = 20,000$.

Example

A mixture is to be made containing x kg of Food A, y kg of Food B and z kg of Food C. Total weight of the mixture to be made is 5 kg. Food A contains 500 units of vitamin per kg and B and C contain 200 and 100 units respectively. The 5 kg mixture is to contain total of 1500 units of vitamin. Food A, B and C contain respectively 300, 600 and 700 calories per kg and 5 kg mixture is to contain a total of 2,500 calories. Derive a general solution for x and y in terms of z so that the 5 kg mixture contains the required 2,500 calories. If the variables x, y and z are not permitted to be negative, find the range of values that z is restricted to.

Solution:

The given information can be written as a system of following equations.

$$x + y + z = 5 \qquad \text{(Weight constraint)}$$

$$500x + 200y + 100z = 1500 \text{ or } 5x + 2y + z = 15 \qquad \text{(Vitamin constraint)}$$

$$300x + 600y + 700z = 2500 \text{ or } 3x + 6y + 7z = 25 \qquad \text{(Calorie constraint)}$$

The coefficient Matrix $$A = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 1 \\ 3 & 6 & 7 \end{bmatrix}$$

$$|A| = 14 + 3 + 30 - 6 - 35 - 6 = 0$$

Thus all the equations are not independent. Dropping (say) third equation, we get a system of two equations in three variables. Let us write them as follows:

$$x + y = 5 - z$$

$$5x + 2y = 15 - z$$

Applying cramer's rule, we can write

$$x = \frac{\begin{vmatrix} 5-z & 1 \\ 15-z & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix}} = \frac{10 - 2z - 15 + z}{2 - 5} = \frac{-(z+5)}{-3} = \frac{z+5}{3}$$

Notes

$$y = \frac{\begin{vmatrix} 1 & 5-z \\ 5 & 15-z \end{vmatrix}}{-3} = \frac{15-z-25+5z}{-3} = \frac{4z-10}{-3} = \frac{10-4z}{3}$$

Since we can assign infinite number of values to z, there are infinite number of solutions.

Since the variables are not permitted to be negative, we can write.

$$x = \frac{z+5}{3} \geq 0 \quad \text{or} \quad z^{3} -5$$

and

$$y = \frac{10-4z}{3} \geq 0 \quad \text{or } 10 - 4z^{3}\ 0 \text{ or } \quad z \leq \frac{5}{2}$$

Since z cannot be negative, therefore the range of values of z is restricted to $0 \leq z \leq \frac{5}{2}$.

Example

Two companies A and B are holding shares in each other, A is holding 20% shares of B and B is holding 10% shares of A. If the separately earned profits of the two companies are ₹ 98,000 and ₹ 49,000 respectively, find total profit of each company using matrix algebra. Also show that the total profits allocated to the outside shareholders is equal to the total of separately earned profits.

Solution:

Let x and y denote the total profit of the two companies A and B respectively.

Then we can write

$$x = 98000 + 0.2y \quad \text{or} \quad x - 0.2y = 98000$$

$$y = 49000 + 0.1x \quad \text{or} \quad -0.1x + y = 49000$$

Writing the above equations as the matrix equation

$$\begin{bmatrix} 1 & -0.2 \\ -0.1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 98000 \\ 49000 \end{bmatrix}$$

This can be solved either by matrix inversion method or Cramer's rule.

Using Cramer's Rule, we get

$$x = \frac{\begin{vmatrix} 98000 & -0.2 \\ 49000 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -0.2 \\ -0.1 & 1 \end{vmatrix}} = \frac{107800.00}{0.98} = 1{,}10{,}000$$

$$y = \frac{\begin{vmatrix} 1 & 98000 \\ -0.1 & 49000 \end{vmatrix}}{0.98} = \frac{58800.00}{0.98} = 60{,}000$$

Notes

Thus x = 1,10,000 and y = 60,000.

Since 90% shares of A and 80% shares of B are allocated to outside shareholders, we can write the vector $S = [0.9 \quad 0.8]$.

Let $P = [x \quad y] = [1{,}10{,}000 \quad 60{,}000]$ be vector of profits of the two companies A and B. Thus, the total profits allocated to the outside shareholders is the scalar product of S and P.

$$S.P = 0.9 \times 1{,}10{,}000 + 0.8 \times 60{,}000 = 1{,}47{,}000.$$

Note that this is equal to 98,000 + 49,000 which is the total of separately earned profits.

Markov Brand-Switching Model

Let there be only two brands, A and B, of a toilet soap available in the market. Let the current market share of brand A be 60% and that of B be 40%. We assume that brand-switching takes place every month such that 70% of the consumers of brand A continue to use it while remaining 30% switch to brand B. Similarly, 80% of the consumers of brand B continue to use it while remaining 20% switch to brand A.

The market shares of the two brands can be written as a row vector, $S = [0.6 \quad 0.4]$ and the given brand switching information can be written as a matrix P of transition probabilities,

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix} \end{array}.$$

Given the current information, we can calculate the shares of the two brands after, say, one month or two months or ...... n months. For example, the shares of the two brands after one month is

$$S(1) = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

Similarly, the shares after the expiry of two months are given by

$$S(2) = S(1) \times P$$

Proceeding in a similar manner, we can write the shares of the two brands after the expiry of n months as

$$\begin{aligned} S(n) &= S(n-1) \times P \\ &= S(n-2) \times P \times P = S(n-2) \times P^2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= S(1) \times P^{n-1} = S(0) \times P^n \end{aligned}$$

where $S(0) = [0.6 \quad 0.4]$ denotes the current market share vector of the two brands.

We note that as $n \to \infty$, the market shares of the two brands will tend to stabilize to an equilibrium position. Once this state is reached, the shares of the two brands become constant. Eventually, we have $S(n) = S(n-1)$. Thus, we can write $S = S.P$, where $S = [s_A \quad s_B]$ is the vector giving the equilibrium shares of the two brands.

The above equation can also be written as

$$S[I - P] = 0$$

Notes

or $$[s_A \quad s_B]\begin{bmatrix} 1-0.7 & -0.3 \\ -0.2 & 1-0.8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$[s_A \quad s_B]\begin{bmatrix} 0.3 & -0.3 \\ -0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note that $I - P$ is a singular matrix and hence, effectively, there is only one equation, given by $0.3s_A - 0.2s_B = 0$.

In order to find s_A and s_B, we need another equation. This equation is provided by the fact that the sum of market shares is unity i.e. $s_A + s_B = 1$. Thus, solving $0.3s_A - 0.2s_B = 0$ and $s_A + s_B = 1$, simultaneously, we get the equilibrium values of the market shares s_A and s_B. In the above example, these values are $s_A = 0.4$ i.e. 40% and $s_B = 0.6$, i.e. 60%.

Example

The price of an equity share of a company may increase, decrease or remain constant on any given day. It is assumed that the change in price on any day affects the change on the following day as described by the following transition matrix:

		Change Tomorrow		
		Increase	Decrease	Unchange
	Increase	0.5	0.2	0.3
Change Today	Decrease	0.7	0.1	0.2
	Unchange	0.4	0.5	0.1

(i) If the price of the share increased today, what are the chances that it will increase, decrease or remain unchanged tomorrow?

(ii) If the price of share decreased today, what are the chances that it will increase tomorrow?

(iii) If the price of the share remained unchanged today, what are the chances that it will increase, decrease or remain unchanged day after tomorrow?

Solution:

(i) Given that the price of the share has increased today, the probability of its going up (today) is 1 and probability of each of events, decreasing or remaining unchanged is equal to zero.

Thus, the initial state vector is $R_0 = [1 \ \ 0 \ \ 0]$. Now the tomorrow's state vector

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.7 & 0.1 & 0.2 \\ 0.4 & 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 & 0.3 \end{bmatrix}$$

Hence, the chances that the price will rise, fall or remain unchanged tomorrow are 50%, 20%, 30% respectively (given that it has increased today).

(ii) The initial state vector is $R_0 = [0 \ \ 1 \ \ 0]$ and the chances of price increase tomorrow are 70%.

(iii) Here $R_0 = [0 \ \ 0 \ \ 1]$

Notes

$$\therefore \qquad R_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.7 & 0.1 & 0.2 \\ 0.4 & 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 & 0.1 \end{bmatrix}$$

$$\text{and} \qquad R_2 = \begin{bmatrix} 0.4 & 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.7 & 0.1 & 0.2 \\ 0.4 & 0.5 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.59 & 0.18 & 0.23 \end{bmatrix}$$

Thus, the chances that the price will increase, decrease or remain unchanged, day-after-tomorrow, are 59%, 18% and 23% respectively.

Example: Two businessmen are trading in shares have three banking company shares as shown in the following table

Merchant	Vijaya Bank	Canara Bank	Corporation Bank
Mr. Jain	200	100	300
Mr. Gupta	250	150	100

The approximate prices of (in ₹) three banking company shares in three stock exchange market are given below.

	Vijaya Bank	Canara Bank	Corporation Bank
Bangalore	39	40	38
Bombay	40	50	35
New Delhi	35	45	42

In which market each of the above businessmen has to sell their shares to get maximum receipt. Solve by matrix multiplication method.

Solution:

$$\text{Let } A = \begin{array}{c} \begin{bmatrix} \text{VB} & \text{CB} & \text{Cor.B} \\ 200 & 100 & 300 \\ 250 & 150 & 100 \end{bmatrix}_{2\times 3} \end{array} \begin{array}{c} \\ \text{Jain} \\ \text{Gupta} \end{array}$$

$$B = \begin{bmatrix} \text{Bang.} & \text{Bomb} & \text{N.Delhi} \\ 39 & 40 & 35 \\ 40 & 50 & 45 \\ 38 & 35 & 42 \end{bmatrix}_{3\times 3} \begin{array}{c} \\ VB \\ CB \\ Cor.\text{B} \end{array}$$

$$\therefore AB = \begin{bmatrix} 200 & 100 & 300 \\ 250 & 150 & 100 \end{bmatrix} \begin{bmatrix} 39 & 40 & 35 \\ 40 & 50 & 45 \\ 38 & 35 & 42 \end{bmatrix}$$

$$= \begin{bmatrix} 200\times 39 + 100\times 40 + 300\times 38 & 200\times 40 + 100\times 50 + 300\times 35 & 200\times 35 + 100\times 45 + 300\times 42 \\ 250\times 39 + 150\times 40 + 100\times 38 & 250\times 40 + 150\times 50 + 100\times 35 & 250\times 35 + 150\times 45 + 100\times 42 \end{bmatrix}$$

Notes

$$= \begin{bmatrix} 7800+4000+11400 & 8000+5000+10500 & 7000+4500+12600 \\ 9750+6000+3800 & 10000+7500+3500 & 8950+6750+4200 \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} \text{Bang.} & \text{Bomb.} & \text{ND} \\ 23200 & 23500 & 24100 \\ 19550 & 21000 & 19900 \end{bmatrix}_{2\times 3} \end{array} \begin{array}{l} \\ \text{Jain} \\ \text{Gupta} \end{array}$$

Jain has to sell his shares in New Delhi and Gupta has to sell his shares in Bombay to get maximum receipt.

Example: Keerthi buys 8 dozen of pens, 10 dozens of pencils and 4 dozen of rubber. Pens cost ₹ 18 per dozen, pencils ₹ 9 per dozen and rubber ₹ 6 per dozen. Represent the quantities bought by a row matrix and prices by a column matrix and hence obtain the total cost.

Solution:

Let A be the row matrix of quantities and B be the column matrix of prices.

$$\therefore\ A = \begin{bmatrix} 8 & 10 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$$

$$\therefore\ AB = \begin{bmatrix} 8 & 10 & 4 \end{bmatrix} \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix} = [144+90+24] = [258]$$

$\therefore$ Total cost is ₹ 258.

Example: Two oil merchants have the following stock of oil (in kg):

Merchant	Groundnut	Sunflower	Coconut
A	250	300	150
B	400	350	100

The approximate prices (in ₹ per kg) of three types of oil in 3 markets are:

Market	Groundnut	Sunflower	Coconut
X	70	50	150
Y	60	55	140
Z	55	60	132

In which market each of the above businessmen has to sell his stocks to get maximum receipt? Solve by matrix multiplication method.

Solution:

Notes

$$\text{Let } P = \begin{array}{cc} \begin{matrix} \text{G.N.} & \text{S.F.} & \text{C.N} \end{matrix} & \\ \begin{bmatrix} 250 & 300 & 150 \\ 400 & 350 & 100 \end{bmatrix} & \begin{matrix} A \\ B \end{matrix} \end{array}$$

$$Q = \begin{array}{cc} \begin{matrix} X & Y & Z \end{matrix} & \\ \begin{bmatrix} 70 & 60 & 55 \\ 50 & 55 & 60 \\ 150 & 140 & 132 \end{bmatrix} & \begin{matrix} GN \\ SF \\ CN \end{matrix} \end{array}$$

$$\therefore PQ = \begin{bmatrix} 250 & 300 & 150 \\ 400 & 350 & 100 \end{bmatrix} \begin{bmatrix} 70 & 60 & 55 \\ 50 & 55 & 60 \\ 150 & 140 & 132 \end{bmatrix}$$

$$\begin{bmatrix} 250 \times 70 + 300 \times 50 + 150 \times 150 & 250 \times 60 + 300 \times 55 + 150 \times 140 & 250 \times 55 + 30 \times 60 + 150 \times 132 \\ 400 \times 70 + 350 \times 50 + 100 \times 150 & 400 \times 60 + 350 \times 55 + 100 \times 140 & 400 \times 55 + 350 \times 60 + 100 \times 132 \end{bmatrix}$$

$$= \begin{bmatrix} 17500 + 15000 + 22500 & 15000 + 16500 + 21000 & 13750 + 18000 + 19800 \\ 28000 + 17500 + 15000 & 24000 + 19250 + 14000 & 22000 + 21000 + 13200 \end{bmatrix}$$

$$= \begin{array}{cc} \begin{matrix} X & Y & Z \end{matrix} & \\ \begin{bmatrix} 55000 & 52500 & 51550 \\ 60500 & 57250 & 56200 \end{bmatrix} & \begin{matrix} A \\ B \end{matrix} \end{array}$$

A has to sell his oil stock in market *X* and *B* also has to sell his oil stock in market *X* to get maximum receipt.

Example: At Bangalore merchant A has 300 bags of Rice, 600 bags of Wheat and 800 bags of Ragi and another merchant *B* has 250 bags, 700 bags and 1000 bags of same foodgrains. The prices (in ₹) at three cities are:

Place	Rice	Wheat	Ragi
Mysore	100	90	80
Mangalore	110	80	70
Kolar	120	70	80

To which city, each merchant will send his supply in order to get maximum gross receipts? Solve by matrix multiplication method.

Solution:

$$\text{Let } P = \begin{bmatrix} 300 & 600 & 800 \\ 250 & 700 & 1000 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$$

$$Q = \begin{bmatrix} 100 & 110 & 120 \\ 90 & 80 & 70 \\ 80 & 70 & 80 \end{bmatrix}$$

Notes

$$\therefore\ PQ = \begin{bmatrix} 300 & 600 & 800 \\ 250 & 700 & 1000 \end{bmatrix} \begin{bmatrix} 100 & 110 & 120 \\ 90 & 80 & 70 \\ 80 & 70 & 80 \end{bmatrix}$$

$$= \begin{bmatrix} 30000 + 54000 + 64000 & 33000 + 48000 + 56000 & 36000 + 42000 + 64000 \\ 25000 + 63000 + 80000 & 27500 + 56000 + 70000 & 30000 + 49000 + 80000 \end{bmatrix}$$

$$= \begin{array}{cc} \begin{array}{ccc} \text{Mysore} & \text{Mangalore} & \text{Kolar} \end{array} & \\ \begin{bmatrix} 148000 & 137000 & 142000 \\ 168000 & 153500 & 179000 \end{bmatrix} & \begin{array}{c} A \\ B \end{array} \end{array}$$

A has to send his supply in order to Mysore and B has to send his supply in order to Kolar to get maximum gross receipts.

Example: If 15 kgs of commodity A and 17 kgs of commodity B together costs ₹ 241 and 25 kgs of A and 13 kgs of B together costs ₹ 279; find the prices of each per kg by using Cramer's Rule method.

Solution:

Let prices of commodities A and B be ₹ x and ₹ y per kg respectively.

$\therefore\ 15x + 17y = 241$ and $25x + 13y = 279$

$$\Delta = \begin{vmatrix} 15 & 17 \\ 25 & 13 \end{vmatrix} = 195 - 425 = -230 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 241 & 17 \\ 279 & 13 \end{vmatrix} = 3133 - 4743 = -1610$$

$$\Delta_2 = \begin{vmatrix} 15 & 241 \\ 25 & 279 \end{vmatrix} = 4185 - 6025 = -1840$$

$$\therefore\ x = \frac{\Delta_1}{\Delta} = \frac{-1610}{-230} = 7$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-1840}{-230} = 8$$

$\therefore$ the price of commodity A is ₹ 7 per kg. and the price of commodity B is ₹ 8 per kg.

Example: The price of 2 kgs of Rice and 5 kgs of Wheat is ₹ 85 and price of 3 kgs of Rice and 8 kgs of Wheat is ₹ 132. Find the prices of Rice and Wheat using Cramer's Rule.

Solution:

Let the price of Rice be ₹ x per kg and the price of Wheat be ₹ y per kg.

$\therefore$ it is given that

$2x+5y=85$ and $3x+8y=132$

$$\Delta = \begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = 16-15=1$$

$$\Delta_1 = \begin{vmatrix} 85 & 5 \\ 132 & 7 \end{vmatrix} = 680-660=20$$

$$\Delta_2 = \begin{vmatrix} 2 & 85 \\ 3 & 132 \end{vmatrix} = 264-255=9$$

$$\therefore\ x=\frac{\Delta_1}{\Delta}=\frac{20}{1}=20$$

$$y=\frac{\Delta_2}{\Delta}=\frac{9}{1}=9$$

$\therefore$ The price of Rice is ₹ 20 per kg and the price of Wheat is ₹ 9 per kg.

4.11 Summary

- To every square matrix A, a real number is associated. This real number is called its determinant.
- It is denoted by $\Delta(A)$.
- The minor of an element of a square matrix A is defined to be the determinant obtained by deleting the row and column in which the element is present.
- The cofactor of an element of a square matrix is defined to be $(-1)^{i+j}\times$ (minor of the element) where i and j are the number of row and column in which the element is present.
- The adjoint of a square matrix A is the transpose of the matrix of the cofactors of the elements of A and is denoted by Adj. A.
- A square matrix A is said to be singular if $|A|=0$ and is said to be non-singular if $|A|\neq 0$.
- Inverse of a square matrix is defined if and only if it is non-singular. The inverse of a non-singular square matrix A is denoted by A^{-1}.
- In this unit we have studied the concepts of determinants and their importance in solving real world problems of business.
- A determinant is a scalar associated with a square matrix.

4.12 Keywords

Cofactor: A cofactor of an element a_{ij}, denoted by C_{ij}, is its minor with appropriate sign.

Determinant: A numeric value that indicate singularity or non-singularity of a square matrix.

Minor: A minor of an element a_{ij} denoted by M_{ij}, is a sub-determinant of $|A|$ obtained by deleting its i^{th} row and j^{th} column.

Notes

4.13 Self Assessment

1. Find the value of $\begin{vmatrix} -1 & 4 \\ -7 & 8 \end{vmatrix}$

   (a) 18

   (b) 20

   (c) 28

   (d) 24

2. Find the value of $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

   (a) $abc + 2fgh - af^2 - bg^2 - ch^2$

   (b) $abc + 2fgc - af^2 - bg^2 - ch^2$

   (c) $abc - 2fgh + af^2 - bg^2 - ch^2$

   (d) $abc + 2fgh + af^2 + bg^2 + ch^2$

3. Find the value of x if $\begin{vmatrix} 2 & 1 & 3 \\ 8 & 4 & x \\ -7 & 5 & 1 \end{vmatrix} = 0$

   (a) 18

   (b) 20

   (c) 12

   (d) 28

4. Find the value of x if $\begin{vmatrix} x & -3 & -3 \\ -3 & 0 & -3 \\ -3 & -3 & x \end{vmatrix} = 0$

   (a) 3

   (b) –3

   (c) –4

   (d) –5

5. Find the adjoint of inverse of $\begin{bmatrix} 2 & -3 & 5 \\ 5 & 2 & 7 \\ -4 & 3 & 1 \end{bmatrix}$

Notes

(a) $\frac{1}{92}\begin{bmatrix} 23 & 18 & 11 \\ 23 & 22 & 39 \\ 23 & 6 & 19 \end{bmatrix}$

(b) $\frac{1}{23}\begin{bmatrix} 92 & 6 & 11 \\ 23 & 22 & 39 \\ 23 & 18 & 19 \end{bmatrix}$

(c) $\frac{1}{92}\begin{bmatrix} 18 & 23 & 23 \\ 22 & 11 & 39 \\ 6 & 6 & 19 \end{bmatrix}$

(d) $\frac{1}{46}\begin{bmatrix} 21 & 18 & 11 \\ 23 & 22 & 39 \\ 23 & 6 & 19 \end{bmatrix}$

Fill in the blanks:

6. .... is a numeric value that indicate singularity or non-singularity of a square matrix.
7. A .... of an element a_{ij} denoted by m_{ij} is a subdeterminant of $|A|$ obtained by deleting its i^{th} row and j^{th} column.
8. A .... of an element a_{ij} denoted by c_{ij} is its minor with appropriate sign.
9. .... of a square matrix A is the transpose of the matrix of the cofactors of the element of A and is denoted by AdjA.
10. Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order ....

4.14 Review Questions

1. Find the adjoint and inverse of the following matrices and verify that $A\,(\text{Adj}\,A) = (\text{Adj}\,A)\,A = |A|\,I.$

   (a) $\begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ -3 & -1 & -2 \end{bmatrix}$

2. Using Cramer's rule, solve the following equations:

   (a) $3x - y + 2z = 13$
   $2x + y - z = 3$
   $x + 3y - 5z = -8$

   (b) $x + 2y + z = -1$
   $x + y + 2z = 3$
   $3x + 2y + 3z = 5$

   (c) $x + 3y + 2z = 5$
   $2x + y + z = 3$
   $5x + 2y + 3z = 6$

   (d) $x + y + 2z = 9$
   $3x + 2y + z = 10$
   $x + 2y + 3z = 14$

Notes

(e) $2x + y - z = 2$
$x - 2y + z = 5$
$x + y + 2z = 3$

3. The following table gives the price per share of two companies A and B during the months of March and April and it also gives the amount in rupees invested by Rakesh during these two months for the purchase of shares of the two companies.

Months	**Company and value per share**		**Total amount invested**
	A	**B**	
March	12	5	116
April	10	9	116

Find the shares of A and B.

4. The cost of 5 kg of Rice, 2 kg of Sugar and 5 kg of Wheat is ₹ 23. The cost of 4 kg of Rice, 4 kg of Sugar and 2 kg of Wheat is ₹ 19. The cost of 3 kg of Rice, 2 kg of Sugar and 4 kg of Wheat is ₹ 18. Find the rate per kg of each of these commodities.

5. There are two families A and B. There are 2 men, 3 women and 1 child in family A and 1 man, 1 woman and 2 children in family B. The recommended daily allowance for calories is:

	Calories	**Proteins**
Men	2400	55 gms
Women	1900	45 gms
Children	1800	33 gms

Represent the above information in the matrix form and calculate the total requirement of calories and proteins for each of the two families.

6. The cost of 2 kg of Wheat and 1 kg of Sugar is ₹ 7. The cost of 1 kg of Wheat and 1 kg of Rice is ₹ 7. The cost of 3 kg of Wheat, 2 kg of Sugar and 1 kg of Rice is ₹ 17. Find the cost of each per kg.

7. Find the value of the following determinants:

(a) $\begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 5 \\ 1 & 3 & 2 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 0 & 0 \\ 4 & 5 & -1 \\ 5 & 6 & 3 \end{vmatrix}$

Answers: Self Assessment

1. (b)
2. (a)
3. (c)
4. (b)
5. (a)
6. Determinant

Notes

7. Minor
8. Cofactor
9. Adjoint
10. n – 1

4.15 Further Readings

Books

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Unit 5: Equations of Straight Lines

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Objectives

After studying this unit, you will be able to:

- Explain the distance between two points
- Discuss the slope of a line
- Understand the various forms of equation of line

Notes

Introduction

In this unit we find the equation of a straight line, when we are given some information about the line. Straight-line equations, or "linear" equations, graph as straight lines, and have simple variable expressions with no exponents on them.. The information could be the value of its gradient, together with the co-ordinates of a point on the line. Alternatively, the information might be the co-ordinates of two different points on the line. There are several different ways of expressing the final equation, and some are more general than others. In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature. If you see an equation with only x and y as opposed to, say x^2 or $sqrt(y)$ - then you're dealing with a straight-line equation.

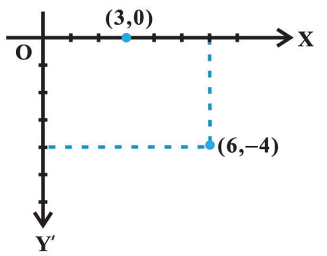
5.1 Distance between Two Points

As we know that coordinates are the pairs of numbers that defining the position of a point on a two-dimensional plane. Given the coordinates of two points, the distance D between the points is given by:

$$D = \sqrt{dx^2 + dy^2}$$

where dx is the difference between the x-coordinates of the points and dy is the difference between the y-coordinates of the points. To review, the location of the points (6, - 4) and (3, 0) in the XY-plane is shown in Figure 5.1. We may note that the point (6, - 4) is at 6 units distance from the y-axis measured along the positive x-axis and at 4 units distance from the x-axis measured along the negative y-axis. Similarly, the point (3, 0) is at 3 units distance from the y-axis measured along the positive x-axis and has zero distance from the x-axis. We also studied there following important formulae.

Figure 5.1



1. Distance between the points P (x_1, y_1) and Q (x_2, y_2) is

   $$D = \sqrt{dx^2 + dy^2}$$

   For example, distance between the points (6, – 4) and (3, 0) is

   $$\sqrt{(3-6)^2 + (0+4)^2} = \sqrt{9+16} = 5 \text{ units.}$$

2. The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y) internally, in the ratio $m: n$ are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$.

   For example, the coordinates of the point which divides the line segment joining A (1, –3) and B (–3, 9) internally, in the ratio 1: 3 are given by $x = \frac{1.(-3) + 3.1}{1+3} = 0$ and $y = \frac{1.9 + 3.(-3)}{1+3} = 0$.

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3. In particular, if $m = n$, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

4. Area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

   $\frac{1}{2}|x_1\ y_2\ y_3\ x_2\ y_3\ y_1\ x_3\ y_1\ y_2|$.

   For example, the area of the triangle, whose vertices are (4, 4), (3, – 2) and (– 3, 16) is

   $\frac{1}{2}|4(-2-16)+3(16-4)+(-3)(4+2)| = \frac{|-54|}{2} = 27.$

Notes If the area of the triangle ABC is zero, then three points A, B and C lie on a line, i.e., they are collinear.

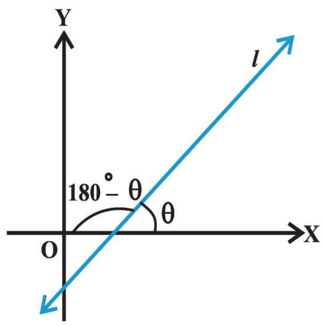
In the this unit, we shall continue the study of coordinate geometry to study properties of the simplest geometric figure – *straight line.* Despite its simplicity, the line is a vital concept of geometry and enters into our daily experiences in numerous interesting and useful ways. Main focus is on representing the line algebraically, for which *slope* is most essential.

5.2 Slope of a Line

As you are already familiar with coordinate geometry. A line in a coordinate plane forms two angles with the x-axis, which are supplementary. The slope of a line is a number that measures its "steepness", usually denoted by the letter m. It is the change in y for a unit change in x along the line. The angle (say) θ made by the line with positive direction of x-axis and measured anti-clock-wise is called the *inclination of the line*. Obviously 0° ≤ θ ≤ 180° (Figure 5.2). If a line passes through two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, its slope is given by: $m = (y_2 - y_1) / (x_2 - x_1)$ with x_2 not equal to x_1

We observe that lines parallel to x-axis, or coinciding with x-axis, have inclination of 0°. The inclination of a vertical line (parallel to or coinciding with y-axis) is 90°.

Figure 5.2



Definition 1: If θ is the inclination of a line l, then tan θ is called the *slope* or *gradient* of the line l.

The slope of a line whose inclination is 90° is not defined.

The slope of a line is denoted by m.

Thus, $m = \tan\theta, \theta \neq 90°$

It may be observed that the slope of x-axis is zero and slope of y-axis is not defined.

Notes

5.2.1 Slope of a Line when Coordinates of any Two Points on the Line are given

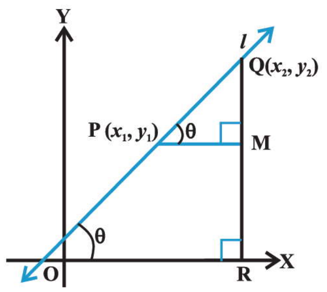
We know that a line is completely determined when we are given two points on it. Hence, we proceed to find the slope of a line in terms of the coordinates of two points on the line.

The slope of a line (also called the gradient of a line) is a number that describes how "steep" it is. If the line slopes downwards to the right, it has a negative slope. As x increases, y decreases. If the line sloped upwards to the right, the slope would be a positive number. Adjust the points above to create a positive slope. The slope of a line can positive, negative, zero or undefined.

Let P(x_1, y_1) and Q(x_2, y_2) be two points on non-vertical line $l$whose inclination is θ. Obviously, $x_1 \neq x_2$, otherwise the line will become perpendicular to x-axis and its slope will not be defined. The inclination of the line l may be acute or obtuse. Let us take these two cases.

Draw perpendicular QR to x-axis and PM perpendicular to RQ as shown in Figures 5.3 (i) and (ii).

Figure 5.3 (i)



Case I: When angle θ is acute:

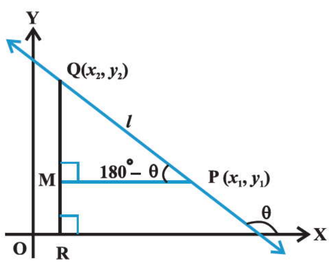
In Figure 5.3 (i), ∠MPQ = θ.

Therefore, slope of line l = m = tan θ. …(1)

But in ΔMPQ, we have $\tan\theta = \dfrac{MQ}{MP} = \dfrac{y_2 - y_1}{x_2 - x_1}$. …(2)

From equations (1) and (2), we have $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Figure 5.3 (ii)



Case II: When angle θ is obtuse: In Figure 5.3 (ii), we have ∠MPQ = 180° – θ.

Therefore, θ = 180° – ∠MPQ.

Notes

Now, slope of the line l

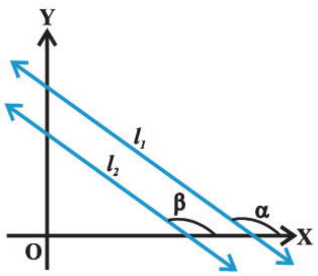
$$m = \tan\theta$$
$$= \tan(180° - \angle MPQ) = -\tan\angle MPQ$$
$$= -\frac{MQ}{MP} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Consequently, we see that in both the cases the slope m of the line through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

5.2.2 Conditions for Parallelism and Perpendicularity of Lines in terms of their Slopes

In a coordinate plane, suppose that non-vertical *lines* l_1 and l_2 have slopes m_1 *and* m_2, respectively. Let their inclinations be α and β, respectively.

Figure 5.4



If the line l_1 is parallel to l_2 (Figure 5.4), then their inclinations are equal, i.e.,

$$\alpha = \beta, \text{ and hence, } \tan\alpha = \tan\beta$$

Therefore $\quad m = m_{2g}$, i.e., their slopes are equal.

Conversely, if the slope of two lines l_1 and l_2 is same, i.e.,

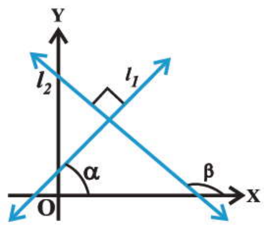
$$m_1 = m_2.$$

Then $\quad \tan\alpha = \tan\beta.$

By the property of tangent function (between 0° and 180°), α = β. Therefore, the lines are parallel.

Hence, two non-vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

Figure 5.5



If the lines l_1 and l_2 are perpendicular (Figure 5.5), then β = α + 90°.

Therefore, $\quad \tan\beta = \tan(\alpha + 90°)$

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$$= -\cot\alpha = \frac{1}{\tan\alpha}$$

i.e., $\quad m_2 = -\frac{1}{m_1} \quad$ or $\quad m_1, m_2 = -1$

Conversely, if $m_1 m_2 = -1$, i.e., $\tan\alpha \tan\beta = -1$.

Then $\tan\alpha = -\cot\beta = \tan(\beta + 90°)$ or $\tan(\beta - 90°)$ Therefore, α and β differ by 90°.

Thus, lines l_1 and l_2 are perpendicular to each other.

Hence, *two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other,*

i.e., $\quad m = -\frac{1}{m_1} \quad$ or $\quad m_1, m_2 = -1.$

Let us consider the following example.

Example: Find the slope of the lines:

1. Passing through the points (3, – 2) and (–1, 4),
2. Passing through the points (3, – 2) and (7, – 2),
3. Passing through the points (3, – 2) and (3, 4),
4. Making inclination of 60° with the positive direction of x-axis.

Solution:

1. The slope of the line through (3, – 2) and (– 1, 4) is

$$m = \frac{4-(-2)}{-1-3} = \frac{6}{-4} = -\frac{3}{2}.$$

2. The slope of the line through the points (3, – 2) and (7, – 2) is

$$m = \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0.$$

3. The slope of the line through the points (3, – 2) and (3, 4) is

$$m = \frac{4-(-2)}{3-3} = \frac{6}{0},\ \text{which is not defined.}$$

4. Here inclination of the line $\alpha = 60°$. Therefore, slope of the line is $m = \tan 60° = \sqrt{3}$.

5.2.3 Angle between Two Lines

Suppose you think about more than one line in a plane, then you find that these lines are either intersecting or parallel. Here we will discuss the angle between two lines in terms of their slopes.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2, respectively. If α_1 and α_2 are the inclinations of lines L_1 and L_2, respectively. Then

$$m_1 = \tan\alpha_1 \text{ and } m_2 = \tan\alpha_2.$$

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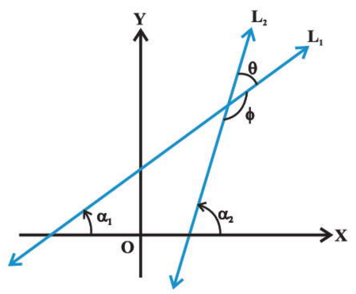
You are know that when two lines intersect each other, they make two pairs of vertically opposite angles such that sum of any two adjacent angles is 180°. Let θ and ϕ be the adjacent angles between the lines L_1 and L_2 (Figure 5.6). Then

$$\theta = \alpha_2 - \alpha_1 \text{ and } \alpha_1, \alpha_2 \neq 90°.$$

Therefore, $\tan\theta = \tan(\alpha_2 - \alpha_1) = \dfrac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2} = \dfrac{m_2 - m_1}{1+m_1m_2}$ (as $1 + m_1m_2 \neq 0$) and $\phi = 180° - \theta$ so that

$$\tan\phi = \tan(180° - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1+m_1m_2}, \text{as } 1+m_1m_2 \neq 0.$$

Figure 5.6



Now, there arise two cases:

Case I: If $\dfrac{m_2 - m_1}{1+m_1m_2}$ is positive, then $\tan\theta$ will be positive and $\tan\phi$ will be negative, which means θ will be acute and ϕ will be obtuse.

Case II: If $\dfrac{m_2 - m_1}{1+m_1m_2}$ is negative, then $\tan\theta$ will be negative and $\tan\phi$ will be positive, which means that θ will be obtuse and ϕ will be acute.

Thus, the acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2, respectively, is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1+m_1m_2}\right|, \text{ as } 1+m_1m_2 \neq 0 \qquad \ldots(1)$$

The obtuse angle (say ϕ) can be found by using $\phi = 180^0 - \theta$.

Example: If the angle between two lines is $\dfrac{\pi}{4}$ and slope of one of the lines is $\dfrac{1}{2}$, find the slope of the other line.

Solution:

We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1+m_1m_2}\right| \qquad \ldots (1)$$

Let $m_1 = \frac{1}{2}, m_2 = m$ and $\theta = \frac{\pi}{4}$.

Now, putting these values in (1), we get

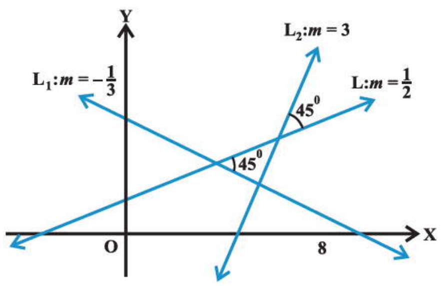
$$\tan\frac{\pi}{4} = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| \text{ or } 1 = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|,$$

which gives $$\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} = 1 \text{ or } \frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} = -1.$$

Therefore $$m = 3 \text{ or } m = -\frac{1}{3}.$$

Hence, slope of the other line is 3 or $\frac{1}{3}$. Figure 5.7 explains the reason of two answers.

Figure 5.7



Example: Line through the points (–2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (x, 24). Find the value of x.

Solution:

Slope of the line through the points (– 2, 6) and (4, 8) is $m_1 = \frac{8-6}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$

Slope of the line through the points (8, 12) and (x, 24) is $m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$

Since two lines are perpendicular, $m_1 m_2 = -1$, which gives $\frac{1}{3} \times \frac{12}{x-8} = -1$ or $x = 4$.

5.2.4 Collinearity of Three Points

You know that slopes of two parallel lines are equal. If two lines having the same slope pass through a common point, then two lines will coincide. Hence, if A, B and C are three points in

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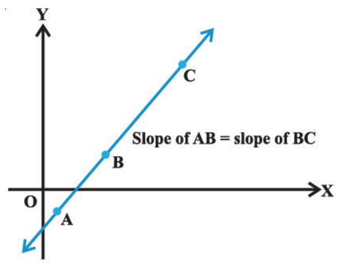
the XY-plane, then they will lie on a line, i.e., three points are collinear (Figure 5.8) if and only if slope of AB = slope of BC.

Figure 5.8



Example: Three points P (h, k), Q (x_1, y_1) and R (x_2, y_2) lie on a line. Show that $(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1)$.

Solution:

Since points P, Q and R are collinear, we have

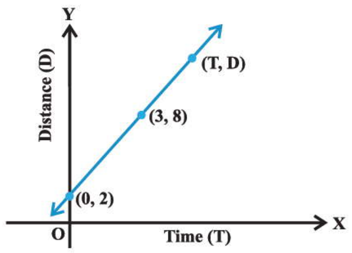
Slope of PQ = Slope of QR, i.e., $\dfrac{y_1 - k}{x_1 - h} = \dfrac{y_2 - y_1}{x_2 - x_1}$

or $\dfrac{k - y_1}{h - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$,

or $(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1)$.

Example: In Figure 5.9, time and distance graph of a linear motion is given. Two positions of time and distance are recorded as, when T = 0, D = 2 and when T = 3, D = 8. Using the concept of slope, find law of motion, i.e., how distance depends upon time.

Figure 5.9



Solution:

Let (T, D) be any point on the line, where D denotes the distance at time T. Therefore, points (0, 2), (3, 8) and (T, D) are collinear so that

$$\frac{8-2}{3-0} = \frac{D-8}{T-3} \quad \text{or} \quad 6(T-3) = 3(D-8)$$

or $D = 2(T + 1)$,

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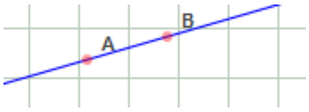
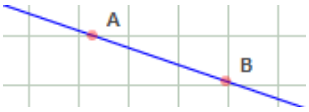
Positive Slope



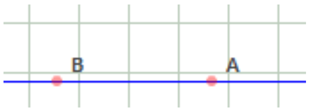
Here, y *increases* as x increases, so the line slopes upwards to the right. The slope will be a positive number. The line on the right has a slope of about +0.3, it goes *up* about 0.3 for every step of 1 along the x-axis.

Negative Slope



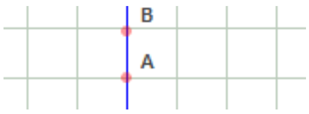
Here, y *decreases* as x increases, so the line slopes downwards to the right. The slope will be a negative number. The line on the right has a slope of about -0.3, it goes *down* about 0.3 for every step of 1 along the x-axis.

Zero Slope



Here, y *does not change* as x increases, so the line in exactly horizontal. The slope of any horizontal line is always zero. The line on the right goes neither up nor down as x increases, so its slope is zero.

Undefined Slope



When the line is exactly vertical, it does not have a defined slope. The two x coordinates are the same, so the difference is zero. The slope calculation is then something like slope

$= \frac{21}{0}$

When you divide anything by zero the result has no meaning. The line above is exactly vertical, so it has no defined slope. We say "the slope of the line AB is undefined". A vertical line has an equation of the form $\boldsymbol{x = a}$, where $\boldsymbol{a}$ is the x-intercept. For more on this see Slope of a vertical line.

Tasks

1. Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4).

2. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P (0, – 4) and B (8, 0).

Notes

5.3 Various Forms of the Equation of a Line

You know that every line in a plane contains infinite poinjts on it.

The general equation of a line can be reduced into various forms of the equation of a line. In all forms, slope is represented by m, the x-intercept by a, and the y-intercept by b. The Following are the different forms of the equation of a line.

- Slope-intercept form
- Intercept form
- Normal form

Notes The standard form coefficients A, B, and C have no particular graphical significance.

As we all know that you can find the equation of the line If two points on the line are given and If one point on the line and the slope is given.

5.3.1 Horizontal and Vertical Lines

The general equation of straight line is given by: Ax + By = C

a - If we set A to zero in the general equation, we obtain an equation in y only of the form

$$By = C$$

which gives y = C/B = k; k is a constant. This is a horizontal line with slope 0 and passes through all points with y coordinate equal to k.

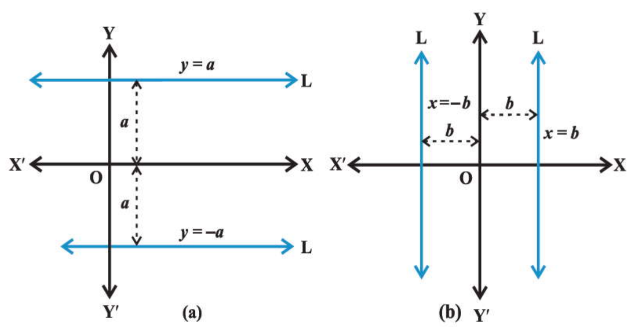
b - If we set B to zero in the general equation, we obtain

$$Ax = C$$

which gives x = C/A = h; h is constant. This is a vertical line with undefined slope and passes through all points with x coordinate equal to h

If a horizontal line L is at a distance a from the x-axis then ordinate of every point lying on the line is either a or $-a$ [Figure 5.10 (a)]. Therefore, equation of the line L is either $y = a$ or $y = -a$. Choice of sign will depend upon the position of the line according as the line is above or below the y-axis. Similarly, the equation of a vertical line at a distance b from the y-axis is either $x = b$ or $x = -b$ [Figure 5.10(b)].

Figure 5.10



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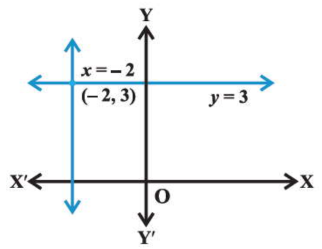
Example: Find the equations of the lines parallel to axes and passing through (– 2, 3).

Solution:

Position of the lines is shown in the Figure 5.11. The y-coordinate of every point on the line parallel to x-axis is 3, therefore, equation of the line parallel to x-axis and passing through (– 2, 3) is $y = 3$. Similarly, equation of the line parallel to y-axis and passing through (– 2, 3) is $x = -2$.

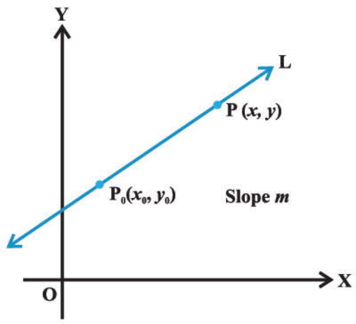
Figure 5.11



5.3.2 Point-slope Form

Assume that P_0 (x_0, y_0) is a fixed point on a non-vertical line L, whose slope is m. Let P (x, y) be an arbitrary point on L (Figure 5.12).

Figure 5.12



Then, by the definition on equation of line through apoint p with the slope of L is given by

$$m = \frac{y - y_0}{x - x_0}, \text{ i.e., } y - y_0 = m(x - x_0) \qquad \ldots(1)$$

Since the point P_0 (x_0, y_0) along with all points (x, y) on L satisfies (1) and no other point in the plane satisfies (1). Equation (1) is indeed the equation for the given line L.

Thus, the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), if and only if, its coordinates satisfy the equation

$$y - y_0 = m(x - x_0)$$

Example: Find the equation of the line through (– 2, 3) with slope – 4.

Solution:

Here $m = -4$ and given point (x_0, y_0) is (– 2, 3).

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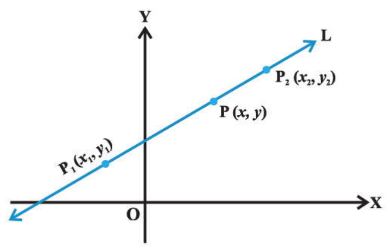
By slope-intercept form formula (1) above, equation of the given line is $y - 3 = -4(x + 2)$ or $4x + y + 5 = 0$, which is the required equation.

5.3.3 Two-point Form

Let the line L passes through two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Let $P(x, y)$ be a general point on L (Figure 5.13).

Figure 5.13



The three points P_1, P_2 and P are collinear, therefore, we have slope of P_1P = slope of P_1P_2

i.e., $\dfrac{y-y_1}{x-x_1} = \dfrac{y_2-y_1}{x_2-x_1}$, or $y - y_1 = \dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$.

Thus, equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \ldots(2)$$

Example: Write the equation of the line through the points (1, –1) and (3, 5).

Solution:

Here $x_1 = 1$, $y_1 = -1$, $x_2 = 3$ and $y_2 = 5$. Using two-point form (2) above for the equation of the line, we have

$$y - (-1) = \frac{5-(-1)}{3-1}(x-1)$$

or $-3x + y + 4 = 0$, which is the required equation.

5.3.4 Slope-intercept Form

The equation of a line with a defined slope m can also be written as follows: $y = mx + b$

where m is the slope of the line and b is the y intercept of the graph of the line.

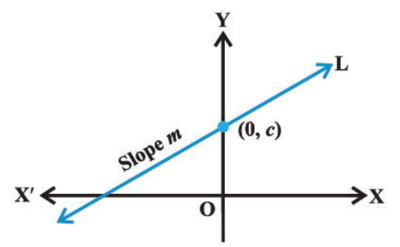
The above form is called the slope intercept form of a line. Sometimes a line is known to us with its slope and an intercept on one of the axes. Then you have to find equations of such lines.

Case I: Suppose a line L with slope m cuts the y-axis at a distance c from the origin (Figure 5.14). The distance c is called the *y-intercept* of the line L. Obviously, coordinates of the point where the line meet the y-axis are $(0, c)$. Thus, L has slope m and passes through a fixed point $(0, c)$. Therefore, by point-slope form, the equation of L is

$y = c + m(x, 0)$ or $y = mx + c$

Notes

Figure 5.14



Thus, the point (x, y) on the line with slope m and y-intercept c lies on the line if and only if

$$y = mx + c \qquad \ldots(3)$$

Note that the value of c will be positive or negative according as the intercept is made on the positive or negative side of the y-axis, respectively.

Case II: Suppose line L with slope m makes x-intercept d. Then equation of L is

$$y = m(x - d) \qquad \ldots(4)$$

Students may derive this equation themselves by the same method as in Case I.

Example: Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and (i) y-intercept is $-\frac{3}{2}$ (ii) x-intercept is 4.

Solution:

(i) Here, slope of the line is $m = \tan\theta = \frac{1}{2}$ and y - intercept $c = -\frac{3}{2}$.

Therefore, by slope-intercept form (3) above, the equation of the line is

$$y = \frac{1}{2}x - \frac{3}{2} \text{ or } 2y - x + 3 = 0,$$

which is the required equation.

(ii) Here, we have $m = \tan\theta = \frac{1}{2}$ and $d = 4$.

Therefore, by slope-intercept form (4) above, the equation of the line is

$$y = \frac{1}{2}(x - 4) \text{ or } 2y - x + 4 = 0,$$

which is the required equation.

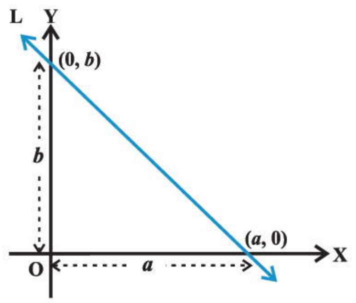
5.3.5 Intercept - Form

Suppose a line L makes x-intercept a and y-intercept b on the axes, and L meets x-axis at the point $(a, 0)$ and y-axis at the point $(0,b)$ (Figure 5.15). By two-point form of the equation of the line, we have $y - 0 = \frac{b-0}{0-a}(x - a)$ or $ay = -bx + ab$,

Notes

i.e., $$\frac{x}{a} + \frac{y}{b} = 1.$$

Figure 5.15



Thus, equation of the line making intercepts a and b on x and y-axis, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{... (5)}$$

Example: Find the equation of the line, which makes intercepts –3 and 2 on the x and y-axis respectively.

Solution:

Here $a = -3$ and $b = 2$. By intercept form (5) above, equation of the line is

$$\frac{x}{-3} + \frac{y}{2} = 1 \text{ or } 2x - 3y + 6 = 0.$$

5.3.6 Normal Form

The equation of a straight line upon which the length of perpendicular from the origin is p and the perpendicular makes an angle with the positive direction of x-axis is given by

x cos? + y sin? = p

Notes In normal form of equation of a straight line p is always taken as positive and a is measured from positive direction of x-axis in anticlockwise direction between 0 and 2n.

Let a non-vertical line is known to us with following data:

(i) Length of the perpendicular (normal) from origin to the line.

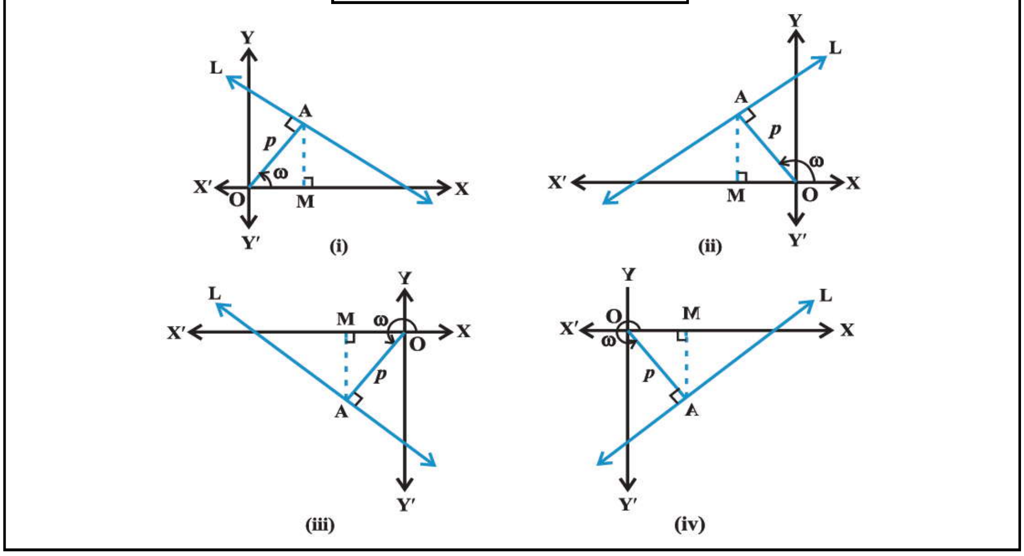
(ii) Angle which normal makes with the positive direction of x-axis.

Let L be the line, whose perpendicular distance from origin O be OA = p and the angle between the positive x-axis and OA be $\angle XOA = \omega$. The possible positions of line L in the Cartesian plane are shown in the Figure 5.16. Now, our purpose is to find slope of L and a point on it. Draw perpendicular AM on the x-axis in each case.

In each case, we have OM = $p \cos \omega$ and MA = $p \sin \omega$, so that the coordinates of the point A are $(p \cos \omega, p \sin \omega)$.

Notes

Figure 5.16



Further, line L is perpendicular to OA. Therefore,

The slope of the line $L = -\dfrac{1}{\text{slope of OA}} = -\dfrac{1}{\tan\omega} = -\dfrac{\cos\omega}{\sin\omega}$.

Thus, the line L has slope $-\dfrac{\cos\omega}{\sin\omega}$ and point A ($p\cos\omega$, $p\sin\omega$)on it. Therefore, by point-slope form, the equation of the line L is

$$y - p\sin\omega = -\frac{\cos\omega}{\sin\omega}(x - p\cos\omega) \quad \text{or} \quad x\cos\omega + y\sin\omega = p\left(\sin^2\omega + \cos^2\omega\right)$$

or $$x\cos\omega + y\sin\omega = p.$$

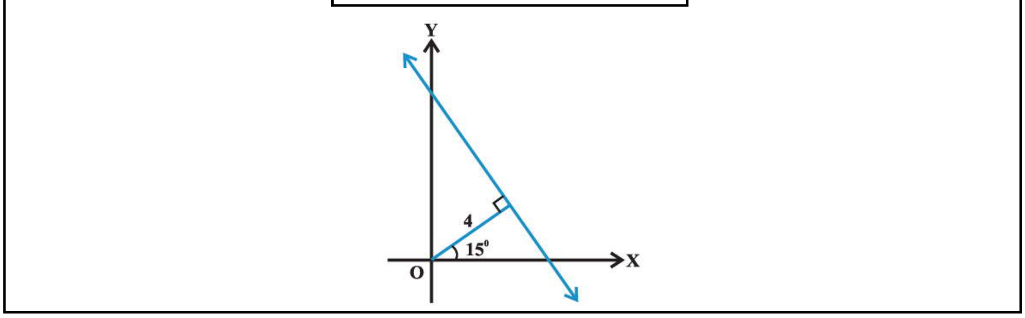
Hence, the equation of the line having normal distance p from the origin and angle ω which the normal makes with the positive direction of x-axis is given by

$$x\cos\omega + y\sin\omega = p \qquad \text{... (6)}$$

Example: Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x-axis is 15°.

Solution: Here, we are given p = 4 and ω = 15° (Figure 5.17)

Figure 5.17



Notes

Now $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

and $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (Why?)

By the normal form (6) above, the equation of the line is

$$x\cos 15° + y\sin 15° = 4 \text{ or } \frac{\sqrt{3}+1}{2\sqrt{2}}x + \frac{\sqrt{3}-1}{2\sqrt{2}}y = 4 \text{ or } \left(\sqrt{3}+1\right)x + \left(\sqrt{3}-1\right)y = 8\sqrt{2}.$$

This is the required equation.

Example: The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that K = 273 when F = 32 and that K = 373 when F = 212. Express K in terms of F and find the value of F, when K = 0.

Solution:

Assuming F along x-axis and K along y-axis, we have two points (32, 273) and (212, 373) in XY-plane. By two-point form, the point (F, K) satisfies the equation

$$K - 273 = \frac{373-273}{212-32}(F-32) \text{ or } K - 273 = \frac{100}{180}(F-32)$$

or
$$K = \frac{5}{9}(F-32) + 273 \qquad \ldots(1)$$

which is the required relation.

When K = 0, Equation (1) gives

$$0 = \frac{5}{9}(F-32) + 273 \text{ or } F - 32 = -\frac{273\times 9}{5} = -491.4 \text{ or } F = -459.4$$

Alternate method: We know that simplest form of the equation of a line is $y = mx + c$. Again assuming F along *x-axis* and K along *y-axis*, we can take equation in the form

$$K = mF + c \qquad \ldots (1)$$

Equation (1) is satisfied by (32, 273) and (212, 373). Therefore

$$273 = 32m + c \qquad \ldots (2)$$

and
$$373 = 212m + c \qquad \ldots (3)$$

Solving (2) and (3), we get

$$m = \frac{5}{9} \text{ and } c = \frac{2297}{9}.$$

Putting the values of m and c in (1), we get

$$K = \frac{5}{9}F + \frac{2297}{9} \qquad \ldots (4)$$

which is the required relation. When K = 0, (4) gives F = – 459.4.

Notes We know, that the equation $y = mx + c$, contains two constants, namely, m and c. For finding these two constants, we need two conditions satisfied by the equation of line. In all the examples above, we are given two conditions to determine the equation of the line.

5.4 General Equation of a Line

As unit, you have studied general equation of first degree in two variables, Ax + By + C = 0, where A, B and C are real constants such that A and B are not zero simultaneously. Graph of the equation Ax + By + C = 0 is always a straight line.

Therefore, any equation of the form Ax + By + C = 0, where A and B are not zero simultaneously is called *general linear equation* or *general equation of a line*.

Different Forms of Ax + By + C = 0

The general equation of a line can be reduced into various forms of the equation of a line, by the following procedures:

Slope-intercept Form

If B ≠ 0, then Ax + By + C = 0 can be written as

$$y = -\frac{A}{B}x - \frac{C}{B} \text{ or } y = mx + c \quad \text{... (1)}$$

where $$m = -\frac{A}{B} \text{ and } c = -\frac{C}{B}.$$

We know that Equation (1) is the slope-intercept form of the equation of a line whose slope is $-\frac{A}{B}$, and y-intercept is $-\frac{C}{B}$.

If B = 0, then $x = -\frac{C}{A}$, which is a vertical line whose slope is undefined and x-intercept is $-\frac{C}{A}$.

Intercept Form

If C ≠ 0, then Ax + By + C = 0 can be written as:

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1 \text{ or } \frac{x}{a} + \frac{y}{b} = 1 \quad \text{...(2)}$$

where $$a = -\frac{C}{A} \text{ and } b = -\frac{C}{B}.$$

We know that equation (2) is intercept form of the equation of a line whose x-intercept is $-\frac{C}{A}$ and y-intercept is $-\frac{C}{B}$.

Notes

If C = 0, then Ax + By + C = 0 can be written as Ax + By = 0, which is a line passing through the origin and, therefore, has zero intercepts on the axes.

Normal Form

Let $x \cos \omega + y \sin \omega = p$ be the normal form of the line represented by the equation Ax + By + C = 0 or Ax + By = – C. Thus, both the equations are:

same and therefore, $$\frac{A}{\cos\omega} + \frac{B}{\sin\omega} = \frac{-C}{P}$$

which gives $$\cos \omega = \frac{Ap}{C} \text{ and } \sin\omega = \frac{Bp}{C}.$$

Now $$\sin^2 \omega + \cos^2 \omega = \left(-\frac{Ap}{C}\right)^2 + \left(-\frac{Bp}{C}\right)^2 = 1$$

or $$p^2 = \frac{C^2}{A^2 + B^2} \text{ or } p = \frac{C}{\sqrt{A^2 + B^2}}$$

Therefore, $$\cos \omega = \pm\frac{A}{\sqrt{A^2 + B^2}} \text{ and } \sin\omega = \pm\frac{B}{\sqrt{A^2 + B^2}}.$$

Thus, the normal form of the equation Ax + By + C = 0 is

$$x \cos \omega + y \sin \omega = p,$$

where $\cos\omega = \pm\frac{A}{\sqrt{A^2 + B^2}}$, $\sin\omega = \pm\frac{B}{\sqrt{A^2 + B^2}}$ and $p = \pm\frac{C}{\sqrt{A^2 + B^2}}$.

Proper choice of signs is made so that p should be positive.

Example: Equation of a line is $3x - 4y + 10 = 0$. Find its (i) slope, (ii) x - and y-intercepts.

Solution:

(i) Given equation $3x - 4y + 10 = 0$ can be written as

$$y = \frac{3}{4}x + \frac{5}{2} \quad \text{... (1)}$$

Comparing (1) with $y = mx + c$, we have slope of the given line as $m = \frac{3}{4}$.

(ii) Equation $3x - 4y + 10 = 0$ can be written as

$$3x - 4y + 10 \text{ or } \frac{x}{-\frac{10}{3}} + \frac{y}{\frac{5}{2}} = 1 \quad \text{...(2)}$$

Comparing (2) with $\frac{x}{a} + \frac{y}{b} = 1$, we have x-intercept as $a = -\frac{10}{3}$ and y-intercept as $b = \frac{5}{2}$.

Notes

Example: Reduce the equation $\sqrt{3}x + y - 8 = 0$ into normal form. Find the values of p and ω.

Solution:

Given equation is:

$$\sqrt{3}x + y - 8 = 0 \quad \text{... (1)}$$

Dividing (1) by $\sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2$, we get

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4 \quad \text{or} \quad \cos 30° \, x + \sin 30° \, y = 4 \quad \text{...(2)}$$

Comparing (2) with $x \cos \omega + y \sin \omega = p$, we get $p = 4$ and $\omega = 30°$.

Example: Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.

Solution:

Given lines are

$$y - \sqrt{3}x - 5 = 0 \text{ or } y = \sqrt{3}x + 5 \quad \text{...(1)}$$

and

$$\sqrt{3}y - x + 6 = 0 \text{ or } y = \frac{1}{\sqrt{3}}x - 2\sqrt{3} \quad \text{...(2)}$$

Slope of line (1) is $m_1 = \sqrt{3}$ and slope of line (2) is $m_2 = \frac{1}{\sqrt{3}}$.

The acute angle (say) θ between two lines is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \quad \text{... (3)}$$

Putting the values of m_1 and m_2 in (3), we get

$$\tan\theta = \left|\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \sqrt{3}\frac{1}{\sqrt{3}}}\right| = \left|\frac{1-3}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$$

which gives $\theta = 30°$. Hence, angle between two lines is either 30°or 180° – 30° = 150°.

Example: Show that two lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$, where $b_1, b_2 \neq 0$ are:

(i) Parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, and

(ii) Perpendicular if $a_1 a_2 + b_1 b_2 = 0$.

Notes

Solution:

Given lines can be written as

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1} \qquad \ldots(1)$$

and

$$y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2} \qquad \ldots(2)$$

Slopes of the lines (1) and (2) are $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$, respectively. Now

(i) Lines are parallel, if $m_1 = m_2$, which gives

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2} \text{ or } \frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

(ii) Lines are perpendicular, if $m_1.m_2 = -1$, which gives

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = 1 \text{ or } a_1b_2 + b_1b_2 = 0$$

Example: Find the equation of a line perpendicular to the line $x - 2y + 3 = 0$ and passing through the point (1, – 2).

Solution:

Given line $x - 2y + 3 = 0$ can be written as

$$y = \frac{1}{2}x + \frac{3}{2} \qquad ...(1)$$

Slope of the line (1) is $m_1 = \frac{1}{2}$. Therefore, slope of the line perpendicular to line (1) is

$$m_2 = -\frac{1}{m_1} = -2$$

Equation of the line with slope – 2 and passing through the point (1, – 2) is

$$y - (-2) = -2(x - 1) \quad \text{or} \quad y = -2x,$$

which is the required equation.

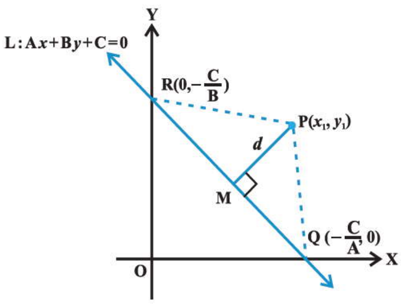
5.5 Distance of a Point From a Line

The distance of a point from a line is the length of the perpendicular drawn from the point to the line. Let L : Ax + By + C = 0 be a line, whose distance from the point P (x_1, y_1) is d. Draw a perpendicular PM from the point P to the line L (Figure 5.18). If the lines meets the x-and y-axes at the points Q and R, respectively. Then, coordinates of the points are $Q\left(-\frac{C}{A}, 0\right)$ and $R\left(0, -\frac{C}{B}\right)$.

Thus, the area of the triangle PQR is given by

Notes

Figure 5.18



$$\text{area }(\Delta PQR) = \frac{1}{2}PM.OR, \text{ which gives } PM = \frac{2\text{ area}(\Delta PQR)}{OR} \quad \text{... (1)}$$

Also,

$$\text{area }(\Delta PQR) = \frac{1}{2}\left|x_1\left(0+\frac{C}{B}\right)+\left(-\frac{C}{A}\right)\left(-\frac{C}{A}-y_1\right)+0(y_1-0)\right|$$

$$= \frac{1}{2}\left|x_1\frac{C}{B}+y_1\frac{C}{A}+\frac{C^2}{AB}\right| \quad \text{... (2)}$$

or

$$\text{area }(\Delta PQR) = \left|\frac{C}{AB}\right|\cdot\left|Ax_1+By_1+C\right|, \text{ and}$$

$$OR = \sqrt{\left(0+\frac{C}{A}\right)^2+\left(\frac{C}{B}-0\right)^2} = \left|\frac{C}{AB}\right|\sqrt{A^2+B^2}$$

Substituting the values of area (ΔPQR) and QR in (1), we get

$$PM = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$

or

$$d = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}.$$

Thus, the perpendicular distance (d) of a line Ax + By + C = 0 from a point (x_1, y_1) is given by

$$d = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}.$$

5.5.1 Distance between Two Parallel Lines

As you have already studied that slopes of two parallel lines are equal.

Therefore, two parallel lines can be taken in the form

$$y = mx + c_1 \quad \text{... (1)}$$

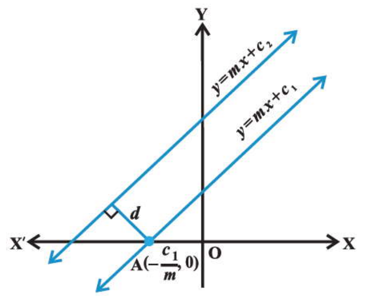
and

$$y = mx + c_2 \quad \text{... (2)}$$

Line (1) will intersect x-axis at the point $A\left(-\frac{c_1}{m},0\right)$ as shown in Figure 5.19.

Notes

Figure 5.19



Distance between two lines is equal to the length of the perpendicular from point A to line (2). Therefore, distance between the lines (1) and (2) is

$$\frac{\left|(-m)\left(-\frac{c_1}{m}\right)+(-c_2)\right|}{\sqrt{1+m^2}} \quad \text{or} \quad d=\frac{|c_1-c_2|}{\sqrt{1+m^2}}.$$

Thus, the distance d between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$$d = \frac{|c_1-c_2|}{\sqrt{1+m^2}}.$$

If lines are given in general form, i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$, then above formula will take the form $d=\frac{|C_1-C_2|}{\sqrt{A^2+B^2}}$

Students can derive it themselves.

Example: Find the distance of the point (3, – 5) from the line $3x - 4y - 26 = 0$.

Solution:

Given line is $\quad 3x - 4y - 26 = 0 \quad$ ... (1)

Comparing (1) with general equation of line $Ax + By + C = 0$, we get A = 3, B = – 4 and C = – 26.

Given point is $(x_1, y_1) = (3, -5)$. The distance of the given point from given line is

$$d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}=\frac{|3.3+(-4)(-5)-26|}{\sqrt{3^2+(-4)^2}}=\frac{3}{5}.$$

Example: Find the distance between the parallel lines $3x - 4y + 7 = 0$ and $3x - 4y + 5 = 0$

Solution:

Here A = 3, B = –4, $C_1 = 7$ and $C_2 = 5$. Therefore, the required distance is $d=\frac{|7-5|}{\sqrt{3^2+(-4)^2}}=\frac{2}{5}$.

Notes

Tasks

1. Reduce the following equations into intercept form and find their intercepts on the axes.

   (i) 3x + 2y – 12 = 0, (ii) 4x – 3y = 6, (iii) 3y +2 = 0.

2. Find the distance of the point (–1, 1) from the line 12(x + 6) = 5(y – 2).

Example: If the lines $2x + y - 3 = 0$, $5x + ky - 3 = 0$ and $3x - y - 2 = 0$ are concurrent, find the value of k.

Solution:

Three lines are said to be concurrent, if they pass through a common point, i.e., point of intersection of any two lines lies on the third line. Here given lines are

$$2x + y - 3 = 0 \quad \text{... (1)}$$

$$5x + ky - 3 = 0 \quad \text{... (2)}$$

$$3x - y - 2 = 0 \quad \text{... (3)}$$

Solving (1) and (3) by cross-multiplication method, we get

$$\frac{x}{-2-3} = \frac{y}{-9+4} = \frac{1}{-2-3} \quad \text{or} \quad x = 1, y = 1.$$

Therefore, the point of intersection of two lines is (1, 1). Since above three lines are concurrent, the point (1, 1) will satisfy equation (2) so that

$$5.1 + k\,.1 - 3 = 0 \text{ or } k = -2.$$

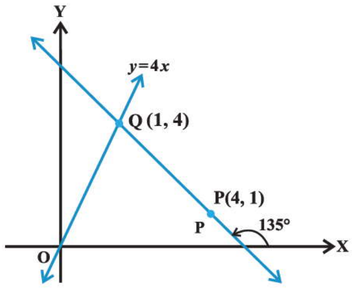
Example: Find the distance of the line $4x - y = 0$ from the point P (4, 1) measured along the line making an angle of 135° with the positive x-axis.

Solution:

Given line is $$4x - y = 0 \quad \text{... (1)}$$

In order to find the distance of the line (1) from the point P (4, 1) along another line, we have to find the point of intersection of both the lines. For this purpose, we will first find the equation of the second line (Figure 5.20). Slope of second line is tan 135° = –1. Equation of the line with slope – 1 through the point P (4, 1) is

Figure 5.20



$$y - 1 = -1\,(x - 4) \text{ or } x + y - 5 = 0 \quad \text{... (2)}$$

Notes

Solving (1) and (2), we get $x = 1$ and $y = 4$ so that point of intersection of the two lines is Q (1, 4).
Now, distance of line (1) from the point P (4, 1) along the line (2)

$$= \text{the distance between the points P (4, 1) and Q (1, 4).}$$

$$= \sqrt{(1-4)^2 + (4-1)^2} = 3\sqrt{2} \text{ units.}$$

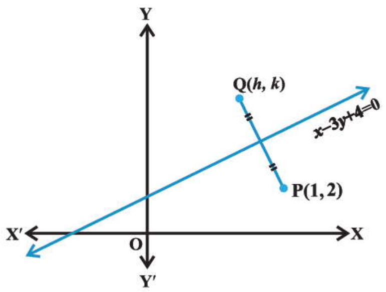
Example: Assuming that straight lines work as the plane mirror for a point, find the image of the point (1, 2) in the line $x - 3y + 4 = 0$.

Solution:

Let Q (h, k) is the image of the point P (1, 2) in the line

$$x - 3y + 4 = 0 \qquad \text{... (1)}$$

Figure 5.21



Therefore, the line (1) is the perpendicular bisector of line segment PQ (Figure 5.21).

Hence Slope of line $\text{PO} = \dfrac{-1}{\text{Slope of line } x - 3y + 4 = 0}$,

so that $$\frac{k-2}{h-1} = \frac{-1}{\frac{1}{3}} \quad \text{or} \quad 3h + k = 5 \qquad \text{... (2)}$$

and the mid-point of PQ, i.e., point $\left(\dfrac{h+1}{2}, \dfrac{k+2}{2}\right)$ will satisfy the equation (1) so that

$$\frac{h+1}{2} - 3\left(\frac{k+2}{2}\right) + 4 = 0 \quad \text{or} \quad \text{h} - 3\text{k} = -3 \qquad \text{...(3)}$$

Solving (2) and (3), we get $h = \dfrac{6}{5}$ and $k = \dfrac{7}{5}$.

Hence, the image of the point (1, 2) in the line (1) is $\left(\dfrac{6}{5}, \dfrac{7}{5}\right)$.

Example: Show that the area of the triangle formed by the lines

$$y = m_1 x + c_1, y = m_2 x + c_2 \text{ and } x = 0 \text{ is } \frac{(c_1 - c_2)^2}{2|m_1 - m_2|}.$$

Solution:

Notes

Given lines are

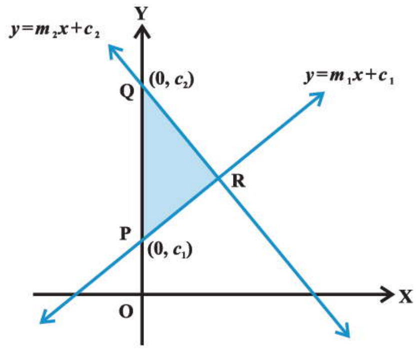
$$y = m_1 x + c_1 \quad \text{... (1)}$$

$$y = m_2 x + c_2 \quad \text{... (2)}$$

$$x = 0 \quad \text{... (3)}$$

We know that line $y = mx + c$ meets the line $x = 0$ (y-axis) at the point $(0, c)$. Therefore, two vertices of the triangle formed by lines (1) to (3) are P $(0, c_1)$ and Q $(0, c_2)$ (Figure 5. 22).

Figure 5.22



Third vertex can be obtained by solving equations (1) and (2). Solving (1) and (2), we get $x = \frac{(c_2 - c_1)}{(m_1 - m_2)}$ and $y = \frac{(m_1 c_2 - m_2 c_1)}{(m_1 - m_2)}$

Therefore, third vertex of the triangle is R $\left(\frac{(c_2 - c_1)}{(m_1 - m_2)}, \frac{(m_1 c_2 - m_2 c_1)}{(m_1 - m_2)}\right)$.

Now, the area of the triangle is:

$$= \frac{1}{2}\left|0\left(\frac{m_1 c_2 - m_2 c_1}{m_1 - m_2} - c_2\right) + \frac{c_2 - c_1}{m_1 - m_2}(c_2 - c_1) + 0\left(c_1 - \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2}\right)\right| = \frac{(c_2 - c_1)^2}{2|m_1 - m_2|}$$

Example: A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain its equation.

Solution:

Given lines are

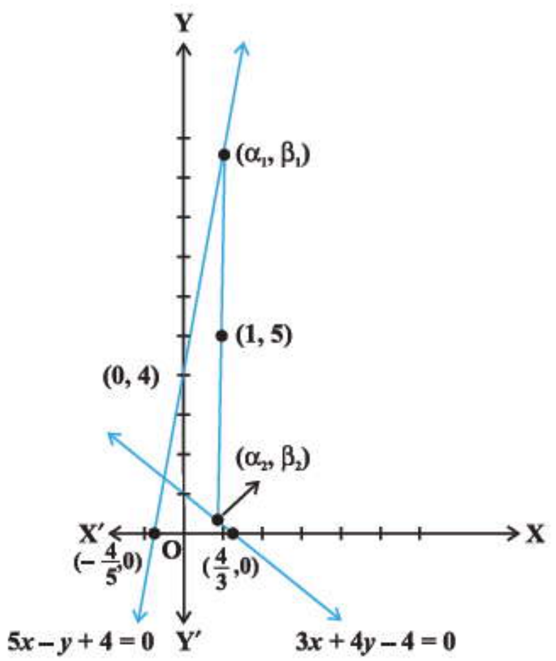
$$5x - y + 4 = 0 \quad \text{... (1)}$$

$$3x + 4y - 4 = 0 \quad \text{... (2)}$$

Let the required line intersects the lines (1) and (2) at the points, (α_1, β_1) and (α_2, β_2), respectively (Figure 5.23). Therefore $5\alpha_1 - \beta_1 + 4 = 0$ and $3\alpha_2 + 4\beta_2 - 4 = 0$

Notes

Figure 5.23



or $$\beta_1 = 5\alpha_1 + 4 \text{ and } \beta_2 = \frac{4 - 3\alpha_2}{4}.$$

We are given that the mid point of the segment of the required line between (α_1, β_1) and (α_2, β_2) is (1, 5). Therefore

$$\frac{\alpha_1 + \alpha_2}{2} = 1 \text{ and } \frac{\beta_1 + \beta_2}{2} = 5,$$

or $$\alpha_1 + \alpha_2 = 2 \text{ and } \frac{5\alpha_1 + 4 + \frac{4 - 3\alpha_2}{4}}{2} = 5,$$

or $$\alpha_1 + \alpha_2 = 2 \text{ and } 20\alpha_1 - 3\alpha_2 = 20 \qquad \text{... (3)}$$

Solving equations in (3) for α_1 and α_2, we get

$$\alpha_1 = \frac{26}{23} \text{ and } \alpha_2 = \frac{20}{23} \text{ and hence, } \beta_1 = 5.\frac{26}{23} + 4 = \frac{222}{23}.$$

Equation of the required line passing through (1, 5) and (α, β) is

$$y - 5 = \frac{\beta_1 - 5}{\alpha_1 - 1}(x - 1) \text{ or } y - 5\frac{\frac{222}{23} - 5}{\frac{26}{23} - 1}(x - 1)$$

or $$107x - 3y - 92 = 0,$$

which is the equation of required line.

Notes

Example: Show that the path of a moving point such that its distances from two lines $3x - 2y = 5$ and $3x + 2y = 5$ are equal is a straight line.

Solution:

Given lines are

$$3x - 2y = 5 \quad \ldots(1)$$

and

$$3x + 2y = 5 \quad \ldots(2)$$

Let (h, k) is any point, whose distances from the lines (1) and (2) are equal. Therefore

$$\frac{|3h-2k-5|}{\sqrt{9+4}} = \frac{|3h+2k-5|}{\sqrt{9+4}} \text{ or } |3h-2k-5| = |3h+2k-5|,$$

which gives $3h - 2k - 5 = 3h + 2k - 5$ or $-(3h - 2k - 5) = 3h + 2k - 5$.

Solving these two relations we get $k = 0$ or $h = \frac{5}{3}$. Thus, the point (h, k) satisfies the equations $y = 0$ or $x = \frac{5}{3}$, which represent straight lines. Hence, path of the point equidistant from the lines (1) and (2) is a straight line.

5.6 Summary

- *Slope* (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}, x_1 \neq x_2$.
- Slope of horizontal line is zero and slope of vertical line is undefined.
- An acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2 is given by $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, 1 + m_1 m_2 \neq 0$.
- Two lines are *parallel* if and only if their slopes are equal.
- Two lines are *perpendicular* if and only if product of their slopes is -1.
- Three points A, B and C are collinear, if and only if slope of AB = slope of BC.
- Equation of the horizontal line having distance a from the x-axis is either $y = a$ or $y = -a$.
- Equation of the vertical line having distance b from the y-axis is either $x = b$ or $x = -b$.
- The point (x, y) lies on the line with slope m and through the fixed point (x_0, y_0), if and only if its coordinates satisfy the equation $y - y_0 = m(x - x_0)$.
- Equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.
- The point (x, y) on the line with slope m and y-intercept c lies on the line if and only if $y = mx + c$.

Notes

- If a line with slope m makes x-intercept d. Then equation of the line is $y = m(x - d)$.
- Equation of a line making intercepts a and b on the x-and y-axis, respectively, is $\frac{x}{a} + \frac{y}{b} = 1$.
- The equation of the line having normal distance from origin p and angle between normal and the positive x-axis ω is given by $x \cos \omega + y \sin \omega = p$.
- Any equation of the form $Ax + By + C = 0$, with A and B are not zero, simultaneously, is called the *general linear equation* or *general equation of a line*.
- The perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.
- Distance between the parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$, is given by $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$.

5.7 Keywords

Condition of Parallelism: When two lines are parallel then their inclination are equal.

Condition of Prependicular: Two non-vertical lines are perpendicular to each other of and only of their slopes are negative reciprocals of each other.

Inclination of line: The angle made by a line with positive direction and measured in anti-clock -wise.

5.8 Self Assessment

Fill in the blanks:

1. A line in a coardinate plane forms two angles with x-axis, which are ………………
2. If θ is the inclination of a line than $\tan \theta$ is called the ……………… of line
3. Two non-vertical lines are parallel if and only if their slopes are ………………
4. If $\frac{m_2 - m_1}{1 + m_1 m_2}$ is positive, then $\tan \theta$ will be ………………
5. Two non-vertical lines are perpendicular to each of and only of their slopes are ……………… of each other.

Choose the Correct Answer:

6. Equation of line is equals to when point (–2, 3) & slope –4
   (a) 4x + y + 5
   (b) 2x + 4y – 5
   (c) 4x + 2y + 6
   (d) 4x + 2y + 3

7. Find the equation whose ⊥ distance from origin is 4 and angles is 15° in positive direction

   (a) x cos 15° + y sin 15° = 4

   (b) x sin 15 + y cos 15° = 4

   (c) x $\sin^{-1}$ 15 + y $\cos^{-1}$ 15° = 4

   (d) x $\sin^{-1}$ 15° + y $\cos^{-1}$ 15° = –4

8. Distance between two parallel lines 3x – 4y + 7 = 0

   (a) 2/3

   (b) 2/4

   (c) 2/5

   (d) 2/6

9. Equation of slope - Intercept form of line is

   (a) y = mx + c

   (b) y = m^2x + cx

   (c) y^2 = m^2x + c

   (d) y = m/2 x +c

10. Distance of the point (3, –5) from the line 3x – 4y – 26 = 0 is

   (a) 3/5

   (b) 4/3

   (c) 3/4

   (d) 5/3

5.9 Review Qustions

1. Find perpendicular distance from the origin of the line joining the points (cos θ, sin θ) and (cos ϕ, sin ϕ).

2. Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - k = 0$.

3. Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.

4. If three lines whose equations are $y = m_1 x + c_1$, $y = m_1 x + c_2$ and $y = m_3 x + c_3$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3 (c_1 - c_2) = 0$.

5. Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line $x - 2y = 3$.

6. Find the image of the point (3, 8) with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.

7. If sum of the perpendicular distances of a variable point P (x, y) from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10. Show that P must move on a line.

8. A ray of light passing through the point (1, 2) reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.

Notes

9. A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find equation of the path that he should follow.

Answers: Self Assessment

1.	Supplementary	2.	Slope
3.	Equal	4.	Positive
5.	Negative reciprocal	6.	(a)
7.	(a)	8.	(c)
9.	(a)	10.	(a)

5.10 Further Readings

Books

D.C. Sanchethi and V.K. Kapoor, Business Mathematics.

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Online links

http://www.tpub.com/math2/2.html

www.ehow.com

www.mathopenry.com

Unit 6: Functions

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define the function and its types
- Discuss weather a function is one–one, many–one, onto or into
- Analysis the graphical representation of functions
- Discuss the composition of two functions
- Explain the inverse of a function

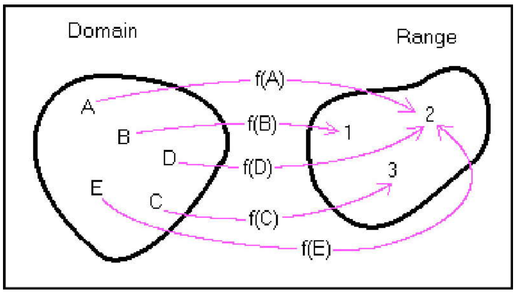
Notes

Introduction

Functions are mathematical ideas that take one or more variables and produce a variable. You can think of a function as a cook that takes one or more ingredients and cooks them up to make a dish. Depending on what you put in, you can get very different things out. Moreover, not all functions are the same. If you give one cook peanut butter, jelly, and bread, he may make a sandwich, whereas another cook may start to sculpt a volcano with the peanut butter, and use the jelly for lava after discarding the bread.

6.1 Functions

In an abstract mathematical sense, a function is a mapping of some domain onto some range. For each item in the domain, there is a corresponding item in the range of the function. Thus, the domain is all of the possible inputs to the function and the range is all of the possible outputs. Each item in the domain corresponds to a specific item in the range. However, an item in the range may correspond to multiple items in the domain.



For example, let's describe a function for album titles. Our function will take as its domain, album titles. Our function, let's call it FL (album title) will output the first letter of the first word in the title of the album. Thus, the range of our function will be all of the inputs.

For most of Algebra, functions are described as things that take a number and put out a number. In higher mathematics, this is described as $R^1 \longrightarrow R^1$. This means that the real number line (R^1) is being mapped to the real number line. If however, we have two inputs and one output, we have a function that is described as $R^2 \longrightarrow R^1$, or the real plane(R^2) is being mapped to the real number line. Generally, we can have a function described by any $R^N \longrightarrow R^M$.

Let's start with an old favorite–the line.

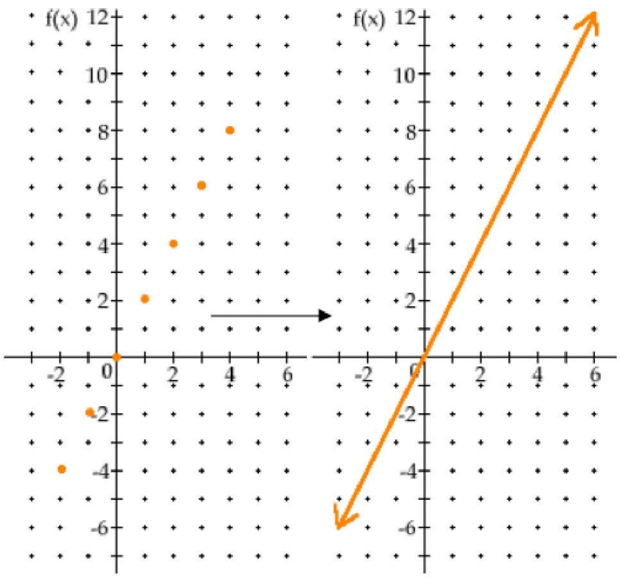
$$f(x) = 2*x$$

Here, f is a function that is defined to take one variable –x. It takes that one variable and doubles it. We can plot this graph on a Cartesian grid by taking x along one axis and f(x) along the other. Because f(x) is simply a constant, that is the number 2, multiplied by x, we know that f(x) is a line. Assuming that we are totally ignorant, let us proceed as though we know nothing at all. To draw a function that is new to us, here is what we normally will do (at least to begin with): We will construct a Table 6.1. In one column, we will list various values for x that we would like to try to see what comes out. In the other column, we will list the values of f that we get when we stuff our values into the function. Next, on a piece of grid paper, we will plot the points, going over on the x–axis to the number we chose for x, and on the y–axis to what we got out for f(x). Finally, we will connect the dots for a rough view of what our function looks like. (More complex functions need lots of dots!) For f(x) = 2*x, here's what we get:

Notes

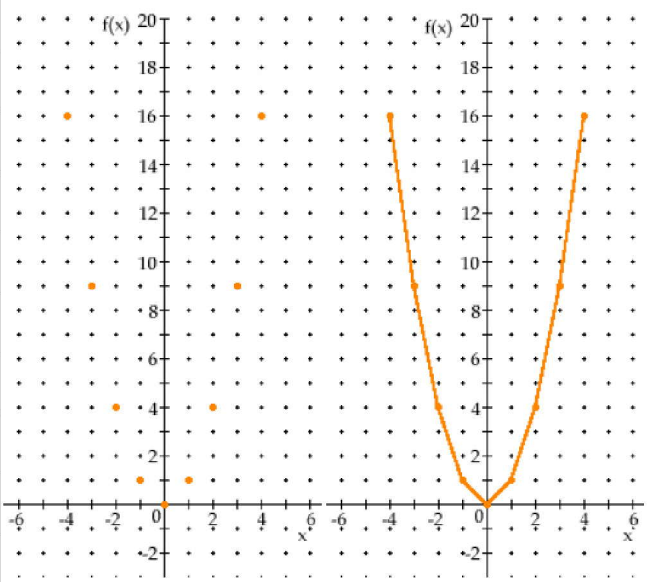
Table 6.1

x	f(x)
–2	–4
–1	–2
0	0
1	2
2	4
3	6
4	8

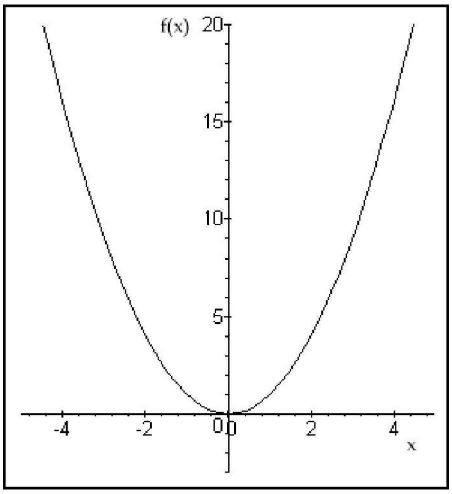


Let's move on to the parabola. A basic parabola formula is: $f(x) = x^2$. Let us try several values to plop into the function to see what comes out:

x	f(x)
–4	16
–3	9
–2	4
–1	1
0	0
1	1
2	4
3	9
4	16



Most of the time, functions come out with nice looking smooth curves. So, if instead of using straight lines to connect out dots, we use a smooth curve, we can get a better approximation of what the function looks like. Hence, the proper parabola looks like the following:



Here are some examples of our function at work.

The concept of a function is essential in mathematics. There are two common notations in use:

(a) $f(x) = x^2 + 2$,

(b) f : x 7!x^2 + 2 .

Part (a) is commonly used. Part (b) is interpreted as the function f maps x to $x^2 + 2$.

Example: If two functions are given as $f(x) = 2x + 3$, and $g(x) = 3 – x^2$, then

1. $f(2) = 2 \times 2 + 3 = 7$
2. $f(–3) = 2 \times (–3) + 3 = –6 + 3 = –3$
3. $g(0) = 3 – (0)^2 = 3$
4. $g(4) = 3 – (4)^2 = 3 – 16 = –13$

Example: Find the numbers which map to zero under the function

h : x 7! $x^2 – 9$.

Solution:

The function can also be written as $h(x) = x^2 – 9$ and if x maps to zero then $h(x) = 0$, i.e.

$$x^2 – 9 = 0$$

$$x^2 = 9$$

since squaring both 3 and 3 gives the value 9.

6.1.1 General Characteristics of a Function

Functions can be classified into different categories according to the nature of their definition or of symbolic expressions. To facilitate this, we first define the following general characteristics of a function.

1. ***Increasing or Decreasing Function***

   Let $y = f(x)$ be a function defined in an interval I and x_1, x_2 be two points of the interval such that $x_1 < x_2$.

Notes

If $f(x_2) \geq f(x_1)$ when $x_1 < x_2$, then $f(x)$ is *increasing*.

If $f(x_2) \leq f(x_1)$ when $x_1 < x_2$, then $f(x)$ is *decreasing*.

If $f(x_2) = f(x_1)$ for all values of x_1 and x_2 in I, then $f(x)$ is *constant*.

If, however, the strict inequality holds in the above statements, then $f(x)$ is strictly increasing (or decreasing) function.

2. ***Monotonic Function***

   A function $y = f(x)$ is said to be monotonic if y is either increasing or decreasing over its domain, as x increases.

   If the function is increasing (decreasing) over its domain, it is called *monotonically increasing (decreasing) function*.

   A monotonic function is also termed as a *one to one function*.

3. ***Implicit and Explicit Function***

   When a relationship between x and y is written as $y = f(x)$, it is said to be an explicit function. If the same relation is written as $F(x, y) = 0$, it is said to be an implicit function. Production possibility function or the transformation function is often expressed as an implicit function.

4. ***Inverse Function***

   If a function $y = f(x)$ is such that for each element of the range we can associate a unique element of the domain (i.e. one to one function), then the inverse of the function, denoted as $x = f^{-1}(y)\ g(y)$, is obtained by solving $y = f(x)$ for x in terms of y. The functions $f(x)$ and $g(y)$ are said to be inverse of each other and can be written as either $g[f(x)] = x$ or $f[g(y)] = y$. We note here that an implicit function $F(x, y) = 0$, can be expressed as two explicit functions that are inverse of each other.

5. ***Symmetry of a Function***

   Symmetry of a function is often helpful in sketching its graph. Following types of symmetry are often useful:

   (i) *Symmetry about y-axis*

   A function $y = f(x)$ is said to be symmetric about y-axis if $f(-x) = f(x)$ for all x in its domain. For example, the function $y = x^2$ is symmetric about y-axis. Such a function is also known as *even function*.

   Similarly, if $g(y) = g(-y)$, then the function $x = g(y)$ is said to be symmetric about x-axis.

   (ii) *Symmetry about the line $x = h$*

   A function $y = f(x)$ is said to be symmetric about the line $x = h$ if $f(h - k) = f(h + k)$ for all real value k.

   (iii) *Symmetry about origin*

   A function $y = f(x)$ is said to be symmetric about origin if $f(-x) = -f(x)$, for all values of x in its domain. For example, the function $y = x^2$ is symmetric about origin. Such a function is also known as *odd function*.

   (iv) *Symmetry about the line $y = x$ (45° line)*

   Two functions are said to be symmetrical about the line $y = x$ (45° line), if the interchange of x and y in one function gives the other function. This type of symmetry

Notes

implies that y as an explicit function of x is exactly of the same form as x as an explicit function of y.

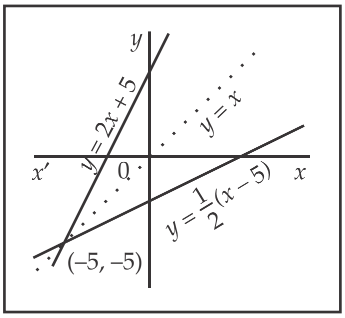
Notes:

(i) Two points with coordinates (a, b) and (b, a) are said to be reflections of one another (or symmetrical) about the line $y = x$.

(ii) Since the in verse function $x = g(y)$ is obtained simply by solving $y = f(x)$ for x, the graphs of these functions remain maltered. However, when we interchange the role of x and y in the function $x = g(y)$ and write as $y = g(x)$, the graph of $y = f(x)$ gets reflected about the $y = x$ line to get the graph of $y = g(x)$.

To illustrate this, we consider $y = f(x) = 2x + 5$ and $y = g(x) = \frac{1}{2}(x - 5)$. Note that (1, 7) is a point on the graph of $y = 2x + 5$ and (7, 1) is a point on the graph of $y = \frac{1}{2}(x - 5)$. The graphs of these functions are shown in Figure 6.1.

Figure 6.1



(iii) The point of intersection of the two functions, that are symmetric about the 45° line, occurs at this line.

(iv) An implicit function $F(x, y) = 0$ is said to be symmetric about the 45° line if an interchange of x and y leaves the function unchanged. For example, the function $xy = a$ is symmetric about the 45° line.

6. ***Composite Function***

If y is a function of u and u is a function of x, then y is said to be a composite function of x. For example, if $y = f(u)$ and $u = g(x)$, then $y = f[g(x)]$ is a composite function of x. A composite function can also be written as $y = (fog)(x)$, where fog is read as f of g.

The domain of $f\{g(x)\}$ is the set of all real numbers x in the domain of g for which $g(x)$ is in the domain of x.

Note: The rules for the sum, difference, product and quotient of the functions f and g are defined below:

$$(f \pm g)(x) = f(x) \pm g(x)$$

$$(fg)(x) = f(x)g(x)$$

Notes

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$$

The domain of each of the resultant function is given by the intersection of the domains of f and g. In the case of quotient, the value of x at which $g(x) = 0$ must be excluded from the domain.

Example

Examine whether the following functions are even or odd.

(a) $y = x^2$ (b) $y = \frac{1}{x^2}$ (c) $y = x^3$ (d) $y = \frac{1}{x}$

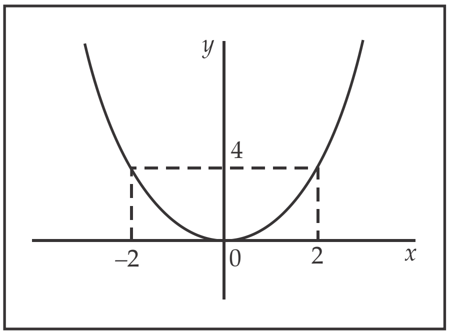
Draw the graph of each function.

Solution:

(a) Let $f(x) = x^2$, then $f(-x) = (x)^2 = x^2 = f(x)$ $\therefore$ $y = x^2$ is an even function. This function is symmetric about y-axis.

To draw graph, we note that when $x = 0$, then $y = 0$. Also y increases as x increases. The graph of the function is shown in Figure 6.2.

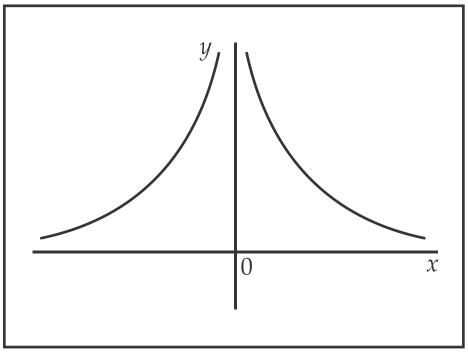
Figure 6.2



(b) Let $f(x) = \frac{1}{x^2}$ then $f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = f(x)$ $\therefore$ $y = \frac{1}{x^2}$ is an even function. This function is also symmetric about y-axis. When $x = 0$, the function is not defined. However, for small values (positive or negative) of x, y approaches ∞ and as x becomes larger and larger y becomes smaller and smaller, i.e. approaches zero, but is never equal to zero. Note that y is positive for all values of x i.e. the whole curve lies above x-axis. Based on the above features, we can draw a broad graph of the function as shown in Figure 6.3.

Notes

Figure 6.3



(c) Let $f(x) = x^2$, then $f(-x) = -x^3 = -f(x)$ $\therefore$ $y = x^3$ is an odd function. This function is symmetric about origin.

When $x = 0$, then $y = 0$, $\therefore$ the graph of the function passes through origin. Further, y is positive (negative) when x is positive (negative). Therefore the graph lies in I and III Quadrants. Note that the values of y increases as x increases. Thus, the function is monotonically increasing in its domain. Based on these features, the broad graph is shown in Figure 6.4.

Figure 6.4

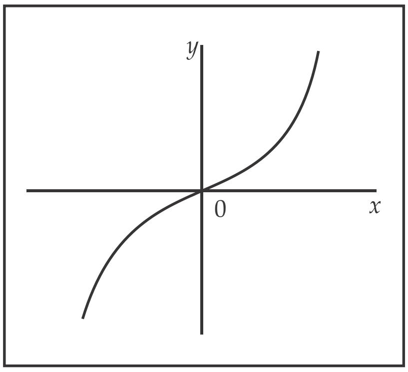
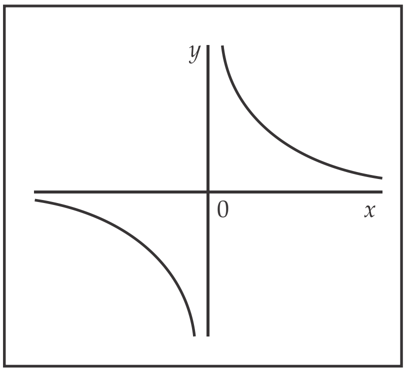


Figure 6.5



Notes

(d) Let $f(x) = \frac{1}{x}$, then $f(-x) = -\frac{1}{x} = -f(x)$ $\therefore$ $y = \frac{1}{x}$ is an odd function and symmetric about origin.

This, function is not defined at $x = 0$. Also the graph of this function lies in I and III Quadrants. When $x > 0$, then y approaches ∞ for small values of x and approaches zero as x approaches ∞.

Similarly, when $x < 0$, then y approaches $-\infty$ as x approaches zero and approaches zero as x approaches $-\infty$. The broad graph of the function is shown in Figure 6.5.

Example

Find inverse of the following functions and show that their graphs are symmetrical about the line $y = x$.

(a) $y = \frac{1}{3}x + 2$ (b) $y = x^2, x \geq 0$

Solution:

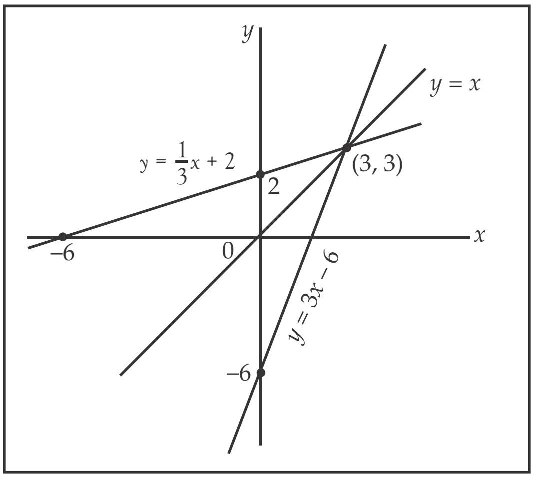
(a) To find inverse of the given function, we solve it for x.

$\therefore$ $x = 3(y - 2)$

To draw graph, we take independent variable on x-axis and dependent variable on y-axis, therefore we interchange x and y in the above equation to get $y = 3(x - 2) = 3x - 6$.

The graphs of the functions $y = \frac{1}{3}x + 2$ and $y = 3x - 6$ are shown in Figure 6.6. These are symmetric about the line $y = x$. Also note that their point of intersection (3, 3) also lies on the line.

Figure 6.6



(b) Solving the given function for x, we get $x = \sqrt{y} = y^{\frac{1}{2}}$, $x \geq 0$.

Notes

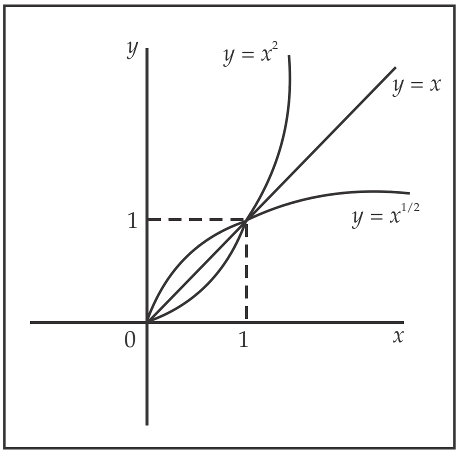
As before interchanging y and x, we can write $y = x^{\frac{1}{2}}$

To draw the graph of the two functions, we note the following points:

	Function	$y = x^2$		$y = x^{1/2}$	
i.	When $x = 0$	$y = 0$	the point lies on the line $y = x$	$y = 0$	the point lies on the line $y = x$
ii.	When $0 < x < 1$	$y < x$	graph lies below the line $y = x$	$y > x$	graph lies above the line $y = x$
iii.	When $x = 1$	$y = x$	the point lies on the line $y = x$	$y = x$	the point lies on the line $y = x$
iv.	When $x > 1$	$y > x$	graph lies above the line $y = x$	$y < x$	graph lies below the line $y = x$

Based on the above, the two graphs are shown in Figure 6.7. Note that if (a, b), (where a and b are + ve) is a point on $y = x^2$, then (b, a) is a point on $y = x^{1/2}$. Hence, the graphs of the two functions are symmetric about the line $y = x$.

Figure 6.7



Example

Show that the function $y = x^2 - 6x - 3$ is symmetric about the line $x = 3$. Draw a broad graph of the function. What is the domain and of the function?

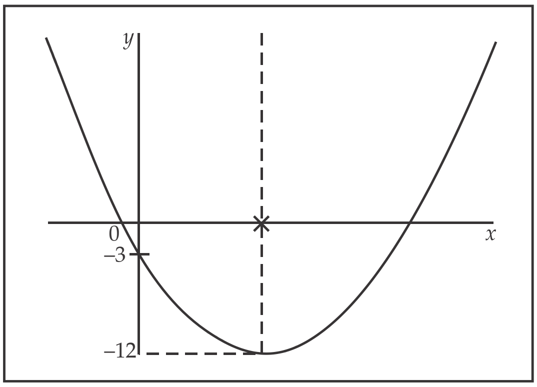
Solution:

A function $y = f(x)$ is symmetric about the line $x = 3$ if $f(3 + k) = f(3 - k)$ for all real values of k.

Now
$$\begin{aligned} f(3+k) &= (3+k)^2 - 6(3+k) - 3 \\ &= 9 + 6k + k^2 - 18 - 6k - 3 \\ &= k^2 - 12 \\ f(3-k) &= (3-k)^2 - 6(3-k) - 3 \\ &= 9 - 6k + k^2 - 18 + 6k - 3 \\ &= k^2 - 12. \end{aligned}$$

Notes

Figure 6.8



Thus, the function is symmetric about the line $x = 3$. To draw the graph, we note that when $x = 3$, then $y = 9 - 18 - 3 = -12 = f(k + 3) = f(k - 3)$, when $k = 0$. Also for large values of x, the behaviour of y is given by the behaviour of x^2 term.

As x approaches $\pm \infty$, y also approaches ∞. Further, (0, –3) is a point on the curve. Based on this information, the graph is shown in Figure 6.8. The domain, of the function is $(-\infty, \infty)$.

6.1.2 Types of Functions

These are names for functions of first, second and third order polynomial functions, respectively. What this means is that the highest order of x (the variable) in the function is 1, 2 or 3.

The generalized form for a linear function (1 is highest power):

$f(x) = ax + b$, where a and b are constants, and a is not equal to 0.

The generalized form for a quadratic function (2 is highest power):

$f(x) = ax^2 + bx + c$, where a, b and c are constants, and a is not equal to 0.

The generalized form for a cubic function (3 is highest power):

$f(x) = ax^3 + bx^2 + cx + d$,

where a, b, c and d are constants, and a is not equal to 0.

The roots of a function are defined as the points where the function $f(x) = 0$. For linear and quadratic functions, this is fairly straight-forward, but the formula for a cubic is quite complicated and higher powers get even more involved. We will see the derivation of the first two now will go over the derivation of the first two now.

A linear equation is very simple to solve for $f(x) = 0$:

$$\begin{aligned} 0 &= ax + b \\ -ax &= b \\ x &= b/(-a) = -b/a \end{aligned}$$

where, a not equal to 0

The equation for the root of a quadratic is only slightly more complex. The idea is to isolate x by putting the left side into the form $(x + q)^2$ and then taking the square root. We do this by some nifty algebra:

Notes

$ax^2 + bx + c = 0$	Try to get $(x+g)^2 = x^2 + (b/a)x + ??$
$x^2 + (b/a)x + c/a = 0$	
$[x^2 + (b/a)x + b^2/4a^2] - b^2/4a^2 + c/a = 0$	
$[x + (b/2a)]^2 - (b^2/4a^2 - c/a) = 0$	
$[x + (b/2a)]^2 = b^2/4a^2 - c/a$	$(x+(1/2)(b/a))^2 = x^2 + 2(1/2)(b/a)x + (1/4)(b^2/a^2)$
$x+(b/2a) = \pm\text{sqrt}(b^2/4a^2-4ac/4a^2)$	$(x+b/2a)^2 = x^2 + (b/a)$
$x = -b/2a \pm\text{sqrt}(b^2-4ac)/2a$	
$x+(b/2a) = \pm\text{sqrt}(b^2/4a^2-4ac/4a^2)$	
$x = (-b \pm\text{sqrt}(b^2-4ac))/2a$	

Even Function

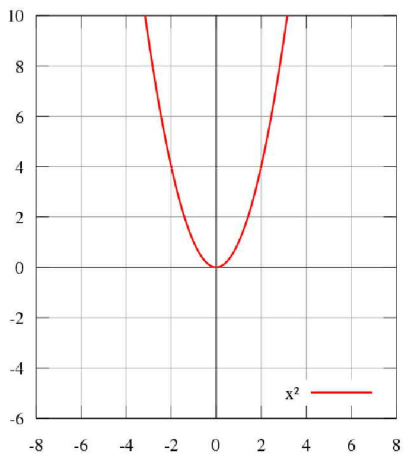
Let f(x) be a real-valued function of a real variable. Then f is even if the following equation holds for all x in the domain of f:

$$f(x) = f(-x)$$

Geometrically, the graph of an even function is symmetric with respect to the y-axis, meaning that its graph remains unchanged after reflection about the *y*-axis.

Examples of even functions are $|x|$, x^2, x^4, cos(x), and cosh(x).

Figure 6.9: Graph of Even Number



Odd Functions

Again, let f(x) is a real-valued function of a real variable. Then f is odd if the following equation holds for all x in the domain of f:

$$-f(x) = f(-x),$$

$$\text{or } f(x)+ f(-x) = 0$$

Geometrically, the graph of an odd function has rotational symmetry with respect to the origin, meaning that its graph remains unchanged after rotation of 180 degrees about the origin.

Examples of odd functions are x, x^3, sin(x), sinh(x), and erf(x).

Notes

6.1.3 Classification of Functions

Depending upon the nature of their symbolic expressions, various functions can be classified into the into different categories. A brief description of some common types of functions is given in the following sections.

Polynomial Functions

A function of the form $y = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where n is a positive integer and ab $a_n \neq 0$, is called a polynomial function of degree n.

(i) If $n = 0$, we have $y = a_0$, a *constant function.*

(ii) If $n = 1$, we have $y = a_0 + a_1x$, a *linear function*.1

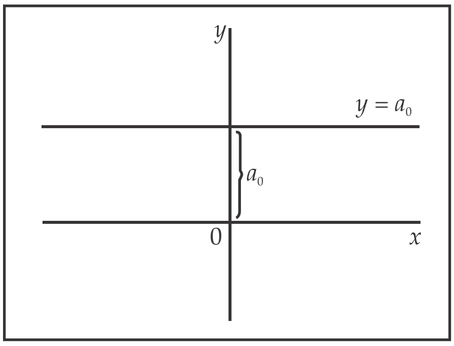
(iii) If $n = 2$, we have $y = a_0 + a_1x + a_2x^2$, a *quadratic or parabolic function*.

(iv) If $n = 3$, we have $y = a_0 + a_1x + a_2x^2 + a^3x^3$, a *cubic function* etc.

Constant Functions

A function of the form $y = f(x) = a_0$ for all real values of x, is a constant function. Graph of such a function is a horizontal straight line with equation $y = a_0$, as shown in Figure 6.10.

Figure 6.10

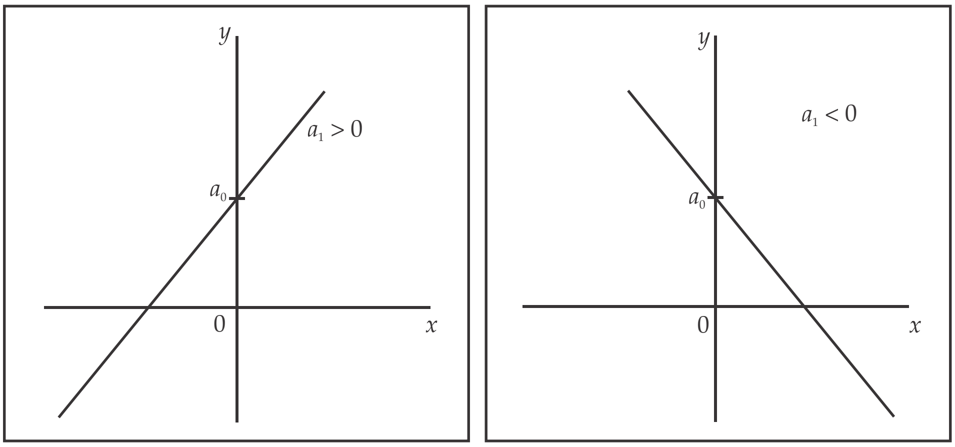


Linear Functions

$y = a_0 + a_1x(a_1 \neq 0)$ is a linear function. The graph of the linear function is a straight line. Here a_0 is the value of y when $x = 0$, known as the intercept of the line on y-axis and a_1 is the slope of the line. If $a_1 > 0$, the line slopes upward and when $a_1 < 0$, the line slopes downward, as shown in Figure 6.11 (a) and 6.11 (b) respectively.

Notes

Figure 6.11



(a) (b)

Alternatively, a linear function or the equation of a straight line can be written as:

(i) $y = mx + c$, where m is the slope and c is the intercept or

(ii) $y - y_1 = m(x - x_1)$ is the equation of a line passing through the point (x_1, y_1) with slope m, or

(iii) $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a line passing through two points (x_1, y_1) and (x_2, y_2).

Note: $\dfrac{y_2 - y_1}{x_2 - x_1}$ is the slope of the line.

Linear Models in Economics

Linear relations are very frequently used in economic analysis very often even when a relationship between economic variables is not linear, we use their linear approximation to comprehend it better. Some applications of linear relations are illustrated in the following examples.

Example

The population of a country was 80 crores in the year 2000 and it became 92 crores in the year 2008. Assuming that the population growth is linear;

(i) Find the relation between population and time.

(ii) Predict the population for the year 2010.

Solution:

(i) Assuming base year as 2000, we take $x = 0$ for the year 2000. Thus, $x = 8$ for 2008.

Let y denote population, thus

Notes

When $x = 0$, $y = 80$ and when $x = 8$, $y = 92$. The equation of a line passing through the two points is $y - 80 = \frac{92-80}{8}(x-0)$. Thus, $y - 80 = \frac{3}{2}x$ or $y = \frac{3}{2}x + 80.$

(ii) For the year 2010, $x = 10$.

$\therefore\ y = \frac{3}{2} \times 10 + 80 = 95$ crores will be the population in the year 2010.

Example

A cinema hall has 2000 seats and the revenue obtained from the sale of tickets in a particular show was ₹ 62000. The price of two categories of seats were ₹ 25 and ₹ 40. Assuming that all the seats were occupied, find the number of seats in each category.

Solution:

Let x be the number seats each with price ₹ 25, then $2000 - x$ will be the number of seats in the other category. Thus, we can write

$$25x + 40(2000 - x) = 62000$$

$$15x = 80000 - 62000$$

$$x = \frac{18000}{15} = 1200$$

$$2000 - x = 2000 - 1200 = 800$$

Thus, the number of seats are 1200 with price ₹ 25 and 800 with price ₹ 40.

Example

An old photocopying machine can copy 10,000 pages in 5 hours. With the help of a new machine, the job can be completed in 2 hours.

(i) How much time would the new machine require to do the job alone?

(ii) How many pages are copied by each machine when each machine is used for three hours?

Solution.

(i) Let x be the time needed for the new machine to complete the job.

Rate of work for the old machine $= \frac{1}{5}$ units of work per hour.

Rate of work for the new machine $= \frac{1}{x}$ units of work per hour

thus the work done by each macine in 2 hours is $\frac{1}{5} \times 2$ and $\frac{1}{x} \times 2$ respectively.

Notes

Hence, we can write

$$\frac{1}{5}\times 2+\frac{1}{x}\times 2 = 1$$

$$2x + 10 = 5x$$

$$\text{or } x = \frac{10}{3}=3\frac{1}{3} \text{ hours.}$$

Thus, the new machine will copy 10,000 papes in $3\frac{1}{3}$ hours.

(ii) Number of pages copied by old machine = $\frac{1}{5}\times 3\times 10000=6000.$

Number of pages copied by new machine = $\frac{3}{10}\times 3\times 10000=9000.$

Example

A retailer purchases two types of tea priced at ₹ 120 and ₹ 160 per kg. He wants to sell the mixture of these two types of tea at a price of ₹ 150 per kg. How much of each should be used to produce 200 kgs of mixture so that there is no change in his revenue?

Solution:

Let x be the quantity of ₹ 120 per kg. tea used then 200 - x will be the quantity of ₹ 160 per kg tea.

$$120x + 160(200 - x) = 150 \times 200$$

$$120x + 3200 - 160x = 30000$$

$$40x = 2000$$

$$x = 50$$

Thus, the retailer should use 50 kg of tea priced at ₹ 120 per kg and 150 kg of sugar priced at ₹ 160 kg.

Example

The total cost of manufacturing x units of a product is assumed to be linear. It consists of a fixed cost plus a variable cost. If the total cost of manufacturing 200 units is ₹ 5,000 and total cost of manufacturing 400 units is ₹ 8,000, find the cost function. What is the fixed cost of production?

Solution:

Let C be the total cost and x be the number of units manufactured. It is given that C = 5,000 when x = 200 and C = 8,000 when x = 400.

Since the cost function is assumed to be linear, therefore we have to find the equation of a straight line passing through the points (200, 5000) and (400, 8000).

Notes

Thus, we have $C - 5000 = \dfrac{8000-5000}{400-200}(x-200) = 15(x-200)$

or $C = 15x + 2000$, is the required cost function.

Further, the fixed cost is ₹ 2,000.

Example

The demand and supply of a commodity are given by $x_d = 81000 - 160p$ and $x_s = -4500 + 125p$, where x denotes quantity and p denotes price. Find the equilibrium price and quantity.

Solution:

We know that $x_d = x_s$ in equilibrium

$\therefore \quad 81000 - 160p = -4500 + 125p$

Thus, $285p = 85500$ or $\left(-\dfrac{\alpha}{\beta}\right) + t$ = ₹ 300

Also equilibrium quantity $x = 81000 - 160 \times 300 = 33000$ units.

Example

When price of a commodity is ₹ 30 per unit, its demand and supply are 600 and 900 units respectively. A price of ₹ 20 per unit changes the demand and supply to 1000 and 700 units respectively. Assuming that the demand and supply equations are linear, find

(i) The demand equation

(ii) The supply equation

(iii) The equilibrium price and quantity

Solution:

Note: *In both the situations of demand or supply, the price is an independent variable and the quantity a dependent variable. However, while plotting them, price is taken on vertical axis and quantity on the horizontal a*xis. This is an exception to the convention followed in most of the other topics of economics as well as in other branches of science, where the independent variable is taken along horizontal axis and the dependent variable along vertical axis.

(i) The demand equation is the equation of line passing through the points (600, 30) and (1000, 20). Thus, we can write

$$p - 30 = \frac{30-20}{600-1000}(x_d - 600)$$

On simplification, we get the demand equation as $x_d = 1800 - 40p$.

(ii) The supply equation is the equation of a line passing through the points (900, 30) and (700, 20). Thus, we can write

$$p - 30 = \frac{30-20}{900-700}(x_s - 900)$$

On simplification, we get the supply equation as $x_s = 300 + 20p$.

Notes

(iii) We have $x_s = x_d$ in equilibrium

$$\therefore \quad 300 + 20p = 1800 - 40p$$

$$\text{or} \quad p = \frac{1500}{60} = ₹\ 25 \text{ (equilibrium price)}$$

Further, equilibrium quantity is $x = 300 + 20 \times 25 = 800$ units.

Break-Even Point

Profit of a firm is given by the difference of its total revenue and total cost. Thus, profit $\pi = TR - TC$.

In general, when a firm starts the production of a commodity it operates at loss when its output is below a certain level, say x, because the total revenue is not large enough to cover fixed costs. However, as the level of output becomes greater than x, the firm starts getting profits. The level of output x is termed as the *break-even point*. Thus, break-even point is the lowest level of output at which the loss of the firm gets eliminated. It is given by the equation $TR - TC = 0$ (or $TR = TC$).

Example

A company decides to set up a small production plant for manufacturing electronic clocks. The cost for initial set up is ₹ 9 lakhs. The additional cost for producing each clock is ₹ 300. Each clock is sold at ₹ 750. During the first month, 1,500 clocks are produced and sold:

(i) Determine the total cost function $C(x)$ for the production of x clocks.

(ii) Determine the revenue function $R(x)$.

(iii) Determine the profit function $P(x)$.

(iv) How much profit or loss the company incurs during the first month when all the 1,500 clocks are sold?

(v) Determine the break-even point.

Solution:

(i) We are given $TFC = 9{,}00{,}000$ and $TVC(x) = 300x$

$\therefore$ Total cost function, $C(x) = 9{,}00{,}000 + 300x$

(ii) Total revenue function, $R(x) = p.x = 750x$

(iii) Profit function, $P(x) = TR - TC = 750x - 9{,}00{,}000 - 300x$

$= 450x - 9{,}00{,}000$

(iv) Profit when $x = 1{,}500$, is given as

$P(1{,}500) = 450 \times 1{,}500 - 9{,}00{,}000 = 6{,}75{,}000 - 9{,}00{,}000 = -2{,}25000$

Note that profit is negative. Thus, the company incurs a loss of ₹ 2,25,000 during first month.

(v) We know that $TR = TC$, at the break-even point

$$\therefore \quad 750x = 9{,}00{,}000 + 300x$$

$$\text{or} \quad 450x = 9{,}00{,}000 \text{ or } x = \frac{9{,}00{,}000}{450} = 2{,}000 \text{ clocks.}$$

Notes

Example

The total cost TC of producing x units of a commodity is given by $TC = 2000 + 4x$. If each unit is sold at ₹ 20 per unit, find the level of output to make sure that the production breaks-even.

Solution:

We can write total revenue of producing x units as $TR = 20x$.

$\therefore$ Profit p = $TR - TC = 20x - 2000 - 4x = 16x - 2000$

The break-even point is given by the level of output at which p = 0.

Thus $16x - 2000 = 0$ or $x = \frac{2000}{16} = 125$

Thus, at least 125 units should be produced to make sure that the firm does not incur losses.

6.1.4 Basic Properties

1. The only function which is both even and odd is the constant function which is identically zero (i.e., f(x) = 0 for all x).
2. The sum of an even and odd function is neither even nor odd, unless one of the functions is identically zero.
3. The sum of two even functions is even, and any constant multiple of an even function is even.
4. The sum of two odd functions is odd, and any constant multiple of an odd function is odd.
5. The product of two even functions is an even function.
6. The product of two odd functions is an even function.
7. The product of an even function and an odd function is an odd function.
8. The quotient of two even functions is an even function.
9. The quotient of two odd functions is an even function.
10. The quotient of an even function and an odd function is an odd function.
11. The derivative of an even function is odd.
12. The derivative of an odd function is even.
13. The composition of two even functions is even, and the composition of two odd functions is odd.
14. The composition of an even function and an odd function is even.
15. The composition of any function with an even function is even (but not vice-versa).
16. The integral of an odd function from A to +A is zero (where A is finite, and the function has no vertical asymptotes between A and A).
17. The integral of an even function from A to +A is twice the integral from 0 to +A (where A is finite, and the function has no vertical asymptotes between A and A).
18. **Series**

    (a) The Maclaurin series of an even function includes only even powers.

Notes

(b) The Maclaurin series of an odd function includes only odd powers.

(c) The Fourier series of a periodic even function includes only cosine terms.

(d) The Fourier series of a periodic odd function includes only sine terms.

6.2 Rational Function

Rational functions and the properties of their graphs such as domain, vertical and horizontal asymptotes, x and y intercepts are explored using an applet. The investigation of these functions is carried out by changing parameters included in the formula of the function. Each parameter can be changed continuously which allows a better understanding of the properties of the graphs of these functions.

6.2.1 Definition and Domain of Rational Functions

A rational function is defined as the quotient of two polynomial functions.

$$f(x) = P(x) / Q(x)$$

Here are some examples of rational functions:

$$g(x) = (x^2 + 1) / (x - 1)$$

$$h(x) = (2x + 1) / (x + 3)$$

The rational functions to explored are of the form

$$f(x) = (ax + b)/(cx + d)$$

where a, b, c and d are parameters that may be changed, using sliders, to understand their effects on the properties of the graphs of rational functions defined above.

6.2.2 Exponential Function

The function e^t.

The exponential function is denoted mathematically by e^t and in matlab by Exponent t. This function is the solution to the world's simplest, and perhaps most important, diferential equation,

$$y\_ = ky$$

This equation is the basis for any mathematical model describing the time evolution of a quantity with a rate of production that is proportional to the quantity itself.

Such models include populations, investments, feedback, and radioactivity. We are using t for the independent variable, y for the dependent variable, k for the proportionality constant, and

$$y = \frac{dy}{dt}$$

for the rate of growth, or derivative, with respect to t. We are looking for a function that is proportional to its own derivative.

Let's start by examining the function

$$y = 2^t$$

We know what 2^t means if t is an integer, 2^t is the t-th power of 2.

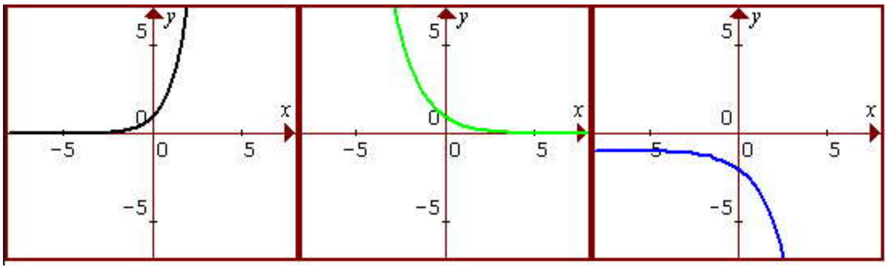
$$2^{-1} = 1/2;\ 2^0 = 1;\ 2^1 = 1;\ 2^2 = 4$$

Notes

We also know what 2^t means if t = p/q is a rational number, the ratio of two integers, $2^{p/q}$ is the q^{th} root of the p^{th} power of 2.

$$2^{1/2} = 2 = 1.4142$$

Figure 6.12: Graphs of Exponents



Exponential graphs share these common features:

1. The graph will level out on the far right or the far left to some horizontal asymptote.
2. The graph "takes off" vertically, but it does not approach a vertical asymptote.
3. Rather, it simply becomes steeper and steeper.
4. The graph will have a characteristic "L" shape, if you zoom out enough.

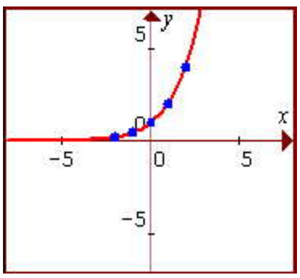
Example: Graph $y = 2^x$. Then use function shift rules to graph

$$y = 1 + 2^x,\ y = 2^{(x-3)},\ \text{and } y = 2^{-x}.$$

If you simply calculate and plot some points for $y = 2^x$, you see that the graph levels out to the horizontal axis and takes off vertically fairly quickly as shown below:

x	0 1 2 3 -1 -2
y	1 2 4 8 ½

Figure 6.13: Graph of $y = 2^x$

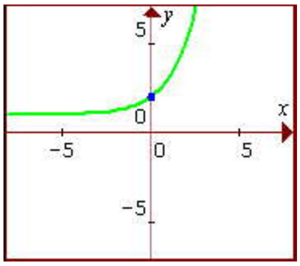


Now, to graph $y = 1 + 2^x$, shift the graph up 1 unit and you get the graph shown below.

Notice that the graph levels out to the horizontal asymptote $y = 1$ instead of $y = 0$. Also, the *y*-intercept (0,1) has been shifted up 1 to (0,2).

Notes

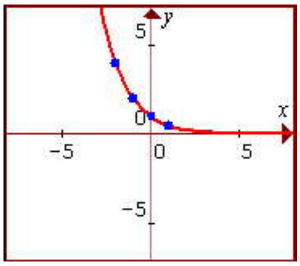
Figure 6.14: Graph of $y = 1 + 2^x$



To graph $y = 2^{(x - 3)}$, shift the graph right 3 unit and you get the graph shown below. The *y*-intercept (0,1) has been shifted right 3 to (3,1).

To graph $y = 2^{-x}$ reflect the graph of $y = 2^X$ across the y-axis as shown below.

Figure 6.15: Graph of $y = 2^{(x - 3)}$



This graph, like $y = 2^X$, levels out to the horizontal asymptote y = 0, except on the right side instead of the left.

Notes To graph exponential functions, you only need to find enough points to generate the "L" shape of the graph. Also, use function shift rules if applicable to save a lot of time.

The Most Common Exponential Base – *e*

Many students assume that either 10 or 2 is the most common base, since those are the bases we use. But, actually, the most common base is *e*, where *e* = 2.71828182846. . . , an irrational number.

Also, *e* is defined exactly as *e* = (1 + 1/m) m as m increases to infinity. You can see how this definition produces *e* by inputting a large value of m like m = 10,000,000 to get (1 + 1/10000000)10000000 = 2.7182817 (rounded), which is very close to the actual n value.

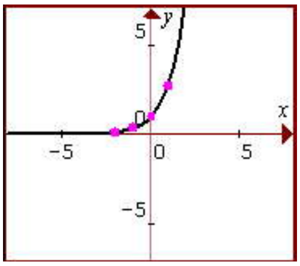
Example: Graph $y = e^X$

To graph this, you would input values much like you did to graph $y = 2^X$. The difference here is that you will have to use a scientific calculator to find the function values. You will need to use your e^X function, which normally requires use of the 2nd function key.

X	0	1	2	3	–1	–2
Y	1	2.718	7.389	20.086	0.368	0.135

Notes

Figure 6.16: Graph of y = 2^x



As you can see, the graph of y = e^x is very similar to y = 2^x. The only difference is that the graph levels off to y = 0 a bit quicker and it gets vertically steeper quicker.

Notes When graphing y = a^x, the value of "a" determines how quickly the graph levels out and takes off vertically. Otherwise, all of the graphs of this form will level out to y = 0 and take off vertically forming an "L" shape.

6.3 Inverse Function

If two functions f(x) and g(x) are defined so that (f o g)(x) = x and (g o f)(x) = x we say that f(x) and g(x) are inverse functions of each other.

6.3.1 Description of the Inverse Function

Functions f(x) and g(x) are inverses of each other if the operations of f(x) reverse all the operations of g(x) in the reverse order and the operations of g(x) reverse all the operations of f(x) in the reverse order.

Example: The function g(x) = 2x + 1 is the inverse of f(x) = (x - 1)/2 since the operation of multiplying by 2 and adding 1 in g(x) reverses the operation of subtracting 1 and dividing by 2. Likewise, the f(x) operations of subtracting 1 and dividing by 2 reverse the g(x) operations of doubling and adding 1.

An invertible function is a function that can be inverted. An invertible function must satisfy the condition that each element in the domain corresponds to one distinct element that no other element in the domain corresponds to. That is, all of the elements in the domain and range are paired-up in monogomous relationships - each element in the domain pairs to only one element in the range and each element in the range pairs to only one element in the domain. Thus, the inverse of a function is a function that looks at this relationship from the other viewpoint. So, for all elements *a* in the domain of f(x), the inverse of f(x) (notation: $f^{-1}(x)$) satisfies:

$$f(a) = b \text{ implies } f^{-1}(b) = a$$

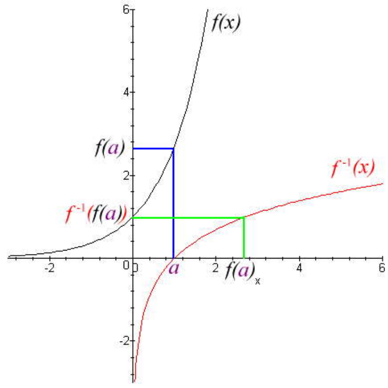
And, if you do the slightest bit of manipulation, you find that:

$$f^{-1}(f(a)) = a$$

Yielding the identity function for all inputs in the domain.

Notes

Figure 6.17: Functions and their Inverse



6.3.2 General Procedure for Finding the Inverse of a Function

Interchange the variables: First exchange the variables. Do this because to find the function that goes the other way, by mapping the old range onto the old domain. So our new equation is x = 2y – 5.

Solution for y: The rest is simply solving for the new *y*, which gives us:

$$2y - 5 = x$$

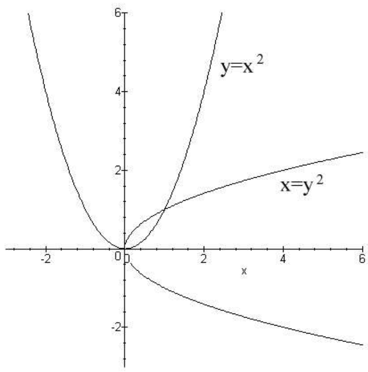
$$2y = x + 5$$

$$y = (x + 5)/2$$

Hence, $y^{-1}(x) = (x + 5)/2$

Find the inverse of the parabola by looking at the graph:

Figure 6.18: Graph of a Parabola showing the Inverse of Function

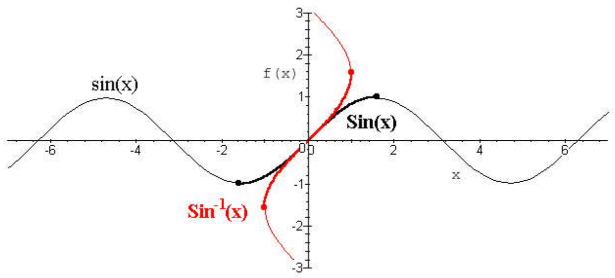


Because a parabola is not a one-to-one the inverse can't exist because for various values of *x* (all x > 0) $f^{-1}(x)$ has to take on two values. To solve this problem in taking inverses, in many cases, people decide to simply limit the domain. For instance, by limiting the domain of the parabola $y = x^2$ to values of x > 0, we can say that the function's inverse is y = +sqrt(x). Sqrt(x) means the square root of *x* or *x* 1/2). This is done to let the trigonometric functions have inverses.

Notes

As you can see, we can't take the inverse of sin(x) because it is not a one-to-one function. However, we can take the inverse of a subset of sin(x) with the domain of π–/2 to π /2. The new function inverse we get is called $Sin^{-1}(x)$ or Arc Sin(x).

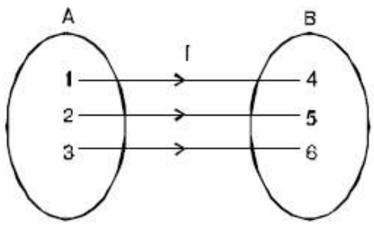
Figure 6.19: Graphical Representation of Inverse of Sin(x)



Inverse Function	Domain	Range
$Sin^{-1}(x)$	{x: –1 ≤ x ≤ 1}	–π/2 ≤ f(x) ≤ π/2
$Cos^{-1}(x)$	{x: –1 ≤ x ≤ 1}	0 ≤ f(x) ≤ π
$Tan^{-1}(x)$	{x: –infinity ≤ x ≤ infinity}	–π/2 ≤ f(x) ≤ π/2π
$Cot^{-1}(x)$	{x: –infinity ≤ x ≤ infinity}	0 ≤ f(x) ≤ π
$Sec^{-1}(x)$	{x: |x| ≥ 1}	0 ≤ f(x) ≤ π, f(x) ≤ π/2
$Cosec^{-1}(x)$	{x: |x| ≥ 1}	0 < |f(x)| ≤ π/2

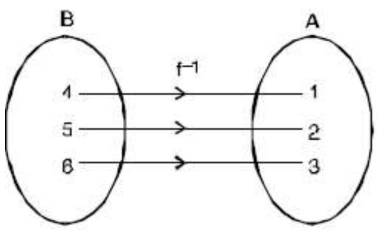
Another Method to Explain

(a) Consider the relation



This is a many-to-one function. Now let us find the inverse of this relation.

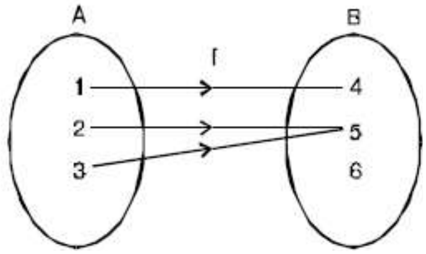
Pictorially, it can be represented as:



Clearly this relation does not represent a function.

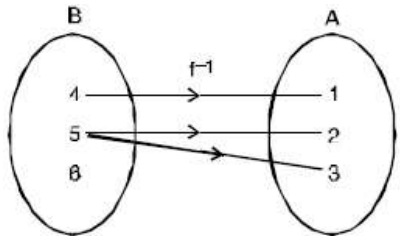
Notes

(b) Now take another relation



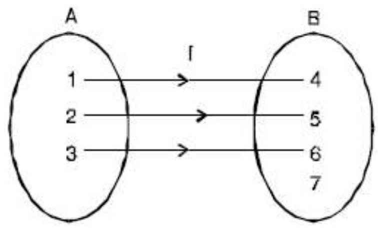
It represents one-to-one onto function.

(c) Now let us find the inverse of this relation, which is represented pictorially as:



This does not represent a function, because element 6 of set B is not associated with any element of A. Also note that the elements of B do not have a unique image.

(d) Let us take the following relation



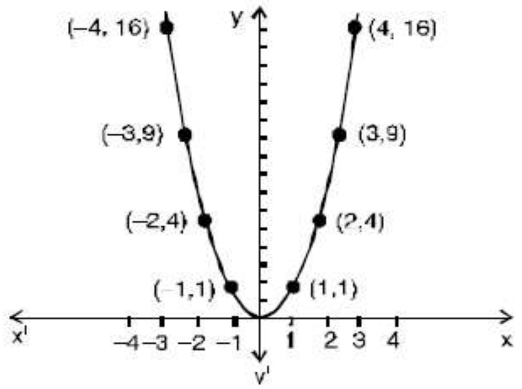
It represent one-to-one into function.

(e) Graphical Representation of Functions

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

$y = x^2$									
x	0	1	1	2	2	3	3	4	4
y	0	1	1	4	4	9	9	16	16

Notes



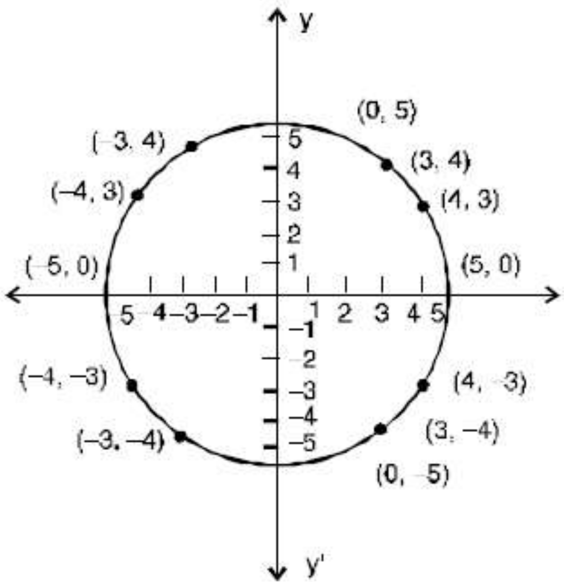
Does this represent a function?

Yes, this represent a function because corresponding to each value of *x* & $ *a is* unique value of *y*.

Now consider the equation $x^2 + y^2 = 25$

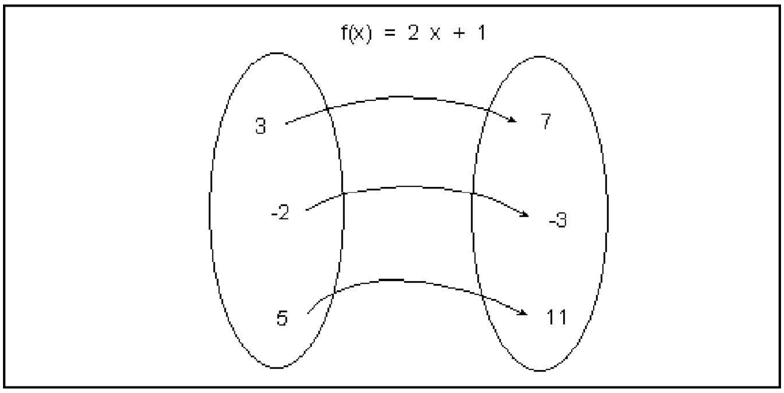
$x^2 + y^2 = 25$

x	0	0	3	3	4	4	5	5	3	3	4	4
y	5	5	4	4	3	3	0	0	4	4	3	3



Consider the function f(x) = 2x + 1. We know how to evaluate f at 3, f(3) = 2*3 + 1 = 7. In this section it helps to think of f as transforming a 3 into a 7, and f transforms a 5 into an 11, etc.

Notes



Now that we think of f as "acting on" numbers and transforming them, we can define the inverse of f as the function that "undoes" what f did. In other words, the inverse of f needs to take 7 back to 3, and take -3 back to -2, etc.

Let g(x) = (x - 1)/2. Then g(7) = 3, g(-3) = -2, and g(11) = 5, so g seems to be undoing what f did, at least for these three values. To prove that g is the inverse of f we must show that this is true for any value of x in the domain of f. In other words, g must take f(x) back to x for all values of x in the domain of f. So, g(f(x)) = x must hold for all x in the domain of f. The way to check this condition is to see that the formula for g(f(x)) simplifies to x.

$$g(f(x)) \;=\; g(2x + 1) = (2x + 1 -1)/2 = 2x/2 = x.$$

This simplification shows that if we choose any number and let f act it, then applying g to the result recovers our original number. We also need to see that this process works in reverse, or that f also undoes what g does.

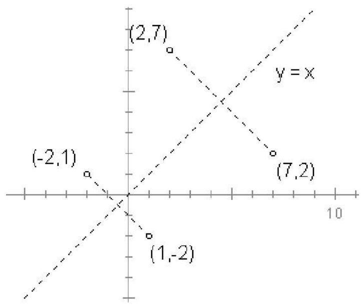
$$f(g(x)) \;=\; f((x - 1)/2) = 2(x - 1)/2 + 1 = x - 1 + 1 = x.$$

Letting f^{-1} denote the inverse of f, we have just shown that $g = f^{-1}$.

6.3.3 Graphs of Inverse Functions

We have seen examples of reflections in the plane. The reflection of a point (a,b) about the x-axis is (a, -b), and the reflection of (a, b) about the y-axis is (-a, b). Now we want to reflect about the line y = x.

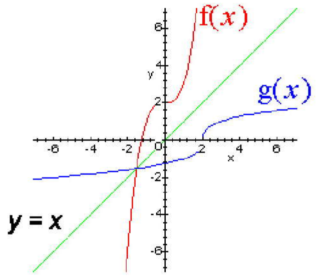
Figure 6.20: The Reflection of the point (a,b) about the line y = x is the point (b, a)



Let $f(x) = x^3 + 2$. Then f(2) = 10 and the point (2, 10) is on the graph of f. The inverse of f must take 10 back to 2, i.e. $f^{-1}(10)=2$, so the point (10, 2) is on the graph of f^{-1}. The point (10, 2) is the reflection in the line y = x of the point (2, 10). The same argument can be made for all points on the graphs of f and f^{-1}.

Notes

Figure 6.21: The graph of f^{-1} is the reflection about the line y = x of the graph of f

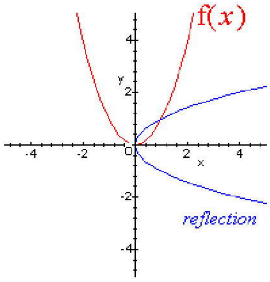


$f(x) = x^3 + 2$ and $g(x) = \sqrt[3]{x-2}$

6.3.4 Existence of an Inverse

Some functions do not have inverse functions. For example, consider $f(x) = x^2$. There are two numbers that f takes to 4, f(2) = 4 and f(-2) = 4. If f had an inverse, then the fact that f(2) = 4 would imply that the inverse of f takes 4 back to 2. On the other hand, since f(-2) = 4, the inverse of f would have to take 4 to -2. Therefore, there is no function that is the inverse of f.

Look at the same problem in terms of graphs. If f had an inverse, then its graph would be the reflection of the graph of f about the line y = x. The graph of f and its reflection about y = x are drawn below.



Note that the reflected graph does not pass the vertical line test, so it is not the graph of a function.

This generalizes as follows: A function f has an inverse if and only if when its graph is reflected about the line y = x, the result is the graph of a function (passes the vertical line test). But this can be simplified. We can tell before we reflect the graph whether or not any vertical line will intersect more than once by looking at how horizontal lines intersect the original graph!

6.3.5 Horizontal Line Test

Let f be a function.

If any horizontal line intersects the graph of f more than once, then f does not have an inverse.

If no horizontal line intersects the graph of f more than once, then f does have an inverse.

Notes

6.3.6 Finding Inverses

Example: First consider a simple example $f(x) = 3x + 2$.

The graph of f is a line with slope 3, so it passes the horizontal line test and does have an inverse.

There are two steps required to evaluate f at a number x. First we multiply x by 3, then we add 2.

Thinking of the inverse function as undoing what f did, we must undo these steps in reverse order.

The steps required to evaluate f^{-1} are to first undo the adding of 2 by subtracting 2. Then we undo multiplication by 3 by dividing by 3.

Therefore, $f^{-1}(x) = (x - 2)/3$.

Steps for finding the inverse of a function f.

1. Replace f(x) by y in the equation describing the function.
2. Interchange x and y. In other words, replace every x by a y and vice-versa.
3. Solve for y.
4. Replace y by $f^{-1}(x)$.

6.4 Logarithmic Function

The logarithmic function is defined as the inverse of the exponential function.

Parameters included in the definition of the logarithmic function may be changed, using sliders, to investigate its properties. The continuous (small increments) changes of these parameters help in gaining a deep understanding of logarithmic functions. The function to be explored has the form

$$f(x) = a*\log B[\ b\ (x+c)\ ] + d$$

a, b, c and d are coefficients and B is the base of the logarithm.

For B > 0 and B not equal to 1, $y = \text{Log } B^x$ is equivalent to $x = B^y$.

Notes The logarithm to the base *e* is written ln(x).

Example:

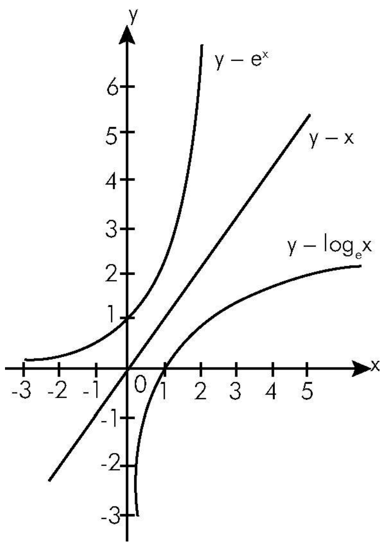
1. f(x) = log2x
2. g(x) = log4x
3. h(x) = log0.5x

Consider the function given below:

$$y = e^x \qquad \ldots(1)$$

Notes

Figure 6.22: Graphical Representation of Logarithmic Functions



We can write it equivalently as:

$$x = \log e^y$$

Thus, $$y = \log e^x \qquad \ldots(2)$$

is the inverse function of $y = e^x$

The base of the logarithm is not written if it is *e* and so $\log e^x$ is usually written as log x.

As $y = e^x$ and $y = \log x$ are inverse functions, their graphs are also symmetric with respect to the line

$$y = x$$

The graph of the function $y = \log x$ can be obtained from that of $y = e^x$ by reflecting it in the line $y = x$.

Some more examples of logarithmic function are given below:

Example: f is a function given by

$$f(x) = \log^2 (x + 2)$$

1. Find the domain of *f* and range of *f*.
2. Find the vertical asymptote of the graph of *f*.
3. Find the *x* and *y* intercepts of the graph of *f* if there are any.
4. Sketch the graph of *f*.

Solution:

1. The domain of *f* is the set of all *x* values such that

   $x + 2 > 0$

   or $x > -2$

Notes

The range of f is the interval (-inf , +inf).

2. The vertical asymptote is obtained by solving

$x + 2 = 0$

which gives

$x = -2$

As x approaches -2 from the right ($x > -2$), f(x) decreases without bound. How do we know this?

Let us take some values:

$$f(-1) = \log^2(-1 + 2) = \log^2(1) = 0$$

$$f(-1.5) = \log^2(-1.5 + 2) = \log^2(1/2) = -1$$

$f(-1.99) = \log^2(-1.99 + 2) = \log^2(0.01)$ which is approximately equal to -6.64.

$f(-1.999999) = \log^2(-1.999999 + 2) = \log^2(0.000001)$ which is approximately equal to -19.93.

3. To find the x intercept we need to solve the equation $f(x) = 0$

$$\log^2(x + 2) = 0$$

Use properties of logarithmic and exponential functions to write the above equation as:

$$2\log^2(x + 2) = 20$$

Then simplify

$$x + 2 = 1$$

$$x = -1$$

The x intercept is at (-1, 0).

The y intercept is given by $(0, f(0)) = (0, \log^2(0 + 2)) = (0, 1)$.

4. So far we have the domain, range, x and y intercepts and the vertical asymptote. We need more points. Let us consider a point at $x = -3/2$ (half way between the x intercept and the vertical asymptote) and another point at $x = 2$.

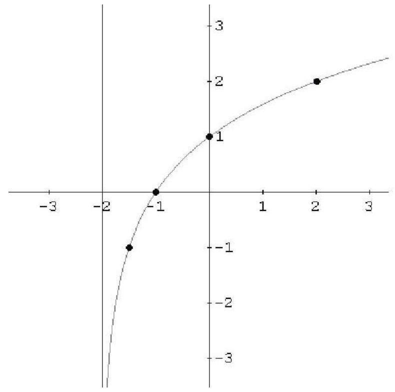
$$f(-3/2) = \log^2(-3/2 + 2) = \log^2(1/2) = \log^2(2 - 1) = -1.$$

$$f(2) = \log^2(2 + 2) = \log^2(22) = 2.$$

We now have more information on how to graph f. The graph increases as x increases. Close to the vertical asymptote $x = -2$, the graph of f decreases without bound as x approaches -2 from the right. The graph never cuts the vertical asymptote. We now join the different points by a smooth curve.

Notes

Figure 6.23: Graph of f(x) = log² (x + 2)



6.5 Composition of Functions

Function is a relation on two sets by a rule. It is a special mapping between two sets. It emerges that it is possible to combine two functions, provided co-domain of one function is domain of another function. The composite function is a relation by a new rule between sets, which are not common to the functions.

We can understand composition in terms of two functions. Let there be two functions defined as:

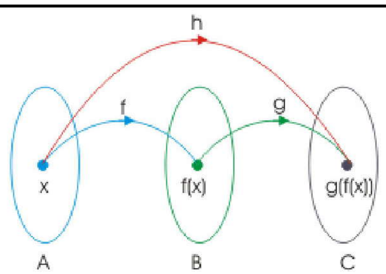
$f : A \rightarrow B$ by $f(x)$ for all $x \in A$

$g : B \rightarrow C$ by $g(x)$ for all $x \in B$

Observe that set "B" is common to two functions. The rules of the functions are given by "f(x)" and "g(x)" respectively. Our objective here is to define a new function $h: A \rightarrow C$ and its rule. Thinking in terms of relation, "A" and "B" are the domain and co-domain of the function "f". It means that every element "x" of "A" has an image "f(x)" in "B".

Similarly, thinking in terms of relation, "B" and "C" are the domain and co-domain of the function "g". In this function, "f(x)" - which was the image of pre-image "x" in "A" - is now pre-image for the function "g". There is a corresponding unique image in set "C". Following the symbolic notation, "f(x)" has image denoted by "g(f(x))" in "C". The figure here depicts the relationship among three sets via two functions (relations) and the combination function.

Figure 6.24: Composition Functions is a Special Relation between sets not Common to two Functions



6.5.1 Composition of Two Functions

For every element, "x" in "A", there exists an element f(x) in set "B". This is the requirement of function "f" by definition. For every element "f(x)" in "B", there exists an element g(f(x)) in set

Notes

"B". This is the requirement of function "g" by definition. It follows, then, that for every element "x" in "A", there exists an element g(f(x)) in set "C". This concluding statement is definition of a new function:

$h : A \rightarrow C$ by $g(f(x))$ for all $x \in A$

By convention, we call this new function as "g o f" and is read as "g circle f" or "g composed with f". g o f(x) = g(f(x)) for all $x \in A$

The two symbolical representations are equivalent.

Example: Let two sets be defined as :

$h : R \rightarrow R$by x^2 for all $x \in R$

$k : R \rightarrow R$by $x + 1$ for all $x \in R$

Determine "h o k" and "k o h".

Solution:

According to definition,

$h \circ k(x) = h(k(x))$

$\Rightarrow \quad h \circ k(x) = h(x + 1)$

$\Rightarrow \quad h \circ k(x) = (x + 1)^2$

Again, according to definition,

$k \circ h(x) = k(h(x))$

$\Rightarrow \quad k \circ h(x) = k(x^2)$

$\Rightarrow \quad k \circ h(x) = (x^2 + 1)$

Importantly note that $h \circ k(x) \neq k \circ h(x)$. It indicates that composition of functions is not commutative.

6.5.2 Existence of Composition Set

In accordance with the definition of function, "f", the range of "f" is a subset of its co-domain "B". But, set "B" is the domain of function "g" such that there exists image g(f(x)) in "C" for every "x" in "A". This means that range of "f" is subset of domain of "g":

Range of "f" $\subset$ Domain of "g".

Clearly, if this condition is met, then composition "g o f" exists. Following this conclusion, "f o g" will exist, if

Range of "g" $\subset$ Domain of "f"

And, if both conditions are met simultaneously, then we can conclude that both "g o f" and "f o g" exist. Such possibility is generally met when all sets involved are set of real numbers, "R".

Example: Let two functions be defined as:

f = {(1,2),(2,3),(3,4),(4,5)}

g = {(2,4),(3,2),(4,3),(5,1)}

Check whether "g o f" and "f o g" exit for the given functions.

Solution:

Here,

$\Rightarrow$ Domain of "f" = {1,2,3,4}

$\Rightarrow$ Range of "f" = {2,3,4,5}

$\Rightarrow$ Domain of "g" = {2,3,4,5}

$\Rightarrow$ Range of "g" = {4,2,3,1} = {1,2,3,4}

Hence,

$\Rightarrow$ Range of "f" $\subset$ Domain of "g" and

$\Rightarrow$ Range of "g" $\subset$ Domain of "f"

It means that both compositions "g o f" and "f o g" exist for the given sets.

Another Method of Explanation

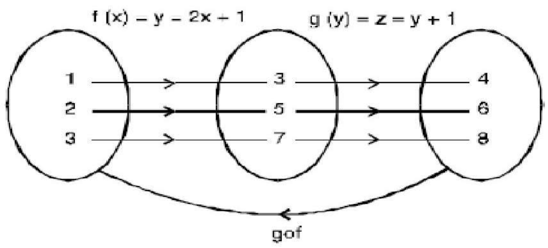
Consider the two functions given below:

$$y = 2x + 1, x \in \{1,2,3\}$$

$$z = y + 1, y \in \{3,5,7\}$$

Then *z* is the composition of two functions *x* and *y* because *z* is defined in terms of *y* and *y* in terms of *x*.

Figure 6.25: Graphical Representation



The composition, say, g o f of function *g* and *f* is defined as function *g* of function *f*.

If f : A $\rightarrow$ B and g : B $\rightarrow$ C

then g o f : A to C

Let $f(x) = 3x + 1$ and $g(x) = x^2 + 2$

Then $f \circ g(x) = f(g(x))$

$$= f(x^2 + 2)$$

$$= 3(x^2 + 2) + 1 = 3x^2 + 7 \quad \ldots(1)$$

$$\text{and } (g \circ f)(x) = g(f(x))$$

$$= g(3x + 1)$$

$$= (3x + 1)^2 + 2 = 9x^2 + 6x + 3 \quad \ldots(2)$$

Check from (1) and (2), if

$$f \circ g = g \circ f$$

Notes

Evidently, f o g # g o f

$$\begin{aligned} \text{Similarly, } (f \circ f)(x) &= f(f(x)) = f(3x + 1) && \text{[Read as function of function } f\text{].} \\ &= 3(3x + 1) + 1 \\ &= 9x + 3 + 1 = 9x + 4 \\ (g \circ g)(x) &= g(g(x)) = g(x^2 + 2) && \text{[Read as function of function } g\text{]} \\ &= (x^2 + 2)^2 + 2 \\ &= x^4 + 4x^2 + 4 + 2 \\ &= x^4 + 4x^2 + 6 \end{aligned}$$

6.6 Summary

- The sum of two odd functions is odd, and any constant multiple of an odd function is odd.
- The product of two even functions is an even function.
- The product of two odd functions is an even function.
- The integral of an odd function from A to +A is zero (where A is finite, and the function has no vertical asymptotes between A and A).
- The integral of an even function from A to +A is twice the integral from 0 to +A (where A is finite, and the function has no vertical asymptotes between A and A).
- A rational function is defined as the quotient of two polynomial functions.

$$f(x) = P(x) / Q(x)$$

6.7 Keywords

Functions: Functions are mathematical ideas that take one or more variables and produce a variable.

Logarithmic Function: It is defined as inverse of exponential functions.

Odd Function: The graph of an odd function has rotational symmetry with respect to origin.

Rational Funciton: It is defined as the quotient of two polynomial functions.

6.8 Self Assessment

1. The graph of even function is ……….. with respect to y-axis, meaning that its graph remain unchanged after reflection about y-axis.
2. The roots of a function are defined as the point where the function ………..
3. The Investigation of ……….. function is carried by changing parameter included in the formula of the function.
4. The graph of an ……….. function has rotational symmetry with respect to the origin.
5. ……….. is denoted mathematicaly by e^t and in matlab by exp(t).
6. An invertible function is a function that can be ………..
7. If f(x) be a real valued function of a real variable then f is even function is equals to ………..

Notes

8. The quotient of two odd function is ....

9. If f(x) is real valued function of a real variable then f is odd function if .... equation holds.

10. The Composition of an even function and an odd function is ....

6.9 Review Questions

1. Which of the following are exponential functions?

   (a) $f(x) = 3e^{-2x}$

   (b) $g(x) = 2^{x/2}$

   (c) $h(x) = x^{3/2}$

   (d) $g(x) = 15/7^x$

2. What is the domain of an exponential function $f(x) = kb^x$? What is the range? Describe the shape of the graph for $b > 1$, and for $b < 1$. What happens to f(x) in each case when x becomes very large (increases without bound) and as x becomes very small (decreases without bound)? Are there any horizontal asymptotes?

3. Solve the following equations. You should not need to use logarithms for the first three.

   (a) $2^x = 32$

   (b) $5^x = 1/125$

   (c) $(1/3)^{2x} = 243$

   (d) $45 = 5^{3x}$

   (e) $500 = 1000e^{-.75x}$

   (f) $56 = 14(1 + e^{.195x})$

4. The population of bacteria in a culture is growing exponentially. At 12:00 there were 80 bacteria present and by 4:00 PM there were 500 bacteria. Find an exponential function f(t) $= ke^{at}$ that models this growth, and use it to predict the size of the population at 8:00 PM.

5. The last nuclear test explosion was carried out by the French on an island in the south Pacific in 1996. Immediately after the explosion, the level of strontium-90 on the island was 100 times the level considered to be "safe" for human habitation. If the half-life of Strontium-90 is 28 years, how long will it take for the island to once again be habitable?

6. Which of the following functions are onto function if $f : R \to R$

   (a) $f(x) = 115x + 49$

   (b) $f(x) = |x|$

7. Which of the following functions are one-to-one functions?

   (a) $f : \{20,21,22\} \to \{40,42,44\}$ defined as $f(x) = 2x$

   (b) $f : \{7,8,9\} \to \{10\}$ defined as $f(x) = 10$

   (c) $f : I \to R$ defined as $f(x) = x^3$

   (d) $f : R \to R$ defined as $f(x) = 2 + x^4$

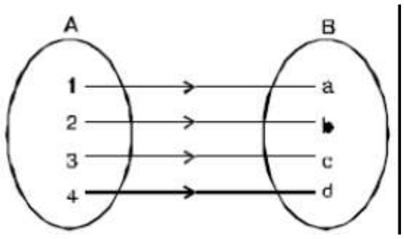
   (e) $f : N \to N$ defined as $f(x) = x^2 + 2x$

Notes

8. Which of the following functions are many-to-one functions?

   (a) f : {–2,–1,1,2} → {2,5} defined as $f(x) = x^2 + 1$

   (b) f : {0,1,2} → {1} defined as f(x) = 1

   (c) 

   (d) f : N → N defined as f(x) = 5x + 7

9. Draw the graph of each of the following functions :

   (a) $y = 3x^2$

   (b) $y = -x^2$

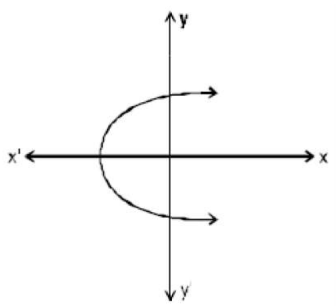
   (c) $y = x^2 - 2$

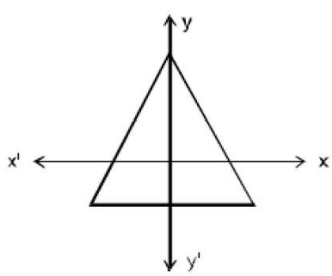
   (d) $y = 5 - x^2$

   (e) $y = 2x^2 + 1$

   (f) $y = 1 - 2x^2$

10. Which of the following graphs represents a functions?

   (a) 

   (b) 

Notes

Answers: Self Assessment

1. Symmetric
2. f(x) = 0
3. Rational Function
4. Odd Function
5. Exponential Function
6. Inverted
7. f(x) = f(-x)
8. Even Function
9. -f(x) = f(-x)
10. Even Function

6.10 Further Readings

Books

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

Smith and Minton, *Calculus Early Trancendental,* Third Edition, McGraw Hill 2008.

Online links

http://www.suitcaseofdreams.net/Trigonometric_Functions.htm

http://library.thinkquest.org/20991/alg2/trigi.html

http://www.intmath.com/trigonometric–functions/5–signs–of–trigonometric–functions.php

Notes

Unit 7: Limits

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss limits of a function
- Explain how to use the basic theorems on limits

Introduction

In the last unit you have studied about functions. In this unit you are going to study limits and continuity. Let f be a function and let c be a real number such that f(x) is defined for all values of x near x = c, except possibly at x = c itself. Suppose that whenever x takes values closer and closer but not equal to c (on both sides of c), the corresponding values of f(x) get very close to and possibly equal to the same real number L. The values of f(x) can be made arbitrarily close to L by taking values of x close enough to c, but not equal to c.

The limit of the function f(x) as x approaches c is the number L.

$$\lim_{x \to c} f(x) = L$$

7.1 Limits and Function Values

Notes

If the limit of a function f as x approaches c exists, this limit may not be equal to f(c). In fact, f(c) may not even be defined.

Non-existence of Limits

The limit of a function f as x approaches c may fail to exist if:

- f(x) becomes infinitely large or infinitely small as x approaches c from either side.
- f(x) approaches L as x approaches c from the right and f(x) approaches M, M ≠ L, as x approaches c from the left.
- f(x) oscillates infinitely many times between two numbers as x approaches c from either side.

Limit of a Constant

If d is a constant, then $\lim_{x\to c} d = d$.

Limit of the Identity Function

For every real number c, $\lim_{x\to c} x = c$

7.1.1 Properties of Limits

If f and g are functions and c, L, and M are numbers such that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

$$\lim_{x\to c}(f+g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

$$= L + M$$

$$\lim_{x\to c}(f-g)(x) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

$$= L - M$$

$$\lim_{x\to c}(f\cdot g)(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

$$= L \cdot M$$

$$\lim_{x\to c}(f/g)(x) = \lim_{x\to c} f(x) / \lim_{x\to c} g(x), \lim_{x\to c} g(x) \neq 0$$

$$= L/M, M \neq 0$$

$$\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}, \text{ f(x)} \geq 0 \text{ for all x near c.}$$

Limits of Polynomial Functions

If f(x)is a polynomial function and c is any real number, then $\lim_{x\to c} f(x) = f(c)$. In other words, the limit is the value of the polynomial function f at x = c.

Limits of Rational Functions

Let f(x) be a rational function and let c be a real number such that f(c) is defined. Then $\lim_{x\to c} f(x) = f(c)$.

Notes

7.1.2 Limit of a Difference Quotient

Example: Difference quotient of a function f is given by $\frac{f(x+h)-f(x)}{h}$

If $f(x) = x^2$, find $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Solution:

$$= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0}(2x+h) = 2x$$

7.1.3 Laws of Limits

Calculating limits using graphs and tables takes a lot of unnecessary time and work. Using the limit laws listed below, limits can be calculated much more quickly and easily.

Let $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist and let c be a constant.

1. $\lim\limits_{x\to a}\left[f(x) \pm g(x)\right] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

2. $\lim\limits_{x\to a}\left[cf(x)\right] = c\lim\limits_{x\to a} f(x)$

3. $\lim\limits_{x\to a}\left[f(x)g(x)\right] = \lim\limits_{x\to a} f(x)\lim\limits_{x\to a} g(x)$

4. $\lim\limits_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x) \neq 0$

5. $\lim\limits_{x\to a}\left[f(x)\right]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$

6. $\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$ if $\sqrt[n]{\lim\limits_{x\to a} f(x)}$

The following properties are special limit laws:

7. $\lim\limits_{x\to a} c = c$

8. $\lim\limits_{x\to a} x = a$

From the limit laws above, comes the property of direct substitution. This property makes it possible to solve most rational and polynomial functions. The property of direct substitution states: For any rational or polynomial function f, if a is in the domain of f then

$$\lim_{x\to a} f(x) = f(a)$$

Notes

Often, the method of direct substitution cannot be used because a is not in the domain of f. In these cases, it is sometimes possible to factor the function and eliminate terms so that the function is defined at the point a.

Consider the function f(x)= $\dfrac{x^2-1}{x-1}$

You can see that the function f(x) is not defined at x = 1 as x–1 is in the denominator. Take the value of x very nearly equal to but not equal to 1 as given in the tables below. In this case x–1 ≠ 0 as x ≠ 1.

∴ We can write f(x) = $\dfrac{x^2-1}{x-1}$ = $\dfrac{(x+1)(x-1)}{(x-1)} = x+1$ = x + 1, because x–1 ≠ 0 and so division by (x – 1) is possible.

x	f(x)
0.5	1.5
0.6	1.6
0.7	1.7
0.8	1.8
0.9	1.9
0.91	1.91
:	:
:	:
0.99	1.99
:	:
:	:
0.9999	1.9999

x	f(x)
1.9	2.9
1.8	2.8
1.7	2.7
1.6	2.6
1.5	2.5
:	:
:	:
1.1	2.1
1.01	2.01
1.001	2.001
:	:
:	:
1.00001	2.00001

In the above tables, you can see that as x gets closer to 1, the corresponding value of f (x) also gets closer to 2.

However, in this case f(x) is not defined at x = 1. The idea can be expressed by saying that the limiting value of f(x) is 2 when x approaches to 1.

Let us consider another function f (x) = 2x. Here, we are interested to see its behavior near the point 1 and at x = 1. We find that as x gets nearer to 1, the corresponding value of f (x) gets closer to 2 at x = 1 and the value of f (x) is also 2.

So from the above findings, what more can we say about the behaviour of the function near x = 2 and at x = 2 ?

In this unit we propose to study the behaviour of a function near and at a particular point where the function may or may not be defined.

7.2 Limits of a Function

In the introduction, we considered the function f(x) = $\dfrac{x^2-1}{x-1}$. We have seen that as x approaches l, f (x) approaches 2. In general, if a function f (x) approaches L when x approaches 'a', we say that L is the limiting value of f (x).

Symbolically it is written as

$$\lim_{x\to a} f(x) = L$$

Notes

Now let us find the limiting value of the function (5x–3) when x approaches 0.

i.e. $\lim\limits_{x \to 0}(5x - 3)$

For finding this limit, we assign values to x from left and also from right of 0.

x	–0.1	–0.01	–0.001	–0.0001..........
5x – 3	–3.5	–3.05	–3.005	–3.0005..........

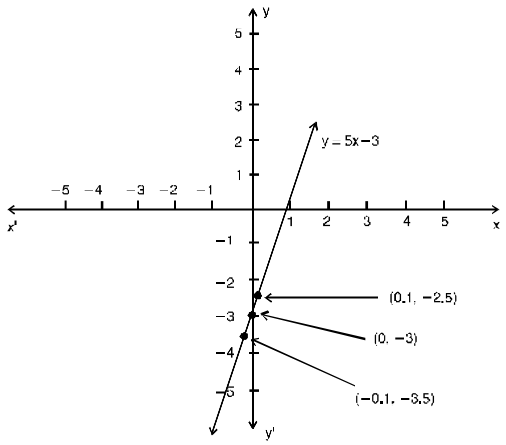
x	–0.1	–0.01	–0.001	–0.0001..........
5x – 3	–2.5	–2.95	–2.995	–2.9995..........

It is clear from the above that the limit of (5x–3) as x $\to$ 0 is –3

i.e., $\lim\limits_{x \to 0}(5x - 3) = -3$

This is illustrated graphically in the Figure 7.1.

Figure 7.1



The method of finding limiting values of a function at a given point by putting the values of the variable very close to that point may not always be convenient.

We, therefore, need other methods for calculating the limits of a function as x (independent variable) ends to a finite quantity.

Consider an example: Find $\lim\limits_{x \to 3} f(x), \text{where } f(x) = \dfrac{x^2 - 9}{x - 3}$

We can solve it by the method of substitution. Steps of which are as follows:

Step 1: We consider a value of x close to a say x = a + h, where h is a very small positive number. Clearly, as x $\to$ a, h $\to$ 0	For f(x) = $\frac{x^2 - 9}{x - 3}$ we write x = 3 + h, so that as x $\to$ 3, h $\to$ 0
Step 2: Simplify f(x) = f(a + h)	Now f(x)= f(3+h) $= \frac{(3+h)^2 - 9}{3 + h - 3}$ $= \frac{h^2 - 6h}{h}$ = h+6

Contd...

Step 3: Put h = 0 and get the required result	$\therefore \quad \lim_{x\to3} f(x) = \lim_{h\to0}(6+h)$ As $x \to 0, h \to 0$ Thus, $\lim_{x\to3} f(x) = 6 + 0 = 6$ by putting h=0

Consider the example:

Find $\lim_{x\to3} f(x)$, where $f(x) = \begin{cases} \dfrac{x^3-1}{x^2-1} & , x \neq 1 \\ 1 & , x = 1 \end{cases}$

Here, for $x \neq 1$, $f(x) = \dfrac{x^3-1}{x^2-1}$

$$= \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$$

It shows that if f(x) is of the form $\dfrac{g(x)}{h(x)}$, then we may be able to solve it by the method of factors .

In such case, we follow the following steps:

Step 1: Factorise g(x) and h (x)	Sol. $\quad f(x) = \dfrac{x^3-1}{x^2-1}$ $= \dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$ ($\because x \neq 1$. $\therefore$ $x-1 \neq 0$ and as such can be cancelled)
Step 2: Simplify f(x)	$\therefore \quad f(x) = \dfrac{x^2+x+1}{x+1}$
Step 3: Putting the value of x, we get the required limit	$\therefore \lim_{x\to1} \dfrac{x^3-1}{x^2-1} = \dfrac{1+1+1}{1+1} = \dfrac{3}{2}$ Also f(1) = 1(given) In this case, $\lim_{x\to1} f(x) \neq f(1)$

Thus, the limit of a function f (x) as $x \to a$ may be different from the value of the function at x = a.

Now, we take an example which cannot be solved by the method of substitutions or method of factors.

Evaluate $\lim_{x\to0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Here, we do the following steps:

Step 1: Rationalise the factor containing square root.

Step 2: Simplify.

Notes

Step 3: Put the value of x and get the required result.

Solution

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \frac{\sqrt{(1+x)^2}-\sqrt{(1-x)^2}}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \frac{1+x-1+x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

[Q $x \neq 0$, \ It can be cancelled]

$$= \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{1+1} = 1$$

Task

Evaluate the following:

1. $\lim_{x\to 0}\dfrac{\sin 4x}{2x}$
2. $\lim_{x\to 0}\dfrac{\sin x^2}{5x^2}$
3. $\lim_{x\to 0}\dfrac{\sin x^2}{x}$
4. $\lim_{x\to 0}\dfrac{\sin ax}{\sin bx}$

7.2.1 Limits of Left and Right Hand

You have already seen that $x \to a$ means x takes values which are very close to 'a', i.e. either the value is greater than 'a' or less than 'a'.

In case x takes only those values which are less than 'a' and very close to 'a' then we say x is approaches 'a' from the left and we write it as $x \to a^-$. Similarly, if x takes values which are greater than 'a' and very close to 'a' then we say x is approaching 'a' from the right and we write it as $x \to a^+$.

Thus, if a function f(x) approaches a limit l_1, as x approaches 'a' from left, we say that the left hand limit of f(x) as $x \to a$ is l_1.

Notes

We denote it by writing

$$\lim_{x \to a^-} f(x) = \ell_1 \quad \text{or} \quad \lim_{h \to 0} f(a-h) = \ell_1, h > 0$$

Similarly, if f(x) approaches the limit l_2, as x approaches 'a' from right we say, that the right hand limit of f(x) as $x \to a$ is l_2.

We denote it by writing

$$\lim_{x \to a^+} f(x) = \ell_2 \quad \text{or} \quad \lim_{h \to 0} f(a+h) = \ell_2, h > 0$$

Working Rules

Finding the right hand limit i.e.,		Finding the left hand limit, i.e.,	
	$\lim_{x \to a^+} f(x)$		$\lim_{x \to a^-} f(x)$
Put	x = a+h	Put	x=a – h
Find	$\lim_{h \to 0} f(a+h)$	Find	$\lim_{h \to 0} f(a-h)$

Limits of a function Y = f(x) at x = a

Example: Find $\lim_{x \to 1} f(x)$, where f(x) = x^2 + 5x +3

Here $\lim_{x \to 1^+} f(x) = \lim_{h \to 0}\left[(1+h)^2 + 5(1+h) + 3)\right]$

$= \lim_{h \to 0}\left[1 + 2h + h^2 + 5 + 5h + 3\right]$

$= 1 + 5 + 3 = 9$ ....(i)

and $\lim_{x \to 1^-} f(x) = \lim_{h \to 0}\left[(1-h)^2 + 5(1-h) + 3)\right]$

$= \lim_{x \to 0}\left[(1 - 2h + h^2 + 5 - 5h + 3\right]$

$= 1 + 5 + 3 = 9$ ....(ii)

From (i) and (ii), $\lim_{x \to 1} f(x) = \lim_{x \to 1^-} f(x)$

Example: Evaluate: $\lim_{x \to 3} \frac{|x-3|}{x-3}$

Here $\lim_{x \to 3^+} \frac{|x-3|}{x-3} = \lim_{h \to 0} \frac{|(3+h)-3|}{[(3+h)-3]}$

$= \lim_{h \to 0} \frac{|h|}{h}$

$= \lim_{h \to 0} \frac{h}{h}$ (as h>0, so | h | = h)

$= 1$ ...(iii)

and $\lim_{x \to 3^-} \frac{|x-3|}{x-3} = \lim_{h \to 0} \frac{|(3-h)-3|}{[(3-h)-3]}$

Notes

$$= \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} \quad \text{(as h>0, so } |-h| = h\text{)}$$

$$= -1 \qquad \text{...(iv)}$$

From (iii) and (iv), $\lim_{x \to 3^+} \frac{|x-3|}{x-3} \neq \lim_{x \to 3^-} \frac{|x-3|}{x-3}$

Thus, in the first example right hand limit = left hand limit whereas in the second example right hand limit ≠ left hand limit.

Hence the left hand and the right hand limits may not always be equal.

We may conclude that

$\lim_{x \to 1} (x^2 + 5x + 3)$ exists (which is equal to 9) and $\lim_{x \to 3} \frac{|x-3|}{x-3}$ does not exist.

7.3 Tangents and Limits

A tangent to a curve is a straight line that touches the curve at a single point but does not intersect it at that point. For example, in the figure to the right, the y-axis would not be considered a tangent line because it intersects the curve at the origin. A secant to a curve is a straight line that intersects the curve at two or more points.

In the figure given below, the tangent line intersects the curve at a single point P but does not intersect the curve at P. The secant line intersects the curve at points P and Q.

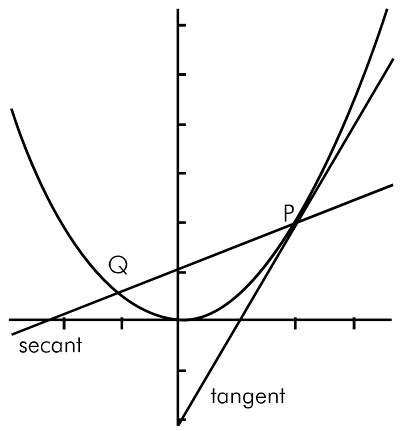
The concept of limits begins with the tangent line problem. We want to find the equation of the tangent line to the curve at the point P. To find this equation, we will need the slope of the tangent line. But how can we find the slope when we only know one point on the line? The answer is to look at the slope of the secant line. It's slope can be determined quite easily since there are two known points P and Q. As you slide the point Q along the curve, towards the point P, the slope of the secant line will become closer to the slope of the tangent line. Eventually, the point Q will be so close to P, that the slopes of the tangent and secant lines will be approximately equal.

A limit of a function is written as:

$$\lim_{x \to a} f(x) = L$$

We want to find the limit of f(x) as x approaches a. To do this, we try to make the values of f(x) close to the limit L, by taking x values that are close to, but not equal to, a. In short, f(x) approaches L as x approaches a.

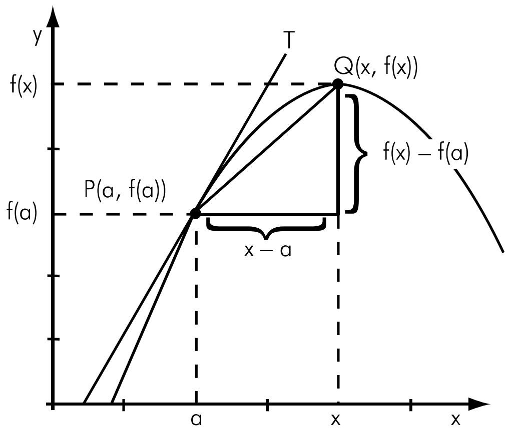
Notes



As explained a tangent to a curve is a line that touches the curve at a single point, P (a,f(a)). The tangent line T is the line through the point P with the slope:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

given that this limit exists. The graph to the right illustrates how the slope of the tangent line is derived. The slope of the secant line PQ is given by f(x)–f(a)/x–a. As x approaches a, the slope of PQ becomes closer to the slope of the tangent line T. If we take the limit of the slope of the secant line as x approaches a, it will be equal to the slope of the tangent line T.



The slope of the tangent line becomes much easier to calculate if we consider the following conditions. If we let the distance between x and a be h, so that x= a + h, and substitute that equality for x in the slope formula, we get:

$$m = \lim_{x \to a} \frac{f(a+h) - f(a)}{h}$$

Notes Either of the limit formulas above can be used to find the slope. You will obtain the same answer using either formula.

Notes

These formulas have many practical applications. They can be used to find the instantaneous rates of change of variables. For example, if we use the formula above, the instantaneous velocity at time t = a is equal to the limit of f(a + h) – f(a)/h as h approaches 0.

Example: Find the equation of the tangent line to the curve $y = \sqrt{2x+5}$ at the point (2, 3).

Solution:

We are given that a = 2 and f(x) = $\sqrt{2x+5}$. Substituting these values into the slope of a tangent line formula, we have

$$m = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to 2}\frac{\sqrt{2x+5}-3}{x-2}$$

$$= \lim_{x\to 2}\frac{\sqrt{2x+5}-3}{x-2}\left(\frac{\sqrt{2x+5}+3}{\sqrt{2x+5}+3}\right)$$

$$= \lim_{x\to 2}\frac{\left(\sqrt{2x+5}-3\right)\left(\sqrt{2x+5}+3\right)}{(x-2)\left(\sqrt{2x+5}+3\right)}$$

$$= \lim_{x\to 2}\frac{\left(\sqrt{2x+5}\right)^2-(3)^2}{(x-2)\left(\sqrt{2x+5}+3\right)}$$

$$= \lim_{x\to 2}\frac{2x+5-9}{(x-2)\left(\sqrt{2x+5}+3\right)}$$

$$= \lim_{x\to 2}\frac{2x-4}{(x-2)\left(\sqrt{2x+5}+3\right)}$$

$$= \lim_{x\to 2}\frac{2(x-2)}{(x-2)\left(\sqrt{2x+5}+3\right)}$$

$$= \lim_{x\to 2}\frac{2\cancel{(x-2)}}{\cancel{(x-2)}\left(\sqrt{2x+5}+3\right)}$$

$$= \frac{2}{\left(\sqrt{2(2)+5}+3\right)}$$

$$= \frac{2}{\left(\sqrt{9}+3\right)}$$

$$= \frac{2}{(3+3)}$$

$$= \frac{1}{3}$$

Notes

Give that the slope of the line is $\frac{1}{3}$ and is passes through the point (2, 3), we can find the equation of the tangent line using the point-slope formula.

$$y - y_1 = m(x - m_1)$$

$$y - 3 = \frac{1}{3}(x-2)$$

$$3y - 9 = x - 2$$

$$x - 3y + 7 = 0$$

$\therefore$ The equation of the tangent line to the curve $y = \sqrt{2x+5}$ at the point (2, 3) is $x - 3y + 7 = 0$.

7.4 The Pinching or Sandwich Theorem

As a motivation let us consider the function

$$f(x) = x^2 \sin\left(\frac{1}{x}\right).$$

When x get closer to 0, the function $g(x) = \sin\left(\frac{1}{x}\right)$ fails to have a limit. So we are not able to use the basic properties discussed in the previous pages. But we know that this function $g(x) = \sin\left(\frac{1}{x}\right)$ is bounded below by –1 and above by 1, i.e.

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

for any real number x. Since $x^2 \ge 0$, we get

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2.$$

Hence when x get closer to 0, x2 and –x2 become very small in magnitude. Therefore any number in between will also be very small in magnitude. In other words, we have

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

This is an example for the following general result:

Theorem: The "Pinching" or "Sandwich" Theorem

Assume that

$$h(x) \le f(x) \le g(x)$$

for any x in an interval around the point a. If

$$\lim_{x\to a} h(x) = L \text{ and } \lim_{x\to a} g(x) = L,$$

then

$$\lim_{x\to a} f(x) = L.$$

Notes

Example: Let f(x) be a function such that $|f(x)| \leq M$, for any $x \neq 0$. The Sandwich Theorem implies

$$\lim_{x \to a} f(x) = 0.$$

Indeed, we have

$$|x f(x)| \leq M|x|$$

which implies

$$-M|x| \leq x f(x) \leq M|x|$$

for any $x \neq 0$. Since

$$\lim_{x \to 0} M|x| = 0 \text{ and } \lim_{x \to 0} -M|x| = 0.$$

then the Sandwich Theorem implies

$$\lim_{x \to 0} x f(x) = 0.$$

Example: Use the Sandwich Theorem to prove that

$$\lim_{x \to 0} \sqrt{x} = \sqrt{a}$$

for any $a > 0$.

Solution:

For any $x > 0$, we have

$$\sqrt{x} - \sqrt{a} = \frac{\left(\sqrt{x} - \sqrt{a}\right)\left(\sqrt{x} + \sqrt{a}\right)}{\sqrt{x} + \sqrt{a}} = \frac{x - a}{\sqrt{x} + \sqrt{a}}.$$

Hence

$$\left|\sqrt{x} - \sqrt{a}\right| = \left|\frac{x - a}{\sqrt{x} + \sqrt{a}}\right| \leq \left|\frac{x - a}{\sqrt{a}}\right|$$

because $\sqrt{a} \leq \sqrt{x} + \sqrt{a}$ for any $x > 0$. In particular, we have

$$-\left|\frac{x - a}{\sqrt{a}}\right| \leq \sqrt{x} - \sqrt{a} \leq \left|\frac{x - a}{\sqrt{a}}\right|.$$

Since

$$\lim_{x \to a} -\left|\frac{x - a}{\sqrt{a}}\right| = \lim_{x \to a} \left|\frac{x - a}{\sqrt{a}}\right| = 0,$$

the Sandwich Theorem implies

$$\lim_{x \to a} \sqrt{x} - \sqrt{a} = 0$$

or equivalently

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}$$

Notes

Example: Use the Sandwich Theorem to prove that

$$\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0.$$

Solution:

For any $x \neq 0$, we have

$$\left|x\cos\left(\frac{1}{x}\right)\right| \le |x|.$$

Hence

$$-|x| \le x\cos\left(\frac{1}{x}\right) \le |x|.$$

Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0}|x| = 0,$$

then the Sandwich Theorem implies

$$\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0.$$

Example: Consider the function

$$f(x) = \begin{cases} 1+2x^2 & \text{if x is rational} \\ 1+x^4 & \text{if x is irrational.} \end{cases}$$

Use the Sandwich Theorem to prove that

$$\lim_{x\to 0} f(x) = 1.$$

Solution:

Since we are considering the limit when x gets closer to 0, then we may assume that $|x| \le 1$. In this case, we have $x^4 \le x^2 \le 2x^2$. Hence for any x, we have

$$1 \le f(x) \le 1 + 2x^2.$$

Since $\lim_{x\to 0} 1+2x^2 = 1,$ then the Sandwich Theorem implies

$$\lim_{x\to 0} f(x) = 1.$$

7.5 Infinite Limits

Some functions "take off" in the positive or negative direction (increase or decrease without bound) near certain values for the independent variable. When this occurs, the function is said to have an infinite limit; hence, you write $\lim_{x\to 0} f(x) = +\infty$ or $\lim_{x\to 0} f(x) = -\infty$. Note also that the function has a vertical asymptote at x = c if either of the above limits hold true.

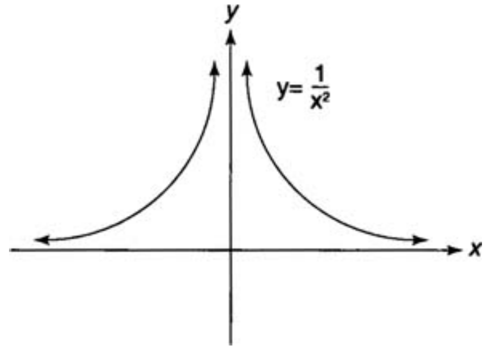
In general, a fractional function will have an infinite limit if the limit of the denominator is zero and the limit of the numerator is not zero. The sign of the infinite limit is determined by the sign of the quotient of the numerator and the denominator at values close to the number that the independent variable is approaching.

Notes

Example: Evaluate $\lim_{x\to 0}\frac{1}{x^{2^*}}$.

As x approaches 0, the numerator is always positive and the denominator approaches 0 and is always positive; hence, the function increases without bound and $\lim_{x\to 0}\frac{1}{x^2}=+\infty$. The function has a vertical asymptote at x = 0 (see Figure 7.2).

Figure 7.2: The Graph of y = 1/ x²



Example: Evaluate $\lim_{x\to 0^*}\frac{x+3}{x-2}$.

As x approaches 2 from the left, the numerator approaches 5, and the denominator approaches 0 through negative values; hence, the function decreases without bound and (x + 3)/(x – 2) = –∞. The function has a vertical asymptote at x = 2.

Example: Evaluate $\lim_{x\to 0^*}\left(\frac{1}{x^2}-\frac{1}{x^3}\right)$.

Rewriting $1/x^2-1/x^3$ as an equivalent fractional expression $(x-1)/x^3$, the numerator approaches −1, and the denominator approaches 0 through positive values as x approaches 0 from the right; hence, the function decreases without bound and $\lim_{x\to 0}\left(1/x^2-1/x^3\right)=-\infty$. The function has a vertical asymptote at x = 0.

A word of caution: Do not evaluate the limits individually and subtract because ±∞ are not real numbers. Using this example,

$$\lim_{x\to 0^*}\left(\frac{1}{x^2}-\frac{1}{x^2}\right)\neq\lim_{x\to 0^*}\frac{1}{x^2}-\lim_{x\to 0^*}\frac{1}{x^3}=(+\infty)-(+\infty)$$

1. Find each of the following limits if it exists. Specify any horizontal or vertical asymptotes of the graphs of the functions.

   (a) $\lim_{x\to\infty}\left(-x^2\right)$.

   (b) $\lim_{x\to\infty}\left(x-x^2\right)$.

Solution

(a) $\lim_{x\to\infty}\left(-x^2\right)=-\infty$. There are no horizontal asymptotes. Since $-x^2$ is defined everywhere, there are no vertical asymptotes.

(b) $\lim_{x\to\infty}\left(x-x^2\right)=\lim_{x\to\infty}\left(x(1-x)\right)=-\infty$. There are no horizontal asymptotes. Since $x-x^2$ is defined for every x, there are no vertical asymptotes.

Notes

2. Find the following limit if it exists. Specify any horizontal or vertical asymptotes of the graph of the function.

$$\lim_{x \to -\infty} \left(\cos^2 x + 1\right).$$

Solution

As $x \to -\infty$, cos x keeps ocillating between 1 and –1, so $\cos^2 x$ keeps oscillating between 0 and 1, thus $\cos^2 x + 1$ keeps oscillating between 1 and 2. Consequently, $\lim_{x \to -\infty} \left(\cos^2 x + 1\right)$ doesn't exist. There are no horizontal asymptotes. As $\cos^2 x + 1$ is defined everywhere, there are no vertical asymptotes.

3. Let $f(x) = \dfrac{\cos^2 x + 1}{x}$.

Determine:

$\lim_{x \to \infty} f(x), \lim_{x \to \infty} f(x)$, and $\lim_{x \to \infty} f(x)$.

Specify horizontal and vertical asymptotes if any.

Solution

As $x \to -\infty$, $\cos^2 x + 1$ keeps oscillating between 1 and 2. So:

$\lim_{x \to +\infty} f(x) = 0, \lim_{x \to -\infty} f(x) = 0.$

Also:

$\lim_{x \to 0+} f(x) = \infty, \lim_{x \to 0-} f(x) = -\infty,$

Thus $\lim_{x \to 0} f(x)$ doesn't exist.

The horizontal asymptote is the x-axis. The vertical asymptote is the y-axis.

4. Let $g(x) = \dfrac{2x - 1}{x + 7}$.

Determine:

$\lim_{x \to \infty} g(x), \lim_{x \to -\infty} g(x)$, and $\lim_{x \to -7} g(x)$.

Specify horizontal and vertical asymptotes if any.

Solution

$$g(x) = \frac{2x - 1}{x + 7} = \frac{\frac{2x - 1}{x}}{\frac{x + 7}{x}} = \frac{\frac{2x}{x} - \frac{1}{x}}{\frac{x}{x} + \frac{7}{x}} = \frac{2 - \frac{1}{x}}{1 + \frac{7}{x}},$$

$$\lim_{x \to \infty} g(x) = \frac{2 - 0}{1 + 0} = 2, \lim_{x \to \infty} g(x) = \frac{2 - 0}{1 + 0} = 2.$$

When $x \to -7+$, we have $2x - 1 \to -15 < 0$ and $x + 7 \to 0$ and $x + 7 > 0$, so:

$$\lim_{x \to -7+} g(x) = \infty.$$

Notes

When $x \to -7-$, we have $2x - 1 \to -15 < 0$ and $x + 7 \to 0$ and $x + 7 < 0$, so:

$\lim_{x \to -7-} g(x) = \infty$.

Thus $\lim_{x \to -7} g(x)$ doesn't exist.

The line y = 2 is the horizontal asymptote. The line x = –7 is a vertical asymptote.

5. Find any horizontal and vertical asymptotes of each of the following functions.

(a) $f(x) = \frac{3}{x-2}$ for all x > 2.

(b) $g(x) = \frac{3}{x-2}$ for all x > 2.

(c) $h(x) = \frac{3}{|x-2|}$ for all $x \neq 2$

Solution

(a) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3}{x-2} = 0;$

the horizontal asymptote of f is the x-axis.

$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} \frac{3}{x-2} = \infty;$

the vertical asymptote of f is the line x = 2.

(b) $\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{3}{x-2} = 0;$

the horizontal asymptote of g is the x-axis.

$\lim_{x \to 2-} g(x) = \lim_{x \to 2-} \frac{3}{x-2} = -\infty;$

the vertical asymptote of g is the line x = 2.

(c) $\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{3}{|x-2|} = 0;$

a horizontal asymptote of h is the x-axis;

7.6 Basic Theorems of Limits

1. $\lim_{x \to a} cx = c \lim_{x \to a} x$, c being a constant.

To verify this, consider the function f(x) = 5x.

We observe that in $\lim_{x \to 2} 5x$, 5 being a constant is not affected by the limit.

$\therefore \quad \lim_{x \to 2} 5x = 5 \lim_{x \to 2} x$

$= 5 \times 2 = 10$

2. $\lim_{x \to a} \left[g(x) + h(x) + p(x) + ... \right] = \lim_{x \to a} g(x) + \lim_{x \to a} h(x) + \lim_{x \to a} p(x) + \ldots$

where g(x), h(x), P(x), ... are any function.

3. $\lim\limits_{x\to a}\left[f(x).g(x)\right] = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$

Notes

To verify this, consider

$$f(x) = 5x^2 + 2x + 3$$

$$\text{and } g(x) = x + 2$$

Then $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}(5x^2 + 2x + 3)$

$$= 5\lim_{x\to 0} x^2 + 2\lim_{x\to 0} x + 3 = 3$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}(x+2) = \lim_{x\to 0} x + 2 = 2$$

$\therefore$ $\lim\limits_{x\to 0}(5x^2 + 2x + 3)\lim\limits_{x\to 0}(x+2) = 6$ ...(i)

Again $\lim\limits_{x\to 0}[f(x).g(x)] = \lim\limits_{x\to 0}[(5x^2 + 2x + 3)(x+2)]$

$$= \lim_{x\to 0}[(5x^3 + 12x^2 + 7x + 6)$$

$$= 5\lim_{x\to 0} x^3 + 12\lim_{x\to 0} x^2 + 7\lim_{x\to 0} x + 6$$

$= 6$ ...(ii)

From (i) and (ii),

$$\lim_{x\to 0}[(5x^2 + 2x + 3)(x+2)] = \lim_{x\to 0}(5x^2 + 2x + 3)\lim_{x\to 0}(x+2)$$

4. $\lim\limits_{x\to a}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ provided $\lim\limits_{x\to a} g(x) \neq 0$

To verify this, consider the function $f(x) = \dfrac{x^2 + 5x + 6}{x+2}$

We have $\lim\limits_{x\to -1}(x^2 + 5x + 6)$ = (–1)2 + 5 (–1) + 6

= 1– 5 + 6

= 2

and $\lim\limits_{x\to -1}(x+2)$ = –1 + 2

= 1

$\dfrac{\lim\limits_{x\to -1}(x^2 + 5x + 6)}{\lim\limits_{x\to -1}(x+2)} = \dfrac{2}{1} = 2$ ....(i)

Also $\lim\limits_{x\to -1}\dfrac{(x^2 + 5x + 6)}{x+2} = \lim\limits_{x\to -1}\dfrac{(x+3)(x+2)}{x+2}$ $\begin{bmatrix} \because x^2 + 5x + 6 \\ = x^2 + 3x + 2x + 6 \\ = x(x+3) + 2(x+3) \\ = (x+3)(x+2) \end{bmatrix}$

$$= \lim_{x\to -1}(x+3)$$

Notes

$$= -1+3 = 2 \qquad \text{....(ii)}$$

$\therefore$ From (i) and (ii),

$$\lim_{x\to -1}\frac{x^2+5x+6}{x+2} = \frac{\lim\limits_{x\to -1}(x^2+5x+6)}{\lim\limits_{x\to -1}(x+2)}$$

We have seen above that there are many ways that two given functions may be combined to form a new function. The limit of the combined function as x → a can be calculated from the limits of the given functions. To sum up, we state below some basic results on limits, which can be used to find the limit of the functions combined with basic operations.

If $\lim\limits_{x\to a} f(x) = \ell$ and $\lim\limits_{x\to a} g(x) = m$, then

(i) $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x) = k\ell$, where k is a constant

(ii) $\lim\limits_{x\to a} [f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = \ell \pm m,$

(iii) $\lim\limits_{x\to a} [f(x).g(x)] = \lim\limits_{x\to a} f(x).\lim\limits_{x\to a} g(x) = \ell.m$

(iv) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{\ell}{m}$, provided $\lim\limits_{x\to a} g(x) \neq 0$

The above results can be easily extended in case of more than two functions.

Example: Find $\lim\limits_{x\to 1} f(x)$, where

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

Solution:

$$f(x) = f(x) = \frac{x^2-1}{x-1}$$

$$= \frac{(x-1)\,(x+1)}{x-1}$$

$$= (x+1) \qquad [\because x \neq 1]$$

$$\therefore \quad \lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1)$$

$$= 1 + 1 = 2$$

Tasks

1. Show that f(x) = $e^{\frac{-2}{3}x}$ is a continuous function.

2. Show that f(x) = e^{3x+2} is a continuous function.

7.6.1 Limits of Important Functions

(i) Prove that $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ where n is a positive integer.

Proof: $\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{h \to 0} \frac{(a+h)^n - a^n}{a+h-a}$

$$= \lim_{h \to 0} \frac{\left(a^n + n\,a^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + ... + h^n\right) - a^n}{h}$$

$$= \lim_{h \to 0} \frac{h\left(n\,a^{n-1} + n\frac{n(n-1)}{2!}a^{n-2}h + ... + h^{n-1}\right)}{h}$$

$$= \lim_{h \to 0} \left[n\,a^{n-1} + n\frac{n(n-1)}{2!}a^{n-2}h + ... + h^{n-1}\right]$$

$$= n\,a^{n-1} + 0 + 0 + ... + 0$$

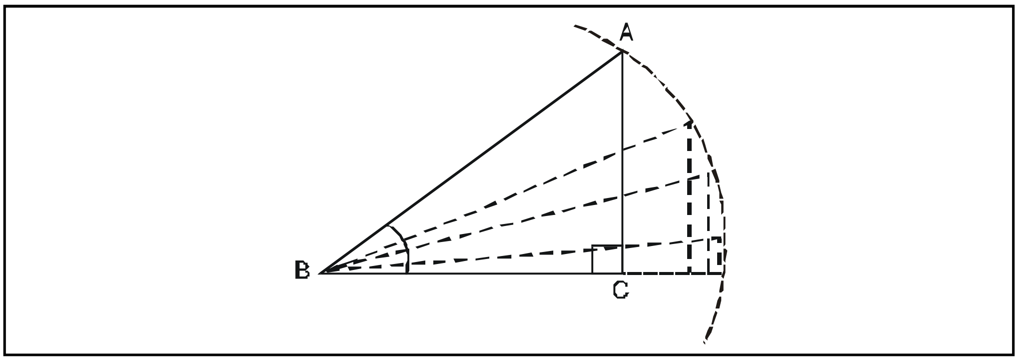
$$= n\,a^{n-1}$$

$\therefore$ $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$

(ii) Prove that

(a) $\lim_{x \to 0} \sin x = 0$ and (b) $\lim_{x \to 0} \cos x = 1$

Proof: Consider a unit circle with centre B, in which ∠C is a right angle and ∠ABC = x radians.



Now sinx = A C and cosx =BC

As x decreases, A goes on coming nearer and nearer to C.

i.e., when $x \to 0$, $A \to C$

or when $x \to 0$, $AC \to 0$

and $BC \to AB$, i.e., $BC \to 1$

$\therefore$ When $x \to 0$ $\sin x \to 0$ and $\cos x \to 1$

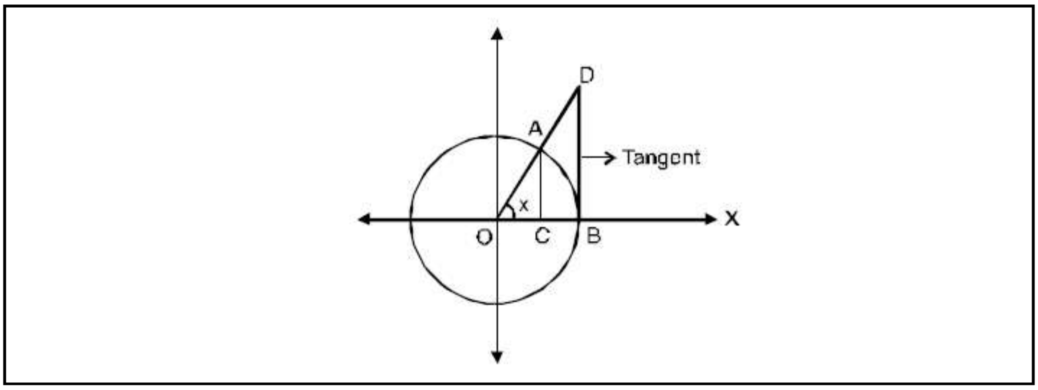
Notes

Thus we have

$$\lim_{x \to 0} \sin x = 0 \text{ and } \lim_{x \to 0} \cos x = 1$$

(iii) Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof: Draw a circle of radius 1 unit and with centre at the origin O. Let B (1,0) be a point on the circle. Let A be any other point on the circle. Draw AC $\perp$ OX.



Let $\angle AOX = x$ radians, where $0 < x < \frac{\pi}{2}$

Draw a tangent to the circle at B meeting OA produced at D. Then BD $\perp$ OX.

Area of $\Delta AOC <$ area of sector OBA $<$ area of ΔOBD.

or $\frac{1}{2}\, OC \times AC < \frac{1}{2}\, x(1)^2 < \frac{1}{2}\, OB \times BD$

$$\left[\because \text{area of triangle} = \frac{1}{2}\text{base} \times \text{height and area of sector} = \frac{1}{2}\theta r^2\right]_{x}$$

$\therefore \frac{1}{2} \cos x \sin x < \frac{1}{2}\, x < \frac{1}{2} \cdot 1 \cdot \tan x$

$$\left[\because \cos x = \frac{OC}{OA}, \sin x = \frac{AC}{OA} \text{ and } \tan x = \frac{BD}{OB}, OA = 1 = OB\right]$$

i.e., $\cos x < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$ $\left[\text{Dividing throughout by } \frac{1}{2}\sin x\right]$

or $\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$

or $\frac{1}{\cos x} > \frac{\sin x}{x} < \cos x$

i.e., $\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$

Taking limit as $x \to 0$, we get

$$\lim_{x \to 0} \cos x < \lim_{x \to 0} \frac{\sin x}{x} < \lim_{x \to 0} \frac{1}{\cos x}$$

Notes

or $1 < \lim_{x \to 0} \frac{\sin x}{x} < 1$ $\left[\because \lim_{x \to 0} \cos x = 1 \text{ and } \lim_{x \to 0} \frac{1}{\cos x} = \frac{1}{1} = 1\right]$

Thus, $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iv) Prove that $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

Proof: By Binomial theorem, when $|x| < 1$, we get

$$(1+x)^{\frac{1}{x}} = \left[1 + \frac{1}{x}.x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2 + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3 + \text{............}\infty\right]$$

$$= \left[1+1+\frac{(1-x)}{2!}+\frac{(1-x)(1-2x)}{3!}+\text{............}\infty\right]$$

$$\therefore \quad \lim_{x \to 0} (1+x)^{\frac{1}{x}} = \lim_{x \to 0}\left[1+1+\frac{(1-x)}{2!}+\frac{(1-x)(1-2x)}{3!}+\text{............}\infty\right]$$

$$= \left[1+1+\frac{1}{2!}+\frac{1}{3!}+\text{............}\infty\right]$$

$$= e \text{ (By definition)}$$

Thus $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

(v) Prove that

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{1}{x}\log(1+x) = \lim_{x \to 0} \log(1+x)^{\frac{1}{x}}$$

$$= \log e \qquad \left(\text{Using } \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e\right)$$

$$= 1$$

(vi) Prove that $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof: We know that

$$e = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{..........}\right)$$

$$\therefore \quad e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{..........} - 1\right)$$

$$= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{..........}\right)$$

$$\therefore \quad \frac{e^x - 1}{x} = \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{..........}\right)}{x}$$

Notes

[Dividing throughout by x]

$$= \frac{x\left(1+\frac{x}{2!}+\frac{x^2}{3!}+..........\right)}{x}$$

$$= \left(1+\frac{x}{2!}+\frac{x^2}{3!}+..........\right)$$

$\therefore$ $$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+..........\right)$$

$$= 1+0+0+......=1$$

Thus, $$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$

Example 6:

Examine the behaviour of the function in each of the following:

(i) $y=\dfrac{3x+1}{2x-4}$ when $x \to 2$, $x \to -\infty$ and $x \to \infty$

(ii) $y=\dfrac{1}{(x-1)^2}$ when $x \to 1$, $x \to -\infty$ and $x \to \infty$

(iii) $y=\dfrac{|a_1x+b_1|}{a_2x+b_2}$ when $x \to +\infty$

Show the behaviour by sketching graph, indicating the asymptotes of the function.

Solution:

Note that y is not defined in each of the above cases.

(i) LHL $=\lim\limits_{x\to 2-}\dfrac{3x+1}{2x-4}=\lim\limits_{h\to 0}\dfrac{3(2-h)+1}{2(2-h)-4}=\lim\limits_{h\to 0}\left(-\dfrac{7}{2h}+\dfrac{3}{2}\right)=-\infty$

RHL $=\lim\limits_{x\to 2+}\dfrac{3x+1}{2x-4}=\lim\limits_{h\to 0}\dfrac{3(2+h)+1}{2(2+h)-4}=\lim\limits_{h\to 0}\left(\dfrac{7}{2h}+\dfrac{3}{2}\right)=\infty$

Also $\lim\limits_{x\to -\infty}\dfrac{3x+1}{2x-4}=\lim\limits_{x\to -\infty}\left(\dfrac{3+1/x}{2-4/x}\right)=\dfrac{3}{2}$

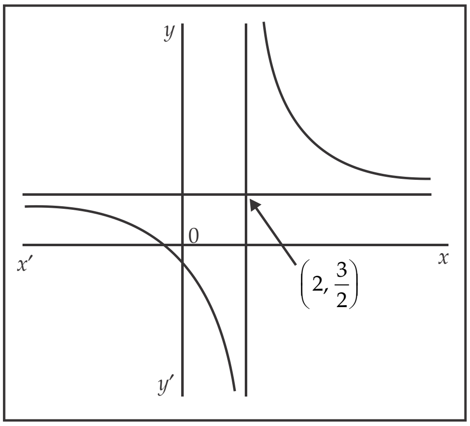
and $\lim\limits_{x\to \infty}\dfrac{3x+1}{2x-4}=\lim\limits_{x\to \infty}\left(\dfrac{3+1/x}{2-4/x}\right)=\dfrac{3}{2}$

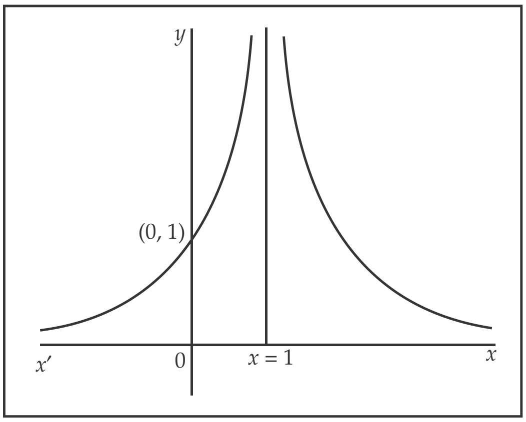
Note that there is a vertical asymptote at $x = 2$ and a horizontal asymptote $y=\dfrac{3}{2}$ to the function.

The behaviour of the function is shown in Fig. 7.3.

Notes

Figure 7.3 **Figure 7.4**





(ii) LHL $= \lim_{x\to 1-} \dfrac{1}{(x-1)^2} = \lim_{h\to 0} \dfrac{1}{(1-h-1)^2} = \lim_{h\to 0} \dfrac{1}{h^2} = \infty$

RHL $= \lim_{x\to 1+} \dfrac{1}{(x-1)^2} = \lim_{h\to 0} \dfrac{1}{(1+h-1)^2} = \lim_{h\to 0} \dfrac{1}{h^2} = \infty$

Further $\lim_{x\to -\infty} \dfrac{1}{(x-1)^2} = 0$ and $\lim_{x\to \infty} \dfrac{1}{(x-1)^2} = 0$

Thus function has x-axis as a horizontal asymptote and $x = 1$ is the vertical asymptote.

The graph of the function is shown in Figure 7.4.

(iii) Let $y = \lim_{x\to\infty} \dfrac{|a_1x+b_1|}{a_2x+b_2}$. This function implies that

$$y = \frac{a_1x+b_1}{a_2x+b_2} \text{ when } x \geq -\frac{b_1}{a_1}$$

$$= -\frac{a_1x+b_1}{a_2x+b_2} \text{ when } x < -\frac{b_1}{a_1}$$

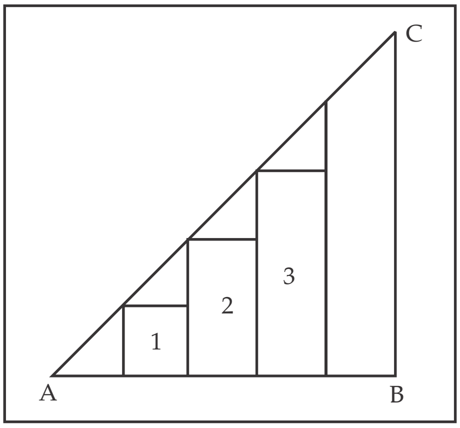
Hence $\lim_{x\to\infty} \dfrac{|a_1x+b_1|}{a_2x+b_2} \Rightarrow \lim_{x\to\infty} \dfrac{a_1x+b_1}{a_2x+b_2} = \lim_{x\to\infty} \dfrac{a_1+b_1/x}{a_2+b_2/x} = \dfrac{a_1}{a_2}$ (Horizontal asymptote).

Example 7:

A right-angled triangle has two equal sides of 1 inch. One of these sides, drawn horizontally, is divided into $(n + 1)$ equal portions. On each of the portion after the first a rectangle is formed with height equal to the vertical distance from the left-hand end of the portion to the hypotenuse of the triangle. Find an expression for the sum of rectangle areas and evaluate the limit of the sum as $n \to \infty$. What is the meaning of the limiting value?

Notes

Figure 7.5



Solution:

ABC is a right-angled triangle which is right angled at *B* and *AB* = *BC* = 1 inch. *AB* can be divided into (n + 1) parts by marking n points on it. We form a rectangle on each part after the first. The width of each rectangle will be $\frac{1}{n+1}$, while the length of first, second, ..., nth rectangle will be $\frac{1}{n+1}, \frac{2}{n+1}, \ldots\ldots, \frac{n}{n+1}$ respectively. The number of such rectangles is n. Let S be the sum of areas of these rectangles.

$$\therefore \qquad S = \frac{1}{(n+1)^2}[1+2+\ldots\ldots+n] = \frac{n}{2(n+1)},$$

$$\text{and } \lim_{n\to\infty} S = \frac{1}{2}\cdot\frac{1}{\left(1+\frac{1}{n}\right)} = \frac{1}{2}$$

This limiting value represents the area of the triangle.

Example 8.

Examine the demand curve $p = \frac{a}{x+b}$, where a and b are positive constants. Show that demand increases from zero to indefinitely large amounts as the price falls. What type of curve is the total revenue curve? Show that total revenue increases to a limiting value. Draw the graphs of demand and total revenue curves to support your argument.

Solution:

Since we have to examine the behaviour of demand as price changes, we write $(x+b) = \frac{a}{p}$ or $x = \frac{a}{p} - b$. The demand equals zero when $p = \frac{a}{b}$ and for small prices it is given by $x = \lim_{p\to 0}\left(\frac{a}{p} - b\right) = \infty$.

On combining the results of these two limits, we can say that demand increases from zero to infinitely large amount as the price falls.

We can write total revenue as $TR = px = \dfrac{ax}{x+b}$

Further, $\lim\limits_{x\to\infty} TR = \lim\limits_{x\to\infty}\left(\dfrac{ax}{x+b}\right) = \lim\limits_{x\to\infty}\left(\dfrac{a}{1+b/x}\right) = a,$

which implies that total revenue increases to a limiting value a.

To draw the graph of demand curve, we note that this is a rectangular hyperbola with centre at $(-b, 0)$ and asymptotes parallel to the axes. Since $a > 0$, the two parts of the curve lie in first and third quadrants, formed by the asymptotes. The part AB of the curve, where x and p are both positive is the relevant demand curve, as shown in Figure 7.6.

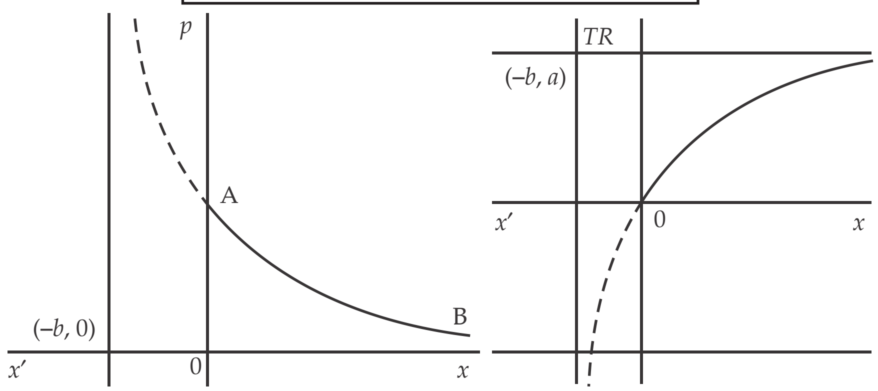
We can write the total revenue function as

$$TR(x + b) - ax = 0$$

or $$TR(x + b) - a(x + b) = -ab$$

or $$(x + b)(TR - a) = -ab$$

Figure 7.6



This is the equation of a rectangular hyperbola with centre at $(-b, a)$ and asymptotes parallel to axes. Since right hand side of the above equation is negative, the two parts of the curve lies in second and fourth quadrants, formed by the asymptotes. The relevant total revenue curve is where TR and x are both positive, as shown in Figure 7.6.

7.7 Summary

- If a function f(x) approaches l when x approaches a, we say that l is the limit of symbolically, it is written as

$$\lim_{x\to a} f(x) = \ell$$

- If $\lim\limits_{x\to a} f(x) = \ell$ and $\lim\limits_{x\to a} g(x) = m$, then

  - $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x) = k\ell$

Notes

- $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = \ell \pm m$
- $\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x)\lim_{x\to a} g(x) = \ell m$
- $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\ell}{m}$, Provided $\lim_{x\to a} g(x) \neq 0$

7.8 Keywords

Limits and Function Values: If the limit of a function f as x approaches c exists, this limit may not be equal to f(c). In fact, f(c) may not even be defined.

Polynomial Functions: If f(x)is a polynomial function and c is any real number, then $\lim_{x\to c} f(x) =$ f(c). In other words, the limit is the value of the polynomial function f at x = c.

7.9 Self Assessment

1. If f(x) = $x^2 + 5x + 3$, $\lim_{h\to 0}$ then value of f(x) is

   (a) 0 (b) 1

   (c) 3 (d) 9

2. Value of $\lim_{x\to 3}\left[\frac{(x-3)}{x-3}\right]$ is equal to

   (a) ∞ (b) 1

   (c) –2 (d) –∞

3. $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ is equal to

   (a) 0 (b) –1

   (c) –∞ (d) ∞

4. If $f(x) = \begin{cases} 1+2x^2 & \text{If } x \text{ is rational} \\ 1+x^4 & \text{If } x \text{ is rational} \end{cases}$ then f(x) will be

   (a) 1/2 (b) –1/2

   (c) 1 (d) –1

5. $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right)$ is equal to

   (a) ∞ (b) –∞

   (c) –1/2 (d) 0

6. $\lim_{x\to\infty}\left(-x^2\right)$ euqal to

   (a) ∞ (b) –∞

   (c) ∞^2 (d) $-\infty^2$

7. f(x) = $x^2 - 2$ of x < 1 lim f(x) is equal to

   (a) 1 (b) 2

   (c) –1 (d) –2

7.10 Review Questions

Notes

1. Give the properties of Functions in relation to limits.
2. (a) Show that f(x) = e^{5x} is a continuous function.

   (d) Show that f(x) = e^{-2x+5} is a continuous function.
3. By means of graph, examine the continuity of each of the following functions:

   (a) $f(x) = x + 1$ (b) $f(x) = \dfrac{x+2}{x-2}$

   (c) $f(x) = \dfrac{x^2-9}{x+3}$ (d) $f(x) = \dfrac{x^2-16}{x-4}$
4. Evaluate the following limits:

   (a) $\lim\limits_{x \to 1} 5$ (b) $\lim\limits_{x \to 0} \sqrt{2}$

   (c) $\lim\limits_{x \to 1} \dfrac{4x^5+9x+7}{3x^6+x^3+1}$ (d) $\lim\limits_{x \to -2} \dfrac{x^2+2x}{x^3+x^2-2x}$

   (e) $\lim\limits_{x \to 0} \dfrac{(x+k)^4-x^4}{k(k+2x)}$ (f) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

   (g) $\lim\limits_{x \to -1} \left[\dfrac{1}{x+1}+\dfrac{2}{x^2-1}\right]$ (h) $\lim\limits_{x \to 1} \dfrac{(2x-3)\sqrt{x}-1}{(2x+3)(x-1)}$

   (i) $\lim\limits_{x \to 2} \dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$ (j) $\lim\limits_{x \to 1} \left[\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right]$

   (k) $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x}$ (l) $\lim\limits_{x \to a} \dfrac{x^2-(a+1)x+a^2}{x^2-a^2}$
5. Find the left hand and right hand limits of the following functions:

   (a) $f(x) = \begin{cases} -2x+3 & \text{if } x \le 1 \\ 3x-5 & \text{if } x > 1 \end{cases}$ as $x \to 1$

   (b) $f(x) = \dfrac{x^2-1}{|x+1|}$ as $x \to 1$

Answers: Self Assessment

1. (d)	2. (b)	3. (a)	4. (c)
5. (d)	6. (b)	7. (c)	

Notes

Answers: Self Assessment

1. (d)	2. (b)	3. (a)	4. (c)
5. (d)	6. (b)	7. (c)	

7.11 Further Readings

Books

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

Smith and Minton, *Calculus Early Trancendental*, Third Edition, McGraw Hill, 2008.

Online links

www.en.wikipedia.org

www.web-source.net

www.webopedia.com

Unit 8: Continuity

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Describe the continuity of a function in an interval.
- Explain how to use the theorem of continuity of function with the help of different examples.

Introduction

Let f be a function that is defined for all x in some open interval containing c. Then f is said to be continuous at x = c under the following conditions:

1. f(c) is defined.

Notes

2. $\lim_{x \to c} f(x)$ exists.

3. $\lim_{x \to c} f(x)$ = f(c).

8.1 Continuity at a Point

Example: Show that the function $f(x) = \frac{\sqrt{x^2 - x + 1}}{x - 5}$ is continuous at x = –3.

1. $f(-3)\frac{\sqrt{(-3)2 - (-3) = 1}}{(-3) - 5} \quad = \quad -\frac{\sqrt{13}}{8}$ f(c) is defined.

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{\sqrt{x^2 - x + 1}}{x - 5}$$

$$= \frac{\lim_{x \to -3} \sqrt{x^2 - x + 1}}{\lim_{x \to -3} (x - 5)} \text{ limit of a quotient}$$

2. $= \frac{\sqrt{\lim_{x \to -3} \left(x^2 - x + 1\right)}}{\lim_{x \to -3} (x - 5)}$ limit of a root $\lim_{x \to c} f(x)$ exists.

$$= \frac{\sqrt{(-3)^2 - (-3) + 1}}{(-3) - 5}$$

$$= \frac{\sqrt{13}}{8}$$

Therefore, $\lim_{x \to -3} f(x) = f(-3)$ and f is continuous at x = –3.

8.1.1 Continuity of Special Functions

- Every polynomial function is continuous at every real number.
- Every rational function is continuous at every real number in its domain.
- Every exponential function is continuous at every real number.
- Every logarithmic function is continuous at every positive real number.
- f(x) = sin x and g(x) = cos x are continuous at every real number.
- h(x) = tan x is continuous at every real number in its domain.

8.1.2 Continuity from the Left and Right

- A function f is continuous from the right at x = a provided that $\lim_{x \to a^+} f(x) = f(a)$.
- A function f is continuous from the right at x = b provided that $\lim_{x \to b^-} f(x) = f(b)$.

8.1.3 Continuity at an End Point

Example: Show that $f(x) = \sqrt{x}$ is continuous from the right at x = 0.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = 0$$

$$f(0) = \sqrt{0} = 0$$

8.1.4 Continuity on an Interval

- A function f is said to be *continuous on an open interval* (a, b) provided that f is continuous at every value in the interval.
- A function f is said to be *continuous on a closed interval* [a, b] provided that f is continuous from the right at x = a, continuous from the left at x = b, and continuous at every value in the open interval (a, b).

8.1.5 Properties of Continuous Functions

If the functions f and g are continuous at x = c, then each of the following functions is also continuous at x = c:

- The sum function f + g
- The difference function f – g
- The product function fg
- The quotient function f/g, g(c) ≠ 0

8.1.6 Properties of Composite Functions

If the function f is continuous at x = c and the function g is continuous at x = f(c), then the composite function g o f is continuous at x = c.

Example: Show that $h(x) = \sqrt{x^3 - 3x^2 + x + 7}$ is continuous at x = 2.

Solution:

$f(x) = x^3 - 3x^2 + x + 7$ and $g(x) = \sqrt{x}$.

$$g \circ f(x) = g\left(x^3 - 3x^2 + x + 7\right) = \sqrt{x^3 - 3x^2 + x + 7}$$

f is continuous at x = 2 and $f(2) = 2^3 - 3 \cdot 22 + 2 + 7 = 5$

g is continuous at 5 since $\lim_{x \to 5} \sqrt{x} = \sqrt{\lim_{x \to 5} x} = \sqrt{5} = g(5)$

Therefore, g o f(x) is also continuous at x = 2.

Notes

Example: Consider the function

$$f(x) = \begin{cases} x^3 + 2 & \text{if } x < 2 \\ 5 & \text{if } x = 2 \\ x^2 + 6 & \text{if } x > 2. \end{cases}$$

The details are left to the reader to see

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^3 + 2 = 10.$$

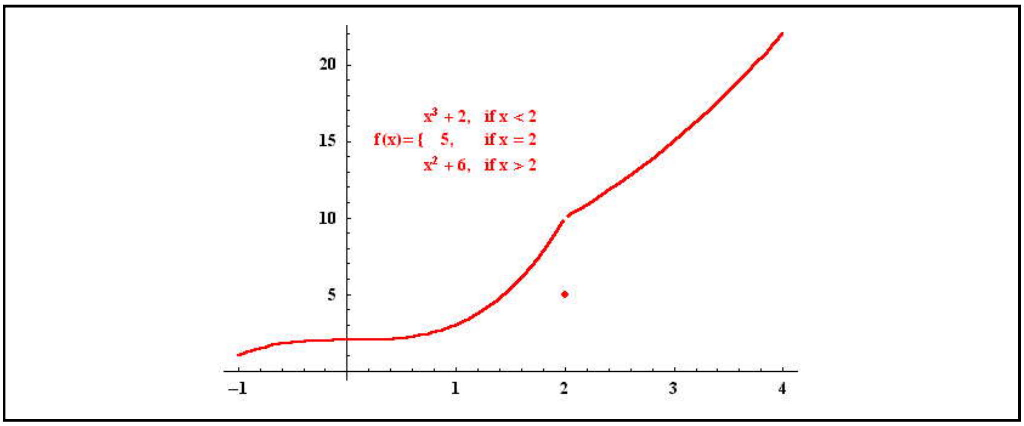
and

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 + 6 = 10.$$

So we have

$$\lim_{x \to 2} f(x) = 10.$$

Since f(2) = 5, then f(x) is not continuous at 2.



Example: Find A which makes the function

$$f(x) = \begin{cases} x^2 - 2 & \text{if } x < 1 \\ Ax - 4 & \text{if } 1 \le x \end{cases}$$

continuous at x=1.

Solution:

We have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 - 2 = -1,$$

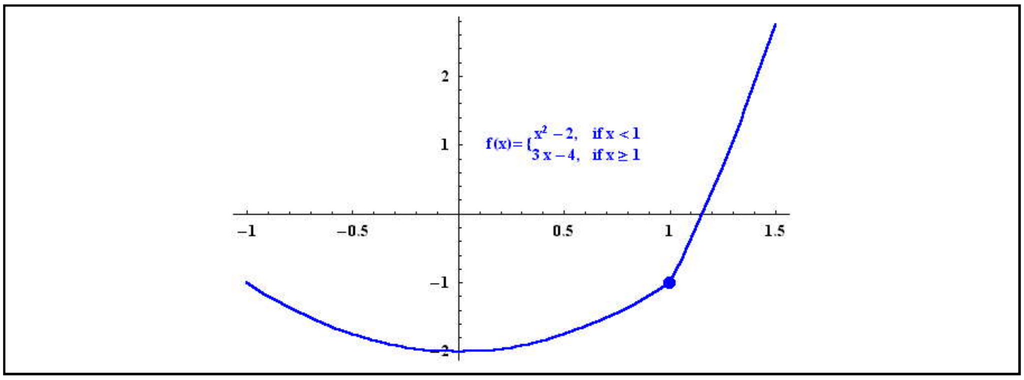
and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} Ax - 4 = A - 4.$$

So f(x) is continuous at 1 if

$$A - 4 = -1 \text{ or equivalently if } A = 3.$$

Notes



Definition: For a function f(x) defined on a set S, we say that f(x) is continuous on S if f(x) is continuous for all $a \in S$.

Example: We have seen that polynomial functions are continuous on the entire set of real numbers. The same result holds for the trigonometric functions sin(x) and cos(x).

The following two exercises discuss a type of functions hard to visualize. But still one can study their continuity properties.

Example: Discuss the continuity of

$$f(x) = \begin{cases} 1 & \text{if x is rational} \\ 0 & \text{if x is irrational.} \end{cases}$$

Solution:

Let us show that for any number a, the limit $\lim_{x \to a} f(x)$ does not exist. Indeed, assume otherwise that

$$\lim_{x \to a} f(x) = L.$$

Then from the definition of the limit implies that for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$| x - a | < \delta \Rightarrow | f(x) - L | < \varepsilon.$$

Set $\varepsilon = \frac{1}{3}$. Then exists $\delta > 0$, such that

$$| x - a | < \delta \Rightarrow |f(x) - L| < \frac{1}{3}$$

or equivalently

$$a - \delta < x < a - < \delta \Rightarrow |f(x) - L| < \frac{1}{3}.$$

Since any open interval contains a rational and an irrational numbers, then we should have

$$|0 - L| < \frac{1}{3} \text{ and } |1 - L| < \frac{1}{3}.$$

Combining the two inequalities we get

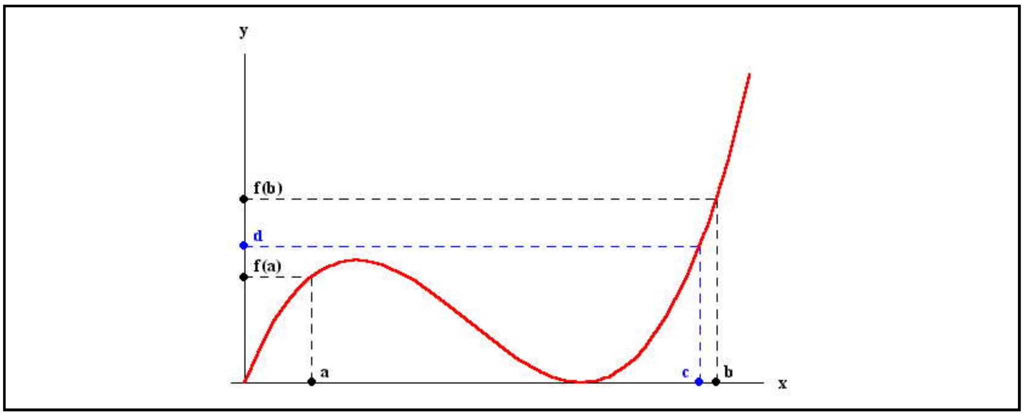
$$|1 - 0| \le |0 - L| + |1 - L| < \frac{2}{3},$$

which leads to an obvious contradiction. Thus, the function is discontinuous at every point a.

Notes

8.2 The Intermediate Value Theorem

Let f (x) be a continuous function on the interval [a, b]. If d ∈ [f (a), f (b)], then there is a c ∈ [a, b] such that f (c) = d.



In the case where f (a) > f (b), [f (a), f (b)] is meant to be the same as [f (b), f (a)]. Another way to state the Intermediate Value Theorem is to say that the image of a closed interval under a continuous function is a closed interval.

Here is a classical consequence of the Intermediate Value Theorem:

Example: Every polynomial of odd degree has at least one real root.

We want to show that if $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ is a polynomial with n odd and an ≠ 0, then there is a real number c, such that P(c) = 0.

First let me remind you that it follows from the results in previous pages that every polynomial is continuous on the real line. There you also learned that

$$\lim_{n\to\infty} \frac{P(x)}{a_n x^n} = 1 \text{ and } \lim_{n\to-\infty} \frac{P(x)}{a_n x^n} = 1.$$

Consequently for | x| large enough, P(x) and anxn have the same sign. But anxn has opposite signs for positive x and negative x. Thus it follows that if an > 0, there are real numbers x0 < x_1 such that $P(x_0) < 0$ and $P(x_1) > 0$. Similarly if an < 0, we can find $x_0 < x_1$ such that $P(x_0) > 0$ and $P(x_1) < 0$. In either case, it now follows directly from the Intermediate Value Theorem that (for d = 0) there is a real number c ∈ $[x_0, x_1]$ with P(c) = 0.

The natural question arises whether every function which satisfies the conclusion of the Intermediate Value Theorem must be continuous. Unfortunately, the answer is no and counterexamples are quite messy. The easiest counterexample is the function:

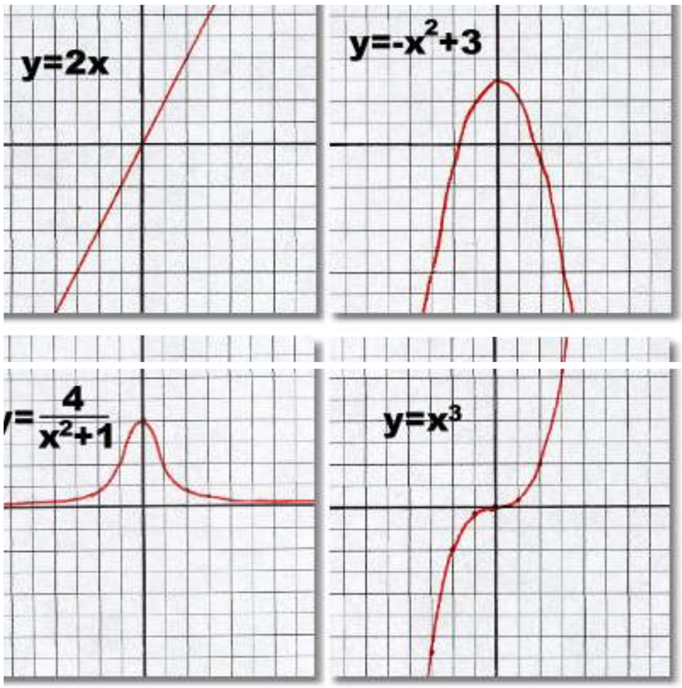
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & , \text{if } x \neq 0 \\ 0 & , \text{if } x = 0 \end{cases}$$

As we found this function fails to be continuous at x = 0. On the other hand, it is not too hard to see that f (x) has the "Intermediate Value Property" even on closed intervals containing x = 0.

Notes

8.2.1 Continuous Functions

A function is continuous if it has no breaks. On this page we'll first look at some common continuous functions, and then show you the discontinuous ones that you're likely to come across in high school mathematics.

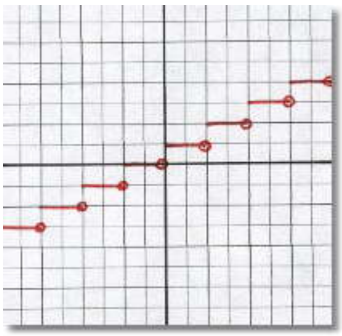


The three functions above are all ones you have seen before: a linear, a quadratic, and a cubic function. The domain of all three is the entire set of Real numbers, and all three functions continue left to right, in both directions, to infinity, without a gap anywhere.

'Continuous' means 'no gaps', or being able to put your finger on the curve and follow it across the grid without having to lift and move your finger.

On the left is a function you may not have seen before ... it's asymptotic to the x-axis, and has a maximum y-value of 4. This function is also continuous ... there are no gaps. (Incidentally, notice that despite the x in the denominator, this Rational expression has no undefined values ... the denominator can never equal zero. Can you see why not?

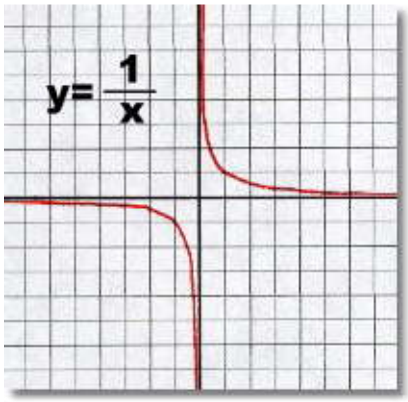
8.2.2 Discontinuous Functions



Notes

This is probably the first discontinuous function you learned about. It's called a step function, and its domain is still the entire set of Real numbers. (The open circles mean that, for example, at x=2, the y-value is no longer 1, but 2).

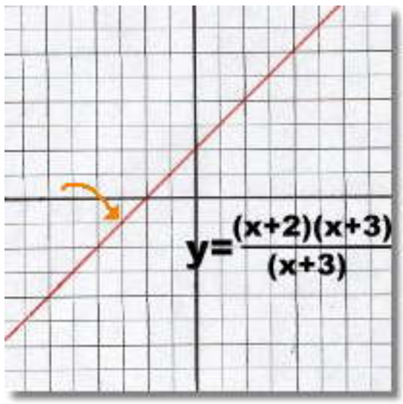
There are clearly gaps when the function jumps to each new value. You can't run your finger along the graph without lifting it to move to the next portion. This function is **discontinuous.**



The next example, at the right, is a Rational expression function where there is an undefined value of x. The value of x can never equal zero, since division by zero is not defined.

As a result, there is an asymptote at x=0; the graph has a break there. On either side of this gap the graph approaches infinity.

You can't run your finger along the graph without lifting it to move to the next portion. This function is **discontinuous.**

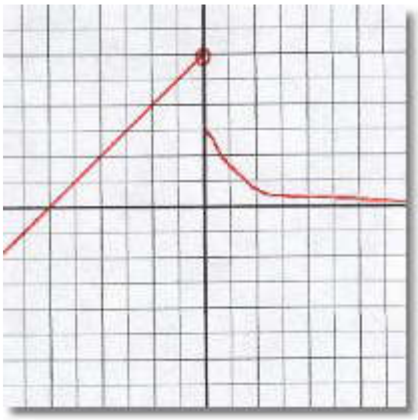


The graph on the left is one you may have come across before. It is very mysterious ... the graph all by itself looks like the simple linear function y = x + 2.

If you examine this function's actual equation, you will notice that it's a Rational expression. The x-value of –3 is undefined. This means there must be a gap at –3, even though you can't see it!

The values of x have corresponding points on the graph right up to –3 on either side, but there is no value for x = –3 itself. This one missing point can't be seen, so although there is a gap, it isn't visible! This function is discontinuous.

Notes



There are many types of discontinuous functions, all of which exhibit one common feature ... there is always a gap.

$$\begin{cases} y = x + 6, & x < 0 \\ y = \dfrac{3}{x^2+1}, & x \geq 0 \end{cases}$$

At the right is a graph made from two different equations:

Again notice that the domain is all Real numbers, but there is still a gap. This function is also discontinuous.

8.2.3 Removing Discontinuous Function

The first way that a function can fail to be continuous at a point a is that

$$\lim_{x \to a} f(x) = L \text{ exists (and is finite)}$$

but f(a) is not defined or $f(a) \neq L$.

Discontinuities for which the limit of f(x) exists and is finite are called removable discontinuities for reasons explained below:

f(a) is not defined.

If f(a) is not defined, the graph has a "hole" at (a, f(a)). This hole can be filled by extending the domain of f(x) to include the point x = a and defining

$$f(a) = \lim_{x \to a} f(x).$$

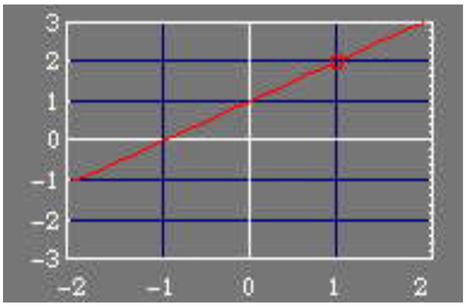
This has the effect of removing the discontinuity.

As an example, consider the function $g(x) = (x^2 - 1)/(x - 1)$. Then $g(x) = x + 1$ for all real numbers except x = 1. Since g(x) and x + 1 agree at all points other than the objective,

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} x + 1 = 2.$$

Notes

If $\lim_{x \to a} f(x) = L$ but f(a) is not defined then the discontinuity at x=a can be removed by defining f(a)=L.



Graph of $(x^2 - 1)/(x - 1)$ If $\lim_{x \to a} f(x) = L$ but $f(a) \neq L$

We can "remove" the discontinuity by filling the hole. The domain of g(x) may be extended to include x = 1 by declaring that g(1) = 2. This makes g(x) continuous at x = 1. Since g(x) is continuous at all other points (as evidenced, for example, by the graph), defining g(x) = 2 turns g into a continuous function.

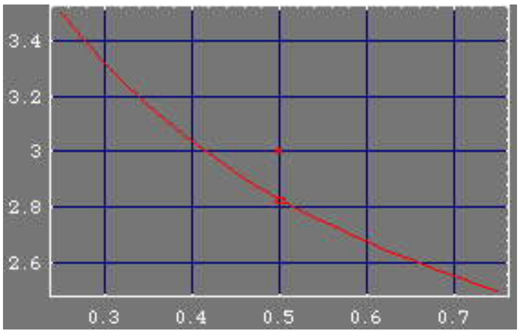
The limit and the value of the function are different.

If the limit as x approaches a exists and is finite and f(a) is defined but not equal to this limit, then the graph has a hole with a point misplaced above or below the hole. This discontinuity can be removed by re-defining the function value f(a) to be the value of the limit.

Then the discontinuity at x = a can be removed by re-defining f(a) = L.

As an example, the piecewise function in the second equipment was given by

$$h(x) = \begin{cases} \text{Undefined} & \text{Unless } 0 < x < 1 \\ 3 & \text{If } x = .5 \\ 1.5 + 1/(x + .25) & 0 < x < 1, x \neq .5 \end{cases}$$



Portion of the graph of h(x)

We can remove the discontinuity by re-defining the function so as to fill the hole. In this case we re-define h(.5) = 1.5 + 1/(.75) = 17/6.

Notes

8.3 Bisection Method

Let f(x) be a continuous function on the interval [a, b]. If d ∈ [f(a), f(b)], then there is a c ∈ [a, b] such that f(c) = d.

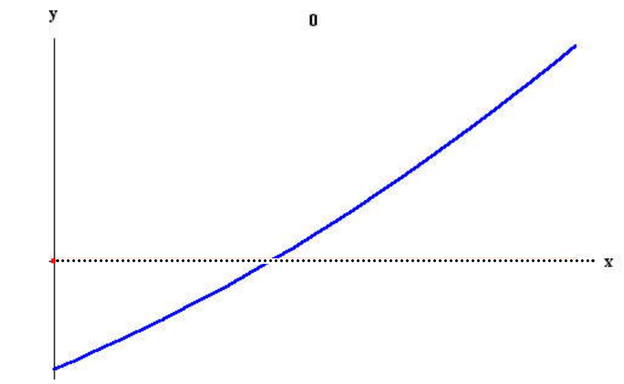
By replacing f(x) by f(x) – d, we may assume that d = 0; it then suffices to obtain the following version: Let f(x) be a continuous function on the interval [a, b]. If f(a) and f(b) have opposite signs, then there is a c ∈ [a, b] such that f(c) = 0.

Here is an outline of its proof: Let's assume that f(a) < 0, while f(b) > 0, the other case being handled similarly. Set $a_0 = a$ and $b_0 = b$.

Now consider the midpoint $m_0 = \frac{a_0 + b_0}{2}$, and evaluate $f(m_0)$. If $f(m_0) < 0$, set $a_1 = m_0$ and $b_1 = b_0$. If $f(m_0) > 0$, set $a_1 = a_0$ and $b_1 = m_0$. (If $f(m_0) = 0$, you're already done.) What situation are we in now? $f(a_1)$ and $f(b_1)$ still have opposite signs, but the length of the interval $[a_1, b_1]$ is only half of the length of the original interval $[a_0, b_0]$. Note also that $a_0 \le a_1$ and that $b_0 \ge b_1$.

You probably guess this by now: we will do this procedure again and again.

Here is the second step: Consider the midpoint $m_1 = \frac{a_1 + b_1}{2}$, and evaluate $f(m_1)$. If $f(m_1) < 0$, set $a_2 = m_1$ and $b_2 = b_1$. If $f(m_1) > 0$, set $a_2 = a_1$ and $b_2 = m_1$. (If $f(m_1) = 0$, you're already done.) What situation are we in now? $f(a_2)$ and $f(b_2)$ still have opposite signs, but the length of the interval $[a_2, b_2]$ is only a quarter of the length of the original interval $[a_0, b_0]$. Note also that $a_0 \le a_1 \le a_2$ and that $b_0 \ge b_1 \ge b_2$.



The red line shows the interval $[a_n, b_n]$.

Continuing in this fashion we construct by induction two sequences:

$$(a_n)_{n=1}^{\infty} \text{ and } (b_n)_{n=1}^{\infty}$$

with the following properties:

1. (a_n) is an increasing sequence, (b_n) is a decreasing sequence.
2. $a_n \le b_n$ for all n.
3. $f(a_n) < 0$ for all n, $f(b_n) > 0$ for all n.
4. $b_n - a_n = 2^{-n}(b - a)$ for all n.

It follows from the first two properties that the sequences (a_n) and (b_n) converge; set

$$\lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b.$$

Notes

The third property and the continuity of the function f(x) imply that f(a) ≤ 0 and that f(b) ≥ 0.

The crucial observation is the fact that the fourth property implies that a = b. Consequently, f (a) = f (b) = 0, and we are done.

Example: Let's compute numerical approximations for the $\sqrt{2}$ with the help of the bisection method. We set $f(x) = x^2 - 2$. Let us start with an interval of length one: $a_0 = 1$ and $b_1 = 2$. Note that $f(a_0) = f(1) = -1 < 0$, and $f(b_0) = f(2) = 2 > 0$. Here are the first 20 applications of the bisection algorithm:

n	a_n	b_n
0	1.	2.
1	1.	1.5
2	1.25	1.5
3	1.375	1.5
4	1.375	1.4375
5	1.40625	1.4375
6	1.40625	1.421875
7	1.4140625	1.421875
8	1.4140625	1.41796875
9	1.4140625	1.416015625
10	1.4140625	1.415039063
11	1.4140625	1.414550781
12	1.4140625	1.414306641
13	1.41418457	1.414306641
14	1.41418457	1.414245605
15	1.41418457	1.414215088
16	1.414199829	1.414215088
17	1.414207458	1.414215088
18	1.414211273	1.414215088
19	1.414213181	1.414215088
20	1.414213181	1.414214134

Bisection is the division of a given curve, figure, or interval into two equal parts (halves).

A simple bisection procedure for iteratively converging on a solution which is known to lie inside some interval $[a, b]$ proceeds by evaluating the function in question at the midpoint of the original interval $x = (a + b)/2$ and testing to see in which of the subintervals $[a, (a + b)/2]$ or $[(a + b)/2, b]$ the solution lies. The procedure is then repeated with the new interval as often as needed to locate the solution to the desired accuracy.

Let a_n and b_n be the endpoints at the ^{n}th iteration (with $a_1 = a$ and $b_1 = b$) and let r_n be the ^{n}th approximate solution. Then the number of iterations required obtaining an error smaller than $\in$ is found by noting that

Notes

$$b_n - a_n = \frac{b-a}{2^{n-1}} \quad \text{...(i)}$$

and that r_n is defined by

$$r_n = \frac{1}{2}(a_n + b_n) \quad \text{...(ii)}$$

In order for the error to be smaller than $\in$,

$$|r_n = r| \leq \frac{1}{2}(b_n - a_n) = 2^{-n}(b-a) < \in \quad \text{...(iii)}$$

Taking the natural logarithm of both sides then gives

$$-n \ln 2 < \ln \in - \ln (b-a), \quad \text{...(iv)}$$

so from 1,2,3 and 4 the result is

$$n > \frac{\ln(b-a) - \ln \in}{\ln 2}$$

8.4 Function at a Point

So far, we have considered only those functions which are continuous. Now we shall discuss some examples of functions which may or may not be continuous.

Example: Show that the function $f(x) = e^x$ is a continuous function.

Solution:

Domain of e^x is R. Let $a \in R$. where 'a' is arbitrary.

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h), \text{where h is a very small number.}$$

$$= \lim_{h\to 0} e^{a+h}$$

$$= \lim_{h\to 0} e^a.e^h$$

$$= e^a \lim_{h\to 0} e^h$$

$$= e^a \times 1 \quad \text{...(i)}$$

$$= e^a \quad \text{...(ii)}$$

$$f(a) = e^a$$

$\therefore$ From (i) and (ii), $\lim_{x\to a} f(x) = f(a)$

$\therefore$ f(x) is continuous at x = a

Since a is arbitary, e^x is a continuous function.

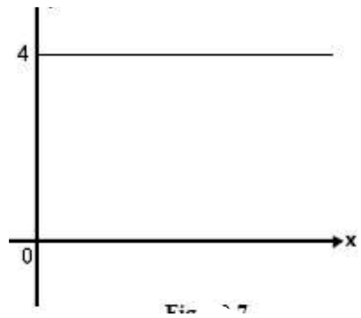
8.4.1 Properties of Continuos Function

1. Consider the function f (x) = 4. Graph of the function f (x) = 4 is shown in the Figure 8.1. From the graph, we see that the function is continuous. In general, all constant functions are continuous.

Notes

2. If a function is continuous then the constant multiple of that function is also continuous.

Figure 8.1



Consider the function f(x) = $\frac{7}{2}x$. We know that x is a constant function. Let 'a' be an arbitrary real number.

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$$

$$= = \lim_{h\to 0} \frac{7}{2}(a+h)$$

$$= = \frac{7}{2}a \qquad \ldots(i)$$

Also $\quad f(a) = \frac{7}{2}a \qquad \ldots(ii)$

From (i) and (ii),

$$\lim_{x\to a} f(x) = f(a)$$

∴ f(x) = $\frac{7}{2}x$ is continuous at x = a.

As $\frac{7}{2}$ is constant, and x is continuous function at x = a, $\frac{7}{2}x$ is also a continuous function at x = a.

3. Consider the function f(x) = $x^2 + 2x$. We know that the function x^2 and 2x are continuous.

Now $\quad \lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$

$$= \lim_{h\to 0}\left[(a+h)^2 + 2(a+h)\right]$$

$$= \lim_{h\to 0}\left[a^2 + 2ah + h^2 + 2a + 2ah\right]$$

$$= a^2 + 2a \qquad \ldots(i)$$

Also $\quad f(a) = a^2 + 2a \qquad \ldots(ii)$

∴ From (i) and (ii), $\lim_{x\to a} f(x) = f(a)$

∴ f(x) is continuous at x = a.

Thus we can say that if x^2 and 2x are two continuous functions at x = a then $(x^2 + 2x)$ is also continuous at x = a.

4. Consider the function $f(x) = (x^2 + 1)(x + 2)$. We know that $(x^2 + 1)$ and $(x + 2)$ are two continuous functions.

Also $f(x) = (x^2 + 1)(x + 2)$

$= x^3 + 2x^3 + x + 2$

As x^3, $2x^2$, x and 2 are continuous functions, therefore.

$x^3 + 2x^2 + x + 2$ is also a continuous function.

$\therefore$ We can say that if $(x^2 + 1)$ and $(x + 2)$ are two continuous functions then $(x^2 + 1)(x + 2)$ is also a continuous function.

5. Consider the function $f(x) = \frac{x^2 - 4}{x + 2}$ at x = 2. We know that $(x^2 - 4)$ is continuous at x = 2. Also (x + 2) is continuous at x = 2.

Again $\lim\limits_{x \to 2} \frac{x^2 - 4}{x + 2} = \lim\limits_{x \to 2} \frac{(x+2)(x-2)}{(x+2)}$

$= \lim\limits_{x \to 2} (x - 2)$

$= 2 - 2 = 0$

Also $f(2) = \frac{(2)^2 - 4}{2 + 2}$

$= \frac{0}{4} = 0$

$\therefore$ $\lim\limits_{x \to 2} f(x) = f(2)$. Thus f(x) is continuous at x = 2.

$\therefore$ If $(x^2 - 4)$ and x + 2 are two continuous functions at x = 2, then $\frac{x^2 - 4}{x + 2}$ is also continuous.

6. Consider the function $f(x) = |x - 2|$. The function can be written as:

$$f(x) = \begin{cases} -(x-2), & x < 2 \\ (x-2), & x \geq 2 \end{cases}$$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{h \to 0} f(2 - h), h > 0$

$= \lim\limits_{h \to 0} [(2 - h) - 2]$

$= 2 - 2 = 0$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{h \to 0} f(2 + h), h > 0$ …(i)

$= \lim\limits_{x \to 2} [(2 + h) - 2]$

$= 2 - 2 = 0$ …(ii)

Also $f(2) = (2 - 2) = 0$ …(iii)

$\therefore$ From (i), (ii) and (iii), $\lim\limits_{x \to 2} f(x) = f(2)$

Notes

Thus, $|x - 2|$ is continuous at x = 2

After considering the above results, we state below some properties of continuous functions.

If f (x) and g (x) are two functions which are continuous at a point x = a, then,

(a) C f(x) is continuous at x = a, where C is a constant.

(b) f(x) ≠ g(x) is continuous at x = a.

(c) f(x) . g(x) is continuous at x = a.

(d) f (x)/g (x) is continuous at x = a, provided g (a) ≠ 0.

(e) $|f(x)|$ is continuous at x = a.

Thus every constant function is a continues function

Tasks

1. Prove that tan x is continuous when $0 \le x < \frac{\pi}{2}$

2. Let f (x) = $f(x) = \begin{cases} x^2 & \text{for } x \le 1, \\ x & \text{for } x \ge 1. \end{cases}$ Show that f is continuous at 1.

8.4.2 Important Result of Constant Function

By using the properties mentioned above, we shall now discuss some important results on continuity.

1. Consider the function f(x) = px + q, x ∈ R

   The domain of this functions is the set of real numbers. Let a be any arbitary real number. Taking limit of both sides of (i), we have

   $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} (px + q) = pa + q \quad [= \text{value of } p\,x + q \text{ at } x = a.]$

   ∴ px + q is continuous at x = a.

   Similarly, if we consider f(x) = $5x^2$ + 2x + 3, we can show that it is a continuous function.

   In general $f(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n$

   where a_0 + a_1 + a_2 … a_n are constants and n is a non-negative integer,

   we can show that $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ are all continuos at a point x = c (where c is any real number) and by property 2, their sum is also continuous at x = c.

∴ f (x) is continuous at any point c.

   Hence every polynomial function is continuous at every point.

2. Consider a function f(x) = $f(x) = \frac{(x+1)(x+3)}{(x-5)}$, f(x) is not defined when x – 5 = 0 i.e, at x = 5.

   Since (x + 1) and (x + 3) are both continuous, we can say that (x + 1) (x + 3) is also continuous. [Using property 3]

∴ Denominator of the function f (x), i.e., (x – 5) is also continuous.

∴ Using the property 4, we can say that the function $f(x) = \frac{(x+1)(x+3)}{(x-5)}$ is continuous at all points except at x = 5.

In general if $f(x) = \frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomial function and $q(x) \neq 0$, then f(x) is continuous if p(x) and q(x) both are continuous.

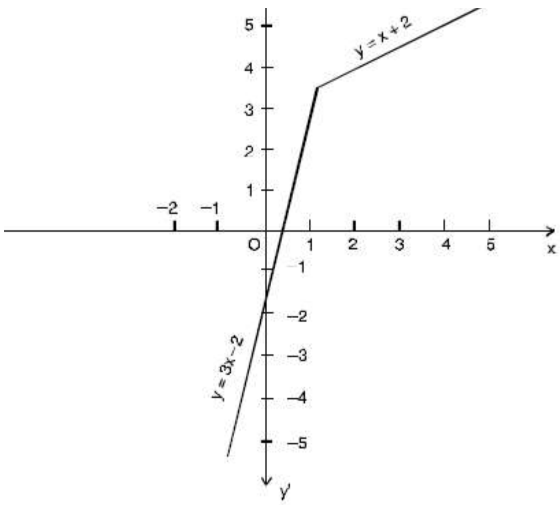
Example: Examine the continuity of the following function at x = 2.

$$f(x) = \begin{cases} 3x - 2 & \text{for } x < 2 \\ x + 2 & \text{for } x \geq 2 \end{cases}$$

Solution:

Since f (x) is defined as the polynomial function 3x – 2 on the left hand side of the point x = 2 and by another polynomial function x + 2 on the right hand side of x = 2, we shall find the left hand limit and right hand limit of the function at x = 2 separately.

Figure 8.2



Left hand limit $= \lim_{x \to 2^-} f(x)$

$= \lim_{x \to 2} (3x - 2)$

$= 3 \times 2 - 2 = 4$

Right hand limit at x = 2;

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (x + 2) = 4$$

Since the left hand limit and the right hand limit at x = 2 are equal, the limit of the function f (x) exists at x =2 and is equal to 4 i.e., $\lim_{x \to 2} f(x) = 4$

Also f(x) is defined by (x + 2) at x = 2

$\therefore \quad f(2) = 2 + 2 = 4.$

Thus, $\lim_{x \to 2} f(x) = f(2)$

Hence f(x) is continuous at x = 2.

Notes

Tasks

1. If $f(x) = \begin{cases} 4x+3, & x \neq 2 \\ 3x+5, & x = 2 \end{cases}$, find whether the function f is continuous at x = 2.

2. Determine whether f(x) is continuous at x = 2, where

$$f(x) = \begin{cases} 4x+3, & x \leq 2 \\ 8-x, & x > 2 \end{cases}$$

3. Examine the continuity of f(x) at x = 1, where

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ x+5, & x > 1 \end{cases}$$

4. Determine the values of k so that the function

$$f(x) = \begin{cases} kx^2, & x \leq 2 \\ 3, & x > 2 \end{cases}$$

is continuous at x = 2.

Example

A travel agency charges Rs 10 per km for travelling upto 300 kms. The agency gives a discount of Rs 2 per km, in revenue, for distance covered in excess of 300 kms. Express the revenue of the company as a function of the distance covered and examine its continuity when distance travelled is 300 kms.

Solution:

Let x be the distance covered in kms and $R(x)$ be the revenue.

$$\therefore \quad R(x) = \begin{cases} 10x & \text{if} \quad 0 < x \leq 300 \\ [10x - 2(x-300)]x & \text{if} \quad x > 300 \end{cases}$$

$$R(x) = \begin{cases} 10x & \text{if} \quad 0 \leq x \leq 30 \\ 8x - 600 & \text{if} \quad x > 300 \end{cases}$$

Now R(300) = 10 × 300 = 3000

LHL = 10 × 300 = 3000

RHL = 8 × 300 + 600 = 3000

Since LHL = RHL = R(300)

The function is continuous at x = 300.

Example

A wholesaler of pencils charges ₹ 30 per dozen on orders of 50 dozens or less. For orders in excess of 50 dozens, the price charged is ₹ 29. If x denote the no. of dozens of pencils, express the revenue function of the wholesaler as a function of x. Is this function continuous everywhere?

Solution:

We can write the revenue function as

$$R(x) = \begin{cases} 30x & \text{if} \quad 0 < x \leq 50 \\ 29x & \text{if} \quad x > 50 \end{cases}$$

First we examine continuity at x = 50

LHL = 30 × 50 = 1500

RHL = 29 × 50 = 1450.

Since LHL ≠ RHL, the function is discontinous at x = 50.

At other values of x the function is a polynomial which is continuous.

8.5 Summary

- Every polynomial function is continuous at every real number.
- Every rational function is continuous at every real number in its domain.
- Every exponential function is continuous at every real number.
- Every logarithmi-c function is continuous at every positive real number.
- f(x) = sin x and g(x) = cos x are continuous at every real number.
- h(x) = tan x is continuous at every real number in its domain.
- A function f is continuous from the right at x = a provided that $\lim_{x \to a^+} f(x) = f(a)$.
- A function f is continuous from the right at x = b provided that $\lim_{x \to b^-} f(x) = f(b)$.
- Let f (x) be a continuous function on the interval [a, b]. If d ∈ [f (a), f (b)], then there is a c ∈ [a, b] such that f (c) = d.

8.6 Keywords

Continuity on an Interval: A function f is said to be *continuous on an open interval* (a, b) provided that f is continuous at every value in the interval.

Infinite Limits: The sign of the infinite limit is determined by the sign of the quotient of the numerator and the denominator at values close to the number that the independent variable is approaching.

8.7 Self Assessment

1. If f(x) = Ax – 4 of 1 ≤ 4 then value of f(x) is
   (a) A – 2
   (b) A + 2
   (c) A – 4
   (d) A + 4
2. $f(x) = \dfrac{\cos^2 x + 1}{x}$, find value of when $\lim_{x \to \infty} f(x)$
   (a) 0
   (b) –∞
   (c) ∞
   (d) 1

Notes

3. If $f(x) = \frac{\sqrt{x^2 - x + 1}}{x - 5}$ is continuous at x = –3 then value of f(x) is

   (a) $\frac{\sqrt{13}}{8}$

   (b) $\frac{\sqrt{8}}{13}$

   (c) $\frac{13}{8}$

   (d) $\sqrt{\frac{13}{8}}$

8.8 Review Questions

1. Find whether the function f(x) = [x] is continuous at

   (a) $x = \frac{4}{3}$ (b) x = 3

   (c) x = –1 (d) $x = \frac{2}{3}$

2. Evaluate the following limits:

   (a) $f(x) = \begin{cases} -2x + 3 & \text{if } x \le 1 \\ 3x - 5 & \text{if } x > 1 \end{cases}$ as $x \to 1$

   (b) $\lim_{x \to 2^+} \frac{|x - 2|}{x - 2}$

   (c) $\lim_{x \to 2^-} \frac{x - 2}{|x - 2|}$

   (d) If $f(x) \frac{(x + 2)^2 - 4}{x}$, prove that $\lim_{x \to 0} f(x) = 4$ through f(0) is not defined.

   (e) Find k so that $\lim_{x \to 2} f(x)$ may exist where

   $$f(x) = \begin{cases} 5x + 2, & x \le 2 \\ 2x + k, & x > 2 \end{cases}$$

   (f) $\lim_{x \to 0} \frac{\sin 7x}{2x}$

   (g) $\lim_{x \to 0} \left[\frac{e^x + e^{-x} - 2}{x^2} \right]$

3. Examine the continuous of the following functions:

   (a) f(x) = |x – 2| at x = 2

   (b) f(x)= |x + 5| at x = –5

   (c) f(x)= |a – x| at x = a

4. Find whether f(x) is continuous at x = 0 or not, where

$$f(x) = \begin{cases} \frac{x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

5. Examine the continuity of the following function at x = 2.

$$f(x) = \begin{cases} 3x - 2, & \text{for } x < 2 \\ x + 2, & \text{for } x \geq 2 \end{cases}$$

Answers: Self Assessment

1. (c) 2. (a) 3. (a)

8.9 Further Readings

Books

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

Smith and Minton, *Calculus Early Trancendental,* Third Edition, McGraw Hill, 2008.

Online links

www.en.wikipedia.org

www.web-source.net

www.webopedia.com

Notes

Unit 9: Differential Calculus

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the derivative of a function
- Explain the certain basic rules that can be used to find the derivatives of various types of composite functions

Introduction

Given a function, we are often interested to know how the change in one variable corresponds to changes in the other. The questions relating to rates of changes require the introduction to the concept of derivatives. The maxima and minima of functions, an important application of derivatives, will also be discussed in this unit.

In this unit, we study various methods of differentiation and its application. It deals with the study of change.

9.1 Differentiation

A function $f(x)$ is said to be differentiable at $x = a$ if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. This limit is denoted by $f'(a)$ and

Notes

$\therefore \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

> *Notes* $f(a)$ exists if the $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists as $x \to a$ through values < a (left hand limit) and $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ through values > a (right hand limit) exist and further they are equal.

9.1.1 Derivative of a Function - Method of First Principles

If $y = f(x)$ is a function then as x changes y also changes.

A change in x is called the increment in x and is denoted by Δx. Corresponding change in y is called increment in y and is denoted by Δy.

$\therefore$ as x changes to $x + \Delta x$, y changes to $y + \Delta y$.

First Principles

Let $\quad y = f(x)$ ...(i)

$\therefore \quad y + \Delta y = f(x + \Delta x)$ ...(ii)

Subtracting (i) from (ii), we get

$\Delta y = f(x + \Delta x) - f(x)$

Divide both sides by Δx

$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$

Taking limits as $\Delta x \to 0$, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If this limit exists then it is called the derivative of y w.r.t., x and is denoted by $\frac{dy}{dx}$.

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$\frac{dy}{dx}$ is also called the differential coefficient of y w.r.t., x.

> *Notes* $\frac{dy}{dx}$ should not be read as the product of d and y divided by the product of d and x.
>
> In fact, $\frac{d}{dx}$ is the symbol for the derivative w.r.t. x or differential coefficient w.r.t. x.

Notes

9.2 General Theorems on Derivatives : (Without Proof)

Theorem 1: $\frac{d(k)}{dx} = 0$ where k is a constant.

Theorem 2: $\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$ where k is a constant and $f(x)$ is a function of x.

or

$$\frac{d}{dx}(ku) = k\frac{du}{dx}$$

Theorem 3: $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

or

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

Theorem 4: $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

or

$$\frac{d}{dx}[u-v] = \frac{du}{dx} - \frac{dv}{dx}$$

Theorem 5: $\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$

This is called the product rule.

or $$\frac{d}{dx}[(uv)] = u\frac{dv}{dx} + v\frac{du}{dx}$$

This can be remembered as

$$\frac{d}{dx}[I\,\text{function} \times II\,\text{function}]$$

$$= (I\,\text{function})\frac{d}{dx}(II\,\text{function}) + (II\,\text{function})\frac{d}{dx}(I\,\text{function})$$

Notes $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

Theorem 6: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$

This is called the Quotient Rule.

or $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

This can be remembered as

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$$\frac{d}{dx}\left[\frac{Nr}{Dr}\right] = \frac{Dr\frac{d}{dx}(Nr) - Nr\frac{d}{dx}(Dr)}{(Dr)^2}$$

where $Nr =$ Numerator, $Dr =$ Denominator

Notes
(1) While doing problems on differentiation, the above theorems should be strictly followed.

(2) The above theorems can be proved using the method of first principles. Since there is no mention of proofs of the theorems in the syllabus, the proofs are not given here. Only Statements are given because they have to be used in the problems.

9.3 Derivatives of Standard Functions

(1) $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

(2) $\frac{d}{dx}(x) = 1$

(3) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(4) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

(5) $\frac{d}{dx}\left(e^x\right) = e^x$

(6) $\frac{d}{dx}\left(a^x\right) = a^x \log_e a$ where $a > 0$ and $a \neq 1$

(7) $\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}$ where $x > 0$

(8) $\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \log_e a}$

(9) $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

(10) $\frac{d}{dx}\left(\frac{1}{ax+b}\right) = -\frac{1}{a} \cdot \frac{1}{(ax+b)^2}$

(11) $\frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$

(12) $\frac{d}{dx}\log_e(ax+b) = a \cdot \frac{1}{ax+b}$

Notes

Example: Solve x^{100}

Solution: $\frac{d}{dx}\left(x^{100}\right) = 100x^{100-1} = 100x^{99}$

Example: Solve $(2x + 3)^2$

Solution: $\frac{d}{dx}(2x+3)^2 = \frac{d}{dx}\left(4x^2+12x+9\right) = 4\frac{d}{dx}\left(x^2\right)+12\frac{d}{dx}(x)+\frac{d}{dx}(9)$

$$= 4(2x)+12(1)+0 = 8x+12$$

Example: Solve $(x + 1)^3$

Solution: $\frac{d}{dx}(x+1)^3 = \frac{d}{dx}\left(x^3+3x^2+3x+1\right)$

$$= \frac{d}{dx}(x^3)+3\frac{d}{dx}(x^2)+3\frac{d}{dx}(x)+\frac{d}{dx}(1)$$
$$= 3x^{3-1}+3(2x)+3(1)+0$$
$$= 3x^2+6x+3$$

Example: Solve $\frac{x^2-5x+1}{\sqrt{x}}$

Solution: $\frac{d}{dx}\left(\frac{x^2-5x+1}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{x^2}{\sqrt{x}}-\frac{5x}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)$

$$= \frac{d}{dx}\left(x^{2-\frac{1}{2}}-5x^{1-\frac{1}{2}}+x^{-\frac{1}{2}}\right)$$

$$= \frac{d}{dx}\left(x^{\frac{3}{2}}-5x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)$$

$$= \frac{d}{dx}(x^{\frac{3}{2}})-5\frac{d}{dx}(x^{\frac{1}{2}})+\frac{d}{dx}(x^{-\frac{1}{2}})$$
$$= \frac{3}{2}x^{\frac{3}{2}-1}-5\cdot\frac{1}{2}x^{\frac{1}{2}-1}+\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$

$$= \frac{3}{2}x^{\frac{1}{2}}-\frac{5}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$$

Task Differentiate w.r.t.

(1) e^{3x} (2) $\log_e(2x + 3)$

Example: Solve $\log_{10}x$

Solution: $\frac{d}{dx}(\log_{10} x) = \frac{d}{dx}\left(\frac{\log_e x}{\log_e 10}\right)$ (using change of base in logarithms)

$$= \frac{1}{\log_e 10} \cdot \frac{d}{dx}(\log_e x)$$

$$= \frac{1}{\log_e 10}\left(\frac{1}{x}\right) = \frac{1}{x \log_e 10}$$

Example: Solve xe^x

Solution: $\frac{d}{dx}(xe^x) = x\frac{d}{dx}e^x + e^x \frac{d}{dx}(x)$ (product rule)

$$= xe^x + e^x(1)$$

$$= (x+1)e^x$$

Example: Solve $x^2 \log x$

Solution: $\frac{d}{dx}(x^2 \log x) = x^2 \frac{d}{dx}(\log x) + (\log x)\frac{d}{dx}(x^2)$ (product rule)

$$= x^2 \cdot \frac{1}{x} + \log x \cdot 2x$$

$$= x + 2x \log x$$

Example: Solve $\left(\frac{x+1}{x-1}\right)$

Solution: $\frac{d}{dx}\left(\frac{x+1}{x-1}\right) = \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$ (Quotient rule)

$$= \frac{(x-1)(1+0)-(x+1)(1-0)}{(x-1)^2}$$

$$= \frac{(x-1)-x-1}{(x-1)^2}$$

$$= \frac{-2}{(x-1)^2}$$

Task Difference $\frac{2x+3}{x-7}$

Notes

Example: Solve $\left[\frac{x^2+x+1}{x^2-x+1}\right]$

Solution: $\frac{d}{dx}\left[\frac{x^2+x+1}{x^2-x+1}\right]$

$$=\frac{\left(x^2-x+1\right)\frac{d}{dx}\left(x^2+x+1\right)-\left(x^2+x+1\right)\frac{d}{dx}\left(x^2-x+1\right)}{\left(x^2-x+1\right)^2}$$

$$=\frac{\left(x^2-x+1\right)(2x+1)-\left(x^2+x+1\right)(2x-1)}{\left(x^2-x+1\right)^2}$$

$$=\frac{2x^3-2x^2+2x+x^2-x+1-2x^3-2x^2-2x+x^2+x+1}{\left(x^2-x+1\right)^2}$$

$$=\frac{-2x^2+2}{\left(x^2-x+1\right)^2}=\frac{2\left(-x^2+1\right)}{\left(x^2-x+1\right)^2}$$

Example: Solve $\frac{e^x+1}{e^x-1}$

Solution: $\frac{d}{dx}\left(\frac{e^x+1}{e^x-1}\right)=\frac{\left(e^x-1\right)\frac{d}{dx}\left(e^x+1\right)-\left(e^x+1\right)\frac{d}{dx}\left(e^x-1\right)}{\left(e^x-1\right)^2}$

$$=\frac{\left(e^x-1\right)\left(e^x\right)-\left(e^x+1\right)\left(e^x\right)}{\left(e^x-1\right)^2}$$

$$=\frac{e^{2x}-e^x-e^{2x}-e^x}{\left(e^x-1\right)^2}$$

$$=\frac{-2e^x}{\left(e^x-1\right)^2}$$

Example: Solve $\frac{e^x}{1-x^2}$

Solution: $\frac{d}{dx}\left(\frac{e^x}{1-x^2}\right)=\frac{\left(1-x^2\right)\frac{d}{dx}e^x-e^x\frac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^2}$

$$=\frac{\left(1-x^2\right)e^x-e^x(-2x)}{\left(1-x^2\right)^2}$$

$$=\frac{e^x\left(1-x^2+2x\right)}{\left(1-x^2\right)^2}$$

Example: Solve $\frac{1}{x^3}$

Solution: $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}\left(x^{-3}\right)$

$$= (-3)x^{-3-1} = -3x^{-4}$$

$$= (-3)\cdot\frac{1}{x^4}$$

$$= \frac{-3}{x^4}$$

Example: Solve $(x^2 + 1)(x^2 - 1)$

Solution: $\frac{d}{dx}\left[\left(x^2+1\right)\left(x^2-1\right)\right] = \frac{d}{dx}\left(x^4-1\right)$ (on multiplication)

$$= \frac{d}{dx}\left(x^4\right) - \frac{d}{dx}(1)$$

$$= 4x^3 - 0 = 4x^3$$

Example: Solve $e^x(x - 5)\log x$

Solution: $\frac{d}{dx}\left[e^x(x-5)\log x\right]$

$$= e^x(x-5)\frac{d}{dx}(\log x) + e^x \log x\frac{d}{dx}(x-5) + (x-5)\log x\frac{d}{dx}\left(e^x\right)$$

$$= e^x(x-5)\frac{1}{x} + e^x \log x(1) + (x-5)\log x\left(e^x\right)$$

$$= e^x\left[1 - \frac{5}{x} + \log x + x\log x - 5\log x\right]$$

$$= e^x\left[1 - \frac{5}{x} + \log x + x\log x - 5\log x\right]$$

$$= e^x\left[1 - \frac{5}{x} - 4\log x + x\log x\right]$$

Task Solve $\left[\frac{e^x \log x}{1+x^2}\right]$

Example: Find $\frac{dy}{dx}$, if, $y = \frac{2x+1}{1-x^2}$

Solution:

$$\therefore \frac{dy}{dx} = \frac{\left(1-x^2\right)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^2}$$

$$= \frac{\left(1-x^2\right)(2) - (2x+1)(-2x)}{\left(1-x^2\right)^2}$$

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$$= \frac{2 - 2x^2 + 4x^2 + 2x}{\left(1 - x^2\right)^2}$$

$$= \frac{2 + 2x + 2x^2}{\left(1 - x^2\right)^2}$$

Example: Find $\frac{dy}{dx}$, if, $y = \frac{x^3 - 2x}{x + 2}$

Solution:

$$\therefore \frac{dy}{dx} = \frac{(x+2)\frac{d}{dx}\left(x^3 - 2x\right) - \left(x^3 - 2x\right)\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$= \frac{(x+2)\left(3x^2 - 2\right) - \left(x^3 - 2x\right)(1+0)}{(x+2)^2}$$

$$= \frac{3x^3 + 6x^2 - 2x - 4 - x^3 + 2x}{(x+2)^2}$$

$$= \frac{2x^3 + 6x^2 - 4}{(x+2)^2}$$

Example: Find $\frac{dy}{dx}$, if, $y = \frac{(x+1)(x+2)}{(x+3)}$

Solution:

$$\therefore \frac{dy}{dx} = \frac{(x+3)\frac{d}{dx}\left[(x+1)(x+2)\right] - (x+1)(x+2)\frac{d}{dx}(x+3)}{(x+3)^2}$$

$$= \frac{(x+3)\left[(x+1)\frac{d}{dx}(x+2) + (x+2)\frac{d}{dx}(x+1)\right] - (x+1)(x+2)(1)}{(x+3)^2}$$

$$= \frac{(x+3)\left[(x+1)(1) + (x+2)(1)\right] - (x+1)(x+2)}{(x+3)^2}$$

$$= \frac{(x+3)\left[(2x+3)\right] - (x+1)(x+2)}{(x+3)^2}$$

$$= \frac{2x^2 + 9x + 9 - x^2 - 3x - 2}{(x+3)^2}$$

$$= \frac{x^2 + 6x + 7}{(x+3)^2}$$

Example: Find $\frac{dy}{dx}$, if, $y = \frac{x-1}{x+1}$

Solution:

$$\therefore \frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)(1-0) - (x-1)(1+0)}{(x+1)^2}$$

$$= \frac{x+1-x+1}{(x+1)^2}$$

$$= \frac{2}{(x+1)^2}$$

Example: Find $\frac{dy}{dx}$, if, $y = \left(x^2+1\right)\left(e^x\right)$

Solution:

$$\therefore \frac{dy}{dx} = \left(x^2+1\right)\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x^2+1\right)$$

$$= \left(x^2+1\right)e^x + e^x(2x)$$

$$= e^x\left(x^2+1+2x\right)$$

$$= e^x(x+1)^2$$

Example: Find $\frac{dy}{dx}$, if, $y = \frac{(x-1)(x-2)}{(x-3)(x-4)}$

Solution:

$$= \frac{x^2-3x+2}{x^2-7x+12}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\left(x^2-7x+12\right)\frac{d}{dx}\left(x^2-3x+2\right) - \left(x^2-3x+2\right)\frac{d}{dx}\left(x^2-7x+12\right)}{\left(x^2-7x+12\right)^2}$$

$$= \frac{\left(x^2-7x+12\right)(2x-3) - \left(x^2-3x+2\right)(2x-7)}{\left(x^2-7x+12\right)^2}$$

$$= \frac{2x^3-14x^2+24x-3x^2+21x-36-2x^3+6x^2-4x+7x^2-21x+14}{\left(x^2-7x+12\right)^2}$$

$$= \frac{-2x^3+6x^2-4x+7x^2-21x+14}{\left(x^2-7x+12\right)^2}$$

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$$= \frac{-4x^2 + 20x - 22}{\left(x^2 - 7x + 12\right)^2}$$

Example: Find $\frac{dy}{dx}$, if, $y = \left(x^2 - 1\right) e^x \log x$

Solution:

$$\therefore \frac{dy}{dx} = \left(x^2 - 1\right) e^x \frac{d}{dx}(\log x) + \left(x^2 - 1\right) \log x \frac{d}{dx} e^x + e^x \log x \frac{d}{dx}\left(x^2 - 1\right)$$

$$= \left(x^2 - 1\right) e^x \cdot \frac{1}{x} + \left(x^2 - 1\right) \log x \cdot e^x + e^x \log x \cdot 2x$$

$$= e^x \left[x - \frac{1}{x} + \left(x^2 - 1\right) \log x + 2x \cdot \log x \right]$$

Example: Find $\frac{dy}{dx}$, if, $y = \frac{e^x - x^2}{1 - \log x}$

Solution:

$$\therefore \frac{dy}{dx} = \frac{(1 - \log x) \frac{d}{dx}\left(e^x - x^2\right) - \left(e^x - x^2\right) \frac{d}{dx}(1 - \log x)}{(1 - \log x)^2}$$

$$\frac{(1 - \log x)(e^x - 2x) - (e^x - x^2)\left(0 - \frac{1}{x}\right)}{(1 - \log x)^2}$$

$$= \frac{e^x (1 - \log x) + 2x (1 - \log x) + \frac{e^x}{x} - x}{(1 - \log x)^2}$$

$$= \frac{e^x \left(1 - \log x + \frac{1}{x}\right) + x - 2x \log x}{(1 - \log x)^2}$$

Example: Find $\frac{dy}{dx}$, if, $y = \frac{x^5 - 4x^2 + 1}{x^3 - 1}$

Solution:

$$\therefore \frac{dy}{dx} = \frac{(x^3 - 1) \frac{d}{dx}(x^5 - 4x^2 + 1) - (x^5 - 4x^2 + 1) \frac{d}{dx}(x^3 - 1)}{(x^3 - 1)^2}$$

$$= \frac{(x^3 - 1)(5x^4 - 8x) - (x^5 - 4x^2 + 1)(3x^2)}{(x^3 - 1)^2}$$

$$= \frac{5x^7 - 5x^4 - 8x^4 + 8x - 3x^7 + 12x^4 - 3x^2}{(x^3 - 1)^2}$$

$$= \frac{2x^7 - x^4 - 3x^2 + 8x}{(x^3-1)^2}$$

Example: Find $\frac{dy}{dx}$, if, $y = \frac{(x^2-1)\log x}{x^2 e^x}$

Solution:

$$\therefore \frac{dy}{dx} = \frac{\left(x^2e^x\right)\frac{d}{dx}\left[\left(x^2-1\right)\log x\right] - \left(x^2-1\right)\log x \frac{d}{dx}\left(x^2e^x\right)}{\left(x^2e^x\right)^2}$$

$$= \frac{\left(x^2e^x\right)\left[\left(x^2-1\right)\frac{1}{x} + \log x(2x)\right] - \left(x^2-1\right)\log x\left(x^2e^x + e^x 2x\right)}{\left(x^2e^x\right)^2}$$

$$= \frac{x^2e^x\left(x - \frac{1}{x}\right) + 2x^3e^x \log x - x^2\left(x^2-1\right)e^x \log x - e^x\left(2x^3 - 2x\right)}{x^4e^{2x}}$$

$$= \frac{-x^2e^x + e^x + 2x^2e^x \log x - x\left(x^2-1\right)e^x \log x}{x^3e^{2x}}$$

9.4 Function of a Function (or Composite Function)

If a function is made up of more than one function then it is called a composite function. A composite function is denoted by the symbol $f(g(x))$, $f(g(h(x)))$ etc.

To Find the Derivative of a Composite Function

Chain Rule

To find the derivative of $f(g(x))$, we use a rule called chain rule.

Let $y = f(g(x))$, $u = g(x)$

$\therefore \quad y = f(u)$ and $u = g(x)$

By differentiating y w.r.t. u, we get $\frac{dy}{du}$ and by differentiating u w.r.t. x, we get $\frac{du}{dx}$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This is called the chain rule.

Similarly, let $y = f(g(h(x)))$, $u = g(h(x)), v = h(x)$

$\therefore y = f(u), u = g(v), v = h(x)$

By differentiating y w.r.t. u, we get $\frac{dy}{du}$, by differentiating u w.r.t. v, we get $\frac{du}{dv}$ and by

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differentiating v w.r.t. x, we get $\frac{dv}{dx}$.

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$ which is chain rule.

Notes **General Rules**

1. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \cdot \frac{d}{dx} f(x)$
2. $\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)} \cdot \frac{d}{dx} f(x)$
3. $\frac{d}{dx}[\log_e f(x)] = \frac{1}{f(x)} \cdot \frac{d}{dx} f(x)$ etc.

Example: Differentiate the following functions w.r.t. x:

1. $\sqrt{x^2+1}$

Solution: Let $y = \sqrt{x^2+1}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x^2+1}} \cdot \frac{d}{dx}\left(x^2+1\right)$$

$$= \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2+1}}$$

2. $\left(ax^2+bx+c\right)^5$

Solution: Let $y = \left(ax^2+bx+c\right)^5$

$$\therefore \quad \frac{dy}{dx} = 5\left(ax^2+bx+c\right)^{5-1} \cdot \frac{d}{dx}\left(ax^2+bx+c\right)$$

$$= 5\left(ax^2+bx+x\right)^4 \cdot (2ax+b)$$

3. e^{2x+3}

Solution: Let $y = e^{2x+3}$

$$\therefore \quad \frac{dy}{dx} = e^{2x+3} \cdot \frac{d}{dx}(2x+3)$$

$$= e^{2x+3} \cdot (2)$$

$$= 2e^{2x+3}$$

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4. $\log\left(2x^2-5x+7\right)$

Solution: Let $y=\log\left(2x^2-5x+7\right)$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{2x^2-5x+7}\cdot\frac{d}{dx}\left(2x^2-5x+7\right)$$

$$=\frac{1}{2x^2-5x+7}.(4x-5)$$

5. $\left(x\log x\right)^{\frac{1}{5}}$

Solution: Let $y=\left(x\log x\right)^{\frac{1}{5}}$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{5}\left(x\log x\right)^{\frac{1}{5}-1}\cdot\frac{d}{dx}\left(x\log x\right)$$

$$=\frac{1}{5}\left(x\log x\right)^{\frac{-4}{5}}\cdot\left(x\cdot\frac{1}{x}+\log x\cdot 1\right)$$

$$=\frac{1}{5}\left(x\log x\right)^{\frac{-4}{5}}\cdot\left(1+\log x\right)$$

6. $\left(x^2e^x\right)^{-2}$

Solution: Let $y=\left(x^2e^x\right)^{-2}$

$$\therefore \quad \frac{dy}{dx}=(-2)(xe^x)^{-2-1}.\frac{d}{dx}(x^2e^x)$$

$$=(-2)\left(x^2e^x\right)^{-3}\cdot\left(x^2e^x+e^x\cdot 2x\right)$$

$$=(-2)\left(x^2e^x\right)^{-3}\cdot e^x\left(x^2+2x\right)$$

$$=(-2)x^{-6}e^{-3x}\cdot e^x\left(x^2+2x\right)$$

$$=-2e^{-2x}\left(x^{-4}+2x^{-5}\right)$$

7. $\sqrt{\dfrac{x^2+1}{x^2-1}}$

Solution: Let $y=\sqrt{\dfrac{x^2+1}{x^2-1}}$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{2\sqrt{\dfrac{x^2+1}{x^2-1}}}\cdot\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$=\frac{1}{2\sqrt{\dfrac{x^2+1}{x^2-1}}}\cdot\frac{\left(x^2-1\right)2x-\left(x^2+1\right)2x}{\left(x^2-1\right)^2}$$

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$$= \frac{\sqrt{x^2-1}\cdot(-4x)}{2\sqrt{x^2+1}\sqrt{x^2-1}\sqrt{x^2-1}\left(x^2-1\right)}$$

$$= \frac{-2x}{\sqrt{x^2+1}\sqrt{x^2-1}\left(x^2-1\right)}$$

$$= \frac{-2x}{\sqrt{\left(x^2+1\right)\left(x^2-1\right)}\cdot\left(x^2-1\right)}$$

$$= \frac{-2x}{\left(x^2-1\right)\sqrt{x^4-1}}$$

8. $\log\left(\frac{2+3x}{2-3x}\right)$

Solution: Let $y = \log\left(\frac{2+3x}{2-3x}\right)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\left(\frac{2+3x}{2-3x}\right)}\cdot\frac{(2-3x)(3)-(2+3x)(-3)}{(2-3x)^2}$$

$$= \frac{(2-3x)}{(2+3x)}\cdot\frac{12}{(2-3x)^2}$$

$$= \frac{12}{(2+3x)(2-3x)}$$

$$= \frac{12}{4-9x^2}$$

9. $\log(\log x)$

Solution: Let $y = \log(\log x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\log x}\cdot\frac{d}{dx}\left(\log x\right)$$

$$= \frac{1}{\log x}\cdot\frac{1}{x} = \frac{1}{x\log x}$$

10. $\sqrt[3]{2+3x}$

Solution: Let $y = \sqrt[3]{2+3x}$

$$y = (2x+3)^{\frac{1}{3}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{3}(2+3x)^{\frac{1}{3}-1}\cdot\frac{d}{dx}(2+3x)$$

Notes

$$= \frac{1}{3}(2+3x)^{\frac{-2}{3}} \cdot (3)$$

$$= \frac{1}{\sqrt[3]{(2+3x)^2}}$$

11. $\log\left(x+\sqrt{1+x^2}\right)$

Solution: Let $y = \log\left(x+\sqrt{1+x^2}\right)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}} . \frac{d}{dx}\left(x+\sqrt{1+x^2}\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}} . \left(1 + \frac{1}{2\sqrt{1+x^2}} . \frac{d}{dx}\left(1+x^2\right)\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}} . \left(1 + \frac{1}{2\sqrt{1+x^2}} . (2x)\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}} . \left(1 + \frac{x}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}} . \left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$= \frac{\left(x+\sqrt{1+x^2}\right)}{\left(x+\sqrt{1+x^2}\right)\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}}$$

Task Differentiate following functions w.r.t. x

(1) $\log(\log(\log_x))$

(2) $\log\left(\frac{\sqrt{1+x^2}}{1-x^2}\right)$

Example

A spherical balloon is being inflated at the rate of 5 cubic feet per second. How fast is the radius of the balloon increasing 15 seconds after the start?

Solution:

Let v denote volume and r denote radius of the balloon. Since v is changing with time, r will also change. Volume of a spherical balloon $v = \frac{4}{3}\pi r^3$. Here v is a function of r and r is a function of

Notes

t. Using chain rule, we can write $\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}$ or $\frac{dr}{dt} = \frac{dv}{dt} \Big/ \frac{dv}{dr}$.

Note that $\frac{dv}{dt} = 5$. Also $\frac{dv}{dr} = 4\pi r^2$, which can be determined if r is known. To find r, we note that volume becomes $15 \times 5 = 75$ cubic feet, after 15 seconds, therefore, we have $\frac{4}{3}\pi r^3 = 75$ or $r = \left(\frac{225}{4\pi}\right)^{1/3}$.

Thus, $\frac{dr}{dt} = \frac{5}{4\pi} \times \left(\frac{4\pi}{225}\right)^{2/3}$ = 0.058 ft/sec.

Rule 5.

Inverse function Rule

If $y = f(x)$ and $x = g(y)$ are inverse functions which are differentiable, then we can write $g[f(x)] = x$.

Differentiating both sides w.r.t. x we have

$$g'[f(x)] \times f'(x) = 1$$

or
$$g'(y) = \frac{1}{f'(x)}$$

or
$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

Example

Find the equation of a tangent at the point (2, 3) to the rectangular hyperbola $xy = 6$. Show that (2, 3) is middle point of the segment of tangent line intercepted between the two axes. What are its intercepts on the two axes?

Solution:

We can write $y = \frac{6}{x}$ or $\frac{dy}{dx} = -\frac{6}{x^2} = -\frac{6}{4} = -1.5$

Equation of tangent is $(y - 3) = -1.5(x - 2)$ or $y = 6 - 1.5x$.

The point of intersection of the tangent with y-axis is obtained by substituting $x = 0$ in the above equation. This point is (0, 6). Similarly (4, 0) is a point of intersection of the tangent with x-axis.

Since (2, 3) is the middle point of the line joining the points (0, 6) and (4, 0), hence the result.

Intercepts of the tangent on x and y axes, are 4 and 6 respectively.

9.5 Implicit Functions

Notes

If a function is in the form $y = f(x)$, then the function is said to be in the **explicit** form. Instead of this, if the variables x and y are related by means of an equation, then the function is said to be in the **implicit** form. In general an implicit function is given by $f(x, y) = c$ where c is a constant.

e.g., $y^2 = 4ax, x^2 + y^2 = a^2, \frac{x^2}{x^2} + \frac{y^2}{b^2} = 1$

To find the derivative of the Implicit Function $f(x, y) = c$

Differentiate $f(x, y) = c$ using the rules of differentiation. Collect all the terms containing $\frac{dy}{dx}$ on the left hand side and the remaining terms on the right hand side. Take the common factor $\frac{dy}{dx}$ on the left hand side. Divide both sides by the coefficient of $\frac{dy}{dx}$ to get $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$, if, $y^2 = 4ax$

Solution: Differentiate both sides w.r.t. x

$$2y\frac{dy}{dx} = 4a(1)$$

$$\therefore \quad \frac{dy}{dx} = \frac{4a}{2y} \text{i.}$$

Example: Find $\frac{dy}{dx}$, if, $x^2 + y^2 = 2xy$

Solution: Differentiate w.r.t. x

$$\therefore \quad 2x + 2y\frac{dy}{dx} = 2\left(x\frac{dy}{dx} + y\right)$$

Cancelling 2 on both sides, we get

$$x + y\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad y\frac{dy}{dx} - x\frac{dy}{dx} = y - x$$

$$\Rightarrow \quad (y - x)\frac{dy}{dx} = (y - x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x}{y - x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 1$$

Notes

Example: Find $\frac{dy}{dx}$, if, $x^2 + y^2 = a^2$

Solution: Differentiate w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{2y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y}$$

Example: Find $\frac{dy}{dx}$, if, $x^2 + y^2 + 2x - 4y + 6 = 0$

Solution: Differentiate w.r.t. x

$$2x + 2y\frac{dy}{dx} + 2 - 4\frac{dy}{dx} = 0$$

Cancelling 2, we get

$$x + y\frac{dy}{dx} + 1 - 2\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx}(y-2) = -x - 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(x+1)}{y-2}$$

Example: Find $\frac{dy}{dx}$, if, $e^{x+y} = e^x$

Solution: Differentiate w.r.t. x

$$e^{x+y} \cdot \frac{d}{dx}(x+y) = e^x$$

$$\text{i.e.,} e^{x+y}\left(1 + \frac{dy}{dx}\right) = e^x$$

$$\therefore \quad 1 + \frac{dy}{dx} = \frac{e^x}{e^{x+y}} = \frac{1}{e^y}$$

$$\text{i.e.,} \frac{dy}{dx} = \frac{1 - e^y}{e^y}$$

Example: Find $\frac{dy}{dx}$, if, $x^{2/3}+y^{2/3}=a^{2/3}$

Solution: Differentiate w.r.t. x

$$\frac{2}{3}x^{\frac{2}{3}-1}+\frac{2}{3}y^{\frac{2}{3}-1}\cdot\frac{dy}{dx}=0$$

Dividing throughout by $\frac{2}{3}$, we get

$$x^{-\frac{1}{3}}+y^{-\frac{1}{3}}\frac{dy}{dx}=0$$

$$\therefore \quad y^{-\frac{1}{3}}\frac{dy}{dx}=-x^{-\frac{1}{3}}$$

$$\therefore \quad \frac{dy}{dx}=-\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}$$

i.e., $\frac{dy}{dx}=\left(\frac{y}{x}\right)^{1/3}$

Example: Find $\frac{dy}{dx}$, if, $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+.........to\infty}}}$

Solution: $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+.........to\infty}}}$

$$\Rightarrow \quad y=\sqrt{\log x+y}$$

Squaring, we get

$$y^2=\log x+y$$

Differentiate w.r.t. x

$$2y\frac{dy}{dx}=\frac{1}{x}+\frac{dy}{dx}$$

$$\Rightarrow \quad 2y\frac{dy}{dx}-\frac{dy}{dx}=\frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}(2y-1)=\frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{x(2y-1)}$$

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Example: Find $\frac{dy}{dx}$, if, $x\sqrt{1+y}+y\sqrt{1+x}=0$ where $x \neq y$, prove that $\frac{dy}{dx}=\frac{-1}{(1+x)^2}$.

Solution: $x\sqrt{1+y}+y\sqrt{1+x}=0$

$\Rightarrow \quad x\sqrt{1+y}=-y\sqrt{1+x}$

Squaring,

$$x^2(1+y)=y^2(1+x)$$

$$\Rightarrow \quad x^2+x^2y=y^2+y^2x$$

$$\Rightarrow \quad x^2-y^2=-x^2y+y^2x$$

$$\Rightarrow \quad (x+y)(x-y)=-xy(x-y)$$

$$\Rightarrow \quad x+y=-xy \text{ (cancelling } (x-y)) \qquad \text{...(i)}$$

Differentiating w.r.t. x

$$1+\frac{dy}{dx}=-\left(x\frac{dy}{dx}+y\right)$$

i.e., $\frac{dy}{dx}+x\frac{dy}{dx}=-1-y$

$$(1+x)\frac{dy}{dx}=-(1+y)$$

$$\therefore \quad \frac{dy}{dx}=-\frac{(1+y)}{1+x} \qquad \text{...(ii)}$$

From (i), $x=-y-xy$

$$\Rightarrow \quad x=-y(1+x)$$

$$\Rightarrow \quad y=\frac{-x}{1+x}$$

Substituting this in (ii), we get

$$\frac{dy}{dx}=\frac{-\left(1-\frac{x}{1+x}\right)}{1+x}$$

$$=-\left[\frac{1+x-x}{(1+x)(1+x)}\right]$$

$$=\frac{-1}{(1+x)^2}$$

Task Find $\frac{dy}{dx}$, if,

1. If $y = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + ...to\infty}}}$, prove that $\frac{dy}{dx} = \frac{-1}{(x+y)^2 + 1}$

2. If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ... + \infty}}}$, prove that $\frac{dy}{dx} = \frac{f'(x)}{2y-1}$.

9.6 Summary

- If x is a real variable, any expression in x is called a function of x. A function is denoted by $y = f(x)$, Where x is independent variable and y is dependent variable.
- The functional value of $f(x)$ at $x = a$ is given by $f(a)$.
- If $f(x)$ gets arbitrarily close to b (a finite number) for x sufficiently close to a, we say that $f(x)$ approaches the limit b as x approaches a, and write $\lim_{x\to a} f(x) = b$
- $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x).\, g(x)] = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$
- A function $y = f(x)$ is said to be differentiable at a point $x = a$, in its domain, $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists.
- Derivative of a constant is zero.
- If $y = e^x$ then $\frac{dy}{dx} = e^x$
- If $y = a^x$, then $\frac{dy}{dx} = a^x \log_e a$

9.7 Keywords

Constant: A quantity whose value remains the same.

Function: A function '*f*' from a set x to set y is a subset of $x.y$, denoted as $\{(x, y)\}$, such that corresponding to each value of x, we can associated one and only one value of y. In such a situation, y is said to be a function of x and is denoted as $y = f(x)$.

Irrational Function: A function which is expressed as a root of a polynomial.

Parametric Function: If the variable x and y are given in terms of a new variable t, then the function is said to be in the parametric form and 't' is called the parameter.

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Polynomial Function: A function of the form $y = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

Rational Function: The ratio of two polynomial functions.

Successive Differentiation: The process of finding higher ordered derivatives is called successive differentiation.

Variable: A quantity whose value changes.

9.8 Self Assessment

1. Find value of $\frac{d(e^{2x-5})}{dx}$

   (a) $2e^{2x-5}$ (b) $2e^{2x+5}$

   (c) $2x^{ex-5x}$ (d) $2x + e^{2x-5}$

2. Find value of $\frac{d}{dx}\log(4x+5)$

   (a) $\frac{4}{4x+5}$ (b) $\frac{2}{2x+5}$

   (c) $\frac{4x}{4x+5}$ (d) $\frac{4x^2}{4x+5}$

3. Find $\frac{d^2y}{dx^2}$, if $xy + 4y = 4x$

   (a) $\frac{5(y-4)}{(x+4)^2}$ (b) $\frac{5(y+4)}{(x-4)^2}$

   (c) $\frac{4(y-4)}{(x-4)^2}$ (d) $\frac{4(y+4)}{(x-2)^2}$

4. Find $\frac{d^2y}{dx^2}$ if $y = (x^2 + 2)\log x$

   (a) $3 + 2\log x - \frac{2}{x^2}$ (b) $2 + 3\log x - \frac{2}{x^2}$

   (c) $2 - 3\log x - \frac{2}{x^2}$ (d) $3 - 2\log x - \frac{2}{x^2}$

Fill in the blanks:

5. A function of the form $y = a_0 + a_1x + a_2x^2 +$ .... $+ a_nx^n$ is called as ....

6. The ratio of two polynomial functions is called ....

7. .... is a function which is expressed as a root of a polynomial.

9.9 Review Questions

1. Find domain and range of the following functions:

   (i) $y = \frac{x^3 - a^3}{x - a}$ (ii) $y = x + \frac{1}{x^2}$

Notes

2. Find the limit of the following functions:

   (i) $\lim\limits_{x \to 3} \dfrac{x^2 - 6x + 9}{x^2 - 2x - 3}$

   (ii) $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$

3. Find derivative of the following functions:

   (i) $y = x^2 - 5x + 10$

   (ii) $y = (x + 1) \cdot e^x$

   (iii) $y = 10(x + 1)(4 - x)$

   (iv) $y = \log x^3$

4. Find $\dfrac{dy}{dx}$ when $y = u^2 + 5$, $u = v^2 + 2$ and $v = x^2 - x$.

Answers: Self Assessment

1. (a)	2. (a)	3. (a)
4. (a)	5. Polynomial Functions	
6. Rational Function	7. Irrational Function	

9.10 Further Readings

Books

D C Sanchethi and V K Kapoor, *Business Mathematics.*

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

R S Bhardwaj, *Mathematics for Economics and Business*, Excel Books, New Delhi, 2005.

Sivayya and Sathya Rao, *An Introduction to Business Mathematics.*

Smith and Minton. *Calculus Early Trancendental,* Third Edition. McGraw Hill.2008

Online links

http://www.suitcaseofdreams.net/Trigonometric_Functions.htm

http://library.thinkquest.org/20991/alg2/trigi.html

http://www.intmath.com/trigonometric–functions/5–signs–of–trigonometric–functions.php

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Unit 10: Logarithmic Differentiation

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss lop logarithmic differentiation
- Explain problem retated to logarithmic differentiation

Introduction

A function which is the product or quotient of a number of functions. A function of the form the [f (x)]g (x) where f (x) and g (x) are both derivable, it is usually advisable to take logarithm of the function first and then differentiate. The process is known as the logarithmic differentiation.

10.1 Logarithmic Differentiation

To differentiate a function of the form $\left[f(x)\right]^{g(x)}$ or $a^{f(x)}$, we use a method called Logarithmic differentiation.

To find the derivative of the functions $\left[f(x)\right]^{g(x)}$ and $a^{f(x)}$.

(i) Let $y = \left[f(x)\right]^{g(x)}$

Taking logarithms, we get

$$\log y = g(x)\log\left[f(x)\right]$$

Differentiate w.r.t. x

$$\frac{1}{y}\cdot\frac{dy}{dx} = g(x)\cdot\frac{1}{f(x)}\cdot f'(x) + \log\left[f(x)\right].g'(x)$$

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$$\therefore \quad \frac{dy}{dx} = y\left[\frac{g(x)}{f(x)} \cdot f'(x) + g'(x) \cdot \log\left(f(x)\right)\right]$$

i.e., $$\frac{dy}{dx} = \left[f(x)\right]^{g(x)}\left[\frac{g(x)}{f(x)} \cdot f'(x) + g'(x) \cdot \log\left[f(x)\right]\right]$$

This method of finding $\frac{dy}{dx}$ is called Logarithmic differentiation.

(ii) Let $y = a^{f(x)}$

Taking logarithms, we get

$$\log y = f(x)\log a$$

Differentiate w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \left(\log a\right)f'(x)$$

$$\therefore \quad \frac{dy}{dx} = y\left[\left(\log a\right)f'(x)\right] = a^{f(x)}\left[\left(\log a\right)f'(x)\right]$$

Example: Differentiate the following w.r.t. x :

1. x^x 2. x^{x^x} 3. $\left(x^x\right)^x$

4. 3^x 5. $x^{x^{x^{x^{\cdot^{\cdot^{\cdot \text{to}\infty}}}}}}$

Solution:

1. Let $y = x^x$

Taking logs, we get

$$\log y = x\log x$$

Differentiate w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\therefore \quad \frac{dy}{dx} = y\left(1 + \log x\right)$$

i.e., $$\frac{dy}{dx} = x^x\left(1 + \log z\right)$$

Notes

2. Let $y = x^{x^x}$

Taking logs, we get

$$\log y = x^x \log x$$

Taking logs again, we get

$$\log(\log y) = x \log x + \log(\log x)$$

Differentiate w.r.t. x

$$\frac{1}{\log y} \cdot \frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1 + \frac{1}{\log x} \cdot \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = y \log y \left[1 + \log x + \frac{1}{x \log x} \right]$$

i.e., $$\frac{dy}{dx} = x^{x^x} \cdot x^x \log x \left[1 + \log x + \frac{1}{x \log x} \right]$$

3. Let $y = \left(x^x\right)^x$

$$\Rightarrow \quad y = x^{x^2}$$

Taking logs, we get

$$\log y = x^2 \log x$$

Differentiate w.r.t. x

$$\frac{1}{y} \frac{dy}{dx} = x^2 \cdot \frac{1}{x} + (\log x) 2x$$

$$\therefore \quad \frac{dy}{dx} = y\left[x + 2x \log x\right]$$

i.e., $$\frac{dy}{dx} = \left(x^x\right)^x \left[x + 2x \log x\right]$$

4. Let $y = 3^x$

Taking logs, we get

$$\log y = x \log 3$$

Differentiating w.r.t. x

$$\frac{1}{y} \frac{dy}{dx} = (\log 3) \cdot 1$$

Notes

$$\therefore \quad \frac{dy}{dx} = y(\log 3)$$

i.e., $\dfrac{dy}{dx} = 3^x(\log 3)$

5. Let $y = x^{x^{x^{\cdot^{\cdot^{to\infty}}}}}$

$$\Rightarrow \quad y = x^y$$

Taking logs, we get

$$\log y = y \log x$$

Differentiating w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1 - y \log x}{y}\right) = \frac{y}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$$

Example: Find $\dfrac{dy}{dx}$, if, $e^y = y^x$

Solution:

Taking logs, we get

$$y \log e = x \log y$$

i.e., $y(1) = x\log y \quad \because \log e = \log_e e = 1$

Differentiating w.r.t. x

$$\frac{dy}{dx} = x \cdot \frac{1}{y}\frac{dy}{dx} + \log y \cdot (1)$$

$$\frac{dy}{dx} - \frac{x}{y}\frac{dy}{dx} = \log y$$

$$\therefore \quad \frac{dy}{dx}\left(1 - \frac{x}{y}\right) = \log y$$

i.e., $\dfrac{dy}{dx}\left(\dfrac{y - x}{y}\right) = \log y$

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$$\therefore \quad \frac{dy}{dx} = \frac{y \log y}{y - x}$$

Example: Find $\frac{dy}{dx}$, if, $y = a^{x+y} \cdot y^x$

Solution:

Taking logs, we get

$$\log y = (x + y)\log a + x \log y$$

Differentiating w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = (\log x)\left(1 + \frac{dy}{dx}\right) + x \cdot \frac{1}{y}\frac{dy}{dx} + (\log y)1$$

i.e., $$\left(\frac{1}{y} - \log a - \frac{x}{y}\right)\frac{dy}{dx} = \log a + \log y$$

$$\therefore \quad \frac{dy}{dx} = \frac{(\log a + \log y)y}{(1 - y \log a - x)}$$

Example: Find $\frac{dy}{dx}$, if, $x^m y^n = (x + y)^{m+n}$

Solution:

Taking logs, we get

$$m \log x + n \log y = (m + n)\log(x + y)$$

Differentiating w.r.t. x

$$m\left(\frac{1}{x}\right) + n\left(\frac{1}{y}\frac{dy}{dx}\right) = (m + n)\frac{1}{x + y} \cdot \left(1 + \frac{dy}{dx}\right)$$

i.e., $$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m + n}{x + y} + \frac{m + n}{x + y} \cdot \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx}\left(\frac{n}{y} - \frac{m + n}{x + y}\right) = \frac{m + n}{x + y} - \frac{m}{x}$$

i.e., $$\frac{dy}{dx}\left(\frac{nx + ny - my - ny}{y(x + y)}\right) = \frac{mx + nx - mx - my}{x(x + y)}$$

i.e., $$\frac{dy}{dx}\left(\frac{nx - my}{y(x + y)}\right) = \frac{(nx - my)}{x(x + y)}$$

i.e., $$\frac{dy}{dx}\frac{(nx - my)}{y(x + y)} = \frac{(nx - my)}{x(x + y)}$$

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i.e., $\frac{dy}{dx} \cdot \frac{1}{y} = \frac{1}{x}$

$\therefore \frac{dy}{dx} = \frac{y}{x}$

Example: Find $\frac{dy}{dx}$, if, $x^y = y^x$

Solution:

Taking logs, we get

$y \log x = x \log y$

Differentiating w.r.t. x

$$y\left(\frac{1}{x}\right) + (\log x)\frac{dy}{dx} = x \cdot \frac{1}{y}\frac{dy}{dx} + (\log y)1$$

i.e., $\frac{dy}{dx}\left(\log x - \frac{x}{y}\right) = \log y - \frac{y}{x}$

i.e., $\frac{dy}{dx}\left(\frac{y \log x - x}{y}\right) = \frac{x \log y - y}{x}$

$\therefore \frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}$

Task Find $\frac{dy}{dx}$, if,

1. $x^y + y^x + 2a = 0$
2. $x^y - y^x + 2ax = 0$

10.2 Logarithmic Differentiation Problems

The following problems illustrate the process of logarithmic differentiation. It is a means of differentiating algebraically complicated functions or functions for which the ordinary rules of differentiation do not apply. For example, in the problems that follow, you will be asked to differentiate expressions where a variable is raised to a variable power. An example and two common incorrect solutions are :

1. $D\{x^{(2x+3)}\} = (2x+3)x^{(2x+3)-1} = (2x+3)x^{(2x+2)}$

and

2. $D\{x^{(2x+3)}\} = x^{(2x+3)}(2)\ln x$

Both of these solutions are wrong because the ordinary rules of differentiation do not apply. Logarithmic differentiation will provide a way to differentiate a function of this type. It requires deft algebra skills and careful use of the following unpopular, but well-known, properties of

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logarithms. Though the following properties and methods are true for a logarithm of any base, only the natural logarithm (base e, where e ≈ 2.718281828), ln, will be used in this problem set.

10.2.1 Properties of the Natural Logarithm

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $\ln y^x = x \ln y$
5. $\ln (xy) = \ln x + \ln y$
6. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

10.2.2 Avoid the Following List of Common Mistakes

1. $\ln (x + y) = \ln x + \ln y$
2. $\ln (x - y) = \ln x - \ln y$
3. $\ln (xy) = \ln x \ln y$
4. $\ln\left(\frac{x}{y}\right) = \frac{\ln x}{\ln y}$
5. $\frac{\ln x}{\ln y} = \ln x - \ln y$

The following exaples range in difficulty from average to challenging:

Example: Differentiate $y = x^x$

Solution:

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation do not apply ! The function must first be revised before a derivative can be taken. Begin with

$$y = x^x$$

Apply the natural logarithm to both sides of this equation getting

$$\begin{aligned} \ln y &= \ln x^x \\ &= x \ln x \end{aligned}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule on the right-hand side. Thus, beginning with

$$\ln y = x \ln x$$

and differentiating, we get

$$\begin{aligned} \frac{1}{y}y' &= x\frac{1}{x} + (1)\ln x \\ &= 1 + \ln x \end{aligned}$$

Notes

Multiply both sides of this equation by y, getting

$$y' = y(1 + \ln x) = x^x (1 + \ln x)$$

Example: Differentiate $y = x^{(ex)}$

Solution:

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation do not apply ! The function must first be revised before a derivative can be taken. Begin with

$$y = x^{(ex)}$$

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln x^{(e^x)}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule on the right-hand side. Thus, beginning with

$$\ln y = e^x \ln x$$

and differentiating, we get

$$\frac{1}{y}y' = e^x\left\{\frac{1}{x}\right\} + e^x \ln x$$

(Get a common denominator and combine fractions on the right-hand side.)

$$= \frac{e^x}{x} + e^x \ln x\left\{\frac{x}{x}\right\}$$

$$= \frac{e^x}{x} + \frac{xe^x \ln x}{x}$$

$$= \frac{e^x + xe^x \ln x}{x}$$

(Factor out e^x in the numerator.)

$$= \frac{e^x(1 + x\ln x)}{x}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{e^x(1 + x\ln x)}{x}$$

$$= x^{(e^x)}\frac{e^x(1 + x\ln x)}{x^1}$$

(Combine the powers of x.)

$$= x\left(e^x - 1\right)e^x(1 + x\ln x)$$

Notes

Example: Differentiate y = $\left(3x^2+5\right)^{1/x}$

Solution:

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation do not apply ! The function must first be revised before a derivative can be taken. Begin with

$$y = \left(3x^2+5\right)^{1/x}$$

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln\left(3x^2+5\right)^{1/x}$$

$$= (1/x)\ln\left(3x^2+5\right)$$

$$= \frac{\ln\left(3x^2+5\right)}{x}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the quotient rule and the chain rule on the right-hand side. Thus, beginning with

$$\ln y = \frac{\ln\left(3x^2+5\right)}{x}$$

and differentiating, we get

$$\frac{1}{y}y' = \frac{x\left\{\frac{1}{3x^2+5}\right\}(6x)-\ln\left(3x^2+5\right)(1)}{x^2}$$

(Get a common denominator and combine fractions in the numerator.)

$$= \frac{\frac{6x^2}{3x^2+5}-\ln\left(3x^2+5\right)\left\{\frac{3x^2+5}{3x^2+5}\right\}}{\frac{x^2}{1}}$$

(Dividing by a fraction is the same as multiplying by its reciprocal.)

$$= \frac{6x^2-\left(3x^2+5\right)\ln\left(3x^2+5\right)}{3x^2+5}\frac{1}{x^2}$$

$$= \frac{6x^2-\left(3x^2+5\right)\ln\left(3x^2+5\right)}{x^2\left(3x^2+5\right)}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{6x^2-\left(3x^2+5\right)\ln\left(3x^2+5\right)}{x^2\left(3x^2+5\right)}$$

$$= \frac{(3x^2+5)^{1/x}\{6x^2-(3x^2+5)\ln(3x^2+5)\}}{x^2(3x^2+5)^1}$$

(Combine the powers of $(3x^2 + 5)$.)

$$= \frac{(3x^2+5)^{(1/x-1)}\{6x^2-(3x^2+5)\ln(3x^2+5)\}}{x^2}$$

Example: Differentiate $y = (\sin x)x^3$

Solution:

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation do not apply ! The function must first be revised before a derivative can be taken. Begin with

$$y = (\sin x)^{x^3}$$

Apply the natural logarithm to both sides of this equation getting

$$\ln y = \ln(\sin x)^{x^3}$$

$$= x^3\ln(\sin x)$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule and the chain rule on the right-hand side. Thus, beginning with truein $\ln y = x^3\ln(\sin x)$ and differentiating, we get

$$\frac{1}{y}y' = x^3\left\{\frac{1}{\sin x}\right\}\cos x + (3x^2)\ln(\sin x)$$

(Get a common denominator and combine fractions on the right-hand side.)

$$= \frac{x^3\cos x}{\sin x} + 3x^2\ln(\sin x)\left\{\frac{\sin x}{\sin x}\right\}$$

$$= \frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{\sin x}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{\sin x}$$

$$= (\sin x)^{x^3}\frac{x^3\cos x + 3x^2\sin x\ln(\sin x)}{(\sin x)^1}$$

(Combine the powers of (sin x).)

$$= (\sin x)^{(x^3-1)}\{x^3\cos x + 3x^2\sin x\ln(\sin x)\}$$

Notes

Example: Differentiate $y = 7x(\cos x)^{x/2}$

Solution:

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation do not apply ! The function must first be revised before a derivative can be taken. Begin with

$$y = 7x (\cos x)^{x/2}$$

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\ln y = \ln\left((7x)(\cos x)^{x/2}\right)$$

$$= \ln (7x) + \ln (\cos x)^{x/2}$$

$$= \ln (7x) + (x/2) \ln (\cos x)$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule and the chain rule on the right-hand side. Thus, beginning with

$$\ln y = \ln (7x) + (x/2) \ln (\cos x)$$

and differentiating, we get

$$\frac{1}{y}y' = \left\{\frac{1}{7x}\right\}7 + (x/2)\left\{\frac{1}{\cos x}\right\}(-\sin x) + (1/2)\ln(\cos x)$$

$$= \frac{1}{x} - \frac{x\sin x}{2\cos x} + \frac{\ln(\cos x)}{2}$$

(Get a common denominator and combine fractions on the right-hand side.)

$$= \frac{1}{x}\left\{\frac{2\cos x}{2\cos x}\right\} - \frac{x\sin x}{2\cos x}\left\{\frac{x}{x}\right\} + \frac{\ln(\cos x)}{2}\left\{\frac{x\cos x}{x\cos x}\right\}$$

$$= \frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2x\cos x}$$

$$= 7x(\cos x)^{x/2}\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2(\cos x)^1}$$

(Divide out a factor of x.)

$$= 7(\cos x)^{x/2}\frac{2\cos x - x^2\sin x + x\cos x\ln(\cos x)}{2(\cos x)^1}$$

(Combine the powers of (cos x).)

$$= (7/2)(\cos x)^{(x/2-1)}\left\{2\cos x - x^2\sin x + x\cos x\ln(\cos x)\right\}$$

Notes

Example: Differentiate $y = \sqrt{x}^{\sqrt{x}} e^{x^2}$

Solution:

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation do not apply ! The function must first be revised before a derivative can be taken. Begin with

$$y = \sqrt{x}^{\sqrt{x}} e^{x^2}$$

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\begin{aligned} \ln y &= \ln\left(\sqrt{x}^{\sqrt{x}} e^{x^2}\right) \\ &= \ln\left(\sqrt{x}^{\sqrt{x}}\right) + \ln\left(e^{x^2}\right) \\ &= \sqrt{x}\ln\left(\sqrt{x}\right) + x^2 \ln(e) \\ &= \sqrt{x}\ln\left(\sqrt{x}\right) + x^2(1) \\ &= \sqrt{x}\ln\left(\sqrt{x}\right) + x^2 \end{aligned}$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x. Use the product rule and the chain rule on the right-hand side. Thus, beginning with

$$\ln y = \sqrt{x}\ln\left(\sqrt{x}\right) + x^2$$

and differentiating, we get

$$\begin{aligned} \frac{1}{y}y' &= \sqrt{x}\left\{\frac{1}{\sqrt{x}}\right\}(1/2)x^{-1/2} + (1/2)x^{-1/2}\ln\left(\sqrt{x}\right) + 2x \\ &= \frac{1}{2\sqrt{x}} + \frac{\ln\left(\sqrt{x}\right)}{2\sqrt{x}} + 2x \end{aligned}$$

(Get a common denominator and combine fractions on the right-hand side.)

$$\begin{aligned} &= \frac{1}{2\sqrt{x}} + \frac{\ln\left(\sqrt{x}\right)}{2\sqrt{x}} + 2x\left\{\frac{2\sqrt{x}}{2\sqrt{x}}\right\} \\ &= \frac{1 + \ln\left(\sqrt{x}\right) + 4x^{1+1/2}}{2\sqrt{x}} \\ &= \frac{1 + \ln\left(\sqrt{x}\right) + 4x^{3/2}}{2\sqrt{x}} \end{aligned}$$

Notes

Multiply both sides of this equation by y, getting

$$y' = y\frac{1+\ln\left(\sqrt{x}\right)+4x^{3/2}}{2\sqrt{x}}$$

$$= \sqrt{x}^{\sqrt{x}} e^{x^2} \frac{1+\ln\left(\sqrt{x}\right)+4x^{3/2}}{2\sqrt{x}^{1}}$$

(Combine the powers of $\sqrt{x}$.)

$$= (1/2)\sqrt{x}^{(\sqrt{x}-1)} e^{x^2}\left\{1+\ln\left(\sqrt{x}\right)+4x^{3/2}\right\}$$

Example: Differentiate y = $x^{\ln x}$ (sec x)^{3x}

Solution:

Because a variable is raised to a variable power in this function, the ordinary rules of differentiation do not apply ! The function must first be revised before a derivative can be taken. Begin with

$$y = x^{\ln x} (\sec x)^{3x}$$

Apply the natural logarithm to both sides of this equation and use the algebraic properties of logarithms, getting

$$\ln y = \ln\left(x^{\ln x}(\sec x)^{3x}\right)$$

$$= \ln x^{(\ln x)} + \ln (\sec x)^{3x}$$

$$= (\ln x)(\ln x) + 3x \ln (\sec x)$$

$$= (\ln x)^2 + 3x \ln (\sec x)$$

Differentiate both sides of this equation. The left-hand side requires the chain rule since y represents a function of x . Use the product rule and the chain rule on the right-hand side. Thus, beginning with

$$\ln y = (\ln x)^2 + (3x) \ln (\sec x)$$

and differentiating, we get

$$\frac{1}{y}y' = 2(\ln x)\left\{\frac{1}{x}\right\}+3x\left\{\frac{1}{\sec x}\right\}(\sec x\tan x)+(3)\ln(\sec x)$$

(Divide out a factor of sec x.)

$$= \frac{2\ln x}{x}+3x\tan x+3\ln(\sec x)$$

(Get a common denominator and combine fractions on the right-hand side.)

$$= \frac{2\ln x}{x}+3x\tan x\left\{\frac{x}{x}\right\}+3\ln(\sec x)\left\{\frac{x}{x}\right\}$$

$$= \frac{2\ln x+3x^2\tan x+3x\ln(\sec x)}{x}$$

Multiply both sides of this equation by y, getting

$$y' = y\frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x}$$

$$= x^{\ln x}(\sec x)^{3x}\frac{2\ln x + 3x^2\tan x + 3x\ln(\sec x)}{x}$$

(Combine the powers of x.)

$$= x^{(\ln x - 1)}(\sec x)^{3x}\left\{2\ln x + 3x^2\tan x + 3x\ln(\sec x)\right\}$$

Logarithmic Differentiation

Example: Determine $\frac{df}{dx}$ of $f : x \to [\cos(x)]^x$

$$f(x) = [\cos(x)]^x$$

$$\ln(f(x)) = \ln[\cos(x)]^x$$

$$\ln(f(x)) = x\cdot\ln[\cos(x)]$$

$$\frac{d}{dx}[\ln(f(x))] = \frac{d}{dx}([x\cdot\ln(f(x))])$$

$$\frac{1}{f}.\frac{df}{dx} = \ln[\cos(x)] + x\left(\frac{1}{\cos(x)}\right)[-\sin(x)]$$

$$\frac{1}{f}.\frac{df}{dx} = \ln[\cos(x)] - x\cdot\tan(x)$$

$$\frac{df}{dx} = f\left[\ln[\cos(x)] - x\cdot\tan(x)\right]$$

$$\frac{df}{dx} = [\cos(x)]^x\ln[\cos(x)] - x[\cos(x)]^x\tan(x)$$

And thus

$$\frac{df}{dx} = \cos^x(x)\ln[\cos(x)] - x\cdot\cos^x(x)\tan(x)$$

Example: Differentiate $y = (2x)^{\sin x}$.

Solution:

Alternate 1

$y = (2x)^{\sin x} = e^{\sin x \ln 2x}$,

$$y' = e^{\sin x\ln 2x}\left(\cos x\ln 2x + (\sin x)\frac{2}{2x}\right) = (2x)^{\sin x}\left(\cos x\ln 2x + \frac{\sin x}{x}\right).$$

Notes

Alternate 2

Using logarithmic differentiation we have:

$$\ln y = \ln (2x)^{\sin x} = \sin x \ln 2x,$$

$$\frac{y'}{y} = \cos x \ln 2x + (\sin x)\frac{2}{2x} = \cos x \ln 2x + \frac{\sin x}{x},$$

$$y' = y\left(\cos x \ln 2x + \frac{\sin x}{x}\right) = (2x)^{\sin x}\left(\cos x \ln 2x + \frac{\sin x}{x}\right).$$

Notes The given function is of the form $(f(x))^{g(x)}$, with $f(x) = 2x$ and $g(x) = \sin x$. The variable appears in both the base and the exponent. Neither the power rule $(d/dx)\, u^a = au^{a-1}u'$ nor the exponent rule $(d/dx)\, a^u = a^u(\ln a)u'$ can be applied directly in this case.

In Solution 1 we transform $(f(x))^{g(x)}$ into the exponential function using the definition $u^v = e^{v \ln u}$, to get $(f(x))^{g(x)} = e^{g(x) \ln f(x)}$. Then we differentiate $e^{g(x) \ln f(x)}$ with respect to x utilizing the exponent rule. This is possible because the base e is a constant. In the answer, we transform $e^{g(x) \ln f(x)}$ back to $(f(x))^{g(x)}$.

In Solution 2 we take the natural logarithm of both sides of the equation $y = (f(x))^{g(x)}$, to obtain $\ln y = \ln (f(x))^{g(x)} = g(x) \ln f(x)$. Then we differentiate implicitly both sides of the resulting equation $\ln y = g(x) \ln f(x)$ with respect to x. Note that $(d/dx) \ln y = (1/y)\, dy/dx = y'/y$, by the chain rule. Next we solve for y'. In the answer, we replace y by $(f(x))^{g(x)}$, since we should express y' in terms of x only, not of x and y. This technique is called **logarithmic differentiation**, since it involves the taking of the natural logarithm and the differentiation of the resulting logarithmic equation. It allows us to convert the differentiation of $(f(x))^{g(x)}$ into the differentiation of a product.

Note that both Alternate 1 and 2 yield the same answer.

Example: Find dy/dx if $y = x^3(\sin x)^{\cos x}$.

Solution:

Alternate 1

$$y = x^3(\sin x)^{\cos x} = x^3 e^{\cos x \ln \sin x},$$

$$\frac{dy}{dx} = 3x^2 e^{\cos x \ln \sin x} + x^3 e^{\cos x \ln \sin x}\left((-\sin x) \ln \sin x + \cos x \frac{\cos x}{\sin x}\right)$$

$$= 3x^2 (\sin x)^{\cos x} + x^3 (\sin x)^{\cos x}(-\sin x \ln \sin x + \cos x \cot x)$$

$$= x^3 (\sin x)^{\cos x}\left(\frac{3}{x} - \sin x \ln \sin x + \cos x \cot x\right).$$

Alternate 2

Utilizing logarithmic differentiation we get:

$$\ln y = \ln x^3 (\sin x)^{\cos x} = 3 \ln x + \cos x \ln \sin x,$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + (-\sin x) \ln \sin x + \cos x \frac{\cos x}{\sin x} = \frac{3}{x} - \sin x \ln \sin x + \cos x \cot x,$$

$$\frac{dy}{dx} = y\left(\frac{3}{x} - \sin x \ln \sin x + \cos x \cot x\right) = x^3 (\sin x)^{\cos x}\left(\frac{3}{x} - \sin x \ln \sin x + \cos x \cot x\right).$$

Notes The given function contains a term of the form $(f(x))^{g(x)}$, with $f(x) = \sin x$ and $g(x) = \cos x$. Hence we use either the equation $(f(x))^{g(x)} = e^{g(x) \ln f(x)}$ as in Alternate 1 or logarithmic differentiation as in Alternate 2. Again, in the answer don't forget to replace $e^{g(x) \ln f(x)}$ by $(f(x))^{g(x)}$, or y by the expression of the given function.

Example: Find:

$$\frac{d}{dx}\frac{(1-x)(2+x)^2(3-x)^3}{(4+x)^4}.$$

Solution:

Let:

$$y = \frac{(1-x)(2+x)^2(3-x)^3}{(4+x)^4}.$$

Employing logarithmic differentiation we obtain:

$$\ln y = \ln(1-x) + 2\ln(2+x) + 3\ln(3-x) - 4\ln(4+x),$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{1-x} + \frac{2}{2+x} - \frac{3}{3-x} - \frac{4}{4+x},$$

$$\begin{aligned}\frac{d}{dx}\frac{(1-x)(2+x)^2(3-x)^3}{(4+x)^4} &= \frac{dy}{dx}\\ &= y\left(-\frac{1}{1-x} + \frac{2}{2+x} - \frac{3}{3-x} - \frac{4}{4+x}\right)\\ &= \frac{(1-x)(2+x)^2(3-x)^3}{(4+x)^4}\left(-\frac{1}{1-x} + \frac{2}{2+x} - \frac{3}{3-x} - \frac{4}{4+x}\right).\end{aligned}$$

Notes Here we have a product and a quotient, but there's no term of the form $(f(x))^{g(x)}$, and we still employ logarithmic differentiation, which therefore isn't exclusive for the form $(f(x))^{g(x)}$. Of course we can use the product and quotient rules, but doing so would be more complicated. Generally, logarithmic differentiation is advantageous when the products and/or quotients are complicated. It enables us to convert the differentiation of a product and that of a quotient into that of a sum and that of a difference respectively.

Example: Differentiate $y = (\sec x)^{\tan x}$ in 2 ways:

1. Express it as natural exponential and then differentiate.
2. Use logarithmic differentiation.

Solution:

1.
$$\begin{aligned} y &= (\sec x)^{\tan x} = e^{\tan x \ln \sec x},\\ y' &= e^{\tan x \ln \sec x}(\sec^2 x \ln \sec x + \tan x\,(1/\sec x)\sec x \tan x)\\ &= (\sec x)^{\tan x}(\sec^2 x \ln \sec x + \tan^2 x).\end{aligned}$$

Notes

2. $\ln y = \ln (\sec x)^{\tan x} = \tan x \ln \sec x,$

$$\frac{y'}{y} = \sec^2 x \ln \sec x + \tan x \frac{\sec x \tan x}{\sec x} = \sec^2 x \ln \sec x + \tan^2 x,$$

$$y' = y(\sec^2 x \ln \sec x + \tan^2 x) = (\sec x)^{\tan x} (\sec^2 x \ln \sec x + \tan^2 x).$$

Example: Find y' using logarithmic differentiation if $y = x^x/(x - 1)^2$.

Solution:

$$\ln y = \ln \frac{x^x}{(x-1)^2} = x \ln x - 2\ln(x-1),$$

$$\frac{y'}{y} = \ln x + \frac{x}{x} - \frac{2}{x-1} = \ln x + 1 - \frac{2}{x-1},$$

$$y' = y\left(\ln x + 1 - \frac{2}{x-1}\right) = \frac{x^x}{(x-1)^2}\left(1 + \ln x - \frac{2}{x-1}\right).$$

Example: Let $f(x) = (x^x)^x$ and $g(x) = x^{(x^x)}$.

1. Which of these functions grows more rapidly for sufficiently large x?
2. Differentiate them.

Solution:

1. $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{(x^x)^x}{x^{(x^x)}} = \lim_{x\to\infty} \frac{x^{(x^2)}}{x^{(x^x)}} = \lim_{x\to\infty} x^{(x^2 - x^x)} = \lim_{x\to\infty} x^{x^2(1-x^{x-2})} = 0,$$

because $\lim_{x\to\infty} x^2(1 - x^{x-2}) = -\infty$. So g grows more rapidly.

2. Using logarithmic differentiation we have:

$$\ln f(x) = \ln(x^x)^x = x \ln x^x = x^2 \ln x,$$

$$\frac{f'(x)}{f(x)} = 2x \ln x + \frac{x^2}{x} = x(2\ln x + 1),$$

$$f'(x) = f(x)x(2\ln x + 1) = (x^x)^x\, x(2\ln x + 1) = x^{x^2+1}(2\ln x + 1),$$

$$\ln g(x) = \ln x^{(x^x)} = x^x \ln x,$$

$$\ln \ln g(x) = \ln(x^x \ln x) = x \ln x + \ln \ln x,$$

$$\frac{g'(x)}{g(x)\ln g(x)} = \ln x + \frac{x}{x} + \frac{1}{x \ln x} = \ln x + 1 + \frac{1}{x\ln x},$$

$$g'(x) = g(x)(\ln g(x))\left(\ln x + 1 + \frac{1}{x\ln x}\right)$$

$$= x^{(x^x)} x^x \ln x\left(\ln x + 1 + \frac{1}{x \ln x}\right)$$

$$= x^{x^x + x}\left(\ln^2 x + \ln x + \frac{1}{x}\right).$$

Notes

Example: Find:

$$\frac{d}{dt}\frac{t^3 \sin^2 t}{(t+1)(t+2)^2}.$$

Solution:

Let:

$$y = \frac{t^3 \sin^2 t}{(t+1)(t+2)^2}.$$

Utilizing logarithmic differentiation we get:

$$\ln y = \ln \frac{t^3 \sin^2 t}{(t+1)(t+2)^2} = 3\ln t + 2\ln \sin t - \ln(t+1) - 2\ln(t+2),$$

$$\frac{1}{y}\frac{dy}{dt} = \frac{3}{t} + 2\frac{\cos t}{\sin t} - \frac{1}{t+1} - \frac{2}{t+2} = 2\cot t + \frac{3}{t} - \frac{1}{t+1} - \frac{2}{t+2},$$

$$\frac{d}{dt}\frac{t^3 \sin^2 t}{(t+1)(t+2)^2} = \frac{dy}{dt}$$

$$= y\left(2\cot t + \frac{3}{t} - \frac{1}{t+1} - \frac{2}{t+2}\right)$$

$$= \frac{t^3 \sin^2 t}{(t+1)(t+2)^2}\left(2\cot t + \frac{3}{t} - \frac{1}{t+1} - \frac{2}{t+2}\right).$$

Example: Find an equation of the tangent line to the curve:

$$y = \frac{\sqrt{1+x}\sqrt{1+2x}\sqrt{1+3x}}{\sqrt{1+6x}}$$

at $x = 0$.

Solution:

$$y = \frac{\sqrt{1+x}\sqrt{1+2x}\sqrt{1+3x}}{\sqrt{1+6x}} = \frac{(1+x)^{1/2}(1+2x)^{1/2}(1+3x)^{1/2}}{(1+6x)^{1/2}}.$$

Employing logarithmic differentiation we obtain:

$$\ln y = \frac{1}{2}\ln(1+x) + \frac{1}{2}\ln(1+2x) + \frac{1}{2}\ln(1+3x) - \frac{1}{2}\ln(1+6x),$$

$$\frac{y'}{y} = \frac{1}{2(1+x)} + \frac{1}{1+2x} + \frac{3}{2(1+3x)} - \frac{3}{1+6x},$$

$$y' = y\left(\frac{1}{2(1+x)} + \frac{1}{1+2x} + \frac{3}{2(1+3x)} - \frac{3}{1+6x}\right).$$

At $x = 0$ we have $y = \left(\sqrt{1+0}\sqrt{1+2(0)}\sqrt{1+3(0)}\right)\Big/\sqrt{1+6(0)} = 1$. Thus the slope of the tangent line is:

$$y'\big|_{x=0,\, y=1} = (1)\left(\frac{1}{2(1+0)} + \frac{1}{1+2(0)} + \frac{3}{2(1+3(0))} - \frac{3}{1+6(0)}\right) = 0.$$

Consequently the equation of the tangent line is $y - 1 = 0(x - 0)$ or $y = 1$.

Notes

Example

Consumer price index of a certain group of workers increases by 15% per year and their quantity index by 6%. What is the annual growth of their expenditure.

Solution:

Let P denotes price index, Q the quantity index and E the expenditure index. We can write $E = P \times Q$.

Taking log of both sides, we get $\log E = \log P + \log Q$

Differentiating w.r.t. t, we get

$$\frac{d\log E}{dt} = \frac{d\log P}{dt} + \frac{d\log Q}{dt}$$

Let us denote $\frac{d\log E}{dt}$, the rate of growth of E, by r_E. Similarly, we denote $\frac{d\log P}{dt} = r_P$ and $\frac{d\log Q}{dt} = r_Q$.

Thus, we can write $r_E = r_P + r_Q = 0.15 + 0.06 = 0.21$.

Hence the rate of growth of expenditure index is 21%.

Example

Agricultural output is the following function of time: $X = K \times a^{b^t}$, where K, a and b are all positive constants with $a < 1$ and $b < 1$.

(i) Show that, starting from an initial level of output X_0, the output is always increasing but is subject to a ceiling which is never, exceeded.

(ii) Show that proportional rate of growth of output is always positive, but declines over time.

Solution:

(i) Note that initial output $X_0 = Ka$

To find $\frac{dX}{dt}$ we take log of both sides i.e. $\log X = \log K + b^t \log a$

Differentiating w.r.t. t, we get $\frac{d\log X}{dt} = \frac{1}{X}\cdot\frac{dX}{dt} = b^t \log a \cdot \log b$

or $\frac{dX}{dt} = K \cdot a^{b^t} \cdot b^t \log a \times \log b > 0$ (Since $\log a, \log b < 0$)

To determine the ceiling on output, we find

$$X = \lim_{t\to\infty} K.a^{b^t} = K \lim_{t\to\infty} a^{b^t} = K$$

(ii) The proportional rate of growth is

$$r = \frac{d\log X}{dt} = b^t \log a \cdot \log b > 0 \therefore \frac{dr}{dt} = b^t \log a(\log b)^2 < 0.$$

Hence proportional rate of growth is positive but declines over time.

10.3 Summary

- $f : x \to [\cos(x)]^x$, which cannot be treated as a power g^n where $g : x \to \cos(x)$ or as an exponent e^x.

  We cannot apply the exponential or power rule for differentiating f.

- Using the properties of the natural logarithm (ln), we can "simplify" some functions to allow us to apply the product rule, and logarithmic rule for differentiating

  $\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f.\frac{dg}{dx}$ and $\frac{d}{dx}[\ln(u)] = \frac{1}{u} \cdot \frac{du}{dx}$.

  - The commonly used property for logarithmic differentiation is $\ln(u^x) = x \cdot \ln(u)$.

- To use logarithmic differentiation we must assume the function with which we take the natural logarithm cannot be less or equal to zero $\ln(f)$ implies that $f > 0$.

  - Functions which output negative values can be solved by taking the absolute value of the function $\ln|f|$.

- To apply logarithmic differentiation we simply take the logarithm on both sides of an equation, simplify, and differentiate implicitly with respect to the independant variable.

$$\begin{aligned} f(x) &= \ldots \\ \ln(f(x)) &= \ln(\ldots) \\ \frac{1}{f(x)} \cdot \frac{df}{dx} &= \frac{d}{dx}[\ln(\ldots)] \\ \frac{df}{dx} &= f_{(x)} \cdot \frac{d}{dx}[\ln(\ldots)] \end{aligned}$$

10.4 Keyword

Logarithmic Differentiation: To differentiate a function of the form $[f(x)]^{g(x)}$ or $a^{f(x)}$, we use a method called Logarithmic differentiation.

10.5 Self Assessment

1. If $y = x^x$ find y′

   (a) $x^x \log ex$ (b) $e^x \log x$

   (c) $x \log ex$ (d) $x \log ex^x$

Notes

2. Value of $\ln\frac{x}{y} =$

   (a) ln x – ln y (b) ln y – ln x

   (c) ln x + ln y (d) ln y × ln x

3. ln (x + y) is equals to

   (a) ln xy + ln yx (b) ln x – ln y

   (c) x ln y – y ln x (d) ln x + ln y

4. $\ln\left(\frac{x}{y}\right)$ equals to

   (a) $\frac{\ln x}{\ln y}$ (b) $\frac{\ln y}{\ln x}$

   (c) $\frac{\ln x^y}{\ln y^x}$ (d) $\ln x \cdot \ln y$

5. If $\frac{\ln x}{\ln y}$ is given then its value equals to

   (a) ln x + ln y (b) ln x – ln y

   (c) ln y – ln x (d) x ln y – y ln x

6. ln 1 equals to

   (a) 0 (b) 1

   (c) –1 (d) ∞

7. Value of lne equals to

   (a) 1 (b) 2

   (c) –1 (d) ∞

8. Value of y^x equals to

   (a) y ln x (b) x ln y

   (c) ln yx (d) ln x^{-y}

9. Value of $D\left\{x^{(2x+3)}\right\}$ equal to

   (a) $x^{(2x+3)2\ln x}$ (b) $x^{(x+3)^2\ln x}$

   (c) $x^{(x+3)\ln x}$ (d) $\ln x^{(3x+2)}, 2\ln x$

10. ln e^x is equal to

   (a) x (b) 1

   (c) 0 (d) –1

Notes

10.6 Review Questions

1. Differentiate the function $y = \dfrac{x^5}{(1-10x)\sqrt{x^2+2}}$.

2. Differentiate y = x^x.

3. Differentiate $y = \dfrac{(\ln x)^x}{2^{3x+1}}$.

4. Differentiate $y = \dfrac{x^{2x}(x-1)^3}{(3+5x)^4}$.

5. Consider the function $f(x) = \dfrac{x^5 e^x (4x+3)}{5^{\ln x}(3-x)^2}$. Find an equation of the line tangent to the graph of f at x = 1.

Answers: Self Assessment

1.	(a)	2.	(a)
3.	(d)	4.	(a)
5.	(b)	6.	(a)
7.	(a)	8.	(b)
9.	(a)	10.	(a)

10.7 Further Readings

Books

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

Smith and Minton, *Calculus Early Trancendental,* Third Edition, McGraw Hill 2008.

Online links

http://www.suitcaseofdreams.net/Trigonometric_Functions.htm

http://library.thinkquest.org/20991/alg2/trigi.html

http://www.intmath.com/trigonometric–functions/5–signs–of–trigonometric–functions.php

Notes

Unit 11: Parametric Differentiation

CONTENTS

Objectives

After studying this unit, you will be able to:

- Differentiate a function defined parametrically
- Find the second derivative of such a function

Introduction

Some relationships between two quantities or variables are so complicated that we sometimes introduce a third quantity or variable in order to make things easier to handle. In mathematics this third quantity is called a parameter. Instead of one equation relating say, x and y, we have two equations, one relating x with the parameter, and one relating y with the parameter. In this unit we will give examples of curves which are defined in this way, and explain how their rates of change can be found using parametric differentiation.

Instead of a function y(x) being defined explicitly in terms of the independent variable x, it is sometimes useful to define both x and y in terms of a third variable, t say, known as a parameter. In this unit we explain how such functions can be differentiated using a process known as parametric differentiation.

11.1 The Parametric Definition of A Curve

In the first example below we shall show how the x and y coordinates of points on a curve can be defined in terms of a third variable, t, the parameter.

Example: Consider the parametric equations

$x = \cos t \qquad y = \sin t \qquad \text{for } 0 \le t \le 2\pi$ (1)

Note how both x and y are given in terms of the third variable t.

Notes

To assist us in plotting a graph of this curve we have also plotted graphs of cos t and sin t in Figure 11.1. Clearly,

when t = 0, x = cos 0 = 1; y = sin 0 = 0

when $t = \frac{\pi}{2}, x = \cos\frac{\pi}{2} = 0; y = \sin\frac{\pi}{2} = 1.$

In this way we can obtain the x and y coordinates of lots of points given by Equations (1). Some of these are given in Table 11.1.

Figure 11.1: Graphs of sin t and cos t

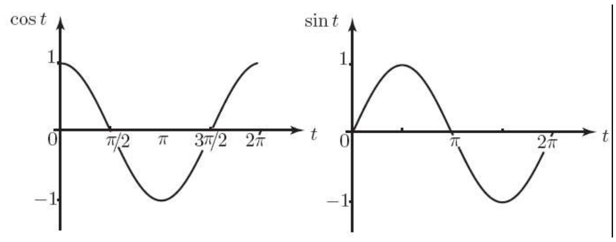
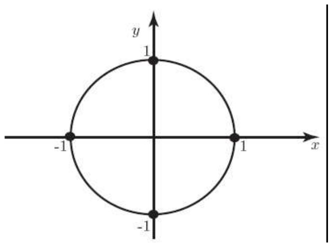


Table 11.1: Values of x and y given by Equations (1)

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	1	0	-1	0	1
y	0	1	0	-1	0

Plotting the points given by the x and y coordinates in Table 1, and joining them with a smooth curve we can obtain the graph. In practice you may need to plot several more points before you can be confident of the shape of the curve. We have done this and the result is shown in Figure 11.2.

Figure 11.2. The parametric equations define a circle centered at the origin and having radius 1



So x = cos t, y = sin t, for t lying between 0 and 2π, are the parametric equations which describe a circle, centre (0, 0) and radius 1.

11.2 Differentiation of A Function Defined Parametrically

It is often necessary to find the rate of change of a function defined parametrically; that is, we want to calculate $\frac{dy}{dx}$. The following example will show how this is achieved.

Notes

Example: Suppose we wish to find $\frac{dy}{dx}$ when x = cos t and y = sin t.

We differentiate both x and y with respect to the parameter, t:

$$\frac{dx}{dt} = -\sin t \qquad \frac{dx}{dt} = \cos t$$

From the chain rule we know that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

so that, by rearrangement

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ provided } \frac{dx}{dt} \text{ is not equal to 0}$$

So, in this case

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t$$

Notes Parametric differentiation: if x = x(t) and y = y(t) then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ provided $\frac{dx}{dt} \neq 0$

Example: Suppose we wish to find $\frac{dy}{dx}$ when $x = t^3 - t$ and $y = 4 - t^2$.

$$x = t^3 - t \qquad y = 4 - t^2$$

$$\frac{dx}{dt} = 3t^2 - 1 \qquad \frac{dy}{dt} = -2t$$

From the chain rule we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

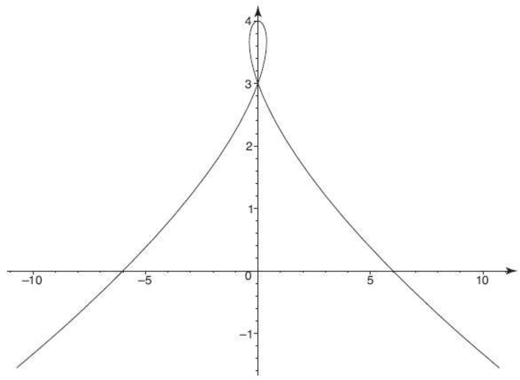
$$= \frac{-2t}{3t^2 - 1}$$

So, we have found the gradient function, or derivative, of the curve using parametric differentiation.

Notes

For completeness, a graph of this curve is shown in Figure 11.3.

Figure 11.3



Example: Suppose we wish to find $\frac{dy}{dx}$ when $x = t^3$ and $y = t^2 - t$.

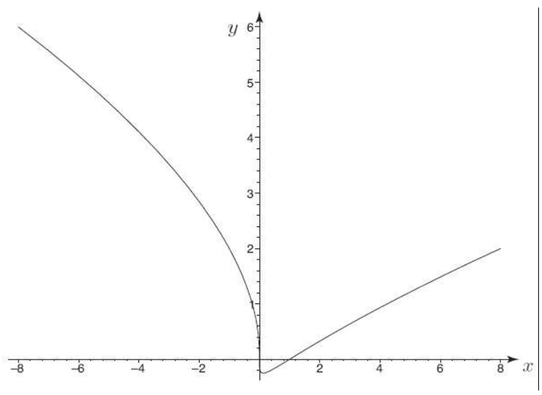
In this Example we shall plot a graph of the curve for values of t between –2 and 2 by first producing a table of values (Table 11.2).

Table 11.2

t	-2	-1	0	1	2
x	-8	-1	0	1	8
y	6	2	0	0	2

Part of the curve is shown in Figure 11.4. It looks as though there may be a turning point between 0 and 1. We can explore this further using parametric differentiation.

Figure 11.4



From

$$x = t^3 \qquad y = t^2 - t$$

we differentiate with respect to t to produce

$$\frac{dx}{dt} = 3t^2 \qquad \frac{dy}{dt} = 2t - 1$$

Notes

Then, using the chain rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{provided} \frac{dx}{dt} \neq 0$$

$$\frac{dy}{dx} = \frac{2t-1}{3t^2}$$

From this we can see that when $t = \frac{1}{2}, \frac{dy}{dx} = 0$ and so $t = \frac{1}{2}$ is a stationary value. When $t = \frac{1}{2}$, $x = \frac{1}{8}$ and $y = -\frac{1}{4}$ and these are the coordinates of the stationary point.

We also note that when $t = 0, \frac{dy}{dx}$ is infinite and so the y axis is tangent to the curve at the point (0, 0).

11.3 Second Derivatives

Example: Suppose we wish to find the second derivative $\frac{d^2y}{dx^2}$ when

$$x = t^2 \qquad y = t^3$$

Differentiating we find

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2$$

Then, using the chain rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{provided } \frac{dx}{dt} \neq 0$$

So that

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$$

We can apply the chain rule a second time in order to find the second derivative, $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Notes

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{3}{2}}{2t}$$

$$= \frac{3}{4t}$$

Notes if x = x(t) and y = y(t) then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example:

Suppose we wish to find $\frac{d^2y}{dx^2}$ when

$$x = t^3 + 3t^2 \qquad y = t^4 - 8t^2$$

Differentiating

$$\frac{dx}{dt} = 3t^2 + 6t \qquad \frac{dy}{dt} = 4t^3 - 16t$$

Then, using the chain rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{provided} \quad \frac{dx}{dt} \neq 0$$

so that

$$\frac{dy}{dx} = \frac{4t^3 - 16t}{3t^2 + 6t}$$

This can be simplified as follows:

$$\begin{aligned}\frac{dy}{dx} &= \frac{4t(t^2-4)}{3t(t+2)} \\ &= \frac{4t(t+2)(t-2)}{3t(t+2)} \\ &= \frac{4(t-2)}{3}\end{aligned}$$

Notes

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}\\ &= \frac{\frac{4}{3}}{3t^2+6t}\\ &= \frac{4}{9t(t+2)}\end{aligned}$$

11.4 Parametric Functions

If the variables x and y are given in terms of a new variable *t,* then the function is said to be in the parametric form and 't' is called the parameter.

In general, the parametric function is given by $x = f(t), y = g(t)$ where $f(t)$ and $g(t)$ are functions of the parameter *t*.

$x = f(t), y = g(t)$ are called the parametric equations.

To find $\frac{dy}{dx}$ when the parametric equations are given

Let $x = f(t)$ and $y = g(t)$ be the parametric equations.

Differentiate $x = f(t)$ w.r.t. *t* to get $\frac{dx}{dt}$

Differentiate $y = g(t)$ w.r.t. *t* to get $\frac{dy}{dt}$

then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

This method of differentiation is called Parametric Differentiation.

Example: Find $\frac{dy}{dx}$, if, $x = at^2, y = 2at$

Solution:

Differentiating both the equations w.r.t. *t*, we get

$$\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

Example: Find $\frac{dy}{dx}$, if, $x=\sqrt{t}, y=1/\sqrt{t}$

Solution:

Differentiating w.r.t. t, we get $\frac{dx}{dt}=\frac{1}{2\sqrt{t}}, \frac{dy}{dt}=\frac{-1}{2t^{3/2}}$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{-1}{2t^{3/2}}}{\frac{1}{2\sqrt{t}}}=-\frac{1}{t}$$

Example: Find $\frac{dy}{dx}$, if, $x=2e^t, y=3e^{-t}$

Solution:

$$\frac{dx}{dt}=2e^t, \frac{dy}{dt}=-3e^{-t}$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$=\frac{-3e^{-t}}{2e^{2t}}$$

$$=\frac{-3}{2e^{2t}}$$

Example: Find $\frac{dy}{dx}$, if, $x=e^t \log t, y=e^{-t}\log t$

Solution:

$$\frac{dx}{dt}=e^t\frac{1}{t}+\log t\cdot e^t$$

$$=e^t\left(\frac{1}{t}+\log t\right)$$

$$\frac{dy}{dt}=e^{-t}\frac{1}{t}+\log\ t\left(-e^{-t}\right)$$

$$=e^{-t}\left(\frac{1}{t}-\log t\right)$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{e^{-t}\left(\frac{1}{t}-\log t\right)}{e^t\left(\frac{1}{t}+\log t\right)}$$

$$=\frac{(1-t\log t)}{e^{2t}(1+t\log t)}$$

Notes

Example: Find $\frac{dy}{dx}$, if, $x = \frac{3at}{1+t^3}$ $y = \frac{3at^2}{1+t^3}$

Solution:

$$\frac{dx}{dt} = \frac{(1+t^3)3a - 3at(3t^2)}{(1+t^3)^2}$$

$$= \frac{3a(1+t^3-3t^3)}{(1+t^3)^2}$$

$$= \frac{3a(1-2t^3)}{(1+t^3)^2}$$

$$y = \frac{3at^2}{1+t^3}$$

$$\frac{dy}{dt} = \frac{(1+t^3)6at - 3at^2 \cdot 3t^2}{(1+t^3)^2}$$

$$\frac{3at(2-t^3)}{(1+t^3)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3at(2-t^3)}{3a(1-2t^3)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{t(2-t^3)}{(1-2t^3)}$$

Example: Find $\frac{dy}{dx}$, if, $x = \log\left(t + \sqrt{l^2+1}\right), y = \sqrt{t^2+1}$

Solution:

$$\frac{dx}{dt} = \frac{1}{t+\sqrt{t^2+1}} \cdot \left(1 + \frac{1}{2\sqrt{t^2+1}} \cdot 2t\right)$$

$$= \frac{1}{\left(t+\sqrt{t^2+1}\right)} \frac{\left(\sqrt{t^2+1}+t\right)}{\sqrt{t^2+1}} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{t^2+1}} \cdot (2t) = \frac{t}{\sqrt{t^2+1}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{t}{\sqrt{t^2+1}}}{\frac{1}{\sqrt{t^2+1}}} = t$$

Notes

> *Task* Find $\frac{dy}{dx}$, *if*
>
> (i) $x = \frac{2-t}{2+t}$
>
> (ii) $y = \frac{2t}{2+t}$ (note $x + y = 1$)

11.5 Summary

- If the variables x and y are given in terms of a new variable *t*, then the function is said to be in the parametric form and 't' is called the parameter.
- In general, the parametric function is given by $x = f(t), y = g(t)$ where $f(t)$ and $g(t)$ are functions of the parameter *t*.

  $x = f(t), y = g(t)$ are called the parametric equations.

11.6 Keywords

Parameter: If the variables x and y are given in terms of a new variable *t*, then the function is said to be in the parametric form and 't' is called the parameter.

Parametric Equations: In general, the parametric function is given by $x = f(t), y = g(t)$ where $f(t)$ and $g(t)$ are functions of the parameter *t*.

$x = f(t), y = g(t)$ are called the parametric equations.

11.7 Self Assessment

1. $x = t + \sqrt{t} \cdot y = t - \sqrt{t}$ then $\frac{dy}{dx}$

   (a) $\frac{3t}{2}$ (b) $-\cot t$

   (c) $\frac{2\sqrt{t}-1}{2\sqrt{t}+1}$ (d) $\frac{4te^t}{1-t}$

2. $x = te^{-t}$, $y = 2t^2 + 1$ then $\frac{dy}{dx}$

   (a) $\frac{5}{4}t^3$ (b) $\frac{2\sqrt{t}-1}{2\sqrt{t}+1}$

   (c) $\frac{4te^t}{1-t}$ (d) $-\cos t$

3. $x = 2t^3 + 1$, $y = te^{-2t}$, determine co-ordinater of the stationary points.

   (a) $\left(\frac{5}{4}, \frac{1}{2e}\right)$ (b) $\left(1+\sqrt{2}, -16\right)$

Notes

(c) $\left(\frac{5}{126}, \frac{-1}{26}\right)$ (d) $\left(\frac{22}{7}, \frac{7}{77}\right)$

4. $x = \sqrt{t} + 1, y = t^3 - 12t$ for $t > 0$, determine co-ordinat of stationary points.

(a) $\left(\frac{5}{4}, \frac{1}{2e}\right)$ (b) $\left(1+\sqrt{2}, -16\right)$

(c) $\left(\frac{5}{28}, \sqrt{2}\right)$ (d) $\left(\sqrt{2}, \sqrt{16}\right)$

5. x = sin t, y = cos t determine $\frac{d^2y}{dx^2}$

(a) $-\sec^3 t$ (b) $-\sec^2 t$

(c) $\cos^2 t$ (d) $\text{cosec}^3 t$

6. $x = e^{-t}$, $y = t^3 + t + 1$, thus $\frac{d^2y}{dx^2}$

(a) $\frac{1}{2t}$ (b) $\left(3t^2 + 6t + 1\right)e^{2t}$

(c) $-\sec^2 t$ (d) $\frac{1}{9t^3}$

11.8 Review Questions

1. Determine $\frac{dy}{dx}$, if $x = t^2 + 1$, $y = t^3 - 1$

2. Explain $\frac{dy}{dx}$, if x = 3 cos t, y = 3 sin t

3. Solve $\frac{dy}{dx}$, if $x = 2t^3 + 1$, $y = t^2 \cos t$

4. Determine the co-ordinates of the stationary points if $x = 5t^4$, $y = 5t^6 - t^5$ for $t > 0$

5. Determine the co-ordinates of the stationary points if $x = t + t^2$, $y = \sin t$ for $0 < t < \pi$

6. Determine $\frac{d^2y}{dx^2}$ in terms of t, if $x = 3t^2 + 1$, $y = t^3 - 2t^2$

7. Explain $\frac{d^2y}{dx^2}$ in terms of t, if $x = \frac{1}{2}t^2 + 2, y = \sin(t+1)$

Notes

Answers: Self Assessment

1.	(c)	2.	(c)
3.	(a)	4.	(b)
5.	(a)	6.	(b)

11.9 Further Readings

Books

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

Smith and Minton, *Calculus Early Trancendental,* Third Edition, McGraw Hill 2008.

Online links

http://www.suitcaseofdreams.net/Trigonometric_Functions.htm

http://library.thinkquest.org/20991/alg2/trigi.html

http://www.intmath.com/trigonometric–functions/5–signs–of–trigonometric–functions.php

Notes

Unit 12: Successive Differentiation

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss successive differentiation
- Differentiate problem related to successive differentiation

Introduction

Differentiation in math terms is the mathematical procedure of taking the derivative of a function. A derivative of a function is a function that gives the slopes of the tangent lines to each point of the curve representative of the function on a graph. We have seen that the derivative of a function of x is in general also a function of x. This new function may also be differentiable, in which case the derivative of the first derivative is called the second derivative of the original function. Similarly, the derivative of the second derivative is called the third derivative; and so on to the n^{th} derivative. Thus, if

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}$$

12.1 Successive Differentiation

If $y = f(x)$ is a differentiable function then by differentiating it w.r.t. x, we get

$$\frac{dy}{dx} = f'(x) \qquad \text{...(i)}$$

Notes

If $\frac{dy}{dx} = f'(x)$ is a differentiable function, then by differentiating it w.r.t. x, we get

$$\frac{d^2y}{dx^2} = f''(x) \qquad ...(ii)$$

Similarly by differentiating it w.r.t. x, we get

$$\frac{d^3y}{dx^3} = f'''(x) \qquad ...(iii)$$

Again by differentiating it w.r.t. x, we get

$$\frac{d^4y}{dx^4} = f^{\text{iv}}(x) \qquad ...(iv)$$

and so on.

This process of finding higher ordered derivatives is called successive differentiation. $\frac{dy}{dx}$ is called first derivative, $\frac{d^2y}{dx^2}$ is called the second derivative, $\frac{d^3y}{dx^3}$ is called the third derivative and $\frac{d^4y}{dx^4}$ is called fourth derivative and so on.

In general, $\frac{d^ny}{dx^n}$ is called the n^{th} derivative, which is obtained by differentiating $\frac{d^{n-1}y}{dx^{n-1}}$ w.r.t. x.

The n^{th} derivative of $y = f(x)$ is denoted by the symbols y_n, $f^{(n)}(x)$, $\frac{d^ny}{dx^n}$, $\frac{d^n}{dx^n}[f(x)]$

Examples: Find the second, third, fourth derivatives of the following functions:

1. $x^4 - 5x^3 + 7x^2 - 2x + \frac{1}{x}$
2. $ax^2 + bx + c$
3. $\frac{ax+b}{cx+d}$
4. $x\log x$
5. xe^x

Solution:

1. Let $y = x^4 - 5x^3 + 7x^2 - 2x + \frac{1}{x}$

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 15x^2 + 14x - 2 - \frac{1}{x^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = 12x^2 - 30x + 14 + \frac{2}{x^3}$$

$$\frac{d^3y}{dx^3} = 24x - 30 - \frac{6}{x^4}$$

$$\frac{d^4y}{dx^4} = 24 + \frac{24}{x^5}$$

Notes

2. Let $y = ax^2 + bx + c$

$\therefore \quad \frac{dy}{dx} = -2ax + b + 0$

$\therefore \quad \frac{d^2y}{dx^2} = 2a + 0$

$= 2a$

$\frac{d^3y}{dx^3} = 0$

$\frac{d^4y}{dx^4} = 0$

3. Let $y = \frac{ax+b}{cx+d}$

$\therefore \quad \frac{dy}{dx} = \frac{(cx+d)a - (ax+b)c}{(cx+d)^2}$

$= \frac{acx + ad - acx - bc}{(cx+d)^2}$

$= \frac{ad - bc}{(cx+d)^2}$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{(ac-bc)(-2c)}{(cx+d)^3}$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{2c(bc-ad)}{(cx+d)^3}$

$\frac{d^3y}{dx^3} = \frac{-6c^2(bc-ad)}{(cx+d)^4}$

$\frac{d^4y}{dx^4} = \frac{24c^3(bc-ad)}{(cx+d)^5}$

4. Let $y = x \log x$

$\therefore \quad \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x = 1 + \log x$

$\therefore \quad \frac{d^2y}{dx^2} = 0 + \frac{1}{x} = \frac{1}{x}$

$\frac{d^3y}{dx^3} = -\frac{1}{x^2}$

$\frac{d^4y}{dx^4} = \frac{2}{x^3}$

Notes

5. Let $y = xe^x$

$\therefore \quad \frac{dy}{dx} = xe^x + e^x \cdot 1$

$\therefore \quad \frac{d^2y}{dx^2} = (x+1)e^x + e^x \cdot 1$

$= (x+2)e^x$

$\frac{d^3y}{dx^3} = (x+2)e^x + e^x.1$

$= (x+3)e^x$

$\frac{d^4y}{dx^4} = (x+3)e^x + e^x.1$

$= (x+4)e^x$

Example: If $x^2 + xy + y^2 = 0$, prove that $\frac{d^2y}{dx^2} = 0$.

Solution: $x^2 + xy + y^2 = 0$

Differentiating w.r.t. x, we get

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x+2y) = -(2x+y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-[2x+y]}{[x+2y]}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \left[\frac{(x+2y)\left(2+\frac{dy}{dx}\right) - (2x+y)\left(1+2\frac{dy}{dx}\right)}{(x+2y)^2}\right]$$

$$= \frac{(x+2y)\left[2 - \frac{2x+y}{x+2y}\right] - (2x+y)\left[1 - 2\frac{(2x+y)}{(x+2y)}\right]}{(x+2y)^2}$$

$$= \left[\frac{(x+2y)3y - (2x+y)(-3x)}{(x+2y)^3}\right]$$

$$= -\left[\frac{3xy + 6y^2 + 6x^2 + 3xy}{(x+2y)^3}\right]$$

Notes

$$= -\left[\frac{6x^2 + 6xy + 6y^2}{(x+2y)^3}\right]$$

$$= -6\left[\frac{x^2 + xy + y^2}{(x+2y)^3}\right] = \frac{-6(0)}{(x+2y)^3} = 0$$

$$\therefore \quad \frac{d^2y}{dx^2} = 0$$

Example: Find $\frac{d^2y}{dx^2}$, if $y = \frac{ax+b}{bx+a}$

Solution: $y = \frac{ax+b}{bx+a}$

$$\therefore \quad \frac{dy}{dx} = \frac{(bx+a)a - (ax+b)b}{(bx+a)^2}$$

$$= \frac{abx + a^2 - abx - b^2}{(bx+a)^2}$$

$$= \frac{a^2 - b^2}{(bx+a)^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = (a^2 - b^2)(-2)(bx+a)^{-2-1}\frac{d}{dx}(bx+a)$$

$$= 2(b^2 - a^2)(bx+a)^{-3}(b) = \frac{2b(b^2 - a^2)}{(bx+a)^3}$$

Example: Find $\frac{d^2y}{dx^2}$, if $y = a^x$.

Solution: $y = a^x$

$$\therefore \quad \frac{dy}{dx} = a^x \log a$$

$$\frac{d^2y}{dx^2} = a^x (\log a)^2$$

Example: If $y = \left(x + \sqrt{x^2+1}\right)^m$, prove that $(x^2+1)y_2 + xy_1 - m^2y = 0$.

Solution: $y = \left(x + \sqrt{x^2+1}\right)^m$

$$\therefore \quad y_1 = m\left(x + \sqrt{x^2+1}\right)^{m-1}\frac{d}{dx}\left(x + \sqrt{x^2+1}\right)$$

Notes

$$= m\left(x+\sqrt{x^2+1}\right)^{m-1}\left[1+\frac{1}{2\sqrt{x^2+1}}(2x)\right]$$

$$= m\left(x+\sqrt{x^2+1}\right)^{m-1}\frac{\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}}$$

i.e., $y_1 = \dfrac{m\left(\sqrt{x^2+1}+x\right)^m}{\sqrt{x^2+1}}$

$\therefore \quad \sqrt{x^2+1}\,y_1 = my$ ...(i)

Differentiating again w.r.t. x, we get

$$\sqrt{x^2+1}\,y_2 + y_1\frac{1}{2\sqrt{x^2+1}}(2x) = my_1$$

Multiplying throughout by $\sqrt{x^2+1}$, we get

$$\left(x^2+1\right)y_2 + xy_1 = my_1\sqrt{x^2+1}$$

$\therefore \quad \left(x^2+1\right)y_2 + xy_1 = m\left(my\right)$ (using (1))

i.e., $\left(x^2+1\right)y_2 + xy_1 = m^2y$

$\therefore \quad \left(x^2+1\right)y_2 + xy_1 - m^2y = 0$

Alternate: Squaring equation (i)

We get $\left(x^2+1\right)y_1^2 = m^2y^2$

Differentiating w.r.t. x,

$\left(x^2+1\right)2y_1y_2 + y_1^2\,(2x) = m^2\,2yy_1$ (cancelling $2y_1$)

$\therefore \quad \left(x^2+1\right)y_2 + xy_1 = m^2y$

i.e., $\left(x^2+1\right)y_2 + xy_1 - m^2y = 0$

Example: If $y = ax^{n+1} + bx^{-n}$, prove that $x^2y_2 - n\left(n+1\right)y = 0$.

Solution: $y = ax^{n+1} + bx^{-n}$

$$\therefore \quad y_1 = (n+1)\,ax^{n+1-1} + b(-n)\,x^{-n-1}$$

$$= (n+1)\,ax^n - bnx^{-n-1}$$

$$y_2 = (n+1)\,nax^{n-1} - bn\,(-n-1)\,x^{-n-1-1}$$

$$y_2 = (n+1)\,nax^{n-1} + bn\,(n+1)\,x^{-n-2}$$

$$\therefore x^2y_2 = (n+1)nax^{n-1}.\,x^2 + bn(n+1)x^{-n-2}.x^2$$

i.e., $\quad x^2y_2 = \left(n+1\right)nax^{n+1} + bn\left(n+1\right)x^{-n}$

Notes

$$= n(n+1)\left[ax^{n+1} + bx^{-n}\right] = n(n+1)y$$

$$\therefore x^2 y_2 - n(n+1)y = 0$$

Example: Find $\dfrac{d^2y}{dx^2}$, if, $x = at^2, 2y = 2at$

Solution:

$$\therefore \qquad \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$

$$\therefore \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx}$$

$$= \frac{-1}{t^2} \cdot \frac{1}{2at} \quad \because \frac{dx}{dt} = 2at$$

$$= \frac{-1}{2at^3}$$

Example: Find $\dfrac{d^2y}{dn^2}$, if, $x = ct, y = \dfrac{c}{t}$

Solution:

$$\frac{dy}{dt} = c, \frac{dx}{dt} = -\frac{c}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{c}{t^2}}{c} = -\frac{1}{t^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{1}{t^2}\right)$$

$$= \frac{d}{dt}\left(-\frac{1}{t^2}\right) \cdot \frac{dt}{dx} = \frac{2}{t^3} \cdot \frac{1}{c} = \frac{2}{ct^3}$$

Example: Find $\dfrac{d^2y}{dx^2}$, if, $x^2 + xy + y^2 = a^2$

Solution:

Differentiate $x^2 + xy + y^2 = a^2$ w.r.t. x

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx}(x+2y) = -(2x+y), \ \therefore \frac{dy}{dx} = -\left[\frac{2x+y}{x+2y}\right]$$

Notes

$$\therefore\ \frac{d^2y}{dx^2} = -\left[\frac{(x+2y)\left(2+\frac{dy}{dx}\right)-(2x+y)\left(1+2\frac{dy}{dx}\right)}{(x+2y)^2}\right]$$

$$= -\left[\frac{2x+4y+(x+2y)\frac{dy}{dx}-2x-y-(4x+2y)\frac{dy}{dx}}{(x+2y)^2}\right]$$

$$= -\left[\frac{3y+\frac{dy}{dx}(x+2y-4x-2y)}{(x+2y)^2}\right]$$

$$= -\left[\frac{3y+3x\left(\frac{2x+y}{x+2y}\right)}{(x+2y)^2}\right]$$

$$= -3\left[\frac{y(x+2y)+x(2x+y)}{(x+2y)^3}\right]$$

$$= -3\left[\frac{xy+2y^2+2x^2+xy}{(x+2y)^3}\right]$$

$$= -\left[\frac{2x^2+2xy+2y^2}{(x+2y)^3}\right]$$

$$= -3(2)\left[\frac{x^2+xy+y^2}{(x+2y)^3}\right]$$

$$= \frac{-6a^2}{(x+2y)^3} \qquad (\because\ x^2+xy+y^2=a^2)$$

Example: Find $\frac{d^2y}{dn^2}$, if, $x^3y^2 = a^5$

Solution:

Differentiating w.r.t. *x*, we get

$$x^3 2y\frac{dy}{dx}+y^2 3x^2 = 0$$

$$\therefore\quad \frac{dy}{dx} = \frac{-3x^2y^2}{2x^3y} = \frac{-3y}{2x}$$

Notes

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{3}{2}\left[\frac{x\dfrac{dy}{dx} - y}{x^2}\right] = \frac{-3}{2x^2}\left[x\left(\frac{-3y}{2x}\right) - y\right]$$

$$= \frac{-3}{2x^2}\left[\frac{-3}{2}y - y\right] = \frac{-3}{2x^2}\left[-\frac{5}{2}y\right] = \frac{15y}{4x^2}$$

Example: Find $\dfrac{d^2y}{dn^2}$, if, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Solution:

Differentiating w.r.t. x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y - x\dfrac{dy}{dx}}{y^2}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{y + x\left(\dfrac{b^2x}{a^2y}\right)}{y^2}\right]$$

$$= -\frac{b^4}{a^2y^2}\left[\frac{a^2y^2 + b^2x^2}{a^2b^2y}\right]$$

$$= \frac{-b^2}{a^2y^2}\left[\frac{a^2y^2 + b^2x^2}{a^2y}\right]$$

$$= -\frac{b^4}{a^2y^3}\left[\frac{x^2}{a^2} + \frac{y^2}{b^2}\right]$$

$$= -\frac{b^4}{a^2y^3}[1] = \frac{-b^4}{a^2y^3}$$

Task Find $\dfrac{d^2y}{dx^2}$, if:

1. $y = (x^2 + 2) \log x$,
2. $y = \dfrac{1}{ax + b}$

12.2 Definition of Successive Differentiation

Notes

Consider,

A one variable function,

y = f(x) (x is independent variable and y depends on x.)

Here if we make any change in x there will be a related change in y.

This change is called derivative of y w.r.t. x. denoted by $f'(x)$ or y_1 or y' or $\frac{dy}{dx}$ called first order derivative of y w.r.t. x.

$f''(x) = (f'(x))' = \frac{d^2y}{dx^2} = y'' = y_2$ is called second order derivative of y w.r.t x.

It gives rate of change in y_1 w.r.t. rate of change in x.

Similarly,

Third derivative of y is denoted by y_3 or $f'''(x)$ or $\frac{d^3y}{dx^3}$ or y''' and

So on…………………

(Above derivatives exist because, If y = f(x), then y_1 = g(x), where g(x) is some function of x depends on f(x) for e.g. if y = sinx then y_1 = cosx, hence y_2 = –sinx and so on………)

Thus,

Derivatives of f(x) (or f) w.r.t. x are denoted by, $f'(x), f''(x), f'''(x), \ldots\ldots f^{(n)}(x), \ldots\ldots$

Above process is called successive differentiation of f (x) w.r.t. x and $f', f'', f''', \ldots, f^{(n)}$ are called successive derivatives of f.

$f^{(n)}(x)$ denotes n^{th} derivative of f.

Notations:

Successive derivatives of y w.r.t. x are also denoted by,

1. $y_1, y_2, y_3, \ldots\ldots y_n, \ldots\ldots\ldots\ldots$ or
2. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots \frac{d^ny}{dx^n}, \ldots\ldots\ldots$ or
3. $f'(x), f''(x), f'''(x), \ldots\ldots, f^{(n)}(x), \ldots\ldots$ or
4. $y', y'', y''', \ldots\ldots y^{(n)}, \ldots\ldots$ or
5. $Dy, D^2y, D^3y, \ldots\ldots, D^ny, \ldots\ldots$

Where D denotes $\frac{d}{dx}$.

Value of n^{th} derivative of y = f(x) at x = a is denoted by,

$f^n(a)$, $y_n(a)$, or $\left(\frac{d^ny}{dx^n}\right)_{x=a}$

(i.e. value can be obtained by just replacing x with a in $f^n(x)$.)

Notes

Notes

Sr no.	Function	n^{th} derivative
01	$y = e^{ax}$	$y_n = a^n e^{ax}$
02	$y = b^{ax}$	$y_n = a^n b^{ax} (\log_e b)^n$
03	$y = (ax + b)^m$	(i) if m is integer greater than n or less than (–1) then, $y_n = m(m-1)(m-2)\ldots(m-n+1)\, a^n(ax+b)^{m-n}$ (ii) if m is less than n then, $y_n = 0$ (iii) if m = n then, $y_n = a^n\, n!$ (iv) if m = -1 then, $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$ (v) if m = -2 then, $y_n = \frac{(-1)^n (n+1)! a^n}{(ax+b)}$
04	$y = \log (ax + b)$	$y_n = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}$

n^{th} derivatives of reciprocal of polynomials (n^{th} derivatives of functions which contain polynomials in denominators) :

Consider

$$y = \frac{ax+b}{cx^2+dx+e} \text{ or } y = \frac{1}{cx^2+dx+e}$$

To find n^{th} derivative of above kind function first obtain partial fractions of f(x) or y.

To get partial fractions:

If $y = \frac{1}{cx^2+dx+e}$ then first factorize $cx^2 + dx + e$.

Let (fx + g) (hx + i) be factors then $y = \frac{1}{(fx+g)(hx+i)}$

Find A & B such that $y = \frac{A}{fx+g} + \frac{B}{hx+i}$

obtain n^{th} derivatives of above fractions separately and add them, answer will give n^{th} derivative of y.

Notes If polynomial in denominator is of higher Degree then we will have more factors. (Do the same process for all the factors).

If $y = \frac{1}{(fx+g)^2(hx+i)}$ then use factors $y = \frac{A}{(fx+g)^2} + \frac{B}{hx+i} + \frac{C}{fx+g}$

Task Obtain n^{th} derivatives of followings:

1. $\dfrac{a-x}{a+x}$ 2. $\dfrac{1}{(x-1)^2(x-2)}$ 3. $\dfrac{x^4}{(x-1)(x-2)}$ 4. $\dfrac{x}{a^2+x^2}$

Notes

Sr no.	**Function**	**n^{th} derivative**
01	y = sin(ax + b)	(i) $y_n = a^n \sin(ax + b + n\,\pi/2)$
		(ii) if b =0 , a =1 then y = sin x & $y_n = \sin(x + n\,\pi/2)$
02	y = cos(ax + b)	(i) $y_n = a^n \cos(ax + b + n\,\pi/2)$
		(ii) if b = 0, a = 1 then y = cos x & $y_n = \cos(x + n\,\pi/2)$
03	$y = e^{ax} \sin(bx + c)$	(i) $y_n = r^n e^{ax} \sin(bx + c + n\Theta)$ Where $r = (a^2 + b^2)^{1/2}$ $\Theta = \tan^{-1}(b/a)$
04	$y = e^{ax} \cos(bx + c)$	(i) $y_n = r^n e^{ax} \cos(bx + c + n\Theta)$ Where $r = (a^2 + b^2)^{1/2}$ $\Theta = \tan^{-1}(b/a)$

Problems Based on Above Formulas :

1. Obtain 4^{th} derivative of sin(3x + 5).
2. Obtain 3^{rd} derivative of $e^{2x}\cos 3x$

Problems Based on Above Formulas :

Obtain n^{th} derivatives of followings:

1. $\sin x \sin 2x$ 2. $\sin^2 x \cos^3 x$ 3. $\cos^4 x$ 4. $e^{2x} \cos x \sin^2 2x$

Task Obtain n^{th} derivatives of followings:

1. $\cos x \cos 2x \cos 3x$ 2. $\sin^4 x$ 3. $e^{-x}\cos^2 x \sin x$

Some Problems (Problems of Special Type) based on Above all (1 & 2) formulas:

1. For $y = \dfrac{x^3}{x^2-1}$

Show that, $$\left(\frac{d^n y}{dx^n}\right)_{x=0} = \begin{cases} 0 & \text{if n is even} \\ (-n) & \text{if } n \text{ is odd integer greater than 1} \end{cases}$$

2. If y = cosh 2x, show that

$y_n = 2^n \sinh 2x$, when n is odd.

$= 2^n \cosh 2x$, when n is even.

Notes

3. Find n^{th} derivative of following:

(a) $\tan^{-1}\left(\frac{1-x}{1+x}\right)$ (b) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(c) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ (d) $\tan^{-1} x$

4. If u = sin nx + cos nx, show that

$$u_r = n^r\left[1+(-1)^r \sin 2nx\right]^{\frac{1}{2}}$$

where u_r denotes the r^{th} derivative of u with respect to x.

5. If $I_n = \frac{d^n}{dx^n}(x^n \log x)$

Prove that $I_n = n\, I_{n-1} + (n-1)!$,

Hence show that

$$I_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}\right)$$

Leibnitz's theorem(only statement):

If y = u .v,

where u & v are functions of x possessing derivatives of n^{th} order then,

$y_n = nC_0 u_n v + nC_1 u_{n-1} v_1 + nC_2 u_{n-2} v_2 + \ldots\ldots\ldots + nC_r u_{n-r} v_r + \ldots\ldots\ldots + nC_n u v_n$

where, $nCr = \frac{n!}{r!(n-r)!}$

Properties:

1. nCr = nCn-r
2. $nC_0 = 1 = nCn$
3. $nC_1 = n = nCn-1$

Notes Generally we can take any function as u and any as v.(If y = u .v) But take v as the function whose derivative becomes zero after some order.

Problems Based on Leibnitz's theorem:

Obtain n^{th} derivatives of followings:

1. $x^3 \log x$ 2. $\frac{x^n}{x+1}$ 3. $x^2 e^x \cos x$

Notes

Task Obtain n^{th} derivatives of followings (using Leibnitz's theorem):

1. $x^2 \log x$ 2. $x^2 e^x$ 3. $x \tan^{-1} x$

Example: Partial differentials–successive differentiation: outline solutions

1. $f(x,y) = \dfrac{x}{x^2+y^2}$

Use the quotient rule

$$\frac{\partial(u/v)}{\partial x} = \frac{\frac{v\partial u}{\partial x} - \frac{u\partial v}{\partial x}}{v^2}$$

$$\frac{\partial f}{\partial x} = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}$$

Use it again to get:

$$\frac{\partial^2 f}{\partial x^2} = \frac{2x^5 - 4x^3y^2 - 6xy^4}{\left(x^2+y^2\right)^4}$$

Same idea with y:

$$\frac{\partial f}{\partial y} = \frac{-2xy}{\left(x^2+y^2\right)^2}$$

AND

$$\frac{\partial^2 f}{dy^2} = \frac{-2x^5 + 4x^3y^2 + 6xy^4}{\left(x^2+y^2\right)^4}$$

Nearly there.

2. (a) $f(x, y) = x^2 \cos y$

This should be easier having done the last one.

First:

$$\frac{\partial f}{\partial x} = 2x \cos y \qquad \frac{\partial f}{\partial x} = -x^2 \sin y$$

Now differentiate again:

$$\frac{\partial^2 f}{\partial x \partial y} = -2x \sin y \qquad \frac{\partial^2 f}{\partial x \partial y} = -2x \sin y$$

which proves it ?

Notes

(b) $f(x, y) = \sin hx \cos y$

To get you started..

$$\frac{\partial f}{\partial x} = \cos hx \cos y \qquad \text{etc.}$$

3. $$v(x, y, z) = \frac{1}{z}\exp\left(\frac{-(x^2+y^2)}{4z}\right)$$

More complicated function but same principle….

$$\frac{\partial v}{\partial x} = \frac{1}{z}\exp\left(-\frac{(x^2+y^2)}{4z}\right)\left(\frac{-2x}{4z}\right)$$

and then:

$$\frac{\partial^2 v}{\partial x^2} = \frac{-x}{2z^2}\exp\left(-\frac{(x^2+y^2)}{4z}\right)\left(\frac{-2x}{4z}\right)+\frac{-1}{2z^2}\exp\left(\frac{-(x^2+y^2)}{4z}\right)$$

Following same steps you should get:

$$\frac{\partial^2 v}{\partial y^2} = \frac{-y}{2z^2}\exp\left(\frac{-(x^2+y^2)}{4z}\right)\left(\frac{-2y}{4z}\right)+\frac{-1}{2z^2}\exp\left(\frac{-(x^2+y^2)}{4z}\right)$$

Turning to $\frac{\partial v}{\partial z}$, you should obtain:

$$\frac{\partial v}{\partial z} = \frac{-1}{z^2}\exp\left(\frac{-(x^2+y^2)}{4z}\right)+\frac{1}{z}\exp\left(\frac{-(x^2+y^2)}{4z}\right)\left(\frac{-(x^2+y^2)}{-4z^2}\right)$$

This is what you should get when you simplify $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$.

Task

1. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$ then prove, $x^2 y_{n+2} + (2n + 1)xy_{n+1} + 2n^2 y_n = 0$
2. If $y = (x^2 - 1)^n$ then prove, $(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n + 1)y_n = 0$
3. If $y = \tan^{-1}\left(\frac{a+x}{a-x}\right)$ then prove, $(a^2 + x^2)y_{n+2} + 2(n + 1)xy_{n+1} + n(n + 1)y_n$

Example: If $y = \sin(\log_e x)$then $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}$ is equal to:

Solution:

$$y = \sin(\log x) \Rightarrow \frac{dy}{dx} = \cos(\log x)\frac{1}{x} \Rightarrow x\frac{dy}{dx} = \cos(\log x)$$

Notes

$$\Rightarrow \quad x\frac{d^2y}{dx^2}+\frac{dy}{dx} = -\sin(\log x)\frac{1}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = -y$$

Example: $f(x) = e^x \sin x \Rightarrow f^{(6)}(x)$ is equal to:

Solution:

$$f(x) = e^{ax} \sin bx$$

$$f^n(x) = (a^2 + b^2)^{n/2} . e^{ax} \sin(bx + n \tan^{-1} b/a)$$

$$a = 1, b = 1, n = 6$$

$$f^6(x) = \left(\sqrt{(1+1)}\right)^6 e^x \sin\left(x+6\tan^{-1}(1)\right)$$

$$= 8e^x \sin\left(\frac{3\pi}{2}+x\right) = -8e^x \cos x$$

Example: If $In = \frac{d^n}{dx^n}\left(x^n \log x\right)$, then $I_n - nI_{n-1}$ is equal to:

Solution:

$$I_n = \frac{d^n}{dx^n}\left(x^n \log x\right)$$

$$y = x^n \log x \Rightarrow y_1 = x^n\left(\frac{1}{x}\right) + nx^{n-1}\log x$$

$$(y_1)_{n-1} = nI_{n-1} + (n-1)!$$

$$\Rightarrow \quad I_n - nI_{n-1} = (n-1)!$$

Example: If $y = ae^x + be^{-x} + c$, where a, b, c are parameters, then y''' is equal to:

Solution:

$$y = ae^x + be^{-x} + c$$

$$y' = ae^x - be^{-x};$$

$$y'' = ae^x + be^{-x}$$

$$y''' = ae^x - be^{-x}$$

$$y''' = y'$$

Example: If $y = a \cos(\log x) + b \sin(\log x)$, where a, b are parameters, then $x^2y'' + xy'$ is equal to:

Solution:

$$y = a \cos(\log x) + b \sin(\log x)$$

$$xy' = -a \sin(\log x) + b \cos(\log x)$$

Notes

$$xy'' + y' = \frac{-a\cos(\log x) - b\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2 y'' + xy' = -y$$

12.3 Economic Applications

12.3.1 Demand Function

We know that demand of a commodity, in a given time period, depends upon its own price, prices of other commodities, income of the consumer etc. In order to understand the behaviour of demand in response to changes in one of the above variables, say price, we assume the remaining variables, income and prices of other commodities etc., as constant. Consequently, we can define three types of relations, given below:

1. The relationship of demand of a commodity with its own price is termed as *the price demand or the law of demand.*
2. The relationship of demand of a commodity with income of the consumer is termed as *income demand.*
3. The relationship of the demand of a commodity with the price of other commodity is termed as *cross demand*.

Price Demand

Other things, like income of the consumer, price of other commodities, taste and habits of the consumer etc., remaining constant, the quantity demanded of commodity (x_d) varies inversely with its price (p). Mathematically we say that x_d is a function of p. Symbolically, we write $x_d = f(p)$.

Since x_d decreases as p increases, we have $\frac{dx_d}{dp} < 0,$ under normal conditions of demand.

Price Elasticity of Demand

The price elasticity of demand or simply the elasticity of demand, is defined as the negative of the ratio of proportionate change in quantity demanded to proportionate change in price. It is denoted by η where $\eta_d = -\frac{d\log x}{d\log p}.$ (The subscript of x is dropped for convenience.)

We can also write $\eta_d = -\frac{d\log x}{dp}\cdot\frac{dp}{d\log p} = -\frac{1}{x}\cdot\frac{dx}{dp}\cdot p = -\frac{dx}{dp}\cdot\frac{p}{x}$

Notes

Notes

(i) The above formula gives the elasticity of demand at a point on the demand curve and hence is also referred to as *point-elasticity formula*.

(ii) Since η_d is a ratio, it is a pure number.

(iii) As per the convention in economics, the elasticity of demand is defined as the negative of the ratio of proportional change in quantity demanded to proportionate change in price.

Income Demand (Engel Function)

Assuming price of a commodity and prices of other commodities etc., as constant, we can say that quantity demanded (x) of a commodity is a function of consumer's income (Y). Symbolically, we can write this as $x = g(Y)$. Note here that $\frac{dx}{dY}$ can be positive or negative.

If $\frac{dx}{dY} > 0$ (< 0), the commodity is said to be *normal* (*inferior*).

Income Elasticity of Demand

The income elasticity of demand, η_Y, is defined as the ratio of proportionate change in quantity demanded to proportionate change in price.

$$\therefore \qquad \eta_Y = \frac{d\log x}{d\log Y} = \frac{d\log x}{dY} \cdot \frac{dY}{d\log Y} = \frac{dx}{dY} \cdot \frac{Y}{x}$$

We note that if $\eta_Y < 0$, the commodity is inferior.

Cross Demand

Let there be two commodities A and B. Assuming other things as constant, we can write demand for A, denoted as x_A, as a function of the price of $B(P_B)$; and also the demand for $B(x_B)$ as a function of the price of $A(P_A)$. Using symbols, we can write

$$x_A = \phi(P_B) \text{ and } x_B = \varphi(P_A)$$

Such functions are termed as cross demand functions. We note here that if $\frac{dx_A}{dp_B} > 0$ and $\frac{dx_B}{dp_A} > 0$, then A and B are termed as substitutes. If both the derivatives are negative, the two commodities are termed as compliments. Nothing can be said about the relationship between A and B, if these derivative are of opposite signs.

Cross Elasticity of Demand

The cross elasticity of demand of commodity A as compared with price of B, denoted by η_{AB}, is the ratio of proportionate change in quantity demanded of A to proportionate change in price of B. Symbolically, we can write

$$\eta_{AB} = \frac{d\log x_A}{d\log p_B} = \frac{dx_A}{dp_B} \cdot \frac{p_B}{x_A}. \text{ Similarly, } \eta_{BA} = \frac{d\log x_B}{d\log p_A} = \frac{dx_B}{dP_A} \cdot \frac{p_A}{x_B}.$$

Notes

12.3.2 Law of Supply

Other things, like prices of other commodities price of factors of production, level of technology etc. remaining constant, the quantity supplied (x_s) of a commodity varies directly with its price. Mathematically, we can say that x_s is a function of p. Using symbols, we can write $x_s = f(p)$. We note that $\frac{dx_s}{dp} > 0.$

Elasticity of Supply

The elasticity of supply η_s is defined as ratio of proportionate change in quantity supplied to proportionate change in price.

$$\eta_s = \frac{d\log x}{d\log p} = \frac{dx}{dp} \cdot \frac{p}{x}$$

Example

(i) Find elasticity of demand of the function, $x = 100 - 5p$ at (a) $p = 10$, (b) $p = 15$.

(ii) Find elasticity of demand of the function $p = -2x^2 + 3x + 150$ at $x = 8$.

(iii) If $p = a - bx$ is the inverse demand function, show that elasticity of demand is different at different points on the demand curve. At what price the demand is unitary elastic?

(iv) $p = f(x)$ is an inverse demand function such that $x \cdot f(x)$ is constant. Show that elasticity of demand is unity at every point on it. Explain the meaning of this result.

(v) Show that elasticity of demand can be expressed as the numerical value of the marginal demand function to average demand function.

Solution:

(i) (a) $x = 100 - 5p \quad \therefore \quad \frac{dx}{dp} = -5$ and $\eta_d = -\frac{dx}{dp} \times \frac{p}{x} = 5 \times \frac{10}{50} = 1$

($x = 50$ when $p = 10$, from the demand equation).

Hence, the elasticity of demand $\eta_d = 1$.

Alternatively, $\eta_d = -\frac{d\log x}{d\log p} = -\frac{d\log x}{dp} \cdot \frac{dp}{d\log p} = \frac{5p}{100 - 5p} = \frac{50}{50} = 1$

(b) When $p = 15$ we have, $\eta_d = \frac{5 \times 15}{25} = 3$

(ii) We have $p = -2x^2 + 3x + 150 \quad \therefore \quad \frac{dp}{dx} = -4x + 3 = -29$ at $x = 8$.

Also, $p = -2 \times 64 + 3 \times 8 + 150 = 46$

$\therefore \quad \eta_d = -\frac{dx}{dp} \cdot \frac{p}{x} = \frac{1}{29} \times \frac{46}{8} = \frac{23}{116} = 0.198$

(iii) We are given $p = a - bx$ $\therefore$ $\dfrac{dp}{dx} = -b$. Thus $\eta_d = \dfrac{1}{b} \cdot \dfrac{p}{x}$

Since elasticity of demand depends upon p (or x) and thus, will be different at different points on the demand curve.

When $\eta_d = 1$, we can write $\dfrac{1}{b} \cdot \dfrac{p}{x} = 1$ or $\dfrac{p}{x} = b$

or $\dfrac{p \times b}{a - p} = b$ $\left(\because \dfrac{1}{x} = \dfrac{b}{a-p}, \text{ from demand equation}\right)$

$\therefore$ $p = a - p$ or $2p = a$ or $p = \dfrac{a}{2}$

Thus, the elasticity of demand is unity when $p = \dfrac{a}{2}$.

(iv) Let $x \cdot f(x) = c$ where c is a constant. Differentiating both sides w.r.t. x, we have $f(x) + xf'(x) = 0$ or $-\dfrac{f(x)}{xf'(x)} = 1$. We note that expression on the left hand side is elasticity of demand of the function $p = f(x)$. Thus $\eta_d = 1$. Since η_d is independent of x (or p), hence elasticity of demand is unity at every point on the demand curve $p = f(x)$.

We note that $x \cdot f(x)$ is the total outlay (or expenditure) of the consumer. Thus when total outlay of the consumer is constant the demand is unitary elastic at every point. It can also be shown that $p = f(x)$, in this case, will represent a rectangular hyperbola with centre at (0, 0) and asymptotes as the axes of the coordinate system.

(v) The elasticity of demand, $\eta_d = -\dfrac{dx}{dp} \cdot \dfrac{p}{x}$, can also be written as

$$\eta_d = -\frac{dx/dp}{x/p} = -\frac{\text{marginal demand function}}{\text{average demand function}}$$

Hence the result.

Example

(i) The price elasticity of demand of a commodity when price = Rs 10 and quantity demanded = 25 units, is given to be 1.5. Find the demand equation of the commodity on the assumption that it is linear.

(ii) Find the elasticity of demand of the inverse demand function $p = 3x^2 - 100x + 800$ when $x = 10$. Find, approximately, the percentage change in demand if price rises by 4%. Also find the elasticity at new price, quantity combination.

Solution.

(i) Given $\eta_d = 1.5$, $\therefore$ $\dfrac{dx}{dp} \cdot \dfrac{p}{x} = -1.5$ or $\dfrac{dx}{dp} \cdot \dfrac{10}{25} = -1.5$

Thus, $\dfrac{dx}{dp} = \dfrac{-1.5 \times 25}{10} = -3.75$

The required demand equation will be a straight line passing through the point (25, 10) with slope = $-\dfrac{1}{3.75}$

Notes

Thus $p - 10 = -\frac{1}{3.75}(x - 25)$

or $x - 25 = -3.75(p - 10)$ or $x = 62.5 - 3.75p$ is the required equation.

(ii) The given demand function is $p = 3x^2 - 100x + 800$

$$\therefore \quad \frac{dp}{dx} = 6x - 100 = 60 - 100 = -40 \text{ when } x = 10$$

When $x = 10$, we have $p = 300 - 1{,}000 + 800 = 100$

$$\therefore \quad \eta_d = -\frac{dx}{dp}\cdot\frac{p}{x} = \frac{1}{40}\times\frac{100}{10} = \frac{1}{4}$$

When price increases by 4%, then the approximate change in demand in given by the formula

$$\eta_d = \frac{-\text{% change in demand}}{\text{% change in price}}$$

or $\quad$ % change in demand $= -\eta_d \times$ % change in price

$$= -\frac{1}{4}\times 0.04 = -0.01$$

i.e. demand falls by 1%.

The new demand $=$ old demand $\times 0.99$

$= 10 \times .99 = 9.9$

New price $= 100 \times 1.04 = 104$

$$\eta_d = \frac{dx}{dp}\cdot\frac{p}{x} = -\frac{1}{(6x-100)}\cdot\frac{p}{x}.$$

$$= -\frac{1}{(6\times 9.9 - 100)}\times\frac{104}{9.9} = 0.2587$$

Example

(i) If $x = 2Y^2$, find income-elasticity of demand.

(ii) If $x_A = \frac{p_B + 1}{p_B - 2}$, find cross-elasticity of demand when $p_B = 5$.

Solution:

(i) $$\eta_Y = \frac{dx}{dY}\cdot\frac{Y}{x} = 4Y\cdot\frac{Y}{2Y^2} = 2$$

(ii) $$\eta_{AB} = \frac{dx_A}{dp_B}\cdot\frac{p_B}{x_A}$$

Now $$\frac{dx_A}{dp_B} = \frac{(p_B - 2) - (p_B + 1)}{(p_B - 2)^2} = \frac{-3}{(p_B - 2)^2} = -\frac{1}{3} \text{ at } p_B = 5$$

Also $x_A = 2$ when $p_B = 5$ $\therefore$ $\eta_{AB} = -\frac{1}{3}\times\frac{5}{2} = -\frac{5}{6} = -0.83.$

Notes

Example

Find the elasticity of supply, η_s, for the following functions:

(i) $x = 2p + p^2$ at (a) $p = 5$ and (b) $p = 7$.

(ii) $p = x^2$ at $x = 5$.

Solution:

(i) (a) The supply function is $x = 2p + p^2$ $\therefore$ $\dfrac{dx}{dp} = 2 + 2p = 12$ when $p = 5$

When $p = 5$, we have $x = 10 + 25 = 35$ $\therefore$ $\eta_s = \dfrac{dx}{dp}\cdot\dfrac{p}{x} = \dfrac{12\times 5}{35} = 1.71.$

(b) When $p = 7$, we have $\dfrac{dx}{dp} = 2 + 14 = 16$ and $x = 14 + 49 = 63$

$$\therefore \qquad \eta_s = \frac{16\times 7}{63} = \frac{16}{9} = 1.77$$

(ii) The supply function is $p = x^2$

$\therefore$ $\dfrac{dp}{dx} = 2x = 10$ and $p = 25$ at $x = 5$ Thus $\eta_s = \dfrac{1}{10}\times\dfrac{25}{5} = 0.5$

Example: Show that for the inverse supply function $p = a + bx(b > 0)$, the supply is elastic if $a > 0$, inelastic if $a < 0$ and unitary elastic if $a = 0$.

Solution:

Given $p = a + bx$, we get $\dfrac{dp}{dx} = b$ $\therefore \eta_s = \dfrac{dx}{dp}\cdot\dfrac{p}{x} = \dfrac{1}{b}\cdot\dfrac{a+bx}{x} = \dfrac{a}{bx} + 1.$

Since, $b > 0$ and x (the quantity) > 0, η_s will be greater than 1 if $\dfrac{a}{bx} + 1 > 1$ or $\dfrac{a}{bx} > 0 \Rightarrow a > 0.$

Similarly, η_s will be less than 1 if $\dfrac{a}{bx} + 1 < 1$ or $\dfrac{a}{bx} < 0 \Rightarrow a < 0.$

Further, $\eta_s = 1$, if $\dfrac{a}{bx} + 1 = 1,$ or $\dfrac{a}{bx} = 0 \Rightarrow a = 0.$

Example

The supply of a certain good is given by $x = a\sqrt{p-b}$, where p ($> b$) is the price and a and b are positive constants. Find an expression for η_s, the elasticity of supply, as a function of price. Show that η_s decreases as price (or supply) increases and becomes unity when $p = 2b$.

Solution:

Given $x = a\sqrt{p-b}$, we get $\dfrac{dx}{dp} = \dfrac{a}{2}(p-b)^{-\frac{1}{2}} = \dfrac{a}{2\sqrt{p-b}}$

Notes

and $\eta_s = \frac{dx}{dp} \cdot \frac{p}{x} = \frac{a}{2\sqrt{p-b}} \cdot \frac{p}{a\sqrt{p-b}} = \frac{p}{2(p-b)}$

Differentiating η_s w.r.t. p, we get $\frac{d(\eta_s)}{dp} = \frac{1}{2}\left[\frac{(p-b)-p}{(p-b)^2}\right] = -\frac{b}{2(p-b)^2}$

Since b is given to be positive, therefore $\frac{d(\eta_s)}{dp} < 0.$ Thus, η_s decreases with increase of price (or supply).

When $p = 2b$, we have $\eta_s = \frac{2b}{2(2b-b)} = 1.$

Example

For a demand function $x = f(p)$, with $\frac{dx}{dp} < 0,$ find $\frac{dx}{dp}$ in terms of elasticity of demand η.

(i) Show that the demand curve is convex from below if $\frac{d\eta}{dp} < 0.$

(ii) If $\frac{d\eta}{dp} > 0,$ show that the demand curve is convex from below provided that $\frac{d\eta}{dp} < \frac{\eta(1+\eta)}{p}.$

Solution:

(i) We can write elasticity of demand, $\eta = -\frac{p}{x} \cdot \frac{dx}{dp}$ $\quad\therefore\quad$ $\frac{dx}{dp} = -\frac{\eta x}{p}$

Differentiating both sides w.r.t. p, we get

$$\frac{d^2x}{dp^2} = -\left[\frac{p\left\{x \cdot \frac{d\eta}{dp} + \eta \cdot \frac{dx}{dp}\right\} - \eta x}{p^2}\right] = \frac{-px \cdot \frac{d\eta}{dp} - p\eta \frac{dx}{dp} + \eta x}{p^2} > 0 \text{ if } \frac{d\eta}{dp} < 0.$$

Thus the demand curve is convex from below.

(ii) If $\frac{d\eta}{dp} > 0,$ then the demand curve will be convex from below if

$$-px\frac{d\eta}{dp} - p\eta \cdot \frac{dx}{dp} + \eta x > 0 \text{ or } px \cdot \frac{d\eta}{dp} < -p\eta \cdot \frac{dx}{dp} + \eta x$$

$$\text{or } \frac{d\eta}{dp} < -\frac{p\eta}{px} \cdot \frac{dx}{dp} + \frac{\eta}{p} = \frac{\eta^2}{p} + \frac{\eta}{p} = \frac{\eta(1+\eta)}{p}$$

12.3 Summary

- It is extension of differentiation of one variable function successive.
- Consider,

  A one variable function,

  y = f(x) (x is independent variable and y depends on x.)

  Here if we make any change in x there will be a related change in y.
- $f^{(n)}(x)$ denotes n^{th} derivative of f.
- Value of n^{th} derivative of y = f(x) at x = a is denoted by,

  $f^n(a)$, $y_n(a)$, or $\left(\frac{d^n y}{dx^n}\right)_{x=a}$

  (i.e. value can be obtained by just replacing x with a in $f^n(x)$.)

12.4 Keyword

Successive Differentiation: If $y = f(x)$ is a differentiable function then by differentiating it w.r.t. x, we get $\frac{dy}{dx} = f'(x)$

12.5 Self Assessment

1. $y = e^{a\sin^{-1}x} \Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1}$ is equal to:

   (a) $(n^2 + n^2)y_n$ (b) $(n^2 - a^2)y_n$

   (b) $(n^2 + a^2)y_n$ (d) $-(n^2 - a^2)y_n$

2. $x = \cos\theta, y = \sin 5\theta \Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$ is equal to:

   (a) $-5y$ (b) $5y$

   (c) $25y$ (d) $-25y$

3. $y = \sin^{-1} x \Rightarrow (1-x^2)\frac{d^2y}{dx^2}$ is equal to:

   (a) $-x\frac{dy}{dx}$ (b) 0

   (c) $x\frac{dy}{dx}$ (d) $x\left(\frac{dy}{dx}\right)^2$

4. If y_k is the k^{th} derivative of y with respect to x, y = cos(sin x) then $y_1 \sin x + y_2 \cos x$ is equal to:

   (a) $y \sin^3 x$ (b) $-y \sin^3 x$

   (c) $y \cos^3 x$ (d) $-y \cos^3 x$

Notes

5. $\frac{d^n}{dx^n}\left(e^x \sin x\right)$ is equal to:

(a) $2^{n/2}.e^x \cos(x + n\pi/4)$ (b) $2^{n/2}.e^x \cos(x - n\pi/4)$

(c) $2^{n/2}.e^x \sin(x + n\pi/4)$ (d) $2^{n/2}.e^x \sin(x - n\pi/4)$

12.6 Review Questions

1. If $y = \sin(m \sin^{-1}x)$

   Then prove, $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0$

2. If $y = \cot^{-1} x$,

   Then prove, $(1 + x^2)y_{n+2} + 2(n + 1)xy_{n+1} + n(n + 1)y_n = 0$

3. If $y^{1/m} + y^{-1/m} = 2x$

   Then prove, $(x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$

4. Let p and q be two real numbers with $p > 0$. Show that the cubic $x^3 + px + q$ has exactly one real root.

5. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

6. Let $f(x) = 1 + 12|x| - 3x^2$. Find the global maximum and the global minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.

Answers: Self Assessment

1.	(c)	2.	(d)	3.	(c)
4.	(d)	5.	(c)		

12.7 Further Readings

Books

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

Smith and Minton, *Calculus Early Trancendental,* Third Edition, McGraw Hill 2008.

Online links

http://www.suitcaseofdreams.net/Trigonometric_Functions.htm

http://library.thinkquest.org/20991/alg2/trigi.html

http://www.intmath.com/trigonometric–functions/5–signs–of–trigonometric–functions.php

Unit 13: Maxima and Minima

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the Extreme-value Theorem
- Explain the points of Inflexion

Introduction

We know that the value of a function is different at different points in its domain. When the function is monotonic, the functional values are either continuously increasing or decreasing. If the function is not monotonic, the functional values may increase (decrease) over a certain subset of the domain and then decrease (increase). This behaviour may be repetitive also.

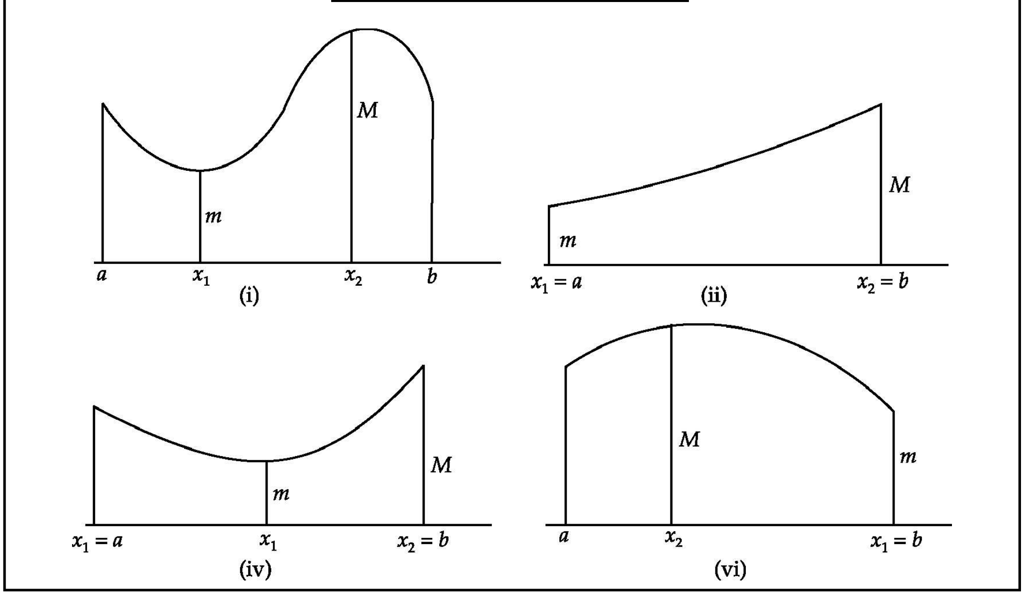
13.1 The Extreme-value Theorem

If a function $f(x)$ is continuous at every point of a closed interval I, then $f(x)$ assumes both an absolute maximum value M and an absolute minimum value m some where in the interval I.

This theorem implies that there always exist two values x_1 and x_2 in I such that $f(x_1) = m$, $f(x_2) = M$ and $m \leq f(x) \leq M$ for other values of x in the interval I.

Notes

Figure 13.1



Some possible situations of absolute maxima and minima are shown, in Fig. 5.1 above, for a function that is continuous in [a, b].

Absolute Maxima/Minima (Definition)

Let $f(x)$ be a function with domain D. Then $f(x)$ has an absolute maxima at a point c in D if $f(x) \leq f(c)$ for all x in D and an absolute minima at a point d in D if $f(x) \geq f(d)$ for all x in D.

Absolute maxima/minima are also called *global* maxima/minima.

Local Maxima/Minima (Definition)

A function $f(x)$ has a local maxima (or minima) at an interior point c in its domain D if $f(x) \leq f(c)$ (or $f(x) \geq f(c)$) for all x in some open interval containing c.

Notes

1. As is evident from Figure (i) the function has a local minima at x_1 which is also absolute minima. Simialrly, the functioin has a local maixma at x_2 which is also an absolute maxima. However, it is not necessary that a local maxima (minima) will always be an absolute maxima (minima) or vice-versa.
2. Suppose a function $f(x) = x$ be defined in [0, 2]. Then this function has a maxima at $x = 2$. However, $f(x) \to 0$ as $x \to 0$, $f(x)$ attains the value 0 and thus it has no minima.

First Derivative Theorem for Local Extrema

If a function $f(x)$ has a local extrema (i.e., maxima or minima) at an interior point c of its domain, and if $f'(c)$ exists, then $f'(c) = 0$.

Critical Point (Definition)

An interior point of the domain of a function $f(x)$ at which $f¢(x)$ is either zero or undefined is termed as a *critical point*.

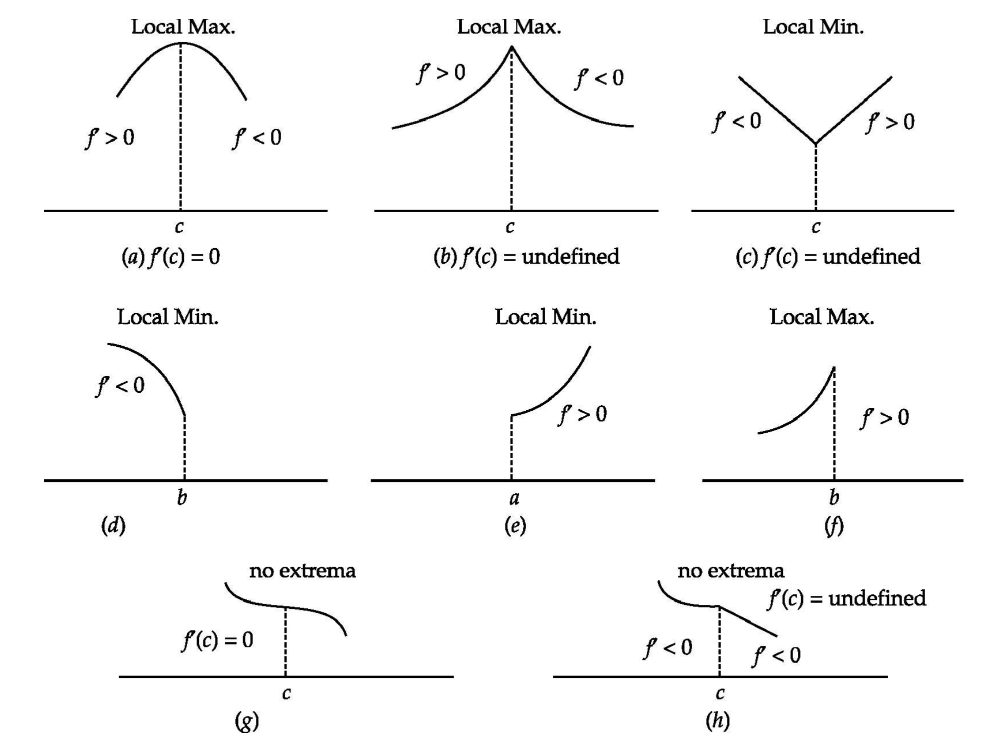
Notes

Notes 1. The points of the domain at which a function can assume extreme values are either *critical point* or *end points*.

2. The end point(s) can also be a local extrema.

13.1.1 First Derivative Criterion for Local Extrema

Figure 13.2



At a point where $f(x)$ has a local maxima (or minima), we note that $f' > 0$ (or < 0) on the interval immediately to the left and $f' < 0$ (or > 0) on the interval immediately to the right of the critical point. If the critical point is an end point (a or b), we consider the interval on the appropriate side of the point. Various possible situations are shown with the help of following figure.

Example: Determine maxima/minima of the following functions, by using only first dirivative:

(a) $y = x^3 - 2x^2 + x + 20$

(b) $y = x^{2/3}(x - 1)$

Solution:

(a) $$\frac{dy}{dx} = 3x^2 - 4x + 1 = 0 \text{ for maxima/minima.}$$

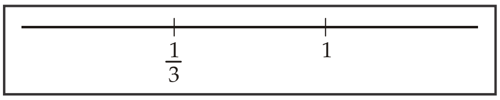
$$3x^2 - 3x - x + 1 = 0 \quad \text{or} \quad 3x(x-1) - 1(x-1) = 0$$

or $$(3x - 1)(x - 1) = 0$$

Notes

The critical points are $x = \frac{1}{3}$ and $x = 1$

These points divide x-axis into intervals on which $\frac{dy}{dx}$ is either positive or negaitve.



When $x < \frac{1}{3}$, say $\frac{1}{4}, \frac{dy}{dx} = 3 \times \frac{1}{16} - 4 \times \frac{1}{4} + 1 = \frac{3}{16} > 0$

When $x > \frac{1}{3}$ say $\frac{2}{3}, \frac{dy}{dx} = 3 \times \frac{4}{4} - 4 \times \frac{2}{3} + 1 = -\frac{1}{3} < 0$

When $x > 1$ say $\frac{4}{3}, \frac{dy}{dx} = 3 \times \frac{16}{9} - 4 \times \frac{4}{3} + 1 = 1 > 0$

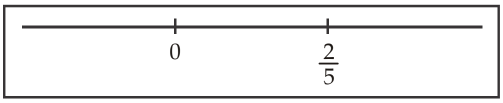
Since $\frac{dy}{dx}$ changes from positive to neative at $x = \frac{1}{3}$, the function has a local maxima at $x = \frac{1}{3}$.

Similarly function has a minima at $x = 1$

(b) We can write $y = x^{5/3} - x^{2/3}$

$$\therefore \qquad \frac{dy}{dx} = \frac{5}{3}x^{-2/3} - \frac{2}{3}x^{-1/3} = \frac{1}{3}x^{-1/3}(5x-2) = \frac{5x-2}{3x^{1/3}}$$

$\frac{dy}{dx} = 0$ at $x = \frac{2}{5}$ and undefined at $x = 0$. These are two critical points.



When $x < 0$, say $x = -1$, $\frac{dy}{dx} = +\frac{7}{3} > 0$

When $0 < x < \frac{2}{5}$ say $x = \frac{1}{5}, \frac{dy}{dx} = \frac{-1}{3\left(\frac{1}{5}\right)^{1/3}} < 0$

When $x > \frac{2}{5}$ say $x = \frac{3}{5}, \frac{dy}{dx} = \frac{1}{3\left(\frac{3}{5}\right)^{\frac{1}{3}}} > 0$

Thus there is local maxima (of the type given in Figure 13.2(b)) at $x = 0$ and minima at $x = \frac{2}{5}$.

Notes

1. If the function $f(x)$ is continuous at the point $x = a$, and $\lim_{x \to a^-} f'(x)$ and $\lim_{x \to a^+} f'(x)$ are both infinite with opposite signs, then the graph of $f(x)$ has a cusp at $x = a$. Note that the graph of the function given in example 1(b) above, has a cusp at $x = 0$.

2. If $\lim_{x \to a^-} f'(x)$ and $\lim_{x \to a^+} f'(x)$ are both infinite with same signs, then the graph of $f(x)$ has a vertical tangent at $x = a$. Note that $f(x) = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$.

Notes

13.1.2 Second Derivative Criterion for Local Extrema

When the function $f(x)$ is twice differentiable at an interior point c of the domain, then

(i) $f(x)$ has a local maxima at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$.

(ii) $f(x)$ has a local minima at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

> *Notes* When $f(x)$ has a maxima (or minima) at c, the curve of $f(x)$ is concave (or convex) from below. This test is inconclusive when $f''(c) = 0$.

Example:

(a) Show that the function $y = x^2 - 2x + 3$ has a minima at $x = 1$. Find the minimum value of the function.

(b) Show that the function $y = x^2 - 2x + 3$ has a maxima at $x = 5/2$. Find the maximum value of y.

Solution:

(a) We have $y = y = x^2 - 2x + 3$ $\quad\therefore\quad \dfrac{dy}{dx} = 2x - 2 = 0$, for maxima or minima.

$\Rightarrow$ $2(x - 1) = 0$ or $x = 1$ is a stationary point (A point at which $\dfrac{dy}{dx} = 0$).

To know whether y is maximum or minimum at $x = 1$, we determine the sign of second derivative at this point.

Since $\dfrac{d^2y}{dx^2} = 2 > 0$, therefore the function has a minima at $x = 1$.

Further, the minimum value of $y = 1^2 - 2 + 3 = 2$.

(b) We have $y = 100 + 15x - 3x^2$ $\;\therefore\; \dfrac{dy}{dx} = 15 - 6x = 0$, for maxima or minima. This implies that

$x = \dfrac{15}{6} = \dfrac{5}{2}$ is a stationary point.

Since $\dfrac{d^2y}{dx^2} = -6 < 0$, therefore, the function has a maxima at $x = \dfrac{5}{2}$. The maximum value of the function is given by $y = 100 + \dfrac{15 \times 5}{2} - \dfrac{3 \times 25}{4} = 118.75.$

Procedure for fuiding absolute extremia

To find absolute extrema of a continuous function $f(x)$ on $[a, b]$

1. Find all critical points of $f(x)$ on $[a, b]$.
2. Evaluate $f(x)$ at each critical points as well as at the end points a and b.
3. The largest-value of $f(x)$, obtained above, is absolute maxima and the smallest-value is absolute minima.

Notes

Example: Find relative maxima and minima of the function $y = x^3 - 4x^2 - 3x + 2$. Also find absolute maxima/minima in [0, 4].

Solution:

$$\text{Given the function } y = x^3 - 4x^2 - 3x + 2, \text{ we have}$$

$$\frac{dy}{dx} = 3x^2 - 8x - 3 = 0, \text{ for maxima or minima.}$$

Rewriting this equation as $3x^2 - 9x + x - 3 = 0$

or $$3x(x - 3) + (x - 3) = 0 \text{ or } (x - 3)(3x + 1) = 0$$

$\Rightarrow$ $$x = 3 \text{ or } x = -\frac{1}{3}$$

Further, $$\frac{d^2y}{dx^2} = 6x - 8 = 10 > 0, \text{ when } x = 3$$

and $$= -10 < 0, \text{ when } x = -\frac{1}{3}$$

Thus, the function has a minima at $x = 3$ and maxima at $x = -\frac{1}{3}$.

To find maxima/minima in [0, 4], we note that there is only one stationary point $x = 3$ in the given interval.

Let $$f(x) = x^2 - 4x^2 - 3x + 2$$

$\therefore$ $$f(0) = 2$$

$$f(3) = 27 - 36 - 9 + 2 = -16$$

$$f(4) = 64 - 64 - 12 + 12 = -10$$

$\therefore$ Function has absolute maxima at $x = 0$, and aboslute minima at $x = 3$

Example: Show that the function $y = x + \frac{1}{x}$ has one maximum and one minimum value and later is larger than the former. Draw a graph to illustrate this.

Solution:

Given $$y = x + \frac{1}{x}, \text{ we have}$$

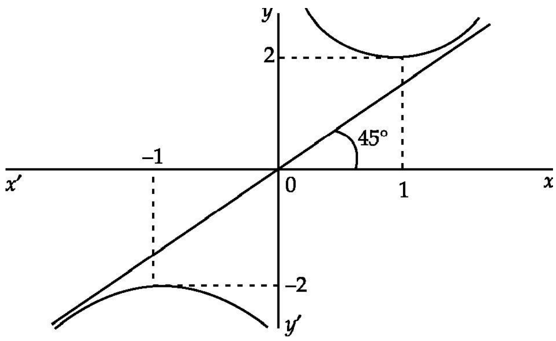
$$\frac{dy}{dx} = 1 - \frac{1}{x^2} = 0 \text{ or } \frac{x^2 - 1}{x^2} = 0 \Rightarrow x^2 = 1$$

or $$x = \pm 1 \text{ are the stationary points.}$$

Further, $$\frac{d^2y}{dx^2} = \frac{2}{x^3}, \text{ which will be positive when}$$

Notes

Figure 13.3



$x = 1$ and negative when $x = -1$. Thus the function has minima at $x = 1$ and maxima at $x = -1$. The minimum value of the function is 2 and the maximum value = –2 which is less than the minimum value. These values are shown in Figure 13.3.

Example: The function $y = \dfrac{ax+b}{(x-1)(x-4)}$ has an extreme point at $A(2, -1)$. Find the values of a and b. What is the nature of the extreme point?

Solution:

Since point $A(2, -1)$ lies on the function, we can write

$$\frac{2a+b}{(2-1)(2-4)} = -1 \text{ or } 2a + b = 2 \qquad \text{... (1)}$$

Further, $$\frac{dy}{dx} = \frac{a\left(x^2-5x+4\right)-(ax+b)(2x-5)}{\left(x^2-5x+4\right)^2} = 0 \text{ for extrema}$$

$$\Rightarrow \quad a[4 - 10 + 4] - (2a + b)(4 - 5) = 0 \text{ or } -2a + 2a + b = 0 \text{ or } b = 0$$

Substituting this value in (1), we get $a = 1$

To check the nature of extreme point at $A(2, -1)$, we find $\dfrac{d^2y}{dx^2}$

Now $$\frac{dy}{dx} = \frac{x^2-5x+4-\left(2x^2-5x\right)}{\left(x^2-5x+4\right)^2} = \frac{-x^2+4}{\left(x^2-5x+4\right)^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{\left(x^2-5x+4\right)^2(-2x)-2\left(4-x^2\right)\left(x^2-5x+4\right)(2x-5)}{\left(x^2-5x+4\right)^4}$$

$$= \frac{\left(x^2-5x+4\right)^2(-2x)}{\left(x^2-5x+4\right)^4} = \frac{-2x}{\left(x^2-5x+4\right)^2} = \frac{-4}{4} = -1 < 0 \text{ at } x = 2$$

Thus the extreme point is a maxima.

Notes

13.2 Points of Inflexion

A point of inflexion marks the change of curvature of a function. Since the curvature may change from convex (from below) to concave (from below) or vice versa, we have two types of points of inflexion which would be termed (for convenience) as type I and type II points of inflexion, as shown in following figures.

Criterion for Point of Inflexion

In order to develop a criterion for the point of inflexion, we have to examine the behaviour of the slope of the function, dy/dx, as we pass through this point.

As is obvious from Figure 13.4, when we approach point A, from its left, the value of $\frac{dy}{dx}$ is increasing and after we cross this point, $\frac{dy}{dx}$ starts declining. Thus, $\frac{dy}{dx}$ is maximum at point A. In a similar way $\frac{dy}{dx}$ is minimum at point B in Figure 13.5.

Figure 13.4

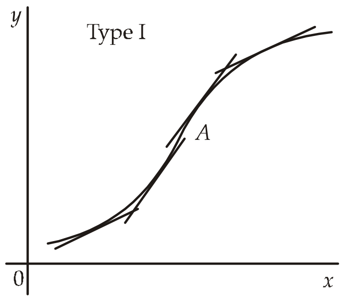
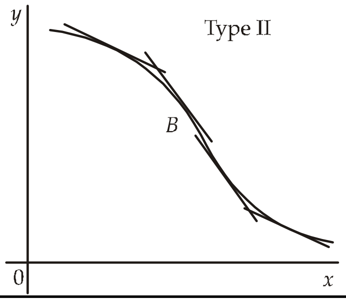


Figure 13.5



Thus, the problem of determination of a point of inflexion is reduced to the problem of determination of the conditions of maxima or minima of $\frac{dy}{dx}$. By suitable modification of the conditions for maxima, minima of y, we can write:

A thrice differentiable function $f(x)$ has a point of inflexion of type I (or II), see Figures 13.4 and 13.5, at an interior point c of the domain if $f'(c) = 0$ and $f''(c) < 0$ (or > 0).

Note that if $f(c)$ is also equal to zero at the point of inflexion, it is termed as a *stationary point of inflexion*.

Example: Find the nature of point of inflexion of the following functions:

(i) $y = x^3 - 15x^2 + 20x + 10$

(ii) $y = 20 + 5x + 12x^2 - 2x^3$

Solution:

(i) $y = x^3 - 15x^2 + 20x + 10 \quad \therefore \quad \frac{dy}{dx} = 3x^2 - 30x + 20$

and $\frac{d^2y}{dx^2} = 6x - 30 = 0$, for the point of inflexion $\Rightarrow x = 5$

Further, $\frac{d^3y}{dx^3} = 6,$ which is positive for all values of x.

$\therefore$ The point of inflexion at $x = 5$ is of type II i.e, curve changes from concave to convex from below.

(ii) $y = 20 + 5x + 12x^2 - 2x^3 \quad \therefore \quad \frac{dy}{dx} = 5 + 24x - 6x^2$

and $\frac{d^2y}{dx^2} = 24 - 12x = 0$, for the point of inflexion $\Rightarrow x = 2$

Further, $\frac{d^3y}{dx^3} = -12 < 0$ $\therefore$ The point of inflexion at $x = 2$ is of type I i.e. the curve changes from convex to concave from below.

Example: Find maxima, minima and the points of inflexion for the following functions and hence trace their curves:

(i) $y = x^3 + 10x^2 + 25x - 40$ (ii) $y = x^4 - 6x^2 + 1$

Solution:

(i) $y = x^3 + 10x^2 + 25x - 40$

First order condition (maxima or minima)

$$\frac{dy}{dx} = 3x^2 + 20x + 25 = 0 \text{ for maxima or minima} \quad \text{or} \quad (3x + 5)(x + 5) = 0$$

Thus, the stationary points are $x = -\frac{5}{3}$ and $x = -5$.

Second order condition

$$\frac{d^2y}{dx^2} = 6x + 20 = 10 > 0, \text{ when } x = -\frac{5}{3}.$$

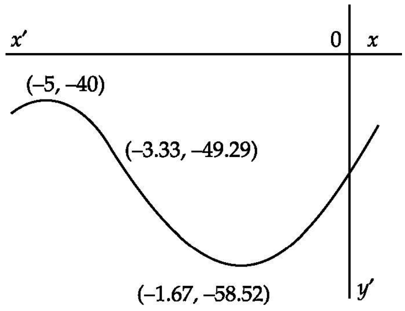
Thus, the function has a minima at $x = -\frac{5}{3} = -1.67$.

The minimum value of the function $f(-1.67) = -58.52$.

When $x = -5$ we have $\frac{d^2y}{dx^2} = -30 + 20 = -10 < 0$. Hence, the function has a maxima at $x = -5$. The maximum value of the function $f(-5) = -40$.

Notes

Figure 13.6



Point of Inflexion

We have $\frac{d^2y}{dx^2} = 6x + 20 = 0 \Rightarrow x = -\frac{10}{3} = -3.33$.

Further, $\frac{d^3y}{dx^3} = 6 > 0$, $\therefore$ the point of inflexion at $x = -3.33$ is of type II. Also $f(-3.33) = -49.29$. Using this information we can trace the curve as shown in the Figure 13.6.

(ii) $y = x^4 - 6x^2 + 1$.

First order condition (maxima or minima)

$$\frac{dy}{dx} = 4x^3 - 12x = 0 \Rightarrow x = 0,\ x = -\sqrt{3}, x = \sqrt{3}$$

Second order condition

$$\frac{d^2y}{dx^2} = 12x^2 - 12 < 0 \text{ when } x = 0$$

$$> 0 \text{ when } x = \sqrt{3} \text{ or } -\sqrt{3}$$

Thus the function has a maxima at $x = 0$, minima at $x = \sqrt{3}$ and $x = -\sqrt{3}$. Also, $f(0) = 1$, and $f(\sqrt{3}) = f(-\sqrt{3}) = -8$.

Point of Inflexion

First order condition

$$\frac{d^2y}{dx^2} = 12x^2 - 12 = 0 \Rightarrow x = \pm 1$$

Second order condition

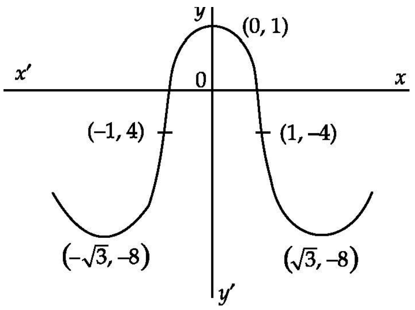
$$\frac{d^3y}{dx^3} = 24x < 0, \text{ when } x = -1$$

and $\quad > 0$ when $x = 1$

Thus, the function has type I point of inflexion at $x = -1$ and type II inflexion at $x = 1$. Also, $f(-1) = f(1) = -4$. Using the above information we can trace the curve as shown in the curve as shown in the Figure 13.7.

Notes

Figure 13.7



Example: Show that the polynomial $y = ax^3 + bx^2 + cx + d$ has only one point of inflexion. Under what conditions

(a) The curvature changes from: (i) convex to concave and (ii) concave to convex?

(b) The point of inflexion is stationary?

Solution:

(a) $$y = ax^3 + bx^2 + cx + d \quad \therefore \quad \frac{dy}{dx} = 3ax^2 + 2bx + c$$

Further, $$\frac{d^2y}{dx^2} = 6ax + 2b = 0 \text{ for point of inflexion} \quad \Rightarrow \quad x = -\frac{b}{3a}$$

Since $$\frac{d^2y}{dx^2} = 0 \text{ at a single value, there is only one point of inflexion.}$$

(i) For change of curvature from convex to concave, we must have

$$\frac{d^3y}{dx^3} = 6a < 0 \quad \Rightarrow \quad a < 0$$

(ii) Similarly, if $a > 0$, the curvature will change from concave to convex.

(b) The point of inflexion is said to be stationary if

$$\frac{dy}{dx} = 3ax^2 + 2bx + c = 0 \text{ at } x = -\frac{b}{3a} \Rightarrow 3a \cdot \frac{b^2}{9a^2} - \frac{2b^2}{3a} + c = 0$$

or $$\frac{b^2}{3a} - \frac{2b^2}{3a} + c = 0 \text{ or } -\frac{b^2}{3a} + c = 0 \text{ or } b^2 = 3ac$$

Example: If $y = \frac{1}{2}(e^x + e^{-x})$ show that

(a) $y(x) = y(-x)$

(b) y has a minima at $x = 0$

(c) The function has no point of inflexion.

Solution:

(a) $$y(-x) = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = y(x)$$

(b) $\dfrac{dy}{dx} = \dfrac{1}{2}\left[e^x - e^{-x}\right] = 0,$ for maxima or minima

$\Rightarrow e^x = e^{-x}$ or $x = -x \Rightarrow 2x = 0 \Rightarrow x = 0$.

Second Order Condition:

$\dfrac{d^2y}{dx^2} = \dfrac{1}{2}\left[e^x + e^{-x}\right] = 1 > 0$ at $x = 0$ $\therefore$ y has a minima at $x = 0$.

(c) Since $\dfrac{d^2y}{dx^2} = \dfrac{1}{2}\left[e^x + e^{-x}\right] > 0$ for all real values of x, the function has no point of inflexion.

13.2.1 N$^{\text{th}}$ Derivative Criterion for Maxima, Minima and Point of Inflexion

The criterion for relative maxima or minima of a function $y = f(x)$, discussed so far, fails if $f'(x) = 0$ at the stationary point. Similarly we cannot determine the nature of the point of inflexion if $f''(x) = 0$ at a point where $f'(x) = 0$. Such situations can be tackled with the help of following n^{th} derivative criterion.

Let us assume that the first non-zero derivative at a point $x = a$, encountered in successive derivation, is $f^n(a)$. Then

(i) $f(a)$ will be a maxima if n is even and $f^n(a) < 0$.

(ii) $f(a)$ will be a minima if n is even and $f^n(a) > 0$.

(iii) $f(a)$ will be a type I point of inflexion if n is odd and $f^n(a) < 0$.

(iv) $f(a)$ will be a type II point of inflexion if n is odd and $f^n(a) > 0$.

Notes

1. If $f(x)$ has a cusp at $x = a$, there is either maxima or minima at $x = a$, although the above criterion is not applicable.
2. If $f(x)$ has a vertical tangent at $x = a$, there is a point of inflexion at $x = a$, although the above criterion is not applicable.

Example: Show that the function $y = \dfrac{1}{(x-1)^3}$ has a point of inflexion at $x = 1$. What is the nature of the point of inflexion?

Solution:

$$y = \frac{1}{(x-1)^3}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{3}{(x-1)^4} \text{ and } \frac{d^2y}{dx^2} = \frac{12}{(x-1)^5}$$

We note that $\dfrac{d^2y}{dx^2}$ is not defined at $x = 1$, therefore, the criterion for point of inflexion is not applicable.

However, since $\dfrac{d^2y}{dx^2} < 0$ when $x < 1$ and, > 0 when $x > 1$, the curve changes from concave to convex and hence the point of inflexion at $x = 1$ is of type II.

Notes

Example: Find the derivative of $y = \sqrt[3]{x^2}$ and show that it is infinite at $x = 0$. Draw a graph of the function and indicate its behaviour in the neighbourhood of origin. Deduce that y has a minimum value at origin which is not a stationary value.

Solution.

$$y = \sqrt[3]{x^2} \quad \therefore \quad \frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} = \infty \text{ at } x = 0$$

To draw graph, we find

$$\frac{d^2y}{dx^2} = -\frac{2}{9}x^{-\frac{4}{3}} = -\frac{2}{9x^{4/3}} < 0 \ \forall x.$$

Thus, the function is concave from below for all values of x.

Further, since $\lim_{x\to 0-} x^{2/3} = \lim_{x\to 0+} x^{2/3} = f(0) = 0,$ the function is continuous at $x = 0$. Since $\frac{dy}{dx} = \infty,$ the function is not differentiable at $x = 0$. This situation is shown in Figure 13.8.

Note that, as we move away from origin on both sides, the value of y becomes greater than its value at $x = 0$. Thus $f(0) = 0$ is a minimum value of $y = \sqrt[3]{x^2}$ which is not a stationary value.

Example: By examining the sign of $\frac{dy}{dx}$, show that $y = \exp\left(x^{\frac{1}{2}} - \frac{2}{5}x\right)$ has a maxima at 25/16 .

Solution.

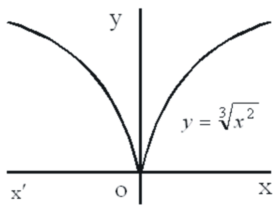
The given function can be written as, $y = e^{x^{\frac{1}{2}} - \frac{2}{5}x}$

$$\frac{dy}{dx} = e^{x^{\frac{1}{2}} - \frac{2}{5}x}\left(\frac{1}{2}x^{-1/2} - \frac{2}{5}\right) = 0$$

$$\Rightarrow \quad \frac{1}{2}x^{-1/2} - \frac{2}{5} = 0 \text{ or } \frac{1}{2x^{1/2}} = \frac{2}{5} \text{ or } x^{1/2} = \frac{5}{4} \text{ or } x = \frac{25}{16}$$

Since $e^{x^{\frac{1}{2}} - \frac{2}{5}x} > 0$ for all values of x, the sign of $\frac{dy}{dx}$ depends on the sign of $\left(\frac{1}{2}x^{-1/2} - \frac{2}{5}\right)$

Figure 13.8



When x is slightly less than $\frac{25}{16}$ say $\frac{24}{16}$, we have

$$\frac{1}{2}\times\sqrt{\frac{16}{24}} - \frac{2}{5} = 0.408 - 0.04 > 0$$

Notes

When x is slightly greater than $\frac{25}{16}$ say $\frac{26}{16}$, we have

$$\frac{1}{2}\times\sqrt{\frac{16}{26}}-\frac{2}{5} \quad = \quad 0.392-0.4<0$$

Since the sign of $\frac{dy}{dx}$ changes from positive to negative as we pass through the point $x=\frac{25}{16}$, the function has a maxima at this point

Example: A rectangular area is to be marked off as a chicken run with one side along an existing wall. The other sides are marked by wire netting of which a given length is available. Show that the area of the run is maximum if one side is twice the other.

Solution:

Let x be the length and y be the breadth of rectangle. Also let l be the length of wire.

$\therefore$ We can write $l = x + 2y$, or $x = l - 2y$.

The area of the rectangle,

$$A = x.y = (l-2y)y = ly-2y^2$$

We want to find y so that A is maximum.

$$\therefore \quad \frac{dA}{dy} = l-4y=0 \text{ or } y=\frac{l}{4}, \text{ for maxima.}$$

Second order condition:

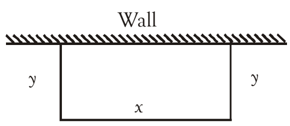
$$\frac{d^2A}{dy^2} = \text{therefore } A \text{ is maximum when } y=\frac{l}{4}.$$

Also $$x = l-2y=l-\frac{l}{2}=\frac{l}{2}.$$

Thus A is maximum when one side is taken as twice the other.

Example: An open box is constructed by removing a small square of side x cms from each corner of the metal sheet and turning up the edges. If the sheet is a square with each side equal to L cms, find the value of x so that volume of the box is maximum. Also find the largest volume of the box.

Figure 13.9



Solution:

After a square of side x cms is removed from each corner, the base of the box will be a square with each side = $L-2x$.

$\therefore$ Volume of the box $V = x\,(L-2x)^2$

Notes

$$\text{Further, } \frac{dV}{dx} = (L-2x)^2 - 4x(L-2x) = 0, \text{ for maximum } V.$$

$$\therefore \quad (L-2x)(L-2x-4x) = \text{ or } (L-2x)(L-6x) = 0$$

$$\text{or} \quad x = \frac{L}{2} \text{ or } x = \frac{L}{6}$$

Second order condition

$$\frac{d^2V}{dx^2} = -4(L-2x) - 4(L-2x) + 8x = -8L + 24x$$

$$= -8L + 12L = 4L > 0, \text{ when } x = \frac{L}{2},$$

$$\text{and} \quad = -8L + 4L = -4L < 0, \text{ when } x = \frac{L}{6}.$$

Thus the volume is largest when $x = \dfrac{L}{6}$.

$$\text{The largest volume} = \frac{L}{6}\left(L - \frac{L}{3}\right)^2 = \frac{2L^3}{27}.$$

Example: A running track of 440 ft. is to be laid out enclosing a football field, the shape of which is a rectangle with a semicircle at each end. If the area of the rectangular portion is to be kept maximum, find the length of its sides.

Solution:

Let x be the length of the rectangular portion and y be the breadth of the football field. Total perimeter of the running track is

$$P = 2x + 2\pi\frac{y}{2} = 2x + py \text{ or } y = \frac{P-2x}{\pi} = \frac{440-2x}{\pi}$$

Let A be the area of the rectangular portion.

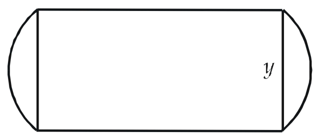
$$\therefore \quad A = x.y = \frac{440x - 2x^2}{\pi}$$

$$\text{Further,} \quad \frac{dA}{dx} = \frac{440-4x}{\pi} = 0 \text{ for maximum } A$$

$$\Rightarrow \quad x = 110 \text{ ft.}$$

$$\text{Also} \quad y = \frac{440-220}{22} \times 7 = 70$$

Figure 13.10



Notes

Further, $\frac{d^2A}{dx^2} = -\frac{4}{\pi} < 0$. Hence second order condition for maxima of A is also satisfied.

13.3 Summary

- Let $f(x)$ be a function with domain D. Then $f(x)$ has an absolute maxima at a point c in D if $f(x) \le f(c)$ for all x in D and an absolute minima at a point d in D if $f(x) \ge f(d)$ for all x in D.

  Absolute maxima/minima are also called *global* maxima/minima.
- A function $f(x)$ has a local maxima (or minima) at an interior point c in its domain D if $f(x) \le f(c)$ (or $f(x) \ge f(c)$) for all x in some open interval containing c.
- If a function $f(x)$ has a local extrema (i.e., maxima or minima) at an interior point c of its domain, and if $f'(c)$ exists, then $f'(c) = 0$.
- When the function $f(x)$ is twice differentiable at an interior point c of the domain, then
  - ❖ $f(x)$ has a local maxima at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$.
  - ❖ $f(x)$ has a local minima at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.
- When $f(x)$ has a maxima (or minima) at c, the curve of $f(x)$ is concave (or convex) from below. This test is inconclusive when $f''(c) = 0$.

13.4 Keywords

Absolute Maxima/Minima (Definition): Let $f(x)$ be a function with domain D. Then $f(x)$ has an absolute maxima at a point c in D if $f(x) \le f(c)$ for all x in D and an absolute minima at a point d in D if $f(x) \ge f(d)$ for all x in D. Absolute maxima/minima are also called *global* maxima/minima.

Local Maxima/Minima (Definition): A function $f(x)$ has a local maxima (or minima) at an interior point c in its domain D if $f(x) \le f(c)$ (or $f(x) \ge f(c)$) for all x in some open interval containing c.

13.5 Self Assessment

1. Determine maxima of $y = x^{\frac{2}{3}}(x-1)$

   (a) $\frac{2}{5}$ (b) $\frac{5}{2}$

   (c) $\frac{2}{3}$ (d) $\frac{2}{6}$

2. Find maximum value of y if $y = x^2 - 2x + 3, x = \frac{5}{2}$ then y is equal to:

   (a) 110.75 (b) 119.12

   (c) 118.75 (d) 111.85

3. If $f(x) = x^2 - 4x^2 - 3x + x$ then find f(x), if x = 4

   (a) 2 (b) –16

   (c) 10 (d) –10

4. Find the nature of point of inffexion if y = 20 + 5x + 12x² – 2x³, then x =

   (a) 2 (b) 3

   (c) 4 (d) -2

5. Find minima if y = x⁴ – 6x² + 1

   (a) $\sqrt{3}$ (b) $\sqrt{2}$

   (c) $\sqrt{10}$ (d) $\sqrt{9}$

13.6 Review Questions

1. Fluid maxima/minima of the following functions, by using only first derivative.

   (i) $y = x^2 + 10x + 15$ (ii) $y = x^3 - 3x^2 - 9x + 20$

   (ii) $y = x^{\frac{4}{3}}(x+2)$

2. Find maxima, minima and point of inflexion, if any, of the following functions:

   (i) $y = x^3 + 6x^2 + 12x + 1$ (ii) $y = 2x^3 + 3x^2 + 4$

   (iii) $y = x^4 + 4x^3 - 8x^2$ (iv) $y = 3x^5 - 5x^3$

   (v) $y = x^4 + 2x^2$ (vi) $y = x^{2/3} + x^{1/3}$

   (vii) $y = \dfrac{x^2}{x^2 - 1}$ (viii) $y = x^4 - 4x^3 + 16x$

   (ix) $y = x^3 - 3x^2 + 5$ (x) $y = \dfrac{1}{2}x^4 - x^2 + 1$

3. Find the absoute maxima/minima of the following functions:

   (i) $y = 8x - x^2$ on [1, 5]

   (ii) $y = \dfrac{3}{4}x - 7$ on [–1, 3]

   (iii) $y = \sqrt{4 - x^2}$ on [–2, 1]

   (iv) $2 - |x|$ on [–2, 2]

   (v) $2 - |x|$ on [4, 7]

   (vi) $y = x^{\frac{1}{2}}(4 - x)$ on [0, 3]

4. (a) If $y = x^4 - 4x^3 + 6x^2 - 4x - 3$, show that y has a minimum at $x = 1$.

   (b) If $y = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 10$, show that y has an inflexional value at $x = -1$.

5. Find maxima, minima and the point of inflexion for the function $y = -x^3 + 3x^2 + 9x - 27$. Show these points on a graph.

   If the domain of the functions is [–2, 2], find maxima/minima.

Notes

6. Find maxima, minima and the point of inflexion of the following function:

   $y = 4r^2x^3 - 4rx^2 + x - 1$, where $0 < r < 1$.

7. Show that the curve $y = 2x - 3 + \frac{1}{x}$ is convex from below for positive values of x and concave from below for negative values of x. Is the same true of the curve $y = ax + b + \frac{c}{x}$?

Answers: Self Assessment

1.	(a)	2.	(c)
3.	(d)	4.	(a)
5.	(a)		

13.7 Further Readings

Books

Husch, Lawrence S. *Visual Calculus, University of Tennessee*, 2001.

Smith and Minton, *Calculus Early Trancendental,* Third Edition, McGraw Hill 2008.

Online links

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http://library.thinkquest.org/20991/alg2/trigi.html

http://www.intmath.com/trigonometric–functions/5–signs–of–trigonometric–functions.php

Unit 14: Business Applications of Maxima and Minima

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss economic applications
- Explain prectical problems related to business applications of maxima and mininma

Introduction

In last unit you studied about maxima and minima. The terms maxima and minima refer to extreme values of a function, that is, the maximum and minimum values that the function attains. Maximum means upper bound or largest possible quantity. The absolute maximum of a function is the largest number contained in the range of the function. That is, if f(a) is greater than or equal to f(x), for all x in the domain of the function, then f(a) is the absolute maximum. In terms of its graph, the absolute maximum of a function is the value of the function that corresponds to the highest point on the graph. Conversely, minimum means lower bound or least possible quantity. The absolute minimum of a function is the smallest number in its range and corresponds to the value of the function at the lowest point of its graph.

14.1 Maximisation of Revenue

We can write total revenue as $TR = p.x$, where p is price and x is quantity. Total revenue will be maximum at a level of output where $\frac{d(TR)}{dx} = 0$ (or $MR = 0$) and $\frac{d^2(TR)}{dx^2} < 0$. The first order

Notes

condition implies that $\frac{d(TR)}{dx} = p + x\frac{dp}{dx} = 0$ or $\frac{p}{x}\cdot\frac{dx}{dp} = -1$ i.e. h = 1. Thus maxima of total revenue occurs at a level of output where elasticity of demand is unity.

Example: The inverse demand function facing a monopolist is $p = \beta - \alpha x$ (α, β > 0). Find the price charged and quantity sold for maximum monopoly revenue. Show that the elasticity at this point is unity.

Solution:

$$TR = p.x = (\beta - \alpha x).x = \beta x - \alpha x^2$$

$$\therefore \quad \frac{d(TR)}{dx} = \beta - 2\alpha x = 0, \text{ for maxima, } \Rightarrow \quad x = \frac{\beta}{2\alpha}.$$

$$\text{The monopoly price } p = \beta - \alpha.\frac{\beta}{2\alpha} = \frac{\beta}{2}$$

Second order condition

$$\frac{d^2(TR)}{dx^2} = -2\alpha < 0.$$

Note that $\frac{d^2(TR)}{dx^2} = \frac{d^2(TR)}{dx^2}$. Thus the second order condition implies that the marginal revenue should be falling at the point $\left(\frac{\beta}{2\alpha}, \frac{\beta}{2}\right)$.

For elasticity of demand at the point $\left(\frac{\beta}{2\alpha}, \frac{\beta}{2}\right)$ on the demand function $p = \beta - \alpha x$, we have $\frac{dp}{dx}$ = $-\alpha$ or $\frac{dx}{dp} = -\frac{1}{\alpha}$.

$$\therefore \quad \eta = -\frac{dx}{dp}\cdot\frac{p}{x} = \frac{1}{\alpha}\cdot\frac{\beta}{2}\times\frac{2\alpha}{\beta} = 1.$$ Hence elasticity of demand is unity.

Example: A wholesaler of pencils charges ₹ 24 per dozen on orders of 50 dozens or less. For orders in excess of 50 dozens, the price is reduced by 20 paise per dozen in excess of 50 dozens. Find the size of the order that maximises his total revenue.

Solution:

Let x be the number of dozens in an order.

When $x \leq 50$, $TR = 24x$

When $x > 50$, the price charged per dozen is given by

$$p = 24 - 0.20(x - 50) = 34 - 0.20x$$

This is the equaton of a straigh line passing through the point (50, 24) with slope = – 0.20.

$$\text{Thus, } TR = p.x = (34 - 0.2x).x = 34x - 0.2x^2$$

We note here that TR will have maxima only when x > 50 because, when x £ 50, TR is a straight line and hence has no maxima.

For maximum *TR*, we have

$$\frac{d(TR)}{dx} = 34 - 0.4x = 0 \text{ or } x = \frac{34}{0.4} = 85 \text{ dozens.}$$

Since $\frac{d^2(TR)}{dx^2} = -4 < 0,$ the second order condition is satisfied.

Alternate Method

Let y be number of dozens ordered in excess of 50 dozens.

Then $p = 24 - 0.20y$ and quantity ordered = 50 + y.

$$\therefore \quad TR = (24 - 0.20y)(50 + y)$$

$$= 1200 + 14y - 0.20y^2$$

$$\frac{d(TR)}{dy} = 14 - 0.40y = 0 \qquad \text{for maxima.}$$

or $y = \frac{14}{0.40} = 35$

$$\frac{d^2(TR)}{dy} = -0.40 < 0$$

Thus, revenue is maximised when (50 + 35) = 85 dozens of pencils are ordered.

Example: A tour operator charges ₹ 200 per passenger for 50 passengers with a discount of ₹ 5 for each 10 passenger in excess of 50. Determine the number of passengers that will maximise the revenue of the operator.

Solution:

Let x be the number of passengers, then revenue from each passenger i.e. price p is given by

$$p = 200 - \frac{5}{10}(x - 50) = 225 - \frac{x}{2}$$

The equation of a straight line passing through the point (50, 200) with slope = $\frac{-5}{10}$.

$$\therefore \quad TR = \left(225 - \frac{x}{2}\right)x = 225x - \frac{x^2}{2}.$$

$$\frac{d(TR)}{dx} = 225 - x = 0 \text{ or}$$

$$\frac{d^2(TR)}{dx^2} = -1 < 0$$

$\therefore$ *TR* is maximised when x = 225 passengers. Alternatively, we can write the revenue function as.

$TR = \left(200 - \frac{y}{2}\right)(50 + y),$ where y is the number of passengers in excess of 50.

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Example: If the demand law is $x = \alpha e^{-\beta p}$, $(\alpha, \beta > 0)$, express marginal revenue as a function of x. At what levels of output and price the total revenue is maximum? Also find maximum total revenue.

Solution:

Taking log of both sides of the demand function, we get

$$\log x = \log\alpha - \beta p \text{ or } p = \frac{1}{\beta}[\log\alpha - \log x]$$

$$\therefore \quad TR = p.x = \frac{1}{\beta}[x.\log\alpha - x.\log x]$$

Now
$$MR = \frac{d(TR)}{dx} = \frac{1}{\beta}[\log\alpha - \log x - 1]$$

$$= \frac{1}{\beta}\left[\log\frac{\alpha}{x} - 1\right] = 0 \text{, for maximum } TR$$

$$\Rightarrow \quad \log\frac{\alpha}{x} - 1 = 0 \text{ or } \log\frac{\alpha}{x} = 1$$

$$\therefore \quad \log\frac{\alpha}{x} = \log e \Rightarrow \frac{\alpha}{x} = e \text{ or } x = \frac{\alpha}{e}$$

Further, $\frac{d^2(TR)}{dx^2} = -\frac{1}{\beta x} < 0$, the second order condition is satisfied. Also price, when $x = \frac{\alpha}{e}$, is given by $p = \frac{1}{\beta}[\log\alpha - \log x] = \frac{1}{\beta}\log\frac{\alpha}{x} = \frac{1}{\beta}$

$$\text{Hence, maximum } TR = \frac{1}{\beta}\cdot\frac{\alpha}{e}.$$

Example: A firm's demand function is : $x = 400\ln\left(\frac{10}{p}\right)$. Find the price and quantity where total revenue is maximum. Also find price elasticity of demand at that price.

Solution:

Note: ln denotes log with base e.

Here it will lie convenient to express total revenue as a function of p.

$$\therefore \quad TR = p.x. = 400p[\ln 10 - \ln p] = 400p.\ln 10 - 400p.\ln p$$

$$\frac{d(TR)}{dp} = 400\ln 10 - 400\ln p - 400p \times \frac{1}{p}$$

$$= 400[\ln 10 - \ln p - 1] = 0 \qquad \text{for maxima.}$$

$$\Rightarrow \quad \ln 10 - \ln p = 1$$

Notes

or $$\ln\left(\frac{10}{p}\right) = 1 \quad \text{or} \quad \frac{10}{p} = e$$

or $$p = 10.e^{-1}$$

Also $$x = 400\ln\left(\frac{10}{p}\right) = 400\ln e = 400$$

Thus $x = 400$ and $p = 10.e^{-1}$ at maximum revenue. To find price elasticity of demand, we write.

$$\ln x = \ln\left[400(\ln 10 - \ln p)\right]$$

$$\frac{d\ln x}{dp} = \frac{1}{400(\ln 10 - \ln p)} \times \left(\frac{-400}{p}\right) = -\frac{1}{400\ln\left(\frac{10}{p}\right)} \times \frac{400}{p}$$

Also $$\frac{d\log p}{dp} = \frac{1}{p}$$

$\therefore$ $$\eta = -\frac{d\ln x}{d\ln p} = \frac{1}{400\ln\left(\frac{10}{p}\right)} \times \frac{400}{p} \times p = \frac{1}{\ln\left(\frac{10}{p}\right)}$$

$$= \frac{1}{\ln e} = 1 \text{ at } p = \frac{10}{e}$$

Example: If $p = f(x)$ is an inverse demand function, find the level of output at which total revenue is maximum. Show that total revenue will always be a maximum if demand curve is downward sloping and concave from below. Is it possible to have maxima of total revenue if the demand curve is convex from below? Discuss.

Solution:

$$TR = xf(x) \therefore \frac{d(TR)}{dx} = f(x) + xf'(x) = 0 \text{ for maxima}$$

$\Rightarrow$ $$f'(x) = -\frac{f(x)}{x} = -\frac{p}{x}$$

Since p and x are always positive, this implies that total revenue is maximum only if $f'(x) < 0$

$$\frac{d^2(TR)}{dx^2} = 2f'(x) + xf''(x) < 0 \text{ (second order condition)}$$

$\Rightarrow$ $$f''(x) < -\frac{2f'(x)}{x}, \text{ which is always satisfied if } f''(x) < 0.$$

When the demand curve is convex from below such that $0 < f''(x) < -\frac{2f'(x)}{x}$, it is possible to have maxima of total revenue.

Notes

Maximisation of Tax Revenue

Let $p = f(x)$ and $p = g(x)$ be the market demand and supply of a commodity and a specific tax of ₹ t per unit be imposed. Then under equilibrium, we can write $f(x) = g(x) + t$.

Let x_t be the equilibrium quantity obtained by solving the above equation for x. We can write the expression for tax revenue T as $T = t.x_t$ (note that x_t is a function of t).

From this we can find t such that T is maximum.

Example: The inverse demand and supply functions of a commodity, in a perfectly competitive market, are given by $p = \beta - \alpha x$ and $p = b + ax$ respectively, where $a, b, a, b > 0$ and $b > b$.

Find the equilibrium values if p and x. If the government imposes a specific tax @ ₹ t per unit, find post-tax equilibrium values. Also find the value of t for maximum tax revenue.

Solution:

We have demand price = supply price, (in equilibrium)

$\therefore$ $\beta - \alpha x = b + ax$ or $x = \dfrac{\beta - b}{\alpha + a}$, is the equilibrium quantity. We substitute this value in demand function to get the equilibrium price.

Thus, $p = \beta - \dfrac{\alpha(\beta - b)}{\alpha + a} = \dfrac{\beta a + \alpha b}{\alpha + a}$

After a specific tax of ₹ t per unit, the equilibrium condition becomes:
demand price = supply price + t

or $\beta - \alpha x = b + ax + t$ $\qquad \therefore\ x_t = \dfrac{\beta - b - t}{\alpha + a}$

The post-tax equilibrium price $p = \beta - \dfrac{\alpha(\beta - b - t)}{\alpha + a} = \dfrac{\beta a + \alpha b + \alpha t}{\alpha + a}$

The tax revenue $T = t.x_t = \dfrac{t(\beta - b - t)}{\alpha + a}$

Thus $\dfrac{dT}{dt} = \dfrac{\beta - b - 2t}{\alpha + a} = 0,$ for maximum T, $\Rightarrow$ $t = \dfrac{1}{2}(\beta - b)$.

Example: The inverse demand and supply of a commodity in a perfectly competitive market are given by $p = f(x)$ and $p = g(x)$, where $f'(x) < 0$ and $g'(x) > 0$. If a specific tax of ₹ t per unit is imposed, show that equilibrium output decreases as tax rate t increases.

Solution:

Let a specific tax of ₹ t per unit be imposed on the commodity with demand and supply function as $p = f(x)$ and $p^s = g(x)$. Where p denotes price paid by the consumer and p^s is the price received by the seller. Thus under equilibrium we have

$$p = p^s + t \text{ or } f(x) = g(x) + t$$

Let x_t (the equilibrium quantity) be the solution of this equation.

Therefore, we can write $f(x_t) = g(x_t) + t$

Since x_t is a function of t, we can differentiate the above equation with respect to t, to get

$$f'(x_t)\cdot\frac{dx_t}{dt} = g'(x_t)\cdot\frac{dx}{dt}+1 \text{ or } \frac{dx_t}{dt} = \frac{1}{f'(x_t)-g'(x_t)}$$

Since demand function is assumed to be downward sloping, the denominator of the above expression is negative. Thus $\frac{dx_t}{dt} < 0,$ which implies that equilibrium output decreases as tax rate increases.

14.2 Maximisation of Output

Assuming that labour is the only variable factor, we can write the production function of a firm as $x = f(L)$, where x denotes total product of labour which will be denoted as TP_L.

The average product of labour is $AP_L = \frac{TP_L}{L} = \frac{f(L)}{L}$, the marginal product of labour is $MP_L = \frac{dx}{dL} = f'(L)$ and necessary condition for maximum output is $\frac{d(TP_L)}{dL} = \frac{dx}{dL} = MP_L = 0$

Often we are interested in finding that level of employment of labour at which its average product is maximum.

For maxima of AP_L, we have

$$\frac{d(AP_L)}{dL} = \frac{L.f'(L)-f(L)}{L^2} = 0$$

$\therefore$ $Lf'(L) = f(L)$ or $f'(L) = \frac{f(L)}{L}$ or $MP_L = AP_L$. Thus, *the marginal and average products of a factor are equal at the maxima of the later.*

Maximisation of Total Revenue Product

If p is price of a unit of output, the total revenue of the firm is, $TR = p.x$. This total revenue, when expressed as a function of L, using production function $x = f(L)$, is called the total revenue product of labour (TRP_L). Units of L, to be employed, for maximum TR is given by the equation $\frac{d(TR)}{dL} = 0$.

The derivative $\frac{d(TR)}{dL}$ is known as the marginal revenue product of labour.

Since TR is a function of x and x is a function of L, using chain rule, we can write an expression for marginal revenue product in terms of marginal revenue and marginal product.

$$MRP_L = \frac{d(TR)}{dL} = \frac{d(TR)}{dx}.\frac{dx}{dL} = MR.MP_L$$

Example: The short-run production function of a manufacturer is given as $x = 11L + 16L^2 - L^3$.

(i) Find the average product function, AP_L, the marginal product function, MP_L, and show that $MP_L = AP_L$ where AP_L is maximum.

Notes

(ii) Find the value of L for which output is maximum.

(iii) Find the value of L at which the total product curve has a point of inflexion and verify that MP_L is maximum at this point. What is the nature of the point of inflexion?

(iv) If the manufacturer sells the product at a uniform price of ₹ 10 per unit, find the maximum total revenue product and associated level of L.

Solution:

(i)
$$AP_L = \frac{x}{L} = 11 + 16L - L^2$$

$$MP_L = \frac{dx}{dL} = 11 + 32L - 3L^2$$

We have $\frac{d(AP_L)}{dL} = 16 - 2L = 0 \therefore AP_L$ is maximum at $L = 8$.

Since $\frac{d^2(AP_L)}{dL^2} = -2 < 0$, the second order condition is satisfied.

The maximum AP_L = 11 + 168 – 8^2 = 75

Further, MP_L when L = 8, is 11 + 32 × 8 – 3 × 8^2 = 75

Thus, AP_L = MP_L, when AP_L is maximum.

(ii) For maximum output:

$$\frac{dx}{dL} = 11 + 32L - 3L^2 = 0$$

or $(11 - L)(1 + 3L) = 0$, \ $L = 11$. The other value, being negative, is dropped.

Since $\frac{d^2x}{dL^2} = 32 - 6L = -34 < 0$, the second order condition for maxima is satisfied.

(iii) For point of inflexion:

$$\frac{d^2x}{dL^2} = 32 - 6L = 0 \therefore L = \frac{16}{3} = 5.33 \text{ and } \frac{d^3x}{dL^3} = -6 < 0.$$

Thus the point of inflexion is of type I, i.e. the curve changes from convex to concave from below.

Since $\frac{d^2x}{dL^2} = \frac{d(MP_L)}{dL} = 0$ and $\frac{d^3x}{dL^3} = \frac{d^2(MP_L)}{dL^2} < 0$,

$\therefore$ MP_L is maximum at L = 5.33.

(iv)
$$TRP_L = p.x = 10\left(11L + 16L^2 - L^3\right).$$

Since TRP_L is a constant multiple of the production function, therefore, maxima of TRP_L will be at the same level of L where x is maximum. Thus, TRP_L will also be maximum at L = 11. The maximum value = $10\left(11 \times 11 + 16 \times 11^2 - 11^3\right)$ = ₹ 7,260.

14.3 Minimisation of Cost

If total cost $C = F(x)$, then we can define $AC = \frac{C}{x} = \frac{F(x)}{x}$, and $MC = \frac{dC}{dx} = F'(x)$.

Very often we are interested in finding the level of output that gives minimum AC. For minima of AC, we have

Notes

$$\frac{d(AC)}{dx} = \frac{xF'(x) - F(x)}{x^2} = 0$$

$$\therefore \quad xF'(x) = F(x) \text{ or } F'(x) = \frac{F(x)}{x} \text{ or } MC = AC.$$

Thus, marginal cost is equal to the average at the minima of the later.

Notes The level of output at which AC is minimum is also known as the *most economic* (or *capacity*) *output*.

Example: The short-run cost function of a food manufacturer is given by

$$C = 1{,}000 + 100x - 10x^2 + x^3.$$

(i) Find AC, AVC and MC functions.

(ii) Show that MC = min. of AC.

(iii) Show that MC = min. of AVC.

(iv) Show that total cost function has a point of inflexion at a level of output where MC is minimum. Find min. MC.

Solution:

(i)
$$AC = \frac{C}{x} = \frac{1{,}000}{x} + 100 - 10x + x^2$$

$$AVC = 100 - 10x + x^2, \ MC = 100 - 20x + 3x^2$$

(ii) For minima of AC, we have

$$\frac{d(AC)}{dx} = -\frac{1000}{x^2} - 10 + 2x = 0 \text{ or } -1000 - 10x^2 + 2x^3 = 0$$

or
$$x^3 - 5x^2 - 500 = 0 \text{ or } (x-10)(x^2 + 5x + 50)$$

Thus $x = 10$ is a stationary point. The other roots, being imaginary, are neglected.

We note that $\frac{d^2(AC)}{dL^2} = \frac{2000}{x^3} + 2 = 4 > 0$, at $x = 10$

Thus AC is minimum at $x = 10$.

Also min. $AC = \frac{1000}{10} + 100 - 10 \times 10 + 10^2 = 200$

and $MC = 100 - 20 \times 10 + 3 \times 10^2 = 200$, at $x = 10$

Thus, MC = min. AC.

(iii) For minima of AVC, we have

$$\frac{d(AVC)}{dx} = -10 + 2x = 0 \text{ or } x = 5, \text{ min.}$$

$$AVC = 100 - 10 \times 5 + 5^2 = 75$$

and $$MC = 100 - 20 \times 5 + 3 \times 5^2 = 75, \text{ at } x = 5$$

Thus, $$MC = \text{min. } AVC$$

(iv) Since $\dfrac{d^2C}{dx^2} = \dfrac{d(MC)}{dx} = -20 + 6x = 0$ and $\dfrac{d^3C}{dx^3} = 6 > 0$, at $x = \dfrac{10}{3}$, the total cost function has a type II point of inflexion.

Since $\dfrac{d(MC)}{dx} = 0$ at $x = \dfrac{10}{3}$, $\therefore$ MC is also minimum at this value.

$$\text{Also, min. } MC = 100 - 20 \times \frac{10}{3} + 3 \times \frac{10^2}{3^2} = \frac{200}{3} = 66.67$$

Example: The cost of fuel consumed per hour in running a train is proportional to the square of its speed (in kms per hour), and it costs ₹ 3,200 per hour at a speed of 40 kms per hour. What is the *most economical speed*, if the fixed charges are ₹ 12,800 per hour?

Solution:

Let F be the cost of fuel and x be the speed of the train per hour. We are given that $F \propto x^2$ or $F = kx^2$, where k is a constant of proportionality.

When $x = 40$, F is given to be 3,200, $\therefore$ $k = \dfrac{3200}{1600} = 2.$

Thus we can write $F = 2x^2$, as the cost of fuel per hour of running the train when its speed is x kms per hour. Now the total cost of running the train for x kms (per hour) is $TC = 12{,}800 + 2x^2$.

$\therefore$ $$\text{Average cost } AC = \frac{12800}{x} + 2x.$$

The most economic speed will be that value of x which minimises AC.

$\therefore$ $$\frac{d(AC)}{dx} = -\frac{12800}{x^2} + 2 = 0, \text{ for minima or}$$

or $$x^2 = \frac{12800}{2} = 6400 \text{ or } x = 80 \text{ kms/hour.}$$

Second order condition

$$\frac{d^2(AC)}{dx^2} = \frac{25600}{x^3} > 0, \text{ when } x = 80.$$

Thus, the second order condition for minima is satisfied.

Coefficients of a Cubic Total Cost Function

Let the cubic total cost function be $TC = ax^3 + bx^2 + cx + d.$ Therefore, the marginal cost function is given by

$$MC = \frac{d(TC)}{dx} = 3ax^2 + 2bx + c$$

Notes

In order that MC curve is U-shaped, the MC function should represent a parabola with axis pointing vertically upward. Further, in order that total cost function makes economic sense, the vertex of the parabola must lie in positive quadrant.

For minima of MC, we have $\frac{d(\text{MC})}{dx} = 6ax + 2b = 0 \Rightarrow x = -\frac{b}{3a}$

Further, $\frac{d^2(\text{MC})}{dx^2} = 6a,$ which should be positive for minima.

This implies that $a > 0$. Also, since x, the output level, should be positive, therefore $b < 0$.

Now min. MC = $3a\left(-\frac{b}{3a}\right)^2 + 2b\left(-\frac{b}{3a}\right) + c = \frac{3ac - b^2}{3a}$

This will be positive only if $b^2 < 3ac$. Since $a > 0$, this condition also implies that $c > 0$. Further, the constant term d, which represents the total fixed cost, is always positive.

14.4 Economic Applications (Continued)

14.4.1 Maximisation of Profits

Profit is the difference between total revenue and total cost of a producer or firm. We know that total revenue as well as total costs are often expressed as functions of level of output, x. If we write $TR = R(x)$ and $TC = C(x)$, then the profit p can be written as $p(x) = R(x) - C(x)$.

We want to find that value of x so that $p(x)$ becomes maximum. The conditions for maxima of $p(x)$ are:

First order condition

$$\pi'(x) = R'(x) - C'(x) = 0, \text{ or } R'(x) = C'(x) \text{ or } MR(x) = MC(x)$$

Let x_e satisfy this equation. Then, we can write $R'(x_e) = C'(x_e)$

Here x_e is termed as the *profit maximising* or *equilibrium output*. Note that the first order condition is also termed as the *equilibrium condition*.

Second order condition

In order that profit $\pi(x)$ is maximum at x_e, we should have $\pi''(x_e) < 0$.

This condition implies that $R''(x_e) < C''(x_e)$ or $R''(x_e) < C''(x_e)$, i.e. the slope of marginal revenue curve must be less than slope of the marginal cost curve at equilibrium point.

Alternatively, we can express total revenue and total cost as functions of price, where price and quantity are related by the demand function $x = \phi(p)$. Thus, we can also express profit of the firm as a function of price. The first and second order condition maximum profits, in this case, can be written as $\pi'(p_e) = 0$ and $\pi''(p_e) < 0$ respectively.

14.4.2 Profit Maximisation by a Firm under Perfect Competition

A firm under perfect competition is a price taker i.e. price is constant. Therefore, the only option before it is to choose that level of output at which its profits are maximised.

Notes

If p is the price at which the firm can sell its output, then total revenue of the firm is $R(x) = p.x$, where x is the level of output. We note that total revenue of the firm is a straight line passing through origin with slope p. Assuming the cost function as $C = C(x)$, we can write the profit of the firm as $p(x) = R(x) - C(x) = px - C(x)$.

$\therefore \qquad \pi'(x) = p - C'(x) = 0,$ for maximum π (note that $MR = p$).

Thus, $\qquad p = C'(x)$ or $p = MC(x)$ is the necessary condition for maximum profits.

Second order condition

$$\pi''(x) = 0 - C''(x) < 0, \text{ for maximum } p.$$

This condition will hold only if $C''(x)$ or $\dfrac{d(MC(x))}{dx} > 0$ at the stationary value i.e. MC must be rising at the stationary point.

Break-Even Point

It can be shown that the break-even point of a profit maximising firm under perfect competition will occur at a level of output where average cost is minimum.

We can write

$$TR = TC \text{ (for break even)}$$

or $\qquad px = TC$ or $p = \dfrac{TC}{x}$

or $\qquad MC = AC$ ($\because$ $p = MC$ in equilibrium)

Starting Point

The starting point of a firm is the minimum level of output at which total variable costs (TVC) of the firm are covered. Therefore we have

$$TR = TVC, \text{ (at the starting point)}$$

or $\qquad px = TVC$ or $p = \dfrac{TVC}{x} = AVC$

or $\qquad MC = AVC$ (in equilibrium)

Thus the starting point occurs at the minima of AVC.

Example: A plant produces x tons of steel per week at a total cost of ₹ $\frac{1}{10}x^3 - 3x^2 + 50x + 300.$ If the market price is fixed at ₹ $33\frac{1}{3}$, find the profit maximising output of the plant and the maximum profit. Will the firm continue production?

Solution:

We can write $\qquad R(x) = \dfrac{100}{3}x$ and $C(x) = \dfrac{1}{10}x^3 - 3x^2 + 50x + 300$

$$\pi(x) = R(x) - C(x) = \frac{100}{3}x - \frac{1}{10}x^3 + 3x^2 - 50x - 300$$

Notes

$$= -\frac{1}{10}x^3 + 3x^2 - \frac{50}{3}x - 300$$

For max. p, we should have $\pi'(x) = 0$ and $\pi''(x) < 0$

Now, $\pi'(x) = -\frac{3}{10}x^2 + 6x - \frac{50}{3} = 0$ or $9x^2 - 180x + 500 = 0$

$\therefore$ $x = \frac{180 \pm \sqrt{180^2 - 4 \times 9 \times 500}}{18} = \frac{180 \pm 120}{18}$

Thus, $x_1 = \frac{300}{18}$ or $\frac{50}{3}$ and $x_2 = \frac{60}{18}$ or $\frac{10}{3}$

Further, $\pi''(x) = -\frac{6}{10}x + 6 = -\frac{6}{10}.\frac{50}{3} + 6 = -4 < 0$ at $x_1 = \frac{50}{3}$

$$= -\frac{6}{10}.\frac{10}{3} + 6 = 4 > 0 \text{ at } x_2 = \frac{10}{3}$$

Therefore, profit maximising output of the plant = $\frac{50}{3}$ or $16\frac{2}{3}$.

Max. profit $= -\frac{1}{10}\left(\frac{50}{3}\right)^2 + 3\left(\frac{50}{3}\right)^2 - \left(\frac{50}{3}\right)^2 - 300 = -207.41$

Thus the firm is incurring loss of ₹ 207.41. Since this loss is less than ₹ 300 (fixed cost), the firm will continue production.

Example: If the total cost of a firm is $C = \frac{1}{3}x^3 - 5x^2 + 30x + 10$, where C is the total cost and x is the level of output, and price under perfect competition is given as ₹ 6, find for what value(s) of x the profit will be maximised? Also find the value of maximum profit and comment on the result.

Solution:

We can write

$$\pi(x) = 6x - \frac{1}{3}x^3 + 5x^2 - 30x - 10 = -\frac{1}{3}x^3 + 5x^2 - 24x - 10$$

We have $\pi'(x) = -x^2 + 10x - 24 = 0$ or $x^2 - 10x + 24 = 0$, for max. p

$\Rightarrow$ $(x-6)(x-4) = 0$ $\therefore$ $x_1 = 6$ and $x_2 = 4$

Further, $\pi''(x) = -2x + 10 = -12 + 10 = -2 < 0$, when $x = 6$

and $= -8 + 10 = 2 > 0$, when $x = 4$

Thus, the profit is maximum when $x = 6$ units.

Maximum profit $= -\frac{1}{3} \times 6^3 + 5 \times 6^2 - 24 \times 6 - 10 = -46$ i.e. loss of ₹ 46.

Since this loss is greater than the loss of ₹ 10, when nothing is produced, the firm will discontinue production.

Notes

Supply Curve of a Firm under Perfect Competition

The supply curve of a firm, under perfect competition, is that portion of the marginal cost curve that lies above the average variable cost curve. Let $p_m = MC$ = min. AVC (average variable cost). We can say that

(i) When $p < p_m$, quantity supplied $x = 0$, and

(ii) When p ³ p_m, quantity supplied is given by the condition $p = MC$. Solving this equation for x gives the supply function of the firm.

Example: The total cost of a firm under perfect competition is given by $C = x^3 - 6x^2 + 15x + 10.$ Find the supply function of the firm.

Solution:

First we find the lowest price p_m below which the supply will be zero.

$$\text{Total variable cost } TVC = x^3 - 6x^2 + 15x \;\therefore\; AVC = x^2 - 6x + 15$$

$$\text{We have } \frac{d(AVC)}{dx} = 2x - 6 = 0, \text{ for minima Þ } x = 3$$

Now, p_m = min. AVC = $3^2 - 6 \times 3 + 15 = 6$, below which quantity supplied will be zero.

Further, we write $p = MC = 3x^2 - 12x + 15$ or $3x^2 - 12x + (15 - p) = 0$

Solving this quadratic equation for x, we have

$$x = \frac{12 \pm \sqrt{144 - 12(15 - p)}}{6} = \frac{12 \pm \sqrt{12p - 36}}{6} = \frac{6 \pm \sqrt{3p - 9}}{3}$$

Thus, the supply function of the firm is

$$x = \frac{6 + \sqrt{3p - 9}}{3} \text{ when } p \geq 6,$$

and $x = 0$ when $p < 6$.

Note that we have ignored the negative sign because this will give values of x lying on that portion of MC which lies below AVC.

Example: The total cost of a firm is $C = \frac{1}{3}x^3 - 6x^2 + 30x + 20.$ Find the equilibrium output if price is fixed at ₹ 10 per unit. What will be the effect of a specific tax of ₹ 3 per unit on the equilibrium output?

Solution:

Profit $\pi(x) = 10x - \frac{1}{3}x^3 + 6x^2 - 30x - 20 = -\frac{1}{3}x^3 + 6x^2 - 20x - 20$

$\pi'(x) = -x^2 + 12x - 20 = 0,$ for maximum profit

$\Rightarrow$ $x^2 - 12x + 20 = 0$ or $(x - 10)(x - 2) = 0 \Rightarrow x = 10$ or 2

Further, $\pi''(x) = -2x + 12 = -2 \times 10 + 12 = -8 < 0,$ when $x = 10$

and $= -2 \times 2 + 12 = 8 > 0,$ when $x = 2$

$\therefore$ Profits are maximised when $x = 10$.

When a tax of ₹ 3 per unit is imposed, the total cost is written as

$$C_t(x) = \frac{1}{3}x^3 - 6x^2 + 30x + 20 + 3x = \frac{1}{3}x^3 - 6x^2 + 33x + 20$$

$$\therefore \quad \text{Profit } \pi_t(x) = 10x - \frac{1}{3}x^3 + 6x^2 - 33x - 20 = -\frac{1}{3}x^3 + 6x^2 - 23x - 20$$

Now $\quad \pi_t'(x) = -x^2 + 12x - 23 = 0,$ for max. $\pi \Rightarrow x^2 - 12x + 23 = 0$

$$\therefore \quad x = \frac{12 \pm \sqrt{144 - 92}}{2} = \frac{12 \pm 7.21}{2}. \text{ Thus, } x = 9.6 \text{ or } 2.4$$

Further it can be shown that $\pi_t''(x) < 0$, when $x = 9.6$. Therefore, post-tax equilibrium occurs at lower level of output.

14.4.3 Profit Maximisation by a Monopoly Firm

Let the firm faces an inverse demand function $p = f(x)$. Then we can write the total revenue of the firm as $R(x) = p.x = x.f(x)$. Assuming the cost function as $C(x)$, we can write the profit function as $p(x) = x.f(x) - C(x)$. As before, the profit maximising conditions are $\pi'(x) = 0$ and $\pi''(x) < 0$.

Notes

1. The equilibrium condition can be written as $MR(x) = MC(x)$ or $p\left(1 - \frac{1}{\eta}\right) = MC(x)$. Thus $p > MC(x)$ when $\eta > 1$ (note that a profit maximising monopolist always operates on the elastic portion of the demand curve). Since $p = MC(x)$ for a perfectly competitive firm, this implies that price charged by a monopolist will be higher for producing the same level of output.
2. Like a perfectly competitive firm, there is no supply curve of a monopoly firm. To show this, we solve the equilibrium condition $p\left(1 - \frac{1}{\eta}\right) = MC(x)$ for x. The solution for x will be a function of p and η. This function can be regarded as a supply function only if η is constant. However, we know that often h is different at different points of the demand curve.

Example: The demand and cost functions of a monopolist are given to be $x = 500 - \frac{1}{2}p$ and $C = x^3 - 59x^2 + 1315x + 2000$ respectively. Find his profit maximising level of output and price.

Solution:

We can write the demand function as $\frac{1}{2}p = 500 - x$ or $p = 1000 - 2x$

Therefore, the profit function of the monopolist is

$$\pi(x) = (1000 - 2x)x - x^3 + 59x^2 - 1315x - 2,000$$

$$= -x^3 + 57x^2 - 315x - 2000$$

We have, $\quad \pi'(x) = 3x^2 + 114x - 315 = 0$ or $x^2 - 38x + 105 = 0$ for max. p

$\Rightarrow \quad (x - 35)(x - 3) = 0 \therefore x = 35$ or 3

Notes

Second order condition:

$$\text{We have } \pi''(x) = -6x+114$$
$$= -6\times 35+114=-96<0\text{, at } x = 35$$
$$= -6\times 3+114=96>0,\ \text{at } x = 3$$

Therefore, profits are maximised when 35 units of the commodity are produced.

Further, the equilibrium price $p=1000-2\times 35$ = ₹ 930.

Example: The total cost of a monopolist is $C=ax^2+bx+c$ (a, b, c > 0) and the inverse demand function is $p=\beta-\alpha x$ $(\alpha,\beta>0)$. Find his equilibrium output, price and net revenue (profit). How will these values change if a tax of ₹ t per unit is levied? Also determine the tax rate that maximises the tax revenue. Find the maximum tax revenue.

Solution:

Profit $\pi(x) = \beta x-\alpha x^2-ax^2-bx-c=-(a+\alpha)x^2-(b-\beta)x-c$

$\therefore$ $\pi'(x) = -2(a+\alpha)x-(b-\beta)=0$, for max. π, $\Rightarrow x=\dfrac{\beta-b}{2(a+\alpha)}$

We note that $\pi''(x) = -2(a+\alpha)<0$. Therefore, $x=\dfrac{\beta-b}{2(a+\alpha)}$ is the profit maximising output.

The equilibrium price

$$p = \beta-\frac{\alpha(\beta-b)}{2(a+\alpha)}=\frac{2a\beta+2\alpha\beta-\alpha\beta+\alpha b}{2(a+\alpha)}=\frac{2a\beta+\alpha(\beta+b)}{2(a+\alpha)}$$

Maximum net revenue $=-(a+\alpha)\dfrac{1}{4}\dfrac{(\beta-b)^2}{(a+\alpha)^2}-\dfrac{(b-\beta)(\beta-b)}{2(a+\alpha)}-c$

$$= -\frac{(\beta-b)^2}{4(a+\alpha)}+\frac{(\beta-b)^2}{2(a+\alpha)}-c=\frac{(\beta-b)^2}{4(a+\alpha)}[-1+2]-c=\frac{(\beta-b)^2}{4(a+\alpha)}-c$$

After a specific tax of ₹ t per unit is imposed, the profit function can be written as $\pi_t(x)=-(a+\alpha)x^2-(b-\beta)x-c-tx$

$\therefore$ $\pi_t'(x) = -2(a+\alpha)x-(b-\beta)-t=0$ or $2(a+\alpha)x=\beta-b-t$ for max. π

$$\Rightarrow \quad x = \frac{\beta-b-t}{2(a+\alpha)}$$

The second order condition is same as before.

The post-tax price $p = \beta-\dfrac{\alpha(\beta-b-t)}{2(a+\alpha)}=\dfrac{2a\beta+\alpha(\beta+b+t)}{2(a+\alpha)}$

The max. net revenue is given by

$$\pi = -(a+\alpha)\frac{(\beta-b-t)^2}{4(a+\alpha)^2}-(b-\beta)\frac{(\beta-b-t)}{2(a+\alpha)}-c-t\frac{(\beta-b-t)}{2(a+\alpha)}$$

$$= -\frac{(\beta-b-t)^2}{4(a+\alpha)} - \frac{(\beta-b-t)}{2(a+\alpha)}(b-\beta+t) - c = -\frac{(\beta-b-t)^2}{4(a+\alpha)} + \frac{(\beta-b-t)^2}{2(a+\alpha)}$$

$$= \frac{(\beta-b-t)^2}{4(a+\alpha)} - c$$

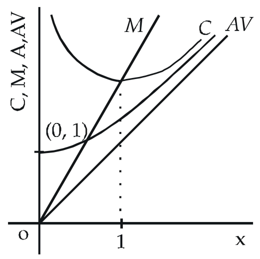
The tax revenue $T = t.x = t.\frac{(\beta-b-t)}{2(a+\alpha)} = \frac{\beta t - bt - t^2}{2(a+\alpha)}$

$\therefore \quad \frac{dT}{dt} = \frac{(\beta-b-2t)}{2(a+\alpha)} = 0,$ for max. $T \Rightarrow t = \frac{1}{2}(\beta-b)$

Second order condition

$$\frac{d^2T}{dt^2} = \frac{-2}{2(a+\alpha)} < 0 \text{ (since } a, a > 0)$$

Figure 14.1



Thus, maximum tax revenue is given by

$$T = \frac{1}{2}(\beta-b)\left[\frac{(\beta-b)-\frac{1}{2}(\beta-b)}{2(a+\alpha)}\right] = \frac{1}{8}\frac{(\beta-b)^2}{(a+\alpha)}$$

Example: A firm under non-perfect competition has the following total cost and demand functions:

$$C = 20 + 2x + 3x^2, \quad p = 50 - x$$

(i) Find the values of p and x that maximise profit.

(ii) An excise tax is imposed @ ₹ 5 per unit. Compute the profit maximising values of p and x in the post-tax situation.

(iii) Find the rate of excise tax t that would fetch maximum tax revenue to the government.

Solution:

(i) Profit $\pi(x) = 50x - x^2 - 20 - 2x - 3x^2 = -4x^2 + 48x - 20$

Now, $\pi'(x) = -8x + 48 = 0,$ for max. $\pi \Rightarrow x = 6$

Second order condition:

$\pi''(x) = -8 < 0, \;\therefore\; x = 6$ is the profit maximising output.

Notes

Also, the profit maximising price $p = 50 - 6 =$ ₹ 44.

(ii) Post-tax situation:

$$\text{Profit } \pi(x) = 50x - x^2 - 20 - 2x - 3x^2 - 5x = -4x^2 + 43x - 20$$

$$\therefore \quad \pi'(x) = -8x + 43 = 0 \Rightarrow x = \frac{43}{8} = 5.375$$

Second order condition:

$\pi''(x) = -8 < 0, \quad \therefore \; x = 5.375$ is the profit maximising output in post-tax situation. Also, price = 50 - 5.375 = ₹ 44.625

(iii) When rate of excise tax is ₹ t per unit

$$\text{Profit } \pi_t(x) = 50x - x^2 - 20 - 2x - 3x^2 - tx = -4x^2 + (48 - t)x - 20$$

$$\therefore \quad \pi_t'(x) = -8x + 48 - t = 0, \text{ for max. } \pi \Rightarrow x = \frac{48 - t}{8}$$

Now tax revenue $T = \left(\frac{48 - t}{8}\right).t = 6t - \frac{t^2}{8}$

We have $\frac{dT}{dt} = 6 - \frac{t}{4} = 0,$ for max. $T \Rightarrow t = 24$

Second order condition:

Since $\frac{d^2T}{dt^2} = -\frac{1}{4} < 0,$ hence, T is maximum when rate of excise-tax is ₹ 24 per unit.

Example: Suppose that the demand and total cost functions of a monopolist are $p = 20 - 4x$ and $C = 4x + 2$ respectively, where p is price x is quantity. If the government imposes tax @ 20% of sales, determine the total tax revenue.

Solution:

We have $p = p_s + 0.2p_s = (1 + 0.2)p_s$ or $p_s = \frac{p}{1.2}$

$$\therefore \quad TR = p_s x = \frac{p}{1.2} \cdot x = \frac{(20 - 4x)x}{1.2}$$

$$\text{Thus, profit } p(x) = \frac{(20 - 4x)x}{1.2} - 4x - 2 = \frac{(20x - 4x^2)}{1.2} - 4x - 2$$

Now $\pi'(x) = \frac{(20 - 8x)}{1.2} - 4 = 0,$ for max. π

$\Rightarrow \quad 20 - 8x = 4.8$ or $x = 15.2/8 = 1.9$

Second order condition

$$\pi''(x) = -\frac{8}{1.2} < 0, \; \therefore \; \pi \text{ is max. at } x = 1.9$$

Now, price $\quad p = 20 - 4 \times 1.9 = 12.4$ (when x = 1.9)

$\therefore \quad$ Tax revenue $T = \dfrac{20}{100} \times \dfrac{12.4}{1.2} \times 1.9 = 3.93$.

Example: Show that a monopolist with constant total cost and downward sloping demand curve will maximise his profits at a level of output where elasticity of demand is unity.

Solution:

Let $p = f(x)$ be the inverse demand function facing a monopolist and c (a constant) be his total cost.

$\therefore \quad$ profit $p(x) = x.f(x)\ c$ and $\pi'(x) = f(x) + x.f'(x) - 0 = 0$ for max. p

$\Rightarrow \quad f(x) = -x.f'(x)$ or $-\dfrac{f(x)}{xf'(x)} = 1$

Thus, h = 1, where h denotes the elasticity of demand.

Second order condition:

For max. p, we should have $\pi''(x) = 2f'(x) + xf''(x) < 0.$

$\Rightarrow \quad f''(x) < -2f'(x)/x$

Since R.H.S of the above inequality is positive, the above result will hold if either the demand curve is concave $\left(f''(x) < 0\right)$ or if convex then $f''(x) < -\dfrac{2}{x} f'(x)$.

Notes Marginal cost of a monopolist, under normal conditions of production, is always non-negative since an additional unit of a commodity can be produced only at some additional cost. Thus we shall always have $MR \geq 0$ at the profit maximising point, implying there by that a profit maximising monopolist will never have his equilibrium on any point that lies on the inelastic portion of the demand curve.

Example: Suppose that the demand facing a monopolist is $x = \alpha p^{-k}$, where $k > 1$, and his total cost function $C = ax^2 + bx + c$.

(i) Find the profit maximising output of the monopolist as $k \to 1$.

(ii) What restrictions on the constants *a, a, b* and *c* are required for the answer to be economically meaningful?

(iii) Find the supply function, if possible? Is this supply function consistent with your answer to part (i)?

Solution:

(i) $\quad$ Total revenue $TR = px = \left(\dfrac{\alpha}{x}\right)^{\frac{1}{k}}.x = \alpha^{\frac{1}{k}}.x^{\frac{k-1}{k}}$

$\therefore \quad$ Profit p $= \alpha^{\frac{1}{k}}.x^{\frac{k-1}{k}} - ax^2 - bx - c$

Notes

Differentiating w.r.t. x, we get

$$\frac{d\pi}{dx} = \alpha^{\frac{1}{k}}.\left(\frac{k-1}{k}\right)x^{-\frac{1}{k}} - 2ax - b = 0, \text{ for maximum profits}$$

Taking limit of the above as $k \to 1$, we get

$$\lim_{k\to 1}\left[\alpha^{\frac{1}{k}}\left(\frac{k-1}{k}\right).x^{-\frac{1}{k}} - 2ax - b\right] = -2ax - b = 0 \text{ or } x = -\frac{b}{2a}$$

The profit at $x = -b/2a$ will be maximum if

$$\frac{d^2\pi}{dx^2} = \lim_{k\to 1}\alpha^{\frac{1}{k}}.\frac{(k-1)}{k}\cdot\left(-\frac{1}{k}\right)x^{-\frac{1}{k}-1} - 2a = -2a < 0 \Rightarrow a > 0$$

(ii) Since x is positive in $x = \alpha p^{-k}$, $\Rightarrow \alpha > 0$.

Further, b must be negative in order that $x = -\dfrac{b}{2a} > 0$ and no restriction is needed for c.

(iii) Since the elasticity of demand is k (constant), we can find the supply function of the monopolist. The supply function is given by the condition $MR = MC$. We have

$$MR = \alpha^{\frac{1}{k}}\cdot\frac{(k-1)}{k}\cdot x^{-\frac{1}{k}} = \left(\frac{\alpha}{x}\right)^{\frac{1}{k}}.\frac{k-1}{k} = p\left(\frac{k-1}{k}\right) \text{ and } MC = 2ax + b$$

$\therefore$ $$p\left(\frac{k-1}{k}\right) = 2ax + b \text{ or } 2ax = p\left(\frac{k-1}{k}\right) - b$$

or $$x = p\left(\frac{k-1}{k}\right)\cdot\frac{1}{2a} - \frac{b}{2a} \text{ is the required supply function.}$$

Since $x = \lim\limits_{k\to 1}\left[p\left(\dfrac{k-1}{k}\right)\cdot\dfrac{1}{2a} - \dfrac{b}{2a}\right] = -\dfrac{b}{2a}$, this supply function is consistent with the answer to part (i).

Example: A monopolist with the cost function $C(x) = \frac{1}{2}x^2$ faces a demand curve $x = 12 - p$.

(i) What will be his equilibrium price and quantity?

(ii) If for some reason the firm behaves as though it were in a perfectly competitive industry, what will equilibrium price and quantity be? How much money will the firm require to forgo monopoly profits and behave competitively instead?

Solution:

(i) Total revenue $TR = px = (12 - x)x = 12x - x^2$

$\therefore$ Profit $\pi = 12x - x^2 - \frac{1}{2}x^2 = 12x - \frac{3}{2}x^2$

$$\frac{d\pi}{dx} = 12 - 3x = 0 \text{ or } x = 4 \text{ for maximum } p.$$

$$\frac{d^2\pi}{dx^2} = -3 < 0, \therefore \text{ the second order condition is satisfied.}$$

Equilibrium price $p = 12-4=8$

(ii) When the firm behaves as a perfectly competitive firm, we can write $TR = px$ where p is constant

$\therefore$ $$p = px - \frac{1}{2}x^2$$

$$\frac{d\pi}{dx} = p - x = 0 \text{ or } p = x \text{ for maximum } \pi$$

Substituting $p = 12 - x$, we get $12 - x = x$ or $x = 6$, also $p = 6$

$$\frac{d^2\pi}{dx^2} = -1 < 0, \therefore \text{ the second order condition is satisfied.}$$

Monopoly profit, $$\pi_m = 12\times 4 - \frac{3}{2}\times 16 = 24 \text{ and}$$

Profit under competition, $\pi_c = 6\times 6 - \frac{1}{2}\times 36 = 18$

Profit forgone = 24 – 18 = ₹ 6

Example: An industry consists of a number of profit maximising firms, each with a long run total cost function $C(x) = x^2 + 1$.

(a) On the same diagram sketch the total cost, average cost, average variable cost and marginal cost curves.

(b) Derive the supply function of the individual firm as a function of price (p). The industry supply X_s is the sum of the firm supplies. Obtain X_s as a function of p if there are n firms in the industry. How does the X_s curve change if the number of firms changes?

(c) The market demand is given by $X_d = 52 - p$. For given n, what will be the (short period) equilibrium price and output for the industry, and what profits will each firm earn?

(d) With free entry, the long run equilibrium price will be the lowest short-run equilibrium price compatible with non-negative profits. Determine this price and the number of firms in the industry in long run equilibrium.

Solution:

(a) (i) The total cost function is $C = x^2 + 1$ or $C - 1 = x^2$. This is a parabola with axis pointing vertically upward and vertex at (0, 1).

Alternatively, we can determine its turning point by equating $\frac{dC}{dx} = 0 \Rightarrow x = 0.$ Also C = 1 when x = 0.

Further, $\frac{d^2C}{dx^2} = 2 > 0$ $\therefore$ C has a minima at x = 0.

(ii) Average cost is $A = x + \frac{1}{x}$

To determine its turning point $\frac{dA}{dx} = 1 - \frac{1}{x^2} = 0$ or $x^2 = 1$ or $x = \pm 1$

Notes

Since $x > 0$, we consider only $x = 1$

$$\frac{d^2A}{dx^2} = \frac{2}{x^3} > 0 \text{ when } x = 1$$

Also $A = 2$ when $x = 1$

$\therefore$ A has a minima at (1, 2).

(iii) Average variable cost is $AV = x$. This is the equation of a straight line passing through origin with slope equal to unity.

(iv) Marginal cost is $M = 2x$ This is the equation of a straight line passing through origin with slope equal to 2. The diagrams of the above function are shown in Figure 14.1.

(b) The supply function of the individual firm is given by the condition $p = MC$ or $p = 2x_i$ (Note that whole of MC lies above AVC)

$\therefore$ $x_i = \frac{p}{2}$ is the supply function of an individual firm.

When there are n firms, the industry supply is

$$X_s = \sum x_i = \frac{n}{2} \cdot p$$

$\frac{dX_s}{dn} = \frac{p}{2}$, which shows that supply increases by a constant $\frac{p}{2}$ with entry of an additional firm.

(c) The condition for the short term equilibrium is $X_d = X_s$

or $52 - p = \frac{n}{2}p$ or $\left(\frac{n+2}{2}\right)p = 52$ or $p = \frac{104}{n+2}$

Further, equilibrium output is $X = \frac{n}{2} \cdot p = \frac{52n}{n+2}$

Profit of the ith firm $\pi_i = TR_i - TC_i$

$$= \frac{104}{n+2} \times \frac{52}{n+2} - \left(\frac{52}{n+2}\right)^2 - 1 = 2\left(\frac{52}{n+2}\right)^2 - \left(\frac{52}{n+2}\right)^2 - 1 = \left(\frac{52}{n+2}\right)^2 - 1$$

(d) In the long run $\pi_i = 0$

or $\frac{(52)^2}{(n+2)^2} = 1$ or $(n+2)^2 = (52)^2$ or $(n+2) = 52$

or $n = 50$ (no. of firms)

Also price $p = \frac{104}{50+2} = 2$

Example: The market demand for a good X is given by the relation $p = \beta - \alpha x$. A monopolist produces X at an average cost $ax + b$ for output x and sells to a merchant at a price p which maximises his profits. The merchant is a monopolist with constant distributive costs and maximises his profits by selling on the market at price p.

Notes

Show that the amount X produced and sold is $x = \frac{\beta - b}{2(a + 2\alpha)}$ and that $\pi = \beta - 2\alpha x$ and $p = \beta - \alpha x$.

Find the output if the producer monopolist sold directly to the market and show that "bilateral monopoly" here restricts output and raises price.

Solution:

Total revenue of the merchant = $(\beta - \alpha x)x = \beta x - \alpha x^2$

Total cost of the merchant = $\pi x + d$

(where d is distributive cost given to be constant.)

Now, profit of the merchant $p_{me} = \beta x - \alpha x^2 - \pi x - d$

We have $\frac{d(P_{me})}{dx} = \beta - 2\alpha x - \pi = 0$, for max. P_{me}

$\Rightarrow \quad \pi = \beta - 2\alpha x$ ... (1)

Second order condition

$\frac{d^2(P_{me})}{dx^2} = -2\alpha < 0$ (since $a > 0$)

The producer monopolist sells the output to the merchant at a price equal $\beta - 2\alpha x$, given by the equilibrium condition (1). Thus, $\pi = \beta - 2ax$ serves as a demand function facing the producer monopolist.

The revenue of the producer monopolist = $(\beta - 2\alpha x)x$, and his total cost = $x(ax+b)$.

$\therefore$ Profit $P_{mo} = \beta x - 2\alpha x^2 - ax^2 - bx$

Thus, $\frac{d(P_{mo})}{dx} = \beta - 4\alpha x - 2ax - b = 0$, for max. profits.

$\Rightarrow \quad x = \frac{\beta - b}{2(a + 2\alpha)}$ ... (2)

Second order condition:

$$\frac{d^2(P_{mo})}{dx^2} = -4\alpha - 2a < 0, \text{ (since } a, a > 0).$$

If the producer monopolist sold direct to the market, we can write his profit function as $P = (\beta - \alpha x)x - (ax^2 + bx)$.

$\therefore \quad \frac{dP}{dx} = \beta - 2\alpha x - 2ax - b = 0$, for max. profits.

or $\quad x = \frac{\beta - b}{2(a + \alpha)}$ ... (3)

Comparing (2) and (3), we conclude that bilateral monopoly restricts output. Since output price are inversely related by the demand function, this also implies that price is higher in bilateral monopoly.

Notes

14.5 Summary

- We can write total revenue as $TR = p.x$, where p is price and x is quantity. Total revenue will be maximum at a level of output where $\frac{d(TR)}{dx}$ = 0 (or MR = 0) and $\frac{d^2(TR)}{dx^2}$ < 0. The first order condition implies that $\frac{d(TR)}{dx}$ = $p + x\frac{dp}{dx} = 0$ or $\frac{p}{x}\cdot\frac{dx}{dp} = -1$ i.e. h = 1. Thus maxima of total revenue occurs at a level of output where elasticity of demand is unity.
- Let $p = f(x)$ and $p = g(x)$ be the market demand and supply of a commodity and a specific tax of ₹ t per unit be imposed. Then under equilibrium, we can write $f(x) = g(x) + t$.
- Let x_t be the equilibrium quantity obtained by solving the above equation for x. We can write the expression for tax revenue T as $T = t.x_t$ (note that x_t is a function of t).
- The average product of labour is $AP_L = \frac{TP_L}{L} = \frac{f(L)}{L}$, the marginal product of labour is MP_L $= \frac{dx}{dL} = f'(L)$ and necessary condition for maximum output is $\frac{d(TP_L)}{dL} = \frac{dx}{dL} = MP_L = 0$
- If total cost $C = F(x)$, then we can define $AC = \frac{C}{x} = \frac{F(x)}{x}$, and $MC = \frac{dC}{dx} = F'(x)$.

14.6 Keywords

Derivative: The rate at which a function changes with respect to its independent variable. Geometrically, this is equivalent to the slope of the tangent to the graph of the function.

Domain: The set, or collection, of all the first elements of the ordered pairs of a function is called the domain of the function.

Function: A set of ordered pairs. It results from pairing the elements of one set with those of another, based on a specific relationship. The statement of the relationship is often expressed in the form of an equation.

Range: The set containing all the values of the function.

14.7 Self Assessment

1. Total number of parallel tangents of $f_1(x) = x^2 - x + 1$ and $f_2(x) = x^3 - x^2 - 2x + 1$ is equal to

   (a) 2

   (b) 3

   (c) 4

   (d) None of these

2. The function $2\tan^3 x - 3\tan^2 x + 12\tan x + 3$, $x \in$ is

   (a) increasing

   (b) decreasing

Notes

(c) increasing in (0, p/4) and decreasing in (p/4, p/2)

(d) none of these

3. Let $f(x) = (4-x^2)^{2/3}$, then f has a

(a) a local maxima at $x = 0$

(b) a local maxima at $x = 2$

(c) a local maxima at $x = -2$

(d) none of these

4. Let $f(x) = x^3 - 6x^2 + 9x + 18$, then f (x) is strictly decreasing in

(a) $(-\infty, 1]$

(b) $[3, \infty)$

(c) $(-\infty, 1] \cup [3, \infty)$

(d) $[1, 3]$

5. The absolute minimum value of $x^4 - x^2 - 2x + 5$

(a) is equal to 5

(b) is equal to 3

(c) is equal to 7

(d) does not exist

6. Equation of the tangent to the curve $y = e^{-|x|}$ at the point where it cuts the line x=1

(a) is $ey + x = 2$

(b) is $x + y = e$

(c) is $ex + y = 1$

(d) does not exist

7. Rolle's theorem holds for the function $x^3 + bx^2 + cx$, $1 \leq x \leq 2$ at the point 4/3, the value of b and c are;

(a) b = 8, c = - 5

(b) b = -5, c = 8

(c) b = 5, c = -8

(d) b = -5, c = -8.

8. The number of value of k for which the equation $x^3 - 3x + k = 0$ has two different roots lying in the interval (0, 1) are

(a) 3

(b) 2

(c) infinitely many

(d) no value of k satisfies the requirement.

Notes

9. From mean value theorem: f(b) – f(a) = (b – a) f′ (x_1); a < x_1 < b if f(x) = 1/x , then x_1 =
   (a) √ab
   (b) a + b/2
   (c) 2ab/q + b
   (d) b – a/b + a
10. The minimum value of ax + by, where xy = r^2, is (r, ab >0)
   (a) 2r √ab
   (b) 2ab √r
   (c) –2r √ab
   (d) None of these

14.8 Review Questions

1. The demand function of a particular commodity is given by $y = 15e^{-x/3}$ for $0 \le x \le 8$ where *y* is price per unit and *x* is the number of units demanded. Determine the price and quantity for which revenue is maximum.
2. Total revenue from the sale of a good X is given by the equation $R = 60Q - Q^2$, for $0 \le Q \le 60$, where *R* is total revenue and *Q* is the quantity sold at price *P*. Calculate the value of *MR* when the point price elasticity of demand is 2.
3. From the demand function $Q = 600/P$, show that total expenditure on the commodity remains unchanged as price falls. Estimate the elasticity of demand at *P* = ₹ 4 and at *P* = ₹ 2.
4. Following are the market demand and market supply equations for a product X:
   $Q_d = 10,000(12 - 2P)$ and $Q_s = 1,000(20P)$
   The government decides to collect a sales tax of ₹ 2.50 per unit sold.
   (a) What effect does it have on the equilibrium price and quantity of commodity X?
   (b) Find total amount of tax collected by the government.
   (c) If the government wants to maximise total tax collections, find the rate of specific tax it should impose.
5. Show that for a competitive market with linear demand and supply functions, the imposition of a specific tax increases the demand price and decreases the supply price by less than the tax rate.
6. Let $x_d = a - bp$ and $x_s = a + bp$ be the demand and supply functions respectively of a good in perfectly competitive market. A specific tax of ₹ *t* per unit is imposed. Find equilibrium price paid by consumer $(\bar{p})$ and that received by seller $(\bar{p}_s)$. Find $\frac{d\bar{p}}{dt}$ and $\frac{d\bar{p}_s}{dt}$ and show that equilibrium the price paid by consumer increases and that received by seller decreases as tax rate is increased.
7. A cultural organisation is arranging a kathakali dance program in a city. It expects that 300 persons would attend the show if the entrance ticket is ₹ 8. It has also estimated that for a unit decrease in entrance fee, 60 additional persons would attend the program. Express the revenue of the organisation as a function of the entrance fee. What should be the entrance fee so that the organisation gets maximum revenue?

Notes

8. A tour operator charges ₹ 136 per passenger upto 100 passengers with a discount of ₹ 4 for each 10 passenger in excess of 100. Determine the number of passengers that will maximise the amount of money the tour operator receives.

9. For a firm under perfect competition, the total cost function is given by $C = \frac{1}{25}q^3 - \frac{9}{10}q^2 + 10q + 12$. If the price of output is ₹ 4 per unit, will the firm continue production?

10. A firm has a revenue function given by $R = 10x$, where R is gross revenue and x is the quantity sold; and a production cost function given by $C = 1,00,000 + 50\left(\frac{x}{1,000}\right)^2$. What is the expression for the profit function $p = R - C$? Find the rate of change of p w.r.t. x at x = 1,00,000 units.

Answers: Self Assessment

1.	(d)	2.	(a)
3.	(a)	4.	(d)
5.	(b)	6.	(a)
7.	(d)	8.	(c)
9.	(a)	10.	(a)

14.9 Further Readings

Books

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