Real Analysis

DMTH401

LOVELY PROFESSIONAL UNIVERSITY

REAL ANALYSIS

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SYLLABUS

Real Analysis

Objectives:

- To allows an appreciation of the many interconnections between areas of mathematics.
- To learn about the countability of sets, metric space, continuity, discontinuities, connectedness and compactness for set of real numbers.

Sr. No.	Content
1	Set Theory Finite, Countable and Uncountable Sets, Metric spaces ;Definition and examples
2	Compactness of k-cells and Compact Subsets of Euclidean, Space R^k, Perfect sets and Cantor's set, Connected sets in a metric space, Connected subset of Real line
3	Sequences I Metric Spaces, Convergent sequences and Subsequences, Cauchy sequence, complete metric space, Cantor's intersection theorem and Baire's Theorem, Branch contraction Principle.
4	Limit of functions, continuous functions, Continuity and compactness, continuity and connectedness, Discontinuities and Monotonic functions
5	Sequences and series; Uniform convergence, Uniform convergence and continuity, Uniform convergence and integration
6	Uniform convergence and differentiation, Equi-continuous families of functions, Arzela's Theorem and Weierstrass Approximation Theorem
7	Reimann Stieltje's integral , Definition and existence of integral, Properties of integration ,R-S integral as a limit of sum
8	Differentiation and integration, fundamental Theorem of Calculus, Mean value Theorems .
9	Lebesgue Measure ;Outer Measure , Measurable sets and Lebesgue measure, A non measurable set, Measurable functions, Littlewood's three principles
10	The Lebesgue Integral of bounded functions, Comparison of Riemann and Lebesgue Integrals,The integral of a non-negative function, General Lebesgue integral, Convergence of measure .

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Unit 1: Sets and Numbers

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Objectives

After studying this unit, you will be able to:

- Discuss the basic concepts of sets and functions
- Explain the system of real numbers
- Describe the representation of real numbers

Introduction

As we know that one of the main features of Mathematics is the identification of the subject matter, its analysis and its presentation in a satisfactory manner. In other words, the language should be a vehicle which carries ideas through the mind without affecting their meaning in any way. Set Theory comes closest to being such a language. Introduced between 1873 and 1895 by a famous German mathematician, George Cantor (1845-1918), Set Theory became the foundation of almost all the branches of Mathematics. Besides its universal appeal, it is quite amazing in its simplicity and elegance.

A rigorous presentation of Set Theory is not the purpose here because we believe that you are already familiar with it. We shall briefly recall some of its basic concepts which are essential for a systematic study of Real Analysis. Closely linked with the sets, is the notion of a function, which also you have learnt in your previous studies. In this unit, we shall review this as well as other related concepts which are necessary for our discussion.

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'Real Analysis' is an important branch of Mathematics which mainly deals with the study of real numbers. What is, then, the system of the real numbers? We shall try to find an answer to this question as well as some other related questions in this unit. Also, we shall give the geometrical representation of the real numbers.

1.1 Sets and Functions

As you all know modern Mathematics is based on the ideas that are expressed in the language of sets and functions. Here you set knowledge of certain basic concepts of Set Theory which are quite familiar to you. These concepts will also serve an important purpose of recalling certain notations and terms that will be used throughout our discussion with you.

1.1.1 Sets

As you are used to the phrases like the 'team' of cricket players, the 'army' of a country, the 'committee' on the education policy, the 'panchayat' of a village, etc. The terms 'team', 'army', 'committee', panchayat', etc., indicate the notion of a 'collection' or 'totality' or 'aggregate' of objects. These are well-known examples of a set.

Therefore, our starting point is an informal description of the term 'set'. A set is treated as an undefined term just as a point in Geometry is undefined. However, for our purpose we say that a set is a well-defined collection of objects. A collection os well-defined of it is possible to say whether a given object belongs to the collection or not.

The following are some examples of sets:

1. The collection of the students registered in Excel Books.
2. The collection of the planets namely Jupiter, Saturn, Earth, Pluto, Venus, Mercury, Mars, Uranus and Neptune.
3. The collection of all the countries in the world. (Do you know how many countries are there in the world?)
4. The collection of numbers, 1, 2, 3, 4, ……

If we consider the collection of tall persons or beautiful ladies or popular leaders, then these collections are not well-defined and hence none of them forms a set. The reason is that the words 'tal' 'beautiful' or 'popular' are not well-defined. The objects constituting a set are called its elements or members or points of the set. Generally, sets are denoted by the capital letters A, B, C etc. and the elements are denoted by the small letters a, b, c etc. If S is any set and x is an element of S, we express it by writing that $x \in S$, where the symbol $\in$ means 'belongs to' or 'is a member of'. If x is not an element an element of a set S, we write $x \notin S$. For example , if S is the set containing 1, 2, 3, 4 only, then $2 \in S$ and $5 \in S$.

You know that there are two method of describing a set. One is known as the Tabular method and the other is the Set-Builder method. In the tabular method we describe a set by actually listing all the elements belonging to it.

Example: If S is the set consisting of all small letters of English alphabet, then we write

$$S = \{a, b, c, ..., x, y, z\}.$$

If N is the set of all natural numbers, then we write

$$N = \{1, 2, 3....\}.$$

Notes

This is also called an explicit representation of a set.

In the set-builder method, a set is described by stating the property which determines the set as a well-defined collection. Suppose p denotes this property and x is an element of a set S. Then

$$S = \{x: x \text{ satisfies } p).$$

Example: The two sets S and N can be written as

S = {x: x is a small letter of English alphabet}

N = {n: n is a natural number).

This is also called an implicit representation of a set.

Note that in the representation of sets, the elements of a set are not repeated. Also, the elements may be listed in any manner.

Example: Write the set S whose elements are all natural numbers between 7 and 12 including both 7 and 12 in the tabular as well as in the set-builder forms.

Solution: Tabular form is S = {7, 8, 9, 10, 11, 12, }.

Set-builder form is $S = \{ n \in N: 7 \leq n \leq 12, \}$.

The following standard notations are used for the sets of numbers:

N = Set of all natural numbers

= {1, 2, 3....}

= {n:n is a natural number)

= Set of all positive integers.

Z = Set of all integers

= { ....–3, –2, –1, 0, 1, 2, 3, ...}

= {p:p is an integer).

Q = Set of all rational numbers

$$= \{x : x = \frac{P}{q}, p \in Z, q \in Z, q \neq 0).$$

R = Set of real numbers

= {x : x is a real number).

We shall, however, discuss the development of the system of real numbers.

A set is said to be finite if it has a finite number of elements. A set is said to be infinite if it is not finite. We shall, however, give a mathematical definition of finite and infinite sets in Unit 2.

Note that an element of a set must be carefully distinguished from the set consisting of this element. Thus, for instance, you must distinguish

$$x, \{x\}, \{\{x\}\}$$

from each other

We talk of equality of numbers, equality of objects, etc.

The question, therefore, arises: What is the notion of the equality of sets?

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Definition 1: Equality of Sets

Any two sets are equal if that are identical. Thus the two sets S and T are equal, written as S = T if they consist of exactly the same elements. When the two sets S and T are unequal, we write

$$S \neq T.$$

It follows from the definition that S = T if any one of $x \in S$ implies $x \in T$ and $y \in T$ implies $y \in S$. Also S is different from T (S +T) if there is at least one element in one of them which is not in the other.

If every member of a given set S is also a member of T, then we say that S is a subset of T or "S is contained in T" and write:

$$S \subset T$$

or equivalently we say that "T contains S" or T is a superset of S, and write

$$T \supset S$$

The relation

$$S \not\subset T$$

means that S is not a subset of T i.e. there is at least one element in T which is not in S.

Thus, you can easily see that any two sets S and T are equal if and only if S is a subset of T and T is a subset of S i.e.

$$S = T \Leftrightarrow S \subset T \text{ and } T \subset S.$$

If $S \subset T$ but $T \not\subset S$, then we say that S is a proper subset of T. Note that $S \subset S$ i.e. every set is a subset of itself.

Another important concept is that of a set having no elements. Such a set, as you know, is called an empty set or a null set or a void set and is denoted by O.

You can easily see that there is only one empty set i.e. O is unique. Further O is a subset of every set.

Now why don't you try an exercise?

Task **Justify the following statements:**

1. The set N is a proper subset of Z.
2. The set R is not a subset of Q.
3. If A, B, C are any three sets such that $A \subset B$, and $B \subset C$, then $A \subset C$.

So far, we have talked about the elements and subsets of a given set. Let us now recall the method of constructing new sets from the given sets.

While studying subsets, we generally fix a set and consider the subsets of this set throughout our discussion. This set is usually called the Universal set. This Universal set may vary from situations to situations. For example, when we consider the subsets of R, then R is the Universal set. When we consider the set of points in the Euclidean plane, then the set of all points in the Euclidean plane is the Universal set. We shall denote the Universal set by X.

Now, suppose that the Universal set X is given as

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$$X = \{1, 2, 3, 4, \ \}$$

and

$$S = \{1, 2, 3\}$$

is a subset of X. Consider a subset of X whose elements do not belong to S. This set is (4, 2}.

Such, a set, as you know is called the complement of S.

We define the complement of a set as follows:

Definition 2: Complement of a Set

Let X be the Universal set and S be a subset of X. The complement of the set S is the set of all those elements of the Universal set X which do not belong to S. It is denoted by S.

Thus, if S is an arbitrary set contained in the Universal Set X, then the complement of S is the set

$$S^c = \{x : x \notin S\}.$$

Associated with each set S is the Power set P(S) of S consisting of all the subsets of S. It is written as

$$P(S) = \{A : A \subset S).$$

Now try the following exercise.

Let us consider the sets S and T given as

$$S = \{1, 2, 3, 4, 5\}, T = \{3, 4, 5, 6, 7\}.$$

Construct a new set {1, 2, 3, 4, 5, 6, 7). Note that all the elements of this set have been taken from S or T such that no element of S and T is left out. This new set is called the union of the sets S and T and is denoted by $S \cup T$.

Thus

$$S \cup T = \{1, 2, 3, 4, 5, 6, 7).$$

Again let us construct another set {3, 4, 5). This set consists of the elements that are common to both S and T i.e. a set whose elements are in both S and T. This set is called the intersection of S and T. It is denoted by $S \cap T$. Thus

$$S \cap T = \{3, 4, 5).$$

These notions of Union and Intersection of 'two sets' can be generalized for any sets in the following way: Note, all the sets under discussion will be treated as subsets of the Universal set X.

Definition 3: Union of Sets

Let S and T be given sets. The collection of all elements which belong to S or T is called the union of S and T. It is expressed as

$$S \cup T = \{x : x \in S \text{ or } x \in T\}.$$

Note that when we say that $x \in S$ or $x \in T$, then it means that x belong to S or x belong to T or x belong to both S and T.

Definition 4: Intersection of Sets

The intersection $S \cap T$ of the sets S and T is defined to be the set of all those elements which belong to both S and T i.e.

$$S \cap T = \{x : x \in S \text{ and } x \in T\}.$$

Notes

Note that the sets are disjoint or mutually exclusive when $S \cap T = 0$ i.e., when their intersection is empty.

You can now verify (or even prove) by means of examples the following laws of union and intersection of sets given in the next exercise.

Also, you can easily see that

$$A \cup A = A, A \cap A = A, A \cup \varnothing = A, A \cup \varnothing = \varnothing.$$

Given any two sets S and T, we can construct a new set in such a way that it contains only those elements of one of the sets which do not belong to the other. Such a set is called the difference of the given sets. There will be two such sets denoted by S—T and T—S. For example, let

$$S = \{2, 4, 8, 10, 11\}, T = \{1, 2, 3.4).$$

Then

$$S - T = \{8, 10, 11), T - S = \{1, 3).$$

Thus, we can define the difference of two sets in the following way.

Definition 5: Difference of two Sets

Given two sets S and T, the difference S – T is a set consisting of precisely those members of S which are not in T.

Thus

$$S - T = \{x : x \in S \text{ and } x \notin T).$$

Similarly, we can define T – S.

Consider a collection of sets S_1, where i varies over some index set J. This simply means that to each element $i \in J$, there is a corresponding set S_i. For example, the collection $\{S_1, S_2, S_3, ...)$ could be expressed as $\{S_i\}_{i \in N}$, where N is the index set.

With the introduction of an index set, the notions of the union and the intersection of sets can be extended to an arbitrary collection of sets. For example,

(i) $\bigcup_{i \in J} S_i = \bigcup_{i \in J} \{x : x \in S_i \text{ for at least one } i \in J\}.$

(ii) $\bigcap_{i \in J} S_i = \bigcap_{i \in J} \{x : x \in S_i \text{ for all } i \in J\}.$

(iii) $(\bigcup S_i)^c_{i \in J} = \bigcap_{i \in J} S_i^c.$

1.1.2 Functions

Let S be the set of Excel Books and let N be the set of all natural numbers. Assign to each book the number of pages the book contains. Here each book corresponds to a unique natural number. In other words, there is a correspondence between the books and the natural numbers, i.e., there is a rule or a mechanism by which we can associate to each book one and only one natural number. Such a rule or correspondence is named as a function or a mapping.

Definition 6: Function

Let S and T be any two non-empty sets. A function f from S to T denoted as $f: S \to T$ is a rule which assigns to each element of the set S, a unique element in the set T.

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The set S is called the domain of the function f and T is called its co-domain. If an elements x in S corresponds to an element y in T under the function f, then y is called the image of x under f. This is expressed by writing $y = f(x)$. The set $\{f(x): x \in S\}$ which is a subset of T is called the range of f. If range of f = co-domain of f, then f is called onto or surjective function; otherwise f is called an into function.

Thus, a function $f: S \to T$ is said to be onto if the range of S is equal to its co-domain T.

Suppose $S = \{1, 2, 3, 4)$ and $T = \{1, 2, 3, 4, 5, 6)$ and $f: S \to T$ is defined by $f(n) = n+l,\ \forall\ n \in S$. Then the range of $f = \{2, 3, 4, 5)$. This shows that f is an into function. On the other hand, if $S = \{1, 2, 3, 4\}$, $T = \{1, 4, 9, 16\}$ and if $f: S \to T$ is defined by $f(n) = n^{Z}$, then f is onto. You can verify that the range of f is, in fact, equal to T.

Refer back to the example on the books in Excel Books. It just possible that two books may have the same number of pages. If it is so, then under this function, two different books shall have the same natural number as their image. However if for a function any two distinct elements in the domain have distinct images in the co-domain, then the function is called one-one or injective.

Thus a function f is said to be one-one if distinct elements in the domain of f have distinct image or in other words, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, for any x_1, x_2 in the domain of f.

A function which is one-one and onto, is called a bijection or a 1-1 correspondence.

Example:

(i) Let $S = \{1, 2, 3)$ and $T = \{a, b, c\}$ and let $f: S \to T$ be defined as $f(1) = a$, $f(2) = b$, $f(3) = c$. Then f is one-one and onto.

(ii) Let $N = \{1, 2, 3, 4, ...\}$ and $f: N \to N$ be defined as $f(n) = n+1$. As 1 does not belong to the range of f, therefore f is not onto. However, f is one-one.

(iii) Let $S = (1, -1, 2, 3, -3)$ and let $T = (1, 4, 9)$. Define $f: S \to T$ by $f(n) = n^2\ \forall\ n \in S$. Then f is not one-one as $f(1) = f(-l) = 1$. However, f is onto.

Definition 7: Identity Function

Let S be any non-empty set. A function $f: S \to S$ defined by $f(x) = x$ for each x in S is called the identity function.

It is generally denoted by I_s. It is easy to see that I_s is one-one and onto.

Definition 8: Constant Function

Let S and T be any two non-empty sets. A function $f: S \to T$ defined by $f(x) = c$, for each x in S, where c is fixed element of T, is called a constant function.

For example $f: S \to R$ defined as $f(x)=2$, for every x in S, is a constant function. Is this function one-one and onto? Verify it.

Definition 9: Equality of Functions

Any two functions with the same domain are said to be equal if for each point of their domain, they have the same image. Thus if f and g are any two functions defined on an non-empty set S, then

$$f = g \text{ if } f(x) = g(x),\ \forall\ x \in S.$$

In other words, f = g if f and g are identical.

Let us now discuss another important concept in this section. This is about the composition or combination of two function. Consider the following situation:

Let S = {1, 2, 3, 4}, T = {1, 4, 9, 16}, N = {1, 2, 3, 4....} be any three sets Let f: S→T be defined by $f(x) = x^2$, $\forall\ x \in S$ and g: T→N be defined by g(x) = x +1, $\forall\ x \in T$. Then, by the function f, an element $x \in S$ is mapped to $f(x) = x^2$. Further by the function g the element f(x) is mapped to $f(x) + 1 = x^2 + 1$. Denote this as g(f(x)). Define a function h: S→N by h(x) = g(f(x)). This function h maps each x in S to some unique elements $g(f(x)) = x^2 + 1$ of N. The function h is called the composition or the composite of the functions f and g. Thus, we have the following definition:

Definition 10: Composite of Functions

Let f: S→T and g: T→V be any two functions. A function h: S→V denoted as h = gof and defined by

$$h(x) = (gof)\ (x) = g(f(x)),\ \ \forall\ x \in S$$

is called the composite of f and g.

Note that the domain of the composite function is the set S and its co-domain is the set V. The set T which contains the range of f is equal to the domain of g.

But in general, the composition of the two functions is meaningful whenever the range of the first is contained the domain of the second.

Example: Let S = T = {1, 2, 3, 4...}, Define

f(x) = 2x and g(x) = x + 5. Then

"gof is defined as (gof) (x) = g(f(x)) = g(2x) = 2x + 5.

Note that we can also define fog the composite of g and f. Here (fog) (x) = f(g(x)) = f(x + 5) = 2 (x + 5) = 2x = 10, Also (fog) (1) = 12 and (gof) (1) = 7. This shows that 'fog' need not be equal to 'gof'.

Let S = {1, 2, 3} and T = {a, b, c). Let f: S→T be f(1) = a, f(2) = b, f(3) = c. Define a function g: T→S as g(a) = 1, g(b) = 2 and g(c) = 3. Under the function g, the element f(x) in T is taken back to the element x in S. This mapping g is called the inverse of f and is given by g(f(x)) = x, for each in S. You may note that f(g(a)) = a, f(g(b)) = b and f(g(c)) = c. Thus, we have the following definition:

Definition 11: Inverse of a Function

Let S and T be two non-empty sets. A function f: S→T is said to be invertible if there exists a function g: T→S such that

$$(gof)(x) = x \text{ for each } x \text{ in } S,$$

and

$$(fog)(x) = x \text{ for each } x \text{ in } T.$$

In this case, g is said to be the inverse of f and we write it as $g = f^{-1}$.

Did u know? **Do all function possess inverses?**

No, all functions do not possess inverses. For example, let S = {1, 2, 3} and T = {a, b). If f: S→T is defined as f(1) = f(2) = a and f(3) = b, then f is not invertible. For, if g: S→T is inverse of f, then

(gof) (1) = g(f(l)) = g(a)

and (gof) (2) = g(f(2)) = g(a).

Therefore, 1 = 2, which is absurd.

This raises another question: Under what conditions a function is as an inverse? If a function f: $S \to T$ is one-one and onto, then it is invertible * conversely, if f is invertible, then f is both one-one and onto. Thus if a function is one-one and onto, then it must have an Inverse.

1.2 System of Real Numbers

You are quite familiar with some number systems and some of their properties. You will, perhaps recall the following properties:

(i) Any number multiplied by zero is equal to zero,

(ii) The product of a positive number with a negative number is negative,

(iii) The product of a negative number with a negative number is positive among takers.

To illustrate these properties, you will most likely use the natural numbers or integers or even rational numbers. The questions, which begin to arise are: What are these various types of numbers? What properties characterise the distinction between these various sets of numbers?

In this section, we shall try to provide answers to these and many other related questions. Since we are dealing with the course on Real Analysis, therefore we confine our discussion to the system of real numbers. Nevertheless, we shall make you peep into the realm of a still larger class of numbers, the so called complex numbers.

The system of real numbers has been evolved in different ways by different mathematicians. In the late 19th Century, the two famous German mathematicians Richard Dedekind [1815-1897] and George Cantor [1845-1918] gave two independent approaches for the construction of real numbers. During the same time, an Italian mathematician, G. Peano [1858-1932] defined the natural numbers by the well-known Peano Axioms. However, we start with the set of natural numbers as the foundation and build up the integers. From integers, we construct the rational numbers and then through the set of rational numbers, we reach the stage of real numbers. This development of number system culminates into the set of complex numbers. A detailed study of the system of numbers leads us to a beautiful branch of Mathematics namely. *The Number Theory*, which is beyond the scope of this course. However, we shall outline the general development of the system of the real numbers in this section. This is crucial to understand the characterization of the real numbers in terms of the algebraic structure to be discussed in Unit 2. Let us start our discussion with the natural numbers.

1.2.1 Natural Numbers

The notion of a number and its counting is so old that it is difficult to trace its origin. It developed much before the time of even the recorded history that its manner of development is based on conjectures and guesses. The mankind, even in the most primitive times, had some number sense. The man, at least, had the sense of recognizing 'more' and 'less', when some objects were added to or taken out from a small collection. Studies have shown that even some animals possess such a sense. With the gradual evolution of society, simple counting became imperative. A tribe had to count how many members it had, how many enemies and how many friends. A shepherd or a cowboy found it necessary to know if his flock of sheep or cows was decreasing or increasing in size. Various ways were evolved to keep such a count. Stones, pebbles, scratches on the ground, notches on a big piece of wood, small sticks, knots in a string or the fingers of hands were used for this purpose. As a result of several refinements of these counting methods, the numbers were expressed in the written symbols of various types called the digits. These digits were written differently according to the different languages and cultures of the societies. In the ultimate development, the numbers denoted by the digits 1, 2, 3, ... became universally acceptable and were named as natural numbers.

Notes

Different theories have been advanced about the origin and evolution of natural numbers. An axiomatic approach, as evolved by G. Peano, is often used to define the natural numbers. Some mathematicians like L. Kronecker [1823-1891] have remarked that the natural numbers are a creation of God while all else is the work of man.

However, we shall not go into the origin of the natural numbers. In fact, we accept that the natural numbers are a gift of nature to the mankind.

We denote the set of all natural numbers as

$$N = \{1,2, 3, ...\}.$$

One of the basic properties of these numbers is that there is a starting number 1. Then for each number there is a next number. This nextness property is an important idea that you may find fascinating with the natural numbers. You may think of any big natural number. Yet, you can always tell its next number. What's the next number after forty nine? After seventy seven? After one hundred twenty three? After three thousand and ninety nine? Thus you have an endless chain of natural numbers.

Some of the basic properties of the natural numbers are concerning the well-known fundamental operations of addition, multiplication, subtraction and division. You know that the symbol '+' is used for addition and the symbol 'x' is used for multiplication. If we add or multiply any two natural numbers, we again get natural numbers. We express it by saying that the set of natural numbers is closed with respect to these operations.

However, if you subtract 2 from 2, then what you get is not a natural number. It is a number which we call zero denoted as '0'. The word, zero, in fact is a translation of the Sanskrit 'shunya'. It is universally accepted that the concept of the number zero was given by the ancient Hindu mathematicians. You come across with certain concrete situations indicating the meaning of zero. For example, the temperature of zero degree is certainly not an absence of temperature.

After having fixed the idea of the number zero, it should not be difficult for you to understand the notion of negative natural numbers. You must have heard the weather experts saying that the temperature on the top of the hills is minus 5 degrees written as $-5°$. What does it mean? The simple and straight explanation is that -5 is the negative of 5 i.e. -5 is a number such that $5 + (-5) = 0$. Hence -5 is a negative natural number. Thus for each natural n, there is a unique number $-n$, called the negative of n such that

$$n + (-n) = 0.$$

1.2.2 Integers

You have seen that in the set N of natural numbers, if we subtract 2 from 2 or 3 from 2, we do not get back natural numbers. Thus set of natural numbers is not closed with respect to the operation of subtraction. After the operation of subtraction is introduced, the need to include 0 and negative numbers becomes apparent. To make this operation valid, we must enlarge the system of natural numbers, by including in it the number 0 and all the negative natural numbers. This enlarged set consisting of all the natural numbers, zero and the negatives of natural numbers, is called the set of integers. It is denoted as

$$Z = \{.... -3, -2, -1, 0, 1, 2, 3 ....\}.$$

Now you can easily verify that the set of integers is closed with respect to the operations of addition, multiplication and subtraction.

The integers 1, 2, 3 .... are also called positive integers which are in fact natural numbers. The integers $-1, -2, -3,....$ are called negative integers which are actually the negative natural numbers.

Notes

The number 0, however, is neither a positive integer nor a negative one. The set consisting of all the positive integers and 0 is called the set of non-negative integers. Similarly we talk of the set of non-positive integers. Can you describe it?

1.2.3 Rational Numbers

If you add or multiply the integers 2 and 3, then the result is, of course, an integer in each case. Also if you subtract 2 from 2 or 2 from 3, the result once again in each case, is an integer. What do you get, when you divide 2 by 3? Obviously, the result is not an integer. Thus if you divide an integer by a non-zero integer, you may not get an integer always. You may get the numbers of the form

$\frac{1}{2}, \frac{1}{3}, \frac{-2}{3}, \frac{-4}{5}, \frac{5}{6} \ldots$ so on.

Such numbers are called rational numbers.

Thus the set Z of integers is inadequate when the operation of division is introduced. Therefore, we enlarge the set Z to that of all rational numbers. Accordingly, we get a bigger set which includes all integers and in which division by non-zero integers is possible. Such a set is called the set of rational numbers. Thus a rational number is a number of the form $\frac{p}{q}$, $q \neq 0$, where p and q are integers. We shall denote the set of all rational numbers by Q. Thus,

$$Q = \{x = \frac{p}{q},\ P \in Z, q \in Z, q \neq 0).$$

Now if you add or multiply any two rational numbers you again get a rational number. Also if you subtract one rational number from another or if you divide one rational number by a non-zero rational, you again get a rational numbers in each case. Thus the set Q of rational numbers looks to be a highly satisfactory system of numbers in the sense that the basic operations of addition, multiplication, subtraction and division are defined on it. However, Q is also inadequate in many ways. Let us now examine this aspect of Q.

Consider the equation $x^2 = 2$. We shall show that there is no rational number which satisfies this equation. In other words, we have to show that there is no rational number whose square is 2. We discuss this question in the form of the following example:

Example: Prove that there is no rational number whose square is 2.

Solution: If possible, suppose that there is a rational number x such that $x^2 = 2$. Since x is a rational number, therefore x must be of the form

$$x = \frac{p}{q},\ p \in Z, q \in Z, q \neq 0.$$

Note that the integers p and q may or may not have a common factor. We assume that p and q have no common factor except 1.

Squaring both sides, we get

$$\frac{p^2}{q^2} = 2.$$

Then we have

$$p^2 = 2q^2.$$

Notes

This means that p^2 is even and hence p is even (verify it). Therefore, we can write p = 2k for some integer k. Accordingly, we will have

$$p_2 = 4k^2 = 2q_2$$

or

$$q^2 = 2k^2.$$

Thus p and q are both even. In other words, p and q have 2 as a common factor. This contradicts our supposition that p and q have no common factor.

Hence there is no rational number whose square is 2.

Why don't you try the following similar exercises?

Thus you have seen that there are numbers which are not rationals. Such numbers are called irrational. In other words, a number is irrational if it cannot be expressed as p/q, $p \in Z$, $q \in Z$, $q \neq 0$. In this way, $\sqrt{2}, \sqrt{3}, \sqrt{5},$ etc. are irrational numbers. In fact, such numbers are infinite. Rather, you will see in Unit 2 that such numbers are even uncountable. Also you should not conclude that all irrational numbers can be obtained in this way. For example, the irrational numbers e and π are not of this form. We denote by I, the set of all irrational numbers.

Thus, we have seen that the set Q is inadequate in the sense that there are number which are not rationals.

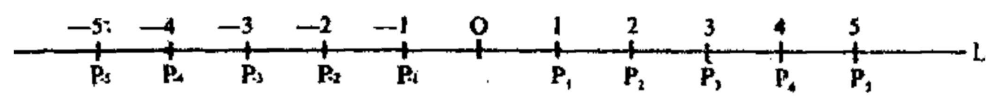
A number which is either rational or irrational is called a real number. The set of real numbers is denoted by R. Thus the set R is the disjoint union of the sets of rational and irrational numbers i.e. $R = Q \cup I$, $Q \cap I = O$.

Now in order to visualise a clear picture of the relationship between the rationals and irrationals, their geometrical representation as points on a line is of great help. We discuss this in the next section.

1.3 The Real Line

Draw a straight line L as shown in the Figure 1.1.

Figure 1.1

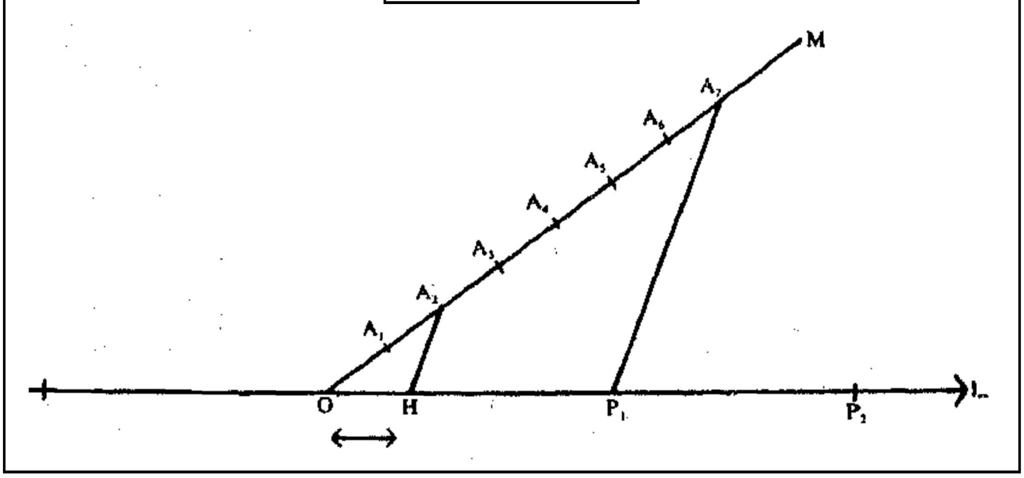


Choose a point O on L and another point P, to the right of O. Associate the number O (zero) to the point O and the number 1 to the point P_1. We take the distance between the points P and P_1 as a unit length. We mark a succession of points P_2, P_3, …… to the right of P_1 each at a unit distance from the preceding one. Then associate the integers 2, 3, ... to the points P_2, P_3, ..., respectively. Similarly, mark the points P_{-1}, P_{-2},..., to the left of the point O, Associate the integers –1, –2,… to the points P_{-1}, P_{-2},…. Thus corresponding to each integer, we have associated a unique point of the line L.

Now associate every rational number to a unique point of L. Suppose you want to associate the rational number $\frac{2}{7}$ to a point on the line L. Then $\frac{2}{7} = 2 \times \frac{1}{7}$ i.e., one unit is divided into seven parts, out of which 2 are to be taken. Let us see how you do it geometrically.

Notes

Figure 1.2



Through O, draw a line OM inclined to the line L. Mark the points A, A,..., A_7 on the line OM at equal distances. Join P_1A,. Now if you draw a line through A, parallel to P_1P_7 to meet the line L in H. Then H corresponds to the rational number $\frac{2}{7}$ i.e., OH = $\frac{2}{7}$.

You can do likewise for a negative rational number. Such points, then, will be to the left of O.

By having any line through O, you can see that the point P does not depend upon chosen line OM. Thus, you have associated every rational number to a unique point on the line L.

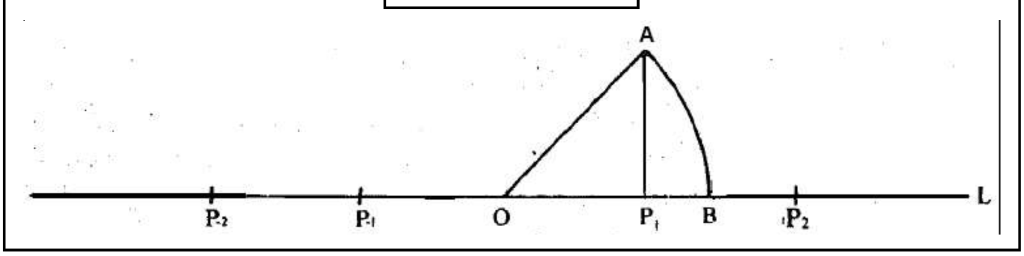
Now arises the important question:

Have you used all the points of the line L while representing rational numbers on it?

The answer to this question is NO. But how? Let us examine this.

At the point P, draw a line perpendicular to the line and mark A such that P_1A = 1 unit. Cut off OB = OA on the line, as shown in the Figure 1.3.

Figure 1.3



Then B is a point which correspond to a number whose square is 2. You have already seen that there is no rational number whose square is 2. In fact, the length OA = $\sqrt{2}$ by Pythagorean Theorem. In other words, the irrational number $\sqrt{2}$ is associated with the point B on the line L. In this way, it can be shown that every irrational number can be associated to a unique point on the line L.

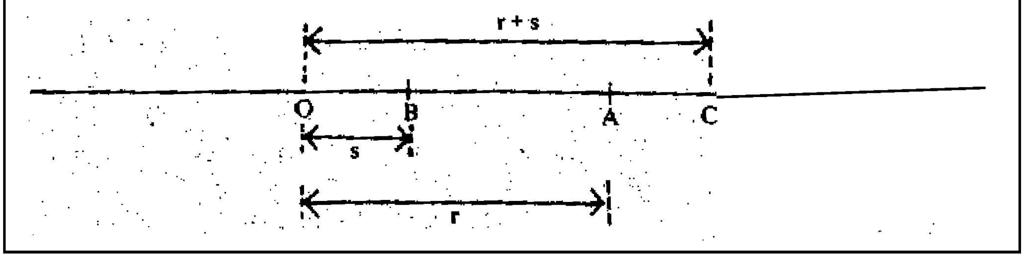
Thus, it can be shown that to every real number, there corresponds a unique point on the line L. In other words, all the real numbers are represented as points on a line. Is the converse true? That is to say, does every point on the line correspond to a unique real number? This is true but we are not going to prove it here. Therefore, hence onwards, we shall say that every real number

Notes

corresponds to a unique point on the line and conversely every point on the line corresponds to a unique real number. In this sense, the line L is called the Real Line.

Now let L be the real line.

Figure 1.4



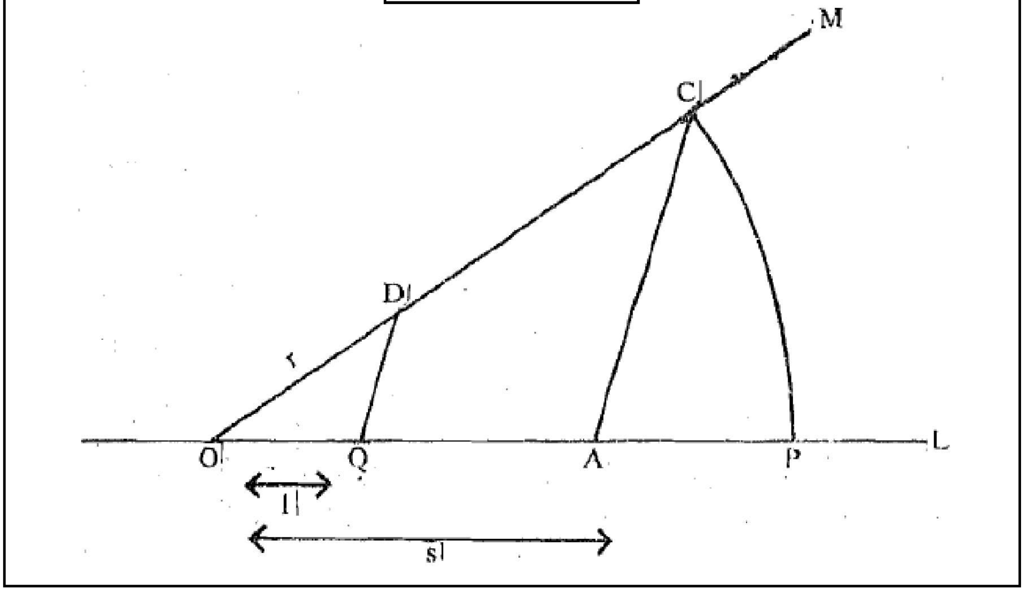
We may define addition (+) and multiplication (.) of real numbers geometrically as follows:

Suppose A represents a real number r and B represents a real number s so that OA = r and OB = s. Shift OB so that O coincides with A. The point C which is the new position of B is defined to represent r + s. See the Figure 1.4.

The construction is valid for positive as well as negative values of r and s. A real number r is said to be positive if r corresponds to a point on the line L on the right of the point O. It is written as r > 0. Similarly, r is said to be negative if it corresponds to a point on the left of the point O and is written as r < 0. Thus if r is a real number then either r is zero or r is positive or r is negative i.e. either r = 0 or r > 0 or r < 0. You should try the following exercise:

What about the product r.s of two real numbers r and s? We shall consider the case when r and s are both positive real numbers.

Figure 1.5



Though O draw some other line OM. On L, let A represent the real number s. On OM take a point D so that OD = r. Let Q be a point on I, so that OQ = 1 unit. Join QD. Through A draw a straight line parallel to QD to meet OM at C. Cut off OP on the line equal to OC. Then F represents the real number r.s.

Notes

Suppose s is a positive real number and r is a negative real number. Then, there exists a number r such that r = –r′ where r′ is a positive real numbers. Therefore, the product rs can be defined on L as

$$rs = (-r')s = -(r's).$$

Similarly you can state that rs = r(–s′) = ⁻ (rs′) where s is negative and s = –s′ for some positive s′, while r is positive.

If both r and s are negative and r = –r′ and s = –s′ where r′ and s′ are positive real numbers, then we define

$$rs = r's' = (-r)\,(-s).$$

We can also similarly define 0, r = r! 0 = 0 for each real number r.

1.4 Complex Numbers

So far, we have discussed the system of real numbers. We have yet, another system of numbers. For example, if you have to find the square root of a negative real number say –5, then you will write at as $\sqrt{-1}, \sqrt{5}$. You know that $\sqrt{5}$ is a real number but what about $\sqrt{-1}$? Again you can verify that a simple equation $x^2 + 1 = 0$ does not have a solution in the set of real numbers because the solution involves the square root of a negative real number. As a matter of fact, the problem is to investigate the nature of the number $\sqrt{-1}$ which we denote by such that $i^2 = -1$. Let us discuss the following example to know the nature of i.

Example: Show that i is not a real number.

We claim that i is not a real number. If it is so, then either i = 0 or i > 0 or I < 0.

If i = 0, then $i^2 = 0$. This implies that –1 = 0 which is absurd. If i > 0, then $i^2 > 0$ which implies that –1 > 0. This is also absurd. Finally, if i < 0, then again $i^2 > 0$ which implies that –1 > 0. This again is certainly absurd. Thus i is not a real number. This number 'i' is called the imaginary number. The symbol 'i' is called 'iota' in Greek language. This generates another class of numbers, the so called complex numbers.

The basic idea of extending the system of real numbers to the system of complex numbers arose due to the necessity of finding the solutions of the equations, like $x^2 + 1 = 0$ or $x^2 + 2 = 0$ and so on. The first contribution to the notion of such a number was made by the most celebrated Indian Mathematician of the 9th century, Mahavira, who showed that a negative real number does not have a square root in the set of real numbers. But it was an Italian mathematician, G. Cardon [1501-1576] who used imaginary numbers in his work without bothering about their existence. Due to notable contributions made by a large number of mathematicians, the system of complex numbers came into existence in the 18th century. Since we are dealing with real numbers, therefore, we shall not go into the detailed discussion of complex numbers. However, we shall give a brief introduction to the system of complex numbers. We denote the set of complex numbers as

$$C = \{z = a + i\,b,\ a \text{ and } b \text{ real numbers}\}$$

In a complex number, z = a + i b, a is called its real part and b is called its imaginary part.

Any two complex numbers $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ are equal if only their corresponding real and imaginary parts are equal.

If $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ are any two complex numbers, then we define addition (+) and multiplication (.) as follows:

Notes

$$z_1 + z_2 = (a_1 + a_2) + i\ (b_1 + b_2)$$

$$z_1.z_2 = (a_1a_2 - b_1b_2) + i\ (a_1b_1 - a_2b_2).$$

The real numbers represent points on a line while complex numbers are identified as points on the plane.

Before concluding this section, we would like to mention yet another classification of numbers as enunciated by some mathematicians. Consider the number $\sqrt{2}$. This is an example of what is called an Algebraic Number because it satisfies the equation

$$x^2 - 2 = 0.$$

A number is called an Algebraic Number if it satisfies a polynomial equation

$$a_0x^n + a_1x^{n-1}\ t\ ....\ + a_{n-1}\ x + a,, + a_n = 0$$

where the coefficients a_0, a_1, a_2,.... a, are integers, a, ≠ 0 and n > 1. The rational numbers are always algebraic numbers. The numbers defined in terms of the square root etc., are also treated as algebraic numbers. But there are some real numbers which are not algebraic. Such numbers are called the Transcendental numbers. The numbers π and ε are transcendental numbers.

You may think that the operations of algebraic operations viz. addition, multiplication, etc. are the only aspects to be discussed about the set of real numbers. But certainly there are some more important aspects of the set of real numbers as points on the real line. We shall discuss these aspects in Unit 3 namely the point sets of the real line called also the topology of the real line. But prior to that, we shall discuss the structure of real numbers in Unit 2.

We conclude this unit by talking briefly about an important hypothesis-closely linked with the system of natural numbers. This is called the Principle of Induction.

1.5 Mathematical Induction

The Principle of Induction and the natural numbers are inseparable. In Mathematics, we often deal with the proofs of various theorems and formulas. Some of these are derived by the direct proofs, while some others can be proved by certain indirect methods. Consider, for example, the validity of the following two statements:

(i) The number 4 divides $5^n - 1$ for every natural number n.

(ii) The sum of the first n natural numbers is $\frac{n(n+1)}{2}$ i.e.

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}.$$

In fact, you can provide most of the verifications for both statements in the following way:

For (i), if n = 1, then $5^n - 1 = 5 - 1 = 4$ which is obviously divisible by,

if n = 2, then $5^2 - 1 = 24$, which is also divisible by 4;

if n = 6, then $5^6 - 1 = 15624$, which is indeed divisible by 4.

Similarly for (ii) if n = 10 then 1 + 2 + ... 4 – 10 = 55, while the formula

$$\frac{n(n+1)}{2} = 55 \text{ when } n = 10.$$

Again, if n = 100; then also you can verify that in each way, the sum of the first hundred natural numbers is 5050 i.e.

$$\frac{n(n+1)}{2} = 5050 \text{ for } n = 100.$$

What do these statements have in common and what do they indicate? The answer is obvious that each statement is valid for every natural number.

Thus to a great extent, a large number of theorems, formulas, results etc. whose statement involves the phrase, "for every natural number n" are those for which an indirect proof is most appropriate. In such indirect proofs, clearly a criterion giving a general approach is applied. One such criterion is known as the criterion of Mathematical Induction. The principle of Mathematical Induction is Stated (without proof) as follows:

Principle of Mathematical Induction

Suppose that, for each $n \in N$, P (n) is a statement about the natural number n. Also, suppose that

(i) P(1) is true,

(ii) if P(n) is true, then P(n + 1) is also true.

Then P(n) is true for every $n \in N$.

Let us illustrate this principle by an example:

Example: The sum of the first n natural numbers is $\frac{n(n+1)}{2}$

Solution: In other words, we have to show that for each $n \in N$,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$S_n = 1 + 2 + 3 + \ldots + n$$

$$= \sum_{k=1}^{n} k.$$

Let P(n) be the statement that

$$S_n = \frac{n(n+1)}{2}$$

We, then, have $S_i = 1$ and $\frac{1(1+1)}{2} = 1$. Hence P(1) is true.

This proves part (i) of the Principle of Mathematical Induction. Now for (ii), we have to verify that if P(n) is true, then P(n + 1) is also true. For this, let us assume that P(n) is true and establish that P(n + 1) is also true. Indeed, if we assume that

$$S_n = \frac{n(n+1)}{2},$$

then we claim that

$$S_{n+1} = \frac{(n+1)(n+2)}{2}$$

Indeed

$$S_{n+1} = 1 + 2 + 3 + \ldots + n + (n+1)$$

$$= S_n + (n + l)$$

Notes

$$= \frac{1}{2} n(n + l) + (n + 1)$$

$$= \frac{(n+1)(n+2)}{2}$$

Thus P(n + 1) is also true.

Similarly, by using the Principle of Induction, you can prove that

(i) the sum of the squares of the first n natural numbers is $\frac{1}{6}$ n(n + l) (2n +1); and

(ii) the sum of the cubes of the first n natural numbers is $\frac{1}{4}$ n^2 $(n + 1)^2$.

Self Assessment

Choose appropriate answer for the following:

1. The complement of the set S is the set of all those element of ....which do not belong to S. It is denoted by S.

   (a) universal set (b) empty set

   (c) union set (d) intersection set

2. Let S and T all two sets. The collection of all elements which belong to S or T is called ....

   (a) universal (b) union

   (c) intersection (d) Difference of two set

3. The intersection .... of sets S and T is defined to be the set of all those elements which belong to both S and T.

   (a) $S \cup T$ (b) $S \neq T$

   (c) $S \cap T$ (d) $S \leq T$

4. If let S = {1, 2, 3} and T = {a, b, c} and let f : S $\rightarrow$ T be defined as f(1) = a, f(2) = b, f(b) = c. Then f is.....................

   (a) one-one (b) onto

   (c) one-one and onto (d) one-one and surjection

5. The set S is called the domain of the function f and T is called its ....

   (a) Range (b) pre-domain

   (c) co-domain (d) bijection

1.6 Summary

- We have recalled some of the basic concepts of sets and functions in section 1.2. A set is a well-defined collection of objects. Each object is an element or a member of the set. Sets are generally designated by capital letters and the members by small letters enclosed with braces. There are two ways to indicate the members of a set. The tabular method or listing method in which we list each element of a set individually and the set-builder method which gives a verbal description of the elements or a property that is common to all the elements of a set.

- A set with a limited number of elements is a Finite set. A set with an unlimited number of elements is an infinite set. A set with no elements is a null-set. A set S is a subset of a set T if every element of S is in T. The set S is said to be a proper subset of T if every element of S is in T and there is at-least one element of T which does not belong to S. The sets S and T are equal if S is a subset of T and T is a subset of S. The null set is a subset of every set and every set is a subset of itself.

- The union of two sets S and T, written as $S \cup T$, includes elements of S and T without repetitions. The intersection of S and T, written as S n T, includes all those elements that are common to both S and T. The complement of a set S in a Universal set X is denoted as S^c and it consists of all those elements of X which do not belong to S. The laws with respect to union, intersection and complement have been asked in the form of exercises. Also, these notions have been extended to an arbitrary family of sets.

- A function f: $S \rightarrow T$ is a rule by which you can associate to each element of S, a unique element of T. The set S is the domain and the set T is the co-domain of f. The set $\{f(x):x \in S\}$ is the Range of f, where f(x) is an image of x under f. The function f is one-one if $f(x_1) = f(x_2)$ $\Rightarrow$ x, = x, for any x, x, in the domain of f. It is said to be onto if the range of f is equal to the domain of f. A function f is said to be a one-one correspondence if it is both one-one and onto. A function i: $S \rightarrow S$ defined by $i(x) = x, \ \forall \ x \in S$ is called an identity function, while a function f; $S \rightarrow T$ is said to be constant if $f(x) = c, x \in S$, c being a fixed element of T.

- Any two functions with the same domain are said to be equal if they have the same image for each point of the domain. The composite of the functions f: $S \rightarrow T$ and g: $T \rightarrow V$ is a function denoted as 'g o f': $S \rightarrow V$ and defined by (g o f) (x) = g(f(x)). The function f: $S \rightarrow T$ is said to be invertible if there exists a function g: $T \rightarrow S$ such that both 'g o f' and 'f o g' are identity functions. Also, a function is invertible if it is both one-one and onto. The inverse of f exists if f is invertible and it is denoted as f.

- We have discussed the development of the system of numbers starting from the set of natural numbers. These are the following:

Natural Numbers (Positive Integers):

$$N = \{1, 2, 4 ....\}$$

Integers:

$$Z = \{.... 3, -2, -1, 0, 1, 2, 3 ....\}$$

Rational Numbers:

$$Q = \{\frac{p}{q} : p \in z, q \in Z, q \neq 0\}$$

Real Numbers:

R = Disjoint Union of Rational and Irrational Numbers

$$R = Q \cup I, Q \cap I = \phi$$

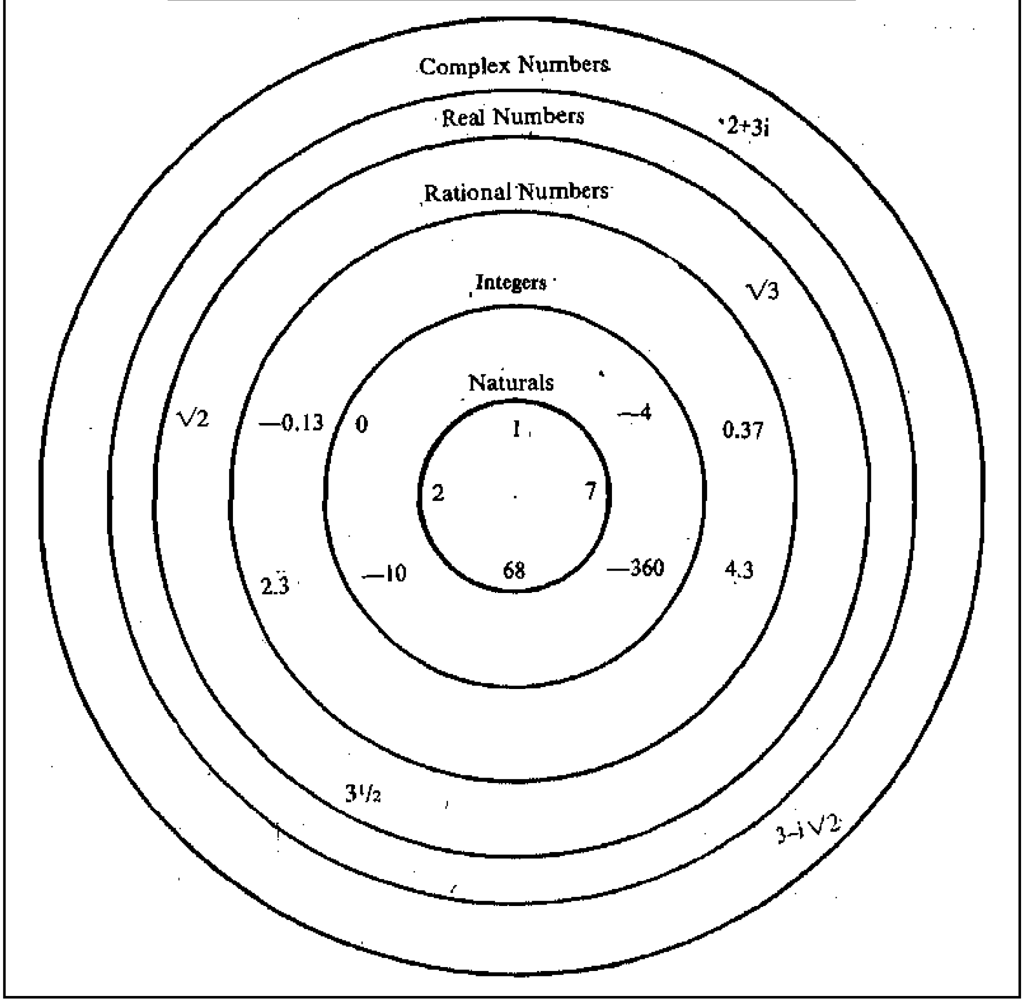
Complex Numbers:

$$C = \{z = x + iy : x \in R, y \in R\}, i = \sqrt{-1}.$$

Notes

A mathematical development of the number systems is depicted in Figure 1.6:

Figure 1.6: A Mathematical Development of Numbers Systems



- We have discussed the geometrical representation of the real numbers and stated the continuum Hypothesis. According to this, every real number can be represented by a unique point on the line and every point on the line corresponds to a unique real number. In view of this, we call this line as the Real Line.

1.7 Keywords

Constant Function: Let S and T be any two non-empty sets. A function f: S→T defined by f(x) = c, for each x in S, where c is fixed element of T, is called a constant function.

Co-domain: The set S is called the domain of the function f and T is called its co-domain.

Finite: A set with a limited number of elements is a Finite set.

Function: A function f: S→T is a rule by which you can associate to each element of S, a unique element of T.

1.8 Review Questions

1. Write the following in the set-builder form:

   A = {2, 4, 6, ...}

   A = {1, 3, 5, ...}

2. Write the following in the tabular form:

   A = {x:x is a factor of 15}

   A = {x:x is a natural number between 20 and 30}

   A = {x:x is a negative integer}

3. Let X be a universal set and let S be a subset of X. Prove that

   (i) P(0) = {∅}

   (ii) $(S^c)^c = S$.

4. Let A, B and C be any three sets. Then prove the following:

   (i) $A \cup B = B \cup A$, $A \cap B = B \cap A$ (Commutative laws).

   (ii) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative laws).

   (iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive laws).

   (iv) $(A \cup B)^C = A^C \cap B^C$, $(A \cap B)^C = A^C \cup B^C$ (DeMorgan laws).

5. Justify that

   (i) N is a proper subset of Z.

   (ii) Z is a proper subset of Q.

Answers: Self Assessment

1.	(a)	2.	(b)
3.	(c)	4.	(c)
5.	(c)		

1.9 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10, Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol : Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik : Mathematical Analysis.

H.L. Royden : Real Analysis, Ch. 3, 4.

Notes

Unit 2: Algebraic Structure and Countability

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the order relation – extended real number system
- Explain the field structure of the set of real numbers
- Describe the order-completeness
- Discuss countability to various infinite sets

Introduction

In Unit 1 we have discussed the construction of real numbers from the rational numbers which, in turn, were constructed from integers. In this unit, we show that the set of real numbers has an additional property which the set of rational numbers does not have, namely it is a complete ordered field. The questions, therefore, that arise are: What is a field? What is an ordered field? What does it mean for an ordered field to be complete? In order to answer these questions we need a few concepts and definitions, e.g., those of order inequalities and intervals in R. We shall discuss these concepts. Also in this unit, we shall explain the extended real number system.

You know that a given set is either finite or infinite. In fact a set is finite, if it contains just n elements where n is some natural number. A set which is not finite is called an infinite set. The elements of a finite set can be counted as one, two, three and so on, while those of an infinite set can not be counted in this way. Can you count the elements of the set of natural numbers? Try it. We shall show that this notion of counting can be extended in certain sense to even infinite sets.

Notes

2.1 Order Relations in Real Numbers

We have demonstrated that every real number can be represented as a unique point on a line and every point on a line represents-% unique real number. This helps us to introduce the notion of inequalities and intervals on the real line which we shall frequently use in our subsequent discussion through out the course.

You know that a real number x is said to be positive if it lies on the right side of O, the point which corresponds to the number 0 (zero) on the real line. We write it as $x > 0$. Similarly, a real number x is negative, if it lies on the left side of O. This is written as $x < 0$. If $x > 0$, then x is a non-negative real number. The real number x is said to be non-positive if $x \leq 0$.

Let x and y be any two real numbers. Then, we say that x is greater than y if $x - y > 0$. We express this by writing $x > y$. Similarly x is less than y if $x - y < 0$ and we write $x < y$. Also x is greater than or equal to y ($x \geq y$) if $x - y \geq 0$. Accordingly, x is less than or equal to y ($x \leq y$) if $x - y \leq 0$. Given any two real numbers x and y, exactly one of the following can hold:

either (i) $x > Y$

or (ii) $x < y$

or (iii) $x = y$.

This is called the law of trichotomy. The order relation ≤ has the following properties:

Property 1

For any x, y, z in R,

(i) If $x \leq y$ and $y | x$, then $x = y$,

(ii) If $x \leq y$ an $y \leq z$, then $x \leq z$,

(iii) If $x \leq y$ then $x + z | y + z$,

(iv) If $x < y$ and $o \leq z$, then $x z \leq y z$.

The relation satisfying (i) is called anti-symmetric. It is called transitive if it satisfies (ii). The property (iii), shows that the inequality remains unchanged under addition of a real number. The property (iv) implies that the inequality also remains unchanged under multiplication by a non-negative real number. However, in this case the inequality gets reversed under multiplication by a non-positive real number. Thus, if $x | y$ and $z \leq 0$, then $xz \geq yz$. For instance, if $z = -1$, we see that

$$-2 \leq 4 \Rightarrow 2(-1) \geq 4(-1) \Rightarrow -2 \geq -4.$$

We state the following results without proof:

- There lie an infinite number of rational numbers between any two distinct rational numbers.
- As a matter of fact, something more is true.
- Between any two real numbers, there lie infinitely many rational (irrational) numbers. Thus there lie an infinite number or real numbers between any two given real numbers.

2.1.1 Intervals

Now that the notion of an order has been introduced in R, we can talk of some special subsets of R defined in terms of the order relation. Before we formally define subset, we first introduce the notion of 'betweenness', which we have already used intuitively in the previous results. If 1, 2, 3 are three real numbers, then we say that 2 lies between 1 and 3. Thus, in general, if a, b and c are any three real numbers such that $a\ 5 \leq b \leq c$ then we say that b lies 'between' a and c. Closely related to notion of betweenness is the concept of an interval.

Notes

Definition 1: Interval

An interval in R is an non-empty subset of R which has the property that, whenever two numbers a and b belong to it, all numbers between a and b also belong to it.

The set N of natural numbers is not an interval because while 1 and 2 belong to N, but 1.5 which lies between 1 and 2, does not belong to N.

We now discuss various forms of an interval.

Let a, b $\in$ R with a $\leq$ b.

(i) Consider the set $\{x \in R : a \leq x \leq b\}$. This set is denoted by]a, b[, and is called a closed interval. Note that the-end points a and b are included in it.

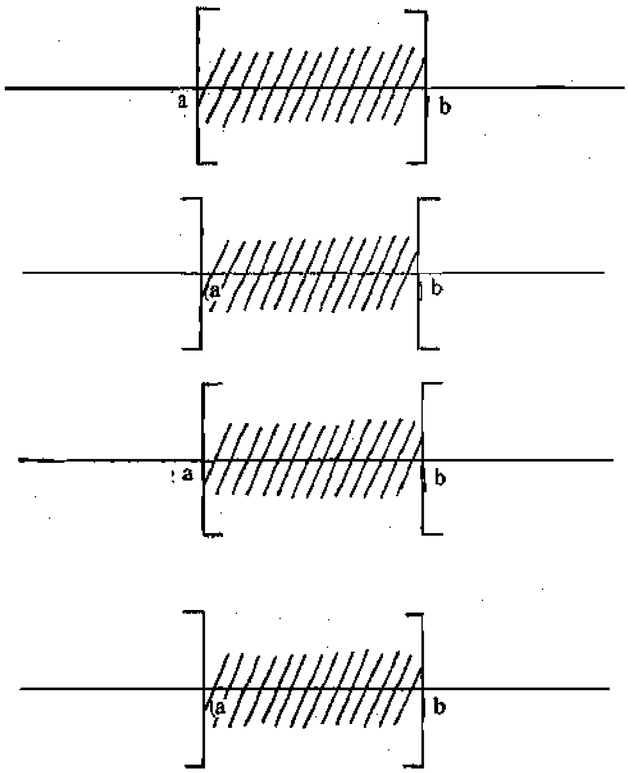
(ii) Consider the set$\{x \in R : a < x < b\}$. This set is denoted by [a, b], and is called an open interval. In this case the end points a and b are not included in it,

(iii) The interval $\{x \in R: a \leq x < b\}$ is denoted by [a, b[.

(iv) The interval $\{x \in R : a < x \leq b\}$ is denoted by]a, b].

You can see the graph of all the four intervals in the Figure 2.1.

Figure 2.1

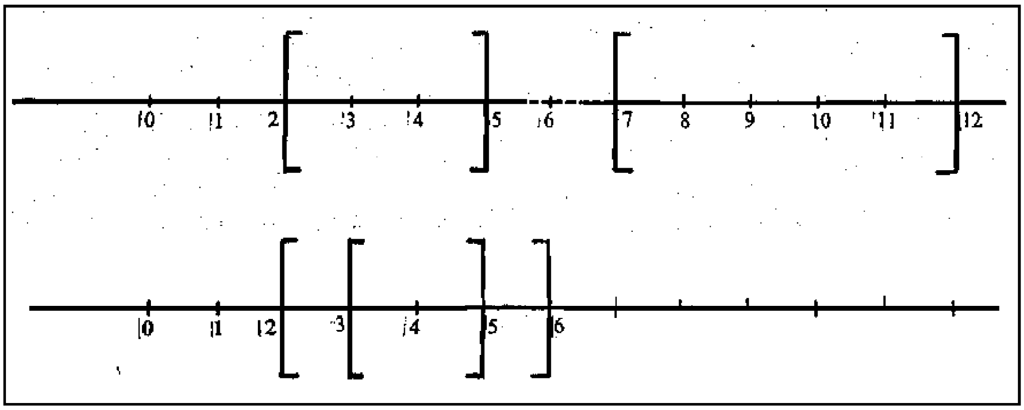


Intervals of these types are called bounded intervals. Some authors also call them finite intervals. But remember that these are not finite sets. In fact these are infinite sets except for the case [a, a] = {a}.

You can easily verify that an open interval]a, b[as well as]a, b] and [a, b[are always contained in the closed interval [a, b].

Example: Test whether or not the union of any two intervals is an interval.

Solution: Let [2, 5] and [7, 12] be two intervals. Then [2, 5] ∪ [7, 12] is not an interval as can be seen on the line in Figure below.



However, if you take the intervals which are not disjoint, then the union is an interval. For example, the union of [2, 5] and [3, 6] is [2, 6] which is an interval. Thus the union of any two intervals is an interval provided the intervals are not disjoint.

2.1.2 Extended Real Numbers

The notion of the extended real number system is important since we need it in this unit as well as in the subsequent units.

You are quite familiar with the symbols + ∞ and – ∞. You often call these symbols are 'plus infinity' and 'minus infinity', respectively. The symbols + ∞ and – ∞ are extremely useful. Note that these are not real numbers.

Let us construct a new set R* by adjoining – ∞ and + ∞ to the set R and write it as

$$R^* = R \cup \{-\infty, +\infty\}.$$

Let us extend the order structure to R* by a relation < as $-\infty < x < +\infty$, for every $x \in R$. Since the symbols – ∞ and + ∞ do not represent any real numbers, you should, therefore, not apply any result stated for real numbers, to the symbols + ∞ and – ∞. The only purpose of using these symbols is that it becomes convenient to extend the notion of (bounded) intervals to unbounded intervals which are as follows:

Let a and b be any two real numbers. Then we adopt the following notations:

$$[a, \infty] = \{X \in R: x \geq a\}$$

$$[a, do] = \{X \in R: x > a\}$$

$$[-\infty, b] = \{x \in R: x \leq b\}$$

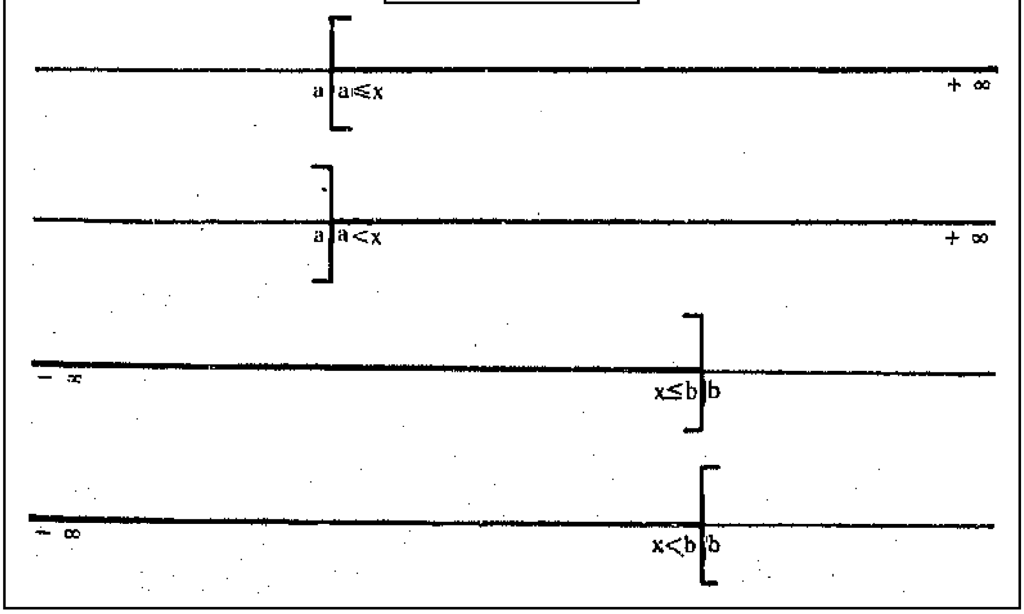
$$[-\infty, b] = \{x \in R: x < b\}$$

$$[-\infty, \infty] = \{X \in R: -\infty < x < \infty\}.$$

Notes

You can see the geometric representation of these intervals in Figure 2.2.

Figure 2.2



All these unbounded intervals are also sometimes called infinite intervals.

You can perform the operations of addition and multiplication involving – ∞ and + ∞ in the following way: For any $x \in R$, we have

$$x + (+\infty) = +\infty,$$

$$x + (+\infty) = -\infty,$$

$$x. (+\infty) = +\infty, \text{ if } x > 0$$

$$x. (+\infty) = -\infty, \text{ if } x < 0$$

$$x. (+\infty) = -\infty, \text{ if } x > 0$$

$$x. (-\infty) = +\infty, \text{ if } x < 0$$

$$\infty + \infty = +\infty, -\infty - \infty = -\infty$$

$$\infty. (-\infty) = -\infty, (-\infty). (-\infty) = +\infty.$$

Note that the operations $\infty - \infty$, $0. \infty$, $\frac{\infty}{\infty}$ are not defined.

2.2 Algebraic Structure

During the 19th Century, a new trend emerged in mathematics to use algebraic structures in order to provide a solid foundation for Calculus and Analysis. In this quest, several methods were used to characterise the red numbers. One of the methods was related to the least upper bound principle used by Richard Dedekind which we discuss in this section.

This leads us to the description of the real numbers as a complete ordered field. In order to define a complete ordered field. We need some definitions and concepts.

You are quite familiar with the operations of addition and multiplication on numbers, union and intersection on the subsets of a universal set. For example, if you add or multiply any two natural numbers, the sum or the product is a natural number. These operations of addition or multiplications on the sets of numbers are examples of a binary operation on a set. In general, we can define a binary operation on a set in the following way:

Definition 2: Binary Operation

Given a non-empty set S, a binary operation on S is a rule which associates with each pair of elements of S, a unique element of S.

We denote this rule by symbols such as ., *, +, etc.

By an Algebraic Structure, we mean a non-empty set together with one or more binary operations defined on it. A field is an algebraic structure which we define, as follows:

Definition 3: Field Structure

A field consists of a non-empty set F together with two binary operations defined on it, denoted by the symbols '+'(addition) and '.' (multiplication) and satisfying the following axioms for any elements x, y, z of the set F.

A_1: $x + y \in F$ (Additive Closure)

A_2: $x + (y + z) = (x + y) + z$ (Addition is Associative)

A_3: $x + y = y + x$ (Addition is Commutative)

A_4: There exists an element in F, denoted by '0' and called the zero or the zero element of F such that $x + 0 = 0 + x = x \ \forall\ x \in F$ (Additive Identity)

A_5: For each $x \in F$, there exists an element $-x \in F$ with the property (Additive Inverse)

$x + (-x) = (-x) + x = 0$

The element $-x$ is called additive inverse of x.

M_1: $x.y \in F$ (Multiplicative Closure)

M_2: $(x.y).z = x. (y.z)$ (Multiplication is Associative)

M_3: $x.y = y.x$ (Multiplication is Commutative)

M_4: There exists an element 1 different from 0 called the unity of F, such that (Multiplicative Identity)

$1.x = x. 1 = x \ \forall\ x \in F$

M_5: For each $x \in F$, $x \neq 0$, there exists an element $x^{-1} \in F$ such that (Multiplicative Inverse)

$x.x^{-1} = .x^{-1} x = 1.$

The element x^{-1} is called the multiplicative inverse of x.

D: $x.(y + z) = x.y + x.z$ (Multiplication is distributive over Addition).

$(x + y) z = x.z + y.z.$

Since the unity is not equal to the zero i.e. $1 \neq 0$ in a field, therefore any field must contain at least two elements. Note that the axioms A_1 (closure under addition) and M_1 (closure under multiplication) are unnecessary because the closures are implied in the definition of a binary operation. However, we include them, for the sake of emphasis.

Notes

Now, you can easily verify that all the eleven axioms are satisfied by the set of rational numbers with respect to the ordinary addition and multiplication. Thus, the set Q forms a field under the operations of addition and multiplication, and so does, the set R of all the real numbers.

We state (without proof) some important properties satisfied by a field. They follow from the field axioms. Can you try?

Property 2

For any x, y, z in F,

1. $x + z = y + z \Rightarrow x = y$,
2. $x, 0 = 0 = 0.x$,
3. $(-x). y = - x,y = x. (-y)$,
4. $(-x), (- y) = x.y$,
5. $x.z = y.z, z \neq 0 \Rightarrow x = y$,
6. $x.y = 0 \Rightarrow$ either $x = 0$ or $y = 0$.

Thus by now you know that the sets Q, R and C form fields under the operations of addition and multiplication.

2.2.1 Ordered Field

We defined the order relation $\leq$ in R. It is easy to see that this order relation satisfies the following properties:

Property 3

Let x, y, z be any elements of R. Then

O_1: For any two elements x and y of R, one and only of the following holds:

(i) $x < y$, (ii) $y < x$, (iii) $x = y$,

O_2: $x \leq y, y \leq x \Rightarrow x \leq z$,

O_3: $x \leq y \Rightarrow x \tau z \leq y + z$,

O_4: $x \leq y, 0 < z \Rightarrow x.z \leq y.z$

We express this observation by saying that the field R is an ordered field (i.e. it satisfies the properties $0_1 - 0_4$). It is easy to see that these properties are also satisfied by the field Q of rational numbers. Therefore, Q is also an ordered field. What about the field C of Complex numbers?

2.2.2 Complete Ordered Field

Although R and Q are both ordered fields, yet there is a property associated with the order relation which is satisfied by R but not by Q. This property is known as the Order-Completeness, introduced for the first time by Richard Dedekind. To explain this situation more precisely, we need a few more mathematical concepts which are discussed as follows:

Consider set S = {1, 3, 5, 7). You can see that each element of S is less than or equal to 7. That is $x \leq 7$, for each $x \in S$. Take another set S, where $S = \{x \in R : x \leq 17)$. Once again, you see that each element of S is less than 18. That is, $x < 18$, for each $x \in S$. In both the examples, the sets have a special property namely that every element of the set is less than or equal to some number.

This number is called an upper bound of the corresponding set and such a set is said to be bounded above. Thus, we have the following definition:

Definition 4: Upper Bound of a Set

Let $S \subset R$. If there is a number $u \in R$ such that $x \leq u$, for every $x \in S$, then S is said to be bounded above. The number u is called an upper bound of S.

Example: Verify whether the following sets are bounded above. Find an upper bound of the set, if it exists.

(i) The set of negative integers

$\{-1, -2, -3, ...\}$.

(ii) The set N of natural numbers.

(iii) The sets Z, Q and R.

Solution:

(i) The set is bounded above with –1 as an upper bound,

(ii) The set N is not bounded above.

(iii) All these sets are not bounded above.

Now consider a set S = {2, 3, 4, 5, 6, 7). You can easily see that this set is bounded above because 7 is an upper bound of S. Again this set is also bounded below because 2 is a lower bound of S. Thus S is both bounded above as well as bounded below. Such a set is called a bounded set. Consider the following sets:

$S_1 = \{... -3, -2, -1, 0, 1, 2, ...\}$,

$S_2 = \{0, 1, 2, ...\}$,

$S_3 = (0, -1, -2, ...\}$.

You can easily see that S, is neither bounded above nor bounded below. The set S_4 is not bounded above while S, is not bounded below. Such sets are known as Unbounded Sets.

Thus, we can have the following definition.

Definition 5: Bounded Sets

A set S is bounded if it is both bounded above and bounded below.

In other words, S has an upper bound as well as a lower bound. Thus, if S is bounded, then there exist numbers u (an upper bound) and v (a lower bound) such that $v \leq x \leq u$, for every $x \in S$.

If a set S is not bounded then S is called an unbounded set. Thus S is unbounded if either it is not bounded above or it is not bounded below.

Example:

(i) Any finite set is bounded.

(ii) The set Q of rational numbers is unbounded.

(iii) The set R of real numbers is unbounded.

(iv) The set P = {sin x, sin 2x, sin 3x,......, sin nx, ...} is bounded because $-1 \leq \sin nx \leq 1$, for every n and x.

Notes

You can easily verify that a subset of a bounded set is always bounded since the bounds of the given set will become the bounds of the subset.

Now consider any two bounded sets say S = {1, 2, 5, 7} and T = {2, 3, 4, 6, 7, 8}. Their union and intersection are given by

$$S \cup T = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

and

$$S \cap T = \{2, 7\}.$$

Obviously $S \cup T$ and $S \cap T$ are both bounded sets. You can prove this assertion in general for any two bounded sets.

> *Task* Prove that the union and the intersection of any two bounded sets are bounded.

Now consider the set of negative integers namely

$$S = \{-1, -3, -2, -4, ...\}.$$

You know that –1 is an upper bound of S. Is it the only upper bound of S? Can you think of some other upper bound of S? Yes, certainly, you can. What about 0? The number 0 is also an upper bound of S. Rather, any real number greater than –1 is an upper bound of S. You can find infinitely many upper bounds of S. However, you can not find an upper bound less than –1. Thus –1 is the least upper bound of S.

It is quite obvious that if a set S is bounded above, then it has an infinite number of upper bounds. Choose the least of these upper bounds. This is called the least upper bound of the set S and is known as the Supremum of the set S. (The word 'Supremum' is a Latin word). We formulate the definition of the Supremum of a set in the following way:

Definition 6: The Supremum of a Set

Let S be a set bounded above. The least of all the upper bounds of S is called the least upper bound or the Supremum of S. Thus, if a set S is bounded above, then a real number m is the supremum of S if the following two conditions are satisfied:

(i) m is an upper bound of S,

(ii) if k is another upper bound of S, then m 5 ≤ k.

> *Task* Give an example of an infinite set which is bounded below. Show that it has an infinite number of lower bounds and hence develop the concept of the greatest lower bound of the set.

The greatest lower bound, in Latin terminology, is called the Infimum of a set.

Let us now discuss a few examples of sets having the supremum and the infimum:

Example: Each of the intervals]a, b[, [a, b]]a, b], [a, b[has both the supremum and the infimum. The number a is the infimum and b is the supremum in each case. In case of [a, b] the supremum and the infimum both belong to the set whereas this is not the case for the set]a, b[. In case of the set]a, b], the infimum does not belong to it and the supremum belongs to it. Similarly, the infimum belongs to [a, b[but the supremum does not belong to it.

Very often in our discussion, we have used the expressions 'the supremum', rather than asupremum. What does it mean? It simply means that the supremum of a set, if it exists, is unique i.e. a set can not have more than one supremum. Let us prove it in the form of the following theorem:

Theorem 1: Prove that the supremum of a set, if it exists, is unique.

Proof: If possible, let there be two supremums (Suprema) say m and m′ of a set $.

Since m is the least upper bound of S, therefore by definition, we have

$$m \leq m'$$

Similarly, since m′ the least upper bound of S, therefore, we must have

$$m' \leq m.$$

This shows that m = m′ which proves the theorem.

You can now similarly prove the following result:

> *Task* Prove that the infimum of a set, if it exists, is unique.

In example 3, you have seen that supremum or the infimum of a set may or may not belong to the set. If the supremum of a set belongs to the set, then it is called the greatest member of the set. Similarly, if the infimum of a set belongs to it, then it is called the least member of the set.

Example:

(i) Every finite set has the greatest as well as the least member.

(ii) The set N has the least member but not the greatest. Determine that number.

(iii) The set of negative integers has the greatest member but not the least member. What is that number?

You have seen that whenever a set S is bounded above, then S has the supremum. In fact this is true in general. Thus, we have the following property of R without proof:

Property 4: Completeness Property

Every non-empty subset S of R which is bounded above, has the supremum.

Similarly, we have

Every non-empty subset S of R that is bounded below, has the infimum.

In fact, it can be easily shown that the above two statements are equivalent.

Now, if you consider a non-empty subset S of Q, then S considered as a subset of R must have, by property, a supremum. However, this supremum may not be in Q. This fact is expressed by saying that Q considered as a field in its down right is not Order-Complete. We illustrate this observation as follows:

Construct a subset S of Q consisting of all those positive rational numbers whose squares are less than 2 i.e.

$$S = \{x \in Q: x > 0, x^2 < 2\}.$$

Since the number 1 ES, therefore S is non-empty. Also, 2 is an upper bound of S because every element of S is less than 2. Thus the set S is non-empty and bounded, above. According to the

Notes

Axiom of Completeness of R, the subset S must have the supremum in R. We claim that this supremum does not belong to Q.

Suppose m is the supremum of the set S. If possible, let m belong to Q. Obviously, then m > 0. Now either $m^2 < 2$ or $m^2 = 2$ or $m^2 > 2$.

Case (i) When $m^2 < 2$. Then a number y defined as

$$y = \frac{4+3m}{3+2m}$$

is a positive rational number and

$$y - m = \frac{2(2-m^2)}{3+2m}$$

Since $m^2 < 2$, therefore $2 - m^2 > 0$, Hence

$$y - m = \frac{3(2-m^2)}{3+2m} > 0$$

which implies that y > m.

Again,

$$y^2 - 2 = \left(\frac{4+3m}{3+2m}\right)^2 - 2$$

$$= \frac{m^2-2}{(3+2m)^2}$$

Since $m^2 < 2$, therefore

$$y^2 - 2 < 0 \text{ i.e. } y^2 < 2.$$

This shows that $y \in S$ and also it is greater than m (the supremum of S). This is absurd. Thus the case $m^2 < 2$ is not possible.

Case (ii) When $m^2 = 2$.

This means there exists a rational number whose square is equal to 2 which is again not possible.

Case (iii) When $m^2 > 2$

In this case consider the positive rational number y defined in case (i). Accordingly, we have

$$y - m = \frac{2(2-m^2)}{3+2m} < 0 \text{ (check yourself)}$$

i.e. y < m.

Also $$2 - y^2 = 2 - \left(\frac{4+3m)}{3+2m}\right) 2 = \frac{2-m^2}{(3+2m)^2}$$

i.e. $$2 - y^2 < 0 \text{ or } y^2 > 2,$$

which shows that y is an upper bound of S.

Thus y is an upper bound of S which does not belong to S. At the same time y is less than the supremum of S. This is again absurd. Thus $m^2 > 2$ is also not possible. Hence none of three possibilities is true. This means there is something wrong with our supposition. In other words, our supposition is false and therefore the set, S does not possess the suprernum in Q.

This justifies that the field Q of rational numbers is not order-complete.

Now you can also try a similar exercise.

2.3 Countability

As we recalled the notion of a set and certain related concepts. Subsequently, we discussed certain properties of the sets of numbers N, Z, Q, R and C. A few more important properties and related aspects concerning these sets are yet to be examined. One such significant aspect is the countability of these sets. The concept of countability of sets was introduced by George Cantor which forms a corner stone of Modern Mathematics.

2.3.1 Countable Sets

You can easily count the elements of a finite set. For example, you very frequently use the term 'one hundred rupees' or 'fifty boxes', 'two dozen eggs', etc. These figures pertain to the number of elements of a set. Denote the number of elements in a finite set S by n (S). If S = {a, b, c, d}, then n (S) = 4. Similarly n (S) = 26, if S is the set of the letters of English alphabet. Obviously, then n (ϕ) = 0, where ϕ is the null set.

You can make another interesting observation when you count the number of elements of a finite set. While you are counting these elements, you are indirectly and perhaps unconsciously, using a very important concept of the one-one correspondence between two sets. Recall the concept of one-one correspondence. Here one of the sets is a finite subset of the set of natural numbers and the other set is the set consisting of the articles/objects like rupees, boxes, eggs, etc. Suppose you have a basket of oranges. While counting the oranges, you are associating a natural number to each of the oranges. This, as you know, is a one-one correspondence between the set of oranges and a subset of natural members. Similarly, when you count the fingers of your hands, you are in fact showing a one-one correspondence between the set of the fingers with a subset, say N_{10} = (1, 2, ... 10) of N.

Although, we have an intuitive idea of finite and infinite sets, yet we give a mathematical definition of these sets in the following way:

Definition 7: Finite and Infinite Sets

A set S is said to be finite if it is empty or if there is a positive integer k such that there is one-one correspondence between the elements of the set S and the set N_K = {1, 2, 3 ..... k}. A set is said to be infinite if it is not finite.

The advantage of using the concept of one-one correspondence is that it helps in studying the countability of infinite sets. Let E = {2, 4, 6, ...} be the set of even natural numbers. If we define a mapping f: N → E as

$$f(n) = 2n \ \forall \ n \in N,$$

then we find that f is a one-one correspondence between N and E.

Consider another examples, Suppose S = {1, 2, ... n} and T = {x_1, X_2, ... x_n}. Define a mapping f: S → T as

$$f(n) = x_n \ n \in S.$$

Notes

Then again f is a one-one correspondence between S and T.

Such sets are known as equivalent sets. We define the equivalent sets in the following way:

Definition 8: Equivalent Sets

Any two sets are equivalent if there is one-one correspondence between them.

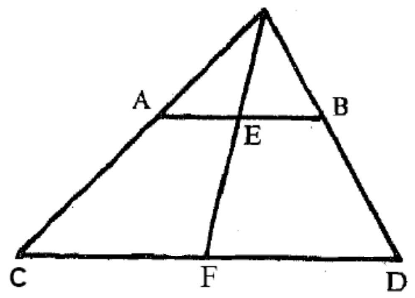
Thus if two sets S and T are equivalent, we write, as S – T.

You can easily show; that S, T and P are any three sets such that S ~ T and T – P, then S – P.

The notion of the equivalent sets is very important because it forms the basis of the 'counting' of the infinite sets.

Now, consider any two line segments AB and CD.

Figure 2.3



Let M denote the set of points on AB and N the set of points on CD. Let us check whether M and N are equivalent.

Join CA and DB to meet in the point P. Let a line through P meet AB in E and CD in F. Define f: M $\rightarrow$ N as f(x) = y where x is any point on AB and y is any point on CD. The construction shows that f is a one-one correspondence. Thus M and N are equivalent sets.

The following are some examples of equivalent sets: Let I be an interval with end points a and b, and J be an interval with end points c and d. Also, we assume that I and J are intervals of the same type. Define f : I $\rightarrow$ J, by

$$f(t) = d + c, \text{ for } t \in I.$$

Then, it is not difficult to see that f is a one-to-one correspondence between intervals I and J. Hence, all the intervals of same type are equivalent to each other.

Now, we introduce the following definition:

Definition 9: Denumerable and Countable Sets

A set which is equivalent to the set of natural numbers is called a denumerable set. Any set which is either finite or denumerable, is called a Countable set.

Any set which is not countable is said to be an uncountable set.

Example:

(i) A mapping f: Z $\rightarrow$ N defined by

$$f(n) = \begin{cases} -2n, \text{ if n is a negative integer} \\ 2n + 1, \text{ if n is non-negative integer} \end{cases}$$

is a one-to-one correspondence. Hence Z – N. Thus the set of integers is a denumerable set and hence a countable set.

(ii) Let E denote the set of all even natural numbers. Then the mapping f: N $\rightarrow$ E defined as f(n) = 2n is a one-one correspondence. Hence the set E of even natural numbers is a denumerable set and hence a countable set.

(iii) Let D denote the set of all odd integers and E the set of even integers. Then the, mapping f: E $\rightarrow$ D, defined as f(n) = n + 1 is a one-one correspondence. Thus E ~ D, But, E – N, therefore D – N. Hence D is a denumerable set and hence a countable set.

We observe that a set S is denumerable if and only if it is of the form {a,, a,, a, ...} for distinct elements a,, a,, a, ... For, in this case the mapping $f(a_n) = n$ is one-one mapping of S onto N i.e. the sets {a,, a,, a_3, ...} and the set N are equivalent.

If we consider the set S_2 = {2, 3, 4, ...}, we find that the mapping f: N $\rightarrow$ S_2 defined as f(n) = n + 1 is one-one and onto. Thus S_2 is denumerable. Similarly if we consider S_3 = {3, 4,....} or S_k = {k, k + 1, ...}, then we find that all these are denumerable sets and hence are countable sets.

We have seen that the set of integers is countable.

Now we discuss the countability of the rational and real numbers. Here is an interesting theorem.

Theorem 2: Every infinite subset of a denumerable set is denumerable.

Proof: Let S be a denumerable set. Then S can be written as

$$S = \{a,, a,, a_3, ...\}.$$

Let A be an infinite subset of S. We want to show that A is also denumerable.

You can see that the elements of S are designated by subscripts 1, 2, 3, ... Let n, be the smallest subscript for which $a_{n1} \in A$. Then consider the set A – $\{a_{n1}\}$. Again, in this new set, let n_2 be the smallest subscript such that $a_{n2} \in$ A – {a,,,}.

Let n_k be the smallest subscript such that

$$a_{nk} \in A - \{a_{n1}, a_{n2}, ..., a_{nk-1}\}.$$

Note that such an element a_{nk} always exists for each k $\in$ N as A is infinite. For, then

$$A = \left\{a_{n_1}, a_{n_2}, ..., a_{n_k}\right\} \neq \varnothing$$

for each k $\in$ N. Thus, we can write

$$A = \left\{a_{n_1}, a_{n_2}, a_{n_3}, ..., a_{n_k}, ...\right\}.$$

Define f: N $\rightarrow$ A by $f(k) = a_{nk}$. Then it can be verified that f is a one-one correspondence. Hence A is denumerable. This completes the proof of the theorem.

Now consider the sets S = {6, 8, 10, 12, ...} and T = {3, 5, 7, 9, 11, ...}, which are both denumerable. Their union S $\cup$ T = {3, 5, 6, 7, 8, 9, ...} is an infinite subset of N and hence its denumerable. Again, if S = (–1, 0, 1, 2} and T = {20, 40, 60, 80, ...}, then we see that S $\cup$ T = (–1, 0, 1, 2, 20, 40, 60, ...} is a denumerable set. Note that in each case S $\cap$ T = 0. In fact, you can prove a general result in the following exercise.

Thus, it follows that the union of any two countable sets is countable.

Indeed, let S and T be any two countable sets. Then S and T are either finite or denumerable.

If S and T are both finite, then S $\cup$ T is also a finite set and hence S $\cup$ T is countable.

Notes

If S is denumerable and T is finite, then also we know that $S \cup T$ is denumerable. Hence $S \cup T$ is countable. Again, if S is finite and T is denumerable, then again $S \cup T$ is denumerable and countable.

Finally, if both S and T are denumerable, then $S \cup T$ is also denumerable and hence countable. In fact, this result can be extended to countably many countable sets. We prove this in the following theorem:

Theorem 3: The union of a countable number of countable sets is countable.

Proof: Let the given sets be A,, A_n, A,,.... Denote the elements of these sets, using double subscripts, as follows:

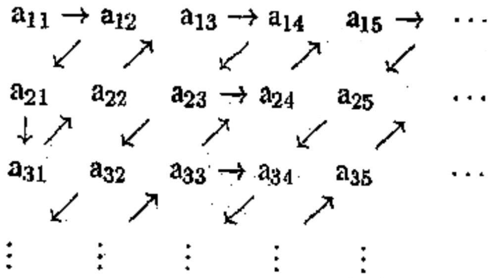
$$A_1 = \{a_{11}, a_{12}, a_{13}, ...\}$$

$$A_2 = \{a_{21}, a_{22}, a_{23}, ...\}$$

$$A_3 = \{a_{31}, a_{32}, a_{33}, ...\},$$

and so on. Note that the double subscripts have been used for the sake of convenience only. Thus a_{ij} is the jth element in the set A. Now, let us try to form a single list of all elements of the union of these given sets.

One method of doing this is by using Cantor's diagonalised counting as indicated by arrows in the following table:



Diagonalised Counting of $\bigcup_{i=1}^{\infty} A_j$.

Enlist the elements as indicated through the arrows. This is a scheme for making a single list of all the elements.

Following the arrows in above table, you can easily arrive at the new single list:

a,,, a,,, a,,, a_{31} a_{22} > a_{13}, a,,, a,,, ...

Note that while doing so, you must omit the duplicates, if any.

Now, if any of the sets A,, A,, ..., are finite, then this will merely shorten the final list. Thus, we have

$\cup_i A_i = \cup_i$ {a,,, a_{i2}, ...}, i = 1, 2, 3, ...

which each element appears only once. This set is countable and, so, complete the proof of the theorem.

We are now in a position to discuss the countability of the sets of rational and real numbers.

2.3.2 Countability of Real Numbers

We have already established that the sets N and Z are countable. Let us, now, consider the case of the set Q of rational numbers. For this we need the following theorems:

Theorem 4: The set of all rational numbers between [0,1] is countable.

Proof: Make a systematic scheme in an order for listing the rational numbers x where * ≤ x ≤ 1, (without duplicates) of the following sets

$$A_1 = \{0, 1\}$$

$$A_2 = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

$$A_3 = \left\{\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots\right\}$$

$$A_4 = \left\{\frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \ldots\right\}$$

....

....

You can see that each of the above sets is countable. Their union is given by

$$A_i = \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \cdots\right\} = [0, 1] \cap Q,$$

which is countable by Theorem 3.

Theorem 5: The set of all positive rational numbers is countable.

Proof: Let Q, denote the set of all positive rational numbers. To prove that Q, is countable, consider the following sets:

$$A_1 = \{1, 2, 3, \ldots\ldots\}$$

$$A_2 = \left\{\frac{1}{2}, \frac{2}{2}, \frac{5}{2}, \ldots\right\}$$

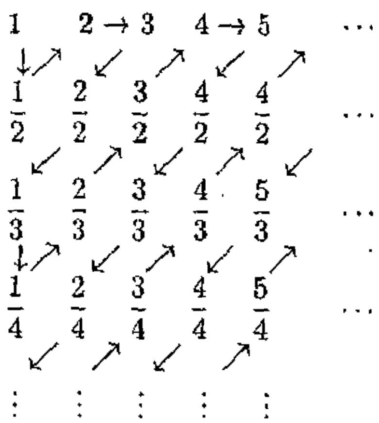
$$A_3 = \left\{\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots\right\}$$

$$A_4 = \left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots\right\}$$

....

....

Enlist the elements of these sets in a manner as you have done in Theorem 3 or as known below:



Notes

You may follow the method of indicating by arrows for making a single list or you may follow another path as indicated here. Accordingly, write down the elements of Q_+ as they appear in the figure by the arrows, while omitting those numbers which are already listed to avoid the duplicates. We will have the following list:

$$Q_+ = \left\{1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, \ldots\right\}$$

$$= \bigcup_i A_i \ (i = 1, 2, 3, \ldots),$$

which is countable by Theorem 3. Thus Q_+ is countable.

Now let Q_- denote the set of all negative rational numbers. But Q_+ and Q_- are equivalent; sets because there is one-one correspondence between Q_+ and Q_-, f: $Q_+ \to Q_-$, given by

$$f(x) = -x, \ \forall \ x \in Q_+.$$

Therefore Q_- is also countable. Further {0} being a finite set is countable. Hence,

$$Q = Q_+ \bigcup_i \{0\} \cup Q_-$$

is a countable set. Thus, in fact, we have proved the following theorem:

Theorem 6: The set Q of all rational numbers is countable.

Proof: You may start thinking that perhaps every finite set is denumerable. This is not true. We have not yet discussed the countability of the set of real numbers or of the set of irrational numbers. To do so, we first discuss the countability of the set of real numbers in an interval with end points 0 and 1, which may be closed or open or semi-closed.

Consider the real numbers in the interval]0, 1[.

Each real number in]0, l[can be expressed in the decimal expansion. This expansion may be non-terminating or may be terminating, e.g.

$$\frac{1}{3} = .333, \ldots$$

is an example of non-terminating decimal expansion, whereas

$$\frac{1}{4} = .25, \ \frac{1}{2} = .5, \ldots,$$

are terminating decimal expansions. Even the terminating expansion can also be expressed as non-terminating expansion in the sense that you can write

$$\frac{1}{4} = .25 = .24999 \ldots$$

Thus, we agree to say that each real number (rational of irrational) in the]0, 1[can be expressed as a non-terminating decimal expansion in terms of the digits from 0 to 9.

Suppose $x \in]0, 1[$. Then it can be written as

$$x = .c_1c_2c_3 \ldots$$

where c_1, c_2,.... take their values from the set {0, l, 2, 3, 4, 5, 6, 7, 8, 9) of ten digits.

Similarly, let y be another, real number in (0, l). Then y can also be expressed as

$$y = .d_1d_2d_3 \ldots$$

We say that x = y if the digits in their corresponding position in the expansions of x and y are identical. Thus, if there is even a single decimal places, say, 10th place such that $d_{10} \neq c_{10}$, then

$$x \neq y.$$

We now discuss the following result due to George Cantor.

Theorem 7: The set of real numbers in the interval]0, 1[is not countable.

Proof: Since the set of numbers in]0, 1[is an infinite set, therefore, it is enough to how that the set of real numbers in]0, 1[is not denumerable.

If possible, suppose the set of real numbers in]0, 1[is denumerable. Then there is a one-one correspondence between N and the elements of]0, 1[i.e. there is a function f: N →]0, 1[which is one-one and onto. Thus, if

$f(1) = x_1, f(2) = x_2, ..., f(k) = x_k, ..., $ then

$]0, 1[= \{x_1, x_2, ..., x_k, ...\}.$

We shall show that there is at least one real number]0, 1[which is not an image of any element of N under f. In other words, there is an element of]0, 1[which is not in the list $x_1, x_2, ...$

Let $x_1, x_2, ...$ be written as

$$x_1 = 0, a_{11}\ a_{12}\ a_{13}\ a_{14}\ ....$$

$$x_2 = 0, a_{21}\ a_{22}\ a_{23}\ a_{24}\ ....$$

$$x_3 = 0, a_{31}\ a_{32}\ a_{33}\ a_{34}\ ....$$

$$x_4 = 0, a_{41}\ a_{42}\ a_{43}\ a_{44}\ ....$$

....

....

....

From this we construct a real number

$$z = b_1\ b_2\ b_3\ b_4\ .....,$$

where $b_1, b_2, ...$ can take any digits from {0, 1, 2,........., 9} in such a way that $b_1 \neq a_{11}, a_2 \neq a_{22}, b_3 = a_{33}, ...$ Thus,

$$z = b_1\ b_2\ b_3\ ....$$

As a real number in]0, 1[such that $z \neq x_1$ because $b_1 \neq a_{11}$, $z \neq x_2$ because $b_2 \neq a_{22}$. In general $z \neq x_n$ because $a_{nn} \neq b_n$. Therefore z is not in the list $\{x_1, x_2\ x_3,....\}$.

Hence]0, 1[is not countable.

We have already mentioned that the intervals [0, 1], [0, 1[,]0, 1] and]0, 1[are equivalent sets. Since the set of real numbers in]0, 1[is not countable, therefore none of the intervals is a countable set of real numbers.

Now you can easily conclude that the set of irrational numbers in]0, 1[is not countable. If possible, suppose that the set of irrational numbers in]0, 1[is countable. Also you know that the set of rational numbers in]0, 1[is countable and that the union of two countable sets is countable. Therefore, the union of the set of rational numbers and the set of irrational numbers]0, 1[is countable i.e. the set of all real numbers in]0, 1[is countable which by above theorem is not so. Hence the set of irrational numbers in]0, 1[is not countable.

In fact, every interval]a, b[or [a, b],]a, b], [a, b[is an uncountable set of real numbers.

Notes

Now what about the countability of the set R of real numbers?

Suppose that R is countable. Then an interval]0, 1[, being an infinite subset of R, must be countable. But then, we have already proved that the set]0, 1[is not countable. Hence R cannot be countable.

Thus even by the method of countability of sets, we have established the much desired distinction between Q and R in the sense that Q is countable but R is not countable.

Also, we observe that although R is not countable, yet it contains subsets which are countable. For example R has subsets as Q, Z and N which are countable. At the same time R is an infinite set. We sum up this observation in the form of the following theorem:

Theorem 8: Every infinite set contains a denumerable set.

Proof: Let S be an infinite set. Consider some element of S. Denote it by n_1. Consider the set S – $\{a_1\}$. Now pick up an element from the new set and denote it by a_2.

Consider the set

$$S = \{a_1, a_2\}.$$

Proceeding in this way, having chosen a_{k-1}, you can have the set

$$S = \{a_1, a_2, ... a_{k-1},\}.$$

This set is always non-empty because S is an infinite set. Hence, we can choose an element in this set. Denote the element by a_k. This can be done for each $k \in N$. Thus the set

$$\{a_1, a_2, ..., a_k, ...\}$$

is a denumerable subset of S and hence a countable subset of S. This proves the theorem.

The importance of this theorem is that it leads us to an interesting area of Cardinality of sets by which we can determine and compare the relative sizes of various infinite sets,

This, however, is beyond the scope of this course.

Self Assessment

Fill in the blanks:

1. Let E denote the set of all even natural numbers. Then the mapping f: N $\rightarrow$ E defined as .... is a one-one correspondence. Hence, the set E of even natural numbers is a denumerable set and hence a countable set.
2. Let D denote the set of all odd integers and E the set of even integers. Then the, mapping f: E $\rightarrow$ D , defined as .... is a one-one correspondence. Thus E ~ D, But, E – N, therefore D – N. Hence D is a denumerable set and hence a countable set.
3. Every .... of a denumerable set is denumerable.
4. The set of all rational numbers between [0, 1] is ....
5. The set of all .... numbers is countable.

2.4 Summary

- We have discussed the order-relations (inequalities) in the set R of real numbers. Given any two real numbers x and y, either x > y or x = y or x < y.

Notes

- This is known as the law of Trichotomy. Then we have stated a few properties with respect to the inequality '≤'. The first property states that the inequality ≤ is antisymmetric. The second states the transitivity of ≤. The third allows us to add or subtract across the inequality, while preserving the inequality. The last property gives the situation in which the inequality is preserved if multiplied by a positive real number, while it is reversed if multiplied by a negative real number.
- We have also defined the bounded and unbounded intervals. The bounded intervals are classified as open intervals, closed intervals, semi-openor semi-closed intervals. The unbounded intervals are introduced with the help of the extended real number system $R \cup \{-\infty, \infty)$ using the symbols $+\infty$ (called plus infinity) and $-\infty$, (called minus infinity).
- There are three important aspects of the real numbers: algebraic, order and the completeness. To describe these aspects, we have specified a number of axioms in each case. In the algebraic aspect, an algebraic structure called the field is used. A field is a non-empty set F having at least two distinct elements 0 and 1 together with two binary operations + (addition) and . (multiplication) defined on F such that both + and . are commutative, associative, 0 is the additive identity, 1 is the multiplicative identity, additive inverse exists for each element of F, multiplicative inverse exists for each element other than 0 and multiplication is distributive over addition. The second aspect is concerned with the Order Structure in which, we use the axioms of the law of trichotomy, the transitivity property, the property that preserve the inequality under addition and the property that preserve the inequality under multiplication by a positive real number.
- In the completeness aspect, we introduce the notions of the supremum (or infimum) of a set and state the axiom of completeness. We find that both Q and R are ordered fields but the axioms of completeness distinguishes Q from R in the sense that Q does not satisfy the axiom of completeness. Thus, we conclude that R is a complete-ordered Field whereas Q is not a complete-ordered field.
- We introduce the notion of the countability of sets. A set is said to be denumerable if it is in one-one correspondence with the set of natural numbers. Any set which is either finite or denumerable is called a countable set. We have shown that the sets N, Z Q are countable sets but the sets 1 (set of irrational numbers) and R are not countable.
- Thus in this unit, we have discussed the algebraic structure, the order structure and the countability of the real numbers.

2.5 Keywords

Countable Set: A set which is equivalent to the set of natural numbers is called a denumerable set. Any set which is either finite or denumerable, is called a Countable set.

Uncountable Set: Any set which is not countable is said to be an uncountable set.

2.6 Review Questions

1. State the properties of order relation in the set R of real numbers with respect to the relation 3 (is greater than or equal to) and illustrate the inequality under multiplication by a negative real number.
2. Give examples to show that the intersection of any two intervals may not be an interval. What happens, if the two intervals are not disjoint? Justify your answer by an example.

Notes

3. Show that the set {0, 1} forms a field under the operations '+' and '.' defined by the following tables:

+	0	1
0	0	1
1	1	0

.	0	1
0	0	0
1	0	1

4. Show that the zero and the unity are unique in a field.

5. Do the sets N (of natural numbers) and Z (set of integers) form fields? Justify your answers. Also verify that the set C of complex numbers is a field.

6. Show that the field C of Complex numbers is not an ordered field.

7. (i) Define a set which is bounded below. Also define a lower bound of a set.

   (ii) Give at least two examples of a set (one of an infinite set) which is bounded below and mention a lower bound in each case.

   (iii) Is the set of negative integers bounded below? Justify your answers.

8. Test which of the following sets are bounded above, bounded below, bounded and unbounded.

   (i) The intervals]a, b], [a, b],]a, b] and [a, b[, where a and b are any two real numbers.

   (ii) The intervals $[2, \infty[$, $]-\infty, 3[$, $]5, \infty[$ and $]-\infty, 4]$.

   (iii) The set {cos e, cos 2 θ, cos 3 e, ...}.

   (iv) $S = \{x \in R : -a \leq x \leq a\}$ for some $a \in R$.

Answers: Self Assessment

1. f(n) = 2n
2. f(n) = n + 1
3. infinite subset
4. countable
5. positive rational

2.7 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 3: Matric Spaces

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define the modulus of a real number
- Describe the notion of a neighbourhood of a point on the line
- Define an open set and give examples
- Discuss the limit points of a set
- Define a closed set and establish its relation with an open set
- Explain the meaning of an open covering of a subset of real numbers

Introduction

You all are quite familiar with an elastic string or a rubber tube or a spring. Suppose you have an elastic string. If you first stretch it and then release the pressure, then the string will come back to its original length. This is a physical phenomenon but in Mathematics, we interpret it differently. According to Geometry, the unstreched string and the stretched string are different since there is a change in the length. But you will be surprised to know that according to another branch of Mathematics, the two positions of the string are identical and there is no change. This branch is known as Topology, one of the most exciting areas of Mathematics.

Notes

The word 'topology' is a combination of the two Greek words 'topos' and 'logos'. The term 'topos' means the top or the surface of an object and 'logos' means the study. Thus 'topology' means the study of surfaces. Since the surfaces are directly related to geometrical objects, therefore there is a close link between Geometry and Topology. In Geometry, we deal with shapes like lines, circles, spheres, cubes, cuboids, etc. and their geometrical properties like lengths, areas, volumes, congruences etc. In Topology, we study the surfaces of these geometrical objects and certain related properties which are called topological properties. What are these topological properties of the surfaces of a geometrical figure? We shall not answer this question at this stage. However, since our discussion is confined to the real line, therefore, we shall discuss this question pertaining to the topological properties of the real line. These properties are related to the points and subsets' of the real line such as neighbourhood of a point, open sets, closed sets, limit points of a set of the real line, etc. We shall, therefore, discuss these notions and concepts in this unit. However, prior to all these, we discuss the modulus of a real number and its relationship with the order relations or inequalities.

3.1 Matric Spaces

Definition

A metric space is an ordered pair (M, d) where M is a set and d is a metric on M, i.e., a function

$$d : M \times M \to \mathbb{R}$$

such that for any x, y, z $\in$ M, the following holds:

1. $d(x, y) \geq 0$ (non-negative),
2. $d(x, y) = 0$ iff $x = y$ (identity of indiscernibles),
3. $d(x, y) = d(y, x)$ (symmetry) and
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The first condition follows from the other three, since:

$$2d(x, y) = d(x, y) - d(y, x) \geq d(x, x) = 0$$

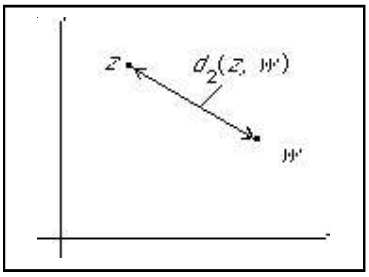
The function d is also called distance function or simply distance. Often, d is omitted and one just writes M for a metric space if it is clear from the context what metric is used.

Example:

1. The prototype: the line R with its usual distance $d(x, y) = |x - y|$.
2. The plane R^2 with the "usual distance" (measured using Pythagoras's theorem):

   $d((x_1, y_1), (x_2, y_2)) = \sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}$.

   This is sometimes called the 2-metric d_2.

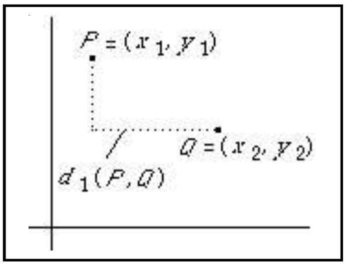


Notes

3. The same picture will give metric on the complex numbers C interpreted as the Argand diagram. In this case the formula for the metric is now: $d(z, w) = |z - w|$ where the | | in the formula represent the modulus of the complex number rather than the absolute value of a real number.

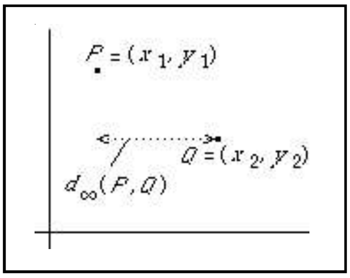
4. The plane with the taxi cab metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.

   This is often called the 1-metric d_1.

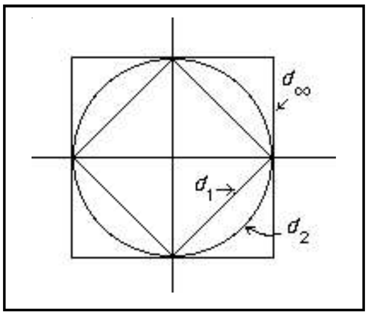


5. The plane with the supremum or maximum metric $d((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$. It is often called the infinity metric d_∞.

   These last examples turn out to be used a lot. To understand them it helps to look at the unit circles in each metric. That is the sets $\{x \in R^2 \mid d(0, x) = 1\}$. We get the following picture:



6. Take X to be any set. The discrete metric on the X is given by: $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise. Then this does define a metric, in which no distinct pair of points are "close". The fact that every pair is "spread out" is why this metric is called discrete.



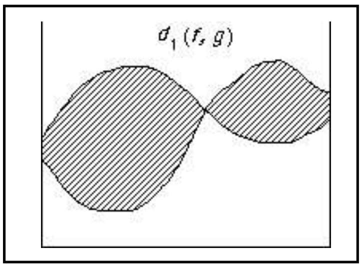
7. Metrics on spaces of functions. These metrics are important for many of the applications in analysis. Let C[0, 1] be the set of all continuous R-valued functions on the interval [0, 1]. We define metrics on by analogy with the above examples by:

   (a) $d_1(f, g) = \int_{D}^{1} |f(x) - g(x)| dx$

   So the distance between functions is the area between their graphs.

Notes

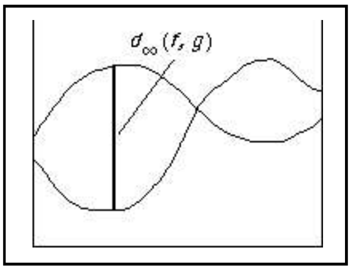
(b) $d_2(f, g) = \sqrt{\left[\int_0^1 f(x) - g(x)^2 dx\right]}$



Although this does not have such case straight forward geometric interpretation as the last example, this case turns out to be the most important in practice. It corresponds to who doing a "least squares approximation".

(c) $d(f, g) = \max \{|f(x) - g(x))| \mid 0 \leq x \leq 1\}$

Geometrically, this is the largest distance between the graphs.



Remarks:

1. The triangle inequality does hold for these metrics
2. As in the R^2 case one may define d_p for any $p \geq 1$ and get a metric.

3.1.1 Space Properties

Metric spaces are paracompact Hausdorff spaces and hence normal (indeed they are perfectly normal). An important consequence is that every metric space admits partitions of unity and that every continuous real-valued function defined on a closed subset of a metric space can be extended to a continuous map on the whole space (Tietze extension theorem). It is also true that every real-valued Lipschitz-continuous map defined on a subset of a metric space can be extended to a Lipschitz-continuous map on the whole space.

Metric spaces are first countable since one can use balls with rational radius as a neighborhood base.

The metric topology on a metric space M is the coarsest topology on M relative to which the metric d is a continuous map from the product of M with itself to the non-negative real numbers.

3.1.2 Distance between Points and Sets; Hausdorff Distance and Gromov Metric

A simple way to construct a function separating a point from a closed set (as required for a completely regular space) is to consider the distance between the point and the set. If (M, d) is a metric space, S is a subset of M and x is a point of M, we define the distance from x to S as

$$d(x, S) = \inf \{d(x, s) : s \in S\}$$

Notes

Then $d(x, S) = 0$ if and only if x belongs to the closure of S. Furthermore, we have the following generalization of the triangle inequality:

$$d(x, S) \leq d(x, y) + d(y, S)$$

which in particular shows that the map is continuous.

Given two subsets S and T of M, we define their Hausdorff distance to be

$$d_H(S, T) = \max \{\sup \{d(s, T) : s \in S\}, \sup \{d(t, S) : t \in T\}\}$$

In general, the Hausdorff distance $d_H(S, T)$ can be infinite. Two sets are close to each other in the Hausdorff distance if every element of either set is close to some element of the other set.

The Hausdorff distance d_H turns the set K(M) of all non-empty compact subsets of M into a metric space. One can show that K(M) is complete if M is complete. (A different notion of convergence of compact subsets is given by the Kuratowski convergence.)

One can then define the Gromov–Hausdorff distance between any two metric spaces by considering the minimal Hausdorff distance of isometrically embedded versions of the two spaces. Using this distance, the set of all (isometry classes of) compact metric spaces becomes a metric space in its own right.

3.1.3 Product Metric Spaces

If (M_1, d_1), ..., (M_n, d_n) are metric spaces, and N is the Euclidean norm on R^n, then (M1 x .. x Mn, N(d1, ..., dn)) is a metric space, where the product metric is defined by

$$N(d1, ..., dn) ((x1, ..., xn), (y1, . . ., yn)) = N(d1(x1, y1), . . ., dn(xn, dn)),$$

and the induced topology agrees with the product topology. By the equivalence of norms in finite dimensions, an equivalent metric is obtained if N is the taxicab norm, a p-norm, the max norm, or any other norm which is non-decreasing as the coordinates of a positive n-tuple increase (yielding the triangle inequality).

Similarly, a countable product of metric spaces can be obtained using the following metric

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{di(x_i, y_i)}{1 + d_i(x_i, y_i)}.$$

An uncountable product of metric spaces need not be metrizable. For example, R^R is not first-countable and thus isn't metrizable.

Continuity of Distance

It is worth noting that in the case of a single space (M, d), the distance map $d: M \times M \to R^+$ (from the definition) is uniformly continuous with respect to any of the above product metrics N(d, d), and in particular is continuous with respect to the product topology of $M \times M$.

Quotient Metric Spaces

If M is a metric space with metric d, and ~ is an equivalence relation on M, then we can endow the quotient set M/~ with the following (pseudo)metric. Given two equivalence classes [x] and [y], we define

$$d'([x], [y]) = \inf \{d(p_1, q_1) + d(p_2, q_2) + . . . + d(p_n, q_n)$$

Notes

where the infimum is taken over all finite sequences $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n)$ with $[p_1] = [x]$, $[q_n] = [y]$, $[q_i] = [p_i + 1]$, $i = 1, 2, \ldots, n - 1$. In general this will only define a pseudometric, i.e. $d'([x], [y]) = 0$ does not necessarily imply that $[x] = [y]$. However for nice equivalence relations (e.g., those given by gluing together polyhedra along faces), it is a metric. Moreover if M is a compact space, then the induced topology on M/~ is the quotient topology.

The quotient metric d is characterized by the following universal property. If $f : (M, d) \to (X, \delta)$ is a metric map between metric spaces (that is, $\delta(f(x), f(y)) \leq d(x, y)$ for all (x, y) satisfying $f(x) = f(y)$ whenever $x \sim y$, then the induced function $\bar{f} : M/\sim \to X$, given by $\bar{f}([x]) = f(x)$, is a metric map $\bar{f} : (M/\sim, d') \to (X, \delta)$. A topological space is sequential if and only if it is a quotient of a metric space

Generalizations of Metric Spaces

- Every metric space is a uniform space in a natural manner, and every uniform space is naturally a topological space. Uniform and topological spaces can therefore be regarded as generalizations of metric spaces.
- If we consider the first definition of a metric space given above and relax the second requirement, or remove the third or fourth, we arrive at the concepts of a pseudometric space, a quasimetric space, or a semi-metric space.
- If the distance function takes values in the extended real number line $R \cup \{+\infty\}$, but otherwise satisfies all four conditions, then it is called an extended metric and the corresponding space is called an ∞-metric space.
- Approach spaces are a generalization of metric spaces, based on point-to-set distances, instead of point-to-point distances.
- A continuity space is a generalization of metric spaces and posets, that can be used to unify the notions of metric spaces and domains.

Metric Spaces as Enriched Categories

The ordered set $(\mathbb{R}, \geq)$ can be seen as a category by requesting exactly one morphism $a \to b$ if $a \geq b$ and none otherwise. By using + as the tensor product and 0 as the identity, it becomes a monoidal category R*. Every metric space (M, d) can now be viewed as a category M* enriched over R*:

- Set $Ob(M^*) := M$
- For each set $X, Y \in M$ set $Hom(X, Y) := d(X, Y) \in Ob(R^*)$.
- The composition morphism $Hom(Y, Z) \otimes Hom(X, Y) \to Hom(X, Z)$ will be the unique morphism in R* given from the triangle inequality $d(y, z) + d(x, y) \geq d(x, z)$.
- The identity morphism $0 \to Hom(X, X)$ will be the unique morphism given from the fact that $0 \geq d(X, X)$.
- Since R* is a strict monoidal category, all diagrams that are required for an enriched category commute automatically.

3.2 Modulus of Real Number

You know that a real number x is said to be positive if x is greater than 0. Equivalently, if 0 represents a unique point 0 on the real line, then a positive real number x lies on the right side of 0. Accordingly, we defined the inequality $x > y$ (in terms of this positivity of real numbers) if

$x - y > 0$. You will recall from Section 2.2 that for the validity of the properties of order relations or the inequalities. Such as the one concerning the multiplication of inequalities, it is essential to specify that some of the numbers involved should be positive. For example, it is necessary that $z > 0$ so that $x > y$ implies $xz > yz$. Again, the fractional power of a number will not be real if the number is negative, for instance $x^{1/2}$ when $x = -4$. Many of the fundamental inequalities, which you may come across in higher Mathematics, will involve such fractional powers of numbers. In this context, the concept of the absolute value or the modulus of a real member is important to which you are already familiar. Nevertheless, in this section, we recall the notion of the modulus of a real number and its related properties which we need for our subsequent discussion.

Defination: Modulus of Real Number

Let x be any real number. The absolute value or the modulus of x denoted by $|x|$ is defined as follows:

$$|x| = x \text{ if } x > 0$$
$$= -x \text{ if } x < 0$$
$$= 0 \text{ if } x = 0.$$

You can easily see that

$$|x| = |x|, \forall x \in R.$$

Not that $|-x|$ is different from $-|x|$.

3.2.1 Properties of the Modulus of Real Number

Since the modulus of a real number is essentially a non-negative real number, therefore the operations of usual addition, subtraction, multiplication and division can be performed on these numbers. The properties of the modulus are mostly related to these operations.

Property 1: For any real number x, $|x|$ = Maximum of (x, – x),

Proof: Since x is any real number, therefore either $x \geq 0$ or $x < 0$. If $x \geq 0$, then by definition, we have

$$|x| = X.$$

Also, $x > 0$ implies that $-x \leq 0$. Therefore, maximum of (x, – x} = x = $|x|$

Again $x < 0$, implies that $-x > 0$. Therefore again maximum of {x, –x} = –x = $|x|$.

Thus,

Maximum (x, – x} = $|x|$

Now consider the numbers $|5|^2$, $|-4.5|$, $\left|\frac{4}{5}\right|$. It is easy to see that

$$|5|^2 = |5| = 5.5 = 5^2 = |-5|^2$$

$$|-4.5| = |-20| = 20 \text{ Also } |-4|.\ |5| = 4.5 = 20$$

i.e. $|-4.5| = |-4|.\ |5|$

and

$$\left|\frac{4}{5}\right| = \frac{4}{5} \text{ and } \frac{|4|}{|5|} = \frac{4}{5} \text{ i.e.}$$

Notes

$$\left|\frac{4}{5}\right| = \frac{|4|}{|5|}.$$

All this lead us to the following properties:

Property 2: For any real number x

$|x|^2 = x^2 = |-x|^2$

Proof: We know that $|x| = x$ for x 2'0.

Thus $|x|^2 = |x|\,|x| = x, x = x$, for $x \geq 0$

Again for $x < 0$, we know that $|x| = -x$. Therefore

$|x^2| = |x|\;|x| = -x. -x = x^2$

Therefore, it follows that

$|x|^2 = x^2$ for any $x \in R$.

Now you should try the other part as an exercise.

Property 3: For any two real numbers x and y, prove that $|x.y| = |x|.|y|$.

Proof: Since x and y are any two real numbers, therefore, either both are positive or one is positive and the other is negative or both are negative i.e. either $x \geq 0$, $y \geq 0$ or $x \geq 0$, $y \geq 0$ or $x \geq 0$, $y \geq 0$ or $x \leq 0$, $y \leq 0$. We discuss the proof for all the four possible cases separately.

Case (i): When $x \geq 0$, $y \geq 0$.

Since $x \geq 0$, therefore, we have, by definition,

$|x| = x$, $|y| = y$

Also $x \geq 0$, $y \geq 0$ simply that $xy \geq 0$ and hence

$|xy| = xy = |x|\,|y|$

which proves the property.

Case (ii): When $x \geq 0$, $y \leq 0$. Then obviously $xy \leq 0$. Consequently by definition, it, follows that

$|x| = x$, $|y| = -Y$, $|xy| = -xy$

Hence

$|xy| = -xy = x(-y) = |x|\,|y|$

which proves the property.

Case (iii): When $x \leq = 0$, $y \geq 0$.

Interchange x and y in (ii).

***Case (iv)*:** When $x \leq 0$, $y \leq 0$, then $xy = 20$. Accordingly, we have

$|x| = -x$, $|y| = -y$, $|xy| = xy$.

Hence

$|xy| = xy = (-x)(-y) = |x|\,|y|$

using the field properties stated. This concludes the proof of the property.

Notes

Alternatively, the proof can be given by using property 2 in following way:

$|xy|^2 = (xy)^2 = x^2 y^2 = |x|^2.|y|^2$

$= (|x|.|y|)^2$

Therefore

$|xy| = \pm(|x|\,|y|)$

Since $|xy|$, $|x|$ and $|y|$ are non-negative, therefore we take the positive sign only and we have

$|xy| = |x|\,|y|$

which proves the property.

You can use any of the two methods to try the following exercise.

The next property is related to the modulus of the sum of two real members. This is one of the most important properties and is known as Triangular Inequality:

Property 4: Triangular Inequality

For any two real numbers x and y, prove that

$|x + y| \le |x| + |y|$.

Proof: For any two real numbers x and y the number $x + y \ge 0$ or $x + y < 0$.

If $x + y \ge 0$, then by definition

$|x + y| = x + y.$ . . .(1)

Also, we know that

$|x| \ge x \quad \forall\, x \in R$

$|x| \ge x \quad \forall\, y \in R$

Therefore

$|x| + |y| \ge x + y$

$x + y \le |x| + |y|.$ . . .(2)

From (1) and (2), it follows that

$|x + y| \le |x| + |y|$

Now, if $x + y < 0$, then again by definition, we have.

$|x + y| = -(x + y)$

or $|x + y| = (-x) + (-y)$ . . .(3)

Also we know that (see property 1)

$-x \le |x|$ and $-y \le |y|$.

Consequently, we get

$(-x) + (-y) \le |x| + |y|$

or $(-x) + (-y) \le |x| + |y|$ . . .(4)

From (3) and (4), we get

$|x + y| \le |x| + |y|$

Notes

This concludes the proof of the property.

You can try the following exercise similar to this property.

Now let us see another interesting relationship between the inequalities and the modulus of a real number.

By definition, $|x|$ is a non-negative real number for any $x \in R$. Therefore, there always exists a non-negative real number u such that

either $(x| < u$ or $|x| > u$ or $|x| = u$.

Suppose $|x| < u$. Let us choose u = 2. Then

$|x| < C \Rightarrow \text{Max.} \{-x, x\} < 2$

$\Rightarrow -x < 2, x < 2$

$\Rightarrow x > -2, x < 2$

$\Rightarrow -2 < x, x < 2$

$\Rightarrow -2 < x < 2.$

i.e. $|x| < 2 \Rightarrow -2 < x < 2$

Conversely, we have

$-2 < x < 2 \Rightarrow -2 < x < 2$

$\Rightarrow 2 > -x, x < 2$

$\Rightarrow -x < 2, x < 2$

$\Rightarrow \text{Max.} \{-x, x\} < 2$

$\Rightarrow |x| < 2.$

i.e.

$-2 < x < \Rightarrow |x| < 2$

Thus, we have shown that

$|x| < \Leftrightarrow -2 < x < 2.$

This can be generalised as the following property.

Property 5: Let x and u be any two real numbers.

$|x| \leq u \Leftrightarrow -u \leq x \leq u.$

Proof: $|x| \leq u \Leftrightarrow \text{Max.} \{-x, x\} \leq u$

$\Leftrightarrow -x \leq u, x \leq u$

$\Leftrightarrow x \geq -u, x \leq u$

$\Leftrightarrow -u \leq x, x \leq u$

$\Leftrightarrow -u \leq x \leq u$

which proves the desired property.

The property 5 can be generalized in the form of the following exercise.

Notes

Task For any real numbers x, a and d,

$|x - a| \le d \Leftrightarrow a - d \le x \le a + d.$

Example: Write the inequality $3 < x < 5$ in the modulus form.

Solution: Suppose that there exists real numbers a and b such that

$a - b = 3, a + b = 5.$

Solving these equations for a and b, we get

$a = 4 > b = 1.$

Accordingly,

$3 < x < 5 \Leftrightarrow 4 - 1 < x < 4 + 1$

$\Leftrightarrow -1 < x - 4 < 1$

$\Leftrightarrow |x - 4| < 1$

Task

1. Write the inequality $2 < x < 7$ in the modulus form.
2. Convert $|x - 2| < 3$ into the corresponding inequality.

3.3 Neighbourhoods

You are quite familiar with the word 'neighbourhood'. You use this word frequently in your daily life. Loosely speaking, a neighbourhood of a given point c on the real line is a set of all those points which are close to c. This is the notion which needs a precise meaning. The term 'close to' is subjective and therefore must be quantified. We should clearly say how much 'close to'. To elaborate this, let us first discuss the notion of a neighbourhood of a point with respect to a (small) positive real number δ.

Let c be any point on the real line and let $6 > 0$ be a real number. A set consisting of all those points on the real line which are at a distance of 6 from c is called a neighbourhood of c. This set is given by

$\{x \in R : |x - c| < 6\}$

$= \{x \in R : c - \delta < x < c + 6)$

$=]c - 6, c + \delta[$

Which is an open interval. Since this set depends upon the choice of the positive real number δ, we call it a 6-neighbourhood of the point c.

Thus, a δ-neighboured of a point c on the real line is an open interval $]c - 6, c + 6[$, $\delta > 0$ while c is the mid point of this neighbourhood. We can give the general definition of neighbourhood of a point in the following way.

Notes

Neighbourhood of a Point

A set P is said to be a Neighbourhood (NBD) of a point V if there exists an open interval which contains c and is contained in P.

This is equivalent to saying that there exists an open interval of the form $]c - \delta, c + \delta[$, for some $6 > 0$, such that

$$]c - 6, c + \delta[\subset P.$$

Example: (i) Every open interval]a, b[is a NBD of each of its points.

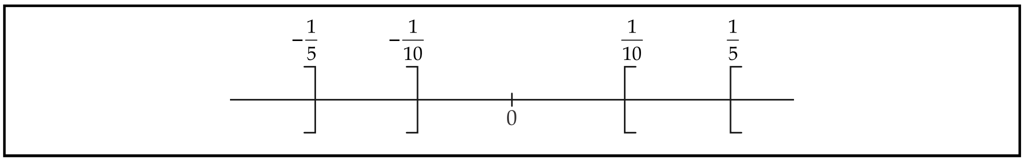
(ii) A closed interval [a, b] is a NBD of each of its points except the end point i.e. [a, b] is not a NBD of the points a and b, because it is not possible to find an open interval containing a or b which is contained in [a, b]. For instance, consider the closed interval [0,1]. It is a NBD of every point in]0, l[. But, it is not a NBD of 0 because for every $\delta > 0$, $]-\delta, \delta[\not\subset [0, 1]$. Similarly [0, 1] is not a NBD of 1.

(iii) The null set 0 is a NBD of each of its point in the sense there is no point in 0 of which it is not a NBD.

(iv) The set R of real numbers is a NBD of each real number x because for every $5 > 0$, the open interval $]x - 6, x + \delta[$ is contained in R.

(v) The set Q of rational numbers is not a NBD of any of its points x because any open interval containing x will also contains an infinite number of irrational numbers and hence the open interval can not be a subset of Q.

Now consider any two neighbourhoods of the point 0 say $]-\frac{1}{10}, \frac{1}{10}[$ and $]-\frac{1}{5}, \frac{1}{5}[$ as shown in the Figure below.



The intersection, of these two neighbourhood is

$$]-\frac{1}{10}, \frac{1}{10}[\cap]-\frac{1}{5}, \frac{1}{5}[=]-\frac{1}{10}, \frac{1}{10}[$$

which is again a NBD of 0. The union of these two neighbourhoods is $]-\frac{1}{5}, \frac{1}{5}[$, which is also a NBD of 0. Let us now examine these results in general.

Example: The intersection of any two neighbourhoods of a point is a neighbourhood of the point.

Solution: Let A and B be any two NBDS of a point c in R. Then there exist open intervals $]c - \delta_1, c + \delta_1]$ and $]c - \delta_2, c + \delta_2$ [such that]$c - \delta_1, c + \delta_1] \subset A$, for some $\delta_1 > 0$, and $]c - \delta_2, c + \delta_2 [\subset B$, for some $\delta_2 > 0$.

Let $6 = \text{Min.} \{\delta_1, \delta_2)$ = minimum of δ_1, δ_2.

This implies that $]c - 6, c + \delta [C\ A \cap B$ which shows that $A \cap B$ is a NBD of c.

Notes

Example: Show that the superset of a NBD of a point is also a NBD of the point.

Solution: Let A be a NBD of a point c. Then there exists an open interval]c – 6, – c + 6[, for some 6 > 0 such that

$]c - 6, c + \delta[\; C \; A.$

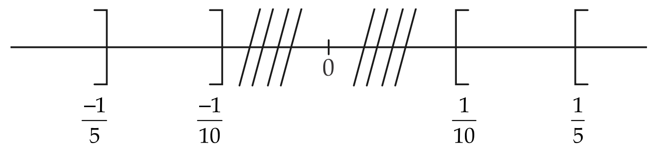
Now let S be a super set which contains A. Then obviously

$A \subset S \Rightarrow [c - \delta, c + \delta[\; \subset S$

which shows that S is also a NBD of c.

For instance, if $] \frac{1}{10}, \frac{1}{10} [$ is a NBD of the point 0.

Then, $] -\frac{1}{5} \; \frac{1}{5} [$ is also a NBD of 0 as can be seen from Figure below.



Is a subset of a NBD of a point also a NBD of the point? Justify your answer.

Now you can try the following exercise.

Task Prove that the Union of any two NBDS of a point is a NBD of the point.

The conclusion of the Exercise, in fact, can be extended to a finite or an infinite or an arbitrary number of the NBDS of a point.

However, the situation is not the same in the case of intersection of the NBDS. It is true that the intersection of a finite number of NBDS of a point is a NBD of the point. But the intersection of an infinity collection of NBDS of a point may not be a NBD of the point. For example, consider the class of NBDS given by a family of open intervals of the form

$I_1 =] -1, 1 [, I_2 =] -\frac{1}{2}, \frac{1}{2} [\; I_3 =] -\frac{1}{3}, \frac{1}{3} [,.$

$I_n =] -\frac{1}{n}, \frac{1}{n} [...$

which are NBDS of the point 0. Then you can easily verify that

$I_1 \cap I_2 \cap I_3 \cap I_4 \cap .... \cap I_n \cap .....$

or $\bigcap_{n=1}^{\infty} I_n = \{0\}$

3.4 Open Sets

You have seen from the previous examples and exercises that a given set may or may not be a NBD of a point. Also, a set may be a NBD of some of its points and not of its other points. A set

Notes

may even be a NBD of each of its points as in the case of the interval]a, b[. Such a set is called an open set.

Definition: X a set S is said to be open if it is a neighbourhood of each of its points.

Thus, a set S is open if for each x in S, there exists an open interval]x – 6, x + δ[, δ > 0 such that

$x \in]\, x - \delta, x + \delta [\cap S.$

It follows at once that a set S is not open if it is not a NBD of even one of its points.

Example: An open interval is an open set

Solution: Let]a, b[be an open interval. Then a < b. Let c∈] a, b[. Then a < c < b and therefore

$c - a > 0$ and $b - c > 0$

Choose

$$\delta = \text{Minimum of } \{b - c, c - a)$$
$$= \text{Min } (b - c, c - a).$$

Note that $b - c > 0$, $c - a > 0$. Therefore $\delta > 0$.

Now $\delta \le c - a \Rightarrow a \le c - \delta$

and $\delta \le b - c \Rightarrow c + \delta < b$.

i.e.

Therefore,]c – 6, c + δ[⊂]a, b[and hence]a, b[is a NBD of c.

Example: (i) The sat R of real numbers is an open set

(ii) The null set f is an open set

(iii) A finite set is not an open set

(iv) The interval]s, b] is not an open set.

Example: Prove that the intersection of any two open sets is an open set.

Solution: Let A and B be any two open sets. Then we have to show that $A \cap B$ is also an open set. If $A \cap B = \phi$, then obviously $A \cap B$ is an open set. Suppose $A \cap B \neq \phi$.

Let x be an arbitrary element of $A \cap B$. Then $x \in A \cap B \Rightarrow x \in A$ and $x \in B$.

Since A and B are open sets, therefore A and B are both NBDS of x. Hence $A \cap B$ is a NBD of x. But $x \in A \cap B$ is chosen arbitrarily. Therefore, $A \cap B$ is a NBD of each of its points and hence $A \cap B$ is an open set. This proves the result. In fact, you can prove that the intersection of a finite number of open sets is an open set. However, the intersection of an infinite number of open sets may not be an open set.

Task

1. Give an example to show that intersection of an infinite number of open sets need not be an open set.
2. Prove that the union of any two open sets is an open set. In fact, you can show that the union of an arbitrary family of open sets is an open set.

3.5 Limit Point of a Set

Notes

You have seen that the concept of an open set is linked with that of a neighbourhood of a point on the real line. Another closely related concept with the notion of neighbourhood is that of a limit point of a set. Before we explain the meaning of limit point of a set, let us study the following situations:

(i) Consider a set S = [1, 2[> Obviously the number 1 belongs to S. In any NBD of the point 1, we can always find points of S which are different from 1. For instance]0-5, l[is a NBD of 1. In this NBD, we can find the point 1.05 which is in S but at the same time we note that $1.05 \neq 1$,

(ii) Consider another set S = $\left\{\frac{1}{n} : n \in N\right)$. The number 0 does not belong to this set.

Take any NBD of 0 say,] –0.1, 0.1 [. The number $\frac{1}{20}$ = 0.05 of S is in this NBD of 0. Note that $0.05 \neq 0$.

(i) Again consider the same set S of (ii) in which the number 1 obviously belongs to S. We can find a NBD of 1, say]0.9, 1.1[in which we can not find a point of S different from 1.

In the light of the three situations, we are in a position to define the following:

Limit Point of a Set

A number p is said to be a limit point of a set S of real numbers if every neighbourhood of p contains at least one point of the set S different from p.

Examples: (i) In the set S = [1, 2[, the number 1 is a limit point of S This limit point belongs to S. The set S = $\left\{\frac{1}{n} : n \in N\right)$ has only one limit point 0. You may note that 0 does not belong to S.

(ii) Every point in Q, (the set of rational numbers), is a limit point of Q, because for every rational number r and $\delta > 0$, i.e.] r – 6, r + δ [has at least one rational number different from r. This is because of the reason that there are infinite rationals between any two real numbers. Now, you can easily see that every irrational number is also a limit point of the set Q for the same reason.

(iii) The set N of natural numbers has no limit point because for every real number a, you can always find $\delta > 0$ such that]a – 6, a + δ[does not contain a point of the set N other than a.

(iv) Every point of the interval]a, b] is its limit point. The end points a and b are also the limit points of]a, b]. But the limit point a does not belong to it whereas the limit point b belongs to it.

(v) Every point of the set [a, ∞[is a limit point of the sets. This is also true for] –∞, a[.

From the foregoing examples and exercises, you can easily observe that

(i) A limit point of set may or may not belong to the set,

(ii) A set may have no limit point,

(iii) A set may have only one limit point.

(iv) A set may have more than one limit point.

Notes

The question, therefore, arises: "How to know whether or not a set has a limit point?" One obvious fact is that a finite set can not have a limit point. Can you give a reason for it? Try it. But then there are examples where even an infinite set may not have a limit point e.g. the sets N and Z do not have a limit point even though they are infinite 'sets. However, it is certainly clear that a set which has a limit point, must necessarily be an infinite set. Thus our question takes the following form:

"What are the conditions for a set to have a limit point?"

This question was first studied by a Czechoslovakian Mathematician, Bemhard Bulzano [1781-1848] in 1817 and he gave some ideas.

Unfortunately, his ideas were so far ahead of their time that the world could not appreciate the full significance of his work. It was only much later that Bulzario's work was extended by Karl Weierstrass [1815-1897], a great German Mathematician, who is known as the "father of analysis". It was in the year 1860 that Weierstrass proved a fundamental result, now known as Bulzano-Weierstrass Theorem for the existence of the limit points of a set. We state and prove this theorem as follows.

3.5.1. Bulzano Weierstrass Theorem

Theorem 1: Every infinite bounded subset of set R has a limit point (in K).

Proof: Let S be an infinite and bounded subset of R. Since A is bounded, therefore A has both a- lower bound as well as an upper bound.

Let m be a lower bound and M be an upper bound of A. Then obviously

$m \leq x \leq M, \ \forall x \in A.$

Construct a set S in the following way:

S = (x∈R: x exceeds at most finite number of the elements of A}. Now, let us examine the following two questions:

(i) Is S a non-empty set?

(ii) Is S also a bounded set?

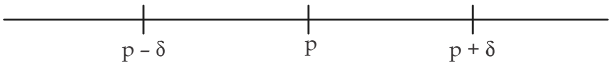
Indeed, S is non-empty because $m \leq x, \leq M, \ \forall x \in A$, implies that $m \in S$. Also M is an upper bound of S because no number greater than or equal to M can belong to S. Note that M cannot belong to S because it exceeds an infinite number of elements of A.

Since the set S is non-empty and bounded above, therefore, by the axiom of completeness, S has its supremum in R. Let p be the supremum of S. We claim that p is a limit point of the set A.

In order to show that p is a limit point of A, we must establish that every NBD of p has at least one point of the set A other than p. In other words, we have to show that every NBP of p has an infinite number of elements of A. For this, it is enough to show that any open interval $]p - \delta, p + \delta[$, for $\delta > 0$, contains an infinite number of members of set A. For this, we proceed as follows.

Since p is the supremum of S, therefore, by the definition of the Supremum of a set, there is at least one element y in S such that $y > p - \delta$, for $\delta > 0$. Also y is a member of S, therefore, y exceeds at the most a finite number of the elements of A. In other words, if you visualise it on the line as shown in the Figure below, the number of elements of A lying on the left of $p - \delta$ is finite at the most. But certainly, the number of elements of A lying on the right side of the point $p - \delta$ is infinite.

Notes



Again since p is the supremum of S, therefore, by definition p + δ can not belong to S. In other Words, p + δ exceeds an infinite number of elements of A. This means that there lie an infinite number of elements of A on the left side of the point p + δ.

Thus we have shown that there lies, an infinite number of elements of A on the right side of p – δ and also there ia an infinite number of elements of A on the left side of p + δ. What do you conclude from this? In other words, what is the number of elements of A in between (i.e., within) the interval]p – δ, p + δ[. Indeed, this number is infinite i.e., there is an infinite number of elements of A in the open interval]p + δ[. Hence the interval] p – δ, p + δ [contains an infinite number of elements of A for some δ > 0. Since δ > 0 is chosen arbitrarily, therefore every interval]p – δ, p + δ [has an infinite number of elements of A. Thus, every NBD-of p contains an infinite number of elements of A. Hence p is a limit point of the set A.

This completes the proof of the theorem.

Example: (i) The intervals [0, 1],]0,1[,] 0, 1], [0,1[are all infinite and bounded sets. Therefore each of these intervals has a limit point. In fact, each of these intervals has an infinite number of limit points because every point in each interval is a limit point of the interval.

(ii) The set [a, ∞[is infinite and unbounded set but has every point as a limit point. This shows that the condition of boundedness of an infinite set is only sufficient in the theorem.

From the previous examples and exercises, it is clear that it is not necessary for an infinite set to be bounded to possess a limit point. In other words, a set may be unbounded and still may have a limit point. However, for a set to have a limit point, it is necessary that it is infinite.

Another obvious fact is that a limit point of a set may or may not belong to the set and a set may have more than one limit point. We shall further study how sets can be characterized in terms of their limit points.

3.6 Closed Sets

You have seen that a limit point of a set may or may not belong to the set. For example, consider the set $S = \{x \in R : 0 \leq 5x < 1\}$. In this set, 1 is a limit point of S but it does not belong to S. But if you take $S = \{x: 0 \leq x \leq 1\}$, then all the limit points of S belong to S. Such a set is called a closed set. We define a closed set as follows:

Definition

A set is said to be closed if it contains all its limit points.

Example: (i) Every closed and bounded interval such as [a, b] and [0, 1] is a closed set.

(ii) An open interval is not a closed set. Check Why?

(iii) The set R is a closed set because every real number is a limit point of R and it belongs to R.

(iv) The null set ϕ is a closed set.

(v) The set $S = \left\{\frac{1}{n} : n \varepsilon N\right\}$ is not a closed set. Why?

(vi) The set]a, ∞[is not a closed set, but] –∞, a] is a closed set.

Notes

You may be thinking that the word open and closed should be having some link. If you are guessing some relation between the two terms, then you are hundred per cent correct. Indeed, there is a fundamental connection between open and closed sets. What exactly is the relation between the two? Can you try to find out? Consider, the following subsets of R:

(i) $]0, 4[$

(ii) $[-2, 5]$

(iii) $]0, = \infty[$

(iv) $] -\infty > 6]$.

The sets (i) and (iii) are open while (ii) and (iv) are closed. If you consider their complements, then the complements of the open sets are closed while those of the closed sets are open. In fact, we have the following concrete situation in the form of following theorem.

Theorem 2: A set is closed if and only if its complement is open.

Proof: We assume that S is a closed set. Then we prove that its complement S^c is open.

To show that S^c is open, we have to prove that S^c is a NBD of each of its points. Let $x \in S^C$. Then, $x \in S^c \Rightarrow x \notin S$. This means x is not a limit point of S because S is given to be a closed set. Therefore, there exists a $\delta > 0$ such that $]x - \delta, x + \delta[$ contains no points of S. This means that $]x - \delta, x + \delta[$ is contained in S^c. This further implies that S^c is a NBD of x. In other words, S^c is an open set, which proves the assertion.

Conversely, let a set S be such that its complement S^c is open. Then we prove that S is closed.

To show that S is closed, we have to prove that every limit point x of S belongs to S. Suppose $x \notin S$, Then $x \notin S^C$.

This implies that S^C is a NBD of x because S^c is open. This means that there exists an open interval $]x - \delta, x + \delta[$, for some $6 > 0$, such that

$$]x - \delta, x + \delta[\in S^c$$

In other words, $]x - \delta, x + \delta[$ contains no point of S. Thus x is not a limit point of S, which is a contradiction. Thus our supposition is wrong and hence, $x \notin S$ is not possible. In other words, the (limit) point x belongs to S and thus S is a closed set.

Note that the notions of open and closed sets are not mutually exclusive. In other words, if a set is open, then it is not necessary that it can not be closed. Similarly, if a set is closed, then it does not exclude the possibility of its being open. In fact, there are sets which are both open and closed and there are sets which are neither open nor closed as you must have noticed in the various examples we have given in our discussion. For example the set R of all the real numbers is both an open sets as well as a closed set. Can you give another example? What about the null set. Again Q, the set of rational numbers is neither open nor closed.

Task Give examples of two sets which are neither closed nor open.

We have discussed the behaviour of the union and intersection of open sets. Since closed sets are closely connected with open sets, therefore, it is quite natural that we should say something about the union and intersection of closed sets. In fact, we have the following results:

Notes

Example: Prove that the union of two closed sets is a closed set.

Solution: Let A and B be any two closed sets. Let $S = A \cup B$, we have to show that S is a closed set. For this, it is enough to prove that the complement S^c is open

Now

$$S^c = (A \cup B)^c = B^c \cap A^c = A^c \cap Bc$$

Since A and B are closed sets, therefore A^c and B^c are open sets. Also, we have proved in the intersection of any two open sets is open. Therefore $A^c \cap B^c$ is an open set and hence S is open.

This result can be extended to a finite number of closed sets. You can easily verify that the union of a finite number of closed sets is a closed set. But, note that the union of an arbitrary family of closed sets may not be closed.

For example, consider the family of closed sets given as

$$S_1 = [l, 2], S_2 = [\frac{1}{2}, 2], S_3 = [\frac{1}{3}, 2], ...$$

and in general

$$S_n = [\frac{1}{n}, 2]..... \text{ for } n = 1, 2, 3, ...$$

Then,

$$\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup S_3 ..... \cup S_n \cup .....$$

$$= I0, 2]$$

which is not a closed set.

Definition: Derived Set

The set of all limit points of a given set S is called the derived set and is denoted by S'.

Example: (i) Let S be a finite set. Then $' = ϕ

(ii) $S = (\frac{1}{n} : n \in N)$, the derived set S' = {0}

(iii) The derived set of R is given by R' = R

(iv) The derived set of Q is given by Q' = R

We define another set connected with the notion of the limit point of a set. This is called the closure of a set.

Definition: Closure of a Set

Let S be any set of real numbers ($S \in R$). The closure of S is defined as the union of the set S and its derived set S. It is denoted by $\bar{S}$, Thus

$$\bar{S} = S \cup S'$$

In other words, the closure of a set is obtained by the combination of the elements of a given set S and its derived set S'.

Notes

For example, $\bar{S}$ of $S = \{\frac{1}{n} : n \in N\}$ is given by.

$$\bar{S} = \{\frac{1}{n}, n \in N\} \cup \{0\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$$

Similarly, you can verify that

$$\vec{Q} = Q \cup Q' = Q \cup R = R$$

$$\vec{R} = R \cup R' = R \cup R = R$$

3.7 Compact Sets

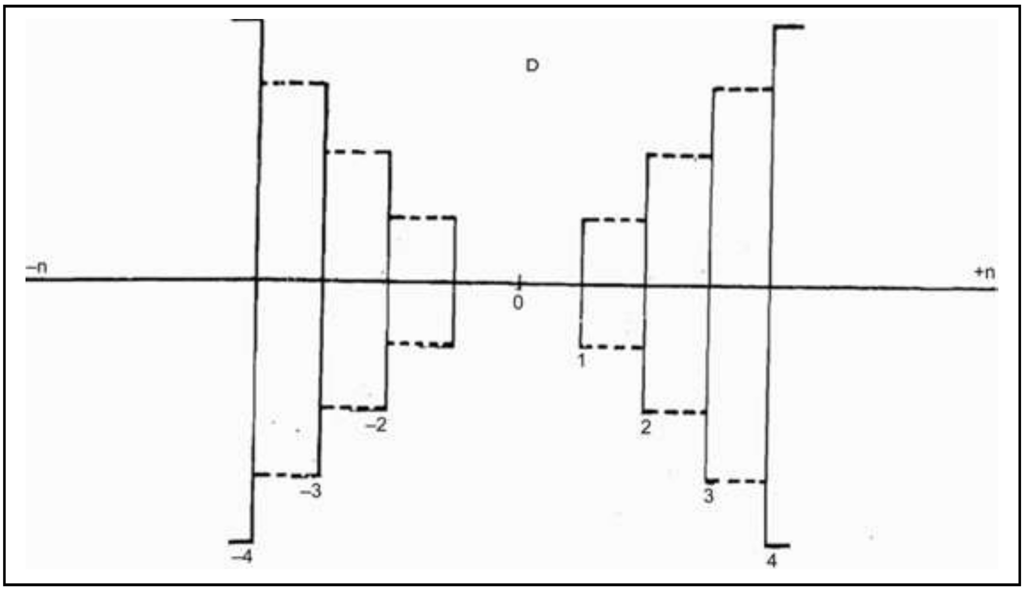
We discuss yet another concept of the so called compactness of a set. The concept of compactness is formulated in terms of the notion of an open cover of a set.

Definition: Open Cover of a Set

Let S be a set and $\{G_\alpha\}$ be a collection of some open subsets of R such that $S \subset \cup G_\alpha$. Then {G} is called an open cover of S.

Example: Verify that the collection $G_n = \{G\}^{\infty}_{n=\infty}$, where $G_n =]-n, n[$ is an open cover of the set R.

Solution:



As shown in the Figure above, we see that every real number belongs to some G_n.

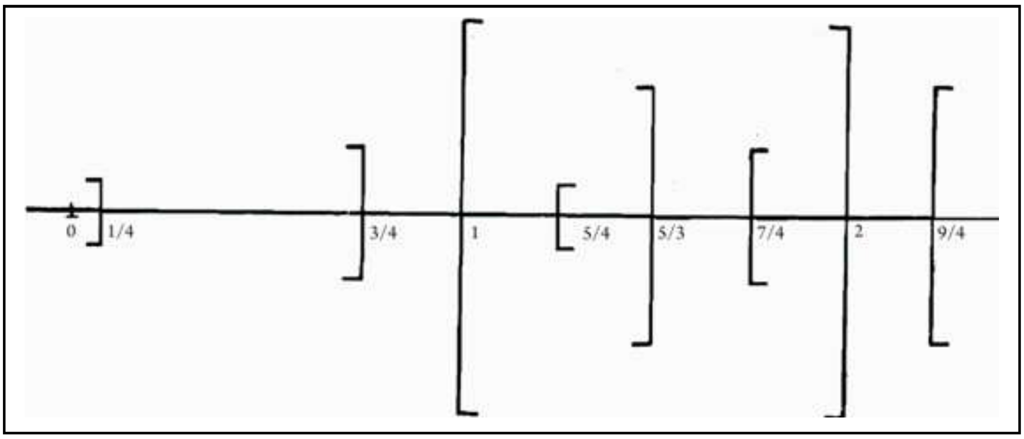
Hence,

$$R = \bigcup_{n=1}^{\infty} G_n$$

Example: Examine whether or not the following collections are open covers of the interval [1, 2].

Notes

Solution: (i) Plot the subsets of G, on the real line as shown in the Figure.



From above figure, it follows that every element of the set S = [1, 2] – {x : 1 ≤ x ≤ 2) belongs to at least one of the subsets of G. Since each of the subsets in G, is an open set, therefore G, is an open cover of S.

(ii) Again plot the subsets of G_j on the real line as done in the case of (i).

You will find that none of the points in the interval $[\frac{5}{4}, \frac{3}{2},]$ belongs to any of the subsets of G_2.

Therefore G_2 is not an open cover of S.

Now consider the set [0, 1] and two classes of open covers of this set namely G, and G given as

$$G_1 = \{] -\frac{1}{n}, 1 + \frac{1}{n} [\}_{n=1}^{\infty}, G_2 = \{] -1 - \frac{1}{2n}, 1 + \frac{1}{2n} [\}_{n=1}^{\infty}.$$

You can see that $G_2 \subset G$,. In this case, we say that G, is a subcover of G. In general, we have the following definition.

Definition: Subcover and finite subcover of a set

Let G be an open cover of a set S. A subcollection E of G is called a subcover of S if E too is a cover of S. Further, if there are only a finite number of sets in E, then we say that E is a finite subcover of the open cover G of S. Thus, if G is an open cover of a set S, then a collection E is a finite subcover of the open cover G of S provided the following three conditions hold.

(i) E is contained in G.

(ii) E is a finite collection.

(iii) E is itself a cover of S.

From the forgoing example and exercise, it follows that an open cover of a set may or may not admit of a finite subcover. Also, there may be a set whose every open cover contains a finite subcover. Such a set is called a compact set. We define a compact set in the following way.

Definition: Compact set

A set is said to be compact if every open cover of it admits of a finite subcover of the set.

For example, consoder the finite set S = {1, 2, 3} and an open cover $\{G_\alpha\}$ of S. Let G^1, G^2, G^3, be the sets in G which contain 1, 2, 3 respectively. Then $\{G^1, G^2, G^3\}$ is a finite subcover of S. Thus S is a compact set. In fact, you can show that every finite set in R is a compact set.

The collection G = {] – n, n[: n ∈N} is an open cover of R but does not admit of a finite subcover of R. Therefore the set R is not a compact set.

Thus you have seen that every finite set is always compact. But an infinite set may or may not be a compact set. The question, therefore, arises, "What is the criteria to determine when a given set is compact?" This question has been settled by a beautiful theorem known as Heine-Borel Theorem named in the honour of the German Mathematician H.E. Heine [1821-1881] and the French Mathematician F.E.E. Borel [1871-1956], both of whom were pioneers in the development of Mathematical Analysis.

We state this theorem without proof.

Theorem: Heine-Borel Theorem

Every closed and bounded subset of R is compact.

The immediate consequence of this theorem is that every bounded and closed interval is compact.

Self Assessment

Fill in the blanks:

1. A number p is said to be a .... of real numbers if every neighbourhood of p contains at least one point of the set S different from p.
2. Let S be an infinite and bounded subset of R. Since A is bounded, therefore A has both a lower bound as well as an ....
3. A set is said to be closed if it contains all its ....
4. A set is closed if and only if its .... is open.
5. Let S be a set and $\{G_\alpha\}$ be a collection of some open subsets of R such that .... Then {G} is called an open cover of S.
6. A set is said to be compact if every open cover of it admits of a .... of the set.

3.8 Summary

- We have defined the absolute value or the modulus of a real number and discussed certain related properties. The modulus of real number x is defined as

  $|x| = x \quad \text{if} \quad x \geq 0$

  $\quad = -x \quad \text{if} \quad x < 0.$

  Also, we have shown that

  $|x - a| < d \Leftrightarrow a - d < x < a + d$
- We have discussed the fundamental notion of NBD of a point on the real line i.e. first we have defined it as a δ – neighbourhood and then, in general, as a set containing, an open interval with the point in it.
- With the help of NBD of a point we have defined, an open set in the sense that a set is open if it is a NBD of each of its points.
- We have introduced the notion of the limit point of a set. A point p is said to be a limit point of a set S if every NBD of p contains a point of S different from p. This is equivalent to saying that a point p is a limit point of S if every NBD of p contains an infinite number of the members of S. Also, we have discussed Bulzano-Weiresstrass theorem which gives

a sufficient condition for a set to possess a limit point. It states that an infinite and bounded set must have a limit point. This condition is not necessary in the sense that an unbounded set may have a limit point.

- The limit points of a set may or may not belong to the set. However, if a set is such that every limit point of the set belongs to it, then the set is said to be a closed set. The concept of a closed set has been discussed. Here, we have also shown a relationship between a closed set and an open set in the sense that a set is closed if and only if its complement is open. Further, we have also defined the Derived set of a set S as the set which consists of all the limit points of the set S. The Union of a given set and its Derived set is called the closure of the set. Note the distinction between a closed set and the closure of a set S.
- Finally, we have introduced another topological notion. It is about the open cover of a given set. Given a set S, a collection of open sets such that their Union contains the set S is said to an open cover of S. A set S is said to be compact if every open cover of S admits of a finite subcover. The criteria to determine whether a given set is compact or not, is given by a theorem named Heine-Borel Theorem which states that every closed and bounded subset of R is compact. An immediate consequence of this theorem is that every bounded and closed interval is compact.

3.9 Keywords

Bulzano Weierstrass Theorem: Every infinite bounded subset of set R has a limit point (in K).

Compact Set: A set is said to be compact if every open cover of it admits of a finite subcover of the set.

Heine-Borel Theorem: Every closed and bounded subset of R is compact.

3.10 Review Questions

1. Prove that $-|x| = \text{Min.}\{x, -x\}$ for any $x \in R$. Deduce that $-|x| \le x$, for every $|x| \in R$. Illustrate it with an example.
2. Prove that $|x|^2 = x^2$, for my $x \in R$.
3. For any two real numbers x and y ($y \neq \theta$), prove that

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

4. Prove that $|x - y| \ge ||x| - |y||$ for any real numbers x and y.
5. Test which of the following are open sets:
   (i) An interval [a, b] far $a \in R$, $b \in R$, $a < b$
   (ii) The intervals [0, l [; and] 0, 1[
   (iii) The set Q of rational numbers
   (iv) The set N of natural numbers and the set Z of integers.
   (v) The set $\left\{\frac{1}{n} : n \in N\right\}$
   (vi) The intervals $]a, \infty[$ and $[a, \infty[$ for $a \in R$.

Notes

Answers: Self Assessment

1. limit point of a set S
2. upper bound
3. limit points
4. complement
5. $S \subset \cup G_\alpha$
6. finite subcover

3.11 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 4: Compactness

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the compactness of a set
- Explain intersection of closed set
- Discuss compactness and continuity
- Describe sequential compactness

Introduction

In last unit you have studied about matric spaces. You all go through concept of open sets, limit points of sets in last unit. This unit will provide you explanations of compactness of a set.

4.1 Compactness

Definition:

- A cover of A is a collection U of sets whose union contains A.
- A subcover is a subcollection of U which still covers A.
- A subcover is open if its members are all open.

Definition: Topological space T is compact if every open cover has finite subcover.

Theorem: (Heine-Borel). Any closed bounded interval $[a, b] \subset \mathbb{R}$ is compact.

Proof: Let U be open cover of [a, b]. Let

$$A = \{x \in [a, b] : [a, x] \text{ covered by finite subfamily of } U\}$$

Notes

Then $a \in A$ so $A \neq \theta$, bounded above by b. Let c = sup A. $a \leq c \leq b$ so $c \in U$ for some $U \in \mathcal{U}$. U open so $\exists\ \delta > 0$ s.t. $(c - \delta, c + \delta) \cdot \subset U$.

c = sup A so $\exists\ x \in A$ s.t. $x > c - \delta$. $[a, c + \delta) \subseteq [a, x] \cup (c - \delta, c + \delta)$ can be covered by finite subfamily of $\mathcal{U}$ so $(c, c + \delta) \cap [a, b] = \theta$ (since any point in here is in A but > c sup A). So c = b.

4.2 Compactness of Subsets

Proposition: Any closed subset C of compact space compact.

Proof: Let $\mathcal{U}$ be cover of C by sets open in T. Adding open T\C get open cover of T. Finite subcover of this cover contains finite subcover of C of sets from $\mathcal{U}$.

Proposition: Compact subspace C of Hausdorff T is closed in T.

Proof: $a \in T\backslash C$. $\forall\ x \in C\ \exists$ disjoint $U_x \ni x$, $V_x \ni a$ open in T since T Hausdorff. U_x open cover of C so has finite subcover $U_{x1},\ldots, U_{xn}$. Then $V = \bigcap_{i=1}^{n} V_{xi}$ open, $a \in V$ and disjoint from C. Hence $a \in (T\backslash C)^{\circ}$ and T\C open.

Proposition: Compact subspace C of metric space M is bounded.

Proof: Let $a \in M$. Balls B(a, r) (r > 0) are open and cover C, so $\exists\ r_1,\ldots, r_n$ s.t. $C \subset \bigcap_{i=1}^{n} B(a, r_i) = B(a, \max\{r_1, \ldots, r_n\})$.

4.3 Intersections of Closed Sets

Theorem: Let F be collection of non-empty closed subsets of compact T s.t. every finite subcollection of F has non-empty intersection. Then intersection of all sets from T non-empty.

Proof: Assume intersection of all sets empty. Let $\mathcal{U}$ be collection of complements. $\mathcal{U}$ covers T by DeMorgan. U open cover so exists finite subcover $U_1,\ldots, U_n$. Then $F_i := T\backslash U_i \in F$ and empty intersection by DeMorgan. This contradicts the assumption of the theorem.

Corollary: Let $F_1 \supset F_2 \supset \ldots$ sequence of non-empty closed subsets of compact T. Then $\bigcap_{k=1}^{\infty} F_k \neq \theta$.

Corollary: Let $F_1 \supset F_2 \supset \ldots$ sequence of non-empty compact subsets of Hausdorff T. Then $\bigcap_{k=1}^{\infty} F_k \neq \theta$.

Proof: By proposition 4.4 compact subsets of Hausdorff space are closed.

4.4 Compactness of Products

Lemma: T, S compact, U open cover of $T \times S$. If $s \in S$ there exists open $V \subset S$, $s \in V$ s.t. $T \times V$ can be covered by finite subfamily of $\mathcal{U}$.

Proof: $\forall\ x \in T$ find $W_x \in \mathcal{U}$ s.t. $(x, s) \in W_x$. Exists open $U_x \subset T$, $V_x \subset S$ s.t. $(x, s) \in U_x \times V_x \subset W_x$. $\{U_x : x \in T\}$ open cover of T so $\exists\ U_{x1},\ldots, U_{xn}$ which cover T. Let $V = \bigcap_{i=1}^{n} V_{xi}$. $V \subset S$ open and

$$T \times V \subset \bigcup_{i=1}^{n} U_{xi} \times V_{xi} \bigcup_{i=1}^{n} W_{xi}$$

Theorem: (Tychonov). S,T compact $\Rightarrow T \times S$ compact.

Proof: By lemma 3.8 $\forall\ y \in S\ \exists\ V_y \subset S$ open s.t. $T \times V_y$ can be covered by finite subfamily of $\mathcal{U}$. S compact, $\{V_y : y \in S\}$ form open cover so $\exists\ V_{yi},\ldots, V_{ym}$ which cover S.

Notes

$T \times S = \bigcup_{j=1}^{m} T \times V_{yi}$. Finite union, each $T \times V_{yj}$ can be covered by finite subfamily of *U*, so $T \times S$ can be covered by finite subfamily of *U*.

4.5 Compactness and Continuity

Proposition: Cts image of compact space compact.

Proof: f: $T \to S$ cts, T compact. *U* open cover of $f(T)$. $f^{-1}(U)$ open $\forall\, U \in U$.

Cover T since $\forall\, x \in T\, f(x)$ in some $U \in U$. Hence $\exists\, f^{-1}(U_1),\dots, f^{-1}(U_n)$ subcover of T. $\forall\, y \in f(T)$ have $y = f(x)$ where $x \in T$ so $x \in f^{-1}(U_i)$ for some *i* so $y \in U_i$. Hence $U_1,\dots,U_n$.

Theorem: Cts bijection of compact T onto Hausdorff S is homeomorphism.

Proof: U open in T, T\U closed so compact.

Therefore $(f^{-1})^{-1}(U) = f(U) = S \setminus f(T \setminus U)$ open, so f^{-1} cts.

Corollary: Let T be compact. Cts $f : T \to \mathbb{R}$ is bdd and attains max and min.

Proof: f(T) compact so closed.

Then $\sup f(T) \in \overline{f(T)} = f(T)$.

Alternative proof: Let $c = \sup_x \in T\, f(x)$. If f not attain c then $f(x) < c\, \forall\, x$ so $\{x : f(x) < r\} = f^{-1}(-\infty, a)$ where $r < c$ s.t. $T \subset \bigcup_{i=1}^{n} \{x : f(x) < r_i\}$. Then $f(x) < \max\{r_1,\dots, r_n\}\, \forall\, x$ so $c = \sup_{x\in T} f(x) \le \max\{r_1,\dots, r_n\} < c$ Contradiction.

Definition: Given cover *U* of metric M, $\delta > 0$ called Lebesgue number of *U* if $\forall\, x \in M\, \exists\, U \in U$ s.t. $B(x, \delta) \subset U$.

Proposition: Every open cover U of compact metric space has a Lebesgue number.

Proof: $\forall\, x \in M$ pick $r(x) > 0$ s.t. $B(x, r(x))$ contained in some set of *U*. Then $M \bigcup_{x\in M} B\left(x, \frac{r(x)}{2}\right)$ so $\exists\, x_1, \dots, x_j$ s.t. $M \subset \bigcup_{i=1}^{j} B\left(x_i, \frac{r(x_i)}{2}\right)$. Let $\delta = \dfrac{\min\{r(x_1),\dots,r(x_j)\}}{2}$. Then $\forall\, x \in M$ pick *i* s.t. $x \in B\left(x_i, \frac{r(x_i)}{2}\right)$ and $B(x, \delta) \subset B(x_i, r(x_i))$ subset of some set from *U*.

Theorem: Cts map of compact metric M to metric N is uniformly cts.

Proof: Let $\in > 0$. Then sets $U_z = f^{-1}(B_N(f(z), \frac{\varepsilon}{2}))$ $z \in M$ open cover of M. Let be Lebesgue number. If $x, y \in M$, $d_M(x, y) < \delta \Rightarrow y \in B(x, \delta) \subset U_z$ some z so $d_N(f(x), f(y)) \le d_N(f(x), z) + d_N(f(y), z) < \varepsilon$.

4.6 Compact Sets in $\mathbb{R}^n$

Theorem: (Heine-Borel). $A \subset \mathbb{R}^n$ compact if f closed and bdd.

Proof: ($\Rightarrow$) Metric spaces are Hausdorff, so A closed.

($\Leftarrow$) $C \subset \mathbb{R}^n$ bdd $\Rightarrow \exists\, [a, b] \subset \mathbb{R}^n$ s.t. $C \subset [a, b] \times \dots \times [a, b]$. This compact by Tychanov. If C closed then closed subset of compact space so compact.

Notes

4.7 Sequential Compactness

Theorem: Metric M is compact if f every sequence in M has convergent subsequence.

Lemma: A_k sequence of subsets of metric M. Then $\forall x \in \bigcap_{j=1}^{\infty}\overline{A_j}$ $\exists x_k \in A_k$ s.t. $x_k \to x$.

Proof: Take $x_k \in A_k \cap B\left(x, \frac{1}{k}\right) \neq \theta$.

Corollary: $x_k \in M$ and $\bigcap_{j=1}^{\infty}\overline{\{x_j, x_{j+1}...\}} \neq \theta$ then x_k have convergent

Proof: Let $x \in \bigcap_{j=1}^{\infty}\overline{\{x_j, x_{j+1}...\}}$. As $\exists k_j \geq j$ s.t. $x_{kj} \to x$. $k_j \to \infty$ so can choose subsequence k_{ji} s.t. $k_{ji+1} > k_{ji}$ (as k_js not necessarily in order). Then x_{kji} subsequence converging to x.

Proof of ($\Rightarrow$) of theorem 3.16. Let $x_k \in M$, $F_j = \overline{\{x_j, x_{j+1}...\}}$. F_j form decreasing sequence of non-empty closed subsets of M.

By corollary 3.6 $\bigcap_{j=1}^{\infty}F_j \neq \theta$ so x_k have convergent subsequence by corollary 3.18.

Notation

U open cover of M. $\forall x \in M$

$$r(x) = \sup\{r \leq 1 : \exists U \in \mathcal{U} \text{ s.t. } B(x, r) \subset U\}$$

Lemma: If $y_k \to x$ $\exists K$ s.t. $y_{k+1} \in B\left(y_k, \frac{r(x)}{2}\right)$ for $k \geq K$.

Proof: Let $U \in \mathcal{U}$ be s.t. $B\left(x, \frac{r(x)}{2}\right) \subset U$. Take K s.t. $d(y_k, x) < \frac{r(x)}{16}$ for $k \geq K$. Then $k \geq K \Rightarrow$ $B\left(y_k, \frac{r(x)}{2} - d(x, y_k)\right) \subset B\left(x, \frac{r(x)}{2}\right) \subset \mathcal{U}$, so $r(y_k) \geq \frac{r(x)}{2} - d(x, y_k) \geq \frac{r(x)}{4}$, so

$$d(y_{k+1}, y_k) \leq d(y_{k+1}, x) + d(y_k, x) < \frac{r(x)}{8} \leq \frac{r(y_k)}{2}$$

$M_1 := M$, $s_1 := \sup_{x\in M1} r(x)$. Find $x_1 \in M_1$ s.t. $r(x_1) > \frac{s_1}{2}$, choose $U_1 \in \mathcal{U}$ s.t. $B\left(x_1, \frac{r(x_1)}{2}\right) \subset U_1$.

If $x_1, ..., x_j$ have been defined,

$$M_{j+1} := M \backslash B\left(x_j, \frac{r(x_j)}{2}\right) = M \setminus \bigcup_{i=1}^{j} B\left(x_j, \frac{r(x_j)}{2}\right)$$

If $M_{j+1} = \theta$ then $M \subset \bigcup_{i=1}^{j} B\left(x_i, \frac{r(x_i)}{2}\right) \subset \bigcup_{i=1}^{j} U_i$ has finite subcover.

If $M_{j+1} \neq \theta$ let $s_{j+1} = \sup_{x\in MJ+1}\{r(x)\}$, find x_{j+1} s.t. $r(x_{j+1}) > \frac{s_{j+1}}{2}$, choose $U_{j+1} \in \mathcal{U}$ s.t. $B\left(x_{j+1}, \frac{r(x_{j+1})}{2}\right) \subset U_{j+1}$.

If procedure stops we have finite subcover. If it runs forever we have infinite sequence x_j s.t. $x_i \notin$

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$B\left(x_j, \frac{r(x_j)}{2}\right)$ for i > j. This has convergent subsequence x_{jk} by assumption, so ∃ k s.t. $B\left(x_{jk}, \frac{r(x_{jk})}{2}\right)$.

This is a contradiction, so the procedure stops.

Self Assessment

Fill in the blanks:

1. T, S compact, U open cover of .... If s ∈ S there exists open V ⊂ S, s ∈ V s.t. T × V can be covered by finite subfamily of *U*.
2. Let T be compact. Cts .... is bdd and attains max and min.
3. Given cover *U* of metric M, δ > 0 called .... of *U* if ∀ x ∈ M ∃ *U* ∈ *U* s.t. B(x, δ) ⊂ U.

4.8 Summary

- Let $F_1 \supset F_2 \supset \ldots$ sequence of non-empty closed subsets of compact T. Then $\bigcap_{k=1}^{\infty} F_k \neq \theta$. Let $F_1 \supset F_2 \supset \ldots$ sequence of non-empty compact subsets of Hausdorff T. Then $\bigcap_{k=1}^{\infty} F_k \neq \theta$.
- ∀ x ∈ T find $W_x \in U$ s.t. (x, s) ∈ W_x. Exists open $U_x \subset T$, $V_x \subset S$ s.t. (x, s) ∈ $U_x \times V_x \subset W_x$. $\{U_x : x \in T\}$ open cover of T so ∃ $U_{x1}, \ldots, U_{xn}$ which cover T. Let $V = \bigcap_{i=1}^{n} V_{xi}$. V ⊂ S open and

  $T \times V \subset \bigcup_{i=1}^{n} U_{xi} \times V_{xi} \bigcup_{i=1}^{n} W_{xi}$
- Let $x \in \bigcap_{j=1}^{\infty} \overline{\{x_j, x_{j+1} \ldots\}}$. ∃ $k_j \geq j$ s.t. $x_{kj} \to x$. $k_j \to \infty$ so can choose subsequence k_{ji} s.t. $k_{ji+1} > k_{ji}$ (as k_js not necessarily in order). Then x_{kji} subsequence converging to x.
- Let $x_k \in M$, $F_j = \overline{\{x_j, x_{j+1} \ldots\}}$. F_j form decreasing sequence of non-empty closed subsets of M. $\bigcap_{j=1}^{\infty} F_j \neq \theta$ so x_k have convergent subsequence.

4.9 Keywords

Space Compact: Cts image of compact space compact.

Homeomorphism: Cts bijection of compact T onto Hausdorff S is homeomorphism.

Lebesgue Number: Every open cover U of compact metric space has a Lebesgue number.

Convergent Subsequence: Metric M is compact iff every sequence in M has convergent subsequence.

4.10 Review Questions

1. Discuss the compactness of a set.
2. Explain intersection of closed set.
3. Discuss compactness and Continuity.
4. Describe sequential compactness.

Notes

Answers: Self Assessment

1. $T \times S$
2. $f : T \to \mathbb{R}$
3. Lebesgue number

4.11 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

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S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 5: Connectedness

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Connectedness
- Discuss the Connectedness in metric spaces
- Explain connected spaces from others
- Describe connected components and Path connected

Introduction

In last unit you have studied about the compactness of the set. As you all come to know about the compactness and continuity. After understanding the concept of compactness let us go through the explanation of connectedness.

5.1 Connected, Separated

Definition: Topological T connected if for every decomposition T = A∪B into disjoint open A, B either A or B is empty.

Definition: $T \subset S$ separated by sets U, $V \subset S$ if $T \subset U \cup V$, $U \cap V \cap T = \theta$, $U \cap T \neq \theta$, $V \cap T \neq \theta$.

Proposition: $T \subset C$ S disconnected if T is separated by some U, $V \subset S$.

Proof: (⇒) If disconnected ∃ A, $B \subset T$, A, $B \neq \theta$ s.t. $T = A \cup B$ and $A \cap B = \theta$. $T \subset S$ so ∃U, V open in S s.t. $A = U \cap T$, $B = V \cap T$. Then U, V separate T.

(⇐) If U, V separate T let $A = U \cap T$, $B = V \cap T$ then T not connected.

Proposition: TFAE:

1. T disconnected

Notes

2. T has subset which is open, closed, different from θ, T
3. T admits non-constant cts function to two point discrete space.

Proof: (1. ⇒ 2.) ∃ decomposition T = A ∪ B with A, B open, non-empty. Hence A = T\B is open and closed, different from θ, = T.

(2. ⇒ 3.) θ, T ≠ A ⊂ T open, closed. Define f : T → {0,1} by $f(x) = \begin{cases} 0 & x \in A \\ 1 & x \notin A \end{cases}$

This cts as pre-images open

(3. ⇒ 1.) f: T → {0, 1} non-constant and cts. Define A = $F^{-1}(0)$, B = f^{-1}

5.2 Connectedness in Metric Spaces

Theorem: T ⊂ M (M metric) disconnected iff ∃ disjoint open U,V ⊂ M s.t. T ∩ U ≠ θ ≠ T → V and T ⊂ U ∪ V.

Proof: (⇐) Clear

(⇒) T = A U B. Let

U = {x ∈ M : d(x, A) < d(x, B)}

V = {x ∈ M : d(x, A) > d(x, B)}

U, V disjoint, open.

Going to prove A ⊂ U: Let x ∈ A. A open in T so ∃ δ > 0 s.t. B(x, δ) ∩ T ⊂ A. B ⊂ T disjoint from A so B(x, δ) ∩ B = θ, so d(x, B) ≥ δ > 0. Since d(x, A) = 0 we have x ∈ U. Similarly B ⊂ V.

Lemma: I ⊂ ℝ is an interval iff ∀ x, y∈ I, ∀ z ∈ ℝ,

x < z < y ⇒ z ∈ I

Proof: Intervals clearly have this property. Conversely suppose I has above property, non-empty, not single point. Let a = inf I, b = sup I.

Show (a, b) ⊂ I: If z ∈ (a, b) ∃ x, y ∈ with x < z < y so z ∈ I. Hence (a, b) ⊂ I ⊂ (a, b) ∪ {a, b}.

Theorem: T ⊂ ℝ connected iff it is an interval.

Proof: (⇒) Suppose I not interval. Then by lemma 4.4 ∃ x, y ∈ I, z ∈ ℝ s.t. x < z < y and z ∉ I. Let A = (–∞, z) ∩ I, B = (z, ∞) ∩ I. A, B disjoint, non-empty, open and I = A ∪ B.

(⇐) Suppose I not connected. Then 3 cts non-constant f : I → {0, 1} where {0, 1} has discrete contradicting IVT.

(⇐) I partitioned into non-empty A, B open. Choose a ∈ A, b ∈ B, a < b. A, B open cover of [a, b]. Let δ be its Lebesgue number. Then $\left[a, a+\frac{\delta}{2}\right] \subset A$, $\left[a+\frac{\delta}{2}, a+\frac{2\delta}{2}\right] \subset A$, ... until we get to an interval containing b. So b ∈ A and A, B not disjoint.

5.3 Connected Spaces from Others

Proposition: Cts image of connected space connected.

Proof: Suppose f : T → S cts, T connected. If f(T) disconnected ∃ U, V ⊂ S open separating f(T). Then $f^{-1}(U)$, $f^{-1}(V)$ open, disjoint, cover T. Contradiction as T connected.

Proposition: If C, C_j ($j \in J$) connected subspaces of topological T and if $c_j \cap \bar{C} \neq \theta \ \forall \ j \in J$ then

$$K = C \cup \bigcup_{j \in J} C_j$$

is connected.

Proof: Suppose K disconnected. Hence $\exists$ U, V $\subset$ T open that separate K.

C connected so cannot be separated by U,V, so does not meet one of them. Suppose w.l.o.g $C \cap V = \theta$. Then $C \subset U$. Since V open $\bar{C} \cap V = \theta$, so $K \cap \bar{C} \subset U$. Then $C_j \cap U \neq \theta \ \forall j$.

C_j connected so $C_j \subset U$ or $C_j \subset V$. $C_j \cap U \neq \theta$ so $C_j \subset U$.

Then $K \subset U$ contradicting $V \cap K \neq \theta$.

Corollary: $C \subset T$ connected and $C \subset K \subset \bar{C}$. Then K connected.

Proof: $K = C \bigcup_{x \in K} \{x\}$ and $\{x\} \cap \bar{C} \neq \theta \ \forall$ x.

Proposition: Product of connected spaces is connected.

Proof: Let T, S connected, so $\in$ S. Define $C = T \times \{s_0\}$ and $C_t = \{t\} \times S$ (for some $t \in T$). Then C, C_t homeomorphic to T and S are connected. $C_t \cap C \neq \theta$ and $T \times S = C \cup \bigcup_{t \in T} C_t$ connected.

Example: $\text{Sin}\left(\frac{1}{t}\right) \cup \underbrace{\{(0, t) \in \mathbb{R}^2 : \in (-1, 1)\}}_{I}$ is connected.

Proof:

$$C = \left\{\left(t, \sin\left(\frac{1}{t}\right)\right) : t > 0\right\}$$

$$D = \left\{\left(t, \sin\left(\frac{1}{t}\right)\right) : t < 0\right\}$$

C, D, I cts images of intervals so connected.

$(0, 0) \in I$ is in $\bar{C}$ as $(t_k) \sin\left(\frac{1}{t_k}\right) \to (0, 0)$ when $t_k = \frac{1}{k\pi}$. Then $I \cup C$ connected. Similarly $I \cup D$.

5.4 Connected Components

Definition: x ~ y if x, y belong to a common connected subspace of T. Equivalence classes are connected components of T.

Are maximal connected subsets of T. Number of connected components is topological invariant.

Property T\{x} connected $\forall \ x \in T$ topological invariant.

5.5 Path Connectedness

Definition: a, b $\in$ T. $\varphi : [0, 1] \to T$ cts with $\varphi(0) = a$, $\varphi(1) = b$ called a path from a to b.

Definition: T path connected if any two points can be joined by a path.

Notes

Proposition: Path connected ⇒ connected.

Proof: a ∈ T. ∀ x ∈ T image C_x of path a to x is connected, and all C_x contain a. Then T = $\bigcup_{x \in T} C_x$ connected by 4.7.

5.6 Open Sets in $\mathbb{R}^n$

Theorem: Any U C $\mathbb{R}^n$ open, connected is path connected.

Proof: Let a ∈ U, V = {x ∈ U : ∃ path from a to x}.

Let z ∈ U ∩ $\bar{V}$. Find δ > 0 s.t. B(z, δ) ⊂ U. z $\bar{V}$ so ∃y ∈ V ∩ B(x, δ).

Then B(z, δ) ⊂ V since join path from a to y to path from y to z.

Theorem: All components of open U ⊂ $\mathbb{R}^n$ open.

Proof: C component of U, x ∈ C. Find δ > 0 with B(x, δ) ⊂ U. B(x, δ) connected and C union of all connected subsets of U containing x so B(x, δ) ⊂ C, so C open.

Theorem: U ⊂ ℝ open iff disjoint union of countably many open intervals.

Proof: (⇐) Any union of open sets open.

(⇒) U ⊂ ℝ open, C_j (j ∈ J) its components. C_j connected and open so are open intervals. For each j ∃ rational r_j ∈ C_j. C^S_j mutually disjoint so j → r_j injection into ℚ, so can order J into a sequence.

Self Assessment

Fill in the blanks:

1. Topological T connected if for every decomposition .... into disjoint open A, B either A or B is empty.
2. T ⊂ M (M metric) disconnected iff ∃ .... U,V ⊂ M s.t. T ∩ U ≠ θ ≠ T → V and T ⊂ U ∪ V.
3. .... is an interval iff ∀ x, y∈ I, ∀ z ∈ ℝ, x < z < y ⇒ z ∈ I
4. Suppose f : T → S cts, T connected. If f(T) disconnected ∃ U, V ⊂ S open separating f(T). Then f^{-1}(U), f^{-1}(V) open, disjoint, cover T. Contradiction as ....
5. C ⊂ T connected and C ⊂ K ⊂ $\bar{C}$. Then ....
6. x ~ y if x, y belong to a common connected subspace of T. .... are connected components of T.

5.7 Summary

- Topological T connected if for every decomposition T = A∪B into disjoint open A, B either A or B is empty.
- T ⊂ S separated by sets U, V ⊂ S if T ⊂ U∪V, U ∩ V ∩ T = θ, U ∩ T ≠ θ, V ∩ T ≠ θ.
- T ⊂ C S disconnected iff T is separated by some U, V ⊂ S.
- Proof. (⇒) If disconnected ∃ A, B ⊂ T, A, B ≠ θ s.t. T = A ∪ B and A ∩ B = θ. T ⊂ S so ∃U, V open in S s.t. A = U ∩ T, B = V ∩ T. Then U, V separate T.
- Suppose K disconnected. Hence ∃ U, V ⊂ T open that separate K.

- C connected so cannot be separated by U,V, so does not meet one of them.
- C component of U, $x \in C$. Find $\delta > 0$ with $B(x, \delta) \subset U$. $B(x, \delta)$ connected and C union of all connected subsets of U containing x so $B(x, \delta) \subset C$, so C open.

5.8 Keywords

Path Connected: T path connected if any two points can be joined by a path.

Topological Invariant: Number of connected components is topological invariant.

5.9 Review Questions

1. Define Connectedness.
2. Discuss the Connectedness in metric spaces.
3. Explain connected spaces from others.
4. Describe connected components and Path connected.

Answers: Self Assessment

1.	$T = A \cup B$	2.	disjoint open
3.	$I \subset \mathbb{R}$	4.	T connected
5.	K connected	6.	Equivalence classes

5.10 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

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T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 6: Completeness

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Completeness
- Discuss the Cauchy
- Explain contraction mapping theorem
- Describe total boundness

Introduction

In earlier unit you have studied about the compactness and connectedness of the set. As you all come to know about the connected components and Path connectedness. After understanding the concept of compactness and connectedness let us go through the explanation of completeness.

6.1 Completeness

This is a concept that makes sense in metric spaces only.

Definition: Metric M is complete if every Cauchy sequence in M converges (to a point of M).

Remark: This is not a topological invariant: (0, 1) – incomplete and $\mathbb{R}$ complete are homeomorphic.

Proposition: Cvgt $\Rightarrow$ Cauchy.

Proof: $\forall \varepsilon > 0 \; \exists N$ s.t. $d(x_n, x) < \frac{\varepsilon}{2}$ for $n \geq N$. If $m, n \geq N$ then

$$d(x_m, x_n) < d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Proposition:

1. Complete subspace S of metric M is closed.
2. Closed subset S of complete M is complete.

Proof:

1. Let $x_n \in S$, $x_n \to x \in M$. (x_n) Cauchy in S so cvgs in S to $y \in S$. $S \leq M$ so $x_n \to y$ in M. By uniqueness of limits $x = y \in S$.
2. Let $(x_n) \subset S$ Cauchy. Cauchy in M so cvgs to point of M which in S as S closed.

Proposition: $\forall$ S B(S) of bdd functions $S \to \mathbb{R}$ with sup norm is complete.

Proof: Let (f_n) Cauchy, $\varepsilon > 0$. 3 N s.t. $||f_m - f_n|| < \varepsilon$ for $n, m \geq N$. Hence for fixed x $(f_n(x))$ Cauchy in $\mathbb{R}$, so cvgs to $f(x) \in \mathbb{R}$.

For $n \geq N$ $|f_m(x) - f_n(x)| \leq \varepsilon \quad \forall m \geq N$. Let $m \to \infty$ then

$|f(x) - f_n(x)| \leq \varepsilon \quad \forall x \, \varepsilon \, S, n \geq N$

Then f bdd and $f_n \to f$.

6.2 Proving Cauchy

Proposition: A sequence $(x_n) \subset M$ is Cauchy iff $\exists$ sequence $\varepsilon_n \geq 0$ s.t. $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ and $d(x_m, x_n) \leq \varepsilon_n$ for $m > n$.

Proof: $(\Rightarrow)$ Suppose (x_n) Cauchy. Then let $\varepsilon_n = \Sigma_{m>n} d(x_m, x_n) \xrightarrow[n\to\infty]{} 0$.

$(\Leftarrow)$ Given $\varepsilon > 0$ find k s.t. $\varepsilon_n < \varepsilon$ for $n \geq k$. Then $d(x_n, x_n) \leq \varepsilon_n < \varepsilon$ for $m > n \geq k$. Exchanging m, n gives $d(x_m, x_n) < \varepsilon \;\forall n, m \geq k$.

Proposition: $(x_n) \subset M$ sequence s.t. $\exists \, \tau_n \geq 0$ with $\Sigma_{n=1}^{\infty} \tau_n < \infty$ and $d(x_n, x_{n+1}) \leq \varepsilon_n \;\forall n$. Then (x_n) is Cauchy.

Proof: Follows from 6.4 with $\varepsilon_n = \Sigma_{n=1}^{\infty} \tau_n$. Then

$$d(x_m, x_n) \underset{\Delta \text{ineq}}{\leq} \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \tau_n \leq \varepsilon_n$$

Example: If K compact topological space then space C(K) with sup norm is complete.

Proof: Each f bdd, attains max. Suffices to show C(K) closed in B(K).

Suppose $f_n \in C(K)$ cvg to $f \in B(K)$. Then $\forall \varepsilon > 0 \; \exists$ N s.t.

$$\sup_{x \in K} |f(x) - f_n(x)| < \varepsilon \quad \forall n \geq N$$

$$\forall a \in \mathbb{R} \; \{x : f(x) > a\} = \bigcup_{\varepsilon > 0} \{x : f_N(x) > a + \varepsilon\}$$

RHS are pre-images of open sets so open. Hence LHS is open. Similarly $\{x : f(x) < a\}$ open. $(-\infty, a)$, (a, ∞) from sub-basis for $\mathbb{R}$ so f cts.

Example: C[0, 1] with norm $||f||_1 = f^1{}_0 |f(x)| dx$ is incomplete.

Notes

Proof:

$$f_n(x) = \begin{cases} \min\left\{\sqrt{n}, \dfrac{1}{\sqrt{x}}\right\} & x > 0 \\ \sqrt{n} & x = 0 \end{cases}$$

so $(f_n) \subset C[0, 1]$.

$$\int_0^1 |f_m(x) - f_n(x)|\,dx = \int_0^{\frac{1}{m}}\left(\sqrt{m} - \sqrt{n}\right)dx + \int_{\frac{1}{m}}^{\frac{1}{n}}\left(\frac{1}{\sqrt{x}} - \sqrt{n}\right)dx$$

$$\le \frac{1}{\sqrt{m}} + \frac{2}{\sqrt{n}}$$

$$\le \frac{3}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

so (f_n) Cauchy.

Let $f \in C[0, 1]$. Find $k \in \mathbb{N}$ s.t. $|f| \le \sqrt{k}$. Then for $n > k$

$$\int_0^1 |f_m(x) - f_n(x)|\,dx = \int_0^{\frac{1}{m}}\left(\sqrt{m} - \sqrt{n}\right)dx + \int_{\frac{1}{m}}^{\frac{1}{n}}\left(\frac{1}{\sqrt{x}} - \sqrt{n}\right)dx$$

$$\ge 2\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{n}}\right) - \frac{1}{\sqrt{k}}$$

$$= \frac{1}{\sqrt{k}} - \frac{2}{\sqrt{n}} \xrightarrow[n\to\infty]{} > 0$$

6.3 Completion

Definition: $S \subset M$ is dense in M if $\overline{S} = M$.

Definition: A completion of metric space M is:

- Complete metric space N s.t. M dense subset of N.
- Complete metric space N and isometry $i: M \to A \subseteq N$ s.t $i(M)$ is dense in N.

Theorem: Any metric M can be isometrically embedded into complete metric space.

Proof: Find isometry of M onto subset of B(M), complete. Fix $a \in M$, define $F : M \to B(M)$ by $F(x)(z) = d(z,x) - d(z, a)$. $|F(x)(z)| \le d(x, a)$ so $F(x) \in B(M)$.

$$|F(x)(z) - F(y)(z)| = |d(z, x) - d(z, y)|$$

$$\le d(x, y)$$

Equality occurs when $z = y$. Then $||F(x) - F(y)|| = d(x, y)$ so F isometry.

Corollary: Any metric space M has a completion.

Proof: Embet M into complete N. Then $\overline{M}$ (closure taken in N) is complete by 6.2, M dense in $\overline{M}$. Then $\overline{M}$ completion of M.

6.4 Contraction Mapping Theorem

Definition: $f : M \to M$ contraction if $\exists$ $k < 1$ s.t.

$$d(f(x), f(y)) \le kd(x, y) \quad \forall\, x, y \in M$$

Theorem: Banach

If f contraction on complete metric M then f has unique fixed point.

Proof: **Uniqueness:** If $f(x) = x$, $f(y) = y$ then

$$d(x, y) = d(f(x), f(y)) \le kd(x, y) \Rightarrow d(x, y) = 0$$

Existence: Pick $x_0 \in M$, $x_{n+1} = f(f_n)$. By repeated application of the contraction property we get that $d(x_j, x_{j+1}) \le k^j d(x_0, x_1)$. $\sum_{j=1}^{\infty} k^j\ d(x_0, x_1) \le \infty$ so (x_n) Cauchy.

M complete so $x_n \to x \in M$, so $f(x_n) \to f(x)$. But also $f(x_n) = x_{n+1} \to x$ so $f(x) = x$.

6.5 Total Boundedness

Definition: Metric M totally bounded if $\forall\, \varepsilon > 0\ \exists$ finite set $F \subset M$ s.t. $M \subset \bigcup_{x \in F} B(x, \varepsilon)$.

Proposition: Subspace M of metric N is totally bounded iff $\forall\, \varepsilon > 0\ \exists$ finite $H \subset N$ s.t. $M \subset \bigcup_{z \in H} B(z, \varepsilon)$.

Proof: ($\Rightarrow$) Obvious.

($\Leftarrow$) Given $\varepsilon > 0$ let $H \subset N$ be finite set s.t. $M \subset \bigcup_{z \in H} B\left(z, \frac{\varepsilon}{2}\right)$. From each non-empty $M \cap B(z, \frac{\varepsilon}{2})$ pick one point. Let F be set of these points.

$F \subset M$ finite.

If $y \in M$ then y in one of $B\left(z, \frac{\varepsilon}{2}\right)$ so $M \cap B\left(z, \frac{\varepsilon}{2}\right) \ne \theta$ so $\exists\, x \in M \cap B\left(z, \frac{\varepsilon}{2}\right)$. Hence $y \in B(x, \varepsilon)$ and $M \subset \bigcup_{z \in F} B(z, \varepsilon)$.

Corollary: Subspace of totally bounded metric space is totally bounded.

Theorem: Metric M totally bounded iff every sequence in M has Cauchy subsequence.

Proof: ($\Rightarrow$) Let $x_n \in M$. M covered by finitely many balls radius 1/2 so $\exists\, B_1$ s.t. $N_1 = \{n \in \mathbb{N} : x_n \in B_1\}$ has $|N_1| = \infty$.

Suppose inductively have defined infinite $N_{k-1} \subset \mathbb{N}$. Since M covered by finitely many balls of radius $\frac{1}{2k}$ $\exists$ one ball B_k s.t. $N_k = \{n \in N_{k-1} : x_n \in B_k\}$ is infinite.

Let n(1) be least element of N_1, n(k) least element of N_k s.t. $n(k) > n(k - 1)$.

Then $\left(x_n(k)\right)_{n=1}^{\infty} \subset \left(x_n\right)_{n=1}^{\infty}$ s.t. $\forall\, k\ x_{n(i)} \in B_k$ for $i \ge k$. Hence $d(x_{n(i)}, x_{n(j)}) < \frac{1}{k}\ \forall i, j \ge k$ so $(x_{n(k)})$ Cauchy.

($\Leftarrow$) Suppose M not totally bounded. Then for some $\varepsilon > 0$ $\nexists$ finite F with all points of M within ε of it. Choose $x_1 \in M$, inductively x_k s.t. $d(x_k, x_i) \ge \varepsilon\ \forall\, i < k$. x_k exists by assumption M not totally bounded.

This gives sequence $(x_k)_{k=1}^{\infty}$ s.t. $d(x_i, x_j) \ge \varepsilon\ \forall\, i \ne j$. Then no subsequence of (x_k) Cauchy.

Notes

Self Assessment

Fill in the blanks:

1. A completion of metric space M is .... N s.t. M dense subset of N.
2. Any metric M can be .... into complete metric space.
3. Metric M totally .... if $\forall\ \varepsilon > 0\ \exists$ finite set $F \subset M$ s.t. $M \subset \bigcup_{x \in F} B(x, \varepsilon)$.
4. Subspace M of .... is totally bounded iff $\forall\ \varepsilon > 0\ \exists$ finite $H \subset N$ s.t. $M \subset \bigcup_{z \in H} B(z, \varepsilon)$.

6.6 Summary

- Complete subspace S of metric M is closed.
- Closed subset S of complete M is complete.
- $\forall$ S B(S) of bdd functions $S \to \mathbb{R}$ with sup norm is complete.
- A sequence $(x_n) \subset M$ is Cauchy iff $\exists$ sequence $\varepsilon_n \geq 0$ s.t. $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ and $d(x_m, x_n) \leq \varepsilon_n$ for m > n.
- $(x_n) \subset M$ sequence s.t. $\exists\ \tau_n \geq 0$ with $\sum_{n=1}^{\infty} \tau_n < \infty$ and $d(x_n, x_{n+1}) \leq \varepsilon_n\ \forall$ n. Then (x_n) is Cauchy.

6.7 Keywords

Cauchy Sequence: Metric M is complete if every Cauchy sequence in M converges (to a point of M).

Cauchy: A sequence $(x_n) \subset M$ is Cauchy iff $\exists$ sequence $\varepsilon_n \geq 0$ s.t. $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ and $d(x_m, x_n) \leq \varepsilon_n$ for m > n.

Completion: Any metric space M has a completion.

6.8 Review Questions

1. Define Completeness.
2. Discuss the Cauchy.
3. Explain contraction mapping theorem.
4. Describe total boundness.

Answers: Self Assessment

1.	Complete metric space	2.	isometrically embedded
3.	bounded	4.	metric N

6.9 Further Readings

Notes

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 7: Convergent Sequence

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define convergent sequence
- Discuss the properties of convergent sequence
- Explain subsequences and compact metric spaces
- Describe subsequence limits
- Explain the Cauchy sequences and convergent sequences

Introduction

In earlier unit you have studied about the compactness and connectedness of the set. After understanding the concept of compactness and connectedness let us go through the explanation of convergent sequence.

7.1 Convergent Sequence

Definition: A sequence $\{p_n\}$ in a metric space (X, d) is said to converge if there is a point $p \in X$ with the following property:

$$(\forall \in > 0)(\exists N)(\forall n > N)\ d(p_n, p) < \in$$

Notes

In this case we also say that $\{p_n\}$ converges to p or that p is the limit of $\{p_n\}$ and we write pn $\to$ p or lim $\lim_{n\to\infty} p_n = p$

If $\{p_n\}$ does not converge we say it diverges

If there is any ambiguity we say $\{p_n\}$ converges/diverges in X

The set of all p_n is said to be the range of $\{p_n\}$ (which may be infinite or finite). We say $\{p_n\}$ is bounded if the range is bounded.

Example: Notice that our definition of convergent depends not only on $\{p_n\}$ but also on X.

For example $\{1/1 : n \in \mathbb{N}\}$ converges in $\mathbb{R}^1$ and diverges in $(0, \infty)$. Consider the following sequence of complex number (i.e. $X = \mathbb{R}^2$)

(a) If $S_n = 1/n$ then $\lim_{n\to\infty} S_n = 0$; the range is infinite, and the sequence is bounded.

(b) If $S_n = n^2$ then the sequence $\{S_n\}$ is divergent; the range is infinite, and the sequence is unbounded.

(c) If $S_n = 1 + [(-1)^n/n]$ then the sequence $\{S_n\}$ converges to 1, is bounded, and has infinite range.

(d) If $S_n = i^n$ the sequence $\{S_n\}$ is divergent, is bounded and has finite range.

(e) If $S_n = 1$ (n = 1, 2, 3, ...) then $\{S_n\}$ converges to 1 is bounded.

7.2 Properties of Convergent Sequences

Theorem:

(a) $\{p_n\}$ converges to $p \in X$ if and only if every neighbourhood of p contains p_n for all but finitely many n.

(b) If $p, p' \in X$ and if $\{p_n\}$ converges to p and to p′ then p = p′

(c) If $\{p_n\}$ converges then $\{p_n\}$ is bounded.

(d) If $E \subseteq X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$

Theorem: Suppose $\{S_n\}$, $\{t_n\}$ are complex sequence with $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} t_n = t$. Then

(a) $\lim_{n\to\infty} (S_n + t_n) = S + t$

(b) $\lim_{n\to\infty} C \cdot S_n = C \cdot S$ and $\lim_{n\to\infty} C + S_n = C + S$ for any number C.

(c) $\lim_{n\to\infty} S_n t_n = S_t$

(d) $\lim_{n\to\infty} \frac{1}{S_n} = \frac{1}{S}$

Theorem:

(a) Suppose $x_n \in \mathbb{R}^k$ $(n \in \mathbb{N})$ and $x_n = (\alpha_1, n, \ldots \alpha_k, n)$. Then $\{x_n\}$ converges to $x = (\alpha_1, \ldots, \alpha_k)$ if and only if

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j (1 \le j \le k)$$

Notes

(b) Suppose $\{x_n\}$, (y_n) are sequence in $\mathbb{R}^k$, $\{\beta_n\}$ is a sequence of real numbers, and $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$. Then

$$\lim_{n\to\infty} (x_n + y_n) = x + y$$

$$\lim_{n\to\infty} (x_n \cdot y_n) = x \cdot y$$

$$\lim_{n\to\infty} \beta_n x_n = \beta x$$

7.2.1 Subsequences

Definition:

Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 < n_3 \ldots$ Then the sequence $\{p_{ni}\}$ is called a subsequence of $\{p_n\}$. If $\{p_{ni}\}$ converges its limit is called a subsequential limit of $\{p_n\}$.

It is clear that $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.

7.3 Subsequences and Compact Metric Spaces

Theorem:

(a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

7.4 Subsequences Limits

Theorem:

The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

7.5 Cauchy Sequence

A sequence $\{p_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for every $\in > 0$ there is an integer N such that $d(p_n, p_m) < \in$ for all $n, m \geq N$.

Definition:

Let E be a non-empty subset of a metric space (X, d), and let $S = \{d(p, q) : p, q \in E\}$. The diameter of E is sup S.

If $\{p_n\}$ is a sequence in X and if E_n consists of the points $p_N, p_{N+1}, \ldots$, it is clear that $\{p_n\}$ is a Cauchy sequence if and only if

$$\lim_{N\to\infty} \text{diam } E_N = 0$$

7.6 Cauchy Sequence and Closed Sets

Theorem:

(a) If $\overline{E}$ is the closure of a set E in a metric space X, then

$$\text{diam } \overline{E} = \text{diam } E$$

(b) If K_n is a sequence of compact sets in X such that $K_n \subset K_{n+1}$ $(n \in \mathbb{N})$ and if $\lim_{n\to\infty} \text{diam } K_n = 0$ then $\bigcap_1^\infty K_n$ consists of exactly one point.

7.7 Cauchy Sequences and Convergent Sequences

Theorem:

(a) In any metric space X, every convergent sequence is a Cauchy sequence.

(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X then $\{p_n\}$ converges to some point of X.

(c) In $\mathbb{R}^k$ every Cauchy sequence converges.

7.7.1 Complete Spaces

Definition:

A metric space is said to be complete if every Cauchy sequence converges.

Notice that all compact metric spaces are complete but there are metric spaces (like $\mathbb{R}^k$) which are complete but not compact.

Lemma

Every closed subset of a complete metric space is complete.

7.8 Increasing/Decreasing Sequences

Definition:

A sequence {Sn} of real numbers is said to be

(a) Monotonically increasing if $S_n \leq S_{n+1}$ for all $n \in \mathbb{N}$

(b) Monotonically decreasing if $S_n \geq S_{n+1}$ for all $n \in \mathbb{N}$

(c) Monotonic if it is monotonically increasing or monotonically decreasing.

Theorem: Suppose $\{S_n\}$ is monotonic. Then $\{S_n\}$ converges if and only if $\{S_n\}$ is bounded.

Self Assessment

Fill in the blanks:

1. If there is any ambiguity we say $\{p_n\}$ .... in X.
2. The set of all p_n is said to be the range of $\{p_n\}$ (which may be infinite or finite). We say $\{p_n\}$ is bounded if the range is ....
3. If $\{p_n\}$ is a sequence in a .... space X, then some subsequence of $\{p_n\}$ converges to a point of X.
4. Every bounded sequence in $\mathbb{R}^k$ contains a ....
5. A metric space is said to be complete if every ....

Notes

7.9 Summary

- A sequence $\{p_n\}$ in a metric space (X, d) is said to converge if there is a point $p \in X$ with the following property:

  $(\forall \in > 0)(\exists N) \qquad (\forall n > N)\ d(p_n, p) < \in$

- In this case we also say that $\{p_n\}$ converges to p or that p is the limit of $\{p_n\}$ and we write $p_n \to p$ or lim $\lim_{n\to\infty} p_n = p$.
- If $\{p_n\}$ does not converge we say it diverges.
- If there is any ambiguity we say $\{p_n\}$ converges/diverges in X.
- The set of all p_n is said to be the range of $\{p_n\}$ (which may be infinite or finite). We say $\{p_n\}$ is bounded if the range is bounded.
- Properties of convergent sequences
  - (a) $\{p_n\}$ converges to $p \in X$ if and only if every neighbourhood of p contains p_n for all but finitely many n.
  - (b) If $p, p' \in X$ and if $\{p_n\}$ converges to p and to p′ then p = p′
  - (c) If $\{p_n\}$ converges then $\{p_n\}$ is bounded.
  - (d) If $E \subseteq X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$
- Theorem of couchy sequences and convergent sequences
  - (a) In any metric space X, every convergent sequence is a Cauchy sequence.
  - (b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X then $\{p_n\}$ converges to some point of X.
  - (c) In $\mathbb{R}^k$ every Cauchy sequence converges.

7.10 Keywords

Subsequential: Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 < n_3 \ldots$ Then the sequence $\{p_{ni}\}$ is called a subsequence of $\{p_n\}$. If $\{p_{ni}\}$ converges its limit is called a subsequential limit of $\{p_n\}$.

Subsequential Limits: The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

Cauchy Sequency: A sequences $\{p_n\}$ in a metric space (X, d) is said to be a Cauchy sequency if for every $\in > 0$ there is an integer N such that $d(p_n, p_m) < \in$ for all $n, m \geq N$.

7.11 Review Questions

1. Define convergent sequence.
2. Discuss the properties of convergent sequence.
3. Explain subsequences and compact metric spaces.
4. Describe subsequence limits.
5. Explain the Cauchy sequences and convergent sequences.

Answers: Self Assessment

Notes

1. converges/diverges
2. bounded
3. compact metric
4. convergent subsequence
5. Cauchy sequence converges

7.12 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 8: Completeness and Compactness

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss Completeness and Compactness
- Describe the Cantor's theorem
- Explain Baire category theorem
- Describe Compactness and Cantor set

Introduction

In earlier unit you have studied about the compactness and connectedness of the set. As you all come to know about the Contraction Mapping Theorem. After understanding the concept of Total boundedness let us go through the explanation of completeness and connectedness.

8.1 Completeness and Compactness

Theorem: Subspace C of complete metric M compact iff closed and totally bounded.

Proof: ($\Rightarrow$) C closed, totally bounded since $\forall\, \varepsilon > 0$ open cover $B(x, \varepsilon)$ $(x \in C)$ has finite subcover.

($\Leftarrow$) Every sequence in C has Cauchy subsequence, converges to point of M since M complete. C closed so limit in C.

Lemma: If M subspace of N totally bounded so is M.

Proof: Fix $\varepsilon > 0$. Let $F \subset cM$ be finite s.t. $M \subset \cup_{x \in F} B(x, \frac{\varepsilon}{2})$. Then

$$\overline{M} \subset \bigcup_{x \in F} \overline{B\left(x, \frac{\varepsilon}{2}\right)} \subset B(x, \varepsilon)$$

Theorem: Subspace S of complete metric M totally bounded iff $\bar{S}$ compact.

Proof: ($\Rightarrow$) $\bar{S}$ totally bounded and so compact.

($\Leftarrow$) $\bar{S}$ totally bounded so is $S \subset \bar{S}$.

8.2 Cantor's Theorem

Definition: Diameter of $0 \neq S \subset M$ defined by

$$\text{diam}(S) = \sup_{x, y \in S} d(x, y)$$

Theorem: Cantor

Let F_n decreasing sequence of non-empty closed subsets of metric M s.t. diam $(F_n) \xrightarrow[n\to\infty]{} 0$.

Then $\bigcap_{n=1}^{\infty} F_n \neq \theta$.

Proof: Pick $x_n \in F_n$. Then $\forall\, i \geq n$, $x_i \in F_i \subset F_n$.

Hence, for i, j $\geq$ n, $d(x_i, x_j) \leq$ diam (F_n). Hence (x_n) Cauchy. Converges to some x as M complete.

F_n closed so $x \in F_n$. Hence $x \in \bigcap_{n=1}^{\infty} F_n$.

8.3 Perfect Set

A set S is perfect if it is closed and every point of S is an accumulation point of S.

Example: Find a perfect set. Find a closed set that is not perfect. Find a compact set that is not perfect. Find an unbounded closed set that is not perfect. Find a closed set that is neither compact nor perfect.

Solution:

- A perfect set needs to be closed, such as the closed interval [a, b]. In fact, every point in that interval [a, b] is an accumulation point, so that the set [a, b] is a perfect set.
- The simplest closed set is a singleton {b}. The element b in then set {b} is not an accumulation point, so the set {b} is closed but not perfect.
- The set {b} from above is also compact, being closed an bounded. Hence, it is compact but not perfect.
- The set $\{-1\} \cup [0, \infty)$ is closed, unbounded, but not perfect, because the element –1 is not an accumulation point of the set.
- The set $\{-1\} \cup [0, \infty)$ from above is closed, not perfect, and also not compact, because it is unbounded.

Example: Is the set {1, 1/2, 1/3, ...} perfect? How about the set $\{1, 1/2, 1/3, ...\} \cup \{0\}$?

Solution: The first set is not closed. Hence it is not perfect.

The second set is closed, and {0} is an accumulation point. However, every point different from 0 is isolated, and can therefore not be an accumulation point. Therefore, this set is not perfect either.

Notes

As an application of the above result, we will see that perfect sets are closed sets that contain lots of points:

8.3.1 Perfect Sets are Uncountable

Every non-empty perfect set must be uncountable.

Proof: If S is perfect, it consists of accumulation points, and therefore can not be finite. Therefore it is either countable or uncountable. Suppose S was countable and could be written as

$$S = \{x_1, x_2, x_3, ...\}$$

The interval $U_1 = (x_1 - 1, x_1 + 1)$ is a neighbourhood of x_1. Since x_1 must be an accumulation point of S, there are infinitely many elements of S contained in U_1.

Take one of those elements, say x_2 and take a neighbourhood U_2 of x_2 such that closure (U_2) is contained in U_1 and x_1 is not contained in closure (U_2). Again, x_2 is an accumulation point of S, so that the neighbourhood U_2 contains infinitely many elements of S.

Select an element, say x_3, and take a neighbourhood U_3 of x_3 such that closure (U_3) is contained in U_2 but x_1 and x_2 are not contained in closure (U_3).

Continue in that fashion: we can find sets U_n and points x_n such that:

- closure $(\cup_{n+1}) \cup_n$
- x_j is not contained in $\cup_n$ for all $0 < j < n$
- x_n is contained in $\cup_n$

Now consider the set

- $V = \cap$ (closure $(\cup_n) \cap S$)

Then each set closure $(\cup_n) \cap S)$ is closed and bounded, hence compact. Also, by construction, (closure $(\cup_{n+1}) \cap S$) (closure $(\cup_n) \cap S$). Therefore, by the above result, V is not empty. But which element of S should be contained in V? It can not be x_1, because x_1 is not contained in closure (U_2). It can not be x_2 because x_2 is not in closure $(\cup_3)$, and so forth.

Hence, none of the elements $\{x_1, x_2, x_3, ...\}$ can be contained in V. But V is non-empty, so that it must contain an element not in this list. That means, however, that S is not countable.

8.4 Cantor Middle Third Set

Start with the unit interval

$S_0 = [0, 1]$

Remove from that set the middle third and set

$S_1 = S_0 \backslash (1/3, 2/3)$

Remove from that set the two middle thirds and set

$S_2 = S_1 \backslash \{(1/9, 2/9)\ (7/9, 8/9)\}$

Continue in this fashion, where

$S_{n+1} = S_n \backslash \{\text{middle thirds of subintervals of } S_n\}$

Then the Cantor set C is defined as

$C = \cap S_n$

The Cantor set gives an indication of the complicated structure of closed sets in the real line. It has the following properties:

Example: The Cantor set is compact.

Solution: The definition of the Cantor set is as follows: let

$$A_0 = [0, 1]$$

and define, for each n, the sets A_n recursively as

$$A_n = A_{n-1} / \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

Then the Cantor set is given as:

$$C = \cap A_n$$

Each set $\bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$ is open. Since A_0 is closed, the sets A_n are all closed as well, which can be shown by induction. Also, each set A_n is a subset of A_0, so that all sets A_n are bounded.

Example: The Cantor set is perfect and hence uncountable.

The definition of the Cantor set is as follows: let

$$A_0 = [0, 1]$$

and define, for each n, the sets A_n recursively as

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

Then the Cantor set is given as:

$$C = \cap A_n$$

From this representation it is clear that C is closed. Next, we need to show that every point in the Cantor set is a limit point.

One way to do this is to note that each of the sets A_n can be written as a finite union of 2^n closed intervals, each of which has a length of $1/3^n$, as follows:

$$A_0 = [0, 1]$$

$$A_1 = [0, 1/3] \cup [2/3, 1]$$

$$A_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$$

...

Note that all endpoints of every subinterval will be contained in the Cantor set. Now take any $x\ C = \cap A_n$. Then x is in A_n for all n. If x is in A_n, then x must be contained in one of the 2^n intervals that comprise the set A_n. Define x_n to be the left endpoint of that subinterval (if x is equal to that endpoint, then let x_n be equal to the right endpoint of that subinterval). Since each subinterval has length $1/3^n$, we have:

$$| x - x_n | < 1/3^n$$

Hence, the sequence $\{x_n\}$ converges to x, and since all endpoints of the subintervals are contained in the Cantor set, we have found a sequence of numbers contained in C that converges to x.

Notes

Therefore, x is a limit point of C. But since x was arbitrary, every point of C is a limit point. Since C is also closed, it is then perfect.

Note that this proof is not yet complete. One still has to prove the assertion that each set A_n is indeed comprised of 2^n closed subintervals, with all endpoints being part of the Cantor set. But that is left as an exercise.

Since every perfect set is uncountable, so is the Cantor.

Hence, C is the intersection of closed, bounded sets, and therefore C is also closed and bounded. But then C is compact.

Example: The Cantor set has length zero, but contains uncountably many points.

Solution: The definition of the Cantor set is as follows: let

$A_0 = [0, 1]$

and define, for each n, the sets A_n recursively as

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

Then the Cantor set is given as:

$C = \cap A_n$

To be more specific, we have:

$A_0 = [0, 1]$

$A_1 = [0, 1] \setminus (1/3, 2/3)$

$A_2 = A_1 \setminus [(1/9, 2/9) \cup (7/9, 8/9)] =$

$[0, 1] \setminus (1/3, 2/3)\,) \setminus (1/9, 2/9) \setminus (7/9, 8/9)$

...

That is, at the n-th stage ($n > 0$) we remove 2^{n-1} intervals from each previous set, each having length $1/3^n$. Therefore, we will remove a total length from the unit interval [0, 1]. Since we remove a set of total length 1 from the unit interval, the length of the remaining Cantor set must be 0.

The Cantor set contains uncountably many points because it is a perfect set.

Example: The Cantor set does not contain any open set

The definition of the Cantor set is as follows: let

$A_0 = [0, 1]$

and define, for each n, the sets A_n recursively as

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

Then the Cantor set is given as:

$C = \cap A_n$

Notes

Another way to write the Cantor set is to note that each of the sets A_n can be written as a finite union of 2^n closed intervals, each of which has a length of $1/3^n$, as follows:

$A_0 = [0, 1]$

$A_1 = [0, 1/3] \cup [2/3, 1]$

$A_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$

...

Now suppose that there is an open set U contained in C. Then there must be an open interval (a, b) contained in C. Now pick an integer N such that

$$1/3^N < b - a$$

Then the interval (a, b) can not be contained in the set A_N, because that set is comprised of intervals of length $1/3^N$. But if that interval is not contained in A_N it can not be contained in C. Hence, no open set can be contained in the Cantor set C.

8.5 Baire Category Theorem

Definition: $S \subset M$ is

- Dense in M if $\bar{S} = M$.
- Nowhere dense in M if $M \setminus \bar{S}$ is dense in M.
- Meagre in M if it is the union of a sequence of nowhere dense sets.

Proposition: $S \subset M$ nowhere dense in M iff $\bar{S}^\circ = \emptyset$

Proof: $\bar{S}^\circ = \emptyset = M \setminus \overline{(M \setminus \bar{S})}$ so if RHS $= \emptyset$ then $M \setminus \bar{S}$ is dense in M so S is nowhere dense.

Conversely if S is nowhere dense in M then $M \setminus \bar{S} = M$ so RHS $= \emptyset$.

Theorem: Baire Category

A complete metric space is not meagre in itself.

I.e. if S_n are the nowhere dense subsets of non-empty complete M then

$$M \setminus \bigcup_{n=1}^{\infty} Sn \neq \emptyset$$

Proof: IDEA: Find decreasing sequence of dense sets with non-empty intersection of their closures by Cantor. Any point in this intersection cannot be in any nowhere dense set.

$G_k := M \setminus \overline{S_k}$ dense in M, open.

Then $G_1 \neq \emptyset$. Choose $x_1 \in G_1$ and $\delta_i > 0$ s.t. $B(x_i, \delta_i) \subset G_1$.

Continue inductively: Having defined x_{k-i}, δ_{k-i} use fact that G_k dense to find $x_k \in G_k \cap B \left(x_{k-1}, \frac{\delta_{k-1}}{2}\right)$. Find $0 < \delta_k < \frac{\delta_{k-1}}{2}$ s.t. $B(x_k, \delta_k) \subset G_k$.

$\delta_k \xrightarrow[k\to\infty]{} 0$ and $\forall$ k, $\overline{B(x_k, \delta_k)} \subset B(x_{k-i}, \delta_{k-i})$.

Notes

Then by Cantor (5.15) $\bigcap_{k=1}^{\infty}\overline{B(x_k, \delta_k)} \neq \emptyset$. Let x be in this intersection. Then $x \in B(x_k, \delta_k) \subset G_k$ $\forall$ k so $x \notin S_k$ $\forall$ k. Hence, there is a point x that is not in the union of all nowhere dense subsets of M, so M cannot be meagre.

Proposition: The Cantor set $\mathfrak{C}$ is uncountable.

Proof: $\forall x \in \mathfrak{C}$ there are points $y \in \mathfrak{C}$, $y \neq x$ arbitrarily close to x. In other words, $\mathfrak{C}\backslash\{x\}$ is dense in $\mathfrak{C}$. Therefore {x} is nowhere dense in $\mathfrak{C}$ as it is closed.

If $\mathfrak{C}$ were countable would have $\mathfrak{C} = \bigcup_j^{\infty}\{x_j\}$ showing $\mathfrak{C}$ meagre in itself. This contradicts Baire's theorem.

Lemma: Let f: $[1, \infty) \to \mathbb{R}$ be cts s.t. for some $a \in \mathbb{R}$ $\exists$ arbitrarily large x with f(x) < a. Then $\forall k \in \mathbb{N}$: $S = \bigcup_{n=k}^{\infty}\{x \in [1,\infty) : f(nx) \geq a\}$ is nowhere x with dense.

Proof: f cts so S closed. Let $1 \leq \alpha < \beta < \infty$. RTP $(\alpha, \beta)\backslash S \neq \emptyset$

For large n, $\frac{n+1}{n} < \frac{\beta}{\alpha}$ so $(n + 1)\alpha < n\beta$. Then $\bigcup_{n=k}^{\infty}(n\alpha, n\beta)$ contains some (r, ∞) and so a point y s.t. f(y) < a.

Find n s.t. $y \in (n\alpha, n\beta)$. Then $x = \frac{y}{n} \in (\alpha, \beta)$ and f(nx) < a so $x \notin S$.

Proposition: Let f: $[1, \infty) \to \mathbb{R}$ be cts s.t. $\forall x \geq 1$, $\lim_{n\to\infty} f(nx)$ exists. Then $\lim_{x\to\infty} f(x)$ exists.

Proof: If $\lim_{x\to\infty} f(x)$ not exist then $\exists$ a, b; a < b s.t. $\exists$ arbitrarily large x,y with f(x) < a, f(y) > b.

Then by previous lemma:

$$\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\{x \in [1,\infty) : f(nx) \geq a\} \cup \bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\{x \in [1,\infty) : f(nx) \leq b\}$$

is meagre. By Baire $\exists x \notin T$.

x not in first union so $\forall k \exists n \geq k$ s.t. f(nx) < a. x not in second union so $\forall k \exists m \geq k$ s.t. f(mx) > b. Hence f(nx) not converge.

Theorem: $\exists f \in C[0, 1]$ not differentiable at any point.

Proof: IDEA: C[0, 1] is complete. Functions with derivative at at least one point form a meagre subset. Result by Baire.

Define S_n:

$$S_n = \{f \in C[0, 1] : (\exists x \in [0, 1])(\forall y \in [0, 1])\ |f(y) - f(x)| \leq n|y - x|\}$$

8.6 Compactness and Cantor Set

Theorem: Every compact metric M is continuous image of Cantor set $\mathfrak{C}$.

Proof: Let $A_k \subset M$ be finite s.t. $\forall x \in M$ $d(A_k, x) \leq 2^{-k}$.

By induction construct sequence of cts functions $f_k : \mathfrak{C} \to M$ s.t. $f_k(\mathfrak{C}) = A_k$, $d(f_k(x), f_{k+1}(x)) \leq 2^k$ $\forall x \in C$.

Notes

(f_k) Cauchy in C($\mathfrak{C}$, M) so converge to cts f : $\mathfrak{C} \to$ M. f($\mathfrak{C}$) dense in M. Also compact, so closed, hence f($\mathfrak{C}$) = M.

Corollary: $\exists$ continuous surjective map f : [0, 1] $\to$ [0, 1].

Proof: Extend surjective cts f : ($\mathfrak{C} \to [0, 1]^2$ linearly to each interval removed during construction of $\mathfrak{C}$.

Self Assessment

Fill in the blanks:

1. A complete .... is not meagre in itself.
2. The Cantor set $\mathfrak{C}$ is ....
3. Let f: [1, ∞) $\to \mathbb{R}$ be cts s.t. for some a $\in \mathbb{R}$ ∃ .... x with f(x) < a. Then $\forall$ k $\in \mathbb{N}$: S = $\bigcup_{n=k}^{\infty}\{x \in [1,\infty) : f(nx) \geq a\}$ is nowhere x with dense.
4. Subspace S of complete metric M totally .... compact.

8.7 Summary

- S_n closed.
- S_n nowhere dense as has dense complement and closed.
- If f'(x) exists for some x then f $\in S_n$ for some n.
- Let $f_k \in S_n$, $f_k \to$ f. Find $x_k \in$ [0, 1] s.t. $\forall$ y $\in$ [0, 1],

  $$|f_k(y) - f_k(x_k)| \leq n|y - x_k|$$

  x_k has convergent subsequence so assume $x_k \to$ x. By uniform convergence

  $$|f(y) - f(x)| \leq n|y - x|$$

  Therefore f $\in S_n$, so S_n closed.
- Let g $\in$ C[0, 1], $\in$ > 0. g uniformly cts so ∃ δ > 0 s.t.

  $$|x - y|\delta \Rightarrow |g(x) - g(y)| < \frac{\varepsilon}{4} \qquad \text{. . . (1)}$$

  Let $x_i \frac{i}{k}$ = φ(x) = kε $\min_{0 \leq i \leq k}|x - x_i|$. Then $0 \leq \varphi \leq \frac{\varepsilon}{2}$ show suffices to show f = φ + g $\notin S_n$.

  Suppose f $\in S_n$ and find x "responsible for it".

  Choose 1 ≤ j ≤ k s.t. x $\in [x_{j-1}, x_j]$. Let y = $\frac{x_{j-1} + x_j}{2}$

  $$\frac{\varepsilon}{2} = |\varphi(y) - \varphi(x_j)|$$

  $$\leq |f(y) - f(x_j)| + |g(y) - g(x_j)|$$

  $$\underset{(1)}{\leq} |f(y) - f(x)| + |f(x_j) - f(x)| + \frac{\varepsilon}{4}$$

Notes

$$\leq |n|y-x| + n|x_j - x| + \frac{\varepsilon}{4}$$

$$\leq \frac{2n}{k} + \frac{\varepsilon}{4}$$

$$\leq \frac{\varepsilon}{2}$$

This is a contradiction. So $f \notin S_n$.

- If f′ (x) exists find δ > 0 s.t. $\forall\, 0 < |y - x| < \delta$,

$$\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| < \left|\frac{f(y)-f(x)}{y-x}\right| < 1 + |f'(x)|$$

Function $y \mapsto \frac{f(y)-f(x)}{y-x}$ is continuous on $[0, 1]\backslash(x - \delta, x + \delta)$ which is compact. Hence the function is bounded, so $\exists\, n \in \mathbb{N}$ s.t.

$$y \in [0, 1] \backslash (x - \delta, x + \delta) \Rightarrow \left|\frac{f(y)-f(x)}{y-x}\right| \leq n$$

May take $n > 1 + |f'(x)|$ so get inequality holding $\forall\, y \in [0, 1]\backslash\{x\}$.

Then $|f(y) - f(x)| \leq n|y - x| \;\; \forall\, y \in [0, 1]$. (This clearly holds for y = x and holds by the above for y ≠ x.) So if $\exists\, f \in C[0, 1]$ s.t. f′(x) exists for some x then $f \in S_n$.

- These three parts together complete the proof, since by Baire (5.17) C[0, 1] is not meagre, so there must be a function which is not differentiable at any point, as any that are differentiable at at least one point are in a nowhere dense subset.

8.8 Keywords

Complete Metric: Subspace C of complete metric M compact iff closed and totally bounded.

Cantor: Let F_n decreasing sequence of non-empty closed subsets of metric M s.t. diam $(F_n) \xrightarrow[n\to\infty]{} 0$. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Continue Inductively: Having defined x_{k-i}, δ_{k-i} use fact that G_k dense to find $x_k \in G_k \cap B\left(x_{k-1}, \frac{\delta_{k-1}}{2}\right)$. Find $0 < \delta_k < \frac{\delta_{k-1}}{2}$ s.t. $B(x_k, \delta_k) \subset G_k$.

$\delta_k \xrightarrow[k\to\infty]{} 0$ and $\forall\, k$, $\overline{B(x_k, \delta_k)} \subset B(x_{k-i}, \delta_{k-i})$.

8.9 Review Questions

1. Discuss Completeness and Compactness.
2. Describe the Cantor's theorem.
3. Explain Baire category theorem.
4. Describe Compactness and Cantor set.

Answers: Self Assessment

Notes

1. metric space
2. uncountable
3. arbitrarily large
4. bounded iff $\overline{S}$

8.10 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 9: Functions

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the different types of algebraic functions
- Explain the trigonometric and the inverse trigonometric functions
- Describe the exponential and logarithmic functions
- Explain some special functions including thus bounded and monotonic functions

Introduction

Real Analysis is often referred to as the Theory of Real Functions. The word 'function' was first introduced in 1694 by L.G. Leibniz [1646-1716], a famous German mathematician, who is also

Notes

credited along with Isacc Newton for the invention of Calculus, Leibniz used the term function to denote a quantity connected with a curve. A Swiss mathematician, L. Euler [1707-1783] treated function as an expression made up of a variable and some constants. Euler's idea of a function was later generalized by an eminent French mathematician J. Fourier [1768-1830]. Another German mathematician, L. Dirichlet (1805-1859) defined function as a relationship between a variable (called an independent variable) and another variable (called the dependent variable). This is the definition which, you know, is now used in Calculus.

The concept of a function has undergone many refinements. With the advent of Set Theory in 1895, this concept was modified as a correspondence between any two non-empty sets. Given any two non-empty sets S and T, a function f from S into T, denoted as $f: S \to T$, defines a rule which assigns to each $x \in S$, a unique element Leonard Euler $y \in T$. This is expressed by writing as $y = f(x)$. This definition, as you will recall, was given in Section 1.2. A function $f\ S \to T$ is said to be a

1. Complex-valued function of a complex variable if both S and T are sets of complex numbers;
2. Complex-valued function of a real variable if S is a set of real numbers and T is a set of complex numbers;
3. Real-valued function of a complex variable if S is a set of complex numbers and T is a set of real numbers;
4. Real-valued function of a real variable if both S and T are some sets of real numbers.

Since we are dealing with the course on Real Analysis, we shall confine our discussion to those functions whose domains as well as co-domains are some subsets of the set of real numbers. We shall call such functions as Real Functions.

In this unit, we shall deal with the algebraic and transcendental functions. Among the transcendental functions, we shall define the trigonometric functions, the exponential and logarithmic functions. Also, we shall talk about some special real functions including the bounded and monotonic functions. We shall frequently use these functions to illustrate various concepts in Blocks 3 and 4.

9.1 Algebraic Functions

In Unit 1, we identified the set of natural numbers and built up various sets of numbers with the help of the algebraic operations of addition, subtraction, multiplication, division etc. In the same way, let us construct new functions from the real functions which we have chosen for our discussion. Before we do so, let us review the algebraic combinations of the functions under the operations of addition, subtraction, multiplication and division on the real-functions.

9.1.1 Algebraic Combinations of Functions

Let f and g be any two real functions with the same domain S C R and their co-domain as the set R of real numbers. Then we have the following definitions:

Definition 1: Sum and Difference of Two Functions

1. The Sum of f and g, denoted as f + g, is a function defined from S into R such that

   $(f + g)(x) = f(x) + g(x)$, If $x \in S$.

2. The Difference of f and g, denoted as f – g, is a function defined from S to R such that

   $(f - g)(x) = f(x) - g(x),\ \forall x \in S$.

Notes

Note that both f(x) and g(x) are elements of R. Hence each of their sum and difference is again a unique member of R.

Definition 2: Product of Two Functions

Let f: S →R and g: S → R be any two functions. The product of f and g, denoted as f, g, is defined as a function f. g: S → R by

$$(f . g) (x) = f(x) . g(x), \ \forall x \in S.$$

Definition 3: Scalar Multiple of a Function

Let f: S → R be a function and k be same fixed real number. Then the scalar multiple of 'f' is a function k f S → R defined by

$$(kf (x) = k. f(x), \ \forall x \in S.$$

This is also called the scalar multiplication.

Definition 4: Quotient of Two Functions

Let f: S → R and g: S → R be any two functions such that g(x) # 0 for each x in S. Then s function $\frac{f}{g}$: S → R defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \forall x \in S$$

is called the quotient of the two functions.

Exercise 1: Let f, g, h be any three functions, defined on S and taking values in R, as f (x) = ax^2, g(x) = bx for every x in S, where a, b, are fixed real numbers. Find f + g, f – g, f, g, f/g and kf, when k is a constant.

9.1.2 Notion of an Algebraic Function

You are quite familiar with the equations ax + b = 0 and $ax^2 + bx + c = 0$, where a, b, c ∈ R, a ≠ 0. These equations, as you know are, called linear (or first degree) and quadratic (or second degree) equations, respectively. The expressions ax + b and $ax^2 + bx + c$ are, respectively, called the first and second degree polynomials in x. In the same way an expression of the form ax′ + bx^2 + cx + d (a # 0, a, b, c, d ER) is called a third degree polynomial (cubic polynomial) in x. In general, an expression of the form $a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ where $a_0 \neq 0$, a ER, i = 0, 1, 2, ..., n, is called an nth degree polynomial in x.

A function which is expressed in the form of such a polynomial is called a polynomial function. Thus, we have the following definition:

Definition 5: Polynomial Function

Let a_1 (i = 0, 1, ..., n) be fixed real numbers where n is some fixed non-negative integer. Let S be a subset of R. A function f: S → R defined by

$$f(x) = a_0 x^n + a_1 x^{a-1} + a_2 x^{n-2} + ... + a_n, \ \forall x \in S, a_0 \neq 0.$$

is called a polynomial function of degree n.

Let us consider some particular cases of a polynomial function on R:

Suppose f : S → R is such that

(i) f(x) = k, ∀ x ∈ S (k is a fixed real number). This is a polynomial function. This is generally called a constant function on S.

Notes

Example:

$f(x) = 2$, $f(x) = -3$, $f(x)= \pi$, $\forall\ x \in R$, are all constant functions.

(ii) One special case of a constant function is, obtained by taking

$k = 0$ i.e. when

$f(x) = 0,\ \forall x \in S$.

This is called the zero function on S.

Let $f: S \rightarrow R$ be such that

(iii) $f(x) = a_0 x + a_1,\ \forall x \in S, a, \# 0$.

This is a polynomial function and is called a linear function on S. For example,

$f(x) = 2x + 3$, $f(x) = -2x + 3$,

$f(x) = 2x - 3$, $f(x) = -2x - 3$, $f(x) = 2x$ for every

$x \in S$ are all linear functions

(iv) The function $f: S \rightarrow R$ defined by

$f(x) = x,\ \forall x \in S$

s called the identity function on S,

(v) $f: S \rightarrow R$ given as .

$f(x),=a_0.x^2 + a_1 x + a_2, \forall x \in R, a_o \# O$.

is a polynomial function of degree two and is called a quadratic function on S.

Example: $f(x) = 2x^2 + 3x - 4$, $f(x) = x^2 + 3$, $f(x) = x^2 + 2x$,

$f(x) = -3x^2$,

for every $x \in S$ are all quadratic functions.

Definition 6: Rational Function

A function which can be expressed as the quotient of two polynomial functions is called a rational function.

Thus a function $f: S \rightarrow R$ defined by

$$f(x) = \frac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^m + b_1\ x^{m-1} + ... + b_m},\ \forall x \in S$$

is called a rational function.

Here $a_0 \# b_0 \# 0$, $a_i, b_j \in R$ where i, j are some fixed real numbers and the polynomial function in the denominator is never zero.

Example: The following are all rational functions on R.

$$\frac{2x+3}{x^2+1}, \frac{4x^2+3x+1}{3x-4}(x \neq \frac{4}{3}) \text{ and } \frac{3x+5}{x-4}(x \neq 4).$$

Notes

The functions which are not rational are known as irrational functions. A typical example of an irrational function is the square root function which we define as follows:

Definition 7: Square Root Function

Let S be the set of non-negative real numbers. A function f: S $\rightarrow$ R defined by

$$f(x) = \sqrt{x}, \quad \forall x \in S$$

is called the square root function.

You may recall that $\sqrt{x}$ is the non-negative real number whose square is x. Also it is defined for all $x \geq 0$.

Polynomial functions, rational functions and the square root function are some of the examples of what are known as algebraic functions. An algebraic function, in general, is defined as follows

Definition 8: Algebraic Function

An algebraic function f : S $\rightarrow$ R is a function defined by y = f(x) if it satisfies identically an equation of the form

$$p_0(x)y^n + p_1(X)y^{n-1} + \ldots + p_{n-1}(x)y + p_n(x) = 0$$

where p(x), $p_1(x)$, ... $p_{n-1}(x)$, $p_n(x)$ are Polynomials in x for all x in S and n is a positive integer.

Example: Show that f: R $\rightarrow$ R defined by

$$f(x) = \sqrt{\frac{x^2 - 3x + 2}{4x - 1}}$$

is an algebraic function.

Solution:

$$\text{Let} \quad y = f(x) = \sqrt{\frac{x^2 - 3x + 2}{4x - 1}}$$

Then $(4x - 1)y^2 - (x^2 - 3x + 2) = 0$

Hence f(x) is an algebraic function.

In fact, any function constructed by a finite number of algebraic operations (addition, subtraction, multiplication, division and root extraction) on the identity function and the constant function, is an algebraic function.

Example: The functions f : R $\rightarrow$ R defined by

(i) $$f(x) = \frac{(x^2 + 2)\sqrt{x - 1}}{x^2 + 4}$$

or $$f(x) = \frac{x^2 + 2x}{\sqrt{x.(3x^2 + 5)}}$$

are algebraic functions.

Notes

Example: Prove that every rational function is an algebraic function.

Solution: Let f : R → R be given as

$$f(x) = \frac{p(x)}{q(x)}, \ \forall x \in R,$$

where p(x) and q(x) are some polynomial functions such that $q(x) \neq 0$ for any $x \in R$.

Then we have

$$y = f(x) = \frac{p(x)}{q(x)}$$

$$q(x)\, y - p(x) = 0$$

which shows that y = f(x) can be obtained by solving the equation

$$q(x)\, y - p(x) = 0.$$

Hence f(x) is an algebraic function.

A function which is not algebraic is called a Transcendental Function. Examples of elementary transcendental functions are the trigonometric functions, the exponential functions and the logarithmic functions, which we discuss in the next section.

9.2 Transcendental Functions

In earlier unit, we gave a brief introduction to the algebraic and transcendental numbers. Recall that a number is said to be an algebraic if it is a root of an equation of the form

$$a_0 x^n + a_1 x^{n-1} + ... x + a_{n-1} x + a_n = 0$$

with integral coefficients and $a_0 \neq 0$, where n is a positive integer. A number which is not algebraic is called a transcendental number. For example the numbers e and IT are transcendental numbers. In fact, the set of transcendental numbers is uncountable. Based on the same analogy, we have the transcendental functions. We have discussed algebraic functions. The functions that are non-algebraic are called transcendental functions. In this section, we discuss some of these functions.

9.2.1 Trigonometric Functions

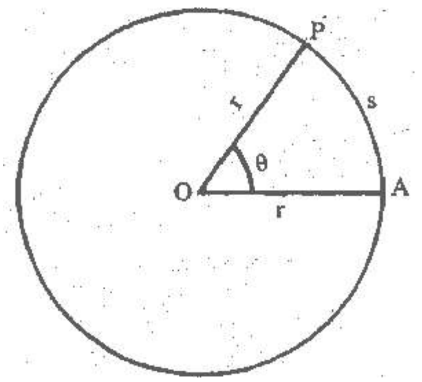
You are quite familiar with the trigonometric functions from the study of Geometry and Trigonometry. The study of Trigonometry is concerned with the measurement of the angles and the ratio of the measures of the sides of a triangle. In Calculus, the trigonometric functions have an importance much greater than simply their use in relating sides and angles of a triangle. Let us review the definitions of the trigonometric functions sin x, cos x and some of their properties. These functions form an important class of real functions.

Consider a circle $x^2 + y^2 = r^2$ with radius r and centre at O. Let P be a point on the circumference of this circle. If θ is the radian measure of a central angle at the centre of the circle as shown in the Figure 9.1 then you know that the lengths of the arc AP is given by

$$s = \theta r.$$

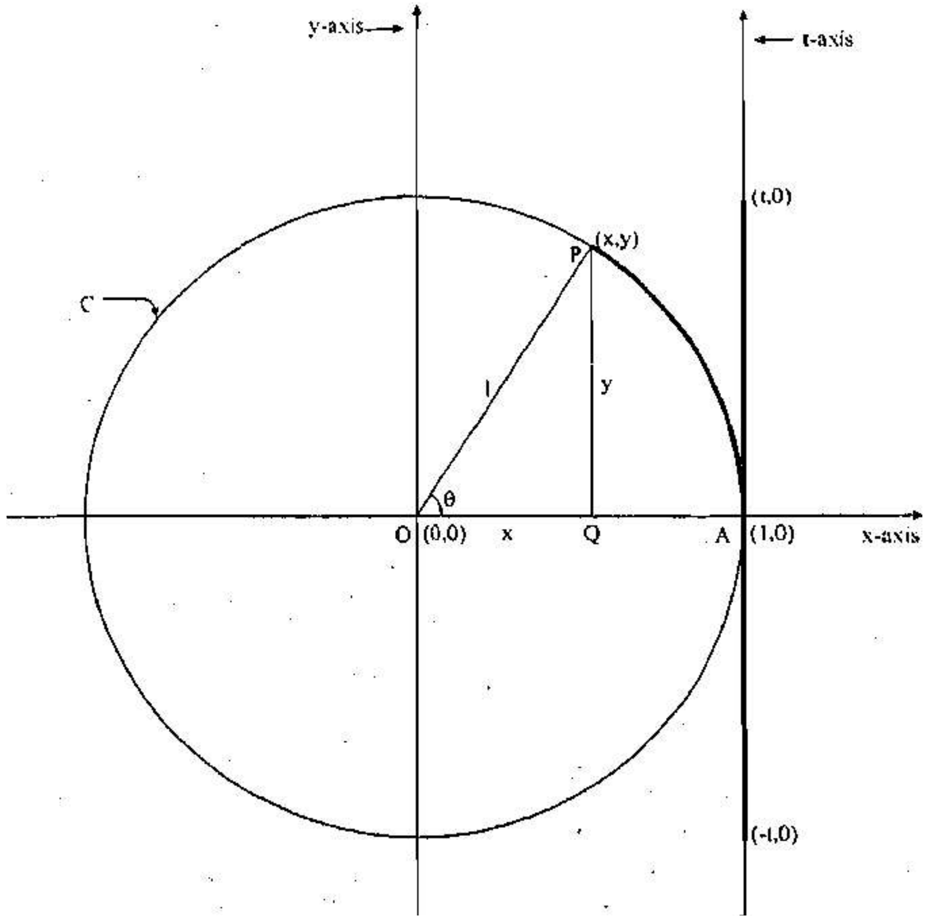
Notes

Figure 9.1



You already know how the trigonometric ratios sin θ, cos θ, etc., are defined for an angle θ measured in degrees or radians. We now define sin x, cos x, etc., for x ∈ R.

Figure 9.2



If we put r = 1 in above relation, then we get θ = s. Also the equation of circle becomes $x^2 + y^2 = 1$. This, as you know, is the Unit Circle. Let C represents this circle with centre O and radius 1. Suppose the circle meets the x-axis at a point A as shown in the Figure 9.2.

Through the point A = (1, 0); we draw a vertical line labeled as t-axis with origin at A and positive direction upwards. Now, let t be any real number and we will think of this as a point on this verticle number line i.e., t-axis.

Imagine this t-axis as a line of thread that can be wrapped around the circle C. Let p(t) = (x, y) be the point where 't' ends up when this wrapping takes place. In other words, the line segment from A to point (t, 0) becomes the arc from A to P, positive or negative i.e., counterclockwise or clockwise, depending on whether $t > 0$ or $t < 0$. Ofcourse, when t = 0, P = A. Then, the trigonometric functions 'sine' and 'cosine', for arbitrary $t \in R$, are defined by

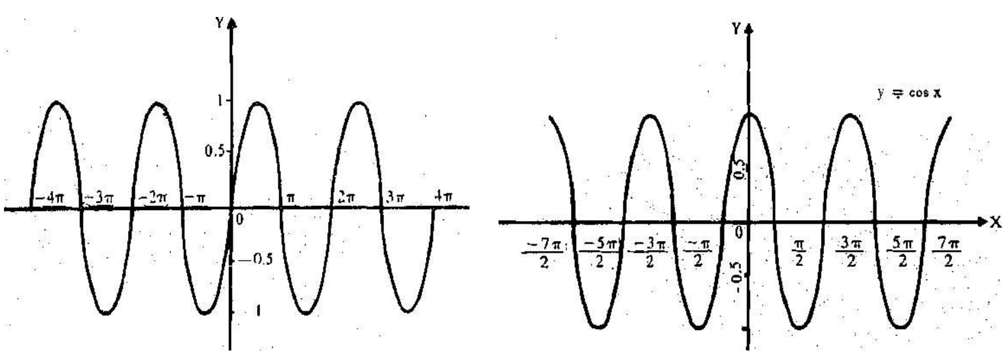
$$\sin t = \sin \theta = y, \text{ and } \cos t = \cos \theta = x,$$

where 'θ' is the radian measure of the angle subtended by the arc AP at the centre of the circle C. More generally, if t is any real number, we may take $(0 < \theta < 2\pi)$ to be the angle (rotation) whose radian measure is t. It is then clear that

$$\sin (t + 2\pi) = \sin t \text{ and } \cos (t + 2\pi) = \cos t.$$

You can easily see that as θ increases from 'θ' to $\pi/2$, PQ increases from 0 to 1 and OQ decreases from 1 to 0. Further, as θ increases from $\frac{\pi}{2}$ to θ, PQ decreases from 1 to 0 and OQ decreases from 0 to –1. Again as θ increases from π to $\frac{3\pi}{2}$, PQ decreases from 0 to –1 and OQ increases from –1 to 0. As θ increases from –1 to 0. The graphs of these functions take the shapes as shown in Figure 9.3.

Figure 9.3



Thus, we define sin x and cos x as follows:

Definition 9: Sine Function

A function $f : R \to R$ defined by

$$f(x) = \sin x, \quad \forall x \in R$$

is called the sine of x. We often write y = sin x.

Definition 10: Cosine Function

A function $f : R \to R$ defined by

$$f(x) = \cos x, \quad \forall x \in R$$

is called the cosine of x and we write y = cos x.

Notes

Note that the range of each of the sine and cosine, is [–1, 1]. In terms of the real functions sine and cosine, the other four trigonometric functions can be defined as follows:

(i) A function $f : S \to R$ defined by

$$f(x) = \tan x = \frac{\sin x}{\cos x}, \cos x \neq 0, \ \forall x \in S = R - \{(2n+1)\frac{\pi}{2}\}$$

is called the cos x Tangent Function. The range of the tangent function is $]-\infty, +\infty[= R$ and the domain is $S = R - \{(2n+1)\frac{\pi}{2}\}$, where n is a non-negative integer.

(ii) A function $f : S \to R$ defined by

$$f(x) = \cot x = \frac{\cos x}{\sin x}, \sin x \neq 0, \ \forall x \in S = S - \{n\pi\},$$

is said to be the Cotangent Function. Its range is also same as its co-domain i.e. range $=]-\infty, \infty[= R$ and the domain is $S = R - \{n\pi\}$ where n is a non-negative integer.

(iii) A function $f : S \to R$ defined by

$$f(x) = \sec x = \frac{1}{\cos x}, \cos x \neq 0, \ \forall x \in S = S - \{2n+1)\frac{\pi}{2}\},$$

is called the Secant Function. Its range is the set

$$S =]-\infty, -1] \cup [1, \infty[\text{ and domain is } S = R - \{2n+1)\frac{\pi}{2}\}.$$

(iv) A function $f : S \to R$ defined by

$$f(x) = \operatorname{cosec} x = \frac{1}{\sin x}, \sin x \neq 0, x \in S = R - \{n\pi\},$$

is called the Cosecant function. Its range is also the set $S =]-\infty, -1] \cup [1, \infty[$ and domain is $S = R - \{n\pi)$,

The graphs of these functions are shown in the Figure 9.4.

Example: Let $S = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Show that the function $f : S \to R$ defined by

$$f(x) = \sin x, \ \forall x \in S$$

is one-one. When is f only onto? Under what conditions f is both one-one and onto?

Solution: Recall from Unit 1 that a function f is one-one if

$$f(X_1) = f(X_2) \Rightarrow X_1 = X_2$$

for every x_1, X_2 in the domain of f.

Therefore, here we have for any $X_1, X_2 \in S$,

$$f(x_1) = f(x_2) \Rightarrow \sin x_1 = \sin x_2$$

$$\Rightarrow \sin x_1 - \sin X_2 = 0$$

$$\Rightarrow 2 \sin \frac{x_1 - x_2}{2} \cos \frac{x_1 + x_2}{2} = 0$$

Notes

$\Rightarrow$ Either sin $\frac{x_1 - x_2}{2}$ = 0, or cos $\frac{x_1 + x_2}{2}$ = 0.

If sin $\frac{x_1 - x_2}{2}$ = 0, then $\frac{x_1 - x_2}{2}$ = 0, ± π, + 2π, ...

If cos $\frac{x_1 + x_2}{2}$ = 0, then $\frac{x_1 + x_2}{2}$ = 0, ± $\frac{\pi}{2}$, ± $\frac{3\pi}{2}$, ...

Since x_1, x_2 ∈ [$-\frac{\pi}{2}$, $\frac{\pi}{2}$]. Therefore we can only have

$$-\frac{\pi}{2} \leq \frac{x_1 - x_2}{2} \leq \frac{\pi}{2}$$

and $$-\frac{\pi}{2} \leq \frac{x_1 - x_2}{2} \leq \frac{\pi}{2}$$

Thus, $\frac{x_1 - x_2}{2}$ = 0 i.e., $x_1 = x_2$. Also, if $\frac{x_1 + x_2}{2}$ = ± $\frac{\pi}{2}$

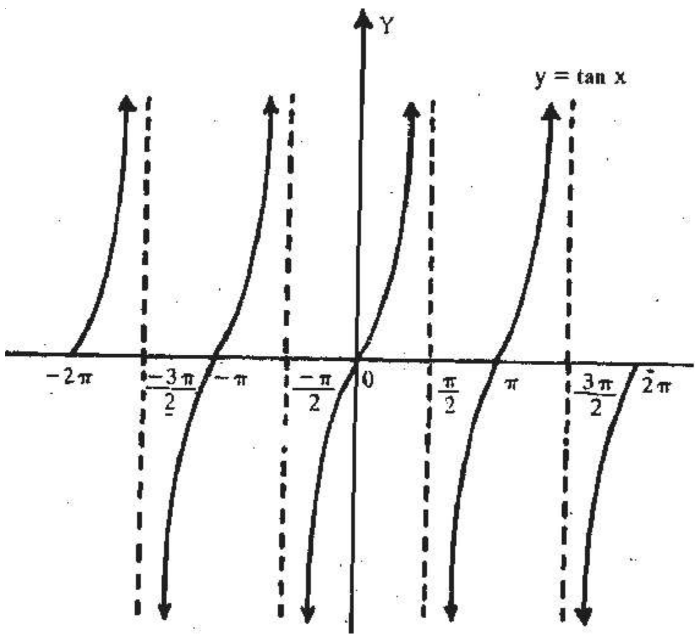
i.e. then $x_1 + x_2 = \pm x$.

Since $x_,, x_, \in [-\frac{x}{2}, \frac{x}{2}]$,

therefore, $x_1 = x_2 = \frac{\pi}{2}$ or $x_, = x_1 = -\frac{\pi}{2}$

Hence $(x_1) = f(x_2) \Rightarrow x_, = x_,$, which proves that f is one-one. Then function f(x) = sin x defined as such, is not onto because you know that the range of sin x is [–1, 1] ≠ R.

Figure 9.4



Notes

Figure 9.5

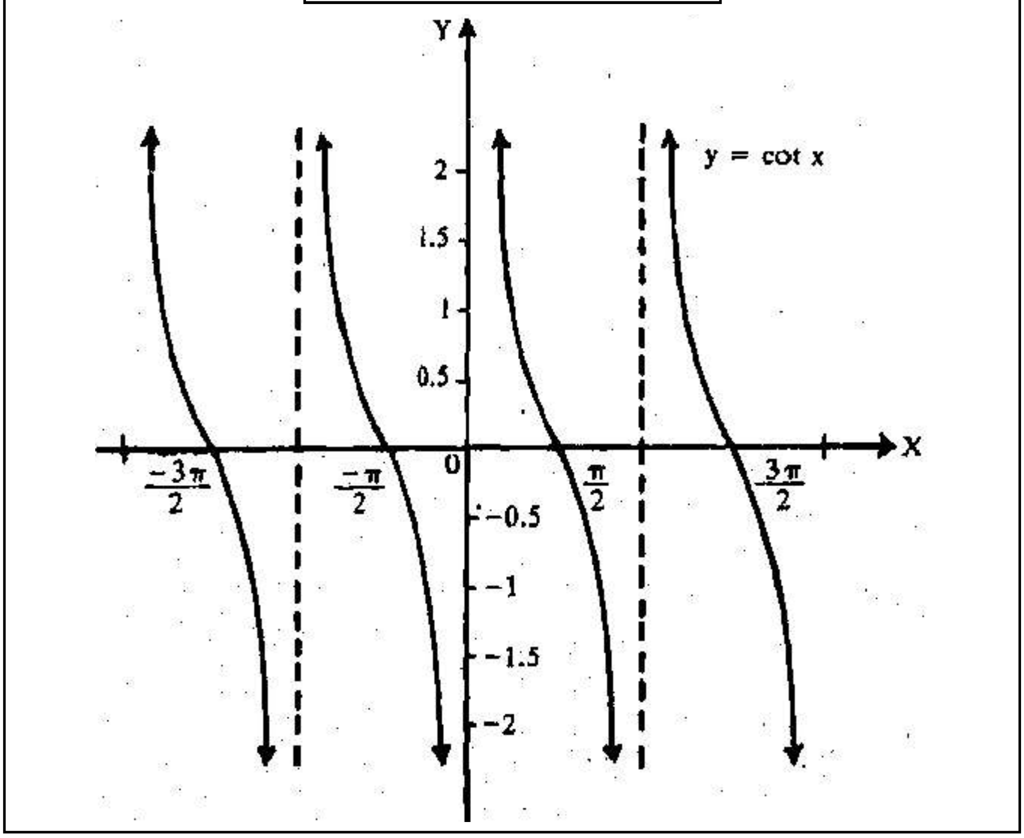
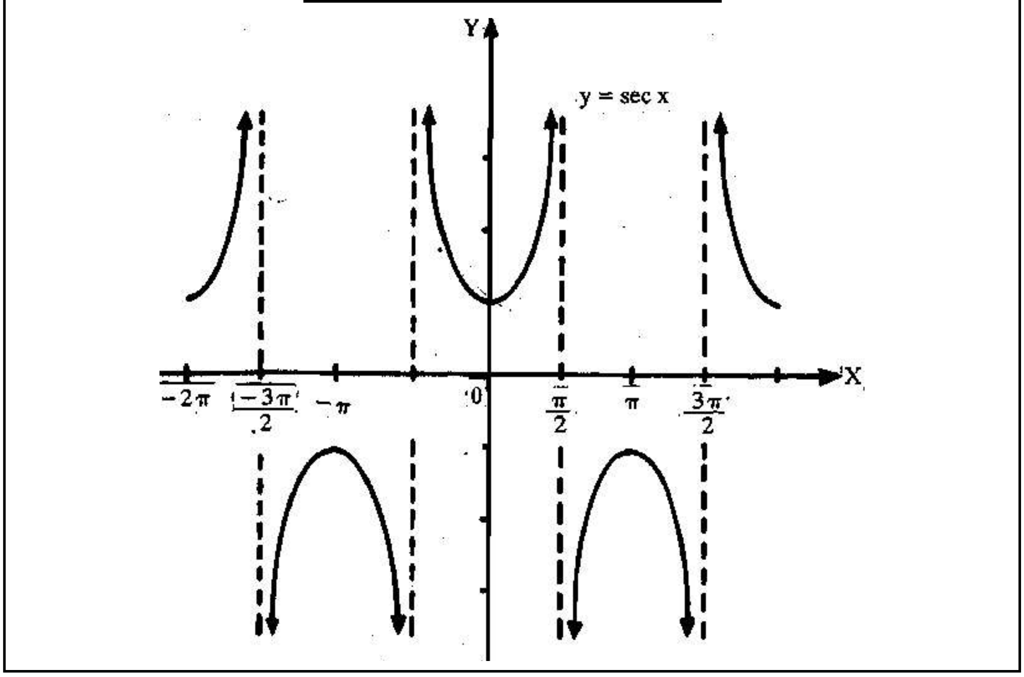
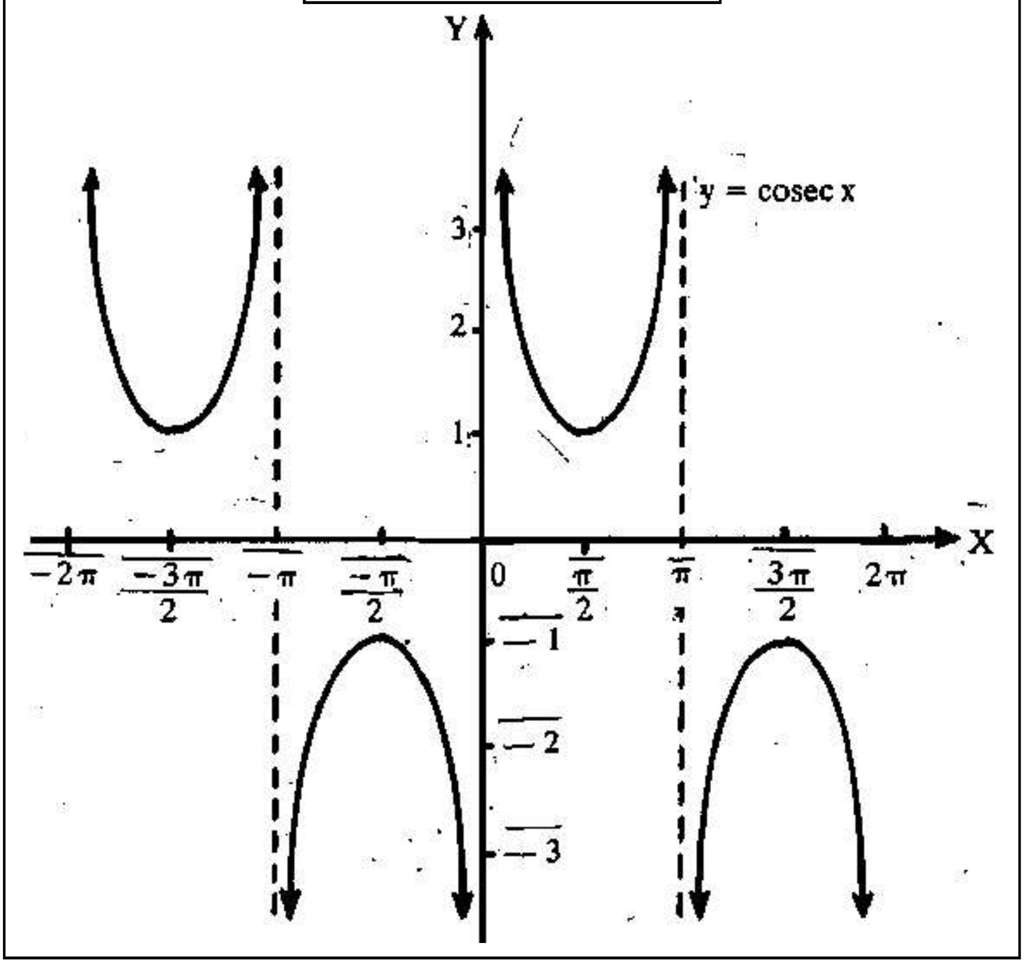


Figure 9.6



Notes

Figure 9.7



If you define f : R → [–1, 1] as

$f(x) = \sin x, \ \forall x \in R$

Then f is certainly onto. But then it is not one-one. However the function.

f : $\frac{\pi}{2}$ [–, $\frac{\pi}{2}$] → [–1, 1] defined by

$f(x) = \sin x, \ \forall x \in R$

is both one-one and onto.

Exercise 2: Two functions g and h are defined as follows:

(i) g : S → R defined by

$g(x) = \cos x, x \in S = [0, \pi]$

(ii) h : S → R defined by

$h(x) = \tan x, x \in S =]-\frac{\pi}{2}, \frac{\pi}{2}[$

Show that the functions are one-one. Under what conditions the function are one-one and onto?

Notes

9.3 Inverse Trigonometric Functions

Here we discussed inverse functions. You know that if a function is one-one and onto, then it will have an inverse. If a function is not one-one and onto, then sometimes it is possible lo restrict its domain in some suitable manner such that the restricted function is one-one and onto. Let us use these ideas to define the inverse trigonometric functions. We begin with the inverse of the sine function.

Refer to the graph of f(x) = sin x in Figure 9.8. The x-axis cuts the curve y = sin x at the points x = 0, x = π, x = 2π. This shows that function f(x) = sin x is not one-one. If we restrict the domain of f(x) = sinx to the interval [–π/2, 71/21], then the function

$$f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1] \text{ defined by}$$

$$f(x) = \sin x, -\frac{71}{21} \le x \le \frac{71}{21}$$

is one-to-one as well as onto. Hence it will have the inverse. The inverse function is called the inverse sine of x and is denoted as sin x. In other words,

$$y = \sin\text{-}'x \Leftrightarrow x = \sin y,$$

where $-\frac{71}{21} \le y \ \frac{\pi}{2}$ and $-1 \le x \le 1$.

Thus, we have the following definition:

Definition 11: Inverse Sine Function

A function $g : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ defined by

$$g(x) = \sin^{-x} x, \quad \forall x \text{ s } [-1, 1]$$

is called the inverse sine function.

Again refer back to the graph of f(x) = cos x in Figure. You can easily see that cosine function is also not one-one. However, if you restrict the domain of f(x) = cos x to the interval [0, π], then the function f : [0, π] → [–1, 1] defined by

$$f(x) = \cos 0, \mid x \le \pi,$$

is one-one and onto. Hence it will have the inverse. The inverse function is called the inverse cosine of x and is denoted by $\cos^{-1} x$ (or by arc cos x). In other words,

$$y = \cos^{-1} x \Leftrightarrow x = \cos y,$$

where 0 I y ≤ π and –1 | x ≤ I.

Thus, we have the following definition:

Definition 12:

A function g : [–1, 1] → {0, π] defined by

$$g(x) = \cos^{-1} x, \quad \forall x \in [-1, 1],$$

is called the inverse cosine function.

You can easily see from Figure that the tangent function, in general, is not one-one. However, again if we restrict the domain of f(x) = tan x to the interval]–π/2, π/2[, then the function.

Notes

$f :]-\frac{\pi}{2}, \frac{\pi}{2}[\rightarrow R$ defined by

$f(x) = \tan x, -\frac{\pi}{2}, < x < \frac{\pi}{2}$

is one-one and onto. Hence it has an inverse. The inverse function is called the inverse tangent of x and is denoted by tan –1 x (or by arctan x). In other words,

$y = \tan^{-1} x \Leftrightarrow x = \tan y,$

where $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $-\infty < x < +\infty$.

Thus, we have the following definition:

Definition 13: Inverse Tangent Function

A function $g : R \rightarrow], -\frac{\pi}{2}, \frac{\pi}{2}[$ defined by

$g(x) = \tan^{-1} x, \; \forall x \in R$

is called the inverse tangent function.

Task Define the inverse cotangent, inverse secant and inverse cosecant function. Specify their domain and range.

Now, before we proceed to define the logarithmic and exponential functions, we need the concept of the monotonic functions. We discuss these functions as follows:

9.3.1 Monotonic Functions

Consider the following functions:

(i) $f(x) = x, \; \forall x \in R.$

(ii) $f(x) = \sin x, \; \forall x \in [-\pi/2, \pi/2].$

(iii) $f(x) = -x^2, \; \forall x \in [0, \infty[,$

(iv) $f(x) = \cos x, \; \forall x \in [0, \pi].$

Out of these functions, (i) and (ii) are such that for any $x_,, x_2$ in their domains,

$x_, < x_2 \Rightarrow f(X_1) \leq f(x_2),$

whereas (iii) and (iv) are such that for any $x_,, x_2$ in their domains,

$x_, < x_2 \Rightarrow f(X_1) \geq f(x_2).$

The functions given in (i) and (ii) are called monotonically increasing while those of (iii) and (iv) are called monotonically decreasing. We define these functions as follows:

Let $f : S \rightarrow R \; (S \subset R)$ be a function

(i) It is said to be a monotonically increasing function on S if

$x_1 < x_, \Rightarrow f(X_1) < f(X_1)$ for any $x_,, x_2 \in S$

(ii) It is said to be a monotonically decreasing function on S if

$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ for any $x_,, x_2 \in S$.

(iii) The function f is said to be a monotonic function on S if it is either monotonically increasing or monotonically decreasing.

(iv) The function f is said to be strictly increasing on S if

$x < x_2 \Rightarrow f(X_1) < f(x_2)$, for $x_,, x_2 \in S$,

(v) It is said to be strictly decreasing on S if

$x_1 < x_2 \Rightarrow f(x_1) > f(X_2)$, for $x_,, x_2 \in S$.

You can notice immediately that if f is monotonically increasing then –f i.e. $-f: R \to R$ defined by $(-f)(x) = -f(x),\ \forall x \in R$

is monotonically decreasing and vice-versa.

Example: Test the monotonic character of the function $f: R \to R$ defined as

$$f(x) = \begin{cases} x^2, x \leq 0 \\ -x^2, x > 0 \end{cases}$$

Solution: For any $X_1, x_2 \in R, X_1 \leq 0; X_2 \leq 0$

$$X_1 < X_2 \Rightarrow X^2_1 > X^2_2 \Rightarrow f(x_1) > f(X_2)$$

which shows that f is strictly decreasing.

Again if $X_1 > 0, X_2 > 0$, then

$$X_1 < X_2 \Rightarrow X^2_1 < x^2_2 \Rightarrow -X^2_1 > -X^2_2 \Rightarrow f(X_1) > f(X_2)$$

which shows that i is strictly decreasing for $x > 0$. Thus f is strictly decreasing for every $x \in R$.

Now, we discuss an interesting property of a strictly increasing function in the form of the following theorem:

Theorem 1: Prove that a strictly increasing function is always one-one.

Proof: Let $f : S \to T$ be a strictly increasing function. Since f is strictly increasing, therefore,

$$X_1 < X_2 \Rightarrow f(x_1) < f(x_2) \text{ for any } X_1, x_2 \in S.$$

Now to show that $f : S \to T$ is one-one, it is enough to show that

$$f(x_1) = f(x_2) \Rightarrow X_1 = X_2.$$

Equivalently, it is enough to show that distinct elements in S have distinct images in T

i.e. $X_1 \neq X_2 \Rightarrow f(x_1) \neq f(x_2)$, for $X_1, X_2 \in S$.

Indeed,

$$x_1 \neq x_2 \Rightarrow x_1 < x_1 \text{ or } x_1 > x_2$$

$$\Rightarrow f(x_1) < f(x_2) \text{ or } f(x_1) > f(x_2)$$

$$\Rightarrow f(x_1) \neq f(x_2)$$

which proves the theorem.

Notes

Example: Let $f : S \to T$ be a strictly increasing function such that f(S) = T. Then prove that f is invertible and $f^1 : T \to S$ is also strictly increasing.

Solution:

Since $f : S \to T$ is strictly increasing, therefore, f is one-one. Further, since f (S) = T, therefore f is onto. Thus f is one-one and onto. Hence f is invertible. In other words, $f^{-1} : T \to S$ exists.

Now, for any $y_1, y_2 \in T$, we have $y_1 = f(x_1)$, $y_2 - f(x_2)$. If $y_1 < y_2$) then we claim $x_1 < x_2$.

Indeed if $x_1 \geq x_2$, then $f(x_1) \geq f(x_2)$ (why?).

This implies that $y_1 \geq y_2$ which contradicts that $y_1 < y_2$.

Hence $y_1 < y_2 \Rightarrow x_1 < x_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)$

which shows that f^{-1} is strictly increasing.

You can similarly solve the following exercise for a strictly decreasing function:

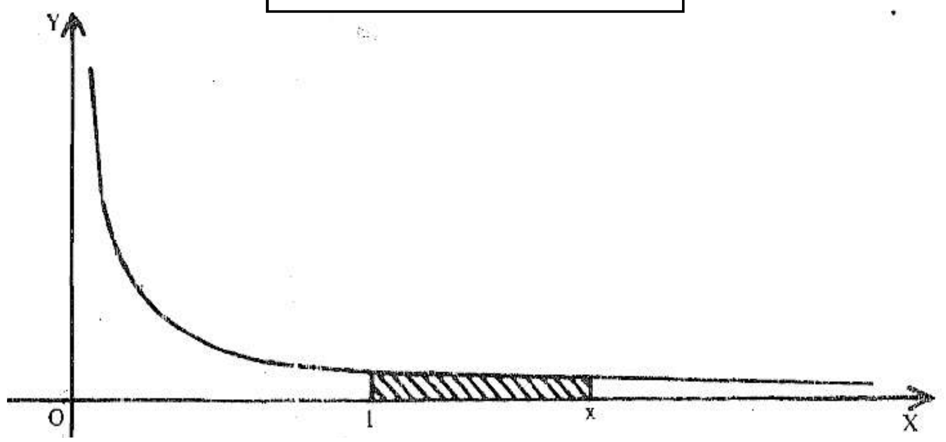
Exercise 3: Let $f : S \to T$ be a strictly decreasing function such that f(S) = T. Show that f is invertible and $f^{-1} : T \to S$ is also strictly decreasing.

9.3.2 Logarithmic Function

You know that a definite integral of a function represents the area enclosed between the curve of the function, X-axis and the two Ordinates. You will now see that this can be used to define logarithmic function and then the exponential function.

We consider the function $f(x) = \frac{1}{x}$ for x > 0, We find that the graph of the function is as shown in the figure 9.8.

Figure 9.8



Definition 14: Logarithmic Function

For $x \geq 1$, we define thus natural logarithmic function log x as

$$\log x = \int_1^x \frac{1}{t} dt$$

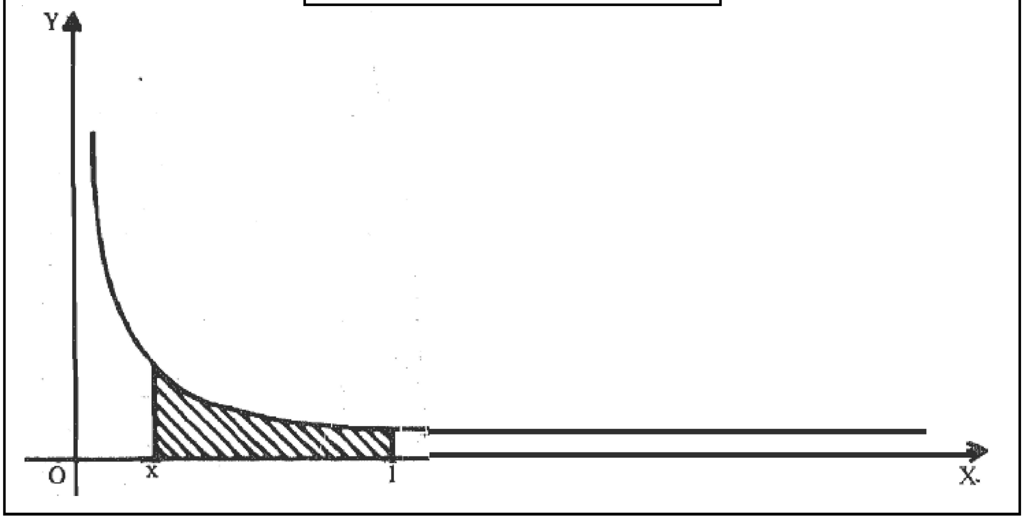
Notes

In the Figure 9.9, log x represents the area between the curve f(t) = $\frac{1}{t}$, x-Axis and the two ordinates at 1 and at x. For 0 < x < 1, we define

$$\log x = \int_x^1 \frac{1}{t} dt$$

This means that for 0 < x < 1. log x is the negative of the area under the graph of f(t) = $\frac{1}{t}$, X-Axis and the two ordinates at x and at 1.

Figure 9.9



We also see by this definition that

$\log x < 0$ if $0 < x < 1$

$\log 1 = 0$

and

$\log x > 0$ if $x > 1$.

It 'also follows by definition that if.

$x_1 > x_2 > 0$, then $\log x_1 > \log x_2$. This shows that log x is strictly increasing. The reason for this is quite clear if we realise by $\log x_1$ as the area under the graph as shown in the Figure 9.10.

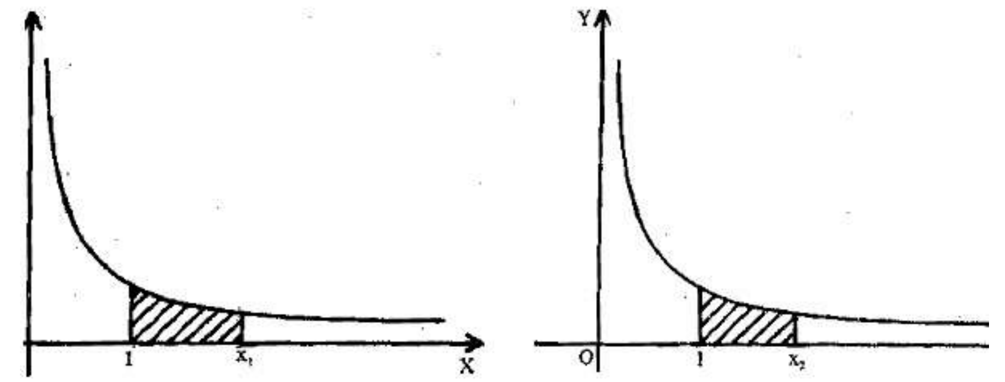
The logarithmic function defined here is called the Natural logarithmic function. For any x > 0, and for any positive real number a ≠ 1, we can define

$$\log x = \frac{\log x}{\log a}$$

This function is called the logarithmic function with respect to the base a. If a = 10, then this function is called the common logarithmic function.

Notes

Figure 9.10



Logarithmic function to the base a has the following properties:

(i) log, $(x_1 x_2)$ = log, X_1 + log, X_2

(ii) log a $\left[\frac{x_1}{x_2}\right] = \log_a x_1 - \log_a x_2$.

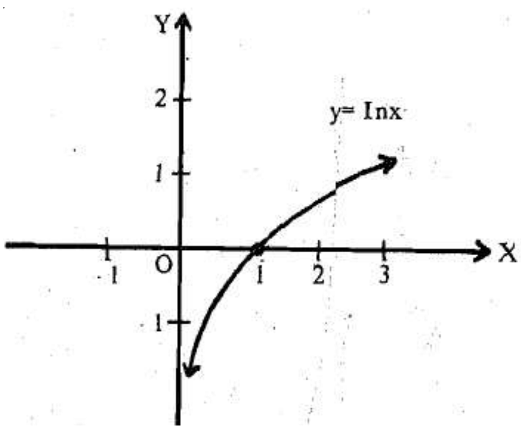
(iii) log, x^m = m log, x for every integer m.

(iv) $\log_a^a$ = I.

(v) $\log_a^1 = 0$

By the definition of log x, we see that log 1 = 0 and as x becomes larger and larger, the area covered by the curve $f(t) = \frac{1}{t}$, X-axis and the ordinates at 1 and x, becomes larger and larger. Its graph is as shown in the Figure 9.11,

Figure 9.11



You already know what is meant, by inverse of s function. You had also seen in Unit 1 that if f is 1 – 1 and onto, then f is invertible. Let us apply that study to logarithmic function.

Notes

9.3.3 Exponential Function

We now come to define exponential function. We have seen that

$\log x :]0, \infty[\to R$

is strictly increasing function. The graph of the logarithmic function also shows that

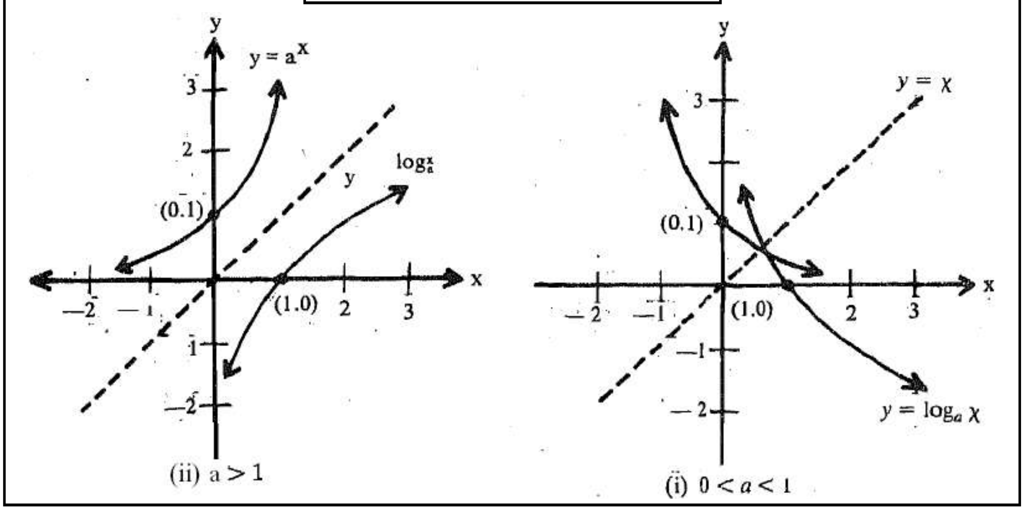
$\log x :]0, \infty[\to R$

is also onto. Therefore this function admits of inverse function. Its inverse function, called the Exponential function, Exp (x) has domain as the set R of all real numbers and range as $]0, \infty[$. If

$\log x = y$, then $\text{Exp}(y) = x$.

The graph of this function is the mirror image of logarithmic function as shown in the Figure 9.12.

Figure 9.12



The Exp (x) satisfies the following properties:

(i) E^p (x + y) = Exp x Exp y

(ii) Exp (x – y) = Exp x/Exp y

(iii) $(\text{Exp } x)^n = \text{Exp}(nx)$

(iv) Exp(0) = 1

We now come to define a^x for $a > 0$ and x any real number. We write

$a^x = \text{Exp}(x \log a)$

If x is any rational number, then we know that $\log a^x = x \log a$. Hence

$\text{Exp}(x \log a) – \text{Exp}(\log a^x) = a^x$. Thus our definition agrees with the already known definition of a in case x is a rational number. The function a^x satisfies the following properties

(i) $a^x a^y = a^{x+y}$

(ii) $\frac{a^x}{a^y} = a^{x-y}$

Notes

(iii) $(a^x)^y = a^{xY}$

(iv) $a^x b^x – (ab)^x$, a > 0, b > 0.

Denote E (I) = e, so that log e = 1. The number e is an irrational number and its approximation say up to five places of decimals is 2.71828. Thus

$$e^x = \text{Exp}\ (x \log e) = \text{Exp}\ (x).$$

Thus Exp (x) is also denoted as e^x and we write for each a > 0, $a^x = e^x \log a$

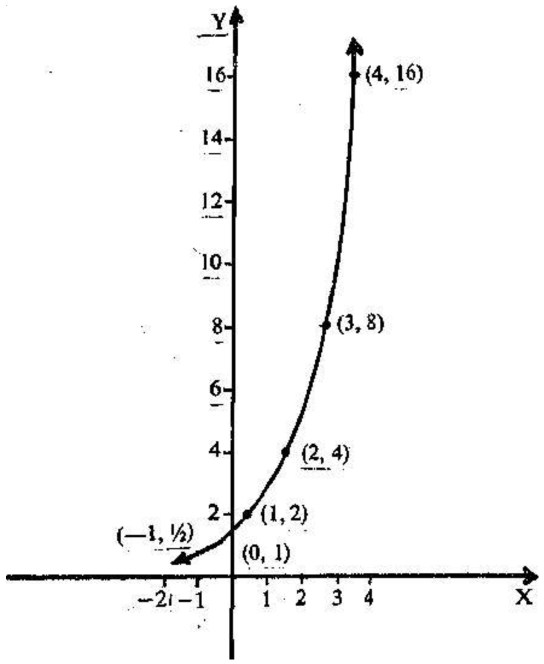
Example: Plot the graph of the function I : R → R defined by $f(x) = 2^x$.

Solution:

x	–2	–1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

The required graph takes the shape as shown in the Figure 9.13.

Figure 9.13



9.4 Some Special Functions

So far, we have discussed two main classes of real functions – Algebraic and Transcendental. Some functions have been designated as special functions because of their special nature and behaviour. Some of these special functions are of great interest to us. We shall frequently use these functions in our discussion in the subsequent units and blocks.

9.4.1 Identity Function

We have already discussed some of the special functions. For example, the Identity function i : R → R, defined as i (x) = x, ∀ x ER has already been discussed as an algebraic function.

Notes

However, this function is sometimes, referred to as a special function because of its special characteristics, which are as follows:

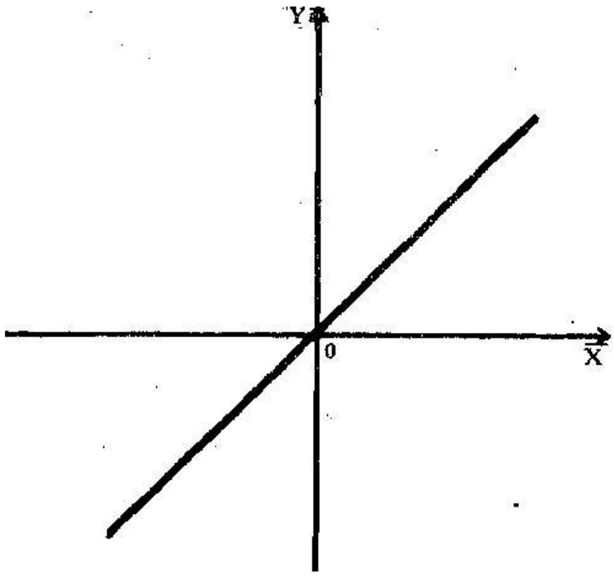
(i) Domain of i = Range of i = Codomain of i

(ii) The function i is one-one and onto. Hence it has an inverse i^{-1} which is also one-one and onto.

(iii) The function i is invertible

(iv) The graph of the identity function is a straight line through the origin which forms an angle of 45° along the positive direction of X-axis as shown in the Figure 9.14.

Figure 9.14



9.4.2 Periodic Function

You know that

$\sin(2\pi + x) = \sin(4\pi + x) = \sin x,$

$\tan(\pi + x) = \tan(2\pi + x) = \tan x.$

This leads us to define a special class of functions, known as Periodic functions. All trigonometric functions belong to this class.

A function $f : S \to R$ is said to be periodic if there exists a positive real number k such that

$f(x + k) = f(x), \ \forall x \in S$

where S C R.

The smallest such positive number k is called the period of the function.

You can verify that the functions sine, cosine, secant and cosecant are periodic each with a period $2n\pi$ while tangent and cotangent are periodic functions each with a period.

9.4.3 Modulus Function

Notes

The modulus or the absolute (numerical) value of a real number has already been defined in Unit 1. Here we define the modulus (absolute value) function as follows:

Let S be a subset of R. A function $f : S \rightarrow R$ defined by

$$f(x) = |x|, \ \forall x \in S$$

is called the modulus function.

In short, it is written as Mod function.

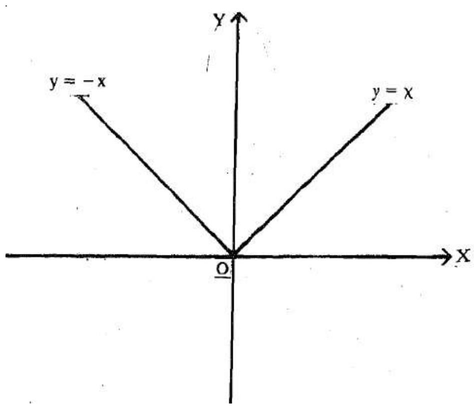
You can easily see the following properties of this function:

(i) The domain of the Modulus function may be a subset of R or the set R itself.

(ii) The range of this function is a subset of the set of non-negative real numbers.

(iii) The Modulus function $f : R \rightarrow R$ is not an onto function (Check why?).

Figure 9.15



(iv) The Modulus function $f : R \rightarrow R$ is not one-one. For instance, both 2 and –2 in the domain have the same image 2 in the range.

(v) The modulus function $f : R \rightarrow R$ does not have an inverse function (why)?

(vi) The graph of the Modulus function is $R \rightarrow R$ given in the Figure 9.15.

It consists of two straight lines:

(i) $y = x \ (y \geq 0)$

and (ii) $y = -x \ (y \geq 0)$

through 0, the origin, making an angle of $\pi/4$ and $3\pi/4$ with the positive direction of X-axis:

9.4.4 Signum Function

A function $f : R \rightarrow R$ defined by

$$f(x) = \begin{cases} |x| \\ x \quad \text{when } x \neq 0 \\ x \quad 0 \text{ if } x = 0 \end{cases}$$

Notes

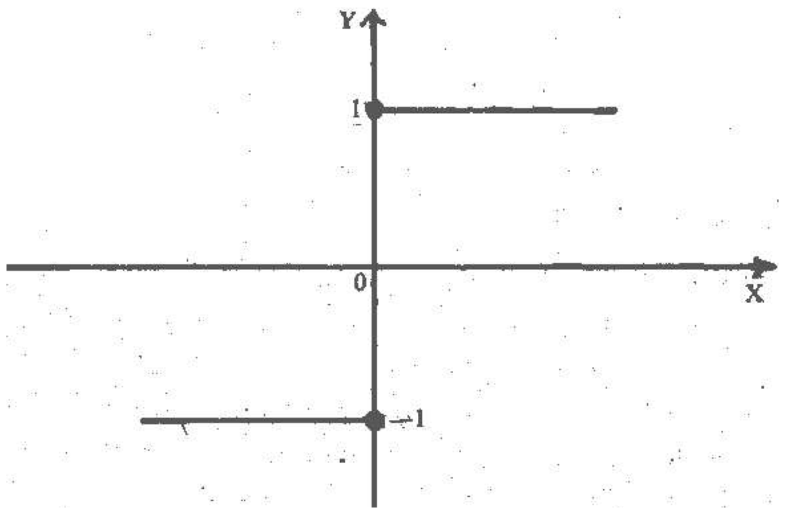
or equivalently by:

$$f(x) = \begin{cases} -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$$

is called the signum function. It is generally written as sgn (x).

Its range set is {–1, 0, 1}. Obviously sgn x is neither one-one nor onto. The graph of sgn x is shown in the Figure 9.16.

Figure 9.16



9.4.5 Greatest Integer Function

Consider the number 4.01. Can you find the greatest integer which is less than or equal to this number? Obviously, the required integer is 4 and we write it as [4.01] = 4.

Similarly, if the symbol [x] denotes the greatest integer contained in x then we have

[3/4] = 0, [5.01] –5,

[–.005] = –1 and [–3.96] = –4.

Based on these, the greatest integer function is defined as follows:

A function $f : R \to R$ defined by

$f(x) = [x], \ \forall x \in R,$

where [x] is the largest integer less than or equal to x is called the greatest integer function.

The following properties of this function are quite obvious:

(i) The domain is R and the range is the set Z of all integers.

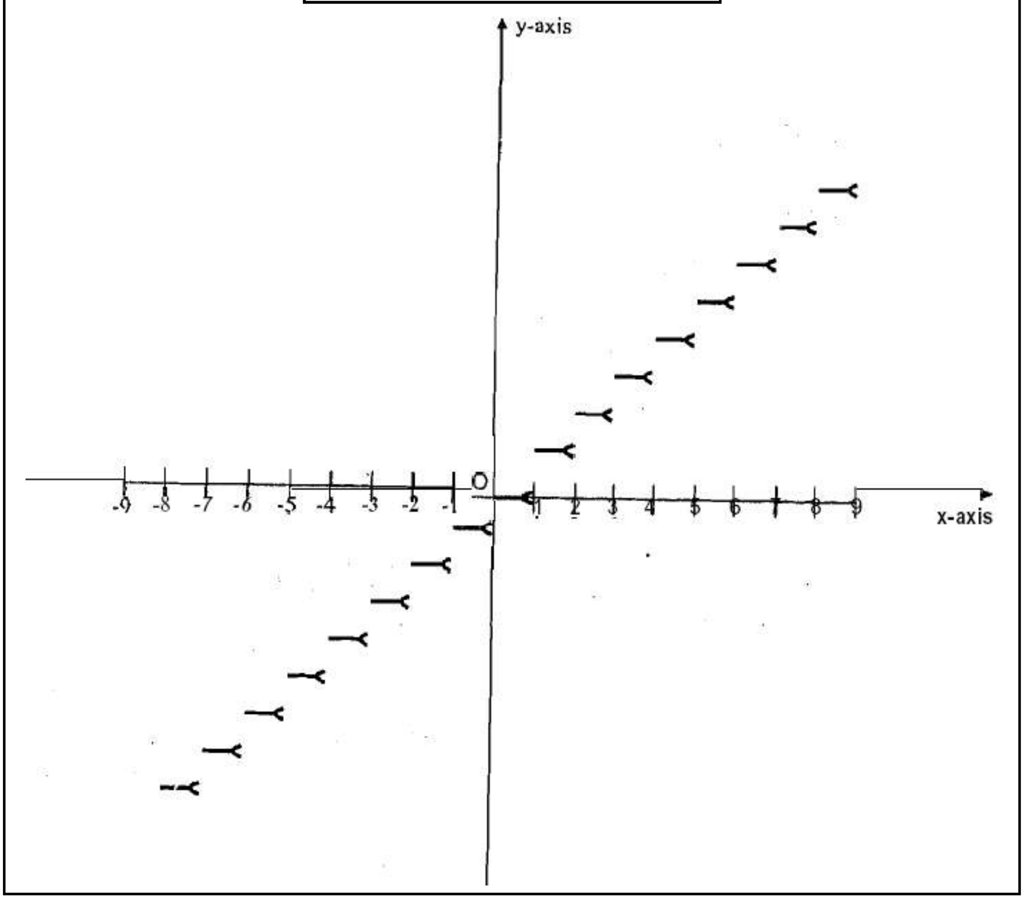
(ii) The function is neither one-one nor onto.

(iii) If n is any integer and x is any real number such that x is greater than or equal to n but less than n + 1 i.e., if $n \le x < n + 1$ (for some integer n), then [x] = n i.e.,

Notes

The graph of the greatest integer function is shown in the Figure 9.17.

Figure 9.17



Example: Prove that

$[x + m] = [x] + m, \ \forall \ x \in R$ and $m \in Z$,

Solution: You know that for every $x \in R$, there exists an integer n such that

$n < x < n + 1.$

Therefore,

$n + m < \ + m < n + 1 + m,$

and hence

$[x + m] = n + m = [x] + m,$

which proves the result.

Task Test whether or not the function $f : R \to R$ defined by $f(x) = x - [x] \ \forall \ x \in R$, is periodic. If it is so, find its period.

Notes

9.4.6 Even and Odd Functions

Consider a function f : R → R defined as

$f(x) = 2x, \ \forall x \in R.$

If you change x to –x, then you have

$f(-x) = 2(-x) = -2x - f(x).$

Such a function is called an odd function.

Now, consider a function f : R → R defined as

$f(x) = x^2 \ \forall x \in R$

Then changing x to –x we get

$q - x) = (-x)^2 = x^2 = f(x)$

Such a function is called an even function.

The definitions of even and odd functions are as follows:

A function f : R → R is called even if f (–x) = f(x), $\forall x \in R$,

It is called odd if f(–x) = –f(x), $\forall x \in R$

Example: Verify whether the function f : R ∈ R defined by

(i) $f(x) = \text{Sin}^2 x + \text{Cos}^3 2x$

(ii) $f(x) = \sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}$ are even or odd.

Solution:

(i) $f(x) + \sin^2 x + \cos^3 2x, \forall x \in R$

$\Rightarrow \sin^2(-x) + \cos^3 2(-x)$

$\sin^2 x + \cos^3 2x = f(x), \ \forall x \in R$

⇒ f is an even function.

(ii) $f(x) = \sqrt{a^2 - ax + x^2} - \sqrt{a^2 - ax + x^2}, \forall x \in R$

$\Rightarrow f(-x) = \sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}$

$= -(x), \ \forall x \in R$

⇒ f is an odd function.

Task Determine which of the following functions are even or odd or neither:

(i) f(x) = x

(ii) A constant function

(iii) sin x, cos x, tan x,

(iv) $f(x) = \dfrac{x-4}{x^2-9} = -(x), \ \forall x \in R$

9.4.7 Bounded Functions

In Unit 2, you were introduced to the notion of a bounded set, upper and lower bounds of a set. Let us now extend these notions to a function.

You know that if $f : S \to R$ is a function, (S C R), then

$\{f(x) : x \in S)\}$

is called the range set or simply the range of the function f.

A function is said to be bounded if its range is bounded.

Let $f : S \to R$ be a function. It is said to be bounded above if there exists a real number K such that

$f(x) \leq k \quad \forall x \in S$

The number K is called an upper bound of it. The function f is said to be bounded below if there exists a number k such that

$f(x) \geq k \quad \forall x \in S$

The number k (is called a lower bound of f).

A function $f : S \to R$, which is bounded above as well as bounded below, is said to be bounded. This implies that there exist two real numbers k and K such that

$k \leq f(x) \leq K \quad \forall x \in S.$

This is equivalent to say that a function $f : S \to R$ is bounded if there exists a real number M such that

$|f(x)| \leq M, \ \forall x \in S.$

A function may be bounded above only or may be bounded below only or neither bounded above nor bounded below.

Recall that sin x and cos x are both bounded functions. Can you say why? It is because of the reason that the range of each of these functions is [–1, 1].

Example: A function $f : R \to R$ defined by

(i) $f(x) = -x^2, \ \forall x \in R$ is bounded above with 0 as an upper bound

(ii) $f(x) = x, \ \forall x \geq 0$ is bounded below with 0 as a lower bound

(iii) $f(x) =$ for $|x|$ –l is bounded because $|f(x)| \leq 1$ for $|x| \leq |$.

Self Assessment

1. Test whether the following are rational numbers:

   (i) $\sqrt{I7}$ (ii) $\sqrt{8}$ (iii) $\sqrt{3} + \sqrt{2}$

2. The inequality $x^2 - 5x + 6 < 0$ holds for

   (i) $x < 2, x < 3$ (ii) $x > 2, x < 3$

   (iii) $x < 2, x > 3$ (iv) $x > 2, x > 3$

Notes

3. Test whether the following statements are true or false:

   (i) The set Z of integers is not a NBD of any of its points.

   (ii) The interval]0, 1] is a NBD of each of its points

   (iii) The set]1, 3[$\cup$] 4, 5[is open.

   (iv) The set [a, ∞[$\cup$] $-\infty$, a] is not open.

   (v) N is a closed set.

   (vi) The derived set of Z is non-empty.

   (vii) Every real number is a limit point of the set Q of rational numbers.

   (viii) A finite bounded set has a limit point.

   (ix) [4, 5] $\cup$ [7, 8] is a closed set.

   (x) Every infinite set is closed.

9.5 Summary

- In this unit, we have discussed various types of real functions. We shall frequently use these functions in the concepts and examples to be discussed in the subsequent units throughout the course.

- We have introduced the notion of an algebraic function and its various types. A function f: $S \to R$ ($S \subset R$) defined as y = f(x), $\forall x \in S$ is said to be algebraic if it satisfies identically an equation of the form

  $p_0(x)\, y^n + p_1(x)\, y^{n-1} + p_2(x)\, y^{n-2} + ... + p(x)\, y + p_n(x) = 0,$

- where $p_0(x)$, $p_1(x)$, $p_n(x)$ are polynomials in x for all $x \in S$ and n is a positive integer. In fact, any function constructed by a finite number of algebraic operations—addition, subtraction, multiplication, division and root extraction – is an algebraic function. Some of the examples of algebraic functions are the polynomial functions, rational functions and irrational functions.

- But not all functions are algebraic. The functions which are not algebraic, are called transcendental functions. Some important examples of the transcendental functions are trigonometric functions, logarithmic functions and exponential functions which have been defined in this section. We have defined the monotonic functions also in this section.

- We have discussed some special functions. These are the identity function, the periodic functions, the modulus function, the signum function, the greatest integer function, even and odd functions. Lastly, we have introduced the bounded functions and discussed a few examples.

9.6 Keywords

Upper Bound: Let f : $S \to R$ be a function. It is said to be bounded above if there exists a real number K such that $f(x) \leq k \quad \forall x \in S$. The number K is called an upper bound of it.

Lower Bound: The function f is said to be bounded below if there exists a number k such that $f(x) \geq k$ $x \in S$. The number k is called a lower bound of f.

Cotangent Function: A function f : $S \to R$ defined.

9.7 Review Questions

Notes

1. Show the graph of f : R → R defined by $F(x) = (\frac{1}{2})^x$

2. Find the period of the function f where $f(x) = |\sin^3 x|$

3. Test which of the following functions with domain and co-domain as R are bounded and unbounded:

   (i) f(x) = tan x

   (ii) f(x) = [x]

   (iii) $f(x) = e^x$

   (iv) f(x) = log x

4. Suppose t : S → R and g : S → R are any bounded functions on S. Prove that f + g and f. g are also bounded functions on S.

5. If a, b, c, d are real numbers such that

   $$a^2 + b^2 = I,\ c^2 + d^2 = 1,$$

   then show that ac + bd ≤ 1.

6. Prove that $|a + b + c| \leq |a|+|b|+|c|$ for all a, b, c, ∈ R.

7. Show that

   $|a_1 + a_2 + ... + a_n| \leq |a_1|+|a_2| + ... +|a_n|$ for $a_1, a_2,.... a, \in R$.

8. Find which of the sets in question 8 are bounded below. Write the infemum if it exists.

9. Which of the sets in question 8 are bounded and unbounded.

10. Justify the following statements:

    (i) The identity function is an odd function.

    (ii) The absolute value function is an even function.

    (iii) The greatest integer function is not onto.

    (iv) The tangent function is periodic with period π.

    (v) The function $f(x) = |x|$ for $-2 \leq x \leq 3$ is bounded.

    (vi) The function $f(x) = e^x$ is not bounded

    (vii) The function f(x) = sin x, for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is monotonically increasing.

    (viii) The function f(x) = cos x for $0 \leq x \leq \pi$ is monotonically decreasing.

    (ix) The function f(x) = tan x is strictly increasing for $x \in [0, \frac{\pi}{2}]$.

    (x) $f(x) = \sqrt{\frac{2x^2 - 3x + 2}{3x - 2}}$ is an algebraic function.

Notes

Answers: Self Assessment

1. None is a rational number
2. For (ii) only since $2 < x < 3$.
3. (i) True (ii) False
   (iii) True (iv) False
   (v) True (vi) False
   (vii) True (viii) False
   (ix) True (x) False

9.8 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 10: Limit of a Function

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define limit of a function at a point and find its value
- Know sequential approach to limit of a function
- Find the limit of sum, difference, product and quotient of functions

Introduction

In earlier unit, we dealt with sequences and their limits. As you know, sequences are functions whose domain is the set of natural numbers. In this unit, we discuss the limiting process for the real functions with domains as subsets of the set R of real numbers and range also a subset of R. What is the precise meaning for the intuitive idea of the values f(x) of a function f tending to or approaching a number A as x approaches the number a? The search for an answer to this question shall enable you to understand the concept of the limit which you have used in calculus. The effect of algebraic operations of addition, subtraction, multiplication and division on the limits of functions.

10.1 Notion of Limit

The intuitive idea of limit was used both by Newton and Leibnitz in their independent invention of Differential Calculus around 1675. Later this notion of limit was also developed by D'Alembert. "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others."

Consider a simple example in which a function f is defined as

$$f(x) = 2x + 3, \quad \forall x \in R, x \neq 1.$$

Give x the values which are near to 1 in the following way:

When $x = 1.5, 1.4, 1.3, 1.2, 1.1, 1.01, 1.001$

Notes

$$f(x) = 6, 5.8, 5.6, 5.4, 5.2, 5.02, 5.002$$

When $x = .5, .6, .7, .8, .9, .99, .999$

$$f(x) = 4, 4.2, 4.4, 4.6, 4.8, 4.98, 4.998$$

You can form a table for these values as follows:

X	.5	.6	.7	.8	.9	.99	.999	1.001	1.01	1.1	1.2	1.3	1.4	1.5
f(x)	4	4.2	4.4	4.6	4.8	4.98	4.998	5.002	5.02	5.2	5.4	5.6	5.8	6

The limit of a function f at a point a is meaningful only if a is a limit point of its domain. That is, the condition $f(x) \to \alpha$ as $n \to a$ would make sense only when there does not exists a nbd. U of a for which the set U n Dom (f)\{a} is empty i.e., $a \in$ (Dom 0))′.

You see that as the values of x approach 1, the values of f(x) approach 5. This is expressed by saying that limit of f(x) is 5 as x approaches 1. You may note that when we consider the limit of f(x) as x approaches 1, we do not consider the value of f(x) at x = 1.

Thus, in general, we can say as follows:

Let f be a real function defined in a neighbourhood of a point x = a except possibly at a. Suppose that as x approaches a, the values taken by f approach more and more closely a value A. In other words, suppose that the numerical difference between A and the values taken by f can be made as small as we please by taking values of x sufficiently close to a. Then we say that f tends to the limit A as x tends to a. We write

$$f(x) \to A \text{ as } x \to a \text{ or } \lim_{x \to a} f(x) = A.$$

This intuitive idea of the limit of a function can be expressed mathematically as formulated by the German mathematician Karl Veierstrass in the 18th Century. Thus, we have the following definition:

Definition 1: Limit of a Function

Let a function f be defined in a neighbourhood of a point ‘a’ except possibly at ‘a’. The function f is said to tend to or approach a number A as x tends to or approaches a number ‘a’ if for any $\varepsilon > 0 >$ there exists a number $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \text{ for } 0 < |x - a| < 6.$$

We write it as $\lim_{x \to a} f(x) = A$. You may note that

$$|f(x) - A| < \varepsilon \text{ for } 0 < |x - a| < 6.$$

can be equivalently written as

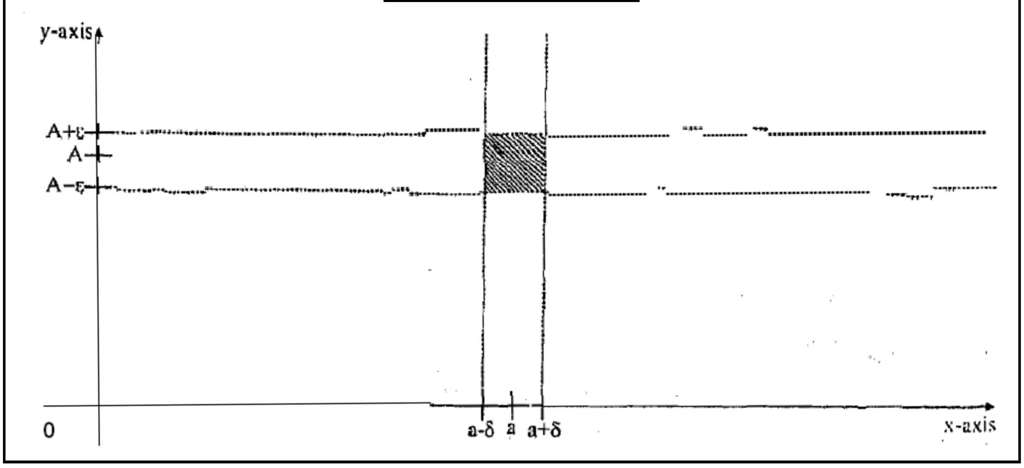
$$f(x) \in]A - \varepsilon, A + \varepsilon[\text{ for } x \in]a - 6, a + \delta[\text{ and } \neq a.$$

Geometrically, the above definition says that, for strip S_A of any given width around the point A, if it is possible to find a strip S_a of some width around the point a such that the values that f(x) takes, for x in the strip S_a $(x \neq a)$, lies in the shaded box formed by the intersection of strips S_A and S_a, then $\lim_{x \to a} f(x) = A$.

This is shown geometrically in Figure 10.1 below. The inequality $0 < |x - a| < 6$ determines the interval $]a - 6, a + \delta[$ minus the point ‘a’ along the x-axis and the inequality $|f(x) - A| < \varepsilon$ determines the interval $]A - \varepsilon, A + \varepsilon[$ along the y-axis.

Notes

Figure 10.1



Example: Let a function f: R → R be defined as

$$f(x) = x^2, \ \forall \ x \in R.$$

Find its limits when x → 2.

Solution: By intuition, it follows that

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 4.$$

In other words, we have to show that for a given $E > 0$, there exists a $\delta > 0$ such that

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < \varepsilon.$$

Suppose that an $\varepsilon > 0$ is fixed. Then consider the quantity $|f(x) - 4|$, which we can write as

$$|f(x) - 4| = |x^2 - 4| = |(x - 2)(x + 2)|.$$

Note that the term $|x - 2|$ is exactly the same that appears in the 6–inequality in the definition. Therefore, this term should be less than δ. In other words,

$$|x - 2| < \delta$$

$$\Rightarrow 2 - \delta < x < 2 + \delta$$

$$\Rightarrow x \in]\, 2 - \delta, 2 + \delta[.$$

We restrict δ to a value 2 so that x lies in the interval $]\, 2 - \delta, 2 + \delta[\subset]\, 0, 4[$. Accordingly, then $|x + 2| < \delta$. Thus, if $\delta \le 2$, then

$$|x - 2| < 2 \Rightarrow 0 < |x + 2| < \delta,$$

and further that

$$|x - 2| < \delta \le 2 \Rightarrow |x + 2| \ |x - 2| < \delta \ |x - 2| < 6\delta.$$

If 6 is small then so is 6δ. In fact it can be made less than ε by choosing δ suitably. Let us, therefore, select 6 such that $\delta = \min.(2, \varepsilon/6)$. Then

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < \delta \ |x - 2| < \delta.\ \delta \le 6.\ \varepsilon/\delta = \varepsilon.$$

Notes

This completes the solution.

Note that the first step is to manipulate the term $|f(x) - A|$ by using algebra. The second step is to use a suitable strategy to manipulate $|f(x) - A|$ into the form

$$|x - a| \text{ (trash)}$$

where the 'trash' is some expression which has the property that: it is bounded provided that 6 is sufficiently small. Why we use the term 'trash, for the expression as a multiple of $|x - a|$? The reason is that once we know that it is bounded, we can replace it by a number and forget about it.

The number 6 arose by virtue of this 'trash'. If you take $6 \leq 3$ (instead of $6 \leq 2$), you can still show that 6 will be replaced by 7. In that case you can set δ as

$$\delta = \min.(3, \varepsilon/7)$$

and the proof will be complete. Thus, there is nothing special about 6. The only thing is that such a number (whether 6 or 7) has to be evaluated by the restriction placed on 6.

Finally, note that in general, 6 will depend upon ε.

Task For a function f: $R \to R$ defined by $f(x) = x^2$, find its limit when x tends to 1 by the $\varepsilon - \delta$ approach.

In Unit 5, we proved that a convergent sequence cannot have more than one limit. In the same way, a function cannot have more than one limit at a single point of its domain. We prove it in the following theorem:

Theorem 1: If $\lim\limits_{x \to a} f(x) = A$, $\lim\limits_{x \to a} f(x) = B$, then A = B.

Proof: In short, we have to show that if $\lim\limits_{x \to a} f(x)$ has two values say A and B, then A = B. Since $\lim\limits_{x \to a} f(x) = A$, $\lim\limits_{x \to a} f(x) = B$, given a number E > 0, there exists numbers δ_1, $\delta_2 > 0$ such that

$$|f(x) - A| < \varepsilon/2 \text{ whenever } 0 < |x - a| < \delta_1$$

and

$$|f(x) - B| < \delta/2 \text{ whenever } 0 < |x - a| < \delta_2.$$

If we take δ equal to minimum of δ_1 and δ_2, then we have

$$|f(x) - A| < \varepsilon/2 \text{ and } |f(x) - B| < \varepsilon/2 \text{ whenever } 0 < |x - a| < \delta.$$

Choose an x_0 such that $0 < |x_0 - a| < \delta$. Then

$$|A - B| = (A - f(x_0) + f(x_0) - B| \leq (A - f(x_0)| + |f(x_0) - B|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

E is arbitrary while A and B are fixed. Hence $|A - B|$ is less than every positive number ε which implies that $|A - B| = 0$ and hence A = B. (For otherwise, if $A \neq B$ then $A - B = C \neq 0$ (say). We can choose $\varepsilon < |C|$ which will be a contradiction to the fact that $|A - B| < \varepsilon$ for every $\varepsilon > 0$.)

In the example considered before defining limit of a function, we allowed x to assume values both greater and smaller than 1. If we consider values of x greater than 1 that is on the right of 1, we see that values of f(x) approaches 5. We say that f(x) tends to 5 as x tends to 1 from the right.

Similarly you see that values of f(x) approach 5 as x tends to 1 from the left i.e. through values smaller than 1. This leads us to define right hand and left hand limits as under:

Definition 2: Right Hand Limits and Left Hand Limits

Let a function f be defined in a neighbourhood of a point 'a' except possibly at 'a'. It is said to tend to a number A as x tends to a number 'a' from the right or through values greater than 'a' if given a number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

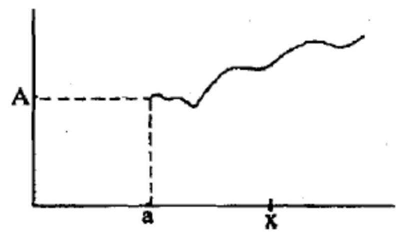
$$|f(x) - A| < \varepsilon \text{ for } a < x < a + \delta.$$

We write, it as

$$\lim_{x \to a+} f(x) = A \text{ or } \lim_{x \to a+0} f(x) = A \text{ or } f(a+) = A.$$

See Figure 10.2(a).

Figure 10.2(a)



The function f is said to tend to a number A as x tends to 'a' from the left or through values smaller than 'a' if given a number $E > 0$, there exists a number $\delta > 0$ such that

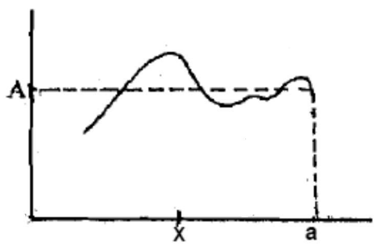
$$|f(x) - A| < \varepsilon \text{ for } a - 6 < x < a.$$

We write it as

$$\lim_{x \to a-} f(x) = A \text{ or } \lim_{x \to a-0} f(x) = A \text{ or } f(a-) = A.$$

See Figure 10.2(b).

Figure 10.2(b)



Since the definition of limit of a function employs only values of x different from 'a' it is totally immaterial what the value of the function is at x = a or whether f is defined at x = a at all. Also it is obvious that $\lim_{x \to a} f(x) = A$ if and only if $f(a+) = A$, $f(a-) = A$.

This we prove in the next theorem. First we consider the following example to illustrate it.

Example: Find the limit of the function f defined by

$$f(x) = \frac{x^2 + 3x}{2x} \text{ for } x \neq 0$$

Notes

when $x \to 0$.

Solution: The given function is not defined at x = 0 since $f(0) = \frac{0}{0}$ which is meaningless.

If $x \neq 0$, then $f(x) = \frac{x+3}{2}$. Therefore

$$\text{Right Hand Limit} = \lim_{x \to 0+0} f(x)$$

$$= \lim_{h \to 0} \frac{(0+h)+3}{2} \quad (h > 0)$$

$$= 3/2.$$

$$\text{Left Hand Limit} = \lim_{x \to 0-0} f(x)$$

$$= \lim_{h \to 0} f(x) = \frac{(0-h)+3}{2} \quad (h > 0)$$

$$= 3/2.$$

Since both the right hand and left hand limits exist and are equal,

$$\lim_{x \to 0} f(x) = 3/2.$$

We, now, discuss the theorem concerning the existence of limit and that of right and the left hand limits.

Theorem 2: Let f be a real function. Then

$$\lim_{x \to a} f(x) = A \text{ if and only if } \lim_{x \to a+} f(x) \text{ and } \lim_{x \to a-} f(x)$$

both exist and are equal to A.

Proof: If $\lim_{x \to a+} f(x) = A$, then corresponding to any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

i.e., $|f(x) - A| < \varepsilon$ whenever $a - \delta < x < a + \delta, x \neq a$

This implies that $|f(x) - A| < \varepsilon$ whenever $a - \delta < x < a$

and $|f(x) - A| < \varepsilon$ whenever $a < x < a + \delta$.

Hence both the left hand and right hand limits exist and are equal to A. Conversely, if f(a+) and f(a–) exist and are equal to A say, then corresponding to E > 0, there exist positive numbers δ_1 and δ_2 such that

$$|f(x) - A| < E \text{ whenever } a < x < a + \delta_1$$

and

$$|f(x) - A| < \varepsilon \text{ whenever } a - \delta_2 < x < a.$$

Let 6 be the minimum of δ_1 and δ_2. Then

$$|f(x) - A| < E \text{ whenever } a - \delta < x < a + \delta, x \neq a$$

i.e. $|f(x) - A| < \varepsilon$ if $0 < |x - a| < \delta$

which proves that

$\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = A$.

Notes

Example: Consider the function I defined by

$$f(x) = \frac{x^2-1}{x-1}, x \in R, x \neq 1$$

Find its limit as $x \to 1$.

Solution: Note that f(x) is not defined at x = 1. (Why?).

For any $x \neq 1$, $$f(x) = \frac{x^2-1}{x-1} = x+1.$$

$$\lim_{x-1+} f(x) = \lim_{x-1+} (x+1) = 2$$

$$\lim_{x-1-} f(x) = \lim_{x-1-} (x+1) = 2$$

Since $\lim_{x-1+} f(x) = \lim_{x-1-} f(x)$, by Theorem 2, $\lim_{x-1} f(x) = 2$,

$\lim_{x-1} f(x) = 2$ can be seen by ε – δ definition as follows:

Corresponding to any number E > 0, we can choose 8 = ε itself. Then, it is clear that

$$0 < |x-1| < \delta = \varepsilon \Rightarrow$$

$$|f(x)-2| = \left|\frac{x^2-1}{x-1} - 2\right| = |x+1-2| = |x-1| < \varepsilon.$$

From Theorem 2, it follows that f(l+) and f(l–) also exist and are both equal to 2.

Example: Let f: $R \to R$ be defined as

$$f(x) = \begin{cases} |x|, x \neq 0 \\ 3, x \approx 0. \end{cases}$$

Find its limit when $x \to 0$.

Solution: You are familiar with the graph of f as given in Unit 4. It is easy to see that $\lim_{x-0} f(x) = 0 = f(0+) = f(0-)$. The fact that f(0) = 3 has neither any bearing on the existence of the limit of f(x) as x tends to 0 nor on the value of the $\lim_{x-0} f(x)$.

Example: Define f on the whole of the real L in ε as follows:

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Find its limit when x tends to 0.

Solution: Since f(x) = 1 for all x > 0,

$$f(0+) = \lim_{x \to 0+} f(x) = +1.$$

Similarly f(0–) = –1. Since $\lim_{x \to 1+} f(x) \neq \lim_{x \to 1-} f(x)$, $\lim_{x \to 0} f(x)$ does not exist.

Now, we give another proof using ε – S definition.

Notes

For, if $\lim_{x \to 0} f(x) = A$, then for a given $\varepsilon > 0$, there must exist some $\delta > 0$, such that $|f(x) - A| < \varepsilon$. Let us choose $x, > 0$, $x, < 0$ such that $|x_1| < S$ and $|x_2| < \delta$. Then

$$
\begin{aligned}
2 &= |f(x_1) - f(x_2)| \\
&\leq |f(x_1) - A| + |A - f(x_2)| \\
&< 2\varepsilon, \qquad \text{(because } |x_1 - 0| < \delta \text{ and } |x_2 - 0| < \delta\text{)}
\end{aligned}
$$

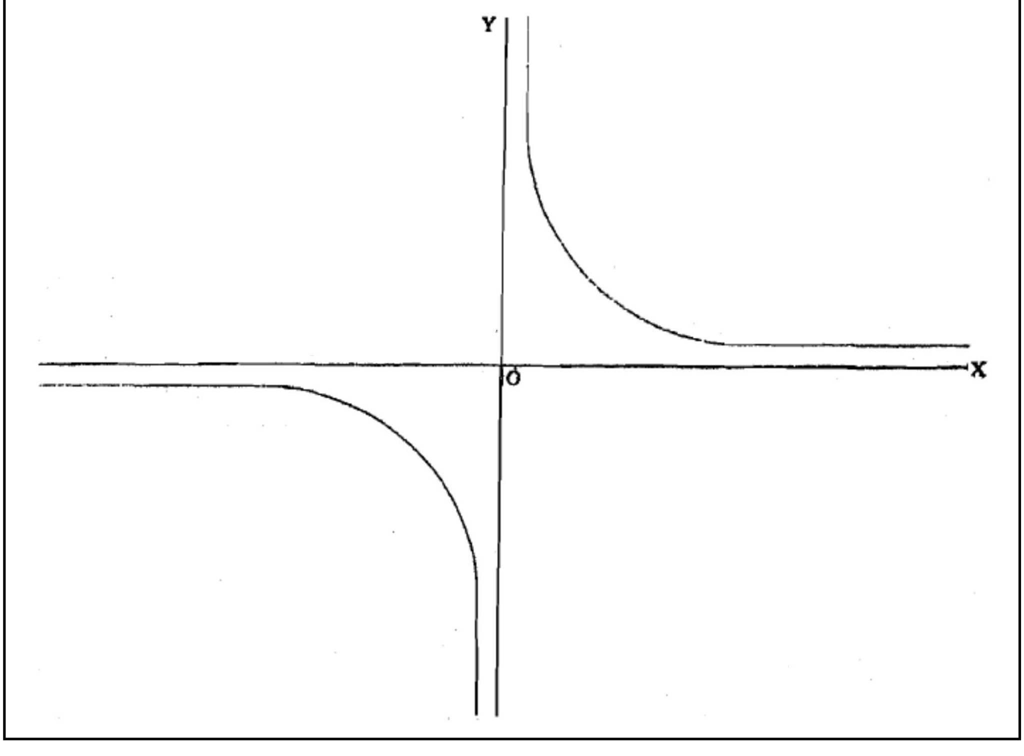
for every ε which is clearly impossible if $\varepsilon < 1$. Non-existence of $\lim_{x \to 0} f(x)$ also follows from Theorem 2, since $f(0+) \neq f(0-)$.

The above example shows clearly that the existence of both f(a+) and f(a–) alone is not sufficient for the existence of $\lim_{x \to 0} f(x)$. In fact, for $\lim_{x \to a} f(x)$ to exist, they both should be equal.

Now consider, the function f defined by $f(x) = \frac{1}{x}$ for $x \neq 0$.

The graph of f looks as shown in the Figure 10.3. You know that it is a rectangular hyperbola. Here none of the $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0+} f(x)$ exists. Hence $\lim_{x \to 0} f(x)$ does not exist.

Figure 10.3



This can be easily seen from the fact that 1/x becomes very large numerically as x approaches 0 either from the left or from the right. If x is positive and takes up larger and larger values, then values of 1/x i.e. f(x) is positive and becomes smaller and smaller. This is expressed by saying that f(x) approaches 0 as x tends to ∞. Similarly if x, is negative and numerically takes up larger and larger values, the values of f(x) is negative and numerically becomes smaller and smaller and we say that f(x) approaches 0 as x tends to $-\infty$. These two observations are related to the notion of the limit of a function at infinity.

Let us now discuss the behaviour of a function f when x tends to ∞.

Let a function f be defined for all values of x greater than a fixed number c. That is to say that f is defined for all sufficiently large values of x. Suppose that as x increases indefinitely, f(x) takes a succession of values which approach more and more closely a value A. Further suppose that the numerical difference between A and the values f(x) taken by the function can be made as small as we please by taking values of x sufficiently large. Then we say f tends to the limit A as x tends to infinity. More precisely, we have the following definition:

Definition 3: A function f tends to a limit A, as x tends to infinity if having chosen a positive number ε, there exists a positive number k such that

$$|f(x - A)| > \varepsilon \ \forall \ x \geq k.$$

The number E can be made as small as we like. Indeed, however small ε we may take, we can always find a number k for which the above inequality holds. We rewrite this definition in the following way:

A function $f(x) \to A$ as $x \to w$ if for every ε > 0, there exists k > 0 such that

$$|f(x) - A| < \varepsilon \text{ for all } x \geq k.$$

We write it as,

$$\lim_{x \to \infty} f(x) = A.$$

This notion of the limit of a function needs a slight modification when x tends to –∞. This is as follows:

We say that $\lim_{x \to -\infty} f(x) = A$, if for a given ε > 0, there exists a number k < 0 such that

$$|f(x) - A| < E \text{ whenever } x \leq k.$$

We write it as $\lim_{x \to -\infty} f(x) = A$.

Instead of f(x) approaching a real number A as x tends to +∞ or –m, we may also have f(x) approaching +∞ or –∞ as x tends to a real number 'a'. For example, if $f(x) = l/x^Z$, $x \neq 0$ and x takes values near 0, the values of f(x) becomes larger and larger. Then we say that f(x) is tending to +∞ as x tends to 0. We can also have f(x) tending to +m or –∞ as x tends to +∞ or –∞. For example f(x) = x tends to +∞ or –∞ as x tends to +∞ or –∞. Again, the function f(x) = –x tends to +∞ or –∞ as x tends to –∞ or +∞. We formulate the following definition to cover all such cases of infinite limits.

Definition 4: Infinite Limits of a Function

Suppose a is a real number. We say that a function f tends to +m when x tends to a, if for a given positive real number M there exists a positive number δ such that

$$f(x) > M \text{ whenever } 0 < |x - a| < \delta.$$

We write it as

$$\lim_{x - a} f(x) = +\infty.$$

In this case we say that the function becomes unbounded and tends to +∞ as x tends to a.

In the same way, f is said to –∞ as x tends to a if for every real number –M, there is a positive number δ such that

$$f(x) < -M \text{ whenever } 0 < |x - a| < \delta.$$

We write it as

$$\lim_{x \to a} f(x) = -\infty.$$

Notes

In this case also f(x) is unbounded and tends to $-\infty$ as x tends to a. You can give similar definitions for $f(a+) = +\infty$, $f(a-) = +\infty$, $f(a+) = -\infty$, $f(a-) = -\infty$.

Now we define $\lim_{x\to\infty} f(x) = \infty$.

f is said to tend to ∞ as x tends to ∞ if given a number $M > 0$, there exists a number $k > 0$ such that

$$f(x) > M \text{ for } x \geq k.$$

We may similarly define

$$\lim_{x\to-\infty} f(x) = +\infty,\ \lim_{x\to+\infty} f(x) = -\infty,\ \lim_{x\to-\infty} f(x) = -\infty.$$

In all such cases we say that the function f becomes unbounded as x tends to $+\infty$ or $-\infty$ as the case may be.

It is easy to see from the definition of limit of a function that the limit of a constant function at any point in its domain is the constant itself. Similarly if $\lim_{x-a} f(x) = A$, then $\lim_{x-a} cf(x) = cA$ for any constant c where c is a real number.

Example: Justify that

$$\lim_{x-2} \frac{1}{(x-2)^2} = \infty.$$

Solution: You have to verify that corresponding to a given positive number M, there exists a positive number 6, such that

$$\frac{1}{(x-2)^2} > M \text{ whenever } 0 < |x-2| < 6.$$

Indeed for $x \neq 2$,

$$\frac{1}{(x-2)^2} > M \Rightarrow (x-2)^2 < \frac{1}{M}$$

$$\Rightarrow |x-2| < \frac{1}{\sqrt{M}}.$$

Take $\delta = \frac{1}{\sqrt{M}}$. Then you can see that

$$\frac{1}{(x-2)^2} > M \text{ whenever } 0 < |x-2| < 6.$$

Hence

$$\lim_{x-2} \frac{1}{(x-2)^2} = \infty.$$

Task

1. Consider $f(x) = |x|$, $x \in R$. Show that $\lim_{x\to+\infty} f(x) = +\infty$, and $\lim_{x\to-\infty} f(x) = +w$ and $f(0+) = f(0-) = 0 = f(0)$

2. Let $f(x) = -|x|$, $x \in R$. Prove that $\lim_{x\to+\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = +\infty$ and $f(0) = f(0+) = f(0-) = 0$.

We have already stated that if a function t is define a by f(x) = 1/x, x ≠ 0, then the limits f(0+) and f(0–) and $\lim_{x \to 0}$ f(x) do not exist. It simply means that these limits do not exist as real numbers. In other words, there is no (finite) real number A such that f (0+) = A f(0–) = A, or $\lim_{x \to 0}$ f(x) = A.

10.2 Sequential Limits

In Unit 5, you studied the notion of the limit of a sequence. You also know that a sequence is also a function but a special type of function. What is special about a sequence? Do you remember it? Recall it from Unit 5. Naturally, you would like to know the relationship of a sequence and an arbitrary real function in terms of their limit concepts. Both require us to find a fixed number A as a first step. Both assume a small positive number ε as a test for closeness. For functions we need a positive number δ corresponding to the given positive number E and for sequences we need a positive integer m which depends on ε. So, then what is the difference between the two notions? The only difference is in their domains in the sense that the domain of a sequence is the set of natural numbers whereas the domain of an arbitrary function is any subset of the set of real numbers. In the case of a sequence, there are natural numbers only which exceed any choice of m. But for a function with a domain as an arbitrary set of real numbers, this is not necessary the case. Thus in a way, the notion of the limit of a function at infinity is a generalization of that of limit of a sequence.

Let us now, therefore, examine the connection between the limit of a function and the limit of a sequence called the sequential limit. We state and prove the following theorem for this purpose:

Theorem 3: Let a function f be defined in a neighbourhood of a point 'a' except possibly at 'a'. Then f(x) tends to a limit A as x tends to 'a' if and only if for every sequence (x_n), $x_n \neq a$ for any natural number n, converging to 'a', f(x_n) converges to A.

Proof: Let, $\lim_{x \to a}$ f(x) = A. Then for a number E > 0, there exists a 6 > 0 such that for 0 < | x – a | < 6 we have

$$|f(x) - A| < \varepsilon$$

Let (x,) be a sequence ($x_n \neq a$ for any n ∈ N) such that (x_n) converges to a i.e. $x_n \to a$.

Then corresponding to δ > 0, there exists a natural number m such that for all n ≥ m

$$|x_n - a| < \delta.$$

Consequently, we have

$$|f(X_n) - A| < \varepsilon, \ \forall \ n \geq m.$$

This implies that f(x_n) converges to A.

Conversely, let f(x_n) converge to A for every sequence x_n which converges to a, $x_n \neq a$ for any n.

Suppose $\lim_{x \to a}$ f(x) ≠ A.

Then there exists at least one ε, say ε = ε_0 such that for any δ > 0 we have an x_δ such that

$$0 < |x_\delta - a| < \delta$$

and

$$|f(x_\delta) - A| \geq \varepsilon_0.$$

Let $$\delta = \frac{1}{n}, n = 1, 2, 3......$$

We get a sequence (x,) such that x, = x_δ where 6 = 1/n and

$$0 < |x_n - a| < \frac{1}{n} \text{ for } n = 1, 2, ...$$

and

$$|f(x_n) - A| \geq \varepsilon_0.$$

$$0 < |x_n - a| \Rightarrow x, \neq a \text{ for any n.}$$

Since $\frac{1}{n} \to 0$ and $|x_n - a| < \frac{1}{n}$, it follows that x, → a.

But $|f(x_n) - A| \geq \varepsilon_0 \Rightarrow f(x_n) \to A$ i.e. $f(x_n)$ does not tend to A.

Therefore $x_n \neq a \ \forall$ a and x_n tends to a as n tends to ∞ whereas $f(X_n)$ does not converge to A, contradicting our hypothesis. This completes the proof of the theorem.

You may note that the above theorem is true even when either a or A is infinite or both a and A are infinite (i.e. +∞ or –∞).

By applying this theorem, we can decide about the existence or non-existence of limit of a function at a point. Consider the following examples:

Example: Let $f(x) = \begin{cases} 0 & \text{if x is rational} \\ 1 & \text{if x is irrational} \end{cases}$

Show that an point a in the real line R $\lim\limits_{x \to R} f(x)$ exists.

Solution: Consider any point 'a' of the real line. Let (p_n) be a sequence of rational numbers converging to the point 'a'. Since p_n is a rational number, $f(P_n) = 0$ for all n and consequently lim $f(P_n) = 0$, Now, consider a sequence (q) of irrational numbers converging to 'a'. Since q, is an irrational number, $f(q_n) = 1$ for all n and consequently lim $f(q_n) = 1$. So for two sequences (p_n) and (q,) converging to 'a'; sequences $(f(p_n))$ and $(f(q_n))$ do not converge to the same limit. Therefore $\lim\limits_{x-a} f(x)$ cannot exist for if it exists and is equal to A, then both $(f(p_n))$ and $(f(q_n))$ would have converged to the same limit A.

Example: Show that for the function f: R → R defined by f(x) = x Q x ∈ R, $\lim\limits_{x-p} f(x)$ exists for every a ∈ R.

Solution: Consider any point a ∈ R. Let (x,) be a sequence of points of R converging to 'a'. Then $f(x_n)$ = x, and consequently lim $f(x_n)$ = lim (x_n) = a. So for every sequence $<x_n>$ converging to 'a' $(f(X_n))$ converges to 'a'. So by Theorem 3, $\lim\limits_{x-a} f(x) = a$. Consequently $\lim\limits_{x-a} f(x)$ exists for every a ∈ R.

> *Task* Show that $\lim\limits_{x \to 1} 2^x = 2$ by proving that for any sequence (x_n), $x_n \neq 1$, converging to 1, 2^{xn} converges to 2.

10.3 Algebra of Limits

We discussed the algebra of limits of sequences. In this section, we apply the same algebraic operations to limits of functions. This will enable us to solve the problem of finding limits of functions. In other words we discuss limits of sum, difference, product and quotient of functions.

Definition 5: Algebraic Operations on Functions

Let f and g be two functions with domain $D \subset R$. Then the sum, difference, product, quotient of f and g denoted by f + g, f – g, fg, f/g are functions with domain D defined by

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x).\ g(x)$$

$$(f/g)(x) = f(x)/g(x)$$

provided in the last case $g(x) \neq 0$ for all x in D.

Now we prove the theorem.

Theorem 4: If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, where A and B are real numbers,

(1) $\lim_{x \to a} (f + g)(x) = A + B = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$,

(ii) $\lim_{x \to a} (f - g)(x) = A - B = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$,

(iii) $\lim_{x \to g} (f \cdot g)(x) = A \cdot B = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$,

(iv) If further $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to g} f/g(x)$ exists and $\lim_{x \to g} \dfrac{f}{g}(x) = A/B = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$.

Proof: Since $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, corresponding to a number $\varepsilon > 0$. There exist numbers

$$\delta_1 > 0 \text{ and } \delta_2 > 0 \text{ such that}$$

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \varepsilon/2 \qquad (1)$$

$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - B| < \varepsilon/2 \qquad (2)$$

Let δ = minimum (δ_1, δ_2). Then from (1) and (2) we have that

$$0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - (A + B)| \leq |f(x) - A| + |g(x) - B|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Which shows that $\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + g(x) = A + B$

This proves part (i).

The proof of (ii) is exactly similar. Try it yourself.

(iii) $$|f(x)\, g(x) - AB| = |(f(x) - A)\, g(x) + A\,(g(x) - B)|$$
$$\leq |f(x) - A|\ |g(x)| + |A|.\ |(g(x) - B)|. \qquad (3)$$

Since $\lim_{x \to a} g(x) = B$ corresponding to 1, there exists a number $\alpha_0 > 0$ such that

$$0 < |x - a| < \alpha_0 \Rightarrow |g(x) - B| < 1.$$

which implies that $|g(x) \leq |g(x) - B| + |B| \leq 1 + |B| = K$ (say) (4)

Since f(x) = A, corresponding to $\varepsilon < 0$, there exists a number $\delta_1 > 0$ such that number $\delta_1 < 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \varepsilon/2K \qquad (5)$$

Notes

Since $\lim_{x \to a} g(x) = B$, corresponding to a number $\varepsilon > 0$, there exists a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \Rightarrow |f(x) - B| < \frac{E}{2(|A|+1)} \quad (6)$$

Let $\delta = \min(a_1, \delta_1, \delta_2)$. Then using (4), (5) and (6) in (3), we have for $0 < |x - a| < \delta$,

$$|f(x)\, g(x) - AB| \leq |f(x) - A|\; |g(x)| + |A|\; |g(x) - B|$$
$$\leq |f(x) - A|\,.\,K + |A|\; |(g(x) - B)|$$
$$< \frac{\varepsilon}{2K}\,.\,K + \frac{\varepsilon}{2(|A|+1)}\,|A| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad (7)$$

Therefore, $\lim_{x \to a} g(x) = AB$ i.e. $\lim_{x \to a} (fg)(x) = AB = \lim_{x \to a} f(x).\ \lim_{x \to a} g(x)$, which proves part (iii) of the theorem.

(iv) First we show that *g* does not vanish in a neighbourhood of a.

$\lim_{x \to a} g(x) = B$ and $B \neq 0$. Therefore $|B| > 0$. Then corresponding to $\frac{|B|}{2}$ we have a number $\mu > 0$ such that for $0 < |x - a| < \mu$, $|g(x) - B| < \frac{|B|}{2}$.

Now by triangle inequality, we have

$$\big|\,|g(x)| - |B|\,\big| \leq |g(x) - B| < \frac{|B|}{2}.$$

i.e.,

$$|B| - \frac{|B|}{2} < |g(x)| < |B| + \frac{|B|}{2}. \quad (8)$$

In other words, $0 < |x - a| < \mu \Rightarrow |g(x)| > \frac{|B|}{2}$.

Again since $\lim_{x \to a} g(x) = B$, for a given number $\varepsilon > 0$, we have a number $\mu' > 0$ such that $0 < |x - a| < \mu'$ implies that

$$|g(x) - B| < \frac{|B|^2 \varepsilon}{2}.$$

Let 6 = min (μ, p′). Then if $0 < |x - a| < \delta$, from (7) and (8) we have

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|B - g(x)|}{|g(x)||B|} < \frac{2|B - g(x)|}{|B|^2} < \frac{2|B|^2 \varepsilon}{2|B|^2} = \varepsilon.$$

This proves that $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{B}$.

Now by part (iii) of this theorem, we get that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x).\frac{1}{g(x)} = \lim_{x \to a} f(x).\lim_{x \to a} \frac{1}{g(x)}$$
$$= A.\,\frac{1}{B} = A/B.$$

i.e.,

$$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = A/B = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

This completes the proof of the theorem. You may note the theorem is true even when a = ±∞. You may also see that while proving (iv), we have proved that if

$$\lim_{x \to a} g(x) = B \neq 0, \text{ then } \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{B}.$$

Before we solve some examples, we prove two more theorems.

Theorem 5: Let f and g be defined in the domain D and let $f(x) \leq g(x)$ for all x in D. Then if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist,

$$\lim_{x \to n} f(x) \leq \lim_{x \to a} g(x).$$

Proof: Let $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} g(x) = B$. If possible, let A > B.

for $\varepsilon = \frac{A - B}{2}$, there exist $\delta_1, \delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < E$$

and $$0 < |x - a| < \delta_2 \Rightarrow |g(x) - B| < E.$$

If 6 = min. (δ_1, δ_2), then for $0 < |x - a| < 6$, $g(x) \in]B - \varepsilon, B + \varepsilon[$ and $f(x) \in]A - \varepsilon, A + \varepsilon[$. But $B + \varepsilon = A - E = \frac{A + B}{2}$. Therefore $g(x) < f(x)$ for $0 < |x - a| < 6$ which contradicts the given hypothesis. Thus $A \leq B$.

Theorem 6: Let S and T be non-empty subsets of the real set R, and let f: S → T be a function of S onto T. Let g: U → R be a function whose domain U C R contains T. Let us assume that $\lim_{x \to a} f(x)$ exists and is equal to b and $\lim_{x \to b} g(y)$ exists and is equal to c. Then $\lim_{x \to a} g(f(x))$ exists and is equal to c.

Proof: Since $\lim_{x \to b} g(y) = c$, given a number E > 0, there exists a number $\alpha_0 > 0$ such that

$$0 < |y - b| < \alpha_0 \Rightarrow |g(y) - c| < \varepsilon.$$

Since $\lim_{x \to a} f(x) = b$, corresponding to $\alpha_0 > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - b| < \alpha_0.$$

Hence, taking y = f(x) and combining the two we get that for

$$0 < |x - a| < \delta, \; |g(f(x)) - c| = |g(y) - c| < \varepsilon$$

$$\text{(since } |f(x) - b| < \alpha_0\text{)}.$$

This completes the proof of the theorem. Finally we give one more result without proof.

Result: If $\lim_{x \to a} f(x) = A$, A > 0 and $\lim_{x \to a} g(x) = B$ where A and B are finite real numbers then

$$\lim_{x \to a} f(x)^{g(x)} = A^B.$$

Now we discuss some examples. You will see how the above results help us in reducing the problem of finding limit of complicated functions to that of finding limits of simple functions.

Example: Find $\lim_{x \to \infty} \frac{(2x+7)(3x-11)(4x+5)}{4x^3 + x - 1}$

Solution:

$$\lim_{x \to \infty} \frac{(2x+7)(3x-11)(4x+5)}{4x^3 + x - 1}$$

Notes

$$= \lim_{x\to\infty} \frac{x^3\left[\left(2+\frac{7}{x}\right)\left(3-\frac{11}{x}\right)\left(4+\frac{5}{x}\right)\right]}{x^3\left(4+\frac{1}{x^2}-\frac{1}{x^3}\right)}$$

We divide the numerator and denominator by x^3 since x^3 is neither zero nor ∞.

$$= \lim_{x\to\infty} \frac{(2x+7)(3x-11)(4x+5)}{4x^3+x-1}$$

$$= \lim_{x\to\infty} \frac{\left(2+\frac{7}{x}\right)\left(3-\frac{11}{x}\right)\left(4+\frac{5}{x}\right)}{4+\frac{1}{x^2}-\frac{1}{x^3}} - \frac{2\times 3\times 4}{4} = .6.$$

Example: Find $\lim_{x\to 3} \frac{x^2-9}{x^2-4x+3}$

Solution: $\lim_{x\to 3} \frac{x^2-9}{x^2-4x+3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x-1)}$

Hence $\lim_{x\to 3} \frac{x^2-9}{x^2-4x+3} = \lim_{x\to 3} \frac{x+3}{x-1}$

$$= \frac{\lim_{x\to 3}(x+3)}{\lim_{x\to 3}(x-1)} = \frac{6}{2} = 3.$$

The function $f(x) = \frac{x^2-9}{x^2-4x+3}$ is not defined at x = 3. But we are considering only the values of the function at those points x in a neighbourhood of 3 for which $x \neq 3$ and hence we can cancel $x - 3$ factor.

Example: Evaluate $\lim_{x\to 0} \frac{(1+x)^{1/2}-1}{(1+x)^{1/3}-1}$.

Solution: To make the problem easier, we make a substitution which enables us to get rid of fractional powers 1/2 and 1/3. L.C.M. of 2 and 3 is 6. So, we put $1 + x = y^6$.

Then we have

$$\lim_{x\to 0} \frac{(1+x)^{1/2}-1}{(1+x)^{1/3}-1} = \lim_{y\to 1} \frac{y^3-1}{y^2-1} = \lim_{y\to 1} \frac{(y-1)(y^2+y+1)}{(y-1)(y+1)}$$

$$= \lim_{y\to 1} \frac{y^2+y+1}{y+1} - \frac{3}{2}.$$

Self Assessment

Fill in the blanks:

1. The intuitive idea of limit was used both by Newton and Leibnitz in their independent invention of Differential Calculus around ....
2. The limit of a function that a point a is meaningful only if a is a limit point of its ....
3. For a function f: R $\to$ R defined by $f(x) = x^2$, find its limit when x tends to 1 by the ....

4. The function f is said to tend to a number A as x tends to 'a' from the left or through values smaller than 'a' if given a number E > 0, there exists a number $\delta > 0$ such that ....
....

10.4 Summary

- We started with the intuitive idea of a limit of a function. Then we derived the rigorous definition of the limit of a function, popularly called $\varepsilon - \delta$ definition of a limit. Further, we gave the notion of right and left hand limits of a function. It has been proved that $\lim_{x \to a} f(x) = A$ if and only if both right hand and left hand limits are equal to A i.e. $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = A$. In the same section we discussed the limit of a function as x tends to $+\infty$ or $-\infty$. Also we discussed the infinite limit of a function.
- We studied the idea of sequential limit of a function by connecting the idea of limit of an arbitrary function with the limit of a sequence. It has been shown how this relationship helps in finding the limits of functions.
- We defined the algebraic operations of sum, difference, product, quotient of two functions. We proved that the limit of the sum, difference, product and quotient of two functions at a point is equal to the sum, difference, product and quotient of the limits of the functions at the point provided in the case of quotient, the limit of the function in the denominator is non-zero. Finally in the same section, the usefulness of the algebra of limits in finding the limits of complicated functions has been illustrated.

10.5 Keywords

Function: A function f tends to a limit A, as x tends to infinity if having chosen a positive number ε, there exists a positive number k such that

$$f(x - A)| > \varepsilon \ \forall \ x \geq k.$$

Infinite Limits of a Function: Suppose a is a real number. We say that a function f tends to +m when x tends to a, if for a given positive real number M there exists a positive number δ such that

$$f(x) > M \text{ whenever } 0 < |x - a| < \delta.$$

10.6 Review Questions

1. Show that $\lim_{x \to 2} \frac{x^2 - x + 18}{3x - 1} = 4$, using the $\varepsilon - \delta$ definition.
2. Find the limit of the function f defined as

   $f(x) = \frac{2x^2 + x}{3x}$, $x \neq 0$ when x tends to 0.
3. Find, if possible, the limit of the following functions.

   (i) $f(x) = \frac{|x - 2|}{x - 2}$, $x \neq 2$

   when x tends to 2.

   (ii) $f(x) = \frac{-1}{e^{1/x} + 1}$, $x \neq 0$

   when x tends to 0.

Notes

4. (i) Let $f(x) = \frac{1}{|x|}$, $x \neq 0$. Show that $\lim_{x \to 0+} f(x) = +\infty$, $\lim_{x \to 0-} f(x) = \infty$ and $\lim_{x - 0} f(x) = +\infty$.

   (ii) Let $f(x) = -\frac{1}{|x|}$, $x \neq 0$. Show that $\lim_{x \to 0+} f(x) = -\infty$, $\lim_{x \to 0-} f(x) = -\infty$ and $\lim_{x - 0} f(x) = -\infty$.

   (iii) Let $f(x) = \frac{1}{x}$, $x \neq 0$. Prove that $\lim_{x \to 0+} f(x) = +\infty$, $\lim_{x \to 0-} f(x) = -\infty$.

   (iv) Let $f(x) = -\frac{1}{x}$, $x \neq 0$. Prove that $\lim_{x \to 0+} f(x) = -\infty$, $\lim_{x \to 0-} f(x) = \infty$.

5. Show that for the function $f: R \to R$ defined by

$$f(x) = x^2,$$

   f(x) exists for every $a \in R$.

6. Find

   (i) $\lim_{x - w} \frac{(2x+3)^3 (3x-2)^2}{x^5 + 5}$

   (ii) $\lim_{x - w} \frac{(x^3+1)^{1/3}}{x+1}$.

7. If $g(x) = \begin{cases} 2x & \text{for } 0 \text{ s } x < 1 \\ 4 & \text{for } x = 1 \\ 5 - 3x & \text{for } 1 < x \leq 2. \end{cases}$

   find $\lim_{x \to 1} g(x)$

8. Find

   (i) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

   (ii) Find $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 + 5x - 14}$

   (iii) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

   (iv) $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$

   (v) $\lim_{x \to \infty} \left(\frac{x-1}{x+1}\right)^x$

   (vi) $\lim_{x \to 0} \left(\frac{\sin 2x}{x}\right)^{1+x}$

   (vii) $\lim_{x \to \infty} \left(\frac{x+1}{2x+1}\right)^{x^2}$

Answers: Self Assessment

1. 1675
2. domain
3. $\varepsilon - \delta$ approach
4. $| f(x) - A | < \varepsilon$ for $a - 6 < x < a$

10.7 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 11: Continuity

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define the continuity of a function at a point of its domain
- Determine whether a given function is continuous or not
- Construct new continuous functions from a given class of continuous functions

Introduction

The values of a function f(x) approaching a number A as the variable x approaches a given point a. When there is break (or jump) in the graph, then this property fails at that point. This idea of continuity is, therefore, connected with the value of $\lim_{x \to a} f(x)$ and the value of the function f at the point a. We define in this unit the continuity of a function at a given point a in precise mathematical language. Therefore extend it to the continuity of a function on a non-empty subset of the domain of f which could be the whole of the domain of f also. We study the effect of the algebraic operations of addition, subtraction, multiplication and division on continuous functions.

Here we discuss the properties of continuous functions and the concept of uniform continuity.

11.1 Continuous Functions

We have seen that the limit of a function f as the variable x approaches a given point a in the domain of a function f does not depend at all on the value of the function at that point a but it depends only on the values of the function at the points near a. In fact, even if the function f is not defined at a then $\lim_{x \to a} f(x)$ may exist.

For example $\lim_{x \to 1} f(x)$ exists when

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ though f is not defined at } x = 1.$$

We have also seen that $\lim_{x \to a} f(x)$ may exist, still it need not be the same as f(a) when it exists. Naturally, we would like to examine the special case when both $\lim_{x \to a} f(x)$ and f(a) exist and are equal. If a function has these properties, then it is called a continuous function at the point a. We give the precise definition as follows:

Definition 1: Continuity of a Function at a Point

A function f defined on a subset S of the set R is said to be continuous at a point a ∈ S, if

(i) $\lim_{x \to a} f(x)$ exists and is finite

(ii) $\lim_{x \to a} f(x) = f(a)$.

Note that in this definition, we assume that S contains some open interval containing the point a. If we assume that there exists a half open (semi-open) interval [a, c[contained in S for some c∈ R, then in the above definition, we can replace $\lim_{x \to a} f(x)$ by $\lim_{x \to a+} f(x)$ and say that the function is continuous from the right of a or f is right continuous at a.

Similarly, you can define left continuity at a, replacing the role of $\lim_{x \to a} f(x)$ by $\lim_{x \to a-} f(x)$. Thus, f is continuous from the right at a if and only if

$$f(a+) = f(a)$$

It is continuous from the left at a if and only if

$$f(a-) = f(a).$$

From the definition of continuity of a function f at a point a and properties of limits it follows that f(a+) = f(a–) = f(a) if and only if, f is continuous at a. If a function is both continuous from the right and continuous from the left at a point a, then it is continuous at a and conversely.

The definition X is popularly known as the Limit-Definition of Continuity.

Since $\lim_{x \to a} f(x)$ is also defined, in terms of ε and δ, we also have an equivalent formulation of the definition X. Note that whenever we talk of continuity of a function f at a in S, we always assume that S contains a neighbourhood containing a. Also remember that if there is one such neighbourhood there are infinitely many such neighbourhoods. An equivalent definition of continuity in terms of ε and δ is given as follows:

Definition 2: (ε, δ)-Definition of Continuity

A function f is continuous at x = a if f is defined in a neighbourhood of a and corresponding to a given number E > 0, there exists some number δ > 0 such that $|x - a| < \delta$ implies $|f(x) - f(a)| < E$.

Note that unlike in the definition of limit, we should have

$$|f(x) - f(a)| < E \text{ for } |x - a| < 6.$$

The two definitions are equivalent. Though this fact is almost obvious, it will be appropriate to prove it.

Theorem 1: The limit definition of continuity and the (ε, δ)-definition of continuity are equivalent.

Proof: Suppose f is continuous at a point a in the sense of the limit definition. Then given ε > 0, we have a δ > 0 such that $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. When x = a, we trivially have

$$|f(x) - f(a)| = 0 < \varepsilon.$$

Hence, $|x - a| < \delta \Rightarrow |f(x) - f(a)| < E$

which is the (ε, δ)-definition.

Conversely we now assume that f is continuous in the sense of (ε, δ)-definition. Then for every E > 0 there exists a δ > 0 such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Notes

Leaving the point 'a', we can write it as

$$0 < (x - a) < \delta \Rightarrow |f(x) - f(a)| < E.$$

This implies the existence, of $\lim_{x \to a} f(x)$ and that $\lim_{x \to a} f(x) = f(a)$.

Note that δ in the definition 2, in general, depends on the given function f, ε and the point a. Also $|x - a| < \delta$ if and only if $a - \delta < x < a + \delta$ and $]a - \delta, a + \delta[$ is an open interval containing a. Similarly $|f(x) - f(a)| < \varepsilon$ if and only if

$$f(a) - \varepsilon < f(x) < f(a) + E.$$

We see that f is continuous at a point a, if corresponding to a given (open) ε-neighbourhood U of f(a) there exists a (open) δ-neighbourhood V of a such that $f(V) \subset U$. Observe that this is the same as $x \in V \Rightarrow f(x) \in U$. This formulation of the continuity at a is more useful to generalise this definition to more general situations in Higher Mathematics.

A function f is said to be continuous on a set S if it is continuous at every point of the set S. It is clear that a constant function defined on S is continuous on S.

Example: Examine the continuity of the following functions:

(i) The absolute value (Modulus) function,

(ii) The signum function.

Solution:

(i) You know that the absolute value function

$f: R \to R$ is defined as $f(x) = |x|$, V $x \in R$.

The function is continuous at every point $x \in R$. For given $\varepsilon > 0$, we can choose $\delta = \varepsilon$ itself. If $a \in R$ be any point them $|x - a| < \delta = \varepsilon$ implies that

$$|f(x) - f(a)| = ||x| - |a|| \leq |x - a| < \varepsilon.$$

(ii) The signum function, as you know a function $f: R \to R$ defined as

$$f(x) = 1 \quad \text{if } x > 0$$
$$= 0 \quad \text{if } x = 0$$
$$= -1 \quad \text{if } x < 0$$

This function is not continuous at the point x = 0. We have already seen that $f(0+) = 1$, $f(0-) = -1$. Since $f(0+) \neq f(0-)$, $\lim_{x \to 0} f(x)$ does not exist and consequently the function is not continuous at x = 0. For every point $x \neq 0$ the function f is continuous. This is easily seen from the graph of the function f. There is a jump at the point x = 0 in the values of f(x) defined in a neighbourhood of 0.

Note that if $f: R \to R$ is defined as,

$$f(x) = 1 \quad \text{if } x \geq 0.$$
$$= -1 \quad \text{if } x < 0.$$

then, it is easy to see that this function is continuous from the right at x = 0 but not from the left. It is continuous at every point $x \neq 0$.

Similarly, if f is defined by $f(x) = 1$ if $x > 0$

$$= -1 \quad \text{if } x \leq 0$$

then f is continuous from the left at x = 0 but not from the right.

Example: Discuss the continuity of the function sin x on the real line R.

Solution: Let f(x) = Sin x $\forall$ x $\in$ R.

We show by the (ε, δ)-definition that f is continuous at every point of R.

Consider an arbitrary point a $\in$ R. We have

$$|f(x) - f(a)| = |\sin x - \sin a| = \left|2 \sin \frac{x-a}{2} \cos \frac{x+a}{2}\right|$$

$$= 2\left|\sin \frac{x-a}{2}\right| \left|\cos \frac{x+a}{2}\right|$$

$$\leq 2\left|\sin \frac{x-a}{2}\right| \left(\text{since} \left|\cos \frac{x+a}{2}\right| \leq I\right)$$

From Trigonometry, you know that $|\text{Sin } \theta| \leq |\theta|$.

Therefore $\left|\sin \frac{x-a}{2}\right| \leq \left|\frac{x-a}{2}\right| = \frac{|x-a|}{2}$

Consequently $|f(x) - f(a)| \leq |x - a|$

$< \varepsilon$ if $|x - a] < \delta$ where $\delta = \varepsilon$.

So f is continuous at the point a. But a is any point of R. Hence Sin x is continuous on the real line R.

Task Discuss the continuity cos x on the real R.

As we have connected the limit of a function with the limit of a sequence of real numbers. In the same way, we can discuss the continuity of a function in the language of the sequence of real numbers in the domain of the function. This is explained in the following theorem.

Theorem 2: A function f: S $\rightarrow$ R is continuous at point a in S if and only for every sequence (x_n), ($x_n \in$ S) converging do a, f(x_n) converges to f(a).

Proof: Let us suppose that f is continuous at a. Then $\lim_{x \to a} f(x) = f(a)$.

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x - a| < 6 \Rightarrow |f(x) - f(a)| < \varepsilon.$$

If x_n is a sequence converging to 'a', then corresponding to $\delta > 0$, there exists a positive integer M such that

$$|x_n - a| < \delta \text{ for } n \geq M.$$

Thus, for n $\geq$ M, we have $|x_n - a| < \delta$ which, in turn, implies that

$$|f(x_n) - f(a)| < \varepsilon,$$

proving thereby f(x_n) converges to f(a).

Notes

Conversely, let us suppose that whenever x_n converges to a, $f(x_n)$ converges to f(a). Then we have to prove that f is continuous at a. For this, we have to show that corresponding to an E > 0, there exists some $\delta > 0$ such that

$$|f(x) - f(a)| \geq \varepsilon, \text{ whenever } |x - a| < \delta.$$

If not, i.e., if f is not continuous at a, then there exists an E > 0 such that whatever $\delta > 0$ we take there exists an x_δ such that

$$|x_\delta - a| < \delta \text{ but } |f(x_\delta) - f(a)| \geq \varepsilon.$$

By taking $\delta = 1, 1/2, 1/3,....$ in succession we get a sequence (x_n), where $x_n = x_\delta$ for $\delta = 1/n$, such that $|f(x_n) - f(a)| \geq \varepsilon$. The sequence (x_n) converges to a. For, if m > 0, these exists M such that $1/n < m$ for $n \geq M$ and therefore $|x_n - a| < m$ for $n \geq M$. But $f(x_n)$ does not converge to f(a), a contradiction to our hypothesis. This completes the proof of the theorem.

Theorem 2 is sometimes used as a definition of the continuity of a function in terms of the convergent sequences. This is popularly known as the Sequential Definition of Continuity which we state as follows:

Definition 3: Sequential Continuity of a Function

Let f be a real-valued function whose domain is a subset of the set R. The function f is said to be continuous at a point a if, for every sequence (x_n) in the domain of f converging to a, we have,

$$\lim_{n\to\infty} f(X_n) = f(a)$$

The next example illustrates this definition.

Example: Let f: R → R be defined as

$$f(x) = 2x^2 + 1, \ \forall \ x \in R$$

Prove that f is continuous on R by using the sequential definition of the continuity of a function.

Solution: Suppose (x_n) is a sequence which converges to a point 'a' of R. Then, we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (2x_n^2 + 1) = 2(\lim_{n-m} x_n)^2 + 1 = 2a^2 + 1 = f(a)$$

This shows that f is continuous at a point $a \in R$. Since a is an arbitrary element of R, therefore, f is continuous everywhere on R.

> *Task* Prove by sequential definition of continuity that the function f : R → R defined by $f(x) = \sqrt{x}$ is continuous at x = 0.

11.2 Algebra of Continuous Functions

As we have proved limit theorems for sum, difference, product, etc. of two functions, we have similar results for continuous functions also. These algebra operations on the class of continuous functions can be deduced from the corresponding theorems on limits of functions, using the limit definition of continuity. We leave this deduction as an exercise for you. However, we give a formal proof of these algebraic operations by another method which illustrates the use of Theorem 2. We prove the following theorem:

Theorem 3: Let f and g be any real functions both continuous at a point $a \in R$. Then,

(i) αf defined by $(\alpha f)(x) = \alpha f(x)$, is continuous for any real number α,

(ii) $f + g$ defined by $(f + g)(x) = f(x) + g(x)$ is continuous at a,

(iii) $f - g$ defined by $(f - g)(x) = f(x) - g(x)$ is continuous at a,

(iv) fg defined by $(fg)(x) = f(x)\,g(x)$ is continuous at a,

(v) f/g defined by $(f/g)(x) = \dfrac{f(x)}{g(x)}$, is continuous at a provided $g(a) \neq 0$.

Proof: Let x_n be an arbitrary sequence converging to a. Then the continuity of f and g imply that the sequences $f(x_n)$ and $g(x_n)$ converge to f(a) and g(a) respectively. In other words, $\lim f(x_n) = f(a)$, $\lim g(x_n) = g(a)$.

Using the algebra of sequences, we can conclude that

$$\lim \alpha f(x_n) = \alpha f(a),$$

$$\lim (f + g)(x_n) = \lim f(x_n) + \lim g(x_n) = f(a) + g(a),$$

$$\lim (f - g)(x_n) = \lim f(x_n) - \lim g(x_n) = f(a) - g(a),$$

$$\lim (f \cdot g)(x_n) = \lim f(x_n) \lim g(x_n) = f(a)\,g(a).$$

If infinite number of x_n's are such that $g(x_n) = 0$, then $g(X_n) - g(a)$ implies that $g(a) = 0$, a contradiction.

This proves the parts (i), (ii), (iii) and (iv). To prove the part (v) we proceed as follows:

Since $g(a) \neq 0$, we can find $a > 0$ such that the interval $]g(a) - \alpha, g(a) + \alpha[$ is either entirely to the right or to the left of zero depending on whether $g(a) > 0$ or $g(a) < 0$. Corresponding to $a > 0$, there exists a $\delta_1 > 0$ such that $|x - a| < \delta_1$ implies $|g(x) - g(a)| < \alpha$, i.e., $g(a) - \alpha < g(x) < g(a) + at$. Thus, for x such that $|x - a| < \delta_i$, $g(x) \neq 0$. If (x_n) converges to a, omitting a finite number of terms of the sequence if necessary, then we can assume that $g(x_n) \neq 0$, for all n. Hence, $\dfrac{f(x_n)}{g(x_n)}$ converges to $\dfrac{f(a)}{g(a)}$ and so $\dfrac{f}{g}$ is continuous at a. This completes the proof of the theorem.

In part (v) if we define f by $f(x) = 1$, then it follows that if g is continuous at 'a' and $g(a) \neq 0$, then its reciprocal function $1/g$ is continuous at 'a'.

Now, we prove another theorem, which shows that a continuous function of a continuous function is continuous.

Theorem 4: Let f and g be two real functions such that the range of g is contained in, the domain of f. If g is continuous at $x = a$, f is continuous at $b = g(a)$ and $h(x) = f(g(x))$, for x in the domain of g, then h is continuous at a.

Proof: Given $\varepsilon > 0$, the continuity of f at $b = g(a)$ implies the existence of an $\eta > 0$ such that for

$$|y - b| < \eta,\ |f(y) - f(b)| < \varepsilon \qquad ...(1)$$

Corresponding to $\eta > 0$, from the continuity of g at $x = a$, we get a $\delta > 0$ such that

$$|x - a| < \delta \text{ implies } |g(x) - g(a)| < \eta \qquad ...(2)$$

Combining (1) and (2) we get that

$|x - a| < \delta$ implies that

$$|h(x) - h(a)| = |f(g(x)) - f(g(a))|$$

$$= |f(y) - f(b)| < \varepsilon,$$

Notes

where we have taken y = g(x). Hence h is continuous at a which proves the theorem.

Let us now study the following example:

Example: Examine for continuity the following functions:

(i) The polynomial function f R → R defined by

$$f(x) = a, + a_1x + a_2x^2 +... + a_xx^n.$$

(ii) The rational function f: R → R defined as

$$f(x) = \frac{f(x)}{q(x)}, \ \forall \ x \text{ for which } q(x) \neq 0.$$

Solution:

(i) It is obvious that the function f(x) = x, x ∈ R, is continuous on the whole of the real line. It follows from theorem 3 that the functions x^Z, x^3,...., are all continuous. The fact that constant functions are continuous, we get that any polynomial f(x) in x, i.e., the function f defined by

$$f(x) = a, + a_1x+ a_2x^2 +... + a_n x^n,$$

is continuous on R.

(ii) It follows from theorem 3(v) that a rational function f, defined by,

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_0 + a_1x + \ldots + a_nx^n}{b_0 + b_1x + \ldots + b_mx^m}$$

is continuous at every point a ∈ R for which q(a) ≠ 0.

11.3 Non-continuous Functions

You have seen that a function may or may not be continuous at a point of the domain of the function. Let us now examine why a function fails to be continuous.

A function f: S → R fails to be continuous on its domain S if it is not continuous at a particular point of S. This means that there exists a point a ∈ S such that, either

(i) $\lim\limits_{x-a} f(x)$ does not exist, or

(ii) $\lim\limits_{x-a} f(x)$ exists but is not equal to f(a).

But you know that a function f is continuous at a point a if and only if

$$f(a+) = f(a-) = f(a).$$

Thus, if f is not continuous at a, then one of the following will happen:

(i) either f(a+) or f(a–) does not exist (this includes the case when both f(a+) and f(a–) do not exist).

(ii) both f(a+) and f(a–) exist but f(a+) ≠ f(a–).

(iii) both f(a+) and f(a–) exist and f(a+) = f(a–) but they are not equal to f(a).

If a function f: S → R is discontinuous for each b ∈ S, then we say that totally discontinuous an S. Functions which are totally discontinuous are often encountered but by no means rare. We give an example.

Example: Examine whether or not the function f: R → R defined as,

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}$$

is totally discontinuous.

Solution: Let b be an arbitrary but fixed real number. Choose $\varepsilon = 1/2$. Let $\delta > 0$ be fixed. Then the interval defined by

$$|x - b| < \delta$$

is $(x: b - \delta < x < b + \delta)$

or $]b - \delta, b + \delta[$

This interval contains both rational as well as irrational numbers. Why?

If b is rational, then choose x in the interval to be irrational, If b is irrational then choose x in the interval to be rational. In either case,

$$0 < |x - b| < \delta$$

and

$$|f(x) - f(b)| = I > \varepsilon.$$

Thus, f is not continuous at b. Since b is an arbitrary element of S, f is not continuous at any point of S and hence is totally discontinuous.

There are certain discontinuities which can be removed. These are known as removable discontinuities. A discontinuity of a given function f: S → R is said to be removable if the limit of f(x) as x tends to a exits and that

$$\lim_{x \to a} f(x) \neq f(a)$$

In other words, f has removable discontinuity at x = a if f(a+) = f(a–) but none is equal to f(a).

The removable discontinuities of a function can be removed simply by changing the value of the function at the point a of discontinuity. For this a function with removable discontinuities can be thought of as being almost continuous. We discuss the following example to illustrate a few cases of removable discontinuities.

Example: Discuss the nature of the discontinuities of the following functions:

(i) $f(x) = \dfrac{x^2 - 4}{x - 4}$, $\quad x \neq 2$

$= 1 \quad x = 2$

at x = 2.

(ii) $f(x) = 3, \quad x \neq 3$

$= 1 \quad x = 3$

at x = 3.

(iii) $f(x) = x^2, \quad x \in] -2, 0 (U) 0, 2 [$

$= 1 \quad x = 0$

at x = 0.

Notes

Solution:

(i) This function is discontinuous at x = 2. This is a removable discontinuity, for if we redefine f(x) = 4, then we can restore the continuity of f at x = 2.

(ii) This is again a case of removable discontinuity at 3. Therefore, if f is defined by f(x) = 3 $\forall$ $x \in R$, then it is continuous at x = 3.

(iii) This function is discontinuous at x = 0. Why? This is a case of discontinuity which is removable. To remove the discontinuity, set f(0) = 0. In other words, define f as

$$f(x) = x^2, x \in]-2, 0[\cup]0, 2[$$
$$= 0, \quad x = 0$$

This is continuous at x = 0. Verify it.

Example: Let a function f: R → R be defined as,

(i) $f(x) = \frac{1}{x}, \quad x \neq 0$

$= 0, \quad x = 0$

(ii) $f(x) = \frac{1}{x}, \quad \text{if } x > 0$

$= 1, \quad \text{if } x < 0$

(iii) $f(x) = \frac{1}{x}, \quad \text{if } x < 0$

$= 1, \quad \text{if } x > 0$

Test the continuity of the function. Determine the type of discontinuity if it exists.

Solution:

(i) Here f(0+) and f(0–) both do not exist (as finite real numbers) and so function is discontinuous. This is not a case of removable discontinuity.

(ii) In this case, f(0) does not exist whereas f(0+) exists and f(0–) = f(0) = 1. This is not a case of removable discontinuity.

Task Prove that the function f defined by f(x) = x sin 1/x if x ≠ 0 and f(0) = 1 has a removable discontinuity at x = 0.

Self Assessment

Fill in the blanks:

1. A function f is said to be .... on a set S if, it is continuous at every point of the set S. It is clear that a constant function defined on S is continuous on S.

2. A function f: S → R is continuous at point a, in S if and only 31 far every sequence (x_n), $(x_n \in S)$ converging do a, $f(x_n)$ .... to f(a).

Notes

3. Let f be a .... whose domain is a subset of the set R. The function f is said to be continuous at a point a if, for every sequence (x_n) in the domain of f converging to a, we have, $\lim_{n\to\infty} f(X_n) = f(a)$.

4. Let f and g be two real functions such that the range of g is contained in, the domain of f. If g is continuous at x = a, f is continuous at b = g(a) and ...., for x in the domain of g, then h is continuous at a.

5. A function f : S → R fails to be .... on its domain S if it is not continuous at a particular point of S.

11.4 Summary

The concept of the continuity of a function at a point of its domain and on a subset of its domain. The limit definition and (ε, –δ)-definition of continuity. It has been proved that both the definitions are equivalent. Sequential definition of continuity has been discussed and illustrations regarding its use for solving problems have been given. The algebra of continuous functions is considered and it has been proved that the sum, difference, product and quotient of two continuous functions at a point is also continuous at the point provided in the case of quotient, the function occurring in the denominator is not zero at the point. In the same section, we have proved that a continuous function of a continuous function is continuous. Finally in Section 9.4, discontinuous and totally discontinuous functions are discussed. Also in this section, one kind of discontinuity that is removable discontinuity has been studied.

11.5 Keywords

Continuity: A function f is continuous at x = a if f is defined in a neighbourhood of a and corresponding to a given number E > 0, there exists some number δ > 0 such that $|x - a| < \delta$ implies $|f(x) - f(a)| < E$.

Sequential Continuity of a Function: Let f be a real-valued function whose domain is a subset of the set R. The function f is said to be continuous at a point a if, for every sequence (x_n) in the domain of f converging to a, we have, $\lim_{n\to\infty} f(X_n) = f(a)$

11.6 Review Questions

1. Examine the continuity of the following functions:

   (i) The function f: R – (0) → R defined as

   $$f(x) = \frac{|x|}{x},$$

   at the point x = 0

   (ii) The function f: R → (1) – R defined as

   $$f(x) = \frac{x^2 - 1}{x - 1},$$

   (iii) The function f: R – (0) – R defined as

   $$f(x) = \frac{1}{x}.$$

2. Examine the continuity of the function f: R → R defined as,

   (i) $f(x) = x^3$ at a point $a \in R$;

   (ii) $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$

3. Show that the function f: R → R defined by

$$f(x) = \begin{cases} 1, & \text{if x is rational} \\ 0, & \text{if x is irrational} \end{cases}$$

   is totally discontinuous. Does f(a+) and f(a–) exist at any point $a \in R$?

4. Prove that the function |f| defined by |f|(x) = |f(x)| for every real x is continuous on R whenever f is continuous on R.

5. (i) Find the type of discontinuity at x = 0 of the function f defined by

$$f(x) = x + 1 \text{ if } x > 0, f(x) = -(x + 1) \text{ if } x < 0 \text{ and } f(0) = 0.$$

   (ii) The function f is defined by

$$f(x) = \sin\frac{1}{x}, \qquad x \neq 0$$
$$= 0, \qquad x = 0$$

   Is f continuous at 0?

Answers: Self Assessment

1.	continuous	2.	converges
3.	real-valued function	4.	h(x) = f(g(x))
5.	continuous		

11.7 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol : Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik : Mathematical Analysis.

H.L. Royden : Real Analysis, Ch. 3, 4.

Unit 12: Properties of Continuous Functions

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the properties of continuous functions on bounded closed intervals
- Explain the important role played by bounded closed intervals in Real Analysis
- Describe the concept of uniform continuity and its relationship with continuity

Introduction

Having studied in the last two units you have studied about limit and continuity of a function at a point, algebra of limits and continuous functions, the connection between limits and continuity, etc., we now take up the study of the behaviour of continuous functions and bounded closed intervals on the real line. You will learn that continuous functions on such intervals are bounded and attain their bounds; they take all values in between any two values taken at points of such intervals. You will also be introduced to the concept of uniform continuity and further you will see that a continuous function on a bounded closed interval is uniformly continuous. This means that continuous functions are well-behaved on bounded closed intervals. Thus, we will see that bounded closed intervals form an important subclass of the class of subsets of the real line which are known as compact subsets of the real line. You will study more about this in higher mathematics at a later stage. We will henceforth call bounded closed intervals of R as compact intervals.

The results of this unit play an important and crucial role in Real Analysis and so for further study in analysis, you must understand clearly the various theorems given in this unit.

It may be noted that an interval of R will not be a compact interval if it is not a bounded or closed interval.

12.1 Continuity on Bounded Closed Intervals

We now consider functions continuous on bounded closed intervals. They have properties which fail to be true when the intervals are not bounded or closed. Firstly, we prove the properties and then with the help of examples we will show the failures of these properties. To prove these properties, we need an important property of the real line that was discussed in Unit 1.

Notes

This property called the completeness property of R states as follows:

Any non-empty subset of the Seal Hue R which is bounded above has the least upper bound or equivalently, any non-empty subset of R which is bounded below has the greatest lower bound.

In the following theorems we prove the properties of functions continuous on bounded closed intervals. In the first two theorems we show that a continuous function on a bounded closed interval is bounded and attains its bounds in the interval. Recall that f is bounded on a set S, if there exists a constant M > 0 such that $|f(x)| \le M$ for all $x \in S$. Note also that a real function f defined on a domain D (whether bounded or not) is bounded if and only if its range f(D) is a bounded subset of R.

Theorem 1: A function f continuous on a bounded and closed interval [a, b] is necessarily a bounded function.

Proof: Let S be the collection of all real numbers c in the interval [a, b] such that f is bounded on the interval [a, c]. That is, a real number c in [a, b] belongs to S if and only if there exists a constant M_c such that $|f(x)| \le M_c$ for all x in [a, c]. Clearly, $S \ne \phi$ since $a \in S$ and b is an upper bound for S. Hence, by completeness property of R, there exists a least upper bound for S. Let it be k (say). Clearly, $k \le b$. We prove that $k \in S$ and k = b which will complete the proof of the theorem.

Corresponding to $\varepsilon = 1$, by the continuity of f at $k(\le b)$ there exists a d > 0 such that

$$|f(x) - f(k)| < \varepsilon = 1 \text{ whenever } |x - k| < d, x \in [a, b].$$

By the triangle inequality we have

$$|f(x)| - |f(k)| \le |f(x) - f(k)| < 1$$

Hence, for all x in [a. b] for which $|x - k| < d$, we have that

$$|f(x)| < |f(k)| + 1 \qquad ...(1)$$

Since k is the least upper bound of S, k – d is not an upper bound of S. Therefore, there is a number $c \in S$ such that

$$k - d < c \le k$$

Consider any t such that $k \le t < k + d$. If x belongs to the interval [c, t] then $|x - k| < d$. For,

$$x \in [c, t] = \Rightarrow c \le x \le t \Rightarrow k - d < c \le x \le t < k + d \qquad ...(2)$$

Now $c \in S$ implies that there exists $M_c > 0$ such that for all

$$x \in [a, c], |f(x)| \le M_c$$

$$x \in [a, t] = [a, c] \cup [c, t] \Rightarrow \text{either } x \in [a, c] \text{ or } x \in [c, t].$$

If $x \in [a, c]$, by (3) we have

$$|f(x)| \le M_c < M_c + |f(k)| + 1.$$

If, however, $x \in [c, t]$ then by (1) and (2) we have

$$|f(x)| < |f(k)| + 1 < M_c + |f(k)| + 1$$

In any case we get that $x \in [a, t]$ implies that

$$|f(x)| < M_c + |f(k)| + 1$$

This shows that f is bounded in the interval [a, t] thus proving that $t \in S$ whenever $k \le t < k + d$. In particular $k \in S$. In such a case k = b. For otherwise we can choose a 't' such that $k < t < k + d$ and $t \in S$ which will contradict the fact that k is an upper bound. This completes the proof of the theorem.

Having proved the boundedness of the : function continuous on a bounded closed interval, we now prove that the function attains its bounds, that is, it has the greatest and the smallest values.

Theorem 2: If f is a continuous function on the bounded closed interval [a, b] then there exists points x_1 and x_2 in [a, b] such that $f(x_1)$ S $f(x) \le f(x_2)$ for all $x \in [a, b]$ (i.e. f attains its bounds).

Proof: From Theorem 1, we know that f is bounded on [a, b].

Therefore there exists M such that $|f(x)| \le M \;\; \forall x \in [a, b]$.

Hence, the collection {f(x) : a a x ≤ b] has an upper bound, since $f(x) \le |f(x)| \le M \;\; \forall x \in [a, b]$.

So by the completeness property of R, the set (f(x) : a ≤ x ≤ b) has a least upper bound.

Let us denote by K the least upper bound of {f(x) : a ≤ x ≤ b].

Then f(x) ≤ K for all x such that a ≤ x ≤ b. We claim that there exists x_2 in [a, b] such that $f(x_2) = K$. If there is no such x_2, then K – f(x) > 0 for all a ≤ x ≤ b. Hence, the function g given by,

$$g(x) = \frac{1}{K - f(x)}$$

is defined for all x in [a, b] and g is continuous since f is continuous. Therefore by Theorem 1, there exists a constant M′ > 0 such that

$$|g(x)| \le M' \;\; \forall x \in [a, b]$$

Thus, we get

$$|g(x)| = \frac{1}{|K - f(x)|} = \frac{1}{K - f(x)} \le M'$$

i.e., $\quad f(x) \le K - \frac{1}{M} \;\; \forall x \in [a, b].$

But this contradicts the choice of K as the least upper bound of the set (f(x) : a | x ≤ b). This contradiction, therefore, proves the existence of an x_2 in [a, b] such that $f(x_2) = K \ge f(x)$ for a ≤ x ≤ b. The existence of x_1 in [a, b] such that $f(x_1) \le f(x)$ for a ≤ x | b can be proved on exactly similar lines by taking the g.l.b. of {f(x) : a ≤ x ≤ b} instead of the l.u.b. or else by considering –f instead of f.

Theorems 1 and 2 are usually proved using what is called the Heine-Borel property on the real line or other equivalent properties. The proofs given in this unit straightaway appeal to the completeness property of the red line (Unit 2) namely that any subset of the real line bounded above has least upper bound. These proofs may be slightly longer than the conventional ones but it does not make use of any other theorem except the property of the real line stated above.

As remarked earlier, the properties of continuous functions fail if the intervals are not bounded or closed, that is, the intervals of the type

]a, b[, 1a, b], [a, b[, [a,′∞[, 1a, ∞[,] –∞, a],] –∞, a [or] –∞, ∞[.

Example: Show that the function f defined by f(x) = 3 V x ∈ [0, ∞[is continuous but not bounded.

Solution: The function f being a polynomial function is continuous in [0, ∞[. The domain of the function is an unbounded closed interval. The function is not bounded since the set of values of the function that is the range of the function is $\{x^2 : x \in [0, \infty[\ \} = [0, \infty[$ which is not bounded.

Example: Show that the function f defined by f(x) = $\frac{1}{x}$ V x ∈] 0, 1[is continuous but not bounded.

Solution: The function f is continuous being the quotient of continuous functions F(x) = 1 and G(x) = x with

$$G(x) \neq 0, x \in]0, 1[$$

Domain of f is bounded but not a closed interval. The function is not bounded since its range is (1/x : x ∈] 0, l [ ] =]1, ∞[which is not a bounded set.

Example: Show that the function f such that f(x) = x ∀ x ∈]0, 1[is continuous but does not attain its bounds.

Solution: As mentioned the identity function f is continuous in]0, 1[. Here the domain of f is bounded but is not a closed interval. The function f is bounded with least upper bound (1.u.b) = 1 and greatest lower bound (g.l.b) = 0 and both the bounds are not attained by the function, since range of f =]0, 1[.

Example: Show that the function f such that

$$f(x) = \frac{1}{x^2} \quad \forall x \in [0, 1[.$$

is continuous but does not attain its g.l.b.

Solution: The function G given by G(x) = x^2 ∀ x ∈]0, 1[is continuous and G(x) ≠ 0 ∀ x ∈]0, 1[therefore its reciprocal function f(x) = $1/x^2$ is continuous in]0, 1[. Here the domain f is bounded but is not a closed interval.

Further l.u.b. of f does not exist whereas its g.l.b. is 1 which is not attained by f.

Task Show that the function f given by f(x) = sin x, x ∈]0, π/2[is continuous but does not attain any of its bounds.

Task Prove that the function f given by f(x) = x^2 ∀ x ∈] –∞, 0[is continuous but does not attain its g.l.b.

We next prove another important property known as the intermediate value property of a continuous function on an interval I. We do not need the assumption that I is bounded and losed. This property justifies our intuitive idea of a continuous function namely as a function f which cannot jump from one value to another since it takes on between any two values f(a) and f(b) all values lying between f(a) and f(b).

Theorem 3: (Intermediate Value Theorem). Let f be a continuous function on an interval containing a and b. If K is any number between f(a) and f(b) then there is a number c, a ≤ c S b such that f(c) = K.

Proof: Either f(a) = f(b) or f(a) < f(b) or f(b) < f(a). If f(a) = f(b) then K = f(a) = f(b) and so c can be taken to be either a or b. We will assume that f(a) < f(b). (The other case can be dealt with similarly.) We can, therefore, assume that f(a) < K < f(b).

Let S denote the collection of all real numbers x in [a, b] such that f(x) < K. Clearly S contains a, so $S \neq \phi$ and b is an upper bound for S. Hence, by completeness property of R, S has least upper bound and let us denote this least upper bound by c. Then $a \leq c \leq b$. We want to show that f(c) = K.

Since f is continuous on [a, b], f is continuous at c. Therefore, given $\varepsilon > 0$, there exists a 6 > 0 such that whenever x is in [a, b] and |x – c| < 6, |f(x) – f(c) (< G,

i.e., $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. .. . (4)

If $c \neq b$, we can clearly assume that c + 6 < b. Now c is the least upper bound of S. So $c - \delta$ is not 'an upper bound' of S. Hence, there exists a y in S such that $c - 6 < y \leq c$. Clearly $|y - c| < \delta$ and so by (4) above, we have

$$f(c) - \varepsilon < f(y) < f(c) + \varepsilon.$$

Since y is in S, therefore f(y) < K. Thus, we get

$$f(c) - S < K$$

If now c = b then $K - \varepsilon < K < f(b) = f(c)$, i.e., K < f(c) + E. If $c \neq b$, then c < b; then there exists an x such that c < x < c + 6, 6, $x \in [a, b]$ and for this x, $f(x) < f(c) + \varepsilon$ by (4) above. Since x > c, $K \leq f(x)$, for otherwise x would be in S which will imply that c is not an upper bound of S. Thus, again we have $K \leq f(x) < f(c) + E$.

In any case,

$$K < f(c) + \varepsilon \qquad ...(6)$$

Combining (5) and (6), we get for every $\varepsilon > 0$

$$f(c) - \varepsilon < K < f(c) + \varepsilon$$

which proves that K = f(c), since ε is arbitrary while K, f(c) are fixed. In fact, when f(a) < K < f(b) and f(c) = K, then a < c < b.

Corollary 1: If f is a continuous function on the closed interval [a, b] and If f(a) and f(b) have opposite signs (i.e., f(a) f(b) < 0), then there is a point x_0 in]a, b[at which f vanishes. (i.e., $f(x_0) = 0$).

Corollary follows by taking K = 0 in the theorem.

Corollary 2: Let f be a continuous function defined on a bounded closed interval [a, b] with values in [a, b]. Then there exists a point c in [a, b] such that f(c) = c. (i.e., there exists a fixed point c for the function f on [a, b]).

Proof: If f(a) = a or f(b) = b then there is nothing to prove. Hence, we assume that $f(a) \neq a$ and $f(b) \neq b$.

Consider the function g defined by g(x) = f(x) – x, $x \in [a, b]$. The function being the difference of two continuous functions, is continuous on [a, b]. Further, since f(a), f(b) are in [a, b], f(a) > a (since $f(a) \neq a$, $f(a) \in [a, b]$) and f(b) < b. (Since $f(b) \neq b$, $f(b) \in [a, b]$). So, g(a) > 0 and g(b) < 0. Hence, by Corollary 1, there exists a c in]a, b[such that g(c) = 0, i.e., f(c) = c. Hence, there exists ac in [a, b] such that f(c) = c.

The above Corollary 1 helps us sometimes to locate some of the roots of polynomials. We illustrate this with the following example.

Example: The equation $x^4 + 2x - 11 = 0$ has a real root lying between 1 and 2.

Solution: The function $f(x) = x^4 + 2x - 11$ is a continuous function on the closed interval [1, 2], f(1) = –8 and f(2) = 9. Hence, by Corollary 1, there exists an $x_0 \in]1, 2[$ such that $f(x_0) = 0$, i.e., x_0 is a real root of the equation $x^4 + 2x - 11 = 0$ lying in the interval]1, 2[.

Notes

Task Show that the equation $16x^4 + 64x^3 - 32x^2 - 117 = 0$ has a real root > 1.

12.2 Pointwise Continuity and Uniform Continuity

Here you will be introduced with the concept of uniform continuity of a function. The concept of uniform continuity is given in the whole domain of the function whereas the concept of continuity is pointwise that is it is given at a point of the domain of the function. If a function f is continuous at a point a in a set A, then corresponding to a number E > 0, there exists a positive number $\delta(a)$ (we are denoting 6 as $\delta(a)$ to stress that 6 in general depends an the point a chosen) such that $|x - a| < \delta(a)$ implies that $|f(x) - f(a)| < \delta$. The number $\delta(a)$ also depends on E. When the point a varies $\delta(a)$ also varies. We may or may not have a 6 which serves for all points a in A. If we have such a 6 common to all points a in A, then we say that f is uniformly continuous on A. Thus, we have the following definition of uniform continuity.

Definition 1: Uniform Continuity of a Function

Let f be a function defined on a subset A contained in the set R of all reals. If corresponding to any number $\varepsilon > 0$, there exists a number $\delta > 0$ (depending only on G) such that

$$|x - y| < \delta, x, y \in A \Rightarrow |f(x) - f(y)| < \varepsilon^*$$

then we say that f is uniformly continuous on the subset A.

An immediate consequence of the definition of uniform continuity is that uniform continuity in a set A implies pointwise continuity in A. This is proved in the following theorem.

Theorem 4: If a function f is uniformly continuous in a set A, then it is continuous in A.

Proof: Since f is uniformly continuous in A, given a positive number E, there corresponds a positive number 6 such that

$$|x - y| < \delta; x, y \in A \Rightarrow |f(x) - f(y)| < \varepsilon \quad \text{... (7)}$$

Let a be any point of A. In the above result (1), take y = a. Then we get,

$$|x - a| < \delta; x \in A \Rightarrow |f(x) - f(a)| < \varepsilon$$

which shows that f is continuous at 'a'. Since 'a' is any point of A, it follows that f is continuous in A.

Now we consider some examples.

Example: Show that the function f : R

$$f(x) = x \text{ V } x \in R,$$

is uniformly continuous on R

Solution: For a given $\varepsilon > 0$, 6 can be chosen to be ε itself so that

$$|x - y| < 6 = G \Rightarrow |f(x) - f(y)| = |x - y| < \varepsilon.$$

Example: Show that the function f : R – R given by

$$f(x) = x^2 \quad \forall x \in R$$

is not uniformly continuous on R.

Notes

Solution: Let ε be any positive number. Let δ > 0 be any arbitrary positive number. Choose $x > \varepsilon/\delta$ and $y = x + \delta/2$. Then

$$|x - y| = \frac{\delta}{2} < \delta.$$

$$(fix) - f(y)| = |x^2 - y^2| = |x + y|\,|x - y|$$

$$= \left(\frac{\delta}{2}\right) |x + y| = \left(\frac{\delta}{2}\right) \left|2x + \frac{\delta}{2}\right|$$

$$> \frac{\delta}{2}\left(\frac{2\varepsilon}{\delta} + \frac{\delta}{2}\right) = \varepsilon + \frac{\delta^2}{4} > \varepsilon$$

That is whatever δ > 0 we choose, there exist real numbers x, y such that $|x - y| < \delta$ but $|f(x) - f(y)| > G$ which proves that f is not uniformly continuous.

But we know that f is a continuous function on R.

Example: In the above example if we restrict the domain of f to be the closed interval [–1, 1], then show that f is uniformly continuous on [–1, 1].

Solution: Given E > 0, choose $\delta < \frac{8}{2}$. If $|x - y| < \delta$ and $x, y \in [-1, 1]$,

then using the triangle inequality for | | we get,

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y|\,|x - y|$$

$$< \in (|x| + |y|)$$

$$\leq 2\delta \text{ (since } |x| \leq 1, |y| \leq 1)$$

You should be able to solve the following exercises:

Task

1. Show that $f(x) = x^n$, n > 1 is not uniformly continuous on R even though for each a > 1, it is a continuous function on R.
2. Show that the function $f(x) = \frac{1}{x}$ for 0 < x < 1 is continuous for every x but not uniformly on]0, 1[.
3. Show that the function $f(x) = \sin \frac{1}{x}$ is not uniformly continuous on the interval]0, 1[even though it is continuous in that interval.
4. Show that f(x) = cx where c is a fixed non-zero real number is a uniformly continuous function on R.

We have seen that the function defined by f(x) = 1/x on the open interval]0, 1 [is not uniformly continuous on]0, 1[even though it is a continuous function on]0, 1[. Similarly the function f defined as $f(x) = x^2$ is continuous on the entire real line R but is not uniformly continuous on R.

However, if we restrict the domain of this function to the bounded closed interval [–1, 1], then it is uniformly continuous. This property is not a special property of the function f, where $f(x) = x^2$

Notes

but is common to all continuous functions defined on bounded closed intervals of the real line. We prove it in the following theorem.

Theorem 5: If f is a continuous function on a bounded and closed interval [a, b] then f is uniformly continuous on [a, b].

Proof: Let f be a continuous function defined on the bounded closed interval [a, b]. Let S be the set of all real numbers c in the interval [a, b] such that for a given $\varepsilon > 0$, there exists positive number d_c such that for points x_1, x_2 belonging to closed interval [a, c],

$|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_2 - x_2| < d_r$.

(In other words f is uniformly continuous on the interval [a, c]. Clearly a ε S so that S is non-empty. Also b is an upper bound of S. Prom completeness property of the real line S has least upper bound which we denote by k. $k \le b$.

f is continuous at k. Hence given E > 0, there exists positive real number d_k such that

$$|f(x) - f(k)| < \varepsilon/2 \text{ whenever } |x - k| < d_k \quad ...(8)$$

Since k is the least upper bound of S, $k - \frac{1}{2}$ is not an upper bound of S.

Therefore there exists a point $c \in S$ such that

$$k - 1/2\, d_k < c \le k. \quad ...(9)$$

Since $c \in S$; from the definition of S we see that there exists d_c such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < d_c, x_1, x_2 \in [a, c], \quad ...(10)$$

Let $\quad d = \min((1/2)\, d_k, d_c)$ and $b' = \min.(k + (1/2)\, d_k, b)$.

Now let $x_1, x_2 \in [a, b']$ and $|x_1 - x_2|$. Then if $x_1, x_2 \in [a, c]$, $|x_1 - x_2| < d \le d_c$ by the choice of d and d_c, then $|f(x_1) - f(x_2)| < \varepsilon$ by (10). If one of x_1 x_2 is not in fa, cl, then both x_1, x_2 belong to the interval $]k - d_k, k + d_k[$. For $x_1 \notin [a, c]$, implies $b' \ge x_1 > c > k - (1/2)d_k > k - d_k$ by (9) above. This means $x_1 \le b'$ implies $x_1 \le k + (1/2)d_k < k - d_k$ by the choice of b′. i.e.

$$k - d_k < k - (1/2)\, d_k < x_1 < k + (1/2)d_k < k + d_k \quad ...(11)$$

$|x_1 - x_2| < d$ implies that $x_1 - (1/2)\, d_k < x_2 < x_1 + (1/2)d_k$ since $d \le (1/2)d_k$ by this choice of d. Thus we get from (11) above that

$$|x_1 - x_2| < x_1 - (1/2)d_k < x_2 < x_1 + (1/2)d_k < k + \left(\frac{1}{2}\right) d_k + \frac{1}{2} d_k = k + d_k \quad ...(12)$$

Then (11) and (12) show that $x_1, x_2 \in]k - d_k, k + d_k[$.

Thus we get that $|x_1 - k| < d_k$ and $|x_2 - k| < d_y$, which in turn implies, by (8) above, that $|f(x_1) - (k)| < \varepsilon/2$ and $|f(x_2) - f(k)| < \varepsilon/2$.

Thus $|f(x_1) - f(x_2)| < |f(x_1) - f(k)| + |f(k) - f(x_2)| < \varepsilon/2 + \varepsilon/2 = E$. In other words, if $|x_1 - x_2| <$ d and x_1, x_2 are in [a, b′] then $|f(x_1) - f(x_2)| < E$ which proves that $b' \in S$ i.e. $b' \le k$. But $k \le b'$ by the choice of b′ since $k \le k + (1/2)\, d_k$ and k a b. Thus we get that k = b′. This can happen only when k = b. For if k < b. i.e. $k = b' = \min(k + (l/2)\, d_k, b) < b$, then it implies that $\min(k + (l/2)\, d_k, b) = (k + (1/2)\, d_k = b'$, where b′ ε S i.e. $k + (1/2)\, d_k$ is in S and is greater than k which is a contradiction to the fact that k is the l.u.b of S. Thus we have shown that k = b ε S. In other words there exists a positive number d_k (corresponding to b) such that $|x_1 - x_2| < d_k, x_1, x_2 \in [a, b]$ implies $|f(x_1) - f(x_2)| < \varepsilon$. Therefore f is uniformly continuous in [a, b].

You may note that uniform continuity always implies continuity but not conversely. Converse is true when continuity is in the bounded closed interval.

Notes

Before we end this unit, we state a theorem without proof regarding the continuity of the inverse function of a continuous function.

Theorem 6: Inverse Function Theorem

Let f : I – J be a function which is both one-one and onto. If f is continuous on I, then $f^{-1} : J \to I$ is continuous on J. For example the function.

$$f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] [-1, 1] \text{ defined by}$$

$$f(x) = \sin x,$$

is both one-one and onto. Besides f is continuous on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Therefore, by Theorem 6, the function

$$f^{-1} : -11 \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \text{ defined by}$$

$$f^{-1}(x) = \sin -x$$

is continuous on [–1, 1].

Self Assessment

1. Give an example of the following:

   (i) A function which is nowhere continuous but its absolute value is everywhere continuous.

   (ii) A function which is continuous at one point only.

   (iii) A linear function which is continuous and satisfies the equation f(x + y) = f(x) + f(y).

   (iv) Two uniform continuous functions whose product is not uniformly continuous.

2. State whether the following are true or false:

   (i) A polynomial function is continuous at every point of its domain.

   (ii) A rational function is continuous at every point at which it is defined.

   (iii) If a function is continuous, then it is always uniformly continuous.

   (iv) The functions e^x and log x are inverse functions for x > 0 and both are continuous for each x > 0.

   (v) The functions cos x and $\cos^{-1} x$ are continuous for all real x.

   (vi) Every continuous function is bounded.

   (vii) A continuous function is always monotonic.

   (viii) The function sin x is monotonic as well as continuous for $s \in [0, 3]$

   (ix) The function cos x is continuous as well as monotonic for every $x \in R$.

   (x) The function $|x|$, $x \in R$ is continuous.

Notes

12.3 Summary

- In this unit you have been introduced to the properties of continuous functions on bounded closed intervals and you have seen the failure of these properties if the intervals are not bounded and closed. These properties have been studied. It has been proved that if a function f is continuous on a bounded and closed interval, then it is bounded and it also attains its bounds. In the same section we proved the Intermediate Value Theorem that is if f is continuous on an interval containing two points a and b, then f takes every value between f(a) and f(b). The notion of uniform continuity is discussed. We have proved that if a function f is uniformly continuous in a set A, then it is continuous in A. But converse is not true. It has been proved that if a function is continuous on a bounded and closed interval, then it is uniformly continuous in that interval. These properties fail if the intervals are not bounded and closed. This has been illustrated with a few examples.

12.4 Keywords

Bounded Function: A function f continuous on a bounded and closed interval [a, b] is necessarily a bounded function.

Boundedness: If f is a continuous function on the bounded closed interval [a, b] then there exists points x_1 and x_2 in [a, b] such that $f(x_1)$ S $f(x) \le f(x_2)$ for all $x \in [a, b]$.

Intermediate Value Theorem: Let f be a continuous function on an interval containing a and b. If K is any number between f(a) and f(b) then there is a number c, a ≤ c S b such that f(c) = K.

12.5 Review Questions

1. Find the limits of the following functions:

   (i) $f(x) = x \cos \frac{1}{x}$, $x \neq 0$, as $x \to 0$.

   (ii) $f(x) = \frac{|x|}{x}$, $x \neq 0$, as $x \to \infty$.

   (iii) $f(x) = \frac{\sin x}{x}$, $x \neq 0$, as $x \to \infty$.

2. For the following functions, find the limit, if it exists:

   (i) $f(x) = \frac{\sqrt{x} - \sqrt{b}}{x - b}$ for $x \neq b$ where $b > 0$, as $x \to b$

   (ii) $f(x) = \frac{1}{1 + e^{-1/x}}$ for $x \neq 0$, as $x \to 0$

   (iii) $f(x) = \begin{cases} 3 - x \text{ when } x \le 1 \\ 2x \text{ when } x > 1 \end{cases}$ as $x \to 1$.

3. Test whether or not the limit exists for the following:

   (i) $f(x) = \begin{cases} 3 - x \text{ when } x > 1 \\ 1 \text{ when } x = 1, \\ 2x \text{ when } x < 1 \end{cases}$ as $x \to 1$.

Notes

(ii) f(x) = $\dfrac{x^2-4}{x^2+4}$, x ∈ R, as x → 1.

(iii) f(x) = $\dfrac{\sqrt{4+x}-2}{x}$, x ≠ O as x → 0.

(iv) f(x) = $\dfrac{1}{x-1}\left(\dfrac{1}{x+3}-\dfrac{2}{3x+5}\right)$ as x → 1.

4. Discuss the continuity of the following functions at the points noted against each.

(i) f(x) = $\begin{cases} x^2 \text{ for } x \neq 1 \\ 0 \text{ for } x = 1 \end{cases}$ as x → 1.

(ii) f(x) = $\begin{cases} 1 \text{ for } 0 \leq x < 1 \\ 0 \text{ otherwise} \end{cases}$ as x → 1.

(iii) f(x) = $\dfrac{x^2-4}{x-1}$ when x ≠ 1.

f(1) = 2

as x → 1.

(iv) f(x) = $\begin{cases} (1+x)^{1/x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ as x → 1.

(v) f(x) = $\begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ as x → 1.

5. Show that the function f : R → R defined as

$\dfrac{1}{1+|x|}$

does not attain its infimum.

6. Show that the function f : R → R such that

f(x) = x is not bounded but is continuous in [1, ∞[.

7. Which of the following functions are uniformly continuous in the interval noted against each? Give reasons.

(i) f(x) = tan x, x ∈[0, π/4J

(ii) f(x) = $\dfrac{1}{x^2-3}$ on [1, 4].

Answers: Self Assessment

1. (i) $\begin{cases} f(x) = 1 & \text{if } x \text{ is rational} \\ = -1 & \text{if } x \text{ is irrational} \end{cases}$

(ii) $\begin{cases} f(x) = x & \text{if } x \text{ is rational} \\ = -x & \text{if } x \text{ is irrational} \end{cases}$

the only point of continuity is 0.

Notes

(iii) $f(x) = Cx, \ \forall x \in R$ where C is a fixed constant.

(iv) $f(x) = x, g(x) = \sin x, \ \forall x \in R$

Both f(x) and g(x) are uniformly continuous but their product

$f(x) g(x) = x \sin x$

is not uniformaly continuous on R.

2. (i) True (ii) True

(iii) False (iv) True

(v) True (vi) False

(vii) False (viii) True

(ix) False (x) True

12.6 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 13: Discontinuities and Monotonic Functions

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Discontinuous Functions
- Describe Classification of Discontinuities
- Explain Monotone Function
- Describe the Discontinuities of Monotone Functions
- Discuss the Discontinuities of Second Kind

Introduction

In mathematics, a monotonic function (or monotone function) is a function that preserves the given order. This concept first arose in calculus, and was later generalized to the more abstract setting of order theory. In calculus, a function f defined on a subset of the real numbers with real values is called monotonic (also monotonically increasing, increasing or non-decreasing), if for all x and y such that x d” y one has $f(x)$ d” $f(y)$, so f preserves the order. Likewise, a function is called monotonically decreasing (also decreasing or non-increasing) if, whenever x d” y, then $f(x)$ e” $f(y)$, so it *reverses* the order.

13.1 Discontinuous Functions

If a function fails to be continuous at a point c, then the function is called **discontinuous** at c, and c is called a **point of discontinuity**, or simply a discontinuity.

Notes

Task **Consider the following functions:**

1. $k(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if} \quad x \neq 3 \\ 1, & \text{if} \quad x = 3 \end{cases}$

2. $h(x) = \begin{cases} 1, & \text{if} \quad x > 0 \\ 0, & \text{If} \quad x = 0 \\ -1, & \text{if} \quad x < 0 \end{cases}$

3. $f(x) = \begin{cases} \sin(1/x), & \text{if} \quad x \neq 0 \\ 0, & \text{If} \quad x \neq 0 \end{cases}$

4. $g(x) = \begin{cases} 1, & \text{if} \quad x = \text{rational} \\ 0, & \text{If} \quad x = \text{irrational} \end{cases}$

Which of these functions, without proof, has a 'fake' discontinuity, a 'regular' discontinuity, or a 'difficult' discontinuity?

13.2 Classification of Discontinuities

Suppose f is a function with domain D and $c \in D$ is a point of discontinuity of f.

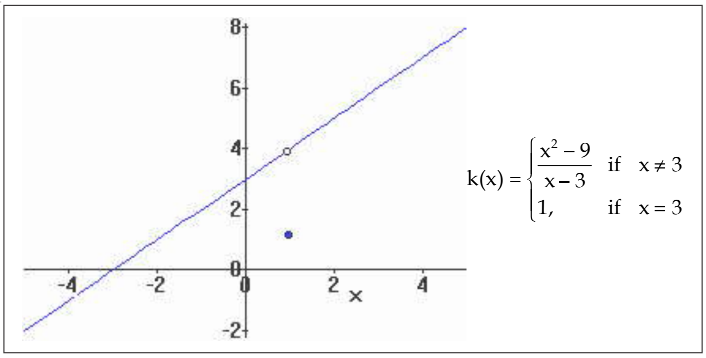
1. If $\lim_{x \to c} f(x)$ exists, then c is called removable discontinuity.

2. If $\lim_{x \to c} f(x)$ does not exist, but both $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist, then c is called a discontinuity of the first kind, or jump discontinuity.

3. If either $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ does not exist, then c is called a discontinuity of the second kind, or essential discontinuity.

Example: Prove that k(x) has a removable discontinuity at x = 3, and draw the graph of k(x).

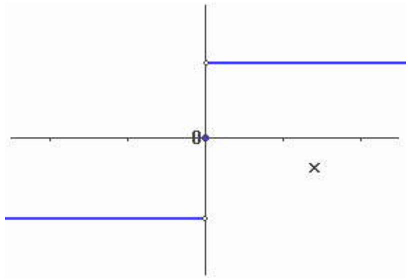
Solution:



We can easily check that the limit as *x* approaches *3* from the right and from the left is equal to *4*. Hence, the limit as *x* approaches *3* exists, and therefore the function has a removable discontinuity at $x = 3$. If we define $k(3) = 4$ instead of $k(3) = 1$ then the function in fact will be continuous on the real line

Example: Prove that $h(x)$ has a jump discontinuity at $x = 0$, and draw the graph of $h(x)$

Solution:

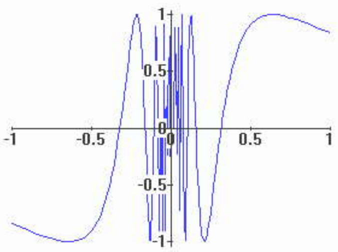


$$h(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{If } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

It is easy to see that the limit of h(x) as x approaches 0 from the left is –1, while the limit of h(x) as x approaches 0 from the right is +1. Hence, the left and right handed limits exist and are not equal, which makes x = 0 a jump discontinuity for this function.

Example: Prove that f(x) has a discontinuity of second kind at x = 0

Solution:



$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{If } x \neq 0 \end{cases}$$

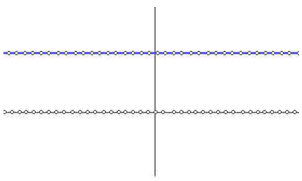
This function is more complicated. Consider the sequence $x_n = 1/(2n\pi)$. As n goes to infinity, the sequence converges to zero from the right. But $f(x_n) = \sin(2n\pi) = 0$ for all k. On the other hand, consider the sequence $x_n = 2/(2n + 1)$. Again, the sequence converges to zero from the right as n goes to infinity. But this time $f(x_n) = \sin((2n + 1)/2)$ which alternates between +1 and –1. Hence, this limit does not exist. Therefore, the limit of f(x) as x approaches zero from the right does not exist.

Since f(x) is an odd function, the same argument shows that the limit of f(x) as x approaches zero from the left does not exist.

Therefore, the function has an essential discontinuity at x = 0.

Example: What kind of discontinuity does the function g(x) have at every point (with proof).

Solution:



$$g(x) = \begin{cases} 1, & \text{if } x = \text{rational} \\ 0, & \text{If } x = \text{irrational} \end{cases}$$

This function is impossible to graph. The picture above is only a poor representation of the true graph. Nonetheless, take an arbitrary point x_0 on the real axis. We can find a sequence $\{x_n\}$ of rational points that converge to x_0 from the right. Then $g(x_n)$ converges to 1. But we can also find a sequence $\{x_n\}$ of irrational points converging to x_0 from the right. In that case $g(x_n)$ converges to 0. But that means that the limit of g(x) as x approaches x_0 from the right does not exist. The same argument, of course, works to show that the limit of g(x) as x approaches x_0 from the left does not exist. Hence, x_0 is an essential discontinuity for g(x).

It is clear that any function is either continuous at any given point in its domain, or it has a discontinuity of one of the above three kinds. It is also clear that removable discontinuities are 'fake' ones, since one only has to define $f(c) = \lim_{x \to c} f(x)$ and the function will be continuous at c.

Of the other two types of discontinuities, the one of second kind is hard. Fortunately, however, discontinuities of second kind are rare, as the following results will indicate.

13.3 Monotone Function

A function f is monotone increasing on (a, b) if $f(x) \leq f(y)$ whenever $x < y$. A function f is monotone decreasing on (a, b) if $f(x) \geq f(y)$ whenever $x < y$.

A function f is called monotone on (a, b) if it is either always monotone increasing or monotone decreasing.

Some basic applications and results

The following properties are true for a monotonic function $f : R \to R$:

- f has limits from the right and from the left at every point of its domain;
- f has a limit at infinity (either ∞ or $-\infty$) of either a real number, ∞, or $-\infty$.
- f can only have jump discontinuities;
- f can only have countably many discontinuities in its domain.

These properties are the reason why monotonic functions are useful in technical work in analysis. Two facts about these functions are:

- if f is a monotonic function defined on an interval I, then f is differentiable almost everywhere on I, i.e. the set of numbers x in I such that f is not differentiable in x has Lebesgue measure zero.
- if f is a monotonic function defined on an interval [a, b], then f is Riemann integrable.

An important application of monotonic functions is in probability theory. If *X* is a random variable, its cumulative distribution function

$$F_X(x) = \text{Prob}\ (X \leq x)$$

is a monotonically increasing function.

A function is *unimodal* if it is monotonically increasing up to some point (the *mode*) and then monotonically decreasing.

Notes

Note If f is increasing if -f is decreasing, and visa versa. Equivalently, f is increasing if

- $f(x)/f(y) \leq 1$ whenever $x < y$
- $f(x) - f(y) \leq 0$ whenever $x < y$

These inequalities are often easier to use in applications, since their left sides take a very nice and simple form. Next, we will determine what type of discontinuities monotone functions can possibly have. The proof of the next theorem, despite its surprising result, is not too bad.

13.4 Discontinuities of Monotone Functions

If f is a monotone function on an open interval (a, b), then any discontinuity that f may have in this interval is of the first kind.

If f is a monotone function on an interval [a, b], then f has at most countably many discontinuities.

Proof: Suppose, without loss of generality, that f is monotone increasing, and has a discontinuity at x_0. Take any sequence x_n that converges to x_0 from the left, i.e. x_n x_0. Then f(x_n) is a monotone increasing sequence of numbers that is bounded above by f(x_0). Therefore, it must have a limit. Since this is true for every sequence, the limit of f(x) as x approaches x_0 from the left exists. The same prove works for limits from the right.

Notes This proof is actually not quite correct. Can you see the mistake? Is it really true that if x_n converges to x_0 from the left then f(x_n) is necessarily increasing? Can you fix the proof so that it is correct?

As for the second statement, we again assume without loss of generality that f is monotone increasing. Define, at any point c, the jump of f at x = c as:

$$j(c) = \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$$

Note that j(c) is well-defined, since both one-sided limits exist by the first part of the theorem. Since f is increasing, the jumps j(c) are all non-negative. Note that the sum of all jumps can not exceed the number f(b) – f(a). Now let J(n) be the set of all jumps c where j(c) is greater than 1/n, and let J be the set of all jumps of the function in the interval [a, b]. Since the sum of jumps must be smaller than f(b) – f(a), the set J(n) is finite for all n. But then, since the union of all sets J(n) gives the set J, the number of jumps is a countable union of finite sets, and is thus countable.

This theorem also states that if a function wants to have a discontinuity of the second kind at a point x = c, then it can not be monotone in any neighbourhood of c.

13.5 Discontinuities of Second Kind

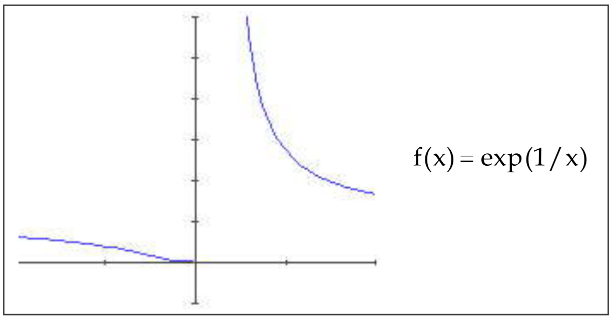
If f has a discontinuity of the second kind at x = c, then f must change from increasing to decreasing in every neighbourhood of c.

Proof: Suppose not, i.e. f has a discontinuity of the second kind at a point x = c, and there does exist some (small) neighbourhood of c where f, say, is always decreasing. But then f is a monotone function, and hence, by the previous theorem, can only have discontinuities of the first kind. Since that contradicts our assumption, we have proved the corollary.

Notes

In other words, f must look pretty bad if it has a discontinuity of the second kind.

Example: What kind of discontinuity does the function f(x) = exp(1/x) have at x = 0?

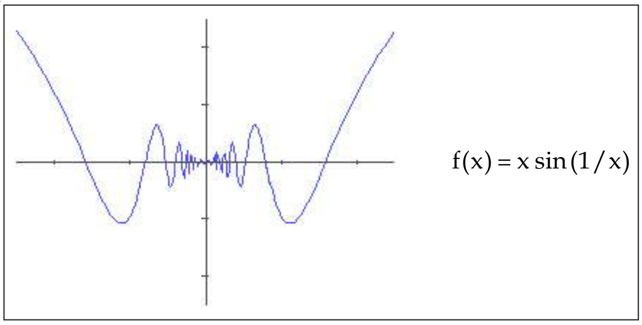


As x approaches zero from the right, 1/x approaches positive infinity. Therefore, the limit of f(x) as x approaches zero from the right is positive infinity.

As x approaches zero from the left, 1/x approaches negative infinity. Therefore, the limit of f(x) as x approaches zero from the left is zero.

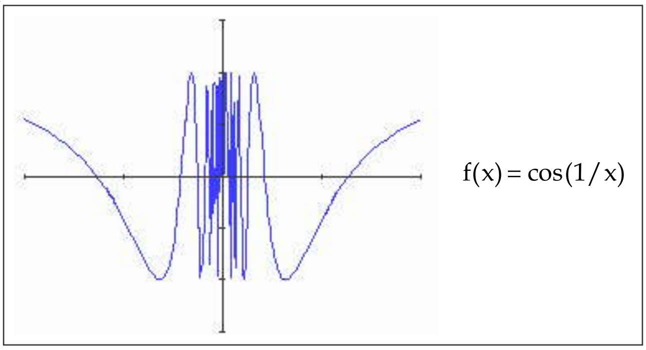
Since the right-handed limit fails to exist, the function has an essential discontinuity at zero.

Example: What kind of discontinuity does the function f(x) = x sin(1/x) have at x = 0?



Since $|x \sin(1/x)| < |x|$, we can see that the limit of f(x) as x approaches zero from either side is zero. Hence, the function has a removable discontinuity at zero. If we set f(0) = 0 then f(x) is continuous.

Example: What kind of discontinuity does the function f(x) = cos (1/x) have at x = 0?



Notes

By looking at sequences involving integer multiples of π or π/2 we can see that the limit of f(x) as x approaches zero from the right and from the left both do not exist. Hence, f(x) has an essential discontinuity at x = 0.

Self Assessment

Fill in the blanks:

1. The class of monotonic functions consists of both the ………………..
2. ……………….. have no discontinuities of second kind.
3. Let f be monotonic on (a; b), then the set of points of (a; b) at which f is ………………… at most countable.
4. A function *f*(*x*) is said to be ……………….. over an interval (*a*, *b*) if the derivatives of all orders of *f* are nonnegative at all points on the interval.
5. The term ……………….. can also possibly cause some confusion because it refers to a transformation by a strictly increasing function

13.6 Summary

- If a function fails to be continuous at a point c, then the function is called discontinuous at c, and c is called a point of discontinuity, or simply a discontinuity.
- In calculus, a function f defined on a subset of the real numbers with real values is called monotonic (also monotonically increasing, increasing or non-decreasing), if for all x and y such that x ≤ y one has f(x) ≤ f(y), so f preserves the order (see Figure 1).
- Suppose f is a function with domain D and ∈ D is a point of discontinuity of f.

  if $\lim_{x\to c} f(x)$ exists, then c is called removable discontinuity.

  if $\lim_{x\to c} f(x)$ does not exist, but both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exit, then c is called a discontinuity of the first kind, or jump discontinuity

  if either $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ does not exist, then c is called a discontinuity of the second kind, or essential discontinuity

  If f is a monotone function on an open interval (a, b), then any discontinuity that f may have in this interval is of the first kind.

  If f is a monotone function on an interval [a, b], then f has at most countably many discontinuities.

13.7 Keywords

Monotonic Transformation: The term monotonic transformation can also possibly cause some confusion because it refers to a transformation by a strictly increasing function.

Monotonically Decreasing: A function is called monotonically decreasing (also decreasing or non-increasing) if, whenever x ≤ y, then f(x) ≥ f(y), so it reverses the order.

Monotonic Function: In mathematics, a monotonic function (or monotone function) is a function that preserves the given order.

Notes

13.8 Review Questions

1. Define Discontinuous Functions.
2. Describe Classification of Discontinuities.
3. Explain Monotone Function.
4. Describe the Discontinuities of Monotone Functions.
5. Discuss the Discontinuities of Second Kind.

Answers: Self Assessment

1. Increasing and decreasing functions
2. Monotonic functions
3. Discontinuous
4. Absolutely monotonic
5. Monotonic transformation

13.9 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol : Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik : Mathematical Analysis.

H.L. Royden : Real Analysis, Ch. 3, 4.

Unit 14: Sequences and Series of Functions

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define sequence and series of functions
- Distinguish between the pointwise and uniform convergence of sequences and series of functions
- Know the relationship of uniform convergence with the notions of continuity, differentiability, and integrability

Introduction

In earlier unit, we have studied about, convergence of the infinite series of real numbers. In this unit, we want to discuss sequences and series whose members are functions defined on a subset of the set of real numbers. Such sequences or series are known as sequences or series of real functions. You will be introduced to the concepts of pointwise and uniform convergence of sequences and series of functions. Whenever they are convergent, their limit is a function called limit function. The question arises whether the properties of continuity, differentiability, integrability of the members of a sequence or series of functions are preserved by the limit function. We shall discuss this question also in this unit and show that these properties are preserved by the Uniform convergence and not by the pointwise convergence.

14.1 Sequences of Functions

As you have studied that a sequence is a function from the set N of natural numbers to a set B. In that unit, sequences of real numbers have been considered in detail. You may recall that for sequences of real numbers, the set B is a sub-set of real numbers. If the set B is the set of real functions defined on a sub-set A of R, we get a sequence called sequence of functions. We define it in the following way:

Definition 1: Sequence of Functions

Let A be a non-empty sub-set of R and let B be the set of all real functions each defined on A. A mapping from the set N of natural numbers to the set B of real functions is called a sequence of functions.

Notes

The sequences of functions are denoted by (f_n), (g) etc., it (f_n) is a sequence of functions defined on A, then its members f_1, f_2, f_3,.... are real functions with domain as the set A. These are also called the terms of the sequence (f_n).

Example: Let $f_n(x) = x^n$, n = 1, 2, 3,...., where $x \in A = \{x : 0 \leq x \leq I\}$. Then (f_n) is a sequence of functions defined on the closed interval [0, 1].

Similarly consider (f_n), where $f_n(x) = \sin_n x$, n = 1, 2, 3,.... $x \in R$. Then $\{f_n\}$, is a sequence of functions defined on the set R of real numbers.

Suppose (f_n) is a sequence of functions defined on a set A and we fix a point x of A, then the sequence $(f_n(x))$, formed by the values of the members of (f_n), is a sequence of real numbers. This sequence of real numbers may be convergent or divergent. For example suppose that $f_n(x) = x^n$, $x \in [-1, 1]$. If we consider the point $x = \frac{1}{2}$, then the sequence $(f_n(x))$ is $((\frac{1}{2})^n)$ which converges to 0. If we take the point x = –1, the sequence $(f_n(x))$ is the constant sequence (1, 1, 1.....) which converges to 1. If x = –1, the sequence (f, (x)) is (–1, I, –1, 1,....) which is divergent.

Thus, you have seen that the sequence $(f_n(x))$ may or may not be convergent. If for a sequence (f_n) of functions defined on a set A, the sequence of numbers $(f_n(x))$ converges for each x in A, we get a function f with domain A whose value f(x) at any point x of A is $\lim_{n\to\infty} fn(x)$. In this case (f_n) is said to be pointwise convergent to f. We define it in the following way:

Definition 2: Pointwise Convergence

A sequence of functions (f_n) defined on a set A is said to be convergent pointwise to f if for each x in A, we have $\lim f_n(x) = f(x)$. Generally, we write $f_n \to f$ (pointwise) on A.

or $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise on A. Also f is called pointwise limit or limit function of (f_n) on A.

Equivalently, we say that a sequence $\{f_n\}$ converges to f pointwise on the set A if, for each $\varepsilon > 0$ and each $x \in A$, there exists a positive integer in depending both on $\in$ and x such that

$$|f_n(x) - f(x)| < \varepsilon, \text{ whenever } n \geq m,$$

Now we consider some examples.

Example: Show that the sequence (f_n) where $f_n(x) = x^n$, $x \in [0, 1]$ is pointwise convergent. Also find the limit.

Solution: If $0 \leq x < 1$, then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$.

If x = 1, then $\lim_{n\to\infty} f(x) = \lim_{n\to\infty} 1 = 1$.

Thus (f,) is pointwise convergent to the limit function f where f(x) = 0 for $0 \leq x \leq 1$ and f(x) = 1 for x = 1.

> *Task* Show that the sequence of functions (f_n) where $f_n(x) = x^n$, for $x \in [-1, 1]$ is not pointwise convergent.

Notes

Example: Define the function f_n, n = 1, 2,....., as follows:

$$f_n(x) = \begin{cases} 0, & \text{if } x = 0 \\ 2n^2 x, & \text{if } 0 < x < \dfrac{1}{2n} \\ 2n - 2n^2 x, & \text{if } \dfrac{1}{2n} \le x \le \dfrac{1}{2n} \\ 0, & \text{if } \dfrac{1}{n} < x \le 1 \end{cases}$$

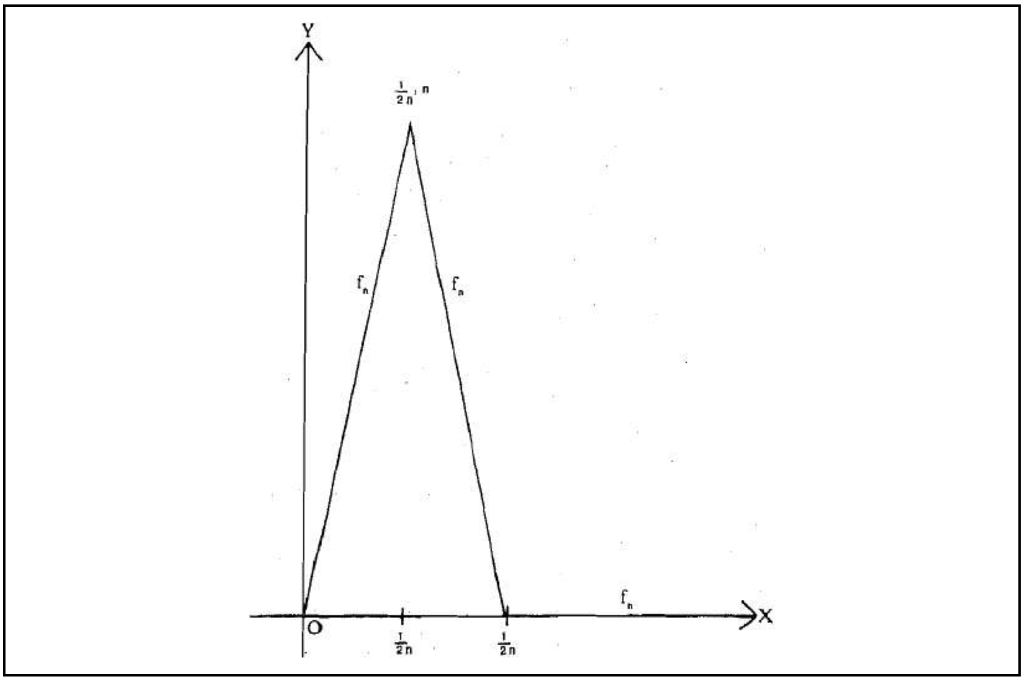
Show that the sequence (f_n) is pointwise convergent.

Solution: The graph of function f_n looks as shown in the Figure.

When x = 0, $f_n(x) = 0$ for n = 1, 2,.....

Therefore, the sequence $(f_n(0))$ tends to 0.

If x is fixed such that 0 < XIP; then choose m large enough so that $\frac{1}{m}$ < x or m > $\frac{1}{X}$. Then $f_m(x) = f_{m+1}(x) = ... = 0$. Consequently the sequence $(f_n(x))$ –> 0 as n – 3 ∞.



Thus, we see that $f_n(x)$ tends to 0 for every x in $0 < x \le 1$ and consequently (f_n) tends pointwise to f where f(x) = 0 $\forall x \in [0, I]$.

Example: Consider the sequence of functions f_n defined by $f_n(x) = \cos nx$ for $-\infty < x < \infty$ i.e. $x \in R$. Show that the sequence is not convergent pointwise for every real x.

Solution: If x = π/4 then $(f_n(x))$ is the sequence

(1/√2, 0, – 1/√2, – 1, – 1/√2, 0,.....) which is not convergent.

> *Task* Show that the sequence (f_n) where $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$, $x \in R$, is pointwise convergent. Also find the pointwise limit.

If the sequence of functions (f_n) converges pointwise to a function f on a subset A of R, then the following question arises: "If each member of (f_n) is continuous, differentiable or integrable, is the limit function f also continuous; differentiable or integrable?". The answer is no if the convergence is only pointwise. For instance each of the functions f_n is continuous (in fact uniformly continuous) but the sequence of these functions converges to a limit function f(x)

$$f(x) = \begin{cases} 0 & \text{for } \; 0 \le x < 1 \\ 1, & \text{for } \; x = 1 \end{cases}$$

which is not continuous. Thus, the pointwise convergence does not preserve the property of continuity. To ensure the passage of the properties of continuity, differentiability or integrability to the limit function, we need the notion of uniform convergence which we introduce in the next section.

14.2 Uniform Convergence

From the definition of the convergence of the sequence or real numbers, it follows that the sequences (f_n) of functions converges pointwise to the function f on A if and only if for each $x \in A$ and for every number $\in > 0$, there exists a positive integer m such that

$$|f_n(x) - f(x)| < \in \text{ whenever } n \ge m.$$

Clearly for a given sequence (f,,) of functions, this m will, in general, depend on the given $\in$ and the point x under consideration. Therefore it is, sometimes, written as m $(\in, x)$. The following example illustrates this point.

Example: Define $f_n(x) = \dfrac{x}{n}$ for $< x < \infty$.

For each fixed x the sequence $(f_n(x))$ clearly converges to zero. For a given $\in > 0$, we must show the existence of an m, such that for all $n \ge m$,

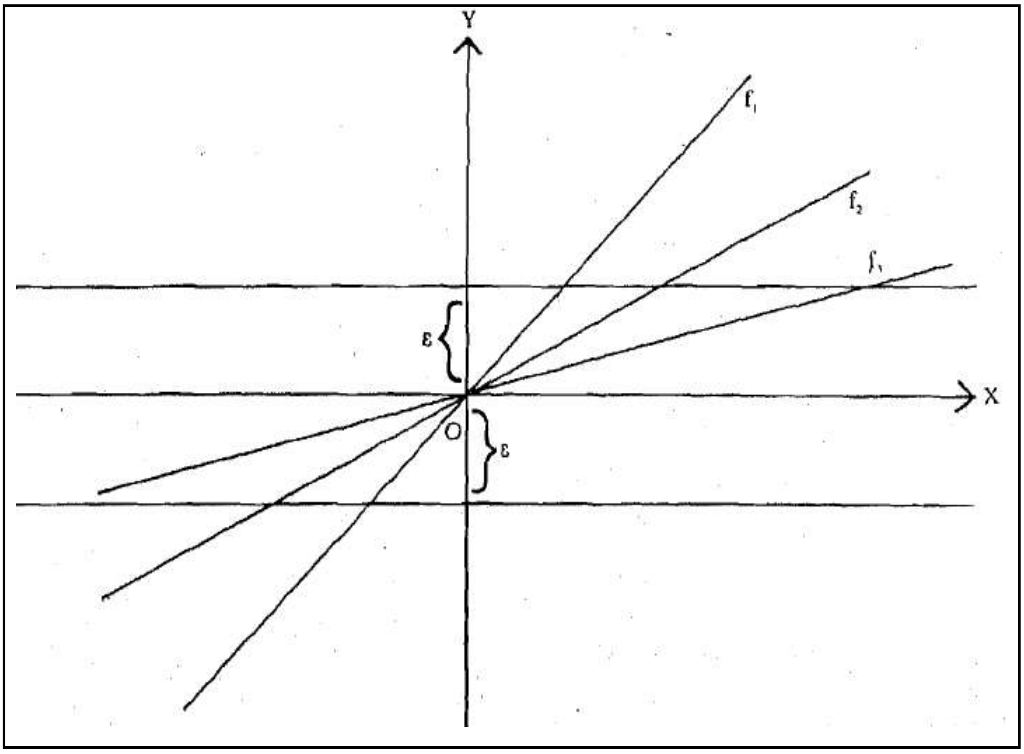
$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{|x|}{n} < \in$$

This can be achieved by choosing $m = \left[\dfrac{|x|}{\in}\right] + 1$ where $\left[\dfrac{|x|}{\in}\right]$ denotes the integral part of $\dfrac{|x|}{\in}$ (i.e. the integer m is next to $|x|$ in the real line). Clearly this choice of m depends both on $\in$ and x.

For example, let $\varepsilon = \dfrac{1}{10^3}$. If $x = \dfrac{1}{10^3}$ then $\dfrac{|x|}{\in} = 1$ and, so, m can be chosen to be 2. If x = 1, then $|x| = 10^3$ and, so, m should be larger than 10^6. Note that it is impossible to find an m that serve for all x. For, if it were, then $\dfrac{|x|}{m} > E$, for all x,

Consequently $|x|$ is smaller than ε m, which is not possible. Geometrically, the f_n's can be described as shown in the Figure.

Notes



By putting $y = f_n(x)$, we see that $y = \frac{1}{?} x$ is the line with slope $\frac{1}{n}$. f_n is the line y = x with slope 1, f_2 is the line with slope $\frac{1}{2}$ and so on. As n tends to ∞, the lines approach the X-axis. But if we take any strip of breadth 2ε around X-axis, parallel to the X-axis as shown in the figure, it is impossible to find a stage m such that all the lines after the stage m, i.e. f_m, f_{m+1},.... lie entirely in this strip.

If it is possible to find m which depends only on ε but is independent of the point x under consideration, we say that (f_n) is uniformly convergent to f. We define uniform convergence as follows:

Definition 3: Uniform Convergence

A sequence of functions (f_n) defined on a set A is said to be uniformly convergent to a function f on A if given a number ∈ > 0, there exists a positive integer m depending only on ∈ such that

$f_n(x) - f(x) | < \in$ for $n \geq m$ and $\forall x \in A$.

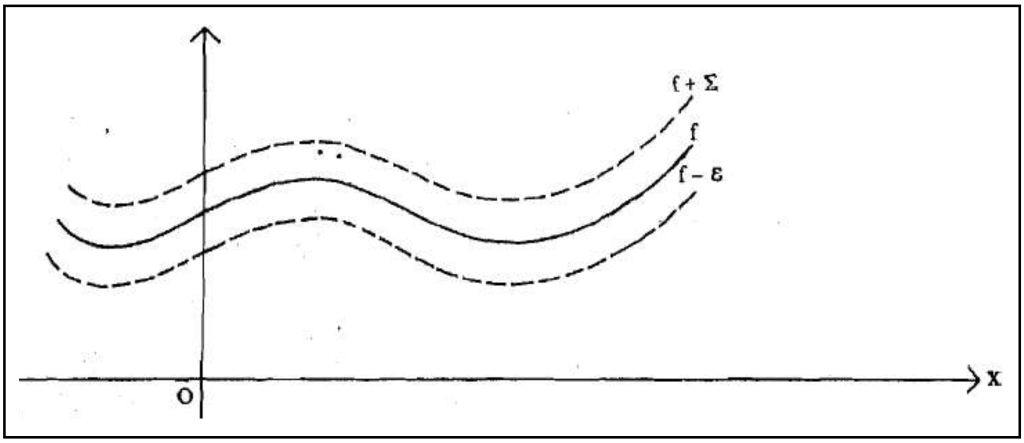
We write it as $f_n \to f$ uniformly on A or lim $f_n(x) = f(x)$ uniformly on A. Also f is called the uniform limit of f on A.

Note that if $f_n \to f$ uniformly on the set A, for a given ∈ > 0, there exists m such that

$f(x) - \in < f_n(x) < f(x) + \in$

for all $x \in A$ and $n \geq m$. In other words, for $n \geq m$, the graph of f_n lies in the strip between the graphs of $f - \in$ and $f + \in$. As shown in the figure below, the graphs of f_n Tor $n \geq m$ will all lie between the dotted lines.

Notes



From, the definition of uniform convergence, it follows that uniform convergence of a sequence of functions implies its pointwise convergence and uniform limit is equal to the pointwise limit. We will show below by suitable examples that the converse is not true.

Example: Show that the sequence (f_n) where $f_n(x) = \frac{x}{n}$, $x \in R$ is pointwise but not uniformly convergent in R.

Solution: You have seen that (f_n) is pointwise convergent to f where $f(x) = 0 \ \forall x \in R$. In the same example, at the end, it is remarked that given $\in > 0$, it is not possible to find a positive integer m such that $\frac{|x|}{n} < \in$ for $n \geq m$ and $\forall x \in R$ i.e., $|f_n(x) - f(x)| < \in$ for $n \geq m$ and $\forall x \in R$. Consequently (f_n) is not uniformly convergent in R.

Example: Show that the sequence (f_n) where $f_n(x) = x^n$ is convergent pointwise but not uniformly on [0, 1].

Solution: You have been shown that (f_n) is pointwise convergent to, f on [0, l] where

$$f(x) = 0 \ \forall x \in [0, 1[\text{ and } f(l) = l$$

Let $\in > 0$ be any number. For x = 0 or x = 1, $|f_n(x) - f(x)| < \in$ for $n \geq 1$.

For $0 < x < l$, $|f_n(x) - f(x)| < \in$ if $x^n < \in$ i.e. $n \log x < \log \in$ i.e. $n > \frac{\log \in}{\log x}$.

since log x is negative for 0 < x < 1. If we choose $m = \left[\frac{\log \in}{\log x}\right] + 1$, then $|f_n(x) - f(x)| < \in$ for $n \in \geq m$.

Clearly m depends upon $\in$ and x.

We will now prove that the convergence is not uniform by showing that it is not possible to find an m independent of x.

Let us suppose that $0 < \in < l$. If there exists m independent of x in [0, 1] so that

$$< |f_n(x) - f(x)| < \in \text{ for all } n \geq m,$$

then $x'' < \in$ for all $n \geq m$, whatever may be x in 0 < x < 1.

If the same m serves for all x for a given $\in > 0$ then $x^m < \in$ for all x, $0 < x < 1$. This implies that $m > \frac{\log \in}{\log x}$ (since log x is negative). This is not possible since log x decreases to zero as x tends to 1 and so $\log \in / \log x$ is unbounded.

Thus we have shown that the sequence (f_n) does not converge to the function f uniformly in [0, 1] even though it converges pointwise.

Example: Show that the sequence (g_r) where $g_n(x) = \frac{x}{1+nx}$, $x \in [0, \infty[$ is uniformly convergent in $[0, \infty[$.

Solution: $\lim_{n\to\infty} g_n(x) = 0$ for all x in the interval $[0, \infty[$. Thus (g_n) is pointwise converge of where $f(x) = 0 \ \forall x \in [0, \infty[$.

Now $|g_n(x) - f(x)| = \frac{x}{1+nx} < \frac{1}{n}$ for all x in $[0, \infty[$.

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, therefore given $\in > 0$, there exists a positive integer m such that $\frac{1}{n} < \in$ for $n \geq m$

Thus m depends only on $\in$. Therefore,

$$|g_n(x) - f(x)| < \in \text{ for } n \geq m \text{ and } \forall x \in [0, \infty[.$$

Therefore $(g_r) \to f$ uniformly in $[0, \infty[$.

Just as you have studied Cauchy's Criterion for convergence of sequence of real numbers, we have Cauchy's Criterion for uniform convergence of sequence of functions which we now state and prove.

Theorem 1: Cauchy's Principle of Uniform Convergence

The necessary and sufficient condition for a sequence of functions (f_n) defined on A to converge uniformly on A is that for every $\in > 0$, there exists a positive integer m such that

$$|f_n(x) - f_k(x)| < \in \text{ for } n > k \geq m \text{ and } \forall x \in A$$

Proof: Condition is necessary. It is given that (f_n) is uniformly convergent on A.

Let f_n – f uniformly on A. Then given $\in > 0$, there exists a positive integer m such that

$$|f_n(x) - f(x)| < \in/2 \text{ for } n \geq m \text{ and } \forall x \in A.$$

$$\therefore |f_n(x) - f_k(x)| = |f_n(x) - f(x)| + |f(x) - f_k(x)|$$

$$< |f_n(x) - f(x)| + |f(x) - f_k(x)| \text{ (By triangular inequality)}$$

$$< \frac{\in}{2} + \frac{\in}{2} + \text{ for } n > k \geq m \text{ and } \forall x \in A$$

This proves the necessary part. Now we prove the sufficient part.

Condition is sufficient: It is given that for every $\in > 0$, there exists a positive integer m such that $|f_n(x) - f_k(x)| < \in$ for $n > k \geq m$ and for all x in A. But by Cauchy's principle of convergence of sequence of real numbers, for each fixed point x of A, the sequence of numbers $(f_n(x))$ converges. In other words, (f_n) is pointwise convergent say to f on A. Now for each $\in > 0$, there exists a positive integer m such that

Notes

$|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for $n > k \geq m$.

Fix k and let $n \to \infty$. Then $f_n(x) \to f(x)$ and we get

$|f(x) - f_k(x)| < \frac{\epsilon}{2}$ i.e., $|f_k(x) - f(x)| < \epsilon$.

This is true for $k \geq m$ and for all x in A. This shows that (f) is uniformly convergent to f on A, which proves the sufficient part.

As remarked in the introduction, uniform convergence is the form of convergence of the sequence of function (f_n) which preserves the continuity, differentiability and integrability of each term f_n of the sequence when passing to the limit function f. In other words if each member of the sequence of functions (f_n) defined on a set A is continuous on A, then the limit function f is also continuous provided the convergence is uniform. The result may not be true if the convergence is only pointwise. Similar results hold for the differentiability and integrability of the limit function f. Before giving the theorems in which these results are proved, we discuss some examples to illustrate the results.

Example: Discuss for continuity the convergence of a sequence of functions (f), where $f(x) = 1 - |1 - x^2|^n$, $x \in \{x \mid |1 - x^2| \leq 1\} = [-\sqrt{2}, \sqrt{2}]$.

Solution: Here $\lim_{n\to\infty} f(x) = \begin{cases} 1, & \text{when } |1-x^2|<1 \\ 0, & \text{when } 1-x^2|=1 \text{ i.e. } x=0\pm\sqrt{2} \end{cases}$

Therefore the sequence (f_n) is pointwise convergent to f where

$$f(x) = \begin{cases} 1, & \text{when } |-x^2|<1 \\ 0, & \text{when } |1-x^2|=1 \end{cases}$$

Now each member of the sequence (f,) is continuous at 0 but f is discontinuous at 0. Here (f_n) is not uniformly convergent in $[-\sqrt{2}, \sqrt{2}]$ as shown below.

Suppose (f_n) is uniformly convergent in $[-\sqrt{2}, \sqrt{2}]$, so that f is its uniform limit.

Taking $\epsilon = \frac{1}{2}$, there exists an integer m such that

$$f(x) = < \frac{1}{2} \text{ for } n \geq m \text{ and } \forall x \in [\sqrt{2}, \sqrt{2}].$$

in particular $|f_m(x) - f(x)| < \frac{1}{2}$ for $x \in [\sqrt{2}, \sqrt{2}]$

Now $|f_m(x) - f(x)| = \begin{cases} |1-x^2|^{m} & \text{when } |1-x^2|<1 \\ 0 & \text{when } |1-x^2|=1 \end{cases}$

Since $\lim_{x\to 0} |1 - x^2|^m = 1$, $\exists$ a + v no. δ such that

$$|1 - x^2|^m - 1| < 1/4 \text{ for } 0 < |x| < \delta$$

$$\text{i.e. } 3/4 < |1 - x^2|^m < 5/4 \text{ for } |x| < \delta$$

So $(1 - x^2|^m > \frac{1}{2}$ for $|x| < \delta$ which is a contradiction.

Consequently (f_n) is not uniformly convergent in $[-\sqrt{2}, \sqrt{2}]$.

Example: Discuss, for continuity, the convergence of the sequence $((f_n)$ where

$$f_n(x) = \frac{x}{1+nx} \quad x \in [0, \infty[.$$

Solution: As you have seen that $(f_n) \to f$ uniformly in $[0, \infty[$ where $f(x) = 0$, $x \in [0, \infty[$.

Here each f_n is continuous in $[0, \infty[$ and also the uniform limit is continuous in $[0, \infty[$.

Example: Discuss for differentiability the sequence (f_n) where

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \quad \forall\, x \in [0, \infty[.$$

Solution: Here $(f_n) \to f$ uniformly where $f(x) = 0 \;\; \forall\, x \in R$. You can see that each f_n and fare differentiable in R and

$$f_n'(x) = \sqrt{n} \cos nx \text{ and } f'(x) = 0 \quad \forall\, x \in R.$$

$$f_n'(0) = \sqrt{n} \to \infty \text{ whereas } f'(0) = 0$$

$$\lim_{n\to\infty} f'_n(0) \neq f'(0)$$

i.e. limit of the derivatives is not equal to the derivative of the limit.

As you will see in the theorem for the differentiability of f and the equality of the limit of the derivatives and the derivative of the limit, we require the uniform convergence of the sequence (f_n).

Example: Discuss for integrability the sequence (f_n) where

$$f(x) = n\,x\,e^{-nx^2}, \; x \in [0, 1],$$

Solution: If x = 0, then $f_n(0) = 0$

and $\lim f_n(0) = 0$. If $x \neq 0$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{nx}{e^{nx^2}}$ which is of the form $\frac{\infty}{\infty}$.

Applying L' Hopital's Rule, we have

$$\lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{x}{2nx\,e^{nx^2}} = 0$$

So $(f_n) \to f$, pointwise, where $f(x) = 0$, $\forall\, x \in [0, 1]$

You may find that $\int_0^1 f_n(x)\,dx = \frac{1}{2}(1 - e^{-n})$ and $\int_0^1 f(x)\,dx = 0$

Notes

Therefore, $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \frac{1}{2} \neq \int_0^1 f(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 \lim_{n\to\infty}(f_n(x))dx$. That is, the integral of the limit is not equal to be limit of the sequence of integrals. In fact, (f_n) is not uniformly convergent to f in [0, 1]. This we prove by the contradiction method. If possible, let the sequence be uniformly convergent in [0, 1]. Then, for $\in = \frac{1}{4}$, there exists a positive integer m such that $|f(x) - f(x)| < \frac{1}{4}$, for $n \geq m$ and $\forall x \in [0, 1]$.

i.e., $\frac{nx}{e^{nx^2}} < \frac{1}{4}$, for $n \geq m$ and $V x \in [0, 1]$

Choose a positive integer $M \geq m$ such that $\frac{1}{M} \in [0, 1]$,

Take n = M and x $\frac{1}{\sqrt{M}}$. We get

$$\frac{1}{\sqrt{M}} < \frac{1}{4} \text{ i.e., } M < \frac{e^2}{16} < 1.$$

which is a contradiction. Hence (f) is not uniformly convergent in [0, 1].

Now we give the theorems without proof which relate uniform convergence with continuity, differentiability and integrability of the limit function of a sequence of functions.

Theorem 2: Uniform Convergence and Continuity

If (f_n) be a sequence of continuous functions defined on [a, b] and $(f_n) \to f$ uniformly on [a, b], then f is continuous on [a, b].

Theorem 3: Uniform Convergence and Differentiation

Let (f_n) be a sequence of functions, each differentiable on [a, b] such that $(f_n(x_0))$ converges for some point x_0 of [a, b]. If (f_n) converges uniformly on [a, b] then (f_n) converges uniformly on [a, b] to a function f such that

$$f'(x) = \lim_{n\to\infty} f'_n(x);\ x \in [a, b].$$

Theorem 4: Uniform Convergence and Integration

If a sequence (f_n) converges uniformly to f on [a, b] and each function f_n is integrable on [a, b], then f is integrable on [a, b] and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx$$

14.3 Series of Functions

Just as we have studied series of real numbers, we can study series formed by a sequence of functions defined on a given set A. The ideas of pointwise convergence and uniform convergence of sequence of functions can be extended to series of functions.

Notes

Definition 4: Series of Functions

A series of the forth $f_1 + f_2 + f_3 + ... + f_n + ...$ where the f_n are real functions defined on a given set ACR is called a series of functions and is denoted by $\sum_{n=1}^{\infty} f_n$. The function f_n is called nth term of the series.

For each x in A, $f_1(x) + f_2(x) + f_3(x) + ... +$ is a series of real numbers. We put $S_n(x) = \sum_{k=1}^{n} f_k(x)$. Then we get a sequence (S,) of real functions defined on A. We say that the given series $f_1 + f_1 + ... + f_n + ...$ of functions converges to a function pointwise if the sequence (S„) associated to the given series of functions converges pointwise to the function f. i.e. (S,, (x)) converges to f(x) for every x in A.

We also say that f is the pointwise sum of the series Σf_n on A.

If the sequence (S_n) of functions converges uniformly to the function f, then we say that the given series $f_1 + f_2 + ... + f_n + ...$, of functions converges uniformly to the function f on A and f is called uniform sum of $\sum_{i=1}^{\infty} f_n$ on A. The function S_n is called the sum of n terms of the given series or the n partial sum of the series and the sequence (S_n) is called the sequence of partial sums of the series $\sum_{i=1}^{\infty} f_n$. To make the ideas clear, we consider some examples.

Example: Let $f_n(x) = x^{n-1}$ where $x_0 = 1$ and $-r \le x \le r$ where $0 < r < 1$. Then the associated series is $1 + x + x^2 + ...$

In this case, $S_n(x) = 1, + x + x^2 + ... + x^{n-1}$. It is clear that $S_n(x) = \frac{1-x^n}{1-x}$.

This sequence (S, (x)) of functions is easily seen to converge pointwise to the function $f(x) = \frac{1}{I-x}$, since $x^n \to 0$ as $n \to \infty$, since $|x| < r < I$ but the convergence is not uniform as shown below:

Let $\varepsilon > 0$ be given.

$$|S_n(x) - f(x)| = \frac{|x|^n}{|1-x|} \le \frac{r^n}{1-r} \text{ if } r^n < \varepsilon(1-r)$$

$$\text{i.e. } n > \frac{\log(\in(1-r))}{\log r}$$

$$\text{If } m = \left[\frac{\log(\in(1-r))}{\log r}\right] = 1, \text{ then}$$

$|s,(x) - f(x)| < \varepsilon$ if $n \ge m$ and for $-r \le x \le r$.

Therefore (S_n) converges uniformly in [–r, r]. Thus the geometric series $1 + x + x^2 + ...$ converges uniformly in [–r, r] to the sum function $f(x) = \frac{I}{1-x}$.

Notes

Example: Let $f_n(x) = n\,x\,e^{-nx^2} - (n-1)\,x\,e^{-(n-1)x^2}$, $x \in [0, 1]$.

Consider the series $\sum_{k=1}^{n} f_n(x)$.

In this case $S_n(x) = \sum_{k=1}^{n} (k\,x^{-kx^2} - (k-1)\,x\,e^{-(k-1)x^2}) = n\,x\,e^{-nx^2}$

As you have seen that this sequence (S_n) is pointwise but not uniformly convergent to the function f where f(x) = 0, $x \in (0, 1)$. Thus the series $\Sigma\, f_n(x)$ is pointwise convergent but not uniformly to the function f where f(x) = 0, $x \in [0, 1]$.

There is a very useful method to test the uniform convergence of a series of functions. In this method, we relate the terms of the series with those of a series with constant terms. This method is popularly called Weierstrass's M-test given by the German mathematician K.W.T. Weierstrass (1815-1 897). We state this test in the form of the following theorem (without proof) and illustrate the method by an example.

Theorem 5: Weierstrass M-Test

Let Σf_n be a series of functions defined on a subset A of R and let (M_n) be a sequence of real numbers such that ΣM_n is convergent and $|f_n(x)| \le M_n$, $\forall$ n and $\forall\, x \in A$. Then Σf_n is uniformly and absolutely convergent on A.

Example: Test the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{x}{n^2(n+1)}$

Solution: Since $|f_n(x)| = \frac{x}{n^2(n+1)} \le \frac{k}{n^3}, \forall n$ and $\in [0, k]$.

Now the series $\Sigma M_n = k\,\Sigma\,\frac{k}{n^3}$ is known to be convergent, by p-test.

Therefore, by Weierstrass M-test, the given series is uniformly convergent in the set [0, k].

Task Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^4 + x^2}$ converges uniformly, $\forall\, x \in R$.

Self Assessment

Fill in the blanks:

1. A sequence of functions (f_n) defined on a set A is said to be convergent pointwise to f if for each x in A, we have given fn (x) = f(x). Generally, we write .... on A.

2. A sequence of functions (f_n) defined on a set A is said to be uniformly convergent to a function f on A if given a number $\in > 0$, there exists a positive integer m depending only on $\in$ such that ....

3. If (f_n) be a sequence of continuous functions defined on [a, b] and $(f_n) \to f$ uniformly on [a, b], then f is continuous on [a, b] is known as ....

4. Let (f_n) be a sequence of functions, each differentiable on [a, b] such that $(f_n(x_0))$ converges for some point x_0 of [a, b]. If (f_n) converges uniformly on [a, b] then (f_n) converges uniformly on [a, b] to a function f such that ....

5. If a sequence (f_n) converges uniformly to f on [a, b] and each function f_n is integrable on [a, b], then f is integrable on [a, b] and ....

14.4 Summary

- In this unit you have learnt how to discuss the pointwise and uniform convergence of sequences and series of functions. Sequence of functions is defined and pointwise convergence of the sequence of functions has been discussed. We say that a sequence of functions (f_n) is pointwise convergent to f on a set A if given a number e > 0, there is a positive integer m such that

  $|f_n(x) - f(x)| < \varepsilon$ for $n \geq m, x \in A$.

  m in general depends on ε and the point x under consideration. If it is possible to find m which depends only on s and not the point x under consideration, then (f_n) is said to be uniformly convergent are f on A. Cauchy's criteria for uniform convergence are discussed. Also in this section you have seen that if the sequence of functions (f_n) is uniformly convergent to a function f on [a, b] and each f_n' is continuous or integrable, then f is also continuous or integrable on [a,b]. Further it has been discussed that if (f_n) is a sequence of functions, differentiable on [a,b] such that $(f_n(x,))$ converges for some point x_0 of [a, b] and if (f_n) converges uniformly on [a, b], then (f_n) converges uniformly to a differentiable function f such that $f'(x) = \lim_{n\to\infty} f'_n(x)$; $x \in [a, b]$.

- Finally pointwise and uniform convergence of series of functions is given. The series of functions is said to be pointwise or uniformly convergent on a set A according as the sequence of partial sums (s_n) of the series is pointwise or uniformly convergent on A.

14.5 Keywords

Uniform Convergence and Continuity: If (f_n) be a sequence of continuous functions defined on [a, b] and $(f_n) \to f$ uniformly on [a, b], then f is continuous on [a, b].

Uniform Convergence and Differentiation: Let (f_n) be a sequence of functions, each differentiable on [a, b] such that $(f_n(x_0))$ converges for some point x_0 of [a, b]. If (f_n) converges uniformly on [a, b] then (f_n) converges uniformly on [a, b] to a function f such that

$$f'(x) = \lim_{n\to\infty} f'_n(x);\ x \in [a, b].$$

Uniform Convergence and Integration: If a sequence (f_n) converges uniformly to f on [a, b] and each function f_n is integrable on [a, b], then f is integrable on [a, b] and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

14.6 Review Questions

1. Examine which of the following sequences of functions converge pointwise

   (i) $f_n(x) = \sin nx, -\infty < x < +\infty$

   (ii) $f_n(x) = \dfrac{mx}{1+n^2x^2}, -\infty < x < +\infty$

Notes

2. Test the uniform convergence of the following sequence of functions in the specified domains

   (i) $f_n(x) = \frac{1}{nx}$ in $0 < x < \infty$

   (ii) $f_n(x) = \frac{nx}{1+n^2x^2}$, $-\infty < < x < \infty$

   (iii) $f_n(x) = \frac{x^n}{1+x^n}$, $0 \le x \le 1$

   (iv) $f_n(x) = \frac{1}{n}$, $0 \le x < \infty$

3. Show that the limit function of the sequence (f,) where $(f_n)(x) = \frac{x}{n}$, $x \in R$, is continuous in R while (f,,) is not uniformly convergent.

4. Show that for the sequence (f,) where $(f_n)(x) = nx(1 - x^2)^n$, $x \in [0, 1]$, the integral of the limit is not equal to the limit of the sequence of integrals.

5. Show that the series

   $\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + ...$ is uniformly convergent in $]k, \infty[$ where k is a positive number.

6. Show that the series $X\frac{x}{n(n+1)}$ is uniformly convergent in [0, k] where k is any positive number but it does not converge uniformly in $[0, \infty]$.

Answers: Self Assessment

1. $f_n \to f$ (pointwise)
2. $f_n(x) - f(x)\,| < \in$ for $n \ge m$ and $\forall\, x \in A$.
3. Convergence and Differentiation
4. $f'(x) = \lim_{n\to\infty} f'_n(x)$; $x \in [a, b]$.
5. $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$

14.7 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 15: Uniform Convergence of Functions

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define uniform convergence
- Explain the testing pointwise and uniform convergence
- Discuss the covers and subcovers
- Explain Dini's theorem

Introduction

In earlier unit as you all studied about sequences in metric spaces. This unit will explain that pointwise convergence of a sequence of functions was easy to define, but was too simplistic of a concept. We would prefer a type of convergence that preserves at least some of the shared properties of a function sequence. Such a concept is uniform convergence.

15.1 Uniform Convergence

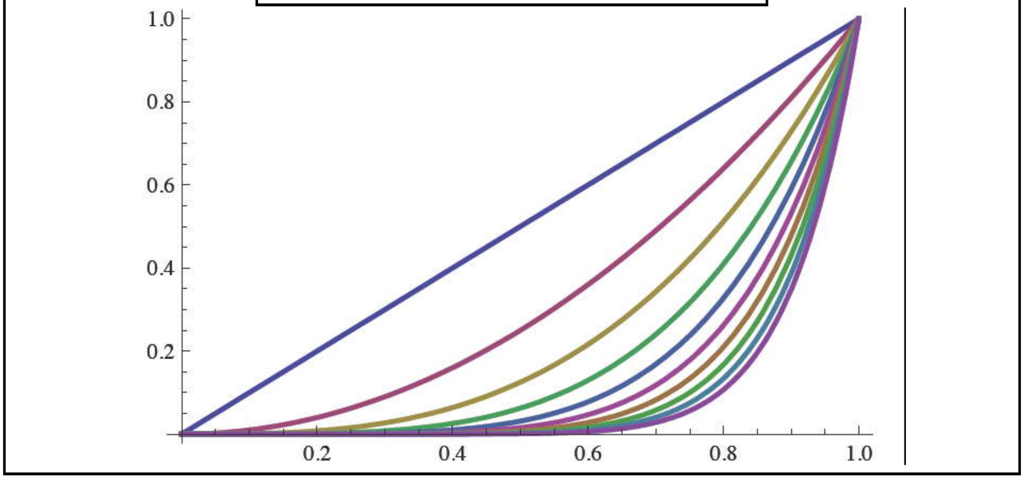
Definition 1: Let $I \subset \mathbb{R}$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on *I*. We say that f_n converges to *f* **pointwise** on *I* as $n \to \infty$ if:

$$\forall x \in I : f_n(x) \to f(x) \text{ as } \to \infty$$

Example: Let *I* = (0, 1) and $f_n(x) = x^n$, $\left(f_1(x) = x^2, f_3(x) = x^3, ...\right)$. It can be observed that $\forall x \in I$: $f_n(x) = x^n \to 0$. So f_n converges pointwise to the zero function on *I*.

Notes

Figure 15.1: Pointwise convergence of $f_n = x^n$. First 10 terms of f_n.

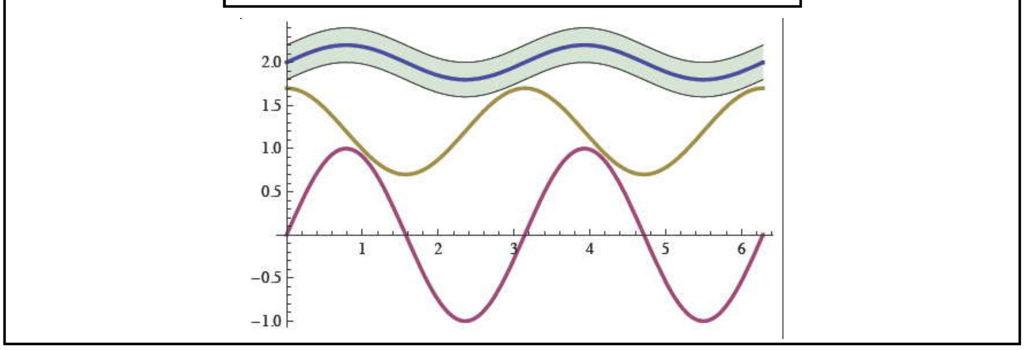


Definition 2: Let $I \subset \mathbb{R}$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on *I*. We say that f_n converges of *f* **uniformly** on *I* if:

$$\forall_{\varepsilon} > 0 \exists N \in \mathbb{N} \forall_{x} \in I : |f_n(x) - f(x)| < \varepsilon$$

Meaning: For any ε-tube around *f* all functions f_n starting from some *N* will be lying inside the tube.

Figure 15.2: Uniform convergence. f_1 in red, f_2 in yellow, *f* in blue, ε-tube arund *f* in green.



Example: Let us take $I = (0,1)$ and $f_n = x^n$. Does it converge to $f(x) = 0$ uniformly on *I*? In this case answer is no. But the converge holds in general.

Theorem 1: If $f_n \to f$ uniformly then $f_n \to f$ pointwise.

Proof: Pointwise convergence (*N* is allowed to depend on *x*):

$$\forall x \in I \ \forall \varepsilon > 0 \ \exists \ N \in \mathbb{N} \forall n \geq N : |f_n(x) - f(x)| < \varepsilon$$

Uniform convergence ($\tilde{N}$ cannot depend on *x*):

$$\forall \tilde{\varepsilon} > 0 \ \exists \tilde{N} \in \mathbb{N} \ \forall n \geq \tilde{N} \ \forall x \in I : |f_n(x) - f(x)| < \varepsilon$$

Notes

Hence, for any given $\varepsilon > 0$ in pointwise definition it suffices to take $N = \tilde{N}_\varepsilon$ taken from uniform definition.

15.2 Testing Pointwise and Uniform Convergence

1. Test the pointwise convergence.
2. If there is no pointwise convergence, there is no uniform convergence.
3. If f_n converges pointwise to some f test the uniform convergence to f.

Example: Let $I = [0,1]$, $f_n(x) = x^n$. We see that f_n converges pointwise to $f(x) = \begin{cases} 0, & x \neq 1 \\ 1, & x = 1 \end{cases}$.

Does it converge uniformly to f on I? the answer is no.

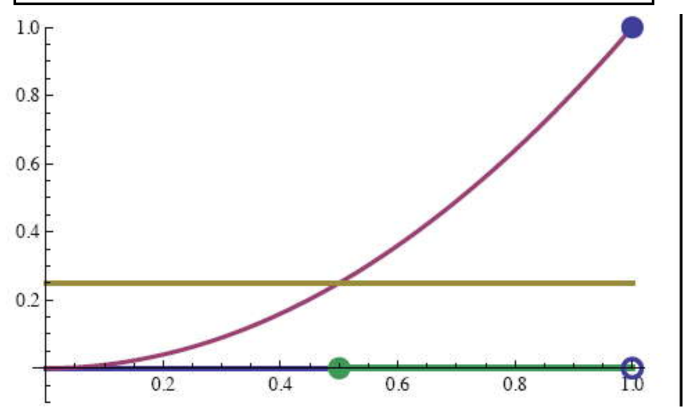
Proof: Negation of uniform convergence is

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists n \geq N \; \exists x \in I : |f_n(x) - f(x)| \geq \varepsilon$$

Take $\varepsilon = \frac{1}{4}$. Then $\forall N \in \mathbb{N}$ we have to find n and x such that $|f_n(x) - f(x)| \geq \varepsilon$. Take $n = N$. Now we want to find x. If we take any $x \in \left(\frac{1}{\sqrt[n]{4}}, 1\right)$ we get

$$|f_n(x) - f(x)| = |f_N(x) - f(x)| = |x^N - 0| = x^N \geq \frac{1}{4} = \varepsilon.$$

Figure 15.3: Suitable x's (green) for function $f_2(x) = x^2$ (red). Limit function f is blue, ε-tube and f in yellow.



Example: Let $I = [0, 1]$ and $f_n(x) = \begin{cases} 1, & x \in [0, 1/n] \\ 0, & x \in [1, 2/n, 1] \end{cases}$. f_n converges pointwise to

$f(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$ but does not converge uniformly to f.

Proof: If $x = 0$ then $f_n(0) \to 1$. If $x \in (0,1)$ then $f_n(x) \to 0$. Therefore $f_n \to f$ pointwise. To prove that f_n does not converge to f uniformly fix $\varepsilon = 1/2$. Then $\forall N \in \mathbb{N}$ choose $n = N$ and $x = 1/2N$. Then

Notes

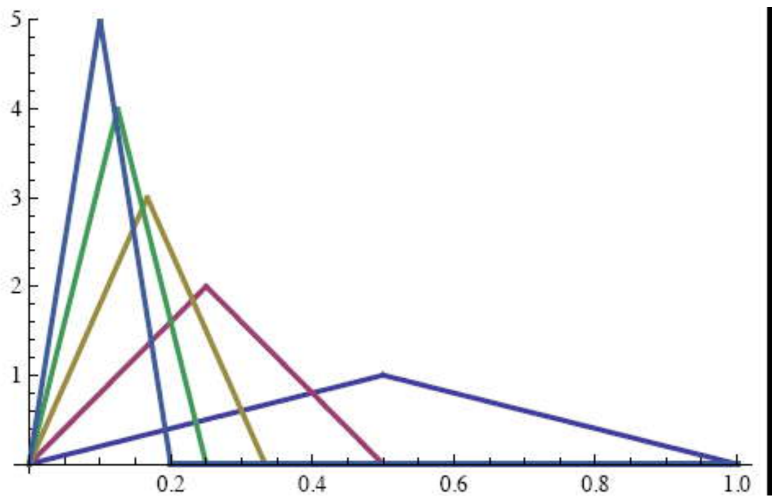
$$|f_n(x)-f(x)| = \left|f_N\left(\frac{1}{2N}\right) - f\left(\frac{1}{2N}\right)\right| = |1-0| \geq \frac{1}{2} = \varepsilon$$

Example: Let $I=[0,1]$ and $f_n(x) = \begin{pmatrix} 0, & x \in [1/n, 1] \\ n - n^2 x & x \in [0, 1/n] \end{pmatrix}$. Then f_x does not converge to any *f* pointwise nor uniformly.

Proof: Look at $x = 0$. Here $f_n(0) \to \infty$. Therefore, f_n does not converge pointwise to any *f*. Contrapositive tells us that f_n does not converge uniformly to any *f*.

Example: Let $I=[0,1]$ and f_n be from Figure 15.4

Figure 15.4: First 5 element of $\{f_n\}_{n=1}^{\infty}$ on [0, 1]. f_n starts at 0, has value of the peak equal of *n* at 1/2*n* and then returns to 0 at 1/*n*, value of the rest of 0.



Proof: Fix $x = 0$. Here $f_n(0) = 0 \to 0$. Now look at $x \in (0,1]$. Put $N = \left[\frac{1}{x}\right] + 1$. Then $\forall n \geq N : n \geq 1/x \Rightarrow x \geq 1/n$ and from image we can see that $f_n(x) \to 0$. Let us prove that f_n does not converge uniformly to the zero function. Take $\varepsilon = 1$. Then $\forall N \in \mathbb{N}$ choose $n = N$ and $x = 1/2N$. We have

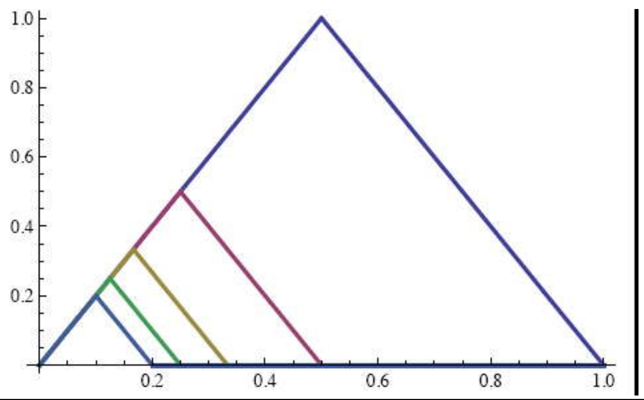
$$|f_n(x) - f(x)| = |f_N(1/2N) - f(1/2N)| = |N - 0| = N \geq 1 = \varepsilon.$$

Example: Let $I = [0, 1]$ and f_n to from Figure 15.5. Then f_n converges to the zero function pointwise and uniformly.

Proof: Pointwise convergence is clear since $\forall n > 1/x : f_n(x) = 0 \to 0$. For proof of uniform convergence of every $\varepsilon > 0$ choose $N = [1/\varepsilon] + 1$. Then $\forall n \geq N$ and $\forall n \in [0,1]$ we have

$$|f_n(x) - f(x)| = f_n \leq 1/n \leq 1/N \leq \varepsilon.$$

Figure 15.5: First 5 $\{f_n\}_{n=1}^{\infty}$ on [0, 1]. f_n is the same one as in Figure 15.4 but having value of the peak equal to 1/*n* instead of *n*.



Example: Let $I = (0, \infty)$ and $f_n(x) = \frac{1}{(x+n)^2}$. Then f_n converges to $f(x) = 0$ pointwise and uniformly.

Proof: Fix $x \in I$. We see that $f_n(x) \to 0$ as $n \to \infty$. Hence f_n converges pointwise. For proof of uniform convergence fix $\varepsilon > 0$, choose $N = [1/\varepsilon] + 1$. Then $\forall n \geq N$ and $\forall x \in I$ we have

$$|f_n(x) - f(x)| = \frac{1}{x+n} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Uniform Convergence and Continuity

If all f_n are continuous and $f_n \to f$ pointwise, does *f* have to be continuous? The answer is no, it suffices to look at Example. Hence, pointwise convergence does not preserve continuity but on the other hand uniform does.

Theorem 2: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $[a, b]$. If all f_n are all continuous and $f_n \to f$ uniformly on $[a, b]$ then the limit function *f* is continuous.

Proof: We need to prove *f* is continuous at every $x \in |a, b|$. That is fix $x \in [a, b]$ and show

$$\forall \varepsilon > 0 \exists \delta > 0 \forall y \in [a, b] : |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$$

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly

$$\exists N \in \mathbb{N} \forall n \geq \forall z \in [a, b] : |f_n(z) - f(z)| < \varepsilon/3.$$

In particular,

$$\forall z \in [a, b] : |f_N(z) - f(z)| < \varepsilon/3.$$

Since f_N is continuous at *x*,

$$\exists \tilde{\delta} > 0 \ \forall y \in [a, b] : |y - x| < \tilde{\delta} \Rightarrow |f_N(y) - f_N(x)| < \varepsilon/3$$

Therefore take $\delta = \tilde{\delta}$ and we get

$$|f(y) f(x)| = |f(y) + f_N(y) - f_N(y) + f_N(x) - f_N(x) - f(x)| \leq$$

Notes

$$|f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

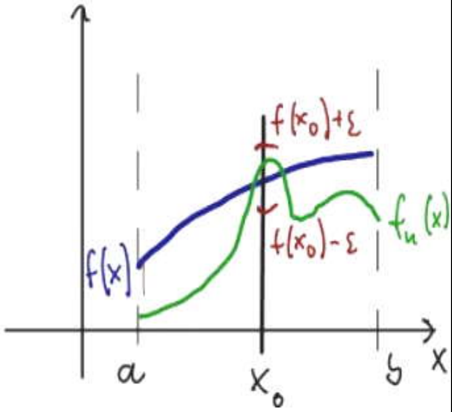
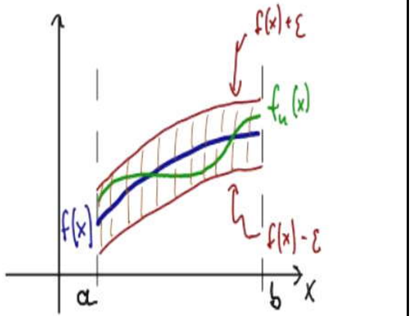
Remark: This theorem also work for f_n defined on any $I \subset \mathbb{R}$. Fix $x \in I$. This can be either isolated point in I or the limit point of I. In both cases we use exactly the same argument as for $[a, b]$ case.

We should compare uniform with pointwise convergence:

- For pointwise convergence we could first fix a value for x and then choose N. Consequently, N depends on both and x.
- For uniform convergence $f_n(x)$ must be uniformly close to f(x) for all x in the domain. Thus N only depends on but not on x.

Let's illustrate the difference between pointwise and uniform convergence graphically:

Table 15.1

Pointwise Convergence	Uniform Convergence
For pointwise convergence we first fix a value x_0. Then we choose an arbitrary neighborhood around $f(x_0)$, which corresponds to a vertical interval centered at $f(x_0)$.	For uniform convergence we draw an ε-neighborhood around the entire limit function f, which results in an "ε-strip" with f(x) in the middle.
	
Finally we pick N so that $f_n(x_0)$ intersects the vertical line $x = x_0$ inside the interval $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$	Now we pick N so that $f_n(x)$ is completely inside that strip for all x in the domain.

Uniform convergence clearly implies pointwise convergence, but the converse is false as the above examples illustrate. Therefore uniform convergence is a more "difficult" concept. The good news is that uniform convergence preserves at least some properties of a sequence.

15.3 Covers and Subcovers

Consider a collection of open intervals $\{I_\alpha\}_{\alpha \in A}$, where A is an index set:

1. Finite collection: $\{I_1, I_2, ..., I_m\}$. In this case $A = \{1, ..., m\}$.

Example: $I_1 = (0, 2),\ I_2 = (4, 5)$.

Notes

2. Infinite collection indexed by $\mathbb{N}: \{I_1, I_2, ...\}$. In this case $A = \mathbb{N}$.

Example: $I_n = (n - 1/3, n + 1/3)$.

3. Infinite collection indexed by $\mathbb{R}$. In this case $A = \mathbb{R}$.

Example: $\{(x-1, x+1)\}_{x \in \mathbb{R}}$. This set contains all open intervals of length 2.

Definition 3: A collection of open intervals $\{I_\alpha\}_{\alpha \in A}$ is a **cover** of a set $S \subset \mathbb{R}$ if $S \subset \cup_{\alpha \in A} I_\alpha$.

Definition 4: Given a set S and its cover $\{I_\alpha\}_{\alpha \in A}$, a **subcover** of $\{I_\alpha\}_{\alpha \in A}$ is a subcollection of $\{I_\alpha\}_{\alpha \in A}$, which itself is a cover for S.

Example: Let $I_1 = (0, 2), I_2 = (4, 5)$. A collection containing these two intervals covers (0, 1) and {1} ∪ (4, 4.5), but does not cover [0, 1).

Example: Let $I_n = (n - 1/3, n + 1/3), n \in \mathbb{N}$. Then $\{I_n\}_{n \in \mathbb{N}}$ covers $\mathbb{N}$, but does not cover $\mathbb{Z}$ or {1/2}. Let $S = \{1\} \cup (3 - 1/4, 3 + 1/4)$. Then $\{I_n\}_{n \in \mathbb{N}}$ is a cover for S. Moreover, $\{I_1, I_3\}$ is a finite subcover. Consider another case where $\{I_n\}_{n \in \mathbb{N}}$ is a cover of $\mathbb{N}$. Here $\{I_n\}_{n \in \mathbb{N}}$ has no finite subcover.

Example: Let $I_n = (-1 + 1/n, 1 - 1/n), n \in \mathbb{N} \setminus \{1\}$. We see that the collection {(–3/4, 3/4)} is a finite subcover for set $S = [-17/24, 17/24]$. Now, consider set $S = (-1, 1)$. Is $\{I_n\}_{n \geq 2}$ a cover for S? The answer is positive and $\{I_n\}_{n \geq 2}$ has no finite subcover.

Example: We can observe that given a set S and its cover $\{I_\alpha\}_{\alpha \in A}$ sometimes it's possible to extract a finite subcover and sometimes isn't.

Theorem 3: (Heine-Borel). Every cover of closed interval $[a, b]$ has a finite subcover.

Proof: Definite a set $B = \{x \in [a, b] : [a, x]$ has a finite subcover. We see that $a \in B$, since $[a, a]$ is a single point. We need to prove that $b \in B$. Define c = sub B. Now, we have to prove two claims:

Claim 1: $c = b$ (Figure 13.6, left).

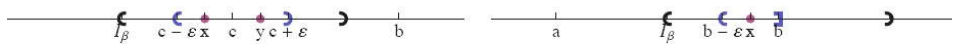
Assume that $c < b$. Then there is an interval I_β s.t. $c \in I_\beta$. Pick $\varepsilon > 0$ s.t. $(c - \varepsilon, c + \varepsilon) \subset I_\beta$ and $(c - \varepsilon, c + \varepsilon) \subset [a, b]$. Since c = sup B there is $x \in B$ s.t. $x \in (c - \varepsilon, c]$. Since $x \in B, [a, x]$ can be covered by finitely many intervals $\{I_{\alpha 1}, ..., I_{\alpha m}\}$. Pick $y \in (c, c + \varepsilon)$ and see that $[a, y]$ is covered by finite subcover $\{I_{\alpha 1}, ..., I_{\alpha m}, I_\beta\}$. Hence $y \in B$. But $y > c$ what implies that $c \neq \sup B$? What is contradiction? Therefore $c = b$.

Claim 2: $b \in B$ (Figure 15.6, right)

Pick I_β covering b. Take $\varepsilon > 0$ s.t. $(b - \varepsilon, b] \subset I_\beta$. Since $b = \sup B$ there exists $x \in B$ s.t. $x \in (b - \varepsilon, b]$. Since $x \in B, [a, x]$ can be covered by finitely many intervals $\{I_{\alpha 1}, ..., I_{\alpha m}\}$. Then $[a, b]$ is covered by finitely many intervals $\{I_{\alpha 1}, ..., I_{\alpha m}, I_\beta\}$.

Notes

Figure 15.6: Claim 1 (left) and Claim 2 (right) visualizations.



15.4 Dini's Theorem

Now, we are above to state in certain sense a converse to Theorem 1.

Theorem 4: (Dini's Theorem). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $[a, b]$. If: (a) $f_n \to f$ pointwise, (b) all f_n are continuous, (c) f is continuous and (d) $\forall x \in [a, b] : \{f_n(x)\}_{n=1}^{\infty}$ is monotone then $f_n \to f$ uniformly.

Proof: Given $\varepsilon > 0$ we want

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq \forall x \in [a,b] : |f_n(x) - f(x)| < \varepsilon.$$

Let $x \in [a, b]$. Since $f_n \to f$ pointwise

$$\exists N(x) \in \mathbb{N}\ \forall n \geq N(x) : |f_n(x)| < \varepsilon/2.$$

In particular,

$$|f_{N(x)}(x) - f(x)| < \varepsilon/2.$$

Let $g(y) = f_{N(x)}(y) - f(y)$. Then $g(y)$ is continuous by (b) and (c). In particular $g(y)$ is continuous at x

$$\exists \delta(x) > 0\ s.t.\ \forall y \in [a, b] : |y - x| < \delta(x) \Rightarrow |g(y) - g(x)| < \varepsilon/2$$

which implies

$$|f_{N(x)}(y) - f(y)| = |g(y)| \leq |g(y) - g(x)| + |g(x)| =$$

$$= |g(y) - g(x)| + |f_{N(x)}(x) - f(x)| < \varepsilon/2 + \varepsilon/2 < \varepsilon.$$

Moreover,

$$\forall n \geq N(x)\ \forall y \in [a, b] : |y - x| < \delta(x) \Rightarrow |f_n(y) - f(y)| < \varepsilon$$

since, by (d)

$$|f_n(y) - f(y)| \leq |f_{N(x)}(y) - f(y)|.$$

Denote $I(x) = (x - \delta(x), x + \delta(x))\ \forall x \in [a,b]$. We have shown

$$\exists N(x) \in \mathbb{N}\ \forall n \geq N(x)\ \forall y \in I(x) \cap [a, b] : |f_n(y) - f(y)| < \varepsilon.$$

Note that, $\{I(x)\}_{x \in [a, b]}$ is a cover for $[a, b]$, since $\forall x$ is covered at least by $I(x)$. By Heine-Borel Theorem 3 there is finite subcover $\{I(x_1), I(x_2), ..., I(x_m)\}$. Choose $N = \max\{N(x_1), ..., N(x_m)\}$ for any $\varepsilon > 0$. Let $n \geq N$ and $x \in [a, b]$. Let $I(x_i)$ be an interval converging x. Then $|x - x_i| < \delta(x_i)$. Since $n \geq N$ we have $n \geq N(x_i)$ and therefore $|f_n(x) - f(x)| < \varepsilon$.

Are all conditions of Dini's Theorem 4 important? The answer is yes, look at following examples.

Example: If (b) is not satisfied: let $I = [0, 1]$ and f_n is in the figure 15.7. We see that:

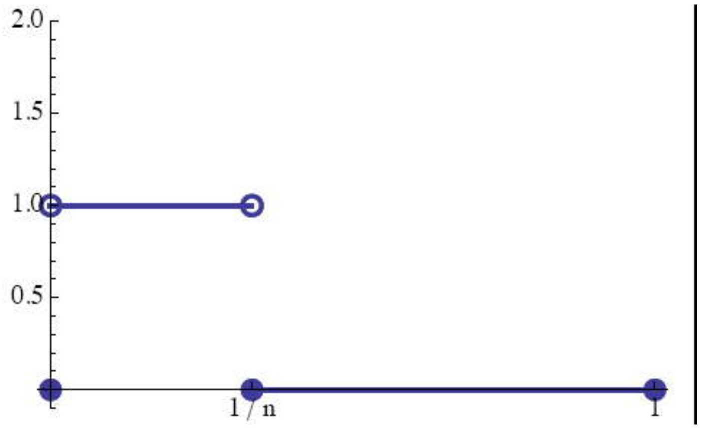
(a) $f_n \to 0$ pointwise.

(c) $f(x) = 0$ is continuous

(d) $\{f_n(x)\} = \{1, 1, 1, 0, 0, ...\}$ is decreasing

But f_n does not converge to $f(x) = 0$ uniformly.

Figure 15.7: Sequence of Real-valued Functions f_n being 1 in (0,1/*n*) and 0 elsewhere on [0, 1].



Example: If (c) is not satisfied: Let $I = [0,1]$ $f_n(x) = x^n$. We see that:

(a) $f_n \to f$. Where $f(x) = \begin{cases} 0, & x \neq 1 \\ 1, & x = 1 \end{cases}$.

(b) All f_n are continuous.

(d) $\{f_n\}$ is decreasing.

But f_n does not coverage to *f* uniformly.

Example: If (d) is not satisfied: Let $I = [0,1]$ and f_n here we see that:

(a) $f_n \to 0$ pointwise

(b) All f_n are continuous

(c) $f(x) = 0$ is continuous

(d) Put $x = 1/4$. Then $f_n(1/4) = \{a, b, c, ...\}$ where $a < b$ and $c < b$. Hence not satisfied.

But f_n does not converge to $f(x) = 0$ uniformly.

Notes

Example: If the interval is not closed. Let $I = (0,1)$ and $f_n(x) = x^n$.

(a) $f_n \to 0$ pointwise

(b) x^n is continuous.

(c) $f(x) = 0$ is continuous

(d) $\{f_n\}$ is monotonic $\forall x \in (0, 1)$.

But f_n does not converge to $f(x) = 0$ uniformly.

Self Assessment

Fill in the blanks:

1. If there is no pointwise convergence, there is no ....
2. If f_n .... to some f test the uniform convergence to f.
3. A collection of open intervals $\{I_\alpha\}_{\alpha \in A}$ is a **cover** of a set ....
4. Given a set S and its cover $\{I_\alpha\}_{\alpha \in A}$, a .... is a subcollection of $\{I_\alpha\}_{\alpha \in A}$, which itself is a cover for S.

15.5 Summary

- Let f_1 f_2 f_3 … be a sequence of functions from one metric space into another, such that for any x in the domain, the images $f_1(x)$ $f_2(x)$ $f_3(x)$ … form a convergent sequence. Let g(x) be the limit of this sequence. Thus the function g is the limit of the functions f_1 f_2 f_3 etc.
- This sequence of functions is uniformly convergent throughout a region R if, for every Σ there is n such that fj(x) is within ε(g(x), for every x in R and for every j $\geq$ n. The functions all approach g(R) together, one n fits all. This is similar to uniform continuity, where one δ fits all.
- If the range space is complex, or a real vector space, the sequence of functions is uniformly convergent iff. All the component functions are uniformly convergent. Given an ε, find f_n that is close to g, and the components of f_n must be close to the components of g, for all x. Conversely, if the components are within ε then the n dimensional function is within nε, for all x.
- Without uniform convergence, g is rather unpredictable. Let the domain be the closed interval [0, 1], and let f_n = x_n. Note that the sequence f approaches a function g that is identically 0, except for g(1) = 1. Each function in the sequence is uniformly continuous, yet the limit function isn't even continuous.

15.6 Keywords

Heine-Borel: Every cover of closed interval $[a, b]$ has a finite subcover.

Dini's Theorem: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $[a, b]$. If: (a) $f_n \to f$ pointwise, (b) all f_n are continuous, (c) f is continuous and (d) $\forall x \in [a, b] : \{f_n(x)\}_{n=1}^{\infty}$ is monotone then $f_n \to f$ uniformly.

Notes

15.7 Review Questions

1. Let I = [1, 2] and $f_n(x)=\begin{cases}1, & x\in[0,1/n] \\ 0, & x\in[1,2/n,1]\end{cases}$. f_n converges pointwise to

   $f(x)$ = $f_n(x)=\begin{cases}1, & x\in[0,1/n] \\ 0, & x\in[1,2/n,1]\end{cases}$. f_n but does not converge uniformly to f.

2. Let $I=[0,1]$ and f_n to from Figure 15.5. Then f_n converges to the zero function pointwise and uniformly.

3. Let $I=(0,\infty)$ and $f_n(x)=\dfrac{1}{(x+n)^2}$. Then f_n converges to $f(x)=0$ pointwise and uniformly.

4. $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $[a,b]$. If all f_n are all continuous and $f_n \to f$ uniformly on $[a,b]$ then the limit function f is continuous.

5. Let $I_n=(n-1/3, n+1/3), n\in\mathbb{N}$. Then $\{I_n\}_{n\in\mathbb{N}}$ covers $\mathbb{N}$, but does not cover $\mathbb{Z}$ or {1/2}. Let $S=\{1\}\cup(3-1/4,\ 3+1/4)$. Then $\{I_n\}_{n\in\mathbb{N}}$ is a cover for S. Moreover, $\{I_1, I_3\}$ is a finite subcover. Consider another case where $\{I_n\}_{n\in\mathbb{N}}$ is a cover of ¥. Here $\{I_n\}_{n\in\mathbb{N}}$ has no finite subcover.

6. Let $I_n=(-1+1/n, 1-1/n), n\in\mathbb{N}\setminus\{1\}$. We see that the collection {(–3/4, 3/4)} is a finite subcover for set $S=[-17/24, 17/24]$. Now, consider set $S=(-1,1)$. Is $\{I_n\}_{n\geq 2}$ a cover for S? The answer is positive and $\{I_n\}_{n\geq 2}$ has no finite subcover.

Answers: Self Assessment

1. uniform convergence
2. converges pointwise
3. $S\subset\mathbb{R}$ if $S\subset\cup_{\alpha\in A} I_\alpha$
4. subcover of $\{I_\alpha\}_{\alpha\in A}$

15.8 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7 (7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10, Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 16: Uniform Convergence and Continuity

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Uniform Convergence preserves Continuity
- Explain the Supremum Norm
- Discuss Sup-norm and Uniform Convergence

Introduction

In earlier unit as you all studied about uniform convergence. Uniform convergence clearly implies pointwise convergence, but the converse is false. Therefore uniform convergence is a more "difficult" concept. The good news is that uniform convergence preserves at least some properties of a sequence. This unit will explain Uniform Convergence preserves Continuity.

16.1 Uniform Convergence Preserves Continuity

If a sequence of functions $f_n(x)$ defined on D converges uniformly to a function f(x), and if each $f_n(x)$ is continuous on D, then the limit function f(x) is also continuous on D.

All ingredients will be needed, that f_n converges uniformly and that each f_n is continuous. We want to prove that f is continuous on D. Thus, we need to pick an x_0 and show that

$$|f(x_0) - f(x)| < \varepsilon \text{ if } |x_0 - x| < \delta$$

Let's start with an arbitrary $\varepsilon > 0$. Because of uniform convergence we can find an N such that

$$|f_n(x) - f(x)| < \varepsilon/3 \text{ if } n \geq N$$

for all $x \in D$. Because all f_n are continuous, we can find in particular a $\delta > 0$ such that

$$|f_N(x_0) - f_N(x)| < \varepsilon/3 \text{ if } |x_0 - x| < \delta$$

But then we have:

$$|f(x_0) - f(x)| \leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f(x)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

as long as $|x_0 - x| < \delta$. But that means that f is continuous at x_0.

Before we continue, we will introduce a new concept that will somewhat simplify our discussion of uniform convergence, at least in terms of notation: we will use the supremum of a function to define a 'norm' of f

16.2 Uniform Convergence and Supremum Norm

Definition 1: The supremum norm of a function $f : I \to \mathbb{R}$ is

$$\|f\|_{\sup} = \|f\|_{\infty} = \sup_{x \in I} |f(x)|.$$

Example: Let $I = \mathbb{R}$ and $f(x) = \sin(x)$. Then $\|f\|_{\sup} = 1.$

Example: Let $I = [0,1]$ and $f(x) = -2x$. Then $\|f\|_{\sup} = 2.$ The norm stays the same even if we change the interval [0, 1] to (0, 1).

Theorem 1: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on *I*. Then $f_n \to f$ uniformly if and only if $\|f_n - f\|_{\sup} \to 0.$ Note that $\|f_n - f\|_{\sup}$ is just a sequence of number.

Proof: $f_n \to f$ uniformly $\Leftrightarrow \forall \varepsilon\ \exists N\ \forall n \geq N\ \forall x \in I : |f_n(x) - f(x)| \leq \varepsilon \Leftrightarrow \forall \varepsilon\ \exists N\ \forall n \geq N : \sup_{x \in I}$ $|f_n(x) - f(x)| \leq \varepsilon \Leftrightarrow \forall \varepsilon\ \exists N\ \forall n \geq N : \|f_n - f\| \leq \varepsilon$

Example: Let $f_n(x) = x^n$ on (0, 1). We can observe that $\|f_n - f\|_{\sup} = \sup_{x \in (0,1)} |x^n - 0|$ = $1 \to 0.$ As $f_n \to 0$ uniformly.

Example: Let $f_n(x) = \begin{cases} 1, & x \geq n \\ 0, & x < n \end{cases}$ on $\mathbb{R}$. Then $\|f\|_{\sup} = \sup_{x \in \mathbb{R}} |f_n(x) - 0| = 1 \to 0.$ So $f_n \to 0$ uniformly.

Using this proposition it can be easy to show uniform convergence of a function sequence, especially if the sequence is bounded. Still, even with this idea of sup-norm uniform convergence can not improve its properties: it preserves continuity but has a hard time with differentiability.

Example: Consider the sequence $f_n(x) = 1/n \sin(nx)$:

- Show that the sequence converges uniformly to a differentiable limit function for all x.
- Show that the sequence of derivatives f_n' does not converge to the derivative of the limit function.

This example is ready-made for our sup-norm because $|\sin(x)| < 1$ for all x. As for our proof: the sequence converges uniformly to zero because:

$$||f_n - f||_D = ||1/n \sin(n\,x) - 0||_D \leq 1/n \to 0$$

Notes

The sequence of derivatives is

$$f'(x) = \cos(nx)$$

Which does not converge (take for example x = π).

Example: Find a sequence of differentiable functions that converges uniformly to a continuous limit function but the limit function is not differentiable

we found a sequence of differentiable functions that converged point wise to the continuous, non-differentiable function f(x) = |x|. Recall:

$$f_n(x)= \begin{cases} -x-\dfrac{1}{2n} & \text{if } -1 \le x \le -\dfrac{1}{n} \\ \dfrac{n}{2}x^2 & \text{if } -\dfrac{1}{n} < x < \dfrac{1}{n} \\ x-\dfrac{1}{2n} & \text{if } \dfrac{1}{n} \le x \le 1 \end{cases}$$

That same sequence also converges uniformly, which we will see by looking at '$||f_n - f||_D$. We will find the sup in three steps:

If $1 \le x -1/n$:

$f_n(x) - f(x)| = |-x - {}^1/_{2n} + x| = {}^1/_{2n}$

If $-1/n < x < 1/n$:

$|f_n(x) - f(x)| \le |{}^n/_2\, x^2| + |x| \le {}^n/_2\, {}^1/_n{}^2 + {}^1/_n = {}^3/_{2n}$

If $1/n \le x \le 1$:

$|f_n(x) - f(x)| = |x - {}^1/_{2n} - x| = {}^1/_{2n}$

Thus, $||f_n - f||_D < {}^3/_{2n}$ which implies that f_n converges uniformly to f. Note that all f_n are continuous so that the limit function must also be continuous (which it is). But clearly f(x) = |x| is not differentiable at x = 0.

16.3 Uniform Convergence and Integrability

Theorem 2: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $[a, b]$. If all f_n are Riemann-integrable and $f_n \to f$ uniformly then *f* is Riemann-integrable and

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

Proof: Recall partition $P: a = t_0 < t_1 < ... < t_n = b$. The upper Darboux sum $U(f, p) = \sum_{i=1}^{n} \sup_{[t_{i-1},\, t_i]}$ $f(x)(t_i - t_{i-1})$ and the lower Darboux sum $L(f, P) = \sum_{i=1}^{n} \inf_{[t_{i-1},\, t_i]} f(x)\,(t_i - t_{i-1})$. f is Riemann-integrable on $[a, b]$ if and only if $\forall \varepsilon > 0\ \exists P U(f, P) - L(f, P) < \varepsilon$.

Claim 1: f is Riemann-integrable.

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly $\exists n$ s.t. $\|f_n - f\|_{\sup} < \dfrac{\varepsilon}{4(b-a)}$. Since f_n is R-integrable $\exists P$ s.t. $U(f_n, P) - L(f_n, P) < \varepsilon/2$. We have

Notes

$$\|f_n - f\|\sup < \frac{\varepsilon}{4(b-a)} \Rightarrow f_n(x) \frac{\varepsilon}{4(b-a)} \leq f(x) \leq f_n(x) + \frac{\varepsilon}{4(b-a)}, \forall x$$

i.e.

$$\sup_{[t_{i-1}, t_i]} f(x) \leq \sup_{[t_{i-1}, t_i]} f_n(x) + \frac{\varepsilon}{4(b-a)}.$$

Then

$$U(f, P) = \sum_{i=1}^{n} \sup_{[t_{i-1}, t_i]} f(x)(t_i - t_{i-1}) \leq \sum_{i=1}^{n} \left(\sup_{[t_{i-1}, t_i]} f_n(x) + \frac{\varepsilon}{4(b-a)} \right) (t_i - t_{i-1}) =$$

$$= U(f_n, P) + \frac{\varepsilon}{4(b-a)} \sum_{i=1}^{n} (t_i - t_{i-1}) = U(f_n, P) + \frac{\varepsilon}{4}.$$

Similarly,

$$L(f, P) \geq L(f_n, P) - \frac{\varepsilon}{4}.$$

So

$$U(f, P) - L(f, P) \geq U(f_n, P) + \frac{\varepsilon}{4} - L(f_n, P) + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

Claim 2: $\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$

Since $\|f_n - f\|_{\sup} \to 0$, given $\varepsilon > 0$ there exists $N \in \mathbb{R}$ s.t. $\forall n \geq N : \|f_n - f\|_{\sup} < \varepsilon/(b-a)$. Therefore, $\forall n \geq N :$

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| = \left| \int_a^b f_n(x)dx \right| \leq \int_a^b \left| f_n(x) - f(x) \right| dx \leq$$

$$\int_a^b \underbrace{\|f_n - f\|_{\sup}}_{\text{number}} dx = \|f_n - f\|_{\sup} (b-a) < \frac{\varepsilon}{(b-a)} (b-a) = \varepsilon.$$

Much more can be said about convergence and integration if we consider the Lebesgue integral instead of the Riemann integral. To focus on Lebesgue integration, for example, we would first define the concept of "convergence almost everywhere":

16.4 Convergence almost Everywhere

A sequence f_n defined on a set D converges (pointwise or uniformly) almost everywhere if there is a set S with Lebesgue measure zero such that f_n converges (pointwise or uniformly) on D\S. We say that f_n converges (point wise or uniformly) to f a.e.

In other words, convergence a.e. means that a sequence converges everywhere except on a set with measure zero. Since the Lebesgue integral ignores sets of measure zero, convergence a.e. is ready-made for that type of integration.

Notes

Example: Let r_n be the (countable) set of rational numbers inside the interval [0, 1], ordered in some way, and define the functions

$$\begin{cases} 1 & \text{if} \quad x = r_1, r_2, r_3, \ldots r_n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) \begin{cases} 1 & \text{if} \quad x = \text{rational} \\ 0 & \text{if} \quad x = \text{irrational} \end{cases}$$

Show the following:

- The sequence g_n converges pointwise to g but the sequence of Riemann integrals of g_n does not converge to the Riemann integral of g.
- The sequence g_n converges a.e. to zero and so does the sequence of Lebesgue integrals of g_n.

Solution:

Each g_n is continuous except for finitely many points of discontinuity. But then each g_n is integrable and it is easy to see that

$$\int_{gn(x)} dx = 0$$

But the limit function is not Riemann-integrable and hence the sequence of Riemann integrals does not converge to the Riemann integral of the limit function.

Please note that while each g_n is continuous except for finitely many points, the limit function g is discontinuous everywhere

On the other hand, each g_n is zero except on a set of measure zero, and so is the limit function. Thus, using Lebesgue integration we have that all integrals evaluate to zero. But then, in particular, the sequence of Lebesgue integrals of g_n converge to the Lebesgue integral of g.

There are many theorems relating convergence almost everywhere to the theory of Lebesgue integration. They are too involved to prove at our level but they would certainly be on the agenda in a graduate course on Real Analysis. For us we will be content stating, without proof, one of the major theorems.

16.5 Lebesgue's Bounded Convergence Theorem

Let $\{f_n\}$ be a sequence of (Lebesgue) integrable functions that converges almost everywhere to a measurable function f. If $|f_n(x)| \le g(x)$ almost everywhere and g is (Lebesgue) integrable, then f is also (Lebesgue) integrable and:

$$\lim_{n \to \infty} \int |f_n - f| dm = 0$$

Self Assessment

Fill in the blanks:

1. Let $I = \mathbb{R}$ and $f(x) = \sin(x)$. Then ....
2. Let $I = [0,1]$ and $f(x) = -2x$. Then .... The norm stays the same even if we change the interval [0, 1] to (0, 1).
3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on *I*. Then $f_n \to f$ uniformly if and only if .... Note that $\|f_n - f\|_{\sup}$ is just a sequence of number.
4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $[a, b]$. If all f_n are Riemann-integrable and $f_n \to f$ uniformly then *f* is .... and

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

16.6 Summary

- If a sequence of functions $f_n(x)$ defined on D converges uniformly to a function f(x), and if each $f_n(x)$ is continuous on D, then the limit function f(x) is also continuous on D.
- All ingredients will be needed, that f_n converges uniformly and that each f_n is continuous. We want to prove that f is continuous on D.
- Before we continue, we will introduce a new concept that will somewhat simplify our discussion of uniform convergence, at least in terms of notation: we will use the supremum of a function to define a 'norm' of f.
- A sequence f_n defined on a set D converges (point wise or uniformly) almost everywhere if there is a set S with Lebesgue measure zero such that f_n converges (pointwise or uniformly) on D\S. We say that f_n converges (pointwise or uniformly) to f a.e.
- Much more can be said about convergence and integration if we consider the Lebesgue integral instead of the Riemann integral. To focus on Lebesgue integration, for example, we would first define the concept of "convergence almost everywhere".
- In other words, convergence a.e. means that a sequence converges everywhere except on a set with measure zero. Since the Lebesgue integral ignores sets of measure zero, convergence a.e. is ready-made for that type of integration.
- Let { f_n } be a sequence of (Lebesgue) integrable functions that converges almost everywhere to a measurable function f. If $|f_n(x)| \le g(x)$ almost everywhere and g is (Lebesgue) integrable, then f is also (Lebesgue) integrable and:

$$\lim_{n\to\infty}\int|f_n - f|dm = 0$$

16.7 Keywords

Convergence almost Everywhere: A sequence f_n defined on a set D converges (pointwise or uniformly) almost everywhere if there is a set S with Lebesgue measure zero such that f_n converges (point wise or uniformly) on D\S. We say that f_n converges (pointwise or uniformly) to f a.e.

Uniform Convergence Preserves Continuity: If a sequence of functions $f_n(x)$ defined on D converges uniformly to a function f(x), and if each $f_n(x)$ is continuous on D, then the limit function f(x) is also continuous on D.

Lebesgue's Bounded Convergence Theorem: Let $\{f_n\}$ be a sequence of (Lebesgue) integrable functions that converges almost everywhere to a measurable function f. If $|f_n(x)| \le g(x)$ almost everywhere and g is (Lebesgue) integrable, then f is also (Lebesgue) integrable and:

$$\lim_{n\to\infty}\int|f_n - f|dm = 0$$

16.8 Review Questions

1. Let $f_n(x) = x^n$ on (0, 1). We can observe that $\|f_n - f\|_{\sup} = \sup_{x\in(0,1)}|x^n - 0| = 1 \to 0.$ By Theorem $f_n \to 0$ uniformly.

2. Let $f_n(x) = \begin{cases} 1, & x \ge n \\ 0, & x < n \end{cases}$ on $\mathbb{R}$. Then $\|f\|_{\sup} = \sup_{x\in\mathbb{R}}|f_n(x) - 0| = 1 \to 0.$ So $f_n \to 0$ uniformly.

Notes

Answers: Self Assessment

1. $\|f\|_{sup} = 1$
2. $\|f\|_{sup} = 2$
3. $\|f_n - f\|_{sup} \to 0$
4. Riemann-integrable

16.9 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 17: Uniform Convergence and Differentiability

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the uniform convergence and differentiability
- Explain the series of the functions
- Describe the central principle of uniform convergence

Introduction

In last unit you have studied about uniform convergence and continuity. Simple or pointwise convergence is not enough to preserve differentiability, and neither is uniform convergence by itself. However, if we combine pointwise with uniform convergence we can indeed preserve differentiability and also switch the limit process with the process of differentiation. It remains to clarify the connection between uniform convergence and differentiability.

17.1 Uniform Convergence and Differentiability

Is it true that if all f_n are differentiable and $f_n \to f$ uniformly then f is differentiable and $f_n' \to f'$? The answer is no. Look at the following examples.

Example: Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n}\sin(n^2 x)$.

We see that all f_n are differentiable and that $f_n \to f$ uniformly $\left(\|f_n - f\|_{\sup} = \left\| \frac{1}{n}\sin(n^2 x) \right\| = 1/n \to 0 \right)$. But $f_n'(x) = n\cos(n^2 x) \to 0$ neither uniformly nor pointwise, although the zero function is differentiable.

Notes

Example: Let $f_n(x) = |x|^{1+1/n}$ be defined on [–1, 1]. See Figure 17.1.

We can observe that all f_n are differentiable, f_n converges uniformly to $f(x) = |x|$ by Dini's Theorem ($f_n(x) \to |x|$ pointwise, $f_n(x)$ are continuous, $f(x)$ is continuous, $\forall x$ in [–1, 1], $\{f_n(x)\}_{n=1}^{\infty}$ is increasing). But $|x|$ is not even differentiable.

The reason why this theorem cannot hold is that the uniform convergence, in fact, does not tell anything about slopes of f_n. See Figure 17.2.

Figure 17.1: First 4 terms of $\{|x|^{1+1/n}\}_{n=1}^{\infty}$

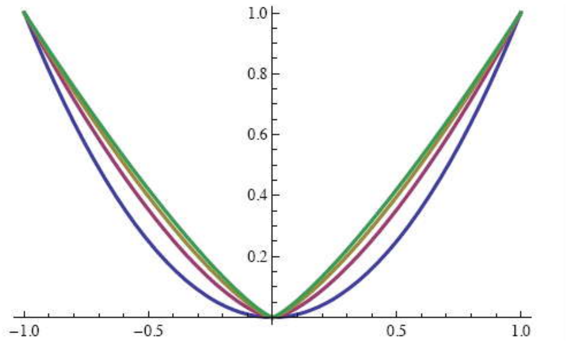
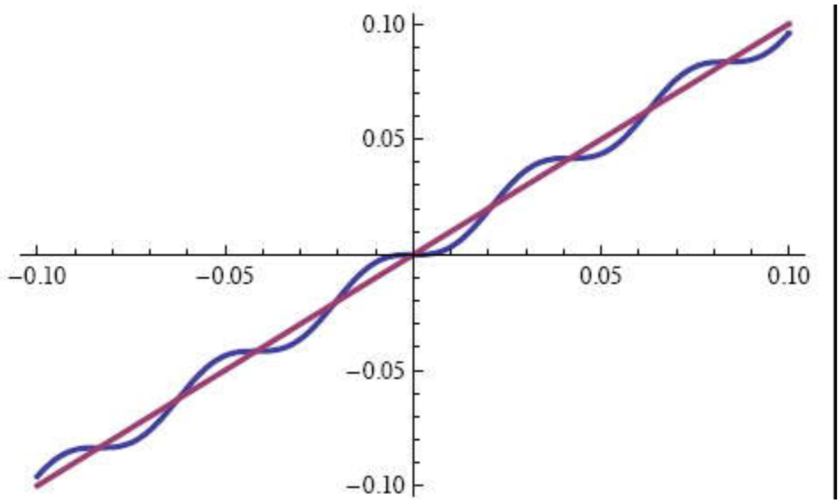


Figure 17.2: Slopes are Independent of Uniform Convergence



Theorem 1: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued function on $[a, b]$. If (a) all f_n are differentiable, (b) all f_n' are continuous, (c) $f_n' \to h$ uniformly, for some function $h:[a, b] \to \mathbb{R}$, (d) $\exists c \in [a, b]$ s.t. $f_n(c)$ converges then f_n converges uniformly to some $f:[a, b] \to \mathbb{R}$, in addition, this uniform limit f is differentiable and $f'(x) = h(x)$.

Proof: Firstly, we define a function $f(x)$ which satisfies $f'(x) = h(x)$ in Part 1 and then we show that $f_n \to f$ uniformly in Part 2.

Notes

Part 1: Define $f(c) := \lim_{n\to\infty} f_n(c)$ and $f(x) := f(c) + \int_c^x h(t)dt.$

Note that $h(t)$ is R-integrable since it is a uniform limit of continuous functions $f_n^{'}$ (Theorem 1). Therefore, according to the definition of $f(x)$ and by the fundamental theorem of calculus we can see that $f'(x) = \frac{d}{dx}(f(c) + \int_c^x h(t)dt) = \frac{d}{dx}(f(c) + \int_a^x h(t)dt + \int_c^a h(t)dt) = \frac{d}{dx}\int_a^x h(t) = h(x), \forall x \in [a, b].$

Part 2: We want to show that $f_n \to f$ uniformly on $[a, b]$. By the fundamental theorem of calculus:

$$f_n(x) = f_n(c) + \int_c^x f_n^{'}(t)dt.$$

Put $\varepsilon > 0$. Since $f_n(c) \to f(c)$ we have $|f_n(c) - f(c)| < \varepsilon/2$ for all $n \ge N_1$. Also $f_n^{'} \to h$ uniformly, so we have $\|f_n^{'} - h\|_{\sup} < \varepsilon/(2(b-a))$ for all $n \ge N_2$. Therefore put $N = \max\{N_1, N_2\}$. Then $\forall n \ge N\ \forall x \in [a, b]$ we get

$$|f_n(x) - f(x)| = \left|f_n(c) - f(c) + \int_c^x f_n^{'}(t)dt - \int_c^x h(t)dt\right| \le$$

$$\le |f_n(c) - f(c)| + \left|\int_c^x f_n^{'}(t) - h(t)dt\right| \le$$

$$\le |f_n(c) - f(c)| + \left|\int_c^x |f_n^{'}(t) - h(t)|dt\right| \le$$

$$\le |f_n(c) - f(c)| + \|f_n^{'} - h\|_{\sup}\left|\int_c^x 1dt\right| =$$

$$= |f_n(c) - f(c)| + \|f_n^{'} - h\|_{\sup}|x - c| \le$$

$$\le |f_n(c) - f(c)| + \|f_n^{'} - h\|_{\sup}(b - a) \le$$

$$< \varepsilon/2 + \frac{\varepsilon}{2(b-a)}(b - a) = \varepsilon$$

Example: Theorem 1 is not true if we replace $[a, b]$ by $\mathbb{R}$. Look at $f_n(x) = \sin(x/n)$ on $\mathbb{R}$. Then $f_n^{'}(x) = \frac{1}{n}\cos(x/n) \to 0$ uniformly $\left\|\frac{1}{n}\cos(x/n)\right\|_{\sup} = 1/n \to 0\}$. Conditions (a) – (d) of Theorem 1 are satisfied but $f_n \to 0$ uniformly.

17.2 Series of Functions

Definition 1: Let $I \subset \mathbb{R}$ and $\{g_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on I. Then

$$\sum_{n=1}^{\infty} g_n$$

Notes

is a series of functions. Partial sums: $f_n(x) = \sum_{i=1}^{n} g_i(x)$. We say that $\sum_{n=1}^{\infty} g_n$ converges pointwise/ uniformly to $f : I \to \mathbb{R}$ if $f_n \to f$ pointwise/unformly.

Example: Consider a series $\sum_{n=0}^{\infty} x^n$ on [–1/2, 1/2] and on (–1, 1). We want to investigate convergence of this series, whether it is pointwise of n uniform and eventually, what is the limit. Partial sums are in this case $f_n(x) = \sum_{i=0}^{n} x^i = 1 + x + x^2 + ... + x^n = \frac{x^{n+1} - 1}{x - 1}$. If we fix $x \in (-1, 1)$ we see that $f_n(x) = \frac{x^{n+1} - 1}{x - 1} \to \frac{1}{1 - x}$ pointwise on (–1, 1) and therefore also on [–1/2, 1/2]. Does not series converge to the $\frac{1}{1-x}$ also uniformly on the same intervals? Look at the modulus

$$|f_n(x) - f(x)| = \left| \frac{x^{n+1} - 1}{x - 1} - \frac{1}{1 - x} \right| = \left| \frac{x^{n+1}}{1 - x} \right|$$

On [–1/2, 1/2] our series converges uniformly to *f* since partial sums do

$$\|f_n - f\|_{\sup} = \sup_{x \in [-1/2,\, 1/2]} \left| \frac{x^{n+1}}{1 - x} \right| \le \frac{1}{2^{n+1}} 2 = \frac{1}{2^n} \to 0.$$

On (–1, 1) the series does not converge to *f* since partial sums do not.

$$\|f_n - f\|_{\sup} = \sup_{x \in (-1,\, 1)} \left| \frac{x^{n+1}}{1 - x} \right| = \infty \text{ if } x \to 1.$$

Let us fix a sequence of real-valued functions $\{g_n\}_{n=1}^{\infty}$ on $[a, b]$.

Theorem 2: Continuity for Series

If (a) all g_n are continuous and (b) $\sum_{n=1}^{\infty} g_n$ converges uniformly then $\sum_{n=1}^{\infty} g_n$ is continuous.

Proof: Consider partial sum $f_n = g_1 + g_2 + ... + g_n$. We see that f_n is continuous, since it is a sum of continuous functions. Also, by (b), $f_n \to \sum_{n=1}^{\infty} g_n$ uniformly. Therefore, the continuity of the uniform limit $\sum_{n=1}^{\infty} g_n$ is Riemann-integrable and we can integrate the series term by term

$$\int_a^b \sum_{n=1}^{\infty} g_n dx = \sum_{n=1}^{\infty} \int_a^b g_n dx.$$

Proof: Since sum of R-integrable functions is R-integrable, we see that the partial sum $\sum_{i=1}^{n} g_i = g_1 + g_2 + ... + g_n$ is R-integrable and by (b) $\sum_{i=1}^{n} g_i \to \sum_{i=1}^{\infty} g_i$

$$\sum_{i=1}^{\infty} \int_a^b g_i dx = \lim_{n \to \infty} \sum_{i=1}^{n} \int_a^b g_i dx = \lim_{n \to \infty} g_i dx = \int_a^b \sum_{i=1}^{\infty} g_i dx$$

Theorem 3: Differentiability for Series

If (a) all g_n are differentiable, (b) all $g_n^{'}$ are continuous, (c) $\sum_{n=1}^{\infty} g_n^{'}$ converges uniformly and (d) $\exists c \in [a,b]$ s.t. $\sum_{n=1}^{\infty} g_n(c) < \infty$ then $\sum_{n=1}^{\infty} g_n$ converges uniformly and

$$\left(\sum_{n=1}^{\infty} gn\right)' = \sum_{n=1}^{\infty} g_n^{'}.$$

Proof: Since sum of differentiable functions is differentiable we observe that the partial sum $f_n = g_1 + g_2 + ... + g_n$ is differentiable. Similarly, all $f_n^{'} = g_1^{'} + g_2^{'} + ... + g_n^{'}$ are continuous. By (c) $f_n^{'} \to \sum_{n=1}^{\infty} g_n^{'}$ uniformly and by (d) $\exists c \in [a,b]$ s.t. $\sum_{n=1}^{\infty} g_n(c)\ \infty$, i.e. $\lim_{n\to\infty} f_n(c) < \infty$. Therefore we can apply Theorem to f_n and observe that $f_n \to \sum_{n=1}^{\infty} g_n$ uniformly and $\left(\sum_{n=1}^{\infty} g_n\right)' = \sum_{n=1}^{\infty} g_n^{'}$

17.3 Central Principle of Uniform Convergence

Definition 2: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $I \subset \mathbb{R}$. Then $\{f_n\}_{n=1}^{\infty}$ is called a uniform Cauchy sequence if:

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \geq N : \|f_n - f_m\|_{\sup} < \varepsilon.$$

Theorem 4: Central Principle of Uniform Convergence, CPUC

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $I \subset \mathbb{R}$. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly of I if and only if $\{f_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence on *I*.

Proof: '$\Rightarrow$' : Suppose f_n converges uniformly to some *f*. Let $\varepsilon > 0$, since $f_n \to f$ uniformly we have

$$\exists N \in \mathbb{N}\ \forall n \geq \forall x \in I : |f_n(x) - f(x)| < \varepsilon/4.$$

Then $\forall_n, m \geq \forall x \in I$:

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f_m(x) + f(x)| \leq$$

$$\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$$

Therefore,

$$\|f_n - f_m\|_{\sup} = \sup_{x \in I} |f_n(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon.$$

'$\Rightarrow$' : Let $\{f_n\}$ be a uniform Cauchy sequence i.e.

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \geq N : \|f_n - f_m(x)\|_{\sup} < \varepsilon/2.$$

In particular $|f_n(x) - f_m(x)| < \varepsilon/2$ for any $x \in I$. Look at the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ which is usual sequence of number and hence converges. Denote its limit $f(x)$. Now let $m \to \infty$ and get

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$$\forall n \geq N \ \forall x \in I : \left| f_n(x) - f(x) \right| \leq \varepsilon/2 < \varepsilon$$

what by definition means that $f_n \to f$ uniformly.

Theorem 5: Weierstrass M-test

Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $I \subset \mathbb{R}$. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of number s.t. $\sum_{n=1}^{\infty} M_n < \infty$. If $|g_n(x)| \leq M_n$, $\forall x \in I \ \forall n \in \mathbb{N}$ then $\sum_{n=1}^{\infty} g_n$ converges uniformly.

Proof: $\sum_{i=1}^{\infty} M_i < \infty$ means $\left\{\sum_{i=1}^{n} M_i\right\}_{n=1}^{\infty}$ converges and therefore is a Cauchy sequence i.e. given $\varepsilon > 0$ we can find $N \in \mathbb{N}$ s.t. without loss of generality

$$\forall n, m \in \mathbb{N}, n > m \geq N : \left| \sum_{i=1}^{n} M_i - \sum_{i=1}^{m} M_i \right| = \left| \sum_{i=m+1}^{n} M_i \right| = \sum_{i=m+1}^{n} M_i < \varepsilon/2.$$

Let's prove that $\left\{\sum_{i=1}^{n} g_i\right\}_{n=1}^{\infty}$ is a uniform Cauchy sequence. Given $\varepsilon > 0$, pick N as above. Then $\forall n > m \geq N \in \mathbb{N} \ \forall x \in I$

$$\left| \sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{m} g_i(x) \right| = \left| \sum_{i=m+1}^{n} g_i(x) \right| \leq$$

$$\leq \sum_{i=m+1}^{n} |g_i(x)| \leq \sum_{i=m+1}^{n} M_i < \varepsilon/2.$$

We get

$$\left\| \sum_{i=1}^{n} g_i - \sum_{i=1}^{m} g_i \right\|_{\sup} \leq \varepsilon/2 < \varepsilon.$$

Therefore $\left\{\sum_{i=1}^{n} g_i\right\}_{n=1}^{\infty}$ is a uniform Cauchy sequence and converges uniformly and so does $\sum_{i=1}^{\infty} g_i$.

Example: Consider series $\sum_{i=1}^{\infty} \frac{\sin(nx)}{2^n}$ on $\mathbb{R}$. By Weierstrass M-test, we see that this series converges uniformly on $\mathbb{R}$ since

$$\left| \frac{\sin(nx)}{2^n} \right| \leq \frac{1}{2^n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

Example: Consider series $\sum_{n=1}^{\infty} \frac{1}{n^2 + x}$ on $[0, \infty)$. By Weierstrass M-test, we can obtain uniform convergence of this series on $[0, \infty)$ since

$$\left| \frac{1}{n^2 + x} \right| \leq \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Example: Look at the uniform convergence of series $\sum_{n=1}^{\infty} x^n$ both on $[-r, r]$, $0 < r < 1$ and (–1, 1). In the first case we see that the series converges uniformly by Weierstrass M-test Since

$$|x^n| \leq r^n \text{ and } \sum_{n=1}^{\infty} r^n < \infty$$

In the second one we will try to show that there is no uniform convergence. Look at the partial sums f_n. If we can prove that $\{f_n\}$ is not a uniform Cauchy sequence then $\{f_n\}$ is not uniformly convergent and therefore the series will not converge uniformly. Often it suffices to look at $\|f_{n+1} - f_n\|_{\sup}$ and show that it does not converge to 0.

$$\|f_{n+1} - f_n\|_{\sup} = \sup_{x \in (-1,1)} \left| \sum_{i=1}^{n+1} x_i - \sum_{i=1}^{n} x^i \right| = \sup_{x \in (-1,1)} |x^{n+1}| = 1.$$

Therefore, take $\varepsilon = 1/2$, and $\forall N \,\varepsilon\, \mathbb{N}$ put $n = N + 1$ and $m = N + 1$. We see that $\|f_n - f_m\|_{\sup} = 1 > 1/2 = \varepsilon$.

In conclusion, use M-test to prove uniform convergence.

17.4 Power Series and Uniform Convergence

Recall, from Analysis 2, that a power series is the series of functions of the form $\sum_{n=1}^{\infty} a_n x^n$, where a_n is sequence of real numbers. We define a radius of convergence R of the series such that $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely on $(-R, R)$ and diverges for $|x| > R$.

Example: Consider $\sum_{n=0}^{\infty} x^n$. The series converges pointwise on (–1, 1), but this convergence is not uniform, whereas on $[-r, r]$ converges uniformly.

Theorem 6: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with a radius of convergence R. Then for any $0 \leq r < R$ the series converges uniformly on $[-r, r]$.

Proof: Fix $r \in (-R, R)$ and define a sequence $M_n = |a_n| r^n$. $\sum_{n=0}^{\infty} M_n$ converges absolutely by our choice of r and we get

$$\forall x \in [-r, r] : |a_n x^n| \leq |a_n| r^n = M_n \text{ and } \sum_{n=0}^{\infty} M_n < \infty.$$

Therefore by Weierstrass M-test, the power series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on [–r, r].

Notes

17.5 Continuous but Nowhere Differentiable Function

Theorem 7: There is a function f : $\mathbb{R} \to \mathbb{R}$ which is continuous but nowhere differentiable.

Proof: The idea of a proof is to find a function with a kind of fractal behaviour. Let g(x) = |x| on [–1, 1] extended by 2-periodicity on $\mathbb{R}$ and let

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n g(4^n x)$$

Figure 17.3

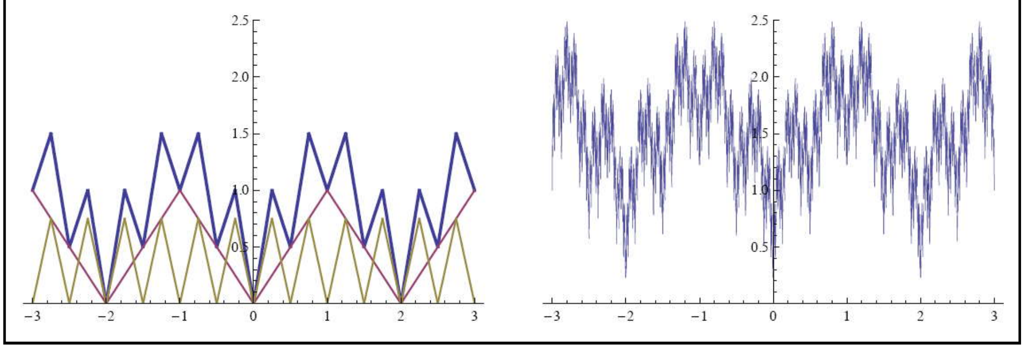

Figure 17.3 denotes the partial sums of f(x) by $s_n(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^i g(4^i x)$. On the left-hand side, we start with the red $s_0(x) = g(x)$. Then refine g(x) to the yellow $\frac{3}{4}g(4x)$. The iteration is obtained by adding these two together into the blue one, that is $s_1(x) = g(x) + \frac{3}{4}g(4x)$. $s_2(x)$ is obtained by adding refinement of $\frac{3}{4}g(4x)$ which is $\frac{9}{16}g(16x)$. Repeat this process at infinitum and get the limit function f(x) visualised on the right-hand side.

Now, we prove that the series is convergent and that the limit function f is continuous, but not differentiable.

Claim 1: The series $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n g(4^n x)$ converges uniformly on $\mathbb{R}$.

Since

$$\left|\left(\frac{3}{4}\right)^n g(4^n x)\right| \le \left(\frac{3}{4}\right)^n \text{ and } \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n < \infty$$

Claim 2: The limit function f is continuous on $\mathbb{R}$.

Firstly, we prove that f is continuous on arbitrary interval [–M, M]. Each practical sum $s_n(x) = \sum_{k=1}^{n}\left(\frac{3}{4}\right)^k g(4^k x)$ is continuous on [–M, M] and $s_n \to f$ uniformly. Then we see, that the limit function f is continuous on [–M, M]. So for any $x \in \mathbb{R}$ take sufficiently large s.t. $x \in (-M, M)$. Continuity of f on [–M, M] implies continuity in x. Therefore, f is continuous on $\mathbb{R}$.

Claim 3: The limit function f is not differentiable.

Let $x \in \mathbb{R}$. Let us show that f is not differentiable at x. We will construct a sequence h_m, such that $h_m \to 0$ and

$$\left|\frac{f(x+h_m)-f(x)}{h_m}\right| \to \infty \text{ as } m \to \infty$$

Consider interval $(4^m x - \frac{1}{2}, 4^m x + \frac{1}{2}]$. Clearly, it is a half-closed interval of length 1 and therefore, can contain, only 1 integer. Define

$$h_m = \begin{cases} +\frac{1}{2}4^{-m}, \text{if there is no integer in } (4^m x,\ 4^m x + \frac{1}{2}) \\ -\frac{1}{2}4^{-m}, \text{if there is no integer in } (4^m x,\ -\frac{1}{2}, 4^m x) \end{cases}$$

We see that $h_m \to 0$ as $m \to \infty$. Let us define a_n as

$$a_n = \left(\frac{3}{4}\right)^n \frac{g(4^n(x+h_m)) - g(4^n x)}{h_m}.$$

Then we can rewrite the derivative as

$$\left|\frac{f(x+h_m)-f(x)}{h_m}\right| = \left|\sum_{n=0}^{\infty} a_n\right|.$$

Note that $|g(x) - g(y)| = |x - y|$, if $x, y \in [k, k+1]$ for $k \in \mathbb{Z}$ and $g|(x) - g(y)| \leq |x - y|$, otherwise see Figure 17.4.

Figure 17.4

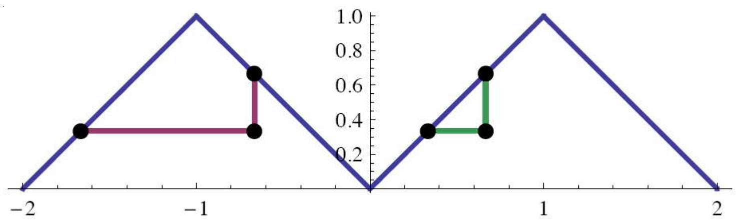

Figure that $|g(x) - g(x)| \leq |x - y|$ with equality when $x, y \in [k, k + 1]$, for some $k \in \mathbb{Z}$.

Let us prove the following three points,

(a) $a_n = 0$, if $n > m$

$$g(4^n(x + h_m)) - g(4^n x \pm \underbrace{\tfrac{1}{2}4^{n-m}}_{\text{div. by 2}}) - g(4^n x) = 0, \text{ due to 2-periodicity.}$$

(b) $|a_n| = 3^m = 3^n$, if $n = m$

$$g(4^n(x + h_m)) - g(4^n x \pm \underbrace{\tfrac{1}{2}4^{n-m}}_{=1} - g(4^n x) = 0,$$

$$= |g(4^m x \pm \frac{1}{2}) - g(4^m x)|.$$

According to the definition of h_m, we see that interval with endpoints $4^m x \pm \frac{1}{2}$ and $4^m x$ does not contain any integer so by Figure 17.4 we obtain

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$$\left|g(4^m x \pm \frac{1}{2}) - g(4^m x)\right| = \left|4^m x \pm \frac{1}{2} - 4^m x\right| = \frac{1}{2}$$

and finally

$$|a_n| = \left(\frac{3}{4}\right)^m \frac{1}{2} \frac{1}{\frac{1}{2}4^m} = 3^m = 3^n$$

(c) $|a_n| \le 3^n$, if $n < m$.

$$|g(4^n(x + h_m)) - g(4^n x)| = |g(4^n x \pm \frac{1}{2} 4^{n-m}) - g(4^n x)| \le |4^m x \pm \frac{1}{2} 4^{n-m} - 4^m x| = \frac{1}{2} 4^{n-m},$$

and therefore

$$|a_n| \le \left(\frac{3}{4}\right)^n 4^{m-n} \frac{1}{2} \frac{1}{\frac{1}{2}4^m} = 3^n$$

Putting all things together we get

$$\left|\frac{f(x+h_m) - f(x)}{h_m}\right| = \left|\sum_{n=0}^{\infty} a_n\right| = \left|a_1 + \dots + a_m \underbrace{+ a_{m+1} + a_{m+2} + \dots}_{= 0 \text{ by (a)}}\right| =$$

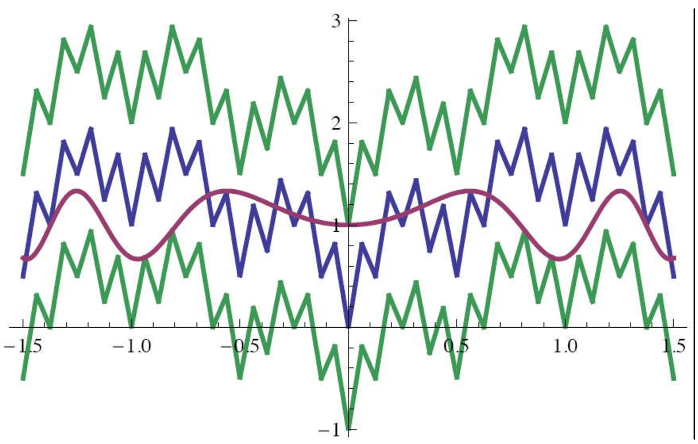
$$= ||a_1 + a_2 + \dots + a_m \ge |a_m| - |a_1 + \dots + a_{m-1}| \ge$$

$$\dots \ge |a_m| - |a_1| - |a_2| - \dots - |a_{m-1}| \overset{\text{by (b)}}{=} 3^m - |a_1| - |a_2| - \dots - |a_{m-1}| \ge$$

$$\overset{\text{by (c)}}{\ge} 3^m - 3^1 - 3^2 - \dots - 3^{m-1} = 3^m - \frac{3^m - 3}{3 - 1} = \frac{3}{2}(3^{m-1} - 1) \to \infty.$$

Remark: In original constructive proof of this theorem in 1872, Karl Weierstrab used f(x) = $\sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ with a a $\in$ (0, 1) and with positive odd integer b both satisfying ab > 1 + 3/4π. Interesting is, that despite of the differentiability of cosine, the limit function will not be differentiable anywhere.

Figure 17.5



Notes

In Figure 17.5, uniform approximation of continuous function f (in blue) by polynomial (in red) in e-tube around f (in green).

Self Assessment

Fill in the blanks:

1. Let $I \subset \mathbb{R}$ and $\{g_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on I. Then $\sum_{n=1}^{\infty} g_n$ is a ....
2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $I \subset \mathbb{R}$. Then .... is called a uniform Cauchy sequence if:

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n, m \geq N : \|f_n - f_m\|_{\sup} < \varepsilon.$$

3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $I \subset \mathbb{R}$. Then $\{f_n\}_{n=1}^{\infty}$ .... of I if and only if $\{f_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence on *I*.
4. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with a .... *R*. Then for any $0 \leq r < R$ the series converges uniformly on $[-r, r]$.
5. There is a function f : $\mathbb{R} \to \mathbb{R}$ which is continuous but nowhere ....

17.6 Summary

- Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued function on $[a, b]$. If (a) all f_n are differentiable, (b)′ all f_n' are continuous, (c) $f_n' \to h$ uniformly, for some function $h:[a, b] \to \mathbb{R}$, (d) $\exists c \in [a, b]$ s.t. $f_n(c)$ converges then f_n converges uniformly to some $f:[a, b] \to \mathbb{R}$, in addition, this uniform limit *f* is differentiable and $f'(x) = h(x)$.
- Let $I \subset \mathbb{R}$ and $\{g_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on I. Then

$$\sum_{n=1}^{\infty} g_n$$

  is a series of functions.
- Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with a radius of convergence *R*. Then for any $0 \leq r < R$ the series converges uniformly on $[-r, r]$.
- There is a function f : $\mathbb{R} \to \mathbb{R}$ which is continuous but nowhere differentiable.

17.7 Keywords

Series of Functions: Let $I \subset \mathbb{R}$ and $\{g_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on I. Then

$$\sum_{n=1}^{\infty} g_n$$

is a series of functions.

Continuity for Series: If (a) all g_n are are continuous and (b) $\sum_{n=1}^{\infty} g_n$ converges uniformly then $\sum_{n=1}^{\infty} g_n$ is continuous.

Notes

Differentiability for Series: If (a) all g_n are differentiable, (b) all g_n' are continuous, (c) $\sum_{n=1}^{\infty} g_n'$ converges uniformly and (d) $\exists c \in [a,b]$ s.t. $\sum_{n=1}^{\infty} g_n(c) < \infty$ then $\sum_{n=1}^{\infty} g_n$ converges uniformly and $\left(\sum_{n=1}^{\infty} gn\right)' = \sum_{n=1}^{\infty} g_n'$.

17.8 Review Questions

1. Prove that $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n}\sin(n^2 x)$.

2. Consider a series $\sum_{n=0}^{\infty} x^n$ on [–1/2, 1/2] and on (–1, 1). We want to investigate convergence of this series, whether it is pointwise of n uniform and eventually, what is the limit.

3. Consider series $\sum_{i=1}^{\infty} \frac{\sin(nx)}{2^n}$ on $\mathbb{R}$. By Weierstrass M-test, we see that this series converges uniformly on $\mathbb{R}$ since

$$\left|\frac{\sin(nx)}{2^n}\right| \le \frac{1}{2^n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

4. Consider series $\sum_{n=1}^{\infty} \frac{1}{n^2 + x}$ on $[0, \infty)$. By Weierstrass M-test, we can obtain uniform convergence of this series on $[0, \infty)$ since

$$\left|\frac{1}{n^2 + x}\right| \le \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Answers: Self Assessment

1. series of functions
2. $\{f_n\}_{n=1}^{\infty}$
3. converges uniformly
4. radius of convergence
5. differentiable

17.9 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 18: Equicontinuous

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Explain the equicontinuity
- Describe the properties of equicontinuous
- Discuss the equicontinuity and uniform convergence
- Define stochastic equicontinuity

Introduction

In last unit, you have studied about the uniform converges and differentiation. This unit provides you the explanation of Equicontinuity. In mathematical analysis, a family of functions is equicontinuous if all the functions are continuous and they have equal variation over a given neighbourhood, in a precise sense described herein. In particular, the concept applies to countable families, and thus sequences of functions.

18.1 Equicontinuity

The equicontinuity appears in the formulation of Ascoli's theorem, which states that a subset of C(X), the space of continuous functions on a compact Hausdorff space X, is compact if and only if it is closed, pointwise bounded and equicontinuous. A sequence in C(X) is uniformly convergent if and only if it is equicontinuous and converges pointwise to a function (not necessarily continuous a-prior). In particular, the limit of an equicontinuous pointwise convergent sequence of continuous functions f_n on either metric space or locally compact space is continuous. If, in addition, f_n are homomorphic, then the limit is also homomorphic.

The uniform boundedness principle states that a pointwise bounded family of continuous linear operators between Banach spaces is equicontinuous.

Notes

18.2 Families of Equicontinuous

Let X and Y be two metric spaces, and F a family of functions from X to Y.

The family F is equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in F$ and all x such that $d(x_0, x) < \delta$. The family is equicontinuous if it is equicontinuous at each point of X.

The family F is uniformly equicontinuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in F$ and all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$.

For comparison, the statement all functions f in F are continuous' means that for every $\varepsilon > 0$, every $f \in F$, and every $x_0 \in X$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $x \in X$ such that $d(x_0, x) < \delta$. So, for continuity, δ may depend on ε, x_0 and f; for equicontinuity, δ must be independent of f; and for uniform equicontinuity, δ must be independent of both f and x_0.

More generally, when X is a topological space, a set F of functions from X to Y is said to be equicontinuous at x if for every $\varepsilon > 0$, x has a neighbourhood U_x such that

$$d_Y(f(y), f(x)) < \varepsilon$$

for all $y \in U_x$ and $f \in F$. This definition usually appears in the context of topological vector spaces.

When X is compact, a set is uniformly equicontinuous if and only if it is equicontinuous at every point, for essentially the same reason as that uniform continuity and continuity coincide on compact spaces.

Some basic properties follow immediately from the definition. Every finite set of continuous functions is equicontinuous. The closure of an equicontinuous set is again equicontinuous. Every member of a uniformly equicontinuous set of functions is uniformly continuous, and every finite set of uniformly continuous functions is uniformly equicontinuous.

Example:

- A set of functions with the same Lipschitz constant is (uniformly) equicontinuous. In particular, this is the case if the set consists of functions with derivatives bounded by the same constant.
- Uniform boundedness principle gives a sufficient condition for a set of continuous linear operators to be equicontinuous.
- A family of iterates of an analytic function is equicontinuous on the Fatou set.

Properties of Equicontinuous

- If a subset $\mathcal{F} \subseteq C(X, Y)$ is totally bounded under the uniform metric, and then $\mathcal{F}$ is equicontinuous.
- Suppose X is compact. If a sequence of functions $\{f_n\}$ in $C(X \mathbb{R}k)$ is equibounded and equicontinuous, then the sequence $\{f_n\}$ has a uniformly convergent subsequence. (Arzelà's theorem)
- Let $\{f_n\}$ be a sequence of functions in C(X, Y). If $\{f_n\}$ is equicontinuous and converges pointwise to a function $f : X \to Y$, then f is continuous and $\{f_n\}$ converges to f in the compact-open topology.

18.3 Equicontinuity and Uniform Convergence

Let X be a compact Hausdorff space, and equip C(X) with the uniform norm, thus making C(X) a Banach space, hence a metric space. Then Ascoli's theorem states that a subset of C(X) is compact if and only if it is closed, pointwise bounded and equicontinuous. This is analogous to the Heine-Borel theorem, which states that subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded. Every bounded equicontinuous sequence in C(X) contains a subsequence that converges uniformly to a continuous function on X.

In view of Ascoli's theorem, a sequence in C(X) converges uniformly if and only if it is equicontinuous and converges pointwise. The hypothesis of the statement can be weakened a bit: a sequence in C(X) converges uniformly if it is equicontinuous and converges pointwise on a dense subset to some function on X (not assumed continuous). This weaker version is typically used to prove Ascoli's theorem for separable compact spaces. Another consequence is that the limit of an equicontinuous pointwise convergent sequence of continuous functions on a metric space, or on a locally compact space, is continuous.

In the above, the hypothesis of compactness of X cannot be relaxed. To see that, consider a compactly supported continuous function g on $\mathbb{R}$ with g(0) = 1, and consider the equicontinuous sequence of functions $\{f_n\}$ on $\mathbb{R}$ defined by $f_n(x) = g(x - n)$. Then, f_n converges pointwise to 0 but does not converge uniformly to 0.

This criterion for uniform convergence is often useful in real and complex analysis. Suppose we are given a sequence of continuous functions that converges pointwise on some open subset G of $\mathbb{R}^n$. As noted above, it actually converges uniformly on a compact subset of G if it is equicontinuous on the compact set.

In practice, showing the equicontinuity is often not so difficult. For example, if the sequence consists of differentiable functions or functions with some regularity (e.g., the functions are solutions of a differential equation), then the mean value theorem or some other kinds of estimates can be used to show the sequence is equicontinuous.

It then follows that the limit of the sequence is continuous on every compact subset of G; thus, continuous on G. A similar argument can be made when the functions are homomorphic. One can use, for instance, Cauchy's estimate to show the equicontinuity (on a compact subset) and conclude that the limit is homomorphism. Note that the equicontinuity is essential here. For example, $f_n(x)$ = arctan nx converges to a multiple of the discontinuous sign function.

18.4 Equicontinuity Families of Linear Operators

Let E, F be Banach spaces, and Γ be a family of continuous linear operators from E into F. Then Γ is equicontinuous if and only if

$$\text{Sup}\{\, ||T|| : T \in \Gamma \,\} < \infty$$

that is, Γ is uniformly bounded in operator norm. Also, by linearity, Γ is uniformly equicontinuous if and only if it is equicontinuous at 0.

The uniform boundedness principle (also known as the Banach-Steinhaus theorem) states that Γ is equicontinuous if it is pointwise bounded; i.e., $\sup\{\, ||T(x)|| : T \in \Gamma \,\} < \infty$ for each $x \in E$. The result can be generalized to a case when F is locally convex and E is a barreled space.

Alaoglu's theorem states that if E is a topological vector space, then every equicontinuous subset of E* is weak-* relatively compact.

Notes

18.5 Equicontinuity in Topological Spaces

The most general scenario in which equicontinuity can be defined is for topological spaces whereas *uniform* equicontinuity requires the filter of neighbourhoods of one point to be somehow comparable with the filter of neighbourhood of another point. The latter is most generally done via a uniform structure, giving a uniform space. Appropriate definitions in these cases are as follows:

A set A of functions continuous between two topological spaces X and Y is **topologically equicontinuous** at the points $x \in X$ and $y \in Y$ if for any open set O about y, there are neighbourhoods U of x and V of y such that for every $f \in A$, if the intersection of f[U] and V is non-empty, $f(U) \subseteq O$. One says A is said to be topologically equicontinuous at $x \in X$ if it is topologically equicontinuous at x and y for each $y \in Y$. Finally, A is equicontinuous if it is equicontinuous at x for all points $x \in X$.

A set A of continuous functions between two uniform spaces X and Y is **uniformly equicontinuous** if for every element W of the uniformity on Y, the set

$$\{(u, v) \in X \times X : \text{for all } f \in A. (f(u), f(v)) \in W\}$$

is a member of the uniformity on X

A weaker concept is that of even continuity:

A set A of continuous functions between two topological spaces X and Y is said to be **evenly continuous at** $x \in X$ and $y \in Y$ if given any open set O containing y there are neighbourhoods U of x and V of y such that $f[U] \subseteq O$ whenever $f(x) \in V$. It is **evenly continuous at** x if it is evenly continuous at x and y for every $y \in Y$, and **evenly continuous** if it is evenly continuous at x for every $x \in X$.

For metric spaces, there are standard topologies and uniform structures derived from the matrices, and then these general definitions are equivalent to the metric-space definitions.

18.6 Stochastic Equicontinuity

Stochastic equicontinuity is a version of equicontinuity used in the context of sequences of functions of random variables, and their convergence.

Let $\{H_n(\theta) : n \geq 1\}$ be a family of random functions defined from, where $\Theta \to \mathbb{R}$ where Θ is any normed metric space. Here $\{H_n(\theta)\}$ might represent a sequence of estimators applied to datasets of size n, given that the data arises from a population for which the parameter indexing the statistical model for the data is θ. The randomness of the functions arises from the data generating process under which a set of observed data is considered to be a realisation of a probabilistic or statistical model. However, in $\{H_n(\theta)\}$, θ relates to the model currently being postulated or fitted rather than to an underlying model which is supposed to represent the mechanism generating the data. Then $\{H_n\}$ is stochastically equicontinuous if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that:

$$\lim_{n\to\infty} \Pr\left(\sup_{\theta\in\Theta} \sup_{\theta'\in B(\theta,\delta)} |H_n(\theta') - H_n(\theta)| > \varepsilon \right) < \delta$$

Here $B(\theta, \delta)$ represents a ball in the parameter space, centered at θ and whose radius depends on.

Self Assessment

Fill in the blanks:

1. The ……………………..states that a pointwise bounded family of continuous linear operators between Banach spaces is equicontinuous.

2. The family F is ……………………$x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in F$ and all x such that $d(x_0, x) < \delta$. The family is equicontinuous if it is equicontinuous at each point of X.

3. Suppose X is compact. If a sequence of functions $\{f_n\}$ in C(X, ℝk) is equibounded and equicontinuous, then the sequence $\{f_n\}$ has a …………………………..

4. The uniform boundedness principle is also known as …………………….. states that Γ is equicontinuous if it is pointwise bounded; i.e., $\sup\{||T(x)|| : T \in \Gamma\} < \infty$ for each $x \in E$. The result can be generalized to a case when F is locally convex and E is a barreled space.

5. …………….. is a version of equicontinuity used in the context of sequences of functions of random variables, and their convergence.

18.7 Summary

- In particular, the limit of an equicontinuous pointwise convergent sequence of continuous functions f_n on either metric space or locally compact space is continuous. If, in addition, f_n are holomorphic, then the limit is also holomorphic.
- The uniform boundedness principle states that a pointwise bounded family of continuous linear operators between Banach spaces is equicontinuous.
- The family F is **equicontinuous at a point** $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in F$ and all x such that $d(x_0, x) < \delta$. The family is **equicontinuous** if it is equicontinuous at each point of X.
- The family F is **uniformly equicontinuous** if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in F$ and all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$.
- If a subset $\mathcal{F} \subseteq C(X, Y)$ is totally bounded under the uniform metric, and then $\mathcal{F}$ is equicontinuous.
- Suppose X is compact. If a sequence of functions $\{f_n\}$ in C(X, ℝk) is equibounded and equicontinuous, then the sequence $\{f_n\}$ has a uniformly convergent subsequence. (Arzelà's theorem)
- Let fn be a sequence of functions in C(X, Y). If $\{f_n\}$ is equicontinuous and converges pointwise to a function $f : X \to Y$, then f is continuous and $\{f_n\}$ converges to f in the compact-open topology.
- The uniform boundedness principle (also known as the Banach-Steinhaus theorem) states that Γ is equicontinuous if it is pointwise bounded; i.e., $\sup\{||T(x)|| : T \in \Gamma\} < \infty$ for each $x \in E$. The result can be generalized to a case when F is locally convex and E is a barreled space.
- Stochastic equicontinuity is a version of equicontinuity used in the context of sequences of functions of random variables, and their convergence.

18.8 Keywords

Stochastic Equicontinuity: Stochastic equicontinuity is a version of equicontinuity used in the context of sequences of functions of random variables, and their convergence

Uniformly Equicontinuous: The family F is uniformly equicontinuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in F$ and all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$.

Notes

Equicontinuous at a Point: The family F is equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in F$ and all x such that $d(x_0, x) < \delta$. The family is equicontinuous if it is equicontinuous at each point of X.

Uniform Boundedness: The uniform boundedness principle states that a pointwise bounded family of continuous linear operators between Banach spaces is equicontinuous.

18.9 Review Questions

1. Explain the Equicontinuity and Families of Equicontinuous.
2. Describe the Properties of equicontinuous.
3. Discuss the Equicontinuity and uniform convergence.
4. Describe Equicontinuity families of linear operators.
5. Explain the Equicontinuity in topological spaces.
6. Define Stochastic equicontinuity.

Answers: Self Assessment

1. uniform boundedness principle
2. equicontinuous at a point
3. uniformly convergent subsequence
4. the Banach-Steinhaus theorem
5. Stochastic equicontinuity

18.10 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 19: Arzelà's Theorem and Weierstrass Approximation Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the Arzelà's Theorem
- Describe the Weierstrass Approximation Theorem

Introduction

In last unit you have studied about the uniform convergence and Equicontinuity. This unit provides you the explanation of Arzelà's Theorem and Weierstrass Approximation theorem. Our setting is a compact metric space *X* which you can, if you wish, take to be a compact subset of R*n*, or even of the complex plane (with the Euclidean metric, of course). Let *C*(*X*) denotes the space of all continuous functions on *X* with values in C (equally well, you can take the values to lie in R). In *C*(*X*) we always regard the distance between functions *f* and *g* in *C*(*X*) to be

$$\text{dist}(f, g) = \max\{|f(x) - g(x)| : x \in X\}$$

19.1 Arzelà–Ascoli Theorem

A sequence $\{f_n\}_{n \in N}$ of continuous functions on an interval I = [a, b] is uniformly bounded if there is a number M such that

$$|f_n(x)| \leq M$$

for every function f_n belonging to the sequence, and every $x \in [a, b]$. The sequence is *equicontinuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \varepsilon \text{ Whenever } |x - y| < \delta$$

for every f_n belonging to the sequence. Succinctly, a sequence is equicontinuous if and only if all of its elements have the same modulus of continuity. In simplest terms, the theorem can be stated as follows:

Consider a sequence of real-valued continuous functions $(f_n)_{n \in N}$ defined on a closed and bounded interval [a, b] of the real line. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence (f_{nk}) that converges uniformly.

Notes

Proof: The proof is essentially based on a diagonalization argument. The simplest case is of real-valued functions on a closed and bounded interval:

Let I = [a, b] ⊂ R be a closed and bounded interval. If F is an infinite set of functions $f : I \to R$ which is uniformly bounded and equicontinuous, then there is a sequence f_n of elements of F such that f_n converges uniformly on I.

Fix an enumeration $\{x_i\}_{i = 1,2,3,...}$ of rational numbers in I. Since F is uniformly bounded, the set of points $\{f(x_1)\}_{f \in F}$ is bounded, and hence by the Bolzano-Weierstrass theorem, there is a sequence $\{f_{n1}\}$ of distinct functions in F such that $\{f_{n1}(x_1)\}$ converges. Repeating the same argument for the sequence of points $\{f_{n1}(x_2)\}$, there is a subsequence $\{f_{n2}\}$ of $\{f_{n1}\}$ such that $\{f_{n2}(x_2)\}$ converges.

By mathematical induction this process can be continued, and so there is a chain of subsequences

$$\{f_{n1}\} \supset \{f_{n2}\} \supset \ldots$$

such that, for each k = 1, 2, 3,…, the subsequence $\{f_{nk}\}$ converges at $x_1,...,x_k$. Now form the diagonal subsequence {f} whose mth term f_m is the mth term in the mth subsequence $\{f_{nm}\}$. By construction, f_m converges at every rational point of I.

Therefore, given any ε > 0 and rational x_k in I, there is an integer $N = N(\varepsilon, x_k)$ such that

$$|f_n(x_k) - f_m(x_k)| < \varepsilon/3, \quad n, m \geq N.$$

Since the family F is equicontinuous, for this fixed å and for every x in I, there is an open interval U_x containing x such that

$$|f(s) - f(t)| < \varepsilon/3$$

for all f ∈ F and all s, t in I such that s, t ∈ U_x.

The collection of intervals U_x, x ∈ I, forms an open cover of I. Since I is compact, this covering admits a finite subcover U_1, ..., U_J. There exists an integer K such that each open interval U_j, 1 ≤ j ≤ J, contains a rational x_k with 1 ≤ k ≤ K. Finally, for any t ∈ I, there are j and k so that t and x_k belong to the same interval U_j. For this choice of k,

$$|f_n(t) - f_m(t)| \leq |f_n(t) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(t)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

for all n, m > N = max{N(ε, x_1), ..., N(ε, x_K)}. Consequently, the sequence $\{f_n\}$ is uniformly Cauchy, and therefore converges to a continuous function, as claimed. This completes the proof.

Theorem 1: Weierstrass Approximation Theorem

Let f: [a, b] → ℝ be continuous. Then there is a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ such that $p_n \to f$ uniformly.

> *Notes* It is important that [a, b] is a closed interval. If it was open, we could take (0, 1) and f(x) = 1/x, which is unbounded. But every polynomial is bounded on (0, 1) and therefore no sequence of polynomials could converge to f uniformly.

It will suffice to prove Weierstrass Approximation Theorem on [0, 1] from which the general case can be easily obtained. Recall the notion of uniform continuity from Analysis 1.

Let I ⊂ ℝ and f be a real-valued function on I. We say that f is uniformly continuous on I if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, y \in I : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Also remind, that a continuous function on [a, b] is always uniformly continuous.

Notes

Definition 1: Define

$$p_{kn}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \ \forall\, n \in \mathbb{N} \text{ and } 0 \le k \le n.$$

Note that, $p_{kn}(x)$ becomes a probability mass function of binomial distribution with probability of successful trial equal to x if $x \in [0, 1]$.

Lemma: (a) $\forall\, n \in \mathbb{N} : \sum_{k=0}^{n} p_{kn}(x) = 1$, (b) $\forall\, n \in \mathbb{N} : \sum_{k=0}^{n} k_{pkn}(x) = nx$, (c) $\forall\, n \in \mathbb{N} : \sum_{k=0}^{n} (k - nx)^2$ $p_{kn}(x) = nx(1 - x)$.

Proof: (a) If $x \in [0, 1]$ the equality follows from normalisation of probability distribution. In general

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

therefore

$$\sum_{k=0}^{n} p_{kn}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} = (x+1-x)^n = 1.$$

(b) If $x \in [0, 1]$, we can define a random variable $Y_{n,x}$ the number of heads observed on unfair x-coin tossed n-times. Then

$$\mathbb{P}(Y_{n,x} = k) = \binom{n}{k} x^k(1-x)^{n-k} = p_{kn}(x).$$

Moreover, we find the following relation with (b)

$$\mathbb{E}[Y_{n,x}] = \sum_{k=0}^{n} k_{p_{kn}}(x) = nx.$$

In general

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = n\binom{n-1}{k-1},$$

so

$$\sum_{k=0}^{n} k_{p_{kn}}(x) = \sum_{k=0}^{n} k\binom{n}{k} x^k(1-x)^{n-k} = \sum_{k=0}^{n} n\binom{n-1}{k-1} x^k(1-x)^{n-k} =$$

$$= \sum_{k=1}^{n} n\binom{n-1}{k-1} x^k(1-x)^{n-k} = nx\sum_{k=1}^{n}\binom{n-1}{k-1} x^{k-1}(1-x)^{n-k} =$$

$$= nx\sum_{i=0}^{n-1}\binom{n-1}{i} x^i(1-x)^{n-1-i} = nx(x+1-x)^{n-1} = nx.$$

(c) If $x \in [0, 1]$, we can rewrite the formula as

$$\mathrm{Var}(Y_{n,x}) = \sum_{k=0}^{n} (k-nx)^2 p_{kn}(x) = nx\,(1-x).$$

In general

$$k(k-1)\binom{n}{k} = n(n-1)\frac{(n-2)!}{(k-2)!((n-2)-(k-2))!} = n(n-1)\binom{n-2}{k-2},$$

Notes

$$\sum_{k=0}^{n} k(k-1)p_{kn}(x) = \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k(1-x)^{n-k}$$

$$= \sum_{k=2}^{n} n(n-1)\binom{n-2}{k-2} x^k(1-x)^{n-k}$$

$$= n(n-1)x^2 \sum_{k=2}^{n}\binom{n-2}{k-2} x^{k-2}(1-x)^{n-k}$$

$$= n(n-1)x^2 \sum_{i=0}^{n-2}\binom{n-2}{i} x^i(1-x)^{n-2-i} = n(n-1)x^2.$$

Hence

$$\sum_{k=0}^{n}(k-nx)^2p_{kn}(x) = \sum_{k=0}^{n}(k^2-2knx+n^2x^2)p_{kn}(x)$$

$$= \sum_{k=0}^{n}(k(k-1)+k-2knx+n^2x^2)p_{kn}(x)$$

$$= n(n-1)x^2 + nx - 2nxnx - n^2x$$

$$n^2x - nx^2 + nx - 2n^2x^2 + n^2x^2 = nx(1-x).$$

Definition 2: For any f: [0, 1] $\to \mathbb{R}$ define its Bernstein polynomials $B_n^f(x)$ such that

$$B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{kn}(x).$$

Theorem 2: Weierstrass Approximation Theorem, special case

Let f be a real-valued function on [0, 1]. If f is continuous then $B_n^f \to f$ uniformly.

Proof: We want

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ \ \forall x \in [0,1] : |\ B_n^f(x) - f(x)\ | < \varepsilon.$$

Let ε > 0 be given. Since f is uniformly continuous on [0, 1]

$$\exists 5 > 0\ \ \forall x, y \in [0,1] : |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon/2.$$

Using this fact we can estimate

$$|B_n^f(x) - f(x)| = \left|\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{kn}(x) - f(x)\underbrace{\sum_{k=0}^{n} p_{kn}(x)}\right|$$

$$= \left|\sum_{k=0}^{n}(f(k/n)-f(x))p_{kn}(x)\right| \leq \sum_{k=0}^{n}\left|(f(k/n)-f(x)\right|p_{kn}(x)$$

$$= \sum_{k:\left|\frac{k}{n}-x\right|<\delta}^{n}\left|(f(k/n)-f(x)\right|p_{kn}(x) + \sum_{k:\left|\frac{k}{n}-x\right|\geq\delta}^{n}\left|(f(k/n)-f(x)\right|p_{kn}(x)$$

$$< \frac{\varepsilon}{2}\sum_{k:\left|\frac{k}{n}-x\right|<\delta}^{n} p_{kn}(x) + 2\|f\|_{sup}\sum_{k:\left|\frac{k}{n}-x\right|\geq\delta}^{n} 1\cdot p_{kn}(x).$$

We used estimate

$$|f(k/n) - f(x)| \leq |f(k/n)| + |f(x)| \leq 2\,\|f\|_{sup}.$$

Now observe that in the second sum we have the following condition on k

Notes

$$|k/n - x| \geq \delta \Rightarrow \left(\frac{k-nx}{n}\right)^2 \geq \delta^2 \Rightarrow \frac{(k-nx)^2}{n^2\delta^2} \geq 1.$$

By using this remark in the second sum and by increasing number of summants in the first sum we get

$$\frac{\varepsilon}{2}\sum_{k=0}^{n} p_{kn}(x) + 2\|f\|_{\sup} \sum_{k:\left|\frac{k}{n}-x\right|\geq\delta}^{n} 1\cdot p_{kn}(x)$$

$$\leq \frac{\varepsilon}{2}\cdot 1 + \frac{2\|f\|_{\sup}}{n^2\delta^2} \sum_{k:\left|\frac{k}{n}-x\right|\geq\delta}^{n} (k-nx)^2 p_{kn}(x)$$

$$\leq \frac{\varepsilon}{2} + \frac{2\|f\|_{\sup}}{n^2\delta^2} \sum_{k=0}^{n} (k-nx)^2 p_{kn}(x) = \frac{\varepsilon}{2} + \frac{2\|f\|_{\sup}}{n^2\delta^2} n \underbrace{x(1-x)}_{\leq 1} \leq \frac{\varepsilon}{2} + \frac{2\|f\|_{\sup}}{n\delta^2}.$$

Let N be such that $\frac{2\|f\|_{\sup}}{N\delta^2} < \frac{\varepsilon}{2}$. Then $\forall n \geq N$ and $\forall x \in [0, 1]$

$$\left|B_n^f(x) - f(x)\right| < \frac{\varepsilon}{2} + \frac{2\|f\|_{\sup}}{n\delta^2} \leq \frac{\varepsilon}{2} + \frac{2\|f\|_{\sup}}{N\delta^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Proof: In this proof we shrink our function f to [0, 1], where it can be approximated uniformly by Bernstein polynomials and then scale these polynomials from [0, 1] to [a, b] where they will approximate the original function. Define $g : [0, 1] \to \mathbb{R}$, $g(t) = f(x(t))$. Where $x(t) = a + (b - a)t$ for $t \in [0, 1]$. We see that g is continuous since it is a composite of two continuous functions. As $B_n^g \to$ g uniformly. Define $q_n(x) = B_n^g(t(x)) = B_n^g\left(\frac{x-a}{b-a}\right)$. Then

$$\|a_n - f\|_{\sup} = \sup_{x\in[a,b]}|q_n(x) - f(x)| = \sup_{x\in[a,b]}\left|B_n^g(t(x)) - g(t(x))\right|$$

$$= \sup_{t\in[0,1]}\left|B_n^g(t) - g(t)\right| = \left\|B_n^g - g\right\|_{\sup} \to 0, \text{ since } B_n^g \to g \text{ uniformly.}$$

Figure 19.1

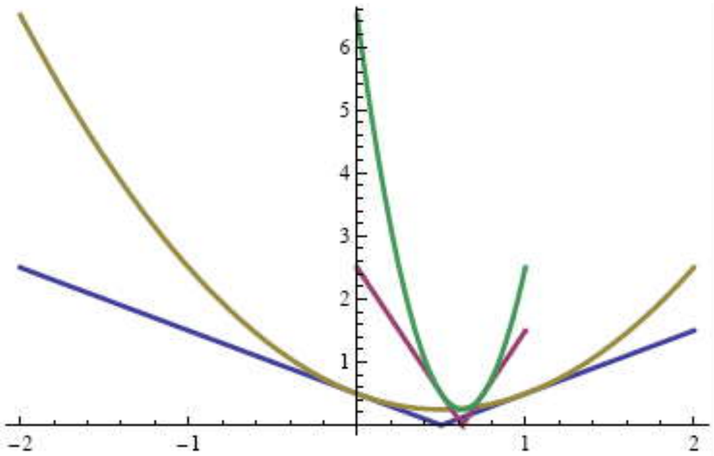
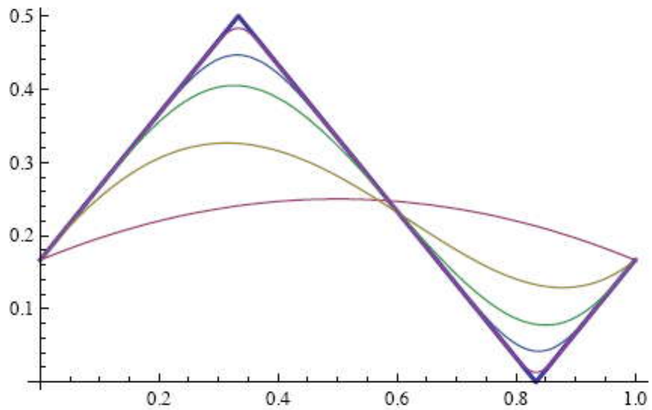


Figure 19.1, the original function f(x) = | x – 1/2 | on [–2, 2], is shrinked to [0, 1], and approximated uniformly by Bernstein polynomials. These polynomials are then scaled to [–2, 2], to approximate uniformly the original function f on [–2, 2].

Notes

Figure 19.2: Approximation of f(x) = | − |x − 1/3| + 1/2| on [0,1] by B2(x),BI(x), B (5(x),B10(x) and Bi500(x)



19.2 Fourier Series

Firstly, let us look at some definitions. We denote the set of all Riemann integrable functions on [a, b] by $\Re[a, b]$.

Definition 3: For any f, g $\in \Re[a, b]$ define inner product of f and g $\langle .,. \rangle$ such that

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

Recall from Linear Algebra that the inner product space is finite-dimensional vector space V equipped with mapping $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ (or $\mathbb{R}$), satisfying these three properties:

1. $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$
2. $\langle v, u \rangle = \overline{\langle v, u \rangle}$
3. $\langle u, u \rangle \in \mathbb{R}$ and $\langle u, u \rangle \geq 0$ with equality $\Leftrightarrow u = 0$

We see that our inner product does not satisfy all three properties since $\langle f, f \rangle =$

$0 \Leftrightarrow f(x) = 0$ does not hold. It suffices to take $f(x) = \begin{cases} 1 & \text{for } x \in [0,1) \\ 0 & \text{for } x = 1 \end{cases}$.

Definition 4: We define the two-norm $\|\cdot\|_2$ on f $\in \Re[a, b]$ such that

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f(x)^2 dx}.$$

Definition 5: A collection of Riemman integrable functions $\{\phi_n\}_{n=1}^{\infty}$ on [a, b] is called an orthogonal system if

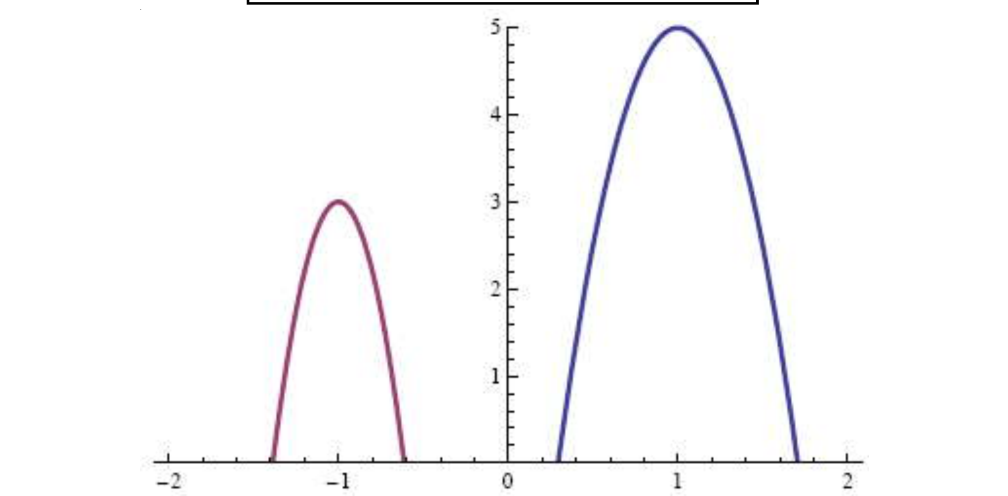
$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)dx = 0, \qquad \forall m \neq n.$$

If in addition $\forall n \in \mathbb{N} : \|\phi_n\|_2 = 1$ we call $\{\phi_n\}_{n=1}^{\infty}$ an orthonormal system.

Example: Consider two continuous functions as on Figure 17.3. We have fg = 0, hence $\langle f,g \rangle = 0$. Therefore, they are orthogonal.

Notes

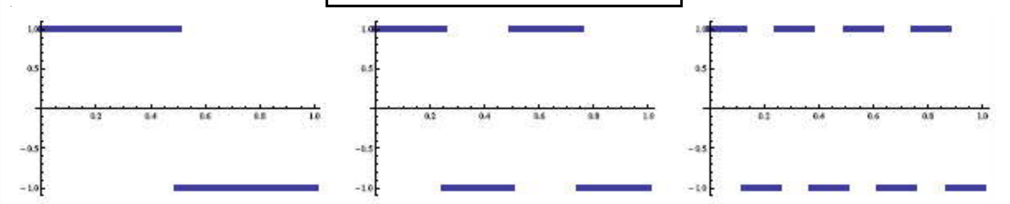
Figure 19.3: Two Orthogonal Continuous Functions on [−2,2]



Example: A collection $\Xi = \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx)\right\}, n \in \mathbb{N}$, is called the trigonometrical orthonormal system on $[-\pi, \pi]$, since $\|f\|_2 = 1$ for all $f \in \Xi$ and $\langle f, g\rangle = 0$ for all $f \neq g$.

Example: For another example of orthonormal system see Figure 19.4.

Figure 19.4



Orthonormal system of functions ϕ_n: [0, 1] → {–1, 1}. Each ϕ_n divides interval [0, 1] into $1/2^n$ subintervals. $\int_0^1 \phi_n(x)^2 dx = 1$ and $\int_0^1 \phi_n(x)\phi_m(x)dx = 0$ if $n \neq m$

Definition 6: Let $\{\phi_n\}_{n=1}^{\infty}$ be an o.n.s. on [a,b] and $f \in \mathfrak{R}[a, b]$. We define Fourier coefficients of f w.r.t. $\{\phi_n\}_{n=1}^{\infty}$ as

$$a_n = \langle f, \phi_n\rangle = \int_a^b f(x)\phi(x)dx, \qquad n \in \mathbb{N}.$$

$\sum_{n=1}^{\infty} a_n\phi_n$ is called the Fourier series of f w.r.t. $\{\phi_n\}_{n=1}^{\infty}$.

Notes

1. $\sum_{n=1}^{\infty} a_n\phi_n$ does not necessarily converge.
2. f(x) is not necessarily equal to its Fourier series.

Notes

Example: Let f(x) = x on [0, 1] and $\{\phi_n\}_{n=1}^{\infty}$. We get Fourier coefficients $a_n = \langle x, \phi_n \rangle = \int_0^1 x\phi(x)dx = -\left(\frac{1}{2^n}\right)^2 \frac{2^n}{2} = -\frac{1}{2^{n+1}}$. Therefore, we can compute Fourier series for f(x) = x which is $-\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \phi_n(x)$.

Self Assessment

Fill in the blanks:

1. For any f: [0, 1] → ℝ define its .... $B_n^f(x)$ such that

$$B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{kn}(x).$$

2. Let f be a real-valued function on [0, 1]. If f is continuous then ....

3. Define the two—norm $\|\cdot\|_2$ on $f \in \Re[a, b]$ such that ....

4. A collection of Riemman integrable functions $\{\phi_n\}_{n=1}^{\infty}$ on [a, b] is called an ....

   $\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)dx = 0, \quad \forall m \neq n.$

19.3 Summary

- Let $I \subset \mathbb{R}$ and f be a real-valued function on I. We say that f is uniformly continuous on I if

  $$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in I : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

  Also remind, that a continuous function on [a, b] is always uniformly continuous.

- $\forall n \in \mathbb{N} : \sum_{k=0}^{n} p_{kn}(x) = 1$, (b) $\forall n \in \mathbb{N} : \sum_{k=0}^{n} k_{pkn}(x) = nx$, (c) $\forall n \in \mathbb{N} : \sum_{k=0}^{n} (k - nx)^2 p_{kn}(x) = nx(1 - x)$.

  If $x \in [0, 1]$, we can define a random variable $Y_{n,x}$ the number of heads observed on unfair x-coin tossed n-times. Then

  $$\mathbb{P}(Y_{n,x} = k) = \binom{n}{k} x^k (1 - x)^{n-k} = p_{kn}(x).$$

  Moreover, we find the following relation with (b)

  $$\mathbb{E}[Y_{n,x}] = \sum_{k=0}^{n} k_{Pkn}(x) = nx.$$

- For any $f, g \in \Re[a, b]$ define inner product of f and g ⟨.,.⟩ such that

  $$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

- Recall from Linear Algebra that the inner product space is finite-dimensional vector space V equipped with mapping $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ (or ℝ).

19.4 Keyword

Fourier Series: For any $f, g \in \Re[a, b]$ define inner product of f and g ⟨.,.⟩ such that

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

Notes

Recall from Linear Algebra that the inner product space is finite-dimensional vector space V equipped with mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ (or $\mathbb{R}$), satisfying these three properties.

19.5 Review Questions

1. The Arzela-Ascoli Theorem is the key to the following result: A subset F of C(X) is compact if and only if it is closed, bounded, and equicontinuous. Prove this.

2. You can think of Rn as (real-valued) C(X) where X is a set containing n points, and the metric on X is the discrete metric (the distance between any two different points is 1). The metric thus induced on Rn is equivalent to, but (unless n = 1) not the same as, the Euclidean one, and a subset of Rn is bounded in the usual Euclidean way if and only if it is bounded in this C(X). Show that every bounded subset of this C(X) is equicontinuous, thus establishing the Bolzano-Weierstrass theorem as a generalization of the Arzela-Ascoli Theorem.

3. Let f(x) = x on [0, 1] and let $\{\phi_n\}_{n=1}^{\infty}$ be as in Ex. 2.3. We get Fourier coefficients $a_n = \langle x, \phi_n \rangle$ $= \int_0^1 x\phi(x)dx = -\left(\frac{1}{2^n}\right)^2 \frac{2^n}{2} = -\frac{1}{2^{n+1}}$, (computation of the integral is left as exercise). Therefore, we can compute Fourier series for f(x) = x which is $-\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \phi_n(x)$.

Answer: Self Assessment

1. Bernstein polynomials
2. $B_n^f \to f$ uniformly
3. $\|f\|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f(x)^2 dx}$.
4. Orthogonal System

19.6 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 20: The Riemann Integration

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define the Riemann Integral of a function
- Derive the conditions of Integrability and determine the class of functions which are always integrable
- Discuss the algebra of integrable functions
- Compute the integral as a limit of a sum

Introduction

You are quite familiar with the words 'differentiation' and distinguishing 'integration'. You know that in ordinary language, differentiation refers to separating things while integration means putting things together. In Mathematics, particularly in Calculus and Analysis, differentiation and integration are considered as some kind of operations on functions. You have used these operations in our study of Calculus.

There are essentially two ways of describing the operation of integration. One way is to view it as the inverse operation of differentiation. The other way is to treat it as some sort of limit of a sum.

The first view gives rise to an integral which is the result of reversing the process of differentiation. This is the view which was generally considered during the eighteenth century.

Accordingly, the method is to obtain, from a given function, another function which has the first function as its derivative. This second function, if it be obtained, is called the indefinite integral of the first function. This is also called the 'primitive' or anti-derivative of the first function. Thus, the integral of a function $f(x)$ is obtained by finding an anti-derivative or primitive function $F(x)$ show that $F'(x) = f(x)$. The indefinite integral of $f(x)$, is symbolized by the notation $\int f(x)\,dx$.

The second view is related to the limiting process. It gives rise to an integral which is the limit of all the values of a function in an interval. This is the integral of a function f(x) over an interval [a,b,]. It is called the definite integral and is denoted by

$$\int_a^b f(x)dx$$

The definite integral is a number since geometrically it corresponds an area of a region enclosed by the graph of a function.

Although both the notions of integration are closely related, yet, you will see later, the definite integral turns out to be a mare fundamental concept. In fact, it is the starting point for some important generalizations like the double integrals, triple integrals, line integrals etc., which you may study on Advanced Calculus.

The integral in the anti-derivative sense was given by Neyrtan. This notion was found to be adequate so long as the functions to be integrated were continuous. But in the early 19th century, Fourier brought to light the need for making integration meaningful for the functions that are not continuous. He came across such functions in applied problems. Cauchy formulated rigorous definition of the integral of a function. He essentially provided a general theory of integration but only for continuous functions. Cauchy's theory of Integration for continuous functions is sufficient for piece-wise continuous functions as well as for the functions having isolated discontinuities. However, it was G.B.F. Riemann [1826-1866] a German mathematician who extended Cauchy's integral to the discontinuous functions also. Riernann answered the question "what is the meaning of $\int f(x)\,dx$?"

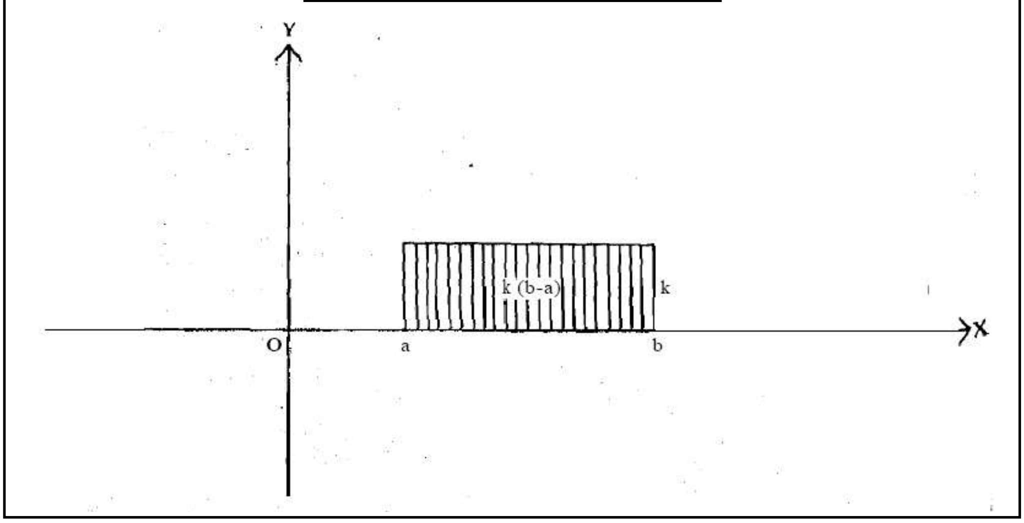
The concept of definite integral was given by Riemann in the middle of the nineteenth century. That is why, it is called Riemann Integral. Towards the end of 19th Century, T.J. Stieltjes [1856-1894] of Holland, introduced a broader concept of integration replacing certain linear functions used in Riemann Integral by functions of more general forms. In the beginning of this century, the notion of the measure of a set of real numbers paved the way to the foundation of modern theory of Lebesgue Integral by an eminent French Mathematician H. Lebesgue [1875-1941], a beautiful generalisation of Riemann Integral which you may study in some advanced courses of Mathematics. In this unit, the Riemann Integral will be defined without bringing in the idea of differentiation. As you have been go through the usual connection between the Integration and Differentiation. Just by applying the definition, it is not always easy to test the integrability of a function. Therefore, condition of integrability will be derived with the help of which it becomes easier to discuss the integrability of functions. Then just as in the case of continuity and derivability, we will also consider algebra of integrable functions. Finally, in this unit, second definition of integral as the limit of a sum will be given to you and you will be shown the equivalence of the two definitions.

20.1 Riemann Integration

The study of the integral began with the geometrical consideration of calculating areas of plane figures. You know that the well-known formula for computing the area of a rectangle is equal to the product of the length and breadth of the rectangle. The question that arises from this formula is that of finding the correct modification of this formula which we can apply to other plane figures. To do so, consider a function defined on a closed interval [a,b] of the real line, which assumes a constant value $K \geq 0$ throughout the interval. The graph of such a function gives rise to a rectangular region bounded by the X-axis and the ordinates $x = a$, $x = b$ as shown in the Figure 20.1.

Notes

Figure 20.1



Obviously, the area enclosed is k (b-a). NOW, suppose that]a, b[is further divided into smaller intervals by inserting points of division, say

$$a = x, \le x_1 \le x_2 \le x_3 \le x_4 = b,$$

and the function f is defined so as to take a constant value at each of the resulting sub-intervals i.e.,

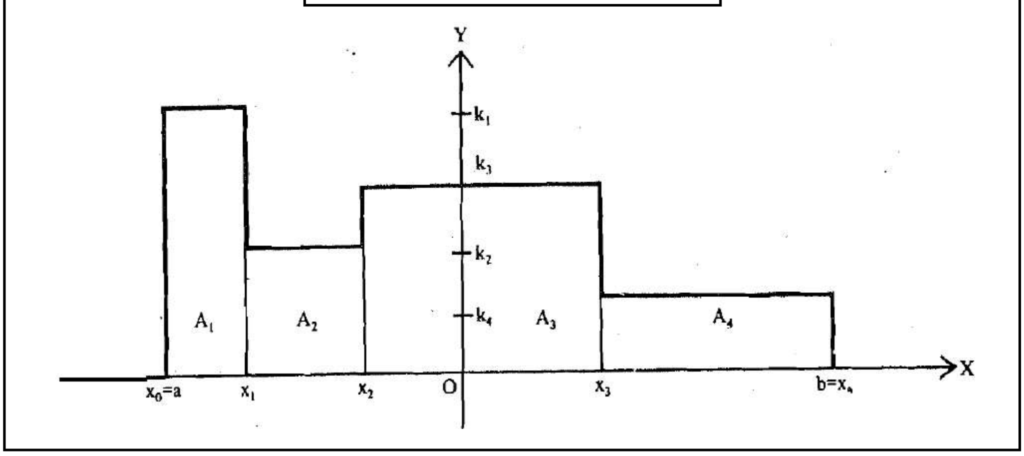
$$f(x) = \begin{cases} k_1, \text{ if } x \in [x_0, x_1[\\ k_2, \text{ if } x \in [x_1, x_2[\\ k_3, \text{ if } x \in [x_2, x_3[\\ k_4, \text{ if } x \in [x_1, x_4] \end{cases}$$

Further, suppose that d_i = length of the ith interval $]x_i, x_{i-1}[$ i.e.,

$d_1 = |x_1 - x_0|, d_2 = |x_2 - x_1|, d_3 = |x_3 - x_2|$, and $d_4 = |x_4 - x_3|$.

Then, we get four rectangular regions and the area of each region is A, = $k_1 d_1$, A, = $k_2 d_2$, A, = $k_3 d_3$, and A, = $k_4 d_4$, as shown in Figure 20.2.

Figure 20.2



Notes

The total area enclosed by the graph of the function, X-axis and the ordinates x=a, x=b is equal to the sum of these areas i.e.

$$\text{Area} = A_1 + A_2 + A_3 + A_4$$

$$= k_1d_1 + k_2d_2 + k_3d_3 + k_4d_4.$$

Note that in the last equation, we have generalized the notion of area. In other words, we are able to compute the area of a region which is not of rectangular shape. How did we get it? By breaking up the region into a series of non-overlapping rectangles which include the totality of the figure and summing up their respective areas. This is simply a slight obstraction of the same process which is used in Geometry.

Since the graph of the function in figure 20.2 consists of 4 different steps, such a function, is called a step function. What we have obtained is the area of a region bounded by

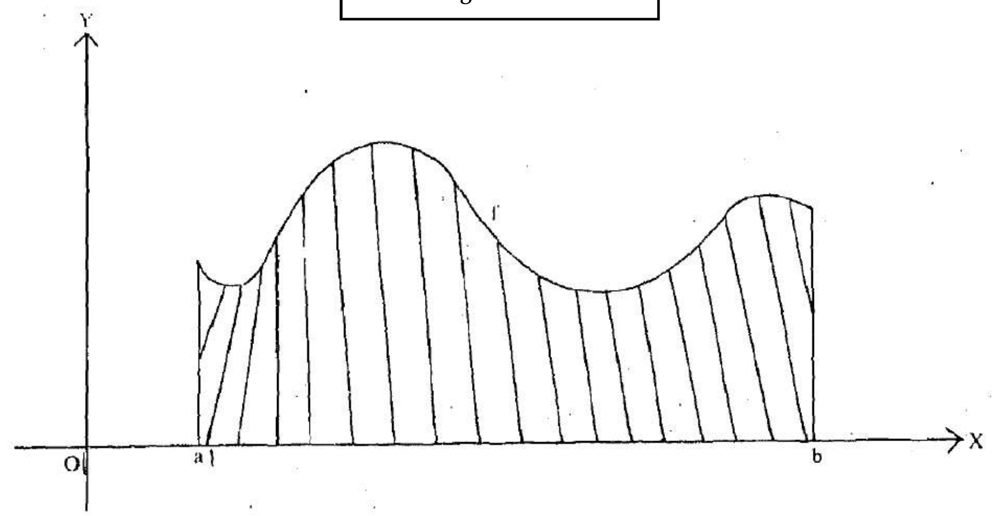
1. a non-negative step function
2. the vertical lines defined by x=a and x=b
3. the X-axis.

This area is just the sum of the areas of a finite number of f non-overlapping rectangles resulting from the graph of the given function. The area is nothing but a real number.

Now suppose that the graph of a given function is as shown in the Figure 20.3.

Does it make any sense to obtain the area of the region under the graph off? If so, how can we compute its value? To answer this question, we introduce the notion of the integral of a function as given by Riemann.

Figure 20.3



To introduce the notion of an integral of a function, we will require such a real number which results for applying the function and which represents the area of the region bounded by the graph off, the vertical lines x=a, x=b and the X-axis. This can be achieved by approximating the given function by suitable step functions. The area of the region will, then, be approximated by the areas enclosed by these step functions, which in turn are obtained as sum of the areas of non-overlapping rectangles as we have computed for the Figure 20.2. This is precisely the idea behind the formal treatment of the integral which we discuss here. First, we introduce some terminology and basic notions which will be used throughout the discussion.

Let f be a real function defined and bounded on a closed interval [a,b].

Recall that a real function f is said to be bounded if the range of f is a bounded subset of R, that is, if there exist numbers m and M such that $m \le f(x) \le M$ for each $x \in [a,b]$. M is an upper bound and m is a lower bound of f in [a,b]. You also know that when f is bounded, its supremum and infimum exist. We introduce the concept of a partition of [a,b] and other related definitions:

Definition 1: Partition

Let [a,b] be a given interval. By a partition P of [a,b] we mean a finite set of points $\{x_0, x_l, ..., x,\}$, where

$$a = x_0, < x_1 < \ldots < x_{n-1} < x_n = b.$$

We write $\Delta x_i = x_i - x_{i-1}$, (i=l, 2, ..., n). So Δx_i is the length of the ith sub-interval given by the partition P.

Definition 2: Norm of a Partition

Norm of a partition P, denoted by $|P|$, is defined by $|P| = \max_{t \le i \le n} Ax,$. Namely, the norm of P is the length of largest sub-interval of [a, b] induced by P. Norm of P is also denoted by $\mu(P)$.

There is a one-to-one correspondence between the partitions of [a,b] and finite subsets of]a, b[. This induces a partial ordering on the set of partitions of [a,b]. So, we have the following definition.

Definltion 3: Refinement of a Partition

Let P, and P, be two partitions of [a,b]. We say that P, is finer than P, or P_2, refines P, or P_2 is a refinement of P_1 if $P_1 \subset P_2$, that is, every point of P_1 is a point of P,.

You may note that, if P, and P_2, are any two partitions of [a,b], then P, $\cup$ P_2 is a common refinement of P, and P_2. For example, if $P_1 = \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$ and $P_2 = \left\{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$ are partitions of [0, 1], then P_2 is a refinement of P_1 and $P_1 \cup P_2 = \left\{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$ is their common refinement.

We now introduce the notions of upper sums and lower sums of a bounded function f on an interval [a, b], as given by Darboux. These are sometimes referred to as Darboux Sums.

Definition 4: Upper and Lower Sums

Let f: $[a,b] \to R$ be a bounded function, and let P = (x_0, X_1 ... x,) be a partition of [a,b]. For i = 1, 2 ....., n, let M_i and m_i be defined by

$$M_i = \text{lub } (f(x) : x_{i-1} \le x \le x_i)$$

$$m_i = \text{glb } (f(x) : x_{i-1} \le x \le x_i)$$

i.e. M_i and m_i be the supremum and infimum of f in the sub-interval $[x_{i-1}, x_i]$.

Then, the upper (Riemann) sum of f corresponding to the partition P, denoted by U (P,f), is defined by

$$U(P,f) = \sum_{i=1}^{n} M_i \, \Delta x_i$$

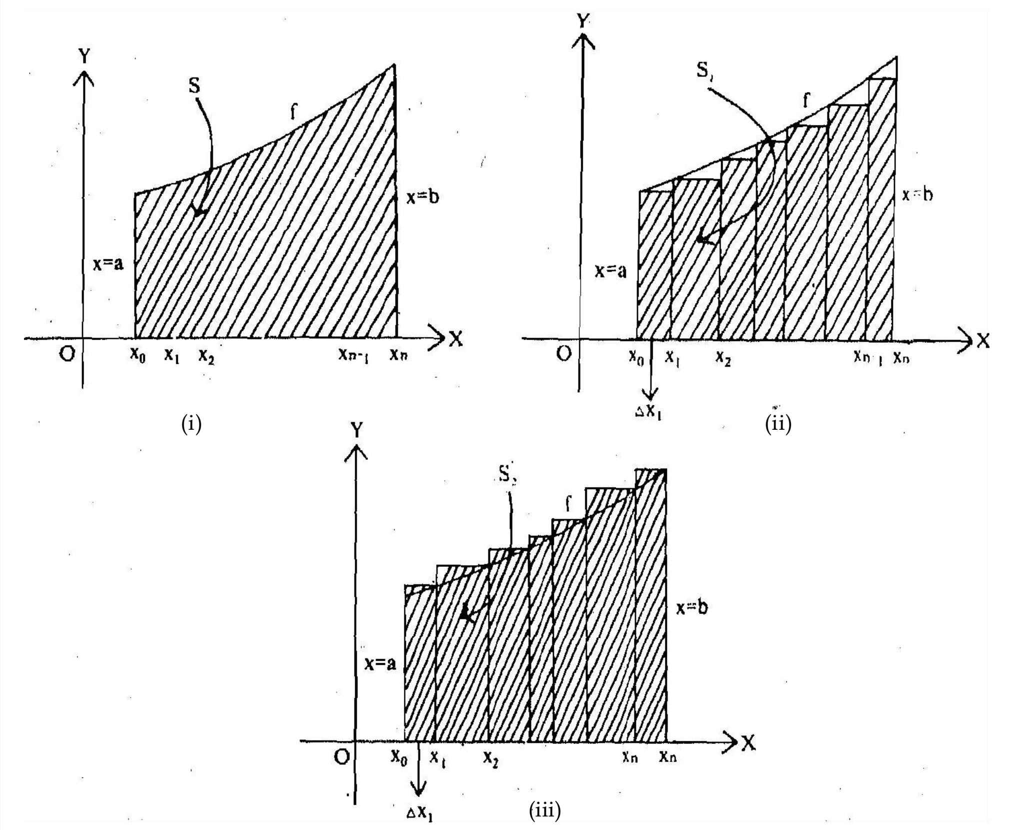
Notes

The lower (Riemann) sum off corresponding to the partition P, denoted by L(P, f), is defined by

$$U(P,f) = \sum_{i=2}^{n} M_i \; \Delta x_i.$$

Before we pass on to the definition of upper and lower integrals, it is good for you to have the geometrical meaning of the upper and lower sums and to visualize the above definitions pictorially. You would, then, have a feeling for what is going on, and why such definitions are made. Refer to Figure 20.4.

Figure 20.4



In figure 20.4(i) the graph off: [a,b] $\rightarrow$ R is drawn. The partition P = $\{x_0, x_{,1}, ... x_n\}$ divides the interval [a,b] into sub-intervals $[x_0, x_1]$, $[x_1, x_2]$, ... $[x_{n-1}, x_n]$. Consider the area S under the graph off. In the first sub-interval $[x_0, x_1]$, m_1 is the g.l.b. of the set of values f(x) for x in $[x_0, x_1]$. Thus m_1 Δx_1 is the area of the small rectangle with sides m_1 and Δx_1 as shown in the figure 20.4(ii). Similarly $m_2 \Delta x_2$ ... m_n A x_n are areas of such small rectangles and $\sum_{i=1}^{n} m,$ Ax, i.e. lower sum L (P,f) is the area S_2 which is the sum of areas of such small rectangles. The area S_1 is less than the area S under the graph of f.

In the same way $M_I \Delta X_1$ I is the area of the Large rectangle with sides M_I and ΔX_1 and $\sum_{i=1}^{n} M_i - 1x,$ i.e., the upper sum U(P, f) is the area S_2 which is the sum of areas or such large rectangles as shown in Figure 20.4(iii). The area S_2 is more than the area S under the graph off. It is intuitively clear that if the points in the partition P are increased, the areas S_1 and S_2 approach the area S.

Notes

We claim that the sets of upper and lower sums corresponding to different partitions of [a,b] are bounded. Indeed, let m and M be the infimum and supremum of f in [a.b].

Then $m \le m_i \le M_i \le M$ and so

$$m \, \Delta x_1 \le m_i \, \Delta x_i \le M_i \, \Delta x_i \le M \, \Delta x_i$$

Putting i = 1, 2, ... n and adding, we get

$$m \sum_{i=1}^{n} \Delta x, \le L(P,f) \le U(P,f) \le M \sum_{i=1}^{n} A x,.$$

$$\sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = b - a$$

Thus $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

For every partition P, there is a lower sum and there is an upper sum. The above inequalities show that the set of lower sums and the set of upper sums are bounded, so that their supremum and infimum exist. In particular, the set of upper sums have an infimum and the set of lower sums have a supremum. This leads us to concepts of upper and lower in tegrals as given by Riemann and popularly known as Upper and Lower Riemann Integrals.

Definition 5: Upper and Lower Riemann Integral

Let f: [a,b] $\to$ R be a bounded function. The infimum or the greatest lower bound of, the set of all upper sums is called the upper (Riemann) integral o f f on [a,b] and is denoted by,

$$\int_a^{\bar{h}} f(x)dx.$$

i.e.

$$\int_a^{\bar{b}} f(x)dx. = \text{g.l.b. } \{U(P,f)\text{: P is a partition of } [a,b]\}.$$

The supremum or the least upper bound of the set of all lower sums is called the lower (Riemann) integral of f on [a,b] and is denoted by

$$\int_a^b f(x)dx$$

i.e.

$$\int_a^b f(x)dx = \text{l.u.b } \{L(P,f)\text{: P is a partition of } [a,b]\}.$$

Now we consider some examples where we calculate upper and lower integrals.

Example: Calculate the upper and lower integrals of the function f defined in [a, b] as follows:

$$f(x) = \begin{vmatrix} 1 \text{ when x is rational} \\ 0 \text{ when x is irrational} \end{vmatrix}$$

Solution: Let P = $\{x_0, x_1 \ldots x_n\}$ be any partition of [a,b]. Let M_i and m_1 be respectively the sup. f and inf. f in $[x_{i-1}, x,]$. You know that every interval contains infinitely many rational as well as irrational numbers. Therefore, m_i = 0 and M, = I for i = 1, 2 ... n. Let us find U(P,f) and L(P,f).

Notes

$$U(P,f) = \sum_{i=1}^{n} M_i \,\Delta x_i = \sum_{i=1}^{n} \Delta x_i = b - a$$

$$U(P,f) = \sum_{i=1}^{n} m_i \,\Delta x_i = 0$$

Therefore U(P,f) = b – a and L(P,f) = 0 for every, partition P of [a,b]. Hence

$\int_a^b f(x)\,dx$ = g.l.b. {U(P,f): P is a partition of [a,b]]

= g.l.b. {b – a] = b – a.

$\int_a^b f(x)dx$ = l.u.b. {L(P,f): P is a partition of [a,b]]

= l.u.b. {0] = 0.

Example: Let f be a constant function defined in [a,b]. Let f(x) = k ∀ x ∈ [a,b]. Find the upper and lower integrals of f.

Solution: With the same notation as in example 1, M_i = k and m_i = k ∀ i.

$$U(P,f) = \sum_{i=1}^{n} M_i \,A\, x_i = \sum_{i=1}^{n} A\, x_i = k(b-a)$$

$$\text{and } L(P,f) = \sum_{i=1}^{n} m_i \,A\, x_i = \sum_{i=1}^{n} A\, xi = k(b-a)$$

Therefore U(P,f) = k(b – a) and L(P,f) = k (b – a) for every partition P of [a,b].

Consequently $\overline{\int_a^b} f(x)dx = k(b-a)$ and $\int_a^b f(x)dx = k(b-a)$

Now try the following exercise.

Exercise

Find the upper and lower Riemann integrals of the function f defined in [a,b] as follows:

$$f(x) = \begin{cases} 1 \text{ when x is ratinal} \\ -1 \text{ when x is irrational} \end{cases}$$

You have seen that sometimes the upper and lower integrals are equal (as in Example) and sometimes they are not equal (as in Example). Whenever they are equal, the function is said to be integrable. So integrability is defined as follows:

Definition 6: Riemann Integral

Let f: [a,b] – R be a bounded function. The function f is said to be Riemann integrable or simply integrable or R-integrable over [a,b] if $\int_a^b f(x)\,dx = \int_{-}^{b} f(x)\,dx$ and iff is Riemann integrable, we denote the common value by $\int_a^b f(x)\,dx.$ This is called the Riemann integral a r simply the integral off on [a, b].

Notes

Example: Show at the function f considered in example is not Riemann integrable.

Solution: As shown in above example, $\int_a^b f(x)\,dx = b - a$ and $\int_a^b f(x)\,dx = 0$ and so $\int_a^{\bar{b}} f(x)\,dx \neq \int_a^b f(x)\,dx$ and consequently f is not Riemann integrable.

Example: Show that a constant function is Riemann integrable over [a,b] and find $\int_1^h f(x)\,dx$.

Solution: As proved in above example, $\int_a^{\bar{b}} f(x)\,dx = k(b-a) = \int_1^b f(x)\,dx$

Therefore, f is Riemann integrable on [a,b] and $\int_1^h f(x)\,dx = k(b-a)$.

Theorem 1: If the partition P_2 is a refinement of the partition P, of [a,b], then $L(P_1,f) \le L(P_2,f)$ and $U(P_2,f) \le U(P_1,f)$.

Proof: Suppose P_2 contains one point more than P,. Let this extra point be c. Let $P_1 = \{x_0, x_,, \ldots, x_n\}$ and $x_{i-1} < c < x_i$. Let M_i and m_i be respectively the sup. f and inf. f in $[x_{i-1}, x_i]$. Suppose sup. f and inf. f in $[x_{i-1}, c]$ are p, and q_1 and those in $[c, x_i]$ are p_2 and q_2, respectively. Then,

$$L(P_2,f) - L(P_1,f) = q_1(c - x_{i-1}) + q_2(x_i - c) - m_i \Delta x_i$$

$$= (q_1 - m_i)(c - x_{i-1}) + (q_2 - m_i)(x_i - c)$$

$\left(\text{since A } x_i = (x_i - c) + (c - x_{i-1})\right)$

Similarly $U(P_2,f) - U(P_1,f) = (p_1 - M_i)(c - x_{i-1}) + (p_2 - M_i)(x_i - c)$

Now $m_i \le q_1 \le p_1 \le M_i$

$m_i \le q_2 \le p_2 \le M_i$

Therefore

$$L(P_2,f) - L(P_1,f) \ge 0 \text{ and } U(P_2,f) - U(P_1,f) \le 0$$

Therefore

$$L(P_1,f) - L(P_2,f) \ge 0 \text{ and } U(P_2,f) - U(P_1,I).$$

Is P_2 contains p points more than P_1, then adding these extra points one by one to P_1 and using the above results, the theorem is proved. We can also write the theorem as

$$L(P_1,f) \le L(P_2,f) \le U(P_2,f) \le U(P_1,f)$$

from which it follows that $U(P_2,f) - L(P_2,f) \le U(P_1,f) - L(P_1,f)$. As an illustration of theorem 1, we consider the following example.

Example: Verify Theorem 1 for the function f(x) = x + 1 defined over [0, 1] and the partition $P_1\left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and $P_2 = \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\}$.

Notes

Solution: For partition P_1, $n = 5, x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{3}, x_1 = \frac{1}{2}, x_4 = \frac{3}{4}, x_4 = 1$ and so $\Delta x_1 = \frac{1}{4}$, $\Delta x_2 = \frac{1}{12}, \Delta x_3 = \frac{1}{6}, \Delta x_4 = \frac{1}{4}, \Delta x_5 = \frac{1}{4}$.

Further $M_i = f(x_i)$ & $m_i = f(x_{i-1})$ for i = 1, 2, 3, 4, 5 and therefore $M_1 = \frac{5}{4}, M_2 = \frac{4}{3}, M_3 = \frac{3}{2}, M_4 = \frac{7}{4}$, $M_5 = 2, m_1 = 1, m_2 = \frac{5}{4}, m_3 = \frac{4}{3}, m_4 = \frac{3}{2}, m_5 = \frac{7}{4}$. We have $L(P_1, f) = \sum_{i=1}^{5} m_i \,\Delta x_i = \frac{25}{18}$ and $U(P_1, f) = \sum_{i=1}^{5} M_1 \, A\, x, = \frac{29}{18}$. Similarly, $L(P_2, f) = \frac{17}{12}$, and $U(P_2, f) = \frac{19}{12}$. Hence $L(P_1, f) \le L(P_2, f)$ and $U(P_2, f) \le U(P_1, f)$.

Theorem 2: $\int_a^b f(x)\,dx \le \int_a^b f(x)\,dx$.

Proof: If P_1 & P_2, are two partitions of [a,b] and $P = P_1 \cup P$, is their common refinement, then using Theorem 1, we have $L(P_1, f) \le L(P, f) \le U(P, f) \le U(P_1, f)$ and

$L(P_2, f) \le L(P, f) \le U(P, f) \le U(P_2, f)$.

Therefore, $L(P_1, f) \le U(P_2, f)$.

Keeping P_2 fixed and taking l.u.b. over all P_1, we get

$$\int_a^b f(x)\,dx \le U(P_2, f)$$

Now taking g.l.b. over all P_2, we obtain

$$\int_a^b f(x)\,dx \le \int_a^b f(x)\,dx$$

This proves the result.

In Theorem 1, we have compared the lower and upper sums for a partition P_1 with those for a finer partition P_2. Next theorem, which we state without proof, gives the estimate of the difference of these sums.

Theorem 3: If a refinement P, of P_1 contains p more points and $|f(x)| \le k$, for all $x \in [a, b]$, then

$$L(P_1, f) \le L(P_2, f) \le L(P_1, f) + 2pk\delta,$$

and $U(P_1, f) \ge U(P_2, f) \ge U(P_1, f) - 2p\,k\,\delta$, where δ is the norm of P_1.

This theorem helps us in proving Darboux's theorem which will enable us to derive conditions of integrability. Firstly, we give Darboux's Theorem.

Theorem 4: Darboux's Theorem

If f: [a,b] → R is a bounded function, then to every $\in > 0$, there corresponds $\delta > 0$ such that

(i) $U(P, f) < \int_a^{\bar{b}} f(x)\,dx + \in$

Notes

(ii) $\quad L(P,f) < \int_a^b f(x)\,dx - \in$

for every partition P of [a,b] with $|P| < \delta$.

Proof: We consider (i). As f is bounded, there exists a positive number k such that $|f(x)| \leq k \ \forall\ x \in [a,b]$. As $\int_a^{\bar{b}} f(x)\,dx$ is the infimum of the set of upper sums, therefore to each $\in > 0$, there is a partition P_1 of [a,b] such that

$$U(P_1, f) < \int_a^{\bar{b}} f(x)\,dx + \frac{\in}{2}, \qquad (1)$$

Let $P_1 = \{x_0, x_1, ..., x_p\}$ and δ be a positive number such that $2\,k\,(p-1)\,\delta = \in/2$. Let P be a partition of [a,b] with $|P| < \delta$. Consider the common refinement $P_2 = P \cup P_1$ of P and P_1.

Each partition has the same end points 'a' and 'b'. So P_2 is a refinement of P having at the most (p – 1) more points than P. Consequently, by Theorem 3,

$$\begin{aligned} U(P,f) - 2(p-1)\,k\,\delta &\leq U(P_2,f) \\ &\leq U(P_2,f) \\ &< \int_a^{\bar{b}} f(x)\,dx + \in/2. \qquad \text{(using (1))} \end{aligned}$$

Thus

$$\begin{aligned} U(P,f) &< \int_a^{\bar{b}} f(x)\,dx + \frac{\in}{2} + 2(p-1)\,k\,\delta \\ &= \int_a^{6} f(x)\,dx + \in, \text{ with } |P| < \delta. \end{aligned}$$

Task Write down the proof of part (ii) of Darboux's Theorem.

As mentioned earlier, Darboux's Theorem immediately leads us to the conditions of integrability. We discuss this in the form of the following theorem:

Theorem 5: Condition of Integrability

First Form: The necessary and sufficient condition for a bounded function f to be integrable over [a,b] is that to every number $\in > 0$ there corresponds $\delta > 0$ such that

$$U(P,f) - L(P,f) < \in, \ \forall\ P \text{ with } |P| < \delta.$$

Proof: We firstly prove the necessity of the condition.

Since the bounded function f is integrable on [a, b], we have

$$\int_a^{\bar{b}} f(x)\,dx = \int_a^{b} f(x)\,dx = \int_a^{b} f(x)\,dx.$$

Let $\in > 0$ be any number. By Darboux Theorem, there is a number $\delta > 0$ such that

Notes

$$U(P,f) < \int_a^b f(x)\,dx + \in/2 \quad = \int_a^b f(x)dx + \in/2 \ \forall\ P \text{ with } |P| < \delta \tag{2}$$

Also,

$$L(P,f) > \int_a^b f(x)\,dx - \in/2 \quad = \int_a^b f(x)\,dx - \in/2 \tag{3}$$

i.e. $-L(P,f) < \int_{-a}^{b} f(x)\,dx + \in/2\ \forall\ P$ with $|P| < \delta$

Adding (2) and (3), we get

$U(P,f) - L(P,f) < \in \forall\ P$ with $|P| < \delta$.

Next, we prove that condition is sufficient.

It is given that, for each number $\in > 0$, there is a number $\delta > 0$ such that

$$U(P,f) - L(P,f) < \in,\ \forall\ P \text{ with } |P| < \delta.$$

Let P be a fixed partition with $|P| < \delta$. Then

$$L(P,f) \le \int_a^b f(x)dx \le \int_a^{\bar{b}} f(x)dx \le U(P,f).$$

Therefore, $\int_a^b f(x)\,dx - \int_a^b f(x)\,dx \le U(P,f) - L(P,f) < \in$.

Since $\in$ is arbitrary, therefore the non-negative number

$$\int_a^{\bar{b}} f(x)\,dx - \int_a^b f(x)\,dx$$

is less than every positive number. Hence it must be equal to zero that is $\int_a^{\bar{b}} f(x)\,dx = \int_a^b f(x)\,dx$ and consequently f is integrable over [a,b].

Second Form: The necessary and sufficient condition for a bounded function f to be integrable over [a,b] is that to every number $\in > 0$, there corresponds a partition P of [a,b] such that

$$U(P,f) - L(P,f) < \in.$$

20.2 Riemann Integrable Functions

As we derived the necessary and sufficient conditions for the integrability of a function, we can now decide whether a function is Riemann integrable without finding the upper and lower integrals of the function. By using the sufficient part of the conditions, we test the integrability of the functions. Here we discuss functions which are always integrable. We will show that a continuous function is always Riemann integrable. The integrability is not affected even when there are finites number of points of discontinuity or the set of points of discontinuity of the function has a finite number of limit points. It will also be shown that a monotonic function is also always Riemann integrable.

We shall denote by R(a,b), the family of all Riemann integrable functions on [a,b]. First we discuss results pertaining to continuous functions in the form of the following theorems.

Notes

Theorem 6: If f: [a, bJ $\to$ R is a continuous function, then f is integrable over [a, b], that is f $\in$ R(a,b).

Proof: If f is a continuous function on [a,b] then f is bounded and is also uniformly continuous.

To show that f $\in$ R [a,b] you have to show that to each number $\in$ > 0, there is a partition P for which

$$U(P,f) - L(P,f) < \in$$

Let $\in$ > 0 be given. Since f is uniformly continuous on [a,b], there is a number δ > 0 such that $|f(x) - f(y)| < \frac{E}{b-a}$ whenever $|x - y| < \delta.$ Let P be any partition of [a,b] with $|P| < \delta.$

We show that, for such a partition P, U (P, f) – L(P, f) < $\in$.

Now,
$$U(P,f) - L(P,f) = \sum_{i=1}^{n} M_i \, \Delta x_i - \sum_{i=1}^{n} m_i \, \Delta x_i$$

$$= \sum_{i=1}^{n} (M_i - m_1)\Delta x_i, \qquad (4)$$

where $\Delta x_i = x_i - x_{i-1}$, and M_i = sup $\{f(x) | x_{i-1} \le x \le x_1\} = f(\xi_1)$ (say), for same $\xi_1 \in [x_{i-1}, x_1]$. Such a ξ_i exists because a continuous function f attains its bounds on $[x_{i-1} - x_1]$.

Similarly, m_i = inf $\{f(x) | x_{i-1} \le x \le x_i\} = f(\eta_i)$ (say), for some $\eta_i \in [x_{i-1}, x_i]$. Hence

$M_i - m_i = f(\xi_i) - f(\eta_i) \le |f(\xi_i) - f(\eta_i)| < \in / b - a$, for all i,

since $|\xi_i - \eta_i| \le A\, x_i < \delta.$ Substituting in (4) we obtain

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$

$$< \frac{\in}{b-a}\left(\sum_{i=1}^{n} \Delta x_i\right)$$

$$\frac{E}{b-a}(b-a) = \in.$$

Thus, every continuous function is Riemann integrable,

But as remarked earlier, even when there are discontinuous of the function, it is integrable. This is given in the next two concepts which we state without proof.

Theorem 7: Let the bounded function f: [a, b] $\to$ R have a finite number of discontinuities. Then f $\in$ R (a,b).

Theorem 8: Let the sec of points of discontinuity of a, bounded function f: [a, b] $\to$ R has a finite number of limit points, then f $\in$ R (a, b).

We illustrate these theorems with the help of examples.

Example: Show that the function f where f(x) = x^2 is integrable in every interval [a,b].

Solution: You know that the function f(x) = x^2 is continuous. Therefore it is integrable in every interval [a,b].

Example: Show that the function f where f(x) = [x] is integrable in [0,2] where [x] denotes the greatest integer not greater than x.

Solution:
$$[x] = \begin{cases} 0 \text{ if } 0 \le x < 1 \\ 1 \text{ if } 1 \le x < 2 \\ 2 \text{ if } x = 2 \end{cases}$$

The points of discontinuity of f in [0,2] are 1 and 2 which are finite in number and so it is integrable in [0,2].

Example: Show that the function F defined on the interval [0,1] by

$$F(x) = \begin{cases} 2rx, \text{ when } \frac{1}{r+1} < x \le \frac{1}{r}, \text{ where r is a positive integer} \\ 0, \text{ elsewhere,} \end{cases}$$

is Riemann integrable.

Solution: The function F is discontinuous at the points $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$. The set of points of discontinuity has 0 as the only limit point. So, the limit points are finite in number and hence the function F is integrable in [0,1], by Theorem 8.

There is one more class of integrable functions and this class is that monotonic functions. This we prove in the following theorem.

Theorem 9: Every monotonic function is integrable.

Proof: We shall prove the theorem for the case where I: [a,b] $\to$ R is a monotonically increasing function. The function is bounded. f(a) and f(b) being g.l.b. and l.u.b. Let $\in > 0$ be given number, Let n be a positive integer such that

$$n > \frac{(b-a)[f(b)-f(a)]}{\in}$$

Divide the interval [a,b] into n equal sub-intervals, by the partition P = {x_0, x_1 …, x_0} of [a, b]. Then

$$U(P,f) - L(P,f) = \sum_{i=1}^{n}(M_i - m_i)(\Delta x_i)$$

$$= \frac{b-a}{n}\sum_{i}^{n}[f(xi) - f(xi-1)]$$

$$= \frac{(b-a)}{n}[f(b) - f(a)] < \in.$$

This proves that f is integrable. Discuss the case of monotonically decreasing function as an exercise. Do it by yourself.

Exercise: Show that a monotonically decreasing function is integrable.

Now we give example to illustrate the theorem.

Notes

Example: Show that the function f defined by the condition $f(x) = \frac{1}{2^n}$

when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, n = 0,1,2 \ldots$

is integrable in [0,1]

Solution: Here we have f (0) = 0,

$$f(x) = 1 \text{ when } \frac{1}{2} < x \le 1$$

$$f(x) = \frac{1}{2} \text{ when } \left(\frac{1}{2}\right)^2 < x \le \frac{1}{2}$$

Clearly f is monotonically increasing in [0, 1]. Hence it is integrable.

20.3 Algebra of Integrable Functions

As we discussed the algebra of the derivable functions. Likewise, we shall now study the algebra of the integrable functions. In the previous class, you have seen that there are integrable as well as non-integrable functions. In this section you will see that the set of all integrable functions on [a,b] is closed under addition and multiplication by real numbers, and that the integral of a sum equals the sum of the integrals. You will also see that difference, product and quotient of two integrable functions is also integrable.

All these results are given in the following theorems.

Theorem 10: If f ∈ R (a , b), and λ is any real number, then λ f ∈ R (a,b) and

$$\int_a^b \lambda\, f(x)\, dx = \lambda \int_a^b f(x)\, dx.$$

Proof: Let P = {x,, x_1,...., x_n) be a partition of [a,b]. Let M_i and m_i be the respective l.u.b. and g.l.b. of the function f in [x_{i-1}, x_i]. Then λ M_i and λ m_i are the respective l.u.b. and g.l.b. of the function λ f in [x_{i-1}, x_i], if A ≥ 0, and λ m_i and λ M_i are the respective l.u.b. and g.l.b. of h f in [x_{i-1}, x_i], if h < 0.

When $\lambda \ge 0$, then $U(P,\lambda f) = \sum_{i=1}^{n} A\, M_i\, \Delta x_i = \lambda \sum_{i=1}^{n} M_i\, Ax, = \lambda\, U(P,f).$

$$\Rightarrow \qquad \int^6 \lambda f(x)\, dx = \lambda \int_a^{\bar{b}} f(x)\, dx.$$

Similarly L(P, λ f) = A L(P,f).

$$\Rightarrow \qquad \int_{\underline{a}}^{b} \lambda f(x)\, dx = \lambda \int_{\underline{a}}^{b} f(x)\, dx$$

If $\lambda < 0, U(P,\lambda f) = \sum_{i=1}^{n} \lambda\, m_i\, \Delta x_i = \lambda\, L(P,f).$

Notes

$\Rightarrow \qquad \overline{\int_a^b} \lambda\, f(x)\, dx = \lambda \underline{\int_a^b} f(x)\, dx$

Similarly L(P, λf) = λ U L(P,f).

$\Rightarrow \qquad \int_a^h \lambda\, f(x)\, dx = A \int_a^b f(x)\, dx$

Since f is integrable in [a,b], therefore

$$\int_a^6 f(x)\, dx = \int_a^b f(x)\, dx = \int_a^b f(x)\, dx.$$

Hence $\int^6 \lambda\, f(x)\, dx = \int^6 \lambda\, f(x)\, dx = \lambda \int_a^b f(x)\, dx,$

whether $\lambda \geq 0$ or $\lambda < 0$.

Hence $\lambda\, f \in R\,[a,b]$ and $\int_a^b \lambda\, f(x)\, dx = \lambda \int_a^b f(x)\, dx.$

Now suppose that λ = –1. In this case the theorem says that if f ∈ R [a,b], then $(-f) \in R[a,b]$

$\int_a^b [-f(x)]dx = \int_{-a}^b f(x)\, dx.$

Theorem 11: If f ∈ R [a,b], g ∈ R [a,b],. then f + g ∈ R [a,b] and

$\int^b (f+g)(x)dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$

Proof: We first show that f+g ∈ R [a,b]. Let ∈ > 0 be a given number. Since f ∈ R [a,b], g ∈ R [a,b], there exist partitions P and Q of [a,b] such that U(P,f) – L(Pλf) < ∈/2 and U (Q,g) – L (Q,g) < ∈/2

If T is a partition of [a,b] which refines both P and Q, then

U(T,f) – L(T,f) < ∈/2 [U(T,f) – L(T,f) ≤ U(P,f) – L(P,f)].

Similarly,

U(T,g) – L(T,g) < ∈/ 2 (5)

Also note that, if M_i = sup {f(x) : $x_{i-1} \leq x \leq x_i$}

and

N_i = sup {g(x): x_{i-1} 6 $x \leq x_i$}

then,

sup {f(x) + g(x): $x_{i-1} \leq x \leq x_i$} ≤ $M_1 + N_i$.

Using this, it readily follows that

U(T, f+g) ≤ U(T,f) + U(T,g)

for every partition T of [a,b]. Similarly

L(T,f+g) ≥ L(T,f) + L(T,g)

for every partition T of [a,b].

Thus U (T,f+g) – L (T,f+g) ≤ [U(T,f) + U(T,g) – L [(T,f) + L(T,g)]

Notes

$= [U(T,f) - L(T,f)] + [U(T,g) - L(T,g)] < \frac{\in}{2} + \frac{\in}{2} = \in$ for T occurring in (5). This shows that $f + g \in R(a,b)$

It remains to show that $\int_a^b [f(x)+g(x)]\,dx = \int_a^b [f(x)\,dx + \int_a^b g(x)$

Now

$$\int^b (f+g)(x)dx = \int_a^{\bar{b}} (f+g)\,(x)\,dx \leq U(P,f+g) \leq U(P,f)+U(P,g) \quad (6)$$

for any partition P of [a,b]. Given any $\in > 0$ we can find a partition P of [a,b] such that

$$U(P,f) < \int_a^b f(x)\,(x)\,dx + \in/2$$

$$U(P,g) < \int_a^b g(x)\,dx + \in/2 \quad (7)$$

Substituting (7) in (6), we obtain

$$\int_a^b (f+g)\,(x)\,dx < \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \in \quad (8)$$

Since (8) holds for arbitrary $\in > 0$, we obtain

$$\int_a^b (f+g)\,(x)\,dx \leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \quad (9)$$

Replacing f and g by –f and –g in (9) we obtain

$$\int^b (-f-g)\,(x)\,dx \leq \int^b \{-f\,(x)\}\,dx + \int^b \{-g(x)\}\,dx$$

or

$$\int_a^b (f+g)\,(x)\,dx \leq -\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

This is equivalent to

$$\int_a^b (f+g)\,(x)\,dx \geq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Combining (9) and (10), we get

$$\int_a^b (f+g)\,(x)\,dx = \int_a^b f(x)\,dx \int_a^b g(x)\,dx$$

Which proves the theorem.

Theorem 12: If $f \in R(a,b)$ and $g \in R\,(a,b)$, then $f - g \in R(a, b)$ and

$$\int_a^b (f-g)\,(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Proof: Since g E R [a,b], therefore $-g \in R\,[a,b]$ and

$$\int^b -[g(x)]dx = -\int^b g(x)\,dx$$

Notes

Now f ∈ R [a,b] and –g ∈ R [a,b] implies that f + (–g) ∈ R [a,b] and therefore,

$$\int_a^b [f+(-g)](x)\,dx = \int_a^b f(x)\,dx + \int_a^b [-g(x)]\,dx$$

that is (f – g) ∈ R [a,b] and

$$\int^b (f-g)\,(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

For the product and quotient of two functions, we state the theorems without proof.

Theorem 13: If f ∈ R(a,b) and g ∈ R(a,b), then f g ∈ R(a,b).

Theorem 14: If f ∈ R(a,b), g ∈ R(a,b) and there exists a number t > 0 such that $|g(x)| \geq t, \forall\ x \in [a,b]$, then f/g ∈ R(a,b).

Now we give some examples.

Example: Show that the function f, where f(x) = x + [x] is integrable is [0, 2].

Solution: The function F(x) = x, being continuous is integrable in [0, 2] and the function G(x) = [x] is integrable as it has only two points namely, 1 and 2 as points of discontinuity. So their sum is, f(x) is integrable in [0, 2].

Example: Give an example of function f and g such that f + g is integrable but f and g are not integrable in [a, b].

Solution: Let f and g be defined in [a, b] such that

$$f(x) = \begin{cases} 0, \text{ when x is rational} \\ 1, \text{ when x is irrational,} \end{cases}$$

$$g(x) = \begin{cases} 1, \text{ when x is rational} \\ 0, \text{ when x is irrational} \end{cases}$$

f and g are not integrable but $(f+g) = 1\ \forall\ x \in [a,b],$ being a constant function, is integrable.

20.4 Computing an Integral

So far, we have discussed several theorems for testing whether a given function is integrable on a closed interval [a,b]. For example, we can see that a function $f(x) = x^2\ \forall\ x \in [0,2]$ is continuous as well as monotonic on the given interval and hence it is integrable over [0,2]. But this information does not give us a method for finding the value of the integral of this function. In practice, this is not so easy as we might think of. The reason is that there are some functions which are integrable by conditions of integrability but it is difficult to find the values of their integrals. For example, suppose a function is given by $f(x) = e^{x^2}$ This is continuous over every closed interval and hence it is integrable. But we cannot find its integral by our usual method of anti derivative since there is no function for which $f(x) = e^{x^2}$ is the derivative. If possible, try to find the anti derivative for this function.

Notes

In such situations, to find the integral of a given function, we use the basic definition of the integral to evaluate its integral. Indeed, the definition of integral as a limit of sum helps us in such situations.

In this section, we demonstrate this method by means of certain examples. We have found the integral $\int_a^b f(x)\,dx$ via the sums U(P,f) and L(P,f). The numbers M_i and m_i which appear in these sums are not necessarily the values of f(x), i f f is not continuous. In fact, we shall now show that $\int f(x)\,dx$ can be considered as limit of sums in which M_i and m, are replaced by values of f. This approach gives us a lot of latitude in evaluating $\int_a^b f(x)\,dx,$ as we shall see in several examples.

Let f: [a,b] $\rightarrow$ R be a bounded function. Let

la = $x_0 < X_1 <$ ...... x_n = b]

be a partition P of [a,b]. Let us choose points t_1, ... t_n, such that

$x_{i-1} \leq t_i \leq x_i$ (i = 1, ... n). Consider the sum

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} f(t_i)\,(x_i - x_{i-1}).$$

Notice that, instead of M_i in U(P, f) and m_i in L(P,f), we have $f(t_i)$ in S(P, f). Since t_i 's are arbitrary points in $[x_{i-1}, x_i]$, S(P, f) is not quite well-defined. However, this will not cause any trouble in case of integrable functions.

S(P,f) is called Riemann Sum corresponding to the partition P.

We say that lim S(P,f) = A

$$|P| - 0$$

or $S(P,f) \rightarrow A$ as $|P| \rightarrow 0$ if for every number $\in < 0\ \exists\ \delta > 0$ such that $|S(P,f) - A| < \in$ for P with $|P| < 6.$

We give a theorem which expresses the integral as the limit of S(P,f).

Theorem 15: If $\lim\limits_{|P|\to 0}$ S(P,f) exists, then f $\in$ R (a,b) and $\lim\limits_{|P|\to 0} S(P,f) = \int_a^b f(x)dx.$

Proof: Let $\lim\limits_{|P|\to 0}$ S(P,f) = A. Then, given a number $\in$ > 0, there exists a number δ > 0 such that

$$|S(P,f) - A| < \in/4, \text{ for P with } |P| < \delta.$$

i.e., $\quad A - \in/4 < S(P,f) < A + \in/4,$ for P with $|P| < \delta.$ (11)

Let P = $\{x_0, x_,, ..., x_n\}$. Suppose the points t, ..., t_n vary in the intervals $[x_0, x,]$, ..., $[x_{n-1}, x_n]$, respectively. Then, the l.u.b. of the numbers S(P,f) are given by

$$\text{l.u.b. } S(P,f) = \text{l.u.b.}\left(\sum_{i=1}^{n} f(t_i)\,\Delta\, x_i\right) = \sum_{i=1}^{n} M_i\ \Delta\, x_i = U(P,f).$$

Similarly, g.1.b. S(P,f) = L(P,f). Then, from (11), we get

$$A - \in/4 \leq L(P,f) \leq U(P,f)\ 5\ A + \in/4 \qquad (12)$$

Therefore,

$$U(P,f) - L(P,f) \leq (A + \in/14) - (A - \in/4)$$

$$= \in/2 < \in.$$

In other words, $f \in R(a,b)$. Thus

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Since $L(P,f) \leq \int_a^b f(x)\,dx \leq \int^h f(x)\,dx \leq U(P,f),$ therefore

$$L(P,f) \leq \int_a^b f(x)\,dx \leq U(P,f). \qquad (13)$$

From (12) and (13), we get

$$A - \in/4 \leq \int_a^b f(x)\,dx \leq A + \in/4.$$

That is,

$$\left|\int_a^b f(x)\,dx - A\right| \leq \in/4 < e.$$

Since $\in$ is arbitrary, therefore $\int_a^b f(x)\,dx - A = 0,$ that is, $\int_a^b f(x)\,dx = A = \lim_{|P|\to 0} S(P,f).$ This completes the proof of the theorem.

To illustrate this theorem, we give two examples.

Example: Show that $\int_a^b dx = \int_a^b 1\,dx = b - a.$

Solution: Here, the function f: [a,b] → R is the constant function f(x) = 1.

Clearly, for any partition $P = (x_0, x_,, ..., x_n)$ of [a,b], we have

$$S(P,f) = (x_1 - x_0)f(t_1) + (x_2 - x_1)\,f(t_2) + ... + (x_n - x_{n-1})f(t_n)$$

$$= (x_1 - x_0)1 + (x_2 - x_1)1 + ... + (x_n - x_{n-1})1 = b - a.$$

Since S(P,f) = b – a, for all partitions, $\int_a^b 1\,dx = \lim_{|P|\to 0} S(P,f) = b - a.$

Example: Show that $\int_a^b x\,dx = \frac{b^2 - a^2}{2}.$

Solution: The function f:[a,b] → R in this example is the identity function f(x) = x.

Let $P = (a = x_,, x_,, \ldots, x_a = b)$ be any partition of [a,b]. Then

$S(P, f) = (x_1 - x_0)\,f(t_1) + (x_2 - x_1)\,f(t_2) + ... + (x_n - x_n)\,f(t_n)$, where $t_1 \in [x_0, x_1,], t_2 \in [x_1, x_2], \ldots$ $t_n \in [x_{n-1}, x_n]$ are arbitrary. Let us choose

$$t_1 = \frac{x_0 + x_1}{2}, t_2 = \frac{x_1 + x_2}{2}, \ldots, t_n = \frac{x_{n-1} + x_n}{2}.$$

Notes

Then, $$S(P,f) = (x_1 - x_0)\frac{x_1 + x_0}{2} + (x_2 - x_1)\frac{x_2 + x_1}{2} + \ldots + (x_n - x_{n-1})\frac{x_n + x_{n-1}}{2}$$

$$= \frac{1}{2}\left[\left(x_1^2 - x_2^0\right) + \left(x_2^2 - x_2^1\right) + \ldots + \left(x_n^2 - x_{n-1}^2\right)\right]$$

$$= \frac{1}{2}\left(x_n^2 - x_0^2\right) = \frac{1}{2}\left(b^2 - b^2\right).$$

Here again, $S(P,f) = \frac{1}{2}\left(b^2 - a^2\right)$, no matter what the partition P we may take, Hence $\int_a^b f(x)\,dx = \int_a^b x\,dx = \lim_{|P|\to 0} S(P,f) = \frac{1}{2}(b^2 - a^2)$.

The converse of Theorem 15 is also true which we state without proof as the next theorem.

Theorem 16: If a function f is Riemann integrable on a closed interval [a,b], then $\lim_{|P|\to 0} S(P,f)$ exists and $\lim_{|P|\to 0} S(P,f) = \int_a^b f(x)\,dx$.

One of the important application of Theorem 16 is in computing the sum of certain power series. For, let us consider a partition P of [a,b] having n sub-intervals, each of length h so that nh = b – a. Then P can be written as P = (a, a + h, a + 2h, ..., a + nh = b).

Let t, = a + ih, i = 1,2 ,...., n. Then

$$S(P,f)\ \sum_{i=1}^{n} f(t_i)\,\Delta\, x_i = h[f(a+h) + f(a+2h) + \ldots + f(a+nh)].$$

When $\lim_{|P|\to 0} S(P,f)$ exists, then

$$\lim_{\substack{n\to\infty \\ h\to 0}} h[f(a+h) + f(a+2h) + \ldots + f(a+nh)] = \int_a^h f(x)\,dx.$$

In the above formulae, we can change the limits of integration from a, b to 0, a, where a ∈ N. For, by changing h to $\frac{b-a}{an}$, it is easy to deduce from above formula that

$$\frac{(b-a)}{\alpha}\lim_{n\to\infty}\frac{1}{n}\sum_{=}^{n} f\left[a + \frac{(b-a)}{\alpha}\frac{r}{h}\right] = \int_a^b f(x)\,dx. \tag{14}$$

But, $\int_a^b f(x)\,dx = \frac{(b-a)}{\alpha}\int_0^\alpha f\left[a + \frac{(b-a)}{\alpha}x\right]dx.$

Therefore, from (14), we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left[a + \frac{(b-a)}{\alpha}\frac{r}{n}\right] = \int_0^\alpha f\left[a + \frac{(b-a)}{\alpha}x\right]dx. \tag{15}$$

In (15), put a = 0, b = α. We get the following result:

If f is integrable in [0, α], then

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^\alpha f(x)\,dx.$$

This gives us the following method for finding the limit of sum of n terms of a series: **Notes**

1. Write the general rth term of the series.

2. Express it as $\frac{1}{n}f\left(\frac{r}{n}\right)$, the product of $\frac{1}{n}$ and a function of $\frac{r}{n}$.

3. Change $\frac{r}{n}$ to x and $\frac{1}{n}$ to dx and integrate between the limits 0 and a. The n value of the resulting integral gives the limit of the sum of n terms of the series.

Since each term of a convergent series tends to 0, the addition or deletion of a finite number of terms of the series does not affect the value of the limit. Similarly, you can verify that

$$\lim_{n\to\infty} \sum_{r=1}^{2n} \left[\frac{1}{n}\phi\left(\frac{r}{n}\right)\right] = \int_0^2 \phi(x)\,dx,$$

$$\lim_{n\to\infty} \sum_{r=1}^{3n} \left[\frac{1}{n}\phi\left(\frac{r}{n}\right)\right] = \int_0^3 \phi(x)\,dx, \text{ and so on.}$$

As an illustration of these results, consider the following examples.

Example: Find the limit, when n tends to infinity, of the series

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{n+n}.$$

Solution: General (rth) term of the series is $\sum_{r=1}^{n} \frac{1}{n+r} = \sum_{r=1}^{n} \frac{1}{n}\left(\frac{1}{1+\frac{r}{n}}\right)$.

Hence, $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}\left(\frac{1}{1+\frac{r}{n}}\right) = \int_0^1 \frac{1}{1+x} = dx = \log 2.$

Example: Find the limit, when n tends to infinity, of the series

$$\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2-1}} + \frac{1}{\sqrt{n^2-2^2}} + \ldots + \frac{1}{\sqrt{n^2-(n-1)^2}}.$$

Solution: Here the rth term $= \sum_{r=1}^{n} \frac{1}{\sqrt{n^2-(r-1)^2}}$

Since it contains (r – 1), we consider its (r + l)th term i.e.,

the term $\sum_{r=0}^{n} \frac{1}{\sqrt{n^2-r^2}} = \sum_{r=0}^{n} \frac{1}{n} \frac{1}{1+\left(\frac{r}{n}\right)^2}$

Notes

Therefore, $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{1}{\sqrt[n]{1+\left(\dfrac{r}{n}\right)^2}} = \int_0^1 -\dfrac{1}{\sqrt{1-x^2}}\,dx.$ because $\lim\limits_{n\to\infty}\dfrac{1}{n}=0.$

The value of this integral, on the r.h.s. of last equality, is $\dfrac{\pi}{2}$,

Example: Find $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{3n}\dfrac{n^2}{(3^n+r)^3}.$

Solution: We have

$$\frac{n^2}{(3^n+r)^3} = \frac{1}{n}\left(\frac{1}{\left(3+\frac{r}{n}\right)^3}\right).$$

Since the number of terms in the summation is 3^n, the resulting definite integral will have the limits from 0 to 3.

$$\text{Therefore, } \lim_{n\to\infty}\sum_{r=1}^{3^n}\frac{n^2}{\left(\overline{3^n}+\bar{r}\right)^3} = \lim_{n\to\infty}\sum_{r=1}^{3^n}\frac{1}{n}\frac{1}{(3+\underline{r})^3} = \int_0^3\frac{dx}{(3+x)^3}$$

This integral you can evaluate easily.

Self Assessment

Fill in the blanks:

1. Let P, and P, be two partitions of [a,b]. We say that P, is finer than P, or P_2, refines P, or P_2 is a refinement of P_1 if ...., that is, every point of P_1 is a point of P,.
2. Let f: [a,b] $\to$ R be a bounded function. The infimum or the greatest lower hound of, the set of ail upper sums is called the upper (Riemann) integral of f on [a, b] and is denoted by,..
3. If the partition P_2 is a refinement of the partition P, of [a,b], then $L(P_1,f) \le L(P_2,f)$ and ....
4. The integrability is not affected even when there are finites number of points of .... or the set of points of discontinuity of the function has a finite number of limit points.
5. If f : [a, bJ $\to$ R is a ...., then f is integrable over [a, b], that is f $\in$ R(a,b).

20.5 Summary

- In this unit, you have been introduced to the concept of integration without bringing in the idea of differentiation. As upper and lower sums and integrals of a bounded function f over closed interval [a,b] have been defined. You have seen that upper and lower Riemann integrals of a bounded function always exist. Only when the upper and lower Riemann integrals are equal, the function f is said to be Riemann integrable or simply integrable over [a,b] and we write it as f $\in$ R [a,b] and the value of the integral of f over [a,b] is denoted by $\int_a^b f(x)\,dx$. Also in this section, it has been shown that in passing from a partition PI to a finer partition P2, the upper sum does not increase and the lower sum does not decrease. Further, you have seen that the lower integrable of a function is less than or equal to the upper integral. Further condition of integrability has been derived with the help of which the integrability of a function can be decided without finding the upper and lower integrals. Using the condition of integrability, it has been shown that a function f is integrable on [a,b] if it is continuous or it has a finite number of points of discontinuities or the set of points of discontinuities have finite number of limit points. Also you have seen that a monotonic function is integrable. As in the case of continuous and derivable functions, the sum, difference, product and quotient of integrable functions is integrable. Riemann sum S(P,f) of a function f for a partition P has been defined and you have been shown that $\lim_{|P|\to 0} S(P,f)$ exists if and only if $f \in R$ [a,b] and $\int_a^b f(x)\,dx = \lim_{|P|\to 0} S(P,f)$. Using this idea a number of problems can be solved.

20.6 Keywords

Partition: Let [a,b] be a given interval. By a partition P of [a,b] we mean a finite set of points $\{x_0, x_1, ..., x,\}$, where

$$a = x_0, < x_1 < \ldots < x_{n-1} < x_n = b.$$

We write $\Delta x_i = x_i - x_{i-1}$, (i=l, 2, ..., n). So Δx_i is the length of the ith sub-interval given by the partition P.

Norm of a Partition: Norm of a partition P, denoted by $|P|$, is defined by $|P| = \max_{t \le i \le n} Ax,$. Namely, the norm of P is the length of largest sub-interval of [a, b] induced by P. Norm of P is also denoted by μ(P).

Darboux's Theorem: If f: [a,b] → R is a bounded function, then to every ∈ > 0, there corresponds δ > 0 such that

(i) $U(P,f) < \overline{\int_a^b} f(x)\,dx + \in$

(ii) $L(P,f) < \int_a^b f(x)\,dx - \in$

20.7 Review Questions

1. Find the upper and lower Riemariri integrals of the function f defined in [a, b] as follows

$$f(x) = \begin{cases} 1 \text{ when x is ratinal} \\ -1 \text{ when x is irrational} \end{cases}$$

2. Show that the function f where f(x) = x[x] is integrable in [0, 2].

3. Show that the function f defined in LO, 21 such that f(x) = 0, when $x = \frac{n}{n+1}$ or $\frac{n+1}{n}$ $(n = 1,2,3,\ldots)$, and f(x) = 1, elsewhere, is integrable.

Notes

4. Prove that the function f defined in [0, 1] by the condition that if r is a positive integer, $f(x) = (-1)^{r-1}$ when $\frac{1}{r+1} < x \le \frac{1}{r}$, and f(x) = 0, elsewhere, is integrable.

5. Show that the function f defined in [0, 1], for integer a > 2, by $f(x) = \frac{1}{a^{r-1}}$, when $\frac{1}{a^r} < x < \frac{1}{a^{r-1}}$ $(r = \frac{1}{a^{r-1}}1,2,3)$, and f(0) = 0, is integrable.

6. Give example of functions f and g such that f – g, fg, f/g are integrable but f and g may not be integrable over [a, b].

7. Find the limit, when n tends 10 infinity, of the series

$$\frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+4)^3}} + \frac{\sqrt{n}}{\sqrt{(n+8)^3}} + \ldots + \frac{\sqrt{n}}{\sqrt{[n+4(n-1)]^3}}$$

8. Find the limit, when 11 tends to infinity, of the series

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n}.$$

Answers: Self Assessment

1. $P_1 \subset P_2$
2. $\int_a^{\bar{b}} f(x)dx.$
3. $U(P_2,f) \le U(P_1,f)$
4. discontinuity
5. continuous function

20.8 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 21: Properties of Integrals

Notes

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Objectives

After studying this unit, you will be able to:

- Identify the properties of the integral and
- Use them to find the Riemann Stieltjes integral of functions

Introduction

In last unit you have studied about Riemann integral. In this unit, we are going to see the properties of Riemann Stieltjes integral.

21.1 Properties of Riemann Integral

As you were introduced to some methods which enabled you to associate with each integrable function f defined on [a,b], a unique real number called the integral $\int_a^b f(x)\,dx$ in the sense of Riemann. A method of computing this integral as a limit of a sum was explained. All this leads us to consider some nice properties which are presented as follows:

Property 1: If f and g are integrable on [a, b] and if

$$f(x) \le g(x)\ \forall\ x \in [a,b],$$

then

$$\int^b f(x)\,dx \le \int^b g(x)\,dx$$

Proof: Define a function h: [a,b] → R as

$$h = g - f .$$

Since f and g are integrable on [a, b], therefore, the difference h is also integrable on [a, b].

Since

$$f(x) \le g(x) \ge g(x) - f(x) \ge 0,$$

therefore h(x) ≥ 0 for all x E [a,b].

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Consequently, if P = {x_0, x_1, … x_n} be any partition of [a,b]

and m, be the inf. of h in [x, 1, x,], then

$$m_i \geq 0 \; \forall \; i = 1, 2, \ldots n$$

$$\Rightarrow \sum_{i=1}^{n} m_i \Delta x_i \geq 0$$

$$\Rightarrow L(P, h) \geq 0$$

Thus for every partition P, the lower sum L(P,h) ≥ 0.

In other words, Sup. (1 (P,h): P is a partition of [a,b]) ≥ 0

or

$$\int_a^h f(x)\, dx$$

Since h is integrable in [a,b], therefore

$$\int_a^b h(x)\, dx = \int_a^b h(x)\, dx = \int_a^b h(x)\, dx.$$

Thus

$$\int_a^b h(x)\, dx \geq 0$$

or

$$\int_a^b (g - f)\, (x)\, dx \geq 0$$

$$\Rightarrow \quad \int_a^b g(x)\, dx \geq \int_a^b f(x)\, dx$$

which proves the property

Property 2: If f, is integrable on [a, b] then $|f|$ is also integrable on

$$[a, b] \text{ and } \left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx$$

Proof: The inequality follows at once from Property 1 provided it is known that $|f|$ is integrable on [a,b]. Indeed, you know that $-|f| \leq f \leq |f|$.

Therefore,

$$\int_{-a}^b |f(x)|\, dx \leq \int^b f(x)\, dx \leq \int_a^b |f(x)|\, dx$$

which proves the required result. Thus, it remains to show that $|f|$ is integrable.

Let ∊ > 0 be any number. There exists a partition P of [a, b] such that

$$U(P,f) - L(P,f) \leq \in$$

Let P = $\{x_0, x_1, x_2, \ldots, x_n\}$.

Let M_i' and m_i' denote the supremum and infimum of $|f|$ and M_i and m_i denote the supremum and infimum of f in $[x_{i-1}, x_i]$.

You can easily check that

$$M_i - m_i \geq M_i' - m_i'.$$

This implies that $\sum_{i=1}^{n} (M_i - m_i) \text{ A } x_i \geq \sum_{i=1}^{n} (M_1' - m_1') \Delta x_i,$

i.e., $U(P,|f|) - L(P,|f|) \leq U(P,f) - L(P,f) < \in,$

This shows that $|f|$ is integrable on [a,b].

Note that the inequality established in Property 2 may be thought of as a Integrability and differentiability generalization of the well-known triangle inequality

$$|a+b| \leq |a| + |b|$$

In other words, the absolute value of the limit of a sum never exceeds the limit of the sum of the absolute values.

You know that in the integral $\int_a^b f(x)\,dx,$ the lower limit a is less than the upper limit b. It is not always necessary. In fact the next property deals with the integral in which the lower limit a may be less than or equal to or greater than the upper limit b.

For that, we have the following definition:

Definition 1: Let f be integrable on [a,b], that is, $\int_a^b f(x)\,dx$ exists when b > a. Then

$$\int_a^b f(x)\,dx = 0, \text{ if } a = b$$

$$= -\int_b^a f(x)\,dx, \text{ if } a > b.$$

Now have the following property.

Property 3: If a function f is integrable in [a,b] and $|f(x)| \leq k \ \forall\ x \in [a,b],$ then $\left|\int_a^b f(x)\,dx\right| \leq k\,|b-a|.$

Proof: There are only three possibilities namely either a < b or a > b or a = b. We discuss the cases as follows:

Case (i): a < b

Since $|f(x)| \leq k \quad \forall\ x \in [a,b],$ therefore

$-k \leq f(x) \leq k \quad \forall\ x \in [a,b]$

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$\Rightarrow \quad \int^{h} -k dx \leq \int_a^b f(x)\, dx \leq \int_a^b k\, dx$ (why?)

$\Rightarrow \quad -k(b-a) \leq \int_a^b f(x)\, dx \leq k(b-a)$

$\therefore \quad \left|\int_a^b f(x) dx\right| \leq k(b-a) = k|b-a|$

which completes the proof of the theorem.

Case (ii): a > b

In this case, interchanging a and b in the Case (i), you will get

$$\left|\int_b^a f(x)\, dx\right| \leq k(a-b)$$

i.e. $\quad \left|-\int^b f(x)\, dx\right| \leq k(a-b)$

i.e. $\quad \left|\int^b f(x)\, dx\right| \leq k(a-b) = k|b-a|.$

Case (iii): a = b

In this case also, the result holds,

since $\int_a^b f(x)\, dx = 0$ for a = b and $k|b-a| = 0$ for a = b.

Let [a,b] be a fixed interval. Let R [a,b] denote the set of all Riemann integrable functions on this interval. We have shown that if f,g C R [a,b], then f + g f.g and λf for A ∈ R belong to R [a,b]. Combining these with Property, we can say that the set R [a,b] of Riemann integrable functions is closed under addition, multiplication, scalar multiplication and the formatian of the absolute value.

If we consider the integral as a function Int: R[a,b] → R defined by

$$\text{Int }(f) = \int_a^b f(x)\, dx$$

with domain R [a,b] and range contained in R, then this function has the following properties:

$$\text{lnt }(f+g) = \text{Int }(f) + \text{Int }(g),\ \text{lnt }(\lambda f) = \lambda \text{ Int }(f)$$

In other words, the function lnt preserves 'Vector sums' and the scalar products. In the language of Linear Algebra, the function lnt acts as a linear transformation. This function also has an additional interesting property such as

$$\text{lnt }(f) \leq \text{lnt}(g)$$

whenever

$$f \leq g\ .$$

We state yet another interesting property (without proof) which shows that the Riemann Integral is additive on an interval.

Property 4: If f is integrable on [a,b] and c ∈ [a,b], then f is integrable on [a,c] and [c,b] and conversely. Further in either case

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

According to this property, if we split the interval over which we are integrating into two parts, the value of the integral over the whole will be the sum of the two integrals over the subintervals. This amounts to dividing the region whose area must be found into two separate parts while the total area is the sum of the areas of the separate portions.

We now state a few more properties of the definite integral $\int_a^b f(x)\,dx$, these are:

(i) $\int_n f(x)\,dx = \int_a f(a-x)\,dx.$

(ii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)dx.$

(iii) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx \text{ if f is an even function} \\ 0 \text{ iff is an old function.} \end{cases}$

(iv) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if f is periodic with period 'a' and n is a positive integer provided the integrals exist.

Self Assessment

Fill in the blanks:

1. If f, is .... on [a, b] then $|f|$ is also integrable on $[a,b]$ and $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

2. The inequality follows at once from Property 1 provided it is known that $|f|$ is .... on [a, b]. Indeed, you know that $-|f| \le f \le |f|$.

3. If a function f is integrable in [a, b] and ...., then $\left|\int_a^b f(x)\,dx\right| \le k|b-a|$.

4. If f is integrable on [a, b] and ...., then f is integrable on [a, c] and [c, b] and conversely.

21.2 Summary

- Sum of two Riemann Stieltjes integrable functions is also Riemann Stieltjes integrable.
- Scalar product of a Riemann Stieltjes integrable function is also Riemann Stieltjes integrable.
- Modulus of a Riemann Stieltjes integrable function is also Riemann Stieltjes integrable.
- Square of a Riemann Stieltjes integrable function is also Riemann Stieltjes integrable.
- If a function is Riemann Stieltjes integrable on an interval, then it is also Riemann Stieltjes integrable on any of its subinterval.

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21.3 Keyword

Riemann Stieltjes Integrable: Sum of two Riemann Stieltjes integrable functions is also Riemann Stieltjes integrable.

21.4 Review Questions

1. Calculate if a < b, $\int_b^a f d\alpha$
2. Suppose f is a bounded valued function on [a, b] and f2 $\in$ R on [a, b]. Does it follow that f $\in$ R on [a, b]?
3. Show that 0∫1 x2dx = 3/5 where α(n) = x3
4. Show that 0∫2 [x]dx = 3/5 where α(x) = x2 = 3.

Answers: Self Assessment

1. integrable	2. integrable
3. $\left\|f(x)\right\| \le k \ \forall \ x \ E[a,b]$	4. c $\in$ [a, b]

21.5 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 22: Introduction to Riemann-Stieltjes Integration, using Riemann Sums

Notes

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Objectives

After studying this unit, you will be able to:

- Discuss Riemann–Stieltjes sums
- Know the Cauchy Criterion for Riemann–Stieltjes Integrability

Introduction

We will approach Riemann-Stieltjes integrals using Riemann-Stieltjes sums instead of the upper and lower sums. The main reasons are to study Riemann-Stieltjes integrals with "integrators" $\alpha(x)$ that are not monotone, but are "of bounded variation," and (most important) here you are able to define Riemann-Stieltjes integrals when the values of my functions belong to an infinite dimensional vector space, where upper and lower sums don't make sense. This makes little difference in the case of real-valued functions, since functions of bounded variation can always be expressed as the difference of two monotone functions. At first, we don't need "bounded variation," so that concept's development will wait until it is needed.

Throughout this note, our functions f(x) will be "finite-valued." They may be real, complex, or vector-valued. Their values will thus lie in a vector space. They can thus be added pointwise, and multiplied by scalars, and their values always have finite "distance from zero," denoted $|f(x)|$, which can denote absolute value or *norm*, such as the length of a vector, or the "L^p norm" and the "L^q norm". In case f(x) is actually a function of t for each x... We always assume that the "absolute value" is *complete*; Cauchy sequences converge.

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22.1 Riemann-Stieltjes Sums

A Riemann-Stieltjes sum for a function f(x) defined on an interval [a, b] is formed with the help of

1. A partition π of [a, b], namely an ordered, finite set of points x_i with $a = x_0 < x_1 < \cdots < x_n = b$ (where n is a positive integer that can be any positive integer, and one that we will often write as $n = n_\pi$),

2. A selection vector $\xi = (\xi_1, ..., \xi_n)$ that has n_π components that must satisfy $x_{i-1} \le \xi_i \le x_i$, for i = 1, 2, ..., n.

   and

3. An integrator $\alpha(x)$, which is a function defined on [a, b] that plays the role of the x in dx ...

   A Riemann-Stieltjes sum for f over [a, b] with respect to the partition π, using the selection vector ξ, and integrator α, may be denoted (in greatest detail!) as follows, and it is given by the value of the sum following it:

4. $$RS\,(f, \alpha, [a, b], \pi, \xi) := \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})).$$

22.2 More Notation: The Mesh (Size) of a Partition

In this definition, as in the Riemann-sums definition, we can write $\Delta x_i := x_i - x_{i-1}$ or $\Delta\alpha_i := \alpha(x_i) - \alpha(x_{i-1})$. These are convenient because they are short and suggest the dx or $d\alpha$ in an integral. But they can cause confusion because they leave out the dependence they have on x_{i-1}. The Δx_i is used in the Riemann-Stieltjes context.

A partition π can be thought of as "dividing" the interval [a, b] into subintervals. We may write $\pi \,|\, [a, b]$ and read this as "π divides [a, b]," or "partitions [a, b]." We will denote the intervals of π by $I_i := [x_{i-1}, x_i]$. When we wish to work with 2 partitions at the same time we will have to distinguish between them somehow, for example we can use y_j to denote the other's points and J_j to denote its intervals, etc.

We measure the fineness of a partition using the length of the longest interval in the partition. This number is written

$$\text{mesh}(\pi) := \max_{1 \le i \le n_\pi} (x_i - x_{i-1}) = \max_{1 \le i \le n_\pi} \Delta x_i.$$

This definition of mesh size is used and not $\max_{1 \le i \le n_\pi} (\alpha(x_i) - \alpha(x_{i-1}))$ even in the Riemann-Stieltjes context.

22.3 The Riemann-Stieltjes – sum Definition of the Riemann-Stieltjes Integral

Definition: A real-valued function f(x) defined on the bounded and closed interval [a, b] is Riemann-Stieltjes integrable on [a, b] with respect to α if there exists a number RSI such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition π of [a, b],

$$\text{mesh}(\pi) < \delta \Rightarrow |RS(f, \pi) - RSI| < \varepsilon.$$

We write

$$\int_a^b f\, d\alpha = \int_a^b f(x)\, d\alpha(x) := RSI$$

and we call this the Riemann-Stieltjes integral of f over [a, b] with respect to α.

If f and α are real-valued and we imagine the set of all numbers RS(f, α, π) that can be formed (using all possible appropriate selection vectors and all possible partitions whose mesh sizes are less than δ), the definition demands that they all lie in the open interval (RSI – ε, RSI + ε). When we had $\alpha(x) = x$ this led to a Theorem.

Theorem: If f is Riemann integrable on [a, b] then f is bounded on [a, b].

This Theorem has to be modified in the Riemann-Stieltjes context! A simple example: suppose that [a, b] is [0, 1] and that $\alpha(x) = 0$ if $0 \leq x \leq c$, where $0 < c < 1$, and $\alpha(x) = 1$ if $c < x \leq 1$. Then every function f(x) that is continuous at c is Riemann-Stieltjes integrable on [0, 1] with respect to this α. In particular the function that is 1/x except at zero, where we define it to be zero, is Riemann-Stieltjes integrable on [0, 1] with respect to this α, but f is not bounded. The difference is that when $\alpha(x)$ was just x, we had $\Delta x_i > 0$ for every i. In our example, $\Delta\alpha_i = 0$ unless I_i contains c and some d with c < d. What we need is that on the set where the function α "really" varies, f must be bounded. To make a definition, we will extend the definitions of f and α beyond the interval [a, b] by setting them equal to their values at the endpoints. Thus we think of f(x) = f(a) if x < a and f(x) = f(b) if x > b, with the same idea used to extend α. We now define the oscillation of f on an interval U by

$$\omega(f, U) := \sup_{x, y \in U} |f(x) - f(y)|.$$

We allow the interval to be open or half-open now!

As before, we will let $\omega_i = \omega_i(f) = \omega(f, I_i)$ when I_i is an interval (closed!) of a partition π. But now we need to use oscillations of α as well.

Definition: If $\alpha(x)$ is defined for $x \in [a, b]$, we denote by $\Omega = \Omega(\alpha, [a, b])$ the set of all $c \in [a, b]$ such that every open interval U that contains c contains $x_1 < c < x_2$ with $|\alpha(x_1) - \alpha(x_2)| > 0$.

Notes Here c can be a or b because of our extension beyond [a, b]! For instance, if for all $\delta > 0$ there exists x_2 such that $a < x_2 < a + \delta$ and $|\alpha(a) - \alpha(x_2)| > 0$, then $a \in \Omega(\alpha, [a, b])$ because for every $x_1 < a$ we have $|\alpha(x_1) - \alpha(x_2)| = |\alpha(a) - \alpha(x_2)| > 0$.

Task Prove that $\Omega(\alpha, [a, b])$ is closed.

Theorem: If f is Riemann-Stieltjes integrable on [a, b] with respect to α then f is bounded on $\Omega(\alpha, [a, b])$.

Proof: There exists a sequence $\{x_n\}$ in $\Omega := \Omega(\alpha, [a, b])$ such that $|f(x_n)| > n$. Since f(x) is finite at every point x in Ω, there are infinitely many distinct x_n, and so some subsequence (that we will still denote $\{x_n\}$) converges to a point x* in Ω. We now choose $\varepsilon = 1$ in the definition of Riemann-Stieltjes integrability, and obtain a corresponding $\delta > 0$. We can then construct a partition π_o with mesh size less than δ in such a way that x* is contained in the interior of some interval I_{i_o} of π_o (unless x* is an endpoint of [a, b]; in that case, we can, by the Note, still use the following argument, with $I_{i_o} = I_1$ or $I_{i_o} = I_{n_\pi}$). We know that every neighbourhood of x* contains infinitely many of the x_n. Now we will refine π_o. We know that $\text{Int}(I_{i_o})$ contains points $\hat{x}_1 < x^* < \hat{x}_2$ with $|\alpha(\hat{x}_1) - \alpha(\hat{x}_2)| > 0$. We add these points to π_o, giving us a new partition π, and mesh(π) < δ. We will now call $[\hat{x}_1, \hat{x}_2]$, which is an interval of π, $\hat{I}$. Next we pick the components ξ_i of a selection vector ξ in an arbitrary way when $I_i \neq \hat{I}$, and we let $\hat{\xi}$ be some $x_N \in \hat{I}$. Then $|RS(f, \pi, \xi) - RSI| < 1$. We next modify ξ by changing only $\hat{\xi} = x_N$ to $\hat{\xi}' := x_M$, where $x_M \in \hat{I}$, and we call the new

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selection vector ξ'. Then $|RS(f, \pi, \xi') - RSI| < 1$, $RS(f,\pi,\xi') - RS(f,\pi,\xi) = (f(x_M) - f(x_N))(\alpha(x_i) - \alpha(x_2))$ and

$$RS(f, p, \xi') - RSI = RS(f, \pi, \xi) - RSI + (f(x_M) - f(x_N))(\alpha(\hat{x}_1) - \alpha(\hat{x}_2)).$$

By choosing M very large compared to N we can arrange that $|f(x_M) - f(x_N)|\,|\alpha(\hat{x}_1) - \alpha(\hat{x}_2)| > 2$. Then

$$1 > |RS(f, \pi, \xi') - RSI| \geq |RS(f, \pi, \xi') - RS(f, \pi, \xi)| - |RS(f, \pi, \xi) - RSI| > 2 - 1 = 1.$$

The definition of Riemann-Stieltjes integrability is contradicted. Hence f is bounded on$\Omega(\alpha, [a, b])$ if f is Riemann-Stieltjes integrable with respect to α.

> *Notes* From now on, we will usually say "f is Riemann-Stieltjes integrable" instead of "f is Riemann-Stieltjes integrable with respect to α."

22.4 A difficulty with the Definition; The Cauchy criterion for Riemann-Stieltjes integrability

In order to tell whether f is Riemann-Stieltjes integrable we have to know $\int_a^b f(x)\, d\alpha(x)$. The idea of a Cauchy sequence leads to the following Theorem, which gives an equivalent definition.

Theorem: Cauchy criterion for Riemann-Stieltjes Integrability

A function defined on [a, b] is Riemann-Stieltjes integrable over [a, b] with respect to α, defined on [a, b], if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all partitions π and π' of [a, b], and for all selection vectors ξ and ξ' associated with π and π', respectively,

$$\text{mesh}(\pi) < \delta \text{ and } \text{mesh}(\pi') < \delta \Rightarrow |RS(f, \alpha, \pi, \xi) - RS(f, \alpha, \pi', \xi')| < \varepsilon.$$

Proof: First we suppose that f is Riemann-Stieltjes integrable over [a, b] with respect to α. Then, using $\varepsilon/2$ in the definition of Riemann-Stieltjes integrability, we obtain $\delta > 0$ and RSI such that for all partitions π of [a, b],

$$\text{mesh}(\pi) < \delta \Rightarrow |RS(\pi) - RSI| < \varepsilon/2$$

Now we suppose that π and π' are partitions of [a, b] and that

$$\text{mesh}(\pi) < \delta \text{ and } \text{mesh}(\pi') < \delta.$$

Then for all selection vectors ξ and ξ' associated with π and π', respectively,

$$|RS(f, \alpha, \pi, \xi) - RS(f, \alpha, \pi', \xi')| \leq |RS(\pi, \xi) - RSI| + |RSI - RS(\pi', \xi')| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This completes half the proof.

Next we suppose that the Cauchy condition, given in the Theorem, is satisfied. We have to find a candidate for $\int_a^b f(x)\, d\alpha(x)$. We first construct a sequence of partitions of [a, b]. We let π_n denote the partition that divides [a, b] into n equal parts $\left(\pi_n \text{ has points } x_{ni} := a + i\frac{b-a}{n}\right)$. Finally we define selection vectors ξ_n by

$$\xi_{ni} := a + i\,\frac{b-a}{n},\ i = 1, ..., n \text{ and define } \sigma_n := \sum_{i=1}^{n} f(\xi_{ni})(\alpha(x_{ni}) - \alpha(x_{n,i-1})),$$

a Riemann-Stieltjes sum ($\sigma_n = RS(f, \alpha, \pi_n, \xi_n)$). Now, given $\varepsilon > 0$, we use $\varepsilon/2$ in the Cauchy criterion, and obtain $\delta > 0$ such that

$$\text{mesh}(\pi) < \delta \text{ and } \text{mesh}(\pi') < \delta \Rightarrow |RS(f, \alpha, \pi, \xi) - RS(f, \alpha, \pi', \xi')| < \varepsilon/2.$$

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Then, if n and n′ are so large that $(b - a)/n < \delta$ and $(b - a)/n' < \delta$, we have

$$\text{mesh}(\pi_n) < \delta \text{ and } \text{mesh}(\pi_{n'}) < \delta \Rightarrow |\sigma_n - \sigma_{n'}| < \varepsilon/2.$$

This means (since ε was arbitrary) that $\{\sigma_n\}$ is a Cauchy sequence in our space. Thus we define

$$\text{RSI} := \lim_{n\to\infty} \sigma_n$$

and it remains to show that if $\pi | [a, b]$ then

$$\text{mesh}(\pi) < \delta \Rightarrow |\text{RS}(\pi) - \text{RSI}| < \varepsilon.$$

This is essentially done. We choose the first n such that $\text{mesh}(\pi_n) < \delta$, and we suppose that $\text{mesh}(\pi) < \delta$. Then

$$|\text{RS}(\pi) - \text{RI}| \le |\text{RS}(\pi) - \sigma_n| + |\sigma_n - \text{RSI}| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

since $\text{RS}(\pi) - \sigma_n = \text{RS}(\pi) - \text{RS}(f, \alpha, \pi_n, \xi_n)$. The proof is complete.

> *Notes* Nothing is said at first about the functions f and α, beside the demand that the integrability definition hold.

If f and α have a discontinuity at the same point, then the Riemann-Stieltjes integral does not exist.

They also show that if the Riemann-Stieltjes integral exists, then this integration-by-parts formula holds:

$$\int_a^b f\, d\alpha = -\int_a^b \alpha\, df + f(b)\alpha(b) - f(a)\alpha(a)$$

(in the applications have far, far back in my mind, the integral on the right would have to be $\int_a^b df\,\alpha$, in order to keep the order of "multiplication" the same). The proof amounts to rearranging the Riemann-Stieltjes sums, adding and subtracting terms in such a way that the ξ_i become partition points and the x_i become selection-vector components when $1 < i < n_\pi$. There are some leftovers, and these turn out to be the "boundary" term $f(x)\alpha(x)|_a^b$.

Wheeden and Zygmund state several properties, routine to prove, about Riemann-Stieltjes integrals:

$$\int_a^b f\, d\alpha \text{ is linear in both f and } \alpha$$

as long as all the integrals involved exist, and if $\int_a^b f\, d\alpha$ exists and $a < c < b$, then both of $\int_a^c f\, d\alpha$ and $\int_c^b f\, d\alpha$ exist, and $\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha$.

What has been covered applies to all Riemann-Stieltjes integrals. That continuity plays a role has already been mentioned.

22.5 Functions of Bounded Variation: Definition and Properties

In what we do from now on, at least one of f and α will be a function of bounded variation, unless otherwise stated. We will begin by discussing real-valued functions of bounded variation. This material can also be found in Measure and Integral, by Wheeden and Zygmund.

Definition: A function $f: [a, b] \to \mathbb{R}$ is a function of bounded variation on [a, b] if

$$V(f, [a, b]) := \sup_{\pi|[a,b]} \sum_{1}^{n_\pi} |f(x_i) - f(x_{i-1})| < \infty \text{ and we say that } f \in BV[a, b].$$

Notes

To go farther it will be useful to have some more notation. If π is a partition of [a, b] we will write

$$V_\pi = V_\pi(f, [a, b]) := \sum_1^{n_\pi} |f(x_i) - f(x_{i-1})|,$$

so that $V = \sup_{\pi|[a,b]} V_\pi$ (here, f and [a, b] are "assumed").

We can call V_π the "π-variation" of f over [a, b]. Since f has a finite value for each $\pi | [a, b]$, V_π is always finite. However, V can be infinite. This is so, for example, if f is the Dirichlet function.

Each V_π pays attention only to the absolute value of the difference between the values at the opposite ends of an interval of the partition π. We will need to take the signs of those differences into account, and they will lead to two new "variations."

For a real number x we define its positive part to be $x^+ := \max\{0, x\}$ and we define its negative part to be $x^- := \max\{0, -x\}$. Both "parts" are non-negative, and we have $x^+ + x^- = |x|$ and $x^+ - x^- = x$.

Example: Prove that for all real numbers x and y, $(x + y)^+ \leq x^+ + y^+$ and $(x + y)^- \leq x^- + y^-$. These are "triangle inequalities!" What can be said about $(xy)^+$ and $(xy)^-$?

We now define the "positive" and "negative" " π-variations" of f over [a, b]:

$$P_\pi = P_\pi(f, [a, b]) := \sum_1^{n_\pi} (f(x_i) - f(x_{i-1}))^+ \text{ and } N_\pi = N_\pi(f, [a, b]) := \sum_1^{n_\pi} (f(x_i) - f(x_{i-1}))^-.$$

Definition: The positive variation, P = P(f, [a, b]) and the negative variation N = N(f, [a, b]) of f over [a, b] are given by $P = \sup_{\pi|[a,b]} P_\pi$ and $N = \sup_{\pi|[a,b]} N_\pi$ respectively.

For example, if f increases on [a, b], $P_\pi = V_\pi = f(b) - f(a)$ and $N_\pi = 0$. If we look at $f(x) := |x|$ on [–1, 1] we will always have $0 \leq P_\pi \leq 1$ and $0 \leq N_\pi \leq 1$, and $0 \leq V_\pi \leq 2$.

Because of how x^+ and x^- were defined, we always have (for any function)

$$P_\pi + N_\pi = V_\pi \text{ and } P_\pi - N_\pi = f(b) - f(a)$$

If τ is a refinement of π, we always have $O_\pi \leq O_\tau$, where O stands for any of the letters N, P or V. This follows from several applications of the triangle inequality.

22.6 Some Properties of Functions of Bounded Variation

If $f \in BV[a, b]$ then f is bounded on [a, b].

Proof: Suppose $a \leq x \leq b$. Then, if we let $\pi := \{a, x, b\}$,

$$|f(x)| = |f(x) - f(a) + f(a)| \leq |f(a)| + |f(x) - f(a)| + |f(b) - f(x)| = |f(a)| + V_\pi \leq |f(a)| + V.$$

The space BV[a, b] is a vector space. For all $c \in \mathbb{R}$ and all $f \in BV[a, b]$, $V(cf, [a, b]) = |c| V(f, [a, b])$. For all $f \in BV[a, b]$ and $g \in BV[a, b]$, $V(f, [a, b]) \leq V(f, [a, b]) + V(g, [a, b])$; $V(f, [a, b]) = 0$ if and only if f is constant.

Proof: The second assertion follows from these facts: for all $\pi | [a, b]$, $V_\pi(cf, [a, b]) = |c| V_\pi(f, [a, b])$; $\sup\{|c| x: x \in E\} = |c| \sup\{x: x \in E\} = |c| \sup E$. The first assertion and the first part of the third one follow from the second one and the triangle inequality. Finally, suppose that $V(f, [a, b]) = 0$ and that $a \leq x \leq b$. Then, with $\pi := \{a, x, b\}$, $|f(x) - f(a)| \leq |f(x) - f(a)| + |f(b) - f(x)| = V_\pi = 0$. Therefore $f(x) \equiv f(a)$.

If $f \in BV[a, b]$ and $a < c < b$ then $f \in BV[a, c]$ and $f \in BV[c, b]$, and conversely. Moreover, $V = V(f, [a, b]) = V(f, [a, c]) + V(f, [c, b])$.

Notes

Proof: If $f \in BV[a, b]$ and $a < c < b$, let partitions $\sigma |\ [a, c]$ and $\tau |\ [c, b]$ be given. Then $\pi := \sigma \cup \tau$ is a partition of $[a, b]$ so $V_\sigma + V_\tau = V_\pi \le V$, hence $V_\sigma \le V$ and $V_\tau \le V$. Thus $f \in BV[a, c]$ and $f \in BV[c, b]$. Conversely, suppose that $a < c < b$ and that $f \in BV[a, c]$ and $f \in BV[c, b]$. Let $\pi | [a, b]$. Then $\pi_c := \pi \cup \{c\}$ is a refinement of π. Therefore $V_\pi \le V_{\pi_c} = V_\sigma + V_\tau$, where $\sigma := \pi_c \cap [a, c]$ and τ is defined similarly. By hypothesis, $V_\pi \le V_{\pi_c} = V_\sigma + V_\tau \le V(f, [a, c]) + V(f, [c, b])$. Thus $V(f, [a, b]) \le V(f, [a, c]) + V(f, [c, b]) < \infty$. This proves part of the asserted equality. To show the other inequality, now that we know $V < \infty$ let partitions $\sigma |\ [a, c]$ and $\tau |\ [c, b]$ be given. We recall that earlier we had $V_\sigma + V_\tau = V_{\pi_c} \le V$, so $V_\sigma + V_\tau \le V$ whenever $\sigma |\ [a, c]$ and $\tau |\ [c, b]$ were arbitrary partitions of $[a, c]$ and $[c, b]$, respectively.

Thus $\sup_{\sigma|[a,c]} (V_\sigma + V_\tau) = V(f, [a, c]) + V_\tau \le V$, and so $\sup_{\tau|[c,b]} (V(f, [a, c]) + V_\tau) = V(f, [a, c]) + V(f, [c, b]) \le V$.

Note The first inequality holds for an arbitrary $\tau | [c, b]$, making the second one valid.

Example: Prove that the equality in (25) holds for every function $f: [a, b] \to \mathbb{R}$, whether f is a function of bounded variation or not.

Motivated by (25), when $f: [a, b] \to \mathbb{R}$ and $a \le x \le b$ we can define the three functions

$$V(x) := V(f, [a, x]),\ P(x) := P(f, [a, x]) \text{ and } N(x) := N(f, [a, x]).$$

Each of these is an increasing function of x. Jordan's Theorem asserts that if $f \in BV[a, b]$ we can represent f in terms of $P(x)$ and $N(x)$.

Theorem: Jordan

A function $f \in BV[a, b]$ if and only if there exist functions g and h, both increasing on $[a, b]$, such that $f(x) = g(x) - h(x)$ for $a \le x \le b$. If this is the case, then $P(x) \le g(x) - g(a)$, $N(x) \le h(x) - h(a)$ and $f(x) = f(a) + P(x) - N(x)$ for $a \le x \le b$.

Proof: Suppose first that $f(t) = g(t) - h(t)$, $t \in [a, b]$, where the functions g and h are both increasing on $[a, b]$. Let $\pi | [a, b]$ (later, we will apply this when $x \in [a, b]$ and $\pi | [a, x]$). Then

$$\Delta f_i = f(x_i) - f(x_{i-1}) = \Delta g_i - \Delta h_i \begin{cases} \le \Delta g_i \\ \ge -\Delta h_i \end{cases}.$$

Thus $-\Delta h_i \le \Delta f_i \le \Delta g_i$, so $|\Delta f_i| \le \max\{\Delta g_i, \Delta h_i\} \le \Delta g_i + \Delta h_i$ for $1 \le i \le n_\pi$. Hence $V_\pi(f) \le V_\pi(g) + V_\pi(h) = g(b) - g(a) + h(b) - h(a) < \infty$, so $f \in BV[a, b]$.

Next, we show that $f(x) = f(a) + P(x) - N(x)$ for $a \le x \le b$. But we will do this just by showing it for $x = b$. Then we can use (25) and let each $x \in [a, b]$ play the role of b. This will show the existence of the functions $g(x)(= f(a) + P(x))$ and $h(x)(= N(x))$. After that is done, we'll prove the P–g and N–h inequalities.

By the definitions of P, N and V we know there exist sequences $\{\pi_k\}$, $\{\rho_k\}$ and $\{\sigma_k\}$ such that $P_{\pi_k} \to P$, $N_{\pi_k} \to N$ and $V_{\sigma_k} \to V$. Let us define $\tau_k := \pi_k \cup \rho_k \cup \sigma_k$. As $P_{\pi_k} \le P_{\tau_k} \le P$. By the Squeeze Principle $P_{\tau_k} \to P$. Similarly, $N_{\tau_k} \to N$ and $V_{\tau_k} \to V$. By Limit Theorems

$P + N = V$ and $P - N = f(b) - f(a)$ and the second is the same as $f(x) = f(a) + P(x) - N(x)$

when $x = b$. By above we can use any $x \in [a, b]$ in place of b by restricting our attention to f on $[a, x]$.

Now suppose that $f(x)$ is defined as the difference of two increasing functions on $[a, b]$: $f(t) = g(t) - h(t)$. We have the following observation: $t_1 \to t^+$ is increasing and $t_1 \to t^-$ is decreasing.

Notes

Therefore, with the help of (28), applied to partitions of [a, x], $(\Delta f_i)^+ \leq (\Delta g_i)^+ = \Delta g_i$ and $\Delta h_i = (-\Delta h_i)^- \geq (\Delta f_i)^-$.

Hence $P_\pi(f, [a, x]) \leq P_\pi(g, [a, x]) = g(x) - g(a)$. Similarly, $h(x) - h(a) = N_\pi(-h, [a, x]) \geq N_\pi(f, [a, x])$. When, in each case, we take the supremum over all $\pi \mid [a, x]$, we get $P(x) \leq g(x) - g(a)$ and $N(x) \leq h(x) - h(a)$. These "say" that there is no "wasted cancellation" in the formula $f(x) = f(a) + P(x) - N(x)$.

Task Prove that if $f(x) \in BV[a, b]$ and f(x) is continuous at $x_o \in [a, b]$ then so are P(x), N(x) and V(x).

Self Assessment

Fill in the blanks:

1. A .... f(x) defined on the bounded and closed interval [a, b] is Riemann-Stieltjes integrable on [a, b] with respect to α if there exists a number RSI such that for all ε > 0 there exists δ > 0 such that for every partition π of [a, b],

$$\text{mesh}(\pi) < \delta \Rightarrow |RS(f, \pi) - RSI| < \varepsilon.$$

2. If f is Riemann integrable on [a, b] then f is ....
3. If α(x) is defined for x ∈ [a, b], we denote by $\Omega = \Omega(\alpha, [a, b])$ the set of all c ∈ [a, b] such that every open interval U that contains c contains $x_1 < c < x_2$ with ....
4. If f is .... on [a, b] with respect to α then f is bounded on Ω(α, [a, b]).
5. If f and α have a .... at the same point, then the Riemann-Stieltjes integral does not exist.

22.7 Summary

- A Riemann-Stieltjes sum for a function f(x) defined on an interval [a, b] is formed with the help of
  (a) A partition π of [a, b], namely an ordered, finite set of points x_i, with $a = x_0 < x_1 < \cdots < x_n = b$ (where n is a positive integer that can be any positive integer, and one that we will often write as $n = n_\pi$),
  (b) A selection vector $\xi = (\xi_1, ..., \xi_n)$ that has n_π components that must satisfy $x_{i-1} \leq \xi_i \leq x_i$, for i = 1, 2, ..., n. and
  (c) An integrator α(x), which is a function defined on [a, b] that plays the role of the x in dx ...

  A Riemann-Stieltjes sum for f over [a, b] with respect to the partition π, using the selection vector ξ, and integrator α, may be denoted (in greatest detail!) as follows, and it is given by the value of the sum following it:

  (d) $RS(f, \alpha, [a, b], \pi, \xi) := \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))$.

  We try to allow context to let us drop some of the items inside the RS(...).
- In this definition, as in the Riemann-sums definition, we can write $\Delta x_i := x_i - x_{i-1}$ or $\Delta\alpha_i := \alpha(x_i) - \alpha(x_{i-1})$. These are convenient because they are short and suggest the dx or dα in an integral. But they can cause confusion because they leave out the dependence they have on x_{i-1}. The Δx_i is used in the Riemann-Stieltjes context.

- A partition π can be thought of as "dividing" the interval [a, b] into subintervals. We may write $\pi \mid [a, b]$ and read this as "π divides [a, b]," or "partitions [a, b]." We will denote the intervals of π by $I_i := [x_{i-1}, x_i]$. When we wish to work with 2 partitions at the same time we will have to distinguish between them somehow, for example we can use y_j to denote the other's points and J_j to denote its intervals, etc.
- A real-valued function f(x) defined on the bounded and closed interval [a, b] is Riemann-Stieltjes integrable on [a, b] with respect to α if there exists a number RSI such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition π of [a, b],

$$\text{mesh}(\pi) < \delta \Rightarrow |RS(f, \pi) - RSI| < \varepsilon.$$

22.8 Keywords

Cauchy Criterion for Riemann-Stieltjes Integrability: A function defined on [a, b] is Riemann-Stieltjes integrable over [a, b] with respect to α, defined on [a, b], if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all partitions π and π' of [a, b], and for all selection vectors ξ and ξ' associated with π and π', respectively,

$$\text{mesh}(\pi) < \delta \text{ and mesh}(\pi') < \delta \Rightarrow |RS(f, \alpha, \pi, \xi) - RS(f, \alpha, \pi', \xi')| < \varepsilon.$$

Jordan: A function $f \in BV[a, b]$ if and only if there exist functions g and h, both increasing on [a, b], such that $f(x) = g(x) - h(x)$ for $a \le x \le b$. If this is the case, then $P(x) \le g(x) - g(a)$, $N(x) \le h(x) - h(a)$ and $f(x) = f(a) + P(x) - N(x)$ for $a \le x \le b$.

22.9 Review Questions

1. Identify the properties of the integral.
2. Use them to find the Riemann stieltjes integral of functions.

Answers: Self Assessment

1. real-valued function
2. bounded on [a, b]
3. $|\alpha(x_1) - \alpha(x_2)| > 0$
4. Riemann-Stieltjes integrable
5. discontinuity

22.10 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 23: Differentiation of Integrals

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Differentiation of Integrals
- Discuss the Theorems on the Differentiation of Integrals

Introduction

In this unit, we are going to study about differentiation of integrals. Suppose $\vee$ is a function of two variables which can be integrated with respect to one variable and which can be differentiated with respect to another variable. We are going to see under what conditions the result will be the same if these two limit process are carried out in the opposite order.

23.1 Differentiation of Integrals

In mathematics, the problem of differentiation of integrals is that of determining under what circumstances the mean value integral of a suitable function on a small neighbourhood of a point approximates the value of the function at that point. More formally, given a space X with a measure μ and a metric d, one asks for what functions $f : X \to R$ does

$$\lim_{r \to u} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f(y)\, d\mu(y) = f(x)$$

for all (or at least μ-almost all) $x \in X$? (Here, as in the rest of the article, $B_r(x)$ denotes the open ball in X with d-radius r and centre x.) This is a natural question to ask, especially in view of the heuristic construction of the Riemann integral, in which it is almost implicit that f(x) is a "good representative" for the values of f near x.

23.2 Theorems on the Differentiation of Integrals

Lebesgue Measure

One result on the differentiation of integrals is the Lebesgue differentiation theorem, as proved by Henri Lebesgue in 1910. Consider n-dimensional Lebesgue measure λ^n on n-dimensional Euclidean space R^n. Then, for any locally integrable function $f : R^n \to R$, one has

Notes

$$\lim_{r \to 0} \frac{1}{\lambda^n(B_r(x))} \int_{B_r(x)} f(y) \, d\lambda^n(y) = f(x)$$

for λ^n-almost all points $x \in R^n$. It is important to note, however, that the measure zero set of "bad" points depends on the function f.

Borel Measures on R^n

The result for Lebesgue measure turns out to be a special case of the following result, which is based on the Besicovitch covering theorem: if μ is any locally finite Borel measure on R^n and $f : R^n \to R$ is locally integrable with respect to μ, then

$$\lim_{r \to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f(y) \, d\mu(y) = f(x)$$

for μ-almost all points $x \in R^n$.

Gaussian Measures

The problem of the differentiation of integrals is much harder in an infinite-dimensional setting. Consider a separable Hilbert space $(H, \langle , \rangle)$ equipped with a Gaussian measure γ. As stated in the article on the Vitali covering theorem, the Vitali covering theorem fails for Gaussian measures on infinite-dimensional Hilbert spaces. Two results of David Preiss (1981 and 1983) show the kind of difficulties that one can expect to encounter in this setting:

- There is a Gaussian measure γ on a separable Hilbert space H and a Borel set $M \subseteq H$ so that, for γ-almost all $x \in H$,

$$\lim_{r \to 0} \frac{\lambda(M \cap B_r(x))}{\gamma(B_r(x))} = 1$$

- There is a Gaussian measure γ on a separable Hilbert space H and a function $f \in L^1(H, \gamma; R)$ such that

$$\lim_{r \to 0} \inf \left\{ \frac{1}{\gamma(B_s(x))} \int_{Ds(x)} f(y) \, d\gamma(y) \middle| x \in II, 0 < s < r \right\} - +\infty$$

However, there is some hope if one has good control over the covariance of γ. Let the covariance operator of γ be $S : H \to H$ given by

$$\langle Sx, y \rangle = \int_H \langle x, z \rangle \langle y, z \rangle \, d\gamma(z)$$

or, for some countable orthonormal basis $(e_i)_{i \in N}$ of H,

$$Sx = \sum_{i \in N} \sigma_1^2 \langle x, e_i \rangle e_i.$$

In 1981, Preiss and Jaroslav Tišer showed that if there exists a constant $0 < q < 1$ such that

$$\sigma_{i+1}^2 \leq q \sigma_i^2,$$

then, for all $f \in L^1(H, \gamma; R)$,

$$\frac{1}{\mu(B_r(x))} \int_{B_r(x)} f(y) \, d\mu(y) \xrightarrow[r \to 0]{\gamma} f(x)$$

Notes

where the convergence is convergence in measure with respect to γ. In 1988, Tišer showed that if

$$\sigma_{i+1}^2 \le \frac{\sigma_i^2}{i^\alpha}$$

for some α > 5/2, then

$$\frac{1}{\mu(B_r(x))}\int_{B_r(x)} f(y)\,d\mu(y) \xrightarrow[r\to 0]{} f(x)$$

for γ-almost all x and all $f \in L^p(H, \gamma; R)$, p > 1.

As of 2007, it is still an open question whether there exists an infinite-dimensional Gaussian measure γ on a separable Hilbert space H so that, for all $f \in L^1(H, \gamma; R)$,

$$\lim_{r\to 0} \frac{1}{\gamma(B_r(x))}\int_{B_r(x)} f(y)\,d\gamma(y) = f(x)$$

for γ-almost all x ∈ H. However, it is conjectured that no such measure exists, since the σ_i would have to decay very rapidly.

Example: If $\alpha \ne 0, \phi(\alpha) = \arctan\left(\frac{1}{\alpha}\right)$

The function $\frac{\alpha}{x^2+\alpha^2}$ is not continuous at the point (x, α) = (0, 0) and the function φ(α) has a discontinuity α = 0, because φ(α) approaches $+\frac{\pi}{2}$ as $\alpha \to 0^+$ and approaches $-\frac{\pi}{2}$ as $\alpha \to 0^-$.

If we now differentiate φ(α) = $\int_0^1 \frac{\alpha}{x^2+\alpha^2}dx$ with respect to α under the integral sign, we get $\frac{d}{d\alpha}\phi(\alpha) = \int_0^1 \frac{x^2-\alpha^2}{x^2+\alpha^2}dx = -\frac{x}{x^2+a^2}\Big|_0^1 = -\frac{1}{1+\alpha^2}$ which is, of course, true for all values of α except α = 0.

Example: The principle of differentiating under the integral sign may sometimes be used to evaluate a definite integral.

Consider integrating $\phi(\alpha) = \int_0^\pi \ln(1-2\alpha\cos(x)+\alpha^2)dx \ (\text{for} |\alpha| > 1)$

Now,

$$\frac{d}{d\alpha}\phi(\alpha) = \int_0^\pi \frac{-2\cos(x)+2\alpha}{-2\alpha\cos(x)+\alpha^2}dx$$

$$= \frac{1}{\alpha}\int_0^\pi \left(1 - \frac{(1-\alpha)^2}{1-2\alpha\cos(x)+\alpha^2}\right)dx$$

$$= \frac{\pi}{\alpha} - \frac{2}{\alpha}\left\{\arctan\left(\frac{1+\alpha}{1-\alpha}\cdot\tan\left(\frac{x}{2}\right)\right)\right\}\Big|_0^\pi$$

Notes

As *x* varies from 0 to π, $\left(\frac{1+\alpha}{1-\alpha}\cdot\tan\left(\frac{x}{2}\right)\right)$ varies through positive values from 0 to ∞ when –1 < α < 1 and $\left(\frac{1+\alpha}{1-\alpha}\cdot\tan\left(\frac{x}{2}\right)\right)$ and varies through negative values from 0 to –∞ when α < –1 or α > 1.

Hence,

$$\arctan\left(\frac{1+\alpha}{1-\alpha}\cdot\tan\left(\frac{x}{2}\right)\right)\Big|_0^\pi = -\frac{\pi}{2} \text{ when } -1<\alpha<1$$

and

$$\arctan\left(\frac{1+\alpha}{1-\alpha}\cdot\tan\left(\frac{x}{2}\right)\right)\Big|_0^\pi = -\frac{\pi}{2} \text{ when } \alpha<-1 \text{ or } \alpha>1.$$

Therefore,

$$\frac{d}{d\alpha}\phi(\alpha) = 0 \text{ when } -1<\alpha<1$$

$$\frac{d}{d\alpha}\phi(\alpha) = \frac{2\pi}{\alpha} \text{ when } \alpha<-1 \text{ or } \alpha>1\,.$$

Upon integrating both sides with respect to α, we get $\phi(\alpha) = C_1$ when –1 < α < 1 and $\phi(\alpha) = 2\pi \ln |\alpha| + C_2$ when α < –1 or α > 1.

C_1 may be determined by setting α = 0 in

$$\phi(\alpha) = \int_0^\pi \ln(1-2\alpha\cos(x)+a^2)dx$$

$$\phi(0) = \int_0^\pi \ln(1)dx$$

$$= \int_0^\pi 0\,dx$$

$$= 0$$

Thus, $C_1 = 0$. Hence, φ(α) = 0 when –1 < α < 1.

To determine C_2 in the same manner, we should need to substitute in $\phi(\alpha) = \int_0^\pi \ln(1-2\alpha\cos(x)+\alpha^2)dx$ a value of α greater numerically than 1. This is somewhat inconvenient. Instead, we substitute, $\alpha = \frac{1}{\beta}$, where $-1<\beta<1$. Then,

$$\phi(\alpha) = \int_0^\pi \ln\left(1-2\beta\cos(x)+\beta^2) - 2\ln|\beta|\right)dx$$

$$= 0 - 2\pi \ln |\beta|$$

$$= 2\pi \ln |\alpha|$$

Therefore, $C_2 = 0$ and φ(α) = 2π ln |α| when α < –1 or α > 1.)

The definition of φ(α) is now complete:

$$\phi(\alpha) = 0 \text{ when } -1<\alpha<1 \text{ and}$$

$$\phi(\alpha) = 2\pi\ln|\alpha| \text{ when } \alpha<-1 \text{ or } \alpha>1$$

Notes

The foregoing discussion, of course, does not apply when α = ±1 since the conditions for differentiability are not met.

Example: Here, we consider the integration of

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{\left(a\cos^2 x + b\sin^2 x\right)} dx$$

where both a, b > 0, by differentiating under the integral sign.

Let us first find $J = \int_0^{\frac{\pi}{2}} \frac{1}{a\cos^2 x + b\sin^2 x} dx$

Dividing both the numerator and the denominator by $\cos^2 x$ yields

$$J = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a + b\tan^2 x} dx$$

$$= \frac{1}{b}\int_0^{\frac{\pi}{2}} \frac{1}{\left(\sqrt{\frac{a}{b}}\right)^2 + \tan^2 x} d(\tan x)$$

$$= \frac{1}{\sqrt{a, b}}\left(\tan^{-1}\left(\sqrt{\frac{b}{a}}\tan x\right)\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{2\sqrt{a, b}}.$$

The limits of integration being independent of a, $J = \int_0^{\frac{\pi}{2}} \frac{1}{a\cos^2 x + b\sin^2 x} dx$ gives us

$$\frac{\partial J}{\partial a} = -\int_0^{\frac{\pi}{2}} \frac{\cos^2 x dx}{\left(a\cos^2 x + b\sin^2 x\right)^2}$$

Whereas $J = \frac{\pi}{2\sqrt{ab}}$ gives us

$$\frac{\partial J}{\partial a} = -\frac{\pi}{4\sqrt{a^3 b}}.$$

Equating these two relations then yields

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\left(a\cos^2 x + b\sin^2 x\right)^2} = \frac{\pi}{4\sqrt{a^3 b}}$$

In a similar fashion, $\frac{\partial J}{\partial b}$ pursuing yields

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\left(a\cos^2 x + b\sin^2 x\right)^2} = \frac{\pi}{4\sqrt{ab^3}}$$

Adding the two results then produces

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{\left(a\cos^2 x + b\sin^2 x\right)^2} dx = \frac{\pi}{4\sqrt{ab}}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Which is the value of the integral I.

Note that if we define

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$$I_n = \int_0^{\frac{\pi}{2}} \frac{1}{\left(a\cos^2 x + b\sin^2 x\right)^n} dx$$

it can easily be shown that

$$\frac{\partial I_{n-1}}{\partial a} + \frac{\partial I_{n-1}}{\partial b} + (n-1)\cdot I_n = 0.$$

Given I_1 this *partial-derivative-based* recursive relation (i.e., integral reduction formula) can then be utilized to compute all of the values of I_n for $n > 1$ (I_1, I_2, I_3, I_4, etc.).

Example: Here, we consider the integral

$$I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos\alpha\,\cos x)}{\cos x} dx .$$

for $0 < \alpha < \pi$.

Differentiating under the integral with respect to α, we have

$$\frac{d}{d\alpha} I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial\alpha}\left(\frac{\ln(1+\cos\alpha\,\cos x)}{\cos x}\right) dx$$

$$= -\int_0^{\frac{\pi}{2}} \frac{\sin\alpha}{1+\cos\alpha\,\cos x} dx$$

$$= -\int_0^{\frac{\pi}{2}} \frac{\sin\alpha}{\left(\cos^2\frac{\pi}{2} + \sin^2\frac{x}{2}\right) + \cos\alpha\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)} dx$$

$$= -\frac{\sin\alpha}{1-\cos\alpha}\int_0^{\frac{\pi}{2}} \frac{1}{\cos^2\frac{x}{2}} \frac{\sin\alpha}{\left[\left(\frac{1+\cos\alpha}{1-\cos\alpha}\right) + \tan^2\frac{x}{2}\right]} dx$$

$$= -\frac{2\sin\alpha}{1-\cos\alpha}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\sec^2\frac{x}{2}}{\left[\left(\frac{2\cos^2\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}}\right) + \tan^2\frac{x}{2}\right]} dx$$

$$= -\frac{2\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)}{2\sin\frac{\alpha}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\left[\left(\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\right)^2 + \tan^2\frac{x}{2}\right]} d\left(\tan\frac{x}{2}\right)$$

$$= -2\cot\frac{\alpha}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\left[\cot^2\frac{\alpha}{2} + \tan^2\frac{x}{2}\right]} d\left(\tan\frac{x}{2}\right)$$

$$= -2\left(\tan - 1\left(\tan\frac{\alpha}{2}\tan\frac{x}{2}\right)\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= -\alpha$$

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Now, when $\alpha = \frac{\pi}{2}$, we have, from

$$I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos\alpha \cos x)}{\cos x} dx, I\left(\frac{\pi}{2}\right) = 0$$

Hence,

$$I(\alpha) = \int_{\frac{\pi}{2}}^{\alpha} -\alpha \, d\alpha$$

$$= -\frac{1}{2}\alpha^2 \Big|_{\frac{\pi}{2}}^{\alpha}$$

$$= \frac{\pi^2}{8} - \frac{\alpha^2}{2},$$

which is the value of the integral I(α).

Example: Here, we consider the integral $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta) d\theta$.

We introduce a new variable ϕ, and rewrite the integral as

$$f(\phi) = \int_0^{2\pi} e^{\phi\cos\theta} \cos(\phi \sin\theta) d\theta$$

Note that for $\phi = 1, f(\phi) = f(1) = \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta) d\theta$

Thus, we proceed

$$\frac{df}{d\phi} = \int_0^{2\pi} \frac{\partial}{\partial\phi}\left(e^{\phi\cos\theta}\cos(\phi\sin\theta)\right) d\theta$$

$$= \int_0^{2\pi} e^{\phi\cos\theta}(\cos\theta \cos(\phi\sin\theta) - \sin\theta \sin(\phi\sin\theta)) d\theta$$

$$= \int_0^{2\pi} \frac{1}{\phi} \frac{\partial}{\partial\theta}\left(e^{\phi\cos\theta}\sin(\phi\sin\theta)\right) d\theta$$

$$= \frac{1}{\phi} \int_0^{2\pi} d\left(e^{\phi\cos\theta}\sin(\phi\sin\theta)\right)$$

$$= \frac{1}{\phi}\left(e^{\phi\cos\theta}\sin(\phi\sin\theta)\right)\Big|_0^{2\pi}$$

$$= 0.$$

From the equation for f(ϕ) we can see f(0) = 2π. So, integrating both sides of $\frac{df}{d\phi} = 0$ with respect to ϕ between the limits 0 and 1, yields

$$\int_{f(0)}^{f(1)} df = \int_0^1 d\phi = 0$$

$$\Rightarrow \quad f(1) - f(0) = 0$$

$\Rightarrow \quad f(1) - 2\pi = 0$

$\Rightarrow \quad f(1) = 2\pi.$

which is the value of the integral $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta$.

Example: Find $\frac{d}{dx}\int_{\sin x}^{\cos x}\cosh t^2 dt$.

In this example, we shall simply apply the above given formula, to get

$$\frac{d}{dt}\int_{\sin x}^{\cos x}\cosh t^2 dt = \cosh(\cos^2 x)\frac{d}{dx}(\cos x) - \cosh(\sin^2 x)\frac{d}{dx}(\sin x) +$$

$$\int_{\sin x}^{\cos x}\frac{\partial}{\partial x}\cosh t^2 dt = -\cosh(\cos^2 x)\sin x - \cosh(\sin^2 x)\cos x$$

Where the derivative with respect to x of hyperbolic cosine t squared is 0. This is a simple example on how to use this formula for variable limits.

Self Assessment

Fill in the blanks:

1. The differentiation of integrals is the Lebesgue differentiation theorem, as proved by Henri Lebesgue in ....
2. The result for .... turns out to be a special case of the following result, which is based on the Besicovitch covering theorem.
3. The problem of .... is that of determining under what circumstances the mean value integral of a suitable function on a small neighbourhood of a point approximates the value of the function at that point.
4. The problem of the differentiation of integrals is much harder in an infinite-dimensional setting. Consider a separable Hilbert space (H, ⟨, ⟩) equipped with a ....

23.3 Summary

- In mathematics, the problem of differentiation of integrals is that of determining under what circumstances the mean value integral of a suitable function on a small neighbourhood of a point approximates the value of the function at that point.
- One result on the differentiation of integrals is the Lebesgue differentiation theorem, as proved by Henri Lebesgue in 1910. Consider n-dimensional Lebesgue measure λ^n on n-dimensional Euclidean space R^n.
- The result for Lebesgue measure turns out to be a special case of the following result, which is based on the Besicovitch covering theorem: if μ is any locally finite Borel measure on R^n and $f : R^n \to R$ is locally integrable with respect to μ, then

$$\lim_{r\to 0}\frac{1}{\mu(B_r(x))}\int_{B_r(x)} f(y)\,d\mu(y) = f(x) \text{ for } \mu\text{-almost all points } x \in R^n.$$

- The problem of the differentiation of integrals is much harder in an infinite-dimensional setting. Consider a separable Hilbert space (H, ⟨, ⟩) equipped with a Gaussian measure γ.

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23.4 Keywords

Differentiation of Integrals: In mathematics, the problem of differentiation of integrals is that of determining under what circumstances the mean value integral of a suitable function on a small neighbourhood of a point approximates the value of the function at that point.

Borel measures on R^n: The result for Lebesgue measure turns out to be a special case of the following result, which is based on the Besicovitch covering theorem: if μ is any locally finite Borel measure on R^n and $f : R^n \to R$ is locally integrable with respect to μ, then

$$\lim_{r \to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f(y) d\mu(y) = f(x)$$

for μ-almost all points $x \in R^n$.

23.5 Review Questions

1. Explain Differentiation of Integrals with the help of example.
2. Discuss the Theorems on the differentiation of integrals.

Answers: Self Assessment

1. 1910
2. Lebesgue measure
3. differentiation of integrals
4. Gaussian measure γ

23.6 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 24: Fundamental Theorem of Calculus

Notes

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Objectives

After studying this unit, you will be able to:

- Discuss the fundamental theorem of calculus
- Explain the primitive of a function

Introduction

In this unit we will discuss about, what is the relationship between the two notions of differentiation and integration? Now we shall try to find an answer to this question. In fact, we shall show that differentiation and Integration are intimately related in the sense that they are inverse operations of each other.

Let us begin by asking the following question: "when is a function f : [a, b] → R, the derivative of some function F : [a, b] → R?"

For example consider the function f : [–1, 1] → R defined by

$$f(x) = \begin{cases} 0 \text{ if } -1 \le x < 0 \\ i \text{ if } 0 \le x < 1 \end{cases}$$

This function is not the derivative of any function F : [–1, 1] → R. Indeed if f is the derivative of a function F : [–1, 1] → R then f must have the intermediate value property. But clearly, the function f given above does not have the intermediate value property.

Hence f cannot be the derivative of any function F : [–1, 1] → R.

However if f : [–1, 1] → R is continuous, then f is the derivative of a function F : [–1, 1] → R. This leads us to the following general theorem.

24.1 Fundamental Theorem of Calculus

Theorem 1: Let f be integrable on [a, b]. Define a function P on [a, b] as

$$F(x) = \int_a^x f(t)\,dt, \forall\, x \in [a, b].$$

Then F is continuous on [a, b]. Furthermore, if f is continuous at a point x, of [a, b], then F is differentiable at x_0 and $F'1(x_0) = f(x_0)$.

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Proof: Since f is integrable on [a,b], it is bounded. In other words, there exists a positive number M such that $|f(x)| \le M, \forall\, x \in [a,b]$.

Let ε > 0 be any number. Choose x,y ∈ [a,b], x < y, such that $|x-y| < \frac{e}{M}$. Then

$$\begin{aligned} |F(y)-F(x)| &= \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right| \\ &= \left|\int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt\right| \\ &= \left|\int_x^y f(t)\,dt\right| \\ &\le \int_x^y |f(t)|\,dt \\ &\le \int_x^y M dt = M(y-x) < \in \end{aligned}$$

Similarly you can discuss the case when y < x. This shows that F is continuous on [a,b]. In fact this proves the uniform continuity of F.

Now, suppose f is continuous at a point x_0 of [a, b]

We can choose some suitable h ≠ 0 such that $x_0 + h \in [a, b]$.

Then,

$$\begin{aligned} F(x_0 + h) - F(x_0) &= \int_a^{x_0+h} f(t)\,dt - \int_a^{x_0} f(t)\,dt \\ &= \int_a^{x_0} f(t)\,dt + \int_{x_0}^{x_0+h} f(t)\,d(t) - \int_a^{x_0} f(t)\,dt = \int_{x_0}^{x_0+h} f(t)\,dt \end{aligned}$$

Thus,

$$F(x_0 + h) - F(x_0) = \int_{xn}^{x_0+h} f(t)\,dt \qquad \ldots(1)$$

Now

$$\begin{aligned} \left|\frac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right| &= \left|\frac{1}{h}\int_{x0}^{x0+h} f(t)\,dt - \frac{1}{h} \times \int_0^{x_0+h} f(x_0)\,dt\right| \\ &= \frac{1}{|h|}\left|\int_{x_0}^{x_0+h} [f(t)-f(x_0)]dt\right|. \end{aligned}$$

Since f is continuous at x_0, given a number ∈ >0, ∃ a number δ > 0 such that $|f(x)-f(x_0)| < \varepsilon/2$, whenever $|x - x_0| < \delta$ and x ∈ [a,b]. So, if $|h| < \delta$, then $|f(t)-f(x_0)| < \in/2$, for $t \in [x_0, x_0+h]$, and consequently

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$$\left|\int_{x_0}^{x_0+h}[f(t)-f(x_0)]dt\right| \le -|h|. \text{ Therefore}$$

$$\left|\frac{F(x_0+h)-F(x_0)}{h}-f(x_0)\right| \le \frac{\varepsilon}{2} < \varepsilon, \text{ if } |h| < \delta.$$

Therefore, $\lim\limits_{h\to 0}\dfrac{F(x_0+h)-F(x)}{h}f(x_0)$, i.e., $F'(x_0)=f(x_0)$

which shows that F is differentiable at x_0 and $F'(x_0) = f(x_0)$. From Theorem 1, you can easily deduce the following theorem:

Theorem 2: Let f: [a, b] → R be a continuous function. Let F : [a, b] → R be a function defined by

$$F(x)=\int_a^x f(t)\,dt,\ x\ E[a,b].$$

Then F'(x) = f(x), a ≤ x ≤ b.

This is the first result which links the concepts of integral and derivative. It says that, if f is continuous on [a,b] then there is a function F on [a, b] such that $F'(x)=f(x), \forall\ x \in [a,b]$.

You have seen that if f: [a, b] → R is continuous, then there is a function F: [a, b] → R such that F' (x) = f(x) on [a, b]. Is such a function F unique? Clearly the answer is 'no'. For, if you add a constant to the function F, the derivative is not altered. In other words, if $G(x)=c+\int_1^x f(t)\,dt$ for a ≤ x ≤ b then also G' (x) = f(x) on [a, b].

Such a function F or G is called primitive off. We have the formal definition as follows:

24.2 Primitive of a Function

If f and F are functions defined on [a,b] such that F′(x) = f(x) for x ∈ [a,b] then F is called a 'primitive' or an 'antiderivative' of f on [a,b].

Thus from Theorem 1, you can see that every continuous function on [a,b] has a primitive. Also there are infinitely many primitives, in the sense that adding a constant to a primitive gives another primitive.

"Is it true that any two primitives differ by a constant?"

The answer to this question is yes. Indeed if F and G are two primitives of f in [a,b], then $F'(x)=G'(x)=f(x)\ \forall\ x \in [a,b]$ and therefore [F(x) – G(x)′ = 0. Thus F(x) – G(x) = k (constant), for x ∈ [a,b].

Let us consider an example.

Example: What is the primitive of f(x) = log x in [1, 2]

Solution: Since $\dfrac{d}{dx}(x\log x - x) = \log x\ \forall\ x \in [1,2]$, therefore F (x) = x log x – x is a primitive of f in [1, 2].

Also G(x) = x lag x – x + k, k being a constant, is a primitive of f.

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According to this theorem, differentiation and integration are inverse operations.

We now discuss a theorem which establishes the required relationship between differentiation and integration. This is called the Fundamental Theorem of Calculus.

It states that the integral of the derivative of a function is given by the function itself.

The Fundamental Theorem of Calculus was given by an English mathematician Isaac Barrow [1630-1677], the teacher of great Isaac Newton.

Theorem 3: Fundamental Theorem of Calculus

If f is integrable on [a,b] and F is a primitive of f on [a,b], then $\int_a^{tb} f(x)\,dx = F(b) - F(a)$.

Proof: Since $f \in R$ [a,b], therefore $\lim_{|P|-0} S(P,f) = \int_a^b f(x)\,dx$

where P = $\{x_0, x_1, x_2, ..., x_n\}$ is a partition of [a,b]. The Riemann sum S(P,f) is given by

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1});\; x_i - 1 \le t_i \le t_i.$$

Since F is the primitive of f on [a, b], therefore F' (x) ≤ f(x), $x \in$ [a, b].

Hence $S(P,f) = \sum_{i=1}^{n} F'(t_i)(x_i - x_{i-1})$. We choose the points t, as follows:

By Lagrange's Mean Value theorem of Differentiability, there is a point t, in $]x_{i-1}, x_i[$ such that

$$F(x_i) - F(x_{i-1}) = F'(t_i)\,(x, - x_{i-1})$$

Therefore, $S(P,f) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = F(x_n) - F(x_0) = F(b) - F(a)$.

Take the limit as $|P| \to 0$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$. This completes the proof.

Alternatively, the Fundamental Theorem of Calculus is also interpreted by stating that the derivative of the integral of a continuous function is the function itself.

If the derivative f of a function f is integrable on [a, b], then $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Applying this theorem, we can find the integral of various functions very easily.

Consider the following example:

Example: Show that $\int_0^t \sin x\,dx = 1 - \cos t$.

Solution: Since g(x) = – cos x is the primitive of f(x) = sin x in the interval [0, t], therefore

$\int_0^t \text{Sin } x\,dx = g(t) - g(o) = 1 - \cos t$.

We have, thus, reduced the problem of evaluating $\int_a^b f(x)\,dx$ to that of finding primitive of f on [a, b]. Once the primitive is known, the value of $\int_a^b f(x)\,dx$ is easily given by the Fundamental Theorem of Calculus.

Notes

You may note that any suitable primitive will serve the purpose because when the primitive is known, then the process described by the Fundamental Theorem is much simpler than other methods. However, it is just possible that the primitive may not exist while you may keep on trying to find it. It is, therefore, essential to formulate some conditions which can ensure the existence of a primitive. Thus now the next step is to find the conditions on the integral, (function to be integrated) which will ensure the existence of a primitive. One such condition is provided by the theorem.

According to theorem 2 if f is continuous in [a, b], then the function F given by

$F(x) = \int_a^x f(t)\,dt\ x \in [a, b]$ is differentiable in [a, b] and $F'(x) = f(x)\ \forall\ x \in [a, b]$

i.e. F is the primitive of f in [a, b]

The following observations are obvious from the theorems 1 and 2:

(i) If f is integrable on [a, b], then there is a function F which is associated with f through the process of integration and the domain of F is the same as the interval [a, b] over which f is integrated.

(ii) F is continuous. In other words, the process of integration generates continuous function.

(iii) If the function f is continuous on [a, b], then F is differentiable on [a, b]. Thus, the process of integration generates differentiable functions.

(iv) At any point of continuity of f, we will have f(c) = f(c) for c ε [a, b]. This means that if f is continuous on the whole of [a, b], then F will be a member of the family of primitives of f on [a, b].

In the case of continuous functions, this leads us to the notion

$$\int f(x)\,dx$$

for the family of primitives of f. Such an integral, as you know, is called an Indefinite integral of f. It does not simply denote one function, but it denotes a family of functions. Thus, a member of the indefinite integral of f will always be an antiderivative for f.

Theorem 3 gives US a condition on the function to be integrated which ensures the existence of a primitive. But how to obtain the primitives, once this condition is satisfied. In the next section, we look for the two most important techniques for finding the primitives. Before we do so, we need to study two important mean-values theorems of integrability.

Self Assessment

Fill in the blanks:

1. This function is not the derivative of any function F : [–1, 1] → R. Indeed if f is the derivative of a function F : [–1, 1] → R then f must have the ....

2. Let f : [a, b] → R be a continuous function. Let F : [a, b] → R be a function defined by ....

3. If f and F are functions defined on [a, b] such that F′(x) = f(x) for x ∈ [a, b] then F is called a 'primitive' or an '...............................' off on [a, b].

Notes

24.3 Summary

- The main thrust of this unit has been to establish the relationship between differentiation and integration with the help of the Fundamental Theorem of Calculus.
- We have discussed some important properties of the Riemann Integral. We have shown that the inequality between any two functions is preserved by their corresponding Riemann integrals; the modulus of the limit of a sum never exceeds the limit of the sum of their module and if we split the interval over which we are integrating a function into two parts, then the value of the integral over the whole will be the sum of the two integrals over the subintervals.
- Let f: [a, b] → R be a continuous function. Let F : [a, b] → R be a function defined by

  $$F(x) = \int_a^x f(t)\,dt,\ x\ E[a,b].$$

  Then F'(x) = f(x), a ≤ x ≤ b.

  This is the first result which links the concepts of integral and derivative. It says that, if f is continuous on [a, b] then there is a function F on [a, b] such that $F'(x) = f(x), \forall\ x \in [a,b]$.

  You have seen that if f: [a, b] → R is continuous, then there is a function F: [a, b] → R such that F' (x) = f(x) on [a, b]. Is such a function F unique? Clearly the answer is 'no'. For, if you add a constant to the function F, the derivative is not altered. In other words, if $G(x) = c + \int_1^x f(t)\,dt$ for a ≤ x ≤ b then also G' (x) = f(x) on [a, b].
- It states that the integral of the derivative of a function is given by the function itself.
- The Fundamental Theorem of Calculus was given by an English mathematician Isaac Barrow [1630-1677], the teacher of great Isaac Newton.
- The following observations are obvious from the theorems 1 and 2.
  (i) If f is integrable on [a, b], then there is a function F which is associated with f through the process of integration and the domain of F is the same as the interval [a, b] over which f is integrated.
  (ii) F is continuous. In other words, the process of integration generates continuous function.
  (iii) If the function f is continuous on [a, b], then F is differentiable on [a, b]. Thus, the process of integration generates differentiable functions.
  (iv) At any point of continuity of f, we will have f(c) = f(c) for c e [a, b]. This means that if f is continuous on the whole of [a, b], then F will be a member of the family of primitives of f on [a, b].

24.4 Keywords

Primitive of a Function: If f and F are functions defined on [a, b] such that F′(x) = f(x) for x∈ [a, b] then F is called a 'primitive' or an 'antiderivative' off on [a, b].

Fundamental Theorem of Calculus: If f is integrable on [a, b] and F is a primitive of f on [a, b], then $\int_a^{tb} f(x)\,dx = F(b) - F(a)$.

24.5 Review Questions

Notes

1. Find the primitive of the function f defined in [0, 2] by

$$f(x) = \begin{cases} x \text{ if } x \in [0,1] \\ I \text{ if } x \in [1,2] \end{cases}$$

2. Find $\int_0^2 f(x)\,dx$ where f is the function given in $f(x) = \begin{cases} x \text{ if } x \leftarrow [0,1] \\ 1 \text{ if } x \leftarrow [1,2] \end{cases}$

3. Evaluate $\int_1^b x^n\,dx$ where n is a positive integer.

Answers: Self Assessment

1. intermediate value property
2. $F(x) = \int_a^x f(t)\,dt,\ x\ E[a,b].$
3. antiderivative

24.6 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

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Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 25: Mean Value Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the first mean value theorem
- Explain the generalized first mean value theorem
- Describe the second mean value theorem

Introduction

In last unit, we discussed some mean-value theorems concerning the differentiability of a function. Quite analogous, we have two mean value theorems of integrability which we intend to discuss here. You are quite familiar with the two well-known techniques of integration namely the integration by parts and integration by substitution which you must have studied in your earlier classes.

25.1 First Mean Value Theorem

Let f : [a, b] → R be a continuous function. Then there exists c ∈ [a, b] such that

$$\int^{b} f(x)\,dx = (b-a)f(c).$$

Proof: We know that

$$m(b-a) \leq \int_{a}^{h} f(x)\,dx \leq M(b-a), \text{ thus}$$

$$m \leq \frac{\int_{a}^{b} f(x)\,dx}{(b-a)} \leq M, \text{ where}$$

m = glb {f(x) : x ∈ [a,b]}, and

M = lub {f(x) : x ∈ [a,b]}.

Notes

Since f is continuous in [a, b], it attains its bounds and it also attains every value between the bounds. Consequently, there is a point c ∈ [a, b] such that

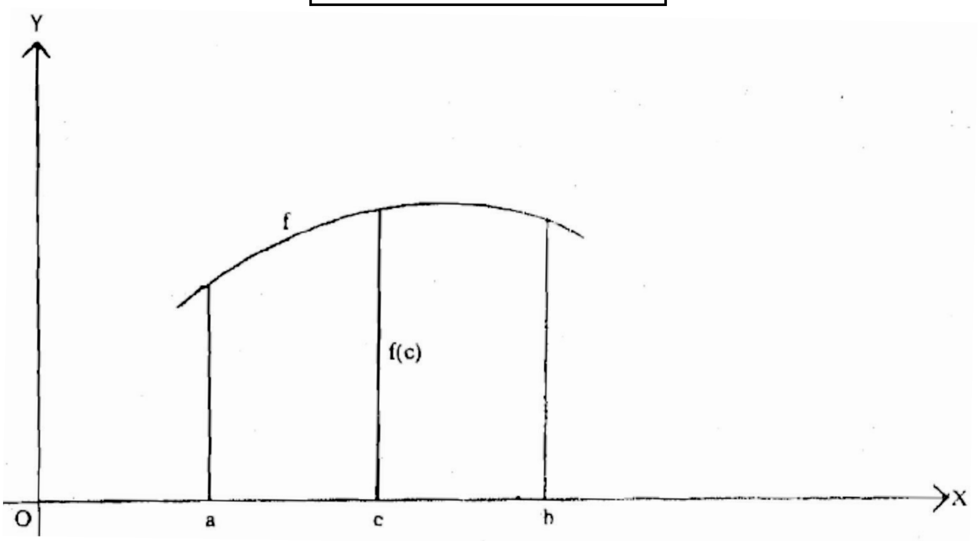
$$\int_a^b f(x)dx = f(c)\,(b-a),$$

which, equivalently, can be written as

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This theorem is usually referred to as the Mean Value theorem for integrals. The geometrical interpretation of the theorem is that for a non-negative continuous function f, the area between f, the lines x = a, x = b and the x-axis can be taken as the area of a rectangle having one side of length (b – a) and the other f(c) for some c ∈ [a, b] as shown in the Figure 25.1.

Figure 25.1



We now discuss the generalized form of the first mean value theorem.

25.2 The Generalised First Mean Value Theorem

Let f and g be any two functions integrable in [a, b]. Suppose g(x) keeps the same sign for all x ∈ [a, b]. Then there exists a number μ lying between the bounds of f such that

$$\int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)\,dx.$$

Proof: Let us assume that g(x) is positive over [a,b]. Since f and g are both integrable in [a, b], therefore both are bounded. Suppose that m and M are the g.l.b. and l.u.b. of f in [a, b]. Then

$$m \le f(x) \le M,\ \forall\ x \in [a,b].$$

Consequently,

$$mg(x) \le f(x)g(x) \le Mg(x),\ \forall\ x \in [a,b].$$

Notes

Therefore,

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le M\int_a^b g(x)\,dx.$$

It then follows that there is a number $\mu \in [m, M]$ such that

$$\int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)\,dx.$$

Corollary: Let f, g be continuous functions on [a,b] and let $g(x) \ge 0$ on [a,b]. Then, there exists a $c \in [a,b]$ such that

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Proof: Since f is continuous on [a,b], so, there exists a point $c \in [a,b]$ such that

$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx,$ where $\mu = f(c)$ is as in Theorem.

We use the first Fundamental Theorem of Calculus for integration by parts. We discuss it in the form of the following theorem.

Theorem 1: If f and g are differentiable functions Qn [a,b] such that the derivatives f'and g' are both integrable on [a,b], then

$$\int_a^b f(x)\,g'\,dx = [f(b)\,g(b) - f(a)\,g(b)] - \int_a^b f'(x)\,g(x)\,dx.$$

Proof: Since f and g are given to be differentiable on [a,b], therefore both f and g are continuous on [a,b]. Consequently both f and g are Riemann integrable on [a,b]. Hence both fg' as well as f' g are integrable.

$$fg' + f'g = (fg)'.$$

Therefore (fg)' is also integrable and consequently, we have

$$\int_a^b (fg)' = \int_a^b fg' + \int_a^b f'g.$$

By Fundamental Theorem of Calculus, we can write

$$\int_a^b (fg)' = |fg|_a^b = f(b)\,g(b) - f(a)\,g(a)$$

Hence, we have

$$\int_a^b fg' = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f'g.$$

i.e.

$$\int_a^b f(x)\,g'(x)\,dx = [f(x)\,g(x)]_a^b - \int_a^b f'(x)\,g(x)\,dx.$$

This theorem is a formula for writing the integral of the product of two functions.

Notes

What we need to know is that the primitive of one of the two functions should be expressible in a simple form and that the derivative of the other should also be simple so that the product of these two is easily integrable. You may note here that the source of the theorem is the well-known product rule for differentiation.

The Fundamental Theorem of Calculus gives yet another useful technique of integration. This is known as method by Substitution also named as the change of variable method. In fact this is the reverse of the well-known chain Rule for differentiation. In other words, we compose the given function f with another function g so that the composite f o g admits an easy integral. We deduce this method in the form of the following theorem:

Theorem 2: Let f be a function defined and continuous on the range of a function g. If g' is continuously differentiable on $|c,d|$, then

$$\int_a^b f(x)\,dx = \int_c^d (f \circ g)\,(x)\,g'(x)\,dx,$$

where a = g(c) and b = g(d).

Proof: Let $F(x) = \int_a^b f(x)\,dt$ be a primitive of the function 1:

Note that the function F is defined on the range of g.

Since f is continuous, therefore, by Theorem 2, it follows that F is differentiable and F′(t) = f(t), for any t. Denote G(x) = (F o g) (x).

Then, clearly G is defined on [c,d] and it is differentiable there because both F and g are so. By the Chain Rule of differentiation, it follows that

$$G'(x) = (F \circ g)'\,(x)\,g'(x) = (f \circ g)\,(x)\,g'(x).$$

Also f o g is continuous since both f and g are continuous. Therefore, f o g is integrable.

Since g′ is integrable, therefore (f o g) g′ is also integrable. Hence

$$\begin{aligned}\int^d (f \circ g)(x)\,g'(x)\,dx &= \int_c^d G'(x)\,dx \\ &= G(d) - G(c) \qquad \text{(Why?)} \\ &= F(g\,(d)) - F(g\,(c)) \\ &= F(b) - F(a) \\ &= \int_a^b f(x)\,dx.\end{aligned}$$

you have seen that the proof of the theorem is based on the Chain Rule for differentiation. In fact, this theorem is sometimes treated as a Chain Rule for Integration except that it is used exactly the opposite way from the Chain Rule for differentiation. The Chain Rule for differentiation tells us how to differentiate a composite function while the Chain Rule for Integration or the change of variable method tells us how to simplify an integral by rewriting it as a composite function.

Thus, we are using the equalities in the opposite directions.

We conclude this section by a theorem (without Proof) known as the Second Mean Value Theorem for Integrals. Only the outlines of the proof are given.

Notes

25.3 Second Mean Value Theorem

Let f and g be any two functions integrable in $|a, b|$ and g be monotonic in $|a, b|$, then there exists $c \in |a, b|$ such that

$$\int_a^b f(x)\,g(x)\,dx \;= g(a)\int_a^c f(x)\,dx + g(b)\int_c^b f(x)\,dx$$

Proof: The proof is based on the following result known as Bonnet's Mean Value Theorem, given by a French mathematician O. Bonnet [1819–1892].

Let f and g be integrable functions in [a,b]. If ϕ is any monotonically decreasing function and positive in [a,b], then there exists a point $c \in$ [a,b] such that

$$\int_a^b f(x)\,\phi\,(x)\,dx \;= \phi\,(a)\int_a^c g(x)\,dx.$$

Let g be monotonically decreasing so that ϕ where $\phi(x) = g(x) - g(b)$, is non-negative and monotonically decreasing in [a,b]. Then there exists a number $c \in$ [a,b] such that

$$\int_a^b f(x)\,[g(x) - g(b)]\,dx \;= [g(a) - g(b)]\int_a^c f(x)\,dx$$

i.e.

$$\int_a^b f(x)\,g(x)\,dx \;= g(a)\int_a^c f(x)\,dx + g(b)\int_c^b f(x)\,dx.$$

Now let g be monotonically increasing so that –g is monotonically decreasing. Then there exists a number $c \in$ [a,b] such that

$$\int_a^b f(x)[-g(x)]\,dx \;= -g(a)\int^c f(x)\,dx - g(b)\int^b f(x)\,dx$$

i.e.

$$\int_a^b f(x)\,g(x)\,dx \;= g(a)\int_a^c f(x)\,dx + g(b)\int_c^h f(x)\,dx.$$

This completes the proof of the theorem.

There are several applications of the Second Mean Value Theorem. It is sometimes used to develop the trigonometric functions and their inverses which you may find in higher Mathematics. Here, we consider a few examples concerning the verification and application of the two Mean Value Theorems.

Example: Verify the two Mean Value Theorems for the functions $f(x) = x$, $g(x) = e^x$ in the interval [–1, 1].

Solution: Verification of First Mean Value Theorem

Since f and g are continuous in [–1, 1], so they are integrable in [–1,1]. Also g(x) is positive in [–1, 1]. By first Mean Value Theorem, there is a number μ between the bounds of f such that

$$\int_{-1}^{1} f(x)\,g(x)\,dx \;= \mu\int_{-1}^{1} g(x)\,dx \text{ i.e., } \int_{-1}^{1} x\,e^x\,dx = \mu\int_{-1}^{1} e^x\,dx.$$

$$\int_{-1}^{1} x\, e^x\, dx = \left|x\, e^x\right|_{-1}^{1} - \int_{-1}^{1} e^x\, dx = \frac{2}{e} \text{ and } \int_{-1}^{1} e^x\, dx = e - \frac{1}{e}.$$

$$\therefore \qquad \frac{2}{e} = \mu\left(e - \frac{1}{e}\right) \text{ i.e., } \mu = \frac{2}{e^2 - 1} = \frac{2}{(2.7)^2 - 1} = \frac{2}{6.29}$$

g.l.b. $\{f(x)|-1 \le x \le 1\} = -1$ and l.u.b. $\{f(x)|-1 \le x \le 1\} = 1$ and, so, $\mu \in [-1,1]$. First Mean Value Theorem is verified.

Verification of Second Mean Value Theorem

As shown above, f and g are integrable in [–1, 1]. Also g is monotonically increasing in [-1, 1]. By second mean value theorem there is a points c ∈ [–1, 1] such that

$$\int_{-1}^{1} f(x)\, g(x)\, dx = g(-1)\int_{-1}^{c} f(x)\, dx + g(1)\int^{1} f(x)\, dx$$

$$\Rightarrow \qquad \int_{-1}^{1} x\, e^x\, dx = \text{'I'}\, x\, dx + e\int_{c}^{1} x\, dx$$

$$\Rightarrow \qquad \frac{2}{e} - \frac{1}{e}\left(\frac{c^2}{2} - \frac{1}{2}\right) + e\left(\frac{1}{2} - \frac{c^2}{2}\right).$$

Therefore $\qquad c^2 = \frac{e^2 - 5}{e^2 - 1} = \frac{2.29}{6.29}$ i.e. $c = \pm\sqrt{\frac{2\ 29}{6\ 29}} \in [-1,1]$

Thus second mean value theorem is verified.

Now we show the use of mean value theorems to prove some inequalities.

Example: By applying the first mean value theorem of Integral calculus, prove that

$$\pi/6 \le \int_{0}^{1/2} \cdot \frac{1}{\sqrt{\left[(1-x)^2(1-k^2\, x^2)\right]}}\, dx \le \frac{\pi}{6}\frac{1}{\sqrt{(1-\frac{1}{4}k^2)}}$$

Solution: In the first mean value theorem, take $f(x) = \frac{1}{\sqrt{(1-k^2\, x^2)}}$, $g(x) = \frac{1}{\sqrt{1-x^2}}$, $x \in \left[0, \frac{1}{2}\right]$. Being continuous functions, f and g are integrable in $\left[0, \frac{1}{2}\right]$.

By the first mean value theorem, there is a number $\mu \in [m, M]$ such that

$$\int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{\left[(1-x^2)(1-k^2\, x^2)\right]}}\, dx = \int_{0}^{\frac{1}{2}} \frac{dx}{1-x^2} = \mu\, \pi/\delta,$$

where in = g.l.b. $\left\{f(x)|0 \le x \le \frac{1}{2}\right\}$ and $M = \text{l.u.b.}\left\{f(x)\middle|0 \le x \le \frac{1}{2}\right\}$. Now m = 1 and $M = \frac{1}{\sqrt{1 - \frac{k^2}{4}}}$.

Notes

Therefore,

$$1 \le \mu \le \frac{1}{4} \text{ i.e. } \frac{\pi}{6} \le \frac{\mu\pi}{6} \le \frac{\pi}{6} - \frac{1}{\sqrt{1-\frac{k^2}{4}}},$$

$$\text{and; so, } \frac{\pi}{6} \le \int_0^{1/2} \frac{1}{\sqrt{\left[(1-x^2)(1-k^2x^2)\right]}}\,dx \le \frac{\pi}{6}\frac{1}{\sqrt{1-\frac{k^2}{4}}}.$$

Example: Prove that $\left|\int_p^q \frac{\sin x}{x}dx\right| \le \frac{2}{p}$, if q > p > 0.

Solution: Let $f(x) = \sin x, \phi(x) = \frac{1}{x}, x \in [p,q]$. Being continuous, these functions are integrable in [p, q]. By Bonnet form of second mean value theorem, there is a point $\xi \in$[p, q] such that

$$\int_p^q f(x)\,\phi(x)\,dx = \phi(p)\int_a^\xi f(x)\,dx$$

$$\text{i.e., } \int_p^q \frac{\sin x}{x}dx = \frac{1}{p}\int_p^\xi \sin x\,dx = \frac{1}{P}(\cos p - \cos\xi).$$

$$\text{Hence } \left|\int_p^q \frac{\sin x}{x}dx\right| \le \frac{1}{P}\left[|\cos p| + |\cos\xi|\right] \le \frac{2}{P}$$

Self Assessment

Fill in the blanks:

1. Let f : [a, b] $\to$ R be a continuous function. Then there exists c $\in$ [a, b] such that ....
2. Since f is continuous in [a, b], it attains its bounds and it also attains every value between the ....
3. The geometrical interpretation of the theorem is that for a .... function f, the area between f, the lines x = a, x = b and the x-axis can be taken as the area of a rectangle having one side of length (b-a) and the other f(c) for some c $\in$ [a, b].
4. If f and g are differentiable functions Qn [a,b] such that the derivatives f' and g' are both .... on [a, b], then

$$\int_a^b f(x)\,g'\,dx = [f(b)\,g(b) - f(a)\,g(b)] - \int_a^b f'(x)\,g(x)\,dx.$$

5. Let f and g be any two functions integrable in $|a,b|$ and g be .... then there exists $c \in |a,b|$ such that

$$\int_a^b f(x)\,g(x)\,dx \quad = g(a)\int_a^c f(x)\,dx + g(b)\int_c^b f(x)\,dx$$

25.4 Summary

- It has been proved that a continuous function has a primitive. Using the idea of a primitive, Fundamental Theorem or Calculus has been proved which shows that differentiation and integration are inverse process.
- Indefinite integral also called the integral function of an integrable function is defined and you have seen that this function is continuous. This function is differentiable whenever the integrable function is continuous. Finally in this section the First and Second Mean Value theorem have been discussed.
- The First Mean Value theorem states that if f is a continuous function in [a,b], then the value of the integral $\int_a^b f(x)\,dx$ is (b – a) times f(c) where c ∈ [a, b]. According to Generalised First Mean Value Theorem, if f and g are integrable in [a, b] and g(x) keeps the same sign, then the value $\int_a^h f(x)\,g(x).dx$ is $\int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)\,dx$ where μ lies between the bounds of f. But in the second mean value theorem, if out of the integrable functions f and g, g is monotonic in [a, b], then the value $\int_a^h f(x)\,g(x)\,dx$ is $g(a)\int_a^c f(x)\,dx + g(b)\int_c^b f(x)\,dx$ where c is point of [a, b].

25.6 Keywords

First Mean Value Theorem: Let f : [a, b] → R be a continuous function. Then there exists c ∈ [a, b] such that

$$\int_a^b f(x)\,dx = (b-a)f(c).$$

The Generalised First Mean Value Theorem: Let f and g be any two functions integrable in [a, b]. Suppose g(x) keeps the same sign for all x ∈ [a, b]. Then there exists a number μ lying between the bounds off such that

$$\int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)\,dx.$$

Second Mean Value Theorem: Let f and g be any two functions integrable in $|a,b|$ and g be monotonic in $|a,b|$, then there exists $c \in |a,b|$ such that

$$\int_a^b f(x)\,g(x)\,dx \;= g(a)\int_a^c f(x)\,dx + g(b)\int_c^b f(x)\,dx$$

25.6 Review Questions

1. Show that the second mean value theorem does not hold good in the interval [–1, 1] for f(x) = g(x) = x^2.
2. What do you say about the validity of the first mean value theorem.

   {1, 2] for f(x) = g(x) = x^3.
3. Show that $\left|\int_a^b \sin x^2\,dx\right| \le \frac{1}{a}$, if b > a > 0.

Notes

Answers: Self Assessment

1. $\int^{b} f(x)\,dx = (b-a)f(c)$
2. bounds
3. non-negative continuous
4. integrable
5. monotonic in $|a, b|$

25.7 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 26: Lebesgue Measure

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the definition of outer measure of sets
- Define outer measure of an interval
- Explain some important properties of outer measure
- Define measurable sets
- Describe measure of countable union of measurable sets
- Measure countable intersection of measurable sets

Introduction

In last unit you have studied about mean value theorems of Riemann Stieltjes integral. In this unit we are going to study about Lebesgue outer measure of a set, measurable sets and Lebesgue measure, their important properties.

We know that the length of an interval is defined to be the difference between two end points. In this unit, we would like to extend the idea of "length" to arbitrary (or at least, as many as possible) subsets of $\mathbb{R}$. To begin with, let's recall two important results in topology.

26.1 Lindelof's Theorem

Proposition: Every open subset V of $\mathbb{R}$ is a countable union of disjoint open intervals.

Proof: For each $x \in V$, there is an open interval I_x with rational endpoints such that $x \in I_x \subseteq V$. Then the collection $\{I_x\}_{x \in V}$ is evidently countable and

$$V = \bigcup_{x \in V} I_x.$$

Notes

Next, we prove it is always possible to have a disjoint collection. Since $\{I_x\}_{x\in V}$ is a countable collection, we can enumerate the open intervals as $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$,.... For each $n \in \mathbb{N}$, define

$$\alpha_n = \inf\{ x \in \mathbb{R} : x \le a_n \text{ and } (x, b_n) \subseteq V\}$$

and

$$\beta_n = \sup\{ x \in \mathbb{R} : x \ge b_n \text{ and } (a_n, x) \subseteq V\}.$$

Then $\{ (\alpha_n, \beta\nu) \}_{n\in\mathbb{N}}$ is a disjoint collection of open intervals with union V.

Theorem 1 (Lindelof's Theorem): Let C be a collection of open subsets of $\mathbb{R}$. Then there is a countable sub-collection $\{O_i\}_{i\in\mathbb{N}}$ of C such that

$$\bigcup_{O\in C} O = \bigcup_{i=1}^{\infty} O_i$$

Proof: Let $U = \bigcup_{o\in C} O$. For any $x \in U$ there is $O \in C$ with $x \in O$. Take an open interval I_x with rational endpoints such that $x \in I_x \subseteq O$. Then $U = \bigcup_{x\in U} I_x$ is a countable union of open intervals. Replace I_x by the set $O \in C$ which contains it, the result follows.

26.2 Lebesgue Outer Measure

As in the Archimedean idea of finding area of a circle (approximated polygons), we define the Lebesgue outer measure $m^*: \mathcal{P}(\mathbb{R}) \to [0, \infty]$ by

$$m^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k \text{ and each } I_k \text{ being open interval in } \mathbb{R}\right\}.$$

Notes By Lindelof's Theorem, the countability of the covering is not important here.

Here are some basic properties of Lebesgue outer measure, all of them can be proved easily by the definition of m*.

(i) $m^*(A) = 0$ if A is at most countable.

(ii) m* is monotonic, i.e. $m^*(A) \le m^*(B)$ whenever $A \subseteq B$.

(iii) $m^*(A) = \inf \{m^*(O): A \subseteq O$ and O is open$\}$. (*Hint:* it suffices to prove $m^*(A) \ge$ R.H.S., which is equivalent to $m^*(A) + \varepsilon >$ R.H.S. for any $\varepsilon > 0$.)

(iv) $m^*(A + x) = m^*(A)$ for all $x \in \mathbb{R}$. (Translation-invariant)

(v) $m^*\left(\bigcup_{k\in\mathbb{N}} A_k\right) \le \sum_{k=1}^{\infty} m^*(A_K)$. (Countable subadditivity)

(vi) If $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$ and $m^*(B\setminus A) = m^*(B)$ for all $B \subseteq R$.

(vii) If $m^*(A \Delta B) = 0$, then $m^*(A) = m^*(B)$.

Notes In (v), even if A_k's are disjoint, the equality may not hold.

Theorem 2: For any interval $I \subseteq R$, $m^*(I) = \ell(I)$.

Proof: We first assume $I = [a, b]$ is a closed and bounded interval. Consider the countable open interval cover $\{(a - \varepsilon, b + \varepsilon)\}$ of I, we have $m^*(I) \le b - a + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $m^*(I) \le b - a$.

To get the opposite result, we need to show for any $\varepsilon > 0$, $m^*(I) + \varepsilon \geq b - a$. Note that there is a countable open interval cover $\{I_k\}_{k\in\mathbb{N}}$ of I satisfying

$$m^*(I) + \varepsilon > \sum \ell(I_k).$$

By Heine-Borel Theorem, there is a finite subcover $\{I_{n_k}\}$ of $\{I_k\}$. Then

$$\sum \ell(I_{n_k}) > b - a \qquad \text{(why?)}$$

and it follows that

$$m^*(I) + \varepsilon > \sum \ell(I_k) \geq \sum \ell(I_{n_k}) > b - a.$$

Letting $\varepsilon \to 0$, $m^*(I) \geq b - a$. Hence, $m^*(I) = b - a$.

Next, we consider the case where I = (a, b), [a, b), or (a, b] which is bounded but not closed. Clearly, $m^*(I) \leq m^*(\overline{I}) = b - a$. On the other hand, if $\varepsilon > 0$ is sufficiently small then there is a closed and bounded interval $I' = [a + \varepsilon, b - \varepsilon] \subseteq I$. By monotonicity, $m^*(I) \geq m^*(I') = b - a - 2\varepsilon$. Letting $\varepsilon \to 0$ gives $m^*(I) \geq b - a$. Hence, $m^*(I) = b - a$.

Finally, if I is unbounded then the result is trivial since in that case I contains interval of arbitrarily large length.

26.3 Non-measurability

Theorem 3: Let $\mathfrak{M} \subseteq \mathcal{P}(\mathbb{R})$ be a translation-invariant σ-algebra containing all er that intervals, and $m : \mathfrak{M} \to [0, \infty]$ be a translation-invariant, countably additive measure such

$$m(I) = \ell(I) \qquad \text{for all interval I.}$$

Then there exists a set $S \notin \mathfrak{M}$.

Proof: Define an equivalent relation $x \sim y$ if and only if $x - y$ is rational. Then $\mathbb{R}$ is partitioned into disjoint cosets $[x] = \{y \in \mathbb{R} : x \sim y\}$.

By Axiom of Choice and Archimedean property of $\mathbb{R}$, there exists $S \subseteq [0,1]$ such that the intersection of S with each coset contains exactly one point.

Enumerate $\mathbb{Q} \cap [-1, 1]$ into $r_1, r_2, \ldots$. Then the sets $S + r_i$ are disjoint and

$$[0, 1] \subseteq \bigcup_{i\in\mathbb{N}} (S + r_i) \subseteq [-1, 2].$$

If S $S \in \mathfrak{M}$, then by monotonicity and countable additivity of m we have

$$1 \leq \sum_{i\in\mathbb{N}} m(S + r_i) \leq 3,$$

which is impossible since $m(S + r_i) = m(S)$ for all $i \in \mathbb{N}$.

26.4 Measurable Sets and Lebesgue Measure

As it is mentioned before, the outer measure does not have countable additivity. One may try to restrict the outer measure m* to a σ-algebra $\mathfrak{M} \not\subseteq \mathcal{P}(\mathbb{R})$ such that the new measure has all the properties we wanted.

Definition (Measurability): A set $E \subseteq \mathbb{R}$ is said to be measurable if, for all $A \subseteq \mathbb{R}$, one has

(1) $$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Since m* is known to be subadditive, (1) is equivalent to

Notes

$$m^*(A) > m^*(A \cap E) + m^*(A \cap E^c).$$

The family of all measurable sets is denoted by $\mathfrak{M}$. We will see later $\mathfrak{M}$ is a σ-algebra and translation-invariant containing all intervals. The set function m: $\mathfrak{M} \to [0, \infty]$ defined by

$$m(E) = m^*(E) \qquad \text{for all } E \in \mathfrak{M}$$

is called Lebesgue measure.

Observe that

- $E \in \mathfrak{M} \Leftrightarrow E^c \in \mathfrak{M}$.
- $\phi \in \mathfrak{M}$ and $\mathbb{R} \in \mathfrak{M}$ because $m^*(A) = m^*(A \cap \phi) + m^*(A \cap \mathbb{R})$ for all $A \subseteq \mathbb{R}$.
- $m^*(E) = 0 \Rightarrow E \in \mathfrak{M}$ because $m^*(A \cap E) + m^*(A \cap E^c) = m^*(A \cap E^c) \leq m^*(A)$ for all $A \subseteq \mathbb{R}$.

Proposition: If $E_1, E_2 \in \mathfrak{M}$ then $E_1 \cup E_2 \in \mathfrak{M}$. (Therefore, $\mathfrak{M}$ is an algebra.)

Proof: For all $A \subseteq R$ one has

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c) \qquad (\because E_1 \in \mathfrak{M}) \\ &= m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c) \qquad \boxed{(\because E_2 \in \mathfrak{M})} \\ &= m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \end{aligned}$$

because m* is subadditive and

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_1^c \cap E_2).$$

Notes Above proposition can be easily extended to a finite union of measurable sets, in fact it can be extended to a countable union. In order to do so, we need the following result.

Lemma 1: Let $E_1, E_2, \ldots, E_n$ be disjoint measurable sets. Then for all $A \subseteq R$, we have

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Proof: Since $E_n \in \mathfrak{M}$, we have

$$\begin{aligned} m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) &= m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap E_n\right) + m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap E_n^c\right) \\ &= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) \end{aligned}$$

Repeat the process again and again, until we get

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Notes If $\{E_i\}_{i \in \mathbb{N}}$ is a sequence of disjoint measurable sets, then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{\infty} E_i\right]\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i).$$

This is because for all $n \in \mathbb{N}$ one has

Notes

$$m^*\left(A \cap \left[\bigcup_{i=1}^{\infty} E_i\right]\right) \geq m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right)$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i).$$

Letting $n \to \infty$ lead to

$$m^*\left(A \cap \left[\bigcup_{i=1}^{\infty} E_i\right]\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i).$$

The opposite inequality follows from countable subadditivity.

Theorem 4: Let $\{E_i\}_{n \in \mathbb{N}}$ be a sequence of measurable sets, then $E = \bigcup_{i=1}^{\infty} E_i$ is also measurable. Moreover, if $E_1, E_2, \ldots$ are disjoint then $m(E) = \sum_{i=1}^{\infty} m(E_i)$.

This is called the countable additivity which can be proved by putting $A = \mathbb{R}$

Proof: We first assume $E_1, E_2, \ldots$ are disjoint. Then for all $A \subseteq R$, $n \in \mathbb{N}$ we have

$$m^*(A) = m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)^c\right)$$

$$\geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^c).$$

Letting $n \to \infty$,

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap E^c)$$

$$= m^*(A \cap E) + m^*(A \cap E^c).$$

This proved E is measurable.

Now, if $E_1, E_2, \ldots$ are not disjoint, we let

$$F_1 = E_1, \qquad F_2 = E_2 \setminus F_1, \qquad F_3 = E_3 \setminus (F_1 \cup F_2),$$

and in general $F_k = E_k \setminus \bigcup_{i=1}^{k-1} F_i$ for $k > 2$. Then $F_1, F_2, \ldots$ are disjoint and $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$. Since $\mathfrak{M}$ is an algebra, $F_1, F_2, \ldots$ are all measurable. So $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ is measurable.

Notes $\mathfrak{M}$ is proved to be a σ-algebra. The next result shows that all Borel sets are measurable. Recall that the family of Borel sets in $\mathbb{R}$ is, by definition, the smallest σ-algebra containing all open subsets of $\mathbb{R}$.

Theorem 5: $\mathfrak{M}$ contains all Borel subsets of $\mathbb{R}$.

Proof: It suffices to show that $(a, \infty) \in \mathfrak{M}$ for all $a \in \mathbb{R}$ (why?). Let $A \in \mathbb{R}$. We need to show that

$$m^*(A) \geq m^*(A \cap (-\infty, a]) + m^*(A \cap (a, \infty)).$$

Without loss of generality, we may assume $m^*(A) < \infty$. For convenience, let $A_1 = A \cap (-\infty, a]$ and $A_2 = A \cap (a, \infty)$. Then we need to show

$$m^*(A) + \varepsilon \geq m^*(A_1) + m^*(A_2) \qquad \text{for all } \varepsilon > 0.$$

Notes

By the definition of m*(A), there is a countable open interval cover $\{I_n\}_{n\in\mathbb{N}}$ of A with

$$m^*(A) + \varepsilon > \sum_{n=1}^{\infty} \ell(I_n).$$

Let $I_n' = I_n \cap [-\infty, a]$ and $I_n'' = I_n \cap (a, \infty)$, then $\{I_n'\}, \{I_n''\}$ are, respectively, interval covers of A_1 and A_2 (note that they may not be open interval covers). Then

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \ell(I_n') + \sum_{n=1}^{\infty} \ell(I_n'')$$

$$= \sum_{n=1}^{\infty} m^*(I_n') + \sum_{n=1}^{\infty} m^*(I_n'') \qquad (\because\ m^* = \ell \text{ for intervals})$$

$$\geq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) + m^*\left(\bigcup_{n=1}^{\infty} I_n''\right) \qquad (\because\ \text{countable subadditivity})$$

$$\geq m^*(A_1) + m^*(A_2) \qquad (\because\ \text{monotonicity})$$

So, $m^*(A) + \varepsilon > \sum_{n=1}^{\infty} \ell(I_n) \geq m^*(A_1) + m^*(A_2)$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, $m^*(A) \geq m^*(A_1) + m^*(A_2)$. This proved that $(a, \infty) \in \mathfrak{M}$.

Notes Since $\mathfrak{M}$ is a σ-algebra, $(-\infty, a] \in \mathfrak{M}$ and $(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - 1/n] \in \mathfrak{M}$. It follows that $(a, b) \in \mathfrak{M}$ since $(a, b) = (-\infty, b) \cap (a, \infty)$. As $\mathfrak{M}$ is a σ-algebra containing all open intervals, it must contain all open sets (recall that every open set is countable union of open intervals by Proposition). Therefore, $\mathfrak{M}$ contains all Borel sets.

Proposition: $\mathfrak{M}$ is translation invariant: for all $x \in \mathbb{R}$, $E \in \mathfrak{M}$ implies $E + x \in \mathfrak{M}$.

Proof: For all $A \in R$, we have

$$m^*(A) = m^*(A - x)$$

$$= m^*((A - x) \cap E) + m^*((A - x) \cap E^c)$$

$$= m^*(((A - x) \cap E) + x) + m^*(((A - x) \cap E^c) + x)$$

$$= m^*(A \cap (E + x)) + m^*(A \cap (E + x)^c)$$

Notes Let $E \in \mathbb{R}$ be given. Then the following statements are equivalent.

1. E is measurable.
2. For any $\varepsilon > 0$, there is an open set $O \supseteq E$ such that $m^*(O \setminus E) < \varepsilon$.
3. For any $\varepsilon > 0$, there is a closed set $F \subset E$ such that $m^*(E \setminus F) < \varepsilon$.
4. There is a $G \in G_\delta$ such that $E \subseteq G$ and $m^*(G \setminus E) = 0$.
5. There is a $F \in F_\sigma$ such that $E \supseteq F$ and $m^*(E \setminus F) = 0$.

   Assume $m^*(E) < \infty$, the above statements are equivalent to

6. For any $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$.

Theorem 6: Littlewood's 1st Principle

Every measurable set of finite measure is nearly a finite union of disjoint open intervals, in the sense

- If E is measurable and $m(E) < \infty$, then for any $\varepsilon > 0$ there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$. (Clearly, the intervals can be chosen to be disjoint.)
- If for any $\varepsilon > 0$ there is a finite union U of open intervals such that $m^*(U \Delta E) < \varepsilon$, then E is measurable. (The finiteness assumption $m^*(E) < \infty$ is not essential.)

Proof: If we can prove (1), (2), and (4) are equivalent, then it is easy to see that (2) and (3) are equivalent, because one implies another by replacing E with E^c. Similarly, (4) and (5) are equivalent.

To show (1) ⇒ (2)

We first consider a simple case $m(E) < \infty$. For any $\varepsilon > 0$, there is a countable open interval cover $\{I_n\}$ of E such that $\sum_{n=1}^{\infty} \ell(I_n) < m(E) + \varepsilon$. Take $O = \bigcup_{n=1}^{\infty} I_n$, we see that O is open and $O \supseteq E$. Also, we have

$$m(O \setminus E) = m(O) - m(E) \leq \sum_{n=1}^{\infty} m(I_n) - m(E) < \varepsilon.$$

Here we use the assumption $m(E) < \infty$ and the countable subadditivity of m.

For the case $m(E) = \infty$, we write $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n = E \cap [-n, n]$. This is a countable union of measurable sets of finite measure. By the above result there is an open set O_n such that $O_n \supseteq E_n$ and $m^*(O_n \setminus E_n) < \frac{\varepsilon}{2^n}$. Take $O = \bigcup_{n=1}^{\infty} O_n$, then O is open and $O \supseteq E$. It remains to show $m(O \setminus E) < \varepsilon$.

Note that $O \setminus E \subseteq \bigcup_{n=1}^{\infty} O_n \setminus E_n$, by countable subadditivity of m we have

$$m(O \setminus E) \leq \sum_{n=1}^{\infty} m(O_n \setminus E_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Hence, we have proved that (1) ⇒ (2).

To show (2) ⇒ (4)

For any $n \in \mathbb{N}$, let O_n be an open set such that $O_n \supseteq E$ and $m^*(O_n \setminus E) < 1/n$. Take $G = \bigcap_{n=1}^{\infty} O_n \in G_\delta$, then

$$m^*(G \setminus E) \leq m^*(O_n \setminus E) < \frac{1}{n}.$$

Letting $n \to \infty$, the result follows.

To show (4) ⇒ (1)

The existence of G guarantees $E = G \setminus (G \setminus E)$ is measurable since both G and $G \setminus E$ are measurable (G is Borel set and $G \setminus E$ is of measure zero).

Hence, (1), (2), (3), (4), (5) are equivalent.

To show (2) ⇒ (6) (with finiteness assumption $m^*(E) < \infty$)

Let $\varepsilon > 0$ be given. Let O be an open set such that $O \supseteq E$ and $m(O \setminus E) < \varepsilon/2$. Write $O = \bigcup_{n=1}^{\infty} I_n$ to be a countable union of disjoint open intervals. By the countable additivity of m, $m(O) = \sum_{n=1}^{\infty} \ell(I_n)$. Let k be a positive integer such that $\sum_{n=1}^{k} \ell(I_n) > m(O) - \varepsilon/2$. (The finiteness assumption has been used here to guarantee that $m(O) < \infty$.)

Notes

Take U = $\bigcap_{n=1}^{k} I_n$. Note that m(O \ U) = m(O) – m(U) < ε/2, so

$$\begin{aligned} m(U \,\Delta\, E) &= m(U\backslash E) + m(E\backslash U) \\ &\le m(O\backslash E) + m(O\backslash U) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The finiteness assumption is essential here. The above result is false if we allow E to have infinite measure. A counter example is E = $\bigcap_{n=1}^{\infty}(2n, 2n+1)$.

To show (6) ⇒ (2) (without finiteness assumption $m^*(E) < \infty$)

Let ε > 0 be given and U be a finite union of open intervals. Then $m^*(E\backslash U) < \varepsilon$, we take an open set $O' \supseteq E\backslash U$ such that $m^*(O') < \varepsilon$ (how to do this?). Then $O = U \cup O'$ is an open set containing E with $m^*(O\backslash E) \le m^*(U\backslash E) + m^*(O') < 2\varepsilon$.

Task Let A ∈ ℝ, prove that there is a measurable set B ⊇ A with $m^*(A) = m^*(B)$.

26.5 Step Functions and Simple Functions

Definition: A function ψ: [a, b] → ℝ is called step function if

$$\psi(x) = c_i \qquad (x_{i-1} < x < x_i)$$

for some partition $\{x_0, x_1, \ldots, x_n\}$ of [a, b] and some constants $c_1, c_2, \ldots, c_n$.

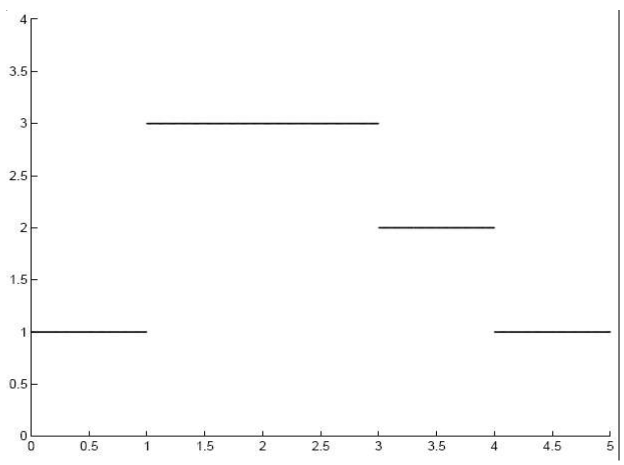
Lemma: Let ψ_1, ψ_2 be step functions on [a, b]. Then $\psi_1 \pm \psi_2$, $\alpha\psi_1 + \beta\psi_2$, $\psi_1 \psi_2$, $\psi_1 \wedge \psi_2$, and $\psi_1 \vee \psi_2$ are all step functions, where α, β ∈ ℝ. Also, if $\psi_2 \neq 0$ on [a,b], then ψ_1/ψ_2 is also step function.

Note $(f \wedge g)(x) = \min\{f(x), g(x)\}$ and $(f \vee g)(x) = \max\{f(x), g(x)\}$.

Lemma: Let ψ be a step function on [a,b] and let ε > 0. Then there is a continuous function g on [a, b] such that ψ = g on [a, b] except on a set of measure less than ε, i.e.

$$m(\{x \in [a, b] : \psi(x) \neq g(x)\}) < \varepsilon.$$

Figure 26.1: An Example of Step Function



Proof: Easy! One can find a piecewise linear function g with the stated property.

Definition: Let E $\in \mathfrak{M}$. A function f: E $\to \mathbb{R}$ is called a simple function if there exists $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $E_1, E_2, \ldots, E_n \in \mathfrak{M}$ such that

(2) $$f = \sum_{i=1}^{k} a_i \chi_{E_i}$$

Note Step function is simple, $\chi_{\mathbb{Q}}$ is simple but not step function.

Proposition: Let f: [a, b] $\to \mathbb{R}$ be a simple function. For any $\varepsilon > 0$, there is a step function ψ: [a, b] $\to \mathbb{R}$ such that f = ψ except on a set of measure less than ε.

Proof: Let f be given by (2), we may assume $E_1, E_2, \ldots, E_n \subseteq E$. By Littlewood's 1st Principle, there is a finite union of disjoint open intervals U_i such that $m(U_i \Delta E_i) < \varepsilon/n$. Then

$$f = \sum_{i=1}^{k} a_i \chi_{U_i} \quad \text{except on } A = \bigcup_{i=1}^{n} (U_i \Delta E_i),$$

where $m(A) < \sum_{i=1}^{n} \varepsilon/n = \varepsilon$.

Notes One can find a continuous function with the same property. Moreover, if f satisfies $m \le f \le M$ on [a, b] then ψ can be chosen such that $m \le \psi \le M$ (reason: replace ψ by $(m \vee \psi) \wedge M$ if necessary).

26.6 Measurable Functions

Definition: A function f: E $\to [-\infty, \infty]$ is said to be measurable (or measurable on E) if E $\in \mathfrak{M}$ and

$$f^{-1}((a, \infty]) \in \mathfrak{M}$$

for all $a \in \mathbb{R}$.

In fact, there is a more general definition for measurability which we will not use here. The definition goes as follows.

Definition: Let X be a measurable space and Y be a topological space. A function f: X $\to$ Y is called measurable if $f^{-1}(V)$ is a measurable set in X for every open set V inY.

Notes Simple functions, step functions, continuous functions and monotonic functions are measurable.

Proposition: Let E $\in \mathfrak{M}$ and f : E $\to [-\infty, \infty]$. Then the following four statements are equivalent:

- $f^{-1}((a, \infty]) \in \mathfrak{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}([a, \infty]) \in \mathfrak{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}([-\infty, a)) \in \mathfrak{M}$ for all $a \in \mathbb{R}$.
- $f^{-1}([-\infty, a]) \in \mathfrak{M}$ for all $a \in \mathbb{R}$.

Notes

Notes The above statements imply $f^{-1}(a) \in \mathfrak{M}$ for all $a \in [-\infty, \infty]$. The converse is not true.

Proof: The first one is clearly equivalent to the fourth one since $f^{-1}([a, \infty]) = E \setminus f^{-1}([-\infty, a])$. Similarly, the second and the third statements are equivalent. It remains to show the first two statements are equivalent, but this follows immediately from

$$f^{-1}([a, \infty]) = \bigcup_{n=1}^{\infty} f^{-1}\left((a - \frac{1}{n}, \infty]\right) \quad \text{and} \quad f^{-1}([a, \infty]) = \bigcup_{n=1}^{\infty} f^{-1}\left([a + \frac{1}{n}, \infty]\right).$$

Proposition: Let $E \in \mathfrak{M}$, $f: E \to [-\infty, \infty]$ and $g: E \to [-\infty, \infty]$. If f = g almost everywhere on E then the measurability of f and g are the same.

Proof: Simply note that

$$m^*(\{x \in E : f(x) > a\} \,\Delta\, \{x \in E : g(x) > a\}) \le m^*(\{x \in E : f(x) \ne g(x)\}) = 0.$$

This implies the measurability of the sets $\{x \in E: f(x) > a\}$ and $\{x \in E : g(x) > a\}$ are the same.

Proposition: Let f, g be measurable extend real-valued functions on $E \in \mathfrak{M}$. Then the following functions are all measurable on E:

$$f + c,\ cf,\ f \pm g,\ fg$$

where $c \in \mathbb{R}$.

Notes One may worry that cf, f ± g, fg may not be defined at some points (for example, if $f = \infty$ and $g = -\infty$ then f + g is meaningless). There are two ways to deal with this problem.

1. Adopt the convention $0 \cdot \infty = 0$.

2. Assume f, g are finite almost everywhere or cf, f ± g, fg are meaningful almost everywhere.

Proof: We only prove f + g and fg are measurable, since the others are easy or similar.

To prove f + g is measurable, one should consider the set

$$\begin{aligned} E_a &= \{x \in E : f(x) + g(x) > a\} \\ &= \{x \in E : f(x) > a - g(x)\} \\ &= \bigcup_{r \in \mathbb{Q}} \{x \in E : f(x) > r > a - g(x)\} \\ &= \bigcup_{r \in \mathbb{Q}} \{x \in E : f(x) > r\} \cap \{x \in E : r > a - g(x)\} \end{aligned}$$

If $f(x) = \infty$ or $g(x) = \infty$ then $x \in E_a$ by convention. Now $E_a \in \mathfrak{M}$ because E_a is countable union of measurable sets.

Next, we prove f^2 is measurable. For $a \ge 0$,

$$\{x \in E: f^2(x) > a\} = \{x \in E : f(x) > \sqrt{a}\} \cup \{x \in E : f(x) < -\sqrt{a}\}$$

is measurable. For $a < 0$, $\{x \in E : f^2(x) > a\} = E$ is also measurable. Therefore, f^2 is measurable and it is valid even if f takes values $\pm\infty$.

Notes

So, if f and g are assumed to be finite, then

$$fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$$

is measurable on E.

Task Find two measurable functions f, g from $\mathbb{R}$ to $\mathbb{R}$ such that f o g is not measurable.

Proposition: Let $\{f_n\}_{n\in\mathbb{N}}$ be measurable extended real-valued functions on a measurable set E. Then

$$f_1 \vee f_2 \ldots \vee f_n, \qquad \sup_{n\in\mathbb{N}} f_n, \qquad \overline{\lim_{n\to\infty}} f_n$$

are all measurable on E. Similar results hold if $\vee$, sup and $\overline{\lim}$ are replaced by $\wedge$, inf, and $\underline{\lim}$.

Proof: Simply note that

$$(f_1 \vee f_2 \cdots \vee f_n)^{-1}((a, \infty)) = \bigcup_{k=1}^{\infty} f_k^{-1}((a,\infty))$$

$$\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}((a,\infty)) = \bigcup_{k=1}^{\infty} f_k^{-1}((a,\infty))$$

$$\overline{\lim_{n\to\infty}} f_n = \inf_{N\in\mathbb{N}}\left(\sup_{k\geq N} f_k\right)$$

Theorem 7: Let $E \in \mathfrak{M}$ with $m(E) < \infty$, $f: E \to [-\infty,\infty]$ be measurable and finite almost everywhere. For any $\varepsilon > 0$, there is a simple function ϕ such that

$|f - \phi| < \varepsilon$ on E except on a set of measure less than ε.

Notes If E = [a, b] is closed and bounded interval, we can find a step function g and a continuous function h play the role of ϕ. This is because simple function can be approximated by step function and step function can be approximated by continuous function.

If f satisfies an additional condition $m \leq f \leq M$, then ϕ, g, and h can be chosen to be bounded below by m and above by M.

The condition $m(E) < \infty$ in Littlewood's 2nd Principle is essential. You can see if this condition is dropped then taking f(x) = x will give a counter example.

To prove Littlewood's 2nd Principle, we introduce a lemma.

Lemma: Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}$ (or any measure space[3]) such that

$$F_1 \supseteq F_2 \supseteq \cdots .$$

Denote $F_\infty = \bigcap_{n\in\mathbb{N}} F_n$. If $m(F_1) < \infty$ then

$$m(F_\infty) = \lim_{n\to\infty} m(F_n).$$

Proof: Write $F_1 = F_\infty \cup (F_1 \backslash F_2) \cup (F_2 \backslash F_3) \cup \cdots$ as disjoint union and use the countable additivity of m.

Notes

> *Notes* Above Lemma 4 is false if the condition $m(F_1) < \infty$ is missing.

Now, we are ready to prove Littlewood's 2nd Principle.

Proof: Let $\varepsilon > 0$ be given. Our proof is divided into two steps.

Step I: Assume $m \le f \le M$ for some $m, M \in \mathbb{R}$.

We divide [m, M] into n subintervals such that the length of each subinterval is less than ε. Symbolically, we take the partition points as follows:

$$m = y_0 < y_1 < \cdots < y_n = M \qquad \text{with } y_i - y_{i-1} < \varepsilon \text{ for } 1 \le i \le n.$$

Let $E_1 = \{x \in E : m \le f(x) \le y_1\}$, $E_2 = \{x \in E : y_1 < f(x) \le y_2\}, \ldots, E_n = \{x \in E : y_{n-1} < f(x) \le M\}$. Now, take $\phi = y_1\chi_{E_1} + y_2\chi_{E_2} + \cdots + y_n\chi_{E_n}$. Since $E_1, E_2, \ldots, E_n$ are all measurable (why?), ϕ is simple and satisfies the inequality $|f - \phi| < \varepsilon$ with no exceptions.

Step II: General case.

We let

$$F_n = \{x \in E : |f(x)| \ge n\}.$$

Then $F_1 \supseteq F_2 \supseteq \cdots$. Note that $m(F_1) \le m(E) < \infty$ and $m(F_\infty) = 0$ by assumption, apply Lemma 4 there exists $N \in \mathbb{N}$ such that

$$m(F_N) < \varepsilon.$$

Now, let $f^* = (-N \vee f) \wedge N$, then $f = f^*$ on E except on a set of measure less than ε. From the result of Step I, there is a simple function ϕ such that $|f^* - \phi| < \varepsilon$ on E. Hence

$$|f - \phi| < \varepsilon \text{ on E except on a set of measure less than } \varepsilon.$$

Corollary: There is a sequence of simple functions ϕ_n such that $\phi_n \to f$ pointwisely almost everywhere on E. If $E = [a, b]$, there are also sequence of step functions and sequence of continuous functions converging to f pointwisely almost everywhere on [a,b].

Proof: Applying Littlewood's 2nd Principle to $\varepsilon = 1/2^n$, there are simple functions ϕ_n and sets A_n with $m(A_n) < 1/2^n$ such that

$$|f - \phi_n| < \frac{1}{2^n} \qquad \text{on } E\backslash A_n.$$

Let $A = \overline{\lim} A_n := \bigcap_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty} A_n\right)$, then $m(A) = 0$ (why?). The proof is completed by noting that $\phi_n \to f$ pointwisely on $E\backslash A$.

> *Notes* In fact, the sequence ϕ_n can be chosen so that $\phi_n \to f$ pointwisely everywhere on E. For example, we can first divide the interval $[-n, n]$ into $2n^2$ subintervals such that each subinterval has length $1/n$, i.e. choose
>
> $$-n = y_0 < y_1 < \cdots < y_{2n^2} = n$$
>
> such that $y_i - y_{i-1} = 1/n$ for all i. Then let
>
> $$\phi_n(x) = \begin{cases} y_i & \text{if } y_i \le f(x) < y_{i+1} \text{ for some i} \\ n & \text{if } f(x) \ge n \\ -n & \text{if } f(x) < -n \end{cases}$$

Notes

Theorem 8: Littlewood's 3rd Principle/Egoroff's Theorem

Let $E \in \mathfrak{M}$ with $m(E) < \infty$, $f: E \to (-\infty, \infty)$ be measurable and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions on E such that

$$f_n \to f \qquad \text{a.e. on } E.$$

Then for any $\eta > 0$ there is a (measurable) subset S of E with $m(S) < \eta$ such that

$$f_n \to f \qquad \text{uniformly on } E \backslash S.$$

Notes Again, the condition $m(E) < \infty$ cannot be dropped. Otherwise $f_n = \chi_{[n, \infty)}$ and $f = 0$ would be a counter example.

Proof: We claim that for any $\varepsilon > 0$ and $\delta > 0$, there exists $A \subseteq E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{whenever } n \geq N \text{ and } x \in E \setminus A.$$

Be careful the above statement is not saying that $f_n \to f$ uniformly on $E \setminus A$ since A depends on ε and δ.

To prove our claim, we let

$$G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$$

and

$$G = \overline{\lim}\, G_n := \bigcap_{n \in \mathbb{N}} E_n, \qquad \text{where } E_n = \bigcup_{k \geq n} G_k.$$

Note that if $x \in G$ then $x \in E_n$ for all $n \in \mathbb{N}$, it follows that $f_n(x) \nrightarrow f(x)$. Since the set of all x such that $f_n(x) \nrightarrow f(x)$ is of measure zero, we have $m(G) = 0$. Note also that $m(E_1) < \infty$ and E_n "decreases" to G, so $\lim m(E_n) = m(G) = 0$ by Lemma 4. There is $N \in \mathbb{N}$ such that $m(E_N) < \delta$. This N, together with $A := E_N$, proved our claim.

Now, let $\eta > 0$ be given. Apply the above result to $\varepsilon = 1/k$ and $\delta = \eta/2^k$, we obtain A_k with $m(A_k) < \eta/2^k$ and $N_k \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{1}{k} \qquad \text{whenever } n \geq N_k \text{ and } x \in E \setminus A_k.$$

Let $S = \bigcup_{k \in \mathbb{N}} A_k$, then $m(S) \leq \sum_{k=1}^{\infty} m(A_k) < \eta$ and $|f_n(x) - f(x)| < 1/k$ whenever $n \geq N_k$ and $x \in E \backslash S$. Hence, $f_n \to f$ uniformly on $E \backslash S$.

Self Assessment

Fill in the blanks:

1. Every open subset V of $\mathbb{R}$ is a .... of disjoint open intervals.
2. The family of all measurable sets is denoted by $\mathfrak{M}$. We will see later $\mathfrak{M}$ is a σ-algebra and translation-invariant containing all intervals. The set function $m: \mathfrak{M} \to [0, \infty]$ defined by

   $$m(E) = m^*(E) \qquad \text{for all } E \in \mathfrak{M}$$

   is called ....
3. Let X be a .... and Y be a topological space. A function $f: X \to Y$ is called measurable if $f^{-1}(V)$ is a measurable set in X for every open set V in Y.
4. Let $E \in \mathfrak{M}$, $f: E \to [-\infty, \infty]$ and $g: E \to [-\infty, \infty]$. If f = g almost everywhere on E then the .... of f and g are the same.

Notes

26.7 Summary

- The definition of outer measure of sets.
- Outer measure of an interval is its length.
- Some important properties of Outer measure.
- The definition of Measurable sets.
- Countable union of measurable sets is also measurable.
- Countable intersection of measurable sets is also measurable.
- Every Borel set is measurable.
- Littlewood's First Principle.

26.8 Keywords

Lindelof's Theorem: Let C be a collection of open subsets of $\mathbb{R}$. Then there is a countable sub-collection $\{O_i\}_{i \in \mathbb{N}}$ of C such that

$$\bigcup_{O \in C} O = \bigcup_{i=1}^{\infty} O_i$$

Lebesgue Measure: The family of all measurable sets is denoted by $\mathfrak{M}$. We will see later $\mathfrak{M}$ is a σ-algebra and translation-invariant containing all intervals. The set function m: $\mathfrak{M} \to [0, \infty]$ defined by

$$m(E) = m^*(E) \qquad \text{for all } E \in \mathfrak{M}$$

is called Lebesgue measure.

$$\bigcup_{O \in C} O = \bigcup_{i=1}^{\infty} O_i$$

Littlewood's 1st Principle: Every measurable set of finite measure is nearly a finite union of disjoint open intervals, in the sense.

Measurable Functions: A function f: E → [-∞, ∞] is said to be measurable (or measurable on E) if $E \in \mathfrak{M}$ and

$$f^{-1}((a, \infty]) \in \mathfrak{M}$$

for all $a \in \mathbb{R}$.

26.9 Review Questions

1. Prove that the family M of measurable sets is an algebra.
2. If E_1, E_2, ….En are measurable, prove that $E_1 \cup E_2 \cup \ldots \cup E_n$ is measurable.
3. If E_1 and E_2 are measurable sets, then prove that $E_1 \cup E_2$ is also measurable.
4. Prove that properties (i) to (v) are equivalent to (vi), if m*E is finite.
5. Show that if E is measurable, then each translate E + y is also measurable.
6. Show that if E_1 and E_2 are measurable, then $m(E_1 \cup E_2) + m(E_1 | E_2) = mE_1 + mE_2$.
7. Let $\{E_i\}$ be a sequence of disjoint measurable sets and A be any set.

   Show that $m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$

Notes

Answers: Self Assessment

1. countable union
2. Lebesgue measure
3. measurable space
4. measurability

26.10 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10, Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 27: Measurable Functions and Littlewood's Second Principle

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define measurable functions
- Discuss the Sum, difference; scalar product and product of measurable functions are measurable
- Explain Littlewood's Theorems

Introduction

In this unit we study the concept of measurability. We shall see that measurable functions are basically very robust (or strong or durable) continuous-like functions. We make "continuous-like" precise in Luzin's Theorem, which is where Littlewood got his second principle. We also study the concept of almost everywhere.

27.1 Measurable Functions

A measurable space is a pair $(X, \mathscr{S})$ where X is a set and, $\mathscr{S}$ is a σ-algebra of subsets of X. The elements of, $\mathscr{S}$ are called measurable sets. Recall that a measure space is a triple $(X, \mathscr{S}, \mu)$ where μ is a measure on $\mathscr{S}$; if we leave out the measure we have a measurable space.

In the discussion at the beginning of this unit we saw that in order to define the integral of a function $f : X \to \overline{\mathbb{R}}$, we needed to require that

$$f^{-1}(I) \in \mathscr{S} \text{ for each } I \in \mathscr{S} \text{ and } f^{-1}(a, \infty] \in \mathscr{S} \text{ for each } a \in \mathbb{R}.$$

If these properties hold, we say that f is measurable. It turns out that we can omit the first condition because it follows from the second. Indeed, since

$$f^{-1}(a, b] = f^{-1}(a, \infty] \backslash f^{-1}(b, \infty],$$

Notes

Notes As a reminder, for any $A \subseteq \overline{\mathbb{R}}$, $f^{-1}(A) := \{x \in X; f(x) \in A\}$, so for instance $f^{-1}(a, \infty] = \{x \in X; f(x) \in (a, \infty)\} = \{x \in X; f(x) > a\}$, or $f^{-1}(a, \infty) = \{f > a\}$ if you wish to be a probabilist.

and $\mathscr{S}$ is a σ-algebra, if both right-hand sets are in $\mathscr{S}$, then so is the left-hand set. Hence, in order to define the integral of f we just need $f^{-1}[a, \infty] \in \mathscr{S}$ for each $a \in \mathbb{R}$. We are thus led to the following definition:

A function $f: X \to \overline{\mathbb{R}}$ is measurable if $f^{-1}[a, \infty] \in \mathscr{S}$ for each $a \in \mathbb{R}$.

We emphasize that the definition of measurability is not "artificial" but is required by Lebesgue's definition of the integral. If X is the sample space of some experiment, a measurable function is called a random variable; thus,

In probability, random variable = measurable function.

We note that intervals of the sort $(a, \infty]$ are not special, and sometimes it is convenient to use other types of intervals.

Proposition: For a function $f: X \to \overline{\mathbb{R}}$, the following are equivalent:

1. f is measurable.
2. $f^{-1}[-\infty, a] \in \mathscr{S}$ for each $a \in \mathbb{R}$.
3. $f^{-1}[a, \infty] \in \mathscr{S}$ for each $a \in \mathbb{R}$.
4. $f^{-1}[-\infty, a] \in \mathscr{S}$ for each $a \in \mathbb{R}$.

Proof: Since preimages preserve complements, we have

$$(f^{-1}[a, \infty])^c = f^{-1}([a, \infty]^c) = f^{-1}[-\infty, a].$$

Since σ-algebras are closed under complements, we have (1) ⇔ (2). Similarly, the sets in (3) and (4) are complements, so we have (3) ⇔ (4). Thus, we just to prove (1) ⇔ (3). Assuming (1) and writing

$$[a, \infty] = \bigcap_{n=1}^{\infty}\left[a - \frac{1}{n}, \infty\right] \Rightarrow f^{-1}[a, \infty] = \bigcap_{n=1}^{\infty} f^{-1}\left[a - \frac{1}{n}, \infty\right],$$

shows that $f^{-1}[a, \infty] \in \mathscr{S}$ since each $f^{-1}\left[a - \frac{1}{n}, \infty\right] \in \mathscr{S}$ and $\mathscr{S}$ is closed under countable intersections. Thus, (1) ⇒ (3). Similarly,

$$[a, \infty] = \bigcup_{n=1}^{\infty}\left[a + \frac{1}{n}, \infty\right] \Rightarrow f^{-1}(a, \infty] = \bigcup_{n=1}^{\infty} f^{-1}\left[a + \frac{1}{n}, \infty\right],$$

shows that (3) ⇒ (1).

As a consequence of this proposition, we can prove that measurable functions are closed under scalar multiplication. Indeed, let $f : X \to \overline{\mathbb{R}}$ be measurable and let $\alpha \in \mathbb{R}$; we'll show that αf is also measurable. Assume that $\alpha \neq 0$ (the $\alpha = 0$ case is easy) and observe that for any $a \in \mathbb{R}$,

$$(\alpha f)^{-1}[a, \infty] = \{x; \alpha f(x) > a\} = \begin{cases} \left\{x; f(x) > \dfrac{a}{\alpha}\right\} & \text{if } \alpha > 0, \\ \left\{x; f(x) < \dfrac{a}{\alpha}\right\} & \text{if } \alpha < 0 \end{cases}$$

$$= \begin{cases} f^{-1}\left[\dfrac{a}{\alpha}, \infty\right] & \text{if } \alpha > 0, \\ f^{-1}\left[-\infty, < \dfrac{a}{\alpha}\right] & \text{if } \alpha < 0. \end{cases}$$

Notes

By Proposition, each set on the right is measurable, thus so is $(\alpha f)^{-1}[a, \infty]$. We'll analyze more algebraic properties of measurable functions in the next section.

We now give some examples of measurable functions.

Example: Let $X = \mathbb{R}^n$ with Lebesgue measure. Then any continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is measurable because for any $a \in \mathbb{R}$, by continuity (the inverse of any open set is open),

$$f^{-1}[a, \infty] = f^{-1}(a, \infty)$$

(where we used that f does not take the value ∞) is an open subset of $\mathbb{R}^n$. Since open sets are measurable, it follows that f is measurable.

Thus, for Lebesgue measure, continuity implies measurability. However, the converse is far from true because there are many more functions that are measurable than continuous. For instance, Dirichlet's function $D : \mathbb{R} \to \mathbb{R}$,

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

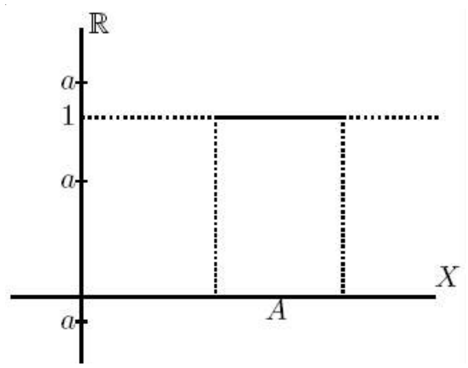
is Lebesgue measurable. Note that D is nowhere continuous. That D is measurable follows from example below and the fact that D is just the characteristic function of $\mathbb{Q} \subseteq \mathbb{R}$, and $\mathbb{Q}$ is measurable.

Example: For a general measure space X and a set $A \subseteq X$, we claim that the characteristic function $\chi_A : X \to \mathbb{R}$ is measurable if and only if the set A is measurable. Indeed, looking at Figure 27.1, we see that

$$\chi_A^{-1}[a, \infty] = \{x \in X; \chi_A(x) > a\} = \begin{cases} X & \text{if } a < 0 \\ A & \text{if } 0 \le a < 1, \\ \varnothing & \text{if } a \ge 1. \end{cases}$$

It follows that $\chi_A^{-1}[a, \infty] \in \mathscr{S}$ for all $a \in \mathbb{R}$ if and only if $A \in \mathscr{S}$, which proves the claim. In particular, there exists a non-Lebesgue measurable function on $\mathbb{R}^n$. In fact, given any non-measurable set $A \subseteq \mathbb{R}^n$, the characteristic function $\chi_A: \mathbb{R}^n \to \mathbb{R}$ is not measurable.

Figure 27.1: Graph of χ_A



Of course, since A is non-constructive, so is χ_A. You will probably never find a non-measurable function in practice. The following example shows the importance of studying extended real-valued functions, instead of just real-valued functions.

Notes

Example: Let $X = S^{\infty}$, where $S = \{0,1\}$, be the sample space for a Monkey-Shakespeare experiment (or any other experiment involving a sequence of Bernoulli trials). Let $f : X \to [0, \infty]$ be the number of times the Monkey types sonnet 18:

$$f(x_1, x_2, x_3,...) = \text{the number of i's such that } x_i = 1.$$

Notice that $f = \infty$ when the Monkey types sonnet 18 an infinite number of times (in fact, as we see that $f = \infty$ on a set of measure). To show that f is measurable, write f as

$$f = \lim_{n\to\infty} f_n,$$

where f_i is the number of i's in 1, 2,..., n such that $x_i = 1$. Notice that $f_1 \le f_2 \le f_3 \le \cdots$ are non-decreasing, so it follows that for any $a \in \mathbb{R}$,

$$f(x) \le a \Leftrightarrow f_n(x) \le a \text{ for all } n \Leftrightarrow x \in \bigcap_{n=1}^{\infty} \{f_n \le a\}. \{f_n < a\}.$$

Thus,

$$f^{-1}[-\infty, a] = \bigcap_{n=1}^{\infty} f_n^{-1}[-\infty, a].$$

The set $\{f_n \le a\}$ is of the form $A_n \times S \times S \times S \times \cdots$ where $A_n \subseteq S^n$ is the subset of S^n consisting of those points with no more than a total of a entries with 1's. In particular, $\{f_n \le a\} \in \mathscr{R}(\mathscr{C})$ and hence, it belongs to $\mathscr{S}(\mathscr{C})$. Therefore, $\{f \le a\}$ also belongs to $\mathscr{S}(\mathscr{C})$, so f is measurable.

We shall return to this example when we study limits of measurable functions.

As we defined simple functions. For a quick review in the current context of our σ-algebra, $\mathscr{S}$, recall that a simple function (or $\mathscr{S}$-simple function to emphasize the σ-algebra, $\mathscr{S}$) is any function of the form

$$s = \sum_{n=1}^{N} a_n \chi_{A_n},$$

where $a_1,..., a_N \in \mathbb{R}$ and $A_1,..., A_N \in \mathscr{S}$ are pairwise disjoint. We know that we don't have to take the A_n's to be pairwise disjoint, but for proofs it's often advantageous to do so.

Theorem 1: Any Simple Function is Measurable

Proof: Let $s = \sum_{n=1}^{N} a_n \chi_{A_n}$ be a simple function where $a_1,..., a_N \in \mathbb{R}$ and $A_1,..., A_N \in \mathscr{S}$ are pairwise disjoint. If we put $A_{N+1} = X \setminus \{A_1 \cup \cdots \cup A_N\}$ and $a_{N+1} = 0$, then

$$X = A_1 \cup A_2 \cup \cdots \cup A_N \cup A_{N+1},$$

a union of pairwise disjoint sets, and $s = a_n$ on A_n, for each n = 1, 2,..., N + 1. It follows that

$$s^{-1}[a, \infty] = \{x \in X; s(x) > a\} = \bigcup_{n=1}^{N+1} \{x \in A_n; s(x) > a\}$$

$$= \bigcup_{n=1}^{N+1} \{x \in A_n; a_n > a\}.$$

Since

$$\{x \in A_n; a_n > a\} = \begin{cases} A_n & \text{if } a_n > a \\ \varnothing & \text{otherwise.} \end{cases}$$

it follows that $s^{-1}[a, \infty]$ is just a union of elements of $\mathscr{S}$. Thus, s is measurable.

Notes

27.2 Measurability and Continuity

We saw earlier that continuity implies measurability, essentially by definition of continuity in terms of open sets. It turns out that we can directly express measurability in terms of open sets.

Theorem 2: Measurability Criterion

For a function $f: X \to \overline{\mathbb{R}}$, the following are equivalent:

1. f is measurable.
2. $f^{-1}(\{\infty\}) \in \mathscr{S}$ and $f^{-1}(\mathcal{U}) \in \mathscr{S}$ for all open subsets $\mathcal{U} \subseteq \mathbb{R}$.
3. $f^{-1}(\{\infty\}) \in \mathscr{S}$ and $f^{-1}(B) \in \mathscr{S}$ for all Borel sets $B \subseteq \mathbb{R}$.

Proof: To prove that (1) ⇒ (2), observe that

$$\{\infty\} = \bigcap_{n=1}^{\infty} [n,\infty] \Rightarrow f^{-1}(\{\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}[n,\infty].$$

Assuming f is measurable, we have $f^{-1}[n, \infty] \in \mathscr{S}$ for each n and since $\mathscr{S}$ is a σ-algebra, it follows that $f^{-1}(\{\infty\}) \in \mathscr{S}$. Also, if $\mathcal{U} \subseteq \mathbb{R}$ is open, then by the Dyadic Cube Theorem we can write $\mathcal{U} = \bigcup_{n=1}^{\infty} I_n$ where $I_n \in \mathscr{S}^1$ for each n. Hence,

$$f^{-1}(\mathcal{U}) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$$

By measurability, $f^{-1}(I_n) \in \mathscr{S}$ for each n, so $f^{-1}(\mathcal{U}) \in \mathscr{S}$.

To prove that (2) ⇒ (3), we don't have to worry about the preimage of ∞, so we just have to prove that $f^{-1}(B) \in \mathscr{S}$ for all Borel sets $B \subseteq \mathbb{R}$.

$$\mathscr{S}_f = \{A \in \mathbb{R};\, f^{-1}(A) \in \mathscr{S}\}$$

is a σ-algebra. Assuming (2) we know that all open sets belong to $\mathscr{S}_f$. Since $\mathscr{S}_f$ is a σ-algebra of subsets of $\mathbb{R}$ and $\mathscr{B}$ is the smallest σ-algebra containing the open sets, it follows that $\mathscr{B} \subseteq \mathscr{S}_f$.

Finally we prove that (3) ⇒ (1). Let $a \in \mathbb{R}$ and note that

$$[a, \infty] = (a, \infty) \cup \{\infty\} \Rightarrow f^{-1}[a, \infty] = f^{-1}(a, \infty) \cup f^{-1}(\{\infty\}).$$

Assuming (3), we have $f^{-1}(\{\infty\}) \in \mathscr{S}$ and since $(a, \infty) \subseteq \mathbb{R}$ is open, and hence is Borel, we also have $f^{-1}(a, \infty) \in \mathscr{S}$. Thus, $f^{-1}(a, \infty] \in \mathscr{S}$, so f is measurable.

We remark that the choice of using +∞ over –∞ in the "$f^{-1}(\{\infty\}) \in \mathscr{S}$" parts of (2) and (3) were arbitrary and we could have used –∞ instead of ∞.

Consider the second statement in the theorem, but only for real-valued functions:

Measurability: A function $f : X \to \mathbb{R}$ is measurable if and only if $f^{-1}(\mathcal{U}) \in \mathscr{S}$ for each open set $\mathcal{U} \subseteq \mathbb{R}$.

One cannot avoid noticing the striking resemblance to the definition of continuity. Recall that for a topological space $(T, \mathscr{T})$, where $\mathscr{T}$ is the topology on a set T.

Continuity: A function $f : T \to \mathbb{R}$ is continuous if and only if $f^{-1}(\mathcal{U}) \in \mathscr{T}$ for each open set $\mathcal{U} \subseteq \mathbb{R}$.

Because of this similarity, one can think about measurability as a type of generalization of continuity. However, speaking philosophically, there are two very big differences between measurable functions and continuous functions as we can see by considering $X = \mathbb{R}^n$ with Lebesgue measure and its usual topology:

(i) There are a lot more measurable functions than continuous functions.

(ii) Measurable functions are closed under a lot more operations than continuous functions are.

To understand Point (i), recall from that all continuous functions on $\mathbb{R}^n$ are measurable; in contrast, there are measurable functions that are highly discontinuous (like Dirichlet's function). There are more measurable functions than continuous functions because there are a lot more measurable sets than there are open sets. For example, not only are open sets measurable but so are points, Cantor-type sets, G_δ sets, F_σ sets, etc. We shall see that, just like continuous functions, measurable functions are closed under all the usual arithmetic operations such as addition, multiplication, etc. What exemplifies Point (ii) is that measurable functions are closed under all limiting operations. For example, a limit of measurable functions is always measurable. This stands in stark contrast to continuous functions. Indeed, that the characteristic function of a Cantor set can be expressed as a limit of continuous functions. The reason that measurable functions are closed under more operations is that measurable sets are closed under operations (e.g. countable intersections and complements) that open sets are not.

Measurable functions are similar to continuous functions, but there are more of them and they are more robust. Littlewood's second principle shows exactly how "similar" measurable functions are to continuous functions.

27.3 Littlewood's Second Principle

We now continue our discussion of Littlewood's Principles where we stated the first principle;

There are three principles, roughly expressible in the following terms: Every [finite Lebesgue] measurable set is nearly a finite union of intervals; every measurable function is nearly continuous; every convergent sequence of measurable functions is nearly uniformly convergent.

—Nikolai Luzin

The third principle is contained in Egorov's theorem, which we'll get to in the next topic. The second principle comes from Luzin's Theorem, named after Nikolai Nikolaevich Luzin (1883-1950) who proved it in 1912 [70], and this theorem makes precise Littlewood's comment that any Lebesgue measurable function is "nearly continuous".

Theorem 3: Luzin's Theorem

Let $X \subseteq \mathbb{R}^n$ be Lebesgue measurable and let $f : X \to \mathbb{R}$ be a Lebesgue measurable function. Then given any $\varepsilon > 0$, there exists a closed set $C \subseteq \mathbb{R}^n$ such that $C \subseteq X$, $m(X\setminus C) < \varepsilon$, and f is continuous on C.

Proof: Here we follow Feldman's [38] proof that only uses Littlewood's First Principle. Luzin's theorem is commonly proved using Egorov's theorem and the fact that every measurable function is the limit of simple functions.

Step 1: We first prove the theorem only requiring that C be measurable; this proof is yet another example of the "$\frac{\varepsilon}{2^k}$–trick." Let $\{\mathcal{V}_k\}$ be a countable basis of open sets in $\mathbb{R}$; this means that every open set in $\mathbb{R}$ is a union of countably many $\mathcal{V}_k$'s. (For example, take the $\mathcal{V}_k$'s as open intervals with rational end points.) Let $\varepsilon > 0$. Then, since $f^{-1}(\mathcal{V}_k)$ is measurable, by Littlewood's First Principle there is an open set $\mathcal{U}_k$ such that

$$f^{-1}(\mathcal{V}_k) \subseteq \mathcal{U}_k \quad \text{and} \quad m(\mathcal{U}_k \setminus f^{-1}(\mathcal{V}_k)) < \frac{\varepsilon}{2^k}.$$

Now put

$$A := \bigcup_{k=1}^{\infty} (\mathcal{U}_k \setminus f^{-1}(\mathcal{V}_k)).$$

Then A is measurable and

$$m(A) \le \sum_{k=1}^{\infty} m(\mathcal{U}_k \setminus f^{-1}(\mathcal{V}_k)) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Notes

If we can prove that

$$g := f|_{X\setminus A} : X \setminus A \to \mathbb{R}$$

is continuous, then we have proven our theorem with $C = X \setminus A$ (modulo the closedness condition). Since $\{\mathcal{V}_k\}$ is a basis for the topology of $\mathbb{R}$ to prove that g is continuous all we have to do is prove that for each k, $g^{-1}(\mathcal{V}_k)$ is open in $X\setminus A$. To prove this, we shall prove that

$$g^{-1}(\mathcal{V}_k) = (X\setminus A) \cap \mathcal{U}_k; \tag{3.1}$$

then, since $\mathcal{U}_k$ is an open subset of $\mathbb{R}^n$, it follows that $g^{-1}(\mathcal{V}_k)$ is open in $X\setminus A$ and we're done. Now to prove the desired equality note that, by definition of g, we have

$$g^{-1}(\mathcal{V}_k) = (X\setminus A) \cap f^{-1}(\mathcal{V}_k) \subseteq (X\setminus A) \cap \mathcal{U}_k,$$

since $f^{-1}(\mathcal{V}_k) \subseteq \mathcal{U}_k$. On the other hand, observe that

$$\begin{aligned} x \in (X\setminus A) \cap \mathcal{U}_k &\Rightarrow x \notin A,\ x \in \mathcal{U}_k \\ &\Rightarrow x \notin (\mathcal{U}_k\setminus f^{-1}(\mathcal{V}_k)),\ x \in \mathcal{U}_k \\ &\Rightarrow x \in f^{-1}(\mathcal{V}_k). \end{aligned}$$

In the second implication we used that $A = \bigcup_{j=1}^{\infty}(\mathcal{U}_j\setminus f^{-1}(\mathcal{V}_j))$ so $x \notin A$ implies, in particular, that $x \notin (\mathcal{U}_k\setminus f^{-1}(\mathcal{V}_k))$. Therefore,

$$(X\setminus A) \cap \mathcal{U}_k \subseteq (X\setminus A) \cap f^{-1}(\mathcal{V}_k),$$

which completes the proof of (3.1).

Step 2: We now require that C be closed. Given $\varepsilon > 0$ by Step 1 we can choose a measurable set $B \subseteq X$ such that $m(X\setminus B) < \varepsilon/2$ and f is continuous on B. By Littlewood's First Principle we can choose a closed set $C \subseteq \mathbb{R}^n$ such that $C \subseteq B$ and $m(B\setminus C) < \varepsilon/2$. Since

$$X\setminus C = (X \setminus B) \cup (B \setminus C),$$

we have

$$m(X \setminus C) \le m(X \setminus B) + m(B \setminus C) < \varepsilon.$$

Also, since $C \subseteq B$ and f is continuous on B, the function f is automatically continuous on the smaller set C. This completes the proof of our theorem.

We shall see that Luzin's theorem holds not just for $\mathbb{R}^n$ but for topological spaces as well.

27.4 Borel Measurability on Topological Spaces

Recall that the collection of Borel subsets of a topological space is the σ-algebra generated by the open sets. For a measurable space $(T, \mathscr{S})$ where T is a topological space with $\mathscr{S}$ its Borel subsets, we call a measurable function $f: T \to \overline{\mathbb{R}}$ Borel measurable to emphasize that the σ-algebra $\mathscr{S}$ is the one generated by the topology and it is not just any σ-algebra on T. For example, a Borel measurable function on $\mathbb{R}^n$ is a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that $f^{-1}(a, \infty] \in \mathscr{B}^n$ for all $a \in \mathbb{R}$.

Proposition: Any continuous real-valued function on a topological space is Borel measurable.

The proof of this proposition follows word-for-word the $\mathbb{R}^n$ case in Example, so we omit its proof. A nice thing about Borel measurability is that it behaves well under composition.

Proposition: If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is Borel measurable and $g : X \to \mathbb{R}$ is measurable, where X is an arbitrary measurable space, then the composition,

$$f \circ g : X \to \overline{\mathbb{R}}$$

is measurable.

Proof: Given $a \in \mathbb{R}$, we need to show that

$$(f \circ g)^{-1}(a, \infty] = g^{-1}(f^{-1}(a, \infty]) \in \mathscr{S}.$$

The function $f : R \to \overline{\mathbb{R}}$ is, by assumption, Borel measurable, so $f^{-1}(a, \infty] \in \mathscr{B}^1$. The function $g : X \to \mathbb{R}$ is measurable, so by Part (3) of Theorem 3.5, $g^{-1}(f^{-1}(a, \infty]) \in \mathscr{S}$. Thus, $f \circ g$ is measurable.

Example: If $g : X \to \mathbb{R}$ is measurable, and $f : \mathbb{R} \to \mathbb{R}$ is the characteristic function of the rationals, which is Borel measurable, then Proposition 3.8 shows that the rather complicated function

$$(f \circ g)(x) = \begin{cases} 1 & \text{if } g(x) \in \mathbb{Q}, \\ 0 & \text{if } g(x) \notin \mathbb{Q}, \end{cases}$$

is measurable. Other, more normal looking, functions of g that are measurable include $e^{g(x)}$, $\cos g(x)$, and $g(x)^2 + g(x) + 1$.

27.5 The Concept of Almost Everywhere

Let $(X, \mathscr{S}, \mu)$ be a measure space. We say that a property holds almost everywhere (written a.e.) if the set of points where the property fails to hold is a measurable set with measure zero. For example, we say that a sequence of functions $\{f_n\}$ on X converges a.e. to a function f on X, written $f_n \to f$ a.e., if $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$ except on a measurable set with measure zero. Explicitly,

$$f_n \to f \text{ a.e.} \Leftrightarrow A := \{x; f(x) \neq \lim_{n\to\infty} f_n(x)\} \in \mathscr{S} \text{ and } \mu(A) = 0.$$

For another example, given two functions f and g on X, we say that $f = g$ a.e. if the set of points where $f \neq g$ is measurable with measure zero:

$$f = g \text{ a.e.} \Leftrightarrow A := \{x; f(x) \neq g(x)\} \in \mathscr{S} \text{ and } \mu(A) = 0.$$

If g is measurable and $f = g$ a.e., then one might think that f must also be measurable. However, as you'll see in the following proof, to always make this conclusion we need to assume completeness.

Proposition: Assume that μ is a complete measure and let $f, g : X \to \overline{\mathbb{R}}$. If g is measurable and $f = g$ a.e., then f is also measurable.

Proof: Assume that g is measurable and $f = g$ a.e., so that the set $A = \{x; f(x) \neq g(x)\}$ is measurable with measure zero. Observe that for any $a \in \mathbb{R}$,

$$\begin{aligned} f^{-1}(a, \infty] &= \{x \in X; f(x) > a\} \\ &= \{x \in A; f(x) > a\} \cup \{x \in A^c; f(x) > a\} \\ &= \{x \in A; f(x) > a\} \cup \{x \in A^c; g(x) > a\} \\ &= \{x \in A; f(x) > a\} \cup (A^c \cap g^{-1}(a, \infty]). \end{aligned}$$

The first set is a subset of A, which is measurable and has measure zero, hence the first set is measurable. g is measurable, so the second set is measurable too, hence f is measurable.

For instance, this proposition holds for Lebesgue measure since Lebesgue measure is complete.

Notes

Self Assessment

Fill in the blanks:

1. A measurable space is a pair (X, $\mathscr{S}$) where X is a set and, $\mathscr{S}$ is a σ-algebra of subsets of X. The elements of, $\mathscr{S}$ are called ....
2. For Lebesgue measure, continuity implies ....
3. A function $f : T \to \mathbb{R}$ is continuous if and only if $f^{-1}(\mathcal{U}) \in \mathscr{T}$ for each open set ....
4. Measurable functions are similar to ...., but there are more of them and they are more robust.
5. .... principle shows exactly how "similar" measurable functions are to continuous functions.
6. Any continuous real-valued function on a topological space is ....
7. If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is Borel measurable and $g : X \to \mathbb{R}$ is measurable, where X is an arbitrary measurable space, then the composition, ... is measurable.

27.6 Summary

- A measurable space is a pair (X, $\mathscr{S}$) where X is a set and, $\mathscr{S}$ is a σ-algebra of subsets of X. The elements of, $\mathscr{S}$ are called measurable sets. Recall that a measure space is a triple (X, $\mathscr{S}$, μ) where μ is a measure on $\mathscr{S}$; if we leave out the measure we have a measurable space.

  In the discussion at the beginning of this chapter we saw that in order to define the integral of a function $f : X \to \overline{\mathbb{R}}$, we needed to require that

  $$f^{-1}(I) \in \mathscr{S} \text{ for each } I \in \mathscr{I}^1 \text{ and } f^{-1}[a, \infty] \in \mathscr{S} \text{ for each } a \in \mathbb{R}.$$

- If these properties hold, we say that f is measurable. It turns out that we can omit the first condition because it follows from the second. Indeed, since

  $$f^{-1}[a, b] = f^{-1}[a, \infty] \setminus f^{-1}[b, \infty].$$

- There are three principles, roughly expressible in the following terms: Every [finite Lebesgue] measurable set is nearly a finite union of intervals; every measurable function is nearly continuous; every convergent sequence of measurable functions is nearly uniformly convergent.
- The third principle is contained in Egorov's theorem, which we'll get to in the next section. The second principle comes from Luzin's Theorem, named after Nikolai Nikolaevich Luzin (1883-1950) who proved it in 1912 [70], and this theorem makes precise Littlewood's comment that any Lebesgue measurable function is "nearly continuous".
- Any continuous real-valued function on a topological space is Borel measurable.
- The proof of this proposition follows word-for-word the $\mathbb{R}^n$ case in Example, so we omit its proof. A nice thing about Borel measurability is that it behaves well under composition.
- If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is Borel measurable and $g : X \to \mathbb{R}$ is measurable, where X is an arbitrary measurable space, then the composition,

  $$f \circ g : X \to \overline{\mathbb{R}}$$

  is measurable.

27.7 Keywords

Notes

Measurable Sets: A measurable space is a pair $(X, \mathscr{S})$ where X is a set and, $\mathscr{S}$ is a σ-algebra of subsets of X. The elements of, $\mathscr{S}$ are called measurable sets.

Measurability Criterion: For a function $f: X \to \overline{\mathbb{R}}$, the following are equivalent:

1. f is measurable.
2. $f^{-1}(\{\infty\}) \in \mathscr{S}$ and $f^{-1}(\mathcal{U}) \in \mathscr{S}$ for all open subsets $\mathcal{U} \subseteq \mathbb{R}$.
3. $f^{-1}(\{\infty\}) \in \mathscr{S}$ and $f^{-1}(B) \in \mathscr{S}$ for all Borel sets $B \subseteq \mathbb{R}$.

Measurable: A function $f : X \to \mathbb{R}$ is measurable if and only if $f^{-1}(\mathcal{U}) \in \mathscr{S}$ for each open set $\mathcal{U} \subseteq \mathbb{R}$.

One cannot avoid noticing the striking resemblance to the definition of continuity. Recall that for a topological space $(T, \mathscr{T})$, where $\mathscr{T}$ is the topology on a set T.

Luzin's Theorem: Let $X \subseteq \mathbb{R}^n$ be Lebesgue measurable and let $f : X \to \mathbb{R}$ be a Lebesgue measurable function. Then given any $\varepsilon > 0$, there exists a closed set $C \subseteq \mathbb{R}^n$ such that $C \subseteq X$, $m(X \setminus C) < \varepsilon$, and f is continuous on C.

Borel Measurable: Any continuous real-valued function on a topological space is Borel measurable.

27.8 Review Questions

1. (a) Prove that a non-negative function f is measurable if and only if for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $0 \le k \le 2^{2n} - 1$, the sets $f^{-1}(k/2^n, (k+1)/2^n]$ and $f^{-1}(2^n, \infty]$, are measurable.

   (b) Prove that an extended real-valued function f is measurable if and only if $f^{-1}(\{\infty\})$ and all sets of the form $f^{-1}(k/2^n, (k+1)/2^n]$, where $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, are measurable.

   (c) If $\{a_n\}$ is any countable dense subset of $\mathbb{R}$, prove that f is measurable if and only if $f^{-1}(\{\infty\})$ and all sets of the form $f^{-1}(a_m, a_n]$, where $m, n \in \mathbb{N}$, are measurable.

2. Here are some problems dealing with non-measurable functions.

   (a) Find a non-Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $|f|$ is measurable.

   (b) Find a non-Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ such that f^2 is measurable.

   (c) Find two non-Lebesgue measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ such that both $f + g$ and $f \cdot g$ are measurable.

3. Here are some problems dealing with measurable functions.

   (a) Prove that any monotone function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

   (b) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be lower-semicontinuous at a point $c \in \mathbb{R}$ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that

   $$|x - c| < \delta \Rightarrow f(c) - \varepsilon < f(x).$$

   Intuitively, f is lower-semicontinuous at c if for x near c, f(x) is either near f(c) or greater than f(c). The function f is lower-semicontinuous if it's lower-semicontinuous at all points of $\mathbb{R}$. (To get a feeling for lower-semicontinuity, show that the functions $\chi_{(0,\infty)}$, $\chi_{(-\infty,0)}$, and $\chi_{(-\infty,0)\cup(0,\infty)}$ are lower-semicontinuous at 0.) Prove that any lower-semicontinuous function is Lebesgue measurable.

   (c) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be upper-semicontinuous at a point $c \in \mathbb{R}$ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that

   $$|x - c| < \delta \Rightarrow f(x) < f(c) + \varepsilon.$$

Notes

Intuitively, f is upper-semicontinuous at c if for x near c, f(x) is either near f(c) or less than f(c). The function f is upper-semicontinuous if it's upper-semicontinuous at all points of $\mathbb{R}$. Prove that any upper-semicontinuous function is Lebesgue measurable.

4. We can improve Luzin's Theorem as follows. First prove the

   (i) Tietze Extension Theorem for $\mathbb{R}$; named after Heinrich Tietze (1880-1964) who proved a general result for metric spaces in 1915 [98]. Let $A \subseteq \mathbb{R}$ be a non-empty closed set and let $f_0 : A \to \mathbb{R}$ be a continuous function. Prove that there is a continuous function $f_1 : \mathbb{R} \to \mathbb{R}$ such that $f_1|_A = f_0$, and if f_0 is bounded in absolute value by a constant M, then we may take f_1 to the have the same bound. Suggestion: Show that $\mathbb{R}\backslash A$ is a countable union of pairwise disjoint open intervals. Extend f_0 linearly over each of the open intervals to define f_1.

   (ii) Using Luzin's Theorem for n = 1, given a measurable function $f : X \to \mathbb{R}$ where $X \subseteq \mathbb{R}$ is measurable, prove that there is a closed set $C \subseteq \mathbb{R}$ such that $C \subseteq X$, $\mathfrak{m}(X\backslash C) < \varepsilon$, and a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that f = g on C. Moreover, if f is bounded in absolute value by a constant M, then we may take g to have the same bound as f.

5. Here are some generalizations of Luzin's Theorem.

   (i) Let μ be a σ-finite regular Borel measure on a topological space X, let $f : X \to \mathbb{R}$ be measurable, and let $\varepsilon > 0$. On "Littlewood's First Principle(s) for regular Borel measures," prove that there exists a closed set $C \subseteq X$ such that $\mathfrak{m}(X\backslash C) < \varepsilon$ and f is continuous on C.

6. Here we present Leonida Tonelli's (1885-1946) integral published in 1924 [100]. Let f : [a, b] $\to \mathbb{R}$ be a bounded function, say $|f| \le M$ for some constant M. f is said to be quasi-continuous (q.c.) if there is a sequence of closed sets $C_1, C_2, C_3, \ldots \subseteq [a, b]$ with $\lim_{n\to\infty} \mathfrak{m}(C_n) = b - a$ and a sequence of continuous functions $f_1, f_2, f_3, \ldots$ where for each n, $f_n : [a, b] \to \mathbb{R}$, $f = f_n$ on C_n, and $|f_n| \le M$.

   (i) Let $f : [a, b] \to \mathbb{R}$ be bounded. Prove that f is q.c. if and only if f is measurable. To prove the "if" statement, use Problem 6.

   (ii) Let $f : [a, b] \to \mathbb{R}$ be q.c. and let $\{f_n\}$ be a sequence of continuous functions in the definition of q.c. for f. Let $R(f_n)$ denote the Riemann integral of f_n and prove that the limit $\lim_{n\to\infty} R(f_n)$ exists and its value is independent of the choice of sequence $\{f_n\}$ in the definition of q.c. for f. Tonelli defines the integral of f as

$$\int_a^b f := \lim_{n\to\infty} R(f_n).$$

   It turns out that Tonelli's integral is exactly the same as Lebesgue's integral.

7. We show that the composition of two Lebesgue measurable function is not necessarily Lebesgue measurable. Let φ and M be the homeomorphism and Lebesgue measurable set, respectively. Let $g = \chi_M$. Show that $g \circ \varphi^{-1}$ is not Lebesgue measurable. Note that both φ^{-1} and g are Lebesgue measurable.

8. Prove the Banach-Sierpinski Theorem, proved in 1920 by Stefan Banach (1892-1945) and Waclaw Sierpinski (1882-1969), which states that if $f : \mathbb{R} \to \mathbb{R}$ is additive and Lebesgue measurable, then f(x) = f(1)x for all $x \in \mathbb{R}$. Suggestion: Observe that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R}; |f(x)| \le n\}.$$

   Prove that for some $n \in \mathbb{N}$, the set $\{x \in \mathbb{R};\ |f(x)| \le n\}$ has positive measure.

Answers: Self Assessment

Notes

1. measurable sets
2. measurability
3. $\mathcal{U} \subseteq \mathbb{R}$
4. continuous functions
5. Littlewood's second
6. Borel measurable
7. $f \circ g : X \to \overline{\mathbb{R}}$

27.9 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10, Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Notes

Unit 28: Sequences of Functions and Littlewood's Third Principle

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the limsups and liminfs of sequences
- Describe operations on measurable functions
- Explain Littlewood's third principle

Introduction

In this unit we continue our study of measurability. We show that measurable functions are very robust in the sense that they are closed under just about any kind of arithmetic or limiting operation that you can imagine: addition, multiplication, division,…, and most importantly, they are closed under just about any conceivable limiting process. We also discuss Littlewood's third principle on limits of measurable functions.

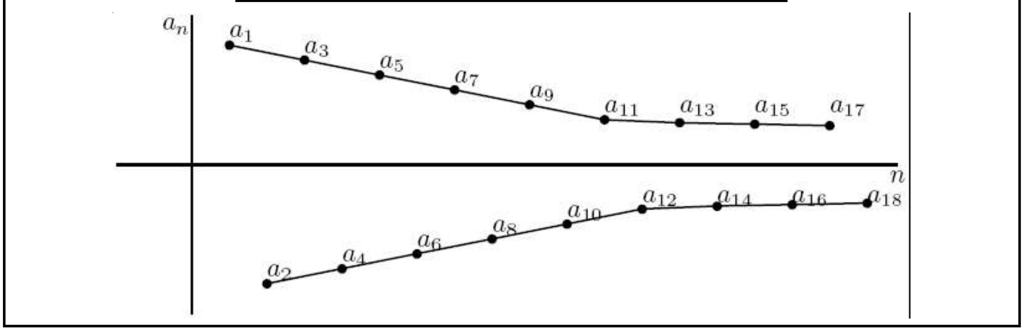
28.1 Limsups and Liminfs of Sequences

Before discussing limits of sequences of functions we need to start by talking about limits of sequences of extended real numbers.

For a sequence $\{a_n\}$ of extended real numbers, we know, in general, that lim a_n does not exist; for example, it can oscillate such as the sequence. However, for the sequence, assuming that the sequence continues the way it looks like it does, it is clear that although limit lim a_n does not exist, the sequence does have an "upper" limiting value, given by the limit of the odd-indexed a_n's and a "lower" limiting value, given by the limit of the even-indexed a_n's. Now, how do we find the "upper" (also called "supremum") and "lower" (also called "infimum") limits of $\{a_n\}$? It turns out there is a very simple way to do so, as we now explain.

Notes

Figure 28.1: A Sequence Bouncing Up and Down



Given an arbitrary sequence $\{a_n\}$ of extended real numbers, put

$$s_1 = \sup_{k\geq 1} a_k = \sup\{a_1, a_2, a_3,...\},$$

$$s_2 = \sup_{k\geq 2} a_k = \sup\{a_2, a_3, a_4,...\},$$

$$s_3 = \sup_{k\geq 3} a_k = \sup\{a_3, a_4, a_5,...\},$$

and in general,

$$s_n = \sup_{k\geq n} a_k = \sup\{a_n, a_{n+1}, a_{n+2},...\}.$$

Note that

$$s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq s_{n+1} \geq \cdots$$

is an non-increasing sequence since each successive s_n is obtained by taking the supremum of a smaller set of elements. Since $\{s_n\}$ is an non-increasing sequence of extended real numbers, the limit $\lim s_n$ exists in $\overline{\mathbb{R}}$; in fact,

$$\lim s_n = \inf_n s_n = \inf\{s_1, s_2, s_3,...\},$$

as can be easily be checked. We define the lim sup of the sequence $\{a_n\}$ as

$$\limsup a_n := \inf_n s_n = \lim s_n = \lim_{n\to\infty} (\sup\{a_n, a_{n+1}, a_{n+2},...\})$$

Note that the term "lim sup" of $\{a_n\}$ fits well because $\limsup a_n$ is exactly the limit of a sequence of supremums.

Example: For the sequence a_n shown in Figure 28.1, we have

$$s_1 = a_1, \quad s_2 = a_3, \quad s_3 = a_3, \quad s_4 = a_5, \quad s_5 = a_5,...,$$

so $\limsup a_n$ is exactly the limit of the odd-indexed a_n's.

We now define the "lower" or "infimum" limit of an arbitrary sequence $\{a_n\}$. Put

$$\iota_1 = \sup_{k\geq 1} a_k = \inf\{a_1, a_2, a_3,...\},$$

$$\iota_2 = \sup_{k\geq 2} a_k = \inf\{a_2, a_3, a_4,...\},$$

$$\iota_3 = \sup_{k\geq 3} a_k = \inf\{a_3, a_4, a_5,...\},$$

Notes

and in general,

$$\iota_n = \sup_{k \geq n} a_k = \inf\{a_n, a_{n+1}, a_{n+2},...\}.$$

Note that

$$\iota_1 \leq \iota_2 \leq \iota_3 \leq \cdots \leq \iota_n \leq \iota_{n+1} \leq \cdots$$

is an non-decreasing sequence since each successive ι_n is obtained by taking the infimum of a smaller set of elements. Since $\{\iota_n\}$ is an non-decreasing sequence, the limit lim ι_n exists, and equals supn ι_n. We define the lim inf of the sequence $\{a_n\}$ as

$$\liminf a_n := \sup_n \iota_n = \lim \iota_n = \lim_{n\to\infty} (\inf\{a_n, a_{n+1}, a_{n+2},...\}).$$

Note that the term "lim inf" of $\{a_n\}$ fits well because lim inf a_n is the limit of a sequence of infimums.

Example: For the sequence a_n shown in Figure 28.1, we have

$$\iota_1 = a_2, \quad \iota_2 = a_2, \quad \iota_3 = a_4, \quad \iota_4 = a_4, \quad \iota_5 = a_6,...,$$

so lim inf a_n is exactly the limit of the even-indexed a_n's.

The following lemma contains some useful properties of limsup's and liminf's. Since its proof really belongs in a lower-level analysis.

Lemma: Let $A \subseteq \overline{\mathbb{R}}$ be non-empty and let $\{a_n\}$ be a sequence of extended real numbers.

1. sup A = –inf(–A) and inf A = –sup(–A), where –A = {–a; a ∈ A}.
2. lim sup a_n = –lim inf($-a_n$) and lim inf a_n = –lim sup($-a_n$).
3. lim a_n exists as an extended real number if and only if lim sup a_n = lim inf a_n, in which case,

$$\lim a_n = \limsup a_n = \liminf a_n.$$

4. If $\{b_n\}$ is another sequence of extended real numbers and $a_n \leq b_n$ for all n sufficiently large, then

$$\liminf a_n \leq \liminf b_n \quad \text{and} \quad \limsup a_n \leq \limsup b_n.$$

28.2 Operations on Measurable Functions

Let $\{f_n\}$ be a sequence of extended-real valued functions on a measure space $(X, \mathscr{S}, \mu)$. We define the functions sup f_n, inf f_n, lim sup f_n, and lim inf f_n, by applying these limit operations pointwise to the sequence of extended real numbers $\{f_n(x)\}$ at each point $x \in X$. For example,

$$\limsup f_n : X \to \overline{\mathbb{R}}$$

is the function defined by

$$(\limsup f_n)(x) := \limsup(f_n(x)) \quad \text{at each } x \in X.$$

We define the limit function lim f_n by

$$(\lim f_n)(x) := \lim_{n\to\infty} (f_n(x))$$

at those points $x \in X$ where the right-hand limit exists.

We now show that limiting operations don't change measurability.

Theorem 1: Limits preserve measurability

If $\{f_n\}$ is a sequence of measurable functions, then the functions

$$\sup f_n, \quad \inf f_n, \quad \limsup f_n, \quad \text{and} \quad \liminf f_n$$

are all measurable. If the limit $\lim_{n\to\infty} f_n(x)$ exists at each $x \in X$, then the limit function $\lim f_n$ is measurable. For instance, if the sequence $\{f_n\}$ is monotone, that is, either non-decreasing or non-increasing, then $\lim f_n$ is everywhere defined and it is measurable.

Proof: To prove that $\sup f_n$ is measurable, we just have to show that $(\sup f_n)^{-1}[-\infty, a] \in \mathscr{S}$ for each $a \in \mathbb{R}$. However, this is easy because by definition of supremum, for any $a \in \mathbb{R}$,

$$\sup\{f_1(x), f_2(x), f_3(x),...\} \leq a \quad \Leftrightarrow \quad f_n(x) \leq a \text{ for all } n,$$

therefore

$$(\sup f_n)^{-1}[-\infty, a] = \{x; \sup f_n(x) \leq a\} = \bigcap_{n=1}^{\infty} \{x; f_n(x) \leq a\}$$

$$= \bigcap_{n=1}^{\infty} f_n^{-1}[-\infty, a].$$

Since each f_n is measurable, we have $f_n^{-1}[-\infty, a] \in \mathscr{S}$, so $(\sup f_n)^{-1}[-\infty, a] \in \mathscr{S}$ as well. Using an analogous argument one can show that $\inf f_n$ is measurable.

To prove that $\limsup f_n$ is measurable, note that by definition of lim sup,

$$\limsup f_n := \inf_n s_n,$$

where $s_n = \sup_{k\geq n} f_k$. Since the sup and inf of a sequence of measurable functions are measurable, we know that s_n is measurable for each n and hence $\limsup f_n = \inf_n s_n$ is measurable. An analogous argument can be used to show that $\liminf f_n$ is measurable (just note that $\liminf f_n = \sup \iota_n$ where $\iota_n = \inf_{k\geq n} f_k$).

If the limit function $\lim f_n$ is well-defined, then by Part (3) of above Lemma we know that $\lim f_n = \limsup f_n$ ($= \liminf f_n$). Thus, $\lim f_n$ is measurable.

In particular, if f is a function on X and if $f = \lim s_n$, where the s_n's are simple function (which are measurable by Theorem 3.4), then f is measurable.

Example: Let $X = S^\infty$, where $S = \{0,1\}$, a sample space for the Monkey-Shakespeare experiment (or any other sequence of Bernoulli trials), and let $f : X \to [0, \infty]$ be the random variable given by the number of times the Monkey types sonnet 18. Then

$$f(x) = \sum_{n=1}^{\infty} x_n$$

That is,

$$f = \sum_{n=1}^{\infty} \chi_{A_n} = \lim_{n\to\infty} \sum_{k=1}^{n} \chi_{A_k},$$

where $A_n = S \times S \times \cdots \times S \times \{1\} \times S \times S \times \cdots$ where $\{1\}$ is in the n-th slot. Since each A_n is measurable, it follows that each χ_{A_n} is measurable and hence so is f.

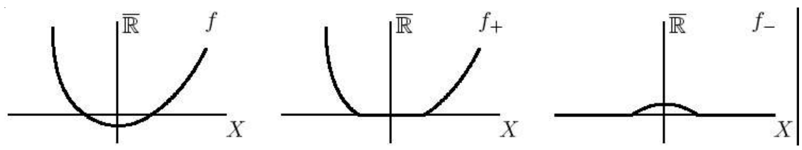
Given $f: X \to \overline{\mathbb{R}}$, we define its non-negative part $f_+: X \to [0, \infty]$ and its non-positive part $f_-: X \to [0, \infty]$ by

$$f_+ := \max\{f, 0\} = \sup\{f, 0\} \quad \text{and} \quad f_- := -\min\{f, 0\} = -\inf\{f, 0\}.$$

Notes

See Figure 28.2 for graphs of $f_{\pm}$. One can check that

Figure 28.2: Graphs of a Function f, f+, and f–



$$f = f_+ - f_- \quad \text{and} \quad |f| = f_+ + f_-.$$

Assuming f is measurable, f_+ and $-f_-$ are also measurable. Also, since measurability is preserves under scalar multiplication $f_- = -(-f_-)$ is measurable. In particular, the equality $f = f_+ - f_-$ shows that any measurable function can be expressed as the difference of non-negative measurable functions.

Theorem 2: Characterization of measurability

A function is measurable if and only if it is the limit of simple functions. Moreover, if the function is nonnegative, the simple functions can be taken to be a non-decreasing sequence of non-negative simple functions.

Proof: Consider first the non-negative case. Let f: X → [0, ∞] be measurable. For each n ∈ ℕ, consider the simple function that we constructed at the very beginning of this chapter:

$$s_n(x) = \begin{cases} 0 & \text{if } 0 \le f(x) \le \frac{1}{2^n} \\ \frac{1}{2^n} & \text{if } \frac{1}{2^n} < f(x) \le \frac{2}{2^n} \\ \frac{2}{2^n} & \text{if } \frac{2}{2^n} < f(x) \le \frac{3}{2^n} \\ \vdots & \vdots \\ \frac{2^{2n}-1}{2^n} & \text{if } \frac{2^{2n}-1}{2^n} < f(x) \le \frac{2^{2n}}{2^n} = 2^n \\ 2^n & \text{if } f(x) > 2^n. \end{cases}$$

See Figure 28.3 for an example of a function f and pictures of the corresponding s_1, s_2, and s_3. Note that s_n is a simple function because we can write

$$s_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \chi_{A_{nk}} + 2^n \chi_{B_n},$$

where

$$A_{nk} = f^{-1}\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right] \quad \text{and} \quad B_n = f^{-1}(2_n, \infty].$$

At least if we look at Figure 28.3, it is not hard to believe that in general, the sequence $\{s_n\}$ is always non-decreasing:

$$0 \le s_1 \le s_2 \le s_3 \le s_4 \le \cdots$$

and $\lim_{n\to\infty} s_n(x) = f(x)$ at every point x ∈ X. Because this is so believable looking at Figure, we leave you the pleasure of verifying these facts.

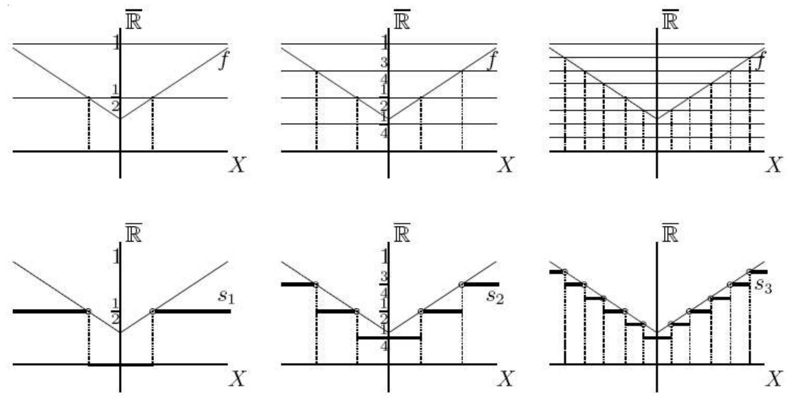
Now let f: X → $\overline{\mathbb{R}}$ be any measurable function; we need to show that f is the limit of simple functions. To prove this, write $f = f_+ - f_-$ as the difference of its non-negative and non-positive parts. Since $f_{\pm}$ are non-negative measurable functions, we know that f_+ and f_- can be written as limits of simple functions, say s_n^+ and s_n^-, respectively. It follows that

$$f = f_+ - f_- = \lim(s_n^+ - s_n^-)$$

is also a limit of simple functions.

Notes

Figure 28.3



Here, f looks like a "V" and is bounded above by 1. The top figures show partitions of the range of f into halves, quarters, then eights and the bottom figures show the corresponding simple functions. It is clear that $s_1 \le s_2 \le s_3$.

Using Theorem 2 on limits of simple functions, it is easy to prove that measurable functions are closed under all the usual arithmetic operations. Of course, the proofs aren't particularly difficult to prove directly.

Theorem 3: If f and g are measurable, then $f + g$, $f \cdot g$, $1/f$, and $|f|^p$ where $p > 0$, are also measurable, whenever each expression is defined.

Proof: We need to add the last statement for f + g and 1/f. For 1/f we need f to never vanish and for f + g we don't want f(x) + g(x) to give a non-sense statement such as $\infty - \infty$ or $-\infty + \infty$ at any point $x \in X$.

The proofs that f + g, f g, 1/f, and $|f|^p$ are measurable are all the same: we just show that each combination can be written as a limit of simple functions. By Theorem 2 we can write $f = \lim s_n$ and $g = \lim t_n$ for simple functions s_n, t_n, n = 1, 2, 3, ... Therefore,

$$f + g = \lim(s_n + t_n)$$

and

$$f\,g = \lim(s_n t_n).$$

Since the sum and product of simple functions are simple, it follows that f + g and f g are limits of simple functions, so are measurable.

To see that 1/f and $|f|^p$ are measurable, write the simple function s_n as a finite sum

$$s_n = \sum_k a_{nk} \chi_{A_{nk}},$$

where $A_{n1}, A_{n2}, \ldots \in \mathscr{S}$ are finite in number and pairwise disjoint, and $a_{n1}, a_{n2}, \ldots \in \mathbb{R}$, which we may assume are all non-zero. If we define

$$u_n = \sum_k a_{nk}^{-1} \chi_{A_{nk}} \quad \text{and} \quad v_n = \sum_k |a_{nk}|^p \chi_{A_{nk}},$$

which are simple functions, then a short exercise shows that

$$f^{-1} = \lim u_n \quad \text{and} \quad |f|^p = \lim v_n,$$

where in the first equality we assume that f is nonvanishing. This shows that f^{-1} and $|f|^p$ are measurable.

In particular, since products and reciprocals of measurable functions are measurable, whenever the reciprocal is well-defined, it follows that quotients of measurable functions are measurable, whenever the denominator is nonvanishing.

28.3 Littlewood's Third Principle

We finally come to the third of Littlewood's principles, which is

Every convergent sequence of [real-valued] measurable functions is nearly uniformly convergent, or, more precisely, in the words of Lebesgue who in 1903 stated this principle as.

Every convergent series of measurable functions is uniformly convergent when certain sets of measure ε are neglected, where ε can be as small as desired.

Lebesgue here is introducing the idea which is nowadays called "convergence almost uniformly." A sequence $\{f_n\}$ of measurable functions is said to converge almost uniformly (or "a.u." for short) to a measurable function f, denoted by

$$f_n \to f \text{ a.u.},$$

if for each $\varepsilon > 0$, there exists a measurable set A such that $\mu(A) < \varepsilon$ and $f_n \to f$ uniformly on $A^c = X \setminus A$. As a quick review, recall that $f_n \to f$ uniformly on A^c means that given any $\delta > 0$,

$$|f_n(x) - f(x)| < \delta, \quad \text{for all } x \in A^c \text{ and n sufficiently large.}$$

Note that $f_n(x)$ and f(x) are necessarily real-valued (cannot take on $\pm\infty$) on A^c. Therefore, Lebesgue is saying that

Every convergent sequence of real-valued measurable functions is almost uniformly convergent.

The following theorem, although stated by Lebesgue in 1903, is named after Dimitri Fedorovich Egorov (1869-1931) who proved it in 1911[34].

Theorem 4: Egorov's Theorem

On a finite measure space, a.e. convergence implies a.u. convergence for real-valued measurable functions. That is, any sequence of real-valued measurable functions that converges a.e. to a real-valued measurable function converges a.u. to that function.

Proof: Let f, f_1, f_2, f_3,... be real-valued measurable functions on a measure space X with $\mu(X) < \infty$, and assume that $f = \lim f_n$ a.e, which means there is a measurable set $A \subseteq X$ with $\mu(X \setminus A) = 0$ and $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in A$. We need to show that $f_n \to f$ a.u.

Step 1: Given $\varepsilon, \eta > 0$ we shall prove that there is a measurable set $B \subseteq X$ and an $N \in \mathbb{N}$ such that

(3.3) $$\mu(B) < \eta \quad \text{and} \quad \text{for } x \in B^c, \quad |f(x) - f_n(x)| < \varepsilon \text{ for all } n > N.$$

Indeed, for each $m \in \mathbb{N}$, put

$$B_m := \bigcup_{n \geq m} \{x \in X; |f(x) - f_n(x)| \geq \varepsilon\}$$

Notice that each B_m is measurable and $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$. Also, since for all $x \in A$, we have $f_n(x) \to f(x)$ as $n \to \infty$, it follows that if $x \in A$, then $|f(x) - f_n(x)| < \varepsilon$ for all n sufficiently large. Thus, there is an m such that $x \notin B_m$, and so, $x \in A \Rightarrow x \notin B_m$ for some m. Taking contrapositives we see that $x \in B_m$ for all $m \Rightarrow x \notin A$, which is to say,

$$\bigcap_{m=1}^{\infty} B_m \subseteq X \setminus A.$$

Thus, $\mu\left(\bigcap_{m=1}^{\infty} B_m\right) \leq \mu(X \setminus A) = 0$ and therefore, since X is a finite measure space, by continuity of measures (from above), we have

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$$\lim_{m\to\infty} \mu(B_m) = 0.$$

Choose N such that $\mu(B_N) < \eta$ and let $B = B_N$. Then by definition of B_N, one can check that holds. This concludes Step 1.

Step 2: We now finish the proof. Let $\varepsilon > 0$. Then by Step 1, for each $k \in \mathbb{N}$ we can find a measurable set $A_k \subseteq X$ and a corresponding natural number $N_k \in \mathbb{N}$ such that

$$\mu(A_k) < \frac{\varepsilon}{2^k} \quad \text{and} \quad \text{for } x \in A_k^c, \quad |f(x) - f_n(x)| < \frac{1}{k} \text{ for all } n > N_k.$$

Now put $A = \bigcup_{k=1}^{\infty} A_k$. Then $\mu(A) < \varepsilon$ and we claim that $f_n \to f$ uniformly on A^c. Indeed, let $\delta > 0$ and choose $k \in \mathbb{N}$ such that $1/k < \eta$. Then

$$x \in A^c = \bigcap_{j=1}^{\infty} A_j^c \Rightarrow x \in A_k^c$$

$$\Rightarrow |f(x) - f_n(x)| < \frac{1}{k} \text{ for all } n > N_k$$

$$\Rightarrow |f(x) - f_n(x)| < \eta \text{ for all } n > N_k.$$

Thus, $f_n \to f$ a.u.

We remark that one cannot drop the finiteness assumption.

Self Assessment

Fill in the blanks:

1. Let $\{f_n\}$ be a sequence of .... on a measure space $(X, \mathscr{S}, \mu)$. We define the functions sup f_n, inf f_n, lim sup f_n, and lim inf f_n, by applying these limit operations pointwise to the sequence of extended real numbers $\{f_n(x)\}$ at each point $x \in X$.
2. If the sequence $\{f_n\}$ is ...., that is, either non-decreasing or non-increasing, then lim f_n is everywhere defined and it is measurable.
3. If f and g are measurable, then ...., and $|f|^p$ where $p > 0$, are also measurable, whenever each expression is defined.
4. Every convergent series of measurable functions is .... when certain sets of measure ε are neglected, where ε can be as small as desired.

28.4 Summary

- For a sequence $\{a_n\}$ of extended real numbers, we know, in general, that lim a_n does not exist; for example, it can oscillate. Assuming that the sequence continues the way it looks like it does, it is clear that although limit lim a_n does not exist, the sequence does have an "upper" limiting value, given by the limit of the odd-indexed a_n's and a "lower" limiting value, given by the limit of the even-indexed a_n's. Now how do we find the "upper" (also called "supremum") and "lower" (also called "infimum") limits of $\{a_n\}$? It turns out there is a very simple way to do so, as we now explain.
- Let $\{f_n\}$ be a sequence of extended-real valued functions on a measure space $(X, \mathscr{S}, \mu)$. We define the functions sup f_n, inf f_n, lim sup f_n, and lim inf f_n, by applying these limit operations pointwise to the sequence of extended real numbers $\{f_n(x)\}$ at each point $x \in X$. For example,

Notes

$$\limsup f_n : X \to \overline{\mathbb{R}}$$

is the function defined by

$(\limsup f_n)(x) := \limsup(f_n(x))$ at each $x \in X$.

- If $\{f_n\}$ is a sequence of measurable functions, then the functions

$$\sup f_n, \quad \inf f_n, \quad \limsup f_n, \quad \text{and} \quad \liminf f_n$$

are all measurable. If the limit $\lim_{n\to\infty} f_n(x)$ exists at each $x \in X$, then the limit function $\lim f_n$ is measurable. For instance, if the sequence $\{f_n\}$ is monotone, that is, either non-decreasing or non-increasing, then $\lim f_n$ is everywhere defined and it is measurable.

- A function is measurable if and only if it is the limit of simple functions. Moreover, if the function is non-negative, the simple functions can be taken to be a non-decreasing sequence of non-negative simple functions.

28.5 Keywords

Limits Preserve Measurability: If $\{f_n\}$ is a sequence of measurable functions, then the functions

$$\sup f_n, \quad \inf f_n, \quad \limsup f_n, \quad \text{and} \quad \liminf f_n$$

are all measurable.

Characterization of Measurability: A function is measurable if and only if it is the limit of simple functions. Moreover, if the function is nonnegative, the simple functions can be taken to be a non-decreasing sequence of nonnegative simple functions.

Uniformly Convergent: Every convergent sequence of real-valued measurable functions is almost uniformly convergent.

Egorov's Theorem: On a finite measure space, a.e. convergence implies a.u. convergence for real-valued measurable functions.

28.6 Review Questions

1. Let $A_1, A_2, \ldots$ be measurable sets and put

$$\limsup A_n := \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \quad \text{and} \quad \liminf A_n := \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k.$$

Let $\bar{f}$ and $\underline{f}$ be the characteristic functions of limsup A_n and liminf A_n, respectively, and for each n, let f_n be the characteristic function of A_n. Prove that

$$\bar{f} = \limsup f_n \quad \text{and} \quad \underline{f} = \liminf f_n.$$

(i) First prove the theorem for simple functions. Suggestion: Let f be a simple function and write $f = \sum_{k=1}^{N} a_k \chi_{A_k}$ where $X = \bigcup_{k=1}^{N} A_k$, the a_k's are real numbers, and the A_k's are pairwise disjoint measurable sets. Given $\varepsilon > 0$, there is a closed set $C_k \subseteq \mathbb{R}^n$ with $\mathfrak{m}(A_k \setminus C_k) < \varepsilon/N$ (why?). Let $C = \bigcup_{k=1}^{N} C_k$.

(ii) We now prove Luzin's theorem for non-negative f. For nonnegative f we know that $f = \lim f_k$ where each f_k, $k \in \mathbb{N}$, is a simple function. By (i), given $\varepsilon > 0$ there is a closed set C_k such that $\mathfrak{m}(X \setminus C_k) < \varepsilon/2^k$ and f_k is continuous on C_k.

Let $K_1 = \bigcap_{k=1}^{\infty} C_k$. Show that $\mathfrak{m}(X \setminus K_1) < \varepsilon$. Use Egorov's theorem to show that there exists a set $K_2 \subseteq K_1$ with $\mathfrak{m}(K_1 \setminus K_2) < \varepsilon$ and $f_k \to f$ uniformly on K_2. Conclude that f is continuous on K_2.

(iii) Now find a closed set $C \subseteq K_2$ such that $m(K_2 \backslash C) < \varepsilon$. Show that $m(X \backslash C) < 3\varepsilon$ and the restriction of f to C is a continuous function.

(iv) Finally, prove Luzin's theorem dropping the assumption that f is non-negative.

2. A sequence $\{f_n\}$ of real-valued measurable functions is said to be convergent in measure if there is a measurable function f such that for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x; |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(Does this remind you of the weak law of large numbers?) Prove that if $\{f_n\}$ converges in measure to a measurable function f, then f is a.e. real-valued, which means $\{x; f(x) = \pm\infty\}$ is measurable with measure zero. If $\{f_n\}$ converges to two functions f and g in measure, prove that f = g a.e. ***Suggestion:*** To see that f = g a.e., prove and then use the "set-theoretic triangle inequality": For any real-valued measurable functions f, g, h, we have

$$\{x;\ |f(x) - g(x)| \geq \varepsilon\} \subseteq \left\{x;\ |f(x) - h(x)| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x;\ |h(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}.$$

3. Here are some relationships between convergence a.e., a.u., and in measure.

(a) (a.u. $\Rightarrow$ in measure) Prove that if $f_n \to f$ a.u., then $f_n \to f$ in measure.

(b) (a.e. $\Rightarrow$ in measure) From Egorov's theorem prove that if X has finite measure, then any sequence $\{f_n\}$ of real-valued measurable functions that converges a.e. to a real-valued measurable function f also converges to f in measure.

(c) (In measure $\not\Rightarrow$ a.u. nor a.e.) Let X = [0,1] with Lebesgue measure. Given $n \in \mathbb{N}$, write $n = 2^k + i$ where k = 0, 1, 2,... and $0 \leq i < 2^k$, and let f_n be the characteristic function of the interval $\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]$. Draw pictures of $f_1, f_2, f_3, \ldots, f_7$. Show that $f_n \to 0$ in measure, but $\lim_{n\to\infty} f_n(x)$ does not exist for any $x \in [0, 1]$. Conclude that $\{f_n\}$ does not converge to f a.u. nor a.e.

4. A sequence $\{f_n\}$ of real-valued, measurable functions is said to be Cauchy in measure if for any $\varepsilon > 0$,

$$\mu\left(\{x;\ |f_n(x) - f_m(x)| \geq \varepsilon\}\right) \to 0, \quad \text{as } n, m \to \infty.$$

Prove that if $f_n \to f$ in measure, then $\{f_n\}$ is Cauchy in measure.

5. In this problem we prove that if a sequence $\{f_n\}$ of real-valued measurable functions is Cauchy in measure, then there is a subsequence $\{f_{n_k}\}$ and a real-valued measurable function f such that $f_{n_k} \to f$ a.u. Proceed as follows.

(a) Show that there is an increasing sequence $n_1 < n_2 < \cdots$ such that

$$\mu\left(\{x; |f_n(x) - f_m(x)| \geq \varepsilon\}\right) < \frac{1}{2^k}, \quad \text{for all } n, m \geq n_k.$$

(b) Let

$$A_m = \bigcup_{k=m}^{\infty} \left\{x; \left|f_{n_k}(x) - f_{n_{k+1}}(x)\right| \geq \frac{1}{2^k}\right\}.$$

Show that $\{f_{n_k}\}$ is a Cauchy sequence of bounded functions on the set A_m^c. Deduce that there is a real-valued measurable function f on $A := \bigcup_{m=1}^{\infty} A_m^c$ such that $\{f_{n_k}\}$ converges uniformly to f on each A_m^c.

(c) Define f to be zero on A^c. Show that $f_n \to f$ a.u.

Notes

Answers: Self Assessment

1. extended-real valued functions
2. monotone
3. f + g, f · g, 1/f
4. uniformly convergent

28.7 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol : Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik : Mathematical Analysis.

H.L. Royden : Real Analysis, Ch. 3, 4.

Unit 29: The Lebesgue Integral of Bounded Functions

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the Lebesgue integral of bounded functions over a set of finite measure
- Explain properties of the Lebesgue integral of bounded functions over a set of finite measure
- Describe bounded convergence theorem

Introduction

After getting basic knowledge of the Lebesgue measure theory, we now proceed to establish the Lebesgue integration theory.

In this unit, unless otherwise stated, all sets considered will be assumed to be measurable.

We begin with simple functions.

29.1 Simple Functions Vanishing Outside a Set of Finite Measure

Recall that the characteristic function χ_A for any set A is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

A function $\varphi : E \to \mathbb{R}$ is said to be simple if there exists $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $E_1, E_2, \ldots, E_n \subseteq E$ such that $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$. Note that here the E_i's are implicitly assumed to be measurable, so a simple function shall always be measurable. We have another characterization of simple functions:

Proposition: A function $\varphi : E \to \mathbb{R}$ is simple if and only if it takes only finitely many distinct values $a_1, a_2, \ldots a_n$ and $\varphi^{-1}\{a_i\}$ is a measurable set for all i = 1, 2,....., n.

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Proof: With the above proposition we see that every simple function φ can be written uniquely in the form

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

where the a_i's are all non-zero and distinct, and the E_i's are disjoint. (Simply take E_i's $\varphi^{-1}\{a_i\}$ for i = 1, 2,...., n where $a_1, a_2, ..., a_n$ are all the distinct values of φ.) We say this is the canonical representation of φ.

We adopt the following notation:

Notation: A function $f : E \to \mathbb{R}$ is said to vanish outside a set of finite measure if there exists a set A with $m(A) < \infty$ such that f vanishes outside A, i.e.

$$f = 0 \quad \text{on} \quad E\backslash A$$

or equivalently f(x) = 0 for all $x \in E\backslash A$. We denote the set of all simple functions defined on E which vanish outside a set of finite measure by $S_0(E)$. Note that it forms a vector space.

We are now ready for the definition of the Lebesgue integral of such functions.

Definition: For any $\varphi \in S_0(E)$ and any $A \subseteq E$, we define the Lebesgue integral of φ over A by

$$\int_A \varphi = \sum_{i=1}^{n} a_i m(E_i \cap A)$$

where $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ is the canonical representation of φ. (From now on we shall adopt the convention that $0 \cdot \infty = 0$. We need this convention here because it may happen that one a_i is 0 while the corresponding $E_iC\backslash A$ has infinite measure. Also note that here A is implicitly assumed to be measurable so m(E_i n A) makes sense. We shall never integrate over non-measurable sets.)

It follows readily from the above definition that

$$\int_A \varphi = \int_A \varphi \chi_A$$

for any $\varphi \in S_0(E)$ and for any $A \subseteq E$.

We now establish some major properties of this integral (with monotonicity and linearity being probably the most important ones). We begin with the following lemma.

Lemma: Suppose $\varphi = \sum_{i=1}^{n} a_i \chi_{Ei} \in S_0(E)$ where the E_i's are disjoint. Then for any $A \subseteq E$,

$$\int_A \varphi = \sum_{i=1}^{n} a_i m(E_i \cap A)$$

holds even if the a_i's are not necessarily distinct.

Proof: If $\varphi = \sum_{j=1}^{n} b_i \chi_{Bj}$ is the canonical representation of φ, we have

1. $B_j = \bigcup_{\{i : a_i = b_j\}}^{m} E_i$

   for j = 1, 2,..., m and

2. $\{1, 2,..., n\} = \bigcup_{j=1}^{m} \{i : a_i = b_j\}$,

   where both unions are disjoint unions. Hence for any $A \subseteq E$, we have

Notes

$$\int_A \varphi = \sum_{j=1}^{m} b_j m(B_j \cap A) \qquad \text{(by definition of the integral)}$$

$$= \sum_{j=1}^{m} b_j m\left(\bigcup_{\{i:a_i=b_j\}} E_i \cap A\right) \qquad \text{(by (1))}$$

$$= \sum_{j=1}^{m} b_j \sum_{\{i:a_i=b_j\}} m(E_i \cap A) \qquad \text{(by finite additivity of m)}$$

$$= \sum_{j=1}^{m} b_j \sum_{\{i:a_i=b_j\}} a_i\, m(E_i \cap A)$$

$$= \sum_{i=1}^{n} a_i m(E_i \cap A) \qquad \text{(by (2))}$$

This complete our proof.

29.2 Properties of the Lebesgue Integral

Proposition: (Properties of the Lebesgue integral)

Suppose $\varphi + \not\varphi \in S_0(E)$. Then for any $A \subseteq E$,

(a) $\int_A(\varphi + \not\varphi) = \int_A \varphi + \int_A \not\varphi$ (Note that $\varphi + \not\varphi \in S_0(E)$ too be the vector space structure

(b) $\int_A \alpha\varphi = \alpha\int_A \varphi$ for all $\alpha \in \Upsilon$. (Note $\alpha\ \varphi \in S_0(E)$ again.)

(c) If $\alpha \le \not\varphi$ a.e. on A then $\int_A \varphi \le \int_A \not\varphi$.

(d) If $\varphi = \not\varphi$ a.e. on A then $\int_A \varphi = \int_A \not\varphi$.

(e) If $\varphi \ge 0$ a.e. on A and $\int_A \varphi = 0$, then $\varphi = 0$ a.e. on A.

(f) $\left|\int_A \varphi\right| \le \int_A |\varphi|$. (Note $|\varphi| \in$ So (E) too.Why?)

Remark: (a) and (b) are known as the linearity property of the integral, while (c) is known as the monotonicity property. Furthermore, Lemma is now seen to hold by the linearity of the integral even without the disjointness assumption on the E_i's.

Proof:

(a) Let $\varphi = \sum_{i=1}^{n} a_i\, \chi_{Ai}$ and $\not\varphi = \sum_{j=1}^{m} b_j\, \chi_{Bj}$ be canonical representations of φ and $\not\varphi$ respectively.

Then noting that $\chi_{A_i} = \sum_{j=1}^{m} \chi_{A_i \cap B_j}$ for all i and $\chi_{B_j} = \sum_{i=1}^{n} \chi_{A_i \cap B_j}$

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i} = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i \chi_{A_i \cap B_j}$$

$$\not\varphi = \sum_{j=1}^{m} b_j\, \chi_{B_j} = \sum_{i=1}^{n}\sum_{j=1}^{m} b_j\, \chi_{A_i \cap B_j}$$

Consequently

$$\varphi + \not\varphi = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i + b_j)\chi_{A_i \cap B_j}.$$

Notes

But the $A_i \cap B_j$'s are disjoint. So by Lemma we have

$$\int_A \varphi = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i m(A_i \cap B_j \cap A)$$

$$\int_A \psi = \sum_{i=1}^{n}\sum_{j=1}^{m} b_i m(A_i \cap B_j \cap A)$$

and

$$\int_A (\varphi + \psi) = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i + b_j) m(A_i \cap B_j \cap A).$$

Hence $\int_A (\varphi + \psi) = \int_A \varphi + \psi$

(b) If $\alpha = 0$ the result is trivial; if not, then let $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ be the canonical representation of φ. We see that $\alpha\varphi = \sum_{i=1}^{n} \alpha a_i \chi_{A_i}$ is the canonical representation of $\alpha\varphi$ and hence the result follows.

(c) Since $\int_A \varphi - \int_A \psi = \int_A (\varphi - \psi)$ by linearity, it suffices to show $\int_A \phi \geq 0$ whenever $\varphi \geq 0$ a.e. on A. This is easy, since if $a_1, a_2, \ldots, a_n$ are the distinct values of ϕ, then

$$\int_A \phi \sum_{\{i:a_i<0\}} a_i m(\phi^{-1}\{a_i\} \cap A) + \sum_{\{i:a_i \geq 0\}} a_i m(\phi^{-1}\{a_i\} \cap A) \geq \sum_{\{i:a_i<0\}} a_i \cdot 0 = 0$$

where the inequality follows from the fact that $m(\phi^{-1}\{a_i \cap A\}) = 0$ for all $a_i < 0$.

(d) This is immediate from (c).

(e) Since it is given that $\varphi \geq 0$ a.e. on A, it suffices to show $m(\{x : \varphi(x) > 0\} \cap A) = 0$.

Suppose not, then there exists $a > 0$ such that $m(\{x : \varphi(x) = a\} \cap A) > 0$ so $\int_A \varphi \geq a \cdot m(\{x : \varphi(x) = a\} \cap A) > 0$. This leads to a contradiction.

(f) This follows directly from monotonicity since $-|\varphi| \leq \varphi \leq |\varphi|$.

29.3 Bounded Measurable Functions Vanishing Outside a Set of Finite Measure

Resembling the construction of the Riemann integral, we define the upper and lower Lebesgue integrals.

Definition: Let $f : E \to \mathbb{R}$ be a bounded function which vanish outside a set of finite measure. For any $A \subseteq C$, we define the upper integral and the lower integral of f on A by

$$\overline{\int_A} f = \inf\left\{\int_A \psi : f \leq \psi \text{ on } A, \psi \in So(E)\right\}$$

$$\underline{\int_A} f = \sup\left\{\int_A \varphi : f \leq \varphi \text{ on } A, \varphi \in So(E)\right\}$$

If the two values agree we denote the common value by $\int_A f$. (Again the set A is implicitly assumed to be measurable so that $\int_A \psi$ and $\int_A \varphi$ make sense.)

Note that both the infimum and the supremum in the definitions of the upper and lower integrals exist because f is bounded and vanishes outside a set of finite measure. It is evident that for the

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functions have their upper and lower integrals both equal to their integral as defined in the last section. In other words, if $\varphi \in S_0(E)$ then $\overline{\int}_A \varphi = \underline{\int}_A \varphi = \int_A \varphi$, where the last integral is as defined in the last section. It is also clear that $-\infty < \underline{\int}_A f = \overline{\int}_A f < \infty$ whenever they are defined; we investigate when $\underline{\int}_A f = \overline{\int}_A f$.

Proposition: Let f be as in the above definition. Then $\underline{\int}_A f = \overline{\int}_A f$ for all $A \subseteq E$ if and only if f is measurable.

Proof: ($\Leftarrow$) Let f be a bounded measurable function defined on E which vanishes outside F with F $\subseteq$ E and m(F) < ∞. Then for each positive integer n there are simple functions φ_n, $\psi_n \in S_0(E)$ vanishing outside F such that $\varphi_n \leq f \leq \psi_n$ and $0 \leq \psi_n - \varphi_n \leq 1/n$ E on E (Why?). Hence for any A $\subseteq$ E, we have

$$0 \leq \overline{\int}_A f - \underline{\int}_A f \quad \text{(subtraction makes sense since both integrals are finite)}$$

$$\leq \overline{\int}_A \psi_n - \underline{\int}_A \varphi_n \quad \text{(definition of } \overline{\int}_A f \text{ and } \underline{\int}_A f\text{)}$$

$$= \int_A (\psi_n - \varphi_n)$$

$$= \int_{A \cap F} (\psi_n - \varphi_n) \quad (\varphi_n = \psi_n = 0 \text{ outside F})$$

$$\leq \int_F (\psi_n - \varphi_n) \quad (\psi_n - \varphi_n \geq 0 \text{ on F and } A \cap F \subseteq F)$$

$$\leq m(F)/n \quad (1,\ \psi_n - \varphi_n\ 1/n \text{ on F})$$

for all n. Letting $n \to \infty$ we have $\underline{\int}_A f = \overline{\int}_A f$. (m(F) < ∞)

($\Leftarrow$) Suppose $\overline{\int}_A f = \underline{\int}_A f$ for any $A \subseteq E$. Then $\overline{\int}_E f = \underline{\int}_E f$. Denote the common value by L. Then for all positive integers n there exists φ_n, ψ_n $S_0(E)$ such that $\varphi_n \leq f \leq \psi_n$ on E and $L - 1/n \leq \int_E \varphi_n \leq \int_E \psi_n \leq L + 1/n$. Let $\varphi = \sup_n \varphi_n$ and $\psi = \inf_n \psi_n$. We shall show $\varphi = \psi$ a.e. on E. (Then the desired conclusion follows since then $\varphi < f < \psi$ on E implies that $\varphi = f = \psi$ a.e. on E and hence f is measurable.) To show that $\varphi = \psi$ a.e. on E, let $\Delta = \{x \in E : \varphi(x) \neq \psi(x)\}$ and $\Delta_i = \{x \in E : \psi(x) - \varphi(x) > 1/i\}$. Then $\Delta = \bigcup_{i=1}^{\infty} \Delta_i$. We wish to show $m(\Delta) = 0$, which will be true if we can show $m(\Delta_i) = 0$ for all i. Now for any i and n, since $\psi_n - \varphi_n \geq \psi - \varphi > 1/i$ on Δ_i, we have

$$\frac{1}{i} m(\Delta_i) = \int_{\Delta_i} \frac{1}{i} \quad \text{(by definition of the integral)}$$

$$\leq \int_{\Delta_i} (\psi_n - \varphi_n)$$

$$\leq \int_E (\psi_n - \varphi_n) \quad (\psi_n - \varphi_n \geq 0 \text{ on E and } \Delta i \subseteq E)$$

$$\leq \int_E \psi_n - \int_E \varphi_n$$

$$\leq 2/n$$

Letting $n \to \infty$ we have $m(\Delta_i) = 0$ for all i, completing our proof.

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Notation: We shall denote the set of all (real-valued) bounded measurable functions defined on E which vanishes outside a set of finite measure by $B_0(E)$.

So from now on for $f \in B_0(E)$, we have

$$\int_A f = \inf\left\{\int_A \varphi : f \le \varphi \in So(E)\right\} = \sup\left\{\int_A \varphi : f \ge \varphi \in So(E)\right\}$$

for any $A \subseteq E$.

Note also that $B_0(E)$ is a vector lattice, by which we mean it is a vector space partially ordered by $\le$ (such that $f \le g$ if and only if $f(x) \le g(x)$ for all $x \in E$) and every two elements of it (say $f, g \in B_0(E)$) have a least upper bound in it (namely $f \vee g \in B_0(E)$). (Why is it a least upper bound?)

We have the following nice proposition concerning the relationship between the Riemann and the Lebsegue integrals.

Proposition: If $f : [a, b] \to \mathbb{R}$ is Riemann integrable on the closed and bounded interval [a, b], then $f \in B_0([a, b])$ and

(3) $\quad (\mathcal{R}) \int_a^b f = (\mathcal{L})\int_{[a,b]} f,$

where the $(\mathcal{R})$ and $(\mathcal{L})$ represents Riemann integral and Lebesgue integral respectively.

Proof: Since step functions defined on closed and bounded interval [a, b] are simple and have the same Lebesgue and Riemann integral over [a, b] (why?), we see from the definitions

$$(\mathcal{R}) = \underline{\int}_a^b f = \sup\left\{\int_a^b \varphi : f \ge \text{step on } [a, b]\right\}$$

$$(\mathcal{L}) = \underline{\int}_{[a,b]} f = \sup\left\{\int_{[a,b]} \varphi : f \ge \varphi \text{ simple on } [a, b]\right\}$$

$$(\mathcal{L}) = \overline{\int}_{[a,b]} f = \inf\left\{\int_{[a,b]} \varphi : f \le \varphi \text{ simple on } [a, b]\right\}$$

$$(\mathcal{R}) = \overline{\int}_a^b f = \inf\left\{\int_a^b \varphi : f \le \varphi \text{ step on } [a, b]\right\}$$

that

(4) $\quad (\mathcal{R}) = \underline{\int}_a^b f \le (\mathcal{L})\underline{\int}_{[a,b]} f \le (\mathcal{L})\overline{\int}_{[a,b]} f \le (\mathcal{R})\overline{\int}_a^b f$

whenever the four quantities exist. Now if f is Riemann integrable over [a, b], then f is bounded on [a,b]. Since [a,b] is of finite measure, we see that all four quantities in (4) exist. In that case $(\mathcal{R})\underline{\int}_a^b f = (\mathcal{R})\overline{\int}_a^b f$ as well so all four quantities in (4) are equal, which implies that f is measurable (so $f \in B_0([a, b])$) and (3) holds.

Proposition: Properties of the Lebesgue integral

Suppose $f, g \in B_0(E)$. Then $f + g$, αf, $|f| \in B_0(E)$, and for any $A \subseteq E$, we have

(a) $\quad \int_A (f+g) = \int_A f + \int_A g$

(b) $\quad \int_A \alpha f = \alpha \int_A f$ for all $\alpha \in \mathbb{R}$.

(c) $\quad \int_A f = \int_E f \chi_A$

(d) $\quad$ If $B \subseteq A$ then $\int_B f + \int_{A \setminus B} f$.

(e) $\quad$ If $B \subseteq A$ and $\int_B f \le \int_A f$

(f) If $f \leq g$ a.e. on A then $\int_A f \leq \int_A g$.

(g) If $f = g$ a.e. on A then $\int_A f = \int_A g$.

(h) If $f \geq 0$ a.e. on A and $\int_A f = 0$, then $f = 0$ a.e. on A.

(i) $\left|\int_A f\right| \leq \int_A |f|$.

Proof: We prove only (h); the others are easy and left as an exercise.

(h) For each positive integer n let $A_n = \{x \in A : f(x) \geq 1/n\}$. Then

$$0 = \int_A f \geq \int_{A_n} f \quad \text{(by (e))}$$

$$\geq \int_{A_n} \frac{1}{n} \quad \text{(by (f))}$$

$$= \frac{1}{n} m(A_n) \quad \text{(by (by definition of the integral)}$$

$$\geq 0$$

so $m(A_n) = 0$. Since this holds for all n, we see from $f^{-1}(0, \infty) \cap A = \bigcup_{n=1}^{\infty} A_n$ that $0 \leq m(f^{-1}(0, \infty) \cap A) \leq \sum_{n=1}^{\infty} m(A_n) = 0$. So $m(f^{-1}(0, \infty) \cap A) = 0$. Together with $f \geq 0$ a.e. on A.

Theorem: Bounded Convergence Theorem

Suppose $m(E) < \infty$, and $\{f_n\}$ is a sequence of measurable functions defined and uniformly bounded on E by some constant $M > 0$, i.e.

$$|f_n| \leq M \quad \text{for all n on E.}$$

If $\{f_n\}$ converges to a function f (pointwisely) a.e. on E, then f is also bounded measurable on E, $\lim_{n\to\infty} \int_E f_n$ exists (in $\mathbb{R}$) and is given by

(5) $$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof: Under the given assumptions it is clear that f, being the pointwise limit of $\{f_n\}$ a.e. on E, is bounded (by M) and measurable on E. We wish to show $\lim_{n\to\infty} \int_E f_n$ exists and (5) holds. The result is trivial if $m(E) = 0$. So assume $m(E) > 0$ and let $\varepsilon > 0$ be given. Then for each natural number i let

$$E_i = \{x \in, E : |f_j(x) - f(x)| \geq \varepsilon/2m(E) \text{ for some } j \geq i\}.$$

Then $\{E_i\}$ is a decreasing sequence of sets with $m(E_1) \leq m(E) < \infty$. So

$$m(E_i) \downarrow m\left(\bigcap_{i=1}^{\infty} E_i\right) = 0,$$

the last equality follows from the fact that

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) \leq m(\{x \in E : f_n(x) \nrightarrow f(x)\}) = 0$$

Choose N large enough such that $m(E_N) < \varepsilon/4M$ and let $A = E_N$. Then $|f_n - f| < \varepsilon/2m(E)$ everywhere on $E\setminus A$ for all $n \geq N$, and hence whenever $n \geq N$ we have

$$\left|\int_E f_n - \int_E f\right| \leq \int_E \left|\int_n - f\right| \quad \text{(by linearity and (i))}$$

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$$= \int_{E\setminus A}|f_n - f| + \int_A |f_n - f| \text{ (by (e))}$$

$$\leq \int_{E\setminus A} \frac{\varepsilon}{2m(E)} + \int_{E\setminus A} 2M \text{ (by our choice of N and that } n \geq N)$$

$$= \frac{\varepsilon m(E \setminus A)}{2m(E)} + 2Mm(A)$$

$$\leq \frac{\varepsilon}{2} + 2M\frac{\varepsilon}{4M}$$

$$= \varepsilon.$$

Hence $\lim_{n\to\infty} \int_E f_n$ exists (in $\mathbb{R}$) and (5) holds.

(Alternatively when $\varepsilon > 0$ is given, by Littlewood's 3rd Principle we can choose a subset A of E with $m(A) < \varepsilon/4M$ such that $\{f_n\}$ converges uniformly to f on $E\setminus A$. Then choose N large enough such that $|f_n - f| < \varepsilon/2m(E)$ everywhere on $E\setminus A$ for all $n \geq N$, we see that whenever $n \geq N$, we have (as in the above)

$$\left|\int_E f_n - \int_E f\right| < \varepsilon.$$

Hence $\lim_{n\to\infty} \int_E f_n$ exists (in $\mathbb{R}$) and (5) holds.)

> *Notes* The first argument is just an adaptation of the proof of Littlewood's 3rd Principal to the present situation.

Self Assessment

Fill in the blanks:

1. A function $\varphi : E \to \mathbb{R}$ is simple if and only if it takes only finitely many distinct values a_1, a_2,.... a_n and $\varphi^{-1}\{a_i\}$ is a .... for all i = 1, 2,....., n.
2. A function $f : E \to \mathbb{R}$ is said to vanish outside a set of .... if there exists a set A with $m(A) < \infty$ such that f vanishes outside A, i.e.

   $f = 0$ on $E\setminus A$
3. Let f be as in the above definition. Then $\underline{\int}_A f = \overline{\int}_A f$ for all $A \subseteq E$ if and only if f is ....
4. If $f : [a, b] \to \mathbb{R}$ is .... on the closed and bounded interval [a, b], then $f \in B_0([a, b])$ and $(\mathcal{R}) \int_a^b f = (\mathcal{L})\int_{[a, b]} f$, where the $(\mathcal{R})$ and $(\mathcal{L})$ represents Riemann integral and Lebesgue integral respectively.
5. Suppose $m(E) < \infty$, and $\{f_n\}$ is a sequence of measurable functions defined and uniformly bounded on E by some constant $M > 0$, i.e. .... for all n on E.

29.4 Summary

- Recall that the characteristic function χ_A for any set A is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

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- A function $\varphi : E \to \mathbb{R}$ is said to be simple if there exists $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $E_1, E_2, \ldots, E_n \subseteq E$ such that $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$. Note that here the E_i's are implicitly assumed to be measurable, so a simple function shall always be measurable. We have another characterization of simple functions:
- A function $\varphi : E \to \mathbb{R}$ is simple if and only if it takes only finitely many distinct values a_1, a_2, ... a_n and $\varphi^{-1}\{a_i\}$ is a measurable set for all i = 1, 2,....., n.

  (a) $\int_A (\varphi + \not\wp) = \int_A \varphi + \int_A \not\wp$ (Note that $\varphi + \not\wp \in S_0(E)$ too by the vector space structure

  (b) $\int_A \alpha\varphi = \alpha \int_A \varphi$ for all $\alpha \in \Upsilon$. (Note $\alpha\ \varphi \in S_0(E)$ again.)

  (c) If $\alpha \leq \not\wp$ a.e. on A then $\int_A \varphi \leq \int_A \not\wp$.

  (d) If $\varphi = \not\wp$ a.e. on A then $\int_A \varphi = \int_A \not\wp$.

  (e) If $\varphi \geq 0$ a.e. on A and $\int_A \varphi = 0$, then $\varphi = 0$ a.e. on A.

  (f) $\left|\int_A \varphi\right| \leq \int_A |\varphi|$. (Note $|\varphi| \in$ So (E) too.Why?)

- Bounded Convergence Theorem Suppose $m(E) < \infty$, and $\{f_n\}$ is a sequence of measurable functions defined and uniformly bounded on E by some constant M > 0, i.e.

  $|f_n| \leq M$ for all n on E.

29.5 Keyword

Bounded Convergence Theorem: Suppose $m(E) < \infty$, and $\{f_n\}$ is a sequence of measurable functions defined and uniformly bounded on E by some constant M > 0, i.e.

$|f_n| \leq M$ for all n on E.

29.6 Review Questions

1. Show that if $A, B \subseteq E$, $A \cap B = \emptyset$ and $\varphi \in S_0(E)$, then $\int_{A \cup B} \varphi = \int_A \varphi + \int_B \varphi$.
2. Show that if $\varphi \in S_0(E)$ vanishes outside F, then $\int_A \varphi = \int_{A \cap F} \varphi$ for any $A \subseteq E$.
3. Show that if $A \subseteq B \subseteq E$ and $0 \leq \varphi \in S_0(E)$, then $\int_A \varphi \leq \int_B \varphi$.
4. Find an example to show that the assumption $m(E) < \infty$ cannot be dropped in the Bounded Convergence Theorem.
5. Prove or disprove the following: Let E be of finite or infinite measure. If $\{f_n\}$ is a sequence of uniformly bounded measurable functions on E which vanishes outside a set of finite measure and converges pointwisely to $f \in B_0(E)$ a.e. on E, then $\lim_{n \to \infty} \int_E f_n = \int_E f$. (Compare with the statement of the Bounded Convergence Theorem.)

Answers: Self Assessment

1.	measurable set	2.	finite measurable
3.	measurable	4.	Riemann integrable
5.	$\lvert f_n \rvert \leq M$		

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29.7 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 30: Riemann's and Lebesgue

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Objectives

After studying this unit, you will be able to:

- Discuss Riemann's and Lebesgue
- Explain the small subsets of R
- Discuss the functions outside small set

Introduction

In last unit you have studied about the Lebesgue integral of bounded functions. In this unit we are going to study about the definition and the difference of Riemann's and Lebesgue.

30.1 Riemann vs. Lebesgue

Measure theory helps us to assign numbers to certain sets and functions to a measurable set we may assign its measure, and to an integrable function we may assign the value of its integral. Lebesgue integration theory is a generalization and completion of Riemann integration theory. In Lebesgue's theory, we can assign numbers to more sets and more functions than what is possible in Riemann's theory. If we are asked to distinguish between Riemann integration theory and Lebesgue integration theory by pointing out an essential feature, the answer is perhaps the following.

Riemann integration theory $\mapsto$ finiteness.

Lebesgue integration theory $\mapsto$ countable infiniteness.

Riemann integration theory is developed through approximations of a finite nature (e.g.: one tries to approximate the area of a bounded subset of $\mathbb{R}^2$ by the sum of the areas of finitely many rectangles), and this theory works well with respect to finite operations– if we can assign numbers to finitely many sets $A_1,\ldots, A_n$ and finitely many functions $f_1,\ldots,f_n$, then we can assign numbers to $A_1\cup\cdots\cup A_n$, $f_1 +\cdots+ f_n$, $\max\{f_1,\ldots, f_n\}$, etc. The disadvantage of Riemann integration theory is that it does not behave well with respect to operations of a countably infinite nature - there may not be any consistent way to assign numbers to $\bigcup_{n=1}^{\infty} A_n$, $\sum_{n=1}^{\infty} f_n$, $\lim_{n\to\infty} f_n$, $\sup\{f_n : n \in \mathbb{N}\}$, etc. even if we can assign numbers to the sets $A_1, A_2, \ldots$, and functions $f_1, f_2 \ldots$. Lebesgue integration theory rectifies this disadvantage to a large extent.

In Riemann integration theory, we proceed by considering a partition of the domain of a function, where as in Lebesgue integration theory, we proceed by considering a partition of the range of the function – this is observed as another difference. Moreover, while Riemann's theory is restricted to the Euclidean space, the ideas involved in Lebesgue's theory are applicable to more general spaces, yielding an abstract measure theory. This abstract measure theory intersects with many branches of mathematics and is very useful. There is even a philosophy that measures are easier to deal with than sets.

30.2 Small Subsets of $\mathbb{R}^d$

It is possible to think about many mathematical notions expressing in some sense the idea that a subset $Y \subset \mathbb{R}^d$ is a small set (or a big set) with respect to $\mathbb{R}^d$. We will discuss this a little as a warm-up. We will also use this opportunity to introduce Lebesgue outer measure.

Suppose you have a certain notion of smallness or bigness for a subset of $\mathbb{R}^d$. Then there are some natural questions. Two sample questions are:

1. If $Y \subset \mathbb{R}^d$ is big, is $\mathbb{R}^d \setminus Y$ small?
2. If $Y_1, Y_2, \ldots \subset \mathbb{R}^d$ are small, is $\bigcup_{n=1}^{\infty} Y_n$ small?

For instance, consider the following two elementary notions. Saying that $Y \subset \mathbb{R}^d$ is unbounded is one way of saying Y is big, and saying that $Y \subset \mathbb{R}^d$ is a finite set is one way of saying Y is small. Note that the complement of an unbounded set can also be unbounded and a countable union of finite sets need not be finite. So here we have negative answers to the above two questions.

Task Find an uncountable collection $\{Y_\alpha : \alpha \in I\}$ of subsets of $\mathbb{R}$ such that Y_α's are pairwise disjoint, and each Y_α is bounded neither above nor below.

To discuss some other notions of smallness, we introduce a few definitions.

Definitions:

(i) We say $Y \subset \mathbb{R}^d$ is a discrete subset of $\mathbb{R}^d$ if for each $y \in Y$, there is an open set $U \subset \mathbb{R}^d$ such that $U \cap Y = \{y\}$. For example, $\{1/n : n \in \mathbb{N}\}$ is a discrete subset of $\mathbb{R}$.

(ii) A subset $Y \subset \mathbb{R}^d$ is nowhere dense in $\mathbb{R}^d$ if $\text{int}[\overline{Y}] = \emptyset$, or equivalently if for any non-empty open set $U \subset \mathbb{R}^d$, there is a nonempty open set $V \subset U$ such that $V \cap Y = 0$. For example, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous map, then its graph $G(f) := \{(x, f(x)) : x \in \mathbb{R}\}$ is nowhere dense in $\mathbb{R}^2$ ($\because$ G(f) is closed and does not contain any open disc).

(iii) A subset $Y \subset \mathbb{R}^d$ is of first category in $\mathbb{R}^d$ if Y can be written as a countable union of nowhere dense subsets of $\mathbb{R}^d$; otherwise, Y is said to be of second category in $\mathbb{R}^d$. For example, $Y = \mathbb{Q} \times \mathbb{R}$ is of first category in $\mathbb{R}^2$ since Y can be written as the countable union $Y = \bigcup_{r\in\mathbb{Q}} Y_r$, where $Y_r := \{r\} \times \mathbb{R}$ is nowhere dense in $\mathbb{R}^2$.

(iv) (The following definition can be extended by considering ordinal numbers, but we consider only non-negative integers). For $Y \subset \mathbb{R}^d$ and integer $n \geq 0$, define the nth derived set of Y inductively as $Y^{(0)} = Y$, $Y^{(n+1)} = \{\text{limit points of } Y^{(n)} \text{ in } \mathbb{R}^d\}$. We say $Y \subset \mathbb{R}^d$ has derived length n if $Y^{(n)} \neq \emptyset$ and $Y^{(n+1)} \neq \emptyset$; and we say Y has infinite derived length if $Y^{(n)} \neq \emptyset$ for every

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integer $n \geq 0$. For example, $\mathbb{Q}$ has infinite derived length (since $\overline{\mathbb{Q}} = \mathbb{R}$), and $\{(1/m,1/n): m,n \in \mathbb{N}\}$ has derived length 2.

(v) We say $A \subset \mathbb{R}^d$ is a d-box if $A = \pi_{j=1}^{d} I_j$, where I_j's are bounded intervals. The d-dimensional volume of a d-box A is $Vol_d(A) = \pi_{j=1}^{d} |I_j|$. For example, $Vol_3([1, 4) \times [0,1/2] \times (-1,3]) = 6$.

(vi) The d-dimensional Jordan outer content $\mu_{j,d}^{*}[Y]$ of a bounded subset $Y \subset \mathbb{R}^d$ is defined as $\mu_{j,d}^{*}[Y] = \{\sum_{n=1}^{k} \inf Vold(An) : k \in \mathbb{N}$, and A_n's are d-boxes with $Y \subset \bigcup_{n=1}^{k} A_n\}$.

(vii) The d-dimensional Lebesgue outer measure $\mu_{L,d}^{*}[Y]$ of an arbitrary set $Y \subset \mathbb{R}^d$ is defined as $\mu_{L,d}^{*}[Y] = \inf\{\sum_{n=1}^{\infty} Vol_d(A_n) : A_n$'s are d-boxes with $Y \subset \bigcup_{n=1}^{\infty} A_n\}$.

We have that $\mu_{L,d}^{*}[Y] \leq \mu_{J,d}^{*}[Y]$ for any bounded set $Y \subset \mathbb{R}^d$, and $\mu_{L,d}^{*}[A] = \mu_{J,d}^{*}[A] = Vol_d(A)$ for any d-box $A \subset \mathbb{R}^d$.

Proof: Any finite union $\bigcup_{n=1}^{k} A_n$ of d-boxes can be extended to an infinite union $\bigcup_{n=1}^{\infty} A_n$ of d-boxes without changing the total volume by taking A_n's to be singletons for $n > k$. This observation yields that $\mu_{L,d}^{*}[Y] \leq \mu_{J,d}^{*}[Y]$. It is easy to see $\mu_{J,d}^{*}[A] = Vol_d(A)$ if A is a d-box. It remains to show $\mu_{L,d}^{*}[A] \geq Vol_d(A)$ when A is a d-box. First suppose A is closed. Then A is compact by Heine-Borel. Let $\varepsilon > 0$ and let $A_1, A_2, \ldots \subset \mathbb{R}^d$ be d-boxes such that $A \subset \bigcup_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} Vol_d(A_n) < \mu_{L,d}^{*}[A] + \varepsilon$. For each $n \in \mathbb{N}$, let B_n be an open d-box with $A_n \subset B_n$ and $Vol_d(B_n) < Vol_d(A_n) + \varepsilon/2^n$. Then $\{B_n: n \in \mathbb{N}\}$ is an open cover for the compact set A. Extracting a finite subcover, we have $Vol_d(A) \leq \sum_{n=1}^{k} Vol_d(B_n) \leq \sum_{n=1}^{\infty} (Vol_d(A_n) + \varepsilon/2^n) < \mu_{L,d}^{*}[A] + 2\varepsilon$. Thus $\mu_{L,d}^{*}[A] = Vol_d(A)$ for closed d-boxes. Now if B is an arbitrary d-box and $\varepsilon > 0$, then there is a closed d-box $A \subset B$ with $Vol_d(B) - \varepsilon < Vol_d(A) = \mu_{L,d}^{*}[A] \leq \mu_{L,d}^{*}[B]$.

Other basic properties of Lebesgue outer measure and Jordan outer content are given below.

(i) $\mu_{L,d}^{*}[\emptyset] = 0$.

(ii) [Monotonicity] $\mu_{L,d}^{*}[X] \leq \mu_{L,d}^{*}[Y]$ if $X \subset Y \subset \mathbb{R}^d$.

(iii) [Translation-invariance] $\mu_{L,d}^{*}[Y + x] = \mu_{L,d}^{*}[Y]$ for every $Y \subset \mathbb{R}^d$ and every $x \in \mathbb{R}^d$.

(iv) [Countable subadditivity] If $Y_1, Y_2, \ldots \subset \mathbb{R}^d$ and $Y = \bigcup_{n=1}^{\infty} Y_n$, then $\mu_{L,d}^{*}[Y] \leq \sum_{n=1}^{\infty} \mu_{L,d}^{*}[Y_n]$.

(v) $\mu_{L,d}^{*}[Y] = 0$ for every countable set $Y \subset \mathbb{R}^d$.

(vi) Forany $Y \subset \mathbb{R}^d$, we have $\mu_{L,d}^{*}[Y]$

$= \{\sum_{n=1}^{\infty} \inf Vold(An) : A_n$'s are closed d-boxes with $Y \subset \bigcup_{n=1}^{\infty} A_n\}$

$= \{\sum_{n=1}^{\infty} \inf Vold(An) : A_n$'s are open d-boxes with $Y \subset \bigcup_{n=1}^{\infty} A_n\}$.

(vii) For any $Y \subset \mathbb{R}^d$, we have $\mu_{L,d}^{*}[Y] = \mu_{L,d}^{*} \inf\{[U] : Y \subset U$ and U is open in $\mathbb{R}^d\}$.

(viii) $\mu_{L,d}^{*}[\mathbb{R}^d] = \infty$.

(ix) If $X,Y \subset \mathbb{R}^d$ are such that $dist(X,Y) := \inf\{||x - y|| : x \in X, y \in Y\} > 0$, then $\mu_{L,d}^{*}[X \cup Y] = \mu_{L,d}^{*}[X] + \mu_{L,d}^{*}[Y]$.

Proof: (i), (ii) and (iii) are clear. To prove (iv), without loss of generality we may assume $\sum_{n=1}^{\infty} \mu_{L,d}^{*}[Y_n] < \infty$. Given $\varepsilon > 0$, there exist d-boxes $A(n,k)$ such that $Y_n \subset \bigcup_{k=1}^{\infty} A(n,k)$ and $\sum_{k=1}^{\infty} Vol_d(A(n,k)) < \mu_{L,d}^{*}[Y_n] + \varepsilon/2^n$. Then $Y \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A(n,k)$ and we have the estimate $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} Vol_d(A(n,k)) \leq \sum_{n=1}^{\infty} (\mu_{L,d}^{*}[Y_n] + \varepsilon/2^n) = (\sum_{k=1}^{\infty} \mu_{L,d}^{*}[Y_n]) + \varepsilon$.

Now (v) follows from (iv) since singletons have Lebesgue outer measure zero (or we can see it directly by noting that singletons are d-boxes with zero volume). The first part of (vi) is clear since any d-box and its closure have equal volume. To get the second part, note that if $A_1, A_2, \ldots$ are d-boxes and $\varepsilon > 0$, there exist open d-boxes $B_1, B_2, \ldots$ such that $A_n \subset B_n$ and $Vol_d(B_n) < Vol_d(A_n) + \varepsilon/2^n$. We may deduce (vii) using part (vi).

Notes

Now we prove (ix). From countable subadditivity, we have $\mu^*_{L,d}[X \cup Y] \leq \mu^*_{L,d}[X] + \mu^*_{L,d}[Y]$. To prove the other inequality, we may assume $\mu^*_{L,d}[X \cup Y] < \infty$. Let $\delta = \text{dist}(X,Y)$. Given $\varepsilon > 0$, find d-boxes $A_1, A_2, \ldots$ such that $X \cup Y \subset \bigcup_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} \text{Vol}_d(A_n) < \mu^*_{L,d}[X \cup Y] + \varepsilon$. By partitioning the d-boxes into smaller d-boxes and throwing away the unnecessary ones, we may assume that $\text{diam}[A_n] < \delta$ and $(X \cup Y) \cap A_n \neq \emptyset$ for every $n \in \mathbb{N}$. Let $\in = \{n \in \mathbb{N}: X \cap A_n \neq \emptyset\}$ and $\Gamma' = \{n \in \mathbb{N}: Y \cap A_n \neq \emptyset\}$. Then $\mathbb{N} = \in \cup \Gamma'$ is a disjoint union, $X \subset \bigcup_{n \in \Gamma} A_n$, and $Y \subset \bigcup_{n \in \Gamma} A_n$. Hence $\mu^*_{L,d}[X] + \mu^*_{L,d}[Y] \leq \sum_{n \in \Gamma} \text{Vol}_d(A_n) + \sum_{n \in \Gamma} \text{Vol}_d(A_n) = \sum_{n=1}^{\infty} \text{Vol}_d(A_n) < \mu^*_{L,d}[X \cup Y] + \varepsilon$.

(i) $\mu^*_{J,d}[\emptyset] = 0$.

(ii) [Monotonicity] $\mu^*_{J,d}[X] \leq \mu^*_{L,d}[Y]$ if $X \subset Y$ are bounded subsets of $\mathbb{R}^d$.

(iii) [Translation-invariance] $\mu^*_{L,d}[Y + x] = \mu^*_{L,d}[Y]$ for every bounded set $Y \subset \mathbb{R}^d$ and every $x \in \mathbb{R}^d$.

(iv) [Finite subadditivity] If $X,Y \subset \mathbb{R}^d$ are bounded subsets, then $\mu^*_{J,d}[X \cup Y] \leq \mu^*_{J,d}[X] + \mu^*_{J,d}[Y]$.

(v) $\mu^*_{J,d}[Y] = 0$ for every finite set $Y \subset \mathbb{R}^d$.

(vi) For any bounded set $Y \subset \mathbb{R}^d$, we have $\mu^*_{J,d}[Y]$

$= \inf\{\sum_{n=1}^{k} \text{Vol}_d(A_n) : k \in \mathbb{N}$, and A_n's are closed d-boxes with $Y \subset \bigcup_{n=1}^{k} A_n\}$

$= \inf\{\sum_{n=1}^{k} \text{Vol}_d(A_n) : k \in N$, and A_n's are open d-boxes with $Y \subset \bigcup_{n=1}^{k} A_n\}$

$= \inf\{\sum_{n=1}^{k} \text{Vol}_d(A_n) : k \in N$, and A_n's are pairwise disjoint d-boxes with $Y \subset \bigcup_{n=1}^{k} A_n\}$.

(vii) If $X,Y \subset \mathbb{R}^d$ are bounded sets with $\text{dist}(X,Y) := \inf\{\|x - y\| : x \in X, y \in Y\} > 0$, then $\mu^*_{J,d}[X \cup Y] = \mu^*_{J,d}[X] + \mu^*_{J,d}[Y]$.

(viii) For any bounded set $Y \subset \mathbb{R}^d$, we have $\mu^*_{J,d}[\overline{Y}] = \mu^*_{J,d}[Y]$.

Proof: To prove (viii), use the first expression for $\mu^*_{J,d}[Y]$ in (vi) and note that a finite union of closed sets is closed.

Example: Let $Y = \mathbb{Q}^d \cap [0, 1]^d$. Note that $\mu^*_{L,d}[Y] = 0 \neq 1 = \mu^*_{L,d}[\overline{Y}]$. But we have $\mu^*_{J,d}[Y] = \mu^*_{J,d}[\overline{Y}] = 1$. So the Jordan outer content of a bounded countable set need, not be zero. This example also shows that $\mu^*_{L,d}[Y] < \mu^*_{J,d}[Y]$ is possible for a bounded set, and that the Jordan outer content does not satisfy countable subadditivity for bounded sets (since the Jordan outer content of a singleton is zero). If $X = [0, 1]^d \setminus Y$, then $\mu^*_{J,d}[X] = 1$ since $\overline{X} = [0, 1]^d$ and hence $\mu^*_{J,d}[X] + \mu^*_{J,d}[Y] = 2 \neq 1 = \mu^*_{J,d}[X \cup Y]$.

Some ways of saying that $Y \subset \mathbb{R}^d$ is a small set:

(i) Y is a countable set.

(ii) Y is a discrete subset of $\mathbb{R}^d$.

(iii) Y is contained in a vector subspace of $\mathbb{R}^d$ of dimension $\leq d - 1$.

(iv) Y is nowhere dense in $\mathbb{R}^d$.

(v) Y is of first category in $\mathbb{R}^d$.

(vi) Y has finite derived length.

(vii) Y is a bounded set with $\mu^*_{J,d}[Y] = 0$.

(viii) $\mu^*_{L,d}[Y] = 0$.

It is good to investigate various possible implications between pairs of notions given above.

> *Task* If Y is a discrete subset of $\mathbb{R}^d$, then Y is countable. [*Hint:* Let $\mathbb{B} = \{B(x, 1/n): x \in \mathbb{Q}^d, n \in \mathbb{N}\}$. Then, $\mathbb{B}$ is countable and for each $y \in Y$, there is $B \in \mathbb{B}$ such that $B \cap Y = \{y\}$.]

Let $K \subset [0,1]$ be the middle-third Cantor set. Then, K is an uncountable, nowhere dense compact set with $\mu_{J,1}[K] = \mu^*_{L,1}[K] = 0$. Moreover, K has no isolated points.

Proof: We recall the construction of K. Let $K_0 = [0,1]$, $K_1 = [0,1/3] \cup [2/3,1]$, $K_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3, 7/9] \cup [8/9,1]$, and so on. That is, K_n is the disjoint union of 2^n closed subintervals of [0,1], each having length $1/3^n$, and K_{n+1} is obtained from K_n by removing the middle-third open intervals from each of these 2^n closed intervals. The middle-third Cantor set K is defined as $K = \bigcap_{n=0}^{\infty} K_n$. Being the intersection of compact sets, K is compact. Since the maximal length of an interval contained in K_n is $(1/3)^n$, K does not contain any open interval, and hence K is nowhere dense. Also, since $K \subset K_n$, the above description yields $\mu^*_{J,1}[K] \le (2/3)^n$. So $\mu^*_{J,1}[K] = 0$ and hence $\mu^*_{L,1}[K] = 0$ also.

It may be verified that $K = \{\sum_{n=1}^{\infty} x_n/3^n : x_n \in \{0, 2\}\}$. That is, K is precisely the set of those $x \in [0,1]$ whose ternary expansion (i.e., base 3 expansion) $x = 0.x_1x_2 \ldots$ contains only 0's and 2's. Hence K is bijective with $\{0, 2\}^{\mathbb{N}}$ which is uncountable.

We show K has no isolated point. Let $x \in K$ and let U be a neighborhood of x. Choose n large enough so that one of the 2^n closed intervals constituting K_n, say J_n, satisfies $x \in J_n \subset U$. Let $y \in J_n \backslash \{x\}$ be an end point of J_n. This end point is never removed in the later construction, so $y \in K_m$ for every $m \ge n$. Thus $y \in K \cap (U \backslash \{x\})$.

> *Notes* It may be noted that for $x \in K$, the base 3 expansion $x = 0.x_1x_2 \ldots$ is eventually constant iff x is an end point of a removed open interval. This helps to see that K contains points other than the end points of the (countably many) removed open intervals.

The following theorem is relevant while considering big and small sets in a topological sense.

> *Task* If Y is contained in a vector subspace W of $\mathbb{R}^d$ with $\dim(W) \le d - 1$, then Y is a nowhere dense subset of $\mathbb{R}^d$. [*Hint:* W is closed in $\mathbb{R}^d$ ($\because$ fix a basis for W and argue with the coefficients of each basis vector separately) and W does not contain any open ball of $\mathbb{R}^d$.]

Baire Category Theorem: Let (X, ρ) be a complete metric space and let $U_n \subset X$ be open and dense in X for $n \in \mathbb{N}$. Then, $\bigcap_{n=1}^{\infty} U_n$ is also dense in X. In particular, $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$.

Proof: Let $V \subset X$ be a nonempty open set. It suffices to show $V \cap (\bigcap_{n=1}^{\infty} Un) \neq \emptyset$. Since U_1 is open and dense, $U_1 \cap V$ is a nonempty open set. Let B_1 be an open ball in X such that $\overline{B_1} \subset U_1 \cap V$ and $\text{diam}[B_1] < 1$. Since U_2 is open and dense, $B_1 \cap U_2 = \emptyset$. Let $B_2 \subset X$ be an open ball with $\overline{B_2} \subset B_1 \cap U_2$ and $\text{diam}[B_2] < 1/2$. In general, let $B_{n+1} \subset X$ be an open ball with $\overline{B_{n+1}} \subset B_n \cap U_{n+1}$ and $\text{diam}[B_{n+1}] < 1/(n+1)$. If x_n is the center of the ball B_n, then we note that for every $n, m \ge k$ we have $x_n, x_m \in B_k$ and hence $\rho(x_n, x_m) \le \text{diam}[B_k] < 1/k$. So (x_n) is a Cauchy sequence. Since (X, ρ) is complete, there is $x \in X$ such that $(x_n) \to x$. Now, for any n, we have $x_m \in \overline{B_n}$ for $m \ge n$ and hence $x \in \overline{B_n}$. Thus $x \in \bigcap_{n=1}^{\infty} \overline{B_n} \in V \cap (\bigcap_{n=1}^{\infty} U_n)$.

Notes

Notes (i) By considering the complements of U_n's in the above, we get the following conclusion: if (X, ρ) is a complete metric space, then X cannot be written as a countable union of nowhere dense (closed) subsets of X. That is, X is of second category in itself. (ii) Since $\mathbb{R}^d$ is a complete metric space with respect to the Euclidean metric, $\mathbb{R}^d$ cannot be written as a countable union of nowhere dense (closed) subsets of $\mathbb{R}^d$. (iii) From a topological point of view, a first category subset is considered as a small set and a dense G_δ subset is considered as a big set because of Baire Category Theorem. However, a set that is topologically big (small) need not be measure theoretically big (small). (iv) The uncountability of the middle-third Cantor set can be proved with the help of Baire Category Theorem also.

We observe in the following the distinction between topological bigness (smallness) and measure theoretical bigness (smallness).

Task For any $Y \subset \mathbb{R}^d$, the set $Y \backslash Y^{(1)}$ is discrete and hence countable. In particular, every uncountable subset of $\mathbb{R}^d$ has a limit point in $\mathbb{R}^d$. [*Hint:* Let $y \in Y \backslash Y^{(1)}$. If $B(y, 1/n) \cap Y$ contains a point other than y for every $n \in \mathbb{N}$, then $y \in Y^{(1)}$, a contradiction.]

(i) For every $\varepsilon > 0$, there is a dense open set $U \subset \mathbb{R}^d$ such that $\mu^*_{L,d}[U] < \varepsilon$.

(ii) There is a dense G_δ subset $Y \subset \mathbb{R}^d$ with $\mu^*_{L,d}[Y] = 0$.

(iii) There is an F_σ set $X \subset \mathbb{R}^d$ of first category with $\mu^*_{L,d}[X] = \infty$ and $\mu^*_{L,d}[\mathbb{R}^d \backslash X] = 0$.

(iv) For every closed d-box A and every $\varepsilon > 0$, there is anywhere dense compact set $K \subset \mathbb{R}^d$ such that $K \subset A$ and $\mu^*_{L,d}[K] > Vol_d(A) - \varepsilon$.

Proof: (i) Write $\mathbb{Q}^d = \{x_1, x_2, \ldots\}$. For each $n \in \mathbb{N}$, let A_n be an open d-box with $x_n \in A_n$ and $Vol_d(A_n) < \varepsilon/2^n$. Put $U = \bigcap_{n=1}^{\infty} A_n$.

(ii) Let $U_n \subset \mathbb{R}^d$ be a dense open subset with $\mu^*_{L,d}[U_n] < 1/n$ and put $Y = \bigcap_{n=1}^{\infty} U_n$.

(iii) Let Y be as in (ii) and take $X = \mathbb{R}^d \backslash Y$.

(iv) LetUbeasin(i)andletK = $A \backslash U$.

The next result shows that the Lebesgue outer measure does not satisfy finite additivity (and hence it does not satisfy countable additivity), even though it satisfies countable subadditivity.

Let $X = \mathbb{Q}^d \cap [0, 1]^d$. Then, there is a subset $Y \subset [0, 1]^d$ satisfying the following:

(i) The translations Y + x are pairwise disjoint for $x \in X$.

(ii) There exist finitely many distinct elements $x_1, \ldots, x_n \in X$ such that $\mu^*_{L,d}[\bigcup_{i=1}^{n}(Y + x_i)] \neq \sum_{i=1}^{n} \mu^*_{L,d}[Y + x_i]$.

Proof: Define an equivalence relation on $[0, 1]^d$ by the condition that $a \sim b$ iff $a - b \in \mathbb{Q}^d$. By the axiom of choice, we can form a set $Y \subset [0, 1]^d$ whose intersection with each equivalence class is a singleton.

(i) We verify that $(Y + r) \cap (Y + s) = \varnothing$ for any two distinct $r, s \in X$. If $(Y + r) \cap (Y + s) = \varnothing$ for $r, s \in X$, then there are $a, b \in Y$ such that $a + r = b + s$. Now we have $a - b = s - r \in \mathbb{Q}^d$, and hence $a \sim b$. So we must have $a = b$ by the definition of Y, and then necessarily $r = s$.

(ii) If $z \in \mathbb{R}^d$, there is $r \in \mathbb{Q}^d$ such that $z - r \in [0, 1]^d$. Then there is $y \in Y$ such that $y \sim z - r$ and so there is $r' \in \mathbb{Q}^d$ such that $y + r' = z - r$ or $z = y + r + r'$. This shows that $\mathbb{R}^d = \bigcup_{r \in \mathbb{Q}^d} (Y + r)$. By [102](iii) and [102](viii), we conclude that $\mu^*_{L,d}[Y] > 0$. Let $\delta = \mu^*_{L,d}[Y]$ and $n \in \mathbb{N}$ be such that $n\delta > 2^d$. Choose distinct elements $x_1,\ldots,x_n \in X$. Then $\sum_{i=1}^{n} \mu^*_{L,d}[Y + x_i] = n\delta > 2^d$ again by translation invariance. On the other hand, $Y + X \in [0,2]^d$ and therefore $\mu^*_{L,d}[\bigcup_{i=1}^{n}(Y+x_i)] < 2^d$.

Note The construction above is due to Vitali, and hence the set Y is called a Vitali set.

30.3 About Functions Behaving Nicely Outside a Small Set

There are a few classical results in Analysis with conclusion of the following form: "… the function has nice behavior outside a small set". We will consider some such results here.

We know that a function that is the pointwise limit of a sequence of continuous functions may not be continuous. For instance, $f : [0, 1] \to \mathbb{R}$ given by $f(1) = 1$ and $f(x) = 0$ for $x < 1$ is the pointwise limit of (f_n), where $f_n : [0, 1] \to \mathbb{R}$ is $f_n(x) = x^n$.

Definition: Let X, Y be metric spaces and let $f: X \to Y$ be a function. Then the oscillation $\omega(f, x)$ of f at a point $x \in X$ is defined as $\omega(f, x) = \lim_{\delta \to 0+} \text{diam}[f(B(x, \delta))]$. Clearly, f is continuous at x iff $\omega(f, x) = 0$.

Task Let X,Y be metric spaces and let $f: X \to Y$ be a function. Then the set $\{x \in X: f$ is continuous at $x\}$ is a G_δ subset of X. [*Hint:* The given set is equal to $\bigcup_{n=1}^{\infty} U_n$, where $U_n = \{x \in X: \omega(f, x) < 1/n\}$, and U_n is open.]

Let (X, ρ_1) be a complete metric space, (Y, ρ_2) be an arbitrary metric space, and let (f_n) be a sequence of continuous functions from X to Y, converging pointwise to a function $f: X \to Y$. Then the set $\{x \in X : f$ is continuous at $x\}$ is a dense G_δ subset of X.

Proof: Let $\varepsilon > 0$ and $D_\varepsilon = \{x \in X : \omega(f, x) > \varepsilon\}$. We know that D_ε is a closed set. We claim that D_ε is nowhere dense in X. Let $U \subset X$ be a nonempty open set. We have to find a nonempty open set $V \subset U$ such that $D_\varepsilon \cap V = \emptyset$.

For $n \in \mathbb{N}$, let $K_n = \{x \in X: \rho_2(f_n(x), fj(x)) \le \varepsilon/8$ for every $j \ge n\}$. Then K_n is a closed set and $X = \bigcup_{n=1}^{\infty} K_n$. The continuity of the distance function ρ_2 implies that $\rho_2(f_n(x), f(x)) \le \varepsilon/8$ for every $x \in K_n$. Let $U_1 \subset X$ be a nonempty open set with $\overline{U_1} \subset U$. Since $(\overline{U_1}, \rho_1)$ is a complete metric space, there is $n \in N$ such that $U_2 := \text{int}[K_n \cap \overline{U_1}] \neq \emptyset$. Let $b \in U_2$ and $V \subset U_2$ be an open set with $\text{diam}[f_n(V)] \le \varepsilon/8$. For any $x, y \in V$, we have $\rho_2(f(x), f(y)) \le \rho_2(f(x), f_n(x)) + \rho_2(f_n(x), f_n(b)) + \rho_2(f_n(b), f_n(y)) + \rho_2(f_n(y), f(y)) \le \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8 = \varepsilon/2$. Hence $\text{diam}[f(V)] \le \varepsilon/2$ and therefore $\omega(f, x) \le \varepsilon/2$ for every $x \in V$. This shows $D_\varepsilon \cap V = \emptyset$, proving our claim.

The claim implies that $D := \bigcup_{n=1}^{\infty} D_1/n$ is an F_σ set of first category in X. This completes the proof since $\{x \in X : f$ is continuous at $x\} = X \backslash D$, and X is a complete metric space.

We know that the derivative of a differentiable real function need not be continuous. However, we can say the following.

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Then there exists a sequence (g_n) of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ converging to f' pointwise. Consequently, $\{x \in \mathbb{R}: f'$ is continuous at $x\}$ is a dense G_δ subset of $\mathbb{R}$.

Notes

Proof: Since f is differentiable, f is continuous. Define $g_n(x) = [f(x + 1/n) - f(x)]/(1/n)$ and use [108].

Now we will show that a monotone real function (increasing or decreasing) is continuous and differentiable at most of the points.

Let $-\infty \le a < b \le \infty$ and let $f: (a, b) \to \mathbb{R}$ be a monotone function. Then, $Y = \{x \in (a, b) : f$ is discontinuous at $x\}$ is a countable set (possibly empty).

Proof: Suppose f is increasing. If $x \in Y$, then necessarily $f(x-) < f(x+)$, and we may choose a rational number between $f(x-)$ and $f(x+)$. This gives a one-one map from Y to Q.

Definition: A collection Γ of non-degenerate intervals is a Vitali cover for a set $X \subset \mathbb{R}$ if for each $\varepsilon > 0$, the subcollection $\{I \in \Gamma : 0 < |I| < \varepsilon\}$ is also a cover for X.

[Vitali's covering lemma] Let $X \subset \mathbb{R}$ be such that $\mu^*_{L,1}[X] < \infty$ and let Γ be a collection of intervals forming a Vitali cover for X. Then,

(i) There are countably many pairwise disjoint intervals $I_1, I_2, \ldots \in \Gamma$ such that $\mu^*_{L,1}[X \setminus U_n I_n] = 0$.

(ii) For every $\varepsilon > 0$, there exist finitely many pairwise disjoint intervals $I_1, \ldots, I_k \in \Gamma$ with the property that $\mu^*_{L,1}[X \setminus \bigcup_{n=1}^{k} I_n] < \varepsilon$.

Proof: Write $\mu^* = \mu^*_{L,1}$ for simplicity.

(i) With out loss of generality assume that every $I \in \Gamma$ is a (non-degenerate) closed interval. Choose an open set $U \subset \mathbb{R}$ such that $X \subset U$ and $\mu^*[U] < \infty$. Every $x \in X$ has a neighbourhood contained in U. Hence $\Gamma' = \{I \in \Gamma : I \subset U\}$ is also a vitali cover for X. We will choose the intervals I_n inductively. Let $\delta_0 = \sup\{|J| : J \in \Gamma'\}$ (note that $\delta_0 < \mu^*[U] < \infty$) and let $I_1 \in \Gamma'$ be any interval with $|I_1| > \delta_0/2$. Suppose that we have chosen pairwise disjoint intervals $I_1, \ldots, I_n \in \Gamma'$. If $X \subset \bigcup_{i=1}^{n} I_i$, then we are done. Else, any $x \in X \setminus \bigcup_{i=1}^{n} I_i$ is at a positive distance from the closed set $\bigcup_{i=1}^{n} I_i$. Let $\delta_n = \sup\{|J| : J \in \Gamma'$ and $I_i \cap J = \emptyset$ for $1 \le i \le n\}$. Then $0 < \delta_n \le \mu^*[U] < \infty$. Let $I_{n+1} \in \Gamma'$ be an interval with $|I_{n+1}| > \delta_n/2$. We will show that the sequence (I_n) does the job.

Observation: For every $J \in \Gamma$, there is $n \in \mathbb{N}$ such that $I_n \cap J \ne \emptyset$ ($\because \Sigma|I_n| \le \mu^*[U] < \infty$ so that $(|I_n|) \to 0$, and hence there is $n \in \mathbb{N}$ such that $|I_n| < |J|/2$).

Let $Y = X \setminus \bigcup_{n=1}^{\infty} I_n$ and $\varepsilon > 0$. We claim that $\mu^*[Y] < \varepsilon$. Let c_n be the midpoint of I_n and let $Y_n \subset \mathbb{R}$ be the closed interval with midpoint c_n and $|Y_n| = 6|I_n|$ (this Y_n may not be in Γ'). Let $k \in \mathbb{N}$ be so that $\sum_{n=k+1}^{\infty} |I_n| < \varepsilon/6$. If $x \in Y$, then in particular x does not belong to the closed set $\bigcup_{n=1}^{k} I_n$. Choose $J \in \Gamma'$ with $x \in J$ and $I_n \cap J = \emptyset$ for $1 \le n \le k$. By our observation above, $I_m \cap J \ne \emptyset$ for some $m \ge k + 1$. Let m be the smallest such number. Then $|J| \le \delta_{m-1} < 2|I_m|$ and hence $|x - c_m| \le |J| + |I_m| \le 3|I_m|$. Therefore, $x \in Y_m$. We have shown that $Y \subset \bigcup_{n=k+1}^{\infty} Y_n$. Since $\sum_{n=k+1}^{\infty} |Y_n| \le 6 \sum_{n=k+1}^{\infty} |I_n| < \varepsilon$, we have proved that $\mu^*[Y] < \varepsilon$.

Now, note that the argument given above actually shows that for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $\mu^*[X \setminus \bigcup_{n=1}^{k} I_n] < \varepsilon$. Hence we have established (ii) also.

When a mathematical problem is difficult, it is a good idea to divide the problem into many subcases and to treat each case separately. If $f : (a, b) \to \mathbb{R}$ is a function, then the four Dini derivatives of f at a point $x \in (a, b)$ are defined as follows.

$$D^+f(x) = \limsup_{h \to 0+} \frac{f(x+h) - f(x)}{h} \qquad \text{[upper right derivative]}$$

$$D_+f(x) = \liminf_{h \to 0+} \frac{f(x+h) - f(x)}{h} \qquad \text{[lower right derivative]}$$

$$D^{-}f(x) = \limsup_{h \to 0-} \frac{f(x+h) - f(x)}{h} \qquad \text{[upper left derivative]}$$

$$D_{-}f(x) = \liminf_{h \to 0-} \frac{f(x+h) - f(x)}{h} \qquad \text{[lower left derivative].}$$

Note Here, $\limsup_{h \to 0+} \frac{f(x+h) - f(x)}{h} := \lim_{y \to 0+} \left[\sup_{0<h<y} \frac{f(x+h) - f(x)}{h} \right]$, and similarly the others.

Example: Let f: (–1,1) → ℝ be f(0) = 0 and f(x) = x sin(1/x) for x = 0. Then, $D^{+}f(0) = 1 = D^{-}f(0)$ and $D_{+}f(0) = -1 = D_{-}f(0)$ so that f is not differentiable at 0.

Notes That f is differentiable at x iff all the four Dini derivatives are equal and real (i.e., different from ±∞). Since $D_{+}f(x) \le D^{+}f(x)$ and $D_{-}f(x) < D^{-}f(x)$ by definition, we also see that f is differentiable at x iff the four Dini derivatives are real numbers satisfying $D^{+}f(x) \le D_{-}f(x)$ and $D^{-}f(x) \infty D_{+}f(x)$.

[Lebesgue's differentiation theorem] Let $-\infty \le a < b \le \infty$, let f : (a, b) → ℝ be a monotone function and let Y = {x ∈ (a, b) : f is not differentiable at x}. Then $\mu^{*}_{L,1}$ [Y] = 0.

Proof: Since (a, b) can be written as a countable union of bounded open intervals, we may as well assume (a, b) itself is bounded. Assume f is increasing and write $\mu^{*} = \mu^{*}_{L,1}$. By the remark above, $Y = Y_1 \cup Y_2$, where $Y_1 = \{x \in (a, b) : D_{-} + f(x) < D^{+}f(x)\}$ and $Y_2 = \{x \in (a, b) : D_{+}f(x) < D^{-}f(x)\}$. We will only show that $\mu^{*}[Y_1] = 0$; the case of Y_2 is similar.

Let $\Gamma = \{(r, s) \in \mathbb{Q}^2 : r < s\}$, let $X(r, s) = \{x \in (a,b): D_{-}f(x) < r < s < D^{+}f(x)\}$ and note that $Y_1 = U_{(rs)\in\Gamma} X(r, s)$. Hence it suffices to show there $\mu^{*}[X(r, s)] = 0$ for every (r, s) ∈ Γ. Fix (r, s) ∈ Γ, write X = X(r, s) and let ε > 0 be arbitrary. Choose an open set U ⊂ (a, b) such that X ⊂ U and $\mu^{*}[U] < \mu^{*}[X] + \varepsilon$.

Since $D_{-}f < r$ on X, for each x ∈ X and δ > 0 we can find a non-degenerate closed interval I(x, δ) = [x – α, x] ⊂ U such that 0 < α < δ and f(x) – f(x – α) < rα. Then Γ = {I(x, δ) : x ∈ X, δ > 0} is a Vitali cover for X. By Vitali's lemma, we can find finitely many pairwise disjoint intervals $I_1,\ldots,I_k \in \Gamma$ such that $\mu^{*}[X \setminus \bigcup_{n=1}^{k} |I_n|] < \varepsilon$.

Let $V = \bigcup_{n=1}^{k} \text{int}[I_n]$. Then, V is open, V ⊂ U, and $\mu^{*}[X] - \varepsilon < \mu^{*}[V] \le \mu^{*}[U] < \mu^{*}[X] + \varepsilon$. Let X′ = V ∩ X. Since $D^{+}f > s$ on X, and hence on X′, for each y ∈ X′ and δ > 0 we can find a non-degenerate closed interval J(y, δ) = [y, y + β] ⊂ V (hence J(y, δ) ⊂ I_n for some n ∈ {1,…,k}) such that 0 < β < δ, and f(y + β) – f(y) > sβ. Then Γ = {J(y, δ) : y ∈ X′, δ > 0} is a Vitali cover for X′. Again by Vitali's lemma, we can find finitely many pairwise disjoint intervals $J_1,\ldots, J_m \in \Gamma$ such that $\mu^{*}[X' \setminus \bigcup_{j=1}^{m} |J_j|] < \varepsilon$. Then $\bigcup_{j=1}^{m} |J_j| \ge \mu^{*}[X'] - \varepsilon \ge \mu^{*}[X] - 2\varepsilon$.

Write $I_n = [x_n - \alpha_n, x_n]$ and $J_j = [y_j, y_j + \beta_j]$. For each n ∈ {1,…,k}, let $D_n = \{j \in \{1,..,m\} : J_j \subset I_n\}$. Then {1,…, m} is the disjoint union of D_n's.

Note that $\sum_{j\in D_n} (f(y_j + \beta_j) - f(y_j)) \le f(xn) - f(x_n - \alpha_n)$ for each n ∈ {1,…,k} since f is increasing. Summing over n, we get $\sum_{j=1}^{m} (f(y_j + \beta_j) - f(y_j)) \le \sum_{n=1}^{k} (f(x_n) - f(x_n - \alpha_n))$, and hence $\sum_{j=1}^{m} s\beta_j < \sum_{n=1}^{k} r\alpha_n$, or $s(\sum_{j=1}^{m} |J_j|) < r(\sum_{n=1}^{k} |I_n|)$. From the earlier estimates we conclude that $s(\mu^{*}[X] - 2\varepsilon) < r(\mu^{*}[X] + \varepsilon)$. Since ε > 0 was arbitrary and r < s, we must have $\mu^{*}[X] = 0$.

The conclusion is, it can be extended to more general class of real functions.

Definition: If f: [a, b] → ℝ is a function and P = {$a_0 = a \le a_1 \le \ldots \le a_{n-1} \le a_n = b$} is a partition of [a, b], let $V_a^b(f, P) = \sum_{i=1}^{n} |f(a_i) - f(a_{i-1})|$. Define the total variation of f as $V_a^b(f) = \sup\{V_a^b(f, P) : P$ is a partition of [a, b]}. We say f is of bounded variation if $V_a^b(f) < \infty$. It is easy to see that if f is of bounded variation, then f is bounded (∵ if x ∈ [a, b], take P = {a ≤ x ≤ b} to see that $|f(x) - f(a)| \le V_a^b(f)$).

Examples:

(i) If f : [a, b] → ℝ is a monotone function, then $V_a^b(f, P) = |f(b) - f(a)|$ for any partition P of [a,b] and hence $V_a^b(f) = |f(b) - f(a)| < \infty$. So f is of bounded variation.

(ii) Suppose f : [a,b] → ℝ is Lipschitz continuous (this happens if f is C^1) with Lipschitz constant λ > 0. Then, it may be seen that $V_a^b(f) \le \lambda(b - a) < \infty$ and hence f is of bounded variation.

Example: A (uniformly) continuous function f : [a, b] → ℝ need not be of bounded variation. Let f: [0, 1] → ℝ be the (uniformly) continuous function defined as f(0) = 0 and f(x) = x sin (1/x) if x ∈ (0,1). Let $a_k = 2/k\pi \in [0,1]$ for k ∈ ℕ. Observe that $|f(a_{2k}) - f(a_{2k-1})| = |0 - a_{2k-1}| = a_{2k-1}$. Let m ∈ ℕ and P = {$0 \le a_{2m} \le a_{2m-1} \le \ldots \le a_1 \le 1$}. Then $V_0^1(f, P) \ge \sum_{k=1}^{m} |f(a_{2k}) - f(a_{2k-1})| = \sum_{k=1}^{m} a_{2k-1} = (2/\pi) \sum_{k=1}^{m} (2K - 1)^{-1} \to \infty$ as m → ∞. Hence $V_0^1(f) = \infty$, and thus f is not of bounded variation. This example also shows that bounded ⇏ bounded variation.

Notes If f, g: [a, b] → ℝ are of bounded variation, r, s ∈ ℝ, and h : [a, b] → ℝ is defined as h = r f(x) + sg(x), then $V_a^b(h) \le |r| V_a^b(f) + |s| V_a^b(g) < \infty$. Hence {f : [a,b] → ℝ : f is of bounded variation } is a real vector space (in fact, it is a normed space with the norm $\|f\| = |f(a)| + V_a^b(f)$).

A function f : [a, b]→ ℝ is of bounded variation iff there exist monotone functions g, h : [a, b]→ ℝ such that f(x) = g(x) – h(x) for every x ∈ [a,b]. Consequently, for any function f : [a, b] → ℝ of bounded variation, we have $\mu_{L,1}^*$ [{x ∈ [a, b] : f is not differentiable at x}] = 0.

Proof: Suppose f = g – h, where g, h are monotone. Since g, h are of bounded variation, f is also of bounded variation since the collection of functions of bounded variation on [a, b] is a vector space. Conversely assume that f is of bounded variation and define g : [a, b] → ℝ as $g(x) = V_a^x(f)$. Then g is monotone increasing. Let h = g – f, and consider points x < y in [a, b]. We have $g(y) - g(x) = V_x^y(f) \ge |f(y) - f(x)| \ge f(y) - f(x)$, and therefore h(y) ≥ h(x). Thus h is also monotone increasing. Clearly, f = g – h.

Let [a, b] be a compact interval. Then for a function f : [a, b] → ℝ, we have the following implications: f is Lipschitz continuous ⇒ f is absolutely continuous ⇒ f is of bounded variation. Consequently, if f is either Lipschitz continuous or absolutely continuous, then $\mu_{L,1}^*[Y] = 0$, where Y = {x ∈ [a, b] : f is not differentiable at x}.

Note However, there is a limit to these type of results; there are continuous functions f : [a, b] → ℝ which are not differentiable at any point.

Now we mention a characterization of Riemann integrable functions in terms of small sets. For simplicity, we restrict ourselves to dimension one, even though the corresponding result is true in higher dimensions as well. If f : [a, b] → ℝ is a bounded function and if P = {$a_0 = a < a_1 \le \cdots \le a_{n-1}$

Notes

$\le a_n = b\}$ is a partition of [a, b], let $M_i = \sup\{f(x) : a_{i-1} \le x \le a_i\}$ and $m_i = \inf\{f(x) : a_{i-1} \le x \le a_i\}$. The upper and lower Riemann sums with respect to the partition P are defined as $U(f,P) = \sum_{i=1}^{n} M_i(a_i - a_{i-1})$ and $L(f, P) = \sum_{i=1}^{n} m_i(a_i - a_{i-1})$. A bounded function $f: [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for every $\varepsilon > 0$ there is a partition P of [a, b] such that $U(f,P) - L(f,P) < \varepsilon$. The following characterization says that a Riemann integrable function is not very different from a continuous function.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $Y = \{x \in [a, b] : f \text{ is not continuous at } x\}$. Then, f is Riemann integrable iff $\mu^*_{L,1}[Y] = 0$.

Proof: Let $\omega(f,x)$ be the oscillation of f at x defined earlier.

$\Rightarrow$: Since $Y = \bigcup_{k=1}^{\infty} Y_k$, where $Y_k = \{x \in [a, b] : \omega(f, x) \ge 1/k\}$, it suffices to show $\mu^*_{L,1}[Y_k] = 0$ for every $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and let $\varepsilon > 0$. Let $P = \{a_0 = a \le a_1 \le \cdots \le a_{n-1} \le a_n = b\}$ be a partition of [a, b] with $U(f,P) - L(f,P) < \varepsilon/2k$. Let $A_i = (a_{i-1}, a_i)$ and $\Gamma = \{1 \le i \le n : Y_k \cap A_i \ne \emptyset\}$. Note that $M_i - m_i \ge 1/k$ for $i \in \Gamma$. Write $Y_k = Y'_k \cap Y''_k$, where $Y'_k = Y_k \cap (\bigcup_{i\in\Gamma} A_i)$ and $Y''_k = Y_k \cap \{a_1, \ldots, a_n\}$. We have $\varepsilon/2k > U(f, P) - L(f, P) \ge \sum_{i\in\Gamma} (M_i - m_i)|A_i| \ge 1/k \sum_{i\in\Gamma} |A_i|$ and hence $\sum_{i\in\Gamma} |A_i| < \varepsilon/2$. And since Y''_k is a finite set, there are finitely many intervals $B_1, \ldots, B_m$ such that $Y''_k \subset \bigcup_{j=1}^{m} |Bj$ and $\sum_{j=1}^{m} |B_j| < \varepsilon/2$. Thus $Y_k \subset [\bigcup_{i\in\Gamma} A_i] \cup [\bigcup_{j=1}^{m} B_j]$ and $\sum_{i\in\Gamma} |A_i| + \sum_{j=1}^{m} |B_j| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\mu^*_{L,1}[Y_k] = 0$.

$\Leftarrow$: Let $\varepsilon > 0$ be given. We have to find a partition P of [a, b] such that $U(f, P) - L(f, P) < \varepsilon$. Let $\lambda = \sup\{|f(x)| : x \in [a, b]\}$ and let $\varepsilon' = \varepsilon/[2\lambda + 2(b - a)]$. For each $x \in [a, b]\setminus Y$, choose an open interval $A(x) \subset \mathbb{R}$ containing x such that $|f(x) - f(z)| < \varepsilon'$ for every $z \in [a, b] \cap \overline{A(x)}$ by continuity. Also choose countably many open intervals B_m such that $Y \subset \bigcup_{m=1}^{\infty} B_m$ and $\bigcup_{m=1}^{\infty} |B_m| < \varepsilon'$. Then $\{A(x) : x \in [a,b]\setminus Y\} \cap \{B_m : m \in \mathbb{N}\}$ is an open cover for the compact set [a, b]. Extract a finite subcover $\{A(x_j) : 1 \le j \le p\} \cup \{B_m : 1 \le m \le q\}$. The end points inside [a, b] of these finitely many intervals determine a partition $P = \{a_0 = a \le a_1 \le \cdots \le a_{n-1} \le a_n = b\}$ of [a, b]. Observe that for each $i \in \{1,\ldots,n\}$, we have $[a_{i-1}, a_i] \subset \overline{A(x_j)}$ for some $j \in \{1,\ldots,p\}$, or $[a_{i-1}, a_i] \subset \overline{B_m}$ for some $m \in \{1,\ldots,q\}$. Let $\Gamma = \{1 \le i \le n : [a_{i-1}, a_i] \subset \overline{A(x_j)}$ for some j$\}$ and $\Gamma' = \{1,\ldots,n\}\setminus\Gamma$. Note that $M_i - m_i < 2\varepsilon'$ if $i \in \Gamma$. Hence $U(f, P) - L(f,P) \le \sum_{i\in\Gamma} (M_i - m_i)(a_i - a_{i-1}) + \sum_{i\in\Gamma} (M_i - m_i)(a_i - a_{i-1}) \le 2\varepsilon' \sum_{i\in\Gamma} (a_i - a_{i-1}) + 2\lambda \sum_{i\in\Gamma} (a_i - a_{i-1}) \le 2\varepsilon' \sum_{i=1}^{n} (a_i - a_{i-1}) + 2\lambda \sum_{m=1}^{q} |B_m| < 2\varepsilon' (b - a) + 2\lambda\varepsilon' = \varepsilon$.

A corollary is that any bounded function $f : [a, b] \to \mathbb{R}$ with at most countably many points of discontinuity (in particular, any continuous function) is Riemann integrable. The higher dimensional generalization can be stated as follows.

Let $A \subset \mathbb{R}^d$ be a d-box, let $f : A \to \mathbb{R}$ be a bounded function and let Y be the set $\{x \in A : f$ is not continuous at $x\}$. Then, f is Riemann integrable iff $\mu^*_{L,d}[Y] = 0$.

Example: Let $f : [0,1] \to \mathbb{R}$ be $f(0) = 0$ and $f(x) = \sin(1/x)$ for $x = 0$. Even though the graph of f has infinitely many ups and downs (in fact, f is not of bounded variation), f is Riemann integrable since f is bounded and is discontinuous only at one point, namely 0.

Definition: Let X be a set and $A \subset X$. The characteristic function $\chi_A : X \to \mathbb{R}$ of the subset A is defined as $\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in X \setminus A. \end{cases}$

Example: We discuss an example that illustrates the main drawback of Riemann integration theory. Write $[0,1] \cap \mathbb{Q} = \{r_1, r_2, \ldots\}$, let $f_n : [0, 1] \to \mathbb{R}$ be the characteristic function of $\{r_1, \ldots, r_n\}$, and let $f : [0,1] \to \mathbb{R}$ be the characteristic function of $[0,1] \cap \mathbb{Q}$. We have $0 \le f_1 \le f_2 \le \cdots \le f \le 1$ and the sequence (f_n) converges to f pointwise. Each f_n is Riemann integrable since f_n is discontinuous only at finitely many points. But f is discontinuous at every point of [0,1], and the Lebesgue outer measure of [0,1] is positive. Hence f is not Riemann integrable by [115]. Thus even the pointwise limit of a uniformly bounded, monotone sequence of Riemann integrable functions need not be Riemann integrable.

Notes

(i) Let f: [0, 1] → $\mathbb{R}$ be the characteristic function of [0, 1] ∩ $\mathbb{Q}$. Since f is not continuous at any point, it is not possible to realize f as the pointwise limit of a sequence of continuous functions from [0, 1] to $\mathbb{R}$, in view of [108].

(ii) Let (f_n) be a sequence of continuous functions from [a, b] to $\mathbb{R}$ converging pointwise to a function f : [a, b] → $\mathbb{R}$, and let Y = {x ∈ [a, b] : f is not continuous at x}. From [108] we know that Y is an F_σ set of first category in [a, b]. But Y can have positive outer Lebesgue measure by [106]. Hence f may not be Riemann integrable. Thus even the pointwise limit of a sequence of continuous functions may not be Riemann integrable (of course, we did not give an example).

(iii) Lebesgue integration theory is developed not just for the sake of making the characteristic function of [0,1] ∩ $\mathbb{Q}$ integrable. The limit theorems in Lebesgue's theory allow us to integrate the pointwise limit of a sequence of integrable functions, and to interchange limit and integration, under very mild hypothesis. Moreover, the powerful tools in Lebesgue's theory make many proofs simpler (e.g.: the proof of the change of variable theorem in d-dimension), and provide us with new ways of dealing with functions (e.g.: L^p spaces). Also, as we will see later, in Lebesgue's theory we have a more satisfactory version of the Fundamental Theorem of Calculus (describing differentiation and integration as inverse operations of each other).

30.4 σ-algebras and Measurable Spaces

A d-box in $\mathbb{R}^d$ has a well-defined d-dimensional volume. We may ask whether it is possible to define the notion of a d-dimensional value for all subsets of $\mathbb{R}^d$. Of course, we would like to have consistency conditions such as monotonicity and countable additivity.

Question: Can we have a function $\mu : P(\mathbb{R}^d) \to [0, \infty]$ such that

(i) $\mu[A] = \text{Vol}_d(A)$ if $A \subset \mathbb{R}^d$ is ad-box,

(ii) [Monotonicity] $\mu[A] \le \mu[B]$ for subsets A, B of $\mathbb{R}^d$ with $A \subset B$,

(iii) [Countable additivity] $\mu[\bigcup_{n=1}^{\infty} A_n] = \sum_{n=1}^{\infty} \mu[A_n]$ if A_n's are pairwise disjoint subsets of $\mathbb{R}^d$?

Notes We know that the Lebesgue outer measure $\mu^*_{L,d}$ does not satisfy countable additivity. The key observation of Lebesgue's theory is that $\mu^*_{L,d}$ will satisfy all the three conditions stated above if we restrict $\mu^*_{L,d}$ to a slightly smaller collection $\mathcal{A} \subset \mathbb{P}(\mathbb{R}^d)$ by discarding some pathological subsets of $\mathbb{R}^d$. In order to describe the structure of this smaller collection $\mathcal{A}$, it is convenient to proceed in an abstract manner, which we do below.

Definition: Let X be a nonempty set. A collection $\mathcal{A} \in \mathbb{P}(X)$ of subsets of X is said to be a σ-algebra on X if the following hold:

(i) $\emptyset, X \in \mathcal{A}$.

(ii) $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$.

(iii) $A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

If $\mathcal{A}$ is a σ-algebra on X, then (X, $\mathcal{A}$) is called a measurable space.

Notes

Example: {Ø, X} and $\mathbb{P}(X)$ are trivial examples of σ-algebras on any nonempty set X. The following are some σ-algebras on $\mathbb{R}^d$ (verify):

$\mathcal{A}_1 = \{A \subset \mathbb{R}^d: A \text{ or } \mathbb{R}^d \setminus A \text{ is countable}\}$,

$\mathcal{A}_2 = \{A \subset \mathbb{R}^d: A \text{ or } \mathbb{R}^d \setminus A \text{ is of first category in } \mathbb{R}^d\}$,

$\mathcal{A}_3 = \{A \subset \mathbb{R}^d: \mu^*_{L,d}[A] = 0 \text{ or } \mu^*_{L,d}[\mathbb{R}^d \setminus A] = 0\}$.

$\mathcal{A}_4 = \{A \subset \mathbb{R}^d: [0, 1]^d \subset A \text{ or } [0, 1]^d \subset \mathbb{R}^d \setminus A\}$.

Definition: Let X be a nonempty set and $\mathcal{C} \subset \mathbb{P}(X)$ be a collection of subsets of X. A σ-algebra $\mathcal{A}$ on X is said to be generated by $\mathcal{C}$ if $\mathcal{A}$ is the smallest σ-algebra on X containing $\mathcal{C}$. Here, $\mathcal{A}$ exists and is unique since $\mathcal{A}$ is precisely the intersection of all σ-algebras on X containing $\mathcal{C}$ (note that there is at least one σ-algebra on X containing $\mathcal{C}$, namely $\mathbb{P}(X)$).

Definition: Let X be a metric space. Then the σ-algebra on X generated by the collection of all open subsets of X is called the Borel σ-algebra on X, and is denoted as $\mathcal{B}(X)$ (or just $\mathcal{B}$, if X is clear from the context). The subsets of X belonging to $\mathcal{B}(X)$ are called Borel subsets of X. For example, open subsets, closed subsets, G_δ subsets and F_σ subsets of X are Borel subsets of X.

[Characterizations of the Borel σ-algebra on $\mathbb{R}^d$] Consider the following collections of subsets of $\mathbb{R}^d$:

$\mathcal{C}_1 = \{A \subset \mathbb{R}^d: A \text{ is closed}\}$,

$\mathcal{C}_2 = \{A \subset \mathbb{R}^d: A \text{ is compact}\}$,

$\mathcal{C}_3 = \{A \subset \mathbb{R}^d: A \text{ is closed d-box}\}$,

$\mathcal{C}_4 = \{A \subset \mathbb{R}^d: A \text{ is an opend-box}\}$,

$\mathcal{C}_5 = \{A \subset \mathbb{R}^d: A \text{ is ad-box}\}$,

$\mathcal{C}_6 = \{A \subset \mathbb{R}^d: A \text{ is an open ball}\}$,

$\mathcal{C}_7 = \{f^{-1}(W) : f : \mathbb{R}^d \to \mathbb{R} \text{ is continuous and } W \subset \mathbb{R} \text{ is open}\}$.

If $\mathcal{A}_i$ is the σ-algebra on $\mathbb{R}^d$ generated by $\mathcal{C}_i$ for $1 \le i \le 7$, then $\mathcal{A}_i = \mathcal{B}(\mathbb{R}^d)$ for $1 \le i \le 7$.

Proof: Clearly $\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1 = \mathcal{B}(\mathbb{R}^d)$. Since $(a, b) = \bigcup_{n=n_0}^{\infty} [a + 1/n, b - 1/n]$ (where n_0 is chosen so that $a + 1/n_0 \le b - 1/n_0$), it follows that any open d-box°is a countable union of closed d-boxes, and therefore $\mathcal{A}_4 \subset \mathcal{A}_3$. Since $[a,b] = \bigcup_{n=1}^{\infty} (a - 1/n, b + 1/n)$, $[a, b) = \bigcup_{n=1}^{\infty} (a - 1/n, b)$, and $(a, b] = \bigcup_{n=1}^{\infty} (a, b + 1/n)$, we deduce that any d-box is a countable intersection of open d-boxes, and hence $\mathcal{A}_4 = \mathcal{A}_5$. Since any open set in $\mathbb{R}^d$ can be written as a countable union of open d-boxes as well as a countable union of open balls, we have $\mathcal{A}_4 = \mathcal{A}_6 = \mathcal{B}(\mathbb{R}^d)$. Thus $\mathcal{A}_i = \mathcal{B}(\mathbb{R}^d)$ for $1 \le i \le 6$.

By the definition of continuity, we have $\mathcal{A}_7 \subset \mathcal{B}(\mathbb{R}^d)$. If $U \subset \mathbb{R}^d$ is an open set different from $\mathbb{R}^d$, let $A = \mathbb{R}^d \setminus U$ and define $f : \mathbb{R}^d \to \mathbb{R}$ as $f(x) = \text{dist}(x, A) := \inf\{ \|x - a\| : a \in A\}$. Then f is continuous, and $A = f^{-1}(0)$ because A is closed. Now, $U = f^{-1}(\mathbb{R}\setminus\{0\})$ and $\mathbb{R}\setminus \{0\}$ is open in $\mathbb{R}$. Hence $\mathcal{B}(\mathbb{R}^d) \subset \mathcal{A}_7$, completing the proof.

Topological Remarks:

(i) If X is a separable metric space, then any base or subbase for the topology of X will generate the Borel σ-algebra $\mathcal{B}(X)$.

(ii) In the above characterization we used implicitly the fact that $\mathbb{R}^d$ is second countable and locally compact. If a metric space X fails to be second countable or locally compact, then the σ-algebra generated by all compact subsets of X will only be a proper sub-collection of $\mathcal{B}(X)$. For example, try to figure out what happens for the spaces ($\mathbb{R}$, discrete metric) (which is not second countable), and ($\mathbb{Q}$, Euclidean metric) (which is not locally compact).

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Next our aim is to determine the cardinality of $\mathcal{B}(\mathbb{R}^d)$. We need some set-theoretic preparation.

Definition: An order $\leq$ on a set X is a partial order if (i) $x \leq x$ for every $x \in X$, (ii) $x \leq y$ and $y \leq x \Rightarrow x = y$ for every $x, y \in X$, (iii) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ for every $x, y, z \in X$. We say $(X, \leq)$ is a totally ordered set if $\leq$ is a partial order and any two elements of X are comparable. We say $(X, \leq)$ is a well-ordered set if $(X, \leq)$ is totally ordered and any nonempty subset $Y \subset X$ has a least element in Y.

Examples:

(i) Let X be the collection of all nonempty subsets of $\mathbb{N}$. Define an order $\leq$ on X as $A \leq B$ iff the minimum of A is less than or equal to the minimum of B. Then this is not a partial order since the second condition fails.

(ii) If X is any nonempty set, then $\mathbb{P}(X)$ with inclusion as order is partially ordered, but in general not totally ordered.

(iii) $\mathbb{R}$ with the usual order is totally ordered, but not well-ordered since the subset (0,1) does not contain a least element.

(iv) $\mathbb{N}$ with the usual order is well-ordered.

Well-ordering principle (equivalent to the axiom of choice): Any nonempty set admits a well-ordering.

Now we describe the construction of some ordinal numbers. Start with an uncountable set X such that card(X) = card($\mathbb{R}$), and let $\leq$ be a well-ordering on X. Let θ denote the least element of X. By adding one extra element to X if necessary, we may also assume that $(X, \leq)$ has a largest element, say θ'. For each $\beta \in X$, let $L_\beta = \{\alpha \in X : \alpha < \beta\}$ be the left section of β in X. Let $Y = \{\beta \in X : L_\beta$ is uncountable$\}$. Then $Y \neq \emptyset$ since $\theta' \in Y$. So Y has a least element, say Ω. Then L_Ω is uncountable, but L_β is countable for every $\beta < \Omega$. Here, Ω is called the first uncountable ordinal, and each $\beta \in L_\Omega$ is called a countable ordinal number since each $\beta \in L_\Omega$ represents the type of a countable well-ordered set through L_β.

Fact: If $A \subset L_\Omega$ is a nonempty countable set, then A has a least upper bound in L_Ω. [*Proof:* If $B = \bigcup_{\beta\in A} L_\beta$, then B is countable and hence $L_\Omega \setminus B \neq \emptyset$. The least element of $L_\Omega \setminus B$ is the least upper bound of A]

If $\alpha \in L_\Omega$, then the least element of the nonempty set $\{\beta \in L_\Omega : \alpha < \beta\}$ will be denoted as $\alpha + 1$. Note that there are no elements between α and $\alpha + 1$ in L_Ω. On the other hand, given $\beta \in L_\Omega$, there may or may not exist $\alpha \in L_\Omega$ such that $\alpha + 1 = \beta$. For example, if $\beta \in L_\Omega$ is the least upper bound of the countable set $\{\theta, \theta + 1, \theta + 2, \ldots\}$ (where recall that θ is the least element of L_Ω), then there is no $\alpha \in X$ with $\alpha + 1 = \beta$. We say $\beta \in L_\Omega$ is a limit ordinal if there is no $\alpha \in L_\Omega$ with $\alpha + 1 = \beta$.

card($\mathcal{B}(\mathbb{R}^d)$) = card($\mathbb{R}$).

Proof: We will use transfinite induction (i.e., induction with respect to ordinal numbers) by using L_Ω described above. Recall that we denoted the least element of L_Ω by the symbol θ. To start the induction process, let $\mathcal{A}_\theta = \{U \subset \mathbb{R}^d : U \text{ is open}\}$. Let $\beta \in L_\Omega$ and assume that we have defined $\mathcal{A}_\alpha$ for every $\alpha \in L_\beta$. If β is a limit ordinal, define $\mathcal{A}_\beta = \bigcup_{\alpha<\beta} A_\alpha$. If $\beta = \alpha + 1$ for some $\alpha \in L_\Omega$, let $\mathcal{A}'_\alpha = \{A \subset \mathbb{R}^d : \mathbb{R}^d \setminus A \in \mathcal{A}_\alpha\}$, and $\mathcal{A}_\beta = \{A \subset \mathbb{R}^d : A$ is a countable union of members from $\mathcal{A}_\alpha \cup \mathcal{A}'_\alpha\}$. This defines $\mathcal{A}_\beta$ for every $\beta \in L_\Omega$. Finally, put $\mathcal{A} = \bigcup_{\beta<\Omega} \mathcal{A}_\beta$. From our construction, it is clear that $\mathcal{A} \subset \mathcal{B}(\mathbb{R}^d)$.

We verify that $\mathcal{A}$ is a σ-algebra on $\mathbb{R}^d$. It suffices to check only the third property. So consider $A_1, A_2, \ldots \in \mathcal{A}$. Then there are $\beta_1, \beta_2, \ldots \in L_\Omega$ such that $A_n \in \mathcal{A}_{\beta_n}$ for every $n \in \mathbb{N}$. By the Fact mentioned above, the countable set $\{\beta_n : n \in \mathbb{N}\}$ has a least upper bound, say δ in L_Ω. Then $A_n \in \mathcal{A}_\delta$

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for every $n \in \mathbb{N}$ and hence $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_{\delta+1} \subset \mathcal{A}$. Thus $\mathcal{A} \subset \mathcal{B}$ is a σ-algebra on $\mathbb{R}^d$ containing all open subsets of $\mathbb{R}^d$. Hence $\mathcal{A} = \mathcal{B}(\mathbb{R}^d)$.

Now, it suffices to show that card($\mathcal{A}$) = card($\mathbb{R}$). Since there is an open ball of radius 1 centered at each point of $\mathbb{R}^d$, we have card($\mathcal{A}$) ≥ card($\mathcal{A}_0$) ≥ card($\mathbb{R}$). So it suffices to establish that card($\mathcal{A}$) ≤ card($\mathbb{R}$). Since card(L_Ω) = card($\mathbb{R}$) and $\mathcal{A} = \bigcup_{\beta<\Omega} \mathcal{A}_\beta$, it is enough to show that card($\mathcal{A}_\beta$) ≤ card($\mathbb{R}$) for each $\beta \in L_\Omega$.

Let $\mathcal{D}$ be the collection of all open balls in $\mathbb{R}^d$ with rational radius and center in $\mathbb{Q}^d$. Then $\mathcal{D}$ is countable, and any open set $U \subset \mathbb{R}^d$ can be written as a countable union of members of $\mathcal{D}$. Hence card($\mathcal{A}_0$) ≤ card($\mathcal{D}^{\mathbb{N}}$) = card($\mathbb{R}$). Let $\beta \in L_\Omega$ and suppose we have proved that card($\mathcal{A}_\alpha$) ≤ card($\mathbb{R}$) for every $\alpha < \beta$. If β is a limit ordinal, then $\mathcal{A}_\beta = \bigcup_{\beta<\Omega} \mathcal{A}_\alpha$ is a countable union and hence card($\mathcal{A}_\beta$) ≤ card($\mathbb{R}$). If there is α with α + 1 = β, then any $A \in \mathcal{A}_\beta$ can be written as $A = \bigcup_{n=1}^{\infty} A_n$ with $A_n \in A_\alpha \cup \mathcal{A}_\alpha'$. This gives a one-one map from $\mathcal{A}_\beta$ into $(A_\alpha \cup \mathcal{A}_\alpha')^{\mathbb{N}}$. Hence card($\mathcal{A}_\beta$) ≤ card($(A_\alpha \cup \mathcal{A}_\alpha')^{\mathbb{N}}$) ≤ card($\mathbb{R}$). This completes the proof.

Corollary: For any uncountable set $Y \subset \mathbb{R}^d$, there is $A \subset Y$ such that A is not a Borel subset of$\mathbb{R}^d$.

Proof: We have card($\mathcal{B}(\mathbb{R}^d)$) = card($\mathbb{R}$) = card(Y) < card($\mathbb{P}$(Y)).

Definition: Let $\mathcal{N}(\mathbb{R}^d) = \{A \subset \mathbb{R}^d: \mu^*_{L,d}[A] = 0\}$. The members of $\mathcal{N}(\mathbb{R}^d)$ are called Lebesgue null sets. The σ-algebra $\mathcal{L}(\mathbb{R}^d)$ on $\mathbb{R}^d$ generated by $\mathcal{B}(\mathbb{R}^d) \cup \mathcal{N}(\mathbb{R}^d)$ is called the Lebesgue σ-algebra on $\mathbb{R}^d$, and members of $\mathcal{L}(\mathbb{R}^d)$ are called Lebesgue measurable subsets of $\mathbb{R}^d$.

card($\mathcal{N}(\mathbb{R}^d)$) = card($\mathcal{L}(\mathbb{R}^d)$) = card($\mathbb{P}(\mathbb{R}^d)$) > card($\mathbb{R}$). Hence, $\mathcal{N}(\mathbb{R}^d) \subsetneq \mathcal{B}(\mathbb{R}^d) \subsetneq \mathcal{L}(\mathbb{R}^d)$.

Proof: Let K be the middle-third Cantor set. Then, for any subset $A \subset K$, we have $\mu^*_{L,1}[A] \le \mu^*_{L,1}[K] = 0$. So $\mu^*_{L,d}[A] = 0$ also. This shows that $\mathbb{P}(K) \subset \mathcal{N}(\mathbb{R}^d) \subset \mathcal{L}(\mathbb{R}^d)$. And card($\mathbb{P}$(K)) = card($\mathbb{P}(\mathbb{R})$) since K is an uncountable subset of $\mathbb{R}$.

[Translation invariance] (i) $A + x \in \mathcal{B}(\mathbb{R}^d)$ for every $A \in \mathcal{B}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

(ii) $A + x \in \mathcal{N}(\mathbb{R}^d)$ for every $A \in \mathcal{N}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

(iii) $A + x \in \mathcal{L}(\mathbb{R}^d)$ for every $A \in \mathcal{L}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Proof: First let us mention a general principle that will be used at many places. To establish that the members of a certain σ-algebra $\mathcal{D}$ on a set X satisfies a certain property P, it suffices to do the following: show that the collection {A ⊂ X: A satisfies property P} is a σ-algebra, and then find a suitable collection $\mathcal{C} \subset \mathbb{P}(X)$ generating $\mathcal{D}$ and show that every member of $\mathcal{C}$ satisfies the property P.

Let $\mathcal{A} = \{A \subset \mathbb{R}^d: A + x \in \mathcal{B}(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d\}$. It is easy to check that $\mathcal{A}$ is a σ-algebra containing all d-boxes. And recall that the collection of all d-boxes generates $\mathcal{B}(\mathbb{R}^d)$. This proves (i). Next, statement (ii) is a consequence of the translation invariance property of the Lebesgue outer measure, and (iii) follows from (i) and (ii) by applying the principle mentioned above.

We will give other characterizations of the Lebesgue measurable sets shortly, and we will also show that $\mathcal{L}(\mathbb{R}^d) \neq \mathbb{P}(\mathbb{R}^d)$.

Self Assessment

Fill in the blanks:

1. .... is developed through approximations of a finite nature (e.g.: one tries to approximate the area of a bounded subset of $\mathbb{R}^2$ by the sum of the areas of finitely many rectangles).
2. While Riemann's theory is restricted to the Euclidean space, the ideas involved in .... are applicable to more general spaces, yielding an abstract measure theory.

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3. The .... $\mu^*_{j,d}[Y]$ of a bounded subset $Y \subset \mathbb{R}^d$ is defined as $\mu^*_{j,d}[Y] = \inf\{\sum_{n=1}^{k} Vol_d(A_n) : k \in \mathbb{N}$, and A_n's are d-boxes with $Y \subset \bigcup_{n=1}^{k} A_n\}$.

4. The construction above is due to Vitali, and hence the set Y is called a ....

5. Let X, Y be metric spaces and let f: X → Y be a function. Then the oscillation ω (f, x) of f at a point x ∈ X is defined as $\omega(f, x) = \lim_{\delta \to 0+} \text{diam}\,[f(B(x, \delta))]$. Clearly, f is .... at x iff ω (f, x) = 0.

6. Let $-\infty \le a < b \le \infty$ and let f: (a, b) → ℝ be a .... Then, Y = {x ∈ (a, b) : f is discontinuous at x} is a countable set (possibly empty).

7. Let X be a metric space. Then the σ-algebra on X generated by the collection of all open subsets of X is called the .... on X, and is denoted as $\mathcal{B}(X)$ (or just $\mathcal{B}$, if X is clear from the context).

8. If $A \subset L_\Omega$ is a .... set, then A has a least upper bound in L_Ω. [*Proof:* If B = $\bigcup_{\beta \in A} L_\beta$, then B is countable and hence $L_\Omega \setminus B \neq \emptyset$. The least element of $L_\Omega \setminus B$ is the least upper bound of A].

30.5 Summary

- Measure theory helps us to assign numbers to certain sets and functions – to a measurable set we may assign its measure, and to an integrable function we may assign the value of its integral. Lebesgue integration theory is a generalization and completion of Riemann integration theory. In Lebesgue's theory, we can assign numbers to more sets and more functions than what is possible in Riemann's theory. If we are asked to distinguish between Riemann integration theory and Lebesgue integration theory by pointing out an essential feature, the answer is perhaps the following.

  (i) We say $Y \subset \mathbb{R}^d$ is a discrete subset of $\mathbb{R}^d$ if for each y ∈ Y, there is an open set $U \subset \mathbb{R}^d$ such that U ∩ Y = {y}. For example, {1/n: n ∈ ℕ} is a discrete subset of ℝ.

  (ii) A subset $Y \subset \mathbb{R}^d$ is nowhere dense in $\mathbb{R}^d$ if $\text{int}[\overline{Y}] = \emptyset$, or equivalently if for any nonempty open set $U \subset \mathbb{R}^d$, there is a nonempty open set V ⊂ U such that V ∩ Y = 0. For example, if f: ℝ → ℝ is a continuous map, then its graph G(f) := {(x, f(x)):x ∈ ℝ} is nowhere dense in $\mathbb{R}^2$ (∵ G(f) is closed and does not contain any open disc).

  (iii) A subset $Y \subset \mathbb{R}^d$ is of first category in $\mathbb{R}^d$ if Y can be written as a countable union of nowhere dense subsets of $\mathbb{R}^d$; otherwise, Y is said to be of second category in $\mathbb{R}^d$. For example, Y = ℚ × ℝ is of first category in $\mathbb{R}^2$ since Y can be written as the countable union $Y = \bigcup_{r \in \mathbb{Q}} Y_r$, where Y_r := {r} × ℝ is nowhere dense in $\mathbb{R}^2$.

  (iv) (The following definition can be extended by considering ordinal numbers, but we consider only non-negative integers). For $Y \subset \mathbb{R}^d$ and integer n ≥ 0, define the nth derived set of Y inductively as $Y^{(0)} = Y$, $Y^{(n+1)}$ = {limit points of $Y^{(n)}$ in $\mathbb{R}^d$}. We say $Y \subset \mathbb{R}^d$ has derived length n if $Y^{(n)} \neq \emptyset$ and $Y^{(n+1)} \neq \emptyset$; and we say Y has infinite derived length if $Y^{(n)} \neq \emptyset$ for every integer n ≥ 0. For example, ℚ has infinite derived length (since $\overline{\mathbb{Q}}$ = ℝ), and {(1/m,1/n): m,n ∈ ℕ} has derived length 2.

  (v) We say $A \subset \mathbb{R}^d$ is a d-box if $A = \prod_{j=1}^{d} I_j$, where I_j's are bounded intervals. The d-dimensional volume of a d-box A is $Vol_d(A) = \prod_{j=1}^{d} |I_j|$. For example, $Vol_3([1, 4) \times [0,1/2] \times (-1,3]) = 6$.

  (vi) The d-dimensional Jordan outer content $\mu^*_{j,d}[Y]$ of a bounded subset $Y \subset \mathbb{R}^d$ is defined as $\mu^*_{j,d}[Y] = \inf\{\sum_{n=1}^{k} Vol_d(A_n) : k \in \mathbb{N}$, and A_n's are d-boxes with $Y \subset \bigcup_{n=1}^{k} A_n\}$.

(vii) The d-dimensional Lebesgue outer measure $\mu^*_{L,d}[Y]$ of an arbitrary set $Y \subset \mathbb{R}^d$ is defined as $\mu^*_{L,d}[Y] = \inf\{\sum_{n=1}^{\infty} Vol_d(A_n) : A_n$'s are d-boxes with $Y \subset \bigcup_{n=1}^{\infty} A_n\}$.

- If $f : [a, b] \to \mathbb{R}$ is a function and $P = \{a_0 = a \le a_1 \le \ldots \le a_{n-1} \le a_n = b\}$ is a partition of [a, b], let $V_a^b(f, P) = \sum_{i=1}^{n} |f(a_i) - f(a_{i-1})|$. Define the total variation of f as $V_a^b(f) = \sup\{V_a^b(f, P) : P$ is a partition of $[a, b]\}$. We say f is of bounded variation if $V_a^b(f) < \infty$. It is easy to see that if f is of bounded variation, then f is bounded ($\because$ if $x \in [a, b]$, take $P = \{a \le x \le b\}$ to see that $|f(x) - f(a)| \le V_a^b(f)$).

30.6 Keywords

Riemann Integration Theory: Riemann integration theory $\mapsto$ finiteness.

Lebesgue Integration Theory: Lebesgue integration theory $\mapsto$ countable infiniteness.

Baire Category Theorem: Let (X, ρ) be a complete metric space and let $U_n \subset X$ be open and dense in X for $n \in \mathbb{N}$. Then, $\bigcap_{n=1}^{\infty} U_n$ is also dense in X. In particular, $\bigcap_{n=1}^{\infty} U_n \ne \emptyset$.

Lebesgue's Differentiation Theorem: Let $-\infty \le a < b \le \infty$, let $f : (a, b) \to \mathbb{R}$ be a monotone function and let $Y = \{x \in (a, b) : f$ is not differentiable at $x\}$. Then $\mu^*_{L,1}[Y] = 0$.

Borel σ-algebra: If X is a separable metric space, then any base or subbase for the topology of X will generate the Borel σ-algebra $\mathcal{B}(X)$.

Well-ordering Principle: Well-ordering principle (equivalent to the axiom of choice): Any non-empty set admits a well-ordering.

30.7 Review Questions

1. If $f, g : [a, b] \to \mathbb{R}$ are of bounded variation, then fg is of bounded variation. [*Hint:* Let $M > 0$ be such that $|f|, |g| \le M$. Now, subtracting and adding the term $f(a_i)g(a_{i-1})$, note that $|(fg)(a_i) - (fg)(a_{i-1})| \le |f(a_i)||g(a_i) - g(a_{i-1})| + |f(a_i) - f(a_{i-1})||g(a_{i-1})|$ and hence $V_a^b(fg) \le M(V_a^b(f) + V_a^b(g))$.]

2. If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then $V_a^b(f) = V_a^c(f) + V_c^b(f)$. [*Hint:* If P_1 is a partition of [a, c] and P_2 is a partition of [c, b], then $V_a^c(f, P_1) + V_c^b(f, P_2) = V_a^b(f, P_1 \cup P_2) \le V_a^b(f)$. Conversely, if P is a partition of [a, b], first refine it by inserting c and then divide into partitions P_1 of [a, c] and P_2 of [c, b]. Check that $V_a^b(f, P) \le V_a^c(f, P_1) + V_c^b(f, P_2) \le V_a^c(f) + V_c^b(f)$.]

3. Let $f: [a, b] \to \mathbb{R}$ be a bounded function. If f is either monotone or of bounded variation, then f is Riemann integrable.

4. If $f, g; [a,b] \to \mathbb{R}$ are Riemann integrable, then $h := \max\{f, g\}$ is also Riemann integrable. [*Hint:* The set of discontinuities of h is contained in $\{x : f$ is not continuous at $x\} \cup \{x : g$ is not continuous at $x\}$.]

5. If $\mathcal{A}$ is a σ-algebra on a set X show that

   (i) $A \backslash B, A \Delta B \in \mathcal{A}$ if $A, B \in \mathcal{A}$,

   (ii) $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$ if $A_1, A_2, \ldots \in \mathcal{A}$.

6. Let $\mathcal{A} = \{A \subset \mathbb{R}^d : A$ is a countable (possibly finite or empty) union of d-boxes$\}$. Is $\mathcal{A}$ a σ-algebra on $\mathbb{R}^d$? [*Hint:* Let d = 1. Consider $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$, or the middle-third Cantor set and its complement.]

7. Are the following σ-algebras on $\mathbb{R}^d$: $\mathcal{A}_1 = \{A \subset \mathbb{R}^d : A$ or $\mathbb{R}^d \backslash A$ is open in $\mathbb{R}^d\}$ and $\mathcal{A}_2 = \{A \subset \mathbb{R}^d : A$ or $\mathbb{R}^d \backslash A$ is dense in $\mathbb{R}^d\}$?

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8. If A_α's are σ-algebras on a set X, then $\bigcap_\alpha \mathcal{A}_\alpha := \{A \subset X : A \subset \mathcal{A}_\alpha \text{ for every } \alpha\}$ is also a σ-algebra on X.

9. Show that $\mathcal{B}(\mathbb{R})$ is generated by each of the following collections: $\{(a, \infty) : a \in \mathbb{R}\}$, $\{[a, \infty) : a \in \mathbb{R}\}$, $\{(-\infty, b) : b \in \mathbb{R}\}$, $\{(-\infty, b] : b \in \mathbb{R}\}$, $\{(a, b) : a < b \text{ and } a, b \in \mathbb{Q}\}$.

10. (i) If card(X) ≤ card($\mathbb{R}$), then card($X^{\mathbb{N}}$) ≤ card($\mathbb{R}$). (ii) If card(J) ≤ card($\mathbb{R}$) and card(X_β) ≤ card($\mathbb{R}$) for each $\beta \in J$, then, card($\bigcup_{\beta \in J} X_\beta$) ≤ card($\mathbb{R}$). [*Hint:* (i) Assume X = (0,1). Define a one-one map $f : (0, 1)^{\mathbb{N}} \to (0,1)$ as follows. If $x = (x_n) \in (0, 1)^{\mathbb{N}}$ and if $x_n = 0.x_{n,1}x_{n,2}\cdots$, then $f(x) = 0.x_{1,1}x_{1,2}x_{2,1}x_{1,3}x_{2,2}x_{3,1}\cdots$. (ii) Let $g: \mathbb{R} \to J$ and $h_\beta : \mathbb{R} \to X_\beta$ be surjections. Then $f: \mathbb{R}^2 \to \bigcup_{\beta \in J} X_\beta$ defined as $f(x, y) = h_{g(y)}(x)$ is a surjection, and card($\mathbb{R}^2$) = card($\mathbb{R}$).]

Answers: Self Assessment

1.	Riemann integration theory	2.	Lebesgue's theory
3.	d-dimensional Jordan outer content	4.	Vitali set
5.	continuous	6.	monotone function
7.	Borel σ-algebra	8.	non-empty countable

30.8 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

Unit 31: The Integral of a Non-negative Function

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the integral of a non-negative function
- Explain Properties of the integral of non-negative functions
- Describe Monotone convergence theorem and
- Definition of Integrable function over a measurable set

Introduction

In this unit we are going to study about the definition and the properties of the integral of non-negative functions and some important theorems.

31.1 Integration of Non-negative Measurable Functions

We integrate non-negative measurable functions through approximation by bounded measurable functions vanishing outside a set of finite measure, which we studied earlier.

Definition: For a non-negative measurable function $f : E \to [0, \infty]$ (where E is a set which may be of finite or infinite measure), we define

$$\int_A f = \sup\{\int_A \varphi : \varphi \leq f \text{ on } A, \varphi \in B_0(E)\}$$

for any $A \subseteq E$.

Note that for non-negative bounded measurable functions vanishing outside a set of finite measure, this definition agrees with the old one. Also note that we allow the functions to take infinite value here.

We verify the monotonicity and linearity of such integrals.

Proposition: Suppose $f, g : E \to [0, \infty]$ are non-negative measurable and $A \subseteq E$.

(a) If $f \leq g$ a.e. on A then $\int_A f \leq \int_A g$.

(b) For $\alpha > 0$, $f + g$ and αf are non-negative measurable functions too and

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$$\int_A (f+g) = \int_A f + \int_A g$$

$$\int_A \alpha f = \alpha \int_A f$$

Proof:

(a) This is clearly true, for if $\varphi \in B_0(E)$ and $\varphi \le f$ on A, then $\varphi \le g$ on A so $\int_A \varphi \le \int_A g$ by definition of $\int_A g$. Taking supremum over all such φ's, we get $\int_A f \le \int_A g$.

(b) The assertion on $\int_A \alpha f$ can be proved using supremum arguments similar to that in (a) by noting that for $\alpha > 0$ and $\varphi \in B_0(E)$, $\varphi/\alpha \le f$ on A whenever $\varphi \le \alpha f$ on A, and $\alpha\varphi \le \alpha f$ on A whenever $\varphi \le f$ on A.

To verify $\int_A (f+g) = \int_A f + \int_A g$, note that if φ, $\psi \in B_0(E)$ and $\varphi \le f$, $\psi \le g$ on A, then $\varphi + \psi \in B_0(E)$ and $\varphi + \psi \le f + g$ on A so

$$\int_A (f+g) \ge \int_A (\varphi + \psi) \quad \text{(by definition of } \int_A (f+g))$$

$$= \int_A \varphi + \int_A \psi$$

take supremum over all such φ's and ψ's we have $\int_A (f+g) \ge \int_A f + \int_A g$. For the opposite inequality, note that if $\phi \in B_0(E)$ with $\phi \le f + g$ on A, then write $\varphi = \min\{\phi, f\}$ and $\psi = \phi - \varphi$ we see that φ, $\psi \in B_0(E)$ (note (i) $-M \le \varphi \le \phi \le M$ if $|\phi| \le M$ so φ is bounded on E; (ii) $\psi = \phi - \varphi$ is bounded on E because both ϕ and φ are; (iii) measurability of φ, ψ is clear; and (iv) from $\varphi = \min\{\phi, f\}$ and $\psi = \max\{0, \phi - f\}$ we see that φ, $\psi = 0$ whenever $\phi = 0$ so φ, ψ vanishes outside a set of finite measure). Further, we have $\varphi \le f$, $\psi \le g$ on A. Hence

$$\int_A \phi = \int_A \varphi + \int_A \psi$$

$$\le \int_A f + \int_A g$$

Taking supremum over all such ϕ's we get $\int_A (f+g) \le \int_A f + g$

Theorem 1: Fatou's Lemma

Suppose $\{f_n\}$ is a sequence of non-negative measurable functions defined on E and $\{f_n\}$ converges (pointwisely) to a non-negative function f a.e. on E. Then

$$\int_E f \le \liminf_{n\to\infty} \int_E f_n$$

Proof: Let $h \in B_0(E)$ and $h \le f$ on E. Then there exists $A \subseteq E$ with $m(A) < \infty$ such that $h = 0$ outside A. Let $h_n = \min\{f_n, h\}$ on A, we have h_n is uniformly bounded and measurable on A : in fact if $|h| \le M$ on E, then $h_n = \min\{f_n, h\} > \min\{0, h\} \ge -M$ and $h_n = \min\{f_n, h\} \le h \le M$ so $|h_n| \le M$ on A Further, with the observation that $\min\{a, b\} = (a + b - |a - b|)/2$ for all real a, b we have

$$h_n = \frac{f_n + h - |f_n - h|}{2} \to \frac{f + h - |f - h|}{2} = \min\{f, h\} = h$$

on A. Since $m(A) < \infty$, we can conclude by Bounded Convergence Theorem that $\int_A h = \lim_{n\to\infty} \int_A h_n$. So assuming $h_n = 0$ on $E\backslash A$, we have

$$\int_E h = \int_A h \lim_{n\to\infty} \int_A h_n = \lim_{n\to\infty} \int_E h_n \le \liminf_{n\to\infty} \int_E f_n$$

where the first equality follows from h = 0 on E/A and the last line $h_n \leq f_n$ on E for all n. Taking supremum over all such h's, we get the desired inequality.

Theorem 2: Monotone Convergence Theorem

If $\{f_n\}$ is an increasing sequence of non-negative measurable functions defined on E (increasing in the sense that $f_n \leq f_{n+1}$ for all n on E) and $f_n \to f$ a.e. on E, then

$$\int_E f_n \uparrow \int_E f$$

by which it means $\{j_E f_n\}$ is an increasing sequence with limit $\int_E f$.

In symbol,

$$0 \leq fn \uparrow f \text{ a.e. on } E \Rightarrow \int_E f_n \uparrow \int_E f$$

Proof:

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n \leq \limsup_{n\to\infty} \int_E f_n \leq \int_E f,$$

the first inequality follows from Fatou's Lemma, the last inequality follows from $f_n \leq f$ on E for all n. Hence $\int_E f_n \uparrow \int_E f$. (That $\int_E f_n$ increases as n increases is immediate from monotonicity of such integrals.)

Corollary: Extension of Fatou's lemma

If $\{f_n\}$ is a sequence of non-negative measurable functions on E, then $\int_E \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_E f_n$.

Proposition: Suppose f is a non-negative measurable function defined on E such that $\int_E f < \infty$. Then for all ε > 0, there is a δ > 0 such that

$$\int_E f < \varepsilon$$

whenever $A \subseteq E$ with m(A) < δ.

Proof: The result clearly holds if f is bounded on E. Suppose now f is not necessarily bounded, we see that $(f \wedge n) \uparrow f$ so by Monotone Convergence Theorem

$$\int_A f = \lim_{n\to\infty} \int_A (f \wedge n)$$

for all $A \subseteq E$. Note that by assumption $\int_E f < \infty$ so both sides of the equality above are finite. Hence if ε > 0 is given, then there is a N such that $\left|\int_A f - \int_A (f \wedge N)\right| < \varepsilon$.

Take δ = ε/2N, we see that

$$\int_A f \leq \left|\int_A f - \int_A f(f \wedge N)\right| + \int_A (f \wedge N) \leq \varepsilon/2 + Nm(A) \leq \varepsilon/2 + N\delta < \varepsilon$$

whenever $A \subseteq E$ with m(A) < δ. So we are done.

31.2 Extended Real-valued Integrable Functions

Here we integrated non-negative measurable functions, and we wish to drop the non-negative requirement. Recall that it is a natural requirement that our integral be linear, and now we can integrate a general non-negative measurable function, so it is tempting to define the integral of a general (not necessarily non-negative) measurable function f to be $\int f^+ - \int f^-$ where f+ = f V0 and f^- = (–f) V0, since f^+, f^- are non-negative measurable and they sum up to f. But it turns out that

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we cannot always do that, because it may well happen that $\int f^+$ and $\int f^-$ are both infinite, in which case their difference would be meaningless. (Remember that ∞ – ∞ is undefined.) So we need to restrict ourselves to a smaller class of functions than the collection of all measurable functions when we drop the non-negative requirement and come to the following definition.

Definition: For f : E → [–∞, ∞], denote f+ = f V0 and f^- = (–f) V0. Then f is said to be integrable if and only if both $\int_E f^+$ and $\int_E f^-$ are finite, in which case we define the integral of f by

$$\int_A f = \int_A f^+ - \int_A f^-$$

for any A ⊆ E

Notation: We shall denote the class of all (extended real-valued) integrable functions defined on E by C(E).

Note that in the above definition, f^+ and f^- are both non-negative measurable, so for any set $A \subseteq E$, $\int_A f^+$ and $\int_A f^-$ are both defined. Furthermore, $\int_A f^+ \le \int_E f^+ < \infty$ and similarly $\int_A f^- < \infty$ so their difference makes sense now. Also note that for non-negative integrable functions this definition agrees with our old one.

We provide an alternative characterization of integrable functions.

Proposition: A measurable function f defined on E is integrable if and only if $\int_E |f| < \infty$ so.

Proof: Just note that $|f| = f^+ + f^-$.

We proceed to investigate the structure of $\mathcal{L}$(E). We want to say it is a vector lattice. But we have to be careful here: Given f, g ∈ $\mathcal{L}$(E) it may well happen that f(x) = + ∞ and g(x) = –∞ for some x ∈ E and then f + g cannot be defined by f(x) + g(x) at that x. Luckily there cannot be too many such x's, in the sense that the set of all such x's is of measure zero. In fact every integrable function is finite. We know that the values of a function on a set of measure zero are not important as far as integration is concerned. (This was observed as in the case of bounded measurable functions vanishing outside a set of finite measure; the reader should verify this for the case of general integrable functions as well.) So that eliminates our previous worries: more precisely, let us agree from now on two functions f,g: E → [–∞, ∞] are said to be equal (write f = g) if and only if they take the same values a.e.on E, and f + g shall mean a function whose value at x is equal to f(x) + g(x) for a.e.x ∈ E. Also say f ≤ g if and only if f(x) ≤ g(x) for a.e. x ∈ E. Then we have the following proposition.

Proposition: $\mathcal{L}$(E) forms a vector lattice (partially ordered by ≤).

Proof: If f,g ∈ $\mathcal{L}$(E), then $\int_E |f+g| \le \int_E |f| + \int_E |g| < \infty$ (we are using linearity and monotonicity and hence f + g ∈ $\mathcal{L}$(E) (the measurability of f + g is previously known). The rest of the proposition is trivial.

With the vector lattice structure of $\mathcal{L}$(E) it is natural to ask whether the integral is linear and monotone or not. We expect it to be true; we verify it below.

Proposition: For any f,g ∈ $\mathcal{L}$(E) and A ⊆ E, we have $\int_A (f+g) = \int_A f + \int_A g$ and $\int_A \alpha f = \alpha \int_A f$. Furthermore, if f ≤ g a.e. on A then $\int_A f \le \int_A g$.

Proof: The parts for monotonicity and $\int_A \alpha f = \alpha \int_A f$ are easy and left as an exercise.

So now let f,g ∈ $\mathcal{L}$(E) and A ⊆ E be given, and we prove $\int_A (f+g) = \int_A f + \int_A g$. By definition of the integral, the LHS is just $\int_A (f+g)^+ - \int_A (f+g)^-$ and the RHS is $\int_A f^+ - \int_A f^- + \int_A g^+ \int_A g^-$, all terms being finite. So it suffices to show

(6) $\int_A (f+g)^+ + \int_A f^- + \int_A g^- = \int_A (f+g)^- + \int_A f^+ + \int_A g^+$,

which will be true if we can show

(7) $(f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+$

a.e. on A because we can then use linearity of Section 3 to conclude that (6) is true. But (7) is clearly true a.e., because $(f + g)^+ – (f + g)^- = f + g = f^+ – f^- + g^+ – g^-$ a.e., all terms being finite a.e. This completes our proof.

Finally we prove the important Generalized Lebesgue Dominated Convergence Theorem.

Theorem 3: If $\{f_n\}$, $\{g_n\}$ are sequences of measurable functions defined on E, $|f_n| \le g_n$, $f = \lim_{n\to\infty} f_n$, $g = \liminf_{n\to\infty} g_n$ and $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$, then $\lim_{n\to\infty} \int_E f_n$ exists and is equal to $\int_E f$.

Proof: Since $|f_n| \le g_n$ implies $g_n \pm f_n$ are non-negative measurable, we see that

$$\int_E g + \int_E f = \int_E \liminf_{n\to\infty} (g_n + f_n) \le \liminf_{n\to\infty} \int_E (g_n + f_n) = \int_E g + \liminf_{n\to\infty} \int_E f_n$$

and similarly

$$\int_E g - \int_E f = \int_E \liminf_{n\to\infty} (g_n - f_n) \le \liminf_{n\to\infty} \int_E (g_n - f_n) = \int_E g + \liminf_{n\to\infty} \int_E f_n$$

So $\int_E f \le \liminf_{n\to\infty} \int_E f_n \le \limsup_{n\to\infty} \int_E f$ (note here we used the assumption that $\int_E g < \infty$) and the desired conclusion follows.

Corollary: Lebesgue Dominated Convergence Theorem

Suppose a sequence of measurable functions $\{f_n\}$ defined on E converges pointwisely a.e. on E to f. If $|f_n| \le g$ on E for some integrable function g, then $\int_E f_n$ converges to $\int_E f$.

A final word of remark: The idea of this section extends readily to complex-valued functions, and the readers who are familar with general measure theory should find that the results in the whole unit is valid on a general measure space without needing the slightest modification.

Self Assessment

Fill in the blanks:

1. For a non-negative measurable function $f : E \to [0, \infty]$ (where E is a set which may be of finite or infinite measure), we define ....
2. For non-negative .... vanishing outside a set of finite measure, this definition agrees with the old one. Also note that we allow the functions to take infinite value here.
3. Suppose $\{f_n\}$ is a sequence of non-negative measurable functions defined on E and $\{f_n\}$ converges (pointwisely) to a .... f a.e. on E. Then $\int_E f \le \lim_{n\to\infty} \inf \int_E f_n$.
4. If $\{f_n\}$ is an .... of non-negative measurable functions defined on E (increasing in the sense that $f_n \le f_{n+1}$ for all n on E) and $f_n \to f$ a.e. on E, then $\int_E f_n \uparrow \int_E f$ by which it means $\{\int_E f_n\}$ is an increasing sequence with limit $\int_E f$.
5. A .... f defined on E is integrable if and only if $\int_E |f| < \infty$ so.
6. For any $f,g \in \mathcal{L}(E)$ and $A \subseteq E$, we have $\int_A (f+g) = \int_A f + \int_A g$ and $\int_A \alpha f = \alpha \int_A f$. Furthermore, if $f \le g$ a.e. on A then ....

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7. If $\{f_n\}$, $\{g_n\}$ are sequences of measurable functions defined on E, $|f_n| \le g_n$, $f = \lim_{n\to\infty} f_n$, $g = \liminf_{n\to\infty} g_n$ and ...., then $\lim_{n\to\infty} \int_E f_n$ exists and is equal to $\int_E f$.

8. Suppose a sequence of measurable functions $\{f_n\}$ defined on E converges pointwisely a.e. on E to f. If $|f_n| \le g$ on E for some integrable function g, then $\int_E f_n$ .... $\int_E f$.

31.3 Summary

- For a non-negative measurable function $f : E \to [0, \infty]$ (where E is a set which may be of finite or infinite measure), we define

  $$\int_A f = \sup\left\{\int_A \varphi : \varphi \le f \text{ on } A, \varphi \in B_0(E)\right\}$$

  for any $A \subseteq E$.

  Note that for non-negative bounded measurable functions vanishing outside a set of finite measure, this definition agrees with the old one. Also note that we allow the functions to take infinite value here.

- Suppose $\{f_n\}$ is a sequence of non-negative measurable functions defined on E and $\{f_n\}$ converges (pointwisely) to a non-negative function f a.e. on E. Then

  $$\int_E f \le \liminf_{n\to\infty} \int_E f_n$$

- If $\{f_n\}$ is an increasing sequence of non-negative measurable functions defined on E (increasing in the sense that $f_n \le f_{n+1}$ for all n on E) and $f_n \to f$ a.e. on E, then

  $$\int_E f_n \uparrow \int_E f$$

  by which it means $\{\int_E f_n\}$ is an increasing sequence with limit $\int_E f$.

- If $\{f_n\}$ is a sequence of non-negative measurable functions on E, then $\int_E \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_E f_n$. The proof is easy and left as an exercise.

  The following proposition is concerned with the absolute continuity of the integral.

- Suppose f is a non-negative measurable function defined on E such that $\int_E f < \infty$. Then for all $\varepsilon > 0$, there is a $\delta > 0$ such that

  $$\int_E f < \varepsilon$$

  whenever $A \subseteq E$ with $m(A) < \delta$.

- Suppose a sequence of measurable functions $\{f_n\}$ defined on E converges pointwisely a.e. on E to f. If $|f_n| \le g$ on E for some integrable function g, then $\int_E f_n$ converges to $\int_E f$.

31.4 Keywords

Fatou's Lemma: Suppose $\{f_n\}$ is a sequence of non-negative measurable functions defined on E and $\{f_n\}$ converges (pointwisely) to a non-negative function f a.e. on E. Then $\int_E f \le \liminf_{n\to\infty} \int_E f_n$.

Monotone Convergence Theorem: If $\{f_n\}$ is an increasing sequence of non-negative measurable functions defined on E (increasing in the sense that $f_n \le f_{n+1}$ for all n on E) and $f_n \to f$ a.e. on E, then $\int_E f_n \uparrow \int_E f$ by which it means $\{\int_E f_n\}$ is an increasing sequence with limit $\int_E f$.

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Lebsegue Dominated Convergence Theorem: Suppose a sequence of measurable functions $\{f_n\}$ defined on E converges pointwisely a.e. on E to f. If $|f_n| \le g$ on E for some integrable function g, then $\int_E f_n$ converges to $\int_E f$.

31.5 Review Questions

1. For a non-negative measurable function f defined on E, show that $\int_A f = \int_E f \chi_A$ for any A Í E. Also show that $\int_A f \le \int_B f$ if $A \subseteq B \subseteq E$.
2. Show that if A, B C E are disjoint and f is a non-negative measurable function defined on E, then $\int_{A \cup B} f = \int_A f + \int_B f$.
3. Show that if f is a non-negative measurable function defined on E and $\int_E f = 0$, then f = 0 a.e. on E.
4. Show that if f is a non-negative measurable function defined on E and $\int_E f < \infty$, then f is finite a.e.
5. Show that w may have strict inequality in Fatou's Lemma.

   (***Hint:*** Consider the sequence {fn} defined by fn(x) = 1 if n x < n + 1,with fn(x) = 0 otherwise.)
6. Show that the monotone convergence theorem need not hold for decreasing sequence of functions.

   (***Hint:*** Let fn(x) = 0, if x < n, fn(x) = 1 for xn.)
7. Show that if f and g are measurable and y $|f| \le |g|$ a.e., and if g is integrable, then prove that f is intergrable.

Answers: Self Assessment

1.	$\int_A f = \sup\left\{\int_A \varphi : \varphi \le f \text{ on } A, \varphi \in B_0(E)\right\}$	2.	bounded measurable functions
3.	non-negative function	4.	increasing sequence
5.	measurable function	6.	$\int_A f \le \int_A g$
7.	$\lim\limits_{n \to \infty} \int_E g_n = \int_E g < \infty$	8.	converges to

31.6 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

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Unit 32: The General Lebesgue Integral and Convergence in Measure

CONTENTS

Objectives

After studying this unit, you will be able to:

- Explain the General Lebesgue integral of a measurable function
- Discuss the Properties of Lebesgue integral
- Discuss Lebesgue convergence theorem
- Explain Generalization of Lebesgue convergence theorem
- Describe convergence in measure of a sequence of measurable functions

Introduction

In this unit, you are going to study about the general Lebesgue integral, some of its properties, convergence in measure and theorems related to them.

32.1 The General Lebesgue Integral

Definition: The positive part of a function f is $f^+ = f \vee 0$ i.e $f^+ (x) = \max \{f(x), 0\}$

The negative part of a function is $f^- = f \wedge 0$. i.e $f^-(x) = \min \{f(x), 0\}$

Hence $f = f^+ - f^-$.

And $|f| = f^+ + f^-$

Definition: A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E.

Then the integral of f is defined as

$$\int_E f = \int_E f^+ - \int_E f^-$$

Theorem 1: Let f and g are integrable over E. Then

(i) The function cf is integrable over E, and $\int_E cf = c\int_E f$.

(ii) The function f + g is integrable over E, and $\int_E f + g = \int_E f + \int_E g$.

(iii) If f g a.e., then $\int_E f \quad \int_E g$

(iv) If A and B are disjoint measurable sets contained in E, then $\int_{A \cup B} f = \int_A f + \int_B f$

Proof:

(i) Since f is integrable over E, both f^+ and f^- are integrable over E and the integral of f is given by

$$\int_E f = \int_E f^+ - \int_E f^-$$

Hence,

both cf^+ and cf^- are integrable over E, and hence, $cf = cf^+ - cf^-$ are integrable over E and

$$\begin{aligned} \int_E cf &= \int_E cf^+ - \int_E cf^- \\ &= c\int_E f^+ - c\int_E f^- \\ &= c[\int_E f^+ - \int_E f^-] \\ &= c\int_E f. \end{aligned}$$

Hence (i) is proved.

(ii) Suppose if f_1 and f_2 are nonnegative integrable functions with $f = f_1 - f_2$,

Then $f^+ - f^- = f_1 - f_2$.

Hence,

$$f^+ + f_2 = f^- + f_1.$$

As you know

$$f^+ + f_2 = f^- - f_1.$$

Therefore,

$$f = f^+ - f^-$$

$$= f_1 - f_2.$$

Since f and g are measurable,

f^+, f^-, g^+, g^- are measurable.

Hence,

$f^+ + g^+, f^- + g^-$ are also measurable.

And $\quad f + g = (f^+ + g^+) - (f^- + g^-)$.

Hence by(1),

$$(f + g) = (f^+ + g^+) - (f^- + g^-)$$

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$= f^+ + g^+ – f^- – g^-$

$= (f^+ – f^-) + (g^+ – g^-)$

$= f + g.$

Hence (ii) is proved.

(iii) Since f g a.e., f^+ - f^- $g^+ – g^-$ a.e.,

Hence, $f^+ + g – g^+ + f^-$ a.e,

$(f^+ + g^-)$ $(g^+ + f^-)$.

Hence

$f^+ + g – g^+ + f^-$.

Hence,

$f^+ – f – g^+ – g^-$

Hence,

f g.

Hence (iii) is proved.

(iv) Consider

$$\begin{aligned}\int_{A\cup B} f &= \int f \cdot \chi_{A\cup B} \\ &= \int f \cdot (\chi_A + \chi_B) \\ &= \int f \cdot \chi_A + \int f \cdot \chi_B \\ &= \int_A f + \int_B f\end{aligned}$$

32.2 Lebesgue Convergence Theorem

Theorem 2: Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n|$ g on E and for almost all x in E we have f(x) = lim fn(x). Then

$$\int_E f = \lim \int_E f_n$$

Proof: Since $|f_n|$ g on E, g – f_n is nonnegative and hence by Fatou's Lemma,

$$\int_E (g - f) \underline{\lim} \int_E (g - f_n) \quad ...(1)$$

Since f(x) = lim $f_n(x)$ a.e. on E and

$|f_n|$ g on E,

$|f|$ g on E.

Hence since g is integrable,

f is also integrable.

$$\int_E (g - f) = \int_E g - \int_E f \quad ...(2)$$

Also,

$$\underline{\lim} \int_E (g - f_n) = \int_E g - \overline{\lim} \int_E f_n \quad ...(3)$$

Substituting (2) and (3) in (1), we get

$$\int_E g - \int_E f \quad \int_E g - \overline{\lim} \int_E f_n$$

Hence

$$\int_E f \quad \overline{\lim} \int_E f_n \qquad ...(4)$$

Similarly by considering g + f_n, we get

$$\int_E f \quad \underline{\lim} \int_E f_n \qquad ...(5)$$

From (4) and (5), we get

$$\overline{\lim} \int_E f_n \quad \int_E f \quad \underline{\lim} \int_E f_n \qquad ...(6)$$

But it is always true that

$$\underline{\lim} \int_E f_n \quad \overline{\lim} \int_E f_n \qquad ...(7)$$

From (6) and (7)

$$\int_E f = \lim \int_E f_n \text{ .}$$

Hence the theorem.

> *Notes* If we replace g by g_n's, we get the following generalization of the Lebesgue Convergence theorem.

Theorem 3: Let $\{g_n\}$ be a sequence of integrable functions which converges a.e to an integrable function g. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \ g_n$ and $\{f_n\}$ converges to f a.e.

If $\int g = \lim \int g_n$,

then $\int f = \lim \int f_n$.

32.3 Convergence in Measure

Definition: A sequence $\{f_n\}$ of measurable functions is said to converge to f in measure if, given $\varepsilon > 0$, there is an N such that for all n N we have

$$m\{x / |f(x) - f_n(x)| \ \varepsilon\} < \varepsilon.$$

Remark: From this definition and littlewood's third principle, it is clear that,

If $\{f_n\}$ is a sequence of measurable functions defined on a measurable set E of finite measure and $f_n > f$ a.e, then $\{f_n\}$ converges to f in measure.

Example: Construct the sequence $\{f_n\}$ as follows:

Let $\quad n = k + 2^v$, $0 \ k < 2^v$, and

Set $\quad f_n(x) = 1$ if $x \in [k2^{-v}, (k + 1) 2^{-v}]$

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And $\quad f_n(x) = 0$ otherwise.

Then $\quad m\{x/\,|fn(x)| > \varepsilon\} = 2^{-\nu} \;\; 2/n \quad [\text{since } 2^{\nu} \;\; n < 2^{\nu} + 1]$

Hence $\quad f_n > 0$ in measure.

> *Notes* That the sequence $\{f_n(x)\}$ has the value 1 for arbitrarily large values of n.
>
> Hence $\{f_n(x)\}$ does not converge for any x in [0, 1].

Theorem 4: Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f.

Then there is a subsequence $\{f_{nk}\}$ that converges to f almost everywhere.

Proof: Since $\{f_n\}$ is a sequence of measurable functions that converges in measure to f,

Given ν, there is an integer n_ν such that for all n nν,

$$m\{x/|f(x) - f_n(x)| \;\; 2^{-\nu}\} < 2^{-\nu} \qquad \text{...(1)}$$

Let $\quad E_\nu = \{x/\,|f_{n\nu}(x) - f(x)| \;\; 2^{-\nu}\}$

Therefore,

if $\quad x \notin \bigcup_{\upsilon=k}^{\infty} E_\upsilon$

then $\quad |f_{n\nu}(x) - f(x)| < 2^{-\nu}$ for νk.

Therefore,

$$F_{n\nu}(x) > f(x).$$

Hence $f_{n\nu}(x) > f(x)$ for any $x \notin A \bigcap_{k=1}^{\infty} \bigcup_{\upsilon=k}^{\infty} E_\upsilon$

But $\quad mA \;\; m\left[\bigcup_{\upsilon=k}^{\infty} E_\upsilon\right]$

$$\sum_{\upsilon=k}^{\infty} mE_\upsilon$$

$$= 2^{-k+1}.$$

Hence $\quad mA = 0$

Theorem 5: Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure.

Then $\{f_n\}$ converges to f in measure if and only if every subsequence of $\{f_n\}$ has in turn a subsequence that converges almost everywhere to f.

Theorem 6: Fatou's lemma and the monotone and Lebesgue Convergence theorem remain valid if 'convergence a.e.' is replaced by 'convergence in measure'.

Self Assessment

Notes

Fill in the blanks:

1. A .... f is said to be integrable over E if f^+ and f^- are both integrable over E.
2. Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n|$ g on E and for almost all x in E we have …………………….
3. Let $\{g_n\}$ be a sequence of ……………….. which converges a.e to an integrable function g.
4. A sequence $\{f_n\}$ of measurable functions is said to ……………….....….. in measure if, given $\varepsilon > 0$, there is an N such that for all nN we have m{x/ | f(x) – fn(x) | ε} < ε.
5. Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f. Then there is a subsequence {nk f} that .... to f almost everywhere.

32.4 Summary

- Definition of General Lebesgue integral of a measurable function
- Properties of Lebesgue integral
- Lebesgue convergence theorem
- Generalization of Lebesgue convergence theorem
- Definition of convergence in measure of a sequence of measurable functions and
- Every sequence of measurable sequence that converges in measure contains a subsequence that converges almost everywhere.

32.5 Keywords

Convergence in Measure: A sequence $\{f_n\}$ of measurable functions is said to converge to f in measure if, given $\varepsilon > 0$, there is an N such that for all n N we have $m\{x/ |f(x) - f_n(x)| \varepsilon\} < \varepsilon$.

Lebesgue Convergence Theorem: Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n|$ g on E and for almost all x in E we have $f(x) = \lim f_n(x)$. Then

$$\int_E f = \lim \int_E f_n.$$

32.6 Review Questions

1. Show that if f is integrable over E, then so is $|f|$ and $\left|\int_E f\right| \le \int_E |f|$. Does the integrability of $|f|$ imply that of f?.
2. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n > f$ a.e with f integrable. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.
3. Show that if f is integrable over E, then $|f|$ is also integrable over E. further $\left|\int_E f\right| \le \int_E |f|$ is the converse true?

Answers: Self Assessment

1.	measurable function	2.	f(x) = lim fn(x).
3.	integrable functions	4.	converge to f
5.	converges		

Notes

32.7 Further Readings

Books

Walter Rudin: Principles of Mathematical Analysis (3rd edition), Ch. 2, Ch. 3. (3.1-3.12), Ch. 6 (6.1 - 6.22), Ch.7(7.1 - 7.27), Ch. 8 (8.1- 8.5, 8.17 - 8.22).

G.F. Simmons: Introduction to Topology and Modern Analysis, Ch. 2(9-13), Appendix 1, p. 337-338.

Shanti Narayan: A Course of Mathematical Analysis, 4.81-4.86, 9.1-9.9, Ch.10,Ch.14, Ch.15(15.2, 15.3, 15.4)

T.M. Apostol: Mathematical Analysis, (2nd Edition) 7.30 and 7.31.

S.C. Malik: Mathematical Analysis.

H.L. Royden: Real Analysis, Ch. 3, 4.

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