Topology

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TOPOLOGY

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SYLLABUS

Topology

Objectives: For some time now, topology has been firmly established as one of basic disciplines of pure mathematics. It's ideas and methods have transformed large parts of geometry and analysis almost beyond recognition. In this course we will study not only introduce to new concept and the theorem but also put into old ones like continuous functions. Its influence is evident in almost every other branch of mathematics.In this course we study an axiomatic development of point set topology, connectivity, compactness, separability, metrizability and function spaces.

Sr. No.	Content
1	Topological Spaces, Basis for Topology, The order Topology, The Product Topology on X * Y, The Subspace Topology.
2	Closed Sets and Limit Points, Continuous Functions, The Product Topology, The Metric Topology, The Quotient Topology.
3	Connected Spaces, Connected Subspaces of Real Line, Components and Local Connectedness,
4	Compact Spaces, Compact Subspaces of Real Line, Limit Point Compactness, Local Compactness
5	The Count ability Axioms, The Separation Axioms, Normal Spaces, Regular Spaces, Completely Regular Spaces
6	The Urysohn Lemma, The Urysohn Metrization Theorem, The Tietze Extension Theorem, The Tychonoff Theorem
7	The Stone-Cech Compactification, Local Finiteness, Paracompactness
8	The Nagata-Smirnov Metrization Theorem, The Smirnov Metrization Theorem
9	Complete Metric Spaces, Compactness in Metric Spaces, Pointwise and Compact Convergence, Ascoli’s Theorem
10	Baire Spaces, Introduction to Dimension Theory

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Unit 1: Topological Spaces

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Objectives

After studying this unit, you will be able to:

- Describe the concept of topological spaces;

- Explain the different kinds of topologies
- Solve the problems on intersection and union of topologies;
- Define open set and closed set;
- Describe the neighborhood of a point and solve related problems;
- Explain the dense set, separable space and related theorems and problems;
- Know the concept of limit point and derived set;
- Define interior and exterior of a set.

Introduction

Topology is that branch of mathematics which deals with the study of those properties of certain objects that remain invariant under certain kind of transformations as bending or stretching. In simple words, topology is the study of continuity and connectivity.

Topology, like other branches of pure mathematics, is an axiomatic subject. In this, we use a set of axioms to prove propositions and theorems.

This unit starts with the definition of a topology and moves on to the topics like stronger and weaker topologies, discrete and indiscrete topologies, cofinite topology, intersection and union of topologies, open set and closed set, neighborhood, dense set, etc.

1.1 Topology and Different Kinds of Topologies

1.1.1 Topology

Definition 1: Let X be a non-empty set. A collection T of subsets of X is said to be a topology on X if

(i) $X \in T, \phi \in T$

(ii) the intersection of any two sets in T belongs to T i.e. $A \in T, B \in T \Rightarrow A \cap B \in T$

(iii) the union of any (finite or infinite) no. of sets in T belongs to T.

i.e. $A_\alpha \in T \;\forall\; \alpha \in \Lambda \Rightarrow \cup A_\alpha \in T$ where Λ is an arbitrary set.

The pair (X, T) is called a Topological space.

Example 1: Let X = {p, q, r, s, t, u} and T_1 = {X, ϕ, {p}, {r, s}, {p, r, s}, {q, r, s, t, u}}

Then T_1 is a topology on X as it satisfies conditions (i), (ii) and (iii) of definition 1.

Example 2: Let X = {a, b, c, d, e} and T_2 = {X, ϕ, {a}, {c, d}, {a, c, e}, {b, c, d}}

Then T_2 is not a topology on X as the union of two members of T_2 does not belong to T_2.

$$\{c, d\} \cup \{a, c, e\} = \{a, c, d, e\}$$

So, T_2 does not satisfy condition (iii) of definition 1.

1.1.2 Different Kinds of Topologies

Stronger and Weaker Topologies

Let X be a set and let T_1 and T_2 be two topologies defined on X. If $T_1 \subset T_2$, then T_1 is called smaller or weaker topology than T_2.

If $T_1 \subset T_2$, then we also say that T_2 is longer or stronger topology than T_1.

Comparable and Non-comparable Topologies

Definitions: The topologies T_1 and T_2 are said to be comparable if $T_1 \subset T_2$ or $T_2 \subset T_1$.

The topologies T_1 and T_2 are said to be non-comparable if $T_1 \not\subset T_2$ and $T_2 \not\subset T_1$.

Example 3: If X = {s, t} then $T_1 = \{\phi, \{s, X\}\}$ and $T_2 = \{\phi, \{t\}, X\}$ are non-comparable as $T_1 \not\subset T_2$ and $T_2 \not\subset T_1$.

Discrete and Indiscrete Topology

Let X be any non-empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

It may be noted that T in above definition satisfy the conditions of definition 1 and so is a topology.

Let X be any non-empty set and T = {X, ϕ}. Then T is called the indiscrete topology and (X, T) is said to be an indiscrete space.

Again, it may be checked that T satisfies the conditions of definition 1 and so is also a topology.

Example 4: If X = {a, b, c} and T is a topology on X with $\{a\} \in T$, $\{b\} \in T$, $\{c\} \in T$, prove that T is the discrete topology.

Solution: The subsets of X are:

$$X_1 = \phi, X_2 = \{a\}, X_3 = \{b\}, X_4 = \{c\}, X_5 = \{a, b\}, X_6 = \{a, c\}, X_7 = \{b, c\}, X_8 = \{a, b, c\} = X$$

In order to prove that T is the discrete topology, we need to prove that each of these subsets belongs to T. As T is a topology, so X and ϕ belongs to T.

i.e. $X_1 \in T$, $X_8 \in T$.

Clearly, $X_2 \in T$, $X_3 \in T$, $X_4 \in T$

Now $X_5 = \{a, b\} = \{a\} \cup \{b\}$

since $\{a\} \in T$, $\{b\} \in T$ (Given)

and T is a topology and so by definition 1, their union is also in T i.e. $X_5 = \{a, b\} \in T$

similarly, $X_6 = \{a, c\} = \{a\} \cup \{c\} \in T$ and $X_7 = \{b, c\} = \{b\} \cup \{c\} \in T$

Hence, T is the discrete topology.

Notes

Cofinite Topology

Let X be a non-empty set, and let T be a collection of subsets of X whose complements are finite along with ϕ, forms a topology on X and is called cofinite topology.

Example 5: Let X = {l, m, n} with topology

$$T = \{\phi, \{l\}, \{m\}, \{n\}, \{l, m\}, \{m, n\}, \{l, n\}, X\}$$

is a cofinite topology since the compliments of all the subsets of X are finite.

Note If X is finite, then topology T is discrete.

Theorem 1: Let X be an infinite set and T be the collection of subsets of X consisting of empty set ϕ and all those whose complements are finite. Show that T is a topology on X.

Proof:

(i) Since $X' = \phi$, which is finite, so $X \in T$.

Also $\phi \in T$ (by definition of T)

(ii) Let $G_1, G_2, \in T$

$\Rightarrow$ G'_1, G'_2 are finite

$\Rightarrow$ $G'_1 \cup G'_2$ is finite

$\Rightarrow$ $(G_1 \cap G_2)'$ is finite (by De-Morgan's law $(G'_1 \cup G'_2 = (G_1 \cap G_2)')$)

$\Rightarrow$ $G_1 \cap G_2 \in T$

(iii) If $\{G_\alpha : \alpha \in \Lambda\}$ is an arbitrary collection of sets in T, then

G'_α is finite $\forall \alpha \in \Lambda$

$\Rightarrow$ $\cap \{G'_\alpha : \alpha \in \Lambda\}$ is finite

$\Rightarrow$ $[\cup \{G_\alpha : \alpha \in \Lambda\}]'$ is finite (by De-Morgan's law)

$\Rightarrow$ $\cup \{G_\alpha : \alpha \in \Lambda\} \in T$

Hence T is a topology for X.

Co-countable Topology

Let X be a non-empty set. Let T be the collection of subsets of X whose complements are countable along with ϕ, forms a topology on X and is called co-countable topology.

Theorem 2: Let X be a non-empty set. Let T be the collection of all subsets of X, whose complements are countable together with empty set ϕ. Show that T is a topology on X.

Proof:

(i) Since $X' = \phi$, which is countable

so, $X \in T$

Also, by definition, $\phi \in T$

Notes

(ii) Let $G_1, G_2 \in T$

$\Rightarrow$ G'_1, G'_2 are countable

$\Rightarrow$ $G'_1, \cup G'_2$ is countable

$\Rightarrow$ $(G_1 \cap G_2)'$ is countable (by De-Morgan's law)

$\Rightarrow$ $G_1 \cap G_2 \in T$

(iii) Let $\{G_\alpha : \alpha \in \Lambda\}$ be an arbitrary collection of members of sets in T.

$\Rightarrow$ G'_α is countable $\forall\ \alpha \in \Lambda$

$\Rightarrow$ $\cap \{G'_\alpha : \alpha \in \Lambda\}$ is countable

$\Rightarrow$ $[\cup \{G_\alpha : \alpha \in \Lambda\}'$ is countable (by De-Morgan's law)

$\Rightarrow$ $\cup \{G_\alpha : \alpha \in \Lambda\} \in T$

Hence, T is a topology for X.

Self Assessment

1. Construct three topologies T_1, T_2, T_3 on a set X = {a, b, c} s.t. $T_1 \subset T_2 \subset T_3$.
2. Let X = {a, b, c} and T = {ϕ, X, {b}, {a, b}. Is T is a topology for X?

1.2 Intersection and Union of Topologies

Intersection of any two topologies on a non-empty set is always topology on that set. While the union of two topologies may not be a topology on that set.

Example 6: Let X = {1, 2, 3, 4}

T_1 = {ϕ, X, {1}, {2}, {1, 2}}

T_2 = {ϕ, X, {1}, {3}, {1, 3}}

$T_1 \cap T_2$ = {ϕ, X, {1}} is a topology on X.

$T_1 \cup T_2$ = {ϕ, X, {1}, {2}, {3}, {1, 2}, {1, 3}} is not a topology on X.

Example 7: If T_1 and T_2 are two topologies defined on the same set X, then $T_1 \cap T_2$ is also a topology on X but $T_1 \cup T_2$ is not a topology on X.

Solution: Part I: Let T_1, T_2 be two topologies on the same set X.

We are to prove that $T_1 \cap T_2$ is a topology on X.

By assumption,

(i) $X \in T_1, X \in T_2$

$\phi \in T_1, \phi \in T_2$

(ii) $A, B \in T_1 \Rightarrow A \cap B \in T_1$

$A, B \in T_2 \Rightarrow A \cap B \in T_2$

Notes

(iii) $G_\alpha \in T_1 \ \forall \ \alpha \in \Delta \Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T_1$

$G_\alpha \in T_2 \ \forall \ \alpha \in \Delta \Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T_2$

Then (I) $X \in T_1 \cap T_2$, $\phi \in T_1 \cap T_2$ by (i)

(II) $A \in T_1 \cap T_2$, $B \in T_1 \cap T_2 \Rightarrow A \cap B \in T_1 \cap T_2$

For $A \in T_1 \cap T_2$, $B \in T_1 \cap T_2$

$\Rightarrow \quad A \in T_1, A \in T_2$ and $B \in T_1, B \in T_2$

$\Rightarrow \quad A \cap B \in T_1, A \cap B \in T_2$ by (ii)

$\Rightarrow \quad A \cap B \in T_1 \cap T_2$

(iv) $G_\alpha \in T_1 \cap T_2 \ \forall \ \alpha \in \Delta$

$\Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T_1 \cap T_2$

For $G_\alpha \in T_1 \cap T_2 \ \forall \ \alpha \in \Delta$

$\Rightarrow \quad G_\alpha \in T_1 \ \forall \ \alpha \in \Delta$ and $G_\alpha \in T_2 \ \forall \ \alpha \in \Delta$

$\Rightarrow \quad \cup G_\alpha \in T_1$ and $\cup G_\alpha \in T_2$ by (iii)

Thus, $T_1 \cap T_2$ is topology on X.

Part II: Let X = {a, b, c}. Then T_1 = {X, ϕ {a}} and T_2 = {X, ϕ, {b}} are topologies on X.

Let $G_1 = \{a\} \in T_1$, $G_2 = \{b\} \in T_2$.

Then $G_1 \cup G_2 = \{a, b\} \notin T_1 \cup T_2$.

Consequently $T_1 \cup T_2$ is not a topology on X.

Self Assessment

3. Prove that the intersection of an arbitrary collection of topologies for a set X is a topology for X.

4. Let T_n be a topology on a set X $\forall$ n $\in \Delta$, Δ being an index set. Then $\cap \{T_n : r \in \Delta\}$ is a topology on X.

1.3 Open Set, Closed Set and Closure of a Set

1.3.1 Definition of Open Set and Closed Set

Let (X, T) be a topological space. Any set A $\in$ T is called an open set and X-A is a closed set.

Example 8: If T = {ϕ, {a}, X} be a topology on X = {a, b} then ϕ, X and {a} are T-open sets.

Example 9: Let X = {a, b, c} and T = {ϕ, {a}, {b, c}, X} be a topology on X.

Since X – {a} = {b, c}

X – {b, c} = {a}

Therefore, T-closed sets are ϕ, {b, c} and X, which are the complements of T-open sets X, {b, c}, {a} and ϕ respectively.

Notes

> *Note* In every topological space, X and ϕ are open as well as closed.

1.3.2 Door Space

A topological space (X, T) is said to be a door space if every subset of X is either T-open or T-closed.

Example 10: Let X = {1, 2, 3) and T = {ϕ, {1, 2}, {2, 3}, {2}, X}

Then, T-closed sets are X, {3}, {1}, {1, 3}, ϕ.

This shows that every subset of X is either T-open or T-closed.

1.3.3 Closure of a Set

Let (X, T) be a topological space and A is a subset of X, then the closure of A is denoted by $\bar{A}$ or Cl (A) is the intersection of all closed sets containing A or all closed superset of A.

Example 11: If T = {ϕ, {a}, {a, b}, {a, c, d}, {a, b, c}, {a, b, c, d}, X} be a topology on X = {a, b, c, d, e} then find the closure of the sets {a}, {b}

Solution: Closed subset of X are

ϕ', {a}′, {a, b}′, {a, c, d}′, (a, b, e}′, {a, b, c, d}′, X′ = X, {b, c, d, e}, {c, d, e}, {b, e}, {c, d}, {e}, ϕ

then { $\bar{a}$ } = X

{ $\bar{b}$ } = X ∩ {b, c, d, e} ∩ {b, e} = {b, e}

Theorem 3: A is closed iff A = $\bar{A}$

Proof: Let us suppose that A is closed

∴ $A \subseteq \bar{A}$ (by definition of closure)

Now also $\bar{A} \subseteq A$ (A is common in all supersets of A)

∴ $\bar{A} = A$

Conversely, let us suppose that A = $\bar{A}$

Since we know that $\bar{A}$ is closed. (by definition of closure of A)

∴ A = $\bar{A}$ is closed

⇒ A is closed

1.3.4 Properties of Closure of Sets

Theorem 4: Let (X, T) be a topological space and let A, B be any two subsets of X. Then

(i) $\bar{\phi} = \phi$, $\bar{X} = X$;

(ii) $A \subset \bar{A}$

(iii) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

Notes

(iv) $\left(\overline{A \cup B}\right) = \overline{A} \cup \overline{B}$

(v) $\left(\overline{A \cup B}\right) \subset \overline{A} \cap \overline{B}$

(vi) $\overline{\overline{A}} = \overline{A}$

Proof:

(i) Since ϕ and X are open as well as closed.

So, ϕ, X being closed, we have

$\overline{\phi} = \phi,\ \overline{X} = X$

(ii) Since we know that $\overline{A}$ is the smallest T-closed set containing A so $A \subset \overline{A}$

(iii) Let $A \subseteq B$

Then $A \subseteq B \subseteq \overline{B}$

i.e. $\overline{B}$ is a closed superset of A. $(\because B \subseteq \overline{B})$

But $\overline{A}$ is the smallest closed superset of A.

$\therefore \quad \overline{A} \subseteq \overline{B}$

Thus, $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.

(iv) We have $A \subseteq A \cup B \Rightarrow \overline{A} \subseteq \overline{A \cup B}$ by (iii)

and $B \subseteq A \cup B \Rightarrow \overline{B} \subseteq \overline{A \cup B}$ by (iii)

Hence $\overline{A} \cup \overline{B} \subseteq \left(\overline{A \cup B}\right)$ … (I)

Since $\overline{A}$, $\overline{B}$ are closed sets, $\overline{A} \cup \overline{B}$ is also closed.

$\Rightarrow$ $A \cup B \subseteq \overline{A} \cup \overline{B}$ … (II)

From (1) & (2), we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(v) We have

$(A \cap B) \subseteq A \Rightarrow \overline{A \cap B} \subseteq \overline{A}$ by (iii)

and $(A \cap B) \subseteq B \Rightarrow \overline{A \cap B} \subseteq \overline{B}$ by (iii)

Hence $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

(vi) We know that if A is a T-closed subset then $\overline{A}$ = A by the theorem: In a topological space (X, T) if A is subset of X then A is closed iff $\overline{A}$ = A.

But $\overline{A}$ is also a T-closed subset.

$\therefore \quad \overline{\overline{A}} = A.$

Notes

Theorem 5: In a topological space, an arbitrary union of open sets is open and a finite intersection of open sets is open. Prove it.

Proof: Let (X, T) be a topological space

Let $G_i \in T \ \forall \ i \in N$

Let $G = \bigcup_{i=1}^{\infty} G_i$, $H = \bigcap_{i=1}^{n} G_i$

We are to prove that G and H are open subsets of X. By definition of topology,

(i) $G_i \in T \ \forall \ i \in N \Rightarrow \bigcup_{i=1}^{\infty} G_i \in T \Rightarrow G \in T$

(ii) $G_i \in T \ \forall \ i \in N \Rightarrow G_1 \cap G_2 \in T$

$G_1 \cap G_2 \in T, G_3 \in T$

$\Rightarrow G_1 \cap G_2 \cap G_3 \in T$

By induction, it follows that

$\bigcap_{i=1}^{n} G_i = H \in T$

Hence proved.

Theorem 6: In a topological space (X, T), prove that an arbitrary intersection of closed sets is closed and finite union of closed sets is closed.

Proof: Let (X, T) be a topological space,

Let $F_i \subset X$ be closed $\forall \ i \in N$

Let $H = \bigcap_{i=1}^{\infty} F_i$, $F = \bigcup_{i=1}^{n} F_i$

We are to prove that F and H are closed sets F_i is closed $\forall \ i \in N$

$\Rightarrow$ $X - F_i$ is open $\forall \ i \in N$

Also, we know, $\bigcup_{i=1}^{\infty} (X - F_i)$ and $\bigcap_{i=1}^{n} (X - F_i)$ are open sets

[$\because$ An arbitrary union of open sets is open and a finite intersection of open sets is open]

$\Rightarrow$ $X - \bigcap_{i=1}^{\infty} F_i$ and $X - \bigcup_{i=1}^{\infty} F_i$ are open sets (by De Morgan's Law)

$\Rightarrow$ $\bigcap_{i=1}^{\infty} F_i$, $\bigcup_{i=1}^{n} F_i$ are closed sets (by definition of closed sets)

i.e. H, F are closed sets.

Hence, proved.

Self Assessment

5. Give two examples of a proper non-empty subset of a topological space such that it is both open and closed and prove your assertion.

Notes

6. On the real line show that every open interval is an open but every open set need not be an open interval.
7. Let (Y, U) be a subspace of a topological space (X, T). Then every U-open set is also T-open iff Y is T-open.

1.4 Neighborhood

Let (X, T) be a topological space. $A \subset X$ is called a neighbourhood of a point $x \in X$ if $\exists$ $G \in T$ with $x \in G$ such that $G \subset A$. The word neighborhood is, in short, written as 'nhd'.

Let G be any open set such that $G \subset X$ with $x \in G$ is also nhd of a point $x \in X$.

Example 12: Let T = {ϕ, X, {b}, {a, b}, {a, b, d}}, be a topology on X = {a, b, c, d}. Find T-nhds of (i) a, (ii) b and (iii) c.

Solution:

(i) T-open sets containing 'a' are X, {a, b}, {a, b, d}.

super set of X is X

supersets of {a, b} are {a, b}, {a, b, c}, {a, b, d}, X

supersets of {a, b, d} are {a, b, d}, X.

T-nhds of 'a' are {a, b}, {a, b, c}, {a, b, d}, X

(ii) T-open sets containing b are

{b}, {a, b}, {a, b, d}, X

supersets of {a, b} are {a, b}, {a, b, c}, {a, b, d}, X

supersets of {a, b, d} are {a, b, d}, X

supersets of {b} are {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {b, c, d}, {a, b, d}, X

T-nhds of 'b' are {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}, X

(iii) T-open set containing 'c' is X.

Hence T-nhd of 'c' is X.

Theorem 7: Let (X, T) be a topological space and $A \subset X$. Then A is T-open $\Leftrightarrow$ A contains T-nhds of each of its points.

Proof: Let (X, T) be a topological space and $A \subset X$.

Step I: Given A is an open set.

To show: A contains T-nhd of each of its points. Clearly $x \in A \subset A$ $\forall$ $x \in A$ and A is an open set. This shows that A contains T-nhd of each of its points.

Step II: Given A contains T-nhd of each of its point, then any $x \in A \Rightarrow \exists$ nhd $N_x \subset X$ such that

$$x \in N_x \subset A \qquad \ldots(1)$$

To show: A is an open set

By definition of nhd, $\exists$ open set G_x s.t.

$$x \in G_x \subset N_x \qquad \ldots(2)$$

Notes

From (1) and (2), we get

$$x \in G_x \subset N_x \subset A \quad \text{...(3)}$$

$\Rightarrow \quad x \in G_x \subset A$

which is true $\forall\ x \in A$

$$\therefore \quad \bigcup_{x \in A} G_x \subset A \quad \text{...(4)}$$

Let G = $\bigcup_{x \in A} G_x$ and an arbitrary union of open sets is open and so G is an open set.

$$\therefore \quad G \subset A \quad \text{...(5) [Using (4)]}$$

for any $x \in A \Rightarrow x \in G_x \subset G \Rightarrow x \in G \Rightarrow A \subset G$ ...(6)

from (5) & (6), we get

$$A = G$$

$\Rightarrow$ A is an open set.

Theorem 8: Let X be a topological space. Then the intersection of two nhds of $x \in X$ is also a nhd of x.

Proof: Let N_1 and N_2 be two nhds of $x \in X$ then $\exists$ open sets G_1 and G_2 such that

$$x \in G_1 \subseteq N_1 \text{ and}$$

$$x \in G_2 \subseteq N_2$$

$$\therefore \quad x \in G_1 \cap G_2 \subseteq N_1 \cap N_2$$

$\therefore$ $G_1 \cap G_2$ is an open set containing x and contained in $N_1 \cap N_2$.

This shows that $N_1 \cap N_2$ is also a nhd of x.

Theorem 9: Let $(y, \bigcup)$ be a subspace of a topological space (X, T). A subset of Y is $\bigcup$-nhd of a point $y \in Y$ iff it is the intersection of Y with a T-nhd of the point $y \in Y$.

Proof: Let $(y, \bigcup) \subset (X, T)$ and $y \in Y$ be arbitrary, then $y \in X$.

Step I: Let N_1 be a $\bigcup$-nhd of y, then

$$\exists\ V \in \bigcup \text{ s.t. } y \in V \subset N_1 \quad \text{...(1)}$$

To show: $N_1 = N_2 \cap Y$ for some T-nhd N_2 of y.

$$y \in V \in \bigcup \Rightarrow G \in T \text{ s.t. } V = G \cap Y$$

$$\Rightarrow y \in G \cap Y \Rightarrow y \in G,\ y \in Y \quad \text{... (2)}$$

Let $\quad N_2 = N_1 \bigcup G$

Then $\quad N_1 \subset N_2,\ G \subset N_2$ ...(3)

From (2) and (3), $y \in G \subset N_2$ where $G \in T$

This shows that N_2 is a T-nhd of y.

$$N_2 \cap Y = (N_1 \cup G) \cap Y = (N_1 \cap Y) \cup (G \cap Y)$$

$$= (N_1 \cap Y) \cup V = N_1 \cup V = N_1 \quad \text{[by (1)]}$$

$\because \quad N_1 \subset Y$ and $V \cup N_1$

Notes

so, N_2 has the following properties

$N_1 = N_2 \cap Y$ and N_2 is a $\cup$-nhd of y.

This completes the proof.

Step II: Conversely Let N_2 be a T-nhd of y so that

$$\exists\, A \in T \text{ s.t. } y \in A \subset N_2 \qquad \ldots(4)$$

To show: $N_2 \cap Y$ is a U-nhd of y.

$$\therefore \quad y \in Y,\ y \in A \Rightarrow y \in Y \cap A \qquad \text{[by (4)]}$$

$$\Rightarrow y \in A \cap Y \subset N_2 \cap Y \qquad \text{[by (3)]}$$

$$A \in T \Rightarrow A \cap Y \in \cup$$

Thus, we have $y \in A \cap Y \subset N_2 \cap Y$, where $A \cap Y \in \cup$.

This shows that $N_2 \cap Y$ is a $\cup$-nhd of y.

Self Assessment

8. Let $T = \{X, \phi, \{p\}, \{p, q\}, \{p, q, t\}, \{p, q, r, s\}, \{p, r, s\}\}$ be the topology on

   $X = \{p, q, r, s, t\}$

   List the nhds of the points r, t.

9. Prove that a set G in a topological space X is open iff G is a nhd of each of its points.

1.5 Dense Set and Boundary Set

1.5.1 Dense Set and No where Dense

Let (X, T) be a topological space and $A \subset X$ then A is said to be dense or everywhere dense in X if $\bar{A} = X$.

Example 13: Consider the set of rational number $Q \subseteq R$, then only closed set containing Q in R, which shows that $\bar{Q} = R$.

Hence, Q is dense in R.

> *Note* Rational are dense in R and countable but irrational numbers are also dense in R but not countable.

Example 14: Prove that A set is always dense in its subset

Solution: Let $A \subset B$ then $A \subset B \subset \bar{B}$

$$\Rightarrow A \subset \bar{B}$$

$$\Rightarrow \bar{B} \supset A$$

$\Rightarrow$ B is dense in A.

Notes

Example 15: If T = {ϕ, {a}, {a, b}, {a, c, d}, {a, b, e}, {a, b, c, d}, X} be a topology on X = {a, b, c, d, e} then which of the set {a}, {b}, {c, e} are dense in X.

Solution: A is called dense in X if $\bar{A} = X$ (By definition)

$$\{\bar{a}\} = \cap \{F : F \text{ is closed subset s.t. } F \supset \{a\}\} = X.$$

$$\{\bar{b}\} = X \cap \{b, c, d, e\} \cap \{b, e\} = \{b, e\}$$

$$\{\overline{c, e}\} = X \cap \{b, c, d, e\} \cap \{c, d, e\} = \{c, d, e\}$$

This shows that {a} is the only dense set in X.

Definition

- A is said to be dense in itself if $A \subset D(A)$.
- A is said to be nowhere dense set in X if int $(\bar{A}) = \phi$ i.e., if the interior of the closure of A is an empty set.

1.5.2 Boundary Set

The Boundary set of A is the set of all those points which belong neither to the interior of A nor to the interior of its complement and is denoted by b(A).

Symbolically, $b(A) = X - A^\circ \cup (X - A)^\circ$.

Elements of b(A) are called bounding points of A. Boundary points are, sometimes called frontier points.

Example 16: Define nowhere dense set and give an example of it.

Solution: $\because D(N) = \phi$

For if a is any real number, then consider a real number $\in > o$, so small that open set $(a - \in, a + \in)$ does not contain any point of N.

$$\therefore \quad Z = \{n : n \in N\} \cup \{0\} \cup \{-n : n \in N\}$$

$$D(Z) = D\{n : n \in N\} \cup D\{0\} \cup D\{-n : n \in N\}$$

$$= \phi \cup \phi \cup \phi$$

$$= \phi \subset Z$$

$$\therefore \quad D(Z) \subset Z \Rightarrow Z \text{ is closed.}$$

$$\Rightarrow \quad Z = \bar{Z}$$

$$\text{Int}(\bar{Z}) = \text{Int}(Z) = \cup \{G \subset R : G \text{ is open}, G \subset Z\}$$

$$= \phi$$

$\therefore$ An open subset of R will be an open interval, say G = (a_1, a_2). This open interval contains all real numbers (rationals and irrationals) x s.t. $a_1 < x < a_2$ and therefore $G \not\subset Z$.

$$\therefore \quad \text{Int}(Z) = \phi$$

This proves that Z is nowhere dense set in R.

Notes

Example 17: Prove that every non-empty subset of an indiscrete space is dense in X.

Solution: Let (X, T) be an indiscrete space.

Let $A \subset X$ be non-empty set.

To show: A is dense in X.

For this, we are to prove $\bar{A} = X$

By definition of an indiscrete topology,

$$T = \{\phi, X\}$$

T-open sets are ϕ, X

T-closed sets are $X - \phi$, X i.e. X, ϕ.

Since $A \neq \phi$ by assumption.

$\therefore$ The only closed superset of A is X,

so that $\bar{A} = X$.

Example 18: Let T = {X, ϕ, {p}, {p, q}, {p, q, t}, {p, q, r, s}, {p, r, s}} be the topology on X = {p, q, r, s, t}

Determine boundary of the following sets

(i) $B = \{q\}$

$$B^\circ = \cup \{\phi\} = \phi$$

$$(X - B)^\circ = \{p, r, s, t\}^\circ = \cup \{f, \{p\}, \{p, r, s\}\}$$

$$= \{p, r, s\}$$

$$b(B) = X - B^\circ \cup (X - B)^\circ$$

$$= X - \phi \cup \{p, r, s\}$$

$$= \{q, t\}$$

Self Assessment

10. In a topological space, prove that:
    (i) A is dense $\Leftrightarrow$ it intersects every non-empty open set.
    (ii) A is closed $\Leftrightarrow$ A contains its boundary.
11. In any topological space, prove that
    $b(A) = \phi \Leftrightarrow$ A is open as well as closed.

1.6 Separable Space, Limit Point and Derived Set

1.6.1 Separable Space

Let X be a topological space and A be subset of X, then X is said to be separable if

(i) $\bar{A} = X$

(ii) A is countable

Notes

Example 19: Let X = {1, 2, 3, 4, 5} be a non-empty set and T = {ϕ, X, {3}, {3, 4}, {2, 3}, {2, 3, 4}} is a topology defined on X. Suppose a subset A = {1, 3, 5} $\subseteq$ X. The closed set are:

X, ϕ, {1, 2, 4, 5}, {1, 2, 5}, {1, 4, 5}, {1, 5}.

So, we have $\bar{A}$ = X. Since A is finite and dense in X. So X is a separable space.

Theorem 10: Show that the cofinite topological space (X, T) is separable.

Solution: Let (X, T) be a cofinite topological space.

(i) When X is countable.

Then X $\subset$ X and $\bar{X} = X$

This shows that X is separable.

(ii) Let A $\subset$ X s.t. A is finite.

By definition of cofinite topological space A′ = X – A is open so that A is closed.

$\Rightarrow$ every finite set A is T-closed and so $\bar{A}$ = X.

Now $\bar{A}$ = X, A is countable.

This shows that (X, T) is separable.

Example 20: A discrete space X is separable iff X is countable.

Solution: As we know that every subset of a discrete space (X, T) is both open and closed. Also, A is said to be everywhere dense in X if $\bar{A}$ = X.

Also, X is separable if $\exists$ A $\subset$ X s.t. $\bar{A}$ = X and A is countable.

So, the only everywhere dense subset of X is X itself.

$\Rightarrow$ X can have a countable dense subset iff X is countable.

Hence, X is separable iff X is countable.

1.6.2 Limit Point or Accumulation Point or Cluster Point

Let (X, T) be a topological space and A $\subset$ X. A point x $\in$ X is said to be the *limit point* or *accumulation point* or *cluster point* of A if each open set containing 'x' contains at least one point of A different from x.

Thus, it is clear from the above definition that the limit point of a set A may or may not be the point of A.

Example 21: Let X = {a, b, c} with topology

T = {ϕ, {a, b}, {c}, X} and A = {a}, then b is the only limit point of A, because the open sets containing b namely {a, b} and X also contains a point of A.

Whereas, 'a' and 'b' are not limit point of C = {c}, because the open set {a, b} containing these points do not contain any point of C. The point c is also not a limit point of C, since then open set {c} containing 'c' does not contain any other point of C different from c. Thus, the set C = {c} has no limit points.

Notes

Example 22: Prove that every real number is a limit point of R.

Solution: Let $x \in R$ then every nhd of x contains at least one point of R other than x

$\therefore$ x is a limit point of R.

But x was arbitrary.

$\therefore$ every real number is a limit point of R.

Example 23: Prove that every real number is a limit point of R – Q.

Solution: Let x be any real number, then every nhd of X contains at least one point of R – Q other than x

$\therefore$ x is a limit point of R – Q.

But x was arbitrary.

$\therefore$ every real number is a limit point of R – Q.

1.6.3 Derived Set

Definition: The set of all limit points of A is called the derived set of A and is represented by D(A).

Example 24: Every derived set in a topological space is a closed.

Solution: Let (X, T) be a topological space and $A \subset X$.

To show: D(A) is a closed set.

As we know that B is a closed set if $D(B) \subset B$.

Hence, D(A) is closed iff $D[D(A)] \subset D(A)$.

Let $x \in D[D(A)]$ be arbitrary, then x is a limit point of D(A) so that

$$(G - \{x\}) \cap D(A) \neq \phi \ \forall \ G \in T \text{ with } x \in G$$

$$\Rightarrow \quad (G - \{x\}) \cap A \neq \phi$$

$$\Rightarrow \quad x \in D(A)$$

Hence proved.

[For every nhd of an element of D(A) has at least one point of A].

Example 25: In any topological space, prove that $A \cup D(A)$ is closed.

Solution: Let (X, T) be a topological space and $A \subset X$.

To prove: $A \cup D(A)$ is a closed set.

Let $x \in X - A \cup D(A)$ be arbitrary then $x \notin A \cup D(A)$ so that $x \notin A$, $x \notin D(A)$

$$x \notin D(A) \Rightarrow \exists \ G \in T \text{ with } x \in G \quad \text{s.t.}$$

$$(G - \{x\}) \cap A = \phi$$

$$\Rightarrow G \cap A = \phi \qquad (\because \quad x \notin A) \qquad \ldots(1)$$

For this G, we also claim

$$G \cap D(A) = \phi$$

Let $y \in G$ be arbitrary.

Now G is an open set containing 'y' s.t.

$$G \cap A = \phi \text{, showing that } y \notin D(A).$$

$\therefore$ any $y \in G \Rightarrow y \notin D(A)$

This shows $G \cap D(A) = \phi$

$\therefore$ $G \cap A = \phi, G \cap D(A) = \phi$

Now, $G \cap [A \cup D(A)] = (G \cap A) \cup [G \cap D(A)]$

$$= \phi \cup \phi = \phi$$

$\Rightarrow$ $G \subset X - A \cup D(A)$

$\therefore$ any $x \in X - A \cup D(A)$

$\Rightarrow$ $G \in T$ with $x \in G$ s.t. $G \subset X - A \cup D(A)$

This proves that x is an interior point of $X - A \cup D(A)$.

Since x is arbitrary point of $X - A \cup D(A)$.

Hence, every point of $X - A \cup D(A)$ is an interior point of $X - A \cup D(A)$.

$\therefore$ $X - A \cup D(A)$ is open.

i.e., $A \cup D(A)$ is closed.

Theorem 11: Let (X, T) be a topological space and $A \subseteq X$, then A is closed iff $A' \subseteq A$ or $A \supseteq D(A)$. A subset A of X in a topological space (X, T) is closed iff A contains each of its limit points.

Proof: Let A be closed $\Rightarrow A^C$ is open.

Let $x \in A^C$

then A^C is open set containing x but containing no point of A other than x. This shows that x is not a limit point of A.

Thus, no point of A^C is a limit point of A. Consequently, every limit point of A is in A and therefore $A' \subseteq A$.

Conversely, Let $A' \subseteq A$.

To show: A is closed.

Let x be an arbitrary point of A^C.

Then $x \in A^C \Rightarrow x \notin A \quad \Rightarrow x \notin A$ and $x \notin A' \quad (\because A' \subseteq A)$

$\Rightarrow$ $x \notin A$ and x is not a limit point of A.

$\Rightarrow$ $\exists$ an open set G such that $x \in G$ and $G \cap A = \phi \quad \Rightarrow \quad G \subseteq A^C$

$\Rightarrow$ $x \in G \subseteq A^C$

$\Rightarrow$ A^C is the nhd of each of its points and therefore A^C is open.

Hence A is closed.

Notes

Theorem 12: In any topological space, prove that

$$\bar{A} = A \cup D(A)$$

Proof: Let (X, T) be a topological space and $A \subset X$.

To show: $\bar{A} = A \cup D(A)$

Since $A \cup D(A)$ is closed and hence

$$\overline{A \cup D(A)} = A \cup D(A) \quad \ldots(1)$$

$\because \quad A \subset A \cup D(A)$

$\therefore \quad \bar{A} \subset \overline{A \cup D(A)} = A \cup D(A)$ [Using (1)]

$$\bar{A} \subset A \cup D(A) \quad \ldots(2)$$

Now, We are to prove that

$$A \cup D(A) \subset \bar{A} \quad \ldots(3)$$

But, $A \subset \bar{A}$ ...(4)

To prove (3), we are to prove

$$D(A) \subset \bar{A} \quad \ldots(5)$$

i.e., to show that

$$D(A) \subset \cap_i \{F_i \subset X : F_i \text{ is closed } F_i \supset A\} \quad \ldots(6)$$

Let $x \in D(A)$ be arbitrary.

$x \in D(A) \Rightarrow$ x is a limit point of A

$\Rightarrow$ x is a limit point of all those sets which contain A.

$\Rightarrow$ x is a limit point of all those F_i appearing on R.H.S. of (6).

$\Rightarrow \quad x \in D(F_i) \subset F_i \quad (\because \ F_i$ is closed)

$\Rightarrow \quad x \in F_i$ for each i

$\Rightarrow \quad x \in \cap_i \{F_i \subset X : F_i \text{ is closed}\}$

$\Rightarrow \quad x \in \bar{A}$

Thus any $x \in D(A) \Rightarrow x \in \bar{A}$

$$D(A) \subset \bar{A}$$

Hence the result (5) proved.

From (4) & (5), we get

$$A \cup D(A) \subset \bar{A} \cup \bar{A} = \bar{A}$$

i.e., $A \cup D(A) \subset \bar{A}$

Hence the result (3) proved.

Combining (2) & (3), we get the required result.

Self Assessment

12. Let X = {a, b, c} and let T = {ϕ, X, {b}, {c}}, find the set of all cluster points of set {a, b}.

13. Let X = {a, b, c} and let T = {ϕ, X, {a}, {b}, {a, b}}, show that D({a}) = {c}, D({c}) = ϕ and find derived sets of other subsets of X.

1.7 Interior and Exterior

1.7.1 Interior Point and Exterior Point

Interior Point: Let X be a topological space and let $A \subset X$.

A point $x \in A$ is called an interior point of A iff $\exists$ an open set G such that $x \in G \subseteq A$.

The set of all interior points of A is known as the interior of A and is denoted by Int (A) or A°. Symbolically,

$$A^\circ = \text{Int}(A) = \cup \{G \in T : G \subset A\}.$$

Example 26: Let T = {ϕ, {a}, {b, c}, {a, b}, {a, b, c}, X} be a topology on X = {a, b, c, d} then

Int (A) = Union of all open subsets of X which are contained in A.

$$\text{Int}[\{a\}] = \phi \cup \{a\} = \{a\}$$

$$\text{Int}[\{a, b\}] = \phi \cup \{a\} \cup \{a, b\} = \{a, b\}$$

Exterior Point: Let X be a topological space and let $A \subset X$.

A point $x \in A$ is called an exterior point of A iff it is an interior point of A^C or X – A.

The set of all exterior points of A is called the exterior of A and is denoted by ext (A).

Symbolically,

$$\text{ext}(A) = (X - A)^\circ \text{ or } (A^C)^\circ.$$

Example 27: Let T = {X, ϕ, {p}, {p, q}, {p, q, t}, {p, q, r, s}, {p, r, s}} be the topology on X = {p, q, r, s, t}

Determine exterior of (i)

$$B = \{q\}$$

Solution:

$$\text{ext}(B) = (X - B)^\circ = \{p, r, s, t\}^\circ$$

$$= \cup \{Q, \{p\}, \{p, r, s\}\}$$

$$= \{p, r, s\}$$

1.7.2 Interior Operator and Exterior Operator

Interior Operator: Let X be a non-empty set and P(X) be its power set. Then, an interior operator 'i' on X is a mapping i : P(X) $\rightarrow$ P(X) which satisfies the following four axioms:

(i) i(X) = X

(ii) i(A) $\subseteq$ A

(iii) i(A $\cap$ B) = i(A) $\cap$ i(B)

(iv) i(i(A) = i(A), where A and B are subsets of X.

Notes

Exterior Operator: Let X be a topological space. Then, an exterior operator on X is a mapping $e : P(X) \to P(X)$ satisfying the following postulates:

(i) $e(\phi) = X$, $e(X) = \phi$

(ii) $e(A) \subseteq A'$

(iii) $e[\{e(A)\}'] = e(A)$

(iv) $e(A \cup B) = e(A) \cap e(B)$ where A and B are subsets of X.

Theorem 13: Prove that $int(A) = \cup \{G : G \text{ is open}, G \subseteq A\}$.

or

Let X be a topological space and let $A \subseteq X$. Then, $A°$ is the union of all open subsets of A.

Proof: Let $x \in A° \leftrightarrow x$ is an interior point of A.

$\leftrightarrow$ A is a nhd of x.

Then $\exists$ an open set G such that $x \in G \subset A$ and hence $x \in \cup \{G : G \text{ is an open subset of } A\}$

Now let $x \in \cup \{G : G \text{ is open}, G \subseteq A\}$ …(1)

$\Rightarrow$ $x \in$ some T-open set G which is contained in A

$\Rightarrow$ $x \in A°$ by definition of $A°$

$\therefore$ $\cup \{G : G \text{ is open}, G \subseteq A\} \subseteq A°$ …(2)

Thus from (1) and (2), we get

$$A° = \cup \{G : G \text{ is open}, G \subseteq A\}$$

Theorem 14: Let X be a topological space and let A be a subset of X. Then int (A) is an open set.

Proof: Let x be any arbitrary point of int (A). Then x is an interior point of A.

This implies that A is a nhd. of x i.e., $\exists$ an open G such that $x \in G \subset A$.

Since G contains a nhd of each of its points, it follows that A is a nhd of each of the point of G.

Thus, each point of G is a interior point of A.

Therefore, $x \in G \subset int(A)$.

Thus, it is shown that to each $x \in A°$, these exists an open set G such that $x \in G \subset int(a)$.

Hence $A°$ is a nhd of each of its point and consequently int (A) is open.

Theorem 15: Let X be a topological space and let $A \subseteq X$. Then $A°$ is the largest open set contained in A.

Proof: Let G be any open subset of A and let x be an arbitrary element of G i.e. $x \in G \subset A$.

Thus A is a nhd of x i.e., x is an interior point of A.

Hence $x \in A°$

$\therefore$ $x \in G \Rightarrow x \in A°$.

Thus $G \subset A° \subset A$.

Hence $A°$ contains every open subset of A and it is, therefore, the largest open subset of A.

Theorem 16: Let X be a topological space and let $A \subseteq X$. Then A is open iff $A° = A$.

Proof: Let A be a T-open set.

Since every T-open set is a T-nhd of each of its point, therefore every point of A is a T-interior point of A. Consequently $A \subset A°$,

Again, since each T-interior point of A belongs to A therefore $A° \subset A$.

Hence, $A = A°$

Consequently, if $A = A°$, then A must be a T-open set for A° is a T-open set.

1.7.3 Properties of Interior

Theorem 17: Let (X, T) be a topological space and $A, B \subset X$. Then

(i) $\phi° = \phi$

(ii) $X° = X$

(iii) $A \subset B \Rightarrow A° \subset B°$

(iv) $(A°)° = A°$ or $A°° = A°$.

Proof: Let (X, T) be a topological space and $A, B \subset X$.

(i) & (ii), By definition of T, $\phi, X \in T$, consequently.

$$\phi° = \phi, \quad X° = X$$

For A is open $\Leftrightarrow A° = A$.

(iii) Suppose $A \subset B$

any $x \in A° \Rightarrow x$ is an interior point of A.

$\Rightarrow \exists$ open set G s.t. $x \in G \subset A$

$\Rightarrow x \in G \subset A \subset B \Rightarrow x \in G \subset B$ & G is open.

$\Rightarrow x \in B°$

$\therefore$ $A° \in B°$.

(iv) We Know that A° is open

Also G is open $\Leftrightarrow G° = G$ …(1)

In view of this, we get

$$(A°)° = A° \quad \text{or} \quad A°° = A° \qquad \text{(on putting } G = A° \text{ in (1))}$$

Theorem 18: Let i be an interior operator defined on a set X. Then these exists a unique topology T on X s.t. for each $A \subset X$.

$$i(A) = \text{T-interior of A}.$$

Proof: Let i be an interior operator on X. Then a map

$$i : P(X) \to P(X) \text{ s.t.}$$

(i) $i(X) = X$

(ii) $i(A) \subset A$

(iii) $i(A \cap B) = i(A) \cap i(B)$

(iv) $i[i(A)] = i(A)$, where $A, B \subset X$

P(X) being power set of X.

Notes

To prove that ∃ unique topology T on X s.t. $i(A) = A^\circ$, where A° = T-interior of A.

Write $T = \{A \subset X : i(A) = A\}$

(1) $X \in T$, for $i(X) = X$

(2) To prove $\phi \in T$

$i(\phi) \subset \phi$, by (ii)

But $\phi \subset i(\phi)$

$i(\phi) \subset \phi$ So that $\phi \in T$

(3) $G_1, G_2, \in T \Rightarrow G_1 \cap G_2 \in T$

For $G_1, G_2, \in T \Rightarrow i(G_1) = G_1, i(G_2) = G_2$

$\Rightarrow i(G_1 \cap G_2) = i(G_1) \cap i(G_2)$ by (iii)

$= G_1 \cap G_2$

$\Rightarrow i(G_1 \cap G_2) = G_1 \cap G_2$

$\Rightarrow G_1 \cap G_2 \in T$

(4) To prove $G_\alpha \in T \ \forall \ \alpha \in \Delta \Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T$

Firstly we shall prove that

$A \subset B \Rightarrow i(A) \subset i(B)$,

where $A, B, \subset X$ …(1)

$A \subset B \Rightarrow A \cap B = A$

$\Rightarrow i(A) = i(A \cap B)$

$= i(A) \cap i(B)$, by (iii)

$\subset I(B)$

$\Rightarrow i(A) \subset i(B)$. Hence the result (1).

Let $G_\alpha \in T \ \forall \ \alpha \in \Delta$ so that

$i(G_\alpha) = G_\alpha$ …(2)

Also let $\cup \{G_\alpha : \alpha \in \Delta\} = G$.

Then $G_\alpha \subset G \Rightarrow i(G_\alpha) \subset i(G)$, by (1)

$\Rightarrow G_\alpha \subset i(G)$, by (2)

$\Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \subset i(G)$

$\Rightarrow G \subset i(G)$

But $i(G) \subset G$, by (ii).

Consequently $i(G) = G$ so that $G \in T$. Hence the result (4). From (1), (2), (3) and (4), it follows that T is a topology on X.

Remains to prove that

$i(A) = A^\circ$.

By (iv), $i[i(A)] = i(A)$

Notes

By construction of T, $\Rightarrow$ i(A) $\in$ T.

Thus, i(A) is T-open set s.t. i(A) $\subset$ A.

Let B be an open set s.t. B $\subset$ A.

$$B \in T, B \subset A \Rightarrow i(B) = (B), i(B) \subset i(A)$$

$$\Rightarrow B \subset i(A)$$

Thus i(A) contains any open set B s.t. B $\subset$ A. It follows that i(A) is the largest open subset of A. Consequently i(A) = A°.

1.7.4 Properties of Exterior

Theorem 19: Let (X, T) be a topological space and A, B $\subset$ X. Then

(i) ext (X) = ϕ

(ii) ext (ϕ) = X

(iii) ext (A) $\subset$ A′

(iv) ext (A) = ext [(ext (A))′]

(v) A $\subset$ B $\Rightarrow$ ext (B) $\subset$ ext (A)

(vi) A° $\subset$ ext [ext (A)]

(vii) ext (A $\cup$ B) = ext (A) $\cap$ ext (B).

Proof:

(i) ext (X) = (X – X)° = ϕ as we know that ext (A) = (X – A)°

(ii) ext (ϕ) = (X – ϕ)° = X° = X

(iii) ext (A) = (X – A)° $\subset$ X – A = A′ or ext (A) $\subset$ A′ for B° $\subset$ B

(iv) [ext (A)]′ = [(X – A)°]′ = X – (X – A)°

or ext [{ext (A)}′] = ext [X – (X – A)°]

= [X – {X – (X – A)°}]°

= [(X – A)°]° = (X – A)°°

= (X – A)° [As B°° = B° $\forall$ B]

= ext (A)

$\Rightarrow$ ext (A) = ext [(ext (A))′]

(v) A $\subset$ B $\Rightarrow$ X – B $\subset$ X – A

$\Rightarrow$ (X – B)° $\subset$ (X – A)°

$\Rightarrow$ ext (B) $\subset$ ext (A)

(vi) ext (A) = (X – A)° $\subset$ X – A

$\Rightarrow$ ext (A) $\subset$ X – A

As A $\subset$ B $\Rightarrow$ ext (B) $\subset$ ext (A), we get

ext (X – A) $\subset$ ext [ext (A)] …(1)

Notes

But ext (X – A) = ext (A′) = (X – A′)° = [X – (X – A)]°

= A°

Now (1) becomes A° ⊂ ext [ext (A)]

(vii) ext (A ∪ B) = [X – (A ∪ B)]° = [(X – A) ∩ (X – B)]°

= (A′ ∩ B′)°

= (A′)° ∩ (B′)°

= ext (A) ∩ ext (B).

Theorem 20: Exterior Operator: The exterior, by definition of interior function 'e' on X is a function

$$e : P(X) \to P(X) \quad \text{s.t.}$$

(i) e (X) = φ

(ii) e (φ) = X

(iii) e (A) ⊂ A′

(iv) e (A) = e [(e (A))′]

(v) e (A ∪ B) = e (A) ∩ e (B)

For any sets A, B ⊂ X. Then there exists a unique topology T on X s.t. e (A) = T-exterior of A.

Proof: Write T = {G ⊂ X : e (G′) = G}

We are to show that T is a topology on X.

(i) e (φ′) = e (X) = φ by (i)

e (X′) = e (φ) = X by (ii)

Now e (φ′) = φ, e (X′) = X ⇒ φ, X ∈ T

(ii) Let $G_1, G_2 \in T$

Then $e(G'_1) = G_1, \quad e(G'_2) = G_2$

But $(G_1 \cap G_2)' = G'_1 \cup G'_2$

$e[(G_1 \cap G_2)'] = e(G'_1 \cup G'_2)$

$= e(G'_1) \cap e(G'_2)$ by (v)

$= G_1 \cap G_2$

$\Rightarrow G_1 \cap G_2 \in T$

(iii) Firstly, we shall show that

$$A \subset B \Rightarrow e(B) \subset e(A) \quad \ldots(1)$$

$$A \subset B \Rightarrow A \cup B = B \Rightarrow e(B) = e(A \cup B)$$

$$= e(A) \cap e(B) \subset e(A)$$

⇒ e (B) ⊂ e (A)

Let $G_\alpha \in T \ \forall \ \alpha \in \Delta$

Then $e(G'_\alpha) = G_\alpha$ …(2)

Let $G = \cup \{G_\alpha : \alpha \in \Delta\}$

Then $G' = \cap \{G'_\alpha : \alpha \in \Delta\}$, (By De Morgan's law)

By (iii), $e(G') \subset G'' = G$ or $e(G') \subset G$ …(3)

$G_a \subset G \Rightarrow G' \subset G'_a \Rightarrow e(G'_a) \subset e(G')$ by (1)

$\Rightarrow$ $G_\alpha \subset e(G')$ by (2)

$\Rightarrow$ $\cup G_\alpha \subset e(G')$

$\Rightarrow$ $G \subset e(G')$ …(4)

From (3) & (4),

$e(G') = G$ so that $G \in T$

So, $G_\alpha \in T \Rightarrow \cup \{G_\alpha : \alpha \in \Delta\} \in T$

This shows that T is a topology on X.

It remains to prove that

$e(A) =$ T-exterior of A.

By (iv), $e(A) = e[(e(A))']$

$\Rightarrow$ $e(A) \in T$ [By (iii)],

$e(A) \subset A'$

Thus, e (A) is an open set contained in A′.

Also, e (A) is the largest open set contained in A′.

$\therefore$ T-interior of $A' = e(A)$

or T-exterior of $A = e(A)$

Self Assessment

14. Let X = {a, b, c} and let T = {ϕ, X, {b}, {a, c}}, find the interior of the set {a, b}.
15. If T = {ϕ, {a}, {a, b}, {a, c, d}, {a, b, e}, {a, b, c, d}, X} be a topology on X = {a, b, c, d, e} then find the interior points of the subset A = {a, b, c} on X.

1.8 Summary

- Topology deals with the study of those properties of certain objects that remain invariant by stretching or bending.
- Let X be any non-empty set and T be the collection of all subsets of X. Then T is called discrete topology.
- Let X be any non-empty set and T = {X, ϕ}, then T is called indiscrete topology.
- Let T be a collection of subset of X where complements are finite along with ϕ, forms a topology on X is called cofinite topology.
- Let (X, T) be a topological space. Any set A ∈ T is called an open set and X – A is called closed set.
- Closure of a set is the intersection of all closed sets containing A where A is subset of X.

Notes

- Let (X, T) be a topological space. $A \subset X$ is called a neighbourhood of a point $x \in X$ if $\exists\, G \in T$ with $x \in G$ such that $G \subset A$.
- Let (X, T) be a topological space. $A \subset X$ is said to be dense or everywhere dense in X if $\bar{A} = X$.
- If A said to be nowhere dense set in X if int $(\bar{A}) = \phi$.
- Let {X, T} be a topological space and $A \subset X$ then X is said to be separable if

  (i) $\bar{A} = X$ (ii) A is countable
- Let (X, T) be a topological space and $A \subset X$.

  A point $x \in X$ is said to be the limit point if each open set containing x contains at least one point of A different form x.
- The set of all limit point of A is called derived set of A.
- Let (X, T) be a topological space and $A \subset X$. A point $x \in A$ is called a interior point of A iff there exists an open set G such that $x \in G \subseteq A$. It is denoted by Int (A) or A°.
- A point $x \in X$ is called an exterior point of A iff it is an interior point of A^C or X – A. It is denoted by ext (A).

1.9 Keywords

Complement: The complement of a set A w.r.t. the universal set X is defined as the set X-A and is denoted by A°.

symbolically, $A° = X - A = \{x \in X : x \notin A\}$.

Intersection: The intersection of two sets A and B, denoted by $A \cap B$, is

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

Subset: If every element of set A is also an element of set B, then A is called a subset of B. It is denoted by the symbol $A \subset B$.

Superset: $A \subset B$ is also expressed by writing, $B \supset A$.

Union: The union of two sets A and B, denoted by $A \cup B$, is

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

1.10 Review Questions

1. Let X = {a, b, c, d, e, f}, which of the following collections of subsets of X is a topology on X? (Justify your answers).

   (a) $T_1 = \{X, \phi, \{c\}, \{b, d, e\}, \{b, c, d, e\}, \{b\}\}$;

   (b) $T_2 = \{X, \phi, \{a\}, \{b, d, e\}, \{a, b, d\}, \{a, b, d, e\}\}$;

   (c) $T_3 = \{X, \phi, \{b\}, \{a, b, c\}, \{d, c, f\}, \{b, d, e, f\}\}$.

2. If X = {a, b, c, d, e, f} and T is the discrete topology on X, which of the following statements are true?

   (a) $X \in T$ (b) $\{X\} \in T$

   (c) $\{\phi\} \in T$ (d) $\phi \in T$

   (e) $\{\phi\} \in X$ (f) $a \in T$

Notes

3. Let (X, T) be any topological space. Verify that the intersection of any finite number of member of T is a member of T.

4. List all possible topologies on the following sets:

   (a) X = {a, b}; (b) Y = {a, b, c}

5. Let X be an infinite set and T a topology on X. If every infinite subset of X is in T, prove that T is the discrete topology.

6. Let (X, T) be a topological space with the property that every subset is closed. Prove that it is a discrete space.

7. Consider the topological space (X, T) where the set X = {a, b, c, d, e}, the topology T = {X, ϕ, {a}, {c, d}, {a, c, d}, {b, c, d, e}}, and A = {a, b, c}. Then b, d, and e are limit points of A but a and c are not limit points of A.

8. Let X = {a, b, c, d, e} and T = {X, ϕ, {a}, {c, d}, {a, c, d}, {b, c, d, e}} show that $\{\overline{b}\}$ = {b, c}, $\{\overline{a, c}\}$ = X, and $\{\overline{b, d}\}$ = {b, c, d, e}.

9. Let X = {a, b, c, d, e, f} and

   T_1 = {X, ϕ, {a}, {c, d}, {a, c, d}, {b, c, d, e, f}},

   (a) Find all the limit points of the following set:

   (i) {a},

   (ii) {b, c},

   (iii) {a, c, d},

   (iv) {b, d, e, f},

   (b) Hence, find the closure of each of the above sets.

10. (a) Let A and B be subsets of a topological space (X, T). Prove carefully that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

    (b) Give an example in which $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

11. Let S be a dense subset of a topological space (X, T). Prove that for every open subset ∪ of X, $\overline{S \cap \cup} = \overline{U}$.

12. Let E be a non-empty subset of a topological space (X, T). Show that $\overline{E}$ = E ∪ d (E), where d (E) is derived set of E.

13. Define interior operator. Explain how can this operator be used to define a topology on a set X.

14. Prove that A subset of topological space is open iff it is nhd of each of its points.

15. (a) Show that A° is the largest open set contained in A.

    (b) Show that the set of all cluster points of set in a topological space is closed.

16. The union of two topologies for a set X is not necessarily a topology for X. Prove it.

17. Let X be a topological space. Let A ⊆ X. Then prove that A ∪ A′ is closed set.

18. Show that A ∪ D(A) is a closed set. Also show that A ∪ D(A) is the smallest closed subset of X containing A.

19. In a topological space, prove that (X – A)° = X – $\overline{A}$. Int A′ = ($\overline{A}$)′. Hence deduce, that A° = ($\overline{A'}$)′.

Notes

20. Let (X, T) be a topological space and $A \subset X$. A point x of A is an interior point of A iff it is not a limit point of X – A.

21. Let T = {X, ϕ, {p}, {p, q}, {p, q, t}, {p, q, r, s}, {p, r, s}} be the topology on X = {p, q, r, s, t}. Determine limit points, closure, interior, exterior and boundary of the following sets:

    (a) A = {r, s, t} (b) B = {p}

22. It T = {ϕ, {a}, {a, b}, {a, c, d}, {a, b, e}, {a, b, c, d}, X} be a topology on X = {a, b, c, d, e} then

    (a) Point out T-open subsets of X.

    (b) Point out T-closed subsets of X.

    (c) Find the closure of the sets {a}, {b}, {c}.

    (d) Find the interior points of the subset A = {a, b, c} on X.

    (e) Which of the sets {a}, {b}, {c, e} are dense in X?

Answers: Self Assessment

1. T_1 = {ϕ, X}, T_2 = {ϕ, X, {b}}, {a, b}, T_3 = {ϕ, X, {a}, {b}, {a, b}}.
2. Yes
8. nhd of *r* are {p, r, s}, {p, q, r, s}

   nhd of t is {p, q, t}
12. D(A) = {c}
13. D ({b}) = D ({a, b}) = D({b, c}) = D({c, a}) = {c}

1.11 Further Readings

Books

J. L. Kelley, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison–Wesley, Mass. 1970.

Unit 2: Basis for Topology

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define the term basis for topology;
- Solve the questions related to basis for topology;
- Describe the sub-base and related theorems;
- State the standard topology.

Introduction

In mathematics, a base or basis $\mathcal{B}$ for a topological space X with topology T is a collection of open sets in T such that every open set in T can be written as a union of elements of $\mathcal{B}$. We say that the base generates the topology T. Bases are useful because many properties of topologies can be reduced to statements about a base generating that topology.

In this unit, we shall study about basis, sub-base, standard topology and lower limit topology.

2.1 Basis for a Topology

Definition: Basis

A collection of subsets $\mathcal{B}$ of X is called a basis or a base for a topology if:

1. The union of the elements of B is X.
2. If $x \in B_1 \cap B_2$, $B_1, B_2, \in \mathcal{B}$, then there exists a B of $\mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

Notes

Another Definition:

$\mathcal{B}$ is said to be a base for the topology T on X if $x \in G \in T \Rightarrow \exists\, B \in \mathcal{B}$ s.t. $x \in B \subset G$.

The elements of $\mathcal{B}$ are referred to as basic open sets.

Example 1:

(1) $\mathcal{S}$, the standard topology on $\mathcal{R}$, is generated by the basis of open intervals (a,b) where a < b.

(2) A basis for another topology on $\mathcal{R}$ is given by half open intervals [a,b), a < b. It generated the lower limit topology $\mathcal{L}$.

(3) The Open intervals (a,b), a < b with a & b rational is a *countable* basis. It generates the same topology as $\mathcal{S}$.

Example 2: Let X = {1, 2, 3, 4}. Let A = {{1, 2}, {2, 4}, {3}}. Determine the topology on X generated by the elements of A and hence determine the base for this topology.

Solution:

Let X = {1, 2, 3, 4} and

A = {{1, 2}, {2, 4}, {3}}.

Finite intersections of the members of A form the class $\mathcal{B}$ given by

$\mathcal{B}$ = {{1, 2}, {3}, {2, 4}, ϕ, {2}, X}.

The unions of the members of $\mathcal{B}$ form the class T given by

T = {{1, 2}, {3}, {2, 4}, ϕ, {2}, X, {1, 2, 3}, {1, 2, 4}, {3, 2, 4}, {3, 2}}.

It can be easily verified that $\mathcal{B}$ is a base for the topology T on X.

2.1.1 Topology Generated by Basis

Lemma 1: Let $\mathcal{B}$ be a basis for a topology T on a set X. Then T equals the collection of all unions of elements of $\mathcal{B}$.

Proof: Each element of $\mathcal{B}$ is open, so arbitrary unions of elements in $\mathcal{B}$ are open i.e., in T. We must show any $\bigcup \in T$ equals a union of basis elements. For each $x \in \bigcup$, choose a set $\mathcal{B}_x \subset \bigcup$ that contains x.

What does the union $\bigcup_x \mathcal{B}_x$ of these basis elements equal? All of $\bigcup$ i.e. $\bigcup_{is}$ a union of basis elements. How to find a basis for your topology.

Lemma 2: Let (X, T) be a topological space. Suppose $\mathcal{B}$ is a collection of open sets of X s.t. $\forall$ open sets $\bigcup$ and $\forall\, x \in \bigcup$, there exists an element $B \in \mathcal{B}$ s.t. $x \in B \subset \bigcup$. Then $\mathcal{B}$ is a basis for T.

Proof: We show the two basis conditions:

1. Since X itself is open in the topology, our hypothesis tells us that $\forall\, x \in X$, there exists $B \in \mathcal{B}$ containing x.

2. Let $x \in B_1 \cap B_2$. Since B_1, B_2 are open, so is $B_1 \cap B_2$; by our hypothesis, there exists $B \in \mathcal{B}$ containing x with $B \subset B_1 \cap B_2$.

So, $\mathcal{B}$ is a basis and generates a topology T′; we must show T′ = T.

Take $\cup \in T$; by hypothesis, there is a set $B \in \mathcal{B}$ with $x \in B \subset \cup$; this is the definition of $\cup$ being an open set in topology T′.

Conversely, take V open in topology T′. Then by the previous lemma, V equals a union of elements of sets in $\mathcal{B}$.

By hypothesis, each set in B is open in topology T; thus V is a union of open sets from T, so it is open in T.

Lemma 3: Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the topologies T and T′, respectively, on X. Then the following are equivalent:

1. T′ is finer than T.
2. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof: (2) ⇒ (1)

Given any element $\cup$ of T,

We are to show that $\cup \in T'$.

Let $x \in \cup$.

Since $\mathcal{B}$ generates T, there is an element $B \in \mathcal{B}$ such that $x \in B \subset \cup$.

Condition (2) tells us ∃ an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Then

$x \in B', \subset \cup,$

so, $\cup \in T'$, by definition

(1) ⇒ (2)

Given $x \in X$ and $B \in \mathcal{B}$, with $x \in B$.

Now B belongs to T by definition

and $T \subset T'$ by condition (1)

∴ $B \in T'$.

Since T′ is generated by $\mathcal{B}'$,

there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

2.1.2 A Characterisation of a Base for a Topology

Theorem 1: Let (X, T) be a topological space. A sub-collection $\mathcal{B}$ of T is a base for T iff every T-open set can be expressed as union of members of $\mathcal{B}$.

or

If T be a topology on X and $\mathcal{B} \subset T$, show that following conditions are equivalent:

(i) Each $G \in T$ is the union of members of $\mathcal{B}$.

(ii) For any x belonging to an open set G, $\exists\, B \in \mathcal{B}$ with $x \in B \subset G$.

Proof: Let $\mathcal{B}$ be a base for the topological space (X, T) so that $x \in G \in T$.

$$\Rightarrow \quad \exists\, B \in \mathcal{B} \text{ s.t. } x \in B \in \mathcal{B} \text{ s.t. } x \in B \subset G \qquad \ldots(1)$$

Notes

To show: $G = \cup \{B : B \in \mathcal{B} \text{ and } B \subset G\}$ …(2)

From (1), the statement (2) at once follows.

Conversely, suppose that $\mathcal{B} \in T$ s.t. (2) holds.

Also, suppose that (X, T) is a topological space.

To prove: statement (1).

Let $x \in X$ be arbitrary and G be an open set s.t. $x \in G$.

Then $x \in G \in T$.

Now (2) suggests that

$$\exists\, B \in \mathcal{B} \text{ s.t. } x \in B \subset G.$$

Hence the result (1).

Self Assessment

1. Let X = {a, b, c, d} and A = {{a, b}, {b, c}, {d}}. Determine a base $\mathcal{B}$ (generated by A) for a unique topology T on X.
2. Let $\mathcal{B}$ be a base for the topology T on X. Let $\mathcal{B}^* \subset T$ s.t. $\mathcal{B} \subset \mathcal{B}^*$. Show that $\mathcal{B}^*$ is a base for the topology T on X.
3. What is necessary and sufficient condition for a family to become a base for a topology?
4. Let B be a base for X and let Y be a subspace of X. Then if we intersect each element of B with Y, the resulting collection of sets is a base for the subspace Y. Prove it.

2.2 Sub-base

Definition: Let (X, T) be a topological space. Let $\mathcal{S} \subset T$ s.t. $S \neq \phi$.

$\mathcal{S}$ is said to be sub, base or open sub-base or semi bases for the topology T on X if finite intersections of the members of $\mathcal{S}$ form a base for the topology T on X i.e. the unions of the members of $\mathcal{S}$ give all the members of T. The elements of $\mathcal{S}$ are referred to as sub-basic open sets.

Example 3: Let $a, b \in \mathcal{R}$ be arbitrary s.t. $a < b$. Clearly $(-\infty, b) \cap (a, \infty) = (a, b)$

The open intervals (a, b) form a base for the usual topology on $\mathcal{R}$. Hence, by definition, the family of infinite open intervals forms a sub-base for the usual topology on $\mathcal{R}$.

Theorem 2: Let $\mathcal{S}$ be a non-empty collection of subsets of a non empty set X. Then $\mathcal{S}$ is a sub-base for a unique topology T for X, i.e., finite intersections of members of $\mathcal{S}$ form a base for T.

Proof: Let $\mathcal{B}$ be the collection of all finite intersections of members of $\mathcal{S}$. Then we have to show that $\mathcal{B}$ is a base for a unique topology on X.

For this, we have to show that $\mathcal{B}$ satisfies conditions (1) and (2).

(1) Since X is the intersection of empty collection of members of $\mathcal{S}$, it follows that

$X \in \mathcal{B}$ and so $X = \bigcup \{B : B \in \mathcal{B}\}$.

(2) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Then B_1, B_2 are finite intersections of members of $\mathcal{S}$. Hence, $B_1 \cap B_2$ is also a finite intersection of members of S and so $B_1 \cap B_2 \in \mathcal{B}$.

Hence, $\mathcal{B}$ is a base for a unique topology on X for which $\mathcal{S}$ is sub-base.

Example 4: Find out a sub-base $\mathcal{S}$ for the discrete topology T on X = {a, b, c} s.t. $\mathcal{S}$ does not contain any singleton set.

Solution: Let X = {a, b, c}. Let T be the discrete topology on X.

If we write $\mathcal{B} = \{\{x\} : x \in X\}$, then by the theorem:

"Let X be an arbitrary set and $\mathcal{B}$ a non empty subset of the power set P(X) of X. $\mathcal{B}$ is a base for some topology on X iff

(i) $\bigcup \{B : B \in \mathcal{B}\} = X$

(ii) $x \in B_1, B_2$ and $B_1, B_2, \in \mathcal{B} \Rightarrow \exists\, B \in \mathcal{B}$ s.t. $x \in B \subset B_1 \cap B_2$.

$\mathcal{B}$ is a base for the topology T on X."

Any family B* of subsets of X. $\mathcal{S}$ does not contain any singleton set. Hence, $\mathcal{S}$ is the required sub-base.

Self Assessment

5. Let $\mathcal{S}$ be a sub-base for the topologies T and T_1 on X. Show that $T = T_1$.
6. Let (Y, $\bigcup$) be a sub-base of (X, T) and $\mathcal{S}$ a sub-base for T on X. Show that the family $\{Y \cap S : S \in \mathcal{S}\}$ is a sub-base for $\bigcup$ on Y.
7. Given a non empty family $\mathcal{S}$ of subsets of a set X, show that $\exists$ weakest topology T on X in which all the members of $\mathcal{S}$ are open sets and $\mathcal{S}$ is a sub-base for T.
8. Let X = {a, b, c, d, e}. Find a sub-base $\mathcal{S}$ for the discrete topology T on X which does not contain any singleton set.

2.3 Standard Topology and Lower Limit Topology

2.3.1 Standard Topology

If $\mathcal{B} = \{(a, b) : a, b \in R \text{ s.t. } a < b\}$ i.e. $\mathcal{B}$ is a collection of open intervals on real line, the topology generated by $\mathcal{B}$ is called standard topology on $\mathcal{R}$.

2.3.2 Lower Limit Topology

If $\mathcal{B}_1 = \{(a, b] : a, b \in R \text{ and } a < b\}$ i.e. $\mathcal{B}_1$ is a collection of semi-open intervals, the topology generated by $\mathcal{B}_1$ is called lower limit topology on $\mathcal{R}$.

When $\mathcal{R}$ is given the lower limit topology, we denote it by $\mathcal{R}_\ell$.

Finally let K denote the set of all numbers of the form $\frac{1}{n}$, for $n \in Z_+$ and let $\mathcal{B}_2$ be the collection of all open intervals (a, b) along with all sets of the form (a, b) –K. The topology generated by $\mathcal{B}_2$ will be called the K-topology on $\mathcal{R}$. When $\mathcal{R}$ is given this topology, we denote it by $\mathcal{R}_K$.

Lemma: The topologies of $\mathcal{R}_l$ and $\mathcal{R}_k$ are strictly finer than the standard topology on $\mathcal{R}$, but are not comparable with one another.

Proof: Let T, T′ and T″ be the topologies or $\mathcal{R}_1$, $\mathcal{R}_\ell$ and $\mathcal{R}_K$, respectively. Given a basis elements (a, b) for T and a point x of (a, b), the basis element [x, b) for T′ contains x and lies in (a, b). On the other hand, given the basis element [x, b) for T′, there is no open interval (a, b) that contains x and lies in [x, d). Thus T′ is strictly finer than T.

Notes

A similar argument applies to $\mathcal{R}_K$. Given a basis element (a, b) for T and a point x of (a, b), this same interval is a basis element for T″ that contains x. On the other hand, given the basis element B = (–1, 1) –K for T″ and the point O of B, there is no open interval that contains O and lies in B.

Now, it can be easily shown that the topologies of $\mathcal{R}_\ell$ and $\mathcal{R}_K$ are not comparable.

Self Assessment

9. Consider the following topologies on $\mathcal{R}$:

   T_1 = the standard topology,

   T_2 = the topology of $\mathcal{R}_K$,

   T_3 = the finite complement topology,

   T_4 = the upper limit topology, having all sets (a, b) as basis,

   T_5 = the topology having all sets $(-\infty, a) = \{x : x < a\}$ as basis

   Determine, for each of these topologies, which of the others it contains.

2.4 Summary

- A base (or basis) $\mathcal{B}$ for a topological space X with topology $\top$ is a collection of open sets in T such that every open set in T can be written as a union of elements of $\mathcal{B}$.
- ***Sub-base:*** Let X be any set and $\mathcal{S}$ a collection of subsets of X. Then $\mathcal{S}$ is a sub-base if a base of X can be formed by a finite intersection of elements of $\mathcal{S}$.
- ***Standard Topology:*** If $\mathcal{B}$ is the collection of all open intervals in the real line $(a, b) = \{x : a < x < b\}$, the topology generated by $\mathcal{B}$ is called standard topology on the real line.
- ***Lower Limit Topology:*** If $\mathcal{B}'$ is the collection of all half-open intervals of the form

  $[a, b) = \{X : a \le x < b\}$,

  where a< b, the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathcal{R}$.

2.5 Keywords

Finer: If $T_1 \subset T_2$, then we say that T_2 is longer or finer than T_1.

Subset: If A and B are sets and every element of A is also an element of B then, A is subset of B denoted by $A \subseteq B$.

Topological Space: It is a set X together with T, a collection of subsets of X, satisfying the following axioms.

(1) The empty set and X are in T.

(2) T is closed under arbitrary union.

(3) T is closed under finite intersection.

2.6 Review Questions

1. Let $\mathcal{B}$ be a basis for a topology on a non empty set X. It $\mathcal{B}_1$ is a collection of subsets of X such that $T \supseteq \mathcal{B}_1 \supset \mathcal{B}$, prove that $\mathcal{B}_1$ is also a basis for T.

2. Show that the collection $\mathcal{B}$ = {(a, b) : a, b ∈ $\mathcal{R}$, a < b} of all open intervals in $\mathcal{R}$ is a base for a topology on $\mathcal{R}$.
3. Show that the collection $\mathcal{C}$ = {[a, b] : a, b ∈ $\mathcal{R}$, a < b} of all closed intervals in $\mathcal{R}$ is not a base for a topology on $\mathcal{R}$.
4. Show that the collection $\mathcal{L}$ = {(a, b] : a, b ∈ $\mathcal{R}$, a < b} of half-open intervals is a base for a topology on $\mathcal{R}$.
5. Show that the collection $\mathcal{S}$ = {[a, b) : a, b ∈ $\mathcal{R}$, a < b} of half-open intervals is a base for a topology on $\mathcal{R}$.
6. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a sub-basis.
7. If $\mathcal{S}$ is a sub-base for the topology T on X, then $\mathcal{S}$ = {X, ϕ} is also a sub-base for T on X.

Answers: Self Assessment

1. $\mathcal{B}$ = {{a, b}, {b, c}, {d}, {b}, ϕ, X}

   T = {$\mathcal{B}$, {a, b, d}, {b, c, d}, {b, d}, {a, b, c}}.

8. $\mathcal{S}$ = {{a, b}, {b, c}, {c, d}, {d, e}, {e, a}}.

2.7 Further Readings

Books

Engelking, Ryszard (1977), *General Topology*, PWN, Warsaw.

Willard, Stephen (1970), *General Topology*, Addison-Wesley. Reprinted 2004, Dover Publications.

Notes

Unit 3: The Order Topology

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the order topology;
- Solve the problems on order topology;
- Describe the open intervals, closed intervals and half-open intervals.

Introduction

If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called the order topology; in this unit, we consider it and study some of its properties.

3.1 The Order Topology

3.1.1 Intervals

Suppose that X is a set having a simple order relation <. Given elements a and b of X such that a < b, there are four subsets of X that are called the intervals determined by a and b. They are the following:

$$(a, b) = \{x \mid a < x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$[a, b] = \{x \mid a \leq x \leq b\}$$

The notation used here is familiar to you already in the case where X is the real line, but these are intervals in an arbitrary ordered set.

- A set of the first type is called an ***open interval*** in X.
- A set of the last type is called a ***closed interval*** in X.
- Sets of the second and third types are called ***half-open intervals***.

> *Note* The use of the term "open" in this connection suggests that open intervals in X should turn out to be open sets when we put a topology on X and so they will.

3.1.2 Order Topology

Definition: Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_o, b)$, where a_o is the smallest element (if any) of X.

(3) All intervals of the form $(a_o, b_o]$, where b_o is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the ***order topology***. If X has no smallest element, there are no sets of type (2), and if X has no largest element, there are no sets of type (3).

> *Notes* One has to check that $\mathcal{B}$ satisfies the requirements for a basis.
>
> (A) First, note that every element x of X lies in at least one element of $\mathcal{B}$: The smallest element (if any) lies in all sets of type (2), the largest element (if any) lies in all sets of type (3), and every other element lies in a set of type (1).
>
> (B) Second, note that the intersection of any two sets of the preceding types is again a set of one of these types, or is empty.

Example 1: The standard topology on $\mathcal{R}$ is just the order topology derived from the usual order on $\mathcal{R}$.

Example 2: Consider the set $\mathcal{R} \times \mathcal{R}$ in the dictionary order; we shall denote the general element of $\mathcal{R} \times \mathcal{R}$ by x × y, to avoid difficulty with notation. The set $\mathcal{R} \times \mathcal{R}$ has neither a largest nor a smallest element, so the order topology on $\mathcal{R} \times \mathcal{R}$ has as basis the collection of all open intervals of the form (a × b, c × d) for a < c, and for a = c and b < d. The subcollection consisting of only intervals of the second type is also a basis for the order topology on $\mathcal{R} \times \mathcal{R}$, as you can check.

Example 3: The positive integers Z_+ form an ordered set with a smallest element. The order topology on Z_+ is the discrete topology, for every one-point set is open : If n > 1, then the one-point set {n} = {n–1, n+1} is a basis element; and if n=1, the one-point set {1} = [1, 2) is a basis element.

Notes

Example 4: The set $X = \{1, 2\} \times Z_+$ in the dictionary order is another example of an ordered set with a smallest element. Denoting $1 \times n$ by a_n and $2 \times n$ by b_n, we can represent X by

$$a_1, a_2, \ldots ; \quad b_1, b_2, \ldots$$

The order topology on X is not the discrete topology. Most one-point sets are open, but there is an exception the one-point set $\{b_1\}$. Any open set containing b_1 must contain a basis element about b_1 (by definition), and any basis element containing b_1 contains points of the a_i sequence.

3.1.3 Rays

Definition: If X is an ordered set, and a is an element of X, there are four subsets of X that are called the rays determined by a. They are the following:

$$(a, +\infty) = \{x \mid x > a\}$$

$$(-\infty, a) = \{x \mid x < a\},$$

$$[a, +\infty) = \{x \mid x \geq a\},$$

$$(-\infty, a] = \{x \mid x \leq a\}.$$

sets of first two types are called open rays; and sets of the last two types are called closed rays.

Notes

(1) The use of the term "open" suggests that open rays in X are open sets in the order topology. And so they are (consider, for example, the ray $(a, +\infty)$. If X has a largest element b_o, then $(a, +\infty)$ equals the basis element $(a, b_o]$. If X has no largest element, then $(a, +\infty)$ equals the union of all basis elements of the form (a, x), for $x > a$. In either case, $(a, +\infty)$ is open. A similar argument applies to the ray $(-\infty, a)$.

(2) The open rays, in fact, form a sub-basis for the order topology on X, as we now show. Because the open rays are open in the order topology, the topology they generate is contained in the order topology. On the other hand, every basis element for the order topology equals a finite intersection of open rays; the interval (a, b) equals the intersection of $(-\infty, b)$ and $(a, +\infty)$, while $[a_o, b)$ and $(a, b_o]$, if they exist, are themselves open rays. Hence the topology generated by the open rays contains the order topology.

3.1.4 Order Topology on the Linearly Ordered Set

The order topology $T_<$ on the linearly ordered set X is the topology generated by all open rays. A linearly ordered space is a linearly ordered set with the order topology.

3.1.5 Lemma (Basis for the Order Topology)

Let $(X, <)$ be a linearly ordered set.

(1) The union of all open rays and all open intervals is a basis for the order topology $T_<$.

(2) If X has no smallest and no largest element, then the set $\{(a, b) \mid a, b \in X, a < b\}$ of all open intervals is a basis for the order topology.

Proof: As we know

$$B_{S<} = \{\text{Finite intersections of S-sets}\}$$

$$= S \cup \{(a, b) \mid a, b \in X, a < b\}$$ is a basis for the topology generated by the sub-basis $S_<$.

If X has a smallest element a_o then $(-\infty, b) = [a_o, b)$ is open. If X has no smallest element, then the open ray $(-\infty, b) = \cup_{a<c} (a, c)$ is a union of open intervals and we do not need this open ray in the basis. Similar remarks apply to the greatest element when it exists.

3.2 Summary

- Open interval : $(a, b) = \{x \mid a < x < b\}$

  Closed interval : $[a, b] = \{x \mid a \le x \le b\}$

- Half open intervals : $(a, b] = \{x \mid a < x \le b\}$

  $[a, b) = \{x \mid a \le x < b\}$

- The order topology $T_<$ on the linearly ordered set X is the topology generated by all open rays. A linearly ordered space is a linearly ordered set with the order topology.

- Open rays : $(a, +\infty) = \{x \mid x > a\}$

  $(-\infty, a) = \{x \mid x < a\}$

- Closed rays : $(-\infty, a] = \{x \mid x \le a\}$

  $[a, +\infty) = \{x \mid x \ge a\}$

3.3 Keywords

Basis: A basis $\mathcal{B}$ for a topological space X with topology T is a collection of open sets in T such that every open set in T can be written as a union of elements of $\mathcal{B}$.

Discrete Space: Let X be any non empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

Open and Closed Set: Any set $A \in T$ is called an open subset of X or simply a open set and X – A is a closed subset of X.

3.4 Review Questions

1. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

2. Show that the dictionary order topology on the set $R \times R$ is the same as the product topology $R_d \times R$, where R_d denotes R in the discrete topology. Compare this topology with the standard topology on R^2.

Notes

3.5 Further Readings

Books

Baker, *Introduction to Topology* (1991).

Dixmier, *General Topology* (1984).

Online links

http://mathforum.org/isaac/problems/bridgesl.html

http://www.britannica.com

mathworld.wolfram.com/ordertopology.html

Unit 4: The Product Topology on X × Y

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Describe the product topology;
- Solve the problems on product topology;
- Define projection mappings;
- Discuss the problems on projection mappings.

Introduction

A product space is the Cartesian product of a family of topological space equipped with a natural topology called the product topology. This topology differs from another, perhaps more obvious, topology called the box topology, which can also be given to a product space and which agrees with the product topology when the product is over only finitely many spaces. However the product topology is 'correct' in that it makes the product space a categorical product of its factors, whereas the box topology is too fine, this is the sense in which the product topology is natural.

4.1 Product Topology

Given two sets X and Y, their product is the set $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$.

For example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and more generally $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.

If X and Y are topological spaces, we can define a topology on X × Y by saying that a basis consists of the subsets U × V as U ranges over open sets in X and V ranges over open sets in Y.

The criterion for a collection of subsets to be a basis for a topology is satisfied since

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

This is called the product topology on X × Y.

Notes

Example 1: A basis for the product topology on $\mathbb{R} \times \mathbb{R}$ consists of the open rectangles $(a_1, b_1) \times (a_2, b_2)$. This is also a basis for the usual topology on $\mathbb{R}^2$, so the product topology coincides with the usual topology.

Example 2: Take the topology T = {ϕ, {a, b}, {a}} on X = {a, b}.

Then the product topology on X × X is

{ϕ, X × X, {(a, a)}, {(a, a), (a, b)}, {(a, a), (b, a)}, {(a, a), (a, b), (b, a)}} where the last open set in the list is not in the basis.

Theorem 1: If (X_1, T_1) and (X_2, T_2) are any two topological spaces, then the collection

$$\mathcal{B} = \{G_1 \times G_2 : G_1 \in T_1, G_2 \in T_2\}$$

is a base for some topology on $X = X_1 \times X_2$.

Proof: Suppose, (X_1, T_1) and (X_2, T_2) be any two topological spaces.

Write $X = X_1 \times X_2$,

$\mathcal{B} = \{U_1 \times U_2 : U_1 \in T_1, U_2 \in T_2\}$.

To show: $\mathcal{B}$ is a base for some topology on X.

(i) To prove: $\cup \{B : B \in \mathcal{B}\} = X$.

$X_1 \in T_1, X_2 \in T_2 \Rightarrow X_1 \times X_2 \in \mathcal{B}$

$\Rightarrow X \in \mathcal{B}$

$\Rightarrow X = \cup \{B : B \in \mathcal{B}\}$

(ii) Let $U_1 \times U_2, V_1 \times V_2 \in \mathcal{B}$ and let

$(x_1, x_2) \in (U_1 \times U_2) \cap (V_1 \times V_2)$

To prove: $\exists\, W_1 \times W_2 \in \mathcal{B}$ s.t.

$(x_1, x_2) \in W_1 \times W_2 \subset (U_1 \times U_2) \cap (V_1 \times V_2)$

$(x_1, x_2) \in (U_1 \times U_2) \cap (V_1 \times V_2)$

$\Rightarrow (x_1, x_2) \in U_1 \times U_2$ and $(x_1, x_2) \in V_1 \times V_2$

$\Rightarrow x_1 \in U_1, x_2 \in U_2 ; x_1 \in V_1, x_2 \in V_2$

$\Rightarrow x_1 \in U_1 \cap V_1 ; x_2 \in U_2 \cap V_2$

$\Rightarrow x_1 \in W_1 ; x_2 \in W_2$

On taking $W_1 = U_1 \cap V_1$,

$W_2 = U_2 \cap V_2$

$\Rightarrow (x_1, x_2) \in W_1 \times W_2$

$U_1 \times U_2 \in \mathcal{B}, V_1 \times V_2 \in \mathcal{B}$

$\Rightarrow U_1 \in T_1, U_2 \in T_2 ; V_1 \in T_1, V_2 \in T_2$

$\Rightarrow U_1 \cap V_1 \in T_1, U_2 \cap V_2 \in T_2$

$\Rightarrow W_1 \in T_1, W_2 \in T_2$

$\Rightarrow W_1 \times W_2 \in \mathcal{B}$

So, we have proved that

$\exists\, W_1 \times W_2 \in B$ s.t $(x_1, x_2) \in W_1 \times W_2$

Now, it remains to prove that

$W_1 \times W_2 \subset (U_1 \times U_2) \cap (V_1 \times V_2)$

Let $(y_1, y_2) \in W_1 \times W_2$ be arbitrary.

$(y_1, y_2) \in W_1 \times W_2 \quad \Rightarrow y_1 \in W_1, y_2 \in W_2$

$\Rightarrow y_1 \in U_1 \cap V_1, y_2 \in U_2 \cap V_2$

$\Rightarrow y_1 \in U_1, y_1 \in V_1$ and $y_2 \in U_2, y_2 \in V_2$

$\Rightarrow (y_1, y_2) \in U_1 \times U_2$ and $(y_1, y_2) \in V_1 \times V_2$

$\Rightarrow (y_1, y_2) \in (U_1 \times U_2) \cap (V_1 \times V_2)$

Finally, any $(y_1, y_2) \in W_1 \times W_2$

$\Rightarrow \quad (y_1, y_2) \in (U_1 \times U_2) \cap (V_1 \times V_2)$

This proves that

$W_1 \times W_2 \subset (U_1 \times U_2) \cap (V_1 \times V_2)$

It immediately follows from (i) and (ii) that B is a base for some topology, say, T on X.

Theorem 2: Let (X_1, T_1) and (X_2, T_2) be two topological spaces and let $\mathcal{B}_1, \mathcal{B}_2$ be bases for T_1 and T_2 respectively.

Let $X = X_1 \times X_2$

Then $\mathcal{B} = \{\mathcal{B}_1 \times \mathcal{B}_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is a base for the product topology T on X.

Proof: Let $\mathcal{C} = \{G_1 \times G_2 : G_1 \in T_1, G_2 \in T_2\}$.

Then $\mathcal{C}$ is a base for the topology T on X (refer theorem 1)

We are to prove that $\mathcal{B}$ is a base for T on X.

By definition of base,

for $(x_1, x_2) \in G \in T$

$\Rightarrow \exists\, G_1 \times G_2 \in \mathcal{C}$ s.t. $(x_1, x_2) \in G_1 \times G_2 \subset G$ ...(1)

Again $(x_1, x_2) \in G_1 \times G_2 \in C$

$\Rightarrow x_1 \in G_1 \in T_1, \quad x_2 \in G_2 \in T_2.$

Applying definition of base,

$x_1 \in G_1 \in T_1 \Rightarrow \exists\, B_1 \in \mathcal{B}_1$, s.t. $x_1 \in B_1 \subset G_1$ ...(2)

$x_2 \in G_2 \in T_2 \Rightarrow \exists\, B_2 \in \mathcal{B}_2$, s.t. $x_2 \in B_2 \subset G_2$ ...(3)

$B_1 \in \mathcal{B}, B_2 \in \mathcal{B}_2 \Rightarrow B_1 \times B_2 \in \mathcal{B}.$

Now (2) and (3) $\Rightarrow \exists\, B_1 \times B_2 \in B$ s.t.

$(x_1, x_2) \in B_1 \times B_2 \subset G_1 \times G_2 \subset G$

or $\quad (x_1, x_2) \in B_1 \times B_2 \subset G$

Notes

Thus, we have shown that

$(x_1, x_2) \in G \in T$

$\Rightarrow \quad \exists\, B_1 \times B_2 \in \mathcal{B} \quad$ s.t. $\quad (x_1, x_2) \in B_1 \times B_2 \subset G$

By definition,

This proves that $\mathcal{B}$ is base for T on X.

Remark: From the theorems (1) and (2), it is clear that

$\mathcal{B} = \{B_1 \times B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$,

$\mathcal{C} = \{G_1 \times G_2 : G_1 \in T_1, G_2 \in T_2\}$

both are bases for the same topology T on X.

Theorem 3: Let (X, T) and (Y, $\mathcal{V}$) be any two topological spaces and let L and M be sub-bases for T_1 and $\mathcal{V}$ respectively. Then the collection $\mathcal{A}$ of all subsets of the form L × Y and X × M, is a sub-base for the product topology T on X × Y, where $L \in \mathcal{L}$, $M \in \mathcal{M}$.

Proof: Now in order to prove that $\mathcal{A}$ is a sub-base for T on X × Y, we are to prove that: the collection $\mathcal{G}$ of finite intersections of members of $\mathcal{A}$ form a base for T on X × Y.

Since the intersection of empty sub collection of $\mathcal{A}$ is X × Y and so X × Y ∈ G.

Next let $\{L_1 \times Y, L_2 \times Y, ..., L_p \times Y\} \cup \{X \times M_1, X \times M_2, ..., X \times M_q\}$ be a non empty finite sub-collection of $\mathcal{A}$. This intersection of these elements belong to $\mathcal{G}$, by construction of $\mathcal{G}$. This element of $\mathcal{G}$ is

$(L_1 \times Y) \cap (L_2 \times Y) \cap ... (L_p \times Y) \cap (X \times M_1) \cap (X \cap M_2) \cap ... \cap (X \times M_q)$

$= [(L_1 \cap L_2 \cap ... \cap L_p) \times Y] \cap [X \times (M_1 \cap M_2 \cap ... \cap M_q)]$

[For $A \times (B \cap C) = (A \times B) \cap (A \cap C)$]

$= [(L_1 \cap L_2 \cap ... \cap L_p) \cap X] \times [(M_1 \cap M_2 \cap ... \cap M_q) \cap Y]$

[For $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$]

$= (L_1 \cap L_2 \cap ... \cap L_p) \times (M_1 \cap M_2 \cap ... \cap M_q)$

[For $L_n \subset X$ and $M_n \subset Y \;\; \forall\; n$]

$$= \left[\bigcap_{n=1}^{p} L_r\right] \times \left[\bigcap_{r=1}^{q} M_r\right] \qquad ...(1)$$

We suppose that $\mathcal{B}$ is base for T_1 on X generated by the elements of $\mathcal{L}$ and $\mathcal{C}$ is a base for $\mathcal{V}$ on Y generated by the elements of $\mathcal{M}$.

As we know that the finite intersections of sub-base form the base for that topology.

In view of the above statements,

$$\bigcap_{r=1}^{p} L_r \in \mathcal{B} \qquad \bigcap_{r=1}^{q} M_r \in \mathcal{C}$$

From (i), it follows that $\mathcal{G}$ is expressible as

$$\mathcal{G} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

Then $\mathcal{G}$ is a base for the product topology T on X × Y. (Refer Theorem 2).

But $\mathcal{G}$ is obtained from the finite intersections of members of $\mathcal{A}$.

It follows that $\mathcal{A}$ is a sub-base for the product topology T on X × Y.

Theorem 4: The product of two second axiom spaces is a second axiom space.

Proof: Let (X, T_1) and (Y, T_2) be two second countable spaces.

Let (X × Y × T) be the product topological space.

To prove that (X × Y × T) is second countable.

Our assumption implies that ∃ countable bases.

$$B_1 = \{B_i : i \in N\} \text{ and } \{C_i : i \in N\}$$

for X and Y respectively. Recall that

$$B = \{G_1 \times G_2; G_1 \in T_1, G_2 \in T_2\}$$

is a base for the topology T on X × Y.

Write

$$C = \{B_i \times C_{j;}\ i, j \in N_1\} = B_1 \times B_2$$

B_1 and B_2 are countable = $B_1 \times B_2$ are countable

$$\Rightarrow C \text{ is countable}$$

By definition of base B

$$\text{any } (x, y) \in N \in T \Rightarrow \exists\, G \times H \in B \text{ s.t. } (x, y) \in G \times H \subset N \quad \ldots(1)$$

$$\Rightarrow x \in G \in T_1, y \in H \in T_2$$

$$\Rightarrow \exists\, B_i \in B_1, C_j \in B_2 \text{ s.t. } x \in B_i \subset G, y \in C_j \subset H$$

This $\Rightarrow (x, y) \in B_i, \times C_j \subset G \times H \subset N.$

Thus any $(x, y) \in N \in T \Rightarrow \exists\, B_i \times C_j \in C$ s.t. $(x, y) \in B_i \times B_j \subset N$. By definition this proves that C is a base for the topology T on X × Y. Also C has been shown to be countable. Hence (X × Y, T) is second countable.

Theorem 5: The product space of two Hausdorff space is Hausdorff space.

Proof: Let (X, T) be a product topological space of two Hausdorff space (X_1, T_1) and (X_2, T_2).

To prove that (X, T) is Hausdorff space.

Consider a pair of distinct elements (x_1, x_2) and (y_1, y_2) in X.

Case I. When $x_1 = y_1$

then $x_2 \neq y_2, \quad \therefore (x_1, x_2) \neq (y_1, y_2)$

By the Hausdorff space property, given a pair of elements

$x_2, y_2 \in X_2$ s.t. $x_2 \neq y_2$, there are disjoint open sets

$$G_2, H_2 \neq X_2 \text{ s.t. } x_2 \in G_2, y_2 \in H_2$$

Then $X_1 \times G_2$ and $X \times H_2$ are disjoint open sets in X. for

$$x_1 \in X_1, x_2 \in G_2 \Rightarrow (x_1, x_2) \in X_1 \times G_2.$$

$$y_1 \in X_1, y_2 \in H_2 \Rightarrow (y_1, y_2) \in X_1 \times H_2.$$

Notes

$\therefore$ Given a pair of distinct elements (x_1, x_2), $(y_1, y_2) \in X$ there are disjoint open subsets $X_1 \times G_2$, $X_1 \times H_2$ of X s.t. $(x_1, x_2) \in X_1 \times G_2$, $(y_1, y_2) \in X_1 \times H_2$.

The leads to the conclusion that (X, T) is a Hausdorff space.

Example 3: Let $T_1 = \{\phi, \{1\}, X_1\}$ be a topology on $X_1 = \{1, 2, 3\}$ and $T_2 = \{\phi , X_2, \{a\}, \{b\} \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ be a topology for $X_2 = \{a, b, c, d\}$.

Find a base for the product topology T.

Solution: Let B_1 be a base for T_1 and B_2 be a base for T_2. Then $B = \{B_1 \times B_2 : B_1 \in B_1, B_2 \in B_2\}$ is a base for the product topology T.

We can take $B_1 = \{\{1\}, X_1\}$

$B_2 = \{\{a\}, \{b\}, \{c, d\}\}$.

The elements of B are

$\{1\} \times\{a\}, \{1\} \times\{b\}, \{1\} \times\{c, d\}, \{1, 2, 3\} \times\{a\}, \{1, 2, 3\} \times\{b\}, \{1, 2, 3\} \times\{c, d\}$.

That is to say

$$B = \begin{Bmatrix} \{(1,a)\}, \{(1,b)\}, \{(1,c),(1,d)\} \\ \{(1,a),(2,a),(3,a)\}, \{(1,b),(2,d),(3,b)\} \\ \{(1,c),(2,c),(3,c),(1,d),(2,d),(3,d)\} \end{Bmatrix}$$

is a base for T.

Self Assessment

1. Let X and X′ denote a single set in the topologies T and T′ respectively let Y and Y′ denote a single set in the topologies U and U′ respectively. Assume these sets are non-empty.
   (a) Show that if $T' \supset T$ and $U' \supset U$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.
   (b) Does the converse of (a) hold? Justify your answer.

4.2 Projection Mappings

Definition:

The mappings,

$\pi_x : X \times Y \to X$ s.t. $\pi_x(x, y) = x \ \forall \ (x, y) \in X \times Y$

$\pi_x : X \times Y \to Y$ s.t. $\pi_y(x, y) = y \ \forall \ (x, y) \in X \times Y$

are called projection maps of X × Y onto X and Y spaces respectively.

Theorem 6: If (X, T) is the product space of topological spaces (X_1, T_1) and (X_2, T_2), then the projection maps π_1 and π_2 are continuous and open.

Proof: Let (X, T) be a product topological space of topological spaces (X_1, T_1) and (X_2, T_2). Then $X = X_1 \times X_2$.

Define maps

$\pi_1 : X \to X_1$ s.t. $\pi_1(x_1, x_2) = x_1 \ \forall \ (x_1, x_2) \in X$

$\pi_2 : X \to X_2$ s.t. $\pi_2(x_1, x_2) = x_2 \ \forall \ (x_1, x_2) \in X$.

Then π_1 and π_2 both are called projection maps on the first and second coordinate spaces respectively.

Step (i): To prove: projection maps are continuous maps.

Firstly, we shall show that π_1 is continuous.

Let $G \subset X_1$ be an arbitrary open set.

$$\begin{aligned}\pi_1^{-1}(G) &= \{(x_1, x_2) \in X : \pi_1(x_1, x_2) \in G\}\\ &= \{(x_1, x_2) \in X : x_1 \in G\}\\ &= \{(x_1, x_2) \in X_1 \times X_2 : x_1 \in G\}\\ &= G \times X_2\\ &= \text{An open set in X.}\end{aligned}$$

For G is open in X_1, X_2 is open in X_2

$\Rightarrow G \times X_2$ is open in X.

$\Rightarrow \pi_1^{-1}(G)$ is open in X.

Thus, we have prove that

any open set $G \subset X_1 \Rightarrow \pi_1^{-1}[G]$ is open in X.

$\Rightarrow \pi_1$ is continuous.

Similarly, we can prove that π_2 is continuous map. Consequently, projection maps are continuous maps.

Step (ii): To prove that projection maps are open maps. We shall first show that π_2 is an open map.

Let $G \subset X$ be an arbitrary open set.

Let $x_2 \in \pi_2[G]$ be arbitrary.

$x_2 \in \pi_2[G] \Rightarrow \exists (u_1, u_2) \in G$ s.t. $\quad \pi_2(u_1, u_2) = x_2$

$\Rightarrow u_2 = x_2 \quad [\because \pi_2(u_1, u_2) = u_2]$

Now $(u_1, x_2) \in G$

Let $\mathcal{B}$ be the base for the topology T on X.

By definition of base,

$(u_1, x_2) \in G \in T \Rightarrow \exists\, U_1 \times U_2 \in \mathcal{B}$ s.t. $\quad (u_1, x_2) \in U_1 \times U_2 \subset G$

$\Rightarrow \pi_2(u_1, x_2) \in \pi_2(U_1 \times U_2) \subset \pi_2(G)$

$\Rightarrow x_2 \in \pi_2(U_1 \times U_2) \subset \pi_2(G)$

$\Rightarrow x_2 \in U_2 \subset \pi_2(G)$.

For $\pi_2(U_1 \times U_2) = \{\pi_2(x_1, x_2) : (x_1, x_2) \in U_1 \times U_2\}$

$= \{x_2 : x_1 \in U_1, x_2 \in U_2\} = U_2$.

$\therefore$ Given any $x_2 \in \pi_2[G] \Rightarrow \exists$ open set $U_2 \subset X_2$ s.t. $x_2 \in U_2 \subset \pi_2[G]$.

This proves that x_2 is an interior point of $\pi_2[G]$. But x_2 is an arbitrary point of $\pi_2[G]$.

Notes

$\therefore$ Every point of $\pi_2[G]$ is an interior point.

This proves that $\pi_2[G]$ is open in X_2.

$\therefore$ Any open set $G \subset X$

$\Rightarrow$ $\pi_2(G)$ is open in X_2.

This proves that the map $\pi_2 : X \to X_2$ is an open map. Similarly, we can show that π_1 is an open map. Consequently, projection maps are open maps.

This completes the proof of the theorem.

Theorem 7: Let (X, T) be the product topological space of (X_1, T_1) and (X_2, T_2).

Let $\pi_1 : X \to X_1$, $\pi_2 : X \to X_2$

be the projection maps on the first and second co-ordinate spaces respectively.

Let $f : Y \to X$ be another map, where Y is another topological space. Show that f is continuous iff π_1 o f and π_2 o f are continuous maps.

Proof: Let $X = X_1 \times X_2$

Let (X, T) be the product topological space of (X_1, T_1) and (X_2, T_2).

Let (Y, U) be another topological space.

Let $\mathcal{B}$ be the base for the topology T on X.

Let $\pi_1 : X \to X_1$,

$\pi_2 : X \to X_2$ be projection maps.

Let $f : Y \to X$ be another map.

Then π_1 o $f : Y \to X_1$

π_2 o $f : Y \to X_2$ are also maps.

Let f be continuous.

To prove that π_1 o f and π_2 o f are continuous maps.

By theorem 6, projection maps are continuous, i.e. π_1 and π_2 are continuous maps.

Also f is given to be continuous.

This means that π_1 o f, π_2 o f are continuous maps. Conversely, suppose that π_1 o f, π_2 o f are continuous maps.

To show that f is continuous.

Let $G \subset X$ be an arbitrary open set.

If we prove that $f^{-1}(G)$ is open in Y, the result will follow.

Let $y \in f^{-1}(G)$ be an arbitrary, then $f(y) \in G$.

$\therefore$ f(y) is an element of $X = X_1 \times X_2$ and hence it can be taken as $f(y) = (x_1, x_2) \in G$

By definition of base,

$(x_1, x_2) \in G \in T \Rightarrow \exists\, U_1 \times U_2 \in \mathcal{B}$ s.t. $(x_1, x_2) \in U_1 \times U_2 \subset G$

$\Rightarrow$ $\pi_1(x_1, x_2) \in \pi_1(U_1 \times U_2) \subset \pi_1(G)$ and

Notes

$\pi_2(x_1, x_2) \in \pi_2(U_1 \times U_2) \subset \pi_2(G)$

$\Rightarrow$ $x_1 \in U_1 \subset \pi_1(G)$ and $x_2 \in U_2 \subset \pi_2(G)$ ...(1)

For $\pi_1(U_1 \times U_2) = \{\pi_1(x_1, x_2) : (x_1, x_2) \in U_1 \times U_2\}$

$= \{x_1 : x_1 \in U_1, x_2 \in U_2\}$

$= U_1$

Similarly, $\pi_2(U_1 \times U_2) = U_2$

$(\pi_1 \text{ o } f)(y) = \pi_1(f(y))$

$= \pi_1(x_1, x_2)$

$= x_1$

Similarly, $(\pi_2 \text{ o } f)(y) = x_2$

Thus, $(\pi_1 \text{ o } f)(y) = x_1, (\pi_2 \text{ o } f)(y) = x_2$

In this event (1) takes the form

$$\left.\begin{array}{l}(\pi_1 \text{ o } f)(y) \in U_1 \subset \pi_1(G) \\ (\pi_1 \text{ o } f)(y) \in U_2 \subset \pi_2(G)\end{array}\right\} \quad ...(2)$$

This $y \in (\pi_1 \text{ o } f)^{-1}(U_1)$ and

$y \in (\pi_2 \text{ o } f)^{-1}(U_2)$

$\Rightarrow$ $y \in [(\pi_1 \text{ o } f)^{-1}(U_1)] \cap [(\pi_2 \text{ o } f)^{-1}(U_2)]$ ...(3)

$\because$ π_1 o f, π_2 o f are given to be continuous and hence $(\pi_1 \text{ o } f)^{-1}(U_1)$ and $(\pi_2 \text{ o } f)^{-1}(U_2)$ are open in Y.

$\Rightarrow$ $[(\pi_1 \text{ o } f)^{-1}(U_1) \cap [(\pi_2 \text{ o } f)^{-1}(U_2)]$ is open in Y.

On taking $(\pi_1 \text{ o } f)^{-1}(U_1) = V_1$, $[\pi_2 \text{ o } f)^{-1}(U_2) = V_2$.

We have $V_1 \cap V_2$ as an open set in Y.

According to (3), $y \in V_1 \cap V_2 = V$ (say)

any $v \in V \Rightarrow v \in V_1$ and $v \in V_2$

$\Rightarrow$ $v \in (\pi_1 \text{ o } f)^{-1}(U_1)$, $v = (\pi_2 \text{ o } f)^{-1}(U_2)$

$\Rightarrow$ $(\pi_1 \text{ o } f)(v) \in U_1$, $(\pi_2 \text{ o } f)(v) \in U_2$

$\Rightarrow$ $(\pi_1 \text{ o } f)(v) \in U_1 \subset \pi_1(G)$ and

$(\pi_2 \text{ o } f)(v) \in U_2 \subset \pi_2(G)$ [from (2)]

$\Rightarrow$ $v \in (\pi_1 \text{ o } f)^{-1}[\pi_1(G)]$ and $v \in (\pi_2 \text{ o } f)^{-1}[\pi_2(G)]$

$\Rightarrow$ $v \in (f^{-1} \text{ o } \pi_1^{-1})[\pi_1(G)]$ and

$v \in (f^{-1} \text{ o } \pi_2^{-1})[\pi_2(G)]$

$\Rightarrow$ $v \in f^{-1}(G)$ and $V \in f^{-1}(G)$

$\therefore$ any $v \in V \Rightarrow v \in f^{-1}(G)$

$\Rightarrow V \subset f^{-1}(G)$

Notes

Thus we have shown that

any $y \in f^{-1}(G) \Rightarrow \exists$ an open set $V \subset Y$ s.t. $y \in V \subset f^{-1}(G)$.

$\Rightarrow$ y is an interior point of $f^{-1}(G)$ and hence every point of $f^{-1}(G)$ is an interior point, showing thereby $f^{-1}(G)$ is open in Y.

Theorem 8: The product topology is the coarser (weak) topology for which projections are continuous.

Proof: Let $(X \times Y, T)$ be product topological space of (X, T_1) and (Y, T_2).

Let $\mathcal{B}$ be a base for T. Then

$\mathcal{B} = \{G_1 \times G_2 : G_1 \in T_1, G_2 \in T_2\}$

The mappings, $\pi_x : X \times Y \to X$ s.t. $\pi_x(x, y) = x$

and $\pi_y : X \times Y \to Y$ s.t. $\pi_y(x, y) = y$

are called projection maps.

These maps are continuous. [Refer theorem (4)]

Let T* be any topology on $X \times Y$ for which π_x and π_y are continuous.

To prove: T is the coarest (weakest) topology for which projections are continuous, we have to show that $T \subset T^*$.

For this, we have to show that

any $G \in T \Rightarrow G \in T^*$

Let $G \in T$, by definition of base,

$G \in T \Rightarrow B_1 \subset B$ s.t. $G = \bigcup\{B : B \in \mathcal{B}_1\}$

$\Rightarrow$ $G = \bigcup\{G_1 \times G_2 : G_1 \times G_2 \in \mathcal{B}_1\}$

$G_1 \subset X \Rightarrow G_1 \cap X = G_1$

$G_2 \subset X \Rightarrow G_2 \cap X = G_2$

Then $G = \bigcup\{(G_1 \cap X) \times (G_2 \cap Y) : G_1 \times G_2 \in \mathcal{B}_1\}$

$= \bigcup\{(G_1 \times G_2) \cap (X \times Y) : G_1 \times G_2 \in \mathcal{B}_1\}$

[For $(a \times b) \cap (c \times d) = (a \cap c) \times (b \cap d)$]

or $G = \{\pi_x^{-1}(G_1) \cap \pi_y^{-1}(G_2) : G_1 \times G_2 \in \mathcal{B}_1\}$ … (1)

$\pi_x : X \times Y \to X, G_1 \in T, \pi_x$ is continuous

$\Rightarrow \pi_x^{-1}(G_1) \in T^*$

Similarly, $\pi_y^{-1}(G_2) \in T^*$

This implies $\pi_x^{-1}(G_1) \cap \pi_y^{-1}(G_2) \in T^*$, be definition of topology.

In this event (1) declares that G is an arbitrary union of T* open sets and hence G is T* open set and so $G \in T^*$.

any $G \in T \Rightarrow G \in T^*$

Notes

Example 4: Let B be a member of the defining base for the product space $X = x_i X_i$, show that the projection of B into any coordinate space is open.

or

Each projection is a continuous map.

Solution: Let B be a member of the defining base for the product space $X = x_i X_i$ so that B is expressible as

$B = x\{X_i : i \neq j_1, j_2, ..., j_m\} \times G_{j_i} \times ... \times G_{j_m}$

where G_{i_k} is an open subset of X_{i_k}.

The projection map π_α is defined as

$$\pi_\alpha : X \to X_\alpha$$

$$\pi_\alpha(B) = \begin{cases} X_\alpha & \text{if} \quad \alpha \neq j_1, j_2, ...j_m \\ G_\alpha & \text{if} \quad \alpha \in \{j_1, j_2, ...j_m\} \end{cases}$$

In either case, $\pi_\alpha(B)$ is an open set.

Theorem 9: Let y_o be a fixed element of Y and let $A = X \times \{y_o\}$. Then the restriction f_x or π_x to A is a homeomorphism of the subspace A of X × Y onto X. Also the restriction f_y of π_y to $B = \{x_o\} \times Y$ into Y is a homeomorphism, where $x_o \in X$.

Proof: Let (X × Y, T) be the product topological space of (X, T_1) and (X, T_2). Let $x \in X$ and $y \in Y$ be arbitrary. Then the projection maps are defined as

$\pi_x : X \times Y \to X$ s.t. $\pi_x (x, y) = x$

and $\pi_y : X \times Y \to X$ s.t. $\pi_y (x, y) = y$.

Let $x_o \in X$ and $y_o \in Y$ be fixed elements.

Let f_x be the restriction of π_x to A so that f_x is a map s.t. $f_x : A \to X$

s.t. $f_x(x, y_o) = x$.

To prove that f_x is a homeomorphism, we have to prove that

(i) f_x is one-one onto

(ii) f_x is continuous

(iii) f_x^{-1} is continuous

$f_x (x_1, y_o) = f_x (x_2, y_o) \Rightarrow x_1 = x_2$, by definition of f_x

$\Rightarrow (x_1, y_o) = (x_2, y_o)$.

Hence f_x is one-one.

Given any $x \in X$, $\exists$ $(x, y_o) \in A$ s.t. $f_x (x, y_o) = x$.

This proves that f_x is onto. Hence the result (i).

π_x is a projection map $\Rightarrow \pi_x$ is continuous.

Also f_x is its restriction $\Rightarrow f_x$ is continuous. Hence (ii).

Notes

(iv) To prove $f_x^{-1} : X \to A$ is continuous. We have to prove: given any V open subset of A.

$[f_x^{-1}]^{-1}$ (V) = f_x (V) is open in X.

Now V is expressible as V = A ∩ B, where B ∈ T.

Let B be a base for T. Then

$$B = \{G \times H : G \in T_1, H \in T_2\}$$

By definition of base,

$$B \in T \Rightarrow \exists\, B_1 \subset B \text{ s.t.}$$

$$B = \bigcup \{G \times H \in T_1 \times T_2 : G \times H \in B_1\}$$

Then $A \cap B = \bigcup \{A \cap (G \times H) : G \times H \in B_1\}$

$$= \bigcup \{(X \times \{y_o\}) \cap (G \times H) : G \times H \in B_1\}$$

$$= \begin{cases} \bigcup \{G \times \{y_o\}) : G \times H \in B_1\} & \text{if } y_o \in H \\ \text{or} \bigcup \{G \times \phi \quad : G \times H \in B_1\} & \text{if } y_o \notin H \end{cases}$$

$$= \begin{cases} \bigcup \{G \times y_o\} : G \times H \in B_1\} & \text{if } y_o \in H \\ \text{or } \phi \qquad : & \text{if } y_o \notin H \end{cases}$$

Moreover ϕ is an open set and an arbitrary union of open sets is open.

In either case, f_x (A ∩ B) is open in X, i.e., f_x (V) is open in X.

Self Assessment

2. Prove that the collection

$$S = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a sub basis for the product topology on X × Y.

3. A map f : X → Y is said to be an open map if for every open set U of X, the set f(U) is open in Y, show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

4.3 Summary

- If X and Y are topological spaces, the product topology on X × Y is the topology whose basis is $\{A \times B : A \in T_X, B \in T_Y\}$.
- Given any product of sets X × Y, there are projections maps π_x and π_y from X × Y to X and to Y given by (x, y) → x and (x, y) → y.
- If (X, Y) is the product space if topological spaces (X_1, T_1) and (X_2, T_2), then the projection maps π_1 and π_2 are continuous and open.

4.4 Keywords

Basis: A collection $\mathcal{B}$ of open sets in a topological space X is called a basis for the topology if every open set in X is a union of sets in $\mathcal{B}$.

Coarser: Let T and T′ are two topologies on a given set X. If T′ ⊃ T, we say that T is coarser than T′.

Hausdorff space: A topological space (X, T) is called a Hausdorff space if a given pair of distinct points x, y ∈ X, ∃ G, H ∈ T s.t. x ∈ G, Y ∈ H, G ⋂ H = ϕ.

Interior point: Let (X, T) be a topological space and $\mathcal{A}$ ⊂ X. A point x ∈ $\mathcal{A}$ is called an interior point of $\mathcal{A}$ iff ∃ an open set G such that x ∈ G ⊆ A.

4.5 Review Questions

1. Let $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n$ be the bases for topological spaces $(X_1, T_1), (X_2, T_2), ..., (X_n, T_n)$ respectively. Then prove that the family $\{O_1 \times O_2 \times ... \times O_n : O_i \in \mathcal{B}_i, i = 1, 2, ..., n\}$ is a basis for the product topology on $X_1 \times X_2 \times ... \times X_n$.
2. Prove that the product of any finite number of indiscrete spaces is an indiscrete space.
3. Let X_1 and X_2 be infinite sets and T_1 and T_2 the finite-closed topology on X_1 and X_2, respectively. Show that the product topology, T on $X_1 \times X_2$ is not the finite-closed topology.
4. Let (X_1, T_1), (X_2, T_2) and (X_3, T_3) be topological spaces. Prove that

$$[(X_1, T_1) \times (X_2, T_2)] \times (X_3, T_3) \cong (X_1, T_1) \times (X_2, T_2) \times (X_3, T_3)$$

5. (a) Let (X_1, T_1) and (X_2, T_2) be topological spaces. Prove that

$$(X_1, T_1) \times (X_2, T_2) \cong (X_2, T_2) \times (X_1, T_1)$$

   (b) Generalise the above result to products of any finite number of topological spaces.

4.6 Further Readings

Books

H.F. Cullen, *Introduction to General Topology,* Boston, MA: Heath.

K.D. Joshi, *Introduction to General Topology,* New Delhi, Wiley.

S. Willard, *General Topology,* MA: Addison-Wesley.

Notes

Unit 5: The Subspace Topology

CONTENTS

Objectives

After studying this unit, you will be able to:

- Describe the concept of subspace of topological space;
- Explain the problems related to subspace topology;
- Derive the theorems on subspace topology.

Introduction

We shall describe a method of constructing new topologies from the given ones. If (X, T) is a topological space and $Y \subseteq X$ is any subset, there is a natural way in which Y can "inherit" a topology from parent set X. It is easy to verify that the set $\cup \cap Y$, as $\cup$ runs through T, is a topology on Y. This prompts the definition of subspace or relative topology.

5.1 Subspace of a Topological Space

Definition: Let (X, T) be a topological space, V be a non empty subset of X and T_Y be the class of all intersections of Y with open subsets of X i.e.

$$T_Y = \{Y \cap \cup : \cup \in T\}$$

Then T_Y is a topology on Y is called the subspace topology (or the relative topology induced on Y by T. The topological space (Y, T_Y) is said to be a subspace of (X, T).

Note Let $A \subset Y \subset X$

(1) It A is open in Y, Y is open in X, then A is open in X.

(2) It A is closed in Y, Y is closed in X, then A is closed in X.

Remark: Consider the usual topology T on R and the relative topology $\mathcal{U}$ on Y = [0, 1]. Then $\left(0, \frac{1}{2}\right)$ is $\mathcal{U}$-open as well as T-open $\left(\frac{1}{2}, 1\right] = \left(\frac{1}{2}, 2\right) \cap [0, 1] = G \cap [0, 1]$

where $\quad G = \left(\frac{1}{2}, 2\right) \in T$

$\therefore \quad \left(\frac{1}{2}, 1\right] = G \cap Y.$

This shows that $\left(\frac{1}{2}, 1\right]$ is $\mathcal{U}$-open but not T-open

$$\left(\frac{1}{2}, \frac{2}{3}\right) = \left(\frac{1}{2}, \frac{2}{3}\right) \cap [0, 1]$$

$$= G \cap Y$$

where $\quad G = \left(\frac{1}{2}, \frac{2}{3}\right) \in T$

$\left(\frac{1}{2}, \frac{2}{3}\right) \in \mathcal{U}$ and also $\left(\frac{1}{2}, \frac{2}{3}\right) \in T.$

Similarly, $\left(0, \frac{1}{2}\right]$ is not $\mathcal{U}$-open as well as it is not T-open.

5.1.1 Solved Examples on Subspace Topology

Example 1: Let X = {a, b, c, d, e, f}

T = {X, ϕ, {a}, {c, d}, {a, c, d}, {b, c, d, e, f}}

and Y = {b, c, e}.

Then the subspace topology on Y is

T_Y = {Y, ϕ, {c}}.

Example 2: Consider the topology

T = {ϕ, {1}, {2, 3}, X} on

X = {1, 2, 3} and a subset Y = {1, 2} of X.

Then $\quad Y \cap \phi = \phi$

$Y \cap \{1\} = \{1\},$

$Y \cap \{2, 3\} = \{2\}$ and

$Y \cap X = V.$

Hence, the relative topology on Y is

T_Y = {ϕ, {1}, {2}, V}.

Notes

Theorem 1: A subspace of a topological space is itself a topological space.

Proof:

(i) $\phi \in T$ and $\phi \cap Y = \phi \Rightarrow \phi \in T_Y$,

$X \in T$ and $X \cap Y = Y \Rightarrow Y \in T_Y$,

(ii) Let $\{H_\alpha : \alpha \in \Lambda\}$ be any family of sets in T_Y.

Then $\forall\ \alpha \in \Lambda\ \exists$ a set $G_\alpha \in T$ such that $H_\alpha = G_\alpha \cap Y$

$$\therefore \quad \bigcup \{H_\alpha : \alpha \in \Lambda\} = \bigcup \{G_\alpha \cap Y : \alpha \in \Lambda\}$$

$$= [\bigcup \{G_\alpha : \alpha \in \Lambda\}] \cap Y \in T_Y$$

since $\bigcup \{G_\alpha : \alpha \in \Lambda\} \in T$

(iii) Let H_1 and H_2 be any two sets in T_Y.

Then $H_1 = G_1 \cap Y$ and $H_2 = G_2 \cap Y$ for some $G_1, G_2 \in T$.

$$\therefore \quad H_1 \cap H_2 = (G_1 \cap Y) \cap (G_2 \cap Y)$$

$$= (G_1 \cap G_2) \cap Y \in T_Y, \text{ since } G_1 \cap G_2 \in T$$

Hence, T_Y is a topology for Y.

Example 3: Let (Y, V) be a subspace of a topological space (X, T) and let (Z, W) be a subspace of (Y, V). Then prove that (Z, W) is a subspace of (X, T).

Solution: Given that $(Y, V) \subset (X, T)$ …(1)

and $(Z, W) \subset (Y, V)$ …(2)

We are to prove that $(Z, W) \subset (X, T)$

From (1) and (2), we get

$$Z \subset Y \subset X \quad \ldots(3)$$

From (1), $V = \{G \cap Y : G \in T\}$ …(4)

and (2), $W = \{H \cap Z : H \in V\}$ …(5)

From (4) and (5), we get $H = G \cap Y$

$$\Rightarrow \quad H \cap Z = (G \cap Y) \cap Z$$

$$= G \cap (Y \cap Z)$$

$$= G \cap Z \quad \text{[Using (3)]}$$

so, $H \cap Z = G \cap Z$ …(6)

Using (6) in (5), we get

$$W = \{G \cap Z : G \in T\}$$

$$\Rightarrow \quad (Z, W) \subset (X, T)$$

Hence, (Z, W) is a subspace of (X, T).

Example 4: If T is usual topology on $\mathcal{R}$, then find relative topology $\cup$ on $\mathcal{N} \subset \mathcal{R}$.

Solution: Every open interval on $\mathcal{R}$ is T-open set.

Let $$G = \left(n - \frac{1}{2}, n + \frac{1}{2}\right), n \in \mathcal{N}.$$

Notes

Then G ∈ T. Now $\mathcal{U} = \{G \cap \mathcal{N} : G \in T\}$

If $G = \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$

then $G \cap \mathcal{N} = \left(n - \frac{1}{2}, n + \frac{1}{2}\right) \cap \mathcal{N}$

$= \{n\}$

Or $\mathcal{U} = \{\{n\} : n \in \mathcal{N}\}$

Every singleton set of $\mathcal{N}$ is $\mathcal{U}$-open set.

As an arbitrary subset of $\mathcal{N}$ is an arbitrary union of singleton sets and so every subset of $\mathcal{N}$ is $\mathcal{U}$-open.

Consequently, $\mathcal{U}$ is a discrete topology on $\mathcal{N}$.

Example 5: Define relative topology. Consider the topology : T = {ϕ, {a}, {b, c}, {a, b, c}, X} on X = {a, b, c, d}. If Y = {b, c, d} is a subset of X, then find relative topology on Y.

Solution: If $\mathcal{U}$ is relative topology on $\mathcal{Y}$, then

$$\mathcal{U} = \{G \cap Y, G \in T\}$$

$$\Rightarrow \quad \mathcal{U} = \{\phi \cap Y, \{a\} \cap Y, \{b, c\} \cap Y, \{a, b, c\} \cap Y, X \cap Y\}$$

$$\Rightarrow \quad \mathcal{U} = \{\phi, \phi, \{b, c\}, \{b, c\}, Y\}$$

$$\Rightarrow \quad \mathcal{U} = \{\phi, Y, \{b, c\}\}$$

Example 6: Let X be a topological space and let Y and Z be subspaces of X such that Y ⊂ Z. Show that the topology which Y has a subspace of X is the same as that which it has as a subspace of Z.

Solution: Let (X, T) be a topological space and Y, Z be subspaces of X such that

$$Y \subset Z \subset X.$$

Further assume $(Y, T_1) \subset (Z, T_2) \subset (X, T)$ …(1)

$(Y, T_3) \subset (X, T)$ …(2)

We are to show that $T_1 = T_3$

By definition (1) declares that

$$T_1 = \{G \cap Y : G \in T_2\} \quad \text{…(3)}$$

$$T_2 = \{H \cap Z : H \in T\} \quad \text{…(4)}$$

$$T_3 = \{P \cap Z : P \in T\} \quad \text{…(5)}$$

Using (4) in (3), we get

$$G \cap Y = (H \cap Z) \cap Y = H \cap (Y \cap Z) = H \cap Y$$

Now, (3) becomes

$$T_1 = \{H \cap Y : H \in T\} \quad \text{…(6)}$$

From (5) and (6), we get $T_1 = T_3$.

Notes

Theorem 2: Let $(Y, \mathcal{U})$ be a subspace of a topological space (X, T). A subset of Y is $\mathcal{U}$-nhd. of a point $y \in Y$ iff it is the intersection of Y with a T-nhd. of the point $y \in Y$.

Proof: Let $(Y, \mathcal{U}) \subset (X, T)$ and $y \in Y$ be arbitrary, then $y \in X$.

(I) Let N_1 be a $\mathcal{U}$-nhd of y, then

$$\exists \quad V \in \mathcal{U} \quad \text{s.t.} \quad y \in V \subset N_1 \qquad \ldots(1)$$

To prove : $N_1 = N_2 \cap Y$ for some T-nhd N_2 of y.

$$y \in V \in \mathcal{U} \Rightarrow \exists\, G \in T \text{ s.t. } V = G \cap Y$$

$$\Rightarrow y \in G \cap Y \Rightarrow y \in G, y \in Y \qquad \ldots(2)$$

Write $N_2 = N_1 \cup G$.

Then $N_1 \subset N_2$, $G \subset N_2$.

so, (2) implies $y \in G \subset N_2$, where $G \in T$

This shows that N_2 is a T-nhd of y.

$$\begin{aligned} N_2 \cap Y &= (N_1 \cup G) \cap Y \\ &= (N_1 \cap Y) \cup (G \cap Y) \\ &= (N_1 \cap Y) \cup V \\ &= N_1 \cup V && \text{[by (1)]} \\ &= N_1 && \because \quad N_1 \subset Y \quad \text{and} \quad V \subset N_1 \end{aligned}$$

Finally, N_2 has the following properties

$$N_1 = N_2 \cap Y \quad \text{and} \quad N_2 \text{ is a T-nhd of y.}$$

This completes the proof.

(II) Conversely, Let N_2 be a T-nhd. of y so that

$$\exists \quad A \in T \quad \text{s.t.} \quad y \in A \subset N_2 \qquad \ldots(3)$$

We are to prove that $N_2 \cap Y$ is a $\mathcal{U}$-nhd of y.

$$\therefore \qquad y \in Y, y \in A \Rightarrow y \in Y \cap A \qquad \text{[by (3)]}$$

$$\Rightarrow y \in A \cap Y \subset N_2 \cap Y \qquad \text{[by (3)]}$$

$$A \in T \Rightarrow A \cap Y \in \mathcal{U}$$

Thus, we have $y \in A \cap Y \subset N_2 \cap Y$, where $A \cap Y \in \mathcal{U}$.

$\Rightarrow N_2 \cup Y$ is a $\mathcal{U}$-nhd of y.

Example 7: Let $(Y, \mathcal{U})$ be a subspace of a topological space (X, T). Then every $\mathcal{U}$-open set is also T-open iff Y is T-open.

Solution: Let $(Y, \mathcal{U}) \subset (X, T)$ and let

any $G \in \mathcal{U} \Rightarrow G \in T \qquad \ldots(1)$

i.e. every $\mathcal{U}$-open set is also T-open set.

To show: Y is T-open, it is enough to prove that $y \in T$.

Let $G \in \cup$ be arbitrary, then $G \in T$, by (1).

We can write $G = H \cap Y$ for some T-open set H.

Now, $G = H \cap Y, G \in T \Rightarrow H \cap Y \in T$

Again $H \cap Y \in T, H \in T$ and $Y \subset X \Rightarrow Y \in T$

Conversely, let $(Y, \cup) \subset (X, T)$ and let $Y \in T$ for any $G \in \cup \Rightarrow G \in T$

$\therefore \quad G \in \cup \Rightarrow \exists A \in T \text{ s.t. } G = A \cap Y$

Again $\quad A \in T, Y \in T \Rightarrow A \cap Y \in T \Rightarrow G \in T$

Finally, any $G \in \cup \Rightarrow G \in T$.

5.1.2 Basis for the Subspace Topology

Example 8: Consider the subset Y = [0, 1] of the real line $\mathbb{R}$, in the subspace topology. The subspace topology has as basis all sets of the form $(a, b) \cap Y$, where (a, b) is an open interval in $\mathbb{R}$, such a set is of one of the following types:

$$(a, b) \cap Y = \begin{cases} (a,b) & \text{if a and b are in Y,} \\ [0,b) & \text{if only b is in Y,} \\ (a,1] & \text{if only a is in Y,} \\ Y \text{ or } \phi & \text{if neither a nor b is in Y.} \end{cases}$$

By definition, each of these sets is open in Y. But sets of the second and third types are not open in the larger space $\mathbb{R}$.

Note that these sets form a basis for the order topology on Y. Thus, we see that in the case of the set Y = [0, 1], its subspace topology (as a subspace of $\mathbb{R}$) and its order topology are the same.

Example 9: Let Y be the subset $[0, 1) \cup \{2\}$ of $\mathbb{R}$. In the subspace topology on Y the one-point set {2} is open, because it is the intersection of the open set $(\frac{3}{2}, \frac{5}{2})$ with Y. But in the order topology on Y, the set {2} is not open. Any basis element for the order topology on Y that contains 2 is of the form

$$\{x \mid x \in Y \text{ and } a < x \leq 2\}$$

for some $a \in Y$; such a set necessarily contains points of Y less than 2.

Lemma 1: If $\mathcal{B}$ is a basis for the topology of X, then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

Proof: Given $\cup$-open in X and given $y \in \cup \cap Y$, we can choose an element B of $\mathcal{B}$ such that

$y \in B \subset \cup$. Then $y \in B \cap Y \subset \cup \cap Y$

Now as we know

"If X is a topological space and $\mathcal{C}$ is a collection of open sets of X such that for each open set $\cup$ of X and each x in $\cup$, there is an element c of $\mathcal{C}$ such that $x \in \mathcal{C} \subset \cup$. The $\mathcal{C}$ is a basis for the topology of X."

Thus, we can say that $\mathcal{B}_Y$ is a basis for the subspace topology on Y.

Lemma 2: Let Y be a subspace of X. If ⋃ is open in Y and Y is open in X, then ⋃ is open in X.

Proof: Since ⋃ is open in Y,

$$\bigcup = Y \cap V \quad \text{for some set V open in X.}$$

Since Y and V are both open in X,

so is $Y \cap V$.

5.1.3 Subspace of Product Topology

Theorem 3: If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof: The set $\bigcup \times V$ is the general basis element for $X \times Y$, where ⋃ is open in X and V is open in Y.

$\therefore$, $(\bigcup \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$.

Now, $(\bigcup \times V) \cap (A \times B) = (\cup \cap A) \times (V \cap B)$.

Since $\bigcup \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B, respectively, the set $(\bigcup \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$.

So, we can say that the bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same.

Hence, the topologies are the same.

5.2 Summary

- A subspace of a topological space is itself a topology space.
- If $\mathcal{B}$ is a basis for the topology of X, then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.
- Let Y be a subspace of X. If ⋃ is open in Y and Y is open in X, then ⋃ is open in X.
- If A is a subspace of X and B is a subspace of Y then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

5.3 Keywords

Basis: Let X be a topological space A set $\mathcal{B}$ of open set is called a basis for the topology if every open set is a union of sets in $\mathcal{B}$.

Closed Set: Let (X, T) be a topological space. Let set $A \in T$. Then X–A is a closed set.

Intersection: The intersection of A and B is written $A \cap B$. $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$.

Neighborhood: Let (X, T) be a topological space. $A \subset X$ is called a neighborhood of a point $x \in X$ if $\exists\, G \in T$ with $x \in G$ s.t. $G \subset A$.

Open set: Let (X, T) be a topological space. Any set $A \in T$ is called an open set.

Product Topology: Let X and Y be topological space. The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $\bigcup \times V$, where ⋃ is an open subset of X and V is an open subset of Y.

Notes

Subset: If A and B are sets and every element of A is also an element of B, then A is subset of B denoted by $A \subseteq B$.

Subspace: Given a topological space (X, T) and a subset S of X, the subspace topology on S is defined by

$$T = \{S \cap U : U \in T\}$$

Topological Space: It is a set X together with T, a collection of subsets of X, satisfying the following axioms. (1) The empty set and X are in T; (2) T is closed under arbitrary union and (3) T is closed under finite intersection. Then collection T is called a topology on X.

5.4 Review Questions

1. Let $X = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\} \subset X$ and

   $T = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{1, 4, 5\}, \{1, 2, 4, 5\}\}$.

   Find relative topology T, on A.

2. Let (X, T) be a topological space and $X^* \subset X$. Let T^* be the collection of all sets which are intersections of X^* with members of T. Prove that T^* is a topology on X^*.

3. Show that if Y is a subspace of X, and $A \subset Y$, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

4. If T and T′ are topologies on X and T′ is strictly finer than T, what do you say about the corresponding subspace topologies on the subset Y of X?

5. Let A be a subset of X. If $\mathcal{B}$ is a base for the topology of X, then the collection

   $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$

   is a base for the subspace topology on A.

6. Let (Y, U) be a subspace of (X, T). If F and F_1 are the collections of all closed subsets of (X, T) and (Y, U) respectively, then $F_1 \subset F \Leftrightarrow Y \in F$.

5.5 Further Readings

Books

Willard, Stephen. *General Topology*, Dover Publication (2004).

Bourbaki, Nicolas, *Elements of Mathematics: General Topology*, Addison-Wesley (1966).

Simmons. *Introduction to Topology and Modern Analysis*.

James & James. *Mathematics Dictionary*.

Notes

Unit 6: Closed Sets and Limit Point

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define closed sets;
- Solve the problems related to closed sets;
- Understand the limit points and derived set;
- Solve the problems on limit points.

Introduction

On the real number line we have a notion of 'closeness'. For example each point in the sequence 1..01..001..0001..00001.. is closer to 0 than the previous one. Indeed, in some sense 0 is a limit point of this sequence. So the interval (0, 1] is not closed as it does not contain the limit point 0. In a general topological space me do not have a 'distance function', so we must proceed differently. We shall define the notion of limit point without resorting to distance. Even with our new definition of limit point, the point 0 will still be a limit point of (0, 1]. The introduction of the notion of limit point will lead us to a much better understanding of the notion of closed set.

6.1 Closed Sets

A subset A of a topological space X is said to be closed if the set X-A is open.

Example 1: The subset [a, b] of R is closed because its complement

$R - [a, b] = (-\infty, a) \cup (b, +\infty)$ is open.

Similarly, $[a, +\infty)$ is closed, because its complement $(-\infty, a)$ is open. These facts justify our use of the terms "closed interval" and "closed ray". The subset [a, b) of R is neither open nor closed.

Example 2: In the discrete topology on the set X, every set is open; it follows that every set is closed as well.

Theorem 1: Let X be a topological space. Then the following conditions hold:

(a) ϕ and X are closed.

(b) Arbitrary intersections of closed sets are closed.

(c) Finite unions of closed sets are closed.

Proof:

(a) ϕ and X are closed because they are the complements of the open sets X and ϕ, respectively.

(b) Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$, we apply De Morgan's law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha).$$

Since the sets $X - A_\alpha$ are open by definition, the right side of this equation represents an arbitrary union of open sets, and is thus open. Therefore, $\cap A_\alpha$ is closed.

(c) Similarly, if A_i is closed for i = 1, …, n, consider the equation

$$X - \bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i).$$

The set on the right side of this equation is a finite intersection of open sets and is therefore open. Hence $\cup A_i$ is closed.

Theorem 2: Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof: Assume that A = C $\cap$ Y, where C is closed in X. Then X – C is open in X, so that (X – C) $\cap$ Y is open in Y, by definition of the subspace topology. But (X – C) $\cap$ Y = Y – A. Hence Y – A is open in Y, so that A is closed in Y. Conversely, assume that A is closed in Y. Then Y – A is open in Y, so that by definition it equals the intersection of an open set U of X with Y. The set X – U is closed in X and A = Y $\cap$ (X – U), so that A equals the intersection of a closed set of X with Y, as desired.

Example 3: Let (Y, U) $\subset$ (X, T) and A $\subset$ Y.

Then A is U-closed iff A = F $\cap$ Y for some T closed set F.

or

A is U-closed iff A is the intersection of Y and a T-closed F.

Solution: Let (Y, U) $\subset$ (X, T) and A $\subset$ Y, i.e. (Y, U) is subspace of (X, T).

To prove that A is U-closed iff

A = F $\cap$ Y for some T-closed set F.

A is U-closed $\Leftrightarrow$ Y – A is U-open.

Then Y – A can be expressed as:

$$Y - A = G \cap Y \text{ for some T-open set } G.$$

From which

$$A = Y - G \cap Y = X \cap Y - G \cap Y$$
$$= (X - G) \cap Y$$
$$= F \cap Y, \text{ where } F = X - G \text{ is a T-closed set.}$$

This completes the proof.

Notes

Self Assessment

1. Show that if A is closed in Y and Y is closed in X, then A is closed in X.
2. Show that if A is closed in X and B is closed in Y, then A × B is closed in X × Y.

6.2 Limit Point

Let (X, T) be a topological space and $A \subset X$. A point $x \in X$ is said to be the limit point or accumulation point of A if each open set containing x contains at least one point of A different from x.

Thus it is clear from the above definition that the limit point of a set A may or may not be the point of A.

Note Limit point is also known as accumulation point or cluster point.

Example 4: Let X = {a, b, c} with topology $T = \{\phi, \{a, b\}, \{c\}, X\}$ and A = {a}, then b is the only limit point of A, because the open sets containing b namely {a, b} and X also contains a point of A.

Where as 'a' and 'b' are not limit point of C = {c}, because the open set {a, b} containing these points do not contain any point of C. The point 'c' is also not a limit point of C, since then open set {c} containing 'c' does not contain any other point of C different from C. Thus, the set C = {c} has no limit points.

Example 5: Prove that every real number is a limit point of R.

Solution: Let $x \in R$

then every nhd of x contains at least one point of R other than x.

$\therefore$ x is a limit point of R.

But x was arbitrary.

$\therefore$ every real number is a limit point of R.

Example 6: Prove that every real number is a limit point of R – Q.

Solution: Let x be any real number, the every nhd of x contains at least one point of R – Q other than x.

$\therefore$ x is a limit point of R – Q

But x was arbitrary

$\therefore$ every real number is a limit point of R – Q.

6.2.1 Derived Set

The set of all limit points of A is called the derived set of A and is denoted by D(A).

Notes

1. In terms of derived set, the closure of a set $A \subset X$ is defined as $A = A + D(A) = A \cup D(A)$.
2. If every point of A is an isolated point of A, then A is known as isolated set.

Example 7: Every derived set in a topological space is a closed.

Solution: Let (X, T) be a topological space and $A \subset X$.

Aim: D(A) is a closed set.

Recall that B is a closed set if $D(B) \subset B$.

Hence D(A) is closed iff $D[D(A)] \subset D(A)$.

Let $x \in D[D(A)]$ be arbitrary, then x is a limit point of D(A) so that $(G - \{x\}) \cap D(A) \neq \phi \forall G \in T$ with $x \in G$.

$\Rightarrow (G - \{x\}) \cap A \neq \phi \Rightarrow x \in D(A)$.

Hence the result.

[For every nhd of an element of D(T) has at least one point of A].

Example 8: Let (X, T) be a topological space and $A \subseteq X$, then A is closed iff $A' \subseteq A$ or $A \supseteq D(A)$.

Solution: Let A be closed.

$\Rightarrow$ A^c is open.

Let $x \in A^c$.

Then A^c is an open set containing x but containing no point of A other than x.

This shows that x is not a limit point of A.

Thus, no point of A^c is a limit point of A.

Consequently, every limit point of A is in A and therefore

$A' \subseteq A$

Conversely, Let $A' \subseteq A$

we have to show that A is closed.

Let x be arbitrary point of A^c.

Then $x \in A^c$

$\Rightarrow$ $x \notin A$

$\Rightarrow$ $x \notin A$ and $x \notin A'$

$\Rightarrow$ $x \notin A$ and x not a limit point of A.

$\Rightarrow$ $\exists$ an open set G such that $x \in G$ and $G \cap A = \phi$

Notes

$\Rightarrow$ $x \in G \subseteq A^c$.

$\Rightarrow$ A^c is the nhd of each of its point and therefore A^c is open.

Hence A is closed.

Example 9: Let (X, T) be a topological space and $A \subset X$. A point x of A is an interior point of A iff it is not a limit point of X – A.

Solution: Let (X, T) be a topological space and $A \subset X$. Suppose a point x of A is an interior point of A so that $x \in A$, $x \in A°$.

To prove that x is not a limit point of X – A i.e., $x \notin D(X - A)$

$$x \in A \Rightarrow \exists\, G \in T \text{ with } x \in G \text{ s.t. } G \subset A$$

$$\Rightarrow G \cap (X - A) = \phi$$

$$\Rightarrow (G - \{x\}) \cap (X - A) = \phi \qquad [\because\ x \notin (X - A)\,]$$

$\therefore$ G is an open set containing set.

$(G - \{x\}) \cap (X - A) = \phi$

This immediately shows that $x \notin D(X - A)$.

Conversely suppose that (X, T) is topological space and $A \subset X$ s.t. a point x of A is not a limit point of (X – A).

To prove that $x \in A°$.

By hypothesis $x \in A$, $x \notin D(X - A)$

$x \notin D(X - A) \Rightarrow \exists\, G \in T$ with $x \in G$ s.t. $(G - \{x\}) \cap (X - A) = \phi$

$\Rightarrow$ $G \cap (X - A) = \phi$ $\qquad [\because\ x \notin X - A]$

$\Rightarrow$ $G \subset A$.

$\therefore$ $x \in A \Rightarrow \exists\, G \in T$ with $x \in G$ s.t. $G \subset A$. This proves that $x \in A°$.

Self Assessment

3. Let x be a topological space and let A, B be subset of x. Then.

   (a) $\phi' = \phi$ or $D(\phi) = \phi$

   (b) $A \subseteq B \Rightarrow A' \subseteq B'$ or $A \subset B \Rightarrow D(A) \subset D(B)$;

   (c) $x \in A' \Rightarrow x \in (G - \{x\})'$;

6.3 Summary

- A subset A of a topological space X is said to be closed if the set X - A is open.
- Let (X, T) be a topological space and $A \subset X$. A point $x \in X$ is said to be the limit point of A if each open set containing x contains at least one point of A different from x.
- The set of all limit points of A is called the derived set of A and is denoted by D(A).

Notes

6.4 Keywords

Discrete Topology: Let X be any non-empty set and T be the collection of all subsets of X. Then T is called discrete topology on the set X.

Open and Closed Set: Let (X, T) be a topological space. Any set A $\in$ T is called an open set and X – A is a closed set.

Subspace: Let (X, T) be a topological space and a subset S of X, the subspace topology on S is defined by $T_S = \{S \cap U \mid U \in T\}$.

6.5 Review Questions

1. Let X be a topological space and A be a subset of X. Then prove that $\bar{A}$ is the smallest closed set containing A.
2. Prove that A is closed iff $A = \bar{A}$.
3. Let (Y, U) $\subset$ (X, S) and A $\subset$ Y. Prove that A point y $\in$ Y is U-limit point of A iff y is a T-limit point of A.
4. Show that every closed set in a topological space is the disjoint union of its set of isolated points and its set of limit points, in the sense that it contains these sets.
5. Show that if U is open in X and A is closed in X, then U – A is open in X, and A – U is closed in X.

6.6 Further Readings

Books

J. L. Kelley, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

Notes

Unit 7: Continuous Functions

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the concept of continuity;
- Define Homeomorphism;
- Define open and closed map;
- Understand the theorems and problems on continuity.

Introduction

The concept of continuous functions is basic to much of mathematics. Continuous functions on the real line appear in the first pages of any calculus look, and continuous functions in the plane and in space follow not far behind. More general kinds of continuous functions arise as one goes further in mathematics. In this unit, we shall formulate a definition of continuity that will include all these as special cases and we shall study various properties of continuous functions.

7.1 Continuity

7.1.1 Continuous Map and Continuity on a Set

Definition: Let (X, T) and (Y, U) be any two topological spaces.

Let $f : (X, T) \to (Y, U)$ be a map.

The map f of said to be continuous at $x_0 \in X$ is given any U-open set H containing $f(x_0)$, $\exists$a T-open set G containing x_0 s.t. $f(G) \subset H$.

If the map in continuous at each $x \in X$ then the map is called a continuous map.

Definition: Continuity on a set. A function

$$f:(X,T)\Rightarrow(Y,U)$$

is said to be continuous on a set $A \subset X$ if it is continuous at each point of A.

Notes The following have the same meaning:

(a) f is a continuous map.

(b) f is a continuous relative to T and U

(c) f is T – U continuous map.

Example 1: Let $\mathcal{R}$ denote the set of real numbers in its usual topology, and let $\mathcal{R}_\ell$ denote the same set in the lower limit topology. Let

$$f:\mathcal{R}\to\mathcal{R}_\ell$$

be the identity function;

$$f(x) = x \text{ for every real number } x.$$

then f is not a continuous function; the inverse image of the open set [a, b) of $\mathcal{R}_\ell$ equals itself, which is not open in $\mathcal{R}$. On the other hand, the identity function.

$$g:\mathcal{R}_\ell\to\mathcal{R}$$

is continuous, because the inverse image of (a, b) is itself, which is open in $\mathcal{R}_\ell$.

7.1.2 Homeomorphism

Definition: A map $f : (X, T) \to (Y, U)$ is said to be homeomorphism or topological mapping if

(a) f is one-one onto.

(b) f and f^{-1} are continuous.

In this case, the spaces X and Y are said to be homeomorphic or topological equivalent to one another and Y is called the homeomorphic image of X.

Example 2: Let T denote the usual topology on R and a any non-zero real number. Then each of the following maps is a homeomorphism

(a) $f : (R, T) \to (R, T)$ s.t. $f(x) = a + x$

(b) $f : (R, T) \to (R, T)$ s.t. $f(x) = ax$

(c) $f : (R, T) \to (R, T)$ s.t. $f(x) = x^3$ where $x \in R$.

Example 3: Show that (R, U) and (R, D) are not homeomorphic.

Solution: Every singleton is D-open and image of a singleton is again singleton which is not U-open. Consequently no one-one D – U continuous map of R onto R can be homeomorphism. From this the required result follows.

Notes

7.1.3 Open and Closed Map

Definition: Open Map

A map f : (X, T) → (Y, U) is called an open or interior map if it maps open sets onto open sets i.e. if

$$\text{any } G \in T \Rightarrow f(G) \in U.$$

Definition: Closed Map

A map f : (X, T) → (Y, U) is called a closed map if

any T-closed set f ⇒ f(F) is U-closed set.

Example 4: (i) Let T denote the usual topology on R. Let a be any non-zero real number, Then each of the following map is open as well as closed.

(a) f : (R, T) → (R, T) s.t. f(x) = a + x

(b) f : (R, T) → (R, T) s.t. f(x) = ax

In this case if a = 0, then this map is closed but not open.

(ii) The identity map f : (X, T) → (X, T) is open and as well as closed.

(iii) A map from an indiscrete space into a topological space is open as well as closed.

(iv) A map from a topological space into a discrete space is open as well as closed.

Note Proof of (i) b,

Let a ≠ 0 and A = (b, c) ∈ T arbitrary.

Then f(b) = ab, f(c) = ac.

∴ f(A) = (ab, ac) ∈ J

i.e., image of an open set is an open set under the map f(x) = ax, a ≠ 0. Hence this map is open.

Similarly $f([b,c]) = [ab, bc]$, i.e. image of a closed set is closed.

∴ f is a closed map

Consider the case in which a = 0

Then f(x) = ax = 0, ∀ x ∈ R

∴ f(x) = 0 ∀ x ∈ R.

Now $f([b,c]) = \{0\} =$ A Finite set = A closed set for a finite set is a T-closed set.

Now the image of a closed set is closed and hence f is a closed map.

Again f (5, 6) = {0} ≠ an open set.

∴ image of an open set is not open.

Consequently, f is not open.

7.1.4 Theorems and Solved Examples

Theorem 1: The function f: (X, J) $\rightarrow$ (Y, U) is continuous iff f^{-1} (V) is open in X for every open set V in Y.

Proof: Let f: (X, J) $\rightarrow$ (Y, U) be a map.

(i) Suppose f is continuous. Let G be an open subset of Y.

To prove that $f^{-1}(G)$ is open in X.

If $f^{-1}(G) = \phi$, then $f^{-1}(G) \in J$.

If $f^{-1}(G) \neq \phi$, then $\exists x \in f^{-1}(G)$ so that $f(x) \in G$.

Continuity of f $\Rightarrow$ f is continuous at x.

$\Rightarrow \exists\ H \in J$ s.t. $x \in H$ and $f(H) \subset G$.

$\Rightarrow x \in H \subset f^{-1}(G)$, $H \subset J$.

Thus we have shown that $f^{-1}(G)$ is a nhd of each of its points and so $f^{-1}(G)$ is J-open.

Conversely, suppose that f : (X, J) $\rightarrow$ (Y, U) is a map such that $f^{-1}(V)$ is open in X for each open set $V \subset Y$.

To prove that f is continuous.

Let $V \in U$ be arbitrary.

Then, by assumption, f^{-1} (V) is open in X.

Take $\quad U = f^{-1}(V)$, so that $U \in J$.

i.e $\quad F(U) = f(f^{-1}(V)) \subset V$, or $f(U) \subset V$.

given any $V \in U$, $\exists\ U \in J$ s.t. $f(U) \subset V$.

This proves that f is a continuous map.

Theorem 2: A map f : X $\rightarrow$ Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$.

A map f : (X, d) $\rightarrow$ (Y, p) be continuous iff $f^{-1}(F)$ is closed in X $\forall\ F \subset Y$ is closed where (X, d) and (Y, p) are metric spaces.

Proof: Let f : X $\rightarrow$ Y be a continuous map.

To prove that $f^{-1}(c)$ is closed in X for each closed set $C \subset Y$.

Let $C \subset Y$ be an arbitrary closed set.

Continuity of f implies that $f^{-1}(Y - C)$ is open in X. (Refer theorem (1))

i.e. $\quad f^{-1}(Y)$-$f^{-1}(C)$ is open in X.

i.e. $\quad X - f^{-1}(C)$ is open in X.

or $\quad f^{-1}(C)$ is closed in X.

Conversely, suppose that f : (X, T) $\rightarrow$ (Y, U) is a map such that $f^{-1}(C)$ is closed for each closed set $C \subset Y$.

To prove that f is continuous.

Let $G \subset Y$ be an arbitrary open set, then $Y - G$ is closed in Y.

By hypothesis, $f^{-1}(Y - G)$ is closed in X.

Notes

i.e., $f^{-1}(Y) - f^{-1}(G)$ is closed in X,

i.e., $X - f^{-1}(G)$ is closed in X,

i.e. $f^{-1}(G)$ is open in X,

$\therefore$ any $G \subset Y$ is open $\Rightarrow f^{-1}(G)$ is open in X

This proves that f is continuous map.

Theorem 3: Let $f : (X, T) \to (Y, U)$ be a map, Let $\mathcal{S}$ be a sub-base for the topology U on Y. Then f is continuous iff $f^{-1}(S)$ is open in X whenever $S \in \mathcal{S}$

or

f is continuous $\Leftrightarrow$ the inverse image of each sub-basic open set is open.

Proof: Let $f : (X, T) \to (Y, U)$ be continuous map. Let $\mathcal{S}$ be a sub-base for the topology U on Y. Let $S \in \mathcal{S}$ be arbitrary.

To prove that $f^{-1}(S)$ is open in X.

$S \in \mathcal{S} \Rightarrow S \in U$ ($\because$ $\mathcal{S} \subset U \Rightarrow S$ is open in Y)

$\Rightarrow f^{-1}(S)$ is open in X, (by Theorem 1).

Conversely, suppose that $f : (X, T) \to (Y, U)$ is a map such that $f^{-1}(S)$ is open in X whenever $S \in \mathcal{S}$, $\mathcal{S}$ being a sub-base for the topology U on Y. Let B be a base for U on Y.

To prove that f is continuous.

Let $G \subset Y$ be an open set, then $G \in U$.

By definition of base,

$$G \in U \Rightarrow \exists\, \mathcal{B}_1 \subset \mathcal{B} \text{ s.t. } G = \cup \{B : B \in \mathcal{B}_1\} \quad \ldots(1)$$

By the definition of sub-base, any $B \in \mathcal{B}$ can be expressed as

$B = \bigcap_{i=1}^{n} S_i$ for same choice of $S_1, S_2, \ldots S_n \in \mathcal{S}$

$$f^{-1}(B) = f^{-1}\left[\bigcap_{i-=1}^{n} S_i\right] = \bigcap_{i=1}^{n} f^{-1}(S_i) \quad \ldots(2)$$

By hypothesis, $f^{-1}(S_i)$ is open in X, Being a finite intersection of open sets in X, $\bigcap_{i=1}^{n} f^{-1}(S_i)$ is open in X, i.e. $f^{-1}(B)$ is open in X

i.e. $f^{-1}(G) = f^{-1}[\cup\{B : B \in \mathcal{B}_1\}]$

$= \cup\left[f^{-1}(B) : B \in \mathcal{B}_1\right]$

= An arbitrary union subsets of X

= open subset of X.

$\therefore$ $f^{-1}(G)$ is open in X.

Thus we have shown that

any $G \subset Y \Rightarrow f^{-1}(G)$ is open in X.

This proves that f is continuous.

Theorem 4: Let (X, T) and (Y, U) be topological spaces.

Let f : (X, T) $\to$ (Y, U) be a map. Then f is continuous iff $f^{-1}(B)$ is open for every B $\in \mathcal{B}$, $\mathcal{B}$ being a base for U on Y.

or

f is continuous iff the inverse image of each basic open set is open.

Proof: Let (X, T) and (Y, U) be topological spaces.

Let $\mathcal{B}$ be a base for U on Y. Let f : X $\to$ Y be a continuous map.

To prove that $f^{-1}(B)$ is open in X for every B $\in \mathcal{B}$

$$B \in B \Rightarrow B \in U$$

($\because$ B $\subset$ U $\Rightarrow$ B is open in Y.)

$\Rightarrow$ $f^{-1}(B)$ is open in X. Then f is continuous.

Conversely, suppose that f : X $\to$ Y is map such that $f^{-1}(B)$ is open in X for each B $\in \mathcal{B}$, $\mathcal{B}$ being a base for the topology U on Y. Let G $\in$ U be arbitrary. Then, by definition of base,

$$\exists \mathcal{B}_1 \subset \mathcal{B} \text{ s.t. } G = \cup\{B : B \in \mathcal{B}_1\}$$

$$\therefore \quad f^{-1}(G) = f^{-1} \cup \{B : B \in \mathcal{B}_1\}$$

$$= \cup\{f^{-1}(B) : B \in B_1\}$$

= An arbitrary union of open subsets of X

[$\because$ $f^{-1}(B)$ is open in X, by assumption]

$\Rightarrow$ An open subset of X.

$\therefore$ $f^{-1}(G)$ is open in X

Starting from an arbitrary open subset G of Y we are able to show that $f^{-1}(G)$ is open in X, showing thereby f is continuous.

Theorem 5: To show that a one-one onto continuous map f : X $\to$ X′ is a homeomorphism if f is either open or closed.

Proof: For the sake of convenience, we take X′ = Y.

Suppose f : (X, T) $\to$ (Y, V) is one-one onto and continuous map. Also suppose that f is either open or closed.

To prove that f is a homeomorphism, it is enough to show that f^{-1} is continuous. For this we have to show that.

$f^{-1}\overline{(B)} \subset \overline{f^{-1}(B)}$. For any set B $\subset$ Y.

$B \subset Y \Rightarrow \overline{f^{-1}(B)} \subset X$ is closed set

Also f is a closed map.

$$\Rightarrow \quad \overline{f\left[f^{-1}(B)\right]} = \overline{\left[f\left(\overline{f^{-1}(B)}\right)\right]}$$

Notes

Evidently

$$f^{-1}(B) \subset \overline{f^{-1}(B)} \qquad \ldots(1)$$

$$\Rightarrow \quad f\{f^{-1}(B)\} \subset f\{\overline{f^{-1}(B)}\}$$

$$\Rightarrow \quad \overline{[f\{f^{-1}(B)\}]} \subset \overline{[f\{\overline{f^{-1}(B)}\}]}$$

$$\Rightarrow \quad \overline{[f\{f^{-1}(B)\}]} \subset f\{\overline{f^{-1}(B)}\} \qquad \text{(on using (1))}$$

$$\Rightarrow \quad ff^{-1}\overline{(B)} \subset f[\overline{f^{-1}(B)}]$$

$$\Rightarrow \quad f^{-1}\overline{(B)} \subset \overline{f^{-1}(B)}$$

$\Rightarrow$ f^{-1} is continuous.

Similarly we can show that if f is open, that f^{-1} is continuous

Theorem 6: A map $f : (X, T) \to (Y, V)$ is closed iff

$\overline{f(A)} \subset f(\overline{A})$ for every $A \subset X$.

Proof: Let $(X, T) \to (Y, V)$ be closed map and $A \subset X$ arbitrary.

To prove $\overline{f(A)} \subset f(\overline{A})$

$\overline{A}$ is closed subset of X, f is closed.

$\Rightarrow$ $f(\overline{A})$ is closed subset of Y.

$$\Rightarrow \quad f(\overline{A}) = \overline{f(\overline{A})} \qquad \ldots(1)$$

But $A \subset \overline{A}$

$\Rightarrow$ $f(A) \subset f(A)$

$\Rightarrow$ $\overline{f(A)} \subset f(\overline{A})$

$\Rightarrow$ $\overline{f(A)} \subset \overline{f(\overline{A})} = f(\overline{A})$, By (1)

$\Rightarrow$ $\overline{f(A)} \subset f(\overline{A})$,

Conversely, suppose $\overline{f(A)} \subset f(\overline{A})\ \forall\ A \subset X$. …(2)

To prove that f is closed.

Let F be a closed subset of X so that $\overline{F} = F$

$$\overline{F} = F \Rightarrow f(\overline{F}) = f(F) \qquad \ldots(3)$$

Also, by (2), $\overline{f(F)} \subset \overline{f(F)}$

Combining this with (3),

$$\overline{f(F)} \subset f(F)$$

But $f(F) \subset \overline{f(F)}$ [For $C \subset \overline{C}$ is true for any set C]

Combining the last two.

$$\overline{f(F)} = f(F).$$

$\Rightarrow$ f(F) is closed.

Thus F is closed. $\Rightarrow$ f(F) is closed.

$\therefore$ f is closed map.

Theorem 7: A function f : (X, T) $\rightarrow$ (Y, V) is continuous iff

$$\left[f^{-1}(B)\right]^{\circ} \supset f^{-1}(B^{\circ}), B \subset Y.$$

or $f^{-1}(B^{\circ}) \subset \left[f^{-1}(B)\right]^{\circ}$

Proof: Let f : (X, T) $\rightarrow$ (Y, V) be a topological map. Let B $\subset$ Y be arbitrary.

(i) Suppose f is continuous.

To prove that $\left[f^{-1}(B)\right]^{\circ} \supset f^{-1}(B^{\circ})$

B $\subset$ Y $\Rightarrow$ B° is open in Y.

$\Rightarrow$ $f^{-1}(B^{\circ})$ is open in X. For f is continuous.

$$\Rightarrow \left[f^{-1}(B^{\circ})\right]^{\circ} = f^{-1}(B^{\circ}) \quad \ldots(1)$$

$$B^{\circ} \subset B \Rightarrow f^{-1}(B^{\circ}) \subset f^{-1}(B)$$

$$\Rightarrow f^{-1}(B) \supset f^{-1}(B^{\circ})$$

$$\Rightarrow \left[f^{-1}(B)^{\circ}\right] \supset \left[f^{-1}(B^{\circ})\right]^{\circ} = f^{-1}(B^{\circ}), \quad \text{[by (1)]}$$

$$\Rightarrow \left[f^{-1}(B)^{\circ}\right] \supset f^{-1}(B^{\circ})$$

Proved.

(ii) Suppose $\left[f^{-1}(B)^{\circ}\right] \supset f^{-1}(B^{\circ})$ …(2)

To prove f is continuous.

Let G be an open subset of Y and hence G = G°

If we show that $f^{-1}(G)$ in open in X, the result will follow:

$$\left[f^{-1}(G)\right]^{\circ} \supset f^{-1}(G^{\circ}), \quad \text{[by (2)]}$$

$$= f^{-1}(G)$$

Notes

$\therefore \quad \left[f^{-1}(G)\right]^{\circ} \supset f^{-1}(G)$

But $\left[f^{-1}(G)\right]^{\circ} \subset f^{-1}(G)$ is always $\left[\text{for } C^{\circ} \subset C \ \forall \ C\right]$

Combining the last two, $[f^{-1}(G)]^{\circ} = f^{-1}(G)$

$\therefore \quad f^{-1}(G)$ is open in X.

Example 5: Let f : R → R be a constant map.

Prove that f is continuous.

Solution: Let f : R → R be a map given by

$$f(x) = c \ \forall \ x \in R. \qquad \ldots(1)$$

Then evidently f is a constant map.

To show that f is continuous.

Let $G \subset R$ be an arbitrary open set.

By definition, $f^{-1}(G) = \left[x \in R : f(x) \in G\right]$ …(2)

From (1) and (2), $f^{-1}(G) = \begin{cases} R \text{ if } c \in G, \\ \phi \text{ if } c \notin G, \end{cases}$

ϕ and R both are open sets in R and hence $f^{-1}(G)$ is open in R.

Given any open set G in R, we are able to show that $f^{-1}(G)$ is open in R. This proves that f is a continuous map.

Example 6: Let T and U be any two topologies on R. Let

$$f : (R, T) \to (R, U)$$

be a map given by $f(x) = 1 \ \forall \ x \in R$.

Then show that f is continuous.

Hint: take C = 1. Instead of writing

"Let $G \subset R$ be an open set", write

"$G \in U$ and $f^{-1}(G) \in T$".

Do these changes in the preceding solution.

7.2 Summary

- Let f : (X, T) → (Y, U) be a map.

  The map f is said to be continuous at $x_0 \in X$ is given any U open set H containing $f(x_0)$, ∃a T-open set G containing x_0 s.t. $f(G) \subset H$.

  If map is continuous at each $x \in X$, then the map is called a continuous map.

- A function f : (X, T) → (Y, U) is said to be continuous on a set $A \subset X$ if it is continuous at each point of A.

Notes

- A map $f : (X, T) \rightarrow (Y, U)$ is said to be homeomorphism or topological mapping if
  (a) f is one-one onto.
  (b) f and f^{-1} are continuous.
- A map $f : (X, T) \rightarrow (Y, U)$ is called an open map if it maps open sets onto open sets i.e. if any $G \in T \Rightarrow f(G) \in U$.
- A map $f : (X, T) \rightarrow (Y, U)$ is called a closed map if any T-closed set F
  $\Rightarrow$ f(F) is U-closed set.

7.3 Keywords

Discrete Space: Let X be any non empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

Indiscrete Space: Let X be any non empty set and $T = \{X, \phi\}$. Then T is called the indiscrete topology and (X, T) is said to be an indiscrete space.

Open and Closed set: Let (X, T) be a topological space. Any set $A \in T$ is called an open set and $X - A$ is a closed set.

7.4 Review Questions

1. In any topological space, prove that f and g are continuous maps $\Rightarrow$ gof is continuous map. Let A, B, C be metric spaces if $f : A \rightarrow B$ is continuous and $g : B \rightarrow C$ is continuous, then $gof : A \rightarrow C$ is continuous.
2. Show that characteristic function of $A \subset X$ is continuous on X iff A is both open and closed in X.
3. Suppose (X, T) is a discrete topological space and (Y, U) is any topological space. Then show that any map
$$f : (X, T) \rightarrow (Y, U)$$
is continuous.
4. Let T be the cofinite topology on R. Let U denote the usual topology on R. Show that the identity map
$$f : (R, T) \rightarrow (R, U)$$
is discontinuous, where as the identity map
$$g : (R, U) \rightarrow (R, T)$$
is a continuous map.
5. Show that the map
$$f : (R, U) \rightarrow (R, U) \text{ given by}$$
$$f(x) = x^2 \ \forall \ x \in R \text{ is not open}$$
U-denotes usual topology.

7.5 Further Readings

Books

J. L. Kelley, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

Notes

Unit 8: The Product Topology

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the product topology;
- Define Cartesian product and box topology;
- Solve the problems on the product topology.

Introduction

There are two main techniques for making new topological spaces out of old ones. The first of these, and the simplest, is to form subspaces of some given space. The second is to multiply together a number of given spaces. Our purpose in this unit is to describe the way in which the latter process is carried out.

Previously, we defined a topology on the product X × Y of two topological spaces. In present unit, we generalize this definition to more general cartesian products. So, let us consider the cartesian products

$$X_1 \times \ldots \times X_n \quad \text{and} \quad X_1 \times X_2 \times \ldots,$$

where each X_i is a topological space. There are two possible ways to proceed. One way is to take as basis all sets of the form $\cup_1 \times \ldots \times \cup_n$ in the first case, and of the form $\cup_1 \times \cup_2 \times \ldots$ in the second case, where $\cup_i$ is an open set of X_i for each i.

8.1 The Product Topology

8.1.1 The Product Topology: Finite Products

Definition: Let $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ be topological spaces. Then the product topology T on the set $X_1 \times X_2 \times \ldots \times X_n$ is the topology having the family $\{O_1 \times O_2 \times \ldots \times O_n, O_i \in T_i, i = 1, \ldots, n\}$

Notes

as a basis. The set $X_1 \times X_2 \times \ldots \times X_n$ with the topology T is said to be the product of the spaces $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ and is denoted by $(X_1 \times X_2 \times \ldots, X_n, T)$ or $(X_1, T_1) \times (X_2, T_2) \times \ldots \times (X_n, T_n)$.

Proposition: Let $B_1, B_2, \ldots, B_n$ be bases for topological spaces $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$, respectively. Then the family $\{O_1 \times O_2 \times \ldots \times O_n : O_i \in B_i, i=1, ..., n\}$ is a basis for the product topology on $X_1 \times X_2 \times \ldots \times X_n$.

Example 1: Let $C_1, C_2, \ldots, C_n$ be closed subsets of the topological spaces $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$, respectively. Then $C_1 \times C_2 \times \ldots \times C_n$ is a closed subset of the product space $(X_1 \times X_2 \times \ldots \times X_n, T)$.

Solution: Observe that

$$(X_1 \times X_2 \times \ldots \times X_n)\backslash(C_1 \times C_2 \times \ldots \times C_n)$$
$$= [(X_1\backslash C_1) \times X_2 \times \ldots \times X_n] \cup [X_1 \times (X_2\backslash C_2) \times X_3 \times \ldots \times X_n] \cup \ldots \cup [X_1 \times X_2 \times \ldots \times X_{n-1} \times (X_n\backslash C_n)]$$

which is a union of open sets (as a product of open sets is open) and so is an open set in $(X_1, T_1) \times (X_2, T_2) \times \ldots \times (X_n, T_n)$. Therefore, its complement, $C_1 \times C_2 \times \ldots \times C_n$, is a closed set, as required.

Notes

(i) We now see that the euclidean topology on R^n, $n \geq 2$, is just the product topology on the set $R \times R \times \ldots R = R^n$.

(ii) Any product of open sets is an open set or more precisely: if $O_1, O_2, \ldots, O_n$ are open subsets of topological spaces $(X_1, T_1), (X_2, T_2) \ldots, (X_n, T_n)$, respectively, then $O_1 \times O_2 \times \ldots O_n$ is an open subset of $(X_1, T_1) \times (X_2, T_2) \times \ldots \times (X_n, T_n)$.

(iii) Any product of closed sets is a closed set.

8.1.2 The Product Topology: Infinite Products

Let $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n), \ldots$ be a countably infinite family of topological spaces. Then the product, $\prod_{i=1}^{\infty} X_i$, of the sets X_i, $i \in N$ consists of all the infinite sequences $\langle x_1, x_2, x_3, \ldots, x_n, \ldots\rangle$, where $x_i \in X_i$ for all i. (The infinite sequence $\langle x_1, x_2, \ldots, x_n, \ldots\rangle$ is sometimes written as $\prod_{i=1}^{\infty} x_i$). The product space, $\prod_{i=1}^{\infty}(X_i, T_i)$, consists of the product $\prod_{i=1}^{\infty} X_i$ with the topology T having as its basis the family

$$B = \left\{\prod_{i=1}^{\infty} O_i : O_i \in T_i \quad \text{and} \quad O_i = X_i \text{ for all but a finite number of i.}\right\}$$

The topology T is called the product topology. So a basic open set is of the form

$$O_1 \times O_2 \times \ldots \times O_n \times X_{n+1} \times X_{n+2} \times \ldots$$

Note It should be obvious that a product of open sets need not be open in the product topology T. In particular, if $O_1, O_2, O_3, \ldots, O_n, \ldots$ are such that $O_i \in T_i$ and $O_i \neq X_i$ for all i, then $\prod_{i=1}^{\infty} O_i$ cannot be expressed as a union of members of B and so is not open in the product space $\left(\prod_{i=1}^{\infty} X_i, T\right)$.

Notes

Example 2: Let $(X_1, T_1), \ldots, (X_n, T_n), \ldots$ be a countably infinite family of topological spaces. Then the box topology T′ on the product $\prod_{i=1}^{\infty} X_i$ is that topology having as its basis the family

$$B' = \left\{\prod_{i=1}^{\infty} O_i : O_i \in T_i\right\}$$

It is readily seen that if each (X_i, T_i) is a discrete space, then the box product $\left(\prod_{i=1}^{\infty} X_i, T'\right)$ is a discrete space. So if each (X_i, T) is a finite set with the discrete topology, then $\left(\prod_{i=1}^{\infty} X_i, T'\right)$ is an infinite discrete space, which is certainly not compact. So, we have a box product of the compact spaces (X_i, T_i) being a non-compact space.

Example 3: Let $(X_i, T_i), \ldots, (Y_i, T_i'), i \in N$, be countably infinite families of topological spaces having product spaces $\left(\prod_{i=1}^{\infty} X_i, T\right)$ and $\left(\prod_{i=1}^{\infty} Y_i, T'\right)$ respectively. If the mapping $h_i: (X_i, T_i) \to (Y_i, T_i)$ is continuous for each $i \in N$, then so is the mapping $h: \left(\prod_{i=1}^{\infty} X_i, T\right) \to \left(\prod_{i=1}^{\infty} Y_i, T'\right)$ given by $h: \left(\prod_{i=1}^{\infty} x_i\right) = \prod_{i=1}^{\infty} h_i(x_i)$; that is, $h(\langle x_1, x_2, \ldots, x_n, \ldots\rangle) = \langle h_1(x_1), h_2(x_2), \ldots, h_n(x_n), \ldots\rangle$.

Solution: It suffices to show that if O is a basic open set in $\left(\prod_{i=1}^{\infty} Y_i, T'\right)$, then $h^{-1}(O)$ is open in $\left(\prod_{i=1}^{\infty} X_i, T\right)$. Consider the basic open set $\cup_1 \times \cup_2 \times \ldots \cup_n \times Y_{n+1} \times Y_{n+2} \times \ldots$ where $\cup_i \in T'$, for $i = 1, \ldots, n$. Then

$$h^{-1}(\cup_1 \times \ldots \times \cup_n \times Y_{n+1} \times Y_{n+2} \times \ldots)$$
$$= h_1^{-1}(\cup_1) \times \ldots \times h_n^{-1}(\cup_n) \times h^{-1}(Y_{n+1}) \times h^{-1}(X_{n+2}) \times \ldots$$

and the set on the right hand side is in T, since the continuity of each h_i implies $h_1^{-1}(\cup_i) \in T_i$, for $i = 1, \ldots, n$. So h is continuous.

8.1.3 Cartesian Product

Definition: Let $\{A_\alpha\}_{\alpha\in J}$ be an indexed family of sets; let $X = \cup_{\alpha\in J} A_\alpha$. The cartesian product of this index family, denoted by $\prod_{\alpha\in J} A_\alpha$, is defined to be the set of all J-tuples $(x_\alpha)_{\alpha\in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \to \bigcup_{\alpha\in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

8.1.4 Box Topology

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space $\prod_{\alpha\in J} X_\alpha$ the collection of all sets of the for $\prod_{\alpha\in J} \cup_\alpha$, where $\cup_\alpha$ is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the box topology.

Example 4: Consider euclidean n-space $\mathbb{R}^n$. A basis for $\mathbb{R}$ consists of all open intervals in $\mathbb{R}$; hence a basis for the topology of $\mathbb{R}^n$ consists of all products of the form

$$(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$$

Since $\mathbb{R}^n$ is a finite product, the box and product topologies agree. Whenever we consider $\mathbb{R}^n$, we will assume that it is given this topology, unless we specifically state otherwise.

Example 5: Consider $\mathbb{R}^w$, the countably infinite product of $\mathbb{R}$ with itself. Recall that

$$\mathbb{R}^w = \prod_{n\in\mathbb{Z}_+} X_n,$$

where $X_n = \mathbb{R}$ for each n. Let us define a function $f : \mathbb{R} \to \mathbb{R}^w$ by the equation

$$f(t) = (t, t, t, \ldots);$$

the n^{th} coordinate function of f is the function $f_n(t) = t$. Each of the coordinate functions $f^n : \mathbb{R} \to \mathbb{R}$ is continuous; therefore, the function f is continuous if $\mathbb{R}^w$ is given the product topology. But f is not continuous if $\mathbb{R}^w$ is given the box topology. Consider, for example, the basic element.

$$B = (-1, 1) \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times \ldots$$

for the box topology. We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. If $f^{-1}(B)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point 0. This would mean that $f((-\delta, \delta)) \subset B$ so that, applying π_n to both sides of the inclusion.

$$f_n((-\delta, \delta)) = (-\delta, \delta) \subset \left(-\tfrac{1}{n}, \tfrac{1}{n}\right)$$

for all n, a contradiction.

Theorem 1: Let $\{X_\alpha\}$ be an indexed family of spaces; Let $A_\alpha \subset X_\alpha$ for each α. If ΠX_α is given either the product or the box topology, then

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}.$$

Proof: Let $x = (x_\alpha)$ be a point of $\prod \bar{A}_\alpha$; we show that $x \in \overline{\prod A_\alpha}$.

Let $\cup = \Pi \cup_\alpha$ be a basis element for either the box or product topology that contains x. Since $x_\alpha \in \bar{A}_\alpha$, we can choose a point $y_\alpha \in \cup_\alpha \cap A_\alpha$ for each α. Then $y = (y_\alpha)$ belongs to both $\cup$ and ΠA_α. Since $\cup$ is arbitrary, it follows that x belongs to the closure of ΠA_α.

Conversely, suppose $x = (x_\alpha)$ lies in the closure of ΠA_α, in either topology. We show that for any given index β, we have $x_\beta \in \bar{A}_\beta$. Let V_β be an arbitrary open set of X_β containing x_β. Since $\prod_\beta^{-1}(V_\beta)$ is open in ΠX_α in either topology, it contains a point $y = (y_\alpha)$ of ΠA_α. Then y_β belongs to $V_\beta \cap A_\beta$. It follows that $x_\beta \in \bar{A}_\beta$.

Theorem 2: Let $f : A \to \Pi_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let ΠX_α have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof: Let π_β be the projection of the product onto its βth factor. The function π_β is continuous, for if $\cup_\beta$ is open in X_β, the set $\pi_\beta^{-1}(\cup_\beta)$ is a sub basis element for the product topology on X_α. Now suppose that $f : A \to \Pi X_\alpha$ is continuous. The function f_β equals the composite π_β of; being the composite of two continuous functions, it is continuous.

Conversely suppose that each co-ordinate function f_α is continuous. To prove that f is continuous, it suffices to prove that the inverse image under f of each sub-basis element is open in A; we remarked on this fact when we defined continuous functions. A typical sub-basis element for the

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product topology on ΠX_α is a set of the form $\pi_\beta^{-1}(\cup_\beta)$, where β is some index and $\cup_\beta$ is open X_β. Now

$$f^{-1}\left(\pi_\beta^{-1}(\cup_\beta)\right) = f_\beta^{-1}(\cup_\beta),$$

because $f_\beta = \pi_\beta$ of. Since f_β is continuous, this set is open in A, as desired.

8.2 Summary

- Let $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ be topological spaces. Then the product topology T on the set $X_1 \times X_2 \times \ldots \times X_n$ is the topology having the family $\{O_1 \times O_2 \times \ldots \times O_n, O_i \in T_i, i = 1, \ldots, n\}$ as a basis. The set $X_1 \times X_2 \times \ldots \times X_n$ with the topology T is said to be the product of the space $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ and is denoted by $(X_1 \times X_2 \times \ldots, X_n, T)$.
- The product space, $\prod_{i=1}^{\infty} (X_i, T_i)$, consists of the product $\prod_{i=1}^{\infty} X_i$ with the topology T having as its basis the family

  $$B = \left\{\prod_{i=1}^{\infty} O_i : O_i \in T_i \quad \text{and} \quad O_i \in X_i \text{ for all but a finite number of i.}\right\}$$

  The topology T is called the product topology.
- The cartesian product of this index family, denoted by $\prod_{\alpha \in J} A_\alpha$, is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$.
- Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$ the collection of all sets of the for $\prod_{\alpha \in J} \cup_\alpha$, where $\cup_\alpha$ is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the box topology.

8.3 Keywords

Discrete Space: Let X be any non empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

Indiscrete Space: Let X be any non empty set and T = {X, ϕ}. Then T is called the indiscrete topology and (X, T) is said to be an indiscrete space.

Open & Closed Set: Any set $A \in T$ is called an open subset of X or simply a open set and X – A is a closed subset of X.

Topological Space: Let X be a non empty set. A collection T of subsets of X is said to be a topology on X if

(i) $X \in T, \phi \in T$

(ii) $A \in T, B \in T \Rightarrow A \cap B \in T$

(iii) $A_\alpha \in T \ \forall \ \alpha \in \Lambda \Rightarrow \cup A_\alpha \in T$ where Λ is an arbitrary set.

8.4 Review Questions

1. If $(X_1, T_1), (X_2, T_2), \ldots, (X_n, T_n)$ are discrete spaces, prove that the product space $(X_1, T_1) \times (X_2, T_2) \times \ldots \times (X_n, T_n)$ is also a discrete space.
2. Let X_1 and X_2 be infinite sets and T_1 and T_2 the finite-closed topology on X_1 and X_2, respectively. Show that the product topology, T, on $X_1 \times X_2$ is not the finite-closed topology.

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3. Prove that the product of any finite number of indiscrete spaces is an indiscrete space.

4. For each $i \in N$, let C_i be a closed subset of a topological space (X_i, T_i). Prove that $\prod_{i=1}^{\infty} C_i$ is a closed subset of $\prod_{i=1}^{\infty}(X_i, T_i)$.

5. Let (X_i, T_i), $i \in N$, be a countably infinite family of topological spaces. Prove that each (X_i, T_i) is homeomorphic to a subspace of $\prod_{i=1}^{\infty}(X_i, T_i)$.

8.5 Further Readings

Books

Dixmier, *General Topology* (1984).

James R. Munkres, *Topology,* Second Edition, Pearson Prentice Hall.

Online links

mathworld.wolfram.com/product topology.html

www.history.mcs.st-and.ac.uk/~john/MT4522/Lectures/L1.5.html

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Unit 9: The Metric Topology

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Objectives

After studying this unit, you will be able to:

- Define metric space and pseudo metric space;
- Understand the definitions of open and closed spheres, boundary set, open and closed set;
- Define convergence of a sequence in a metric space and interior, closure and boundary of a point;
- Define neighborhood and limit point;
- Solve the problems on metric topology.

Introduction

The most important class of topological spaces is the class of metric spaces. Metric spaces provide a rich source of examples in topology. But more than this, most of the applications of topology to analysis are via metric spaces. The notion of metric space was introduced in 1906 by Maurice Frechet and developed and named by Felix Hausdorff in 1914.

One of the most important and frequently used ways of imposing a topology on a set is to define the topology in terms of a metric on the set. Topologies given in this way lie at the heart of modern analysis. For example, In this section, we shall define the metric topology and shall give a number of examples. In the next section, we shall consider some of the properties that metric topologies satisfy.

9.1 The Metric Topology

9.1.1 Metric Space

Let $X \neq \phi$ be any given space.

Let $x, y, z \in X$ be arbitrary.

A function $d : X \times X \to R$ having the properties listed below:

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ iff $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality)

is called a **distance function** or **a metric** for X. Instead of saying, "Let X be a non-empty set with a metric d defined on it". We always say, "Let (X, d) be a metric space".

Evidently, d is a real valued map and d denotes the distance between x and y. A set X, together with a metric defined on it, is called metric space.

Example 1:

(1) Let $X = R$ and $\rho(x, y) = |x - y|$ $\forall x, y \in X$. Then ρ is a metric on X. This metric is defined as usual metric on R.

(2) Let $x, y \in R$ be arbitrary

Let $$\rho(x, y) = \begin{cases} 0 & \text{iff} \quad x = y \\ 1 & \text{iff} \quad x \neq y \end{cases}$$

Then ρ is a metric on R.

This metric is defined as trivial metric or discrete metric on R.

9.1.2 Pseudo Metric Space

Let $X \neq \phi$ be any given space. Let $x, y, z \in X$ be arbitrary. A function $d : X \times X \to R$ having the properties listed below:

(i) $d(x, y) \geq 0$,

(ii) $d(x, y) = 0$ if $x = y$,

(iii) $d(x, y) = d(y, x)$,

(iv) $d(x, y) + d(y, z) \geq d(x, z)$,

Where $x, y, z \in X$

is called pseudo metric on x. The set X together with the pseudo metric d is called pseudo metric space. Pseudo metric differs from metric in the sense that.

$$d(x, y) = 0 \text{ even if } x \neq y$$

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Thus for a pseudo metric

$$x = y \Rightarrow d(x, y) = 0$$

but not conversely.

Remark: Thus every metric space is a pseudo metric space but every pseudo metric space is not necessarily metric space.

Example 2: Consider a map $d : R \times R \rightarrow R$ s.t.

$$d(x, y) = \left|x^2 - y^2\right| \quad \forall\ x, y \in R$$

Evidently $d(x, y) = 0 \Rightarrow x = +y$.

It can be shown that d is a pseudo metric but not metric.

9.1.3 Open and Closed Sphere

Let (X, ρ) be a metric space.

Let $x_o \in X$ and $r \in R^+$. Then set $\{ x \in X : \rho (x_o, x) < r \}$ is defined as open sphere (or simply sphere) with centre x_o and radius r.

The following have the same meaning:

Open sphere, closed sphere, open ball and open disc.

We denote this **open sphere** by the symbol $S(x_o, r)$ or by $S_r(x_o)$ or by $B_r(x_o, d)$ or $B(x_o, r)$. This open sphere is also called as Spherical neighborhood of the point x_o or r-nhd of the point x_o.

We denote closed sphere by $S_r[x_o]$ and is defined as

$$S_r[x_o] = \{ x \in X : \rho (x, x_o) \leq r \}.$$

The following have the same meaning:

Closed sphere, closed ball, closed cell and disc.

Examples on Open Sphere

In case of usual metric, we see that

(i) If $X = R$, then $S_r(x_o) = (x_o - r, x_o + r)$ = open interval with x_o as centre.

(ii) If $X = R^2$, then $S_r(x_o)$ = open circle with centre x_o and radius r.

(iii) If $X = R^3$, then $S_r(x_o)$ = open sphere with centre x_o and radius r.

9.1.4 Boundary Set, Open Set, Limit Point and Closed Set

Boundary Set

Let (X, d) be a metric space and $A \subset X$. A point x in X is called a boundary point of A if each open sphere centered at x intersects A and A′. The boundary of A is the set of all its boundary points and is denoted by b(A). It has following properties.

(1) b(A) is a closed set

(2) $b(A) = A \cap A'$

(3) A is closed $\Leftrightarrow$ A contains its boundary.

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Open Set

Let (X, ρ) be a metric space.

A non-empty set $G \subset X$ is called an open set if any $x \in G \Rightarrow \exists\, r \in R^+$ s.t. $S_r(x) \subset G$.

Limit Point

Let (X, ρ) be a metric space and $A \subset X$. A point $x \in X$ is called a limit point or limiting point or accumulation point or cluster point if every open sphere centered on x contains a point of A other than x, i.e., $x \in X$ is called limit point of A if $(S_{r(x)} - \{x\}) \cap A \neq \phi$, $r \in R^+$.

The set of all limiting points of a set A is called **derived set of A** and is denoted by D(A).

Closed Set

Let (X, ρ) be a metric space and $A \subset X$. A is called a closed set if the derived set of A i.e., $D(A) \subset A$ i.e., if every limit point of A belongs to the set itself.

9.1.5 Convergence of a Sequence in a Metric Space

Let $<x_n>$ be a sequence in a metric space (X, ρ). This sequence is said to coverage to $x_0 \in X$, if given any $\varepsilon > 0$, $\exists\, n_o \in N$ s.t. $n \geq n_0 \Rightarrow \rho(x_n, x_0) < \varepsilon$ or equivalently, given any $\varepsilon > 0$, $\exists\, n_0 \in N$ s.t. $n \geq n_0 \Rightarrow x_n \in S_\varepsilon(x_0)$.

9.1.6 Theorems on Closed Sets and Open Sets

Theorem 1: In a metric space (X, ρ), ϕ and X are closed sets.

Proof: Let (X, ρ) be a metric space.

To prove that ϕ and X are closed sets.

$\because \quad D(\phi) = \phi \subset \phi$

$\therefore \quad D(\phi) \subset \phi.$

$\Rightarrow \phi$ is a closed set.

All the limiting points of X belong to X. For X is the universal set.

i.e., any $\quad x \in D(x) \Rightarrow x \in X$

$\because \quad D(X) \subset X$

$\Rightarrow$ X is a closed set.

Theorem 2: Let (X, d) be a metric space. Show that $F \subset X$, F is closed $\Leftrightarrow F'$ is open.

Proof: Let (X, d) be a metric space.

Let F be a closed subset of X, so that $D(F) \subset F$.

To prove that F' is open in X.

Let $x \in F'$ be arbitrary. Then $x \notin F$.

$$D(F) \subset F,\ x \notin F \Rightarrow x \notin D(F)$$

$$\Rightarrow (S_{r(x)} - \{x\}) \cap F = \phi \text{ for some } r > 0$$

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$$\Rightarrow S_{r(x)} \cap F = \phi \qquad [\because x \notin F]$$

$$\Rightarrow S_{r(x)} \subset X - F$$

$$\Rightarrow S_{r(x)} \subset F'.$$

$\therefore$ Given $x \in F'$, $\exists$ any open sphere $S_{r(x)}$ s.t.

$$S_{r(x)} \subset F'$$

By definition, this proves that F′ is open.

Conversely suppose that F′ is open in X.

To prove that F is closed in X.

Let $x \in F'$ be arbitrary, then $x \notin F$.

$\because$ F′ is open, $\exists_r \subset R^+$ s.t., $S_{r(x)} \subset F'$

$$\Rightarrow S_{r(x)} \cap F = \phi$$

$$\Rightarrow (S_{r(x)} - \{x\}) \cap F = \phi$$

$$\Rightarrow x \notin D(F).$$

Thus, any $x \in F' \Rightarrow x \notin D(F)$

i.e. any $x \in X - F \Rightarrow x \in X - D(F)$

$$\Rightarrow X - F \subset X - D(F) \text{ or } D(F) \subset F$$

$$\Rightarrow F \text{ is closed.}$$

Theorem 3: In any metric space (X, d), each open sphere is an open set.

Proof: Let (X, d) be a metric space. Let $S_{r_0(x_0)}$ be an open sphere in X.

To prove that $S_{r_0(x_0)}$ is an open set.

Let $x \in S_{r_0(x_0)}$ be arbitrary, then $d(x, x_0) < y_0$

Write $r = r_0 - d(x, x_0)$ ...(1)

By definition $S_{r_0(x_0)} = \{y \in X : d(y, x_0) < r_0\}$

$$S_{r(x)} = \{y \in X : d(y, x) < r\}.$$

We claim $S_{r(x)} \subset S_{r_{(0)}(x_0)}$

Let $y \in S_{r(x)}$ be arbitrary

Then $d(x, y) < r$

$$d(y, x_0) \leq d(y, x) + d(x, x_0)$$

$$< r + d(x, x_0) = r_0. \qquad \text{[on using (1)]}$$

$\therefore$ $d(y, x_0) < r_0$

$$\Rightarrow y \in S_{r_0(x_0))}$$

and $y \in S_{r(x)} \Rightarrow y \in S_{r_0(x_0)}$

$$\Rightarrow S_{r(x)} \subset S_{r_0(x_0)}$$

Thus we have shown that for given any $x \in S_{r_0(x_0)}$, $\exists\, r > 0$ s.t. $S_{r(x)} \subset S_{r_0(x_0)}$.

By definition, this proves that $S_{r_0(x_0)}$ is an open set.

Theorem 4: In any metric space, any closed sphere is a closed set.

Proof: Let $S_{r_0[x_0]}$ denote a closed sphere in a metric space (X, d).

To prove that $S_{r_0[x_0]}$ is a closed set.

For this we must show that $S'_{r_0[x_0]}$ is open in X.

Let $x \in S'_{r_0[x_0]}$ be arbitrary,

$$x \in S'_{r_0[x_0]} \Rightarrow x \notin S_{r_0[x_0]}$$

$$\Rightarrow d(x, x_0) > r_0, \qquad \left[\because S_{r_0[x_0]} = \{y \in X : d(y, x_0) \le r_0\}\right]$$

$$\Rightarrow d(x, x_0) - r_0 > 0$$

$$\Rightarrow r > 0, \text{ on taking } r = d(x, x_0) - r_0 \qquad ...(1)$$

We claim $S_{r(x)} \subset S'_{r_0[x_0]}$.

Let $y \in S_{r(x)}$ be arbitrary, so that, $d(y, x) < r$.

$\therefore \quad d(x, x_0) \le d(x, y) + d(y, x_0)$.

$\therefore \quad d(y, x_0) \ge d(x, x_0) - d(x, y) > d(x, x_0) - r = r_0$ [on using (1)]

$\therefore \quad d(y, x_0) > r_0 \Rightarrow y \notin S_{r_0[x_0]}$

Thus, any $\quad y \in S_{r(x)} \Rightarrow y \in S'_{r_0[x_0]}$

$$\Rightarrow S_{r(x)} \subset S'_{r_0[x_0]}$$

$\therefore$ Given any $x \in S'_{r_0[x_0]}, \exists r > 0$ s.t. $S_{r(x)} \subset S'_{r_0[x_0]}$

This prove that $S'_{r_0[x_0]}$ is open in x.

Example 3: Give an example to show that the union of an infinite collection of closed sets in a metric space is not necessarily closed.

Solution: Let $\{[\frac{1}{n}, 1] : n \in N\}$ be the infinite collection for the usual metric space ($\mathcal{R}$, d).

Now each member of this collection is a closed set, being a closed interval.

But $\cup\{[\frac{1}{n}, 1] : n \in N\} = \{1\} \cup [\frac{1}{2}, 1] \cup [\frac{1}{3}, 1] \cup \ldots =]0, 1]$.

Since]0, 1] is not closed, it follows that the union of an infinite collection of closed sets is not closed.

Example 4: Show that every closed interval is a closed set for the usual metric on $\mathcal{R}$.

Solution: Let $x, y \in \mathbb{R}$ where $x < y$. We shall show that [x, y] is closed.

Now $\quad \mathcal{R} - [x, y] = \{a \in \mathcal{R} : a < x \text{ or } a > y\}$

$$= \{a \in \mathcal{R} : a < x\} \cup \{a \in \mathcal{R} : a > y\}$$

$$=]-\infty, x [\cup] y, \infty [$$

Notes

which is open, being a union of two open sets.

Hence [x, y] is closed.

Example 5: Give an example of two closed subsets A and B of the real line $\mathcal{R}$ such that d(A, B) = 0 but $A \cap B = \phi$.

Solution: Let

$$A = \{2, 3, 4, 5, \ldots\}$$

$$B = \left\{2\tfrac{1}{2}, 3\tfrac{1}{3}, 4\tfrac{1}{4}, \ldots\right\},$$

Clearly $A \cap B = \phi$.

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$$

If $n \in A$ and $n + \frac{1}{n} \in B$

$$d(A, B) = \lim_{n\to\infty} d\left(n, n + \tfrac{1}{n}\right)$$

$$= \lim_{n\to\infty} \tfrac{1}{n} \qquad [\because \text{ d is usual metric for } \mathcal{R}]$$

$$= 0$$

9.1.7 Interior, Closure and Boundary of a Point

Interior

Let (X, d) be a metric space and $A \subset X$.

A point $x \in A$ is called an interior point of A if $\exists\, r \in R^+$ s.t. $S_{r(x)} \subset A$.

The set of all interior point of A is called the interior of A and is denoted by A°, or by Int. (A).

Thus $\quad A° = \text{int. } (A) = \{x \in A : S_{r(x)} \subset A \text{ for some } r)$

Alternatively, we define

$$A° = \bigcup (S_{r(x)} : S_{r(x)} \subset A).$$

Evidently

(i) A° is an open set.

For an arbitrary union of open sets is open.

(ii) A° is the largest open subset of A.

Closure

Let (X, d) be a metric space and $A \subset X$.

The closure of A, denoted by $\bar{A}$, is defined as the intersection of all closed sets that contain A. Symbolically

$$\bar{A} = \cap \{F \subset X : F \text{ is closed}, F \supset A\} \qquad ...(1)$$

Evidently

(i) $\bar{A}$ is closed set

For an arbitrary intersection of closed sets is closed.

(ii) $\bar{A} \supset A$.

(iii) $\bar{A}$ is the smallest closed set which contain A.

Alternatively we define

$$\bar{A} = A \cup D(A) \qquad ...(2)$$

A point $x \in \bar{A}$ is called a point of closure of A.

Alternatively, a point $x \in X$ is called a point of closure of A iff $x \in A$ or $x \in D(A)$.

Boundary of a Point

Let (X, d) be a metric space. Let $A \subset X$

(i) ***Boundary or Frontier*** of a set A is denoted by b(A) or $F_r(A)$ and is defined as

$$b(A) = F_r(A) = X - A^\circ \cup (X - A)^\circ.$$

Elements of b(A) are called boundary points of A.

(ii) ***The exterior of A*** is defied as the set $(X - A)^\circ$ and is denoted by ext (A).

Symbolically ext (A) = $(X - A)^\circ$.

(iii) A is said to be *dense or everywhere dense* in X if $\bar{A} = X$.

(iv) A is said to be *somewhere dense* if $(\bar{A})^\circ \neq \phi$ i.e., if closure of A contains some open set.

(v) A is said to be *nowhere dense* (or non where dense set) if $(\bar{A})^\circ = \phi$.

(vi) A metric space (X, d) is said to be separable if $\exists\, A \subset X$ s.t. A is countable and $\bar{A} = X$.

(vii) A is said to be *dense* in itself of $A \subset D(A)$.

Example 6:

(1) To find the boundary of set of integers Z and set of rationals Q.

$$Z^\circ = \cup \{G \subset R : G \text{ is open ad } G \subset Z\} = \phi$$

For every sub set of R contains fractions also.

Similarly $(R - Z)^\circ = \phi$

$$b(Z) = R - Z^\circ \cup (R - Z)^\circ = R - \phi \cup \phi = R$$

$$b(Z) = R.$$

Similarly $b(Q) = R$.

(2) Give two examples of limit points

(i) If $A = \left\{1 + \frac{1}{n} : n \in N\right\}$,

i.e. $A = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, ...\right\}$, then

1 is limit point of A. For $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right) = 1$.

Notes

(ii) If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\} = \left\{\frac{1}{n} : n \in N\right\}$

then 0 is the limit point of A

For $\lim_{n \to \infty} \frac{1}{n} = 0$

9.1.8 Neighborhood

Let (X, d) be a metric space and $x \in X$, $A \subset X$

A subset A of X is called a neighborhood (nbd) of x if $\exists$ open sphere $S_{r(x)}$ s.t. $S_{r(x)} \subset A$

This means that A is nbd of a point x iff x is an interior point of A.

From the definition of nbd, it is clear that:

(1) Every superset of a nbd of a point is also a nbd.

(2) Every open sphere $S_{r(x)}$ is a nbd of x.

(3) Every closed sphere $S_{r(x)}$ is a nbd of x.

(4) Intersection of two nbds of the same point is given a nbd of that point.

(5) A set is open if it contains a nbd of each of its points.

(6) Nbd of a point need not be an open set.

9.1.9 Theorems and Solved Examples

Theorem 5: A subset of a metric space is open iff it is a nbd of each of its point.

Proof: Let A be a subset of a metric space (X, d).

Step I: Given A is a nbd of each of its points.

Aim: A is an open set

Recall that a set N is called nbd of a point $x \in X$ if $\exists$ open set $G \subset X$ s.t. $x \in G \subset N$.

Let $p \in A$ be arbitrary, then by assumption, A is a nbd of p. By definition of nbd, $\exists$ open set $G_p \subset X$ s.t. $p \in G_p \subset A$.

It is true $\forall p \in A$

Take $A = \cup \{G_p : p \in G_p, G_p \text{ is an open set}, G_p \subset A\}$

= An arbitrary union of open sets

= open set

$\therefore$ A is an open set.

Step II: Let A be an open subset of X.

Aim: A is a nbd of each of its points. By assumption, we can write $p \in A \subset A \;\forall\, p \in A$.

$\Rightarrow$ A is a nbd of each of its points.

Problem: Every set of discrete metric space is open.

Solution: Let (X, d) be a discrete metric space. Let x, y ∈ X be arbitrary. By definition of discrete metric,

$$d(x, y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

Let r be any positive real number s.t. r ≤ 1.

Then $S_{r\,(x)} = \{y \in X : d\,(y, x) < r \leq 1\}$

$= \{y \in X : d\,(y, x) < 1\}$

$= \{y \in X : d\,(y, x) = 0\}$ (by definition of d)

$= \{y \in X : y = x\} = \{0\}$

or $S_{r\,(x)} = \{0\}$

But every open sphere is an open set.

∴ {x} is an open set is X ∀ x ∈ X.

If $A = \{x_1, x_2, ..., x_n\}$ = finite set ⊂ X, then

$$A = \bigcup_{r=1}^{n} \{x_r\} = \text{finite union of open sets.}$$

= open set.

Hence every finite subset of X is open set. ...(1)

If $B = \{x_1, x_2, x_3, ...\} \subset X$, then

B is an infinite subset of X.

Now $B = \bigcup_{r=1}^{\infty} \{x_r\}$

= Arbitrary union of open sets

= Open set,

∴ B is an open set. ...(2)

From (1) and (2), it follows that every subset (finite or infinite) is an open set in X.

Problem: A finite set in any metric space has no limit point.

Solution: Let A be a finite subset of a metric space (X, d). We know that "x ∈ X is a limit point of any set B if every open sphere $S_{r\,(x)}$ contains an infinite number of points of B other than *x*."

This condition can not be satisfied here as A is finite set.

Hence A has no limit point.

Theorem 6: Let (X, d) be a metric space. A subset A of X is closed if given any x ∈ X – A, d (x, A) ≠ 0.

Proof: Let (X, d) be a metric space and A ⊂ X be an arbitrary closed set.

To prove that

Given any x ∈ X – A, d (x, A) ≠ 0

Notes

A is closed ⇒ X – A is open.

By definition of open set,

any $x \in X - A \Rightarrow \exists_r \in R^+$ s.t. $S_{r_{(x)}} \subset X - A \Rightarrow S_{r_{(x)}} \cap A = \phi$

$\Rightarrow d(X, A) \geq r \Rightarrow d(x, A) \neq 0.$

Conversely let A be any subset of a metric space (X, d).

Let any $x \in X - A \Rightarrow d(x, A) \neq 0.$

To prove that A is closed.

Let $x \in X - A$ be arbitrary so that, by assumption

$$d(x, A) = r \neq 0 \Rightarrow S_{r_{(x)}} \cap A = \phi \Rightarrow S_{r_{(x)}} \subset X - A$$

$\therefore$ $x \in X - A \Rightarrow \exists\, r \in R^+$ s.t. $S_{r_{(x)}} \in X - A.$

By definition, this implies X – A is open

⇒ A is closed

Proved.

Problem: In any metric space, show that

$$X - \bar{A} = (X - A)^\circ$$

or $$(\bar{A})' = (A')^\circ.$$

Solution: $$(\bar{A})' = X - \bar{A}$$

$= X$ – Intersection of all closed super sets of A

$= X - \bigcap_i F_i$ where F_i is closed and $F_i \supset A$

$= \bigcup_i (X - F_i)$ where $X - F_i$ is open and $X - F_i \subset X - A$

= Union of open subsets of X – A = A′

$= (A')^\circ.$

Proved.

Problem: In any metric space (X, d), prove that A is open ⇔ A° = A.

Solution: Let A be a subset of a metric space (X, d). By definition of interior,

$$A^\circ = \cup\{S_{r_{(x)}} : S_{r_{(x)}} \subset A\} \quad ...(1)$$

since every open sphere is an open set and arbitrary union of open sets is open.

Consequently,

A° is an open set. ...(2)

By (1), it is clear that A° ⊂ A ...(3)

and A° is largest open subset of A. ...(4)

Notes

(i) Given $A = A°$ ...(5)

Aim: A is an open set

(2) and (5) $\Rightarrow$ A is an open set.

(ii) Given A is an open set. ...(6)

Aim: $A = A°$

(4) and (6) $\Rightarrow A = A°$.

9.1.10 Uniform Convergence

A sequence defined on a metric space (X, d) is said to be uniformly convergent if given $\varepsilon > 0$, $\exists$ $n_0 \in N$ s.t. $n \geq n_0 \Rightarrow d(f_n(x), f(x)) < \varepsilon \;\forall \in X$.

Theorem 7: Let $<f_n(x)>$ be a sequence of continuous functions defined on a metric space (X, d). Let this sequence converge uniformly to f on X. Then f(x) is continuous on X.

OR

Uniform limit of a sequence of continuous function is continuous.

Proof: Since $< f_n(x)>$ converges uniformly to f on (X, d). Hence given $\varepsilon > 0$, $\exists n_0 \in N$ independent of $x \in N$ s.t. $n \geq n_0$.

$$\Rightarrow d(f_n(x), f(x)) < \varepsilon/3 \quad ...(1)$$

Let $a \in X$ be arbitrary. To prove that f is continuous on X, we have to prove that f is continuous at x = a, for this we have to show that given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $d(x, a) < \delta$

$$\Rightarrow d(f(x), f(a)) < \varepsilon. \quad ...(2)$$

Continuity of f_n at $a \in X$

$$\Rightarrow d(f_n(x), f_n(a)) < \frac{\varepsilon}{3} \text{ for } d(x, a) < \delta \quad ...(3)$$

By (1), $d(f_n(a), f(a)) < \frac{\varepsilon}{3} \forall\, n \geq n_0$ ...(4)

If $d(x, a) < \delta$, then

$$d(f(x), f(a)) \leq d[f(x), f_n(x)] + d[f_n(x), f_n(a)] + d[f_n(a), f(a)]$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ by (1), (3) and (4)}$$

or $d(f(x), f(a) < \varepsilon$ for $d(x, a) < \delta$. Hence the result (2).

Theorem 8: **Frechet space.** Let F be the set of infinite sequences of real numbers.

Let $x, y, z \in F$, then

$$x = <x_n> = <x_1, x_2 ...>, y = <y_n>, z = <z_n>$$

where $x_n, y_n, z_n \in R$

we define a map

$d : F \times F : \to R$ s.t.

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{[1 + |x_n - y_n|]}$$

Notes

To show that d is metric on F.

(i) d (x, y) ≥ 0. For $|x_n - y_n| \geq 0 \ \forall n$

(ii) d (x, y) = 0 ⇔ x = y

For
$$d(x, y) = 0 \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|} = 0$$

$$\Leftrightarrow \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1+|x_n - y_n|} = 0 \ \forall n$$

$$\Leftrightarrow |x_n - y_n| = 0 \ \forall_n \Leftrightarrow x_n = y_n \forall n$$

$$\Leftrightarrow x = y$$

(iii) $$d(x, y) = d(y, x)$$

For $$|x_n - y_n| = |y_n - x_n|$$

(iv) $$d(x, y) \geq d(x, z) + d(z, y)$$

Here we use the fact that

$$\frac{|\alpha+\beta|}{1+|\alpha+\beta|} \geq \frac{|\alpha|}{1+|\alpha|} + \frac{|\beta|}{1+|\beta|}$$

In view of this, we have

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{1+|x_n - y_n|} \cdot \frac{1}{2^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|(x_n - z_n) + (z_n - y_n)|}{1+|(x_n - z_n) + (z_n - y_n)|}$$

$$\geq \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - z_n|}{1+|x_n - z_n|} + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|z_n - y_n|}{1+|z_n - y_n|}$$

$$= d(x, z) + d(z, y)$$

Thus d is metric on F. The fair (F, d) is a metric space and this metric space is called **Frechet space**.

Example 7: In a metric space (X, d), prove that

F is closed ⇔ D (F) ⊂ F.

Prove that a subset F of a metric space X contains all its limit points iff X – F is open.

Solution: Let (X, d) be a metric space and F ⊂ X.

We know that F is closed ⇔ X – F is open.

Step I: Let X – F be open so that F is closed,

Aim: D (F) ⊂ F.

Let $x \in X - F$ be arbitrary. Then $X - F$ is an open set containing x s.t. $(X - F) \cap F = \phi$.

$\Rightarrow$ x is not a limit point of F

$$\Rightarrow x \notin D(F) \Rightarrow x \in X - D(F)$$

Thus, $\forall\ x \in X - F \Rightarrow x \in X - D(F)$

$\therefore$ $X - F \subset X - D(F)$

or, $D(F) \subset F$

Step II: Given $D(F) \subset F$. ...(1)

To prove F is closed.

Let $y \in X - F$, then $y \notin F$

$$y \notin F, D(F) \subset F \Rightarrow y \notin D(F)$$

$$\Rightarrow \exists \text{ open set } G \text{ with } y \in G \text{ s.t.}$$

$$(G - \{y\}) \cap F = \phi$$

$$\Rightarrow G \cap F = \phi \text{ as } y \notin F$$

$$\Rightarrow G \subset X - F$$

Thus we have show that

$$\text{any } y \in X - F \Rightarrow \exists \text{ open set } G \text{ with } y \in G \text{ s.t. } G \subset X - F$$

$$\Rightarrow X - F \text{ is open} \Rightarrow F \text{ is closed.}$$

9.2 Summary

- Let $X \neq \phi$ be any given space. Let x, y, z, $\in X$ be arbitrary. A function $d : X \times X \to R$ having the properties listed below:

  (i) $d(x, y) \geq 0$

  (ii) $d(x, y) = 0$ iff $x = y$

  (iii) $d(x, y) = d(y, x)$

  (iv) $d(x, y) + d(y, z) \geq d(x, z)$

  is called a distance function or a metric for X.

- Let $X \neq \phi$ be any given space. Let $x, y, z \in X$ be arbitrary. A function $d : X \times X \to R$ having the properties listed below:

  (i) $d(x, y) \geq 0$

  (ii) $d(x, y) = 0$ if $x = y$

  (iii) $d(x, y) = d(y, x)$

  (iv) $d(x, y) + d(y, z) \geq d(x, z)$, where $x, y, z \in X$ is called pseudo metric on X.

- Let (X, ρ) be a metric space. Let $x_0 \in X$ and $r \in R^+$. Then set $\{x \in X : \rho(x_0, x) < r\}$ is defined as open sphere with centre x_0 and radius r.

- Closed sphere:

  $S_r[x_0] = \{x \in X : \rho(x, x_0) \leq x\}$

Notes

- Let (X, ρ) be a metric space. A non empty set $G \subset X$ is called an open set if any $x \in G \Rightarrow \exists$ $r \in R^+$ s.t. $S_{r(x)} \subset G$.
- Let (X, ρ) be a metric space and $A \subset X$. A point $x \in X$ is called a limit point if every open sphere centered on x contains a point of A other than x, i.e. $x \in X$ is called the limit point of A if $\left(S_{r_{(x)}} - \{x\}\right) \cap A \neq \phi$, $r \in R^+$.
- Let (X, ρ) be a metric space and $A \subset X$. A is called a closed set if the derived set of A i.e. $D(A) \subset A$ i.e. if every limit point of A belongs to the set itself.
- Set $<x_n>$ be a sequence in a metric space (X, ρ). This sequence is said to converge to $x_0 \in X$, if given any $\in > 0$, $\exists\, n_o \in N$ s.t. $n \geq n_0 \Rightarrow \rho(x_n, x_0) < \in$.
- A point $x \in A$ is called an interior point of A if $\exists\, r \in R^+$ s.t. $S_{r_{(x)}} \subset A$.
- The closure of A, denoted by $\bar{A}$, is defined as the intersection of all closed sets that contain A.
- Boundary of a set A is denoted by b(A) is defined as $b(A) = X - A° \cup (X - A)°$.
- The exterior of A is defined as the set $(X - A)°$ and is denoted by ext (A).
- A is said to be dense or everywhere dense in X if $\bar{A} = X$.
- A is said to be nowhere dense if $(\bar{A})° = \phi$.
- A metric space (X, d) is said to be separable if $\exists\, A \subset X$ s.t. A is countable and $\bar{A} = X$.
- A sequence defined on a metric space (X, d) is said to be uniformly convergent if given $\in > 0$, $\exists\, n_0 \in N$ s.t. $n \geq n_0$

  $\Rightarrow d(f_n(x), f(x)) < \in \quad \forall\ x \in X$.

9.3 Keywords

Frechet Space: A topology space (X, T) is said to satisfy the T_1 – axiom of separation if given a pair of distinct point $x, y \in X$.

$\exists\, G, H \in T$ s.t. $x \in G$, $y \notin G$; $y \in H$, $x \notin H$.

In this case the space (X, T) is called Frechet Space.

Intersection: The intersection of two sets A and B, denoted by $A \cap B$, is defined as the set containing those elements which belong to A and B both. Symbolically

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

Union: The union of two sets A ad B, denoted by $A \cup B$, is defined as the set of those elements which either belong to A or to B. Symbolically

$A \cup B = \{x : x \in A \text{ or } x : B\}$

9.4 Review Questions

1. In any metric space (X, d), show that

   (a) an arbitrary intersection of closed sets is closed.

   (b) any finite union of closed sets is closed.

Notes

2. Let R be the set of all real numbers and let

$$d(x, y) = \frac{|x - y|}{1 + |x - y|} \text{ for all } x, y \in R.$$

Prove that d is a metric for R.

3. Every derived set in a metric space is a closed set.

4. Let A and B is disjoint closed set in a metric space (X, d). Then $\exists$ disjoint open sets G, H s.t. $A \subset G$, $B \subset H$.

5. Let $X \neq \phi$ and let d be a real function of ordered pairs of X which satisfies the following two conditions:

$$d(x, y) = 0 \Leftrightarrow x = y$$

and $d(x, y) \leq d(x, z) + d(z, y)$.

Show that d is a metric on X.

6. Give an example of a pseudo metric which is not metric.

7. Let X be a metric space. Show that every subset of X is open $\Leftrightarrow$ each subset of X which consists of single point is open.

8. In a metric space prove that

(a) $(\bar{A}) = \text{Int}(A')$,

(b) $\bar{A} = \{x : d(x, A) = 0\}$.

9.5 Further Readings

Books

B. Mendelson, *Introduction to Topology*, Dover Publication.

J. L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

Notes

Unit 10: The Quotient Topology

CONTENTS

Objectives

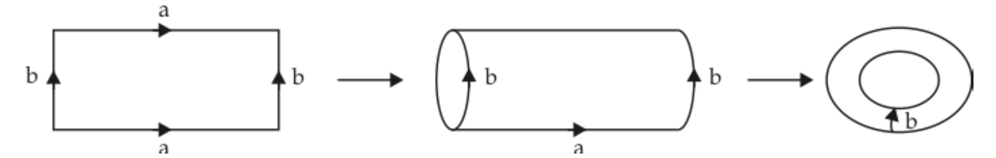
After studying this unit, you will be able to:

- Understand the quotient map, open map and closed map;
- Explain the quotient topology;
- Solve the theorems and questions on quotient topology.

Introduction

The quotient topology is not a natural generalization of something. You have already studied in analysis. Nevertheless, it is easy enough to motivate. One motivation comes from geometry, where one often has occasion to use 'cut-and-paste' techniques to construct such geometric objects as surfaces. The torus (surface of a doughnut), for example can be constructed by taking a rectangle and 'pasting' its edges together appropriately in Figure 10.1.

Figure 10.1



Formalizing these constructions involves the concept of quotient topology.

10.1 The Quotient Topology

10.1.1 Quotient Map, Open and Closed Map

Quotient Map

Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

The condition is stronger than continuity, some mathematicians call it 'strong continuity'. An equivalent condition is to require that a subset A of Y be closed in Y if and only if $p^{-1}(A)$ is closed in X. Equivalence of the two conditions follow from equation

$$f^{-1}(Y - B) = X - f^{-1}(B).$$

Open map: A map $f : X \to Y$ is said to be an open map if for each open set U of X, the set f(U) is open in Y.

Closed Map: A map $f : X \to Y$ is said to be a closed map if for each closed set A of X, the set f(A) is closed in Y.

Example 1: Let X be the subspace $[0, 1] \cup [2, 3]$ of $\mathbb{R}$ and let Y be the subspace [0, 2] of $\mathbb{R}$. The map $p : X \to Y$ defined by

$$p(x) = \begin{cases} x & \text{for} \quad x \in [0,1], \\ x-1 & \text{for} \quad x \in [2,3] \end{cases}$$

is readily seen to be surjective, continuous and closed. Therefore, it is a quotient map. It is not, however, an open map; the image of the open set [0, 1] of X is not open in Y.

> *Note* If A is the subspace $[0, 1] \cup [2, 3]$ of X, then the map $q : A \to Y$ obtained by restricting p is continuous with surjective but it is not a quotient map. For the set [2, 3] is open in A and is saturated w.r.t q, but its image is not open in Y.

Example 2: Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection onto the first coordinate, then π_1 is continuous and surjective. Furthermore, π_1 is an open map. For if U × V is a non-empty basis element for $\mathbb{R} \times \mathbb{R}$, then $\pi_1(U \times V) = U$ is open in $\mathbb{R}$; it follows that π_1 carries open sets of $\mathbb{R} \times \mathbb{R}$ to open sets of $\mathbb{R}$. However, π_1 is not a closed map. The subset

$$C = \{x \times y \mid xy = 1\}$$

of $\mathbb{R} \times \mathbb{R}$ is closed, but $\pi_1(C) = \mathbb{R} - \{0\}$, which is not closed in $\mathbb{R}$.

> *Note* If A is the subspace of $\mathbb{R} \times \mathbb{R}$ that is the union of C and the origin {0}, then the map $q : A \to \mathbb{R}$ obtained by restricting π_1 is continuous and surjective, but it is not a quotient map. For the one-point set {0} is open in A and is saturated with respect to q. But its image is not open in $\mathbb{R}$.

Notes

10.1.2 Quotient Topology

If X is a space and A is a set and if p : X → A is a surjective map, than there exists exactly one topology T on A relative to which p is a quotient map; it is called the quotient topology induced by p.

The topology T is of course defined by letting it consists of those subsets U of A such that $p^{-1}(U)$ is open in X. It is easy to check that T is a topology. The sets ϕ and A are open because $p^{-1}(\phi) = \phi$ and $p^{-1}(A) = X$. The other two conditions follow from the equations

$$p^{-1}\left(\bigcup_{\alpha\in J} U_\alpha\right) = \bigcup_{\alpha\in J} p^{-1}(U_\alpha),$$

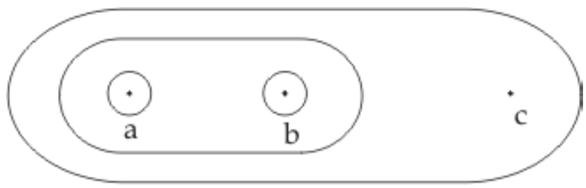
$$p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i)$$

Example 3: Let p be the map of the real line ℝ onto the three point set A = {a, b, c} defined by

$$p(x) = \begin{cases} a & \text{if} \quad x > 0 \\ b & \text{if} \quad x < 0 \\ c & \text{if} \quad x = 0 \end{cases}$$

You can check that the quotient topology on A induced by p is the one indicated in figure (10.2) below

Figure 10.2



10.1.3 Quotient Space

Let X be a topological space and let X* be a partition of X into disjoint subsets whose union is X. Let p : X → X* be the surjective map that carries each point of X to the element of X* containing it. In the quotient topology induced by p, the space X* is called a quotient space of X.

Given X*, there is an equivalence relation on X of which the elements of X* are the equivalence classes. One can think of X* as having been obtained by 'identifying' each pair of equivalent points. For this reason, the quotient space X* is often called an identification space, or a decomposition space of the space X.

We can describe the topology of X* in another way. A subset U of X* is a collection of equivalence classes, and the set $p^{-1}(U)$ is just the union of the equivalence classes belonging to U. Thus the typical open set of X* is a collection of equivalence classes whose union is an open set of X.

Example 4: Let X be the closed unit ball $\{x \times y \mid x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$, and let X* be the partition of X consisting of all the one-point sets $\{x \times y\}$ for which $x^2 + y^2 < 1$, along with the set $S^1 = \{x \times y\} \mid x^2 + y^2 = 1\}$. One can show that X* is homeomorphic with the subspace of $\mathbb{R}^3$ called the unit 2-sphere, defined by

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

Theorem 1: Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p.

(1) If A is either open or closed in X, then q is a quotient map.

(2) If p is either an open map or a closed map, then q is a quotient map.

Proof: *Step (1):* We verify first the following two equations:

$$q^{-1}(V) = p^{-1}(V) \qquad \text{if } V \subset p(A);$$

$$p(U \cap A) = p(U) \cap p(A) \qquad \text{if } U \subset X$$

To check the first equation, we note that since $V \subset p(A)$ and A is saturated, $p^{-1}(V)$ is contained in A. It follows that both $p^{-1}(V)$ and $q^{-1}(V)$ equal all points of A that are mapped by p into V. To check the second equation, we note that for any two subsets U and A of X, we have the inclusion

$$p(U \cap A) \subset p(U) \cap p(A)$$

To prove the reverse inclusion, suppose $y = p(u) = p(a)$, for $u \in U$ and $a \in A$. Since A is saturated, A contains the set $p^{-1}(p(a))$, so that in particular A contains u. They $y = p(u)$, where $u \in U \cap A$.

Step (2): Now suppose A is open or p is open. Given the subset V of p(A), we assume that $q^{-1}(V)$ is open in A and show that V is open in p(A).

Suppose first that A is open. Since $q^{-1}(V)$ is open in A and A is open in X, the set $q^{-1}(V)$ is open in X. Since $q^{-1}(V) = p^{-1}(V)$, the latter set is open in X, so that V is open in Y because p is a quotient map. In particular, V is open in p(A).

Now suppose p is open. Since $q^{-1}(V) = p^{-1}(V)$ and $q^{-1}(V)$ is open in A, we have $p^{-1}(V) = U \cap A$ for some set U open in X.

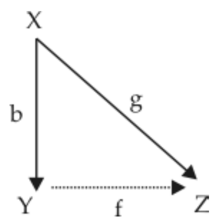
Now $p(p^{-1}(V) = V$ because p is surjective, then

$$V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$$

The set p(U) is open in Y because p is an open map; hence V is open in p(A).

Step (3): The proof when A or p is closed is obtained by replacing the word 'open' by the word 'closed' throughout step 2.

Theorem 2: Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.



Notes

Proof: For each y ∈ Y, the set $g(p^{-1}(\{y\})$ is a one-point set in Z (since g is constant on $p^{-1}(\{y\})$. If we let f(y) denote this point, then we have defined a map f : Y → Z such that for each x ∈ X, f(p(x)) = g(x). If f is continuous, then g = f o p is continuous. Conversely, suppose g is continuous. Given an open set V of $\overline{Z}$, $g^{-1}(V)$ is open in X. But $g^{-1}(V) = p^{-1}(F^{-1}(V))$; because p is a quotient map, it follows that $f^{-1}(V)$ is open in Y. Hence f is continuous. If f is a quotient map, then g is the composite of two quotient maps and is thus a quotient map. Conversely, suppose that g is a quotient map. Since g is subjective, so is f.

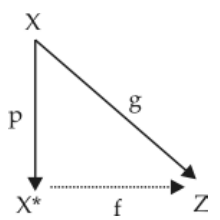
Let V be a subset of Z; we show that U is open in Z if $f^{-1}(V)$ is open in Y. Now the set $p^{-1}(f^{-1}(V))$ is open in X because p is continuous. Since this set equals $g^{-1}(V)$, the latter is open in X. Then because g is a quotient map, V is open in Z.

Corollary (1): Let g : X → Z be a surjective continuous map. Let X* be the following collection of subsets of X:

$$X^* = \{g^{-1}(\{z\}) \mid z \in Z\}$$

Give X* the quotient topology.

(a) The map g induces a bijective continuous map f : X* → Z, which is a homeomorphism if and only if g is a quotient map.



(b) If Z is Hausdorff, so is X*.

Proof: By the preceding theorem, g induces a continuous map f : X* → Z; it is clear that f is bijective. Suppose that f is a homeomorphism. Then both f and the projection map p : X → X* are quotient map. So that their composite q is a quotient map. Conversely, suppose that g is a quotient map. Then it follows from the preceding theorem that f is a quotient map. Being bijective, f is thus a homeomorphism.

Suppose Z is Hausdorff. Given distinct points of X*, their images under f are distinct and thus possess disjoint neighbourhoods U and V. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighbourhoods of the two given points of X*.

10.2 Summary

- Let X and Y be topological spaces; let p : X → Y be a surjective map. The map p is said to be a quotient map provided a subset U of Y is open in y if and only if $p^{-1}(U)$ is open in X.
- A map f : X → Y is said to be an open map if for each open set U of X, the set f(U) is open in Y.
- A map f : X → Y is said to be a closed map if for each closed set A of X, the set f(A) is closed in Y.
- If X is a space and A is a set and if p : X → A is a surjective map, then there exists exactly one topology T on A relative to which p is a quotient map; it is called the quotient topology induced by p.

- Let X be a topological space and let X* be a partition of X into disjoint subsets whose union is X. Let p : X → X* be the surjective map that carries each point of X to the element of X* containing it. In the quotient topology induced by p, the space X* is called a quotient space of X.

10.3 Keywords

Equivalence relation: A relation R in set A is an equivalence relation iff it is reflexive, symmetric and transitive.

Homeomorphism: A map f : (X, T) → (Y, U) is said to be homeomorphism if (i) f is one-one onto (ii) f and f^{-1} are continuous.

10.4 Review Questions

1. Prove that the product of two quotient maps needs not be a quotient map.
2. Let p : X → Y be a continuous map. Show that if there is a continuous map f : Y → X such that p o f equals the identify map of Y, then p is a quotient map.
3. Show that a subset G of Y is open in the quotient topology (relative to f : X → Y) iff $f^{-1}(G)$ is an open subset of X.
4. Show that if f is a continuous, open mapping of the topological space X onto the topological space Y, then the topology for Y must be the quotient topology.
5. Show that Y, with the quotient topology, is a T_1-space iff $f^{-1}(y)$ is closed in X for every $y \in Y$.
6. Show that if X is a countably compact T_1-space, then Y is countably compact with the quotient topology.
7. Show that if f is a continuous, closed mapping of X onto Y, then the topology for Y must be the quotient topology.
8. Show that a subset F of Y is closed in the quotient topology (relative to f : X → Y) iff $f^{-1}(F)$ is a closed subset of X.

10.5 Further Readings

Books

J.L. Kelley, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

Notes

Unit 11: Connected Spaces, Connected Subspaces of Real Line

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define connected spaces;
- Solve the questions on connected spaces;
- Understand the theorems and problems on connected subspaces of the real line.

Introduction

The definition of connectedness for a topological space is a quite natural one. One says that a space can be "separated" if it can be broken up into two "globs" – disjoint open sets. Otherwise, one says that it is connected. Connectedness is obviously a topological property, since it is formulated entirely in terms of the collection of open sets of X. Said differently, if X is connected, so is any space homeomorphic to X.

Now how to construct new connected spaces out of given ones. But where can we find some connected spaces to start with? The best place to begin is the real line. We shall prove that R is connected, and so are the intervals.

11.1 Connected Spaces

Definition: A topological space X is said to be disconnected iff there exists two non-empty separated sets A and B such that $E = A \cup B$.

In this case, we say that A and B form a partition or separation of E and we write, $E = A | B$.

A topological space X is said to be connected if it cannot be written as the union of two disjoint non-empty open sets.

A subspace Y of a topological space X is said to be connected if it is connected as a topological space it its own right.

Notes

Note A set is said to be connected iff it has no separation

Example 1:

(1) Let X be an indiscrete topological space. Then X is connected since the indiscrete topology consists of the empty set ϕ and the whole space X only.

(2) Let X be a discrete topological space with at least two elements. Then X is disconnected since if A is any non-empty proper subset of X, then A and A^c are disjoint non-empty open subsets of X such that $X = A \cup A^c$.

(3) ϕ is connected. Since ϕ cannot be expressed as the union of two non-empty separated sets. So ϕ has no separation and is therefore connected.

Theorem 1: In a topological space X the following statements are equivalent:

(i) X is connected;

(ii) The empty set ϕ and the whose space X are the only subsets of X that are both open and closed in X i.e. X has no non-trivial subset that is both open and closed in X;

(iii) X cannot be represented as the union of two non-empty disjoint closed sets.

(iv) X cannot be represented as the union of two non-empty separated sets.

Proof: We shall prove the theorem by showing that

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii)

Let X be connected.

Suppose A is a non-trivial subset of X that is simultaneously open and closed in X. Then $B = A^c$ is non-empty, open and $X = A \cup B$, $A \cap B = \phi$

This is contrary to the given hypothesis that X is connected and accordingly (ii) must be true

(ii) $\Rightarrow$ (iii)

Let (ii) be true.

Suppose X = A U B, where A and B are two disjoint non-empty closed sets.

Then $A = B^c$ is a non-trivial subset of X that is open as well as closed in X. This contradicts the given hypothesis (ii) and thus (ii) must be true.

(iii) $\Rightarrow$ (iv)

Let (iii) be true.

Suppose $X = A \cup B$

where $A \neq \phi$, $B \neq \phi$, $A \cap \overline{B} = \phi = \overline{A} \cap B$.

Then clearly $X = \overline{A} \cup \overline{B}$

where $\overline{A}$ and $\overline{B}$ are non-empty closed sets.

Also $A \cap \overline{B} = \phi$

Notes

$\Rightarrow \overline{B} \subseteq A^C$

$\overline{A} \cap B = \phi \Rightarrow \overline{A} \subseteq B^C$

$\Rightarrow \overline{A} \cap \overline{B} \subseteq B^C \cap A^C = (B \cup A)^C = X^C = \phi$

i.e., $\overline{A} \cap \overline{B} = \phi$

Thus X can be represented as the union of two disjoint non-empty closed sets.

This contradicts the given hypothesis (iii) and thus (iv) must be true.

(iv) $\Rightarrow$ (i)

Let (iv) be true

Suppose that X is disconnected.

Then there exist disjoint non-empty open sets G and H such that $X = G \cup H$.

Since G and H are open and $G \cap H = \phi$, is follows $G \cap \overline{H} = \phi$ and $\overline{G} \cap H = \phi$.

This contradicts the given hypothesis (iv) and thus (i) must be true.

Hence the proof of the theorem.

Theorem 2: The closure of a connected set is connected

OR

If A is connected subset then show that A is also connected.

Proof: Let (X, T) be a topological space and A be a subset of X.

If A is connected, then we have to show that $\overline{A}$ is also connected.

If $\overline{A}$ is not connected then it has a separation.

Let $\overline{A} = G \mid H$

So by theorem, Let (X, T) be a topological space and let E be a connected subset of (X, T). If E has a separation $X = A \mid B$, then either $E \subseteq A$ or $E \subseteq B$, we have

$$\overline{A} \subseteq G \text{ or } \overline{A} \subseteq H$$

If $\overline{A} \subseteq G$

$\Rightarrow \overline{\overline{A}} \subseteq \overline{G}$

$\Rightarrow \overline{A} \subseteq \overline{G}$

$\Rightarrow \overline{A} \cap H \subseteq \overline{G} \cap H$

$\Rightarrow \overline{A} \cap H = \phi$ (∵ G and H are separated.) ...(1)

Also $\overline{A} = G \cup H$ ...(2)

$\Rightarrow H \subset \overline{A}$

Notes

Now from (1) and (2), we get

H = ϕ,

which contradicts the given fact that H is non-empty.

Hence $\overline{A}$ is also a connected set.

Theorem 3: If every two points of a set E are contained in same connected subset of E, then E is connected.

Proof: Let us suppose that E is not connected.

Then, it must a separation E = A | B

i.e. E is the union of non-empty separated sets A and B.

Since A and B are non-empty, let a $\in$ A and b $\in$ B.

Then, A and B being disjoint

$\Rightarrow$ a, b are two distinct points of E.

So, by given hypothesis there exists a connected subset C of E such that a, b $\in$ c

But, C being a connected subset of a disconnected set E with the separation E = A | B,

we have C $\subseteq$ A or C $\subseteq$ B.

This is not possible, since A and B are disjoint and C contains at least one point of A and one that of B, which leads to a contradiction.

Hence E is connected.

Theorem 4: A topological space (X, T) is connected iff the only non-empty subset of X which is open and closed is X itself.

Proof: Let (X, T) be a connected space.

Let A be a non-empty subset of X that is both open and closed. Then A^c is both open and closed.

$\therefore \quad \overline{A} = A$ and $\overline{A}^C = A^C$

Thus $A \cap A^C = \phi$

$\Rightarrow \overline{A} \cap A^C = \phi$ and $A \cap \overline{A}^C = \phi$

Also $X = A \cup A^C$

Therefore A and A^C are two separated sets whose union in X.

Now if $A \neq \phi$ and $A^C \neq \phi$, then we have separation $X = A \mid A^C$, which leads to the contradiction as X is connected.

So either $A = \phi$ or $A^C = \phi$

But $A = \phi$ or $A^C = \phi$

But $A \neq \phi$

So $A^C = \phi$

$\therefore \quad X = A \cup A^C = A \cup \phi = A$

This shows that the only non-empty subset of X that is both open and closed is X itself. Conversely, let the only subset of X which is both open and closed be X itself.

Then, there exists no non-empty proper subset of X which is both open and closed.

Hence (X, T) is not disconnected and therefore, it is connected.

Notes

Theorem 5: A continuous image of connected space is connected.

Proof: Let $f : X \rightarrow Y$ be a continuous mapping of a connected space X into an arbitrary topological space Y.

We shall show that f[X] is connected as a subspace of Y.

Let us suppose f[X] is disconnected.

Then there exists G and H both open in Y such that

$G \cap f[X] \neq \phi$, $H \cap f[X] \neq \phi$

$(G \cap f[X]) \cap (H \cap f[X]) = \phi$

and $(G \cap f[X]) \cup (H \cap f[X]) = f[X]$

It follows that

$$
\begin{aligned}
\phi &= f^{-1}[\phi] \\
&= f^{-1}[(G \cap f[X]) \cap (H \cap f[X])] \\
&= f^{-1}((G \cap H) \cap f[X]) \\
&= f^{-1}[G] \cap f^{-1}[H] \cap f^{-1}(f[X]) \\
&= f^{-1}[G] \cap f^{-1}[H] \cap X \\
&= f^{-1}[G] \cap f^{-1}[H]
\end{aligned}
$$

and

$$
\begin{aligned}
X &= f^{-1}(f[X]) \\
&= f^{-1}[(G \cap f[X]) \cup (H \cap f[X])] \\
&= f^{-1}[(G \cup H) \cap f[X])] \\
&= f^{-1}[G \cup H] \cap f^{-1}(f[X]) \\
&= f^{-1}[G] \cup f^{-1}[H] \cap X \\
&= f^{-1}[G] \cup f^{-1}[H]
\end{aligned}
$$

Since f is continuous and G and H are open in Y both intersecting f[X].

If follows that $f^{-1}[G]$ and $f^{-1}[H]$ are both non-empty open subsets of X.

Thus X has been expressed as union of two disjoint open subsets of X and consequently X is disconnected, which is a contradiction.

Hence f[X] must be connected.

Example 2: Show that (X, T) is connected space if X = {a, b, c, d} and T = {X,ϕ, {a, b}}.

Solution: T-open sets are X, ϕ, {a, b}.

T-closed sets are ϕ, X, {c, d}

For X-{a, b} = {c, d}

Thus $\exists$ non-proper subset of X which is both open and closed. Consequently (X, T) is not disconnected. It follows that (X, T) is connected.

Notes

Example 3: Show that every indiscrete space is connected.

Solution: Let (X, T) be an indiscrete space so that T = {ϕ, X}. Then T-open sets are ϕ, X. T-closed sets are X, ϕ. Hence the only non-empty subset of X which is both open and closed is X.

∴ X is connected, by theorem (4).

Self Assessment

1. Prove that the closure of connected set is connected.
2. Prove that a continuous image of a connected space is a connected set.
3. Prove that connectedness is preserved under continuous map.

11.2 Connected Subspaces of Real Line

Theorem 6: The set of real numbers with the usual metric is a connected space.

Proof: Let if possible (R, U) be a disconnected space. Then there most exist non-empty closed subsets A and B of R such that

$$A \cup B = R \text{ and } A \cap B = \phi$$

Since A and B are non-empty, $\exists$ a point $a \in A$ and $b \in B$

Since $A \cap B = \phi$

∴ $a \neq b$

Thus $a < b$ or $a > b$

Let $a < b$

We have $[a, b] \subseteq p$

$\Rightarrow [a, b] \subseteq A \cup B$

Thus $x \in [a, b] \Rightarrow x \in A$ or $x \in B$

Let $p = \sup([a, b] \cap A)$

Then $a \leq p \leq b$

Since A is closed, $p \in A$

Again $A \cap B = \phi$ and $p \in B$

$\Rightarrow p < b$

Also by definition of p

$p + \varepsilon \in B \ \forall \in > 0$

∴ $p + \varepsilon \leq b$

Again since B is closed, $p \in B$.

Thus, we get

$p \in A$ and $p \in B \Rightarrow p \in A \cap B$

But $A \cap B = \phi$

Thus we get a contradiction. Hence R is connected.

Notes

Theorem 7: A subspace of the real line R is connected iff it is an interval. In particular, R is connected.

Proof: Let E be a subspace of R.

We first prove that if E is connected, then it is an interval. Let us suppose that E is not an interval. Then there exists real numbers a, b, c with $a < c < b$ such that $a, b \in E$ but $c \notin E$.

Let $A =]-\infty, C[$ and $B=]c, \infty[$.

Then A and B are open subset of R such that $a \in A$ and $b \in B$.

Now, $E \cap A \neq \phi$ and $E \cap B \neq \phi$, since $a \in E \cap A$ and $b \in E \cap B$.

Also, $(E \cap A) \cap (E \cap B) = E \cap (A \cap B) = \phi$ $(\because A \cap B = \phi)$

and $(E \cap A) \cup (E \cap B) = E \cap (A \cup B) = E \cap R - \{c\} = E$

Thus, $A \cup B$ forms a disconnection of E i.e., E is disconnected, a contradiction.

Hence E must be an interval.

Conversely, Let E be an interval and if possible let E is disconnected.

Then E is the union of two non-empty disjoint sets G and H, both closed in E, i.e. $E = G \cup H$.

Let $a \in G$ and $b \in H$

Since $G \cap H = \phi$, we have $a \neq b$

So either $a < b$ or $b < a$

Without any loss of generality we may assume that $a < b$.

Since $a, b \in E$ and E is an interval, we have $[a, b] \subset E = G \cup H$.

Let $p = \sup\{G \cap [a, b]\}$, then clearly $a \leq p \leq b$

Consequently, $p \in E$.

But, G being closed in E, the definition of p shows that $p \in G$ and therefore, $p \neq b$.

Consequently, $p < b$

Moreover, the definition of p shows that $p + \varepsilon \in H$ for each $\varepsilon > 0$ for which $p + \varepsilon \leq b$.

This shows that every nhd. of p contains at least one point of H, other than p. So, p is a limit point of H. But H being closed, we have $p \in H$.

Thus, $p \in G \cap H$ and therefore $G \cap H \neq \phi$, which is a contradiction.

Hence E must be connected

Theorem 8: Prove that the real line is connected.

Proof: Let R be an interval and if possible let R is disconnected. Then R is the union of two non-empty disjoint sets G and H, both closed in R, i.e. $R = G \cup H$.

Let $a \in G$ and $b \in H$.

Since $G \cap H = \phi$, we have $a \neq b$

So either $a < b$ or $b < a$

Without any loss of generality, we may assume that $a < b$.

Since $a, b \in R$ and R is an interval, we have $[a, b] \subset R = G \cup H$

Notes

Let p = sup{G ∩ [a, b]}, then clearly $a \leq p \leq b$

Consequently $p \in R$

But G being closed in R, the definition of p shows that $p \in G$ and therefore $p \neq b$.

Consequently, $p < b$

Moreover the definition of p shows that $p + \varepsilon \in H$ for each $\in > 0$ for which $p + \in \leq b$.

This shows that every nhd. of p contains at least one point of H other than p. So p is a limit point of H.

But H being closed, we have $p \in H$

Thus $p \in G \cap H$ and therefore $G \cap H \neq \phi$, which is a contradiction.

Hence R must be connected.

Example 4: Show that if X is a connected topological space and f is a non-constant continuous real function defined on X then X is uncountably infinite.

Solution: $f : X \to \mathcal{R}$ is continuous and X is connected, so f (X) is a connected subspace of $\mathcal{R}$.

Suppose that f (X) is not connected, there exists a non-empty proper subset E of f (X) such that E is both open and closed in f (X).

As f is continuous

$\Rightarrow f^{-1}$ (E) is non-empty proper subset of X which is both open and closed in X.

This contradicts the fact that X is connected. Hence f (X) must be a connected subspace of $\mathcal{R}$.

Also f is non-constant, there exist $x, y \in X$ such that $f(x) \neq f(y)$

Let a = f (x) and b = f (y).

Without any loss of generality we may suppose that $a < b$. Now $a, b \in f(X)$, f (x) is a connected subspace of $\mathcal{R}$

$\Rightarrow [a, b] \subseteq f(X)$.

[∵ a subspace E of real line $\mathcal{R}$ is connected iff E is an interval
i.e. if $a, b \in E$ and $a < c < b$ then $c \in E$. In particular $\mathcal{R}$ is connected.]

Since [a, b] is uncountably infinite, it follows that f (X) is uncountably infinite and consequently X must be uncountably infinite.

Example 5: Show that the graph of a continuous real function defined on an interval is a connected subspace of the Euclidean plane.

Solution: Let $f : 1 \to \mathcal{R}$ be continuous and let G be the graph of f.

Then $G = I \times f(I) \subseteq \mathcal{R}^2$.

Now since I is connected by the theorem "A subspace E of the real line $\mathcal{R}$ is connected iff E is an interval."

Also, f is continuous, it follows that f (I) is a connected subspace of R since continuous image of a connected space is connected. Also we know that connectedness is a product invariant property, hence G is connected.

Notes

Example 6: The spaces $\mathcal{R}^n$ and C^n are connected.

Solution: We know that $\mathcal{R}^n$ is a topological space can be regarded as the product of n replicas of the real line $\mathcal{R}$. But $\mathcal{R}$ is connected therefore $\mathcal{R}^n$ is connected since the product of any non-empty class of connected spaces is connected.

We next prove that C^n and $\mathcal{R}^{2n}$ are essentially the same as topological spaces by taking a homomorphism f of C^n onto $\mathcal{R}^{2n}$.

Let $z = (z_1, z_2, \ldots, z_n)$ be an arbitrary element in C^n.

Let us suppose that each coordinate z_K is of the form

$$z_K = a_K + ib_K$$

where a_K and b_K are its real and imaginary parts.

Let us define f by

$$f(z) = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n).$$

f is clearly a one-to-one mapping of C^n onto $\mathcal{R}^{2n}$ and if we observe that $||f(z)|| = ||z||$, then f is a homeomorphism which shows that $\mathcal{R}^{2n}$ is connected. Hence C^n is also connected.

Self Assessment

4. Show that if f is continuous map of a connected space X into R, then f(X) is an interval.
5. Show that a subset $\mathcal{A}$ of the real line that contains at least two distinct points is connected if and only if it is an interval.

11.3 Summary

- A topological space X is said to be connected if it cannot be written as the union of two disjoint non-empty open sets.
- The closure of a connected set is connected.
- If every two points of a set E are contained in some connected subset of E, then E is connected.
- A continuous image of connected space is connected.
- The set of real numbers with the usual metric is a connected space.
- A subspace of the real line R is connected iff it is an interval. In particular, R is connected.

11.4 Keyword

Separated set: Let A, B be subsets of a topological space (X, T). Then the set A and B are said to be separated iff

(i) $A \neq \phi, B \neq \phi$

(ii) $A \cap \overline{B} = \phi, \overline{A} \cap B = \phi$

11.5 Review Questions

1. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \phi$ for all n. Show that UA_n is connected.
2. Let p : X → Y be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected and if Y is connected, then X is connected.
3. Let Y ⊂ X ; let X and Y be connected. Show that if A and B form a separation of X – Y, then Y ∪ A and Y ∪ B are connected.
4. Let (X, T) be a topological space and let E be a connected subset of (X, T). If E has a separation X = A | B, then either E ⊆ A or E ⊆ B.
5. Prove that if a connected space has a non-constant continuous real map defined on it, then it is uncountably infinite.
6. Show that a set is connected iff A is not the union of two separated sets.
7. Let f : S′ → R be a continuous map. Show there exists a point x of S′ such that f(x) = f(–x).
8. Prove that connectedness is a topological property.
9. Prove that the space R^n and C^n are connected.

11.6 Further Readings

Books

William W. Fairchild, Cassius Ionescu Tulcea, *Topology*, W.B. Saunders Company.

B. Mendelson, *Introduction to Topology*, Dover Publication.

Online links

www.mathsforum.org

www.history.mcs.st/andrews.ac.uk/HistTopics/topology/in/mathematics.htm

Notes

Unit 12: Components and Local Connectedness

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the term components of a topological space;
- Solve the problems on components of a topological space;
- Define locally connectedness;
- Solve the problems on locally connectedness.

Introduction

Given an arbitrary space X, there is a natural way to break it up into piece that are connected. We consider that process now. Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the components or the "connected components" of X.

Connectedness is a useful property for a space to possess. But for some purposes, it is more important that the space satisfy a connectedness condition locally. Roughly speaking, local connectedness means that each point has "arbitrary small" neighbourhoods that are connected. So, in this unit, we shall deal with two important topics components and local connectedness.

12.1 Components of a Topological Space

Definition: A subset E of a topological space X is said to be a component of X if

1. E is a connected set and
2. E is not a proper subset of any connected subspace of X i.e. if E is a maximal connected subspace of X.

Notes

Notes

(i) Let (X, J) be a topological space and E be a subset of X. If $x \in E$, then the union of all connected sets containing x and contained in E, is called component of E with respect to x and is denoted by C (E, x).

(ii) Since the union of any family of connected sets having a non-empty intersection is a connected set, therefore the component of E with respect of x i.e. C (E, x) is a connected set.

(iii) If E is a component of X, the $E \neq \phi$.

Example 1:

(i) If X is a connected topological space, then X has only one component, namely X itself.

(ii) If X is a discrete topological space, then each singleton subset of X is its component.

Theorem 1: In a topological space (X, T) each point in X is contained in exactly one component of X.

Proof: Let x be any point of X

Let $A_x = \{A_i\}$ be the class of all connected subspaces of X which contains x

$$A_x \neq \phi \text{ as } \{x\} \in A_x$$

Also (i) $A_i \neq \phi$ since $x \in \bigcap_i A_i$

Therefore by theorem, Let X be a topological space and $\{A_i\}$ be a non-empty class of connected subspaces of X such that $\bigcap_i A_i \neq \phi$ then $A = U_i\ A_i$ is connected subspace of $X, \bigcup_i A_i = C_x$ (say) is connected subspace of X.

Further, $x \in C_x$ and if B is any connected subspace of X containing x, then $B \in A_x$ and so $B \subseteq C_x$.

Therefore C_x is a maximal connected subspace i.e. a component of X containing x.

Now we shall prove that C_x is the only component which contains x.

Let C_x^* be any other component of X which contain x. The C_x^* is one of the A_i's and is therefore contained in C_x. But C_x^* is maximal as a connected sub-space of X, therefore we must have $C_x^* = C_x$ i.e. C_x is unique in the sense that each point $x \in X$ is contained in exactly one component C_x of X.

Theorem 2: In a topological space each components is closed.

Proof: Let (X, T) be a topological space and let C be a component of X.

By the definition of component, C is the largest connected set containing x. Then, $\bar{C}$ is also a connected set containing x.

Thus $\bar{C} \subset C$

Also $C \subseteq \bar{C}$

Therefore $C = \bar{C}$

Hence C is closed.

Notes

Theorem 3: In a topological space X each connected sub space of X is contained in a component of X.

Proof: Let E be any connected subspace of X.

If $E = \phi$, then E is contained in every component of X.

Let $E \neq \phi$, and let $x \in E$

Then $x \in X$

Let E_x be the union of all connected subsets of X containing x. Then, E_x is a component of X containing x.

Now, E is a connected set containing x and E_x is the largest connected set containing x. So $E \subseteq E_x$.

Theorem 4: In a topological space (X, T), a connected subspace of X which is both open and closed, in a component of X.

Proof: Let G be a connected subspace of X which is both open and closed.

If G = ϕ, then G is contained in every component.

If G ≠ ϕ, then G contains a point $x_1 \in X$ and so

$$G \subset C\,(X, x_i) \;=\; C$$

We shall show that G = C

In order to show that G = C, let us assume that G is a proper subset of C, so that

$G \cap C \neq \phi$ and $G' \cap C \neq \phi$ where $G' = X - C$.

Since G is both open and closed, G' is also both open and closed.

Also $$(G \cap C) \cap (G' \cap C) = (G \cap G') \cap$$

$$= \phi \cap C = \phi$$

and $$(G \cap C) \cup (G' \cap C) = (G \cup G') \cap C = X \cap C = X$$

which shows that C is disconnected, which is a contradiction of the given fact that C is connected

Hence G = C.

Theorem 5: The product of any non-empty class of connected topological spaces is connected i.e. connectedness is a product invariant property.

Proof: Let {x_i} be a non-empty connected topological spaces and $X = \Pi_i\, X_i$ be the product space.

Let $a =< a_i > \in X$ and E be a component of a.

We claim that $X \subseteq \overline{E} = E$ $\quad$ ($\because$ E is closed)

Let $x = < x_i >$ be any point of X and let $G = \Pi\{X_i : i \neq i_1, ...i_m\} X G_1\ X....X G_i$

be any basic open set containing x.

Now $H = \Pi\{\{a_i\}; i \neq i_1, i_2, ..., i_m\} \times X_{i_1} \times X_{i_a} \times .... \times X_{i_m}$ is homeomorphic to $X_{i_1} \times X_2 \times ...X_{i_m}$ and is therefore connected

($\because$ connectedness is a topological property)

Further $a \in H$, H connected and E a component of a implies that H is a subset of E. But $G \cap H \neq \phi$, so that G contains a point of H and hence of E.

Thus we have shown that every basic nhd of x contains a point of E.

Consequently every nhd of x will contain a point of E and therefore $x \in \overline{E}$.

Thus $x \in X \Rightarrow x \in \overline{E} = E$, so that $X \subseteq E$ But $E \subseteq X$.

Hence X = E and is therefore connected.

Theorem 6: The component of a topological space X form a position of X i.e. any two components of X are either disjoint or identical and the union of all the components is X.

Proof: For each $x \in X$, let C (X, x) the union of all connected sets containing x.

Then C (X, x) is a component of X.

Clearly, the family $\{C_x : x \in X\}$ consists of all components of X and $X = \cup \{C_x : x \in X\}$. Now let $C (X, x_1)$ and $C (X, x_2)$ be the components of X with respect of x_1 and x_2 respectively, $x_1 \neq x_2$

If $C(X, x_1) \cap C (X, x_2) = \phi$, we are done

so, let $C(X, x_1) \cap C (X, x_2) = \phi$

Let $x \in C (X, x_1) \cap C (X, x_2)$

them $x \in C (X, x_1)$ and $x \in C (X, x_2)$

Now $C (X, x_1)$ and $C (X, x_2)$ are connected sets containing x and C (X, x) is a component containing x, therefore

$$C (X, x_1) \subseteq C (X, x)$$

and $$C (X, x_2) \subseteq (X, x)$$

But $C (X, x_1)$ and $C (X, x_2)$ being components, they cannot be contained in a larger connected subset of X.

Therefore $C (X, x_1) = C (X, x_2) = C (X, x)$

Thus, any two components of X one either disjoint or identical.

Hence, the components of X form a partition of X.

Self Assessment

1. Prove that the components of E corresponding to different points of E are either equal or disjoint.

12.2 Local Connectedness

12.2.1 Locally Connected Spaces

A topological space X is said to locally connected at a point $x \in X$ if every nhd. of x contains a connected nhd. of x i.e. if N is any open set containing x then there exists a connected open set G containing x such that $G \subseteq N$

or

A topological space (X, T) is said to be locally connected iff for every point $x \in X$ and every nhd. G of x, there exists a connected nhd. H such that $x \in H \subset G$. Thus the space (X, T) is locally connected iff the family of all open connected sets is a base for T.

Notes

Example 2: Each interval and each ray in the real line in both connected and locally connected. The subspace $[-1, 0) \cup (0, 1]$ of $\mathbb{R}$ is not connected, but it is locally connected.

12.2.2 Locally Connected Subset

Let (X, T) be a topological space and let (Y, T_y) be a sub-space of (X, T)

The subset $Y \subseteq X$ is said to be locally connected if (Y, T_y) is a locally connected space.

12.2.3 Theorems and Solved Examples

Theorem 7: Every discrete space is locally connected.

Solution: Let x be an arbitrary point of a discrete space X. We know that every subset of a discrete space is open and that every singleton set is connected. Hence {x} is a connected open nhd. of x. Also every open nhd. of x must contain {x}.

Hence X is locally connected.

Example 3: Give two examples of locally connected space which are not connected.

Or

Is locally connected space always connected? Justify.

Solution:

1. Let X be a discrete space containing more than one point.

   Let $x \in X$. Then {x} is an open connected set and is obtained in every open set containing x. So, X is locally connected at each point of X. Also, every singleton subset of X is a non-empty proper subset of X which is both open and closed. So X is disconnected.

2. Consider the usually topological space (R, U)

   Let $A \subset R$, which is the union of two disjoint open intervals.

   Then A is not a interval and therefore it is not connected.

   To show that A is locally connected.

   Let x be an arbitrary point of A and G_x be a set open in A such that $x \in G_x$. Then there exists an open interval I_x such that $x \in I_x \subseteq G_x$. But I_x being an interval, it is connected in R and therefore in A.

   Thus every open nhd. of x in A contains a connected open nhd. of x in A.

   Hence A is locally connected.

Example 4: Give example of a space which is connected but not locally connected.

Solution: Consider the subspace $A \cup B$ of the Euclidean Plane R^2, where

$$A = \{(0, y) : -1 \le y \le 1\}$$

and
$$B = \left\{(x, y) : y = \sin\left(\frac{1}{x}\right), 0 < x \le 1\right\}$$

The $A \cap B = \phi$ and each point of A is a limit point of B and so A and B are not separated. Consequently, $A \cup B$ is connected.

But $A \cup B$ is not locally connected at (0, 1), since the open disc with centre (0, 1) and radius $\left(\frac{1}{4}\right)$ does not contain any connected open subset of R^2 containing (0, 1).

Hence $A \cup B$ is connected but not locally connected.

Theorem 8: Every component of a locally connected space is open.

Proof: Let (X, T) be a locally connected space and E be a component of X.

We shall show that E is an open set.

Let x be any element of E.

Since X is locally connected, there exists a connected space set G_x which contains x. Since E is a component, we have $x \in G_x \subset E$ clearly, $E = \cup \{G_x : x \in E\}$.

Therefore E, being a union of open sets, is an open set.

Theorem 9: A topological space X is locally connected iff the components of every open subspace of X are open in X.

Proof: Let X be locally connected and Y be an open subspace of X.

Let E be a component of Y.

We are to show that E is open in X i.e. if x is any element of E then there exists a nhd. G of x such that $G \subseteq E$.

Now $E \subseteq$, Y, Y open in X, $x \in Y$ and X is locally connected implies that there exists a connected open set G containing x such that $G \subseteq Y$.

Since the topology which G has as a subspace of Y is the same as that it has as a subspace of X, therefore G is also connected as a subspace of Y and consequently $G \subseteq E$ as E is a component of Y.

Conversely, let the components of every open subspace of X be open in X, Let $x \in X$ and Y an open subset of X containing x. Let E_x be a component of Y containing x. Then by hypothesis, E_x is open and connected in Y and therefore in X.

Example 5: Give an example of locally connected space which is totally disconnected.

Solution: Every discrete space is locally connected as well as totally disconnected.

Let x be an arbitrary point of a discrete space X.

We know that every subset of a discrete space is open and that every singleton set is connected.

Hence {x} is a connected open nhd. of x. Also every open nhd. of x must contain {x}.

Hence X is locally connected.

To prove X is totally disconnected.

Let x, y be any two distinct points of a discrete space X.

The $G = \{x\}$ and $H = X - \{x\}$ are both non-empty open disjoint sets whose union is X such that $x \in G$ and $y \in H$. It follows that X is totally disconnected.

Theorem 10: Local connectedness neither implies nor is implied by connectedness.

Proof: The union of two disjoint open intervals on the real line forms a space which is locally connected but not connected. Example of a space which is connected but not locally connected.

Let X be the subspace of Euclidean plane defined by

$$X = A \cup B \text{ where}$$

Notes

$A = \{(x, y) : x = 0, y \in [-1, 1]\}$

and $B = \{(x, y) : 0 \le x \le 1 \text{ and } y = \sin \frac{1}{x}\}$.

Since B is the image of (0, 1] under a continuous mapping f give by

$$f(x) = \left(x, \sin\frac{1}{x}\right)$$

So B is connected.

($\because$ Continuous image of a connected space is connected).

Since $X = \overline{B}$, therefore X is connected. But it is not locally connected because each point $x \in A$ has a nhd. which does not contain any connected nhd. of x.

Theorem 11: The image of a locally connected space under a mapping which is both open and continuous is locally connected. Hence locally connectedness is a topological property.

Proof: Let X be a locally connected space and Y be an arbitrary topological space.

Let $f : X \to Y$ be a map which is both open and continuous. Without any loss of generality we may assume that f is onto. We shall show that Y = f (X) is locally connected.

Let y = f (x), $x \in X$, be any point of Y and G be any nhd. of y. Since f is continuous.

$\Rightarrow f^{-1}(G)$ is open in X containing $f^{-1}(y) = x$.

Thus, $f^{-1}(G)$ is open, nhd. of x.

Now X being locally connected, these exists a connected open set H such that $x \in H \subseteq f^{-1}(G)$.

$\therefore y = f(x) \in f(H) \subseteq f[f^{-1}(G)] \subseteq G$,

where f (H) is open, since f is open.

Moreover, the continuous image of a connected set is connected, it follows that f (H) is connected.

This shows that f (X) is locally connected at each point.

Hence, f (X) is locally connected.

Self Assessment

2. Show that a connected subspace of a locally connected space has a finite number of components.

3 Show that the product X × Y of locally connected sets X and Y is locally connected.

12.3 Summary

- A subset of E of a topological space X is said to be a component of X if
  - (i) E is a connected set &
  - (ii) E is not a proper subset of any connected subspace of X i.e. if E is a maximal connected subspace of X.
- A topological space X is said to locally connected at a point $x \in X$ if every nhd of x contains a connected nhd. of x i.e. if N is any open set containing x then there exists a connected open set G containing x such that $G \subseteq N$.
- Let (X, T) be a topological space and let (Y, T_y) be a subspace of (X, T). The subset $y \subseteq X$ is said to be locally connected if (y, T_y) is a locally connected space.

12.4 Keywords

Notes

Connected: A topological space X is said to be connected if it cannot be written as the union of two disjoint non-empty open sets.

Discrete Space: Let X be any non empty set of T be the collection of all subsets of X. Then T is called the discrete topology on the set X. The topological space (X, T) is called a discrete space.

Open Set: Let (X, T) be a topological space. Any set $A \in T$ is called an open set.

Partition: A topological space X is said to be disconnected if there exists two non-empty separated sets A and B such that $E = A \cup B$. In this case, we say that A and B form a partition of E and we write $E = A/B$.

12.5 Review Questions

1. Let $p : X \to Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected.
2. A space X is said to be weakly locally connected at x if for every neighbourhood U of x, there is a connected subspace of X contained in U that contains a neighbourhood of x. Show that if X is weakly locally connected at each of its points, then X is locally connected.
3. Prove that a space X is locally connected if and only if for every open set U of X and each component of U is open in X.
4. Prove that the components of X are connected disjoint subspaces of X whose union is X, such that each non-empty connected subspace of X intersects only one of them.

12.6 Further Readings

Books

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

Notes

Unit 13: Compact Spaces and Compact Subspace of Real Line

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define open covering of a topological space;
- Understand the definition of a compact space;
- Solve the problems on compact spaces and compact subspace on real line.

Introduction

The notion of compactness is not nearly so natural as that of connectedness. From the beginning of topology, it was clear that the closed interval [a, b] of the real line had a certain property that was crucial for proving such theorems as the maximum value theorem and the uniform continuity theorem. But for a long time, it was not clear how this property should be formulated for an arbitrary topological space. It used to be thought that the crucial property of [a, b] was the fact that every infinite subset of [a, b] has a limit point, and this property was the one dignified with the name of compactness. Later, mathematicians realized that this formulation does not lie at the heart of the matter, but rather that a stranger formulation, in terms of open coverings of the space, is more central. The latter formulation is what we now call compactness. It is not as natural of intuitive as the former; some familiarity with it is needed before its usefulness becomes apparent.

13.1 Compact Spaces

Definition: A collection $\mathcal{A}$ of subsets of a space X is said to *cover X*, or to be a covering of X, if the union of the elements of A is equal to X. It is called an *open covering* of X if its elements are open subsets of X.

Definition: A space X is said to be *compact* if every open covering $\mathcal{A}$ of X contains a finite sub-collection that also covers X.

Notes

Example 1: The real line ℝ is not compact, for the covering of ℝ by open intervals

$$\mathcal{A} = \{(n, n+2) / n \in \mathbb{Z}\}$$

contains no finite sub-collection that covers ℝ.

Example 2: The following subspace of ℝ is compact

$$X = \{0\} \cup \{1/n \in \mathbb{Z}_+\}.$$

Given an open covering $\mathcal{A}$ of X, there is an element U of $\mathcal{A}$ containing O. The set U contains all but finitely many of the point 1/n; choose, for each point of X not in U, an element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$, along with the element U, is a finite sub-collection of $\mathcal{A}$ that covers X.

Lemma (i): Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite sub-collection covering Y.

Proof: Suppose that Y is compact and $\mathcal{A} = \{A_x\}_{\alpha \in T}$ is a covering of Y by sets open in X. Then the collection

$$\{A_x \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y; hence a finite sub-collection

$$\{A_{\alpha_1} \cap Y, ..., A_{\alpha_n} \cap Y\}$$

covers Y. Then $\{A_{\alpha_1}, ..., A_{\alpha_n}\}$ is a sub-collection of $\mathcal{A}$ that covers Y.

Conversely, suppose the given condition holds; we wish to prove Y compact. Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α, choose a set A_α open in X such that

$$A'_\alpha = A_\alpha \cap Y$$

The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X. By hypothesis, some finite sub-collection $\{A_{\alpha_1}, ..., A_{\alpha_n}\}$ covers Y. Then $\{A'_{\alpha_1}, ..., A'_{\alpha_n}\}$ is a sub-collection of $\mathcal{A}'$ that covers Y.

Theorem 1: Every closed subspace of a compact space is compact.

Proof: Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets open in X, let us form an open covering $\mathcal{B}$ of X by $\mathcal{A}$ joining to $\mathcal{A}$ the single open set X – Y that is

$$B = \mathcal{A} \cup \{X - Y\}$$

Some finite sub-collection of $\mathcal{B}$ covers X. If this sub-collection contains the set X – Y, discard X – Y; otherwise, leave the sub-collection alone. The resulting collection is a finite sub-collection of $\mathcal{A}$ that cover Y.

Theorem 2: Every compact subspace of a Hausdorff space is closed.

Proof: Let Y be a compact subspace of the Hausdorff space X. We shall prove that X – Y is open. So that Y is closed. Let x_0 be a point of X – Y. We show there is a neighborhood of x_0 that is disjoint from Y. For each point y of Y, let us choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively (using the Hausdorff condition). The collection $\{V_y / y \in Y\}$ is a covering of Y by sets in X; therefore, finitely many of them $V_{y1}, ..., V_{yn}$ cover Y. The open set

$$V = V_{y1} \cup .... \cup V_{yn}$$

Notes

contains Y, and it is disjoint from the open set

$$U = U_{y1} \cap ... \cap U_{yn}$$

formed by taking the intersection of the corresponding neighborhoods of x_0. For if z is a point of V, then $z \in V_{yi}$ for some i, hence $z \notin U_{yi}$ and so $z \notin U$. Then U is a neighbourhood of x_0, disjoint from Y, as desired.

Theorem 3: The image of a compact space under a continuous map is compact.

Proof: Let $f : X \to Y$ be continuous; let X be compact. Let $\mathcal{A}$ be a covering of the set f(X) by sets open in Y. The collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is a collection of sets covering X; these sets are open in X because f is continuous. Hence finitely many of them. Say

$$f^{-1}(A_1), ..., f^{-1}(A_n), \text{ cover X, then the sets } A_1 ...., A_n \text{ cover } f(X)$$

Note Use of the proceeding theorem is as a tool for verifying that a map is a homeomorphism

Theorem 4: Let $f : X \to Y$ be a bijective function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof: We shall prove that images of closed sets of X under f are closed in Y; this will prove continuity of the map f^{-1}. If A is closed in X, then A is compact by theorem (1). Therefore by the theorem just proved f(A) is compact. Since Y is Hausdorff, f(A) is closed in Y by theorem (2)

Example 3: Show by means of an example that a compact subset of a topological space need not be closed.

Solution: Suppose (X, I) is an indiscrete topological space such that X contains more than one element. Let A be a proper subset of X and let (A, I_1) be a subspace of (X, I). Here, we have $I_1 = \{\phi, A\}$. For $I = \{\phi, X\}$. Hence, the only I_1 – open cover of A is {A} which is finite. Hence A is compact. But A is not I-closed. For the only I-closed sets are ϕ, X. Thus A is compact but not closed.

Theorem 5: A closed subset of a countably compact space is countably compact.

Proof: Let Y be a closed subset of a countably compact space (X, T).

Let $\{G_n : n \in N\}$ be a countable T-open cover of Y, then

$$Y \subset \bigcup_n G_n.$$

But $\quad X = Y' \cup Y$

Hence $\quad X = Y' \cup \{G_n : n \in N\}$

This shows that the family consisting of open sets $Y', G_1, G_2, G_3, ...$ forms an open countable cover of X which is known to be countably compact. Hence this cover must be reducible to a finite subcover, say

$Y', G_1, G_2, ..., G_n$ so that $X = Y' \cup \left[\bigcup_{i=1}^{n} G_i\right]$

$\Rightarrow \quad Y \subset \bigcup_{i=1}^{n} G_i$

It means that $\{G_i : 1 \le i \le n\}$ is finite subcover of the countable cover

$$\{G_n : n \in N\}$$

Hence Y is countably compact

Self Assessment

1. Prove that a topological space is compact if every basic open cover has a finite sub-cover.
2. Show that every cofinite topological space (X, T) is compact.
3. Show that if (Y, T_1) is a compact subspace of a Hausdorff space (X, T), then Y is T-closed.

13.2 Compact Subspaces of the Real Line

The theorems of the preceding section enable is to construct new compact spaces from existing ones, but in order to get very far we have to find some compact spaces to start with. The natural place to begin is the real line.

Application include the extreme value theorem and the uniform continuity theorem of calculus, suitably generalised.

Theorem 6: *Extreme Value Theorem*

Let f: X → Y be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \le f(x) \le f(d)$ for every $x \in X$.

The extreme value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Proof: Since f is continuous and X is compact, the set A = f(X) is compact. We show that A has a largest element M and a smallest element m. Then since m and M belong to A, we must have m = f(c) and M = F(d) for some points c and d of X.

If A has no largest element, then the collection

$$\{(-\infty, a) \mid a \in A\}$$

forms an open covering of A. Since A is compact, some finite subcollection

$$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$

covers A. If a_i is the largest of the elements $a_1, \ldots, a_n$, then a_i belongs to none of these sets, contrary to the fact that they cover A.

A similar argument shows that A has a smallest element.

Definition: Let (X, d) be a metric space; let A be a non-empty subset of X. For each $x \in X$, we define the *distance from x to A* by the equation

$$d(x, A) = \inf \{d(x, a) \mid a \in A\}.$$

It is easy to show that for fixed A, the function d (x, A) is continuous function of x.

Given $x, y \in X$, one has the inequalities

$$d(x, A) \le d(x, a) \le d(x, y) + d(y, a),$$

for each $a \in A$. It follows that

$$d(x, A) - d(x, y) \le \inf d(y, a) = d(y, A),$$

so that

$$d(x, A) - d(y, A) \leq d(x, y).$$

The same inequality holds with x and y interchanged, continuity of the function d(x, A) follows.

Now we introduce the notion of Lebesgue number. Recall that the diameter of a bounded subset A of a metric space (X, d) is the number

$$\sup \{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

Lemma (1) (The Lebesgue number Lemma): Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it.

The number δ is called a Lebesgue number for the covering $\mathcal{A}$.

Proof: Let $\mathcal{A}$ be an open covering of X. If X itself is an element of $\mathcal{A}$, then any positive number is a Lebesgue number of $\mathcal{A}$. So assume X is not an element of $\mathcal{A}$.

Choose a finite subcollection $\{A_1, ..., A_n\}$ of $\mathcal{A}$ that covers X. For each i, set $C_i = X - A_i$, and define f: $X \to \mathbb{R}$ be letting f(x) be the average of the numbers $d(x, C_i)$. That is,

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} (x, c_i)$$

We show that f(x) > 0 for all x. Given $x \in X$, choose i so that $x \in A_i$. Then choose $\in$ so $\in$-neighborhood of x lies in A_i. Then $d(x, c_i) \geq \in$, so that $f(x) \geq \in/n$.

Since f is continuous, it has a minimum value δ_i we show that δ is our required Lebesgue number. Let B be a subset of X of diameter less that δ. Choose a point x_0 of B; then B lies in the δ-neighborhood of x_0. Now

$$\delta \leq f(x_0) \leq d(x_0, C_m),$$

where $d(x_0, C_m)$ is the largest of the number $d(x_0, C_i)$. Then the δ-neighborhood of x_0 is contained in the element $A_m - X - C_m$ of one covering $\mathcal{A}$.

Definition: *Uniformly Continuous*

A function F from the metric space (X, d_X) to the metric (Y, d_Y) is said to be uniformly continuous if given $\in > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_x(x_0, x_1) < d \Rightarrow d_Y(f(x_0), f(x_1)) < \in.$$

Theorem 7: *Uniform Continuity Theorem*

Let f: $X \to Y$ be a continuous map of the compact metric space (X, d_x) to be metric space (Y, d_y). Then f is uniformly continuous.

Proof: Given $\in > 0$, take the open covering of Y by balls B $(y, \in/2)$ of radius $\in/2$. Let A be the open covering of X by the inverse images of these balls under f. Choose δ to be a Lebesgue number for the covering A. Then if x_1 and x_2 are two points of X such that $dx(x_1, x_2) < \delta$, the two point set $\{x_1, x_2\}$ has diameter less than δ. So that its image $\{f(x_1), f(x_2)\}$ lies in some ball B $(y, \in/2)$. Then dy $(f(x_1), f(x_2) < \in$, as desired.

Finally, we prove that the real numbers are uncountable. The interesting thing about this proof is that it involves no algebra at all-no decimal or binary expansions of real numbers or the like-just the other properties of $\mathbb{R}$.

Theorem 8: Every closed and bounded interval on the real line is compact.

Proof: Let $I_1 = [a, b]$ be a closed and bounded interval on $\mathcal{R}$. If possible, let I_1 be not compact. Then there exists an open covering $\mathcal{C} = \{G_i\}$ of I_1, having no finite sub covering.

Let us write $$I_1 = [a, b] = \left[a, \frac{a+b}{2}\right] \cup \left[\frac{a+b}{2}, b\right] \quad \ldots(1)$$

Since I_1 is not covered by a finite sub-class of $\mathcal{C}$ and therefore at least one of the intervals of the union in (1) cannot be covered by any finite sub-class of $\mathcal{C}$.

Let us denote such an interval by $I_2 = [a_1, b_1]$.

Now writing $$I_2 = [a_1, b_1] = \left[a_1, \frac{a_1+b_1}{2}\right] \cup \left[\frac{a_1+b_1}{2}, b_1\right] \quad \ldots(2)$$

As argued before, at least one of the intervals in the union of (2) cannot be covered by a finite sub-class of $\mathcal{C}$.

Let us denote such an interval by $I_3 = [a_2, b_2]$.

On continuing this process we obtain a nested sequence $\langle I_n \rangle$ of closed intervals such that none of these intervals I_n can be covered by a finite sub-class of $\mathcal{C}$.

Clearly the length of the inverval.

$$I_n = \frac{a-b}{2^n}$$

Thus $\lim |I_n| = 0$.

Hence, by the nested closed interval property, $\cap I_n \neq \phi$.

Let $p \in \cap I_n$, then $p \in I_n \ \forall n \in N$.

In particular $p \in I_1$.

Now since $\mathcal{C}$ is an open covering of I_1, there exists some A_{α_0} in C such that $p \in A_{\alpha_0}$.

Since A_{α_0} is open there exists an open interval $(p - \varepsilon, p + \varepsilon)$ such that $p \in (p - \varepsilon, p + \varepsilon) \subseteq A_{\alpha_0}$.

Since $\ell(I_n) \to 0$ as $n \to \infty$, there exists some

$$I_{n_0} \subseteq (p - \varepsilon, p + \varepsilon) \subseteq A_{\alpha_0}.$$

This contradicts our assumption that no I_n is covered by a finite number of members of $\mathcal{C}$.

Hence $[a, b]$ is compact.

Example 4: The real line is not compact.

Solution: Let $\mathcal{C} = \{]-n, n[: n \in N\}$.

Then each member of $\mathcal{C}$ is clearly an open interval and therefore, a U-open set.

Also if p is any real number, then there exists a positive integer n_p such that $n_p > |p|$.

Then clearly $p \in]-n_p, n_p[\in \mathcal{C}$.

Thus each point of $\mathcal{R}$ is contained in some member of $\mathcal{C}$ and therefore $\mathcal{C}$ is an open covering of $\mathcal{R}$.

Notes

Now if $\mathcal{C}^*$ is a fanily of finite numbr of sets in $\mathcal{C}$, say

$$\mathcal{C}^* = \{]-n_1, n_1[,]-n_2, n_2[, \ldots,]-n_K, n_K[\}$$

and if $n^* = \max\{n_1, n_2, \ldots, n_K\}$, then

$$n^* \notin \bigcup_{i=1}^{K} (]-n_i, n_i[)$$

Thus it follows that no finite sub-family of $\mathcal{C}$ cover $\mathcal{R}$.

Hence ($\mathcal{R}$, U) is not compact.

Theorem 9: A closed and bounded subset (subspace) of $\mathcal{R}$ is compact.

Proof: Let $I_1 = [a_1, b_1]$ be a closed and bounded subset of $\mathcal{R}$. Let $G = \{(c_i, d_i) : i \in \Delta\}$ be an open covering of I_1.

To prove that $\exists$ finite subcover of the original cover G.

Suppose the contrary.

Then $\exists$ no finite subcover of the cover G.

Divide I_1 into two equal closed intervals.

$$\left[a_1, \frac{a_1+b_1}{2}\right] \text{ and } \left[\frac{a_1+b_1}{2}, b_1\right].$$

Then, by assumption, at least one of these two intervals will not be covered by any finite subclass of the cover G. Call that interval by the name I_2.

Write $I_2 = [a_2, b_2]$

Then $[a_2, b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2}, b_1\right]$.

Divide I_2 into two equal closed intervals $\left[a_2, \frac{a_2+b_2}{2}\right]$ and $\left[\frac{a_2+b_2}{2}, b_2\right]$. Again by assumption, at least one of these two intervals will not be covered by any finite sub-family of the cover G. Call that interval by the name I_3.

Write $I_3 = [a_3, b_3]$.

Repeating this process an infinite number of times, we get a sequence of intervals $I_1, I_2, I_3, \ldots$ with the properties.

(i) $I_n \supset I_{n+1}$ $\forall$ n $\in$ N.

(ii) I_n is closed $\forall$ n $\in$ N.

(iii) I_n is not covered by any finite sub-family of G.

(iv) $\underset{n\to\infty}{\text{lt}} [I_n] = 0$, where $|I_n|$ denotes the length of the interval I_n and similar is the meaning of $|[a_r, b_r]|$.

Evidently the sequence of intervals $\langle I_n \rangle$ satisfies all the conditions of nested closed interval property.

This $\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$

So that $\exists$ a number $p_0 \in \bigcap_{n=1}^{\infty} I_n$.

Self Assessment

Notes

4. Prove that if X is an ordered set in which every closed interval is compact, the X has the least upper bound property.
5. Let X be a metric space with metric d; let $A \subset X$ be non-empty. Show that d(x, A) = 0 if and only if $x \in \overline{A}$.

13.3 Summary

- A collection A of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of A is equal to X. It is called an open covering of X if its element are ope subsets of X.
- A space X is said to be compact if every open covering A of X contains a finite subcollection that also cover X.
- Let A be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of A continuing it. The number δ is called a Lebesgue number for the covering A.

13.4 Keywords

Closed Open Set: Let (X, T) be a topological space. Any set $A \in T$ is called an open set and X – A is a closed set.

Countably Compact: A topological space (X, T) is said to be countably compact iff every countable T-open cover of X has a finite subcover.

Homeomorphism: A map f : (X, T) → (Y, U) is said to be homeomorphism if (i) f is one-one onto (ii) f and f^{-1} are continuous.

Indiscrete Topology: Let X be any non-empty set ad T = {X, ϕ}. Then T is called the indiscrete topology.

13.5 Review Questions

1. Let T and T′ be two topologies on the set X; suppose that $T' \supset T$. What does compactness of X under one of these topologies imply about compactness under the other?
2. Show that if X is compact Hausdorff under both T and T′, then either T and T′ are equal or they are not comparable.
3. Show that a finite union of compact subspaces of X is compact.
4. Let A and B be disjoint compact subspace of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.
5. Let Y be a subspace of X. If $Z \subset Y$, then show that Z is compact as a subspace of Y $\Leftrightarrow$ it is compact as a subspace of X.
6. Prove that a closed subset of a compact space is compact.

Notes

13.6 Further Readings

Books

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York

S. Willard, *General Topology*, Addison-Wesley Mass. 1970.

Unit 14: Limit Point Compactness

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define limit-point compactness and solve related problems;
- Define the term sequentially compact and solve questions on it.

Introduction

In this unit, we introduce limit point compactness. In some ways, this property is more natural and intuitive than that of compactness. In the early days of topology, it was given the name "compactness", while the open covering formulation was called "bicompactness". Later, the word "compact" was shifted to apply the open covering definition, leaving this one to search for a new name. It still has not found a name on which everyone agrees. On historical grounds, some call it "Frechet compactness" others call it the "Bolzano–Weierstrass property". We have invented the term "limit point compactness". It seems as good a term as any at least it describes what the property is about.

14.1 Limit Point Compactness and Sequentially Compact

14.1.1 Limit Point Compactness

A space X is said to be limit point compact if every infinite subset of X has a limit point.

Theorem 1: Compactness implies limit point compactness, but not conversely.

Proof: Let X be a compact space. Given a subset A of X, we wish to prove that if A is infinite, then A has a limit point. We prove the contra positive – if A has no limit point, then A must be finite.

So suppose A has no limit point. Then A contains all its limit points, so that A is closed. Further more, for each $a \in A$, we can choose a neighborhood U_a of a such that U_a intersects A in the point a alone. The space of X is covered by the open set X – A and the open sets U_a; being compact, it can be covered by finitely many of these sets. Since X – A does not intersect A, and each set U_a contains only one point of A, the set A must be finite.

Notes

Example 1: Let Y consist of two points; give Y the topology consisting of Y and the empty set. Then the space $X = Z_+ \times Y$ is limit point compact, for every non-empty subset of X has a limit point. It is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ has no finite subcollection covering X.

14.1.2 Sequentially Compact

Let X be a topological space. If (x_n) is a sequence of points of X, and if

$$n_1 < n_2 < ... < n_i < ....$$

is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{ni}$ is called a subsequence of the sequence (x_n). The space X is said t_o be sequentially compact if every sequence of points of X has a convergent subsequence.

Theorem 2: Let X be a metrizable space. Then the following are equivalent:

1. X is compact
2. X is limit point compact
3. X is sequentially compact

Proof: We have already proved that (1) ⇒ (2). To show that (2) ⇒ (3), assume that X is limit point compact. Given a sequence (x_n) of points of X, consider the set $A = \{x_n n \in Z_+\}$. If the set A is finite, then there is a point x such that $x = x_n$ for infinitely many values of n. In this case, the sequence (x_n) has a subsequence that is constant, and therefore converges trivially. On the other hand, if A is infinite, then A has a limit point of x. We define a subsequence of (x_n) converging to x as follows.

First choose n_1 so that

$$x_{n_1} \in B(x, 1)$$

Then suppose that the positive integer n_{i-1} is given. Because the ball B (x, 1/i) intersects A in infinitely many points, we an choose an index $n_i > n_{i-1}$ such that

$$x_{ni} \in B(x, 1/i)$$

Then the subsequence $x_{n1}, x_{n2}, ...,$ converges to x.

Finally, we show that (3) ⇒ (1). This is the hardest part of the proof.

First, we show that if X is sequentially compact, then the Lebesgue number lemma holds for X. (This would follow from compactness, but compactness is what we are trying to prove.) Let $\mathcal{A}$ be an open covering of X. We assume that there is no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathcal{A}$ containing it, and derive a contradiction.

Our assumption implies in particular that for each positive integer n, there exists a set of diameter less than 1/n that is not contained in any element of $\mathcal{A}$; let C_n be such a set. Choose a point $x_n \in C_n$, for each n. By hypothesis, some subsequence (x_{ni}) of the sequence (x_n) converges, say to the point a. Now a belongs to some element A of the collection $\mathcal{A}$; because A is open, we may choose an $\in > 0$ such that $B(a, \in) \subset A$. If i is large enough that $1/n_i < \in/2$, then the set C_{ni} lies in the $\frac{\in}{2}$ neighborhood of x_{ni}; if i is also chosen large enough that $d(x_{ni}, a) < \in/2$, then C_{ni} lies in the $\in$-neighborhood of a. But this means that $C_{ni} \subset A$, contrary to hypothesis.

Second, we show that if X is sequentially compact, then given $\in > 0$, there exists a finite covering of X by open $\in$-balls. Once again, we proceed by contradiction. Assume that there exists an $\in > 0$ such that X cannot be covered by finitely many $\in$-balls. Construct a sequence of points x_n of X as follows: First, choose x_1 to be any point of X. Noting that the ball $B(x_1, \in)$ is not all of X

(Otherwise X could be covered by a single $\in$-ball), choose x_2 to be a point of X not in $B(x_1, \in)$. In general, given $x_1, ..., x_n$, choose x_{n+1} to be a point in the union

$$B(x_1, \in) \cup .....\cup B(x_n, \in)$$

using the fact that these balls do not cover X. Note that by construction $d(x_{n+1}, x_i) \geq \in$ for $i = 1, ..., n$. Therefore, the sequence (x_n) can have no convergent subsequence; in fact, any ball of radius $\in/2$ can contain x_n for at most one value of n.

Finally, we show that if X is sequentially compact, then X is compact. Let $\mathcal{A}$ be an open covering of X. Because X is sequentially compact, the open covering $\mathcal{A}$ has a Lebesgue number δ. Let $\in = \delta/3$; use sequential compactness of X to find a finite covering of X by open $\in$- balls. Each of these balls has diameter at most $2\delta/3$, so it lies in an element of $\mathcal{A}$. Choosing one such element of $\mathcal{A}$ for each of these $\in$-balls, we obtain a finite subcollection of $\mathcal{A}$ that covers X.

Example 2: Prove that a continuous image of a sequentially compact set is sequentially compact.

Solution: Let (X, T) be a sequentially compact topological space so that every sequence $\langle x_n \rangle$ in X has a convergent subsequence $\langle x_{i_k} : K \in N \rangle$ and let this subsequence converge to x_{i_0}, i.e.,

$$x_{i_K} \to x_{i_0} \in X.$$

Let f : (X, T) $\to$ (Y, U) be a continuous map.

To prove that f (X) is sequentially compact set.

f is continuous map $\Rightarrow$ f is sequentially continuous

Furthermore $x_{i_K} \to x_{i_0}$.

This implies that $f(x_{i_K}) \to f(x_{i_0})$.

Showing thereby f (X) is sequentially compact.

Example 3: A finite subset of a topological space is necessarily sequentially compact.

Solution: Let (X, T) be a topological space and $A \subset X$ be finite and $\langle x_n \rangle$ be a sequene in A so that $x_n \in A \ \forall$ n. Also $\langle x_n \rangle$ contains infinite number of terms. It follows that at least one element of A, say x_0 must appear infinite number of times in $\langle x_n \rangle$. Thus $\langle x_0, x_0, x_0, \ldots \rangle$ is a subsequence of $\langle x_n \rangle$ and this subsequence converges to $x_0 \in A$, showing thereby A is a sequentially compact.

Theorem 3: A metric space is sequentially compact iff it has the Bolzano Weierstrass Property.

Proof I: Let (X, d) be a sequentially compact metric space. To prove that (X, d) has Bolzano Weierstrass Property,

Let $A \subset X$ be an infinite set.

If we show that A has a limit point in X, the result will follow.

A is an infinite set $\Rightarrow$ A contains an enumerable set, say $\{x_n : n \in N\}$

$\Rightarrow \langle x_n \in A, n \in N \rangle$ is a sequence with infinitely many distinct points.

By the assumption of sequential compactness, the sequence $\langle x_n \rangle$ has a convergent subsequence $\langle x_{in} : n \in N \rangle$ (say). Let this convergent sequence $\langle x_{in} : n \in N \rangle$ converge to x_0. Then $x_0 \in X$ and $\langle x_n \rangle$ also converges to x_0, i.e., $x_n \to x_0$. Consequently x_0 is a limit point of the set $\{x_n : n \in N\}$.

Evidently $\{x_n : n \in N\} \subset A$.

Notes

So that $D(\{x_n : n \in N\}) \subset D(A)$.

But $x_0 \in D\{x_n : n \in N\}$ and hence $x_0 \in D(A)$, i.e., A has a limit point $x_0 \in X$.

Proof II: Conversely, suppose that the metric space (X, d) has Bolzano Weierstrass property.

To prove that X is sequentially compact.

By the assumption of Bolzano Weierstrass property, every infinite subset of X has a limit point in X. Let $\langle x_n \rangle$ be an asbitrary sequence in X.

Case (i): If the sequence $\langle x_n \rangle$ has an element x which is infinitely repeated, then it has a constant subsequence $\langle x, x, \ldots, x, \ldots \rangle$ which certainly converges to x.

Case (ii): If the sequence $\langle x_n \rangle$ has infinitely many distinct points then by assumption, the set $\{x_n : x \in N\}$ has a limit point, say $x_0 \in X$. Consequently x_0 is a limit of the sequence $\langle x_n : n \in N \rangle$ with infinitely many distinct points so that this sequence contains a subsequence $\langle x_{in} : n \in N\}$ which also converges to X_0.

$\therefore$ In either case, we have shown that every sequence in X contains a convergent subsequence so that X is sequentially compact.

Hence the result.

14.2 Summary

- A space X is said to be limit point compact if every infinite subset of X has a limit point.
- Compactness implies limit point compactness, but not conversely.
- A topological space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.

14.3 Keywords

BWP: A topological space (X, T) is said to have Bolzano Weierstrass Property denoted by BWP if every infinite subset has a limit point.

Compact Space: A space X is said to be compact if every open covering A of X contains a finite subcollection that also covers X.

Lebesgue Covering Lemma: Every open covering of a sequentially compact space has a Lebesgue number.

Lebesgue Number: Let $\{G_i : i \in \Delta\}$ be an open cover for a metric space (X, d), a real number $\delta > 0$ is called a Lebesgue number for the cover if any $A \subset X$ s.t. $d(A) < \delta \Rightarrow A \subset G_{i0}$ for at least one index $i_0 \in \Delta$.

Metrizable: Any topological space (X, T), if it is possible to find a metric on ρ on X which induces the topology T i.e. the open sets determined by the metric ρ are precisely the members of δ, then X is said to the metrizable.

Open Cover: Let (X, T) be a topological space and $A \subset X$. Let G denote a family of subsets of X. G is called a cover of A if $A \subset U \{G : G \in G\}$. If every member of G is a open set, then the cover G is called an open cover.

14.4 Review Questions

Notes

1. Show that [0, 1] is not limit point compact as a subspace of $\mathbb{R}_l$.
2. Let X be limit point compact.
   (a) If f : X → Y is continuous, does it follow that f(X) is limit point compact?
   (b) If A is a closed subset of X, does it follow that A is limit point compact?
   (c) If X is a subspace of the Hausdorff space Z, does it follow that X is closed in Z?
3. A space X is said to be countably compact if every countable open covering of X contains a finite subcollection that covers X. Show that for a T_1 space X, countable compactness is equivalent to limit point compactness.

   [***Hint:*** If no finite subcollection of U_n covers X, choose $x_n \notin U_1 \cup ... \cup U_n$, for each n.]

14.5 Further Readings

Books

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley Mass. 1970.

Unit 15: Local Compactness

CONTENTS

Objectives

After studying this unit, you will be able to:

- Describe the local compactness;
- Solve the problems on local compactness;
- Explain the theorems on local compactness.

Introduction

In this unit, we study the notion of local compactness and we prove the theorems that every continuous image of a locally compact space is locally compact and many other theorems.

15.1 Locally Compact

Let (X, T) be a topological space and let $x \in X$ be arbitrary. Then X is said to be locally compact at x if the closure of any neighborhood of x is compact.

X is called locally compact if it is compact at each of its points, but need not be compact as whole.

Alternative definition: A topological space (X, T) is locally compact if each element $x \in X$ has a compact neighborhood.

Example 1: Show that $\mathbb{R}$ is locally compact.

Solution: Let $x \in R$ be arbitrary.

Evidently, $\overline{S_r(x)} = S_r[x]$

$S_r[x]$ is compact, being closed and bounded subset of $\mathbb{R}$. Thus the closure of the neighborhood $S_r(x)$ of x is compact and hence the result.

Example 2: Show that compactness $\Rightarrow$ locally compact.

Solution: Let (X, T) be a compact topological space. To prove that X is locally compact.

For this, we must show that the closure of any neighborhood of any point $x \in X$ is compact. This follow from the fact that X is the neighborhood of each of its points and $X = \overline{X}$, X is compact.

Notes

Theorem 1: Let (X, T) and (Y, $\mathcal{U}$) be topological spaces and f : (X, T) $\xrightarrow{\text{onto}}$ (Y, U) be a continuous open map. Then if X is locally compact, then Y is also.

Or

Every open continuous image of a locally compact space is locally compact.

Proof: Let f : (X, T) $\rightarrow$ (Y, U) be a continuous open map and X a locally compact space.

We claim Y is locally compact.

Let $y \in Y$ be arbitrary and $U \subset Y$ a nbd of y.

$y \in Y$, f : X $\rightarrow$ Y is onto $\Rightarrow \exists\, x \in X$ s.t. f(x) = y

$\because$ f is continuous

$\therefore$ Given any nbd U of y, $\exists$ a nbd $V \subset X$ of x s.t. $f(V) \subset U$. X is locally compact.

$\Rightarrow$ X is locally compact at x and V is a nbd of x.

$\Rightarrow$ $\exists$ is compact set A s.t. $x \in A^o \subset A \subset V$

$\Rightarrow$ $f(x) \in f(A^o) \subset f(A) \subset f(V) \subset U$

$\Rightarrow$ $y \in f(A^o) \subset f(A) \subset U$ ...(1)

Now, f is open, $A^o \subset X$ is open.

$\Rightarrow$ $f(A^o) \subset Y$ is open

$\Rightarrow$ $f(A^o) = [f(A^o)]^o$ ...(2)

From (1), $f(A^o) \subset f(A)$

Thus $[f(A^o)]^o \subset [f(A)]^o$

$\Rightarrow$ $f(A^o) \subset [f(A)]^o$, (on using (2))

$\Rightarrow$ $f(A^o) \subset [f(A)]^o \subset f(A)$

Using this in (1),

$y \in f(A^o) \subset [f(A)]^o \subset f(A) \subset U$

or $y \in [f(A)]^o \subset f(A) \subset U$

Taking B = f(A) = continuous image of compact set A

= compact set

We obtain $y \subset B^o \subset B \subset U$, B is compact.

Finally, we have shown that given any $y \in Y$ and a nbd U of y, $\exists$ a compact set $B \subset Y$, s.t. $y \subset B^o \subset B \subset U$.

Hence Y is locally compact at y so that Y is locally compact.

Theorem 2: Every locally compact T_2-space is a regular space.

Proof: Let (X, T) be a locally compact T_2-space. To prove that (X, T) is a regular space. Let $x \in X$ be arbitrary and G a nbd of x.

By definition of locally compact space,

$\exists$ a compact set $A \subset X$ s.t. $x \in A^o \subset A \subset G$.

A is compact, X is T_2-space $\Rightarrow$ A is closed.

$\Rightarrow \quad (\overline{A^o}) \subset \overline{A} = A$

$\Rightarrow \quad (\overline{A^o}) \subset A$ ...(1)

$\therefore \quad x \in A^o \subset A \subset G$

$\therefore \quad x \in A^o \subset (\overline{A^o}) \subset A \subset G,$ [by (1)]

Taking $A^o = U$

$x \in U \subset \overline{U} \subset G$

Thus we have shown that given any nbd G of x, $\exists$ a nbd U of x s.t.

$x \in U \subset \overline{U} \subset G$

Consequently X is regular.

Theorem 3: Any open subspace of a locally compact space is a locally compact.

Proof: Let (Y, U) be an open subspace of a locally compact space (X, T) so that Y is open in X.

To prove that Y is locally compact.

Let $x \in Y \subset X$ be arbitrary and G a U-nbd of x in Y, then $x \in X, G \subset Y$.

X is locally compact $\Rightarrow$ X is locally compact at x.

G is a U-nbd of x in Y $\Rightarrow \exists\, G_1 \in U$ s.t. $x \in G_1 \subset G$

$\Rightarrow \quad G_1 \in T$ s.t. $x \in G_1 \subset G$. For Y is open in X.

$\Rightarrow$ G is a T-nbd of x in X.

Also X is locally compact $\Rightarrow \exists$ a compact set $A \subset X$ s.t. $x \in A^o \subset A \subset G$. But $G \subset Y$.

$\Rightarrow \quad x \in A^o \subset A \subset G \subset Y$

Thus (i) $A \subset Y$, A is U-compact.

For A is T-compact $\Rightarrow$ A is U-compact.

(ii) G is a nbd of x in Y s.t. $x \in A^o \subset A \subset G$.

This proves that Y is locally compact at any $y \in Y$ and hence the result follows. Proved.

Theorem 4: Every closed subspace of a locally compact space is locally compact.

Proof: Let (Y, U) be a closed subspace of a locally compact space (X, T), then Y is T-closed set. Let $y \in Y \subset X$ be arbitrary.

To prove that Y is locally compact, we have to prove that Y is locally compact at y.

X is locally compact $\Rightarrow$ X is locally compact at y

$\Rightarrow \quad \exists$ T-open nbd N of x s.t. $\overline{N}$ is T-compact.

$\Rightarrow \quad N \cap Y$ is U-open nbd of y.

$N \cap Y \subset N \Rightarrow \overline{N \cap Y} \subset \overline{N}$.

Thus $\overline{N \cap Y}$ is a closed subset of a compact set $\overline{N}$. Hence $\overline{N \cap Y}$ is compact.

Y is T-closed ⇒ T-closure of $N \cap Y$ = U-closure of $N \cap Y$.

Thus $N \cap Y$ is U-open nbd of y s.t. $\overline{N \cap Y}$ is compact, showing thereby Y is locally compact at y.

15.2 Summary

- A topological space (X, T) is locally compact if each element $x \in X$ has a compact neighborhood.
- Any open subspace of a locally compact space is a locally compact.
- Every locally compact T_2-space is a regular space.
- Every closed subspace of a locally compact space is locally compact.

15.3 Keywords

Closure: Let (X, T) be a topological space and $A \subset X$. The closure of $\mathcal{A}$ is defined as the intersection of all closed sets which contain A and is denoted by the symbol $\overline{A}$.

Compact set: Let (X, T) be a topological space and $A \subset X$. A is said to be a compact set if every open covering of A is reducible to finite sub-covering.

Interior point: A point $x \in A$ is called an interior point of A if $\exists\, r \in R^+$ s.t. $S_r(x) \subset A$.

Neighborhood: Let $\varepsilon > 0$ be any real number. Let x_0 be any point on the real line. Then the set $\{x \in R : |x - x_0| < \varepsilon\}$ is defined as the $\in$-neighborhood of the point x_0.

Regular space: A regular space is a topological space in which every nbd of a point contains a closed neighborhood of the same point.

T_2-space: A T_2-space is a topological space (X, T) fulfilling the T_2-axiom: every two points $x, y \in X$ have disjoint neighborhoods.

15.4 Review Questions

1. Show that the rationals Q are not locally compact.
2. Let X be a locally compact space. If $f : X \to Y$ is continuous, does it follow that f(x) is locally compact? What if f is both continuous and open? Justify your answer.
3. If $f : X_1 \to X_2$ is a homeomorphism of locally compact Hausdorff spaces, show f extends to a homeomorphism of their one-point compactifications.
4. Is every open subspace of a locally compact space is locally compact? Give reasons in support of your answer.
5. Show by means of an example that locally compact space need not be compact.
6. Show that local compactness is a closed hereditary property.
7. X_1, X_2 are L-compact if and only if $X_1 \times X_2$ is L-compact.

Notes

15.5 Further Readings

Books

Kelley, John (1975), *General Topology,* Springer.

Sbeen, Lynn Arthur, Seebach, J. Arthur Jr. (1995), *Counter examples in Topology* (Dover reprint of 1978 ed.) Berlin, New York.

Willard, Stephan (1970), *General Topology,* Addison-Wesley (Dover Edition).

Online links

www.math.stanford.edu

Unit 16: The Countability Axioms

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define countability axioms;
- Understand and describe the theorems on countability axioms;
- Discuss the theorems on countability axioms related to the metric spaces.

Introduction

The concept we are going to introduce now, unlike compactness and connectedness, do not arise naturally from the study of calculus and analysis. They arise instead from a deeper study of topology itself. Such problems as imbedding a given space in a metric space or in a compact Hausdorff space are basically problems of topology rather than analysis. These particular problems have solutions that involve the countability and separation axioms. In this unit, we shall introduce countability axioms and explore some of their consequences.

16.1 Countability Axioms

16.1.1 First Axiom of Countability

Let (X, T) be a topological space. The space X is said to satisfy the first axiom of countability if X has a countable local base at each $x \in X$. The space X, in this case, is called first countable or first axiom space.

Example 1: Consider $x \in R$

$$A_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \forall\, x \in N$$

Notes

Take $B_x = \{A_n : n \in N\}$.

Evidently, B_x is a local base at $x \in X$ for the usually topology on R.

Clearly, $B_x \sim N$ under the map $A_n \to n$.

Therefore, B_x is a countable local base at $x \in X$. But $x \in X$ is arbitrary.

Hence R with usual topology is first countable.

16.1.2 Second Axiom of Countability

Let (X, T) be a topological space. The space X is said to satisfy the second axiom of countability if $\exists$ a countable base for T on X.

In this case, the space X is called second countable or second axiom space.

Note A second countable space is also called completely separable space.

Example 2: The set of all open intervals (r, s) and r with s as rational numbers forms a base, say B for the usual topology $\cup$ of R. Since Q, Q × Q are countable sets and so B is a countable base for $\cup$ on R.

$\therefore$ (R, $\cup$) is second countable.

16.1.3 Hereditary Property

Let (X, T) be a topological space. A property P of X is said to be hereditary if the property is possessed by every subspace of X.

E.g. first countable, second countable are hereditary properties, where as closed sets, open sets, are not hereditary properties.

16.1.4 Theorems and Solved Examples on Countability Axioms

Theorem 1: Let (X, T) be a second axiom space and let C be any collection of disjoint open subsets of X. Then C is a countable collection.

Proof: Let (X, T) be a second countable space, then $\exists$ a countable base

$$\mathcal{B} = \{B_n : n \in \Delta\} \text{ for topology T on X.}$$

Let C be a collection of disjoint open subsets of X.

Let $A \in C$ be arbitrary, then $A \in T$.

By definition of base, $\exists\, B_n \in \mathcal{B}$ st. $B_n \subset A$.

We associate with A, a least positive integer n s.t. $B_n \subset A$.

Members of C are disjoint

$\Rightarrow$ distinct integers will be associated with distinct member of C.

If we now order the members of C according to the order of associated integers, then we shall get a sequence containing all the members of C. Hence, C is a countable collection.

Theorem 2: Let (X, T) be a first axiom space. Then $\exists$ is a nested (monotone decreasing) local base at every point of X.

Proof: Let (X, T) be first axiom space, then $\exists$ is a countable local base

$$B(x) = \{B_n : n \in N\} \text{ at every point } x \in X.$$

Write $C_1 = B_1, B_2 = B_1 \cap B_2, C_3 = B_1 \cap B_2 \cap B_3, \ldots,$

$$C_n = \bigcap_{i=1}^{n} B_i.$$

Then $C_1 \supset C_2 \supset C_3 \supset \ldots \supset C_n$.

$$x \in B_n \in \mathcal{B} \quad \forall n \quad \Rightarrow x \in C_n \in T \quad \forall n.$$

It follows that $C(x) = \{C_n : n \in N\}$ is a nested local base at x.

Theorem 3: A second countable space is always first countable space.

Or

Prove that second axiom of countability $\Rightarrow$ first axiom of countability.

Proof: Let (X, T) be a topological space which satisfies the second axiom of countability so that (X, T) is second countable.

To prove that (X, T) also satisfies the first axiom of countability.

i.e., to prove that (X, T) is first countable.

By hypothesis, $\exists$ a countable base $\mathcal{B}$ for topology T on X.

$\mathcal{B}$ is countable $\Rightarrow \mathcal{B} \sim N$

This show that $\mathcal{B}$ can be expressed as

$$\mathcal{B} = \{B_n : n \in N\}$$

Let $x \in X$ be arbitrary.

Write $L_x = \{B_n \in \mathcal{B} : x \in B_n\}$

(i) L_x, being a subset of a countable set $\mathcal{B}$, is countable.

(ii) Since members of $\mathcal{B}$ are T open sets and hence the members of L_x. For $L_x \subset \mathcal{B}$.

(iii) Any $G \in L_x \Rightarrow x \in G$, according to the construction of L_x.

(iv) Let $G \in T$ for arbitrary s.t. $x \in G$.

Then, by definition of base,

$$x \in G \in J \Rightarrow B_r \in \mathcal{B} \quad \text{s.t. } x \in B_r \subset G,$$

$$\Rightarrow \exists B_r \in L_x \quad \text{s.t. } B_r \subset G,$$

For $B_r \in \mathcal{B}$ with $x \in B_r \Rightarrow B_r \in L_x$.

Finally $x \in G \in T \Rightarrow \exists B_r \in L_x$ s.t. $B_r \subset G$. ...(1)

From (i), (ii), (iii), (iv) and (1), it follows that L_x is a countable local base at $x \in X$. Hence, by definition, X is first countable.

Theorem 4: To prove that first countable space does not imply second countable space. Give an example of a first countable space which does not imply second countable space.

Proof: We need only give an example of a space which does satisfy the first axiom of countability but not the second axiom of countability.

Let T be a discrete topology on an infinite set X so that every subset of X is open in X and hence in, particular, each singleton set {x} is open in X for each $x \in X$.

Write $\mathcal{B} = \{\{x\} : x \in X\}$.

Then it is easy to verify that $\mathcal{B}$ is a base for the topology T on X and $\mathcal{B}$ is not countable. For X is not countable. Hence X is not second countable.

If we take $L_x = \{x\}$ then evidently L_x is a countable local base at $x \in X$ as it has only one number.

For any $G \in T$ with $x \in G$, $\exists\{x\}$ s.t. $x \in \{x\} \subset G$. From what has been done, it follows that X is first countable but not second countable.

Theorem 5: Show that the property of a space being first countable is hereditary.

Proof: Let $(Y, \bigcup)$ be a subspace of a first countable space (X, T).

If we show that $(Y, \bigcup)$ is first countable, we can conclude the required result.

Let $y \in Y$ be arbitrary, then $y \in X$. $[\because Y \subset X]$

X is first countable $\Rightarrow \exists$ a countable local base at each $x \in X$ and hence, in particular, $\exists$ a countable local base $\mathcal{B}$ at $y \in X$.

Members of $\mathcal{B}$ can be enumerated as $B_1, B_2, B_3, B_4, \ldots$

i.e. $\mathcal{B}_n = \{B_n : n \in N\}$.

Evidently, $y \in B_n \ \forall \ n \in N$.

Write $B_1 = \{Y \cap B_n : n \in N\}$ ...(1)

$y \in Y, y \in B_n \ \forall \ n \in N \Rightarrow y \in Y \cap B_n \ \forall \ n \in N$ ...(2)

$B_n \in \mathcal{B} \ \forall \ n \in N \Rightarrow B_n \in T \Rightarrow Y \cap B_n \in \bigcup$ ...(3)

We claim $\mathcal{B}_1$ is a countable local base at y for $\bigcup$ on Y.

(i) Evidently $N \sim B_1$ under the map $n \to Y \cap B_n$. Hence B_1 is countable. ...(4)

(ii) any $G \in \mathcal{B}_1 \Rightarrow y \in G$ ...(5)

(iii) $\mathcal{B}_1$ is family of all $\cup$ - open sets. ...(6)

(iv) let $G \in \bigcup$ be arbitrary s.t.

$y \in G$, then $\exists H \in T$ s.t. $G = H \cap Y$.

$y \in H$. For $y \in G = H \cap Y$.

By definition of local base.

$$y \in H \in T \Rightarrow \exists B_r \in \mathcal{B} \text{ s.t. } y \in B_r \subset H$$

or $$y \in H \cap Y \in \bigcup \Rightarrow \exists B_r \cap Y \in \mathcal{B}_1 \text{ s.t. } B_r \cap Y \subset H \cap Y$$

or $$y \in G \in \bigcup \Rightarrow \exists B_r \cap Y \in \mathcal{B}_1 \text{ s.t. } y \in B_r \cap Y \subset G \quad \text{...(7)}$$

The result (1), (4), (5), (6) and (7) taken together imply that $\mathcal{B}_1$ is a local base at $y \in Y$ for the topology $\cup$ on Y and hence $(Y, \bigcup)$ is first countable.

Theorem 6: Show that the property of a space being second countable is hereditary.

or

Prove that every subspace of a second countable space is second countable.

Notes

Proof: Let (Y, $\mho$) be a sub-space of a topological space (X, T) which is second countable so that there exists a countable base $\mathcal{B}$ for the topology T.

If we show that (Y, $\mho$) is second countable, the result will follow

$$\mathcal{B} \text{ is countable } \Rightarrow \mathcal{B} \sim N$$

$$\Rightarrow \mathcal{B} \text{ is expressible as}$$

$$\mathcal{B} = \{B_n : N \in N\}$$

Write

(i) Evidently $\mathcal{B}_1 \sim N$ under the map $Y \cap B_n \to n$.

$\therefore$ $\mathcal{B}_1$ is countable.

(ii) $\mathcal{B}_1$ is a family of all $\mho$–open sets.

For $B_n \in \mathcal{B} \Rightarrow B_n \in T$,

$\therefore B \subset T \Rightarrow Y \cap B_n \in \mho$

(iii) any $y \in G \in \cup \Rightarrow \exists\, B_r \cap Y \in \mathcal{B}_1$

s.t. $y \in Y \cap B_r \subset G$.

For proving this let $G \in \mho$

s.t. $y \in G$, then $\exists\, H \in T$ s.t. $G = H \cap Y$.

$y \in G \Rightarrow y \in H \cap Y \Rightarrow y \in H$ and $y \in Y$.

By definition of base,

any $y \in H \in T \Rightarrow \exists\, B_r \in \mathcal{B}$ s.t. $y \in B_r \subset H$

from which any $y \in H \cap Y \in \mho$

$$\Rightarrow \exists\, Y \cap B_r \subset \mathcal{B}_1 \text{ s.t. } y \in Y \cap B_r \in G$$

i.e. any $y \in G \in \cup \Rightarrow \exists\, Y \cap B_r \in B_1$

s.t. $y \in Y \cap B_r \subset G$.

Thus it follows that $\mathcal{B}_1$ is a countable base for the topology $\mho$ and Y. Consequently, (Y, $\mho$) is second countable.

Theorem 7: A second countable space is always separable.

Proof: Let (X, T) be a second countable space.

To prove: (X, T) is separable.

Since X is second countable and hence $\exists$ a countable base $\mathcal{B}$ for the topology T on X. Members of $\mathcal{B}$ may be enumerated as $B_1, B_2, B_3, \ldots$.

Choose an element x_i from each B_i and take A as the collection of all these x_i's.

That is to say, $\quad x_i \in B_c \in \mathcal{B} \;\; \forall \;\; i \in N$ ...(1)

and $\quad A = \{x_i : i \in N\}$ ...(2)

Evidently $\quad N \sim A \quad$ under the map $i \to x_i$

Therefore, $\quad$ A is enumerable.

Notes

Clearly, $A \subset X$

We claim $\bar{A} = X$.

Suppose not, then $X - \bar{A} \neq \phi$ ...(3)

Let $y \in X - \bar{A}$ be arbitrary. $\bar{A}$ is closed and hence $X - \bar{A}$ is open. It amounts to saying that

$$y \in X - \bar{A} \in T.$$

By definition of base

$$y \in X - \bar{A} \in T \Rightarrow \exists\, B_y \in \mathcal{B} \text{ s.t. } y \in B_y \subset X - \bar{A}.$$

In particular $\Rightarrow$

$$x_{n_o} \in X - \bar{A} \in T$$

$\Rightarrow$ $\exists\, B_{n_o} \in \mathcal{B}$ s.t. $x_{n_o} \subset X - \bar{A}$.

Now $x_{n_o} \in X - \bar{A} \;\Rightarrow\; x_{n_o} \notin \bar{A} \supset A$

$\Rightarrow\; x_{n_o} \notin A$ ...(4)

$x_{n_o} \in B_n \Rightarrow x_{n_o} \in A$, according to (1) and (2), Contrary to (4).

Hence our assumption $X - \bar{A} \neq \phi$ is wrong.

Consequently $X - \bar{A} = \phi$ i.e. $X = \bar{A}$

Thus, we have shown that

$\exists\, A \subset X$ s.t. $\bar{A} = X$ and X is enumerable set. By definition, this proves that X is separable.

Theorem 8: Every second axion space is hereditarily separable.

Proof: Let $(Y, \cup)$ be a subspace of second axion, i.e. second countable space (X, T).

To prove the required result, we have to show that $(Y, \cup)$ is second countable and separable since every second countable space is separable. [Refer theorem (7)].

Now it remains to show that $(Y, \cup)$ is second countable. Now write the proof of Theorem (6).

Example 3: Prove that $(R, \bigcup)$ is a second axiom space (Second countable.).

Solution: We know that Q is a countable subset of R. If we write

$$\mathcal{B} = \{(a, b) : a < b \text{ and } a, b \in Q\}$$

Then $\mathcal{B}$ forms a countable base for the usual topology $\bigcup$ and R so that R is second countable.

Example 4: Prove that $(R^2, \bigcup)$ is second countable.

Solution: If we write

$$\mathcal{B} = \{S_r(x) : x, r \in Q\}$$

then $\mathcal{B}$ forms a countable base for the usual topology $\bigcup$ on R^2. Hence $(R^2, \cup)$ is second countable space.

16.1.5 Theorems Related to Metric Spaces

Theorem 9: A metric space is second countable iff it is separable.

Proof:

(i) Let (x, ρ) be a metric space. Let T be the metric topology on X corresponding to the metric ρ. Let (X, T) be second countable. To prove that X is separable.

Here write the complete proof of the theorem (6).

(ii) Conversely, suppose that (x, ρ) is a metric space and T is a metric topology on X corresponding to the metric ρ. Also, suppose that X is separable, so that

$\exists\, A \subset X$ s.t. $\bar{A} = X$ and A is countable.

A is countable $\Rightarrow$ A is expressible as

$A = \{a_n : n \in N\}$

To prove that X is second countable.

We know that each open sphere forms an open set.

Let $a_n \in A$ be arbitrary.

Write $\mathcal{B} = \{S_r(a_n) : r \in Q^+, n \in N\}$.

Q is an enumerable set

$\Rightarrow$ Q^+ is an enumerable set

$\therefore$ $Q^+ \subset Q$

Then $\mathcal{B}$ is a countable base for the topology T on X.

$\therefore$ X is second countable.

Let $G \in T$ be arbitrary s.t. $x \in G$.

x being an arbitrary point of X.

By definition of open set in a metric space,

$\exists$ a positive real number $\in$ s.t. $S_{(x, \in)} \subset G$ ...(1)

Since A is dense in X and so there will exist a point $a \in A$ s.t.

$$\rho(a,x) < \frac{\in}{3} \qquad ...(2)$$

Since Q is dense in R for the usual topology on R and hence its subset Q^+ is also dense in R with usual topology so that $\exists\, r \in Q^+$ s.t.

$$\frac{\in}{3} < r < \frac{2\in}{3}$$

Aim: $S_{r(a)} \subset S_{\in(x)} \subset G$.

Also let $y \in S_{(a,r)}$ be arbitrary so that $\rho(y,a) < r$ ...(3)

$$\rho(x,y) \le \rho(x,a) + \rho(a,y)$$

$$< \frac{\in}{3} + r < \frac{\in}{3} + \frac{2\in}{3}, \qquad \text{from (3)}$$

$\Rightarrow$ $\rho(x,y) < \in \Rightarrow y \in S_{(x,\in)}$

Notes

Finally, any $y \in S_{(x,r)} \Rightarrow y \in S_{(x,\in)}$

$$\therefore \quad S_{(a, r)} \subset S_{(x, \in)}$$

From (2) and (3), $\rho(a,x) < r$, so that $x \in S_{(a,r)}$.

Thus, we have shown that

$$x \in S_{(a,r)} \subset S_{(x,\in)} \subset G.$$

from which $x \in S_{(a,r)} \subset G$.

Thus, $\quad x \in G \in T \Rightarrow \exists\, r \in Q^+$ s.t. $x \in S_{(a,r)}$

i.e. $\quad x \in G \in T \Rightarrow \exists S_{(a,r)} \in \mathcal{B}$ s.t. $x \in S_{(a,r)} \subset G$.

This proves that $\mathcal{B}$ is a base for the topology T on X. From what has been done, it follows that $\mathcal{B}$ is enumerable base for the topology T on X and hence X is second countable.

Example 5: Every separable metric space is second countable.

Solution: Refer second part of the above theorem.

Theorem 10: A metric space is first countable.

Proof: Let (X, ρ) be a metric space. Let T be metric topology on X, corresponding to the metric ρ on X. Let ρ ∈ X be arbitrary.

To prove that (X, T) is first countable, it suffices to show that ∃ a countable local base at p for the topology T on X.

Write $L_p = \{S_{(p, r)} : r \in Q^+\}$.

Q is enumerable and hence its subset Q^+,

Q^+ is enumerable ⇒ L_p is enumerable.

Let G ∈ T be arbitrary s.t. ρ ∈ G.

Then, by definition of an open set.

$$\exists\, s \in R^+ \text{ s.t. } S_{(p, s)} \subset G.$$

Choose a positive rational number r s.t. r < s.

Then $\quad S_{(p,r)} \subset S_{(p,s)} \subset G$

or $\quad S_{(p,r)} \subset G$.

Given any G ∈ T with ρ ∈ G.

$$\exists\, r \in Q^+ \text{ s.t. } S_{(p, r)} \subset G.$$

Now L_p has the following properties:

(i) every member of L_p is an open set containing p.

∵ each open sphere forms an open set.

(ii) L_p is enumerable set.

(iii) Given any $G \in T$ with $p \in G$, $\exists\, r \in Q^+$ s.t.

$S_{(p,\, r)} \subset G$.

From what has been done, it follows that L_p is an enumerable local base at p of the topology T on X.

16.2 Summary

- Let (X, T) be a topological space. The space X is said to satisfy the first axiom of countability if X has a countable local base at each $x \in X$.
- Let (X, T) be a topological space. The space X is said to satisfy the second axiom of countability if $\exists$ a countable base for T on X.
- Let (X, T) be a topological space. A property P of X is said to be hereditary if the property is possessed by every subspace of X.

16.3 Keywords

Base: $\mathcal{B}$ is said to be a base for the topology T on X if $x \in G \in T \Rightarrow \exists\, B \in \mathcal{B}$ s.t. $x \in B \subset G$.

Local Base: A family $\mathcal{B}_x$ of open subsets of X is said to be a local base at $x \in X$ for the topology T on X if

(i) any $B \in \mathcal{B}_x \Rightarrow x \in B$

(ii) any $G \in T$ with $y \in G \Rightarrow \exists\, B \in \mathcal{B}_x$ s.t. $y \in B \subset G$.

Open Sphere: Let (X, ρ) be a metric space. Let $x_0 \in X$ and $r \in R^+$. Then set $\{x \in X : \rho(x_0, x) < r\}$ is defined as open sphere with centre x_0 and radius r.

Separable: Let X be a topological space and A be a subset of X, then X is said to be separable if

(i) $\bar{A} = X$

(ii) A is countable.

16.4 Review Questions

1. Prove that the property of being a first axiom space is a topological property.
2. For each point x in a first axiom T_1 – space,

   $\{x\} = \bigcap_{n \in N} B_n(x)$
3. Prove that the property of being a second axiom space is a topological property.
4. In a second axiom T_1 – space, a set is compact iff it is countable compact.
5. Show that in a second axiom space, every collection of non empty disjoint open sets is countable.
6. Give an example of a separable space which is not second countable.
7. Show that every separable metric space is second countable. Is a separable topological space is second countable? Justify your answer.
8. Every sub-space of a second countable space is second countable and hence show that it is also separable.

Notes

16.5 Further Readings

Books

G.F. Simmons, *Introduction to Topology and Modern Analysis,* McGraw Hill International Book Company, New York, 1963.

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

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Unit 17: The Separation Axioms

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define T_0-axiom and solve related problems;
- Explain the T_1-axiom and related theorems;
- Describe the T_2-axiom and discuss problems and theorems related to it.

Introduction

The topological spaces we have been studying thus far have been generalizations of the real number system. We have obtained some interesting results, yet because of the degree of generalization many intuitive properties of the real numbers have been lost. We will now consider topological spaces which satisfy additional axioms that are motivated by elementary properties of the real numbers.

17.1 T_0-Axiom or Kolmogorov Spaces

A topological space X is said to be a T_0-space if for any pair of distinct points of X, there exist at least one open set which contains one of them but not the other.

In other words, a topological space X is said to be a T_0-space if it satisfy following axiom for any $x, y \in X$, $x \neq y$, there exist an open set $\mathcal{U}$ such that $x \in U$ but $y \notin U$.

Example 1: Let X = {a, b, c} with topology T = {ϕ, X, {a}, {b}, {a, b}} defined on X, then (X, T) is a T_0-space because

(i) for a and b, there exist an open set {a} such that $a \in \{a\}$ and $b \notin \{a\}$

(ii) for a and c, there exist an open set {b} and $b \in \{b\}$ and $c \notin \{b\}$

Examples of T_0-space

(i) Every metric space is T_0-space.

(ii) If (X, T) is cofinite topological space, then it is T_0-space.

(iii) Every discrete space is T_0-space.

(iv) An indiscrete space containing only one point is a T_0-space.

17.1.1 T_1-Axiom of Separation or Frechet Space

A topological space (X, T) is said to satisfy the T_1-Axiom of separation if given a pair of distinct points $x, y \in x$

$$\exists G, H \in T \text{ s.t. } x \in G, y \notin G, y \in H, x \notin H$$

In this case the space (X, T) is called T_1-space or Frechet space.

Example 2: Let X = {a, b, c} with topology T = {ϕ, X, {a}, {b}, {a, b}} defined on X is not a T_1-space because for $a, c \in X$, we have open sets {a} and X such that $a \in \{a\}$, $c \notin \{a\}$. This shows that we cannot find an open set which contains c but not a, so (X, T) is not a T_1-space. But we have already showed that (X, T) is a T_0-space. This shows that a T_0-space may not be a T_1-space. But the converse is always true.

Theorem 1: A topological space (X, T) is a T_1-space if f(x) is closed for each $x \in X$. In a topological space, show that T_1-space $\Leftrightarrow$ each point is a closed set.

Proof: (i) Let (X, T) be a topological space s.t. {x} is closed $\forall\ x \in X$.

To prove that X is T_1-space.

Consider $x, y \in X$ s.t. $x \neq y$.

Then, by hypothesis, {x} and {y} are disjoint closed sets. This means that X-{x} and X-{y} are T-open sets.

Write G = X-{y}, H = X – {x},

Then $G, H \in T$ s.t. $x \in G, y \in G, y \in H, x \notin H$.

This proves that (X, T) is a T_1-space.

(ii) Conversely, suppose that (X, T) is a T_1-space.

To prove that {x} is closed $\forall\ x \in X$.

Since X is a T_1-space.

$\therefore$ Given a pair of distinct points $x, y \in X, \exists\ G, H \in T$.

s.t. $x \in G, y \notin G$ and $y \in H, x \notin H$.

Evidently, $G \subset X - \{y\}$, $H \subset X - \{x\}$.

Given any $x \in X - \{y\} \Rightarrow \exists\ G \in T$ s.t. $x \in G \subset x - \{y\}$.

This proves that every point x of X – {y} is an interior point of X – {y}, meaning thereby X – {y} is open, i.e., {y} is closed. Furthermore, given any $y \in X - \{x\} \Rightarrow \exists\ H \in T$ s.t. $y \in H \subset X - \{x\}$.

This implies that every point y of X – {x} is an interior point of X – {x}. Hence X – {x} is open , i.e.,

{x} is closed.

Notes

Finally {x}, {y} are closed sets in X.

Generalising this result.

{x} is closed $\forall$ x $\in$ X.

Example 3: Prove that in a T_1-space all finite sets are closed.

Solution: Let (X, T) be a T_1-space.

To prove that {x} is closed $\forall$ x $\in$ X.

Now write (ii) part of the proof of the theorem 1

Let A be an arbitrary finite subset of X.

Then A = $\cup$ {{x}} : x $\in$ A}

= finite union of closed sets = closed set.

$\therefore$ A is a closed set.

Example 4: A topological space (X, T) is a T_1-space iff T contains the cofinite topology on X.

Solution: Let (X, T) be a T_1-space.

To prove that T contains cofinite topology on X, we have to show that T contains subsets A of X s.t. X – A is finite.

Here we shall make use of the fact that

X is T_1-space $\Rightarrow$ {x} is closed $\forall$ x $\in$ X

$\Rightarrow$ X – {x} is open subset of X $\Rightarrow$ X – {x} $\in$ T

Thus X – {x} $\in$ T $\Rightarrow$ X – (X – {x}) = {x} = finite set.

This is true $\forall$ x $\in$ X.

Hence by definition T contains cofinite topology on X.

Conversely, suppose that T contains cofinite topology on X.

To prove that (X, T) is T_1-space.

{x} is a finite subset of X.

Also T contains cofinite topology.

Consequently X – {x} $\in$ T so that

{x} is closed $\forall$ x $\in$ X

$\Rightarrow$ (X, T) is T_1-space.

Theorem 2: A topological space X is a T_1-space of X iff every singleton subset {x} of X is closed.

Proof: Let X be a T_1-space and x $\in$ X.

By the T_1-axiom, we know that if y $\neq$ x $\in$ X, than there exists an open set G_y which contain y but not x i.e.

y = $G_y \subseteq \{x\}^c$

Notes

Then $\{x\}^c = \cup \{y : y \neq x\} \subseteq \{G_y : y \neq x\} \subseteq \{x\}^c$.

Therefore $\{x\}^c = \cup \{G_y : y \neq x\}$.

Thus $\{x\}^c$ being the union of open sets is an open set. Hence {x} is a closed set.

Conversely, let us suppose that {x} is closed.

We have to prove that X is a T_1-space.

Let x and y be two distinct points of X.

Since {x} is a closed set, $\{x\}^c$ is an open set which contains y but not x.

Similarly $\{y\}^c$ is an open set which contains x but not y.

Hence X is a T_1-space.

Theorem 3: The property of being a T_1-space is preserved by one-to-one onto, open mappings and hence is a topological property.

Proof: Let (X, T) be a T_1-space and let (Y, V) be a space homomorphic to the topological space (X, T).

Let f be a one-one open mapping of (X, T) onto (Y, V).

We shall prove that (Y, V) is also a T_1-space.

Let y_1, y_2 be any two distinct points of y.

Since the mapping f is one-one onto, there exist, points x_1 and x_2 in X such that

$$x_1 \neq x_2 \text{ and } f(x_1) = y_1 \text{ and } f(x_2) = y_2$$

Since (X, T) is a T_1-space, there exists T-open sets G and H such that

$$x_1 \in G \text{ but } x_2 \in G$$

$$x_2 \in H \text{ but } x_1 \in H$$

Again, since f is an open mapping, f[G] and f[H] are V-open subsets such that

$$f(x) \in f[G] \text{ but } f(x_2) \notin f[G]$$

$$\text{and } f(x_2) \in f[H] \text{ but } f(x_1) \in f[x]$$

Hence (Y, V) is also a T_1-space.

Thus, the property of being a T_1-space is preserved under one-one onto, open mappings.

Hence it is a topological property.

Theorem 4: Every subspace of T_1-space is a T_1-space i.e. the property being a T_1-space is hereditary.

Proof: Let (X, T) be a T_1-space and let (X*, T*) be a subspace of (X, T).

Let x_1 and x_2 be two distinct point of X*. Since $X^* \subset X$, x_1 and x_2 are also distinct points of X. But (X, T) is a T_1-space, therefore there exist T-open sets G and H such that

$x_1 \in G$ but $x_2 \notin G$

and $x_2 \in H$ but $x_1 \notin H$

Then $G_1 = G \cap X^*$

and $H_1 = H \cap X^*$ are T*-open sets such that

$X_1 \in G_1$ but $x_2 \notin G_1$

Notes

and $x_2 \in H_2$ but $x_1 \notin H_1$

Hence (X*, T*) is a T_1-space.

Self Assessment

1. Show that any finite T_1-space is a discrete space. Is a discrete space T_1 space? Justify your answer.
2. If (X, T) is a T_0-space and T_1 is finer than T, then (X, T_1) is also T_0-space.
3. A finite subset of a T_1-space has no cluster point.
4. If (X, T) is a T_1-space and $T^* \geq T$, then (X, T*) is also a T_1-space.

17.2 T_2-Axiom of Separation or Hausdorff Space

A topological space (X, T) is said to satisfy the T_2-axiom or separation if given a pair of distinct points $x, y \in X$.

$\exists$ G, H $\in$ T s.t. $x \in G$, $y \in H$, $G \cap x = \phi$

In this case the space (X, T) is called a T_2-space or Hausdorff space or separated space.

Example 5: Let X = {1, 2, 3} be a non-empty set with topology T = P(X) (all the subsets of X, powers set or discrete topology). Hence

For 1, 2 $\quad 1 \in \{1\}, 2 \notin \{1\}$

For 2, 3 $\quad 2 \in \{2\}, 3 \notin \{2\}$

For 3, 1 $\quad 3 \in \{3\}, 1 \notin \{3\}$ and (X, T) is a T_2-space

For 1, 2 $\quad 1 \in \{1\}, 2 \in \{2\} \Rightarrow \{1\} \cap \{2\} = \phi$

For 2, 3 $\quad 2 \in \{2\}, 3 \in \{3\} \Rightarrow \{2\} \cap \{3\} = \phi$

For 3, 1 $\quad 3 \in \{3\}, 1 \in \{1\} \Rightarrow \{3\} \cap \{1\} = \phi$

Example 6: Show that every T_2-space is a T_1-space.

Solution: Let (X, T) be a T_2-space.

Let x, y be any two distinct points of X. Since the space is T_2, then there exist open nhd. G and H of x and y respectively such that $G \cap H = \phi$.

Thus G and H are open sets such that

$x \in G$ but $y \notin G$

and $y \in H$ but $x \notin H$

Hence the space is T_1.

Example 7: Prove that every T_2-space is a T_1-space but converse is not true. Justify.

Solution: Let (X, T) be a T_2-space.

Let x, y be any two distinct points of X.

Since the space is T_2, $\exists$ open nhds G and H of x and y respectively such that $G \cap H = \phi$

Notes

Thus, G and H are open sets such that

$x \in G$ but $y \notin G$

and $y \in H$ but $x \notin H$

Hence, the space (X, T) is a T_1-space.

Conversely, let us consider the cofinite topology T on an infinite set X.

Let x be an arbitrary point of X.

by definition of T,

X – {x} is open, for {x} is finite set and so {x} is T-closed.

Thus, every singleton subset of X is closed.

It follows that the space (X, T) is a T_1-space. Now we shall show that the space (X, T) is not a T_2-space.

For this topology, no two open subsets of X can be disjoint.

Let if possible G and H be two open disjoint subsets of X, then

$$G \cap H = \phi$$

$$\Rightarrow \quad (G \cap H)' = \phi'$$

$$\Rightarrow \quad G' \cup H' = X \qquad \text{(by De-Morgan's law)}$$

Here $G' \cup H'$ being the union of two finite sets is finite, where as X is infinite.

Hence for this topology no two open sets can be disjoint i.e. no two distinct points can be separated by open sets.

Hence, (X, T) is not T_2-space.

Theorem 5: Every subspace of a T_2-space is a T_2-space

or

Prove that every subspace of a Hausdorff space is also Hausdorff.

Proof: Let (X, T) be a Hausdorff space and (Y, T_y) be a subspace of it.

Let x and y be any two distinct points of Y.

Then x and y are distinct points of X.

But (X, T) is a Hausdorff space, $\exists$ T-open nhds. G and H of x and y respectively such that

$$G \cap H = \phi$$

Consequently, $Y \cap G$ and $Y \cap H$ are T_y-open nhds of x and y respectively.

Also $x \in G, x \in Y \Rightarrow x \in Y \cap G$

and $y \in H, y \in y \Rightarrow y \in Y \cap H$

and since $G \cap H = \phi$, we have

$(Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = Y \cap \phi = \phi$

This shows that (Y, T_y) is also a T_2-space. Hence, every subspace of a Hausdorff space is also a Hausdorff space.

Notes

Theorem 6: The property of being a Hausdorff space is a topological invariant.

or

The property of being a Hausdorff space is preserved by one-one onto open mapping and hence is a topological property.

Proof: Let (X, T) be a T_2-space and let (Y, T_y) be any topological space.

Let f be a one-one open mapping of X onto Y. Let y_1, y_2 be two distinct elements of Y. Since f is one-one onto map, there exists distinct elements x_1 and x_2 of X such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

Since (X, T) is a T_2-space, ∃ T-open nhds. G and H of x_1 and x_2 such that $G \cap H = \phi$

Now, f being open, it follows that f(G) and f(H) are open subsets of Y such that

$$y_1 = f(x_1) \in f(G)$$

$$y_2 = f(x_2) \in f(H)$$

and $f(G) \cap f(H) = f(G \cap H) = f(\phi) = \phi$

This shows that (Y, T_y) is also a T_2-space.

Since a property being a T_2-space is preserved under one-one, onto, open maps, it is preserved under homeomorphism.

Hence, it is a topological property.

Theorem 7: Prove that every compact subset of Hausdorff space is closed.

Proof: Let (Y, T*) be a compact subset of Hausdorff space (X, T).

In order to prove that Y is T-closed, we have to show that X – Y is T-open.

Let x be an arbitrary element of X – Y.

Since (X, T) is a T_2-space, then for each $y \in Y$, ∃ T-open sets G_y and H_y such that

$$x \in G_y,\ y \in H_y \text{ and } G_y \cap H_y = \phi$$

Now consider the class

$\mathcal{C} = \{H_y \cap Y : y \in Y\}$

Clearly, $\mathcal{C}$ is T*-open cover of Y.

Since (Y, T*) is a compact subset of (X, T), there must exist a finite sub cover of $\mathcal{C}$ i.e. ∃ n points $y_1, y_2, ..., y_n$ in Y such that

$\{H_{yi} \cap Y : i \in T_n\}$ is a finite sub cover of $\mathcal{C}$.

Thus $Y \subset \bigcup_{i=1}^{n} \{H_{y_i}\}$

Let $N = \bigcap_{i=1}^{n} \{G_{yi}\}$, then N is T-nhd of x, and $N \cap \left[\bigcup_{i=1}^{n} \{H_{y_i}\}\right] = \phi$.

Thus, $N \cap Y = \phi \Rightarrow N \subset X - Y$

i.e. X – Y contains a T-nhd of each of its points.

Hence, X – Y is T-open i.e. Y is T-closed.

Notes

Example 8: Show that every convergent sequence in Hausdorff space has a unique limit.

Solution: Let (X, T) be a Hausdorff space.

Let $<x_n>$ be a sequence of points of Hausdorff space X.

Let $\underset{n\to\infty}{\text{Lt}}\ x_n = x$

Suppose, if possible,

$\underset{n\to\infty}{\text{Lt}}\ x_n = y$, where $x \neq y$.

Since X is a Hausdorff space, ∃ open sets G and H such that $x \in G$, $y \in H$

and $G \cap H = \phi$ ...(1)

Since $x_n \to x$ and $x_n \to y$

and G, H are nhds of x and y respectively, ∃ positive integers n_1 and n_2 such that

$x_n \in G\ \forall\ n \geq n_1$ and

$x_n \in H\ \forall\ n \geq n_2$

Let $n_0 = \max(n_1, n_2)$, then $x_n \in G \cap H\ \forall n \geq n_0$

This contradicts (1).

Hence, the limit of the sequence must be unique.

Note Converse of the above theorem is not true.

Example 9: Show that each singleton subset of a Hausdorff space is closed.

Solution: Let X be a Hausdorff space and let $x \in X$.

Let $y \in X$ be any arbitrary point of X other than x i.e. $x \neq y$.

Since X is a T_2-space, ∃ a nhd of y which does not contain x.

It follows that y is not a limit point of {x} and consequently $D(\{x\}) = \phi$

Hence $\{\bar{x}\} = x$.

This shows that {x} is T-closed.

Example 10: Show that every finite T_2-space is discrete.

Solution: Let (X, T) be a finite T_2-space. We know that every singleton subset of X is T-closed. Also a finite union of closed sets is closed. It follows that every finite subset of X is closed.

Hence, the space is discrete.

Theorem 8: A first countable space in which every convergent sequence has a unique limit is a Hausdorff space.

Notes

Proof: Let (X, T) be a first countable space in which every convergent sequence has a unique limit. If possible, let (X, T) be not a Hausdorff space.

Then given $x, y \in X$, $x \neq y$, $\exists$ open sets G and H

$$\text{such that } x \in G, y \in H, G \cap H \neq \phi$$

Now (X, T) being first countable, there exists monotone decreasing local bases

$\mathcal{B}_x = \{B_n(x) : x \in N\}$ and

$\mathcal{B}_y = \{B_n(y) : n \in N\}$ at x and y respectively.

Clearly, $B_n(x) \cap B_n(y) \neq \phi \qquad \forall\ n \in N$

[$\because$ $B_n(x)$ and $B_n(y)$ are open nhds. of x and y respectively]

Let $\quad x_n \in B_n(x) \cap B_n(y) \qquad \forall\ n \in N$

But $B_n(x)$ and $B_n(y)$ being monotone decreasing local bases at x and y respectively, $\exists$ a positive integer n_0 such that

$$n > n_0 \Rightarrow B_n(x) \subset G \quad \text{and}$$

$$B_n(y) \subseteq H$$

$\Rightarrow \qquad x_n \in B_n(x) \subseteq G$ and

$\qquad\quad x_n \in B_n(y) \subseteq H$

$\Rightarrow \quad x_n \in G$ and $x_n \in H$

$\therefore \quad x_n \to x$ and $x_n \to y$

But, this contradicts the fact that every convergent sequence in X has a unique limit.

Hence, (X, T) must be a Hausdorff space.

Theorem 9: The product space of two Hausdorff spaces is Hausdorff.

Proof: Let X and Y be two Hausdorff spaces. We shall prove that X × Y is also a Hausdorff spaces.

Let (x_1, y_1) and (x_2, y_2) be any two distinct points of X × Y.

Then either $x_1 \neq x_2$ or $y_1 \neq y_2$

Let us take $x_1 \neq x_2$

Since X is a Hausdorff space, $\exists$ T open nhds. G and H of x_1 and x_2 respectively such that $x_1 \in G$, $x_2 \in H$ and $G \cap H = \phi$

Then G × Y and H × Y are open subsets of X × Y such that

$(x_1, y_1) \in G \times Y$,

$(x_2, y_2) \in H \times Y$ and

$$(G \times Y) \cap (H \times Y) = (G \cap H) \times Y$$

$$= \phi \times Y = \phi$$

Thus, in this case, distinct points (x_1, y_1) and (x_2, y_2) of X × Y have disjoint open nhds.

Similarly, when $y_1 \neq y_2$ $\exists$ disjoint open nhds of (x_1, y_1) and (x_2, y_2)

Hence X × Y is Hausdorff.

Notes

Self Assessment

5. Show that one-to-one continuous mapping of a compact topological space onto a Hausdorff space is a homeomorphism.
6. The product of any non-empty class of Hausdorff spaces is a Hausdorff space. Prove it.
7. Show that if (X, T) is a Hausdorff space and T* is finer than T, then (X, T*) is a T_2-space.
8. Show that every finite Hausdorff space is discrete.

17.3 Summary

- T_0-axiom of separation:

  A topological space (X, T) is said to satisfy the T_0-axiom

  If for $x, y \in X$, either $\exists\, G \in T$ s.t. $x \in G, y \notin G$

  or $\exists\, H \in T$ s.t. $y \in H, x \notin H$
- T_1-axiom:

  A topological space (X, T) is said to satisfy the T_1-axiom if

  for $x, y \in X\ \exists\, G, H \in T$

  s.t. $x \in G, y \notin G; y \in H, x \notin H$
- T_2-axiom:

  A topological space (X, T) is said to satisfy the T_2-axiom if for $x, y \in X$

  $\exists\, G, H \in T$ s.t. $x \in G, y \in H, G \cap H = \phi$

17.4 Keywords

Cofinite topology: Let X be a non-empty set, and let T be a collection of subsets of X whose complements are finite along with ϕ, forms a topology on X and is called cofinite topology.

Compact: A compact space is a topological space in which every open cover has a finite sub cover.

Discrete: Let X be any non-empty set and T be the collection of all subsets of X. Then T is called the discrete topology on the set X.

Indiscrete space: Let X be any non-empty set and T = {X, ϕ}. Then T is called the indiscrete topology and (X, T) is said to be an indiscrete space.

Limit point: A point $x \in X$ is said to be the limit point of $A \subset X$ if each open set containing x contains at least one point of A different from x.

17.5 Review Questions

1. Show that A finite subset of a T_1-space has no limit point.
2. Prove that for any set X there exists a unique smallest T such that (X, T) is a T_1-space.
3. (X, T) is a T_1-space iff the intersection of the nhds of an arbitrary point of X is a singleton.
4. Show that a topological space X is a T_1-space iff each point of X is the intersection of all open sets containing it.

Notes

5. For any set X, there exists a unique smallest topology T such that (X, T) is a T_1-space.
6. A T_1-space is countably compact iff every infinite open covering has a proper subcover.
7. If (X, T) is a T_1-space and $T^* \geq T$, then (X, T*) is also a T_1-space.
8. If (X, T_1) is a Hausdorff space, (X, T_2) is compact and $T_1 \leq T_2$ than $T_1 = T_2$.
9. If f and g are continuous mappings of a topological space X into a Hausdorff space, then the set of points at which f and g are equal is a closed subset of X.
10. If f is a continuous mapping of a Hausdorff space X into itself, show that the set of fixed points; i.e. {x : f(x) = x}, is closed.
11. Show that every infinite Hausdorff space contains an infinite isolated set.
12. If (X, T) is a T_2-space and $T^* \geq T$, then prove that (X, T*) is also a T_2-space.

17.6 Further Readings

Books

Eric Schechter (1997), *Handbook of Analysis and its Foundations,* Academic Press.

Stephen Willard, *General Topology,* Addison Wesley, 1970 reprinted by Dover Publications, New York, 2004.

Unit 18: Normal Spaces, Regular Spaces and Completely Regular Spaces

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define normal space;
- Solve the problems on normal space;
- Discuss the regular space;
- Describe the completely regular space;
- Solve the problems on regular and completely regular space.

Introduction

Now we turn to a more through study of spaces satisfying the normality axiom. In one sense, the term "normal" is something of a misnomer, for normal spaces are not as well-behaved as one might wish. On the other hand, most of the spaces with which we are familiar do satisfy this axiom, as we shall see. Its importance comes from the fact that the results one can prove under the hypothesis of normality are central to much of topology. The Urysohn metrization theorem and the Tietze extension theorem are two such results; we shall deal with them later. We shall study about regular spaces and completely regular spaces.

18.1 Normal Space

A topological space (X, T) is said to be normal space if given a pair of disjoint closed sets C_1, $C_2 \subset X$.

$\exists$ disjoint open sets $G_1, G_2 \subset X$ s.t. $C_1 \subset G_1$, $C_2 \subset G_2$.

Example 1: Metric spaces are normal.

Solution: Before proving this, we need a preliminary fact. Let X be a metric space with metric d. Given a subset $A \subset X$ define the distance d(x, A) from a point $x \in X$ to A to the greatest lower bound of the set of distances d(x, a) from x to points $a \in A$. Note that $d(x, A) \geq 0$, and $d(x, A) = 0$ iff x is in the closure of A since $d(X, A) = 0$ is equivalent to saying that every ball $B_r(x)$ contains points of A.

Example 2: A compact Hausdorff space is normal.

Solution: Let A and B be disjoint closed sets in a compact Hausdorff space X. In particular, this implies that A and B are compact since they are closed subsets of a compact space. By the argument in the proof of the preceding example we know that for each $x \in A$, $\exists$ disjoint open sets U_x and V_x with $x \in U_x$ and $B \subset V_x$. Letting x very over A, we have an open cover of A by the sets U_x.

So, there is a finite subcover. Let U be the union of the sets U_x in this finite subcover and let V be the intersection of the corresponding sets V_x. Then U and V are disjoint open nhds. of A and B.

Example 3: A closed sub-space of a normal space is a normal space.

Solution: Let (X, T) be a topological space which is normal and (Y, U) a closed sub-space of (X, T) so that Y is closed in X. To prove that Y is a normal space.

Let $F_1, F_2 \subset Y$ be disjoint sets which are closed in Y. Y is closed in X, a subset F of Y is closed in Y iff F is closed in X.

$\therefore$ F_1 and F_2 are disjoint closed sets in X.

By the property of normal space (X, T).

$$\exists\, G_1, G_2 \in T \text{ s.t. } F_1 \subset G_1, F_2 \subset G_2, G_1 \cap G_2 = \phi$$

$$F_1 \subset G_1 \Rightarrow F_1 \cap Y \subset G_1 \cap Y \Rightarrow F_1 = F_1 \cap Y \subset G_1 \subset Y$$

$$\Rightarrow F_1 \subset G_1 \cap Y.$$

Similarly $F_2 \subset G_2 \Rightarrow F_2 \subset G_2 \subset Y$.

By definition of relative topology,

$$G_1, G_2 \in T \Rightarrow Y \cap G_1, Y \cap G_2 \in U$$

Also $(G_1 \cap Y) \cap (G_2 \cap Y) = (Y \cap Y) \cap (G_1 \cap G_2) = Y \cap \phi = \phi$.

Finally given a pair of disjoint closed sets F_1, F_2 in Y, $\exists$ disjoint sets.

$$G_1 \cap Y, G_2 \cap Y \in U \text{ s.t. } F_1 \subset G_1 \cap Y, F_2 \subset G_2 \cap Y.$$

This proves that (Y, U) is a normal space.

Self Assessment

1. Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.
2. Give an example of a normal space with a subspace that is not normal.
3. Show that paracompact space (X, T) is normal.

18.2 Regular Space

A topological space (X, T) is said to be regular space if: given an element $x \in X$ and closed set $F \subset X$ s.t. $x \notin F$, $\exists$ disjoint open sets $G_1, G_2 \subset X$ s.t. $x \in G_1$, $F \subset G_2$.

Notes

> *Notes* A regular T_1-space is called a T_3-space.
>
> A normal T_1-space is called a T_4-space.

Examples of Regular Space

1. Every discrete space is regular.
2. Every indiscrete space is regular.

Example 4: Give an example to prove that a regular space is not necessarily a T_1-space.

Solution: Let X = {a, b, c} and let T = {T^ϕ, X, {c}, {a, b}} be a topology on X.

The closed subsets of X are ϕ, X, {c}, {a, b). Clearly this space (X, T) satisfies the R-axiom and it is a regular space. But it is not a T_1-space, for the singleton subset {b} is not a closed set.

Thus, this space (X, T) is a regular but not a T_1-space.

Example 5: Give an example of T_2-space which is not a T_3-space.

Solution: Consider a topology T on the set $\mathcal{R}$ of all real numbers such that the T-nhd. of every non-zero real number is the same as its $\cup$-nhd but T-nhd. of 0 are of the form

$$G - \left\{\frac{1}{n} : n \in N\right\}$$

where G is a $\cup$-nhd. of 0.

Then T is finer than $\cup$.

Now, (R, $\cup$) is Hausdorff and $\cup \subset$ T, so (R, T) is Hausdorff.

But $\left\{\frac{1}{n} : n \in N\right\}$ being T-closed, cannot be separated from 0 by disjoint open sets.

Hence, (R, T) is not a regular space.

Thus, (R, T) is T_2 but not T_3.

Theorem 1: A topological space (X, T) is a regular space iff each nhd. of an element x $\in$ X contains the closure of another nhd. of x.

Proof: Let (X, T) be a regular space.

Then for a given closed set F and x $\in$ X such that x $\notin$ F there exist disjoint open sets G, H such that

$x \in G$ and $F \subset H$.

Now $x \in G \Rightarrow G$ is a nhd. of x ($\because$ G is open)

Again, $G \cap H = \phi$

$\Rightarrow G \subset X - H$

$\Rightarrow \overline{G} \subset (\overline{X - H}) = X - H$ (Since H is open and so X–H is closed)

Notes

$\Rightarrow \bar{G} \subset X - H$

$\Rightarrow \bar{G} \subset X - F$

$\Rightarrow \bar{G} \subset X - F = M$ (say)

$\Rightarrow \bar{G} \subset M$.

Since F is a closed set, M is an open set and

$x \notin F \Rightarrow x \in X - F$.

$\Rightarrow x \in M$, thus M is a nhd. of x.

Hence, if M is a nhd. of x, there exists a nhd. G of x such that

$x \in G \subset \bar{G} \subset M$.

Conversely, Let N_1 and N_2 be the nhds. of $x \in X$.

If $\overline{N_2} \subset N_1$, then we have to show that (X, T) is a regular space.

Let F be a closed subset of X and let x be an element of X such that $x \notin F$.

Now F is closed and $x \notin F$.

$\Rightarrow x \in X - F$ and $X - F$ is open.

$\Rightarrow X - F$ is a nhd. of x.

Let $X - F = N_1$, then by hypothesis

$x \in N_2 \subset \overline{N_2} \subset X - F$ $\quad\quad (\because \overline{N_2} \subset N_1)$

Let us write $N_2 = G_1$ and

$X - \overline{N_2} = G_2$

Then $\quad G_1 \cap G_2 = N_2 \cap (X - \overline{N_2})$

$= (N_2 \cap X) - (N_2 \cap \overline{N_2})$

$= N_2 - N_2$

$= \phi$.

$\therefore G_1 \cap G_2 = \phi$.

Also $x \in N_2 \Rightarrow x \in G_1$

and $\overline{N_2} \subset X - F \Rightarrow F \subset X - \overline{N_2}$

or $F \subset G_2$

Since $\overline{N_2}$ is a closed set, therefore G_2 is open.

Thus, we have proved that for a given closed subset F of X and $x \in X$ such that $x \notin F$ there exist disjoint open subsets G_1, G_2 such that

$x \in G$, and $F \subset G_2$

Hence X is a regular space.

Notes

Theorem 2: Prove that a normal space is a regular space i.e. to say, X is a T_4-space $\Rightarrow$ X is a T_3-space.

Proof: Let (X, T) be a T_4-space so that

(i) X is a T_1-space

(ii) X is a regular space

To prove that X is a T_3-space. For this we must show that

(iii) X is a T_1-space

(iv) X is a regular space

Evidently (i) $\Rightarrow$ (iii)

If we show that (ii) $\Rightarrow$ (iv), the result will follow. Let $F \subset X$ be a closed set and $x \in X$ s.t. $x \notin F$. X is a T_1-space $\Rightarrow$ $\{x\}$ is closed in X.

By normality, given a pair of disjoint closed sets $\{x\}$ and F in X, $\exists$ disjoint open sets G, H in X s.t. $\{x\} \subset G$, $F \subset X$, i.e. given a closed set $F \subset X$ and $x \in X$ s.t. $x \notin F$. $\exists$ disjoint open sets G, H in X s.t. $\{x\} \subset G$, $F \subset H$. This proves that (X, T) is a regular space.

Example 6: Show that the property of a space being regular is hereditary property.

Solution: Let $(Y, \mathcal{U})$ be a subspace of a regular space (X, T). We claim that the property of regularity is hereditary property. If we show that $(Y, \mathcal{U})$ is regular, the result will follow.

Let F be a $\mathcal{U}$-closed set and $p \in Y$ s.t. $p \notin F$.

Let $\bar{F}^T$ = closure of F w.r.t. the topology T. and $\bar{F}^{\mathcal{U}}$ = closure of F w.r.t. the topology $\mathcal{U}$ we know that $\bar{F}^{\mathcal{U}} = \bar{F}^T \cap Y$.

Since F is a $\mathcal{U}$-closed set $\Rightarrow F = \bar{F}^{\mathcal{U}} \Rightarrow F = \bar{F}^T \cap Y$.

$p \notin F \Rightarrow p \notin \bar{F}^T \cap Y \Rightarrow p \notin \bar{F}^T$ or $p \notin Y$

$\Rightarrow p \notin \bar{F}^T$ for $p \in Y$.

$\bar{F}^T$ is a T-closed set.

$\because$ closure of any set is always closed.

By the regularity of (X, T), given a closed set $\bar{F}^T$ and a point $p \in X$. s.t. $p \notin \bar{F}^T$; $\exists$ disjoint sets G, $H \in T$ with $p \in G$, $\bar{F}^T \subset H$.

Consequently, $F = \bar{F}^T \cap Y \subset H \cap Y$, $p \in G \cap Y$

$(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap (Y \cap Y) = \phi \cap Y = \phi$

Thus, we have shown that given a $\mathcal{U}$-closed set F and a point $p \in Y$ s.t. $p \notin F$, we are able to find out the disjoint open sets $G \cap Y$, $H \cap Y$ in Y s.t. $p \in G \cap Y$, $F \subset H \cap Y$.

This proves that $(Y, \mathcal{U})$ is regular. Hence proved.

Self Assessment

4. Show that the usual topological space $(R, \cup)$ is regular.

5. Show that every T_3-space is a T_2-space.

6. Give an example to show that a normal space need not be a regular.
7. Prove that regularity is a topological property.

18.3 Completely Regular Space

A topological space (X, T) is called a completely regular space if given a closed set $F \subset X$ and a point $x \in X$ s.t. $x \notin F$, $\exists$ a continuous map $f : X \to [0, 1]$ with the property,

$$f(x) = 0, f(F) = \{1\}$$

Example 7: Every metric space is a completely regular space.

Solution: Let (X, d) be a metric space.

Let $a \in X$ and F be a closed set in X not containing a.

Define $F : X \to R$ by

$$f(x) = \frac{d(x,a)}{d(x,a) + d(x,F)} \ \forall \ x \in X,$$

where $d(x,F) = \inf\{d(x, y) : y \in F\}$,

$$d(x, F) = 0 \Leftrightarrow x \in \overline{F} = F,$$

Consequently $d(x, a) + d(x, F) \neq 0$ as $a \notin F$.

Thus we see that $f \in \mathcal{C}(X, R)$, $0 \leq f(x) \leq 1$ for every $x \in X$, $f(a) = 0$ and $f(F) = \{1\}$.

Theorem 3: Every subspace of a completely regular space is completely regular i.e. complete regularity is hereditary property.

Proof: Let (Y, T_Y) be a subspace of a completely regular space (X, T).

Let F be a T_Y-closed subset of Y and $y \in Y - F$. Since F is a T_Y-closed, there exists a T-closed subset F* of X such that

$$F = Y \cap F^*$$

Also $y \notin F \Rightarrow y \notin Y \cap F^*$

$\Rightarrow \quad y \notin F^* \qquad (\because y \in Y)$

and $y \in Y \Rightarrow y \in X$.

It follows that F* is a T-closed subset of X and $y \in X - F^*$.

Since X is completely regular, there exists a continuous real valved function $f : X \to [0, 1]$, such that

$$f(y) = 0 \text{ and } f(F^*) = \{1\}.$$

Let g denote the restriction of f to Y. Then g is a continuous mapping of Y into [0, 1].

Now by the definition of g.

$$g(x) = f(x) \ \forall \ x \in Y.$$

Hence $f(y) = 0 \Rightarrow g(y) = 0$

and $f(x) = 1 \ \forall \ x \in F^*$

Notes

and $F \subset F^* \Rightarrow g(x) = f(x) = 1 \ \forall \ x \in F$

$\therefore \ g(F) = \{1\}$.

Hence for every T_Y-closed subset F of Y and for each point $y \in Y - F$, there exists a continuous mapping g of Y into [0, 1] such that

$$g(y) = 0 \quad \text{and} \quad g(F) = \{1\}.$$

Hence (Y, T_Y) is also completely regular.

Theorem 4: A completely regular space is regular.

Proof: Let (X, T) be a completely regular space, then given any closed set $F \subset X$ and $p \in X$ s.t. $p \notin F$; $\exists$ continuous map $f : X \to [0, 1]$ with the property that

$$f(p) = 0, \quad f(F) = \{1\}.$$

To prove that (X, T) is a regular space.

Consider the set [0, 1] with usual topology. It is easy to verity that [0, 1] is a T_2-space, then we can find out disjoint open sets G, H in [0, 1] s.t. $0 \in G$, $1 \in H$.

By hypothesis, f is continuous, hence $f^{-1}(G)$, $f^{-1}(H)$ are open in X.

$$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(H \cap G) = f^{-1}(\phi), = \phi$$

$$f^{-1}(G) = \{x \in X : f(x) \in G\}.$$

Furthermore,
$$f(p) = 0 \in G \Rightarrow f(p) \in G \Rightarrow p \in f^{-1}(G)$$

$$f(F) = 1 \in H \Rightarrow f(F) = \{1\} \subset H$$

$$\Rightarrow f(F) \subset H$$

$$\Rightarrow F \subset f^{-1}(H).$$

Given any closed set $F \subset X$ and $p \in X$ s.t. $p \notin F$; $\exists$ disjoint open sets $f^{-1}(G)$, $f^{-1}(H)$ in X s.t. $p \in f^{-1}(G)$, $F \subset f^{-1}(H)$, in X s.t. $p \in f^{-1}(G)$, $F \subset f^{-1}(H)$, showing thereby X is regular.

Theorem 5: A Tychonoff space is a T_3-space. Or Completely regular space $\Rightarrow$ regular space.

Proof: Let (X, T) be a Tychonoff space, then

(i) X is a T_1-space

(ii) X is a completely regular space.

To prove that (X, T) is a T_3-space, it suffices to show that

(iii) X is a T_1-space.

(iv) X is a regular space

Evidently (i) $\Rightarrow$ (iii)

Prove as in Theorem (1)

Hence the result.

Example 8: Prove that a topological space (X, T) is completely regular iff for every $x \in X$ and every open set G containing x there exists a continuous mapping f of X into [0, 1] such that

$$f(x) = 0 \quad \text{and} \quad f(Y) = 1 \ \forall \ y \in X - G.$$

Solution: Let (X, T) be a topological space for which the given conditions hold. Let F be a T-closed subset of X and let x be a point of X such that $x \notin F$. Then X – F is a T-open set containing x. By the given condition there exits a continuous mapping $f : X \rightarrow [0, 1]$ such that

$$f(x) = 0 \quad \text{and} \quad f(Y) = 1 \ \forall \ y \in X - (X - F) \text{ i.e. } y \in F.$$

Hence the space is completely regular.

Conversely, Let (X, T) be a completely regular space and let G be an open subset of X containing x.

Then X – G is a closed subset of X such that $x \notin X - G$. Since X is completely regular there exists a continuous mapping $f : X \rightarrow [0, 1]$ such that

$$f(x) = 0 \quad \text{and} \quad f(X - G) = \{1\}$$

Self Assessment

8. Let F be a closed subset of a completely regular space (X, T) and $x_0 \in F'$, then prove that there exists a continuous map $f : X \rightarrow [0, 1]$ s.t. $f(x_0) = 1$, $f(F) = \{0\}$.
9. Prove that a normal space is completely regular iff it is regular.

18.4 Summary

- A topological space (X, T) is said to be normal space if: given a pair of disjoint closed sets $C_1, C_2 \subset X$. $\exists$ disjoint open sets $G_1, G_2 \subset X$ s.t. $C_1 \subset G_1$, $C_2 \subset G_2$.
- Matric spaces are normal.
- A closed subspace of a normal space is a normal space.
- A topological space (X, T) is said to be regular space if: given an element $x \in X$ and closed set $F \subset X$ s.t. $x \notin F$, $\exists$ disjoint open sets $G_1, G_2 \subset X$ s.t. $x \in G_1$, $F \subset G_2$.
- A regular T_1-space is called a T_3-space.
- A normal T_1-space is called a T_4-space.
- A normal space is a regular.
- A topological space (X, T) is called a completely regular space if : given a closed set $F \subset X$ and a point $x \in X$ s.t. $x \notin F$, $\exists$ a continuous map $f : X \subset [0, 1]$ with the property, $f(x) = 0$, $f(F) = \{1\}$.
- Every metric space is a completely regular space.
- Complete regularity is hereditary property.
- A completely regular space is regular.

18.5 Keywords

Compact: A topological space (X, T) is called compact if every open cover of X has a finite sub cover.

Hausdorff Space: It is a topological space in which each pair of distinct points can be separated by disjoint neighbourhoods.

Metric Space: Any metric space is a topological space, the topology being the set of all open sets.

Tychonoff Space: Tychonoff space is a Hausdorff space (X, T) in which any closed set A and any $x \notin A$ are functionally separated.

Notes

18.6 Review Questions

1. Prove that regularity is a hereditary property.
2. Prove that normality is a topological property.
3. Prove that complete regularity is a topological property.
4. Show that if X is completely regular, then every pair of disjoint subsets A and B such that A is compact and B is closed, there exists a real valued continuous mapping F of X such that f(A) = {0} and f(B) = {1}.
5. Show that a closed subspace of a normal space is normal.
6. Show that a completely regular space is regular and hence a Tychonoff space is a T_3-space.
7. Give an example of Hausdorff space which is not normal.
8. Show that a topological space X is normal iff for any closed set F and an open set G containing F there exists an open set H such that

$$F \subset H,\ \bar{H} \subset G \text{ i.e. } F \subset H \subset \bar{H} \subset G.$$

18.7 Further Readings

Books

A.V. Arkhangel'skii, V.I. Ponomarev, *Fundamentals of General Topology: Problems and Exercises*, Reidel (1984).

J.L. Kelly, *General Topology*, Springer (1975).

Stephen Willard (1970), *General Topology*.

Unit 19: The Urysohn Lemma

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- State Urysohn's lemma;
- Understand the proof of Urysohn's lemma;
- Solve the problems on Urysohn's lemma.

Introduction

Saying that a space X is normal turns out to be a very strong assumption. In particular, normal spaces admit a lot of continuous functions. Urysohn's lemma is sometimes called "the first non-trivial fact of point set topology" and is commonly used to construct continuous functions with various properties on normal spaces. It is widely applicable since all metric spaces and all compact Hausdorff spaces are normal. The lemma is generalized by (and usually used in the proof of) the Tietze Extension Theorem.

19.1 Urysohn's Lemma

In topology, Urysohn's lemma is a lemma that states that a topological space is normal iff any two disjoint closed subsets can be separated by a function.

This lemma is named after the mathematician Pavel Samuilovich Urysohn.

19.1.1 Proof of Urysohn's Lemma

Urysohn's Lemma: Consider the set R with usual topology where $R = \{x \in R : 0 \le x \le 1\}$

A topological space (X, T) is normal iff given a pair of disjoint closed sets $A, B \subset X$, there is a continuous functions.

$$f : X \to R \text{ s.t. } f(A) = \{0\} \text{ and } f(B) = \{1\}.$$

Notes

Proof:

(1) Let R denote the set of all real numbers lying in the closed interval [0, 1] with usual topology. Let (X, T) be a topological space and let given a pair of disjoint closed sets $A, B \subset X$; $\exists$ a continuous map $f : X \to R$ s.t.

$f(A) = \{0\}, f(B) = \{1\}$.

To prove that (X, T) is a normal space.

Let $a, b \in R$ be arbitrary s.t. $a \leq b$

write $G = [0, a), H = (b, 1]$.

Then G and H are disjoint open sets in R.

Continuity of f implies that $f^{-1}(G)$ and $f^{-1}(H)$ are open in X.

Then our assumption says that

$f(A) = \{0\}, f(B) = \{1\}$

$f(A) = \{0\} \Rightarrow f^{-1}(\{0\} = f^{-1}(f(A)) \supset A$

$\Rightarrow f^{-1}(\{0\}) \supset A \Rightarrow A \subset f^{-1}(\{0\})$

Similarly $B \subset f^{-1}\{1\}$.

Evidently

$$\{0\} \subset [0, a) \Rightarrow f^{-1}(\{0\}) \subset f^{-1}([0, a))$$

$$\Rightarrow A \subset f^{-1}(\{0\}) \subset f^{-1}([0, a)]$$

$$\Rightarrow A \subset f^{-1}([0, a)) \Rightarrow A \subset f^{-1}(G)$$

$$\{1\} \subset (b, 1] \Rightarrow B \subset f^{-1}(\{1\}) \subset f^{-1}((b, 1])$$

$$\Rightarrow B \subset f^{-1}((b, 1]) \Rightarrow B \subset f^{-1}(H)$$

$$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$$

Given a pair of disjoint closed sets $A, B \subset X$, we are able to discover a pair of disjoint open sets,

$$f^{-1}(G), f^{-1}(H) \subset X \text{ s.t. } A \subset f^{-1}(G), B \subset f^{-1}(H).$$

This proves that (X, T) is a normal space.

(2) Conversely, suppose that R is a set of real numbers lying in the interval [0, 1] with usual topology. Also suppose that A, B are disjoint closed subsets of a normal space (X, T).

To prove that $\exists$ a continuous map.

$$f : (X, T) \to R \text{ s.t. } f(A) = \{0\}, f(B) = \{1\}.$$

Step (i): Firstly, we shall prove that $\exists$ a map

$$f : (X, T) \to R \text{ s.t. } f(A) = \{0\}, f(B) = \{1\}.$$

Write $T = \left\{t : t = \frac{m}{2^n}, \text{where } m, n \in N \text{ s.t. } m \leq 2^n\right\}$

Throughout the discussion we treat $t \in T$.

Making use of the fact that m takes 2^n values for a given value of n, we have

$$\sup(T) = \sup(t) = \sup\left(\frac{m}{2^n}\right) = \frac{1}{2^n}\sup(m) = \frac{2^n}{2^n} = 1 \sup(T) = 1$$

Notes

$t = \frac{1}{2}, 1,$ for $n = 1$

$t = \frac{1}{2^2}, \frac{2}{2^2}, \frac{3}{2^2}, \frac{4}{2^2},$ for $n = 2$

$A \cap B = \phi \Rightarrow A \subset X - B$

$\because$ $X - B$ is an open set containing a closed set A. Using the normality, we can find an open set $G \subset X$ s.t.

$$A \subset G \subset \overline{G} \subset X - B \quad ...(1)$$

Writing $G = H_{1/2}$, $X - B = H_1$, we get

$$A \subset H_{1/2} \subset \overline{H}_{1/2} \subset H_1$$

This is the first stage of our construction

Consider the pairs of sets $(A, H_{1/2})$, $(\overline{H}_{1/2}, H_1)$

Using normality, we obtain open sets, $H_{1/4}$, $H_{3/4} \subset X$ s.t.

$$A \subset H_{1/4} \subset \overline{H}_{1/4} \subset H_{1/2}$$

$$\overline{H}_{1/2} \subset H_{3/4} \subset \overline{H}_{3/4} \subset H_1$$

Combining the last two relations, we have

$$A \subset H_{1/4} \subset \overline{H}_{1/4} \subset H_{1/2} \subset \overline{H}_{1/2} \subset H_{3/4} \subset \overline{H}_{3/4} \subset H_1$$

This is the second stage of our construction.

If we continue this process of each dyadic rational m of the function $t = m/2^n$, where

$$n = 1, 2, ... \text{ and } m = 1, 2 ... 2^n - 1,$$

Then open sets H_t will have the following properties:

(i) $A \subset H_t \subset \overline{H}_t \subset H_1 \ \forall \ t \in T$

(ii) Given $t_1, t_2 \in T$ s.t.

$t_1 < t_2 \Rightarrow A \subset H_{t_1} \subset \overline{H}_{t_1} \subset H_{t_2} \subset \overline{H}_{t_2} \subset H_2$

Construct a function $f : X \to R$ s.t. $f(x) = 0 \ \forall \ x \in H_t$

and $f(x) = \{t : x \notin H_t\}$ otherwise

In both cases $x \in X$.

$f(x) = \sup\{t : x \in H_t\} = \sup\{t\} = \sup(T) = 1$

Thus $f(x) = 0 \ \forall \ x \in H_t$

and $f(x) = 1 \ \forall \ x \notin H_t$ otherwise

$f(x) = 0 \ \forall \ x \in H_t, A \subset H_t \ \forall \ t \in T \Rightarrow f(x) = 0 \ \forall x \in A \Rightarrow f(A) = \{0\}$

$f(x) = 1 \ \forall \ x \notin H_t, H_t \subset H_1 \ \forall \ t \in T \Rightarrow f(x) = 1 \ \forall \ x \notin H_1$

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$\Rightarrow f(x) = 1 \ \forall \ x \in X - B \ (\because X - B = H_1)$

$\Rightarrow f(x) = 1 \ \forall \ x \in B$

$\Rightarrow f(B) = \{1\}$

Thus we have shown that $f(A) = \{0\}$, $f(B) = \{1\}$.

Step (ii): Secondly, we shall prove that f is continuous. Let $a \in R$ be arbitrary then [0, a) and (a, 1] are open sets in R with usual topology.

Write $G_1 = f^{-1}([0, a))$, $G_2 = f^{-1}((a, 1])$.

Then G_1, G_2 can also be expressed as

$$\begin{aligned} G_1 &= \{x \in X : f(x) \in [0, a)\} \\ &= \{x \in X : 0 \leq f(x) < a\} \\ &= \{x \in X : f(x) < a\} \end{aligned}$$

$\because$ According to the construction of f

$0 \leq f(x) \leq 1 \ \forall \ x \in X$

$$\begin{aligned} G_2 &= \{x \in X : f(x) \in (a, 1)\} \\ &= \{x \in X : a < f(x) \leq 1\} \\ &= \{x \in X : a < f(x)\} \\ &= \{x \in X : f(x) > a\} \end{aligned}$$

Finally, $G_1 = \{x \in X : f(x) < a\}$, $G_2 = \{x \in X : f(x) > a\}$

We claim $G_1 = \bigcup_{t<a} H_1, \quad G_2 = \bigcup_{t>a} \left(\overline{H}_t\right)$

Any $x \in G_1 \Rightarrow f(x) < a \Leftrightarrow x \in H_t$ for some $t < a$

This proves that $G_1 = \bigcup_{t<a} H_t$

$x \in G_2 \Rightarrow f(x) > a \Leftrightarrow x$ is out side of $\overline{H}_t$ for $t > a$

$\Leftrightarrow x \in \bigcup_{t>a} \left(\overline{H}_t\right)'$

Hence we get $G_2 = \bigcup_{t>a} \left(\overline{H}_t\right)'$

Since an arbitrary union of open sets is an open set and hence

$\bigcup_{t<a} H_t, \ \bigcup_{t>a} \left(\overline{H}_t\right)' \subset X$

are open i.e., $G_1, G_2 \subset X$ are open, i.e.,

$f^{-1}([0, a))$, $f^{-1}((a, 1])$ are open in X.

$\therefore$ f is continuous

19.1.2 Solved Examples

Notes

Example 1: If F_1 and F_2 are T-closed disjoint subsets of a normal space (x, T), then there exist a continuous map g of X into [0, 1] such that

$$g(x) = \begin{cases} 0 & \text{if } x \in F_1 \\ 1 & \text{if } x \in F_2 \end{cases}$$

$f(F_1) = \{0\}$ and $g(F_2) = \{1\}$.

Solution: Here write the proof of step II of the Urysohn's Theorem.

Example 2: If F_1 and F_2 are T-closed disjoint subsets of a normal space (X, T) and [a, b] is any closed interval on the real line, then there exists a continuous map f of X into [a, b] such that

$$f(x) = \begin{cases} a & \text{if } x \in F_1 \\ b & \text{if } x \in F_2 \end{cases}$$

i.e., $f(F_1) = \{a\}, f(F_2) = \{b\}$

This problem in known as **general form of Urysohn's lemma.**

Solution: Let F_1 and F_2 be disjoint closed subset of (X, T).

To prove that $\exists$ a continuous map

$$f : X \to [a, b] \text{ s.t. } f(F_1) = \{a\}, f(F_2) = \{b\}$$

By Urysohn's lemma, $\exists$ a continuous map

$$g : X \to [0, 1] \text{ s.t. } g(F_1) = \{0\}, g(F_2) = \{1\}.$$

Define a map $h : [0, 1] \to [a, b]$ s.t.

$$h(x) = \frac{(b-a)x}{1-0} + a$$

i.e., $h(x) = x(b - a) + a$

[This is obtained by writing the equation of the straight line joining (0, a) and (1, b) and then putting y = h(x)].

Evidently $h(0) = a$, $h(1) = b - a + a = b$

Also h is continuous

Write f = hg

$g : X \to [0, 1]$, $h : [0, 1] \to [a, b]$

$\Rightarrow hg : X \to [a, b] \Rightarrow f : X \to [a, b]$

Product of continuous functions is continuous

Therefore $f(F_1) = (hg)(F_1) = h[g(F_1)] = h(\{0\}) = \{a\}$

$f(F_2) = (hg)(F_2) = h[g(F_2)] = h(\{1\}) = \{b\}$

Notes

Thus ∃ a continuous map.

$f : X \to [a, b]$ s.t. $f(F_1) = \{a\}$, $f(F_2) = \{b\}$

19.2 Summary

- Urysohn's lemma is a lemma that states that a topological space is normal iff any two disjoint closed subsets can be separated by a function.
- Urysohn's lemma is sometimes called "the first non-trivial fact of point set topology."
- Urysohn's lemma: If A and B are disjoint closed sets in a normal space X, then there exists a continuous function $f : X \to [0, 1]$ such that $\forall\ a \in A, f(a) = 0$ and $\forall\ b \in B, f(b) = 1$.

19.3 Keywords

Continuous map: A continuous map is a continuous function between two topological spaces.

Disjoint: A and B are disjoint if their intersection is the empty set.

Normal: A topological space X is a normal space if, given any disjoint closed sets E and F, there are open neighbourhoods U of E and V of F that are also disjoint.

Separated sets: A and B are separated in X if each is disjoint from the other's closure. The closures themselves do not have to be disjoint from each other.

19.4 Review Questions

1. Prove that every continuous image of a separable space is separable.
2. (a) Prove that the set of all isolated points of a second countable space is countable.

   (b) Show that any uncountable subset A of a second countable space contains at least one point which is a limit point of A.
3. (a) Let f be a continuous mapping of a Hausdorff non-separable space (X, T) onto itself.

   Prove that there exists a proper non-empty closed subset A of X such that f(A) = A.

   (b) Is the above result true if (X, T) is separable?
4. Examine the proof of the Urysohn lemma, and show that for given r,

$$f^{-1}(r) = \bigcap_{p>r} U_p - \bigcup_{q<r} U_q,$$

   p, q rational.
5. Give a direct proof of the Urysohn lemma for a metric space (X, d) by setting

$$f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$$

6. Show that every locally compact Hausdorff space is completely regular.
7. Let X be completely regular, let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

19.5 Further Readings

Notes

Books

G. F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw Hill.

J. L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

Online links

www.planetmath.org.

www.amazon.ca/lemmas-pumping...urysohns

Notes

Unit 20: The Urysohn Metrization Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Describe the Metrization;
- Explain the Urysohn Metrization Theorem;
- Solve the problems on Metrizability;
- Solve the problems on Urysohn Metrization Theorem.

Introduction

With Urysohn's lemma, we now want to prove a theorem regarding the metrizability of topological space. The idea of this proof is to construct a sequence of functions using Urysohn's lemma, then use these functions as component functions to embed our topological space in the metrizable space.

20.1 Metrization

Given any topological space (X, T), if it is possible to find a metric ρ on X which induces the topology T i.e. the open sets determined by the metric ρ are precisely the members of T, then X is said to the metrizable.

Example 1: The set $\mathbb{R}$ with usual topology is metrizable. For the usual metric on $\mathbb{R}$ induces the usual topology on $\mathbb{R}$. Similarly $\mathbb{R}^2$ with usual topology is metrizable.

Example 2: A discrete space (X, T) is metrizable. For the trivial metric induces the discrete topology T on X.

Example 3: Prove that if a set is metrizable, then it is metrizable in an infinite number of different ways.

Solution: Let X be a metrizable space with metric d.

Then $\exists$ a metric d on X which defines a topology T on X.

Notes

write $d_1(x, y) = \dfrac{d(x,y)}{1+d(x,y)}$ $\quad \forall\, x, y \in X$

Then d_1 is a metric on X.

Again $d_2(x, y) = \dfrac{d_1(x,y)}{1+d_1(x,y)}$ $\quad \forall\, x, y \in X$

Then d_2 is also a metric on X.

Continuing like this, we can define an infinite number of metrics on X.

Urysohn Metrization Theorem

Statement: Every regular second countable T_1-space is metrizable.

or

Every second countable normal space is metrizable.

Proof: Let (X, T) be regular second countable T_1-space.

To prove: (X, T) is metrizable.

X is regular and second countable.

$\Rightarrow$ X is normal.

Since (X, T) is second countable and hence there exists countable base $\mathcal{B}$ for the topology T on X. The elements of $\mathcal{B}$ can be enumerated as $B_1, B_2, B_3, \ldots$, where $\phi \neq B_n \in T$. Let $x \in X$ be arbitrary and $x \in U \in B$.

By normality of X,

$\exists\, V \in \mathcal{B}$ s.t. $x \in \overline{U} \subset V$

Write $C = \{(U, V) : U \times V \in \mathcal{B} \times \mathcal{B}$ s.t. $\overline{V} \subset V\}$

$\mathcal{B}$ is countable. $\Rightarrow \mathcal{B} \times \mathcal{B}$ is countable.

$\Rightarrow$ every subset of $\mathcal{B} \times \mathcal{B}$ is countable.

$\Rightarrow \mathcal{C}$ is countable.

For $\mathcal{C} \subset \mathcal{B} \times \mathcal{B}$

$\overline{U} \subset V \Rightarrow \overline{U} \cap (X - V) = \phi$

Also $\overline{U}$ and $X - V$ are closed in the normal space (X, T).

Hence, by Urysohn's lemma,

$\exists$ continuous map $f : X \to [0, 1] = I$, s.t.

$f(\overline{U}) = \{0\}$, $f(X - V) = \{1\}$

This implies $\quad f(x) = 0 \quad$ iff $\quad x \in \overline{U}$

and $\quad f(x) = 1 \quad$ iff $\quad x \in X - V$

Notes

Since continuous map f can be determined corresponding to every element (U, V) of $\mathcal{C}$. Take $\mathcal{F}$ as the collection of all such continuous maps.

$\mathcal{C}$ is countable $\Rightarrow$ $\mathcal{F}$ is countable.

To prove that $\mathcal{F}$ distinguishes points and closed sets. For this, let H is closed subset of X and $x \in X - H$.

Now X – H is a nhd of x so that

$$\exists B_j \in \mathcal{B} \text{ s.t. } x \in B_j \subset X - H$$

Regularity of X $\Rightarrow \exists G \in B$ s.t. $x \in G \subset \overline{G} \subset B_j$.

By definition of base, we can choose $B_j \in \mathcal{B}$ s.t. $x \in B_j \subset G$

Thus, $x \in B_i \subset \overline{B_i} \subset B_j \subset X - H$

or $\quad x \in \overline{B_i} \subset B_j \subset X - H$

This implies $(B_i, B_j) \subset \mathcal{C}$

If f be corresponding member of $\mathcal{F}$, then

$f(\overline{B_i}) = \{0\}, f(X - B_j) = \{1\}$

$B_j \subset X - H$

$\Rightarrow H \subset X - B_j$

$\Rightarrow f(H) \subset f(X - B_j) = \{1\}$

$\Rightarrow f(H) \subset \{1\}$

$\Rightarrow \overline{f(H)} \subset \overline{\{1\}} = \{1\}$.

{For {1} is closed in I = [0, 1] for the usual topology on I and so $\overline{\{1\}} = \{1\}$].

This implies $\overline{f(H)} \subset \{1\} \Rightarrow \overline{f(H)} = \{1\}$

Also $f(X - B_j) = \{1\}$.

Hence, $f(X - B_j) = \{1\} = \overline{f(H)}$

Also $f(\overline{B_i}) = \{0\}$

$$f(x) = 0 \notin \{1\} = f(X - B_j) = \overline{f(H)}$$

$$\Rightarrow \quad f(x) \notin \overline{f(H)} \qquad ...(1)$$

$\overline{f(H)}$ is closed subset of X.

Equation (1) shows that $\mathcal{F}$ distinguishes points and closed sets. Also, we have seen that $\mathcal{F}$ is countable family of continuous maps $f : X \to [0, 1]$.

If follows that X can be embedded as a subspace of the Hilbert Cube I^N which is metrizable.

Also, every subspace of metrizable space is metrizable.

This proves that (X, T) is metrizable.

Notes

Example 4: A compact Hausdorff space is separable and metrizable if it is second countable.

Solution: Let (X, T) be a compact Hausdorff space which is second countable.

To prove that X is separable and metrizable.

Firstly, we shall show that X is regular.

X is a Hausdorff space. $\Rightarrow$ X is a T_2-space.

$\Rightarrow$ X is also a T_1-space.

$\Rightarrow \{x\}$ is closed $\forall\ x \in X$.

Let $F \subset X$ be closed and $x \in X$ s.t. $x \notin F$.

Then F and {x} are disjoint closed subsets of X.

X is a compact Hausdorff space.

$\Rightarrow$ X is a normal space.

As we know that "A compact Hausdorff space is normal".

By definition of normality,

We can find a pair of open set $G_1, G_2 \subset X$

s.t. $\{x\} \subset G_1$, $F \subset G_2$, $G_1 \cap G_2 = \phi$

i.e. $x \in G_1$, $F \subset G_2$, $G_1 \cap G_2 = \phi$

$\therefore$ Given a closed set f and a point $x \in X$ s.t. $x \notin F$ implies that $\exists$ disjoint open sets $G_1, G_2 \subset X$ s.t. $x \in G_1$, $F \subset G_2$.

This implies X is a regular space. ...(2)

X is a second countable. [A second countable space is always separable] ...(3)

$\Rightarrow$ X is separable.

From (1), (2) and (3), it follows that (X, T) is a regular second countable T_1-space.

And so by Urysohn's theorem, it will follow that X is metrizable. ...(4)

From (3) and (4), it follows that X is separable and metrizable.

Hence the result.

Theorem 1: Every metrizable space is a normal Frechet space.

Proof: Let X be a metrizable space so that $\exists$ a metric d on X which defines a topology T on X.

Step (i): To prove that (X, T) is a Frechet space i.e. T_1 space.

Let (X, d) be a metric space. Let $x, y \in X$ be arbitrary s.t. $d(x, y) = 2r$. Let T be a metric topology.

We know that every open sphere is T open. Then $S_{r(x)}$, $S_{r(y)}$ are open sets s.t.

$x \in S_{r(x)}$, $y \notin S_{r(x)}$

$x \in S_{r(y)}$, $x \notin S_{r(y)}$

Hence (x, d) is a T_1-space.

Notes

Step (ii): To prove (X, T) is a normal space.

It follows by the theorem.

"Every metric space is normal space" proved in Unit -17.

Example 5: Every subspace of a metrizable is metrizable.

Solution: Let (Y, ρ) be a subspace of a metric space (X, d) which is metrizable so that

(i) ∃ a topology T on X defined by the metric d on X.

(ii) $Y \subset X$ and $\rho(x, y) = d(x, y) \quad \forall\ x, y \in X$

Then the map ρ is a restriction of the map 'd' of Y. Consequently ρ defines the relative topology $\mathcal{U}$ on Y, showing thereby Y is metrizable.

20.2 Summary

- Given any topological space (X, T), if it is possible to find a metric ρ on X which induces the topology T then X is said to be the metrizable.
- The set R with usual topology is metrizable.
- Urysohn metrization theorem: Every second countable normal space is metrizable.
- Every metrizable space is a normal Frechet space.

20.3 Keywords

Compact: X is compact iff every open cover of X has a finite subcover.

Hausdorff : A topological space (X, T) is a Hausdorff space if given any two points x, y ∈ X, ∃ G, H ∈ T s.t. x ∈ G, y ∈ H, G ∩ H = φ.

Normal: Let X be a topological space where one-point sets are closed. Then X is normal if two disjoint sets can be separated by open sets.

Regular: Let X be a topological space where one-point sets are closed. Then X is regular if a point and a disjoint closed set can be separated by open sets.

T_1 space: A topological space X is a T_1 if given any two points x, y ∈ X, x ≠ y, there exists neighbourhoods U_x of x such that y ∉ Ux.

20.4 Review Questions

1. Give an example showing that a Hausdorff space with a countable basis need not be metrizable.
2. Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.
3. Let X be a locally compact Hausdorff space. Let Y be the one-point compactification of X. Is it true that if X has a countable basis, then Y is metrizable? Is it true that if Y is metrizable, then X has a countable basis?
4. Let X be a compact Hausdorff space that is the union of the closed subspaces X_1 and X_2. If X_1 and X_2 are metrizable, show that X is metrizable.

Notes

5. A space X is locally metrizable if each point x of X has a neighbourhood that is metrizable in the subspace topology. Show that a compact Hausdorff space X is metrizable if it is locally metrizable.
6. Let X be a locally compact Hausdorff space. Is it true that if X has a countable basis, then X is metrizable? Is it true that if X is metrizable, then X has a countable basis?
7. Prove that the topological product of a finite family of metrizable spaces is metrizable.
8. Prove that every metrizable space is first countable.

20.5 Further Readings

Books

Robert Canover, *A first Course in Topology*, The Williams and Wilkins Company 1975.

Michael Gemignani, *Elementary Topology*, Dover Publications 1990.

Notes

Unit 21: The Tietze Extension Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- State the Tietze Extension Theorem;
- Understand the proof of Tietze Extension Theorem.

Introduction

One immediate consequence of the Urysohn lemma is the useful theorem called the Tietze extension theorem. It deals with the problem of extending a continuous real-valued function that is defined on a subspace of a space X to a continuous function defined on all of X. This theorem is important in many of the applications of topology.

21.1 Tietze Extension Theorem

Suppose (X, T) is a topological space. The space X is normal iff every continuous real function of defined point a closed subspace F of X into a closed interval [a, b] has a continuous extension.

$$f^* : X \to [a, b]$$

Proof:

(i) Suppose (X, T) is a topological space s.t. Every continuous real valued function $f : F \to [a, b]$ has a continuous extended function $g : X \to [a, b]$ where F is a closed subset of X, [a, b] being closed interval.

To prove X is a normal space.

Let F_1 and F_2 be two closed disjoint subsets of X.

Define a map $f : F_1 \cup F_2 \to [a, b]$

s.t. f(x) = a if $x \in F_1$ and f(x) = b is $x \in F_2$.

This map f is certainly continuous over the subspace $F_1 \cup F_2$. By assumption, f can be extended to a continuous map

$$g : X \to [a, b] \text{ s.t.}$$

$$g(x) = \begin{cases} a & \text{if } x \in F_1 \\ b & \text{if } x \in F_2 \end{cases}$$

The map g satisfies Urysohn's lemma and hence (X, T) is normal.

(ii) Conversely, suppose that (X, T) is a normal space.

Let f : F → [a, b] be a continuous map. F being a closed subset of X.

To prove that ∃ a continuous extension of f over X. For convenience, we take a = – 1, b = 1

Now we define a map f_0 = F → [–1, 1] s.t.

$$f_0(x) = f(x) \ \forall \ x \in F.$$

Suppose A_0 and B_0 are two subsets of F. s.t.

$$A_0 = \left\{x : f_0(x) \le -\frac{1}{3}\right\}, B_0 = \left\{x : f_0(x) \ge \frac{1}{3}\right\}$$

Then A_0 and B_0 are closed in X.

For F is closed in X. Applying general from of Urysohn's lemma, ∃ a continuous function

$$g_0 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right] \text{s.t. } g_0(A_0) = -\frac{1}{3}, g_0(B_0) = \frac{1}{3}$$

Write $f_1 = f_0 - g_0$

Then $|f_1(x)| = |(f_0 - g_0)(x)| = |f_0(x) - g_0(x)| \le \frac{2}{3}$

Let

$$A_1 = \left\{x : f_1(x) \le \left(-\frac{1}{3}\right), \left(\frac{2}{3}\right)\right\},$$

$$B_1 = \left\{x : f_1(x) \ge \frac{1}{3}, \frac{2}{3}\right\},$$

Then A_1, B_1 are non-empty disjoint closed sets in X and hence ∃ a continuous function s.t.

$$g_1 : X \to \left[-\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right]$$

$$g_1 (A_1) = -\frac{1}{3}, \frac{2}{3}, g_1(B_1) = \frac{1}{3}, \frac{2}{3}$$

Again we define a function f_2 and F s.t.

$$f_2 = f_1 - g_1 = f_0 - g_0 - g_1 = f_0 - (g_0 + g_1)$$

Then $|f_2(x)| = |f_0(x) - (g_0 + g_1)(x)| \le \left(\frac{2}{3}\right)^2$

Continuing this process, we get a sequence of function.

$$\langle f_0, f_1, f_2, \ldots, f_n, \ldots \rangle$$

defined on F s.t. $|f_n(x)| \le \left(\frac{2}{3}\right)^n$

and a sequence $\langle g_0, g_1, g_2, \ldots. \rangle$

defined on X s.t. $|g_n(x)| \le \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n$

Notes

$$f_n = f_0 - (g_0 + g_1 + ... + g_{(n-1)})$$

Write $$S_n = \sum_{r=0}^{n-1} g_r$$

Now S_n can be regarded as partial sums bounded continuous function defined on X. Since the space of bounded real valued function is complete and

$$|g_n(x)| \leq \frac{1}{3}.\left(\frac{2}{3}\right)^n \text{ and } \sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^n = 1,$$

the sequence $\langle S_n \rangle$ converges confirmly on X to g (say) when $|g(x)| \leq 1$.

$|f_n(x)| \leq \left(\frac{2}{3}\right)^n \Rightarrow \langle S_n \rangle$ converges uniformly on F to f_0 say

Hence g = f on F.

Thus g is a continuous extension of f to X which satisfies the given conditions.

21.2 Summary

- Tietze extension theorem:

  Suppose (X, ρ) is a topological space. The space X is normal iff every continuous real function f defined on a closed subspace F of X into a closed interval [a, b] has a continuous extension f^* $X \to [a, b]$

21.3 Keywords

Closed Set: A subset A of a topological space X is said to be closed if the set X – A is open.

Continuous Map: A function $f : R \to R$ is said to be continuous if for each $a \in R$ and each positive real number $\in$, there exists a positive real number δ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \in$.

Normal Space: A topological space (X, T) is said to be a normal space iff it satisfies the following axioms of Urysohn: If F_1 ad F_2 are disjoint closed subsets of X then there exists a two disjoint subsets one containing F_1 and the other containing F_2.

21.4 Review Questions

1. Show that the Tietze extension theorem implies the Urysohn lemma.
2. Let X be metrizable. Show that the following are equivalent:
   (a) X is bounded under every metric that gives the topology of X.
   (b) Every continuous function $\phi : X \to R$ is bounded.
   (c) X is limit point compact.

21.5 Further Readings

Notes

Books

J.F. Simmons, *Introduction to Topology and Modern Analysis*. McGraw Hill International Book Company, New York 1963.

A.V. Arkhangel'skii, V.I. Ponomarev, *Fundamentals of General Topology: Problems and Exercises*, Reidel (1984).

Online links

www.mathword. wolfram.com

http://www.answers.com/topic/planetmath

Notes

Unit 22: The Tychonoff Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define finite intersection property;
- Solve the problems on finite intersection;
- Understand the proof of Tychonoff's theorem.

Introduction

Like the Urysohn Lemma, the Tychonoff theorem is what we call a "deep" theorem. Its proof involves not one but several original ideas; it is anything but straightforward. We shall prove the Tychonoff theorem, to the effect that arbitrary products of compact spaces are compact. The proof makes use of Zorn's lemma. The Tychonoff theorem is of great usefulness to analysts we apply it to construct the Stone-Cech compactification of a completely regular space and in proving the general version of Ascoli's theorem.

22.1 Finite Intersection Property

Let X be a set and f a family of subsets of X. Then f is said to have the finite intersection property if for any finite number $F_1, F_2, ..., F_n$ of members of f.

$$F_1 \cap F_2 \cap .... F_n \neq \phi$$

Proposition: Let (X, T) be a topological space. Then (X, T) is compact if any only if every family f of closed subsets of X with the finite intersection property satisfies $\cap_{F \in f} F \neq \phi$.

Proof: Assume that every family f of closed subsets of X with the finite intersection property satisfies $\cap_{F \in f} F \neq \phi$. Let $\mathcal{U}$ be any open covering of X. Put f equal to the family of complements of members of $\mathcal{U}$. So each $F \in f$ is closed in (X, f). As $\mathcal{U}$ is an open covering of X, $\cap_{F \in f} F = \phi$. By our assumption, then, f does not have the finite intersection property. So for some $F_1, F_2,...F_n$ in f, $F_1 \cap F_2 \cap...., \cap F_n = \phi$.

Thus $\bigcup_1 \cup \bigcup_2 \cup ... \cup \bigcup_n = X$, where

$\cup_i = X \backslash F_i$, i = 1,...., n.

So $\mathcal{U}$ has a finite subcovering. Hence, (X, T) is compact.

The converse statement is proved similarly.

Notes

Example 1: Let X be a topological space and let $\mathcal{B}$ be a closed sub-base for X and let $\{B_i\}$ be its generated closed base i.e. the class of all finite union of members of $\mathcal{B}$ if every class of B_i's with the finite intersection property (FIP) has a non-empty intersection then X is compact.

Solution: Under the given hypothesis, we shall prove that X is compact. In order to prove the required result it is sufficient to show that every basic cover of X has a finite sub-cover.

Let $\{O_j\}$ be any basic open cover of X. Then $X = \bigcup_j O_j$.

Now, $\{B_i^C\}$ being an open base for X implies that each O_j is a union of certain B_i^C's and the totality of all such B_i^C's that arise in this way is a basic open cover of X. By De-Morgan's law, the totality of corresponding B_i's has empty intersection and therefore by the given hupothesis this totality does not have FIP. This implies that there exist finitely many B_i's, say,

$$B_{i_1}, B_{i_2} \ldots B_{i_n} \text{ such that } \bigcap_{K=1}^{n} B_{i_K} = \phi.$$

Taking complements on both sides, we set

$$\bigcup_{K=1}^{n} B_{i_K}^C = X. \qquad \text{(By De-Morgan's Law)}$$

For each $B_{i_K}^C$ (K = 1, 2, …, n) we can find a O_{j_K} such that $B_{i_K}^C \subseteq O_{j_K}$.

Thus $X = \bigcup_{K=1}^{n} O_{j_K}$.

Thus, we have shown that every basic open cover of X has a finite sub-cover.

Example 2: Let X be a non-empty set. Then every class $\{B_j\}$ of subsets of X with the FIP is contained in some maximal class with the FIP.

Solution: Let $\{B_j\}$ be a class of subsets of X with the FIP and let P be the family of all classes of subsets of X that contains $\{B_j\}$ and have the FIP.

For any F_λ, $F_\mu \in P$, define $F_\lambda \le F_\mu$ so that $F_\lambda \subseteq F_\mu$.

Then (P, ≤) is a partially ordered set. Let $\top$ be any totally ordered subset of (P, ≤). Then, the union of all classes in $\top$ has an upper bound for $\top$ in P.

Thus (P, ≤) is a partially ordered set in which every totally ordered subset has an upper bound.

Hence by Zern's lemma, P possesses a maximal element i.e., there exist a class $\{B_K\}$ of subsets of X such that $\{B_j\} \subseteq \{B_K\}$, $\{B_K\}$ has the FIP and any class of subsets of X which properly contains $\{B_K\}$ does not have the FIP.

Tychonoff's Theorem

Before proving Tychonoff's theorem, we shall prove two important lemmas.

Lemma 1: Let X be a set; Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. Then there is a collection D of subsets of X such that D contains $\mathcal{A}$ and D has the finite intersection property, and no collection of subsets of X that properly contains D has this property.

We often say that a collection D satisfying the conclusion of this theorem is maximal with respect to the finite intersection property.

Notes

Proof: As you might expect, we construct D by using Zorn's lemma. It states that, given a set A that is strictly partially ordered, in which every simply ordered subset has an upper bound, A itself has a maximal element.

The set A to which we shall apply Zorn's lemma is not a subset of X, nor even a collection of subsets of X, but a set whose elements are collections of subsets of X. For purpose of this proof, we shall call a set whose elements are collections of subsets of X a "superset" and shall denote it by an outline letter. To summarize the notation:

c is an element of X.

C is a subset of X.

$\mathcal{C}$ is collection of subset of X.

$\mathbb{C}$ is a superset whose elements are collections of subsets of X.

Now by hypothesis, we have a collection $\mathcal{A}$ of subsets of X that has the finite intersection property. Let $\mathbb{A}$ denote the superset consisting of all collections $\mathcal{B}$ of subsets of X such that $\mathcal{B} \supset \mathcal{A}$ and $\mathcal{B}$ has the finite intersection property. We use proper inclusion $\not\subseteq$ as our strict partial order of $\mathbb{A}$. To prove our lemma, we need to show that $\mathbb{A}$ has a maximal element D.

In order to apply Zorn's lemma, we must show that if $\mathbb{B}$ is a "sub-superset" of $\mathbb{A}$ that is simply ordered by proper inclusion, then $\mathbb{B}$ has a upper bound in $\mathbb{A}$. We shall show in fact that the collection

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathrm{B}} \mathcal{B},$$

which is the union of the collections belonging to $\mathbb{B}$, is an element of $\mathbb{A}$; the it is the required upper bound on $\mathbb{B}$.

To show that $\mathcal{C}$ is an element of $\mathbb{A}$, we must show that $\mathcal{C} \in \mathcal{A}$ and the $\mathcal{C}$ has the finite intersection property. Certainly $\mathcal{C}$ contains $\mathcal{A}$, since each element of B contains $\mathcal{A}$. To show that $\mathcal{C}$ has the finite intersection property, let $C_1, ..., C_n$ be elements of $\mathcal{C}$. Because $\mathcal{C}$ is the union of the elements of $\mathbb{B}$, there is, for each i, an element $\mathcal{B}_i$ of $\mathbb{B}$ such that $C_i \in \mathcal{B}_i$. The superset $\{\mathcal{B}_i,...., \mathcal{B}_n\}$ is contained in $\mathbb{B}$. So it has a largest element; that is, there is an index K such that $B_i \subset B_K$ for i = 1, ..., n. then all the sets $C_1...., C_n$ are elements of $\mathcal{B}_k$. Since $\mathcal{B}_k$ has the finite intersection property, the intersection of the sets $C_1, ..., C_n$ is non-empty, as desired.

Lemma 2: Let X be a set; Let D be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:

(a) Any finite intersection of elements of D is a element of D.

(b) If A is a subset of X that intersects every element of D, then A is an element of D.

Proof:

(a) Let B equal the intersection of finitely many elements of D. Define a collection of E by adjoining B to D, so that E = D ∪ {B}. We show that E has the finite intersection property; then maximality of D implied that E = D, so that B ∈ D as desired.

Take finitely many elements of E. If none of them is the set B, then their intersection is non-empty because D has the finite intersection property. If one of them is the set B, then their intersection is of the form

$$D_1 \cap ,... D_m \cap B.$$

Since B equals a finite intersection of elements of D, this set is non-empty.

(b) Given A, define E = D ∩ {A}. We show that E has the finite intersection property from which we conclude that A belongs to D. Take finitely many elements of E. If none of them is the set A, their intersection is automatically non-empty. Otherwise, it is of the form

$$D_1 \cap ... \cap D_n \cap A.$$

Notes

Now $D_1 \cap ... \cap D_n$ belongs to D, by (a); therefore this intersection is non-empty, by hypothesis.

***Theorem 1:* (*Tychonoff theorem*):** An arbitrary product of compact spaces is compact is the product topology:

Proof: Let

$$X = \prod_{\alpha \in J} X_\alpha,$$

where each space X_α is compact. Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. We prove that the intersection

$$\bigcap_{A \in \mathcal{A}} \bar{A}$$

is non-empty. Compactness of X follows:

Applying *Lemma 1*, choose a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D} \supset \mathcal{A}$ and $\mathcal{D}$ is maximal with respect to the finite intersection property. It will suffice to show that the intersection $\cap_{D \in \mathcal{D}} \bar{D}$ is non-empty.

Given $\alpha \in J$, let $\pi_\alpha : X \to X_\alpha$ be the projection map, as usual. Consider the collection

$$\{\pi_\alpha (D) \mid D \in \mathcal{D}\}$$

of subset of X_α. This collection has the finite intersection property because $\mathcal{D}$ does. By compactness of X_α, we can for each α choose a point x_α of X_α such that

$$x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}.$$

Let x be the point $(x_\alpha)_\alpha \in J$ of X. We shall show that for $x \in \bar{D}$ for every $D \in \mathcal{D}$; then our proof will be finished.

First we show that if $\pi_\beta^{-1} (U_\beta)$ is any sub-basis element (for the product topology on X) containing x, then $\pi_\beta^{-1} (U_\beta)$ intersects every element of $\mathcal{D}$. The set U_β is a neighbourhood of x_β in X_β. Since $x_\beta \in \overline{\pi_\beta(D)}$ by definition, U_β intersects $\pi_\beta (D)$ in some point $\pi_\beta (y)$, where $y \in D$. Then it follows that $y \in \pi_\beta^{-1} (U_\beta) \cap D$.

It follows from (b) of Lemma 2, that every sub-basis element containing x belongs to D. And then it follows (a) of the same lemma that every basis element containing x belongs to $\mathcal{D}$. Since $\mathcal{D}$ has the finite intersection property, this means that every basis element containing x intersects every element of $\mathcal{D}$; hence $x \in \bar{D}$ for every $D \in \mathcal{D}$ as desired.

22.2 Summary

- Let X be a set and f a family of subsets of X. Then f is said to have the finite intersection property if for any finite number $F_1, F_2, ... F_n$ of members of f. $F_1 \cap F_2 \cap ... \cap F_n \neq \phi$.
- Let (X, T) be a topology space. Then (X, T) is compact iff every family f of closed subsets of X with the finite intersection property satisfies $\cap_{F \in f} F \neq \phi$.
- An arbitrary product of compact spaces is compact in the product topology.

Notes

22.3 Keywords

Compact Set: Let (X, T) be a topological space and $A \subset X$. A is said to be a compact set if every open covering of A is reducible to finite sub-covering.

Maximal: Let (A, ≤) be a partially ordered set. An element $a \in A$ is called a maximal element of A if ∃ no element in A which strictly dominates a, i.e.

$$x \leq a \text{ for every comparable element } x \in A.$$

Projection Mappings: The mappings

$$\pi_x ; X \times Y \to X \text{ s.t. } \pi_x(x, y) = x \ \forall (x, y) \in X \times Y$$

$$\pi_y ; X \times Y \to Y \text{ s.t. } \pi_y(x, y) = y \ \forall (x, y) \in X \times Y$$

are called projection maps of X × Y onto X and Y space respectively.

Tychonoff Space: It is a completely regular space which is also a T_1-space i.e. $T_{3\frac{1}{2}} = [CR] + T_1$.

Upper bound: Let $A \subset R$ be any given set. A real number b is called an upper bound for the set A if.

$$x \leq b \ \forall \ x \in A.$$

22.4 Review Questions

1. Let X be a space. Let D be a collection of subsets of X that is maximal with respect to the finite intersection property.
   (a) Show that $x \in \bar{D}$ for every $D \in \mathcal{D}$ if any only if every neighbourhood of x belongs to $\mathcal{D}$. Which implication uses maximality of $\mathcal{D}$?
   (b) Let $D \in \mathcal{D}$. Show that if $A \supset D$, then $A \in \mathcal{D}$.
   (c) Show that if X satisfies the T_1 axion, there is at most one point belonging to $\cap_{D \in \mathcal{D}} \bar{D}$.
2. A collection $\mathcal{A}$ of subsets of X has the countable intersection property if every countable intersection of elements of $\mathcal{A}$ is non-empty. Show that X is a Lindelöf space if any only if for every collection $\mathcal{A}$ of subsets of X having the countable intersection property,

$$\bigcap_{A \in \mathcal{A}} \bar{A}$$

   is non-empty.

22.5 Further Readings

Books

Bimmons, *Introduction to Topology and Modern Analysis*.

Nicolas Bourbaki, *Elements of Mathematics*.

Online links

www.planetmath.org

www.jstor.org

Unit 23: The Stone-Cech Compactification

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Describe the compactification;
- Define the Stone-Cech compactification;
- Explain the related theorems.

Introduction

We have already studied one way of compactifying a topological space X, the one-point compactification; it is in some sense the minimal compactification of X. The Stone-Cech compactification of X, which we study now, is in some sense the maximal compactification of X. It was constructed by M. Stone and E. Cech, independently, in 1937. It has a number of applications in modern analysis. The Stone-Cech compactification is defined for all Tychonoff Spaces and has an important extension property.

23.1 Compactification

A compactification of a space X is a compact Hausdorff space Y containing X as a subspace such that $\bar{X} = Y$. Two compactifications Y, and Y_2 of X are said to be equivalent if there is a homeomorphism $h : Y_1 \rightarrow Y_2$ such that h(x) = x for every $x \in X$.

Remark: If X has a compactification Y, then X must be completely regular, being a subspace of completely regular space Y. Conversely, if X is completely regular, then X has a compactification.

Lemma 1: Let X be a space; suppose that $h : X \rightarrow X$ is an imbedding of X in the compact Hausdorff space Z. Then there exists a corresponding compactification Y of X; it has the property that there is an imbedding $H : Y \rightarrow Z$ that equals h on X. The compactification Y is uniquely determined up to equivalence.

We call Y the compactification induced by the imbedding h.

Proof: Given h, let X_0 denote the subspace h(X) of Z, and let Y_0 denote its closure of Z.

Notes

Then Y_0 is a compact Hausdorff space and $\bar{X}_0 = Y_0$; therefore, Y_0 is a compactification of X_0.

We now construct a space Y containing X such that the pair (X, Y) is homeomorphic to the pair (X_0, Y_0). Let us choose a set A disjoint from X that is in bijective correspondence with set $Y_0 - X_0$ under map $K : A \to Y_0 - X_0$.

Define $Y = X \cup A$, and define a bijective correspondence $H : Y \to Y_0$ by the rule

$$H(x) = h(x) \quad \text{for } x \in X,$$

$$H(a) = k(a) \quad \text{for } a \in A.$$

Then topologize Y by declaring $\cup$ to be open in Y if and only if $H(\cup)$ is open in Y_0. The map H is automatically a homeomorphism; and the space X is a subspace of Y because H equals the homeomorphism 'h' when restricted to the subspace X of Y. By expanding the range of H, we obtain the required imbedding of Y into Z.

Now suppose Y_i is a compactification of X and that $H_i : Y_i \to Z$ is an imbedding that is an extension of h, for i = 1, 2. Now H_i maps X onto $h(X) = X_0$. Because H_i is continuous, it must map Y_i into $\bar{X}_0$; because $H_i(Y_i)$ contains X_0 and is closed (being compact), it contains $\bar{X}_0$. Hence, $H_i(Y_i) = \bar{X}_0$ and $H_2^{-1} \circ H_1$ defines a homeomorphism of Y_1 with Y_2 that equals the identity on X.

Theorem 1: The collection of all compactifications of a topological space is partially ordered by $\geqq$. If (f, Y) and (g, Z) are Hausdorff compactifications of a space and $(f, Y) \geqq (g, Z) \geqq (f, Y)$, then (f, Y) and (g, Z) are topologically equivalent.

Proof: If $(f, Y) \geqq (g, Z) \geqq (h, U)$, where these are compactification of a space X, then there are continuous functions j on Y to Z and K on Z to U such that $g = j \circ f$ and $h = k \circ g$ and hence $h = k \circ j \circ f$ and $(f, Y) \geqq (h, U)$. Consequently $\geqq$ partially orders the collection of all compactifications of X. If (f, Y) and (g, Z) are Hausdorff compactifications each of which follows the other relative to the ordering $\geqq$, then both $f \circ g^{-1}$ and $g \circ f^{-1}$ have continuous extensions j and k to all of Z and Y respectively.

Since $k \circ j$ is the identity map on the dense subset g [X] of Z and Z is Hausdorff $k \circ j$ is the identity map of Z onto itself and similarly $j \circ k$ is the identity map of Y onto Y. Consequently (f, Y) and (g, Z) are topologically equivalent.

23.1.1 One Point Compactification

Definition: Let X be a locally compact Hausdorff space.

Take some objects outside X, denoted by the symbol ∞ for convenience and adjoin it to X, forming the set

$$Y = X \cup \{\infty\}.$$

Define topology $\cup$ on Y as follows:

(i) $G \in \cup$ if T

(ii) $Y - C \in \cup$ if C is a compact subset of X.

The space Y is called one point compactification of X.

Theorem 2: Let X be a locally compact Hausdorff space which is not compact. Let Y be one point compactification of X. Then Y is compact Hausdorff space : X is a subspace of Y : the set Y – X consists of a single point and $\bar{X} = Y$.

Proof:

Notes

1. To show that X is a subspace of Y and $\bar{X}$ = Y.

   Let ∪ be a topology on Y. Let H ∈ ∪, then

   H ∩ X = H

   and so H ∈ T. Also (Y – C) ∩ X = X – C

   and so X – C ∈ T. Conversely any open set in X is of the type (1) and therefore open in Y. Since X is not compact, each open set Y – C containing ∞ intersects X, meaning thereby ∞ is a limit point of X, so that $\bar{X}$ = Y.

2. To show that Y is compact.

   Let G be an ∪-open covering of Y. The collection G must contain an open set of the type Y – C. Also G contains set of the type G, where G ∈ T, each of these sets does not contain the point ∞. Take all such sets of G different from Y – C, intersect them with X, they form a collection of open sets in X covering C.

   As C is compact, hence a finite number of these members will cover C; the corresponding finite collection of elements of G along with the elements of Y – C cover all of Y.

   Hence Y is compact.

3. To show that y is Hausdorff.

   Let x, y ∈ Y.

   If both of them lie in X and X is known to be compact so that ∃ disjoint open sets ∪, V in X

   s.t. x ∈ ∪, y ∈ V.

   On the other hand if

   x ∈ X

   and y = ∞.

We can chose compact set C and X containing a nbd ∪ of x.

The ∪ and Y – C are disjoint nbds of x and ∞ respectively in Y.

Theorem 3: If (X^*, T^*) be a one point compactification of a non-compact topological space (X, T), then (X^*, T^*) is a Hausdorff space iff (X, T) is locally compact.

Proof: Assuming that X is a Hausdorff space, each pair of distinct points in X^*, all of which belong to X can be separated by open subsets of X. Thus it is sufficient to show that any pair (x, ∞) ∈ X, can be separated by open subsets of X^*. Now X is locally compact

⇒ any x ∈ X, has a nbd N whose closure N in X is compact

⇒ N and $\bar{N}'$ are disjoint open subsets of X^* s.t. x ∈ N and ∞ ∈ N′

⇒ distinct points x, ∞ of X^* have disjoint nbds

⇒ (X^*, T^*) is Hausdorff.

Conversely if (X^*, T^*) is Hausdorff, then

X is a subspace of X^* ⇒ X is Hausdorff, since Hausdorffness is hereditary.

Notes

Now we claim that X is locally compact. It will be so if every point of it has a nbd whose closure is compact.

$x \in X$ is fixed and distinct $x, \infty \in X^*$ (Hausdorff) $\Rightarrow \exists$ disjoint open sets A_1^*, A_2^* in X^* s.t. $x \in A_1^*$ and $\infty \in A_2^*$.

But an open set containing ∞ must be of the form

$$A_2^* = \{\infty\} \cup A$$

where A is an open set in X containing x s.t. its complement is compact.

Also $\infty \in A_1^* = A_1^*$ is an open set in X containing x, whose closure is contained in A

$\Rightarrow A_1^*$ is compact

$\Rightarrow$ every point of X has a nbd whose closure is compact

$\Rightarrow$ X is locally compact.

23.1.2 Stone-Cech Compactification

The pair (e, $\beta(X)$), where X is a Tychonoff space and $\beta(X)\left(=\overline{e(x)}\right)$ is called Stone-Cech compactification of X. e is a map from X into $\beta(X)$.

For each completely regular space X, let us choose, once and for all, a compactification of X satisfying the extension condition i.e. For a completely regular space X, $\exists$ a compactification Y of X having the property that every bounded continuous map $f : X \to \mathcal{R}$ extends uniquely to a continuous map of Y into $\mathcal{R}$.

We will denote this compactification of X by $\beta(X)$ and call it the Stone-Cech compactification of X. It is characterized by the fact that any continuous map $f : X \to C$ of X into a compact Hausdorff space C extends uniquely to a continuous map $g : \beta(X) \to C$.

Theorem 4: Let X be a Tychonoff space, (e, $\beta(X)$) its stone-cech compactification and suppose $f : X \to [0, 1]$ is continuous. Then there exists a map $g : \beta(X) \to [0, 1]$ such that $g \circ e = f$, i.e. g is an extension of f to $\beta(X)$, if we identify X with e(X).

Proof: Let $\exists$ be the family of all continuous functions from X into [0, 1]. Then $\beta(X) \subset [0, 1]^{\exists}$ we define g on the entire cube $[0, 1]^{\exists}$ by $g(\lambda) = \lambda(f)$ for $\lambda \in [0, 1]^{\exists}$.

This is well defined because an element of $[0, 1]^{\exists}$ is a function from $\exists$ into [0, 1] and can be evaluated at f since $f \in \exists$. Equivalently, g is nothing but the projection f from $[0, 1]^{\exists}$ onto [0, 1], and hence is continuous. Now if $x \in X$ then, by definition of the evaluation map, $e(x) \in [0, 1]^{\exists}$ is the function $e(x) : \exists \to [0, 1]$ such that

$$g \circ e(x)(h) = h(x) \quad \text{for } h \in \exists.$$

Now $$g \circ e(x) = g(e(x)) = e(x)(f) = f(x) \ \forall\ x \in X$$

So $$g \circ e = f.$$

Thus, we extended f not only to $\beta(x)$ but to the entire cube $[0, 1]^{\exists}$. Its restriction to $\beta(X)$ proves the theorem.

Theorem 5: A continuous function from a Tychonoff space into a compact Hausdorff space can be extended continuously over the stone-cech compactification of the domain. Moreover such an extension is unique.

Proof: Let X be a Tychonoff space, $\beta(X)$ its stone-cech compactification and $f : X \to Y$ a map where Y is a compact Hausdorff space.

Notes

Let $\exists_1$, $\exists_2$ be respectively the families of all continuous functions from X, Y respectively to the unit interval [0, 1] and let e, e′ be the embedding of X, Y into $[0,1]^{\exists_1}$ and $[0,1]^{\exists_2}$ respectively. For any $g \in \exists_2$ let $\pi g : [0,1]^{\exists_2} \to [0, 1]$ be the corresponding projection.

Then π_g o e′ of is a map from X into [0, 1] and so it has an extension say θ_g to β(X). Then θ_g o e π_g o e′ o f.

Now consider the family $\{\theta_g = g \in \exists_2\}$ of maps from β(X) into [0, 1]. Let $\theta : \beta(X) \to [0, 1]^{\exists}$ be the evaluation map determined by this family. We claim that θ o e = e′ o f. Let x ∈ X. Them θ(e(x)) is an element of $[0,1]^{\exists_2}$ given by

$$\theta(e(x))(g) = \theta g(e(x)) \qquad \text{[by the definition of the evaluation functions]}$$

$$\text{But } \theta_g(e(x)) = \pi_g(e'f(x)) = e'(f(x))(g)$$

Thus for all $g \in \exists_2$

$$[\theta_o e(x)](g) = [e'_o f(x)](g) \quad \text{and so}$$

$$\theta \text{ o } g = e'_o f \text{ as claimed.}$$

Now $\theta(e(x)) = e'(f(X)) \subset e'(Y)$.

Since Y is compact, e′(Y) compact and hence a closed subset of $[0,1]^{\exists_2}$.

So $\overline{\theta(e(X))} \subset e'(Y)$.

But since θ is continuous,

$$\theta(B(X)) = \overline{\theta(e(X))} \subset \overline{\theta(e(X))}$$

Thus we see that θ maps β(X) into e′(Y). Since e′ is an embedding, there exists a map $e_1 : e'(Y) \to Y$ which is an inverse to e′ regarded as a map from Y onto e′(Y). Then e_1 o e′ o f = f.

Uniqueness of the extension is immediate in view of the fact that Y is a Hausdorff space and e(X) is dense in β(X).

23.2 Summary

- A compactification of a space X is a compact Hausdorff space Y containing X as a subspace such that $\bar{X} = Y$.
- Two compactifications Y_1 and Y_2 of X are said to be equivalent if there is a homeomorphism $h : Y_1 \to Y_2$ such that h(x) = x for every x ∈ X.
- If X has a compactification Y, then X must be completely regular, being a subspace of completely regular space Y.
- If X is completely regular, then X has a compactification.
- The pair (e, β(X)), where X is a Tychonoff space and β(X) $\left(=\overline{e(x)}\right)$ is called Stone-Cech compactification of X, e is a map from X into β(x).
- The Stone-Cech compactification is defined for all Tychonoff spaces.

Notes

23.3 Keywords

Connected Spaces: A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

Hausdorff Space: It is a topological space in which each pair of distinct points can be separated by disjoint neighbourhoods.

Homeomorphism: A map $f : (X, T) \to (Y, \cup)$ is said to be homeomorphism if:

(i) f is one-one onto.

(ii) f and f^{-1} are continuous.

23.4 Review Questions

1. Let (X, T) be a Tychonoff space and (βX, T′) its stone-cech compactification. Prove that (X, T) is connected if and only if (βX, T′) is connected.

   [***Hint:*** Firstly verify that providing (X, T) has at least 2 points it is connected if and only if there does not exist a continuous map of (X, T) onto the discrete space {0, 1}.]

2. Let (X, T) be a Tychonoff space and (βX, T′) its stone-cech compactification. If (A, T_1) is a subspace of (βX, T′) and $A \supseteq X$, prove that (βX, T′) is also the stone-cech compactification of (A, T_1).

3. Let (X, T) be a dense subspace of a compact Hausdorff space (Z, T_1). If every continuous mapping of (X, T) into [0, 1] can be extended to a continuous mapping of (Z, T_1) into [0, 1], prove that (Z, T_1) is the Stone-Cech compactification of (X, T).

4. Let Y be an arbitrary compactification of X; let β(X) be the Stone-Cech compactification. Show that there is a continuous surjective closed map $g : \beta(X) \to Y$ that equals the identity on X.

5. Under what conditions does a metrizable space have a metrizable compactification?

23.5 Further Readings

Books

S. Lang, *Algebra* (Second Edition), Addison-Wesley, Menlo Park, California 1984.

S. Willard, *General Topology*, MA : Addison-Wesley.

Online links

www.planetmath.org

www.jstor.org

Unit 24: Local Finiteness and Paracompactness

Notes

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Objectives

After studying this unit, you will be able to:

- Define local finiteness and solve problems on it;
- Define countably locally finite, open refinement and closed refinement;
- Understand the paracompactness and theorems on it.

Introduction

In this unit we prove some elementary properties of locally finite collections and a crucial lemma about metrizable spaces.

The concept of paracompactness is one of the most useful generalization of compactness that has been discovered in recent years. It is particularly useful for applications in topology and differential geometry. Many of the spaces that are familiar to us already are paracompact. For instance, every compact space is paracompact; this will be an immediate consequence of the definition. It is also true that every metrizable space is paracompact; this is a theorem due to A.H. Stone, which we shall prove. Thus the class of paracompact space includes the two most important classes of spaces we have studied. It includes many other spaces as well.

24.1 Local Finiteness

Definition: Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be a locally finite in X if every point of X has a neighbourhood that intersects only finitely many elements of $\mathcal{A}$.

Example 1: The collection of intervals

$$\mathcal{A} = \{(n, n + 2) \mid n \in \mathbb{Z})\}$$

is locally finite in the topological space $\mathbb{R}$, on the other hand, the collection

$$\mathcal{B} = \{0, 1/n\} \mid n \in \mathbb{Z}_+\}$$

Notes

is locally finite in (0, 1) but not in $\mathbb{R}$, as in the collection

$$\mathcal{C} = \{(1/(n+1), 1/n) \mid n \in \mathbb{Z}_+\}.$$

Lemma 1: Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then:

(a) Any sub collection of $\mathcal{A}$ is locally finite.

(b) The collection $\mathcal{B} = \{\bar{A}\}_{A \in \mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.

(c) $\cup_{A \in \mathcal{A}} A = \cup_{A \in \mathcal{A}} \bar{A}$.

Proof: Statement (a) is trivial. To prove (b), note that any open set U that intersects the set $\bar{A}$ necessarily intersects A. Therefore, if U is a neighbourhood of x that intersects only finitely many elements A of $\mathcal{A}$, then U can intersect at most the same number of sets of the collection $\mathcal{B}$. (It might intersect fewer sets of $\mathcal{B}$, $\bar{A}_1$ and $\bar{A}_2$ can be equal even though A_1 and A_2 are not).

To prove (c), let Y denote the union of the elements of $\mathcal{A}$:

$$\bigcup_{A \in \mathcal{A}} A = Y.$$

In general, $\cup \bar{A} \subset \bar{Y}$; we prove the reverse inclusion, under the assumption of local finiteness. Let $x \in \bar{Y}$; let U be a neighbourhood of x that intersects only finitely many elements of $\mathcal{A}$, say $A_1, \ldots, A_k$. We assert that x belongs to one of the sets $\bar{A}_1, \ldots, \bar{A}_k$ and hence belongs to $\cup \bar{A}$. For otherwise, the set $U - \bar{A}_1 - \ldots - \bar{A}_k$ would be a neighbourhood of x that intersect no element of $\mathcal{A}$ and hence does not intersect Y, contrary to the assumption that $x \in \bar{Y}$.

24.1.1 Countably Locally Finite

Definition: A collection $\mathcal{B}$ of subsets of X is said to be countably locally finite of $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.

24.1.2 Open Refinement and Closed Refinement

Definition: Let $\mathcal{A}$ be a collection of subsets of the space X. A collection $\mathcal{B}$ of subsets of X is said to be a refinemet of $\mathcal{A}$ (or is said to refine $\mathcal{A}$) if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containing B. If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ an open refinement of $\mathcal{A}$; if they are closed sets, we call $\mathcal{B}$ a closed refinement.

Lemma 2: Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X, then there is an open covering E of X refining $\mathcal{A}$ that is countably locally finite.

Proof: We shall use the well-ordering theorem in proving this theorem. Choose a well-ordering, < for collection $\mathcal{A}$. Let us denote the elements of $\mathcal{A}$ generically by the letters U, V, W,.... .

Choose a metric for X. Let n be a positive integer, fixed for the moment. Given an element U of $\mathcal{A}$, let us define S_n (U) to be the subset of U obtained by "shrinking" U a distance of 1/n. More precisely, let

$$S_n(U) = \{x \mid B(x, 1/n) \subset U\}.$$

(It happens that $S_n(U)$ is a closed set, but that is not important for our purposes.) Now we use the well-ordering < of $\mathcal{A}$ to pass to a still smaller set. For each U in $\mathcal{A}$, define

$$T_n(\cup) = S_n(U) - \bigcup_{V<U} V.$$

Notes

The situation where $\mathcal{A}$ consists of the three sets $U < V < W$. The sets we have formed are disjoint. In fact, they are separated by a distance of at least 1/n. This means that if V and W are distinct elements of $\mathcal{A}$, then $d(x, y) \geq 1/n$ whenever $x \in T_n(V)$ and $y \in T_n(W)$.

To prove this fact, assume the notation has been so chosen that $V < W$. Since x is in $T_n(V)$, then x is in $S_n(V)$, so the 1/n-neighbourhood of x lies in V. On the other hand since $V < W$ and y is in $T_n(W)$, the definition of the latter set tells us that y is not in V. If follows that y is not in the 1/n-neighbourhood of x.

The sets $T_n(U)$ are not yet the ones we want, for we do not know that they are open sets. (In fact, they are closed.) So let us expand each of them slightly to obtain an open set $E_n(\cup)$. Specifically, let $E_n(U)$ be the 1/3 n-neighbourhood of $T_n(U)$; that is, let $E_n(U)$ be the union of the open balls $B(x, 1/3n)$, for $x \in T_n(U)$.

In case $U < V < W$, we have the situation. The sets we have formed are disjoint. Indeed, if V and W are distinct elements of $\mathcal{A}$, we assert that $d(x, y) \geq 1/3n$ whenever $x \in E_n(V)$ and $y \in E_n(W)$; this fact follows at once from the triangle inequality. Note that for each $V \in \mathcal{A}$, the set $E_n(V)$ is contained in V.

Now let us define

$$\varepsilon_n = \{E_n(\cup) \mid \cup \in \mathcal{A}\}.$$

We claim that E_n is a locally finite collection of open sets that refines $\mathcal{A}$. The fact that E_n refines $\mathcal{A}$ comes from the fact that $E_n(V) \subset V$ for each $V \in \mathcal{A}$. The fact E_n is locally finite comes from the fact that for any x in X, the 1/6n - neighbourhood of x can intersect at most one element of E_n.

Of course, the collection ε_n, will not cover X. But we assert that the collection

$$E = \bigcup_{n \in \mathbb{Z}_+} \varepsilon_n$$

does cover X.

Let x be a point of X. The collection $\mathcal{A}$ with which we began covers X; let us choose U to be the first element of $\mathcal{A}$ (in the well-ordering <) that contains x. Since U is open, we can choose n so that $B(x, 1/n) \subset U$. The, by definition, $x \in S_n(U)$. Now because U is the first element of $\mathcal{A}$ that contains x, the point x belongs to $T_n(U)$. Then x also belongs to the element $E_n(U)$ of E_n, as desired.

Self Assessment

1. Many spaces have countable bases; but no T_1 space has a locally finite basis unless it is discrete. Prove this fact.
2. Find a non-discrete space that has a countably locally finite basis but does not have a countable basis.

24.2 Paracompactness

Definition: A space X is paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement B that covers X.

Example 2: The Space $\mathbb{R}^n$ is paracompact. Let $X = \mathbb{R}^n$. Let $\mathcal{A}$ be an open covering of X. Let $B_0 = \phi$, and for each positive integer m, let B_m denote the open ball of radius m centered at the origin. Given m, choose finitely many elements of $\mathcal{A}$ that cover $\overline{B}_m$ and intersect each one with the open set $X - \overline{B}_{m-1}$; let this finite collection of open sets be denoted $\mathcal{C}_m$. Then the collection $\mathcal{C} = \cup\mathcal{C}_m$ is a refinement of $\mathcal{A}$. It is clearly locally finite, for the open set B_m intersects only finitely many elements of $\mathcal{C}$, namely those elements belonging to the collection $\mathcal{C}_1 \cup ... \cup \mathcal{C}_m$. Finally,

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$\mathcal{C}$ covers X. For, given x let m be the smallest integer such that $x \in \overline{B}_m$. Then x belongs to a element of $\mathcal{C}_m$, by definition.

> *Note* Some of the properties of a paracompact space are similar to those of a compact space. For instance, a subspace of a paracompact space is not necessarily paracompact; but a closed subspace is paracompact. Also, a paracompact Hausdorff space is normal. In other ways, a paracompact space is not similar to a compact space; in particular, the product of two paracompact spaces need not be paracompact.

Theorem 1: Every paracompact Hausdorff space X is normal.

Proof: The proof is somewhat similar to the proof that a compact Hausdorff space is normal. First one proves regularity. Let a be a point of X and let B be a closed set of X disjoint from a. The Hausdorff condition enables is to choose for each b in B, an open set $\cup_b$ about b whose closure is disjoint from a. Cover X by the open sets $\cup_b$, along with the open set X – B; take a locally finite open refinement $\mathcal{C}$ that covers X. Form the subcollection D of $\mathcal{C}$ consisting of every element of $\mathcal{C}$ that intersects B. The $\mathcal{D}$ covers B. Furthermore, if $D \in \mathcal{D}$, then $\overline{D}$ is disjoint from a. For D intersect B, so it lies in some set $\cup_b$, whose closure is disjoint from a. Let

$$V = \bigcup_{D \in \mathcal{D}} D;$$

then V is an open set in X containing B. Because $\mathcal{D}$ is locally finite,

$$\overline{V} = \bigcup_{D \in \mathcal{D}} \overline{D},$$

so that $\overline{V}$ is disjoint from a. Thus regularity is proved.

To prove normality, one merely repeats the same argument, replacing a by the closed set A throughout and replacing the Hausdorff condition by regularity.

Theorem 2: Every closed subspace of a paracompact space is paracompact.

Proof: Let Y be a closed subspace of the paracompact space X; let $\mathcal{A}$ be a covering of Y by sets open in Y.

For each $A \in \mathcal{A}$, choose an open set A′ of X such that $A' \cap Y = A$. Cover X by the open sets A′, along with the open set X – Y.

Let $\mathcal{B}$ be a locally finite open refinement of this covering that covers X.

The collection $\mathcal{C} = \{B \cap Y : B \in \mathcal{B}\}$

is the required locally finite open refinement of $\mathcal{A}$.

Example 3: A paracompact subspace of a Hausdorff space X need not be closed in X.

Solution: Indeed, the open interval (0, 1) is pracompact, being homeomorphic to $\mathbb{R}$, but it is not closed in $\mathbb{R}$.

Lemma 3: Let X be regular. Then the following conditions on X are equivalent:

Every open covering of X has a refinement that is:

1. An open covering of X and countably locally finite.
2. A covering of X and locally finite.

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3. A closed covering of X and locally finite.

4. An open covering of X and locally finite.

Proof: It is trivial that (4) ⇒ (1).

What we need to prove our theorem is the converse. In order to prove the converse, we must go through the steps (1) ⇒ (2) ⇒ (3) ⇒ (4)

anyway, so we have for convenience listed there conditions in the statement of the lemma.

(1) ⇒ (2).

Let $\mathcal{A}$ be an open covering of X. Let $\mathcal{B}$ be an open refinement of $\mathcal{A}$ that covers X and is countably locally finite; let

$$\mathcal{B} = \cup \mathcal{B}_n$$

where each $\mathcal{B}_n$ is locally finite.

Now we apply essentially the same sort of shrinking trick, we have used before to make sets from different $\mathcal{B}_n'$ disjoint. Given i, let

$$V_i = \bigcup_{U \in \mathcal{B}_i} U$$

Then for each $n \in Z_+$ and each element U of $\mathcal{B}_n$, define

$$S_n(U) = U - \bigcup_{i<n} V_i$$

[Note that $S_n(U)$ is not necessarily open, nor closed.]

Let $\mathcal{C}_n = \{S_n(U) : U \in \mathcal{B}_n\}$

Then $\mathcal{C}_n = \cup \mathcal{C}_n$. We assert that $\mathcal{C}$ is the required locally finite refinement of $\mathcal{A}$, covering X.

Let x be a point of X. We wish to prove that x lies in an element of $\mathcal{C}$, and that x has a neighbourhood intersecting only finitely many elements of $\mathcal{C}$. Consider the covering $\mathcal{B} = \cup \mathcal{B}_n$; let N be the smallest integer such that x lies in an element of $\mathcal{B}_N$. Let $\cup$ be an element of $\mathcal{B}_N$ containing x. First, note that since x lies in no element of $\mathcal{B}_i$ for $i < N$, the point x lies in the element $S_N(\cup)$ of $\mathcal{C}$. Second, note that since each collection $\mathcal{B}_n$ is locally finite, we can choose for each n = 1, ..., N a neighbourhood W_n of x that intersects only finitely many elements of $\mathcal{B}_n$. Now if W_n intersects the element $S_n(V)$ of $\mathcal{C}_n$, it must intersect the element V of $\mathcal{B}_n$, since $S_n(V) \subset V$. Therefore, W_n intersects only finitely many elements of $\mathcal{C}_n$. Furthermore, because U is in $\mathcal{B}_N$, U intersects no element of $\mathcal{C}_n$ for n > N. As a result, the neighbourhood

$$W_1 \cap W_2 \cap ... \cap W_n \cap \cup$$

of x intersects only finitely many elements of $\mathcal{C}$.

(2) ⇒ (3). Let $\mathcal{A}$ be an open covering of X. Let $\mathcal{B}$ be the collection of all open sets U of X such that $\overline{U}$ is contained in an element of $\mathcal{A}$. By regularity, $\mathcal{B}$ covers X. Using (2), we can find a refinement $\mathcal{C}$ of $\mathcal{B}$ that covers X and is locally finite. Let

$$\mathcal{D} = \{\overline{C} : C \in C\}$$

Then $\mathcal{D}$ also covers X; it is locally finite by lemma (1) and it refines $\mathcal{A}$.

(3) ⇒ (4): Let $\mathcal{A}$ be an open covering of X. Using (3), choose $\mathcal{B}$ to be a refinement of $\mathcal{A}$ that covers X and is locally finite. (We can take $\mathcal{B}$ to be closed refinement if we like, but that is irrelevant.) We seek to expand each element B of $\mathcal{B}$ slightly to an open set, making the expansion slight enough that the resulting collection of open sets will still be locally finite and will still refine $\mathcal{A}$.

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This step involve a new trick. The previous trick, used several times, consisted of ordering the sets in some way and forming a new set by subtracting off all the previous ones. That trick shrinks the sets; to expand them we need something different. We shall introduce an auxiliary locally finite closed covering $\mathcal{C}$ of X and use it to expand the element of $\mathcal{B}$.

For each point x of X, there is a neighbourhood of x that intersects only finitely many elements of $\mathcal{B}$. The collection of all open sets that intersect only finitely many element of $\mathcal{B}$ is thus an open covering of X. Using (3) again, let $\mathcal{C}$ be a closed refinement of this covering that covers X and is locally finite. Each element of $\mathcal{C}$ intersect only finitely many elements of $\mathcal{B}$.

For each element B of $\mathcal{B}$, let

$$\mathcal{C}(B) = \{C : C \in \mathcal{C} \text{ ad } C \subset X - B\}$$

Then define
$$E(B)X = X - \bigcup_{C \in \mathcal{C}(B)} C$$

Because $\mathcal{C}$ is locally finite collection of closed sets, the union of the elements of any subcollection of $\mathcal{C}$ is closed by lemma, therefore the set E(B) is an open set. Furthermore, E(B) ⊃ B by definition.

Now we may have expanded each B too much; the collection {E(B)} may not be a refinemet of $\mathcal{A}$. This is easily remedied. For each B ∈ $\mathcal{B}$, choose an element F(B) of $\mathcal{A}$ containing B. Then define

$$\mathcal{D} = \{E(B) \cap F(B) \mid B \in \mathcal{B}\}.$$

The collection $\mathcal{D}$ is a refinement of A. Because B ⊂ (E(B) ∩ F(B)) and $\mathcal{B}$ covers X, the collection $\mathcal{D}$ also covers X.

We have finally to prove that $\mathcal{D}$ is locally finite. Given a point x of X, choose a neighbourhood W of x that intersects only finitely may elements of $\mathcal{C}$, say $C_1, ..., C_k$. We show that W intersects only finitely many elements of $\mathcal{D}$. Because $\mathcal{C}$ covers X, the set W is covered by $C_1,...C_K$. thus, it suffices to show that each element C of $\mathcal{C}$. Now if C intersects the set E (B) ∩ F(B), then it intersects E(B), so by definition of E(B) it is not contained in X–B; hence C must intersect B. Since C intersects, only finitely many elements of $\mathcal{B}$, it can intersect at most the same number of elements of the collection $\mathcal{D}$.

Theorem 3: Every metrizable space is paracompact.

Proof: Let X be a metrizable space. We already know from Lemma 2 that, given an open covering $\mathcal{A}$ of X, it has an open refinement that covers X and is countably locally finite. The preceding lemma then implies that $\mathcal{A}$ has an open refinement that covers X and is locally finite.

Example 4: The product of two paracompact spaces need not be paracompact. The space $\mathbb{R}_\ell$ is paracompact, for it is regular and Lindelöf. However, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not paracompact, for it is Hausdorff but not normal.

Self Assessment

3. Show that Paracompactness is a topological property.

4. If every open subset of a paracompact space is paracompact, then every subset is paracompact. Prove it.

24.3 Summary

- Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be locally finite in X if every point of X has a neighbourhood that intersects only finitely many elements of $\mathcal{A}$.

Notes

- A collection $\mathcal{B}$ of subsets of X is said to be countably locally finite is $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.
- Let $\mathcal{A}$ be a collection of subsets of space X. A collection $\mathcal{B}$ of subsets of X is said to be a refinement of $\mathcal{A}$ if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containing B. If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ an open refinement of $\mathcal{A}$; if they are closed sets, we call $\mathcal{B}$ a closed refinement.
- A space X is paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X.

24.4 Keywords

Metrizable: Any topological space (X, T) if it a possible to find a metric ρ on X which induces the topology T i.e. the open sets determined by the metric ρ are precisely the members of T, then X is said to the metrizable.

Open Cover: Let (X, T) be a topological space and $A \subset X$ let G denote a family of subsets of X. G is called a cover of A if $A \subset \bigcup \{G : G \in G\}$.

24.5 Review Questions

1. Give an example to show that if X is paracompact, it does not follow that for every open covering $\mathcal{A}$ of X, there is a locally finite subcollection of $\mathcal{A}$ that covers X.
2. (a) Show that the product of a paracompact space and a compact space is paracompact. [***Hint:*** Use the tube lemma.]

   (b) Conclude that S_Ω is not paracompact.
3. Is every locally compact Hausdorff space paracompact?
4. (a) Show that if X has the discrete topology, then X is paracompact.

   (b) Show that if $f : X \to Y$ is continuous and X is paracompact, the subspace f(X) of Y need not be paracompact.
5. (a) Let X be a regular space. If X is a countable union of compact subspace of X, then X is paracompact.

   (b) Show $\mathbb{R}^\infty$ is paracompact as a subspace of $\mathbb{R}^w$ in the box topology.
6. Let X be a regular space.

   (a) Show that if X is a finite union of closed paracompact subspaces of X, then X is paracompact.

   (b) If X is a countable union of closed paracompact subspaces of X whose interiors cover X, show X is paracompact.
7. Find a point-finite open covering $\mathcal{A}$ of $\mathbb{R}$ that is not locally finite (The collection $\mathcal{A}$ is point finite if each point of $\mathbb{R}$ lies in only finitely many elements of $\mathcal{A}$).
8. Give an example of a collection of sets $\mathcal{A}$ that is not locally finite, such that the collection $\mathcal{B} = \{\bar{A} / A \in \mathcal{A}\}$ is locally finite.
9. Show that if X has a countable basis, a collection $\mathcal{A}$ of subsets of X is countably locally finite if and only if it is countable.
10. Consider $\mathbb{R}^w$ in the uniform topology. Given n, let $\mathcal{B}_n$ be the collection of all subsets of $\mathbb{R}^w$ of the form ΠA_i; where $A_i = \mathbb{R}$ for $i \le n$ and A_i equals either {0} or {1} otherwise. Show that collection $\mathcal{B} = \bigcup \mathcal{B}_n$ is countably locally finite, but neither countable nor locally finite.

Notes

24.6 Further Readings

Books

J.L.. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

S. Willard, *General Topology*, Addison-Wesley, Mass. 1970.

Unit 25: The Nagata-Smirnov Metrization Theorem

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define G_δ set;
- State "The Nagata-Smirnov Metrization Theorem";
- Understand the proof of "The Nagata Smirnov Metrization Theorem".

Introduction

Although Urysohn solved the metrization problem for separable metric spaces in 1924, the general metrization problem was not solved until 1950. Three mathematicians, J. Nagata, Yu. M. Smirnov, and R.H. Bing, gave independent solutions to this problem. The characterizations of Nagata and Smirnov are based on the existence of locally finite base, while that of Bing requires a discrete base for the topology.

We will prove the regularity of X and the existence of a countably locally finite basis for X are equivalent to metrizability of X.

25.1 The Nagata Smirnov Metrization Theorem

25.1.1 G_δ Set

A subset A of a space X is called a G_δ set in X if it equals the intersection of a countable collection of open subsets of X.

Example 1: In a metric space X, each closed set is a G_δ set- Given $A \subset X$, let $U(A, \in)$ denote the $\in$ – neighbourhood of A. If A is closed, you can check that

$$A = \bigcap_{n \in Z_+} U(A, 1/n)$$

Notes

Lemma 1: Let X be a regular space with a basis $\mathcal{B}$ that is countably locally finite. Then X is normal, and every closed set in X is a G_δ set in X.

Proof: *Step I:* Let W be open in X. We show there is a countable collection $\{U_n\}$ of open sets of X such that

$$W = \bigcup U_n = \bigcup \overline{U}_n$$

since the basis $\mathcal{B}$ for X is countable locally finite, we can write $\mathcal{B} = \bigcup \mathcal{B}_n$, where each collection $\mathcal{B}_n$ is locally finite. Let $\mathcal{C}_n$ be the collection of those basis elements $\mathcal{B}$ such that $\mathcal{B} \in \mathcal{B}_n$ and $\overline{B} \subset W$. Then $\mathcal{C}_n$ is locally finite, being a subcollection of $\mathcal{B}_n$.

Define $U_n = \bigcup_{B \in \mathcal{C}_n} B$

Then U_n is an open set, and by Lemma "Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then:

(a) Any subcollection of $\mathcal{A}$ is locally finite.

(b) The collection $\mathcal{B} = \{\overline{A}\}_{A \in \mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.

(c) $\overline{U_{A \in \mathcal{A}} A} = U_{A \in \mathcal{A}} \overline{A}$.

$\overline{U}_n = \bigcup_{B \in C_n} \overline{B}$

Therefore, $\overline{U}_n \subset W$, so that

$\bigcup U_n \subset \bigcup \overline{U}_n \subset W.$

We assert that equality holds. Given $x \in W$, there is by regularity a basis element $B \in \mathcal{B}$ such that $x \in B$ and $\overline{B} \subset W$. Now $B \in \mathcal{B}_n$ for some n. Then $B \in \mathcal{C}_n$ by definition, so that $x \in U_n$. Thus $W \subset \bigcup U_n$, as desired.

Step II: We show that every closed set C in X is a G_δ set in X. Given C, let W = X – C, by Step I, there are sets U_n in X such that $W = \bigcup \overline{U}_n$. Then

$$C = \bigcap (X - \overline{U}_n),$$

so that C equals a countable intersection of open sets of X.

Step III: We show X is normal. Let C and D be disjoint closed sets in X. Applying step I to the open set X – D, we construct a countable collection $\{U_n\}$ of open sets such that $\bigcup U_n = \bigcup \overline{U}_n = X - D.$

Then $\{U_n\}$ covers C and each set $\overline{U}_n$ is disjoint from D. Similarly there is a countable covering $\{V_n\}$ of D by open sets whose closures are disjoint from C.

Now we are back in the situation that arose in the proof that a regular space with a countable basis is normal. We can repeat that proof. Define

$$U'_n = U_n - \bigcup_{i=1}^{n} \overline{V}_i \text{ and } V'_n = V_n - \bigcup_{i=1}^{n} \overline{U}_i$$

Then the sets

$$U' = \bigcup_{n \in Z_+} U'_n \text{ and } V' = \bigcup_{n \in Z_+} V'_n$$

are disjoint open sets about C and D, respectively.

Lemma 2: Let X be normal, let A be a closed G_δ set in X. Then there is a continuous function f : X → [0, 1] such that f(x) = 0 for x ∈ A and f(x) > 0 for x ∉ A.

Proof: Write A as the intersection of the open sets U_n, for n ∈ Z_+. For each n, choose a continuous function $f_n : X \to [0, 1]$ such that f(x) = 0 for x ∈ A and f(x) = 1 for x ∈ X – U_n. Define $f(x) = \Sigma f_n(x)/2^n$. The series converges uniformly, by comparison with $\Sigma 1/2^n$, so that f is continuous. Also, f vanishes on A and is positive on X – A.

25.1.2 Nagata–Smirnov Metrization Theorem

Statement: A space X is metrizable if and only if X is regular and has a basis that is countably locally finite.

Proof: *Step I:* Assume X is regular with a countably locally finite basis $\mathcal{B}$. Then X is normal, and every closed set in X is a G_δ set in X. We shall show that X is metrizable by imbedding X in the metric space $(\mathcal{R}^J, \bar{\rho})$ for some J.

Let $\mathcal{B} = \bigcup \mathcal{B}_n$, where each collection $\mathcal{B}_n$ is locally finite. For each positive integer n, and each basis element $\mathcal{B} \in \mathcal{B}_n$, choose a continuous function

$$f_{n,B} : X \to \left[0, \frac{1}{n}\right]$$

such that $f_{n,B}(x) > 0$ for x ∈ B and $f_{n,B}(x) = 0$ for x ∉ B. The collection $[f_{n,B}]$ separates points from closed sets in X: Given a point x_0 and a neighbourhood U of x_0, there is basis element B such that $x_0 \in B \subset U$. Then $B \in \mathcal{B}_n$ for some n, so that $f_{n,B}(x_0) > 0$ and $f_{n,B}$ vanishes outside U.

Let J be the subset of $Z_+ \times \mathcal{B}$ consisting of all pairs (n, B) such that B is an element of $\mathcal{B}_n$.

Define $F : X \to [0, 1]^J$

by the equation $F(x) = (f_{n,B}(x))_{(n,b) \in J}$.

Relative to the product topology on $[0, 1]^J$, the map F is an imbedding.

Now we give $[0, 1]^J$ the topology induced by the uniform metric and show that F is an imbedding relative to this topology as well. Here is where the condition $f_{n,B(x)} < \frac{1}{n}$ comes in. The uniform topology is finer (larger) than the product topology. Therefore, relative to the uniform metric, the map $\mathcal{F}$ is injective and carries open sets of X onto open sets of the image space $\mathcal{Z}$ = F(x). We must give a separate proof that F is continuous.

Note that on the subspace $[0, 1]^J$ of $\mathcal{R}^J$, the uniform metric equals the metric

$$\rho((x_\alpha), (y_\alpha)) = \sup\{|x_\alpha - y_\alpha|\}$$

To prove continuity, we take a point x_0 of X and a number ∈ > 0, and find a neighbourhood W of x_0 such that

$$x \in W \Rightarrow \rho(F(x), F(x_0)) < \in$$

Notes

Let n be fixed for the moment. Choose a neighbourhood U_n of x_0 that intersects only finitely many elements of the collection $\mathcal{B}_n$. This means that as B ranges over $\mathcal{B}_n$, all but finitely many of the functions $f_{n,B}$ are identically equal to zero on U_n. Because each function $f_{n,B}$ is continuous, we can now choose a neighbourhood V_n of x_0 contained in U_n on which each of the remaining functions $f_{n,B}$ for $B \in \mathcal{B}_n$, varies by at most $\in/2$.

Choose such a neighbourhood V_n of x_0 for each $n \in Z_+$. Then choose N so that $\frac{1}{N} \leq \frac{\in}{2}$, and define $W = V_1 \cap \ldots \cap V_n$. We assert that W is the desired neighbourhood of x_0. Let $x \in W$. If $n \leq \mathcal{N}$, then

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq \in/2$$

because the function $f_{n,B}$ either vanishes identically or varies by at most $\in/2$ on W. If n > N, then

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq Y_n < \in/2$$

because $f_{n,B}$ maps X into $\left[0, \frac{1}{n}\right]$. Therefore,

$$\rho(F(x), F(x_0)) \leq \in/2 < \in,$$

as desired.

Step II: Now we prove the converse.

Assume X is metrizable. We know X is regular; let us show that X has a basis that is countably locally finite.

Choose a metric for X. Given m, let $\mathcal{A}_m$ be the covering of X by all open balls of radius $\frac{1}{m}$. There is an open covering $\mathcal{B}_m$ of X refining $\mathcal{A}_m$ such that $\mathcal{B}_m$ is countably locally finite. Note that each element of $\mathcal{B}_m$ has diameter at most $\frac{2}{m}$. Let $\mathcal{B}$ be the union of the collections $\mathcal{B}_m$, for $m \in \mathcal{Z}_+$. Because each collection $\mathcal{B}_m$ is countably locally finite, so is $\mathcal{B}$. We show that $\mathcal{B}$ is a basis for X.

Given $x \in X$ and given $\in > 0$, we show that there is an element B of $\mathcal{B}$ containing x that is contained in $B(x, \in)$. First choose m so that $\frac{1}{m} < \frac{\in}{2}$. Then, because $\mathcal{B}_m$ covers X, we can choose an element B of $\mathcal{B}_m$ that contains x. Since B contains x and has diameter at most $\frac{2}{m} < \in$, it is contained in $B(x, \in)$, as desired.

25.2 Summary

- A subset A of a space X is called a G_δ set in X if it equals the intersection of a countable collection of open subsets of X.
- Let X be a regular space with a basis $\mathcal{B}$ that is countably locally finite. Then X is normal, and every closed set in X is a G_δ set in X.
- A space X is metrizable iff X is regular and has a basis that is countably locally finite.

25.3 Keywords

Basis: A collection of subsets B of X is called a basis for a topology if:

(1) The union of the elements of B is X.

(2) If $x \in B_1 \cap B_2$, $B_1, B_2 \in B$, then there exists a B_3 of B such that $x \in B_3 \subset B_1 \cap B_2$.

Notes

Metrizable: A topological X is metrizable if there exists a metric d on set X that induces the topology of X.

Neighbourhood: An open set containing x is called a neighbourhood of x.

Product topology: Let X, Y be sets with topologies T_x and T_y. We define a topology $T_{X \times Y}$ on X × Y called the product topology by taking as basis all sets of the form U × W where $U \in T_X$ and $W \in T_Y$.

25.4 Review Questions

1. Many spaces have countable bases; but no T_1 space has a locally finite basis unless it is discrete. Prove this fact.

2. Find a non-discrete space that has a countably locally finite basis does not have a countable basis.

3. A collection $\mathcal{A}$ of subsets of X is said to be locally discrete if each point of X has a neighbourhood that intersects at most one elements of $\mathcal{A}$. A collection $\mathcal{B}$ is countably locally discrete if it equals a countable union of locally discrete collections. Prove the following:

   Theorem (Being Metrization Theorem):

   A space X is metrizable if and only if it is regular and has a basis that is countably locally discrete.

4. A topological space is called locally metrizable iff every point is contained in an open set which is metrizable. Prove that if a normal space has a locally finite covering by metrizable subsets, then the entire space is metrizable.

25.5 Further Readings

Books

Lawson, Terry, *Topology*: *A Geometric Approach*, New York, NY: Oxford University Press, 2003.

Patty. C. Wayne (2009), *Foundations of Topology* (2nd Edition) Jones and Barlett.

Robert Canover, *A First Course in Topology*, The Willams and Wilkins Company 1975.

Notes

Unit 26: The Smirnov Metrization Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Understand the locally metrizable space;
- Explain the Smirnov Metrization theorem.

Introduction

The Nagata-Smirnov metrization theorem gives one set of necessary and sufficient conditions for metrizability of a space. In this, unit we prove a theorem that gives another such set of conditions. It is a corollary of the Nagata-Smirnov theorem and was first proved by Smirnov. This unit starts with the definitions of paracompact and locally metrizable space. After explaining these terms, proof of "The Smirnov Metrization Theorem" is given.

26.1 Locally Metrizable Space

A space X is locally metrizable if every point x of X has a neighborhood $\cup$ that is metrizable in the subspace topology.

The Smirnov Metrization Theorem

Statement: A space X is metrizable if and only if it is a paracompact Hausdorff space that is locally metrizable.

Proof: Suppose that X is metrizable.

Then X is locally metrizable; it is also paracompact. [Every metrizable space is paracompact].

Conversely, suppose that X is a paracompact Hausdorff space that is locally metrizable.

We shall show that X has a basis that is countably locally finite. Since X is regular, it will then follow from the Nagata – Smirnov theorem that X is metrizable.

Cover X by open sets that are metrizable; then choose a locally finite open refinement $\mathcal{C}$ of this covering that covers X. Each element C of $\mathcal{C}$ is metrizable, let the function $d_C : C \times C \to R$ be a metric that gives the topology of C. Given $x \in C$, let $B_C(x, \in)$ denote the set of all points y of C such that $d_C(x, y) < \in$. Being open in C, the set $B_C(x, \in)$ is also open in X.

Notes

Given $m \in Z_+$, let $\mathcal{A}_m$ be the covering of X by all these open balls of radius $\frac{1}{m}$; that is, let

$$\mathcal{A}_m = \left\{B_C\left(x, \frac{1}{m}\right) : x \in C \quad \text{and} \quad C \in \mathcal{C}\right\}$$

Let $\mathcal{D}_m$ be a locally finite open refinement of $\mathcal{A}_m$ that covers X. (Here we use paracompactness).

Let $\mathcal{D}$ be the union of the collections $\mathcal{D}_m$.

Then $\mathcal{D}$ is countably locally finite.

We assert that $\mathcal{D}$ is a basis for X; our theorem follows.

Let x be a point of X and let $\cup$ be a neighbourhood of x. We seek to find an element D of $\mathcal{D}$ such that $x \in D \subset \cup$.

Now x belongs to only finitely many elements of $\mathcal{C}$ say to $C_1, \ldots, C_K$. Then $\cup \cap C_i$ is a neighbourhood of x in the set C_i, so there is an $\in_i > 0$ such that

$$B_{C_i}(x, \in) \subset (\cup \cap C_i).$$

Choose m so that $\frac{2}{m} < \min.\{\in_1, \ldots, \in_K\}$.

Because the collection $\mathcal{D}_m$ covers X, there must be an element D of $\mathcal{D}_m$ containing x.

Because $\mathcal{D}_m$ refines $\mathcal{A}_m$, there must be an element $B_C\left(y, \frac{1}{m}\right)$ of $\mathcal{A}_m$, for some $C \in \mathcal{C}$ and some $y \in C$ that contains D. Because $x \in D \subset B_C\left(y, \frac{1}{m}\right)$, the point $x \in C$, so that C must be one of the sets $C_1, \ldots, C_K$. Say $C = C_i$. Since $B_C\left(y, \frac{1}{m}\right)$ has diameter at most $\frac{2}{m} < \in_i$, it follows that

$$x \in D \in B_{C_i}\left(y, \frac{1}{m}\right) \subset B_{C_i}(x, \in_i) \subset \cup, \text{ as desired.}$$

26.2 Summary

- A space X is locally metrizable if every point x of X has a neighbourhood $\cup$ that is metrizable in the subspace topology.
- A space X is metrizable iff it is a paracompact Hausdorff space that is locally metrizable.

26.3 Keywords

Hausdorff Space: A topological space X is a Hausdorff space if given any two points $x, y \in X$, $x \neq y$, there exists neighbourhoods $\cup_x$ of x, $\cup_y$ of y such that $\cup_x \cap \cup_y \neq \phi$.

Metrizable: A topological X is metrizable if there exists a metric d on set X that induces the topology of X.

Paracompact: A space X is paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X.

Notes

Regular: Let X be a topological space where one-point sets are closed. Then X is regular if a point and a disjoint closed set can be separated by open sets.

26.4 Review Questions

1. If a separable space is also metrizable, then prove that the space has a countable base.
2. Show that any finite subset of metrizable space is always discrete.
3. Show that a topological space X is metrizable ⇔ there exists a homeomorphism of X onto a subspace of some metric space Y.
4. A compact Hausdorff space is separable and metrizable if it is:

   (a) second countable (b) not second countable

   (c) first countable (d) none

26.5 Further Readings

Books

Lawson, Terry, *Topology : A Geometric Approach*. New York, NY: Oxford University Press, 2003.

Robert Canover, *A first course in topology*. The Williams and Wilkins Company, 1975.

Unit 27: Complete Metric Spaces

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define Cauchy's sequence;
- Solve the problems on Cauchy's sequence;
- Define complete metric space;
- Solve the problems on complete metric spaces.

Introduction

The concept of completeness for a metric space is basic for all aspects of analysis. Although completeness is a metric property rather than a topological one, there are a number of theorems involving complete metric spaces that are topological in character. In this unit, we shall study the most important examples of complete metric spaces and shall prove some of these problems.

27.1 Cauchy's Sequence

A sequence $<x_n>$ in a metric space X is said to be a Cauchy sequence in X if given $\in > 0$ there exists a positive integer n_o such that

$$d(x_m, x_n) < \in \quad \text{where } m, n \geq n_o.$$

Alternative definition: A sequence $<x_n>$ is Cauchy if given $\in > 0$, there exists a positive integer n_o such that

$$d(x_{n+p}, x_n) < \in \quad \text{for all } n \geq n_o \text{ and for all } p \geq 1.$$

Theorem 1: Every convergent sequence in a metric space is a Cauchy sequence.

Proof: Let (X, d) be a metric space.

Let $<x_n>$ be a convergent sequence in X.

Suppose $\lim\limits_{n\to\infty} x_n = x$.

Notes

Then given $\in > 0$, there exist a positive integer n_o such that $m, n \geq n_o \Rightarrow d(x_m, x) < \frac{\in}{2}$ and $d(x_n, x) < \frac{\in}{2}$. Therefore, $m, n \geq n_o \Rightarrow d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\in}{2} + < \frac{\in}{2} = \in$.

Hence, $<x_n>$ is a Cauchy sequence.

Note The converse of this theorem is not true i.e., Cauchy sequence need not be convergent.

To prove this, consider the following example.

Let $X = \mathcal{R} - \{0\}$.

Let $d(x, y) = |x - y|$

Consider the sequence $x_n = \frac{1}{n}, n \in \mathcal{N}$

We shall show that

$<x_n>$ is a Cauchy sequence but it does not converge in X. Let $\in > 0$ be given and n_o be a positive integer such that $n_o > \frac{2}{\in}$.

Now
$$\begin{aligned} d(x_m, x_n) &= |x_m - x_n| \\ &= |x_m + (-x_n)| \\ &= |x_m| + |x_n| \\ &= \frac{1}{m} + \frac{1}{n} \end{aligned}$$

If $m \geq n_o \Rightarrow m > \frac{2}{\in}$ and so $\frac{1}{m} < \frac{\in}{2}$

Similarly, $\frac{1}{n} < \frac{\in}{2}$

$$\therefore \quad d(x_m, x_n) \leq \frac{1}{m} + \frac{1}{n} < \frac{\in}{2} + \frac{\in}{2} = \in$$

Thus $d(x_m, x_n) < \in$.

Hence $<x_n>$ is a Cauchy sequence.

Clearly, the limit of this sequence is 0 (zero) which does not belong to X.

Thus x_n does not converge in X.

Example 1: Let $<a_n>$ be a Cauchy sequence in a metric space (X, ρ) and let $<b_n>$ be any sequence in X s.t. $\rho(a_n, b_n) < \frac{1}{n} \; \forall \; n \in N$.

Show that

(i) $<b_n>$ is a Cauchy sequence.

(ii) $<a_n>$ converges to a point $p \in X$ iff $<b_n>$ converges in p.

Solution: Let $<a_n>$ be a Cauchy sequence in a metric space (X, ρ) so that

given $\in, K > 0 \quad \exists\, n_o \in N$ s.t.

$$n, m \geq n_o \Rightarrow \rho(a_n, a_m) < \in K \qquad \ldots(1)$$

Also let $<b_n>$ be a sequence in X s.t.

$$\rho(a_n, b_n) < \frac{1}{n} \;\; \forall\, n \in N \qquad \ldots(2)$$

Step (i): To prove that $<b_n>$ is a Cauchy sequence.

Let ε, K > 0 any given real numbers.

Then $\exists\, m_o \in N$ s.t. $\frac{1}{m_o} < \varepsilon k$. …(3)

Set $K_o = \max.\, (n_o, m_o)$.

Then $K_o \geq n_o, m_o$, so that

$$\frac{1}{K_o} \leq \frac{1}{n_o}, \frac{1}{m_o} \qquad \ldots(4)$$

$$\frac{1}{m_o} < \varepsilon K, \frac{1}{K_o} \leq \frac{1}{m_o} \Rightarrow \frac{1}{K_o} \leq \frac{1}{m_o} < \varepsilon K \Rightarrow \frac{1}{K_o} < \varepsilon K. \qquad \ldots(5)$$

If $n \geq K_o$, then $\rho(a_n, b_n) < \frac{1}{n} \leq \frac{1}{K_o} < \varepsilon K$,

i.e., $\rho(a_n, b_n) < \varepsilon K \;\forall\, n, m \geq K_o$ …(6)

For $n, m \geq K_o$, we have

$$\rho(b_n, b_m) \leq \rho(b_n, a_n) + p(a_n, a_m) + p(a_m, b_m)$$

$$< \varepsilon K + \varepsilon K + \varepsilon K = 3\,\varepsilon K.$$

Choosing initially $K = \frac{1}{3}$, we get

$$\rho(b_n, b_m) < \varepsilon \;\forall n \geq K_o.$$

This proves that $<b_n>$ is a Cauchy sequence.

Step (ii): Let $a_n \to p \in X$.

To prove that $b_n \to p$.

$$a_n \to p \Rightarrow \text{given } \varepsilon, K > 0, \exists\, m_o \in N \text{ s.t.}$$

$$n \geq m_o \Rightarrow \rho(a_n, p) < \varepsilon K.$$

We have seen that $<a_n>$ and $<b_n>$ are Cauchy Sequences and therefore given $\varepsilon, K > 0, \exists\, n_o \in N$ s.t.

$$\forall\, m, n \geq n_o \Rightarrow \rho(a_n, a_m) < \varepsilon K, \rho(b_n, b_m) < \varepsilon K.$$

Notes

Choose $K_o = \max.\ (n_o, m_o)$

$$\rho(b_n, p) \leq \rho(b_n, b_m) + \rho(b_m, a_m) + \rho(a_m, p)$$

$$< \varepsilon K + \varepsilon K + \varepsilon K = 3\,\varepsilon K \quad \forall m, n \geq K_o.$$

Choosing initially $K = \frac{1}{3}$, we get

$$\rho(b_n, p) < \varepsilon \ \forall n \geq K_o$$

This $b_n \to p$.

Conversely if $b_n \to p$, then by making parallel arguments, we can show that $a_n \to p$. Hence the result.

Self Assessment

1. In any metric space, prove that every Cauchy sequence is totally bounded.
2. Let a subsequence of a sequence $<a_n>$ converge to a point p. Prove that $<a_n>$ also converges to p.

27.2 Complete Metric Space

A metric space X is said to be complete if every Cauchy sequence of points in X converges to a point in X.

Example 2: The complex plane C is complete.

Solution: Let $<z_n>$ be a Cauchy sequence of complex numbers, where $Z_n = x_n + i\,y_n$.

Here $<x_n>$ and $<y_n>$ are themselves Cauchy sequences of real numbers,

$$|x_m - x_n| \leq |z_m - z_n|$$

and $$|y_m - y_n| \leq |z_m - z_n|$$

But the real line being a complete metric space, there exists real numbers x and y such that $x_n \to x$ and $y_n \to y$.

Thus, taking z = x + iy, we find $z_n \to z$ as

$$|z_n - z| = |(x_n + i\,y_n) - (x + i\,y)|$$

$$= |(x_n - x) + i(y_n - y)|$$

$$\leq |x_n - x| + |y_n - y|$$

$$\to 0 \text{ as } n \to \infty$$

$\therefore$ $$|z_n - z| = 0 \Rightarrow z_n \to z.$$

Hence if the real line is a complete metric space, then the complex plane is also a complete metric space.

27.3 Theorems and Solved Examples

Theorem 2: Let X be a complete metric space and Y be a subspace of X. Show that Y is closed iff it is complete.

Proof: Let Y be closed.

Let $<x_n>$ be a Cauchy sequence in Y. This implies that it is a Cauchy sequence in X.

Since X is complete, $<x_n>$ converges to some point $x \in X$.

Let A be the range of $<x_n>$.

If A is finite, then x is that term of $<x_n>$ which is infinitely repeated and therefore $x \in X$. If A is infinite, then x, being limit of $<x_n>$, is a limit point of its range A. Since $A \subset Y$, so, x is a limit point of Y. But Y is closed, therefore, $x \in Y$.

This implies that $<x_n>$ is convergent in Y. Hence Y is complete.

Conversely, let Y be complete.

Here we are to prove that Y is closed.

Let x be a limit point of Y.

Then, for each positive integer n, $\exists$ an open sphere $S\left(x, \frac{1}{n}\right)$ containing at least one point x_n of Y, other than x.

Let $\in > 0$ be given.

$\exists$ a positive integer n_o such that $\frac{1}{n_o} < \in$. We have $\frac{1}{n} < \in$ for all $n \geq n_o$.

Since $x_n \in S\left(x, \frac{1}{n}\right)$,

$$d(x_n, x) < \frac{1}{n}.$$

Therefore $d(x_n, x) < \in \ \forall\ n \geq n_o$.

This implies that $<x_n>$ converges to x in X. Therefore $<x_n>$ is a Cauchy sequence in X, So it is a Cauchy sequence in Y.

But Y is complete.

Therefore $<x_n>$ is convergent in Y.

This implies that $x \in Y$, because limit of convergent sequence is unique. Hence, Y is closed.

Theorem 3: Cantor's Intersection Theorem.

Let X be a complete metric space. Let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$ as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Proof: Let $F = \bigcap_{n=1}^{\infty} F_n$.

For $n \in N$, let $x_n \in F_n$, we prove that $<x_n>$ is a Cauchy sequence.

Notes

Let $\in > 0$ be given.

$d(F_n) \to 0$, therefore there exists a positive integer n_o of such that $d(F_{n_o}) < \in$.

Since $<F_n>$ is a decreasing sequence,

$\therefore \quad m, n \geq n_o \Rightarrow F_m, F_n \subseteq F_{n_o}$

$\Rightarrow x_m, x_n \in F_{n_o}$

$\Rightarrow d(x_m, x_n) < d(F_{n_o})$

$\Rightarrow d(x_m, x_n) < \in \qquad [\because \; d(F_{n_o}) < \in]$

$\Rightarrow <x_n>$ is a Cauchy sequence.

Since the space X is complete, $<x_n>$ must converge to some point, say x in X i.e. $x_n \to x \in X$.

We shall prove that

$$x \in \bigcap_{n=1}^{\infty} F_n.$$

If possible, let $x \notin \bigcap_{n=1}^{\infty} F_n$.

$\Rightarrow \quad x \notin F_k$ for some $k \in N$.

Since each F_n is a closed set, F_k is also a closed set, therefore x cannot be a cluster point of F_k, and so $d(x, F_k) \neq 0$.

Let $\quad d(x, F_k) = r > o$ so that

$$d(x, y) \geq r \;\forall\; y \in F_k.$$

This shows that $\quad F_k \cap S\left(x, \frac{1}{2}r\right) = \phi.$

Now, $\quad n > k \Rightarrow F_n \subset F_k$

$\Rightarrow x_n \in F_k \qquad (\because x_n \in F_n \subset F_k)$

$\Rightarrow x_n \notin S\left(x, \frac{1}{2}r\right) \qquad [\because F_k \cap S\left(x, \frac{1}{2}r\right) = \phi]$

This contradicts the fact that $x_n \to x$.

Therefore $x \in \bigcap_{n=1}^{\infty} F_n$ and hence $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

Example 3: Show that every compact metric is complete.

Solution: Let (X, d) be a compact metric space.

To prove : X is complete.

Let $<a_n>$ be an arbitrary Cauchy sequence in X. If we show that $<a_n>$ converges to a point in X, the result will follow.

X is compact $\Rightarrow$ X is sequentially compact.

$\Rightarrow$ Every sequence in X has a convergent subsequence.

$\Rightarrow$ In particular, every Cauchy sequence in X has a convergent subsequence.

$\Rightarrow <a_n>$ has a subsequence $<a_{in} : n \in N>$ which converges to a point $a_{io} \in X$

$\Rightarrow <a_n>$ also converges to the point $a_{i_o} \in X$.

Theorem 4: A metric space is compact iff it is totally bounded and complete.

Proof: Let (X, d) be a compact metric space.

To prove that X is complete and totally bounded.

X is compact. ⇒ X is sequentially compact. …(1)

⇒ X is totally bounded. …(2)

X is sequentially compact. ⇒ every sequence in X has convergent subsequence.

⇒ In particular, every Cauchy sequence in X has a convergent subsequence

⇒ Every Cauchy sequence in X converges to some point in X.

⇒ X is complete. …(3)

From (2) & (3) the required result follows.

Conversely, suppose that a metric space (X, d) is complete and totally bounded.

To prove that X is compact.

Consider an arbitrary sequence

$$S_1 = \langle x_{11}, x_{12}, x_{13}, \ldots \rangle$$

X is totally bounded ⇒ ∃ finite class of open spheres, each of radius 1, whose union is X.

From this we can deduce that S_1 has a subsequence

$$S_2 = \langle x_{21}, x_{22}, x_{23}, \ldots \rangle$$

all of whose points be in some open sphere of radius $\frac{1}{2}$.

Similarly we can construct a subsequence S_3 of S_2 s.t.

$$S_3 = \langle x_{31}, x_{32}, x_{33}, \ldots \rangle$$

all of whose points be in some open sphere of radius $\frac{1}{3}$.

We continue this process to from successive subsequences. Now we suppose that

$$S = \langle x_{11}, x_{22}, x_{33}, \ldots \rangle.$$

Then S is a diagonal subsequence to form successive subsequence. Now we suppose that $S = \langle x_{11}, x_{22}, x_{33} \ldots \rangle$. Then S is a diagonal subsequence of S_1. By nature of this construction, S is clearly Cauchy subsequence of S_1.

X is complete ⇒ every Cauchy sequence in X is convergent.

⇒ in particular, the Cauchy sequence S is convergent.

Finally, the sequence, S_1 has a convergent subsequence S. Since the sequence S_1 in X is arbitrary and hence every sequence in X has a convergent subsequence, meaning thereby X is sequentially compact and hence X is compact.

Theorem 5: Let A be a subset of a complete metric space (X, d). Prove that A is compact ⇔ A is closed and totally bounded.

Proof: Let A be a compact subset of complete metric space (X, d).

To prove that A is closed and totally bounded.

Notes

X is a metric space ⇒ X is a Hausdroff space w.r.t. the metric topology.

Being a compact subset of a Hausdroff space, A is closed.

A is compact. ⇒ A is sequentially compact.

⇒ A is totally bounded.

Finally, we have shown that A is closed and totally bounded.

Conversely, suppose that A is closed and totally bounded subset of complete metric space (X, d).

To prove that A is compact.

A is complete, being a closed subset of a complete metric space (X, d). Thus A is complete and totally bounded.

Self Assessment

3. Let X be a metric space and Y is a complete metric space, and let A be dense subspace of X. If f is a uniformly continuous mapping of A into Y, then f can be extended uniquely to a uniformly continuous map of X into Y.

4. Let A be subspace of a complete metric space and show that $\bar{A}$ is compact ⇔ A is totally founded.

5. If $<A_n>$ is a sequence of nowhere dense sets in a complete metric space X, then there exist a point in X which is not in any of the A_n's.

27.4 Summary

- A sequence $<x_n>$ is Cauchy if given $\in > 0$, ∃ a positive integer n_o such that

  $d(x_{n+p}, x_n) < \in$ for all $n \geq n_o$ and for all $p \geq 1$.

- A metric space X is said to be complete if every Cauchy sequence of points in X converges to a point in X.

- A metric space is compact iff it is totally bounded and complete.

27.5 Keywords

Closed Set: A set A is said to be closed if every limiting point of A belongs to the set A itself.

Cluster Point: Let (X, T) be a topological space and $A \subset X$. A point $x \in X$ is said to be the cluster point if each open set containing x contains at least one point of A different from x.

Convergent Sequence: A sequence $<a_n>$ is said to converge to a, if $\varepsilon > 0$, $\exists\, n_o \in N$, s.t. $n \geq n_o \Rightarrow |a - a_n| < \varepsilon$.

Sequentially Compact: A metric space (X, d) is said to be sequentially compact if every sequence in X has a convergent subsequence.

27.6 Review Questions

1. If a Cauchy sequence has a convergent subsequence, then prove that it is itself convergent.

2. Show that every compact metric space is complete.

Notes

3. Show that the metric space ($\mathcal{R}$, d) is complete, where d is usual metric on $\mathcal{R}$.
4. Show that the set $\mathcal{C}$ of complex numbers with usual metric is complete metric space.
5. Prove that every closed subset of a complete metric space is complete.
6. Prove that Frechet space is complete.
7. Show that a metric space is complete iff every infinite totally bounded subset has a limit point.

27.7 Further Readings

Books

Dmitre Burago, Yu D Burago, Sergei Ivanov, *A Course in Metric Geometry*, American Mathematical Society, 2004.

Victor Bryant, *Metric Spaces; Iteration and Application*, Cambridge University Press, 1985.

Notes

Unit 28: Compactness in Metric Spaces

CONTENTS

Objectives

After studying this unit, you will be able to:

- Know the Bolzano Weierstrass theorem and BWP;
- Define sequentially compact and lebesgul measure;
- Define totally bounded set;
- Describe the compactness in metric spaces;
- Solve the related problems.

Introduction

We have already shown that compactness, limit point compactness and sequentially compact are equivalent for metric spaces. There is still another formulation of compactness for metric spaces, one that involves the notion of completeness. We study it in this unit. As an application, we shall prove a theorem characterizing those subspaces of $\mathcal{C}(X, R^n)$, that are compact in the uniform topology.

28.1 Bolzano Weierstrass Theorem

A closed and bounded infinite subset of R contains a limit point.

Bolzano Weierstrass Property: A metric space (X, d) is said to have the Bolzano weierstrass property if every infinite subset of X has a limit point.

In brief, 'Bolzano Weierstrass Property' is written as B.W.P. A space with B.W.P. is also called Frechet compact space.

28.1.1 Sequentially Compact

Notes

A metric space (X, d) is said to be sequentially compact if every sequence in X has a convergent sub-sequence.

Example 1: The set of all real numbers in (0, 1) is not sequentially compact.

For the sequence $\left\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$ in (0, 1) converges to $0 \notin (0, 1)$, on the other hand [0, 1] is sequentially compact.

28.1.2 Lebesgue Number

Let $\{G_i : i \in \Delta\}$ be an open cover for a metric space (X, d). A real number $\delta > 0$ is called a Lebesgue number for the cover if any $A \subset X$ s.t. $d(A) < \delta \Rightarrow A \subset G_{i_0}$ for at least one index $i_0 \in \Delta$.

Lebesgue Covering Lemma

Every open covering of a sequentially compact space has a lebesgue number.

28.1.3 Totally Bounded Set

Let (X, d) be a metric space. Let $\in > 0$ be any given real number. A set $A \subset X$ is called an $\in$ - net if

(i) A is finite set

(ii) $X = U\{S_{\varepsilon(a)} : a \in A\}$

The metric space (X, d) is said to be topology bounded if it contains an $\in$ – net for every $\in > 0$. Here (ii) $\Rightarrow$ given any point $p \in X$, $\exists$ at least one point $a \in A$ s.t. $d(p, a) < \in$.

28.1.4 Compactness in Metric Spaces

If (X, d) be a metric space and $A \subset X$, then the statement that A is compact, A is countably compact and A is sequentially compact are equivalent.

28.2 Theorems and Solved Examples

Theorem 1: A metric space is sequentially compact iff it has the Bolzano Weierstrass Property.

Proof: Let X be a metric space.

Let us suppose that it is sequentially compact.

Let A be an infinite subset of X.

Since A is infinite so let $\langle x_n \rangle$ be any sequence of distinct points of A. Since X is sequentially compact, so there exists a convergent subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$. Let x be its limit and B be its range.

Since $\langle x_n \rangle$ is a sequence of distinct points, B is infinite.

We know that if the range of a convergent sequence is infinite then its limit point is the limit point of the range.

Notes

Thus, x is the limit point of B.

$\Rightarrow$ x is a limit point of A, as $B \subset A$.

Hence X has the Bolzano Weierstrass Property.

Conversely, let X has the Bolzano Weierstrass Property. Let $\langle x_n \rangle$ be a sequence in X. Let A be the range of $\langle x_n \rangle$. If A is infinite, then there is some term of $\langle x_n \rangle$ which is infinitely repeated and that gives us a convergent subsequence of $\langle x_n \rangle$. If A is infinite then by our assumption the set A has a limit point, say x.

Since A is infinite and x is a limit point of A, therefore there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \to x$.

Thus proves that X is sequentially compact.

Theorem 2: Every compact metric space has the Bolzano Weierstrass Property.

Proof: Let X be a compact metric space.

To prove: X has Bolzano Weierstrass Property.

Let A be an infinite subset of X. Suppose that A has no limit point. Then to each $x \in X$, there exists an open sphere S_x which contains no other point of A other than its centre x.

Thus, the class $\{S_x\}$ of all such open spheres is an open cover of X.

But X is compact, therefore its open cover is reducible to a finite subcover say

$\{S_{x_i} : i = 1, 2, ..., n\}$, so that

$$A \subset \bigcup_{i=1}^{n} S_{x_i}.$$

Each S_{x_i} contains no point of A other than its centre x_i, i = 1,2,...,n

$\therefore \quad A = \{x_1, x_2, ..., x_n\}$

$\Rightarrow \quad$ A is finite.

This contradicts the fact that A is infinite.

Hence A must have a limit point.

Thus, the compact metric space X has BWP.

Theorem 3: A compact metric space is separable.

Proof: Let (X, d) be a compact metric space.

To prove that (X, d) is separable.

Fix a positive integer n.

Each open sphere forms an open set.

Consider the family $\{S_{(x, 1/n)} : x \in X\}$

Clearly it is an open cover of X which is known to be compact.

Hence this cover must be reducible to a finite sub cover, say

$\{(S_{x_{nr}}, 1/n) : r = 1, 2, ..., K_n\}$

Notes

Write $A_n = \{(x_{nr} : r = 1, 2, \ldots, K_n\}$.

The set A_n can be constructed for each $n \in N$.

A_n has the following properties:

(i) A_n is a finite set,

(ii) given $x \in X$; $\exists\, x_{nr} \in A_n$ s.t. $d(x, x_{nr}) < \frac{1}{n}$.

Write $A = \bigcup_{n \in N} A_n$

Being a countable union of countable sets, A is enumerable

Clearly $A \subset X$

Taking closure of both sides, $\overline{A} \subset \overline{X} = X$ i.e.

$\overline{A} \subset X$ [$\because$ X is closed in X]

We claim $\overline{A} = X$

For this it is enough to show that $X \subset \overline{A}$.

Let $x \in X$ be arbitrary and let $G \subset X$ be an open set s.t. $x \in G$.

By the property (ii) of A_n,

Given, $x \in X$, $\exists\, x_{nr} \in A_n \subset A$ s.t. $d(x_{nr}, x) < \varepsilon$ on taking $\frac{1}{n} < \varepsilon$. By the definition of open set in a metric space.

$X \in G$, G is open $\Rightarrow \exists$ positive real number r, $S_{(x,r)} \subset G$

$\Rightarrow$ in particular $S_{(x, \in)} \subset G$

$d(x, x_{nr}) < \varepsilon \Rightarrow x_{nr} \in S_{(x, \in)} \subset G$

$\Rightarrow x_{nr} \in G$

$\Rightarrow$ G contains some points of A other than x.

$\Rightarrow (G - \{x\}) \cap A \neq \phi$

$\Rightarrow x \in D(A) \subset \overline{A}$

$\Rightarrow x \in \overline{A}$

Thus we have shown that

any $x \in X \Rightarrow x \in \overline{A}$

This proves that $X \subset \overline{A}$

Finally we have shown that

$\exists\, A \subset X$ s.t. A is enumerable and $\overline{A} = X$.

This proves that X is separable.

Notes

Example 2: If a metric space (X, d) is totally bounded, then X is bounded.

Solution: Let (X, d) be a totally bounded metric space so that it contains an ε - net for every ε > 0. Let $A \subset X$ be an ε - net then:

(i) A is finite

(ii) $X = \bigcup \{S_\varepsilon(a) : a \in A\}$

(i) ⇒ A is bounded ⇒ d(A) is finite.

(ii) ⇒ $d(X) \leq d(A) + 2\varepsilon$ = a finite quantity.

⇒ d(X) ≤ a finite quantity

⇒ X is a bounded set. Hence proved.

Example 3: Every totally bounded metric space is separable.

Solution: Let (X, d) be totally bounded metric space so that X contains an ε - net A_n $\forall$ $\varepsilon_n > 0$.

To prove that X is separable.

A_n is ε - net ⇒ A_n is finite and $X = \bigcup\{S(a, \varepsilon_n) : a \in A_n\}$.

Write $A = \bigcup\{A_n : n \in N\}$

Being an enumerable union of finite sets A is enumerable.

$A \subset X \Rightarrow \overline{A} \subset \overline{X} = X \Rightarrow \overline{A} \subset X$. ...(1)

Let $x \in X$ be arbitrary and let G be an open set s.t. $x \in G$.

By definition of open set

$G \subset S_{(x, \in_n)}$ ...(2)

Also $A_n \bigcap S_{(x, e_n)} \neq \phi$. For A_n is ε - net.

This $S_{(x, e_n)} \bigcap A \neq \phi$

⇒ $G \bigcap A \neq \phi$ [by (2)]

⇒ $x \in \overline{A}$

∴ Any $x \in X \Rightarrow x \in \overline{A}$

Consequently $X \subset \overline{A}$

In view of (1), this $X = \overline{A}$

This leads to the conclusion that X is separable.

Theorem 4: Lebesgue covering lemma: Every open cover of sequentially compact metric space has a Lebesgue number.

Proof: Let $\{G_i : i \in \Delta\}$ be an open cover for a metric space (X, d). A real number δ > 0 is called a Lebesgue number for the cover if any $A \subset X$ s.t. $d(A) < \delta \Rightarrow A \subset G_{i_0}$ for at least one index $i_0 \in \Delta$.

Let $\{G_i : i \in \Delta\}$ be an open cover of a sequentially compact metric space (X, d).

To prove that the cover $\{G_i\}_{i \in \Delta}$ has a Lebesgue number.

Notes

Suppose the contrary.

Then ∃ no Lebesgue number for the cover $\{G_i\}_{i \in D}$. Then for each $n \in N$, ∃ a set $B_n \subset X$ with the property that $0 < d(B_n) < \frac{1}{n}$

and $B_n \not\subseteq G_i \ \forall \ i \in \Delta$ ...(1)

Choose a point $b_n \in B_n \ \forall \ n \in N$ and consider the sequence $<b_n>$. By the assumption of sequential compactness, the sequence $<b_n : n \in N>$ contains a subsequence $<b_{in} : n \in N>$ which converges to $b \in X$.

But $\{G_i\}$ is an open cover of X so that

∃ open set G_{i_0} s.t. $b \in G_{i_0}$. By definition of open set

$$S_{\varepsilon(b)} \subset G_{i_0} \quad ...(2)$$

$\because \quad b_{i_n} \to b$

$\therefore \quad$ Given any $\varepsilon > 0$, $\exists \ n_0 \in N$ s.t. $\forall \ i_n \geq n_0 \Rightarrow b_{i_n} \in S\frac{\varepsilon}{2}(b)$. ...(3)

Choosing a positive integer K_0 ($\geq n_0$) such that

$$\frac{1}{K_0} < \frac{\varepsilon}{2} \quad ...(4)$$

From (3), $i_n \geq K_0 \Rightarrow b_{i_n} \in S_{\varepsilon/2}(b)$

$\Rightarrow$ In particular $bK_0 \in S_{\varepsilon/2}(b)$ ...(5)

In accordance with (1)

$$b_{K_0} \in B_{K_0}, 0 < d(B_{K_0}) < \frac{1}{K_0} \quad ...(6)$$

On using (4)

$0 < d(B_{K_0}) < \varepsilon/2$ ...(7)

From (5) and (6), if follows that

$B_{K_0} \cap S_{\varepsilon/2}(b) \neq \phi$ ...(8)

From (7) and (8), if follows that B_{K_0} is a set of diameter $< \frac{\varepsilon}{2}$ and it intersects $S\frac{\varepsilon}{2}.2.(b)$, Showing thereby

$$B_{K0} \subset S\frac{\varepsilon}{2}.2.(b)$$

i.e., $B_{K0} \subset S_\varepsilon(b)$.

In view of (2), this gives $B_{K_0} \subset G_{i_0}$ ...(9)

In accordance with (1), $B_{K_0} \not\subset G_{i_0}$, $i_0 \in \Delta$

In particular, $B_{K_0} \subseteq G_{i_0}$, $i_0 \in \Delta$

Contrary to (9).

Hence the required results follows.

Notes

Theorem 5: Every compact subset of a metric space is closed and bounded.

Proof: Let Y be a compact subset of a metric space (X, d). If f is finite, then it is certainly bounded and closed.

Consider the case in which Y is not finite.

Y is compact $\Rightarrow$ Y is sequentially compact.

To prove that Y is bounded. Suppose not. Then Y is not bounded. Then it is possible to find a pair of points of Y at large distance apart. Let $y_1 \in Y$ be arbitrary.

Then we take $y_2 \in Y$

s.t. $d(y_1, y_2) > 1$

Now we can select a point y_3 s.t.

$d(y_1, y_3) > l + d(y_1, y_2)$

Continuing this process, we get a sequence

$\langle y_n \rangle \in Y$

with the property that $d(y_1, y_m) > 1 + d(y_1, y_{m-1}) \ \forall \ n \in N$

$\therefore \quad d(y_1, y_m) > 1 + d(y_1, y_n)$ for $m > n$ (1)

This $d(y_m, y_n) \geq |d(y_1, y_m) - d(y_1, y_n)| > 1$

Above relation shows that $\langle y_n \rangle$ has no convergent subsequence contrary to the fact that Y is sequentially compact. Hence Y is bounded.

Aim: Y is closed.

Let y be a limit point of Y, $\exists$ sequence

$\langle y_n \rangle \in Y$ s.t. $\lim y_n = y$

Every sequence of $\langle y_n \rangle$ converges to y. For Y is sequentially compact and so every sequence in Y must converge in Y.

Hence $y \in Y$

Thus $y \in D(Y) \Rightarrow y \in Y$

or $D(Y) \subset Y$ or Y is closed.

Theorem 6: Every sequentially compact metric space is compact.

Proof: Let (X, d) be a sequentially compact metric space. To prove that X is compact.

Since X is sequentially compact metric space.

X is totally bounded. Let $\varepsilon > 0$ be an arbitrary real number fixed.

X is totally bounded $\Rightarrow$ X has ε - net.

Let us denote the set $\in$ net by A.

Then A is finite subset of X with the property

$$X = \cup\{S_{\varepsilon(a)} : a \in A\} \quad ...(1)$$

Since A is finite and hence we can write

$$A = \{x_1, x_2, x_3, ..., x_n\}$$

In this event (1) takes the form

$$X = \bigcup_{i=1}^{n} S_{\varepsilon(x_i)} \qquad ...(2)$$

Let $\{G_i : i \in \Delta\}$ be an open cover of X which is known to be sequentially compact so that, by theorem (lebesgue covering lemma), ∃ a Lebesgue number, say, δ for the cover $\{G_i\}_{i \in \Delta}$. Set $\delta = 3\varepsilon$.

The diameter of an open sphere of radius r is less than 2r.

i.e., $d(S_\varepsilon(x_i)) < 2\varepsilon = 2.\frac{\delta}{3} < \delta$

∴ $d(S_\varepsilon(x_i) < \delta$

By definition of Lebesgue number, ∃ an open set

$G_{ik} \in \{G_i : i \in \Delta\}$ s.t. $S_\varepsilon(x_k) \subset G_{ik}$ for $1 \leq k \leq n$.

From which we get $\bigcup_{k=1}^{n} S_{\varepsilon(x_k)} \subset \bigcup_{k=1}^{n} G_{ik}$

On using (2), $X \subset \bigcup_{k=1}^{n} G_{ik}$ ...(3)

But X is a universal set,

$$\bigcup_{k=1}^{n} G_{ik} \subset X \qquad ...(4)$$

Combining (3) and (4), we get $X = \bigcup_{k=1}^{n} G_{ik}$.

This implies that the family $\{G_{ik} : 1 \leq K \leq n\}$ is an open cover of X.

Thus the open cover $\{G_i : i \in \Delta\}$ of X is reducible to a finite the subcover $\{G_{ik} : 1 \leq K \leq n\}$ showing thereby X is compact.

Sequentially compact ⇒ compact ⇒ Countably compact

Theorem 7: A metric space (X, d) is compact iff it is complete and totally bounded.

Proof: If X is a compact metric space then X is complete. The fact that X is totally bounded is a consequence of the fact that the covering of X by all open ∈ - balls must contain a finite subcovering.

Conversely, Let X be complete and totally bounded.

To prove: X is sequentially compact.

Let $<x_n>$ be sequence of points of X. We shall construct a subsequence of $<x_n>$ i.e. a Cauchy sequence, so that it necessarily converges.

First cover X by finitely many balls of radius 1. At least one of these balls, say B_1, contains x_n for infinitely many values of n. Let J_1 be the subset of Z_+ consisting of those indices n for which $x_n \in B_1$.

Notes

Next, cover X by finitely many balls of radius $\frac{1}{2}$. Because J_1 is infinite, at least one of these balls, say B_2, must contain x_n for infinitely many values of n in J_1. Choose J_2 to be the set of those indices n for which $n \in J_1$ and $x_n \in B_2$. In general, given on infinite set J_k of positive integers, choose J_{k+1} to be an infinite subset of J_k such that there is a ball B_{k+1} of radius $\frac{1}{k+1}$ that contains x_n for all $n \in J_{k+1}$.

Choose $n_1 \in J_1$. Given n_k, choose $n_{k+1} \in J_{k+1}$ such that $n_{k+1} > n_k$; this we can do because J_{k+1} is an infinite set. Now for i, j ≥ k, the indices n_i and n_j both belong to J_k (because $J_1 \supset J_2 \supset \ldots$ is nested sequence of sets). Therefore, for all i, j ≥ k, the points x_{n_i} and x_{n_j} are contained in a ball B_k of radius $\frac{1}{k}$. It follows that the sequence $\langle x_{n_i} \rangle$ is a Cauchy sequence, as desired.

Theorem 8: Let X be a space; let (Y, d) be a metric space. If the subset $\mathcal{F}$ of $\mathcal{C}(X, Y)$ is totally bounded under the uniform metric corresponding to d, then $\mathcal{F}$ is equicontinuous under d.

Proof: Assume $\mathcal{F}$ is totally bounded. Give $0 < \in < 1$, and given x_0, we find a nhd U of x_0 such that $d(f(x), f(x_0)) < \in$ for $x \in U$ and $f \in F$.

Set $\delta = \varepsilon/3$; Cover $\mathcal{F}$ by finitely many open δ - balls.

$B(f_1, \delta)$, ..., $B(f_n, \delta)$ in $\mathcal{C}(X, Y)$. Each function f_i is continuous; therefore, we can choose a nhd of x_0 such that for i = 1 ,.., n.

$$d(f_i(x), f_i(x_0)) < \delta$$

whenever $x \in U$.

Let f be an arbitrary element of F. Then f belongs to at least one of the above δ balls say to $B(f_i, \delta)$. Then for $x \in U$, we have

$$\bar{d}(f(x), f_i(x)) < \delta,$$

$$d(f_i(x), f_i(x_0)) < \delta$$

$$\bar{d}\,(f_i(x_0), f(x_0)) < \delta.$$

The first and third inequalities hold because $\bar{p}(f, f_i) < \delta$, and the second holds because $x \in U$.

Since δ > 1, the first and third also hold if $\bar{d}$ is replaced by d; then the triangle inequality implies that for all $x \in U$, we have d(f(x)), f(x0) < ε, as desired.

Example 4: Let E be a subspace of a metric space X. Show that E is totally bounded $\Leftrightarrow \bar{E}$ is totally bounded.

Solution: Let E be totally bounded and ε > 0 be given.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be an $\frac{\varepsilon}{2}$ net for E so that

$$E \subseteq \bigcup_{i=1}^{n} S\left(a_i, \frac{\varepsilon}{2}\right) \qquad \ldots(1)$$

Let y be any element of $\bar{E}$.

Then there exists $x \in E$ such that

$$d(x, y) < \frac{\varepsilon}{2} \qquad \ldots(2)$$

Notes

$$x \in E \Rightarrow x \in S\left(a_i, \frac{\varepsilon}{2}\right) \text{ for some i, } 1 \le i \le n \text{ by (1)}$$

$$\Rightarrow d(x, a_i) < \frac{\varepsilon}{2} \; (1 \le i \le n) \qquad \ldots(3)$$

Hence $\quad d(y, a_i) \le d(y, x) + d(x, a_i)$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ by (2) and (3).}$$

$\Rightarrow y \in S(a_i, \varepsilon)\ (1 \le i \le n)$

Thus $y \in \bar{E} \Rightarrow y \in S(a_i, \varepsilon)$ for some i, $1 \le i \le n$.

$\Rightarrow \bar{E} \subseteq \bigcup_{i=1}^{n} S(a_i, \varepsilon)$

$\Rightarrow A = \{a_1, a_2, \ldots, a_n\}$ is an ε-net for $\bar{E}$

$\Rightarrow \bar{E}$ is totally bounded.

Conversely, let $\bar{E}$ be totally bounded. Then since $E \subseteq \bar{E}$, E is totally bounded since every subspace of a totally bounded metric space is totally bounded.

Example 5: Let A be a compact subset of a metric space (X, d). Show that for any $B \subset X$ there is a point $p \in A$ such that

$$d(p, B) = d(A, B).$$

Solution: By the definition, we have

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

Let $d(A, B) = \varepsilon$.

$\therefore \varepsilon = \inf\{d(a, b) : a \in A, b \in B\} \le d(a, b)$,

$a \in A$, $b \in B$ being arbitrary which follows that

$\forall n \in N$, $a_n \in A$ and $b_n \in B$ such that

$\varepsilon \in d(a_n, b_n) < \varepsilon + \frac{1}{n}$.

Since A is compact, it is also sequentially compact and so the sequence $\langle a_n \rangle$ has a subsequence $\langle a_{n_i} \rangle$ which converges to a point $p \in A$.

We claim that $d(p, B) = \varepsilon$

Let, if possible, $d(p, B) > \varepsilon$

Let $d(p, B) = \varepsilon + \varepsilon'$ where $\varepsilon' > 0$

Since $\langle a_{n_i} \rangle$ converges to p there must exist a natural number n_0 such that

$$d(p, a_{n_0}) < \frac{\varepsilon'}{2}$$

and $$d(a_{n_0}, b_{n_0}) < \varepsilon + \frac{1}{n_0}$$

Notes

$$< \varepsilon + \frac{\varepsilon'}{2}$$

$$\therefore \quad d(p, a_{n_0}) + d(a_{n_0}, b_{n_0}) < \frac{1}{2}\varepsilon' + \varepsilon + \frac{1}{2}\varepsilon'$$

$$< \varepsilon + \varepsilon' = d(p, B)$$

$$\leq d(p, b_{n_0}) \quad \text{since } b_{n_0} \in B.$$

or $\quad d(p, a_{n_0}) + d(a_{n_0}, b_{n_0}) < d(p, b_{n_0})$

This contradicts the triangle inequality.

Thus d(p, B) = d(A, B).

28.3 Summary

- A closed and bounded infinite subset of R contains a limit point.
- A metric space (X, d) is said to have the BWP if every infinite subset of X has a limit point.
- A metric space (X, d) is said to sequentially compact if every sequence in X has a convergent subsequence.
- Let $\{G_i : i \in \Delta\}$ be an open cover for a metric space (X, d). A real number $\delta > 0$ is called a Lebesgue number for the cover if any $A \subset X$ s.t. $d(A) < d \Rightarrow A \subset G_{i_0}$ for at least one index $i_0 \in \Delta$.
- Every open covering of a sequentially compact space has a lebesgue number.
- If (X, d) be a metric space and $A \subset X$, then the statement that A is compact, A is countably compact and A is sequentially compact are equivalent.

28.4 Keywords

Cauchy sequence: Let $<x_n>$ be a sequence in a metric space (X, d). Then $<x_n>$ is called a cauchy sequence if given $\varepsilon > 0$, $\exists\, n_0 \in N$ s.t. $m, n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$.

Compact: Let (X, T) be a topological space and $A \subset X$. A is said to be a compact set if every open covering of A is reducible to finite sub covering.

Complete metric space: Let (X, d) a metric space then (X, d) is complete if cauchy sequence of elements of X converges to some elements (belonging to X).

Equicontinuous: A collection of real valued functions.

$A = \{f_n : f_n : X \to R\}$ defined on a metric space (X, d) is said to be equicontinuous if

given $\varepsilon > 0$, $\exists\, \delta = \delta(\in) > 0$ s.t.

$d(x_0, x_1) < \delta \Rightarrow |\, f(x_0) - f(x_1)\, | < \varepsilon \;\; \forall\; f \in A$.

Finite subcover: If $\exists\, G_1 \subset G$ s.t. G_1 is a finite set and that $\{G : G \in G_1\}$ is a cover of A, then G_1 is called a finite subcover of the original cover.

Open cover: If every member of G is an open set, then the cover G is called an open cover.

Notes

28.5 Review Questions

1. A finite subset of a topological space is necessarily sequentially compact. Prove it.
2. Prove that if X is sequentially compact, then it is countably compact.
3. Let A be a compact subset of a metric space (X, d). Show that for every $B \subset X$, $\exists\, p \in A$ s.t. $d(p, B) = d(A, B)$.
4. Let A be a compact subset of a metric space (X, d) and let $B \subset X$, be closed. Show that $d(A, B) > 0$ if $A \cap B = \phi$.

28.6 Further Readings

Books

John Kelley (1955), *General Topology*, *Graduate Texts in Mathematics*, Springer-Verlag.

Dmitre Burago, Yu D Burgao, Sergei Ivanov, *A course in Metric Geometry,* American Mathematical Society, 2004.

Notes

Unit 29: Pointwise and Compact Convergence

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define pointwise convergence and solve related problems;
- Understand the concept of compact convergence and solve problems on it;
- Discuss the compact open topology.

Introduction

There are other useful topologies on the spaces Y^X and $\mathcal{C}(X, Y)$, in addition to the uniform topology. We shall consider three of them here: they are called the topology of pointwise convergence, the topology of compact convergence, and the compact-open topology.

29.1 Pointwise and Compact Convergence

29.1.1 Pointwise Convergence

Definition: Given a point x of the set X and an open set U of the space Y, let

$S(x, U) = \{f \mid f \in Y^X \text{ and } f(x) \in U\}$

The sets S(x, U) are a sub-basis for topology on Y^X, which is called the topology of pointwise convergence (or the point open topology).

Example 1: Consider the space $\mathbb{R}^I$, where I = [0, 1]. The sequence (f_n) of continuous functions given by $f_n(x) = x^n$ converges in the topology of pointwise convergence to the function f defined by

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}$$

Notes

This example shows that the subspace $\mathcal{C}(I, \mathbb{R})$ of continuous functions is not closed in $\mathbb{R}^I$ in the topology of pointwise convergence.

29.1.2 Compact Convergence

Definition: Let (Y, d) be a metric space; let X be a topological space. Given an element f of Y^X, a compact subspace C of X, and a number $\in > 0$, let $B_C(f, \in)$ denote the set of all those elements g of Y^X for which

$$\sup \{d(f(x), g(x)) \mid x \in C\} < \in$$

The sets $B_C(f, \in)$ form a basis for a topology on Y^X. It is called the topology of compact convergence (or sometimes the "topology of uniform convergence on compact sets").

It is easy to show that the sets $B_C(f, \in)$ satisfy the conditions for a basis. The crucial step is to note that if $g \in B_C(f, \in)$, then for

$$\delta = \in - \sup\{d(f(x), g(x)) \mid x \in C\},$$

we have $B_C(g, \delta) \subset B_C(f, \in)$

Note The topology of compact convergence differs from the topology of pointwise convergence in that the general basis element containing f consists of functions that are "close" to f not just at finitely many points, but at all points of some compact set.

29.1.3 Compactly Generated

Definition: A space X is said to be compactly generated if it satisfies the following condition. A set A is open in X if $A \cap C$ is open in C for each compact subspace C of X.

This condition is equivalent to requiring that a set B be closed in X if $B \cap C$ is closed in C for each compact C. It is a fairly mild restriction on the space; many familiar spaces are compactly generated.

Lemma 1: If X is locally compact, or if X satisfies the first countability axiom, then X is compactly generated.

Proof: Suppose that X is locally compact. Let $A \cap C$ be open in C for every compact subspace C of X. We show A is open in X. Given $x \in A$, choose a neighbourhood U of x that lies in a compact subspace C of X. Since $A \cap C$ is open in C by hypothesis, $A \cap U$ is open in U, and hence open in X. Then $A \cap U$ is a neighbourhood of x contained in A, so that A is open in X.

Suppose that X satisfies the first countability axiom. If $B \cap C$ is closed in C for each compact subspace C of X, we show that B is closed in X. Let x be a point of $\overline{B}$; we show that $x \in B$. Since X has a countable basis at x, there is a sequence (x_n) of points of B converging to x. The subspace

$$C = \{x\} \cup \{x_n \mid n \in \mathbb{Z}_+]$$

is compact, so that $B \cap C$ is by assumption closed in C. Since $B \cap C$ contains x_n for every n, it contains x as well. Therefore, $x \in B$, as desired.

Lemma 2: If X is compactly generated, then a function $f : X \to Y$ is continuous if for each compact subspace C of X, the restricted function $f \mid C$ is continuous.

Proof: Let V be an open subset of Y; we show that $f^{-1}(V)$ is open in X. Given any subspace C of X.

$$f^{-1}(V) \cap C = (f \mid C)^{-1}(V)$$

Notes

If C is compact, this set is open in C because f|C is continuous. Since X is compactly generated, it follows that $f^{-1}(V)$ is open in X.

Theorem 1: Let X be a compactly generated space. Let (Y, d) be a metric space. Then $\mathcal{C}(X, Y)$ is closed in Y^X in the topology of compact convergence.

Proof: Let $f \in Y^X$ be a limit point of $\mathcal{C}(X, Y)$; we wish to show f is continuous. It suffices to show that f |C is continuous for each compact subspace C of X. For each n_1 consider the neighbourhood $B_c(f, 1/n)$ of f; it intersects $\mathcal{C}(X, Y)$, so we can choose a function $f_n \in C(X, Y)$ lying in this neighbourhood. The sequence of functions $f_n \mid C : C \to Y$ converges uniformly to the function f|C, so that by the uniform limit theorem, f | C is continuous.

29.1.4 Compact-open Topology

Definition: Let X and Y be topological spaces. If C is a compact subspace of X and U is an open subset of Y, define

$$S(C, U) = \{f \mid f \in \mathcal{C}(X, Y) \text{ and } f(C) \subset U\}$$

The sets S(C, U) form a sub-basis for a topology on $\mathcal{C}(X, Y)$ that is called the compact-open topology.

Theorem 2: Let X be a space and let (Y, d) be a metric space. On the set $\mathcal{C}(X, Y)$, the compact-open topology and the topology of compact convergence coincide.

Proof: If A is a subset of Y and $\in > 0$, let $U(A, \in)$ be the $\in$ - neighbourhood of A. If A is compact and V is an open set containing A, then there is an $\in > 0$ such that $U(A, \in)$. Indeed, the minimum value of the function d(a, X – V) is the required $\in$.

We first prove that the topology of compact convergence is finer than the compact-open topology. Let S(C, U) be a sub-basis element for the compact-open topology, and let f be an element of S(C, U). Because f is continuous, f(C) is a compact subset of the open set U. Therefore, we can choose $\in$ so that $\in$ - neighbourhood of f(C) lies in U. Then, as desired.

$$B_C(f, \in) \subset S(C, U)$$

Now we prove that the compact-open topology is finer than the topology of compact convergence. Let $f \in \mathcal{C}(X, Y)$. Given an open set about f in the topology of compact convergence, it contains a basis element of the form $B_C(f, \in)$. We shall find a basis element for the compact-open topology that contains f and lies in $B_C(f, \in)$.

Each point x of X has a neighbourhood V_x such that $F(V_x)$ lies in an open set U_x of Y having diameter less than $\in$. [For example, choose V_x so that $f(V_x)$ lies in the $\in/4$-neighbourhood of f(x). Then $f(V_x)$ lies in the $\in/3$-neighbourhood of f(x), which has diameter at most $2\in/3$]. Cover C by finitely many such sets V_x, say for $x = x_1,...,x_n$. Let $C_x = V_x \cap C$. Then C_x is compact, and the basis element.

$$S(C_{x_1}, U_{x_1}) \cap ... \cap S(C_{x_n}, U_{x_n})$$

Theorem 3: Let X be locally compact Hausdorff; let e (X, Y) have the compact-open topology. Then the map

$$e : X \times e(X, Y) \to Y$$

defined by the equation

$$e(x, f) = f(x)$$

is continuous.

The map e is called the evaluation map.

Proof: Given a point (X, f) of X × *e* (X, Y) and an open set V in Y about the image point e (x, f) = f (x), we wish to find an open set about (x, f) that e maps into V. First, using the continuity of f and the fact that X is locally compact Hausdorff, we can choose an open set ⋃ about x having compact closure $\bar{U}$, such that f carries $\bar{U}$ into V. Then consider the open set ⋃ × S ($\bar{U}$, V) in X × *e* (X, Y). It is an open set containing (x, f). And if (x′, f′) belongs to this set, then e (x′, f′) = f′ (x′) belongs to V, as defined.

Theorem 4: Let X and Y be spaces, give *e* (X, Y) the compact-open topology. If f : X × Z → Y is continuous, then so is the induced function F : Z → *e* (X, Y). The coverse holds if X is locally compact Hausdorff.

Proof: Suppose first that F is continuous and that X is locally compact Hausdorff. It follows that f is continuous, since f equals the composite.

$$X \times Y \xrightarrow{i_x \times F} X \times e\,(X \times Y) \xrightarrow{e} Y,$$

where i_x is the identity map of X.

Now suppose that f is continuous. To prove continuity of F, we take a point Z_0 of Z and a sub-basic element S (*e*, ⋃) for C (X, Y) containing F (Z_0) and find a neighborhood W of Z_0 that is mapped by F into S (C, ⋃). This will suffice.

The stalement that F (Z_0) lies in S (C, ⋃) means simply that (F (Z_0)) (x) = f (x, Z_0) is in ⋃ for all x ∈ C. That is, f (C × Z_0) ⊂ ⋃. Continuity of f implies that f^{-1} (⋃) is an open set in X × Z containing C × Z_0. Then

$$f^{-1}\,(\cup) \cap (C \times Z)$$

is an open set in the subspace C × Z containing the slice C × Z_0.

The tube lemma implies that there is a neighborhood W of Z_0 in Z such that the entire tube C × W lies in f^{-1} (⋃). Then for Z ∈ W and x ∈ C, we have f (x, z) ∈ ⋃. Hence F (W) ⊂ S (C, ⋃), as desired.

29.2 Summary

- Give a point x of the set X and an open set U of the space Y, let

$$S(x, U) = \{f \mid f \in Y^X \text{ and } f(x) \in U\}$$

  The sets S(x, U) are a sub-basis for topology on Y^X, which is called the topology of pointwise convergence.

- Let (Y, d) be a metric space; let X be a topological space. Given an element f of Y^X, a compact subspace C of X, and a number ∈ > 0, let $B_C(f, \in)$ denote the set of all those elements g of Y^X for which

$$\sup\{d(f(x), g(x)) \mid x \in C\} < \in$$

  The sets $B_C(f, \in)$ form a basis for a topology of Y^X. It is called the topology of compact convergence.

- A space X is said to be compactly generated if it satisfies the following condition. A set A is open in X if A ∩ C is open in C for each compact subspace C of X. This condition is equivalent to requiring that a set B be closed in X if B ∩ C is closed in C for each compact C. It is a fairly mild restriction on the space; many familiar spaces are compactly generated.

- Let X and Y be topological spaces if C is a compact subspace of X and U is an open subset of Y, define S(C, U) = {f | f ∈ C(x, y) and f(C) ⊂ U}.

Notes

29.3 Keywords

Compact set: Let (X, T) be a topological space and $A \subset X$. A is said to be a compact set if every open covering of A is reducible to fine sub-covering.

Locally compact: Let (X, T) be a topological space and let $x \in X$ be arbitrary. Then X is said to be locally compact at x if the closure of any neighbourhood of x is compact.

Subbase: Let (X, T) be a topological space. Let $S \subset T$ s.t. $S \neq \phi$

S is said to be a sub-base or open sub-base for the topology T on X if finite intersections of the members of S form a base for the topology T on X i.e. the unions of the members of S give all the members of T. The elements of S are referred to as sub-basic open sets.

29.4 Review Questions

1. Show that the set $\mathcal{B}(\mathbb{R}, \mathbb{R})$ of founded functions $f : \mathbb{R} \to \mathbb{R}$ is closed in $\mathbb{R}^{\mathbb{R}}$ in the uniform topology, but not in the topology of compact convergence.

2. Consider the sequence of functions

$$f_n : (-1, 1) \to \mathbb{R}, \text{ defined by}$$

$$f_n(x) = \sum_{k=1}^{n} Kx^k$$

   (a) Show that (f_n) converges in the topology of compact convergence, conclude that the limit function is continuous.

   (b) Show that (f_n) does not converge in the uniform topology.

3. Show that in the compact-open topology, $\mathcal{C}(X, Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular.

   [***Hint:*** If $\overline{U} \subset V$, then $\overline{S(C, U)} \subset S(U, V)$]

4. Show that if Y is locally compact Hausdorff then composition of maps

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$$

   is continuous, provided the compact open topology is used throughout.

29.5 Further Readings

Books

J.L. Kelly, *General Topology*, Van Nostrand, Reinhold Co., New York.

J. Dugundji, *Topology*, Prentice Hall of India, New Delhi, 1975.

Unit 30: Ascoli's Theorem

CONTENTS

Objectives

After studying this unit, you will be able to:

- Define equicontinuous and uniformly equicontinuous;
- Understand the proof of Ascoli's theorem;
- Solve the problems on Ascoli's theorem.

Introduction

Ascoli's theorem deals with continuous functions and states that the space of bounded, equicontinuous functions is compact. The space of bounded "equimeasurable functions," is compact and it contains the bounded equicontinuous functions as a subset. Giulio Ascoli is an Italian Jewish mathematician. He introduced the notion of equicontinuity in 1884 to add to closedness and boundedness for the equivalence of compactness of a function space. This is what is called Ascoli's theorem.

30.1 Ascoli's Theorem

30.1.1 Equicontinuous

A family F of functions on a metric space (X, d) is called equicontinuous if

$\forall$ x $\in$ X, $\forall \in > 0$, $\exists\ \delta > 0$ s.t. $\forall$ y $\in$ X with d(x, y) < δ we have $|f(x) - f(y)| < \in$ for all f $\in$ F.

30.1.2 Uniformly Equicontinuous

A family F of functions on a metric space (X, d) is called uniformly equicontinuous if $\forall \in > 0$, $\exists\ \delta > 0$ s.t. $\forall$ x, y $\in$ X with d(x, y) < δ. We have $|f(x) - f(y)| < \in$ for all f $\in$ F.

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Theorem 1: Let f_n be an equicontinuous sequence of functions on (X, d). Suppose that $f_n(x) \to f(x)$ pointwise. Then f(x) is continuous.

Proof: Let $x \in X$ and $\in > 0$, choose $\delta > 0$ so that $d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \in/2$ for any n.

Then $$|f(x) - f(y)| = \underset{n\to\infty}{\text{lt}} |f_n(x) - f_n(y)|$$

$$\leq \sup_n |f_n(x) - f_n(y)|$$

$$\leq \frac{\in}{2} < \in.$$

30.1.3 Statement and Proof of Ascoli's Theorem

Statement: Let $\mathcal{A}$ be a closed subset of the function space C [0, 1]. Then $\mathcal{A}$ is compact iff $\mathcal{A}$ is uniformly bounded and equicontinuous.

Proof: Let $\mathcal{A}$ be closed subset of the function space C [0, 1].

Step I: Let $\mathcal{A}$ be compact.

To prove : $\mathcal{A}$ is uniformly bounded and equicontinuous.

$\mathcal{A}$ is compact $\Rightarrow$ $\mathcal{A}$ is totally bounded

$\Rightarrow$ $\mathcal{A}$ is bounded.

Now $\mathcal{A}$ is a bounded subset of C [0, 1] and each member of C [0, 1] is uniformly continuous. It means that $\mathcal{A}$ is uniformly bounded as a set of functions. Remains to show that $\mathcal{A}$ is equicontinuous.

By definition of totally bounded, $\mathcal{A}$ has an $\in$-net Denote this $\in$-net by $\mathcal{B}$. We can take

$B = \{f_1, f_2, \ldots, f_m\}$ s.t. for any

$f \in \mathcal{A}$, $\exists\, f_{i_o} \in \mathcal{B}$ s.t. $\|f - f_{i_o}\| < \in k$, where k > 0

where $$\|f - f_{i_o}\| = \sup\{|f(x) - f_{i_o}(x)| : x \in [0,1]\}$$

$$\Rightarrow \quad |f(x) - f_{i_o}(x)| < \in k \ \forall\ x \in [0, 1]. \qquad \ldots(1)$$

Let $x, y \in [0, 1]$ and $f \in \mathcal{A}$ be arbitrary.

$$|f(x) - f(y)| = |f(x) - f_{i_o}(x) + f_{i_o}(x) - f_{i_o}(y) + f_{i_o}(y) - f(y)|$$

$$< |f(x) - f_{i_o}(x)| + |f_{i_o}(x) - f_{i_o}(y)| + |f_{i_o}(y) - f(y)|$$

Using (1), $$|f(x) - f(y)| < \in k + |f_{i_o}(x) - f_{i_o}(y)| + \in k \qquad \ldots(2)$$

$f_{i_o} \in \mathcal{B} \Rightarrow f_{i_o} \in \mathcal{A} \Rightarrow f_{i_o}$ is uniformly continuous on [0, 1].

$$\therefore \quad \exists\ \delta_i > 0 \text{ s.t. } |x - y| < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \in k \qquad \ldots(3)$$

Take $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$. Then, by (3), we get

$|x - y| < \delta \Rightarrow |f_{i_o}(x) - f_{i_o}(y)| < \in k$, Using this in (2),

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or $|f(x) - f(y)| < \in' k$, for $|x - y| < \delta$, $f \in \mathcal{A}$ where $k = \frac{1}{3}$.

This proves that $\mathcal{A}$ is equicontinuous.

Step II: Suppose $\mathcal{A}$ is uniformly bounded and equicontinuous.

To prove: $\mathcal{A}$ is compact.

Since C [0, 1] is complete and $\mathcal{A}$ is a closed subset of it and so $\mathcal{A}$ is complete. Hence we need only to show that $\mathcal{A}$ is totally bounded.

[As we know that "A metric space is compact iff it is totally bounded and complete."]

Given $\in > 0$, $\exists$ positive integer n_o s.t.

$$|x - y| < \frac{1}{n_o} \Rightarrow |f(x) - f(y)| < \frac{\in}{5} \ \ \forall f \in \mathcal{A}$$

for each $f \in \mathcal{A}$, we can construct a polygon arc p_f s.t. $\|f - p_f\| < \in$ and p_f connects points belonging to

$$P = \left\{(x, y) : x = 0, \frac{1}{n_o}, \frac{2}{n_o}, \ldots, 1, y = \frac{n \in}{5}, n \text{ is an integer}\right\}.$$

Write $\mathcal{B} = \{p_f : f \in \mathcal{A}\}$

We want to show that $\mathcal{B}$ is finite and hence an $\in$-net for $\mathcal{A}$.

A is uniformly bounded.

$\Rightarrow$ B is uniformly bounded.

Hence a finite number of points in $\mathcal{A}$ will appear in the polygonal arcs in $\mathcal{B}$. It means that there can only be a finite number of arcs in $\mathcal{B}$, showing thereby $\mathcal{B}$ is an $\in$-net for $\mathcal{A}$ and so $\mathcal{A}$ is totally bounded. Also $\mathcal{A}$ is complete. Consequently $\mathcal{A}$ is compact.

Remark: Ascoli's theorem is also sometimes called Arzela-Ascoli's theorem.

Theorem 2: Every compact metric space is separable.

Proof: Let (X, d) be a compact metric space.

Let m be a fixed positive number.

Let $\mathcal{C} = \left\{S\left(x, \frac{1}{m}\right) : x \in X\right\}$ be a collection of open spheres.

($\because$ each open sphere forms an open set.)

Then $\mathcal{C}$ is clearly an open cover of X. Since X is compact and hence its open cover is reducible to a finite sub cover say

$$\mathcal{C}' = \left\{S\left(x_{m_i}, \frac{1}{m}\right) : i = 1, 2, \ldots, k\right\}$$

Let $A_m = \{x_{m_i} : i = 1, 2, \ldots, k\}$.

Thus for each $m \in N$, we can construct A_m in above defined manner.

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Also, each such set is finite and for each $x \in X$, there is an element $x_{m_i} \in A_m$ such that $d(x, x_{m_i}) < \frac{1}{m}$.

Then $A = \bigcup_{m \in N} A_m \subset X$ is countable as it is the union of countable sets.

Now $\qquad A \subset X \Rightarrow \bar{A} \subset \bar{X}$

$\Rightarrow \qquad \bar{A} \subset X$ since X is closed $\Rightarrow \bar{X} = X$.

In order to show that (X, d) is separable, it is sufficient to show that $\bar{A} = X$, for which it is sufficient to show that each point of X is an adherent point of A.

So, let x be an arbitrary point of X and G be any open nhd. of x, $\exists$ an open sphere $S\left(x, \frac{1}{m}\right)$ for some positive integer m such that,

$$x \in S\left(x, \frac{1}{m}\right) \subset G \qquad \ldots(1)$$

But for each $x \in X$, $\exists\ x_{m_i} \in A_m \subset A$ such that $d(x, x_{m_i}) < \frac{1}{m}$

or $$x_{m_i} \in S\left(x, \frac{1}{m}\right) \qquad \ldots(2)$$

Then from (1) and (2), we get

$$x_{m_i} \in S\left(x, \frac{1}{m}\right) \subset G.$$

Thus, every open nhd. of x contains at least one point of A and therefore, x is an adherent point of A.

This shows that every point of X is an adherent point of A.

$\therefore \qquad X \subset \bar{A}$ and therefore

$\qquad \bar{A} = X$

which follows that A is countable dense subset of X and hence X is separable.

30.2 Summary

- A family $\mathcal{F}$ of functions on a metric space (X, d) is called equicontinuous if $\forall\ x \in X$, $\forall \in > 0$, $\exists\ \delta > 0$ s.t. $\forall\ y \in X$ with $d(x, y) < \delta$, we have

  $|f(x) - f(y)| < \in$ for all $f \in \mathcal{F}$.

- A family $\mathcal{F}$ of functions on a metric space (X, d) is called uniformly equicontinuous if $\forall \in > 0$, $\exists\ \delta > 0$, s.t. $x, y \in X$ with $d(x, y) < \delta$, we have

  $|f(x) - f(y)| < \in$ for all $f \in \mathcal{F}$.

- ***Ascoli's Theorem:*** Let $\mathcal{A}$ be a closed subset of the function space C [0, 1]. Then $\mathcal{A}$ is compact iff $\mathcal{A}$ is uniformly bounded and continuous.

Notes

30.3 Keywords

Adherent Point: A point $x \in X$ is called an adherent point of A iff every nhd of x contains at least one point of A.

Compact Metric Space: If (X, d) be a metric space and $A \subset X$, then the statement that A is compact, A is countably compact and A is sequentially compact are equivalent.

Complete Metric Space: A metric space X is said to be complete if every Cauchy sequence of points in X converges to a point in X.

Open Sphere: Let (X, ρ) be a metric space. Let $x_o \in X$ and $r \in R^+$. Then set $\{x \in X : \rho(x_o, x) < r\}$ is defined a open sphere with centre x_o and radius r.

Separable Space: Let X be a topological space and $A \subset X$, then X is said to be separable if

(i) $\bar{A} = X$ (ii) A is countable

Totally Bounded: A metric space (X, d) is said to be totally bounded if for every $\in > 0$, there is a finite covering of X by $\in$-balls.

30.4 Review Questions

1. Prove that A subset T of $\mathcal{C}$ (X) is compact if and only if it is closed, bounded and equicontinuous.

2. Prove the following:

   Theorem: If X is locally compact Hausdorff space, then a subspace T of $\mathcal{C}$ (X, R^n) in the topology of compact convergence has compact closure if and only if T is pointwise bounded and equicontinuous under either of the standard metric on R^n.

3. Let (Y, d) be a metric space; let $f_n : X \to Y$ be a sequence of continuous functions; let $f : X \to Y$ be a function (not necessarily continuous). Suppose f_n converges to f in the topology of pointwise convergence. Show that if $\{f_n\}$ is equicontinuous, then f is continuous and f_n converges to f in the topology of compact convergence.

4. Prove the following:

   Theorem (Arzela's theorem, general version). Let X be a Hausdorff space that is σ-compact; let f_n be a sequence of functions $f_n : X \to R^k$. If the collection $\{f_n\}$ is pointwise bounded and equicontinuous, then the sequence f_n has a subsequence that converges, in the topology of compact convergence, to a continuous function.

30.5 Further Readings

Books

H.F. Cullen, *Introduction to General Topology*, Boston, M.A.

Stephen Willard, *General Topology*, (1970).

Notes

Unit 31: Baire Spaces

CONTENTS

Objectives

After studying this unit, you will be able to:

- Know about the Baire spaces;
- Understand the Baire's category theory;
- Understand the Baire's category theorem.

Introduction

In this unit, we introduce a class of topological spaces called the Baire spaces. The defining condition for a Baire space is a bit complicated to state, but it is often useful in the applications, in both analysis and topology. Most of the spaces we have been studying are Baire spaces. For instance, a Hausdorff space is a Baire space if it is compact, or ever locally compact. And a metrizable space X is a Baire space if it is topologically complete, that is, if there is a metric for X relative to which X is complete.

Then we shall give some applications, which ever if they do not make the Baire condition seem any more natural, will at least show what a useful tool it can be in feet, it turns out to be a very useful and fairly sophisticated tool in both analysis and topology.

31.1 Baire Spaces

31.1.1 Definition – Baire Space

A space X is said to be a Baire space of the following condition holds. Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\cup A_n$ also has empty interior in X.

Example 1: The space Q of rationals is not a Baire space. For each one-point set in Q is closed and has empty interior in Q; and Q is the countable union of its one-point subsets. The space Z_+, on the other hand, does form a Baire space. Every subset of Z_+ is open, so that there

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exist no subsets of Z_+ having empty interior, except for the empty set. Therefore Z_+ satisfies the Baire condition vacuously.

Lemma 1: X is a Baire space iff gives any countable collection $\{ \cup_n \}$ of open sets in X, each of which is dense in X their intersection $\cap \cup_n$ is also dense in X.

Proof: Recall that a set $\mathcal{C}$ is dense in X if $\bar{C} = X$. The theorem now follows at once from the two remarks.

1. A is closed in X iff X-A is open in X.
2. B has empty interior in X if and only if X-B is dense in X.

Lemma 2: Any open subspace Y of a Baire space X is itself a Baire space.

Proof: Let A_n be a countable collection of closed set of Y that have empty interiors in Y. We show that $\cup A_n$ has empty interior in Y.

Let $\bar{A}_n$ be the closure of A_n in X; then $\bar{A}_n \cap Y = A_n$. The set $\bar{A}_n$ has empty interior in X. For it $\cup$ is a non empty open set of X contained in $\bar{A}_n$, then $\cup$ must intersect A_n. Then $\cup \cap Y$ is a non-empty open set of Y contained in A_n, contrary to hypothesis.

If the union of the sets A_n contains the non empty open set W of Y, then the union of the sets $\bar{A}_n$ also contains the set W, which is open in X because Y is open in X. But each set $\bar{A}_n$ has empty interior in X, contradicting the fact that X is a Baire space.

31.1.2 Baire's Category Theory

Let (X, d) be a metric space and $A \subset X$. The set A is called of the **first category** if it can be expressed as a countable union of non dense sets. The set A is called of the second category if it is not of the first category.

Definition: A metric space is said to be totally of **second category** if every non empty closed subset of X is of the second category.

Example 2: Let $q \in \mathcal{Q}$ be arbitrary.

$$\{\bar{q}\} = \{q\} \cup D(\{q\}), \qquad [\because \bar{A} = A \cup D(A)]$$

$$= \{q\} \cup \phi = \{q\}$$

$$\therefore \quad \text{int}\{\bar{q}\} = \text{int}\{q\}$$

$$= \cup \{G \subset \mathcal{R} : G \text{ is open}, G \subset \{q\}\} = \phi.$$

For every subset of $\mathcal{R}$ contains rational as well irrational numbers.

Thus, $\quad \text{int}\{\bar{q}\} = \phi.$

This proves that {q} is a non-dense subset of $\mathcal{Q}$.

$$\mathcal{Q} = \cup \{\{q\} : a \in \mathcal{Q}\}.$$

Furthermore $\mathcal{Q}$ is enumerable.

$\therefore$ $\mathcal{Q}$ is an enumerable union of non-dense sets.

From what has been done it follows that $\mathcal{Q}$ is of the first category.

Example 3: Consider a sequence $<f_n(x)>$ of continuous functions defined from I = [0, 1] into $\mathcal{R}$ s.t. $f_n(x) = x_n \ \forall x \in N$.

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Then $<f_n>$ converges pointwise to $g : \mathcal{I} \rightarrow \mathcal{R}$ s.t.

$$g(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

Evidently g is not continuous.

31.1.3 Baire Category Theorem

Theorem 1: Every complete metric space is of second category.

Proof: Let (X, d) be a complete metric space.

To prove that X is of second category.

Suppose not. Then X is not of second category so that X is of first category. By def., X is expressible as a countable union of nowhere dense sets arranged in a sequence $<A_n>$. Since A_1 is non-dense and so $\exists$ a closed sphere K_1 of radius $r_1 < \frac{1}{2}$ s.t. $K_1 \cap A_1 = \phi$.

Let the open sphere with same centre and radius as r_1 be denoted by S_1. In S_1, we can find a closed sphere K_2 of radius $r_2 < \left(\frac{1}{2}\right)^2$ s.t.

$$K_1 \cap A_2 = \phi \quad \text{and so } K_2 \cap A_1 = \phi$$

Continuing like this we construct a nested sequence $<K_n>$ of closed spheres having the following properties:

(i) For each positive integer n, K_n does not intersect $A_1, A_2, \ldots, A_n$.

(ii) The radius of K_n tends to zero as $n \rightarrow \infty$. For $\frac{1}{2^n} \rightarrow 0$ as $n \rightarrow \infty$.

Since (X, d) is complete and so by Cantor's intersection theorem, $\bigcap_n K_n$ contains a single point x_o.

$$\therefore \quad x_o \in \bigcap_{n=1}^{\infty} K_n \Rightarrow x_o \in K_n \ \forall n$$

$$\Rightarrow x_o \notin A_n \ \forall n \text{ (according to (i))}$$

$$\Rightarrow x_o \notin \bigcup_{n=1}^{\infty} A_n = X$$

$$\Rightarrow x_o \notin X. \text{ A contradiction}$$

For X is universal set.

Hence X is not of first category. A contradiction. Hence the required result follows.

Remarks: The theorem 1 can also be expressed in the following ways:

1. If $<A_n>$ is a sequence of nowhere dense sets in a complete metric space (X, d), then $\exists$ a point in X, which is not in A_n's.
2. If a complete metric space is the union of a sequence of its subsets, then the closure of at least one set in the sequence must have non-empty interior.

Theorem 2: Let X be a space; let (Y, d) be a metric space. Let $f_n : X \rightarrow Y$ be a sequence of continuous functions such that $f_n(x) \rightarrow f(x)$ for all $x \in X$, where $f : X \rightarrow Y$. If X is a Baire space, the set of points at which f is continuous is dense in X.

Proof: Given a positive integer N and given $\varepsilon > 0$, define

$$A_N(\varepsilon) = \{x | d(f_n(x), f_m(x)) \leq \varepsilon \text{ for all } n, m \geq N\}.$$

Note that $A_N(\varepsilon)$ is closed in X. For the set of those x for which $d(f_n(x), f_m(x)) \leq \varepsilon$ is closed in X, by continuity of f_n and f_m and $A_N(\in)$ is the intersection of these sets for all $n, m \geq N$.

For fixed $\in$, consider the sets $A_1(\in) \subset A_2(\in) \subset \ldots$. The union of these sets is all of X. For, given $x_0 \in X$, the fact that $f_n(x_0) \to f(x_0)$ implies that the sequence $f_n(x_0)$ is a Cauchy sequence; hence $x_0 \in A_N(\in)$ for some N.

Now let

$$\cup(\in) = \bigcup_{N \in \mathbb{Z}_+} \text{Int} A_N(\in).$$

We shall prove two things:

(1) $\cup(\in)$ is open and dense in X.

(2) The function f is continuous at each point of the set

$$\mathcal{C} = \cup(1) \cap \cup(1/2) \cap \cup(1/3) \cap \ldots.$$

Our theorem then follows from the fact that X is a Baire space. To show that $\cup(\in)$ is dense in X, it suffices to show that for any non-empty open set V of X, there is an N such that the set $V \cap \text{Int } A_N(\in)$ is non-empty. For this purpose, we note first that for each N, the set $V \cap A_N(\in)$ is closed in V. Because V is a Baire space by the preceding lemma, at least one of these sets, say $V \cap A_M(\in)$, must contain a non-empty open set W of V. Because V is open in X, the set W is open in X; therefore, it is contained in Int $A_M(\varepsilon)$.

Now we show that if $x_0 \in \mathcal{C}$, then f is continuous at x_0. Given $\varepsilon > 0$, we shall find a neighborhood W of x_0 such that $d(f(x), f(x_0)) < \varepsilon$ for $x \in W$.

First, choose K so that $1/K < \varepsilon/3$. Since $x_0 \in \mathcal{C}$, we have $x_0 \in \cup(1/K)$ therefore, there is an N such that $x_0 \in \text{Int } A_N(1/K)$. Finally, continuity of the function f_N enables us to choose a neighborhood W of x_0, contained in $A_N(1/K)$, such that

(*) $d(f_N(x), f_N(x_0)) \leq \varepsilon/3$ for $x \in W$.

The fact that $W \subset A_N(1/K)$ implies that

(**) $d(f_n(x), f_N(x)) \leq 1/K$ for $n \geq N$ and $x \in W$.

Letting $n \to \infty$, we obtain the inequality

(***) $d(f(x), f_N(x)) \leq 1/K < \varepsilon/3$ for $x \in W$.

In particular, since $x_0 \in W$, we have

$$d(f(x_0), f_N(x_0)) < \varepsilon/3$$

Applying the triangle inequality (*), (**) and (***) gives us our desired result.

Theorem 3: If Y is a first category subset of a Baire space (X, T) then the interior of Y is empty.

Proof: As Y is first category, $Y = \bigcup_{n=1}^{\infty} Y_n$, where each Y_n, $n \in N$ is nowhere dense.

Let $\cup \in T$ be such that $\cup \subseteq Y$. Then $\cup \subseteq \bigcup_{n=1}^{\infty} Y_n \subseteq \bigcup_{n=1}^{\infty} \overline{Y}_n$. So $X \backslash \cup \geq \bigcap_{n=1}^{\infty} (X \setminus \overline{Y}_n)$, and each of the sets $X \setminus \overline{Y}_n$ is open and dense in (X, T). As (X, T) is Baire. $\bigcap_{n=1}^{\infty} (X \setminus \overline{Y}_n)$ is dense in (X, T). So the closed set $X \backslash \cup$ is dense in (X, T). This implies $X \backslash \cup = \overset{n=1}{X}$. Hence $\cup = \phi$. This completes the proof.

Notes

31.2 Summary

- A space X is said to be a Baire space if the following condition holds: Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\cup A_n$ also has empty interior in X.
- Let (X, d) be a metric space and $A \subset X$. The set A is called of the first category if it can be expressed as a countable union of non dense sets. The set A is called of the second category if it is not of the first category.

31.3 Keywords

Complete Metric Space: A metric space X is said to be complete if every Cauchy sequence of points in X converges to a point in X.

Dense: A said to be dense in X if $\bar{A} = X$.

Nowhere Dense: A is said to be nowhere dense if $(\bar{A})^\circ = \phi$.

31.4 Review Questions

1. Show that if every point x of X has a neighborhood that is a Baire space, then X is a Baire space.

   [***Hint:*** Use the open set formulation of the Baire Condition].
2. Show that every locally compact Hausdorff space is a Baire space.
3. Show that the irrationals are a Baire space.
4. A point x in a topological space (X, T) is said to be an isolated point if $\{x\} \in T$. Prove if (X, T) is a countable T_1-space with no isolated points. Then it is not a Baire space.
5. Let (X, T) be any topological space and Y and S dense subsets of X. If S is also open in (X, T), prove that $S \cap Y$ is dense in both X and Y.
6. Let (X, T) and (Y, T_1) be topological space and f : (X, T) $\to$ (Y, T_1) be a continuous open mapping. If (X, T) is a Baire space. Show that an open continuous image of a Baire space is a Baire space.
7. Let (Y, T_1) be an open subspace of the Baire space (X, T). Prove that (Y, T) is a Baire space. So an open subspace of a Baire space is a Baire space.
8. Let B be a Banach space where the dimension of the underlying vector space is countable. Using the Baire Category Theorem, prove that the dimension of the underlying vector space is, in fact, finite.

31.5 Further Readings

Books

A.V. Arkhangal'skii, V.I. Ponomarev, *Fundamentals of General Topology: Problems and Exercises*, Reidel (1984).

J. Dugundji, *Topology*, Prentice Hall of India, New Delhi.

Online link

www.springer.com/978-3642-00233-5

Unit 32: Introduction to Dimension Theory

Notes

CONTENTS

Objectives

After studying this unit, you will be able to:

- Know about the dimensional theory;
- Define Hausdorff dimension of measures;
- Define pointwise dimension;
- Solve the problems on the dimensional theory.

Introduction

For many familiar objects there is a perfectly reasonable intuitive definition of dimension: A space is d-dimensional if locally it looks like a patch R^d. This immediately allows us to say: The dimension of a point is zero; the dimension of a line is 1; the dimension of a plane is 2; the dimension of R^d is d.

There are several different notions of dimension for more general sets, some more easy to compute and others more convenient in applications. We shall concentrate on Hausdorff dimension. Hausdorff introduced his definition of dimension in 1919. Further contributions and applications, particularly to number theory, were made by Besicovitch.

Hausdorff's idea was to find the value at which the measurement changes from infinite to zero. Dimension is at the heart of all fractal geometry, and provides a reasonable basis for an invariant between different fractal objects.

32.1 Introduction to Dimension Theory

Before we begin defining Hausdorff and other dimensions, it is a good idea to clearly state our objectives. What should be the features of a good definition of dimension? Based on intuition,

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we would expect that the dimension of an object would be related to its measurement at a certain scale. For example, when an object is scaled by a factor of 2.

- for a line segment, its measure will increase by $2^1 = 2$
- for a rectangle, its measures will increase by $2^2 = 4$
- for a parallelepiped, its measures will increase by $2^3 = 8$

In each case, we extract the exponent and consider this to be the dimension. More precisely, dim F = log $\Delta\mu(F)$/log 1/p where p is the precision (1/p is the scaling factor) and $\Delta\mu(F)$ is the change in the 'measure' of F when scaled by 1/p. Falconer suggests that most of following criteria also be net [Falc[2]], by any thing called a dimension:

1. ***Smooth manifolds:*** If F is any smooth, n-dimensional manifold, dim F = n.
2. ***Open Sets:*** For an open subset $F \subset \mathcal{R}^n$, dim F = n.
3. ***Countable Sets:*** dim F = 0 if F is finite or countable.
4. ***Monotonicity:*** $E \subset F \Rightarrow \dim E \leq \dim F$.
5. ***Stability:*** $\dim (E \cup F) = \max (\dim E, \dim F)$.
6. ***Countable Stability:*** $\dim\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup_i\{\dim F_i\}$.
7. ***Lipschitz Mapping:*** If $f : E \to \mathcal{R}^m$ is lipschitz, then $\dim f(E) \leq \dim (E)$.
8. ***Bi-lipschitz Mapping:*** If $f : E \to \mathcal{R}^m$ is Bi-lipschitz, then $\dim f(E) = \dim (E)$.
9. ***Geometric Invariance:*** dim f(F) = dim F, if f is a similarity or affine transformation.

Recall that $f : E \to \mathcal{R}^m$ is ***Lipschitz*** iff $\exists$ c such that

$$|f(x) - f(y)| \leq c|x - y| \quad \forall\, x, y \in E;$$

and that f is ***Bi-lipschitz*** iff $\exists$ c_1, c_2 such that

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y| \quad \forall\, x, y \in E;$$

and f is a ***Similarity*** iff $\exists$ c such that

$$|f(x) - f(y)| = c|x - y| \quad \forall\, x, y \in E;$$

32.1.1 Hausdorff Dimension of Measures

Let μ denote a probability measure on a set of X. We can define the Hausdorff dimension μ in terms of the Hausdorff dimension of subsets of A.

Definition: For a given probability measure μ we define the Hausdorff dimension of the measure by

$$\dim_H(\mu) = \inf\{\dim_H(X) : \mu(X) = 1\}.$$

We next want to define a local notion of dimension for a measure μ at a typical point $x \in X$.

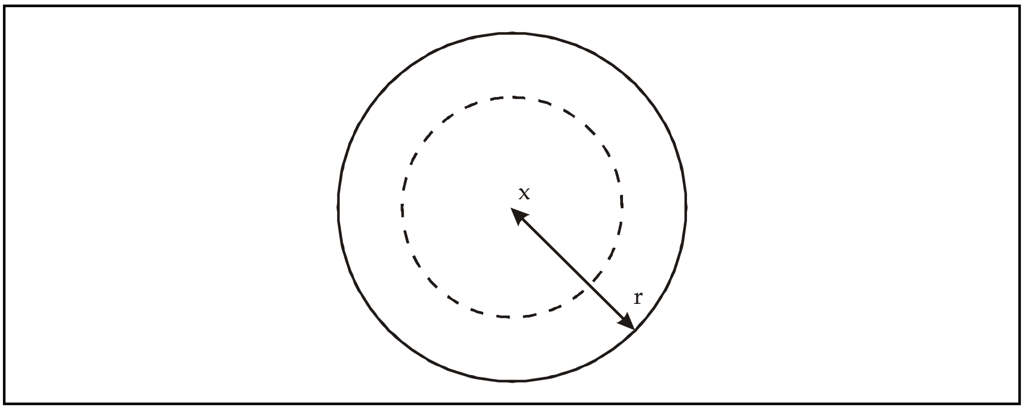
32.1.2 Pointwise Dimension

Definition: The upper and lower pointwise dimensions of a measure μ are measurable functions,

$\overline{d}_\mu, \underline{d}_\mu : X \to R \cup \{\infty\}$ defined by $\overline{d}_\mu(x) = \lim_{r\to 0} \sup \dfrac{\log \mu(B(x,r))}{\log r}$ and $\underline{d}_\mu(x) = \lim_{r\to 0} \inf \dfrac{\log \mu(B(x,r))}{\log r}$

where B(x, r) is a ball of radius r > 0 about x.

Notes



The pointwise dimensions describe how the measure μ is distributed. We compare the measure of a ball about x to its radius r, as r tends to zero.

These are interesting connections between these different notions of dimension for measure.

Theorem 1: If $\underline{d}_\mu(x) \geq d$ for a.e.(μ) x ∈ X then $din_H(\mu) \geq d$.

Proof: We can choose a set of full μ measure $X_o \subset X$ (i.e. $\mu(X_o) = 1$). Such that $\underline{d}_\mu(x) \geq d$ for all $x \in X_o$. In particular for any ∈ > 0 and x ∈ X we have $\limsup_{r \to 0} \mu(B(x, r))/r^{d-\epsilon}$ Fix C > 0 and δ > 0, and let us denote

$$X_\delta = \{x \in X_i : \mu(B(x,r)) \leq C_r^{d-\epsilon}, \forall 0 < r < \delta\}.$$

Let $\{U_i\}$ be any δ-cover for X. Then if $x \in U_i$, $\mu(U_i) \leq C \text{ diam } (U_i)^{d-\epsilon}$. In particular

$$\mu(X_\delta) \leq \sum_{\cup_i \cap X_\delta} \mu(U_i) \leq C \sum_i \text{diam}(U_i)^{d-\epsilon}.$$

Thus, taking the infimum over all such cover we have $\mu X_\delta \leq CH^{d-\epsilon}(X_\delta) \leq CH^{d-\epsilon}(X)$. Now letting $\delta \to 0$ we have that $1 = \mu(X_o) \leq CH^{d-\epsilon}(X)$. Since C > 0 can be chosen arbitrarily large we deduce that $H^{d-\epsilon}(X) = +\infty$. In particular $\dim_H(X) \geq d-\epsilon$ for all ∈ > 0. Since ∈ > 0 is arbitrary, we conclude that $\dim_H(X) \geq d$.

We have the following simple corollary, which is immediate from the definition of $\dim_H(\mu)$.

Corollary: Given a set $X \in \mathcal{R}^d$, assume that there is a probability measure μ with μ(X) = 1 and $\underline{d}_\mu(x) \geq d$ for a. ∈ (μ) x ∈ X. Then $\dim_H(X) \geq d$.

In the opposite direction we have that a uniform bound on pointwise dimensions leads to an upper on the Hausdorff Dimension.

Theorem 2: If $\overline{d}_\mu(x) \leq d$ for a. ∈ (μ) x ∈ X then $\dim_H(\mu) \leq d$. Moreover, if there is a probability measure μ with μ(X) = 1 and $\overline{d}_\mu(x) \leq d$ for every x ∈ X then $\dim_H(X) \leq d$.

Proof: We begin with the second statement. For any ∈> 0 and x ∈ X we have $\limsup_{r\to 0} \mu(B(x, r))/r^{d+\epsilon} = \infty$. Fix C > 0. Given δ > 0, consider the cover μ for X by the balls

$$\{B(x, r) : 0 < r \leq \delta \quad \text{and} \quad \mu(B(x, r)) > C_r^{d+\epsilon}\}.$$

We recall the following classical result.

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32.1.3 Besicovitch Covering Lemma

There exists $N = N(d) \geq 1$ such that for any cover by balls we can choose a sub-cover $\{U_i\}$, such that any point x lies in at most N balls.

Thus we can bound

$$H_\delta^{d+\epsilon}(X) \leq \sum_i \text{diam}(U_i)^{d+\epsilon} \leq \frac{1}{C}\sum_i \mu(B_i) \leq \frac{N}{C}.$$

Letting $\delta \to 0$ we have that $H^{d+\epsilon}(X) \leq \frac{N}{C}$. Since $C > 0$ can be chosen arbitrarily large we deduce that $H^{d+\epsilon}(X) = 0$. In particular, $\dim_H(X) \leq d + \epsilon$ for all $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, we deduce that $\dim_H(X) \leq d$.

The proof of the first statement is similar, except that a replace X by a set of full measure for which $\overline{d}_\mu(x) \leq d$.

Example 1: If $L : X_1 \to X_2$ is a surjective Lipschitz map i.e. $C > 0$ such that

$$|L(x) - L(y)| \leq C|x - y|,$$

then $\dim_H(X_1) \leq \dim_H(X_2)$.

Example 2: If $L : X_1 \to X_2$ is a bijective bi-Lipschitz map i.e. $\exists\, C > 0$ such that

$$\left(1/C\right)|x - y| \leq |L(x) - L(y)| \leq C|x - y|,$$

then $\dim_H(X_1) = \dim_H(X_2)$.

Solution: For part 1, consider an open cover $\mathcal{U}$ for X_1 with $\dim(U_i) \leq \epsilon$ for all $U_i \in \mathcal{U}$. Then the images $\mathcal{U}' = \{L(U) : U \in \mathcal{U}\}$ are a cover for X_2 with $\dim(L(U_i)) \leq L_\epsilon$ for all $U \in \mathcal{U}'$. Thus, from the definitions, $H^\delta_{L_\epsilon}(X_2) \geq H^\delta_\epsilon(X_1)$. In particular, letting $\epsilon \to 0$ we see that $H^\delta(X_1) \geq H^\delta(X_2)$. Finally, from the definitions $\dim_H(X_1) \leq \dim_H(X_2)$.

For part 2, we can apply the first part a second time with $\mathcal{L}$ replaced by L^{-1}.

32.1.4 Bernoulli's Measures

Example 3: For an iterated function scheme $T_1, \ldots\ldots, T_k : U \to U$ we can denote as before

$$\Sigma = \left\{\underline{x} = (x_m)_{m=0}^{\infty} : x_m \in \{1,\ldots,k\}\right\}$$

with the Tychonoff product topology. The shift map $\sigma : \Sigma \to \Sigma$ is a local homeomorphism defined by $(\sigma x)_m = x_{m+1}$. The kth level cylinder is defined by,

$$[x_0,\ldots,x_{k-1}] = \left\{(i_m)_{m=0}^{\infty} \in \Sigma : i_m = x_m \text{ for } 0 \leq m \leq k-1\right\}$$

(i.e., all sequence which begin with $x_0, \ldots, x_{k-1}$). We denote by $W_k = \{(x_0, \ldots, x_{k-1}]\}$ the set of all kth level cylinders (of which there are precisely k^n).

Notation: For a sequence $\underline{i} \in \Sigma$ and a symbol $r \in \{1, \ldots, k\}$ we denote by $k_r(\underline{i}) = \text{card}\{0 \leq m \leq k-1 : i_m = r\}$ the number of occurrences of r in the first k terms of $\underline{i}$.

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Consider a probability vector $\underline{p} = (p_0, \ldots, p_{n-1})$ and define the Bernoulli measure of any level cylinder to be,

$$\mu([i_0, \ldots, i_{k-1}]) = p_0^{k_0(\underline{i})} p_1^{k_1(\underline{i})} \ldots p_{n-1}^{k_{n-1}(\underline{i})}.$$

A ***probability measure*** μ on σ is said to be ***invariant*** under the shift map if for any Borel set $B \subset X$, $\mu(B) = \mu(\sigma^{-1}(B))$. We say that μ is ergodic if any Borel set $B \subseteq \Sigma$ such that $\sigma^{-1}(X) = X$ satisfies $\mu(X) = 0$ or $\mu(X) = 1$. A Bernoulli measure is both invariant and ergodic.

32.2 Summary

- Criteria for defining a dimension
  - (i) When X is a manifold then the value of the dimension is an integer which coincides with the usual notion of dimension;
  - (ii) For more general sets X we can have "fractional" dimensional; and
  - (iii) Points and countable unions of points, have zero dimension.
- For a given probability measure μ, we define the Hausdorff dimension of the measure by

  $$\dim_H(\mu) = \inf \{\dim_H(X) : \mu(X) = 1\}.$$
- The upper and lower pointwise dimensions of a measure μ are measurable functions, $\overline{d}_\mu$, $\underline{d}_\mu : X \to \mathcal{R} \cup \{\infty\}$ defined by

$$\overline{d}_\mu(x) = \lim_{r \to 0} \sup \frac{\log \mu(B(x,r))}{\log r} \text{ and}$$

$$\underline{d}_\mu(x) = \lim_{r \to 0} \inf \frac{\log \mu(B(x,r))}{\log r}$$

32.3 Keywords

Countable Set: A set is countable if it is non-empty and finite or if it is countably infinite.

Hausdorff Space: A topological space (X, T) is called Hausdorff space if given a pair of distinct points $x, y \in X$,

$$\exists\, G, H \in T \quad \text{s.t.} \quad x \in G, y \in H, G \cap H = \phi.$$

Iterated Function Scheme: An iterated function scheme on an open set $\cup \subset R^d$ consists of a family of contractions $T_1, \ldots, T_k : \cup \to \cup$.

Open Set: Any set $A \in T$ is called an open set.

Subcover: Let (X, T) be a topological space and $A \subset X$. Let G denote a family of subsets of X. If $\exists$ $G_1 \subset G$ s.t. G_1 is a finite set and that $\{G : G \in G_1\}$ is a cover of A then G_1 is called a finite subcover of the original cover.

32.4 Review Questions

1. Write a short note on Dimension Theory.
2. State Besicovitch covering lemma.

Notes

3. If $\dim_H(X) < d$ then show that the (d-dimensional) Lebesgue measure of X is zero.
4. Let $\Lambda_1, \Lambda_2 \subset \mathcal{R}$ and let

$$\Lambda_1 + \Lambda_2 = \{\lambda_1 + \lambda_2 : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2\}$$

then prove that $\dim_H(\Lambda_1 + \Lambda_2) \leq \dim_H(\Lambda_1) + \dim_H(\Lambda_2)$.

5. If we can find a probability measure μ satisfying the above hypothesis then prove that $\dim_H(X) \geq d$.

32.5 Further Readings

Books

Rogers, M. (1998), *Hausdorff Measures*, Cambridge University Press.

Lapidus, M. (1999), *Math 209A – Real Analysis Mid-term*, UCR Reprographics.

Online links

en.wikipedia.org/wiki/E8-mathematics

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