Differential and Integral Equation

DMTH504

LOVELY PROFESSIONAL UNIVERSITY

DIFFERENTIAL AND INTEGRAL EQUATION

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SYLLABUS

Differential and Integral Equation

Objectives: The objective of the course is to know different methods to solve ordinary and partial differential equations and also to solve Integral equation of Fredholm and Voltera type.

Sr. No.	Content
1	Bessel functions, Legendre polynomials, Hermite polynomials, Laguerre polynomials, recurrence relations, generating functions, Rodrigue formula and orthogonality .
2	Existence theorem for solution of the equation dy/dx= f(x,y) [Picard's methods as in Yoshida], general properties of solutions of linear differential equations of order n, total differential equations, simultaneous differential equations, adjoint and self-adjoint equations.
3	Green's function method, Sturm Liouville's boundary value problems, Sturm comparison and separation theorems, orthogonality of solutions.
4	Classification of partial differential equations, Cauchy's problem and characteristics for first order equations, Classification of integrals of the first order partial differential equations.
5	Lagrange's methods for solving partial differential equations, Charpit's method for solving partial differential equations, Jacobi's method for solving partial differential equations, higher order equations with constant coefficients and Monge's method.
6	Classification of second order partial differential equations, Solution of Laplace's equation, Wave and diffusion equations by separation of variable (axially symmetric cases).
7	Integral equations and algebraic system of linear equations, Volterra equation & L2_Kernels and functions.
8	Volterra equations of the first kind, Volterra integral equations and linear differential equations.
9	Fredholm equations, Solutions by the method of successive approximations.
10	Neumann's series, Fredholm's equations with Poincere Goursat Kernels, the Fredholm theorems.

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Unit 1: Bessel's Functions

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Objectives

After studying this unit, you should be able to:

- Deduce Bessel's Differential equation from Laplace equation
- Obtain singular and non-singular points of Bessel's equations
- Obtain series solutions of Bessel's equation by Frobenius Method
- Establish recurrence relations between various Bessel's Co-efficient
- Obtain the formula for $J_n(x)$ from its generating functions
- Obtain zeroes of Bessel Functions.

Introduction

In this unit we shall be dealing with the various forms of Laplace differential equation involving Cartesian, Cylindrical and Spherical polar Co-ordinates.

Bessel's functions play a very important and central place in optical phenomenical and in applied mathematical process. Just as a Fourier series, power series, Bessel's functions are quite useful in solving problems involving laplace equations in cylindrical co-ordinates. In this unit the importance is given to the following aspects of the Bessel's functions:

1. Solution of Bessel's functions $J_n(x)$, $Y_n(x)$ for various values of n as well as for different expansions involving x or $(1/x)$.

Notes

2. Recurrence relations are quite useful as they help in finding whole class of $J_n(x)$ in terms of two or three $J_n(x)$ of lower values of n i.e., $n = 0, 1, 2$.
3. Generating function for $J_n(x)$ is introduced so that certain formulas involving Bessel functions can be deduced. With the help of generating functions we can deduce recurrence relations or certain other formulas straight away.
4. Finally we also discuss the zeros of Bessel functions as they will lead us to the completeness as well as orthogonality properties of Bessel's Functions.

1.1 Bessel's Differential Equations from Laplace Equations

In dealing with the theory of potential problems in electrostatics or in gravitational field we commonly use Laplace equations

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(1)$$

Here V is a function of the Cartesian Co-ordinates. Any solution V_n of this equation, which is a homogeneous polynomial of degree n in x, y, z is called the solid spherical Harmonies.

Depending upon the symmetry of the problem we can express Laplace equation in cylindrical co-ordinates (r, θ, z) or spherical polar co-ordinates (r, θ, Φ). You must be knowing that the relations between x, y, z and r, θ, z are

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \qquad ...(2)$$

Also the relation between x, y, z and r, θ, Φ are

$$\left.\begin{aligned} x &= r\sin\theta\cos\Phi \\ y &= r\sin\theta\sin\Phi \\ z &= r\cos\theta \end{aligned}\right\} \qquad ...(3)$$

1.2 Bessel's Differential Equations

To define Bessel functions we first of all obtain Bessel's Differential equation from Laplace's equation. To do that we write Laplace's equations (1) in cylindrical co-ordinates as

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial y}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(4)$$

We assume that V as a function of r, θ and z can be written as

$$V = R(r)\,\Theta'(\theta)\,z'(z) \qquad ...(5)$$

Where R, Θ', Z' are functions of r, θ, z alone respectively. Substituting in (4) we get

$$\Theta' Z'\frac{d^2R}{dr^2}+\frac{1}{r}\Theta' Z'\frac{dR}{dr}+\frac{RZ'}{r^2}\frac{d^2\Theta'}{d\Theta^2}+R\Theta'\frac{d^2Z}{dz^2} = 0$$

Or

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{rR}\frac{dR}{dr}+\frac{1}{r^2}\cdot\frac{1}{\Theta'}\frac{d^2\Theta}{d\theta^2}+\frac{1}{Z'}\frac{d^2Z'}{dZ^2} = 0 \qquad ...(6)$$

Notes

Since the first three terms are independent of z, therefore the fourth term must also be independent of z. Let it be a constant c, so that

$$\frac{1}{Z'}\frac{d^2Z'}{dz^2} = c$$

Or
$$\frac{d^2Z'}{dz^2} = cZ' \qquad ...(7)$$

Similarly, the third term in equation (6) must be free from θ i.e.

$$\frac{1}{\Theta'}\frac{d^2\Theta}{d\theta^2} = d$$

Or
$$\frac{d^2\Theta}{d\theta^2} = d\Theta' \qquad ...(8)$$

With the help of (7) and (8) equation (6) becomes

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2}d+c = 0$$

or
$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(d+cr^2)R = 0 \qquad ...(9)$$

Let us put $kr = x$, so that

$$\frac{dR}{dr} = k\frac{dR}{dx}$$

$$\frac{d^2R}{dr^2} = k^2\frac{d^2R}{dx^2}$$

By putting these values in (9), we get

$$k^2r^2\frac{d^2R}{dx^2}+kr\frac{dr}{dx}+\left(d+\frac{cx^2}{k^2}\right)R = 0$$

Putting $c = k^2$ and $D = -n^2$, we get

$$x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}+(x^2-n^2)R = 0$$

Again put $R = y$ we have

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y = 0$$

This is Bessel's differential equation. The solution of this equation is called cylindrical function or Bessel's function of order n, denoted as $J_n(x)$.

In this unit we shall be using certain properties of gamma function $\Gamma(x)$:

(i) $\Gamma(n)$ is defined by the integral

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx, \qquad n > 0$$

Notes

(ii) $\Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2n-1}$

(iii) $\Gamma(1) = 1$

(iv) $\Gamma(1/2) = \sqrt{\pi}$

(v) $\Gamma(n + 1) = n\,\Gamma(n)$, $n > 0$

(vi) $\Gamma(n + 1) = 1.\,2.\,3.\,...n = n!$ for n a +ve integer

(vii) $\Gamma(n)\,\Gamma(1 - n) = \dfrac{\pi}{\sin n\pi}$

(viii) $\Gamma(m) = \infty$ if $m = 0$ or $-$ *ve* integer

(ix) $\Gamma(2n) = \dfrac{2^{2n-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(n+1/2)$

1.3 On Second Order Differential Equation of the Fuchs Type

Consider Bessel's equation for any n:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y = 0$$

Or

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y = 0 \qquad ...(A)$$

Let

$$\left.\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 1-\frac{n^2}{x^2} \end{aligned}\right\} \qquad ...(B)$$

Thus $p(x)$ has a pole at $x = 0$ and $q(x)$ has a double pole at $x = 0$. Thus $x = 0$ is a singular point of Bessel's Differential Equation. Since

$$x\,p(x) \text{ and } x^2\,q(x), \qquad ...(C)$$

are finite at $x = 0$, the point $x = 0$ is a regular singular point of Bessel Differential equation. Also by putting $x = 1/r$ as independent variable we can show that $x = \infty$ is an irregular singular point. To see this put

$$x = \frac{1}{r}, \quad r = \frac{1}{x}$$

Then

$$\frac{dy}{dx} = \frac{dy}{dr}\frac{dr}{dx} = -\frac{dy}{dr}\left(\frac{1}{x^2}\right) = -r^2\frac{dy}{dr}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-r^2\frac{dy}{dr}\right)$$

Notes

$$= -r^2 \frac{d}{dr}\left(-r^2 \frac{dy}{dr}\right)$$

$$= r^2\left[2r\frac{dy}{dr} + r^2\frac{d^2y}{dr^2}\right]$$

$$\frac{d^2y}{dx^2} = r^4\frac{d^2y}{dr^2} + 2r^3\frac{dy}{dr}$$

So the Bessel's equation becomes

$$\frac{1}{r^2}\left(r^4\frac{d^2y}{dr^2} + 2r^3\frac{dy}{dr}\right) + \frac{1}{r}\left(-r^2\frac{dy}{dr}\right) + \left(\frac{1}{r^2} - n^2\right)y = 0$$

Or $$r^2\frac{d^2y}{dr^2} + r\frac{dy}{dr} + \left(\frac{1}{r^2} - n^2\right)y = 0$$

∴ $$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \left(\frac{1}{r^4} - \frac{n^2}{r^2}\right)y = 0$$

Thus $r \to 0$ or $x = \infty$ is an irregular singular point. Since the singular points for the Bessel's equation are only 0 and ∞, therefore we can get a series solution of the Bessel's equation in powers of x which converges for $0 < x < \infty$. According to Fuchs theorem, the point is regular singular point provided $p(x)$, $q(x)$ satisfy conditions (C).

Fuchs theorem states that for $x = x_0$ to be a regular singular point, it is necessary and sufficient that $p(x)$ has at most a pole of order 1 and $q(x)$ at most a pole of order 2.

1.3.1 Series Solution of Bessel's Differential Equation

Bessel's differential equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad ...(10)$$

Here we shall apply Frobenius method which assumes the solution to be of the form

$$y = x^k \sum_{r=0}^{\infty} x^r c_r \qquad ...(11)$$

Substituting in equation 10 we have

$$\sum_{r=0}^{\infty}\left\{C_r(k+r)(k+r-1)x^{k+r} + C_r(k+r)x^{k+r} + C_r(x^2-n^2)x^{n+r}\right\} = 0$$

or $$\sum_{r=0}^{\infty} C_r x^{k+r}\left\{(k+r)(k+r-1) + (k+r) + (x^2-n^2)\right\} \equiv 0 \qquad ...(12)$$

Equating to zero the lowest power of x i.e. x^k to zero we have

$$C_0\left\{k(k-1) + k - n^2\right\} = 0$$

or $$C_0\left\{k^2 - n^2\right\} = 0 \qquad ...(13)$$

Notes

As $C_0 \neq 0$, we have

$$K^2 - n^2 = 0 \quad ...(14)$$

The equation (14) is called *indicial equation*.

So $k = n$ or $k = -n$

We first consider the case $k = n$, next equate the co-efficient of x^{k+1} to zero i.e.

$$C_1 [(k+1)^2 - n^2] = 0$$

For $k = n, (k+1)^2 - n^2 \neq 0$

So we have $C_1 = 0$ ...(15)

Putting the co-efficient of x^{k+2} to zero, we get

$$C_2 \{(k+2)(k+1) + k + 2 - n^2\} + C_0 = 0$$

or $$C_2 [(k+2)^2 - n^2] + C_0 = 0$$

or $$C_2 = -\frac{C_0}{(k+2)^2 - n^2}$$

$$= -\frac{C_0}{(n+2)^2 - n^2} \quad \text{for } k = n$$

or $$= -\frac{C_0}{(2n+2)(2)} = -\frac{C_0}{(n+1)2^2} \quad ...(16)$$

Putting the co-efficient of x^{k+3} to zero, we get

$$C_3 [(k+3)^2 - n^2] + C_1 = 0$$

or $$C_3 = -\frac{C_1}{(n+3)^2 - n^2} = 0, \text{ as } C_1 = 0$$

Putting the co-efficient of x^{k+4} to zero, we get

$$C_4 [(k+4)^2 - n^2] + C_2 = 0$$

or $$C_4 = -\frac{C_2}{(n+4)^2 - n^2}$$

$$= -\frac{C_2}{(2n+4)(4)}$$

$$= -\frac{C_2}{(n+2)2, 2^2} = \frac{(-1)^2 \quad C_2}{(n+1)(n+2)1.2(2)^4}$$

Proceeding in the same way we get

$$C_1 = 0 = C_3 = C_5 = C_7 = ... \quad ...(17)$$

And $$C_{2k} = \frac{(-1)^k \quad C_0}{(n+1)(n+2)...1.2...(2k) \quad 2^{2k}} \quad \text{for } 1, 2, 3 \quad ...(18)$$

Notes

So $$C_6 = \frac{(-1)^3 \quad C_0}{(n+1)(n+2)(n+3)\underline{|3}\,(2)^6}$$

$$C_8 = \frac{(-1)^4 \quad C_0}{(n+1)(n+2)(n+3)(n+4)\underline{|4}\,(2)^8}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

Substituting the values of k, C_0, C_1, C_2,... in equation for y we get

$$y = x^n \left\{ C_0 - \frac{k_0}{(n+1)\cdot 1}\left(\frac{x}{2}\right)^2 + \frac{k_0}{(n+1)(n+2)\underline{|2}}\left(\frac{x}{2}\right)^4 .... \right\}$$

$$= C_0\, x^n \left\{ 1 - \frac{1}{(n+1)}\frac{1}{1}\left(\frac{x}{2}\right)^2 + \frac{1}{(n+1)(n+2)\underline{|2}}\left(\frac{x}{2}\right)^4 .... \right\} \qquad ...(19)$$

If we now take C_0 to be

$$C_0 = \frac{1}{2^n\,\Gamma(n+1)} \qquad ...(20)$$

Where $\Gamma(n)$ is a gamma function.

As you know the properties of gamma functions $n\,\Gamma(n) = \Gamma(n+1)$, for any value of n, so we get various values of $\Gamma(n)$. The equation for y becomes

$$y = \frac{x^n}{2^n\,\Gamma(n+1)}\left\{ 1 - \frac{1}{(n+1)\cdot 1}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{(n+1)(n+2)1.2}\left(\frac{x}{2}\right)^4 + ... \right\}$$

$$= \left\{ \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n - \frac{1}{\Gamma(n+2)\cdot 1}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{\Gamma(n+3)\cdot\underline{|2}} - \left(\frac{x}{2}\right)^4 + ... \right\}$$

or $$y = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+1+s)\underline{|s}}\left(\frac{x}{2}\right)^{n+2s} \qquad ...(21)$$

Here we have used the fact that

$(n + 1)\,\Gamma(n + 1) = \Gamma(n + 2)$,

$(n + 2)\,\Gamma(n + 2) = \Gamma(n + 3)$ and so on.

Also 1, 2, 3... $s = \underline{|s} = \Gamma(s+1)$

The above solution is called Bessel's function $J_n(x)$. Thus

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+1+s)}\frac{1}{\Gamma(s+1)}\left(\frac{x}{2}\right)^{n+2s} \qquad ...(22)$$

For $k = -n$ and if n is not an integer then the other solution for $k = -n$ is obtained from the equation of $J_n(x)$ by replacing $n \to -n$ i.e.

Notes

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1-n)\,\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s-n} \qquad ...(23)$$

Thus the general solution of Bessel's equation is

$$y = A\,J_n(x) + B\,J_{-n}(x) \qquad ...(24)$$

Where A, B are arbitrary constants.

Example: Proceeding as above shown that for $n = 0$

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(\underline{|2})^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(\underline{|3})^2}\left(\frac{x}{2}\right)^6 + \ldots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \qquad ...(25)$$

Prove for integer n

$$J_{-n}(x) = J_n(x)\,(-1)^n \qquad ...(26)$$

To prove this consider the expression for $J_n(x)$ i.e.

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+s+1)\Gamma(s+1)}\left(\frac{x}{2}\right)^{n+2s}$$

Thus $$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1-n)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s-n}$$

$$= \sum_{s=0}^{n-1} \frac{(-1)^s}{\Gamma(s+1-n)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s-n} + \sum_{s=n}^{\infty} \frac{(-1)^s}{\Gamma(s+1-n)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s-n} \qquad ...(27)$$

In the first term we have the argument of

$$\Gamma(s + 1 - n),$$

To be negative i.e.

$$S + 1 - n$$

is $-ve$ for $s = 0$ to $n - 2$ and it is zero for $s = n - 1$. From the properties of gamma functions

$$\Gamma(s + 1 - n) \text{ is } \infty \text{ for } s + 1 - n \le 0 \qquad ...(28)$$

So the first series for $J_{-n}(x)$ is zero and the expression for $J_{-n}(x)$ becomes

$$J_{-n}(x) = \sum_{s=n}^{\infty} \frac{(-1)^s}{\Gamma(s+1-n)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s-n}$$

Putting $S = r + n$, we have for

$s = n, n + 1, \ldots \infty$

$r = 0, 1, 2, \ldots \infty$

Thus $$\mathrm{J}_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^{r+n}}{\Gamma(r+n+1-n)\Gamma(r+n+1)}\left(\frac{x}{2}\right)^{2r+n}$$

Notes

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\Gamma(r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

or $\quad J_{-n}(x) = (-1)^n J_n(x) \quad$ ...(26)

Thus $J_{-n}(x)$ is not independent of $J_n(x)$

1.3.2 Solution of Bessel's Differential Equation when *n* is a Non-negative Integer

We had seen that when n is not an integer there are two independent solutions i.e. $J_n(x)$ and $J_{-n}(x)$.

When n is a non-negative integer

$$J_{-n}(x) = (-1)^n J_n(x) \quad \text{...(26)}$$

And so it is dependent on $J_n(x)$. To find a second solution we introduce Neumann Function

$$Y_v(x) = \frac{J_v(x)\cos \pi v - J_{-v}(x)}{\sin \pi v} \quad \text{...(29)}$$

If v is not an integer, then $Y_v(x)$ and $J_v(x)$ form a general solution of the Bessel's equation. If v is a non-negative integer, then from equation (26), equation (29) becomes an indeterminate form. To calculate the limit of (29) for $v \to n$, differentiate both the numerator and denominator with respect to v. Then setting $v \to n$, we have

$$Y_n(x) = \lim_{\upsilon \to n} Y_\upsilon(x) = \lim_{\upsilon \to n} \frac{-\pi \sin \pi_\upsilon J_\upsilon(x) + \cos \pi\upsilon J'_\upsilon(x) - J'_{v}(x)}{\pi \cos \pi\upsilon}$$

$$= \frac{1}{\pi}\frac{\partial J_\upsilon(x)}{\partial \upsilon}\Bigg)_{\upsilon=n} - \frac{(-1)^n}{\pi}\frac{\partial J_{-\upsilon}(x)}{\partial \upsilon}\Bigg)_{\upsilon=n} \quad \text{...(29a)}$$

Now from equation (21)

$$J_v(x) = \sum_{s=0}^{\infty} \frac{(-1)s}{\Gamma(\upsilon+s+1)\underline{|s}} \left(\frac{x}{2}\right)^{\upsilon+2s}$$

$$\therefore \quad \frac{\partial J_\upsilon(x)}{\partial \upsilon} = \sum_{s=0}^{\infty} \frac{(-1)s}{\underline{|s}} \left(\frac{x}{2}\right)^{2s} \left\{ \frac{\left(\frac{x}{2}\right)^\upsilon \log \frac{x}{2}}{\Gamma(\upsilon+s+1)} - \frac{\Gamma'(\upsilon+s+1)}{[\Gamma(u+s+1)]^2}\left(\frac{x}{2}\right)^\upsilon \right\}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)s}{\underline{|s}} \frac{(x/2)^{\upsilon+2\upsilon}}{\Gamma(\upsilon+s+1)} \left\{ \log\left(\frac{x}{2}\right) - \Psi(\upsilon+s+1) \right\}]$$

where

$$\psi(\upsilon+s+1) = \frac{\Gamma'(\upsilon+s+1)}{\Gamma(\upsilon+s+1)} \quad \text{...(30)}$$

thus $$\lim_{\upsilon \to n} \frac{\partial J_\upsilon(x)}{\partial \upsilon} = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{\underline{|s}\Gamma(n+s+1)} \left[\log\left(\frac{x}{2}\right) - \psi(n+s+1) \right] \quad \text{...(31)}$$

Notes

The expression for $J_{-n}(x)$ is from (27)

$$J_{-\upsilon}(x) = \sum_{s=0}^{n-1} \frac{(-1)^s}{\Gamma(s+1-\upsilon)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s-\upsilon} + \sum_{s=n}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(s+1-\upsilon)}\left(\frac{x}{2}\right)^{2s-\upsilon} \qquad ...(27)$$

As you know from the properties of gamma functions

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad ...(32)$$

From (27) we obtain

$$\frac{\left(\frac{x}{2}\right)^{-\upsilon+2s}}{\Gamma(-\upsilon+s+1)} = \frac{\left(\frac{x}{2}\right)^{-\upsilon+2s}\Gamma(\upsilon-s)\sin(\upsilon-s)\pi}{\pi} \qquad ...(33)$$

Differentiating (33), we see that for $0 \le s \le n$

$$\frac{d}{d\upsilon}\left\{\frac{(x/2)^{\upsilon+2s}\,\Gamma(\upsilon-s)\sin(\upsilon-s)\pi}{\pi}\right\}\Bigg|_{\upsilon=n} = \left[\left(\frac{1}{2}x\right)^{-\upsilon+2s}\Gamma(\upsilon-s)\left\{\pi^{-1}\psi(\upsilon-s)\sin(\upsilon-s)\pi + \right.\right.$$

$$\left.\left. + \cos(\upsilon-m)\pi - \pi^{-1}\log(x/2)\sin(\upsilon-s)\pi\right]\right|_{\upsilon=n}$$

$$= \left(\frac{x}{2}\right)^{-n+2m}\Gamma(n-m)\cos(n-m)\pi$$

Therefore as $\upsilon \to n$, $\frac{\partial J_{-\upsilon}(x)}{\partial x}$ tends to

$$\sum_{s=0}^{n-1}\frac{(-1)^n\;\Gamma(\upsilon-s)(x/2)^{-n+2s}}{\Gamma(s+1)} + \sum_{s=n}^{\infty}\frac{(-1)^s(x/2)^{-\upsilon+2s}}{\lfloor s\,\Gamma(-n+s)}\{-\log(x/2)+\psi(-n+s+1)\}$$

$$= (-1)^n\sum_{s=0}^{n-1}\frac{(n-s-1)}{\lfloor s}\left(\frac{x}{2}\right)^{-n+2s} + (-1)^{n-1}\sum_{s=n}^{\infty}(-1)^m(\tfrac{1}{2}x)^{-n+2s}\left[\log\frac{x}{2}-\psi(s+1)\right] \qquad ...(34)$$

Using (31) and (34) we get for Neumann Function $Y_n(x)$ with n being a non-negative integer the following

$$Y_n(x) = \frac{2}{\pi}J_n(x)\log\frac{x}{2} - \frac{1}{\pi}\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{n+2s}\left\{\frac{\psi(s+1)+\psi(n+s+1)}{s!(n+s)!}\right\}$$

$$-\frac{1}{\pi}\sum_{s=0}^{n-1}\frac{(n-1-s)!}{s!}\left(\frac{x}{2}\right)^{-n+2s} \qquad ...(35)$$

For $n = 0$, the last term does not appear. Thus $J_n(x)$ and $Y_n(x)$ form the general solution.

Thus we see that the Neumann Function $Y_n(x)$ defined by the relation

$$Y_n(x) = \frac{J_\upsilon(x)\cos\pi\upsilon - J_{-\upsilon}(x)}{\sin\pi\upsilon}$$

converges uniformly to $Y_n(x)$ given by equation (35) as $v \to n$ is any bounded closed domain in the complex x plane except for the origin. Formula (35) for $Y_n(x)$ is known as Hankel Formula.

Hankel Functions: The Hankel Function, or the Bessel Functions of the third kind are defined by

$$H_\upsilon^{(1)}(x) = J_\upsilon(x) + i\,Y_\upsilon(x)$$

$$H_\upsilon^{(2)}(x) = J_\upsilon(x) + i\,Y_\upsilon(x)$$

Prove that

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x \qquad \text{...(29)}$$

Proof: $J_n(x)$ is given by

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+1+s)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s+n}$$

So $$J_{1/2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+3/2)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s+1/2}$$

$$= \left(\frac{x}{2}\right)^{1/2} \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+3/2)\Gamma(s+1)}\left(\frac{x}{2}\right)^{2s}$$

Expanding

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\Gamma(3/2)\Gamma(1)}\left(\frac{x}{2}\right)^{o} - \frac{1}{\Gamma(5/2)\Gamma(2)}\left(\frac{x}{2}\right)^{2} + \frac{1}{\Gamma(7/2)\Gamma(3)}\left(\frac{x}{2}\right)^{4} ....\right\}$$

or $$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\frac{1}{\Gamma(3/2)}\left\{1 - \frac{1}{3/2\cdot\Gamma(2)}\left(\frac{x}{2}\right)^{2} + \frac{1}{3/2\cdot 5/2\cdot\Gamma(3)}\left(\frac{x}{2}\right)^{4} - ...\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2}\frac{1}{\Gamma(1/2)\Gamma(1/2)}\left\{1 - \frac{x^2\,(2)}{3\cdot 2^2} + \frac{2.2}{1\cdot 2\cdot 3\cdot 5}\,\frac{x^4}{(2)^4} - ...\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2}\frac{1}{(1/2)\Gamma(1/2)}\left\{1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}...\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2}\frac{2}{\Gamma(1/2)}\left(\frac{1}{x}\right)\left\{x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5}...\right\}$$

$$= \frac{1}{\Gamma(1/2)}\left(\frac{2}{x}\right)^{1/2}\sin x$$

Here $\Gamma(1/2) = \sqrt{\pi}$

Notes

Self Assessment

1. Prove that

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x$$

Show that when n is any integer positive or negative

$$J_n(-x) = (-1)^n J_n(x)$$

The expression for $J_n(x)$ is given by

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s+1)!}\left(\frac{x}{2}\right)^{n+2s}$$

Case I let n be a positive integer. Replacing $x \to -x$ in the above equation we have

$$J_n(-x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s+1)!}\left(\frac{-x}{2}\right)^{n+2s}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s\,(-1)^{n+2s}}{s!\,(n+s+1)!}\left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s\,(-1)^{2s}}{s!\,(n+s+1)!}\left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n J_n(x) \qquad \left[as\,(-1)^{2s} = 1\right]$$

Thus $\quad J_n(-x) = (-1)^n J_n(x)$

1.4 Recurrence Formulas for $J_n(x)$

Some of the recurrence relations involving Bessel functions are as follows:

I. $\boldsymbol{x\,J'_n(x) = n\,J_n(x) - x\,J_{n+1}(x)}$,

where $\boldsymbol{J'_n(x) = \frac{d}{dx} J_n(x)}$

To prove the above relation, we start from the series expansion of $J_n(x)$ as follows:

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

Differentiating it w.r.t. x and multiplying by x on both sides, we have

$$x\,J'_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s\,(n+2s)\,x^{n+2s}}{s!(n+s)!\,2^{n+2s}}$$

Notes

$$= n\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} + x\sum_{s=1}^{\infty}\frac{(-1)^s\ (x/2)^{n+2s-1}}{(s-1)!\,(n+s)!}$$

$$= n\,J_n(x) + x\sum_{s=1}^{\infty}\frac{(-1)^s}{(s-1)!(n+s)!}\left(\frac{x}{2}\right)^{n+2s-1}$$

In the last sum, let us replace s by r as

$s = r + 1$, then

$$x\,J'_n(x) = n\,J_n(x) + x\sum_{r=0}^{\infty}\frac{(-1)^{r+1}\left(\frac{x}{2}\right)^{n+1+2r}}{r!(n+1+r)!}$$

or $$x\,J'_n(x) = n\,J_n(x) - x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+1+r)!}\left(\frac{x}{2}\right)^{n+1+2r}$$

$= n\,J_n(x) - x\ \ J_{n+1}(x)$

As the last sum is equal to $J_{n+1}(x)$. Thus

$$\boldsymbol{x\,J'_n(x) = n\,J_n(x) - x\ \ J_{n+1}(x)}$$

II. $\boldsymbol{x\,J'_n(x) + n\,J_n(x) = x\ \ J_{n-1}(x)}$

Again, we have

$$x\,J'_n(x) = \sum_{s=0}^{\infty}\frac{(-1)^s\ (n+2s)}{s!(n+s)!}\frac{x^{n+2s}}{2^{n+2s}}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^s\ (2n+2s-n)}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^s\ (2n+2s)}{s!(n+s)!}\left(\frac{x}{2}\right)^{n-1+2s}\cdot\frac{x}{2} - n\,J_n(x)$$

$$= x\sum_{s=0}^{\infty}\frac{(-1)^s\ (n+s)}{s!(n+s)!}\left(\frac{x}{2}\right)^{n-1+2s} - n\,J_n(x)$$

$$= x\sum_{s=0}^{\infty}\frac{(-1)}{s!(n-1+s)!}\left(\frac{x}{2}\right)^{n-1+2s} - n\,J_n(x)$$

{As $(n + s)! = (n + s)(n - 1 + s)!$}

Thus identifying the sum with $J_{n-1}(x)$, we have

$x\,J'_n(x) = x\,J_{n-1}(x) - n\,J_n(x)$

Notes

or rearranging terms we have

$$x\,J'_n(x) + n\,J_n(x) = x\,J_{n-1}(x)$$

III. $\mathbf{2n\,J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]}$

To prove this we just make use of the above two recurrence relations I and II, here we have

$$x\,J'_n(x) = n\,J_n(x) - x\,J_{n+1}(x)$$

and $\quad x\,J'_n(x) + n\,J_n(x) = x\,J_{n-1}(x)$

Substracting we get

$$-n\,J_n(x) = n\,J_n(x) - x\,J_{n+1}(x) - x\,J_{n-1}(x)$$

or $\quad 2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$.

Again rearranging terms we have

$$2n\,J_n(x) = x\,J_{n+1}(x) + x\,J_{n-1}(x)$$

or $\quad 2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$

You can see that relation III is not independent. It depends upon I and II recurring relations.

IV. $\mathbf{2\,J'_n(x) = J_{n-1}(x) - J_{n+1}(x)}$

Hint: Add recurrence relations I and II and simplify the result.

From recurrence relation I, we can show that

$$J'_0(x) = -J_1(x)$$

V. $\quad \boldsymbol{\frac{d}{dx}\left[x^{-n}\,J_n(x)\right] = -x^{-n}\,J_{n+1}(x)}$

Now, the left hand side is

$$\frac{d}{dx}\left[x^{-n}\,J_n(x)\right] = -n\,x^{-n-1}\,J_n(x) + x^{-n}\,J'_n$$

$$= x^{n-1}\,[-n\,J_n(x) + x\,J'_n(x)]$$

$$= x^{n-1}\,[-n\,J_n(x) + n\,J_n(x) - x\,J_{n+1}(x)] \qquad \text{\{From recurrence relation I\}}$$

$$= x^{n-1}\,[-x\,J_{n+1}(x)]$$

$$= -x^{-n}\,J_{n+1}(x) = R.H.S$$

Self Assessment

2. Prove

$$\frac{d}{dx}\left[x^n\,J_n(x)\right] = x^n\,J_{n-1}(x)$$

Notes

1.5 Generating Function for $J_n(x)$

Prove that when n is positive integer $J_n(x)$ is the Co-efficient of t^n in the expansion of

$$e^{\frac{x}{2}(t-1/t)} \qquad ...(A)$$

in ascending and descending powers of t. Also show that $J_n(x)$ multiplied by $(-1)^n$ is the co-efficient of t^{-n} in the expansion of the above expression.

Proof:

Expanding $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ in powers of x *i.e.*

$$e^{\frac{x}{2}(t-1/t)} = \left(e\frac{xt}{2}\right)\left(e\frac{-x}{2t}\right)$$

$$= \left\{1+\frac{xt}{2}+\frac{x^2t^2}{\lfloor 2}+\frac{x^3t^3}{3!}+.......\right\}\times\left\{1-\left(\frac{x}{2t}\right)+\left(\frac{-x}{2t}\right)^2,\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}\left(\frac{-x}{2t}\right)^3+.......\right\} \qquad (B)$$

In the above expansion, collecting the co-efficients of t^n, we have

$$\rightarrow \quad \frac{1}{\lfloor n}\left(\frac{x}{2}\right)^n\cdot 1 - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+1}+\frac{1}{(n+2)!}\frac{1}{\lfloor 2}\left(\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^{n+2}-...$$

$$= \frac{1}{\lfloor n}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{(n+2)!\lfloor 2}\left(\frac{x}{2}\right)^{n+4}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^s}{\lfloor s\,\lfloor n+s}\left(\frac{x}{2}\right)^{n+2s} \equiv J_n(x) \qquad ...(C)$$

Similarly co-efficients of t^{-n} in the above product is

$$= 1.\ \frac{(-1)^n}{\lfloor n}\left(\frac{x}{2}\right)^n+\frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+1}\left(\frac{x}{2}\right)+\frac{\left(\frac{x}{2}\right)(-1)^{n+2}}{\lfloor 2\,(n+2)!}\left(\frac{x}{2}\right)^{n+2}+...$$

$$= (-1)^n\left[\frac{1}{\lfloor n}\left(\frac{x}{2}\right)^n+\frac{(-1)}{\lfloor n+1\,\lfloor 1}\left(\frac{x}{2}\right)^{n+2}+\frac{(-1)^2}{\lfloor 2\ \lfloor n+2}\left(\frac{x}{2}\right)^{n+4}+...\right]$$

$$= (-1)^n\sum_{s=0}^{\infty}\frac{(-1)^s}{\lfloor s\,\lfloor n+s}\left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n J_n(x)$$

In the above product the co-efficient of t^0 is

$$= 1-\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^4\frac{1}{2^2}-\left(\frac{x}{2}\right)^6\frac{1}{2^2\cdot 3^2}+\left(\frac{x}{2}\right)^8\cdot\frac{1}{2^2\cdot 3^2\cdot 4^2}....$$

$$= 1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\frac{x^8}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}....$$

$$= J_0(x)$$

Notes

Thus in the expansion of t,

$$e\frac{x}{2}\left(t-\frac{1}{t}\right) = J_0 + \left(t-\frac{1}{t}\right)J_1 + \left(t^2+\frac{1}{t^2}\right)J_2 + \left(t^3-\frac{1}{t^3}\right)J_3 + \ldots + \ldots.. + \left(t^n+(1)^n\frac{1}{t^n}\right)J_n + \ldots$$

$$= J_0(x) + t\left[J_1(x)+J_{-1}(x)\right] + t^2\,\big]\big[J_2(x)+J_{-2}(x)\big] + \ldots$$

$$= \sum_{n=-\infty}^{+\infty} t^n J_n(t)$$

Here we have used the result $J_{-n}(x) = (-1)^n J_n(x)$

(A) ***Trigonometric Expansions involving Bessel's Functions***

Show that

(a) $\cos(x\sin\phi) = J_0(x) + 2\cos 2\phi\, J_2 + 2\cos 4\phi\, J_4 + \ldots.$

(b) $\sin(x\sin\phi) = 2\sin\phi\, J_1 + 2\sin 3\phi\, J_3 + \ldots.$

(c) $\cos(x\cos\phi) = J_0 - 2\cos\phi\, J_2 + 2\cos 4\phi\, J_4 - \ldots.$

(d) $\sin(x\cos\phi) = 2\cos\phi\, J_1 - 2\cos 3\phi\, J_3 + 2\cos 5\phi\, J_5 + \ldots$

(e) $\cos x = J_0 - 2J_2 + 2J_3 - 2J_4 + \ldots$

(f) $\sin x = 2J_1 - 2J_3 + 2J_5 \ldots.$

Proof: We know from generating function that

$$e\frac{x}{2}\left(t-\frac{1}{t}\right) = \sum_{n=\infty}^{+\infty} t^n J_n(x)$$

$$= \sum_{n=0}^{\infty} t^n J_n(x) + \sum_{n=-1}^{\infty} t^n J_n(x)$$

$$= \sum_{n=0}^{\infty} t^n J_n(x) + \sum_{n=-1}^{\infty} t^{-n} J_{-n}(x)$$

$$= J_0 + \sum_{n=1}^{\infty} (t^n + (-1)^n t^{-n}) J_n(x) \qquad \{\text{since } J_{-n}(x) = (-1)\, J_n(x)\}$$

Thus,

$$e^{\frac{x}{2}}\left(t-\frac{1}{t}\right) = J_0 + (t-t^{-1})J_1 + (t^2+t^{-2})J_2 + (t^3-t^{-3})J_3 + \qquad \ldots\text{(i)}$$

Let us put $t = e^{i\phi}$, $t^n = e^{in\phi}$

then (i) becomes $e^{\frac{x}{2}}(e^{i\phi}-e^{-i\phi}) = J_0 + (e^{i\phi}-e^{-i\phi})J_1 + (e^{2i\phi}+e^{2i\phi})J_2 + (e^{3i\phi}-e^{-3i\phi})J_3 + \ldots \qquad$ (ii)

Since $\cos n\phi = \frac{1}{2}\left(e^{in\phi}+e^{-in\phi}\right)$

$\sin n\phi = \frac{1}{2i}\left(e^{in\phi}-e^{-in\phi}\right)$

Notes

So (ii) may be written as

$$e^{ix \sin \phi} = J_0 + 2^i \sin \phi\, J_1 + 2 \cos 2\phi\, J_2 + 2\, i \sin 3\phi\, J_3 + ... \quad ...(iii)$$

comparing real and imaginary part on both sides

we have

(a) $\cos (x \sin \phi) = J_0 + 2 \cos 2\phi\, J_2 + 2 \cos 4\phi\, J_4 + ...$ ...(iv)

(b) $\sin (x \sin \phi) = 2 \sin \phi\, J_1 + 2 \sin 3\phi\, J_3 + ...$ ...(v)

Replacing ϕ by $\pi/2 - \phi$ in (iv) and (v) and using $\sin \phi \rightarrow \sin (\pi/2 - \phi) = \cos \phi$, we get

(c) $\cos (x \cos \phi) = J_0 - 2 \cos 2\phi\, J_2 + 2 \cos 4\phi\, J_4 ...$ ...(vi)

(d) $\sin (x \cos \phi) = 2 \cos \phi\, J_1 - 2 \cos 3\phi\, J_3 + 2 \cos 5\phi\, J_5 ...$ ...(vii)

Replacing ϕ by 0 in (iv) and (vii) we get

(e) $\cos x = J_0 - 2\, J_2 + 2\, J_4 ...$ ...(viiii)

and

(f) $\sin x = 2\, J_1 - 2\, J_3 + 2\, J_5 ...$ ...(ix)

1.6 On the Zeros of Bessel Functions $J_n(x)$

We know that Bessel function $J_n(x)$ satisfies the equation

$$x^2 \frac{d^2 J_n(x)}{dx^2} + x \frac{d J_n(x)}{dx} + (x^2 - n^2) J_n(x) = 0$$

Here n is a positive integer

let us put $x = \lambda v$,

$$\frac{d J_n}{dx} = \frac{1}{\lambda} \frac{d J_n}{dv}$$

$$\frac{d^2 J_n}{dx^2} = \frac{1}{\lambda^2} \frac{d^2 J_n}{dv^2}$$

So equation (1) becomes

$$v^2 \frac{d^2 J_n(\lambda v)}{dv^2} + v \frac{d J_n(\lambda v)}{dv} + (\lambda^2 v^2 - n^2) J_n(\lambda v) = 0 \quad ...(ii)$$

which may be written as

$$\frac{d}{dv}\left[v \frac{d J_n(\lambda v)}{dv} \right] + \left[\frac{-n^2}{v} + 2\lambda v \right] J_n(\lambda v) = 0 \quad ..(iii)$$

let us put $R = v$, $P = v$, $Q = -\frac{n^2}{v}$

Then $$\frac{d}{dv}\left(R \frac{d J_n(\lambda v)}{dv} \right) + \left[Q + 2\lambda p \right] J_n(\lambda v) \quad ...(iv)$$

Notes

Here due to $R = 0$, it can be shown that for some a i.e. $0 \le x \le a$, $J_n(\lambda\nu)$ satisfies the Boundary Condition $J_n(\lambda a) = 0$ ...(v)

And so the solutions of (iii) form an orthonormal set w.r.t. weight function $P = \nu$.

So zeros of $J_n(\lambda\nu)$ if denoted by $\propto_{in}$ $i = 1, 2,...$

Let

$$\alpha_{1a} < \alpha_{2a} < \alpha_{3a}... \alpha_m....$$

So $\lambda a = a_{mn}$

thus $\lambda = \dfrac{\alpha_{mn}}{a} \equiv \lambda_{mn}$

Since both J_n and $\dfrac{dJ_n}{d\nu}$ are continuous at $\nu = 0$, therefore for each fixed $n = 0, 1, 2...$ the Bessel function $J_n(\lambda_{mn})$ $(m = 1, 2,...)$ with $\lambda_{mn} = \dfrac{\alpha_m}{a}$, form an orthogonal set on the internal $0 \le x \le a$ w.r.t. weight $P = \nu$ i.e.

$$\int_0^a \nu\, J_n(\lambda_{mn}\nu) J_n(\lambda_{pn}\nu) = 0 \text{ for } p \ne m$$

So zeros of $J_n(x)$ are useful in obtaining orthogonal properties of $J_n(x)$. The details of the above discussion will be given in the later units.

Example: Prove that $J_n(x) = 0$ has no repeated roots except at $x = 0$.

Solution: If possible let $\propto$ be a repeated root of

$$J_n(x) = 0 \quad \text{at} \quad x = \alpha \qquad ...(i)$$

Thus $J_n(\alpha) = 0$ as well as $J'_n(\alpha) = 0$ ...(ii)

Now from recurrence formulae I and II,

$$x J'_n(x) = n J_n(x) - x J_{n+1}(x),$$

$$x J'_n(x) + n J_n(x) = x J_{n-1}(x),$$

We have

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J'_n(x) \qquad ...(iii)$$

$$J_{-n-1}(x) = \frac{n}{x} J_n(x) + J'_n(x) \qquad ...(iv)$$

As $J_n(x) = 0$ and $J'_n(\alpha) = 0$, we have from III and IV $J_{-n+1}(\alpha) = 0$ and $J_{n-1}(\alpha) = 0$, i.e. for the same value of $x = \alpha$, $J_n(x)$, $J_{n+1}(x)$, $J_{n-1}(x)$ are all zero x, which is absurd as we cannot have two power series having the same sum function. Then $J_n(x) = 0$ cannot have repeated roots except $x = 0$.

1.7 Illustrative Examples

Example 1: Show that

(i) $x \sin x = 2\,(2^2 J_2 - 4^2 J_4 + 6^2 J_6 - ...)$

(ii) $x \cos x = 2\,(1^2 J_1 - 3^2 J_3 + 5^2 J_5 - ...)$

Solution: (i) We know that

$$\cos(x \sin \phi) = J_0 + 2\,J_2 \cos 2\phi + 2\,J_4 \cos 4\phi + ... \qquad ...(i)$$

Differentiating w.r.t. 'ϕ' we get

$$-\sin(x \sin \phi).\, x \cos \phi = 0 - 2.2\,J_2 \sin 2\phi - 2.4\,J_4 \sin 4\phi\ ... \qquad ...(ii)$$

Differentiating (ii) w.r.t. 'ϕ', we have

$$-\cos(x \sin \phi).\,(x \cos \phi)^2 + \sin(x \sin \phi)\,(x \sin \phi)$$

$$= -2.2^2.J_2 \cos 2\phi - 2.4^2\,J_4 \cos 4\phi - 2.6^2\,J_0 \cos 6\phi\ ... \qquad ...(iii)$$

Replacing ϕ by π/2 in (iii), we get

$$x \sin x = 2\,(2^2 J_2 - 4^2 J_4 + 6^2 J_6\ ...)$$

(ii) Start with

$$\sin(x \sin \phi) = 2J_1 \sin \phi + 2J_3 \sin 3\phi + ...$$

Differentiate this twice w.r.t. 'ϕ' as in part (i) and then replace ϕ by π/2. Thus we can get the required answer.

Example 2: Show that when n is integral

(a) $$\pi\,J_n = \int_0^{\pi} \cos(n\theta - x \sin\theta)\,d\theta$$

(b) $$\pi\,J_0 = \int_0^{\pi} \cos(x \cos\phi)\,d\phi$$

$$= \int_0^{\pi} \cos(x \sin\phi)\,d\phi$$

and hence deduce that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + ...$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{\left(2^r.r!\right)^2}$$

Solution: We know that

$$\cos(x \sin\theta) = J_0 + 2J_2 \cos 2\theta + ... + 2J_{2m} \cos 2m\phi + ... \qquad ...(i)$$

and $\sin(x \sin\theta) = 2 \sin\theta.J_1 + 2 \sin 3\theta\, J_3 + ...$

$$+2J_{2m+1} \sin(2m+1)\,\theta + ... \qquad ...(ii)$$

Notes

Multiplying both sides of (i) by cos $2m\theta$ and then integrating between the limits 0 to π, we get

$$\int_0^\pi \cos(x\sin\theta)\cos 2m\theta\, d\theta$$

$$= J_0\int_0^\pi \cos 2m\theta\, d\theta + 2J_2\int_0^\pi \cos 2\theta \cos 2m\theta\, d\theta + ... + 2J_{2m}\int_0^\pi \cos^2 2m\theta\, d\theta + ...$$

$$= 0 + 0 + ... + J_{2m}\int_0^\pi (1+\cos 4m\theta)\, d\theta + ...$$

$$= \pi J_{2m}.$$

Similarly, we can prove that

$$\int_0^\pi \cos(x\sin\theta)\cos(2m+1)\theta\, d\theta = 0$$

Again multiplying both sides of (ii) by sin $(2m + 1)\theta$ and then integrating between the limits 0 to π, we get

$$\int_0^\pi \sin(x\sin\theta)\sin(2m+1)\theta\, d\theta$$

$$= 2J_1\int_0^\pi \sin\theta.\sin(2m+1)\theta\, d\theta + 2J_2\int_0^\pi \sin 3\theta \sin(2m+1)\theta\, d\theta +$$

$$+ ... + 2J_{2m+1}\int_0^\pi \sin^2(2m+1)\theta\, d\theta + ...$$

$$= 0 + 0 + ... + 2J_{2m+1}\int_0^\pi \{1-\cos 2(2m+1)\theta\}\, d\theta + ...$$

$$= J_{2m+1}\ [\theta]_0^\pi = \pi J_{2m+1}$$

Similarly,

$$\int_0^\pi \sin(x\sin\theta)\sin 2m\theta\, d\theta = 0$$

Therefore

$$\int_0^\pi \cos(2m\theta - x\sin\theta)\, d\theta = \int_0^\pi \cos 2m.\cos(x\sin\theta)d\theta$$

$$+ \int_0^\pi \sin 2m\theta.\sin(x\sin\theta)\, d\theta$$

$$= \pi J_{2m}$$

Also

$$\int_0^\pi \cos\left[(2m+1)\theta - x\sin\theta\right] d\theta$$

$$= \int_0^\pi \cos(2m+1)\theta.\cos(x\sin\theta)\,d\theta + \int_0^\pi \sin(2m+1)\theta\sin(x\sin\theta)\,d\theta$$

$$= \pi J_{2m+1}$$

Hence for all positive integral n, we get

$$\int_0^\pi \cos(\pi\theta - x\sin\theta)\,d\theta = \pi J_n.$$

If n is negative, say $n = -m$, where m is positive, then

$$\int_0^\pi \cos(\pi\theta - x\sin\theta)\,d\theta$$

$$= \int_0^\pi \cos(-m\theta - x\sin\theta)\,d\theta$$

$$= -\int_\pi^0 \cos\left\{-m(\pi-\phi) - x\sin(\pi-\phi)\right\} d\pi \qquad \text{Putting } \theta = \pi - \phi$$

$$= \int_0^\pi \cos\left\{-m\pi + (m\phi - x\sin\phi)\right\} d\phi$$

$$= \int_0^\pi \{\cos m\pi \cos(m\phi - x\sin\phi) + \sin m\pi \sin(m\phi - x\sin\phi)\,d\theta$$

$$= (-1)^m \int_0^\pi \cos(m\phi - x\sin\phi)\,d\phi$$

$$= (-1)^m \pi J_m(x) \qquad \text{Since } J_{-m}(x) = (-1)^m J_m(x)$$

$$= \pi J_n(x)$$

Hence for all integral values of n

$$\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta = \pi J_n$$

(b) Putting $\theta = \pi/2 + \phi$ in the value of $\cos(x\sin\theta)$ from (i), we have

$$\cos(x\cos\phi) = J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi - \ldots$$

$$\therefore \qquad \int_0^\pi \cos(x\cos\phi)\,d\phi = J_0\int_0^\pi d\theta - 2J_2\int_0^\pi \cos 2\phi\,d\phi + \ldots$$

$$= \pi J_0$$

Notes

From (i) we have

$$\cos(x\sin\phi) = J_0 + 2J_2\cos 2\phi + 2J_4\cos 4\phi + ...$$

$$\therefore \qquad \int_0^{\pi}\cos(x\sin\phi)d\phi = J_0\int_0^{\pi}d\phi + 2J_2\int_0^{\pi}\cos 2\phi\, d\phi + ...$$

$$= \pi J_{0.}$$

Deduction: We have to prove that

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{x^2\cos^2\phi}{2!} + \frac{x^4\cos^4\phi}{4!} - \frac{x^6\cos^6\phi}{6!} + ...\right)d\phi \qquad ...(iii)$$

Since $\displaystyle\int_0^{\pi}\cos^{2r}\phi\, d\phi = \frac{1.3.5...(2r-1)}{2.4.6...(2r)}\cdot\pi$

from definite integrals.

$\therefore$ from (iii), we get

$$J_0(x) = \frac{1}{\pi}\left[\pi - \frac{x^2}{2!}\cdot\frac{1}{2}\pi + \frac{x^4}{4!}\cdot\frac{1.3}{2.4}\pi - \frac{x^6}{6!}\cdot\frac{1.3.5}{2.4.6}\pi + ...\right]$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + ...$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot(2!)^2} - \frac{x^6}{2^6\cdot(3!)} + ...$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r.r!)}$$

Self Assessment

3. Verify directly from the representation

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\phi)d\phi$$

that $J_0(x)$ satisfies Bessel's equation in which $n = 0$

Example 3: Prove

$$\int_0^{\infty}e^{-ax}j_0(bx)dx = \frac{1}{\sqrt{\{(a^2+b^2)\}}}, a > 0.$$

Solution: From example above, we have **Notes**

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\phi)dx$$

$$\therefore \int_0^{\infty} e^{-ax} j_0(bx)dx = \int_0^{\infty} e^{-ax}\left\{\frac{1}{x}\int_0^{\pi}\cos(bx\sin\phi)d\phi\right\}dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\int_0^{\infty} e^{-ax}\cos(bx\sin\phi)dx\right]d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\int_0^{\infty} e^{-ax}\frac{e^{i(bx\sin\phi)}+e^{-i(bx\sin\phi)}}{2}dx\right]d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\int_0^{\infty}\left\{e^{-(a-ib\sin\phi)x}+e^{-(a+ib\sin\phi)x}\right\}dx\right]d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\frac{e^{-(a-ib\sin\phi)x}}{-(a-ib\sin\phi)}-\frac{e^{-(a+ib\sin\phi)x}}{(a+ib\sin\phi)}\right]_0^{\infty}d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\frac{1}{a-ib\sin\phi}+\frac{1}{a+ib\sin\phi}\right]d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{2a\,d\phi}{a^2+b^2\sin^2\phi}$$

$$= 2.\frac{a}{\pi}\int_0^{\pi/2}\frac{\operatorname{cosec}^2\phi\,d\phi}{b^2+a^2\operatorname{cosec}^2\phi}$$

$$= 2.\frac{a}{\pi}\int_0^{\pi/2}\frac{\operatorname{cosec}^2\phi\,d\phi}{(a^2+b^2)+a^2\cot^2\phi}$$

$$= 2.\frac{a}{\pi}\left[\frac{1}{a\sqrt{\{(a^2+b^2)\}}}\cot^{-1}\frac{a\cot\phi}{\sqrt{\{(a^2+b^2)\}}}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi\sqrt{\{(a^2+b^2)\}}}\left[\cot^{-1}0-\cot^{-1}\infty\right]$$

$$= \frac{1}{\sqrt{\{(a^2+b^2)\}}}$$

Notes

Example 4: Using generating function or otherwise, show that

$$J_n(-x) = (-1)^n J_n(x)$$

Solution: We have

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$$

Replacing x by $-x$ in (i), we get

$$\sum_{n=-\infty}^{\infty} J_n(-x) z^n = e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{\frac{x}{2}\left(-z-\frac{1}{-z}\right)}$$

$$= \sum_{n=-\infty}^{\infty} J_n(x).(-z)^n \qquad \text{[by (i)]}$$

$$\sum_{n=-\infty}^{\infty} J_n(-x) z^n = \sum_{n=-\infty}^{\infty} J_n(x).(-1)^n z^n$$

Equating the coefficient of z^n from both sides of (ii) gives

$$J_n(-x) = (-1)^n J_n(x).$$

Example 5: If $n > -1$, show that

$$\int_0^x x^{n+1} J_n(x) dx = x^n J_{n-1}(x)$$

Solution: From recurrence formula I, we have

$$\frac{d}{dx}\left\{x^n J_n(x)\right\} = x^n J_{n-1}(x) \qquad \text{...(i)}$$

Replacing n by $(n + 1)$ in (i), we get

$$\frac{d}{dx}\left\{x^{n+1} J_{n+1}(x)\right\} = x^{n+1} J_n(x) \qquad \text{...(ii)}$$

Integrating (i) w.r.t. 'x' between the limits 0 and x, we get

$$\left[x^{n+1} J_{n+1}(x)\right]_0^x = \int_0^x x^{n+1} J_n(x) dx$$

or $$\int_0^x x^{n+1} J_n(x) dx = x^{n+1} J_{n+1}(x)$$

Notes

Example 6: Show that

(a) $\int_0^x x^{-n} J_{n+1}(x)\,dx = \frac{1}{2^n \Gamma(n+1)'} - x^{-n} J_n,\ n > 1.$

(b) $\int_0^\infty x^{-n} J_{n+1}(x) dx = \frac{1}{2^n \Gamma(n+1)},\ n > -\frac{1}{2}.$

Solution:

(a) From recurrence formula II, we have

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = x^{-n} J_{n+1}(x) dx \qquad ...(i)$$

Integrating (i) w.r.t. 'x' between the limits 0 and x, we get

$$\left[x^{-n} J_n(x)\right]_0^x = \int_0^x x^{-n} J_{n+1}(x) dx$$

$$\therefore\ x^{-n} J_n(x) - \lim_{x\to 0} \frac{J_n(x)}{x^n} = -\int_0^x x^{-n} j_{n+1}(x) dx \qquad ...(ii)$$

But $\lim_{x\to 0} \frac{J_n(x)}{x^n} = \lim_{x\to 0} \frac{1}{x^n} . \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2.2.(n+1)} + ...\right] = \frac{1}{2^n \Gamma(n+1)}$

Hence (ii) may be written as

$$\int_0^x x^{-n} J_{n+1}(x) dx = \frac{1}{2^n \Gamma(n+1)} - x^{-n} J_n(x)$$

(b) Integrating (i) w.r.t. 'from 0 to ∞' we get

$$\left[x^{-n} J_n(x)\right]_0^\infty = -\int_0^\infty x^{-n} J_{n+1}(x) dx$$

$$\therefore\ \lim_{x\to 0} \frac{J_n(x)}{x^n} - \lim_{x\to 0} \frac{J_n(x)}{x^n} = -\int_0^\infty x^{-n} J_{n+1}(x) dx \qquad ...(iii)$$

As in part (a), $\lim_{x\to 0} \frac{J_{n+1}(x)}{x^n} = \frac{1}{2^n \Gamma(n+1)}$ ...(iv)

We know that for large values of x the approximate value of $J_n(x)$ is given by

$$J_n(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left\{x - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right\}, n > -\frac{1}{2}$$

Notes

Using (v), $\lim_{x \to \infty} \frac{J_n(x)}{x^n} = 0$

Using (iv) and (vi), (iii) reduces to

$$\int_0^{\infty} x^{-n} J_{n+1}(x) dx = \frac{1}{2^n \Gamma(n+1)}$$

1.8 Summary

- Bessel Differential equation is seen to have $x = 0$ as regular singular point
- $x = \infty$ is irregular singular point of the Bessel Differential.
- Bessel Differential equation is deduced from Laplace equation.
- Bessel Differential equation is of Fuchs Type and so Frobenius method of expanding solution of Bessel's equation as power series in x is valid.
- The generating function of Bessel function is given by

$$e^{\frac{x}{z}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{+\infty} t^n J_n(x)$$

- With the help of generating function we obtain recurrence relations
- It is seen that $J_n(x)$ does not have repeated zeroes except at $x = 0$.

1.9 Keywords

Ordinary point of a Differential equation is such that the solution can be expressed in terms of a power series.

Regular singular point $x = x_0$ is such that $p(x)$, $q(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

behave as

$$(x - x_0)p(x) = \text{finite} \lim_{x \to x_0}, \; (x - x_0)^2 q(x) = \text{finite} \lim_{x \to x_0}$$

Recurrence relation is a relation involving a few Bessel functions i.e. it involves $J_n(x), J_{n-1}(x), J_{n+1}(x)$ and $\frac{dJ_n(x)}{dx}$.

Generating function is such a function which on expansions gives the values of $J_n(x)$.

Fuchs type differential equation satisfies the properties as given above.

Indicial equation gives the values of the parameter appearing in power series expansion of $J_n(x)$.

1.10 Review Questions

Notes

Prove that:

1. $J_2(x) = \frac{d^2 J_0(x)}{dx^2} - x^{-1}\frac{dJ_0}{dx}(x)$

2. $J_2(x) - J_0(x) = 2\frac{d^2 J_0(x)}{dx^2}$

3. $J_2(x) + 3\frac{dJ_0}{dx} + 4\frac{d^2 J_0}{dx^2}(x) = 0$

4. $2\frac{d}{dx}J_n(x) = J_{n-1}(x) - J_{n+1}(x)$

5. Solve the Differential equation

   $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{9}{4}\right)y = 0$$

   and show that

   $$y = A\ J_{\frac{3}{2}}(x) + B\,J_{\frac{-3}{2}}(x)$$

1.11 Further Readings

G. N. Watson, A Treatise on the Theory of Bessel Functions

Louis A. Pipes and L.R. Harvill, Applied Mathematics for Engineers and Physicists

K. Yosida, Lectures on Differential and Integral Equations

Jai Dev Anand, P.K. Mittal and Ajay Wadhwa, Mathematical Physics Part II

Notes

Unit 2: Legendre's Polynomials

CONTENTS

Objectives

After studying this unit, you should be able to:

- Observe that Legendre's differential equation is obtained from the Laplace differential equation
- Obtain the Legendre's polynomial $P_n(x)$ as a power series having x^n as a maximum powerterm for $n > 0$ integer
- See recurrence relations of $P_n(x)$ help in finding all $P_n(x)$ in terms of two or three lower $P_n(x)$.
- See that a generating function is found by which various $P_n(x)$ are found.
- See that orthogonal properties of $P_n(x)$ help in expressing any function $f(x)$ in terms of various $P_n(x)$.

Introduction

The Legendre's polynomials $P_n(x)$ play an important role in potential problems i.e. in electrostatics and gravitational field. It is therefore important to study the properties of $P_n(x)$.

1. First of it is important to study the solution of Legendre's equations so that more insight to $P_n(x)$ can be seen.

2. Recurrence relations derived in this unit help us in finding unknown $P_n(x)$ in terms of two or three known Legendre polynomial

- The Legendre's polynomials $P_n(x)$ have zeroes at some $x = x_i, i = 1, 2, ...$ i.e. $P_2(x)$ has two zeroes, $P_3(x)$ has three and so on.
- Legendre polynomials are quite suited in numerical evaluations of certain integrals.

2.1 Legendre's Differential Equation from Laplace's Equation

Laplace's equation in spherical polar coordinates is

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \qquad ...(A)$$

Let us put

$$V = r^n F_n(\theta, \phi) \qquad ...(B)$$

Here $F_n(\theta, \phi)$ is a function of θ and ϕ. So

$$\frac{\partial V}{\partial r} = n\, r^{n-1}\, F_n$$

$$\frac{\partial V}{\partial\theta} = r^n \frac{\partial F_n}{\partial\theta}$$

$$\frac{\partial^2 V}{\partial\phi^2} = r^n \frac{\partial^2 F_n}{\partial\phi^2}$$

Substituting in Laplace equation, we get

$$\frac{\partial}{\partial r}\left(n\, r^{n+1} F_n\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(r^n \sin\theta\frac{\partial F_n}{\partial\theta}\right) + \frac{r^n}{\sin^2\theta}\frac{\partial^2 F_n}{\partial\phi^2} = 0$$

or

$$n(n+1)r^n\, F_n + \frac{r^n}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_n}{\partial\theta}\right) + \frac{r^n}{\sin^2\theta}\frac{\partial F_n}{\partial\phi^2} = 0$$

Dividing by r^n, we have

$$n(n+1)F_n + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 F_n}{\partial\phi^2} = 0 \qquad ...(C)$$

Next consider the case when $F_n(\theta,\phi)$ is independent of ϕ, so

$$n(n+1)\, F_n + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_n}{\partial\theta}\right) = 0 \qquad ...(D)$$

Let us put the independent variable θ in terms of x given by

$$x = \cos\theta$$

$$\frac{d}{d\theta} F_n = \frac{\partial F_n}{\partial x}\frac{\partial x}{\partial\theta} = -\sin\theta\frac{\partial F_n}{\partial x}$$

Notes

$$\sin\theta\frac{\partial}{\partial\theta}F_n = -\sin^2\theta\frac{\partial F_n}{\partial x}$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}F_n\right) = \frac{\partial}{\partial\theta}\left(-\sin^2\theta\frac{\partial F_n}{\partial x}\right)$$

$$= -2\sin\theta\cos\theta\frac{\partial F_n}{dx}+\sin^3\theta\frac{d^2F_n}{dx^2}$$

Substituting in equation (D) we have

$$n(n+1)F_n - 2\cos\theta\frac{\partial F_n}{\partial x}+\sin^2\theta\frac{d^2F}{dx^2} = 0$$

or $$n(n+1)F_n - 2x\frac{dF_x}{dx}+(1-x^2)\frac{d^2F}{dx^2} = 0$$

Rewriting it as:

$$(1-x^2)\frac{d^2F_n}{dx^2}-2x\frac{dF_n}{dx}+n(n+1)F_n = 0 \quad \text{...(E)}$$

This equation (E) is known as Legendre's differential equations. The solution of equation (E) for positive integer values of n are known as Legendre Polynomial.

Putting Legendre equation in Fuchs form we have

$$\frac{d^2F_n}{dx^2}-\frac{2x}{(1-x^2)}\frac{dF_n}{dx}+\frac{n(n+1)}{(1-x^2)}F_n = 0 \quad \text{...(F)}$$

Here let coefficients of $\frac{dF_n}{dx}$ and F_n be

$$\left.\begin{aligned} p(x) &= -\frac{2x}{(1-x^2)} \\ q(x) &= \frac{n(n+1)}{(1-x^2)} \end{aligned}\right\} \quad \text{..(G)}$$

At $x=1$ and $x=-1$, both $p(x)$ and $q(x)$ have poles of the first order. So the points $x = 1$ and $x = -1$ are regular singular points of the Legendre's equations. Let us investigate the behaviour of the equation for $x=\infty$. For this purpose let us put

$$x = \frac{1}{r},\ F_n = y \quad \text{...(H)}$$

$$\frac{dy}{dx} = \frac{dy}{dx}\frac{dr}{dx} = -r^2\frac{dy}{dr}$$

Notes

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -x^2\frac{d}{dr}\left(-r^2\frac{dy}{dr}\right)$$

$$= r^2\left(2r\frac{dy}{dx}\right) + r^4\frac{d^2y}{dr^2}$$

The equation (F) becomes

$$r^4\frac{d^2y}{dr^2} + \frac{2r^2}{r\left(1-\frac{1}{r^2}\right)}\frac{dy}{dx} - \frac{n(n+1)}{\left(1-\frac{1}{r^2}\right)}y = 0$$

$$r^4\frac{d^2y}{dr^2} + \frac{2r^3}{(x^2-1)}\frac{dy}{dx} - \frac{n(n+1)r^2}{(r^2-1)}y = 0$$

or
$$\frac{d^2y}{dr^2} + \frac{2}{r(r^2-1)}\frac{dy}{dr} - \frac{n(n+1)}{r^2(r^2-1)}y = 0 \qquad \text{(I)}$$

Thus $r = 0$ or $x = \infty$ is a regular singular point of the differential equation (Legendre's). Thus we can find a solution of Legendre's equation in terms of a power series in x as well as in powers of $\frac{1}{x}$.

2.1.1 Power Series Solution of Legendre's Equation in Ascending Powers of x

$$(1-x^2)\frac{d^2F_n}{dx^2} - 2x\frac{dF_n}{dx} + n(n+1)F_n = 0 \qquad \text{...(A)}$$

As in the case of Bessel's differential equation we assume a solution of the form:

$$F_n = x^s\sum_{r=0}^{\infty} C_r x^r$$

or
$$F_n = \sum_{r=0}^{\infty} C_r x^{r+s} \qquad \text{...(B)}$$

For (B) to be a solution of (A) it is necessary that when equation (B) is substituted into (A), the coefficients of every power of x vanish. So we have

$$(1-x^2)\sum_{r=0}^{\infty}(r+s)(r+s-1)C_r x^{r+s-2} - 2x\sum_{r=0}^{\infty}C_r(r+s)x^{r+s-1} + n(n+1)\sum_{r=0}^{\infty}C_r x^{r+s} \equiv 0$$

or
$$\sum_{r=0}^{\infty}\left[(r+s)(r+s-1)C_r(x^{r+s-2} - x^{r+s}) - 2C_r(r+s)x^{r+s} + x^{r+s}n(n+1)C_r\right] \equiv 0$$

or
$$\sum_{r=0}^{\infty}\left\{(r+s)(r+s-1)C_r x^{r+s-2} + C_r x^{r+s}[n(n+1) - 2(r+s) - (r+s)(r+s-1)]\right\} \equiv 0$$

$$\sum_{r=0}^{\infty}\left[(r+s)(r+s-1)C_r x^{r+s-2} + C_r x^{r+s}(n-r-s)(n+r+s+1)\right] \equiv 0 \qquad \text{...(C)}$$

Notes

Equating coefficients of x^{r+s-2} we get

$$C_r(r+s)(r+s-1)+(n-r-s+2)(n+r+s-1)C_{r-2} \quad = \quad 0$$

for $r=0,\ 1,\ldots$ ...(D)

Since the leading term is C_o so that

$C_{-1}=0, C_{-2}=0.$ Thus C_o satisfies

$$C_0(s)(s-1) \quad = \quad 0 \qquad \text{...(E)}$$

Since $C_0 \neq 0,$ so the indicial equation is

$$s(s-1) \quad = \quad 0 \qquad \text{...(F)}$$

giving the value $s=0$ and $s=1.$

Next putting $r=1,$ we have

$$(s+1)s\ C_1 \quad = \quad 0 \qquad \text{...(G)}$$

So for $s=0,$ C_0 and C_1 are both arbitrary. Thus for s = 0, equation (D) becomes

$$C_r \quad = \quad \frac{(n-r+2)(n+r-1)}{r(r-1)}C_{r-2} \qquad \text{...(H)}$$

From equation (H),

$$C_2 \quad = \quad -\frac{n(n+1)}{1.2}C_0$$

$$C_3 \quad = \quad -\frac{(n-1)(n+2)}{3.2}C_1$$

$$C_4 \quad = \quad \frac{-(n-2)(n+3)}{4.3}C_2 = \frac{n(n-2)(n+1)(n+3)}{1.2.3.4}C_0$$

$$C_5 \quad = \quad \frac{-(n-3(n+4)}{5.4}C_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{1.2.3.4.5}C_1$$

....

....

....

and so

Substituting the above values of C_{r0} in equation (B) and using s = 0 value we have

$$F_n(x) \quad = \quad C_0\left[1-\frac{n(n+1)}{\underline{|2}}x^2+\frac{n(n-2)(n+1)(n+3)x^4}{1.2.3.4}\ldots\right]+$$

$$+C_1\left[x-\frac{(n-1)(n+2)}{1.2.3}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{1.2.3.4.5}x^4\ldots\right] \qquad \text{...(I)}$$

By applying ratio test it may be shown that above two series converge in the interval (-1, 1)

As a problem, one can show that for $s = 1$, we can get second series by the above procedure. Since equation (I) contains two arbitrary constants so equation (I) is the general solution of Legendre's equation (A). Now if we give arbitrary coefficients C_0 and C_1 such numerical value that the polynomial (I) becomes equal to one when x is unity, we obtain for n the values 0, 1, 2, 3, ..., and obtain the following system of polynomials:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = (3x^2 - 1)/2$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x);\ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \quad ...(J)$$

The general polynomial $P_n(x)$ which satisfies Legendre's equation is given by the series

$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n (n-r)!(n-2r)!} x^{n-2x} \quad ...(K)$$

Where $N = n/2$ for even n and $N = (n-1)/2$ for n odd.

2.1.2 Solution of Legendre's Equation in Descending Powers of x

The Legendre's Equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad ...(A)$$

Let us assume

$$y = \sum_{r=0}^{\infty} C_r x^{s-r} \quad ...(B)$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} (s-r) C_r x^{s-r-1} \quad ...(C)$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (s-r)(s-r-1)\, C_r x^{s-r-2} \quad ...(D)$$

Substituting in (A), we get

$$(1-x^2)\sum_{r=0}^{\infty} C_r (s-r)(s-r-1)x^{s-r-2} - 2x\sum_{r=0}^{\infty} C_r (s-r)x^{s-r-1} + n(n+1)\sum_{r=0}^{\infty} C_r x^{s-r} = 0$$

or

$$\sum_{r=0}^{\infty} C_r \left\{(s-r)(s-r-1)x^{s-r-2} + \left[n(n+1) - (s-r)(s-r+1)\right]x^{s-r}\right\} = 0 \quad ...(E)$$

Simplifying (E) we have

$$\sum_{r=0}^{\infty} C_r \left\{(s-r)(s-r-1)x^{s-r-2} + (n-s+r)(n+s-r+1)x^{s-r}\right\} \equiv 0 \quad ...(F)$$

Notes

Equation (F) being identity, we can equate to zero the coefficients of various powers of x. Equating to zero the coefficients of highest powers of x i.e. of x^s, we have

$$C_0(n-s)(n+s+1) = 0 \quad ...(G)$$

Since $C_0 \neq 0,$ so the indicial equation is

$$(n-s)(n+s+1) = 0 \quad ...(H)$$

The solutions of equation (H) are

$$s = n \text{ and } s = -n-1 \quad ...(I)$$

Equating to zero the coefficient of the next lower power of x i.e. of $x^{s-1},$ we have

$$a_1(n-s+1)(n+s) = 0 \quad ...(J)$$

So $a_1 = 0,$ as its coefficient is not zero for both $s = n$ and $s = -n-1.$

Again equating to zero the coefficient of the general term i.e. of $x^{k-r},$ we have

$$C_{s-2}(s-r+2)(s-r+1)+(n-s+r)(n+s-r+1)\,C_r = 0$$

or

$$C_r = -\frac{(s-r+2)(s-r+1)}{(n-s+r)(n+s-r+1)}C_{r-2} \quad ...(K)$$

Putting $r = 2$

$$C_2 = -\frac{(s)(s-1)}{(n-s+2)(n+s-1)}C_0$$

Putting $r = 3$

$$C_3 = -\frac{(s-1)(s-2)C_1}{(n-s+3)(n+s-2)} = 0, \text{ as } C_1 = 0$$

Thus

$$C_1 = C_3 = C_5 = ... = 0 \quad (L)$$

Now there are two values for s i.e.

$$s = n \text{ and } s = -n-1 \quad (I)$$

We first take $s = n$, then the general recurrence relation (K) becomes

$$C_r = \frac{-(n-r+2)(n-r+1)}{r(2n-r+1)}C_{r-2} \quad (M)$$

Putting $r = 2, 4, 6, ...$ we obtain the coefficients $C_2, C_4, C_6...$ in terms of C_o i.e.

$$C_2 = -\frac{n(n-1)}{2(2n-1)}C_0$$

$$C_4 = -\frac{(n-2)(n-3)}{n(2n-3)}C_2$$

$$= -\frac{(n-3)(n-2)(n-1)n}{1.2.4.(2n-3)(2n-1)} - C_0$$

$$C_6 = -\frac{(n-4)(n-5)}{6(2n-5)}C_4$$

$$= -\frac{(n-5)(n-4)(n-3)(n-2)(n-1)n}{2.4.6(2n-5)(2n-3)(2n-1)}C_0$$

....

....

....

Substituting these values of C's in equation (B) we have for $s = n$

$$y = C_0\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{x-2} + \frac{n(n-1)(n-2)(n-3)}{2.4\,(2n-1)(2n-3)}x^{n-4}...\right\} \qquad ...(N)$$

$= y_1$ (say)

For the second value of $s = -n-1,$ we have from equation (K)

$$C_r = -\frac{(-n-r+1)(-n-r)}{(2n+r+1)(-r)}C_{r-2}$$

or $$C_r = \frac{(n+r-1)(n+r)}{r(2n+r+1)}C_{r-2} \qquad ...(O)$$

Putting the values of $r = 2, 4, 6, ...$ in equation (O)

$$C_2 = \frac{(n+1)(n+2)}{2(2n+3)}C_0$$

$$C_4 = \frac{(n+3)(n+4)}{4(2n+5)}C_2$$

$$= \frac{(n+4)(n+3)(n+2)(n+1)}{2.4(2n+3)(2n+5)} - C_0$$

....

....

....

Substituting these values of C's in equation (B) we have for $s = -n-1$

$$\text{y} = C_0x^{-n-1} + C_2x^{-n-3} + C_4x^{-n-5} + ...$$

$$= C_0\left\{x^{-n-1} + \frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5} + ...\right\}$$

$$= y_2 \qquad ...(P)$$

Notes

So the two solutions of Legendre's equations form the general solution

$$y = A\,y_1 + B\,y_2 \qquad ...(Q)$$

In particular, if we take constant C_0 to be

$$C_0 = \frac{1.3.5...(2n-1)}{n!}$$

in equation (N), we get the solution

$$P_n(x) = \frac{1.3.5-(2n-1)}{\underline{|n}}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4\,(2n-1)(2n-3)}x^{n-4}...\right\} \qquad ...(R)$$

denoted by $P_n(x)$, and is called Legendre's function of first kind.

Legendre's Functions of the Second Kind

When n is a positive integer and putting the value of C_o, as

$$C_o = \frac{n!}{1.3.5....(2n+1)} \qquad ...(S)$$

in the second solution (P) we get the Legendre's function of the second kind denoted by $Q_n(x)$ i.e.

$$Q_n(x) = \frac{n!}{1.3.5...(2n+1)}\left\{x^{-n-1} + \frac{(n+1)\,(n+2)}{2.(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.\,(2n+3)\,(2n+5)}x^{-n-5} + ...\right\} \qquad ...(T)$$

As is seen from equation (T), $Q_n(x)$ is an infinite or non-terminating series.

Thus the general solution of Legendre's equation is

$$y = A\,P_n(x) + B\,Q_n(x) \qquad ...(U)$$

2.2 Rodrigue's Formula for Legendre Polynomials

An other formula for $P_n(x)$ can be obtained from the Legendre's differential equation. Here we start with

$$u = (x^2 - 1)^n \qquad ...(A)$$

Then
$$\frac{du}{dx} = 2\,nx(x^2-1)^{n-1}$$

Multiplying both sides by $(x^2 - 1)$ and transposing to left hand side, we get

$$(x^2-1)\frac{du}{dx} - 2nx(x^2-1)^n = 0$$

or
$$(x^2-1)\frac{du}{dx} - 2nx\,u = 0$$

Differentiating the above equation with respect to x, we get

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + 2nu + 2nx\frac{du}{dx} = 0$$

$$(1-x^2)\frac{d^2u}{dx^2} + 2(n-1)x\frac{du}{dx} + 2nu = 0 \qquad ...(B)$$

We now apply Leibnitz theorem to differentiate equation r times. Here Leibnitz theorem states that the r^{th} differentiation of product of two functions is given by

$$\frac{d^r}{dx^r}(fg) = f\frac{d^r g}{dx^r} + r\left(\frac{df}{dx}\right)\frac{d^{r-1}}{dx^{r-1}}g + \frac{r(r-1)}{\underline{|2}}\frac{d^2 f}{dx^2}\frac{d^{r-2}g}{dx^{n-2}} + ... \quad ...(C)$$

So differentiating equation (B) r times we get

$$(1-x^2)\frac{d^{r+2}u}{dx^{r+2}} + r\frac{d^{r+1}u}{dx^{r+1}}.(-2x) + \frac{r(r-1)}{2}\frac{d^r u}{dx^r}(-2) + 2(n-1)\left[x\frac{d^{r+1}u}{dx^{r+1}} + r.\frac{d^r u}{dx^r}\right] + 2n\frac{d^r u}{dx^r} = 0$$

or rearranging terms

$$(1-x^2)\frac{d^{r+2}u}{dx^{r+2}} + 2x(n-1-r)\frac{d^{r+1}u}{dx^{r+1}} + \frac{d^r u}{dx^r}\{-r(r-1) + 2r(n-1) + 2n\} = 0 \quad ...(D)$$

Simplifying the above equation and putting

$$u_r = \frac{d^r u}{dx^r}, \quad ...(E)$$

We get

$$(1-x^2)\frac{d^2 u_r}{dx^2} + 2x(n-1-r)\frac{du_r}{dx^r} + (r+1)(2n-r)u_r = 0$$

We now put $r = n$ and get

$$(1-x^2)\frac{d^2 u_n}{dx^2} + 2x(-1)\frac{du_n}{dx} + (n+1)(n)u_n = 0$$

This is Legendre's equation. Hence for $r = x$, u_n satisfies Legendre's equation. Thus the Legendre's polynomial are given by

$$P_n(x) = \frac{d^n}{dx^n}(x^2-1)^n(C) \quad ...(F)$$

Where C is a constant. To evaluate C we compare the coefficients of x^n on both sides of (F) i.e.

$$\frac{(2n)!\,x^n}{2^n(n!)^2} = C\frac{d^n}{dx^n}x^{2n} = C(2n)(2n-1)...(n+1)x^{2n}$$

$$= C\frac{(2n)!}{n!}x^n$$

Thus

$$\frac{1}{(n!)2^n} = C$$

Thus

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

This is Rodrigue's formula for the Legendre's polynomials. We can again find a few Legendre polynomials from this formula.

Notes

Self Assessment

1. Find

 $P_1(x), P_2(x), P_3(x)$

 from Rodrigue formula

2.3 Generating Function for Legendre Polynomials

In the following we will show that $P_n(x)$ is the coefficient of h^n in the expansion of

$$(1-2xh+h^2)^{\frac{-1}{2}}$$

for $|x|\leq|, |h|<|$

i.e.
$$(1-2xh+h^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x) \quad \text{...(A)}$$

Now
$$(1-2hx+h^2)^{\frac{-1}{2}} = [1-h(2x-h)]^{\frac{-1}{2}}$$

$$= 1-\frac{1}{2}(-h)(2x-h)+\frac{1}{2}\cdot\frac{3}{2}\,\frac{1}{2}h^2(2x-h)^2+\ldots$$

$$+\ldots+\frac{1.3\ldots(2n-3)}{2.4.6(2n-2)}h^{n-1}(2x-h)^{n-1}+$$

$$+\frac{1.3.\ \ldots(2n-1)}{2.4.6\ldots.(2n)}h^n(2x-h)^n+\ldots$$

Therefore the coefficients of h^n are

$$= \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}(2x)^2+\frac{1.3.5.\ \ldots(2n-3)}{2.4.6\ \ldots(2n-2)}(2x)^{n-2}n-1_{C_1}+\frac{1.3.5\ldots(2n-5)}{2.4.6\ldots(2n-4)}n-2_{C_2}(2x)^{n-4}+\ \text{...(B)}$$

$$= \frac{1.3.5\ldots(2n-1)}{\underline{|n}}\left\{x^n-\frac{2n}{2n-1}(n-1)\frac{x^{n-2}}{2^2}+\frac{(2n)(2n-2)(n-2)(n-3)}{(2n-1)(2n-3)\,\underline{|2}\,2^4}+\ldots\right\}$$

$$= \frac{1.3.5\ldots(2n-1)}{\underline{|n}}\left\{x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-2)}x^{n-4}+\ldots\right\}$$

$$= P_n(x) \quad \text{...(C)}$$

Thus

$$(1-2xh+h^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Where $P_n(x)$ is given by

$$P_n(x) = \frac{1.3.5\ldots(2n-1)}{\underline{|n}}\left\{x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{(n-2)(n-1)\,n\,(n-3)}{2.4.(2n-1)(2n-2)}x^{n-4}-\ldots\right\} \quad \text{...(D)}$$

Notes

Also it can be written as

$$P_n(x) = \sum_{r=0}^{n} \frac{(-1)^r (2n-2r)!}{2^n (n-r)!(n-2r)!} x^{n-2r} \quad \text{...(E)}$$

Example 1: From the relation

$$(1-2xh+h^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Obtain $P_0(x), P_1(x), P_2(x), P_3(x)$ and $P_4(x)$.

i.e.

Prove

$$P_0(x) = 1,\ P_1(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2-1),\ P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3).$$

Example 2: Express $P(x) = x^4+2x^3+2x^2-x-3$ in terms of Legendre's polynomials.

Solution: From Example 1, we have

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{(3x^2-1)}{2},$$

$$P_3(x) = \frac{(5x^3-3x)}{2}, P_4(x) = \frac{35x^4-30x^2+3}{8}$$

from $P_4(x) = \frac{1}{8}(35x^4-30x^2+3),$

$$x^4 = \frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35},$$

from $P_3(x) = \frac{1}{2}(5x^3-3x),\ x^3 = \frac{2}{5}P_3(x)+\frac{3}{5}x,$

from $P_2(x) = \frac{1}{2}(3x^2-1),\ x^2 = \frac{2}{5}P_2(x)+\frac{1}{3}$

and $x = P_1(x);\quad 1 = P_0(x)$

Substituting these values, we have

$$P(x) = \frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35}+2x^3+2x^2-x-3$$

$$= \frac{8}{35}P_4(x)+2x^3+\frac{20}{7}x^2-x-\frac{108}{35}$$

Notes

$$= \frac{8}{35}P_4(x)+2\left[\frac{2}{5}P_3(x)+\frac{3}{5}x\right]+\frac{20}{7}x^2-x-\frac{108}{35}$$

$$= \frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{20}{7}x^2+\frac{1}{5}x-\frac{108}{35}$$

$$= \frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{20}{7}\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right]+\frac{1}{5}x-\frac{108}{35}$$

$$= \frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{40}{21}P_2(x)+\frac{1}{5}x-\frac{224}{105}$$

$$= \frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{40}{21}P_2(x)+\frac{1}{5}P_1(x)-\frac{224}{105}P_0(x)$$

Example 3: Prove $1+\frac{1}{3}P_1(\cos\theta)+\frac{1}{3}P_2(\cos\theta)+...$

$$+... = \log\left[\left(1+\sin\frac{\theta}{2}\right)\Big/\left(\sin\frac{\theta}{2}\right)\right]$$

Solution: From the generating function, we have

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2hx+h^2)^{-1/2} \quad ...(i)$$

Integrating w.r.t. h from 0 to 1, we get

$$\sum_{n=0}^{\infty}\int_0^1 h^n P_n(x)\,dh = \int_0^1 \frac{dh}{\sqrt{(1-2hx+h^2)}} \quad ...(ii)$$

Replacing x by cos θ on both sides, (ii) gives

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\int_0^1 h^n\,dh = \int_0^1 \frac{dh}{\sqrt{(1-2h\cos\theta+h^2)}}$$

or
$$\sum_{n=0}^{\infty} P_n(\cos\theta)\left[\frac{h^{n+1}}{n+1}\right]_0^1 = \int_0^1 \frac{dh}{\sqrt{\left[(h-\cos\theta)^2+\sin^2\theta\right]}}$$

or
$$\sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{n+1} = \log(h-\cos\theta)+\sqrt{\left[(h-\cos\theta)^2+\sin^2\theta\right]_0}$$

$$= \log\{(1-\cos\theta)+\sqrt{[(1-\cos\theta)^2+\sin^2\theta]}\}-\log(1-\cos\theta)$$

$$= \log\{(1-\cos\theta)+\sqrt{[2(1-\cos\theta)\}}]-\log(1-\cos\theta)$$

$$= \log\frac{(1-\cos\theta)+\sqrt{2}\sqrt{(1-\cos\theta)}}{(1-\cos\theta)}$$

$$= \log\frac{\sqrt{\{(1-\cos\theta)\}}\sqrt{\{(1-\cos\theta)\}}+\sqrt{2}\sqrt{\{(1-\cos\theta)\}}}{\sqrt{\{(1-\cos\theta)\}}\sqrt{\{(1-\cos\theta)\}}}$$

Notes

$$= \log \frac{\sqrt{[(1-\cos\theta)]}+\sqrt{2}}{\sqrt{[(1-\cos\theta)]}}$$

$$= \log \frac{\sqrt{\left\{\left(2\sin^2\frac{\theta}{2}\right)\right\}}+\sqrt{2}}{\sqrt{\left\{\left(2\sin^2\frac{\theta}{2}\right)\right\}}}$$

$$= \log \frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\therefore \qquad \frac{P_0(\cos\theta)}{1}+\frac{1}{2}P_1(\cos\theta)+\frac{1}{3}P_2(\cos\theta)+\ldots = \log \frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

or $$1+\frac{1}{2}P_1(\cos\theta)+\frac{1}{3}P_2(\cos\theta)+\ldots = \log \frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} \qquad [\because P_0(\cos\theta)=1]$$

Example 4: Show that

$$(a)\ P_n(1) = 1$$

$$(b)\ P_n(-x) = (-1)^n P_n(x)$$

Hence deduce that $P_n(-1) = (-1)^n$.

Solution:

(a) We know that

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$$

Putting $x = 1$

$$\sum_{n=0}^{\infty} h^n P_n(1) = (1-2h+h^2)^{-1/2}$$

$$= (1-h)^{-1}$$

$$= 1+h+h^2+\ldots+h^n+\ldots$$

$$= \sum_{n=0}^{\infty} h^n$$

Equating the coefficients of h^n, we get $P_n(1)=1$.

(*b*) we have

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Notes

Now, $$(1+2xh+h^2)^{-1/2} = \{1-2x(-h)+(-h)^2\}^{-1/2}$$

$$= \sum_{n=0}^{\infty} (-h)^n P_n(x)$$

$$= \sum_{n=0}^{\infty} (-1)^n h^n P_n(x) \quad ...(i)$$

Again $$(1+2xh+h^2)^{-1/2} = \{1-2(-x)+h+h^2\}^{-1/2}$$

$$= \sum_{n=0}^{\infty} h^n P_n(-x) \quad ...(ii)$$

From (i) and (ii) we have

$$= \sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x)$$

Equating the coefficients of h^n from both sides, we get

$$P_n(-x) = (-1)^n P_n(x).$$

Deduction: Putting $x = 1$, we have

$$P_n(-1) = (-1)^n P_n(1)$$

$$= (-1)^n [\because P_n(1) = 1].$$

Example 5: Prove that $(a)\, P_n'(1) = \frac{1}{2} n(n+1)$

$$(b)\ P_n'(-1) = (-1)^{n-1} \frac{1}{2} n(n+1)$$

Solution: $P_n(x)$ satisfies Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \text{ putting } y = P_n(x)$$

$$\because \quad (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \quad ...(i)$$

(a) Putting $x = 1$, in (i) we have

$$-2P_n''(1) + n(n+1)P_n(1) = 0$$

$$\therefore \quad P_n'(1) = \frac{1}{2} n(n+1) P_n(1)$$

$$= \frac{1}{2} n(n+1) \quad [\because P_n(1) = 1].$$

(b) Putting $x = -1$ in (i), we get

$$2P_n'(-1)+n(n+1)P_n(-1) = 0$$

or $$P_n'(-1) = -\frac{1}{2}n(n+1)P_n(-1)$$

$$= (-1)^{n-1}.\frac{1}{2}n(n+1) \qquad [\because P_n(-1)=(-1)^n].$$

Example 6: Prove that $P_n(0)=0,$ for n odd and

$$P_n(0) = \frac{(-1)^{n/2}n!}{2^n\{n/2!\}^2}, \text{ for } n \text{ even.}$$

Solution:

(i) We know that

$$P_n(x) = \frac{1.3.5...(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - ...\right]$$

when n is odd, say $n=(2m+1),$ then

$$P_{2m+1}(x) = \frac{1.3.5...\{2(2m+1)-1\}}{(2m+1)!}\times\left[x^{2m+1} - \frac{(2m+1)(2m+1-1)}{2.\{2(2m+1)-1\}}x^{2m+1-2} + ...\right]$$

Putting $x=0,$ we get $P_{2m+1}(0)=0,$

i.e., $P_n(0) = 0$ when n is odd.

Also, we have

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$$

or $$\sum_{n=0}^{\infty} h^n P_n(0) = (1+h^2)^{-1/2} = \left\{1-(-h)^2\right\}^{-1/2}$$

$$= 1+\frac{1}{2}.(-h^2)+\frac{1.3}{2.4}(-h^2)^2+\frac{1.3.5}{2.4.6}(-h^2)^3+...+\frac{1.3.5...(2r-1)}{2.4...2r}(-h^2)^r+...$$

Hence all powers of h on the R.H.S. are even.

Equating the coefficient of h^{2m} on both sides, we have

$$P_{2m}(0) = \frac{1.3.5...(2m-1)}{2.4.6...2m}(-1)^m$$

$$= (-1)^m\frac{(2m)!}{2^{2m}(m!)^2}$$

i.e. when $n = 2\,m$, then

$$P_n(0) = \frac{(-1)^{n/2}n!}{2^n\{(n/2)!\}^2}$$

Notes

Example 7: Prove that $(1-2xz+z^2)^{\frac{-1}{2}}$ is a solution of the equation

$$z\frac{\partial^2(z\nu)}{\partial z^2}+\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial\nu}{\partial x}\right\} = 0$$

Where

$$\nu = (1-2xz+z^2)^{-1/2}$$

Solution: Let

$$\nu = (1-2xz+z^2)^{-1/2}=\sum_{n=0}^{\infty}z^nP_n$$

or

$$z\nu = \sum_{n=0}^{\infty}z^{n+1}P_n$$

∴

$$z\frac{\partial^2}{\partial z^2}(z\nu) = \sum_{n=0}^{\infty}(n+1)nz^nP_n.$$

Also

$$\frac{\partial\nu}{\partial x} = \sum_{n=0}^{\infty}z^nP_n'$$

∴

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial\nu}{\partial x}\right\} = \frac{\partial}{\partial x}\left[(1-x^2)\sum_{n=0}^{\infty}z^nP_n'\right]$$

$$= (1-x^2)\sum_{n=0}^{\infty}z^nP_n''-2x\sum_{n=0}^{\infty}z^np_n'$$

Substituting this in the L.H.S. of the given equation, we get

$$z.\frac{\partial^2(z\nu)}{\partial z^2}+\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial\nu}{\partial x}\right\} = \sum_{n=0}^{\infty}[(n+1)nz^nP_n+(1-x^2)z^nP_n''-2xz^nP_n']$$

$$= \sum_{n=0}^{\infty}z^n[(1-x^2)P_n''-2xP_n'+n(n+1)P_n]$$

$= 0$ since P_n is a solution of Legendre's equation.

Self Assessment

2. Show that

$$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)z^x.$$

Laplace's First Integral for $P_n(x)$: when n is a positive integer. Show that

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x\pm\sqrt{(x^2-1)}\cos\phi\right]^x d\phi.$$

Proof: From integral calculus, we have

$$\int_0^{\pi}\frac{d\phi}{a\pm b\cos\phi} = \frac{\pi}{\sqrt{\left[(a^2-b^2)\right]}},\ \text{where } a^2>b^2 \qquad \text{...(i)}$$

Putting $a = 1 - hx$ and $b = h\sqrt{(x^2-1)}$

so that $$a^2 - b^2 = (1-hx)^2 - h^2(x^2-1) = 1 - 2xh + h^2$$

Thus, we have from (i)

$$\pi(1-2xh+h^2)^{-1/2} = \int_0^{\pi}\left[1 - hx \pm h\sqrt{(x^2-1)}\cos\phi\right]^{-1} d\phi$$

$$= \int_0^{\pi}\left[1 - h\left(x \pm \sqrt{x^2-1}\right)\cos\phi\right]^{-1} d\phi$$

$$= \int_0^{\pi}(1-ht)^{-1} d\phi \qquad \text{where } t = x \pm \sqrt{x^2-1}\cos\phi$$

or $$\pi\sum h^n P_n(x) = \int_0^{\pi}(1 - ht + h^2t^2 + ... + h^n t^n + ...)\, d\phi$$

Equating coefficient of h^n we get

$$\pi p_n(x) = \int_0^{\pi} t^n d\phi = \int_0^{\pi}\left[x \pm \sqrt{x^2-1}\cos\phi\right]^n d\phi$$

$\therefore$ $$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{(x^2-1)}\cos\phi\right]^n d\phi$$

Deductions

(i) Putting $x = \cos\theta$ in above relation, we get

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^n d\phi.$$

(ii) If we take n = 1 and +ve sign, then we get

$$P_1(x) = \frac{1}{\pi}\int_0^{\pi}\left[x + \sqrt{[(x^2-1)]}\cos\phi\right] d\phi.$$

Laplace's Second Integral for $P_n(x)$: When n is a Positive Integer. Show that

$$\boldsymbol{P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left[x \pm \sqrt{(x^2-1)}\cos\phi\right]^{n+1}}}$$

Proof: From integral calculus, we have

$$\int_0^{\pi}\frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{(a^2-b^2)}}, \text{ where } a^2 > b^2. \qquad ...(i)$$

Let $a = xh - 1$ and $b = h\sqrt{(x^2-1)}$

so that $$a^2 - b^2 = 1 - 2xh + h^2$$

Notes

By putting these values in (i) we have

$$\pi(1-2xh+h^2)^{-1/2} = \int_0^{\pi}\left[-1+xh\pm h\sqrt{(x^2-1)}\cos\phi-1\right]^{-1}d\phi$$

or $$\frac{\pi}{h}\left[1-2x.\frac{1}{h}+\frac{i}{h^2}\right]^{-1/2} = \int_0^{\pi}\left[h\{x\pm\sqrt{(x^2-1)}\cos\phi-1\right]^{-1}d\phi$$

or $$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^{\pi}(t-1)^{-1}d\phi \qquad \text{where } t=h\left\{x\pm\sqrt{(x^2-1)}\cos\phi\right\}$$

$$= \int_0^{\pi}\frac{1}{t}\left[1-\frac{1}{t}\right]^{-1}d\phi$$

$$= \int_0^{\pi}\frac{1}{t}\left[1+\frac{1}{t}+\frac{1}{t^2}+\ldots+\frac{1}{t^n}+\ldots\right]d\phi$$

$$= \int_0^{\pi}\left[\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots+\frac{1}{t^{n+1}}\right]d\phi$$

$$= \int_0^{\pi}\sum_{n=0}^{\infty}\frac{1}{t^{n+1}}d\phi$$

$$= \sum_{n=0}^{\infty}\int_0^{\pi}\frac{d\phi}{h^{n+1}\left[x\pm\sqrt{(x^2-1)}\cos\theta\right]^{n+1}}$$

Equating the coefficient of $\frac{1}{h^{n+1}}$, we get

$$\pi P_n(x) = \int_0^{\pi}\frac{d\phi}{\left\{x\pm\sqrt{(x^2-1)}\cos\phi\right\}^{n+1}}\ ]$$

$$P_n(x) = \frac{1}{x}\int_0^{\pi}\frac{d\phi}{\left\{x\pm\sqrt{(x^2-1)}\cos\phi\right\}^{n-1}}$$

Deductions: Replacing n by $-(n+1)$ in above relation, we get

$$P_{-(n+1)}(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left\{x\pm\sqrt{(x^2-1)}\cos\phi\right\}^{-n}}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left\{x\pm\sqrt{(x^2-1)}\cos\phi\right\}^{x}d\phi$$

$$= P_n(x)$$

$\therefore$ $$P_n(x) = P_{-n-1}(x)$$

2.4 Recurrence Relations for Legendre Polynomials

Notes

I. ***Prove that***

$$\mathbf{(2n+1)xP_n(x) = (n+1)P_{n+1}(x)+nP_{n-1}(x)}$$

We now have from generating function

$$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

Differentiating both sides w.r.t. h we have

$$-\frac{1}{2}(-2x+2h)(1-2hx+h^2)^{-3/2} = \sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

Multiplying both sides by $(1-2hx+h^2)$; we get

$$(x-h)(1-2hx+h^2)^{-1/2} = (1-2hx+h^2)\sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

or

$$(x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2hx+h^2)\sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

Expanding

$$(x-h)\left[P_0(x)+h\,P_1(x)+h^2\,P_2(x)+\ldots\right] \equiv (1-2hx+h^2)\left[P_1(x)+2h\,P_2(x)+3h^2\,P_3(x)+\ldots\right]$$

Comparing the coefficients of h^n on both sides, we have

$$xP_n(x)-P_{n-1}(x) = (n+1)P_{n+1}(x)-2xn\,P_n(x)+(n-1)P_{n-1}(x)$$

Rearranging terms, we have

$$xP_n(x)+2xn\,P_n(x) = (n+1)\,P_{n+1}(x)+(n-1+1)P_{n-1}(x)$$

or

$$\mathbf{(2n+1)\times P_n(x) = (n+1)P_{n+1}(x)+nP_{n-1}(x)}$$

II. ***Prove that***

$$\mathbf{nP_n(x) = xP_n'(x)-P_{n-1}'(x)}$$

Proof:

Consider the relation

$$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^x P_n(x) \qquad \ldots(A)$$

Differentiating w.r.t. h, we have

$$\left(-\frac{1}{2}\right)(-2x+2h)(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

Notes

or

$$(x-h)(1-2hx+h^2)^{-3/2} = \sum_{n=0}^{\infty} nh^{n-1}P_n(x) \quad \text{...(B)}$$

Differentiating (A) again by x, we have

$$(-2h)(-1/2)(1-2hx+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x)$$

or

$$h(1-2hx+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x) \quad \text{...(C)}$$

Multiplying (B) by h and (C) by $(x-h)$, and subtracting we get

$$h\sum_{n=0}^{\infty} n\, h^{n-1}P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) \quad \text{...(D)}$$

Now comparing the coefficients of h^n on both sides we have

$$\mathbf{(n)P_n(x) = xP_n'(x) - P_{n-1}'(x)}$$

which is the recurrence relation II

III. ***Prove that***

$$\mathbf{(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)}$$

Proof:

From recurrence relation I

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

Differentiating w.r.t. x we have

$$(2n+1)xP_n'(x) + (2n+1)P_n(x) = (n+1)P_{n+1}'(x) + nP_n'(x) \quad \text{...(A)}$$

From recurrence formula II

$$xP_n'(x) = nP_n(x) + P_{n-1}'(x) \quad \text{...(B)}$$

Substituting in (A) we have

$$(2n+1)\left[nP_n(x) + P_{n-1}'(x)\right] + (2n+1)P_n(x) = (n+1)P_{n+1}'(x) + nP_n'(x)$$

$$(2n+1)\left[(n+1)P_n(x) + P_{n-1}'(x)\right] = (n+1)P_{n+1}'(x) + np_{n-1}'(x)$$

or rearranging

$$(2n+1)(n+1)P_n(x) = (n+1)P_{n+1}'(x) - (n+1)P_{n-1}'(x)$$

Removing common factor we have

$$\mathbf{(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)}$$

Notes

Self Assessment

3. ***Prove***

$$(n+1)P_n(x) = P'_{n+1}(x) - xP_n(x)$$

4. ***Prove that***

$$(1-x)^2 P'_n(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$$

2.5 Orthogonal Properties of Legendre Polynomials

Prove that

(i) $\int_{-1}^{+1} P_m(x) P_n(x)\, dx = 0$ if $m \neq n$ and

(ii) $\int_{-1}^{+1} [P_n(x)]^2 dx = \dfrac{2}{2n+1}$

Proof:

From Legendre equation $P_n(x)$ being solution of it so we have

$$(1-x^2)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dt} + n(n+1)P_n(x) = 0$$

or

$$\frac{d}{dx}\left[(1-x)^2\frac{dP_n(x)}{dx}\right] + n(n+1)\,P_n(x) = 0 \quad \text{...(A)}$$

In the same way, we have

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_m(x)}{dx}\right] + m(m+1)\,P_m(x) = 0 \quad \text{...(B)}$$

Multiplying equation (A) by $P_m(x)$ and (B) by P_n and subtracting

$$P_m\frac{d}{dx}\left[(1-x^2)\frac{d\,P_n(x)}{dx}\right] - P_n\left\{\frac{d}{dx}\left[(1-x^2)\frac{d\,P_m}{dx}(x)\right]\right\} + [n(n+1) - m(m+1)]\,P_m(x)\,P_n(x) = 0 \quad \text{...(C)}$$

Integrating equation (C) between the limits –1 to 1, we have

$$\int_{-1}^{+1} P_m(x)\,\frac{d}{dx}\left[(1-x^2)\frac{d\,P_n(x)}{dx}\right]dx - \int_{-1}^{+1} P_n(x)\frac{d}{dx}\left[(1-x^2)\frac{d\,P_m}{dx}\right]dx + (n-m)(n+m+1)$$

$$\times\int_{-1}^{+1} P_m(x)P_n(x)dx = 0$$

Integrating by parts we have

$$\left[P_m(x)(1-x^2)\frac{d\,P_n}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1}\frac{dP_m(x)}{dx}(1-x^2)\frac{d\,P_n(x)}{dx}dx - \left[P_n(x)(1-x^2)\frac{dP_m(x)}{dx}\right]_{-1}^{+1} +$$

$$+\int_{-1}^{+1}\frac{d}{dx}P_n(x)(1-x^2)\frac{d\,P_m(x)}{dx}dx + (n-m)(n+m+1)\int_{-1}^{+1} P_m(x)P_n(x)\,dx = 0$$

Notes

or

$$0-\int_{-1}^{+1}\frac{dP_m(x)}{dx}\frac{dP_n(x)}{dx}(1-x^2)\,dx-0+\int_{-1}^{+1}\frac{d\,P_n(x)}{dx}\frac{dP_m(x)}{dx}(1-x^2)dx+$$

$$+(n-m)(n+m+1)\int_{-1}^{+1}P_m(x)\,P_n(x)dx = 0$$

or

$$(n-m)(n+m+1)\int_{-1}^{+1}P_m(x)\;P_n(x)dx = 0$$

Thus

$$\int_{-1}^{+1}P_m(x)P_n(x)dx = 0 \quad \text{for } m\neq n \qquad \text{...(D)}$$

This proves the first part.

To prove (ii)

We have

$$(1-2xh+h^2)^{-1} = (1-2xh+h^2)^{-1/2}.(1-2x+h^2)^{-1}$$

$$= \left[\sum_{n=0}^{\infty}h^n P_n(x)\right]\left[\sum_{m=0}^{\infty}h^m\,P_m(x)\right]$$

$$= \sum_{n=0}^{\infty}h^{2n}\,P_n^2(x)+2\sum_{\substack{m=0\\n=0\\m\neq n}}^{\infty}h^{m+n}\,P_n(x)\,P_m(x) \qquad \text{...(E)}$$

Integrating between the limits −1 to +1, we have

$$\int_{-1}^{+1}\sum_{n=0}^{\infty}h^{2n}P_n^2(x)\,dx+2\int_{-1}^{+1}\sum_{\substack{m=0\\n=0\\m\neq n}}^{\infty}h^{m+n}P_n(x)P_m(x)dx = \int_{-1}^{+1}\frac{dx}{(1-2hx+h^2)}$$

Thus

$$\sum_{n=0}^{\infty}h^{2n}\int_{-1}^{+1}P_n^2(x)\,dx = \int_{-1}^{+1}\frac{dx}{(1-2xh+h^2)^{1/2}} \qquad \text{...(F)}$$

$$= -\frac{1}{2h}\log\left(\frac{1-h}{1+h}\right)^2=\frac{1}{h}\log\left(\left|\frac{1+h}{1-h}\right|\right)$$

Expanding the R.H.S. in powers of h, we have

$$\sum_{n=0}^{\infty}h^{2n}\int_{-1}^{+1}P_n^2(x)\,dx = \frac{1}{h}\left\{h+\frac{h^2}{2}+\frac{h^3}{3}+\frac{h^4}{4}+...-\left(-h+\frac{h^2}{2}-\frac{h^3}{3}+\frac{h^4}{4}...\right)\right\}$$

$$= \frac{2}{h}\left\{h+\frac{h^3}{3}+\frac{h^5}{5}+\frac{h^7}{7}+...\right\}$$

$$= 2\sum_{n=0}^{\infty}h^{2n}\left(\frac{1}{2n+1}\right)$$

Notes

So comparing the coefficients of h^{2n} on both sides we have

$$\int_{-1}^{+1} P_n^2(x)\,dx = \frac{2}{2n+1} \quad \text{...(G)}$$

Thus
$$\int_{-1}^{+1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{2}{2n+1} & \text{for } m = n \end{cases}$$

From the properties of Legendre's polynomials we can prove certain results.

2.6 Expansion of a *f*(*x*) in terms of Legendre's Polynomials

Since $P_0(x), P_1(x), P_2(x)...$ a set Legendre polynomials are orthogonal in the range of $x, (-1,1)$, any function $f(x)$ can be expressed in terms an expansion series involving $P_n(x)$ i.e.

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x) \quad \text{for } x \text{ in the range } -1 \leq x \leq 1 \quad \text{...(i)}$$

Multiplying equation (i) by $P_m(x)$ and integrating over the limit −1 to 1, we have

$$\int_{-1}^{+1} f(x)P_m(x)\,dx = \sum_{n=0}^{\infty} C_n \int_{-1}^{+1} P_m(x)P_n(x)\,dx \quad \text{...(ii)}$$

Now

$$\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases} \quad \text{...(iii)}$$

Substituting in (i) we have

$$\int_{-1}^{+1} f(x)P_m(x)\,dx = C_m\left(\frac{2}{2m+1}\right) \quad \text{...(iv)}$$

Example: Expand $f(x)$ in the form

$$\sum_{r=0}^{\infty} C_r P_r(x),$$

Where

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases} \quad \text{...(i)}$$

We know

$$f(x) = \sum_{r=0}^{\infty} C_r P_r(x) \quad \text{...(ii)}$$

Notes

where

$$C_r = \left(\frac{2r+1}{2}\right)\int_{-1}^{+1} f(x)\,P_r(x)\,dx$$

$$\therefore \qquad C_r = \frac{(2r+1)}{2}\int_0^1 1.\,P_r(x)\,dx \qquad \text{for } r = 1, 2, \ldots \qquad \ldots\text{(iii)}$$

Putting $r = 0, 1, 2, 3, \ldots.$

$$C_0 = \frac{1}{2}\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 1.\,dx = \frac{1}{2}x\Big|_0^1 = \frac{1}{2}$$

$$C_1 = \frac{3}{2}\int_0^1 P_1(x)\,dx = \frac{3}{2}\int_0^1 x\,dx = \frac{3}{2}.\frac{x^2}{2}\Big|_0^1 = \frac{3}{4}$$

$$C_2 = \frac{5}{2}\int_0^1 P_2(x)\,dx$$

$$= \frac{5}{2}\int_0^1 \frac{1}{2}(3x^2 - 1)\,dx$$

$$= \frac{5}{4}\left[\frac{\cancel{3}x^3}{\cancel{3}} - x\right]_0^1 = 0$$

$$C_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx$$

$$= \frac{7}{2}\int_0^1 \frac{1}{2}(5x^3 - 3x)\,dx$$

$$= \frac{7}{4}\left[\frac{5x^4}{4} - \frac{3x^2}{2}\right]_0^1$$

$$= \frac{7}{4}\left[\frac{5}{4} - \frac{3}{2}\right]$$

$$= \frac{7}{4}\left[\frac{5-6}{4}\right] = -\frac{7}{6}$$

So $$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \ldots$$

Self Assessment

Notes

5. Obtain the first three terms in the expansion of the function

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$$

in terms of Legendre's Polynomials and show that

$$f(x) = \frac{1}{4}P_0(x) + \frac{3}{4}P_1(x) + \frac{25}{48}P_2(x) + \dots$$

Prove that all the roots of $P_n(x) = 0$ are distinct

Solution: If the roots of $P_n(x) = 0$ are not all different, then at least two of them must be equal.

Let α be their common value. Then

$$P_n(\alpha) = 0 \qquad \text{(i)}$$

and $$P_n'(\alpha) = 0 \qquad \left[\text{Here} \frac{dp}{dx} = P'\right]$$

Since $P_n(x)$ is the solution of Legendre's equation

$$(1-x^2)\frac{d^2}{dx^2}P_n(x) - 2x\frac{d\,P_n(x)}{dx} + n(n+1)P_n(x) = 0 \qquad \text{...(ii)}$$

Differentiating (ii) r times by Leibnitz's theorem, we get

$$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x) - 2x^r c_1 \frac{d^{n+1}}{dx^{n+1}}P_n(x) - 2^r c_2 \frac{dr}{dx^r}P_n(x)$$

$$-2\left[x\frac{d^{r+1}}{dx^{r+1}}P_n(x) + 1.c_1^r \frac{d^r}{dx^r}P_n(x)\right] + n(n+1)\frac{dr}{dx^r}P_n(x) = 0$$

or $$(1-x)^2\frac{d^{r+2}}{dx^{r+2}}P_n(x) - 2x\left(r_{C_1}+1\right)\frac{d^{r+1}}{dx^{r+1}}P_n(x) - \left[2r_{C_2} + 2r_{C_1} - n(n+1)\right]\frac{d^r P_n(x)}{dr} = 0 \qquad \text{...(iii)}$$

Putting $r = 0,\ x = \alpha$

$$(1-\alpha^2)\left[\frac{d^2}{dx^2}P_n(x)\right]_{x=\alpha} - 2\alpha\left[\frac{d}{dx}P_n(x)\right]_{x=\alpha} + n(n+1)P_n(\alpha) = 0 \qquad \text{...(iv)}$$

Since $\left.\frac{d}{dx}P_n(x)\right|_{x=\alpha} = 0$ and $P_n(\alpha) = 0, so$

$$\left[\frac{d^2 P_n(x)}{dx^2}\right]_{x=\alpha} = 0 \qquad \text{...(v)}$$

Notes

Similarly putting r = 1, 2, ... in (iii) and simplifying stepwise, we have

$$P_n'''(\alpha) = 0 = P_n^{iv}(\alpha) = 0 = 0... = P_n^n(\alpha) = 0 \quad ...(vi)$$

But since

$$P_n^n(x)\Big|_{x=\alpha} = \frac{1.3...(2n-1)}{\lfloor n}.n! \neq 0 \quad ...(vii)$$

Therefore our assumption that $P_n(\alpha) = 0$ has a repeated root is not correct.

Hence all the roots of $P_n(x) = 0$ are distinct.

Example: Find the roots of $P_2(x) = 0$

As $$P_2(x) = 0 = \frac{1}{2}(3x^2 - 1)$$

$$P_2(\alpha) = 0 = \frac{1}{2}(3x^2 - 1)$$

$$\therefore \quad 3\alpha^2 = 1$$

$$\alpha = \pm 1/\sqrt{3}$$

So the roots are

$$\alpha_1 = -1/\sqrt{3}, d_2 = \frac{1}{\sqrt{3}}$$

Self Assessment

6. Show that the roots of $P_3(x) = 0$ are

$$-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}$$

2.7 Summary

- Legendre's Differential equation is obtained from Laplace equation in spherical polar co-ordinates.
- Legendre's Differential equation has $x = \pm 1,$ as well as $x = \infty$ as regular singular points.
- So Legendre's Differential equation is solved as a power series.
- It is found that Legendre polynomial $P_n(x)$ is a finite power series having x^n as the highest power of x.
- The generating function for $P_n(x)$ is found to be $(1 - 2h + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

- Rodrigue's formula for Legendre polynomials help us to find a few $P_n(x)$ i.e. $P_0(x), P_1(x), P_2(x), \ldots\ldots$.
- Orthogonal properties of $P_n(x)$ are obtained. It is seen that $\{P_n(x)\} n = 0, 1, \ldots$ form a complete set in the range $-1 \le x \le 1$.
- Just as Fourier series we show that a function in the range $-1 \le x \le 1$ is expanded in terms of $P_n(x)$'s.

2.8 Keywords

Regular singular points of Legendre equations are $x = \pm 1$ and $x = \infty$.

Legendre polynomial $P_n(x)$ is a terminating series with highest power of x as x^n.

Generating function of the Legendre polynomial is $(1 - 2hx + h^2)^{-1} = \sum_{n=0}^{\infty} h^n P_n(x)$

Rodrigue's formula has been obtained and certain properties of $P_n(x)$ are obtained in a straight forward manner.

Recurrence relations between various Legendre's polynomials obtained are useful in expressing higher polynomials in terms of $P_0(x)$ and $P_1(x)$.

Orthogonality properties of the Legendre Polynomials obtained, help us in evaluating certain integrals easily.

2.9 Review Questions

Show that

1. $P_n'(x) - P_{n-2}(x) = (2n-1) P_{n-1}(x)$

2. $\int_{-1}^{+1} x \, P_n(x) \, P_{n-1}(x) \, dx = \dfrac{2n}{4n^2 - 1}$

3. $x P_9'(x) = P_8'(x) + 9 P_9(x)$

4. Show that all the roots of $P_n(x) = 0$ are real and lie between –1 and +1.

5. Prove that

$$x^4 - 3x^2 + x \equiv \frac{8}{35} P_4(x) + \frac{6}{35} P_2(x) + P_1(x)$$

Notes

2.10 Further Readings

Books

Piaggio H.T.H., Differential Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 3: Hermite Polynomials

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Solve second order differential equation like Hermite equation.
- Familiarize yourself with the properties of Hermite Polynomials through generating function.
- Obtain certain relations involving Hermite polynomials with the help of Rodrigue formula.
- Solve certain integrals. You can express any function $f(x)$ in terms of Hermite polynomials $H_n(x)$.
- Relate some Hermite polynomials in terms of others with the help of recurrence relations.

Introduction

In the previous two units you have learnt the method of Frobenius in solving second order differential equations in power series. This method will help us to solve Hermite differential equation. In this unit we will be able to solve the equation for $-\infty < x < \infty$ range.

Just as the generating functions were introduced in the previous chapter, here in this chapter also it will be introduced for Hermite polynomials. Also orthogonal properties and recurrence relations are very important in understanding the properties of Hermite polynomials.

3.1 Power Series Solution of Hermite Polynomials

Consider the following equation, containing a parameter λ,

$$\frac{d}{dx}\left(e^{-x^2}\frac{dy}{dx}\right)+2\lambda\, e^{-x^2} y = 0 \qquad \text{...(A)}$$

Notes

On the infinite open interval $(-\infty,\infty)$. Here we take as boundary conditions the following: as $x \to -\infty$, and as $x \to +\infty$, $y(x)$ tends to infinity of an order not greater than a certain finite power of x, i.e.

$$y(x) = 0\left(x^k\right) \text{ as } x \to \pm\infty \qquad \text{...(B)}$$

The equation (i) is written as

$$e^{-x^2}\frac{d^2y}{dx^2} - 2x\,e^{-x^2}\frac{dy}{dx} + 2\lambda\,e^{-x^2}y = 0$$

or

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\lambda y = 0 \qquad \text{...(i)}$$

From the coefficients of $\frac{dy}{dx}$ and y, it is clear that there are no singular points except $x = \pm\infty$.

Hence its solution can be given by a power series by Frobenius method

$$y(x) = \sum_{r=0}^{\infty} a_r x^{r+k} \qquad \text{...(ii)}$$

Which converges for $|x| < \infty$.

$$\frac{dy}{dx} = \sum_{r=0}^{\infty}(r+k)a_r x^{k+r-1}$$

and
$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$$

Substituting in (i), we have

$$\sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)x^{k+r-2} - 2(k+r)x^{k+r} + 2\lambda x^{k+r}\right] = 0,$$

or
$$\sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)x^{k+r-2} - 2(k+r-\lambda)x^{k+r}\right] = 0 \qquad \text{...(iii)}$$

Now (iii) being an identity, we can equate to zero the coefficients of various powers of x.

Equating to zero the coefficient of lowest power of x, i.e., of x^{k-2}, we get

$$a_0\,k(k-1) = 0.$$

Now $a_0 \neq 0$, as it is the coefficient of the first term with which the series is started.

$\therefore$ either $k = 0$

or $k = 1$ ...(iv)

Equating the coefficient of x^{k-1} in (iii) to zero, we get

$$a_1(k+1)k = 0 \qquad \text{...(v)}$$

Notes

which implies that $a_1 = 0$ or $k = 0$ or both are zero, since $k+1 \neq 0$ for any value of k given by (iv).

Now equating to zero the coefficient of general term, i.e., x^{k+r} in (iii), we get

$$a_{r+2}(k+r+2)(k+r+1) = 2a_r(k+r-\lambda) = 0$$

or
$$a_{r+2} = \frac{2(k+r-\lambda)}{(k+r+2)(k+r+1)}.a_r$$

or
$$a_{r+2} = \frac{2(k+r)-2\lambda}{(k+r+2)(k+r+1)}.a_r \qquad \text{...(vi)}$$

Now two cases arise—

Case I: when $k = 0$, then from (vi), we have

$$a_{r+2} = \frac{2r-2\lambda}{(r+2)(r+1)}a_r \qquad \text{...(vii)}$$

Putting $r = 0, 2, 4$, etc. in (vii), we have

$$a_2 = \frac{-2\lambda}{2.1}a_0 = -\frac{2\lambda}{2!}a_0$$

$$a_4 = \frac{4-2\lambda}{4.3}a_2 = -\frac{(4-2\lambda).2\lambda}{4.3.2!}a_0$$

$$= \frac{2^2(-2+\lambda)\lambda}{4!}a_0 = \frac{2^2\lambda(\lambda-2)}{4!}a_0$$

and so on.

$$\therefore \quad a_{2m} = \frac{(-2)^m\lambda(\lambda-2)\ldots(\lambda-2m+2)}{(2m)!}a_0.$$

Again putting $r = 1, 3, 5$, etc.

$$a_3 = \frac{2-2\lambda}{3.2}a_1 = -\frac{2(\lambda-1)}{3!}a_1$$

$$a_5 = \frac{6-2\lambda}{5.4}a_3$$

$$= \frac{-2(6-2\lambda)(\lambda-1)}{5.4.3.2}a_1$$

$$= (-2)^2\frac{(\lambda-1)(\lambda-3)}{5!}a_1$$

and so on.

$$\therefore \quad a_{2m+1} = \frac{(-2)^m(\lambda-1)(\lambda-3)\ldots(\lambda-2m+1)}{(2m+1)!}a_1$$

Notes

Now if $a_1 \neq 0$, then we have

$$y = \sum_{r=0}^{\infty} a_r x^r$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 \left[1 - \frac{2\lambda}{2!} x^2 + \frac{2^2 \lambda(\lambda-2)}{4!} x^4 + \dots + \frac{(-2)^m \lambda(\lambda-2)\dots(\lambda-2m+2)}{(2m)!} x^{2m} + \dots \right]$$

$$+ a_1 \left[x - \frac{2(\lambda-1)}{3!} x^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5!} x^5 + \dots + \right.$$

$$\left. \frac{(-2)^m(\lambda-1)(\lambda-3)\dots(\lambda-2m+1)}{(2m+1)!} x^{2m+1} + \dots \right] \quad \text{...(viii)}$$

and if $a_1 = 0$, then we have

$$y = a_0 \left[1 - \frac{2\lambda}{2!} x^2 + \frac{2^2 \lambda(\lambda-2)}{4!} x^4 + \dots + \frac{(-2)^m \lambda(\lambda-2)\dots(\lambda-2m+2)}{(2m)!} x^{2m} + \dots \right] \quad \text{...(ix)}$$

$$= y_1 \text{ (say)}.$$

Case II: When $k = 1$, from (vi), we have

$$a_{r+2} = \frac{2(r+1) - 2\lambda}{(r+3)(r+2)} a_r.$$

Putting $r = 1, 3, \dots$ etc.

$$a_3 = a_5 = \dots = 0 \text{ (each)}.$$

Since in this case from (iv), $a_1 = 0$

Putting $r = 0, 2, 4, \dots$ etc.

$$a_2 = \frac{2-2\lambda}{3.2} a_0 = -\frac{2(\lambda-1)}{3!} a_0$$

$$a_4 = \frac{6-2\lambda}{5.4} a_2 = \frac{2(\lambda-1)(\lambda-3)}{3!} a_0$$

and so on.

$$\therefore \quad a_{2m} = \frac{(-2)^m(\lambda-1)(\lambda-3)\dots(\lambda-2m+1)}{(2m+1)!} a_0$$

$\therefore$ we have

$$y = \sum_{r=0}^{\infty} a_r x^{r+1}$$

$$= a_0 x + a_2 x^3 + a_4 x^5 + \dots + a_{2m} x^{2m+1} + \dots$$

Notes

$$= a_0\left[x - \frac{2(\lambda-1)}{3!}x^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5!}x^5 + \ldots + \right.$$

$$\left. \frac{(-2)^m(\lambda-1)(\lambda-3)\ldots(\lambda-2m+1)}{(2m+1)!}x^{2m+1} + \ldots\right]$$

$$= y_2 \text{ (say)} \qquad \ldots(x)$$

From (viii) and (x) it is obvious that (x) is the part of solution, given by (viii). But as the two are the solutions of the same equations so (x) must not be the part of solution (viii).

∴ $a_1 = 0$ and the solution in the case $k = 0$ must be given by (ix).

Hence the general solution of Hermite's equation is

$$y = Ay_1 + By_2,$$

where A and B are arbitrary constants and y_1, y_2 are given by (ix) and (x).

Hermite's Polynomials

When λ is an even integer, equation (ix) gives an even polynomial of degree n.

Let $\lambda = n$, n being an even integer and let

$$a_0 = (-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!}$$

∴ Coefficient of x^n in (ix) is

$$(-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!}\cdot\frac{(-2)^{n/2}n(n-2)\ldots(n-n+2)}{n!} = \frac{2^n.\frac{n}{2}\left(\frac{n}{2}-1\right)\ldots 1}{(n/2)!} = 2^n.$$

Similarly coefficient of x^{n-2}

$$= (-1)^{n/2}\frac{n!}{(n/2)!}\frac{(-2)^{(n-2)/2}n(n-2)\ldots(n-n+2+2)}{(n-2)!}$$

$$= -\frac{2^{n-2}n(n-1)n/2(n/2-1)\ldots 2}{(n/2)!}$$

$$= -\frac{n(n-1)}{1!}2^{n-2}$$

and so on.

So value of y is given by

$$y_n = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \ldots + (-1)^{n/2}\frac{n!}{(n/2)!}$$

Notes

This value of y_n is known as the Hermite's polynomial of degree n and is written as

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \ldots + (-1)^{n/2}\frac{n!}{(n/2)!}$$

or $$H_n(x) = \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)!}(2x)^{n-2r}$$

where $$\left(\frac{n}{2}\right) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases}$$

A first few $H_n(x)$ are given as follows

$$H_0(x) = 1, H_1(x) = 2x$$

$$H_2(x) = (2x)^2 - 2 = 4x^2 - 2$$

$$H_3(x) = (2x)^3 - \frac{3.2}{1}(2x) = 4x\left(2x^2 - 3\right)$$

$$H_4(x) = (2x)^4 - \frac{12}{1}(2x)^2 + \frac{4.3.2.1}{2}(1)$$

$$= 16x^4 - 48x^2 + 12$$

Self Assessment

Fill in the blanks:

1. Hermite polynomial $H_n(x)$ is a .... series.
2. As $x \to \infty, H_4(x)$ tends to infinity of an order not greater than .... power of x.
3. $H_3(x)$ satisfies equation (i) for λ = ...
4. The value of $H_4(o)$ is ....

We now give some of the properties of Hermite polynomials like generating functions, Rodrigue formula, orthogonality relations and the recurrence formulae.

3.2 Generating Functions of Hermite Polynomials $H_n(x)$

To prove that

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x)$$

or

show that $\frac{H_n(x)}{\lfloor n}$ are the coefficients of t^n in the expansion of the function e^{2xt-t^2} (known as generating function for $H_n(x)$), We have

$$e^{2tx-t^2} = e^{2tx}.e^{-t^2}$$

$$= \sum_{r=0}^{\infty}\frac{(2tx)^r}{r!}\sum_{s=0}^{\infty}\frac{\left(-t^2\right)^s}{s!}$$

$$= \sum_{r=0,s=0}^{\infty}(-1)^s\frac{(2x)^r}{r!s!}.t^{r+2s}$$

Coefficient of t^n (for fixed value of s)

[obtained by putting $r + 2s = n$, i.e., $r = n - 2s$]

$$= (-1)^s\frac{(2x)^{n-2s}}{(n-2s)!s!}$$

The total value of t^n is obtained by summing over all allowed values of s, and since $r = n - 2s$

$\therefore$ $$n-2s \geq 0 \text{ or } s \leq n/2$$

Thus if n is even s goes from 0 to $n/2$ and if n is odd, s goes from 0 to $(n - 1)/2$.

$\therefore$ $$\text{Coefficient of } t^n = \sum_{s=0}^{(n/2)}(-1)^s\frac{(2x)^{n-2s}}{(n-2s)!s!}$$

$$= \frac{H_n(x)}{n!}$$

$$\text{Hence } e^{2tx-t^2} = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)$$

$$e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x).$$

Other form for the Hermite Polynomials

Prove

$$H_n(x) = 2^n\left\{\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)x^n\right\} \qquad \text{...(i)}$$

We have

$$\frac{1}{2}\frac{d}{dx}e^{2tx} = te^{2tx}$$

$$\frac{d}{dx}\left(\frac{1}{2}\frac{d}{dx}e^{2tx}\right) = 2t^2e^{2tx}$$

Notes

$\therefore$ $$\frac{1}{2}\frac{d}{dx}\left(\frac{1}{2}\frac{d}{dx}e^{2tx}\right) = t^2 e^{2tx}$$

or $$\left(\frac{1}{2}\frac{d}{dx}\right)^2 e^{2tx} = t^2 e^{2tx}$$

$\therefore$ $$\left(\frac{1}{2}\frac{d}{dx}\right)^n e^{2tx} = t^n e^{2tx}$$

Hence

$$\left\{\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}e^{2tx} = \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)^n\right]e^{2tx}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{1}{2}\frac{d}{dx}\right)^{2n} e^{2tx}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}e^{2tx} \text{ [from (ii)]}$$

$$= e^{2tx}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}$$

$$= e^{2tx}.e^{-t^2} = e^{(2tx-t^2)}$$

or $$\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}\sum_{n=0}^{\infty}\frac{1}{n!}(2tx)^n = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)$$

Equating the coefficient of t^n from the two sides, we have

$$\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}\frac{1}{n!}2^n x^n = \frac{1}{n!}H_n(x)$$

or $$H_n(x) = 2^n\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}x^n.$$

Self Assessment

5. Obtain the expression for $H_2(x)$ from generating function e^{2xt-t^2}.

6. Obtain the expression for $\frac{d}{dx}H_n(x)$ from the generating function e^{2xt-t^2}.

7. Show that for odd n.

$$H_n(0) = 0$$

8. $H_1(x) - 2xH_0(x)$ is

   (a) positive

   (b) zero

   (c) negative

   (d) none of the above

Notes

3.3 The Rodrigue's Formula for $H_n(x)$

To Prove

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad ...(i)$$

Proof:

We have
$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

or
$$e^{x^2-(t-x)^2} = \frac{H_0(x)}{0!}t^0 + \frac{H_1(x)}{1!}t + \frac{H_2(x)}{2!}t^2 + \ldots + \frac{H_n(x)}{n!}t^n + \frac{H_{n+1}(x)}{(n+1)!}t^{n+1} + \ldots$$

Differentiating both sides, partially with respect to t, n times and then putting t = 0, we have

$$\frac{H_n(x)}{n!}n! = \left[\frac{\partial^n}{\partial t^n} e^{-(t-x)^2}\right]_{t=0} e^{x^2} \quad ...(ii)$$

Now let $t - x = \mu$, i.e., at $t = 0, x = -\mu$

$\therefore \quad \frac{\partial}{\partial t} \equiv \frac{\partial}{\partial \mu}$

or
$$\left[\frac{\partial^n}{\partial t^n} e^{-(t-x)^2}\right]_{t=0} = \frac{\partial^n}{\partial \mu^n}\left(e^{-\mu^2}\right)$$

$$= (-1)^n \frac{\partial^n}{\partial x^n}\left(e^{-x^2}\right)$$

$$= (-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$\therefore \quad H_n(x) = (-1)^n . e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad ...(A)$$

Notes

First few Hermite Polynomials from Rodrigue's Formula

From Rodrigue's Formula for $H_n(x)$

$$H_n(x) = (-1)^n e^{x^2} \cdot \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Putting $n = 0, 1, 2, 3, \ldots$ we get

$$H_0(x) = e^{x^2} \cdot e^{-x^2} = 1$$

$$H_1(x) = (-1)e^{x^2} \frac{d}{dx}\left(e^{-x^2}\right) = 2x$$

$$H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2}\left(e^{-x^2}\right) = e^{x^2} \frac{d}{dx}\left(-2xe^{-x^2}\right)$$

$$= e^{x^2}\left(4x^2 e^{-x^2} - 2e^{-x^2}\right)$$

$$= \left(4x^2 - 2\right).$$

$$H_3(x) = (-1)^3 e^{x^2} \frac{d^3}{dx^3}\left(e^{-x^2}\right)$$

$$= -e^{x^2} \frac{d}{dx}\left\{\left(4x^2 - 2\right)e^{-x^2}\right\}$$

$$= -e^{x^2}\left\{-2x\left(4x^2 - 2\right)e^{-x^2} + 8xe^{-x^2}\right\}$$

$$= -e^{x^2}\left[\left(-8x^3 + 12x\right)e^{-x^2}\right] = 8x^3 - 12x.$$

Similarly, $H_4(x) = 16x^4 - 48x^2 + 12$ etc.

3.4 Orthogonal Properties of Hermite Polynomials

Prove

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \sqrt{\pi}\, 2^2 (n)! & \text{if } m = n \end{cases}$$

We have $$e^{-t^2 + 2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

and $$e^{-s^2 + 2sx} = \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!}$$

$\therefore$ $$e^{-t^2 + 2tx} e^{-s^2 + 2sx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!}$$

Notes

$\therefore \qquad \frac{1}{n!m!}H_n(x)H_m(x) \;=\;$ Coeff. of $t^n s^m$ in the expansion of $e^{-t^2+2tx}e^{-s^2+2sx}$

$\therefore \quad \int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx$ is equal to $n!\ m!$ times the coefficient of $t^n s^m$ in the expansion of

$$\int_{-\infty}^{\infty} e^{-x^2}.e^{-t^2+2tx}.e^{-s^2+2sx}dx$$

Now, $\int_{-\infty}^{\infty} e^{-x^2}.e^{-t^2+2tx}.e^{-s^2+2sx}dx$

$$= e^{-t^2-s^2}\int_{-\infty}^{\infty} e^{-x^2+2tx+2sx}dx$$

$$= e^{-t^2-s^2}\int_{-\infty}^{\infty} e^{-\left[x^2-(t+s)^2+(t+s)^2\right]dx}$$

$$= e^{2ts}\int_{-\infty}^{\infty} e^{-[x-(t+x)]2dx}$$

$$= e^{2ts}\int_{-\infty}^{\infty} e^{-u^2}du, \text{ putting } x-(t+s)=u$$

$$= e^{2ts}\sqrt{\pi}, \qquad \left(\text{since} \int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}\right)$$

$$= \sqrt{\pi}\left[1+2ts+\frac{(2ts)^2}{2!}+\ldots+\frac{(2ts)^n}{n!}+\ldots\right]$$

Coefficient of $t^n s^m$ in the expansion of

$$\int_{-\infty}^{\infty} e^{-x^2}e^{-t^2+2tx}e^{-s^2+2sx}dx$$

is 0 if $m \neq n$

and $\frac{2^n\sqrt{\pi}}{n!}$. if $m = n$.

We can also write it as follows

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx = \sqrt{\pi}2^n n!\delta_{mn},$$

where δ_{mn} is Kronecker delta defined as

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

Notes

Self Assessment

9. Using Rodrigue's Formula derive the Hermite's polynomials $H_2(x)$ and $H_3(x)$

10. Evaluate

$$\int_{-\infty}^{+\infty} x\, e^{-x^2} H_2(x)\, H_1(x) dx$$

11. Evaluate

$$\int_{-\infty}^{+\infty} x\, e^{-x^2} H_2(x)\, dx$$

3.5 Recurrence Formula for Hermite Polynomials

(I) Prove

$$\frac{d}{dx} H_n(x) = 2n\, H_{n-1}(x) \qquad \text{for } n \geq 1$$

We have from generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} = e^{2xt-t^2} \qquad \text{...(i)}$$

Differentiating both sides with respect to x, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \frac{dH_n(x)}{dx} = 2t\, e^{2xt-t^2}$$

$$= 2t \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!}$$

$$= 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} \qquad \text{Let } n+1 = n'$$

$$= 2 \sum_{n'=1}^{\infty} \frac{H_{n'-1}(x) t^{n'}}{(n'-1)!}$$

or

$$\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \frac{dH_n(x)}{dx} = 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(x) t^n}{(n-1)!} \qquad \text{...(ii)}$$

Comparing t^n on both sides we have

$$\frac{H'_n(x)}{n!} = 2 \frac{H_{n-1}(x)}{(n-1)!} \qquad \left[\text{Here } \frac{d H_n(x)}{dx} = H'_n(x)\right]$$

or

Notes

or $$\boldsymbol{H_n'(x) = 2n\, H_{n-1}(x)}$$

(II) $$\boldsymbol{2x\, H_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)}$$

we have $$\sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} = e^{-t^2+2^t x}$$

Differentiating both sides with respect to t, we get

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} n\, t^{n-1} = e^{-t^2+2tx}(-2t+2x)$$

or $$\sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1} = 2t\, e^{-t^2+2tx} + 2x\, e^{-t^2+2tx}$$

or $$\sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1} = -2t\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n + 2x\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

(Since term of L.H.S. Corresponding to n = 0 is zero)

or $$2x\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = 2\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} + \sum_{n=0}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1}$$

or $$2x\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = 2\sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^n + \sum_{n=0}^{\infty} \frac{H_{n+1}(x)}{n!} t^n$$

Equating the coefficient of t^n, on both sides, we have

$$2x\frac{H_n(x)}{n!} = 2\frac{H_{n-1}(x)}{(n-1)!} + \frac{H_{n+1}(x)}{n!}$$

or $$\boldsymbol{2x\, H_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x)}$$

(III) $$\boldsymbol{H_n'(x) = 2x\, H_n(x) - H_{n+1}(x)}$$

Writing recurrence formulae I and II, we have

$$\boldsymbol{H_n'(x) = 2n\, H_{n-1}(x)} \quad \text{...(i)}$$

and $$2x\, H_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x) \quad \text{...(ii)}$$

Subtracting (ii) from (i), we have

$$\boldsymbol{H_n'(x) = 2x\, H_n(x) - H_{n+1}(x)}$$

(IV) $$\boldsymbol{H_n''(x) - 2x\, H_n'(x) + 2nH_n(x) = 0}$$

Hermite's differential equation is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

Notes

Since $H_n(x)$ is the solution of (i), hence, we have

$$\boldsymbol{H_n''(x) - 2x\, H_n'(x) + 2nH_n(x) = 0}$$

Illustrative Examples

Example 1: Evaluate

$$\int_{-\infty}^{\infty} xe^{-x^2} H_n(x) H_m(x)\,dx$$

Solution: From recurrence formula II, we have

$$xH_n(x) = nH_{n-1}(x) + \frac{1}{2}H_{n+1}(x)$$

$$\therefore \quad \int_{-\infty}^{\infty} xe^{-x^2} H_n H_m(x)\,dx = \int_{-\infty}^{\infty} e^{-x^2}\left\{nH_{n-1}(x) + \frac{1}{2}H_{n+1}(x)\right\} H_m(x)\,dx$$

$$= n\int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x)H_m(x)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} H_{n+1}(x)H_m(x)\,dx$$

$$= n\sqrt{\pi}\,2^{n-1}(n-1)!\,\delta_{n-1,m} + \frac{1}{2}\sqrt{\pi}\,2^{n+1}(n+1)!\,\delta_{n+1,m}$$

$$= \sqrt{\pi}\,2^{n-1}n!\,\delta_{n-1,m} + \sqrt{\pi}\left(2^n\right)(n+1)!\,\delta_{n+1,m}$$

where δ is Kronecker delta.

Example 2: Prove that $H_n'' = 4n(n-1)H_{n-1}$

Solution: From recurrence formula I, we have

$$H_n' = 2nH_{n-1} \quad \text{...(i)}$$

Differentiating with respect to x, we have

$$H_n'' = 2nH_{n-1}' \quad \text{...(ii)}$$

Replacing n by $(n - 1)$ in (i), we have

$$H_{n-1}' = 2(n-1)H_{n-2} \quad \text{...(iii)}$$

∴ From (ii) and (iii), we have

$$\boldsymbol{H_n'' = 4n(n-1)H_{n-1}}$$

Example 3: Prove that, if $m < n$

$$\frac{d^n}{dx^m}\{H_n(x)\} = \frac{2^m n!}{(n-m)!}H_{n-m}(x)$$

Solution: We have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{-t^2+2tx} \qquad \text{...(i)}$$

$$\therefore \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^m}{dx^m}\{H_n(x)\} = \frac{d^m}{dx^m} e^{-t^2+2tx}$$

$$= (2t)^m e^{-t^2+2tx}$$

$$= (2t)^m \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \qquad \text{(from (i))}$$

$$= 2^m \sum_{n=0}^{\infty} \frac{1}{n!} t^{n+m} H_n(x)$$

Putting $n+m=r$, $n=r-m$, for $n=0$;

$r=m$, for $n=\infty$, $r=\infty$,

$$\sum_{n=0} \frac{t^n}{n!} \frac{d^m}{dx^m} H_n(x) = 2^m \sum_{r=m}^{\infty} \frac{1}{(r-m)!} t^r \, H_{r-m}(x) \qquad \text{...(ii)}$$

Equating the coefficient of t^n from the two sides, we have

$$\frac{1}{n!} \frac{d^n}{dx^m}\{H_n(x)\} = 2^m \frac{1}{(n-m)!} H_{n-m}(x)$$

$$\therefore \qquad \frac{d^n}{dx^m}\{H_n(x)\} = \frac{2^m n!}{(n-m)!} H_{n-m}(x) \text{ Q.E.D.}$$

Example 4: Prove that $H_{2n}(0) = (-1)^n . \frac{(2n)!}{n!}$ and (ii) $H_{2n+1}(0) = 0$

Solution: We have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{-t^2+2tx}$$

Putting $x = 0$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(0) = e^{-t^2}$$

$$= \left\{1 - t^2 + \frac{(t^2)^2}{2!} + ... + (-1)^n \frac{(t^2)^n}{n!} + ...\right\} \qquad \text{...(1)}$$

Notes

(i) Equating the coefficients of t^{2n}, on both sides, we have

$$\frac{1}{(2n)!}H_{2n}(0) = (-1)^n \frac{1}{n!}$$

or

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$$

(ii) Again equating the coefficients of t^{2n+1}, on both sides of (i), we have

$$\frac{1}{(2n+1)!}H_{2n+1}(0) = 0$$ [Since R.H.S. of (i) does not involve odd powers of t]

Hence $$H_{2n+1}(0) = 0.$$

Example 5: Prove that

$$P_n(x) = \frac{2}{\sqrt{\pi}\, n!}\int_0^\infty t^n e^{-t^2} H_n(xt)\,dt.$$

Solution: We have

$$H_n(x) = \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)}(2x)^{n-2r}$$

$$\therefore \quad H_n(xt) = \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)}(2xt)^{n-2r}$$

$$\therefore \quad \frac{2}{\sqrt{\pi}n!}\int_0^\infty t^n e^{-t^2} H_n(xt)dt = \frac{2}{\sqrt{\pi}.n!}\int_0^\infty t^n e^{-t^2}\sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)}(2xt)^{n-2r}.dt$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1}(-1)^r x^{n-2r}}{\sqrt{\pi}\, r!(n-2r)!}\int_0^\infty e^{-t^2} t^{2\left(n-r+\frac{1}{2}\right)-1} dt$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1}(-1)^r x^{n-2r}}{\sqrt{\pi}\, r!(n-2r)!}\frac{1}{2}\Gamma\left(n-r+\frac{1}{2}\right)$$

$$\left[\text{Since } 2\int_0^\infty e^{-t^2} t^{(2n-1)}dt = \Gamma(n)\right]$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r}(-1)^r x^{n-2r}\left[2(n-r)\right]!}{\sqrt{\pi}\, r!(n-2r)!2^{2(n-r)}(n-r)!}\sqrt{\pi}$$

$$\left[\text{Since } \Gamma\left(x+\frac{1}{2}\right) = \frac{(2x)!}{2^{2x}}\frac{\sqrt{\pi}}{x!}\right]$$

Notes

$$= \sum_{n=0}^{(n/2)} (-1)^r \frac{(2n-2r)!x^{n-2\pi}}{2^n (r)!(n-2r)!(n-r)!} = P_n(x)$$

Hence,

$$P_n(x) \frac{2}{\sqrt{\pi} n!} \int_0^{\infty} t^n e^{-t^2} H_n(xt) dt.$$

Self Assessment

12. From recurrence relation II Obtain the value of $H_3(x)$. Given that

$$H_2(x) = 4x^2 - 2; H_1(x) = 2x$$

13. Prove that

$$H_n''(x) - 4nx\, H_{n-1}(x) + 2n\, H_n(x) = 0$$

14. Prove that

$$\frac{dH_3(x)}{dx} = 6H_2(x)$$

3.6 Summary

- Hermite differential equation has no finite singular points except $x = \pm\infty$. Therefore Frobenius method involving a power series solution is obtained.
- There are two independent solutions corresponding to two different values of indicial power.
- For $\lambda = n$ a polynomial solution called Hermite polynomial is obtained.
- Hermite polynomials are seen to be generated by a generating function.
- Orthogonal properties of Hermite polynomials are obtained. It helps in expressing any polynomial in terms of $H_n(x)$.
- Recurrence relations established help in expressing every polynomial as well as its derivatives in terms of two or three Hermite polynomials.

3.7 Keywords

Boundary Conditions are the behaviour of the solution of the differential equations in the initial value of the independent variable as well as at the final value of independent variable.

Frobenius Method: At an ordinary point as well as at regular singular point, helps in evaluating the solution as a power series.

Orthogonality relations of Hermite polynomials are relations involving integrals of two Hermite polynomials. These relations help us to see that $H_n(x)$ form a complete set.

Recurrence Relations are relations between two or three polynomials for all values of n and x.

Rodrigue Formula Expresses $H_n(x)$ in an alternative way than that of finding a solution of differential equations.

Notes

3.8 Review Questions

1. Use the Rodrigue's formula to drive the Hermite polynomials $H_2(x)$ and $H_3(x)$
2. Evaluate

$$\int_{-\infty}^{+\infty} x\, e^{-x^2} H_2(x) H_3(x) dx$$

3. Show that

$$H_1(x) = 2xH_0(x)$$

4. For what value of $n, H_n(0) = 0$?
5. From generating function show that

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

Answers: Self Assessment

1. Terminating
2. Finite
3. n
4. 12
5. $H_2(x) = (4x^2 - 2)$
6. $\frac{dH_n(x)}{dx} = 2nH_{n-1}(x)$
8. b
9. $H_2(x) = 4x^2 - 2, H_3(x) = 8x^3 - 12x$
10. $4\sqrt{\pi}$
11. Zero

3.9 Further Readings

Books K. Yosida, Lectures on Differential and Integral Equations

L.D. Landau and E.M. Lifshitz, Quantum Mechanics

Unit 4: Laguerre Polynomials

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Use generating function which helps you to familiarise with more properties of Laguerre polynomials.
- Use Rodrigue formula which is quite helpful in making you more familiar with properties of Laguerre polynomials.
- Employ of orthogonal properties to evaluate certain integrals.
- Use recurrence relations to correct one set of polynomials into another.

Introduction

Laguerre polynomials are shown to satisfy Laguerre differential equation. This equation has $x = 0$ as **regular singular** point whereas $x = \infty$ is an **irregular singular point.** A power series solution is obtained by Frobenius method.

Generating function is obtained wherein it will be seen that most properties of Laguerre polynomials are obtained orthogonal properties, recurrence relations Rodrigue's formula for Laguerre polynomials are very important and almost all properties of $L_n(x)$ are obtained from the above relations.

4.1 Solution of Laguerre's Differential Equation

Consider the following differential equation containing a parameter λ.

$$\left(x\, e^{-x} y'\right)' + \lambda\, e^{-x} y = 0$$

Notes

On the infinite interval $(0,\infty)$, we take as boundary conditions the following:

$y(x)$ remains finite as $x \to 0$,

$y(x)$ tends to infinity as $0(x^{\alpha})$ as $x \to \infty$.

The above equation when expanded is equal to

$$x\,e^{-x}y'' - e^{-x}(x-1)y' + \lambda\,e^{-x}y = 0$$

or

$$x\,y'' + (1-x)y' + \lambda\,y = 0 \qquad \text{...(i)}$$

Here $$y' = \frac{dy}{dx}.$$

Equation (i) has only one finite regular singular point $x = 0$ whereas $x \to \infty$ is irregular singular point. So we can apply Frobenius method to express the solution of (i) as a power series:

$$y = \sum_{n=0}^{\infty} a_r\,x^{k+r} \qquad \text{...(ii)}$$

$$\therefore \qquad \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k=r-2}$$

Substituting in (i), we get

$$\sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)x^{k+r-1} + (1-x)(k+r)x^{k+r-1} + \lambda x^{k+r}\right] = 0$$

or

$$\sum_{r=0}^{\infty} a_r\left[(k+r)^2 x^{k+r-1} - (k+r-\lambda)x^{k+r}\right] = 0 \qquad \text{...(iii)}$$

Now (iii) being an identity, we can equate the coefficients of various powers of x to zero.

Equating to zero the coefficient of lowest power of x, i.e., of x^{k-1}, we have

$$a_0\,k^2 = 0$$

Now, $a_0 \neq 0$, as it is coefficient of the first term with which the series is started.

$$\therefore \qquad k = 0.$$

Equating to zero the coefficient of general term, i.e., of x^{k+r}, we have

$$a_{r+1}(k+r+1)^2 - a_r(k+r-\lambda) = 0$$

Notes

$\therefore \qquad a_{r+1} = \dfrac{(k+r-\lambda)}{(k+r+1)^2} a_r$

for $\qquad k = 0$

$$a_{r+1} = \frac{r-\lambda}{(r+1)^2} a_r \qquad \text{...(iv)}$$

Putting $r = 0, 1, 2, ...,$ in (iv), we have

$$a_1 = -\frac{\lambda}{1} a_0 = (-1)\lambda a_0$$

$$a_2 = \frac{1-\lambda}{2^2} a_1 = (-1)^2 \frac{\lambda(\lambda-1)}{(2!)^2} a_0$$

$$a_3 = \frac{2-\lambda}{3^2} \cdot a_2 = (-1)^3 \frac{\lambda(\lambda-1)(\lambda-2)}{(3!)^2} a_0 \text{ etc.}$$

Hence $\qquad a_r = (1)^r \dfrac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-r+1)}{(r!)^2} a_0$

$\therefore$ From (ii), we have

$$y = \sum_{r=0}^{\infty} a^r x^r = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \ldots$$

$$= a_0 \left[1 - \lambda x + \frac{\lambda(\lambda-1)}{(2!)^2} x^2 - \frac{\lambda(\lambda-1)(\lambda-2)}{(3!)^2} x^3 \right.$$

$$\left. + \ldots + (-1)^r \frac{\lambda(\lambda-1)\ldots(\lambda-r+1)}{(r!)^2} x^r + \ldots \right] \qquad \text{...(v)}$$

If $\lambda = n$

$$y = a_0 \left[1 - \frac{n}{1^2} . x + \frac{n(n-1)}{(2!)^2} x^2 + \ldots + (-1)^2 \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} \right]$$

$$= a_0 \sum_{r=0}^{n} (-1)^r \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} x^r$$

$$= a_0 \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r$$

Laguerre Polynomials

The standard solution of Laguerre equation for which $a_0 = 1$ is called the Laguerre polynomial of order n and is denoted by $L_n(x)$.

Notes

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r \qquad ...(vi)$$

The first few Laguerre polynomials are:

$$L_0(x) = 1, L_1(x) = 1 - x$$

$$L_2(x) = \frac{1}{2}\left(2 - 4x + x^2\right)$$

$$L_3(x) = \frac{1}{6}\left(6 - 18x + 9x^2 - x^3\right)$$

Self Assessment

1. The value of $L_n(0)$ is

   (a) 0 (b) 1

   (c) −1 (d) None of these

2. $L_2(x)$ satisfies Laguerre's differential equation for λ equal to

   (a) −1 (b) 3

   (c) 2 (d) 1

3. Fill in the blanks:

   The Laguerre polynomial tends to infinity as a .... power of x as $x \to \infty$.

4. Laguerre polynomial $L_n(x)$ is a polynomial having a leading power of x equal to

   (a) n (b) Zero

   (c) One (d) None of the above

4.2 Generating Function for Laguerre Polynomials $L_n(x)$

To prove $\frac{1}{1-t} e^{-tx/(1-t)} = \sum_{r=0}^{\infty} t^n L_n(x)$.

We have

$$\frac{1}{1-t} e^{-tx/(1-t)} = \frac{1}{1-t} \sum_{r=0}^{\infty} \frac{1}{r!} \left(-\frac{xt}{1-t}\right)^r$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{x^r t^r}{(1-t)^{r+1}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r (1-t)^{-(r+1)}$$

Notes

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r \left[1 + (r+1)t + \frac{(r+1)(r+2)}{2!} t^2 + \ldots\right]$$

$$= \sum_{r=0}^{\infty} \left[\frac{(-1)^r}{r!} x^r t^r \sum_{s=0}^{\infty} \frac{(r+s)!}{r!s!} . t^s\right]$$

$$= \sum_{r,\, s=0}^{\infty} (-1)^r \frac{(r+s)!}{(r!)^2 s!} x_t^r r + s$$

Putting $s + r = n$, or $s = n - r$, we get the coefficient of t^n, for a fixed value of r as

$$(-1)^r \frac{n!}{(r!)^2 (n-r)!} . x^r$$

Therefore the total coefficient of t^n is obtained by summing over all allowed values of r, since $s = n - r$ and $s \geq 0$

$\therefore \quad n - r \geq 0$ or $r \leq n$.

Hence the coefficient of t^n is

$$\sum_{r=0}^{n} (-1)^r \frac{n!}{(r!)^2 (n-r)!} x^r = L_n(x)$$

Hence $$\frac{1}{(1-t)} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n L_n(x).$$

Self Assessment

5. Obtain the expression for $L_1(x)$ and $L_2(x)$ from the generating function

$$\frac{1}{(1-t)} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n L_n(x)$$

6. Show that from the generating function

$$L_n(0) = 1 \text{ for } n = 0, 1, 2, \ldots$$

7. Obtain the expression for $L_3(x)$ from the generating function

$$\frac{1}{(1-t)} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n L_n(x)$$

8. Whether $2L_2(x) - x^2 + 4x$ is equal to

(a) 0 (b) 1

(c) 2 (d) −2

Notes

4.3 Rodrigue's Formula for Laguerre Polynomials $L_n(x)$

To prove

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right) \text{ for } n = 0, 1, 2, \ldots \qquad \text{...(i)}$$

Proof: Using Leibnitz's theorem we have

$$\frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right) = \frac{e^x}{n!}\Big[x^n(-1)^n e^{-x} + n\cdot n\cdot x^{n-1}(-1)^{n-1}e^{-x} + $$

$$+\frac{n(n-1)}{2}\cdot n(n-1)x^{n-2}(-1)^{n-2}e^{-x} + \ldots + n!\,e^{-x}\Big]$$

$$= \frac{e^x e^{-x}}{n!}\left[(-1)^n x^n + (-1)^{n-1}\frac{n\cdot n!}{(n-1)!}x^{n-1} + \ldots + n!\right] \qquad \text{...(ii)}$$

$$= (-1)^n \frac{n!}{(n!)^2}x^n + (-1)^{n-1}\frac{n!}{\{(n-1)!\}^2 .i!}x^{n-1} + \ldots + \frac{n!}{n!}$$

$$= \sum_{r=0}^{n}(-1)^r \frac{n!}{(r!)^2(n-r)!}x^r = L_n(x) \qquad \text{...(iii)}$$

Hence $$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right).$$

First few Laguerre Polynomials from Rodrigue's Formula

We have from Rodrigue's formula

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

Putting $n = 0$

$$L_0(x) = \frac{e^x}{0!}\frac{d^0}{dx^0}(x^0 e^{-x}) = 1$$

Putting $n = 1$

$$L_1(x) = \frac{e^x}{1!}\frac{d}{dx}(xe^{-x}) = e^x\left(e^{-x} - xe^{-x}\right) = 1 - x$$

Putting $n = 2$

$$L_2(x) = \frac{e^x}{2!}\frac{d^2}{dx^2}\left(x^2 e^{-x}\right) = \frac{e^x}{2!}\frac{d}{dx}\left(2xe^{-x} - x^2 e^{-x}\right)$$

$$= \frac{e^x}{2!}\left(2e^{-x} - 4x\,e^{-x} + x^2 e^{-x}\right)$$

$$= \frac{1}{2!}\left(2-4x+x^2\right)$$

Similarly,

$$L_3(x) = \frac{1}{3!}\left(6-18x+9x^2-x^3\right)$$

$$L_4(x) = \frac{1}{4!}\left(24-96x+72x^2-16x^3+x^4\right), \ldots etc.$$

Self Assessment

9. Show that

$$L_2(x) = \frac{e^x}{2}\frac{d^2}{dx^2}\left(x^2 e^{-x}\right)$$

10. Show that x^3 is given by

$$x^3 = 6\left[L_0(x) - 3L_1(x) + 3L_2(x) - L_3(x)\right]$$

11. From Rodrigue's formula show that

$$\frac{dL_2(x)}{dx} = -L_1(x) - L_0(x)$$

4.4 Orthogonality Property of Laguerre Polynomials $L_n(x)$

To prove

$$\int_0^\infty e^{-x} L_n(x) L_m(x) dx = \delta_{mn} = \begin{cases} 0 \text{ if } m \neq n \\ 1 \text{ if } m = n \end{cases} \quad \ldots\text{(i)}$$

We have from the generating function of Laguerre polynomial, that

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{\left(-tx/(1-t)\right)}$$

and

$$\sum_{m=0}^{\infty} s^m L_m(x) = \frac{1}{1-s} e^{-xs/(1-s)}$$

$$\therefore \quad \sum_{m,n=0}^{\infty} e^{-x} t^n s^m L_n(x) L_m(x) = e^{-x} \frac{1}{(1-t)(1-s)} e^{-tx/(1-t)-\frac{sx}{(1-s)}} \quad \ldots\text{(ii)}$$

Thus

$$\int_0^\infty e^{-x} L_n(x) L_m(x) dx = \text{Coeff. of } s^m\, t^n \text{ in the expansion of } \int_0^\infty e^{-x} \frac{1}{(1-t)(1-s)} e^{-tx/(1-t) e^{-sx/(1-s)}} dx$$

Notes

$$= \frac{1}{(1-t)(1-s)} \int_0^\infty \left[e^{\left[-x\left(1+\frac{t}{1-t}+\frac{s}{1-s}\right)\right]} \right] dx$$

$$= \frac{1}{(1-t)(1-s)} \frac{1}{\left[1+\frac{t}{(1-t)}+\frac{s}{(1-s)}\right]} \times \left[e^{\left[-x\left(1+\frac{t}{1-t}+\frac{s}{1-s}\right)\right]} \right]_0^\infty$$

$$= -\frac{1}{(1-t)(1-s)} \frac{(1-t)(1-s)}{\{(1-t)(1-s)+t(1-s)+s(1-t)\}} [-1]$$

$$= \frac{1}{1-st} = (1-st)^{-1} = \left[1+st+(st)^2+(st)^3+\ldots(st)^n+\ldots\right] \qquad \text{...(iii)}$$

In which coefficient of $s^m \; t^n$

is 0 if $m \neq n$ ...(iv)

and is 1 if $m = n$

Hence

$$\int_0^\infty e^{-x} L_m(x) L_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

or

$$\int_0^\infty e^{-x} L_m(x) L_n(x) dx = \delta_{mn} \text{ (where } m, n, = 1, 2, 3, \ldots) \qquad \text{...(v)}$$

Self Assessment

12. Whether $\int_0^\infty e^{-x} L_2(x) L_3(x) dx$ is equal to

    (a) 1 (b) 5

    (c) –1 (d) 0

13. Find out

$$\sum_{m=0}^{\infty} \delta_{mn} L_m(x)$$

14. Prove that

$$\int_0^\infty e^{-x} L_1(x) L_2(x) dx = 0$$

Notes

4.5 Recurrence Formulae for Laguerre Polynomials $L_n(x)$

I. $$(n+1)L_{n+1}(x) = (2n+1-x)\,L_n(x) - n\,L_{n-1}(x)$$

We have $$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{(1-t)} e^{\left[-\frac{-tx}{(1-t)}\right]}$$

Differentiating both sides with respect to t, we have

$$\sum_{n=0}^{\infty} n\, t^{n-1} L_n(x) = \frac{1}{(1-t)^2}\left(1-\frac{x}{1-t}\right) e^{\left[-\frac{-tx}{(1-t)}\right]}$$

or $$(1-t)^2 \sum_{n=0}^{\infty} n\, t^{n-1} L_n(x) = (1-t)\frac{e^{-tx/(1-t)}}{(1-t)} - x.\frac{1}{1-t} e^{-tx/(1-t)}$$

or $$(1-t)^2 \sum_{n=0}^{\infty} n\, t^{n-1} L_n(x) = (1-t)\sum_{n=0}^{\infty} t^n L_n(x) - x \sum_{n=0}^{\infty} t^n L_n(x)$$

or $$\left(1-2t+t^2\right)\sum_{n=1}^{\infty} n\, t^{n-1} L_n(x) = (1-t)\sum_{n=1}^{\infty} t^n L_n(x) - x \sum_{n=0}^{\infty} t^n L_n(x)$$

or $$\sum_{n=1}^{\infty} n\, t^{n-1} L_n(x) - 2\sum_{n=1}^{\infty} n\, t^n L_n(x) + \sum_{n=1}^{\infty} n\, t^{n+1} L_n(x)$$

$$= \sum_{n=0}^{\infty} t^n L_n(x) - \sum_{n=0}^{\infty} t^{n+1} L_n(x) - x\sum_{n=0}^{\infty} t^n L_n(x)$$

Equating the coefficient of t^n on both sides, we have

$$(n+1)L_{n+1}(x) - 2n - L_n(x) + (n-1)L_{n-1}(x)$$

$$= L_n(x) - L_{n-1}(x) - xL_n(x)$$

or $$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - n\,L_{n-1}(x)$$

II. $$xL_n'(x) = nL_n(x) - nL_{n-1}(x)$$

We have

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{(1-t)} e^{\left[\frac{-tx}{(1-t)}\right]}$$

Differentiating with respect to x, we have

$$\sum_{n=0}^{\infty} t^n L_n'(x) = \frac{1}{(1-t)} e^{-tx/(1-t)}\left(-\frac{t}{1-t}\right)$$

or $$(1-t)\sum_{n=0}^{\infty} t^n L_n'(x) = -t.\frac{1}{1-t} e^{-tx/(1-t)}$$

Notes

or $$(1-t)\sum_{n=0}^{\infty} t^n L_n'(x) = -t\sum_{n=0}^{\infty} t^n L_n(x)$$

or $$\sum_{n=0}^{\infty} t^n L_n'(x) - \sum_{n=0}^{\infty} t^{n+1} L_n'(x) = -\sum_{n=0}^{\infty} t^{n+1} L_n(x)$$

Equating the coefficients of t^n, on both sides, we get

$$L_n'(x) - L_{n-1}'(x) = -L_{n-1}(x)$$

or $$L_n'(x) = L_{n-1}'(x) - L_{n-1}(x) \quad \text{...(i)}$$

Differentiating recurrence formula I with respect to x, we get

$$(n+1)L_{n+1}'(x) = (2n+1-x)\, nL_n'(x) - L_n(x) - nL_{n-1}'(x) \quad \text{...(ii)}$$

Replacing n by $(n + 1)$ in (i), we get

$$L_{n+1}'(x) = L_n'(x) - L_n(x)$$

Also from (i) $$L_{n-1}'(x) = L_n'(x) + L_{n-1}(x)$$

Substituting these values in (ii), we have

$$(n+1)\{L_n'(x) - L_n(x)\} = (2n+1-x)L_n'(x) - L_n(x) - n\{L_n(x) + L_{n-1}(x)\}$$

or $$\mathbf{xL_n'(x) = nL_n(x) - n\, L_{n-1}(x)}$$

III $$\mathbf{L_n'(x) = -\sum_{r=0}^{n-1} L_r(x)}$$

We have $$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)}$$

Differentiating with respect to x, we have

$$\sum_{n=0}^{\infty} t^n L_n'(x) = \frac{-t}{1-t}\sum_{r=0}^{\infty} t^r L_r(x) \quad \text{(as in II)}$$

$$= -t(1-t)^{-1}\sum_{r=0}^{\infty} t^r L_r(x)$$

$$= -t(1+t+t^2+\ldots)\sum_{r=0}^{\infty} t^r L_r(x)$$

$$= -t\sum_{s=0}^{\infty} t^s \sum_{r=0}^{\infty} t^r L_r(x)$$

$$= -\sum_{s=0,\, r=0}^{\infty} t^{r+s+1} L_r(x) \quad \text{...(i)}$$

Notes

For fixed values of r, the coefficient of t^n on the R.H.S. is $-L_r(x)$, obtained by putting $r+s+1=n$ or $s=n-r-1$.

Total Coefficient of t^n is obtained by summing over all allowed values of r.

Since $s=n-r-1$ and $r \geq 0$

Therefore $n-r-1 \geq 0$ or $r \leq (n-1)$.

$\therefore$ Coefficient of t^n on the R.H.S. $= -\sum_{r=0}^{n-1} L_r(x)$

Therefore equating coefficient of t^n, on both sides of (i), we have

$$\boldsymbol{L_n'(x) = -\sum_{r=0}^{n-1} L_r(x).}$$

Illustrative Examples

Example 1: Prove that $L_n(0) = 1$.

Solution: We have

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{\left[\frac{-tx}{(1-t)}\right]}$$

Putting $x = 0$, we have

$$\sum_{n=0}^{\infty} t_n L_n(0) = \frac{1}{(1-t)} = (1-t)^{-1}$$

$$= 1 + t + t^2 + \ldots + t^n + \ldots$$

$$= \sum_{n=0}^{\infty} t^n$$

$$L_n(0) = 1$$

Example 2: Expand $x^3 + x^2 - 3x + 2$ in a series of Laguerre polynomials.

Solution: We know that $L_n(x)$ is a polynomial of degree n. Since $x^3 + x^2 - 3x + 2$ is a polynomial of degree 3, we may write

$$x^3 + x^2 - 3x + 2 = \sum_{r=0}^{3} C_r L_r(x) \qquad \ldots\text{(i)}$$

Putting values of $L_0(x), L_1(x), L_2(x)$ and $L_3(x)$ from section 4.3, we have

$$x^3 + x^2 - 3x + 2 = c_0 + c_1(1-x) + c_2 . \frac{1}{2!}\left(2 - 4 + x^2\right) + \frac{c_3}{3!}\left(6 - 18x + 9x^2 - x^3\right)$$

Notes

or $$x^3 + x^2 - 3x + 2 = (c_0 + c_1 + c_2 + c_3) - (c_1 + 2c_2 + 3c)x + \left(\frac{c_2}{2} + \frac{3}{2}c_3\right)x^2 - \frac{c_3}{6}x^3 \qquad \text{...(ii)}$$

Equating coefficients of like powers of x on both sides of (ii), we get

$$c_0 + c_1 + c_2 + c_3 = 2$$

$$c_1 + 2c_2 + 3c_3 = 3$$

$$\frac{1}{2}c_2 + \frac{3}{2}c_3 = 1 \text{ and } -\frac{c_3}{6} = 1$$

Solving these, we get,

$$c_3 = -6, C_2 = 20, c_1 = -19, c_0 = 7 \qquad \text{...(iii)}$$

Putting these values in (i) we get

$$x^3 + x^2 - 3x + 2 = 7L_0(x) - 19L_1(x) + 20L_2(x) - 6L_3(x).$$

Example 3: Prove that

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0$$

and hence deduce that

$$L_n'(0) = -n$$

Solution: Since $L_n(x)$ satisfies the Laguerre's equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0, \text{ for } \lambda = n$$

$$\therefore \quad xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0.$$

Putting $x = 0$, we have

$$L_n'(0) = -nL_n(0)$$

or $$L_n'(0) = -n \qquad (\text{since } L_n'(0) = -1)$$

Self Assessment

15. Express $L_4(x)$ in terms of $L_3(x)$ and $L_2(x)$

16. Show that

$$L_n'(x) - L_{n+1}'(x) = L_n(x)$$

17. Show that

$$L_n''(1) + nL_n(1) = 0$$

4.6 Summary

- Laguerre differential equation has $x=0$ as a regular singular point. Thus Frobenius method is applied to get a power series.
- For $\lambda = n, n$ being a positive integer we obtain a finite power series solution known as Laguerre polynomials $L_n(x)$. The highest power of $L_n(x)$ is x^n.
- Like in the previous units here we show a generating function, Rodrigue formula for $L_n(x)$.
- $L_n(x)$ for $n = 0,1,2\ldots$ form an orthogonal set of functions and satisfy orthogonality property.
- Various recurrence relations are obtained that help in understanding Laguerre polynomials.

4.7 Keywords

Laguerre Polynomials are a finite power series in x.

Frobenius Method: Laguerre differential equation has $x=0$ as regular singular point. So Frobenius method on application gives a power series solution.

Orthogonal Relations of Laguerre polynomials are relations involving integrals of two Hermite polynomials. Due to these relations $L_n(x)$ for $n = 0,1,2,\ldots$ form an orthogonal set of functions.

4.8 Review Questions

1. Discuss the nature of singularities of the differential equation

$$xy'' + y' - xy = 0$$

2. Find all the singular points of the differential equation

$$\left(1-x^2\right)y'' - xy' + x^2 y = 0$$

3. Show from recurrence relation III

$$L_n'(x) = -\sum_{r=0}^{n-1} L_r(x)$$

Prove that

$$\int_0^\infty e^{-x} \frac{dL_n(x)}{dx} \cdot L_n(x) dx = 0, \text{ for } n = 1,2,\ldots$$

4. Show that $L_3(x), L_2(x)$ and $L_1(x)$ are related as

$$3L_3(x) = (5-x)L_2(x) - L_1(x)$$

Notes

Answers: Self Assessment

1. (b)
2. (c)
3. finite
4. n
5. $L_1(x) = 1 - x, L_2(x) = \frac{1}{2}(2 - 4x + x^2)$
7. $L_3(x) = \frac{1}{6}\left(6 - 18x + 9x^2 - x^3\right)$
8. (c)
12. (d)
13. $L_n(x)$

4.9 Further Readings

Books K. Yosida, Lectures on differential and Integral Equations

L.D. Landau and E.M. Lifshitz, Quantum Mechanics

Unit 5: Differential Equations

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know that before dealing with any equation we classify various differential equation.
- Notice that the order of a differential equation is the highest differential coefficient.
- Show how to develop a differential from a certain relation between independent variable x and dependent variable y.
- See that this unit on differential equations is helpful for the next few units.

Introduction

This chapter gives an introduction to the various types of differential equations. Some methods of solving a linear ordinary first order differential equations are given. In the next unit we will discuss the existence of the solution of a first order differential equation.

5.1 Classifications of Differential Equations

Quite often we come across a function $f(x)$ of a variable x. Whenever the value of x, known as independent variable changes, it brings about a change in the value of $f(x)$, known as dependent variable (say y). This dependence of y on x can be translated in mathematical terms by means of a differential equation. This differential equation involves y, x, and the derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ..... Sometimes $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ..... are denoted by y', y'', y'''..... respectively. Sometimes we may be concerned by more than one dependent variables. Some examples of differential equations are:

$$\frac{dy}{dx} + \sin y + x = 0 \quad \text{...(i)}$$

Notes

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{...(ii)}$$

$$\frac{d^2y}{dx^2} + m^2y = 0 \quad \text{...(iii)}$$

$$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0 \quad \text{...(iv)}$$

$$a\frac{dy}{dx} + a_2\frac{d^2y}{dx^2} + a_3\frac{d^3y}{dx^3} + ... + an\frac{d^ny}{dx^n} = Q(x) \quad \text{...(v)}$$

$$\frac{d^2y}{dx^2} + \alpha\left(\frac{dy}{dx}\right)^2 + \beta\frac{dy}{dx} + \gamma y = g(x) \quad \text{...(vi)}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0 \quad \text{...(vii)}$$

and so on.

In equations (i), (ii) only $\frac{dy}{dx}$ is present and are known as equations of first order, whereas (iii), (iv) and (vi) are known as second order differential equations. The highest power of the derivative gives the degree of differential equation. So equation (vi) is of second degree equation. Equation (v) is known as nth order differential equation involving y. The equation (vii) involves three independent variables x, y, z and one dependent variable and is known as partial differential equation. The right hand side of equations (i), (iii), (iv), (vii) is zero and they are called homogeneous equation. The coefficients a_1, a_2.....a_n of equation (v) may be constant or variables dependent on the independent variable x. The dependent variable y in equation (ii), (iii), (iv), (v) is only of first power so they are called linear differential equations.

5.2 Examples of Differential Equations

1. Find the differential equation for the equation of a circle given by

   $$x^2 + y^2 = a^2 \quad \text{...(i)}$$

   Here y is dependent and x an independent variable. Differentiation (I) w.r.t. x we have:

   $$2x + 2y\frac{dy}{dx} = 0$$

   or $y\frac{dy}{dx} + x = 0$

2. If y is given by

   $$y = Ae^x + Be^{3x} \quad \text{...(ii)}$$

   find its differential equation.

   Differentiating (ii) we have

   $$\frac{dy}{dx} = Ae^x + 3Be^{3x} \quad \text{...(iii)}$$

Differentiating again we have:

$$\frac{d^2y}{dx^2} = Ae^x + 9Be^{3x} \qquad \text{...(iv)}$$

Subtracting (iii) from (iv) we have

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 6Be^{3x} \qquad \text{...(v)}$$

Subtracting (ii) from (iii)

$$\frac{dy}{dx} - y = 2Be^{3x} \qquad \text{...(vi)}$$

Thus $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3\frac{dy}{dx} - 3y$

or $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$ ...(vii)

3. The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(viii)}$$

Find its differential equation

Differentiating (viii) we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \qquad \text{...(ix)}$$

Differentiating again, we have

$$\frac{2}{a^2} + \frac{2}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{2y}{b^2}\frac{d^2y}{dx^2} = 0 \qquad \text{...(x)}$$

From (ix), we have

$$\frac{b^2}{a^2} = -\frac{y}{x}\frac{dy}{dx} \qquad \text{...(xi)}$$

From (x) we have

$$\frac{b^2}{a^2} = -\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2} \qquad \text{...(xii)}$$

From (xi) and (xii) we have

$$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{y}{x}\right)$$

or $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx} = 0$ ...(xiii)

Notes

Thus we see from above examples that if there is one constant, the resultant differential equation is of first order. If there are two arbitrary constants the differential equation is of second order.

So from the above it is clear that nth order differential equation involves n arbitrary constant in its solution.

Self Assessment

1. Find the differential equation for the Harmonic equation

   $y = A \sin wt + B \cos wt$

   Where A, B and w are constants.

2. Find the differential equation of the following curves

   $x^2 + cy + x = e^x$

   where c is an arbitrary constant.

3. Discuss the nature of the equation

   $$a\frac{d^2y}{dx^2} + b\left(\frac{dy}{dx}\right)^2 + c\frac{dy}{dx} + dy = Q(x)$$

   where a, b and c are constants.

4. Find the differential equation of the curve

   $y = e^x(c_1 \sin x + c_2 \cos x)$

5.3 Linear Ordinary Differential Equations of First Order

The most general first order differential equation can be put in the form

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \qquad \text{...(1)}$$

where f is any arbitrary function of x, y and $\frac{dy}{dx}$. Various cases arise due to the nature of the function $f\left(x, y, \frac{dy}{dx}\right) = 0$. In the following we consider a few of them with some examples.

(A) ***The equation with separable variables. Here the equation can be put in the form***

$$M(x)\,dx + N(y)\,dy = 0 \qquad \text{...(2)}$$

where $M(x)$ is a function of x and $N(y)$ a function of y. So integrating (2) we have

$$\int M(x)dx + \int N(y)dy = a \qquad \text{...(3)}$$

where a is an arbitrary constant

Example 1: Solve

$$(y - xy)dx + x^2 dy = 0$$

Solution:

$$y(1-x)dx + x^2 dy = 0$$

or $$\frac{1-x}{x^2}dx + \frac{dy}{y} = 0, \text{ Integrating}$$

$$\int \frac{1-x}{x^2}dx + \int \frac{dy}{y} = a$$

or $$\left(-\frac{1}{x} - \log x\right) + \log y = a$$

or $$\log y - \log x - \frac{1}{x} = a$$

or $$\log\left(\frac{y}{x}\right) - \frac{1}{x} = a$$

Example 2: Solve

$$\frac{dy}{dx} + xy = 8x$$

Solution:

$$\frac{dy}{dx} + xy - 8x = 0$$

or $$\frac{dy}{dx} + x(y-8) = 0$$

or $$\frac{dy}{y-8} + xdx = 0$$

Integrating we have

$$\log(y-8) + \frac{x^2}{2} = a$$

Here a being an arbitrary constant.

(B) ***The Exact Equation***

Consider the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \quad \text{...(i)}$$

with the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{...(ii)}$$

Notes

Then equation (i) is known as exact equation. Here $\frac{\partial}{\partial X}, \frac{\partial}{\partial y}$ are partial derivative w.r.t. x and y respectively. Let us introduce $U(x, y)$ such that

$$M(x, y)\,dx + N(x, y)\,dy = dU(x, y) = 0 \quad \text{...(iii)}$$

also $$dU(x, y) = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \quad \text{...(iv)}$$

From (iii) and (iv) we have

$$\frac{\partial U(x,y)}{\partial x} = M(x,y)$$

$$\frac{\partial U(x,y)}{\partial y} = N(x,y)$$

Now $$\frac{\partial^2 U(x,y)}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence we have

$$dU(x, y) = 0$$

or $U(x, y) = a$ (a being a constant) is a solution

Example 3: Solve

$$(2xy+1)dx + (x^2+4y)dy = 0$$

Consider a function $U(x, y)$ such that

$$dU = (2xy+1)dx + (x^2+4y)dy = 0$$

$$= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

So $2xy + 1 = M$, $x^2 + 4y = N$; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\frac{\partial U}{\partial x} = 2xy + 1 \quad \text{...(i)}$$

$$\frac{\partial U}{\partial y} = x^2 + 4y \quad \text{...(ii)}$$

From (i) $U = x^2y + a(y) + b = \text{constant}$

From (ii) $U = x^2y + \frac{4y^2}{2} + c = \text{constant}$

Comparing we have $a(y) = 2y^2$ and $c = b$ so

$$U = x^2y + 2y + a$$

Notes

Example 4:

$$2x\log y dx + \frac{x^2}{y}dy = 0$$

Here $M = 2x\log y$

$$N = \frac{x^2}{y}$$

$$\frac{\partial M}{\partial y} = \frac{2x}{y}$$

$$\frac{\partial N}{\partial x} = \frac{2x}{y}$$

So $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$

Thus $2x\log y dx + \frac{x^2}{y}dy = dU(x,y) = 0$

$$\frac{\partial U}{\partial x} = 2x\log y$$

$$\frac{\partial U}{\partial y} = \frac{x^2}{y}$$

$$U = x^2\log y + C_1$$

Also $U = x^2\log y + C_2$

$\therefore x^2\log y = a$ is the solution where a is a constant.

(C) ***Integrating Factors***

Let us consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad \text{...(i)}$$

whose equation can be put into the form

$$U(x,y) = a \qquad \text{...(ii)}$$

where a is a constant, now from (ii)

$$dU(x,y) = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \qquad \text{...(iii)}$$

Notes

Thus equation (i) is exact if

$$\frac{\partial U}{\partial x} = M,$$

$$\frac{\partial U}{\partial y} = N$$

or $$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{...(iv)}$$

On the other hand if (iv) is not satisfied then we can multiply equation (i) by a function $f(x, y)$ such that

$$\left.\begin{aligned} \frac{\partial U}{\partial x} &= f(x,y)M = M' \\ \frac{\partial U}{\partial y} &= f(x,y)N = N' \end{aligned}\right\} \qquad \text{...(v)}$$

and so $$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \qquad \text{...(vi)}$$

and $f(x, y)\, M(x, y)\, dx + f(x, y)\, N(x, y)\, dy = 0$ ...(vii)

is an exact equation. Here $f(x, y)$ is known as integrating factor.

Example 5: Solve the differential equation by suitable integrating factor

$$xdy - ydx + x^2 dx = 0 \qquad \text{...(i)}$$

Solution:

Here $\left(x^2 - y\right)dx + xdy = 0$

so $M = x^2 - y$ and $N = x$

Now $\frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1$

so $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now multiplying equation (i) by $\frac{1}{x^2}$, so that

$$\frac{xdy - ydx}{x^2} + \frac{x^2 dx}{x^2} = 0 \qquad \text{...(ii)}$$

or $$d\left(\frac{y}{x}\right) + dx = 0$$

$$d\left(\frac{y}{x}+x\right)=0 \qquad ...(iii)$$

so that $\frac{y}{x}+x$ = constant $=a$ (say a) ...(iv)

so the solution of (i) is equation (iv)

(D) ***Linear Equation of the first order (non-homogeneous)***

Let the differential equation of the first order has the form

$$\frac{dy}{dx}+f_1(x)y=f_2(x) \qquad ...(i)$$

In order to solve (i), put

$$y=u(x)v(x) \qquad ...(ii)$$

Then $\frac{dy}{dx}=u\frac{dv}{dx}+\frac{du}{dx}v$

So equation (i) becomes

$$u\frac{dv}{dx}+v\frac{du}{dx}+f_1(x)u(x)v(x)=f_2(x)$$

or $$v\left[\frac{du}{dx}+f_1(x)u(x)\right]+u\frac{dv}{dx}=f_2(x) \qquad ...(iii)$$

we choose u such that

$$\frac{du}{dx}+f_1(x)u(x)=0$$

or $$\frac{du}{u}+f_1(x)dx=0 \qquad ...(iv)$$

Solving (iv) we have

$$\log u+\int f_1(x)dx=a \qquad ...(v)$$

The simplest solution is when a = 0, so that

$$u=e^{-\int f_1(x)dx} \qquad ...(vi)$$

From (iii), (iv) and (vi) we have

$$e^{-\int f_1(x)dx}\frac{dv}{dx}=f_2(x)$$

or $$\frac{dv}{dx}=f_2(x)e^{\int f_1(x)dx}$$

Thus $$v=\int e^{+\int f_1(x)dx}f_2(x)dx+a_2 \qquad ...(vii)$$

so $$y=uv=\left\{a_2e^{-\int f_1(x)dx}+e^{-\int f_1(x)dx}\int e^{+\int f_1(x)dx}f_2(x)dx\right\} \qquad ...(viii)$$

Notes

Example 6: solve

$$\frac{dy}{dx} - \frac{y}{x^2+1} = -\frac{1}{x^2+1}$$

Let $y = uv$

Here $u = e^{-\int \frac{dx}{1+x^2}} = e^{\tan^{-1}(x)}$

so that $v = \int e^{-\tan^{-1}(x)} \left[\frac{-dx}{1+x^2}\right]$

Let $\tan^{-1} x = t$

$$dt = \frac{dx}{1+x^2}$$

$$v = -\int e^{-t} dt = e^{-t} = e^{-\tan^{-1}}(x)$$

Thus $y = uv$

$$= a_2 e^{-\tan^{-1}(x)} + e^{-\tan^{-1} x} . e^{-\tan^{-1}(x)}$$

$$= a_2 e^{\tan^{-1}(x)} + 1$$

Self Assessment

5. Solve by the method (A):

$$\frac{dy}{dx} + y\cos x = 0$$

6. Solve by the method (B):

$$2x \sin y dx + x^2 \cos y dy = 0$$

7. Solve by the method (C):

$$xdy - ydx - xydx = 0$$

8. Solve by the method (D):

$$\frac{dy}{dx} - \frac{2}{x} y = x^2 \cos 3x$$

5.4 Summary

- Various differential equations are introduced for their classification.
- Some differential equations are set up after illuminating the constants from the equations relating x and y.
- Some methods of solving differential equations are given.

Notes

5.5 Keywords

The ***first order*** differential equation is an equation when the relation between x, y involves $\frac{dy}{dx}$ only. The degree of differential equation is the highest power of its derivatives.

The ***Linear differential*** equation is an equation involving linear power of independent variable.

5.6 Review Questions

1. The equation of a parabola is

   $$y^2 = 4a(x-b)$$

   where a and b are constants form a differential equation

2. If $x^2 + y^2 = a^2$, prove that

   $$\frac{dy}{dx} = -\frac{x}{y}$$

3. The equation of a straight line is

   $$y = mx + c$$

   show that its differential equation is

   $$\frac{d^2y}{dx^2} = 0$$

4. Solve by the method (A)

   $$ydx - xdy = xydx$$

5. Show by the method (B), the solution of

   $$xdx + ydy + \frac{xdy - ydx}{x^2+y^2} = 0, \text{ is } x^2 + y^2 - 2\tan^{-1}\left(\frac{x}{y}\right) = K \text{ (}K\text{ being a constant)}$$

Answers: Self Assessment

1. $\frac{d^2y}{dx^2} + w^2y = 0$

2. $x\frac{dy}{dx} - y + x^2 + (x+1)e^x = 0$

3. The equation is of second order, second degree non-homogeneous and non-linear.

4. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

5. $y = ae^{-\sin x}$, a being an arbitrary constant.

Notes

6. $x^2 \sin y = a$

7. $\log\left(\frac{y}{x}\right) - x = a$

8. $y = ax^2 + \frac{x^2}{3}\sin 3x$

5.7 Further Readings

Books

H.T.H. Piaggio, Differential Equations,

Ince E.L., Ordinary Differential Equations

N.M. Kapoor, Differential Equations

K. Yosida, Lectures in Differential and Integral Equations

Notes

Unit 6: Existence Theorem for the Solution of the Equation $\frac{dy}{dx} = f(x, y)$

CONTENTS

Objectives

After studying this unit, you will be able to:

- Discuss the existence and the uniqueness of the solution of the first order equation.
- Employ Picard's method of finding the solution. The method consists in successive approximation. It also leads to integral equations under certain conditions.
- Learn that the method is not so famous as it involves a lengthy set of solving integrals.

Introduction

The Picard's method of finding the existence of the solution of first order equation is well explained in Yosida's book.

The method is quite general and can be applied to a system of n coupled first order differential equations as well as equations of nth order. The case of nth order differential equation will be taken up in the next unit.

6.1 On the Solution of a Differential Equation

In the previous units we have been studying different types of differential equations and their solutions. Those differential equations chosen were for special purposes of studying certain functions like Bessel function, Legendre polynomials, Hermite polynomials and Laguerre polynomials. We also studied some differential equations which were easily soluble. In this unit we want to study whether a given differential equation has a solution or not. We shall see under what conditions the solution does exist.

An ordinary differential equation involves the dependent variable y, its derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2} ..... \frac{d^ny}{dx^n}$, and independent variable x in the form of a functional relation

Notes

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots\ldots \frac{d^ny}{dx^n}\right) = 0 \qquad \ldots(1)$$

The general solution of an nth order differential equation involves n arbitrary constants a_1, a_2, ... a_n. In the following we shall study the existence of an ordinary first order differential equation. The ordinary differential equation of the first order is generally written in the form

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0 \qquad \ldots(2)$$

we shall study the solution of the equation (2) with the initial conditions i.e. at

$$x = x_0, \quad y = y_0 \qquad \ldots(3)$$

We can vary x in a certain range i.e.

$$x_0 - h \leq x \leq x_0 + h \qquad \ldots(4)$$

where h is an increment to x. The above range of x is in a domain D. When x varies in the above range we want to see how y changes from the initial value y_0. Let us assume that y varies in the range

$$y_0 - k \leq y \leq y_0 + k \qquad \ldots(5)$$

So let D be a domain in (x, y) plane given by (4) and (5). Let the set of points in (4) are given by $x_0, x_1, \ldots x_n, \ldots$ and set of points in (5) are given by $y_0, y_1, \ldots y_n, \ldots$. We want to study the existence and uniqueness of the solution of equation (2). There are various forms of (2). We in particular study the equation in the form

$$\frac{dy}{dx} = f(x, y) \qquad \ldots(6)$$

subject to the initial conditions (3).

6.2 Picard's Method

Our purpose is to find a solution of equation (6) subject to the initial condition (3). To formula the problem we have to make the following assumptions concerning $f(x, y)$. The behaviour of $f(x, y)$ will decide the solution of (6).

Assumption 1: The function $f(x, y)$ is real-valued and continuous on a domain D of the (x, y) plane given by

$$x_0 - h \leq x \leq x_0 + \text{h}, \; y_0 - k \leq y \leq y_0 + k \qquad \ldots(7)$$

Here h, k are positive numbers.

Assumption 2: $f(x, y)$ satisfies the Lipschitz condition with respect to y in D, that is, there exists a positive constant k such that

$$|f(x, y_1) - f(x, y_2)| \leq k\,|y_1 - y_2| \qquad \ldots(8)$$

for every pair of points (x, y_1), (x, y_2) of D.

If $f(x, y)$ has a continuous partial derivative $\frac{\partial t(x, y)}{\partial y}$ then assumption 2 is satisfied. Now since D is a bounded closed domain and $\left|\frac{\partial f(x,y)}{\partial y}\right|$ is continuous in D so $\left|\frac{\partial f(x,y)}{\partial y}\right|$ is bounded. Put

Notes

$$k = \sup_{(x,y)\varepsilon D} \left|\frac{\partial t(x,y)}{\partial y}\right| \qquad ...(9)$$

where k is a limit superior.

Then the mean value theorem implies that (8) holds for $f(x, y)$. By Assumption 1, $f(x, y)$ is continuous on the bounded domain D, therefore $|f(x, y)|$ is bounded on D, that is,

$$\text{SUP}_{(x,y)\varepsilon D} |f(x,y)| = M < \infty \qquad ...(10)$$

Set

$$\delta = \text{Min}\ (h, \text{k/m}) \qquad ...(11)$$

Let us define a sequence of functions $\{y_n(x)\}$ for $|x - x_0| \le \delta$,

successively by

$$\left.\begin{aligned} &y_0(x) = y_0 \\ &y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)dt \\ &y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1)dt \\ &\text{...} \\ &\text{...} \\ &\text{...} \\ &y_n(x) = y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)dt \end{aligned}\right\} \qquad ...(12)$$

Theorem: That $\{y_n(x)\}$ converges uniformly on the internal $|x - x_0| \le \delta$, and the limit $y(x)$ of the sequence is a solution of (5) which satisfies (3).

Picard's Method of Successive Approximation

The above theorem is proved by Picard's method of successive approximation as follows. We here give this proof as shown by K. Yosida.

Proof: According to (10) and (11), we obtain

$$|y_1(x) - y_0| \le \delta M \leqq \text{k}$$

for $|x - x_0| \to \delta$. Therefore $\int_{x_0}^{x} f(t, y_1(t)dt$ can be defined for $|x - x_0| \le h$, and

$$|y_2(x) - y_0| \leqq \delta M \le K$$

In the same manner, we can define $y_3(x)$,......$y_n(x)$ for $|x - x_0| \leqq \delta$ and obtain

$$|y_k(x) - y_0| \leqq \delta M \leqq K, \text{ for } K = 1, 2, ...n$$

using assumption (2), we have

$$|y_{k+1}(x) - y_k(x)| \leqq K \left|\int_{x_0}^{x} |y_k(t) - y_{k-1}(t)|\, dt\right.$$

for $|x - x_0| \leqq \delta$. Therefore, if we assume that for $k = 1, 2,$ ......n

Notes

$$|y_l(x) - y_{l-1}(x)| \leqq \frac{h|K|x - x_0|^{l-1}}{(l-1)!} \text{ for } |x - x_0| \leqq \delta \qquad ...(13)$$

We obtain for $l = n + 1$,

$$|y_{n+1}(x) - y_n(x)| \leqq \frac{k|K|x - x_0|^n}{n!} \text{ for } |x - x_0| \leqq \delta \qquad ...(14)$$

Since (13) holds for $n = 1$ as mentioned above, we see, by mathematical induction, that (14) holds for every n. Thus for $m > n$, we obtain

$$|y_m(x) - y_n(x)| \leqq \left|\sum_{l=n}^{m-1} y_{l+1}(x) - y_l(x)\right| \leqq k\sum_{l=n}^{m-1}\frac{(k\delta)^l}{l!} \qquad ...(15)$$

Since the right hand side of (15) tends to zero as $n \to \infty$, $\{y_n(x)\}$ converges uniformly to a function $y(x)$ on the interval $|x - x_0| \leqq \delta$. As the convergence is uniform, $y(x)$ is continuous and more over, evidently, $y(x_0)$ to y_0. To prove that $y(x)$ is the solution, we know that as the sequence of functions $\{y_n(x)\}$ converges uniformly and $y_n(x)$ is continuous on the interval $|x - x_0| \leqq \delta$, then the lim and integral can be interchanged. Thus

$$\lim_{n\to\infty}\int_{x_0}^{x} y_n(x)dx \to \int_{x_0}^{x}\lim_{n\to\infty} y_n(x)dx$$

Hence we obtain

$$\begin{aligned} y(x) &= \lim_{n\to\infty} y_{n+1}(x) \\ &= y_0 + \lim_{n\to\infty}\int_{x_0}^{x} f(t, y_n(t))dt \\ &= y_0 + \int_{x_0}^{x}[\lim_{n\to\infty} f(t, y_n(t))]dt \\ &= y_0 + \int_{x_0}^{x} f(t, y(t))dt \end{aligned}$$

that is,

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \qquad ...(16)$$

The integrand $f(t, y(t))$ on the right side of (16) is a continuous function, hence $y(x)$ is differentiable with respect to x, and its derivative is equal to $f(x, y(x))$.

Hence the proof.

Integrating from x_0 to x, we see that a solution $y(x)$ of (6) satisfying the initial conditions (3), must satisfy the integral equation (16). The above proof also shows that the integral equation can be solved by the method of successive approximation.

Uniqueness of Solution

In the above treatment we have obtained by the method of successive approximation, a solution $y(x)$ of (6) satisfying the initial condition (3). We have yet to show the uniqueness of the above solution.

Proof:

If the solution $y(x)$ is not unique, let $z(x)$ be another solution of (6), such that $z(x_0) = y_0$. Then

$$z(x) = y_0 + \int_{x_0}^{x} f\left(t, z(t)\right)dt.$$

Notes

By assumption 2, we obtain

$$|y(x) - z(x)| \leqq \int_{x_0}^{x} |y(t) - z(t)dt| \quad ...(17)$$

Therefore we also obtain for $|x - x_0| \leqq \delta$.

$$|y(x) - z(x)| \leqq KN|x - x_0|$$

where

$$N = \text{SUP}_{|x-x_0|\delta} \; |y(x) - z(x)|.$$

Substituting the above estimate for $|y(t) - z(t)|$ on the right side of (17), we obtain

$$|y(x) - z(x)| \leqq N\,(K|x - x_0)^2|2!$$

for $|x - x_0| \leqq \delta$. Substituting this estimate for $|y(t) - z(t)|$ once more on the right side of (17), we have

$$|y(x) - z(x)| \leqq N(K|x - x_0|)^3/3! \text{ for } |x - x_0| \leqq \delta.$$

Repeating this substitution, we obtain

$$|y(x) - z(x)| \leqq N(K|x - x_0|)^m/m!,\ m = 1, 2, ... \quad ...(18)$$

for $|x - x_0| \leqq \delta$. The right side of the above inequality tends to zero as $m \to \infty$. This means that

$$N = \text{SUP}_{|x-x_0| \leqq \delta} \; |y(x) - z(x)|$$

is equal to zero.

Hence $y(x)$ given by (16) is a unique solution.

Example 1: Solve

$$\frac{dy}{dx} = xy \quad ...(1)$$

with the initial conditions $x = 0.0$, $y(0) = 0.1$

Now $y_0(x) = 0.1$

$$y_1(x) = 0.1 + \int_0^x x\, y_0(x)dx$$

$$= 0.1 + \int_0^x x\,(0.1)dx$$

$$= 0.1 + 0.1\frac{x^2}{2} = 0.1\left(1 + \frac{x^2}{2}\right)$$

$$y_2(x) = 0.1 + \int_0^x x\, y_2(x)dx$$

$$= 0.1 + 0.1\int_0^x x\left(1 + \frac{x^2}{2}\right)dx$$

$$= 0.1 + 0.1\left(\frac{x^2}{2} + \frac{x^4}{2.4}\right)$$

$$= 0.1 + \left(1 + \frac{x^2}{2} + \frac{x^4}{2.4}\right)$$

Notes

$$y_3(x) = 0.1 + 0.1\int_0^x x\left(1 + \frac{x^2}{2} + \frac{x^4}{2.4}\right)dx$$

$$= 0.1 + 0.1\left(\frac{x^2}{2} + \frac{x^4}{2.4} + \frac{x^6}{2.4.6}\right)$$

$$= 0.1\left(1 + \frac{x^2}{2} + \frac{x^4}{2.4} + \frac{x^6}{2.4.6}\right)$$

....

....

....

$$y_k(x) = 0.1\left(1 + \frac{x^2}{2} + \frac{1}{2^2.1.2}(x^2)^2 + ... + \frac{(x^2)^k}{2^k k!}\right) \quad ...(2)$$

So the solution of equation (1) is $y(x)$

$$y(x) = \lim_{k \to \infty} y_k(x) = 0.1\left[1 + \frac{x^2}{2} + \frac{1}{2^2 2!}(x^2)^2 + \frac{1}{2^3 3!}\left(\frac{x^2}{2}\right)^3 + ...\right] \quad ...(2)$$

The above series is a convergent series

Example 2: Solve the following by Picard's method of integrating by successive approximation

$$\frac{dy}{dx} = z,$$

$$\frac{dy}{dx} = x^3(y + z)$$

where $y = 1$ and $z = \frac{1}{2}$ when $x = 0$

Here $\quad y = 1 + \int_0^x z\,dx$ and $z = \frac{1}{2} + \int_0^x x^3(y + z)dx$

The first approximation gives us

$$y = 1 + \int_0^x \left(\frac{1}{2}\right)dx = 1 + \frac{x}{2},$$

$$z = \frac{1}{2} + \int_0^x x^3\left(1 + \frac{1}{2}\right)dx = \frac{1}{2} + \frac{3}{2}.\frac{x^4}{4}$$

Second approximation

$$y = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8}x^4\right)dx = 1 + \frac{x}{2} + \frac{3}{40}x^5$$

$$z = \frac{1}{2} + \int_0^x x^3\left(\frac{3}{2} + \frac{x}{2} + \frac{3}{8}x^4\right)dx = \frac{1}{2} + \frac{3}{8}x^4 + \frac{1}{10}x^5 + \frac{3}{64}x^8$$

Third approximation

$$y = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8}x^4 + \frac{1}{10}x^5 + \frac{3}{64}x^8\right)dx$$

Notes

$$= 1 + \frac{x}{2} + \frac{3}{40}x^5 + \frac{x^6}{60} + \frac{x^9}{192}$$

$$z = \frac{1}{2} + \int_0^x x^3\left(\frac{3}{2} + \frac{x}{2} + \frac{3}{8}x^4 + \frac{7}{40}x^5 + \frac{3}{64}x^8\right)dx$$

$$= \frac{1}{2} + \frac{3}{8}x^4 + \frac{x^5}{10} + \frac{3}{64}x^8 + \frac{7}{360}x^9 + \frac{x^{12}}{256}$$

and so on. So the series solution of y and z are convergent for $x < 1$.

Self-Assessment

1. Solve the differential equation

$$\frac{dy}{dx} = y$$

under the initial conditions $y = 1$ for $x = 1$ by the method of successive approximations.

2. Solve the differential equation

$$\frac{dy}{dx} = x + y^2$$

under the initial condition $y = 0$ when $x = 0$.

6.3 Remark on Approximate Solutions

On letting $m \to \infty$ in equation (15), we obtain

$$|y(x) - y_n(x)| \leqq K \sum_{k=n}^{\infty} \frac{(K\delta)^k}{\underline{|\delta}} \quad ...(1)$$

for $|x - x_0| \leqq \delta$. The equation (17) is an estimate of the error of the nth approximate solution $y_n(x)$. The method of successive approximation may be used, in principle. However this method is not always practical because it requires one to repeat the evaluation of indefinite integrals many times.

We shall now consider another method which is sometimes rather useful. Suppose that $g(x, y)$ is a suitable approximation to $f(x, y)$ such that we can find the solution $z(x)$ of the differential equation

$$\frac{dz}{dx} = g(x, y) \quad ...(2)$$

On the interval $|x - x_0| \leqq \delta$ satisfying the initial condition $z(x_0) = y_0$. We put

$$\text{SUP}_{(x,y)\varepsilon D} |f(x, y) - g(x, y)| \leqq \varepsilon \quad ...(3)$$

Let $y(x)$ be the unique solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \quad ...(4)$$

on the interval $|x - x_0| \leqq h$ satisfying the initial condition $y(x_0) = y_0$. Then from (2) it follows that

$$y(x) - z(x) = \int_{x_0}^{x} (f(t, y(t)) - g(t, z(t))dt.$$

Notes

We obtain by assumption 2,

$$|y(x) - z(x)| = \left|\int_{x_0}^{x} \{f(t, z(t) - g(t, z(t)\}dt + \int_{x_0}^{x} \{f(t, y(t) - f(t, z(t)\}dt\right|$$

$$\leqq \left|\int_{x_0}^{x} \{f(t, z(t) - g(t, z(t)\}dt\right| + K\left|\int_{x_0}^{x} |y(t) - z(t)|\,dt\right|$$

$$\leqq \varepsilon|x - x_0| + K|\int_{x_0}^{x} |y|(t) - z(t)dt \quad \text{...(5)}$$

Therefore setting

$$\text{SUP}_{|x - x_0| \leqq \varepsilon} |y(x) - z(x)| = M',$$

We have

$$|y(x) - z(x)| \leqq \varepsilon\ |x - x_0| + KM'\ |x - x_0|$$

for $|x - x_0| \leqq \delta$. Substituting this estimate for $|y(t) - z(t)|$ on the right hand side of (5), we obtain

$$|y(x) - z(x)| \leqq \frac{M'K^2|x - x_0|^2}{\underline{|2}} + \varepsilon\sum_{m=1}^{2}\frac{K^{m-1}|x - x_0|^m}{m!}$$

for $|x - x_0| \leqq \delta$. Repeating this substitution, we obtain, for each $n = 1, 2, 3, ...$,

$$|y(x) - z(x)| \leqq \frac{M'K^n|x - x_0|^n}{n!} + \varepsilon\sum_{m=1}^{n}\frac{K^{m-1}|x - x_0|^m}{m!}$$

for $|x - x_0| \leqq \delta$. As $n \to \infty$ the first term on the right hand side converges to zero uniformly on the interval $|x - x_0| \leqq \delta$. The second term is less than

$$\varepsilon K^{-1}\{\exp(K|x - x_0|) - 1\}$$

Accordingly, the estimate of the error of the appropriate solution $z(x)$ in the interval $|x - x_0| \leqq \delta$ is given by

$$|y(x) - z(x)| \leqq (\varepsilon K)\ (\exp(K(x - x_0) - 1) \quad \text{...(6)}$$

6.4 Solutions by Power Series Expansion

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{...(1)}$$

in the case when $f(x, y)$ is a complex valued function of complex variables x and y. We assume that $f(x, y)$ can be expanded in a convergent power series in $(x - x_0)$ and $(y - y_0)$ in a domain D′ of the complex (x, y) space given by

$$|x - x_0| < a',\ |y - y_0| < b'.$$

Notes

In other words, $f(x, y)$ is regular function in the domain D'. From this assumption it follows that $\frac{\partial f(x,y)}{dy}$ is also regular in D'. Therefore, for any positive numbers a, b such that $a < a'$ and $b < b'$, both $|f(x, y)|$ and $\frac{\partial f(x,y)}{dy}$ are continuous on the closed domain D given by

$$|x - x_0| \leqq a, \; |y - y_0| \leqq b$$

Thus there exist positive numbers M and K such that

$$\left.\begin{aligned} &\text{SUP}_{(x,y)\varepsilon D} |f(x,y)| = M < \infty \\ &\text{SUP}_{(x,y)\varepsilon D} \left|\frac{\partial f(x,y)}{\partial y}\right| = K < \infty \end{aligned}\right\} \quad \text{...(2)}$$

Integrating $\frac{\partial f(x,y)}{\partial y}$ along the segment connecting y_1 and y_2, we obtain

$$f(x, y_1) - f(x, y_2) = \int_{y_1}^{y_2} \frac{\partial f(x,y)}{\partial y} dy.$$

Hence the Lipschitz condition

$$|f(x, y_2) - f(x, y_1)| \leqq K|y_2 - y_1| \quad \text{...(3)}$$

holds on D. Therefore, under the above assumption, we can apply to the equation (1), the method of successive approximations and the domain

$$|x - x_0| \leqq h = \min|a, b/M| \quad \text{...(4)}$$

as follows, we write

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)dt$$

....

....

$$y_n(x) = y_0 + \int_{x_0}^{x} f\{x, y_{n-1}(t)\}dt$$

where the integration means complex integration along a smooth curve connecting x_0 and x in the domain (4). Since $f(x, y_0)$ is regular in the domain $|x - x_0| < h$, the first integral is well-defined, independent of the curves, and hence so is y_1. Taking the first integral along the segment connecting x_0 and x, we obtain,

$$|y, (x) - y_0| \leqq hM \leqq b$$

Hence $f\{x, y_1(x)\}$ is well defined for $|x - x_0| < h$ as a function of x.

Since $y_1(x)$ is given by the integral of the regular function $f(x, y_0)$, $y_1(x)$ is regular in the domain $|x - x_0| < h$. Hence $f\{(f, y_1(x)\}$ is also regular. Therefore the second integral is well defined and hence $y_2(x)$ is well defined and regular. Taking the integral along the segment connecting x_0 and x, we obtain further

$$|y_2(x) - y_0(x)| \leqq hM \leqq b.$$

Notes

In this way we can define $y_3(x)$, $y_4(x)$,...... successively in the domain $|x - x_0| < h$. The functions $f_n(x)$, $n = 1, 2, 3$, ... all regular in the domain $|x - x_0| < h$ and

$$|y_n(x) - y_0| \leqq b.$$

So taking the integral along the segment connecting x_0 and x we can prove that the sequence of regular functions $|y_n(x)|$ converges uniformly in the domain $|x - x_0| < h$ and that the limit function $y(x)$ satisfies

$$y(x_0) = y_0 \text{ and } \frac{dy(x)}{dx} = f(x, y)$$

in the domain $|x - x_0) < h$. As $y(x)$ being the uniform limit of the sequence of regular functions is also regular.

The Method of Undetermined Coefficients

Since in the previous section we have guaranteed the existence of the regular solution $y(a)$, we can calculate this solution by the method of undetermined coefficients as follows. By virtue of its regularity, $y(x)$ can be expanded in a power series

$$y(x) = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_{x_0} + \frac{(x-x_0)^2}{\underline{|2}}\frac{d^2y}{dx^2} + ...$$

in the domain $|x - x_0| < h$. Substituting this expansion for y on the right hand side of the equation and differentiating we obtain

$$\frac{dy}{dx} = f(x, y)$$

$$\frac{d^2y}{dx^2} = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx}$$

....

....

setting in these equations $x = x_0$ and $y = y_0$ we can determine successively the expansion coefficients

$$\left.\frac{dy}{dx}\right|_{x_0}, \left.\frac{d^2y}{dx^2}\right|_{x_0}, \left.\frac{\partial^3 y}{\partial x^3}\right|_{x_0} ......$$

6.5 Summary

- Picard method of finding the conditions under which the solution of the first order differential equation is described.
- The method involves on the successive approximation and proving the uniform convergence of the series. It also reduces to an integral equation.
- The Picard method of successive approximation does not find favour of the method of existence as compared to Cauchy's method of comparison test or other numerical methods like Runge's method.

6.6 Keyword

Notes

The method of finding the conditions for the existence of the solution of the ***first order differential equation*** is quite appealing but sometimes cumbersome.

6.7 Review Questions

1. Solve $\frac{dy}{dx} = x - y$.

   when $x = 0$, $y = 1$, by Picard method up to fifth successive approximation

2. Solve $\frac{dy}{dx} = 3x + y^2$

   given $x = 0$, $y = 1$.

   up to third successive approximation.

Answers: Self-Assessment

1. $y = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + ...$

   $= \sum_{n=0}^{\infty} \frac{x^n}{\underline{|n}}.$

2. $y = \frac{1}{2}x^2 + \frac{1}{20}x^5 + \frac{1}{160}x^8 + \frac{1}{4400}x^{11}$.

6.8 Further Readings

Books

Yosida, K., Lectures in Differential and Integral Equations

Piaggio, H.T.H., Differential Equations

Notes

Unit 7: General Properties of Solutions of Linear Differential Equations of Order *n*

CONTENTS

Objectives

After studying this unit, you should be able to:

- Deal with a differential equation of order n, and there are lots of properties to be kept in mind before actually solving any problem.
- Discuss Picard method of existence and uniqueness of the linear differential equation before solving any problem.
- Know some properties of linear differential equation of nth order with constant coefficients and the solutions obtained both for complementary functions (C.F.) and Particular Integral (P.I.)

Introduction

The method of proof of the existence of the solution of nth order differential equation is similar to that of first order one.

Some properties of the differential equations are listed and later used to find the solutions of a class of nth order differential equations.

7.1 Existence and Uniqueness of the Solution of a System of Differential Equations

An nth order linear differential equation involving dependent variable y and independent variable x can be written as

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + ay \quad = \quad 0$$

Notes

Assuming that $a_n \neq 0$, we can write the above equation in the form

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^{n-1}y}{dx^{n-1}}\right) \quad ...(1)$$

We are interested in solving the equation (1) under the initial conditions

$$y(x_0) = y_0, \frac{dy}{dx}(x_0) = y_0', ..., \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_0^{n-1} \quad ...(2)$$

Let us define

$$\left.\begin{aligned} \frac{dy}{dx} &= y_1 \\ \frac{dy_1}{dx} &= y_2 \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ \frac{dy_{n-2}}{dx} &= y_{n-1} \end{aligned}\right\} \quad (3)$$

$$\frac{dy_{n-1}}{dx} = y_n = f(x, y, y_1, ..., y_{n-1})$$

with the initial conditions

$$y(x_0) = y_0, y_1(x_0) = y_0', y_2(x_0) = y_0''....y_{n-1}(x_0) = y_0^{(n-1)} \quad ...(4)$$

We may consider more generally, the system of ordinary differential equations

$$\left.\begin{aligned} \frac{dz_1}{dx} &= f_1(x, z_1, z_2, ...z_n) \\ \frac{dz_2}{dx} &= f_2(x, z_1, z_2, ...z_n) \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ \frac{dz_n}{dx} &= f_n(x, z_1, z_2, ..., z_n) \end{aligned}\right\} \quad (5)$$

with the initial conditions

$$z_m(x_0) = y_0^{(m-1)}, m = 1, 2, ..., n$$

where $y_0^{(0)} = y_0$. For this problem we shall prove the following theorem 1.

Theorem 1: Let

$$f_1(x, z_1, z_2, ...z_n), f_2(x, z_1, z_2, ...z_n), ... \quad f_n(x, z_1, z_2, ...z_n) \quad ...(6)$$

be real valued and continuous on a Domain of the real $(x, z_1, z_2, ..., z_n)$ space given by

$$|x - x_0| \leqq a, \left|z_m - y_0^{(m-1)}\right| \leqq b, \quad m = 1, 2, ...n \quad ....(7)$$

Notes

Assume that Lipschitz condition with respect to z_1, z_2, ... z_n is satisfied in *D*, that is, there exists positive constant *k* such that for every pair of points $(x, \varepsilon_1, \varepsilon_2 ... \varepsilon_n), (x, \eta_{1,} \eta_{2,...} \eta_n)$ in *D*

$$|f_i(x, \varepsilon_1, \varepsilon_2, ... \varepsilon_3) - f_i(x, \eta_1, \eta_2, ..., \eta_n)| \leqq K \sum_{m=1}^{n} |(\varepsilon_m - \eta_m)|$$

for every i = 1, 2, ..., *n*. Further let

$$\left.\begin{aligned} h &= \min(a, b/m) \\ M &= \sup_{\substack{(x, z_1, ..., z_n) \in D \\ i=1,2,3,....m}} |f_i(x, z_1, z_2, z_3, ...z_n)| \end{aligned}\right\} \quad ...(8)$$

Then there exists one and only one set of solution $z_1(x), z_2(x) ... z_n(x)$ of (5) on the interval

$$|x - x_0| \leqq \text{h} \quad ...(9)$$

satisfying the initial conditions (6).

This theorem implies the following:

Assume that $f(x, z_1, z_2, ...z_n)$ is real valued and continuous on the domain D and satisfies the Lipschitz condition on D, that is for every pair of points $(x, \varepsilon_1, \varepsilon_2, ..., \varepsilon\ ), (x, \eta_1, \eta_2, ...\eta_n)$ of D,

$$|f(x, \varepsilon_1, \varepsilon_2, ...\varepsilon_n) - f(x, \eta_1, ...\eta_n)| \leqq K \sum_{m=1}^{n} |\varepsilon_n - \eta_m|.$$

Then there exists one and only one solution *y*(*x*) of the equation (1) satisfying the initial conditions (2) on the interval.

$$|x - x_0| \leqq h.$$

where $h = \min(a, b/m)$ and $m = \sup_{(x, z_1, z_{2,...zn}) \in D} |f(x, z_1, z_2, ...z_n)|$

Proof of the theorem 1

The proof of the theorem 1 is entirely the same as in the case of the first order differential equation in unit 6. The initial value problem for (5) with (6) can be reduced to the system of integral equations.

$$z_m(x) = y_0^{(m-1)} + \int_{x_0}^{x} f_m\left(t, z, (t), z_2(t), ... z_n(t) dt\right) \qquad (m = 1, 2, ... n)$$

and solved by the method of successive approximations. In this case the successive approximation functions are defined by

$$z_{m,1}(x) = y_0^{(m-1)} + \int_{x_0}^{x} f_m(t, y_0, y_0^1, y_0^2, ..., y_0^{(n-1)}) dt$$

$$z_{m,2}(x) = y_0^{(m-1)} + \int_{x_0}^{x} f_m(t, z_1, (t), z_1, 1(t), ...z_{n,1}(t) dt$$

....

$$z_{m,k}(x) = y_0^{(m-1)} + \int_{x_0}^{x} f_m\left(t, z_{1,k-1}, (t)\ z_{2,k-1}(t), z_{3,k-1}(t) ... z_{n,k-1}(t)\right) dt$$

Then by virtue of the Lipschitz condition, we obtain

Notes

$$\sum_{n=1}^{m} \left| z_{m,k}(x) - z_{m,k-1}(x) \right| \leqq K \left| \int_{x_0}^{x} \sum_{m=1}^{n} \left| z_{m,k-1}(t) - z_{m,k-2}(t) \right| dt \right|$$

From this we obtain, for $k > s$

$$\sum_{m=1}^{n} \left| z_{mk}(x) - z_{m,s}(x) \right| \leqq nb \sum_{t=s}^{k-1} \frac{(K|x - x_0|)^t}{\underline{|t}} \qquad ...(10)$$

On the interval (9), provided that $z_{m,l}(x) = y_0^{m-l}$. This suffices to prove the theorem.

7.2 General Properties of Solution of Linear Differential Equations of Order *n*

We now discuss some of the properties of the solution of *n*th order linear differential equations. For this purpose write down the differential equation in the form

$$\frac{d^n}{dx^n} y + p_1(x) \frac{d^{n-1}y}{dx^{n-1}} + ... + p_n y = p_n y = q(x) \qquad ...(1)$$

The equation (1) is said to homogeneous if $q(x) = 0$, otherwise it is called inhomogeneous. We assume that the coefficients $p_1, p_2, ...p_n$, $q(x)$ are all continuous on a domain D. We state that

(1) If $y_1(x)$ and $y_2(x)$ are any two non-zero solutions of equation (1) then $y_1(x) + y_2(x)$ is also a solution.

(2) In fact if $y_1(x)$, $y_2(x)$, $y_3(x)...y_n(x)$ are solutions of equation (1) then any linear combination

$$y = \sum_{i=1}^{m} c_i y_i \qquad ...(2)$$

of these solutions with arbitrary coefficients c_1, c_2, ..., c_m is also a solution of (1). This fact is called the *principle of superposition*.

(3) Let $y_1(x)$, y_2,...y_{n+1} be an arbitrary set of $n + 1$ solutions of equation (1), then there exist $n + 1$ numbers c_1, c_2, ..., c_{n+1} not all zero such that

$$\sum_{i=1}^{n+1} c_i y_i(x) = 0 \qquad ...(3)$$

that means that the set of $n + 1$ functions y_1, y, ...y_{n+1} is a dependent set.

Thus if we have a set of n independent functions y_1,.... y_n then the most general solution of equation (1) is written as

$$y = \sum_{i=1}^{n} c_i y_i \qquad ...(4)$$

So a set of n solutions of $y_1(x)$, $y_2(x)$,...$y_n(x)$, which are linearly independent is called a *fundamental system of the solutions* of equation (1) (or general solution)

Notes

(4) ***Relations between the solution and the coefficients***

Let $y_1(x)$, $y_2(x)$,...$y_n(x)$ be a fundamental system of the solutions of (1). If every $y_i(x)$ (i = 1, 2, ... n) satisfies another equation

$$\frac{d^n y}{dx^n} + r_1 \frac{d^{n-1} y_i}{dx^{n-1}} + ... + r_n y_i = 0$$

With continuous coefficients $r_i(x)$, i = 1, 2, ... n in the domain D then we have

$$r_i(x) \equiv p_i(x), \qquad i = 1, 2, ...n.$$

This fact may be stated as follows:

The coefficients of a linear differential equation of the nth order are determined uniquely by an arbitrary chosen fundamental system of the solutions, provided the coefficient of $\frac{d^n y}{dx^n}$ is identically one.

Let us write equation (1) as

$$y^n + p_i y^{n-1} + p_2 y^{n-2} + ...p_n y = 0 \qquad ...(5)$$

with conditions

$$y(x_0) = \eta, y'(x_0) = \eta', ...y''(x_0) = \eta^n, ...y^n(x_0) = \eta^n \qquad ...(6)$$

(5) ***Wronskian. Liouville's formula***

We shall enter into the details of the relations between the solutions and the coefficients mentioned above. We denote by $W(y, y_1, y_2,....y_n)$ the determinant

$$\begin{vmatrix} y & y_1 & y_2 & \cdots & y_n \\ y' & y_1' & y_2' & \cdots & y_n' \\ y'' & y_1'' & y_2'' & \cdots & y_n'' \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

which is called the Wronskian of the $n + 1$ functions $y, y_1, y_2, ..., y_n$. We consider the linear differential equation

$$W(y, y_1(x), y_2(x), ..., y_n(x)) = 0 \qquad ...(i)$$

where y is unknown and $y_1(x), y_2(x), ..., y_n(x)$ is a fundamental system of the solutions of (5). Since

$$W(y_i(x), y_1(x), y_2(x), ..., y_n(x)) = 0 \qquad (i = 1, 2, ..., n)$$

every $y_i(x)$ satisfies the equation (i). Furthermore, as will be shown shortly, the coefficient

$$(-1)^n W(y_1(x), y_2(x), ..., y_n(x)) \qquad ...(ii)$$

of $y^{(n)}$ in (i) does not vanish at any point in the domain D. Therefore, we obtain the following identity

$$y^{(n)} + p_1(x) y^{(n-1)} + ... + p_{n-1}(x) y' + p_n(x) y = \frac{(-1)^n W(y, y_1(x), y_2(x), ..., y_n(x))}{W(y_1(x), y_2(x), ..., y_n(x))} \qquad ...(iii)$$

This gives the relations between the solutions and the coefficients.

Notes

Now we shall prove that (ii) does not vanish at any point in D. Suppose that there exists a point x_0 in D for which

$$W(y_1(x_0), y_2(x_0), ..., y_n(x_0)) = 0 \quad ...(iv)$$

Then the system of linear equations with the coefficients $y_i^{(j)}(x_0)$

$$C_1 y_1(x_0) + C_2 y_2(x_0) + ... + C_n y_n(x_0) = 0$$

$$C_1 y_1'(x_0) + C_2 y_2'(x_0) + ... + C_n y_n'(x_0) = 0$$

....

$$C_1 y_1^{(n-1)}(x_0) + C_2 y_2^{(n-1)}(x_0) + ... + C_n y_n^{(n-1)}(x_0) = 0$$

has solutions C_1, C_2, ..., C_n, not all zero. The linear combination

$$y(x) = \sum_{i=1}^{n} C_i \, y_i(x)$$

of $y_i(x)$ with these coefficients C_i obviously satisfies the equation (5) and the initial conditions (6) at the point x_0 in D. Therefore, we have

$$y(x) = \sum_{i=1}^{n} C_l \, y_i(x) \equiv 0$$

This contradicts the fact that $y_1(x), y_2(x), ..., y_n(x)$ are linearly independent. Therefore, the Wronskian of linearly independent solutions $y_1(x), y_2(x), ..., y_n(x)$ does not vanish at any point in D.

Next we shall consider the Wronskian $W\left(y_1(x), y_2(x), ..., y_n(x)\right)$ of n solutions $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ where $y_1(x), y_2(x), ... y_n(x)$ are not necessarily linearly independent. Differentiating $W\left(y_1(x), y_2(x), ..., y_n(x)\right)$ with respect to x, we obtain

$$\frac{dW\left(y_1(x), y_2(x), ..., y_n(x)\right)}{dx} = \begin{vmatrix} y_1(x), & y_2(x), & ..., & y_n(x) \\ y_1'(x), & y_2'(x), & ..., & y_n'(x) \\ \multicolumn{4}{c}{..} \\ y_1^{(n-1)}(x), & y_2^{(n-2)}(x), & ..., & y_n^{(n-2)}(x) \\ y_1^{(n)}(x), & y_2^{(n)}(x), & ..., & y_n^{(2)}(x) \end{vmatrix} \quad ...(v)$$

Since $y_l(x)$ satisfies the equation (5)

$$y_l^{(n)}(x) = -\sum_{k=1}^{n-1} p_k(x) y_l^{(n-k)}(x) - p_n(x) y_i(x)$$

Substituting this in the above determinant, we obtain

$$= \frac{dW\left(y_1(x), y_2(x), ..., y_n(x)\right)}{dx} \quad ...(vi)$$

$$= -p_i(x) W\left(y_1(x), y_2(x), ..., y_n(x)\right)$$

Notes

Accordingly, $W(y_1(x), y_2(x)..., y_n(x))$ transpose is a solution of the linear homogeneous equation (vi) with coefficients continuous in D. Therefore, if $W(y_1(x), y_2(x)..., y_n(x))$ vanishes at a point in D, then, $W(y_1(x), y_2(x)..., y_n(x))$ is identically zero in the whole domain D. This proves the following theorem.

Theorem 1: Either the Wronskian of n solutions of (5) is identically zero or it never vanishes at any point in D.

By integration of the equation (vi), we obtain

$$W(y_1(x), y_2(x)..., y_n(x)) = W(y_1(x_0), y_2(x_0)..., y_n(x_0))\exp\left(-\int_{x_0}^{x} p_1(t)dt\right), x \in D \quad ...(vii)$$

which is called *Liouville's formula*. From (3), it follows immediately that, if n solutions $y_1(x)$, $y_2(x)$, ... $y_n(x)$ of (5) are linearly dependent, then the Wronskian $W(y_1(x), y_2(x), ..., y_n(x))$ is identically zero on D. Thus we obtain the following:

Theorem 2: Let $y_1(x)$, $y_2(x)$, ... $y_n(x)$ be n solutions of the equation (5). Then these solutions are linearly independent if and only if the Wronskian $W(y_1(x), y_2(x), ..., y_n(x))$ does not vanish at any point in D. Further, these solutions are linearly dependent if and only if their Wronskian is identically zero in D.

(6) ***Lagrange's method of variation of constants and D' Alembert's method of reduction of order***

We shall be concerned with the inhomogeneous linear differential equation (1). Let $y_1(x)$, $y_2(x)$ be solutions of (1). Then, clearly, $y(x) = y_1(x) - y_2(x)$ is a solution of the associated homogeneous equation (5). This proves the following theorem.

Theorem 1: The general solution of (1) is written as the sum of a particular solution of (1) and the general solution of (5).

However, if we know a fundamental system of the solutions of (5), then we can obtain a particular solution of (1) by the *method of variation of constants* which is due to Lagrange. Accordingly, in order to solve linear differential equations, it is sufficient to solve the associated homogeneous equations.

The method of variation of constants. Let $y_1, y_2, ..., y_n$ be a fundamental system of the solutions of (5). Then the general solution of (5) is written in the form

$$y(x) = \sum_{i=1}^{n} C_i\, y_n(x) \quad ...(i)$$

Now we regard these constants C_t as functions of x, and try to determine them in such a way that

$$y(x) = \sum_{i=1}^{n} C_t(x) y_i(x)$$

satisfies (1). As was shown by Lagrange, if $C_1(x), C_2(x), ..., C_n(x)$ satisfy the system of linear equations

$$\begin{aligned} y_1(x)C_1'(x) + y_2(x)C_2'(x) + ... + y_n(x)C_n'(x) &= 0 \\ y_1'(x)C_1'(x) + y_2'(x)C_2'(x) + ... + y_n'(x)C_n'(x) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \quad ...(ii) \\ y_1^{(n-2)}(x)C_1'(x) + y_2^{(n-2)}(x)C_2'(x) + ... + y_n^{(n-2)}C_n'(x) &= 0 \end{aligned}$$

Notes

$$y_1^{(n-1)}(x)C_1'(x) + y_2^{(n-1)}(x)C_2'(x) + ... + y_n^{(n-1)}C_n'(x) \quad = \quad q(x)$$

then $\Sigma_{i=1}^{n} C_l(x) y_i(x)$ satisfies (1).

In fact, if there exist $C_1(x)$, $C_2(x)$, ..., $C_n(x)$ satisfying (ii), then, by differentiation and by making use of (ii), we obtain successively

$$y(x) \quad = \quad \sum_{i=1}^{n} C_i(x) y_i(x)$$

$$y'(x) \quad = \quad \sum_{i=1}^{n} C_i(x) y_i'(x)$$

....

$$y^{(n-1)}(x) \quad = \quad \sum_{i=1}^{n} C_i(x) y_i^{(n-1)}(x)$$

$$y^{(n)}(x) \quad = \quad \sum_{i=1}^{n} C_i(x) y_i^{(n)}(x) + q(x)$$

Since $y_i(x)$ satisfies (5), $y(x)$ is certainly a solution of (1).

Now we consider the system (ii). According to Theorem 2, the Wronskian $W(y_1(x), y_2(x), ..., y_n(x))$ of the fundamental system $\{y_i(x)\}$ never vanishes at any point in the domain D, in which the coefficients $p_1(x)$, $p_2(x)$, ..., $p_n(x)$ of (5) are continuous. Therefore, there exists one and only one set of solutions $C_i'(x), C_2'(x), ..., C_n'(x)$ of (ii), which is written as

$$dC_i(x)/dx \quad = \quad q(x)W_i(x)/W(y_i(x), y_2(x), ..., y_n(x)) \qquad ...(iii)$$

$$= \quad Z_i(x), \qquad (i = 1, 2, ..., n)$$

where $W_i(x)$ is the cofactor of $y_i^{(n-1)}(x)$ in $W(y_1(x), y_2(x), ..., y_n(x))$. Integrating (iii), we obtain

$$C_i(x) \quad = \quad \int_{x_0}^{x} Z_i(t)\,dt + \bar{C}_t, \qquad (i = 1, 2, ..., n) \qquad ...(iv)$$

where $\bar{C}_t$ is a constant of integration. Consequently, a particular solution of the equation (1) is

$$y(x) \quad = \quad \sum_{x}^{n} \left(\int_{x_0}^{x} Z_i(t)\,dt + \bar{C}_i \right) y_i(x) \qquad ...(v)$$

The method of reduction of order. If a particular solution $y_1(x)$, not identically zero, of the nth order linear differential equation (5) is known, then, by setting

$$y \quad = \quad y_1 z$$

(5) can be reduced to a linear differential equation of the $(n - 1)$ order with respect to dz/dx. This procedure is called the *method of reduction of order* and is due to D′ Alembert.

Notes

In fact, Leibnitz's formula yields

$$y^{(p)} = y_1 z^{(p)} + p y_1' z^{(p-1)} + \cdots + y_1^{(p)} z \qquad (p = 1, 2, ..., n)$$

Substituting these in (5), we see that the coefficient of $z^{(n)}$ is y_1, and that of z is zero. Thus (5) becomes an equation of the $(n-1)$ order with respect to z',

$$y_1 z^{(n)} + q_1(x) z^{(n-1)} + q_2(x) z^{(n-1)} + ... + q_{n-1}(x) z' = 0 \qquad \text{(vi)}$$

In particular, when $n = 2$, the reduced equation (vi) can be solved. Hence, by virtue of this method, we obtain the general solution

$$y(x) = y_1(x) \int^x y_1(t)^{-2} \exp\left(-\int^i p_1(\tau)\, d\tau \right) dt \qquad \text{...(viii)}$$

$y_1(x)$ being a particular solution of (5) with $n = 2$. This method is useful in the practical treatment of the linear differential equations.

Self Assessment

1. Consider the second order differential equations

   $$y'' + p_1(x) y' + p_2(x) y = 0$$

   having two independent solutions y_1 and y_2. Find a relation between p_1, p_2 in terms of y_1, y_2 and their derivatives.

2. Obtain the particular solution of the differential equation

   $$y'' - y = e^{2x}$$

   by the method of variation of constants.

7.3 Solution of the Linear Equation with Constant Coefficients

To solve the equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + P_n y = 0, \qquad \text{...(i)}$$

where P_0, P_1,...., P_n are constants.

Substitute $y = e^{mx}$ on a trial basis,

Then $$e^{mx}(P_0 m^n + P_1 m^{n-1} + ... + P_n) = 0 \qquad \text{...(ii)}$$

Now, e^{mx} is a solution of (i) if m is a root of the algebraic equation

$$P_0 m^n + P_1 m^{n-1} + ... + P_n = 0 \qquad \text{...(iii)}$$

Auxiliary Equation

The equation (iii) is called the *auxiliary equation.* Therefore if m have a value say m_1 that satisfies (iii), $y = e^{m_1 x}$ is an integral of (i), and if the n roots of (iii) be m_1, m_2, m_3, ...m_n the complete solution of (i) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}.$$

This will be the case when all the roots, $m_1, m_2, m_3, \ldots m_n$ of the auxiliary equation are real, distinct and different.

Auxiliary Equation having Equal Roots

If the auxiliary equation has two equal roots, say m_1 and m_2, the solution of the given equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y = 0$$

will be $$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_2 x} + \ldots + c_n e^{m_n x}$$

or $$y = c e^{m_1 x} + c_3 e^{m_2 x} + \ldots + c_n e^{m_n x}$$

where $$c_1 + c_2 = c.$$

This is not the general solution of (i), because it contains $(n - 1)$ arbitrary constants while the order of the equation is n. To obtain the general solution of (i) in this case, we proceed as follows:

Consider the repeated factor as $\left(\frac{dy}{dx} - m_1\right)^2 y = 0.$ This can be written as $(D - m_1)^2 y = 0,$ where $D = \frac{d}{dx}$.

Put $$(D - m_1) y = v;$$

then $$(D - m_1) v = 0.$$

Therefore $$\frac{dv}{dx} = m_1 v$$

or $$\frac{dv}{v} = m_1 \, dx$$

Integrating, we have $\log \frac{v}{c_2} = m_1 x$

Hence $$v = c_2 e^{m_1 x}.$$

or $$(D - m_1) y = c_2 e^{m_1 x}$$

or $$\frac{dy}{dx} - m_1 y = c_2 e^{m_1 x}$$

This is a linear differential equation and we will have

$$y e^{-m_1 x} = c_1 + \int c_2 e^{m_1 x} . e^{-m_1 x} \, dx$$

$$= c_1 + c_2 x$$

$$\therefore \quad y = (c_1 + c_2 x) e^{m_1 x}.$$

Notes

This consequently means that if two roots of the auxiliary equation are equal, the general solution of (i) will be

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_2 x} + ... + c_n e^{m_n x}.$$

In general, if r roots of the auxiliary equation $P_0 m^n + P_1 m^{n-1} + ... + P_n = 0$ are equal to m_1 say, the general solution of (i) will be

$$y = (c_1 + c_2 x + c_3 x^2 + ... + c_r x^{r-1}) e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + ... + c_n e^{m_n x}.$$

Auxiliary Equation having Complex Roots

If some of the roots of auxiliary equation are complex, then we shall follow the procedure as given below:

Let $\alpha \pm i\beta$ be the roots of the auxiliary equation; then the corresponding part shall become

$$= c_1 e^{(\alpha + i\beta)} + c_2 e^{(\alpha - i\beta)x}$$

$$= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$$

$$= e^{ax}(c_1 \cos \beta x + i c_1 \sin \beta x) + e^{ax}(c_2 \cos \beta x - i c_2 \sin \beta x)$$

$$= e^{ax}[(c_1 + c_2)\cos \beta x + (ic_1 - ic_2)\sin \beta x]$$

$$= e^{ax}[A \cos \beta x + b \sin \beta x],$$

where A and B are arbitrary constants.

Therefore the solution is

$$y = e^{ax}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_2 x} + ... + c_n e^{m_n x}$$

Example 1: The expression $e^{ax}(A \cos \beta x + b \sin \beta x)$ can be also written as

$$c_1 e^{ax} \cos(\beta x \pm c_2) \text{ or } c_1 e^{ax} \sin(\beta x \pm c_2),$$

Example 2: if the auxiliary equation has two equal pairs of complex roots, say $\alpha \pm i \beta$ occurring twice, then the portion of the solution corresponding to these roots, is

$$e^{\alpha x}\left[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x\right]$$

Example 3: If the auxiliary equation has the roots as $\alpha \pm \sqrt{\beta}$, then the portion of the solution corresponding to these roots is

$$c_1 e^{ax} \cos h\left(x\sqrt{\beta} + c_2\right) \text{ or } c_1 e^{ax} \sin h\left(x\sqrt{\beta} + c_2\right)$$

Solution of equations of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + P_n y = 0.$$

will have the following properties. **Notes**

Nature of the roots	Solution
1. Real and distinct i.e., $m_1, m_2, ...m_n$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
2. Real and equal, each m_1 (say)	$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{m_1 x}$
3. Non-repeated roots as $\alpha \pm i\beta$	$y = (c_1 \cos \beta x + c_2 \sin \beta x)\, e^{ax}$ or $y = c_1 e^{\alpha x} \cos(\beta x + c_2)$
4. Repeated roots $\alpha \pm i\beta$, r times	$y = [(c_1 + c_2 x + \cdots + c_r x^{r-1}) \cos\beta x + (c_1' + c_2' x + \cdots + c_r' x^{r-1}) \sin \beta x]\, e^{\alpha x}$
5. Irrational roots as $\alpha \pm \sqrt{\beta}$	$y = c_1 e^{ax} \cos h(x\sqrt{\beta} + c_2)$ or $y = c_1 e^{\alpha x} \sin h(x\sqrt{\beta} + c_2)$

Example 4: The symbol D is used for $\frac{d}{dx}$ for D^n for $\frac{d^n}{dx^n}$. It should be kept in mind that D and D^{-1} are the inverse operations, i.e., as D *means differentiations*, D^{-1} *means integrations*.

Illustrative Examples

Example 1: Solve: $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 44y = 0.$

Solution: The equation can be written as $(D^2 - 7D - 44)\, y = 0$

The auxiliary equation is

$$m^2 - 7m - 44 = 0 \quad \text{or} \quad (m - 11)(m + 4) = 0$$

$\therefore$ $m = 11, -4$, which are real and distinct. Hence solution of the given equation is

$$y = c_1 e^{11x} + c_2\, e^{-4x}.$$

Example 2: Solve: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0.$

Solution: The given equation is

$$(D^2 - 4\,D + 1) = 0$$

The auxiliary equation is

$$m^2 - 4\,m + 1 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Hence general solution is

$$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$$

It can also be written in the form

$$y = e^{2x}(c_1 e^{\sqrt{3x}} + c_2 e^{(-\sqrt{3x})})$$

Notes

or $$y = e^{2x}\left(c_1 \cosh\sqrt{3x} + c_2 \sin h\sqrt{3x}\right).$$

Example 3: Solve: $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0.$

Solution: The given equation is

$$(D^3 - 2D^2 - 4D + 8)\,y = 0$$

Auxiliary equation is

$$m^2 - 2m^2 - 4m + 8 = 0$$

or $$(m-2)(m^2-4) = 0;\ m = 2, -2.$$

∴ General solution is

$$y = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x}.$$

Example 4: Solve: $\frac{d^2y}{dx^2} + 4y = 0.$

Solution: The given equation is

$$(D^2 + 4)\,y = 0.$$

Auxiliary equation is

$$m^2 + 4 = 0 \text{ or } m = \pm 2i.$$

The general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x.$$

Self Assessment

3. Solve

$$\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$$

4. Solve

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$$

5. Solve

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$$

6. Solve

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$$

Notes

7.4 Particular Integral

Let $$\frac{1}{f(D)}Q \qquad \text{...(i)}$$

denote some function of x which when operated upon by $f(D)$ gives Q. This function of x is a particular solution of the differential equation.

$$f(D)y = Q \qquad \text{...(ii)}$$

As $f(D)$ and $f(D)^{-1}$ are inverse operations, therefore

$$D\{D^{-1}(Q)\} = Q \qquad \text{(Particular case)}$$

or $$\frac{d}{dx}\{D^{-1}(Q)\} = Q$$

$$D^{-1}(Q) = \int Q\,dx$$

Example: Properties of $\frac{1}{f(D)}$.

1. If $Q = u_1 + u_2 + u_3 + \cdots + u_n$ then

$$\frac{1}{f(D)}Q = \frac{1}{f(D)}u_1 + \frac{1}{f(D)}u_2 + \cdots + \frac{1}{f(D)}u_n.$$

2. $\frac{1}{f(D)}(k\,Q) = k.\frac{1}{f(D)}Q.$ where k is a constant

3. $\frac{1}{f(D)}$ can be resolved into factors.

4. $\frac{1}{f(D)}$ can be broken into partial fractions.

5. $\frac{1}{f(D)}$ Q is a particular integration.

To show that $\frac{1}{D-\alpha}Q = e^{\alpha x}\int e^{-ex}Q\,dx$

Let $$\frac{1}{(D-\alpha)}Q = V$$

Therefore $$(D-\alpha)\,V = Q$$

or $$\frac{dv}{dx} - \alpha V = Q$$

This is a linear differential equation. The solution is

Notes

$$V e^{-ax} = \int Q e^{-ax}\, dx + c$$

$$V = e^{ax} \int Q e^{-ax} dx + c e^{ax}.$$

Now c can be taken zero, for we want only a particular solution.

Hence $$V = e^{ax} \int Q e^{-ax} dx.$$

or $$\frac{1}{(D-\alpha)} Q = e^{ax} \int Q\, e^{-ax} dx.$$

We are now in a position to evaluate

$$\{f(D)\}^{-1} Q$$

Let on factorization

$$f(D) = (D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)$$

Then $$(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)y = Q$$

It follows that

$$(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)y = (D-\alpha_1)^{-1} Q$$

$$= e^{\alpha_1 x} \int e^{-\alpha_1 x} Q\, dx$$

Therefore

$$(D-\alpha_3)\cdots(D-\alpha_n)y = (D-\alpha_2)^{-1} e^{\alpha_1 x} \int e^{-\alpha_1 x} Q\, dx$$

or $$(D-\alpha_3)\cdots(D-\alpha_n)y = e^{\alpha_2 x} \int e^{(\alpha_1-\alpha_2)x} \int e^{-\alpha_1 x} Q\, dx$$

and so on.

Hence, we get generally

$$y = e^{\alpha_n x} \int e^{(\alpha_{n-1}-a_n)x} \int \cdots \int e^{(\alpha_1-\alpha_2)x} \int e^{-\alpha_1 x} Q\, dx \cdots dx.$$

This is the required particular integral.

Note: In case $f(D)$ fails to give real linear factors, we may use imaginary factors and use the above method and finally put the result in a real form.

Let $\dfrac{1}{f(D)}$ be capable of resolving into partial fractions. Thus

$$\frac{1}{f(D)} = \frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots + \frac{A_n}{D-\alpha_n}$$

Now, particular integral

$$= \frac{1}{f(D)} Q = \frac{A_1}{D-\alpha_1} Q + \frac{A_2}{D-\alpha_2} Q + \ldots + \frac{A_n}{D-\alpha_n} Q.$$

Notes

$$A_1 e^{\alpha_1 x}\int e^{-\alpha_1 x} Q\,dx + A_2 e^{\alpha_2 x}\int e^{-\alpha_2 x} Q\,dx$$

$$+\cdots + A_n e^{\alpha_n x}\int e^{-\alpha_n x}\,Q\,dx.$$

To evaluate $\frac{1}{f(D)}e^{\alpha\, x}$, where

$$f(D) = P_0 D^n + P_1 D^{n-1} + \cdots + P_n,$$

and $f(\alpha) \neq 0$.

We know that

$$D(e^{ax}) = ae^{ax}$$

$$D^2(e^{ax}) = a^2 e^{ax}$$

....

....

$$D^n(e^{ax}) = a^n\, e^{ax}.$$

Therefore,

$$f(D)e^{ax} = (P_0 D^n + P_1\, D^{n-1} + \cdots + P_n)\, e^{ax}$$

$$= P_0\, D^n\, e^{ax} + P_1\, D^{n-1}\, e^{ax} + \cdots + P_n\, e^{ax}$$

$$= P_0 a^n\, e^{ax} + P_1\, a^{n-1}\, e^{ax} + \cdots + P_n\, e^{ax}$$

$$= (P_0 a^n + P_1 a^{n-1} + \cdots + P_n)e^{ax}$$

Now, $f(D)e^{ax} = f(a)e^{ax}$.

Operating upon both sides with $\frac{1}{f(D)}$ we have

$$\frac{1}{f(D)} f(D)e^{ax} = \frac{1}{f(D)} f(a)e^{ax},$$

$$e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$$

$$\therefore \qquad \frac{e^{ax}}{f(a)} = \frac{1}{f(D)}e^{ax}, \text{ provided } f(a) \neq 0.$$

Illustrative Examples

Example 1: Solve the following equation

$$(D^2 - 3D + 2)y = e^{5x}.$$

Notes

Solution: The given equation is

$$(D^2 - 3D + 2)y = e^{5x}$$

Auxiliary equation is

$$m^2 - 3m + 2 = 0 \text{ or } (m-1)(m-2) = 0$$

$$\therefore \quad m = 1, 2$$

$$\therefore \quad \text{C.F.} = c_1 e^x + c_2 e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} e^{5x}$$

$$= \frac{1}{25 - 3.5 + 2} e^{5x} = \frac{1}{12} e^{5x}$$

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^x + c_2 e^{2x} + \frac{1}{12} e^{5x}$$

Example 2: Solve: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{-x}$.

Solution: Here the auxiliary equation is

$$m^2 + m + 1 = 0, \qquad \therefore \quad m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore \quad \text{C.F.} = e^{-\frac{1}{2}x}\left[A\cos\frac{1}{2}\sqrt{3}x + B\sin\frac{1}{2}\sqrt{3}x\right]$$

Also

$$\text{P.I.} = \frac{1}{D^2 + D + 1} e^{-x}$$

$$= \frac{1}{(-1)^2 + (-1) + 1} e^{-x} = e^{-x}$$

Hence the general solution of the given equation is

$$y = e^{-\frac{1}{2}x}\left\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right\} + e^{-x}$$

Self Assessment

Solve the following differential equations:

7. $(D^2 + 5D + 6)y = e^{2x}$.

8. $(D^3 - D^2 - 4D + 4)y = e^{3x}$.

9. $(4D^2 + 4D - 3)y = e^{2x}$

10. $(D^3 + 1)y = (e^x + 1)^2$

Notes

To evaluate $\frac{1}{f(D)}\sin ax$, where $f(D) = P_0 D^n + P_1 D^{n-1} + ... P_n$.

Case I. When $f(D)$ contains even powers of D

Let $f(D^2) = P_0(D^2)^n + P_1(D^2)^{n-1} + \cdots + P_{n.}$

We notice that $D^2 \sin ax = -a^2 \sin ax.$

$D^4 \sin ax = (-a^2)^2 \sin ax$

$D^6 \sin ax = (-a^2)^3 \sin ax$

....

....

$(D^2)^n \sin ax = (-a^2)^n \sin ax$

Therefore $f(D^2)\sin ax = P_0(D^{2n} + P_1 D^{2n-2} + \cdots + P_n)\sin ax$

or $f(D^2)\sin ax$

$$= P_0 D^{2n} \sin ax + P_1 D^{2n-2} \sin ax + \cdots + P_n \sin ax$$

$$= P_0(-a^2)^n \sin ax + P_1(-a^2)^{n-1} \sin ax + ... + P_n \sin ax$$

$$= f(-a^2)\sin ax.$$

Operating on both sides with $\frac{1}{f(D^2)}$, we have

$$\frac{1}{f(D^2)} f(D^2)\sin ax = \frac{1}{f(D)^2} f(-a^2)\sin ax$$

or $\sin ax = f(-a^2).\frac{1}{f(D^2)}\sin ax.$

Dividing both sides by $f(-a^2)$, we have

$$\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax.$$

Case II. When $f(D)$ contains odd powers of D.

Let it be put in the form $f_1(D^2) + Df_2(D^2)$; then

$$\frac{1}{f(D)}\sin ax = \frac{1}{f_1(D^2) + D f_2(D^2)}\sin ax$$

$$= \frac{1}{f(-a^2) + D f_2(-a^2)}\sin ax$$

$$= \frac{1}{m + nD}\sin ax \text{ say}$$

Notes

[where $m = f_1(-a^2), n = f_2(-a^2)$]

$$= (m - nD)\left\{\frac{1}{(m-nD)}.\frac{1}{m+nD}\sin ax\right\}$$

Since $(m-nD), \dfrac{1}{(m-nD)}$ are inverse operations.

$$= (m - nD)\left\{\frac{1}{(m^2 - n^2 D^2)}\sin ax\right\}$$

$$= (m - nD)\frac{1}{m^2 + n^2 a^2}\sin ax$$

$$= \frac{m \sin ax - na \cos ax}{m^2 + n^2 a^2}$$

$$= \frac{f_1(-a^2)\sin ax - f_2(-a^2)a\cos ax}{\left\{f_1(-a^2)\right\}^2 + a^2\left\{f_2(-a^2)\right\}^2}$$

Note Similar results are true for $\dfrac{1}{f(D)}\cos ax$.

Illustrative Examples

Example 1: Solve: $(D^2 + D + 1)\, y = \sin 2x$.

Solutions: Here C.F. $= e^{-x/2}\left(c_1 \cos\dfrac{\sqrt{3}}{2}x + c_2 \sin\dfrac{\sqrt{3}}{2}x\right)$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}\sin 2x$$

$$= \frac{1}{-(2)^2 + D + 1}\sin 2x$$

$$= \frac{1}{D-3}\sin 2x$$

$$= \frac{D+3}{D^2 - 9}\sin 2x$$

$$= \frac{D(\sin 2x) + 3\sin 2x}{-4-9}$$

$$= -\frac{1}{13}(2\cos 2x + 3\sin 2x)$$

Therefore the general solution is

Notes

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{-x/2}\left\{c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right\} - \frac{1}{13}(2\cos 2x + 3\sin 2x)$$

Self Assessment

11. Solve the following differential equations

    $(D^2 - D - 2)y = \sin 2x$

12. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin 3x$

7.5 Summary

- The unit starts with the existence the uniqueness of the solution of nth order differential equation.
- Here the nth order linear differential equation is reduced to a system of n first order equations and the method of last unit applied.
- Some of the properties listed, help us in finding the general solution of the equation when the coefficients are constant.

7.6 Keywords

Complementary functions are the solutions of the nth order differential equation without the non-homogeneous term and involves n arbitrary constants.

Particular Integral (P.I.): It is the solution of non-homogeneous, nth order differential equation without having any arbitrary constants.

7.7 Review Questions

1. Solve

   $$9\frac{d^2y}{dx^2} + 18\frac{dy}{dx} - 16y = 0$$

2. Solve

   $$\frac{d^4y}{dx^4} + y = 0$$

3. Solve

   $(D^4 - D^3 - 9D^2 - 11D - 4)\ y = 0$

4. Solve

   $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$$

Notes

5. Solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$$

6. $$\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$$

Answers: Self Assessment

1. $p_1 = \dfrac{(y_1'' y_2 - y_2'' y_1)}{(y_1 y_2' - y_1' y_2)}$, $p_2 = \dfrac{(y_1' y_2'' - y_1' y_2')}{(y_1' y_2 - y_1 y_2')}$

2. Particular integral, P.I. = $\dfrac{e^{2x}}{3}$

3. $y = c_1 e^x + c_2 e^{3x} + c_3 e^{5x}$

4. $y = e^{-4x}(c_1 \cos 3x + c_2 \sin 3x)$

5. $y = (c_1 + c_2 x) e^x + c_3 e^{2x}$

6. $y = (c_1 + c_2 x)e^x + c_3 \cos 2x + c_4 \sin 2x$

7. $y = c_1 e^{-2x} + c_2 e^{-3x} + \dfrac{1}{20} e^{2x}$

8. $y = c_1 e^x + c_2 e^{2x} + c_3 e^{-2x} + \dfrac{1}{10} e^{3x}$

9. $y = c_1 e^{x/2} + c_2 e^{\frac{-3x}{2}} + \dfrac{1}{21} e^{2x}$

10. $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + \dfrac{1}{4}e^{2x} + e^x + 1$

11. $y = c_1 e^{2x} + c_2 e^{-x} + \dfrac{1}{20}(\cos 2x - 3\sin 2x)$

12. $y = c_1 e^{2x} + c_2 e^{3x} + \dfrac{1}{78} x(5 \cos 3x - \sin 3x)$

7.8 Further Readings

Books

Yosida, K., Lectures in Differential and Integral Equations

Piaggio, H.T.H., Differential Equations

Unit 8: Total Differential Equations, Simultaneous Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Deal with equations which are total differentials as well as simultaneous differential equations involving more than one dependent variable and one independent variable.
- See whether total differential equations are integrable and study the condition of integrability as well its uniqueness of the solution.

Introduction

The total differential equations are seen to be integrable with some illustrated examples. There are four differential methods of obtaining the solution of total differential equations. The conditions when the total differential is exact are obtained.

8.1 Total Differential Equation

An equation of the form

$$P\,dx + Q\,dy + R\,dz = 0 \qquad \text{...(i)}$$

Where, P, Q, R are functions of x, y, z is known as 'total differential equation'. The equation (i) is said to be integrable if there exists a relation of the form

$$u(x,y,z) = c, \qquad \text{...(ii)}$$

which on differentiation gives the above differential equation (i). The relation (ii) is called the complete integral or solution of the given differential equation.

Now consider equation (i). If (ii) is the integral of (i) and since

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz, \qquad \text{...(iii)}$$

Notes

du = 0, gives on comparison with (i) the relations

$$\frac{\frac{\partial u}{\partial x}}{P} = \frac{\frac{\partial u}{\partial y}}{Q} = \frac{\frac{\partial u}{\partial z}}{R} = \lambda \qquad \text{(say)} \qquad ...(iv)$$

So we get

$$\frac{\partial u}{\partial x} = \lambda P, \frac{\partial u}{\partial y} = \lambda Q, \frac{\partial u}{\partial z} = \lambda R \qquad ...(v)$$

8.2 Condition of Integrability of Total Differential Equation

Now differentiating these three equations (v), first with respect to y and z, second with respect to z and x and third with respect to x and y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = P\frac{\partial \lambda}{\partial y} + \lambda\frac{\partial P}{\partial y}, \frac{\partial^2 u}{\partial z \partial x} = P\frac{\partial \lambda}{\partial z} + \lambda\frac{\partial P}{\partial z}$$

$$\frac{\partial^2 u}{\partial x \partial y} = Q\frac{\partial \lambda}{\partial x} + \lambda\frac{\partial Q}{\partial x}, \frac{\partial^2 u}{\partial z \partial y} = Q\frac{\partial \lambda}{\partial z} + \lambda\frac{\partial Q}{\partial z}$$

$$\frac{\partial^2 u}{\partial x \partial z} = R\frac{\partial \lambda}{\partial x} + \lambda\frac{\partial R}{\partial x}, \frac{\partial^2 u}{\partial y \partial z} = R\frac{\partial \lambda}{\partial y} + \lambda\frac{\partial R}{\partial y},$$

equating the values of $\frac{\partial^2 u}{\partial x \partial y}$ etc., and rearranging

$$\left.\begin{aligned} \lambda\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] &= Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y} \\ \lambda\left[\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right] &= R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z} \\ \lambda\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] &= P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x} \end{aligned}\right\} \qquad ....(vi)$$

Now multiplying the above three equations by R, P, Q respectively and adding, we get

$$R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] + P\left[\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right] + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] = 0 \qquad ...(vii)$$

which is the required condition.

Sufficiency of the Condition (vii)

Now if (vii) holds for the coefficients of (i), a similar relation holds for coefficients of

$$\mu P\,dx + \mu Q\,dy + \mu R\,dz = 0 \qquad ...(viii)$$

where μ is a function of x, y, z. Now consider $P\,dx + Q\,dy$. If it is not an exact differential with respect to x, y an integrating factor μ can be found for it. So $P\,dx + Q\,dy$ can be regarded as an exact differential.

Notes

Now μ $P\,dx + Q\,dy$ is an exact differential,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

and if

$$V = \int [P\,dx + Q\,dy]$$

$$\left.\begin{aligned} \frac{\partial V}{\partial x} &= P \text{ and } \frac{\partial V}{\partial y} = Q \\ \frac{\partial P}{\partial z} &= \frac{\partial^2 V}{\partial z dx}, \frac{\partial Q}{\partial z} = \frac{\partial^2 V}{\partial z \partial y} \end{aligned}\right\} \quad \text{...(ix)}$$

Putting these values in (vii)

$$\frac{\partial V}{\partial x}\left\{\frac{\partial^2 V}{\partial z \partial y} - \frac{\partial R}{\partial y}\right\} + \frac{\partial V}{\partial y}\left[\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z \partial x}\right] = 0$$

or

$$\frac{\partial V}{\partial x}\frac{\partial}{\partial y}\left[\frac{\partial V}{\partial z} - R\right] - \frac{\partial V}{\partial y}\frac{\partial}{\partial x}\left[\frac{\partial V}{\partial z} - R\right] = 0$$

or

$$\begin{vmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial z} - R\right) \\ \dfrac{\partial V}{\partial y} & \dfrac{\partial}{\partial y}\left(\dfrac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0$$

This equation shows that a relation independent of x and y exists between

$$V \text{ and } \frac{\partial V}{\partial z} - R.$$

Therefore $\frac{\partial V}{\partial z} - R$ can be expressed as a function of z and V alone.

Suppose

$$\frac{\partial V}{\partial z} - R = \phi\,(z, V)$$

Since

$$P\,dx + Q\,dy + R\,dz = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz + \left(R - \frac{\partial V}{\partial z}\right)dz \quad \text{...(x)}$$

Equation (i) may be written, on taking into account (x) as

$$dV - \phi(z, V)\,dz = 0 \quad \text{...(xi)}$$

The equation is an equation in two variables. Its integration will lead to an equation of the form

$$F(V, z) = c.$$

Notes

Hence the condition (vii) is necessary and sufficient both. In the vector form the equation (i) can be written as

$$\vec{A}.\vec{dr} = 0$$

where

$$\vec{A} = P\hat{i} + Q\hat{j} + R\hat{k} \text{ and}$$

$$\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

The necessary and sufficient condition then becomes $\vec{A}.$ Curve $\vec{A} = 0$ i.e.

$$\begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0$$

Self Assessment

1. Show that the differential equation

   $xz^3dx - z\,dy + 2y\,dz = 0$

   is integrable.

2. Show that the differential equation

   $yz(y+z)dx + zx(z+x)dy + xy(x+y)dz = 0$

   is integrable.

8.3 Methods for Solving the Differential Equations

$$P\,dx + Q\,dy + R\,dx = 0 \quad ...(1)$$

The condition for integrability of the above equation is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad ...(2)$$

If the differential equation (1) is exact differential then its integral is of the form

$$u(x,y,z) = c, \quad ...(3)$$

Now

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad ...(4)$$

Giving us the conditions

$$P = \frac{\partial u}{\partial x}, Q = \frac{\partial u}{\partial y}, R = \frac{\partial u}{\partial z}$$

Notes

Now $$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

or $$\left.\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} \\ \text{Similarly} \quad \frac{\partial Q}{\partial z} &= \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \end{aligned}\right\} \quad ...(5)$$

There are various methods of solving equation (1) which are shown below.

Method I : Solution by Inspection

If the conditions of integrability are satisfied, then sometimes by rearranging the terms of the given equation and/or by dividing by some suitable function, the given equation may be changed to a form containing several parts, all of which are exact differential. Then integrating it, the integral can be obtained directly.

Note: Certain common exact differentials, which may occur in the transformed total differential equation are as follows:

$$x\,dy + y\,dx = d(xy)$$

$$xy\,dz + xz\,dy + yz\,dz = d(x\,y\,z)$$

$$\frac{x\,dy - y\,dx}{x^2} = d(y/x);$$

$$\frac{y\,dx - x\,dy}{y^2} = d(x/y)$$

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = d(\tan^{-1}(y/x))$$

$$\frac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\frac{1}{2}\log(x^2 + y^2)\right]$$

$$\frac{d\,f(x,y,z)}{f(x,y,z)} = d\left[\log f(x,y,z)\right]$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2} = d\left[\frac{1}{2}\log(x^2 + y^2 + z^2)\right]$$

Example 1: Solve

$$(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0 \quad ...(1)$$

Notes

Solution:

Let

$$P = y^2 + yz, Q = z^2 + zx, R = y^2 - xy \quad ...(2)$$

The condition for integrability of equation (1) is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = 0 \quad ...(3)$$

Now

$$\frac{\partial Q}{\partial z} = 2z + x, \frac{\partial R}{\partial y} = 2y - x$$

$$\frac{\partial R}{\partial x} = -y, \qquad \frac{\partial P}{\partial z} = y$$

$$\frac{\partial P}{\partial y} = 2y + z, \qquad \frac{\partial Q}{\partial x} = z$$

Substituting in equation (3) we get

$$(y^2 + yz)(2z + 2x - 2y) + (z^2 + zx)(-y - y) + (y^2 - xy)(2y + z - z)$$

or $$y^2(2z + 2x - 2y + 2y) + yz(2z + 2x - 2y) - 2y(z^2 + zx) - 2xy^2$$

$$= 2y^2z + 2xy^2 + 2yz^2 + 2xyz - 2y^2z - 2yz^2 - 2xyz - 2xy^2 = 0 = \text{R.H.S.}$$

So condition of integrability is verified.

Let z be constant, so that $dz = 0$. So from (1) we get

$$(y^2 + yz)dx + (z^2 + zx)dy = 0 \quad ...(4)$$

So $$\frac{dx}{x + z} + \frac{z\,dy}{y^2 + yz} = 0$$

or $$\frac{dx}{x + z} + \left\{\frac{1}{y} - \frac{1}{z + y}\right\}dy = 0 \quad ...(5)$$

Integrating we get

$$\log(x + z) + \log\frac{y}{y + z} = \text{Constant}$$

or $$\log\left\{\frac{(x + z)y}{y + z}\right\} = \text{constant} \quad ...(6)$$

$$= \log \phi \quad \text{(say)}$$

so $$\frac{y(x + z)}{y + z} = \phi \quad ...(7)$$

Where ϕ is only a function of z. Taking the differential of both the sides, we get

$$\frac{(y+z)\left[y(dx+dz)+(x+z)dy\right]-y(x+z)(dy+dz)}{(y+z)^2} = d\phi$$

or

$$\frac{(y^2+yz)dx+dy(z^2+zx)+dz(y^2+zy-yx-yz)}{(y+z)^2} = d\phi \qquad ...(8)$$

Now from (1) and (8) we have,

$$d\phi = 0 \qquad \text{or} \qquad \phi = k \text{ (constant)}$$

Thus from (7)

$$\frac{y(x+z)}{y+z} = \text{k}$$

or the solution is

$$y(x+z) = k(y+z) \qquad \text{Q.E.D.}$$

Example 2: Solve

$$(x^2y-y^3-y^2z)dx+(xy^2-x^2z-x^3)dy+(xy^2+x^2y)dz = 0 \qquad ...(1)$$

Let $\qquad \text{P} = x^2y-y^3-y^2z, Q = xy^2-x^2z-x^3,\ R = xy^2+x^2y$

The condition of integrability is

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left[\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right]+R\left[\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right] = 0 \qquad ...(2)$$

So

$$\frac{\partial Q}{\partial z} = -x^2,\ \frac{\partial R}{\partial y} = 2xy+x^2$$

$$\frac{\partial R}{\partial x} = y^2+2xy,\ \frac{\partial P}{\partial z} = -y^2$$

$$\frac{\partial P}{\partial y} = x^2-3y^2-2yz,\ \frac{\partial Q}{\partial x} = y^2-2xz-3x^2$$

Substituting in (2) we have

$$= (x^2y-y^3-y^2z)\left[-x^2-2xy-x^2\right]+\left[xy^2-x^2z-x^3\right](y^2+2xy+y^2)+$$

$$+(xy^2+x^2y)\left[x^2-3y^2-2yz-y^2+2xz+3x^2\right]$$

$$= y\left[(x-y)(x+y)-yz\right][-2x](x+y)+\left[x(y-x)(y+x)-x^2z\right](2y)(x+y)+$$

$$+2xy(x+y)[2x^2-2y-yz+xz]$$

Notes

$$= 2yx(x+y)[-x^2+y^2+yz+y^2-x^2-xz+2x^2-2y^2-yz+xz]$$

$$= 2xy(x+y)[0]=0 \qquad ...(3)$$

So integrability condition is satisfied.

Now dividing by x^2y^2 eq. (1) we have

$$\left(\frac{1}{y}-\frac{y}{x^2}-\frac{z}{x^2}\right)dx+\left(\frac{1}{y}-\frac{z}{y^2}-\frac{x}{y^2}\right)dy+\left(\frac{1}{x}+\frac{1}{y}\right)dz = 0$$

or $$\frac{y\,dx-x\,dy}{y^2}+\frac{x\,dy-y\,dx}{x^2}+\frac{x\,dz-z\,dx}{x^2}+\frac{y\,dz-z\,dy}{y^2} = 0$$

or $$d\left(\frac{x}{y}\right)+d\left(\frac{y}{x}\right)+d\left(\frac{z}{x}\right)+d\left(\frac{z}{y}\right) = 0 \qquad ...(4)$$

Integrating (4) we have

$$\frac{x}{y}+\frac{y}{x}+\frac{z}{x}+\frac{z}{y} = 0 \qquad \text{(say)} \qquad ...(5)$$

or

$$x^2+y^2+z(x+y) = cxy$$ is the solution of equation (1).

Self Assessment

3. Solve the differential equation

   $2yz\,dx+zx\,dy-xy(1+z)dz$

4. Solve the differential equation

   $x\,dx+y\,dy-\sqrt{a^2-x^2-y^2}\,dz=0$

Method II: Regarding one Variable as Constant

If the differential equation satisfies the condition of integrability and any two terms say $Pdx+Qdy=0$ can easily be integrated, then the third variable (say z) may be regarded as constant so that $dz=0$.

Note that we should choose such a variable constant so that the remaining equation may be integrated easily.

So the given differential equation will reduce to the integrable form

$$Pdx+Q\,dy = 0 \qquad ...(1)$$

suppose its solution is

$$u = c \qquad \text{(constant)} \qquad ...(2)$$

Notes

i.e. not involving x, y. Now we take

$$u = \phi(z) \qquad ...(3)$$

where $\phi(z)$ is the function of z alone as the solution of the given equation. Now taking the differential of both sides of equation (3), we must get the given equation.

On equating the two, we may get the value of $\frac{d\phi}{dz}$. Eliminating x, y from the value of $\frac{d\phi}{dz}$, using (3), and then integrating we can obtain the value of $\phi(z)$. Substituting the value of ϕ in (3), we get required solution.

Example 1: Solve

$$3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0 \qquad ...(1)$$

by regarding one variable as constant.

Solution:

Let z be constant so that

$$dz = 0 \qquad ...(2)$$

Then (1) gives

$$3x^2dx + 3y^2dy = 0 \qquad ...(3)$$

This gives

$$x^3 + y^3 = \text{constant} = \phi(z) \quad \text{(say)} \qquad ...(4)$$

Taking the differential of (4) we have

$$3x^2dx + 3y^2\,dy = d\phi \qquad ...(5)$$

Comparing (5) with (1) we have

$$dz(x^3 + y^3 + e^{2z}) = d\phi \qquad ...(6)$$

or eliminating x, y from (6) we have

$$(\phi + e^{2z}) = \frac{d\phi}{dz}$$

or $$\frac{d\phi}{dz} - \phi = e^{2z} \qquad ...(7)$$

This equation is linear in ϕ, whose $I.F. = e^{-z}$. So

$$\phi e^{-z} = \int e^{2z}.e^{-z}dz + \text{constant}$$

$$= \int e^{z}\,dz + C \quad \text{(say)}$$

Thus $$\phi(z) = e^{2z} + Ce^{z}$$

Notes

Now from (4) we have

$$x^3 + y^3 = e^{2z} + Ce^z \quad ...(8)$$

which is the required solution

Example 2: Solve

$$(2x^2 + 2xy + 2xz^2 + 1)dx + dy + dz.2z = 0 \quad ...(1)$$

by regarding one variable as constant.

Solution: Let x be constant, so that

$$dx = 0 \quad ...(2)$$

Then $$dy + 2z\,dz = 0$$

or $$d(y + z^2) = 0$$

so $$y + z^2 = \text{constant}$$

$$= \phi(x) \quad \text{(say)} \quad ...(3)$$

Taking differential of (3) we have

$$dy + 2z\,dz = d\phi(x) \quad ...(4)$$

Comparing (4) with (1) we have

$$-(2x^2 + 2xy + 2xz^2 + 1)\,dx = d\phi(x)$$

$$-\frac{d\phi}{dx} = 2x^2 + 1 + 2x(y + z^2)$$

or $$-\frac{d\phi}{dx} = 2x^2 + 1 + 2x\phi$$

So $$\frac{d\phi}{dx} + 2x\phi = -2x^{2-1} \quad ...(5)$$

The equation (5) is linear in ϕ, so I.F. is $e^{+\int 2x\,dx} = e^{x^2}$.

Thus $$\phi e^{x^2} = -\int (2x^2 + 1)\, e^{x^2} dx + C$$

$$= -\int x\left[2x e^{x^2}\right] dx - \int e^{x^2} dx + C$$

$$= -x\, e^{x^2} + \int e^{x} dx - \int e^{x^2} dx + C$$

$$= -x\, e^{x^2} + C$$

So $\phi = -x + C e^{-x^2}$. Thus $y + z^2 = -x + C e^{-x^2}$ Q.E.D.

Self Assessment

5. Solve the differential equation

$$yz\,dx^{+2}zx\,dy - 3xy\,dz = 0$$

6. Solve

$$2(y+z)dx - (x+z)dy + (2y - x + z)dz = 0$$

Method III: For Homogeneous Equations

Consider the equation

$$P\,dx + Q\,dy + R\,dy = 0 \quad \text{...(1)}$$

If the functions P, Q and R are homogeneous functions of x, y, z then one variable say z, can be separated from the other variables by substituting $x = z\,u$ and $y = zv$, so that

$$dx = z\,du + u\,dz,$$

$$dy = z\,dv + v\,dz, \quad \text{...(2)}$$

in the given equation. Then transformed equation can be integrated as

$$\frac{du\,f_1(u,v) + f_2(u,v)dv}{F(u,v)} + \frac{dz}{z} = 0 \quad \text{...(3)}$$

Now to integrate the first term, we find $d[F(u,v)]$ and add and subtract it to numerator. After doing so, the first term will also be integrable.

Example 1: Solve

$$(yz + z^2)dx - xz\,dy + xy\,dz = 0 \quad \text{...(1)}$$

Here $yz + z^2, -xz$ and xy are homogeneous in x, y, z. Let us put $x = uz$, and $y = vz$, so that

$$\left.\begin{aligned} dx &= z\,du + u\,dz \\ dy &= z\,dv + v\,dz \end{aligned}\right\} \quad \text{...(2)}$$

Substituting (2) in (1) we have

$$(vz^2 + z^2)(z\,du + u\,dz) - uz^2(z\,dv + v\,dz) + uvz^2dz = 0$$

$$z\left[(v+1)du - u\,dv\right] + \left[u(v+1)\begin{matrix}-uv\\+uv\end{matrix}\right]dz = 0 \quad \text{...(3)}$$

or

$$\frac{(v+1)du - u\,dv}{u(v+1)} + \frac{dz}{z} = 0 \quad \text{...(4)}$$

Simplifying we have

$$\frac{du}{u} - \frac{dv}{1+v} + \frac{dz}{z} = 0 \quad \text{...(5)}$$

Notes

Integrating

$$\log u - \log(1+v) + \log z = \log c \qquad (c \text{ being constant})$$

or $$\frac{uz}{1+v} = c$$

or $$u\,z^2 = c(z+zv)$$

or $$xz = c(y+z) \qquad ...(6)$$

is the solution of the equation (1).

Example 2: Solve

$$z(z-y)dx + z(z+x)dy + x(x+y)dz = 0 \qquad ...(1)$$

Here $$P = z(z-y),\ Q = z(z+x),\ R = x(x+y) \qquad ...(2)$$

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= -z,\ \frac{\partial Q}{\partial x} = z \\ \frac{\partial R}{\partial x} &= 2x+y,\ \frac{\partial D}{\partial z} = 2z-y \\ \frac{\partial Q}{\partial z} &= 2z+x,\ \frac{\partial R}{\partial y} = x \end{aligned}\right\} \qquad ...(3)$$

The integrability condition

$$P\left[\frac{dQ}{dz} - \frac{\partial R}{\partial y}\right] + Q\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + R\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = 0 \qquad ...(4)$$

L.H.S. of equation (4) is

$$= z(z-y)[2z+x-x] + z(z+x)[2x+y-2z+y] + x(x+y)[-z-z]$$

$$= 2z^2(z-y) + z(z+x)(2x+2y-2z) - 2zx(x+y)$$

$$= 2z^3 - 2z^2y + 2z^2x + 2zx^2 + 2yz^2 + 2xyz - 2z^3 - 2z^2x - 2zx^2 - 2xyz = 0 = \text{R.H.S.}$$

So condition (4) is satisfied

Let

$$\left.\begin{aligned} x &= uz,\ dx = z\,du + udz \\ y &= vz,\ dy = z\,dv + vdz \end{aligned}\right\} \qquad ...(5)$$

Substituting in equation (1)

$$z^2(1-v)[z\,du + u\,dz] + z^2(1+u)[z\,dv + v\,dz] + z^2u(u+v)\,dz = 0$$

or $$(1-v)z\,du + z(1+u)\,dv + \left[u(1-v) + v(1+u) + u(u+v)\right]dz = 0$$

Notes

or $$\frac{(1-v)\,du+(1+u)dv}{(u+v)(1+u)}+\frac{dz}{z} = 0$$

$$\frac{[1+u-u-v]du}{(u+v)(1+u)}+\frac{dv}{u+v}+\frac{dz}{z} = 0$$

or $$\left(\frac{1}{u+v}-\frac{1}{1+u}\right)du+\frac{dv}{u+v}+\frac{dz}{z} = 0$$

$$\frac{du+dv}{u+v}-\frac{du}{1+u}+\frac{dz}{z} = 0$$

Integrating we have

$$\log(u+v)-\log(1+u)+\log z = \log\left(\frac{1}{c}\right) \quad \left(c \text{ being constant}\right)$$

or $$cz(u+v) = 1+u$$

or $$c(x+y)z = z+x \qquad \text{...(6)}$$

is the solution of the equation (1).

Self Assessment

7. Solve the differential equation

   $z^2dx+(z^2-2yz)dy+(2y^2-yz-zx)dz=0$

8. Solve

   $(y^2+z^2-x^2)dx-2xy\ dy-2xz\ dz=0$

Method IV: Method of Auxiliary Equations

Let the given equation

$$P\ dx+Q\ dy+R\ dz = 0 \qquad \text{...(1)}$$

be integrable. Then we must have

$$P\left[\frac{dQ}{dz}-\frac{\partial R}{\partial y}\right]+Q\left[\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right]+R\left[\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right] = 0 \qquad \text{...(2)}$$

Comparing these two, we obtain

$$\frac{dx}{\left(\frac{dQ}{dz}-\frac{\partial R}{\partial y}\right)} = \frac{dy}{\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)}=\frac{dz}{\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)}$$

These equations are called auxiliary equations and can be solved as shown in the two examples below.

Notes

Example 1: Solve

$$(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0 \qquad ...(1)$$

Here put

$$P = y^2 + yz + z^2, Q = z^2 + zx + x^2$$

$$R = x^2 + xy + y^2 \qquad ...(2)$$

Now

$$\frac{\partial Q}{\partial z} = 2z + x, \frac{\partial R}{\partial y} = 2y + x$$

$$\frac{\partial R}{\partial x} = 2x + y, \frac{\partial P}{\partial z} - 2z + y$$

$$\frac{\partial P}{\partial y} = 2y + z, \frac{\partial Q}{\partial x} = 2x + z$$

The auxiliary equations are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}} \qquad ...(3)$$

or

$$\frac{dx}{2(z-y)} = \frac{dy}{2(x-z)} = \frac{dz}{2(y-x)} \qquad ...(4)$$

so

$$\frac{dx + dy + dz}{z - y + x - z + y - x} = \frac{dx + dy + dz}{0} \qquad ...(5)$$

Thus

$$dx + dy + dz = 0$$

or

$$x + y + z = \text{constant} = u \quad \text{(say)} \qquad ...(6)$$

Also from (4)

$$\frac{(z+y)dx}{z^2 - y^2} = \frac{(x+z)dy}{x^2 - z^2} = \frac{(y+x)dz}{y^2 - x^2}$$

So

$$\frac{(z+y)\,dx + (x+z)\,dy + (y+x)\,dz}{0} \qquad ...(7)$$

Gives us

$$(z+y)\,dx + (x+z)\,dy + (y+x)dz = 0 \qquad ...(8)$$

or

$$y\,dx + x\,dy + z\,dy + y\,dz + z\,dx + x\,dz = 0$$

or

$$d(xy + yz + zx) = 0$$

So

$$xy + yz + zx = \text{constant} = v \text{ (say)} \qquad ...(9)$$

Let the solution of (1) is

$$A du + B\, dv \qquad \text{...(10)}$$

then

$$A\, du + B dv = 0 \qquad \text{...(11)}$$

is identical to (1) i.e.

$$A(dx + dy + dz) + B\left[(z + y)dx + (x + z)dy + (y + x)dz\right] = 0 \qquad \text{...(12)}$$

$$\left[A + B(z + y)\right]dx + \left[A + B(x + z)\right]dy + \left[A + B(y + x)\right]dz = 0 \qquad \text{...(12}'\text{)}$$

Comparing (12′) with (1) we have

$$\left.\begin{aligned} A + B(y + z) &\equiv y^2 + yz + z^2 \\ A + B(x + z) &\equiv z^2 + zx + x^2 \\ A + B(x + y) &\equiv x^2 + xy + y^2 \end{aligned}\right\} \qquad \text{...(13)}$$

From (13) we have $B = x + y + z = u$ ...(14)

And

$$A = -(xy + yz + xz) = -v \qquad \text{...(15)}$$

Hence

$$Au + Bv = 0 \qquad \text{...(16)}$$

becomes

$$-v\, du + u\, dv = 0$$

or

$$-\frac{du}{u} + \frac{dv}{v} = 0$$

on integrating

$$\log\left(\frac{u}{v}\right) = \log k$$

or

$$\frac{u}{v} = k \qquad \text{...(17)}$$

From (6) and (9) we have

$$\frac{x + y + z}{xy + yz + zx} = k \qquad \text{...(18)}$$

which is the solution of equation (1).

Example 2: Solve

$$(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0 \qquad \text{...(1)}$$

Solution: By the method of forming auxiliary equations

Here $P = 2xz - yz$, $Q = 2yz - zx$, $R = -x^2 + xy - y^2$

Notes

The set of Auxiliary equations are

$$\frac{dx}{\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)} = \frac{dy}{\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)}=\frac{dz}{\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)} \quad ...(2)$$

$$\frac{\partial Q}{\partial z} = 2y-x, \frac{\partial R}{\partial y}=z+x-2y$$

$$\frac{\partial R}{\partial x} = -2x+y, \frac{\partial P}{\partial z}=2x-y$$

$$\frac{\partial P}{\partial y} = -z, \frac{\partial Q}{\partial x}=-z$$

$$\left.\begin{aligned}\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y} &= 2y-x-x+2y\\ \frac{\partial R}{\partial x}-\frac{\partial P}{\partial z} &= -2x-2x\\ \frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x} &= -z+z=0\end{aligned}\right\} \quad ...(3)$$

Thus substituting (3) into equation (2) we have

$$\frac{dx}{(2y-x)-(x-2y)} = \frac{dy}{(y-2x)-(2x-y)}=\frac{dz}{-z-(-z)}$$

$$\frac{dx}{2(2y-x)} = \frac{dy}{2(y-2x)}=\frac{dz}{0} \quad ...(4)$$

Last equation gives $dz = 0$

or $\quad z = a = u \quad$ (say) ...(5)

From first two members of equation (4) we have

$$\frac{dx}{2y-x} = \frac{dy}{y-2x}$$

or $\quad (y-2x)dx = (2y-x)dy$

Re-arranging we have

$$y\,dx+x\,dy-2x\,dx-2y\,dy = 0$$

or $\quad d(xy-d(x^2)-d(y^2) = 0$

$$d(xy-x^2-y^2) = 0$$

Thus $\quad xy-x^2-y^2 = \text{constant} = v$ (say) ...(6)

Notes

Let the given equation (1) be identical to

$$A\,du + B\,dv = 0 \qquad ...(7)$$

From (5) $du = dz.$

From (6) and (7) we have

$$A\,dz + B\,d(xy - x^2 - y^2) = 0$$

or $$A\,dz + B(x\,dy + y\,dx - 2x\,dx - 2y\,dy) = 0 \qquad ...(8)$$

Rearranging in (8) we have

$$(By - 2x\,B)dx + (x - 2y)B\,dy + A\,dz = 0 \qquad ...(9)$$

Comparing (9) with (1) we have

$$By - 2xB = 2xz - yz,\ \ i.e\ B = -z = -u \qquad ...(10)$$

And $$A = xy - x^2 - y^2 \equiv v \qquad ...(11)$$

Hence (7) gives

$$v\,du - u\,dv = 0 \qquad ...(12)$$

Integrating (12)

$$\frac{du}{u} - \frac{dv}{v} = 0$$

or $$\log u - \log v = \text{constant} = \log c \qquad \text{(say)}$$

Therefore

$$\frac{u}{v} = c$$

or $$\frac{z}{xy - x^2 - y^2} = c$$

is the solution of equation (1).

Self Assessment

9. Solve

   $(a - z)(y\,dx + x\,dy) + xy\,dz = 0$

10. Solve

   $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$

8.4 Simultaneous Differential Equations

In the unit 5 we have discussed differential equations involving two variables i.e. one independent variable and another dependent variable. There is quite a lot of situations in which we have to deal with a number of dependent variables that depend on one independent variable. In the above sections also we have been dealing with more than two variables. So in these cases we can take one variable as independent and solve the equations for the other remaining variables. We illustrate these by means of examples.

Notes

Example 1: Solve

$$\frac{dx}{dt}+wy = 0 \quad ...(1)$$

$$\frac{dy}{dt}-wx = 0 \quad ...(2)$$

Differentiate (1) by t, we have

$$\frac{d^2x}{dt^2}+w\frac{dy}{dt} = 0 \quad ...(3)$$

Substituting the value of $\frac{dy}{dt}$ from (2) into (3) we have

$$\frac{d^2x}{dt^2}+w^2x = 0 \quad ...(4)$$

The solution of (4) is

$$x = A\cos wt + B\sin wt \quad ...(5)$$

Where A, B are constants. Substituting this value of x in (1) we have

$$-wA\sin wt + wB\cos wt + wy = 0$$

or $$y = -A\sin wt + B\cos wt \quad ...(6)$$

Example 2: Solve

$$\frac{dx}{dt}+4x+3y = t \quad ...(1)$$

$$\frac{dy}{dt}+2x+5y = e^t \quad ...(2)$$

Introducing D operator, $D=\frac{d}{dt}$ in (1) and (2) we have

$$(D+4)x+3y = t \quad ...(3)$$

$$(D+5)y+2x = e^t \quad ...(4)$$

Operating equation by $(D + 5)$,

$$(D+5)(D+4)x+3(D+5y) = (D+5)t$$

or $$(D+5)(D+4)x+3(D+5)y = 5t+1 \quad ...(5)$$

Eliminating y from (5)

$$(D+5)(D+4)x+3(e^t-2x) = 5t+1$$

or $$(D^2+9D+20)x-6x = 5+1-3e^t$$

Notes

or $\quad (D^2+9D+14)x = 1+5t-3e^t \quad ...(6)$

or $\quad (D+7)(D+2)x = 1+5t-3e^t \quad ...(7)$

C.F. is $\quad C_1e^{-7t}\ C_2e^{-2t}$

The particular integral, P.I. is given by

$$P.I. = \frac{1}{[14+9D+D^2]}\{1+5t-3e^t\}$$

$$= \frac{1}{14}\left(1+\frac{9D+D^2}{14}\right)^{-1}\left\{1+5t-3e^t\right\}$$

$$= \frac{1}{14}\left(1-\frac{9D}{14}\right)(1+5t)-\frac{3e^t}{14+9(1)+(1)^2}$$

$$= \frac{1}{14}\left(1+5t-\frac{45}{14}\right)-\frac{3e^t}{24}$$

$$= \frac{1}{14}\left(-\frac{31}{14}+5t\right)-\frac{e^t}{8} \quad ...(8)$$

So the complete solution is

$$C_1e^{-7t} + C_2e^{-2t}+\frac{5}{14}t-\frac{31}{196}-\frac{e^t}{8} \quad ...(9)$$

Self Assessment

11. Solve $\frac{dx}{dt}-7x+y=0$

$\frac{dy}{dt}-2x-5y=0$

12. Solve $\frac{dx}{dt}+2\frac{dy}{dt}-2x+2y=3e^t$

$3\frac{dx}{dt}+\frac{dy}{dt}+2x+y=4e^{3t}$

The equation of the type

$$\left.\begin{aligned} P_1dx+Q_1dy+R_1dz &= 0 \\ P_2dx+Q_2dy+R_2dz &= 0 \end{aligned}\right\} \quad ...(1)$$

Where P_1, P_2, Q_1, Q_2 and R_1, R_2 are functions of x, y, z

We can write these equations as

$$P_1\frac{dx}{dz}+Q_1\frac{dy}{dz}+R_1 = 0$$

Notes

$$P_2\frac{dx}{dz}+Q_2\frac{dy}{dz}+R_2 = 0$$

Solving for $\frac{dx}{dz}$ and $\frac{dy}{dz}$

$$\frac{dx}{dz} = \frac{Q_1R_2-Q_2R_1}{P_1Q_2-Q_1P_2},\ \frac{dy}{dz}=\frac{R_1P_2-P_1R_2}{P_1Q_2-Q_1P_2}$$

hence

$$\frac{dx}{Q_1R_2-Q_2R_1} = \frac{dy}{R_1P_2-P_1R_2}=\frac{dz}{P_1Q_2-P_2Q_1} \quad ...(2)$$

i.e. equations (1) can be put in the form

$$\frac{dx}{P} = \frac{dy}{Q}=\frac{dz}{R} \quad ...(3)$$

Hence forth the equations (3) will be taken as the standard form of a pair of ordinary simultaneous equations of the first order and of the first degree.

Solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

We have

$$\frac{dx}{P} = \frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{l\,P+mQ+n\,R} \quad ...(4)$$

and if $l\,P+mQ+n\,R = 0$ ...(5)

then $l\,dx+m\,dy+n\,dz = 0$ ...(6)

and if (5) is an exact differential, say du, then $u = a$ is one equation of the complete solution.

Similarly choosing l', m' and n' such that

$$l'P+m'Q+n'R = 0.$$

then $l'dx+m'dy+n'dz = dv=0$ ...(7)

Whence $v = b$ is another equation of the complete solution.

This method may be used with advantage in some examples to obtain a zero denominator and a numerator that is an exact differential or a non-zero denominator of which the numerator is the differential.

Example 1: Solve

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)}=\frac{dz}{x^2+y^2} \quad ...(1)$$

Each fraction is equal to

$$= \frac{x\,dx-y\,dy-z\,dz}{xz(y+x)-yz(x-y)-z(x^2+y^2)}=\frac{x\,dx-y\,dy-z\,dz}{0}$$

Therefore

$$x\,dx - y\,dy - z\,dz = 0 \qquad ...(2)$$

or $$d\left(\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2}\right) = 0$$

or $$x^2 - y^2 - z^2 = \text{constant} = c_1 \qquad ...(3)$$

Similarly

$$\frac{y\,dx + x\,dy - z\,dz}{yz(y+x) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x dy - zdz}{0}$$

Thus

$$y\,dx + x\,dy - z\,dz = 0$$

Thus $$xy - \frac{z^2}{2} = \text{constant} = c_2 \qquad ...(4)$$

So the two integrals (3), (4) are complete integrals of (1) Q.E.D.

Example 2: Solve

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z} \qquad ...(1)$$

Solution: From the first two members

$$\frac{dx+dy}{x^2+y^2+2xy} = \frac{dz}{(x+y)z}$$

or

$$\frac{dx+dy}{x+y} = \frac{dz}{z} \qquad ...(2)$$

Integrating (2) we have

$$\log(x+y) = \log z + \log c$$

∴ $$x + y = cz \qquad ...(3)$$

Also from (i)

$$\frac{dx+dy}{(x+y^2)} = \frac{dx-dy}{(x-y)^2} \qquad ...(4)$$

Integrating (4) we have

$$-(x+y)^{-1} = -(x-y)^{-1} - c_2 \qquad (c_2 \text{ being a constant}) \qquad ...(5)$$

or $$\frac{1}{x+y} = \frac{1}{x-y} + c_2$$

Notes

or $$\frac{1}{x-y}-\frac{1}{x+y}+c_2 = 0$$

$$\frac{x+y-x+y}{(x^2-y^2)}+c_2 = 0$$

$\therefore$ $$2y+c_2(x^2-y^2) = 0$$

So $$c_2 = \frac{2y}{y^2-x^2}$$

So complete solution is

$$\phi(c_1,c_2) = 0=\phi\left(\frac{x+y}{z},\frac{zy}{y^2-x^2}\right)=0 \qquad ...(6)$$

Example 3: Solve

$$\frac{dx}{xy} = \frac{dy}{y^2}=\frac{dz}{xyz-2x^2} \qquad ...(1)$$

Solution:

From the first two members

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$$\frac{dx}{x} = \frac{dy}{y} \qquad ...(2)$$

Integrating (2) we have

$$\log x = \log y+\log c_1$$

or $$x = c_1 y \qquad ...(3)$$

From the second and third member of (1) we have

$$\frac{dy}{y^2} = \frac{dz}{xyz-2x^2} \qquad ...(4)$$

Putting the value of x from (3) we have from (4)

$$\frac{dy}{y^2} = \frac{dz}{[zc_1y^2-2c_1^2y^2]}$$

or $$dy = \frac{dz}{(c_1z-2c_1^2)} \qquad ...(5)$$

Integrating (5) we have

$$\int dy = \int\frac{dz}{c_1(z-2c_1)}+\frac{c_2}{c_1}$$

or $$y = \frac{1}{c_1}\log(z-2c_1)+\frac{c_2}{c_1}$$

or $$c_1 y = \log(z-2c_1)+c_2 \quad ...(6)$$

Substituting value of c_1 from (3)

$$x = \log\left(z-\frac{2x}{y}\right)+c_2 \quad ...(7)$$

Thus from (3), (7) we have

$$\left.\begin{aligned} c_1 &= \frac{x}{y} \\ c_2 &= x-\log\left(\frac{zy-2x}{y}\right) \end{aligned}\right\} \quad ...(8)$$

So equation (8) form the complete integral of the set of equations.

Self Assessment

13. Solve

$$\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z}$$

14. Solve

$$\frac{dx}{x^2-y^2-yz}=\frac{dy}{x^2-y^2-yz}=\frac{dz}{z(x-y)}$$

Geometrical Meaning of

$$\frac{dx}{P} = \frac{dy}{Q}=\frac{dz}{R} \quad ...(1)$$

We know that the direction ratio of the tangent to a curve at any point (x, y, z) on it are proportional to dx, dy, dz at that point. Hence geometrically the given equations represent a system of curves in space, such that the direction ratios of the tangent to any one of these curves in space, at that point (x, y, z) on it are proportional to P, Q and R at that point. If $u = a, v = b$ are the general solutions of (1), then system of curves must be the curves of intersection of the surfaces $u = a, v = b$. It is also clear that since a, b are arbitrary constants, the system of curves represented by the equations is doubly infinite.

8.5 Summary

- Total differential equations can be solved under certain conditions.
- Simultaneous Differential equations are also shown to be solved by the above method.
- Illustrated examples are solved so that the technique of solving by various methods is clear.

Notes

8.6 Keywords

Exact Differential: An equation

$$P\,dx + Q\,dy + R\,dz = 0, \qquad ...(1)$$

is an exact differential if its integral is found in the form

$$u(x, y, z) = c, \qquad \text{(c being a constant)}$$

Exact Differential Equation: When equation (1) is put into the form

$$d\,u(x, y, z) \equiv P\,dx + Q\,dy + R\,dz = 0,$$

it is called Exact Differential Equation

Integrable: A differential equation when solved is said to be integrable.

8.7 Review Questions

1. Solve $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$
2. Solve $yz\log y\ dx - z\,x\log z\ dy + xy\ dz = 0$
3. Solve $(y+b)(z+c)\,dx + (x+a)(z+c)\,dy + (x+a)(y+b)\,dz = 0$
4. Solve $yz^2(x^2 - yz)\,dx + zx^2(y^2 - xz)\,dy + xy^2(z - xy)\,dz = 0$
5. Solve $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$

Answers: Self Assessment

3. $x^2y = cze^2,$ (*c* being a constant)
4. $(a^2 - x^2 - y^2)^{1/2} = C - Z,$ (*c* being a constant)
5. $xy^2 = cz^3,$ (*c* being a constant)
6. $(x+z)^2 = c(y+z)$ (*c* being a constant)
7. $z(x+y) - y^2 = cz^2$ (*c* being a constant)
8. $x^2 + y^2 + z^2 = cx$ (*c* being a constant)
9. $xy = c(a - z)$ (*c* being an arbitrary constant)
10. $xy + yz + zx = c(x+y+z),$ (*c* being a constant)
11. $x = e^{6t}$ (A cos t + B sin t)

    $y = e^{6t}[(A-B)\cos t + (A+B)\sin\ t]$

12. $x = c_1\left[-\frac{6}{5}t\right] + \frac{e^{2t}}{2} - \frac{3e^t}{11}$

$y = c_2 e^{-t} - \frac{c_1}{8}\exp\left[-\frac{6}{5}t\right]$

13. $x + y + z = c_1 z$

$\frac{x(2+x)}{y(2+y)} = c_2$

14. $x - y - z = c_1$

$x^2 - y^2 = c_2 z^2$

8.8 Further Readings

Books

H.T. Piaggio, Differential Equations

E.L. Ince, Ordinary Differential Equations

Unit 9: Adjoint and Self-Adjoint Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- See that adjoint and self-adjoint operators play an important part in the solution of certain types of equations.
- Observe that the properties of the solutions as well as the values of certain parameter are obtained in a systematic manner.
- Notice that the self-adjoint equations when solved under certain boundary conditions yield values of the solutions known as eigenfunctions corresponding to certain eigenvalues.

Introduction

In this unit the method of putting an equation into a self-adjoint form is dealt with. This method and the Sturm–Liouville's method leads us to the solutions of the differential equations which are orthogonal.

The solutions form a set of eigenfunctions which are complete and so any function on the given interval can be expanded in terms of these eigenfunctions.

9.1 Adjoint and Self-adjoint Operators

In this unit we are interested in solving inhomogeneous boundary value problems for linear, second order differential equations. We will now develop an approach that is based upon the idea of linear algebra. We shall work with the simplest possible type of linear differential operator L, $C^2[a, b\} \to C\{a, b\}$ being in self-adjoint form:

$$L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \qquad ...(1)$$

where $p(x) \in C^1[a, b]$ and is strictly non-zero for all $x \in (a, b)$, and $q(x) \in C'[a, b]$. The reasons for referring to such an operator as self-adjoint will become clear later in this unit.

This definition encompasses a wide class of second order differential operators.

For example, if

$$L^1 \equiv a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x) \quad ...(2)$$

is non-singular on [a, b], we can write it in self-adjoint form by defining

$$p(x) = \exp\left(\int^x \frac{a_1(t)}{a_2(t)}dt\right), q(x) = \frac{a_0(x)}{a_2(x)}\exp\left(\int^x \frac{a_1(t)}{a_2(t)}dt\right) \quad ...(3)$$

Note that $p(x) \neq 0$ for $x \in [a, b]$. By studying inhomogeneous boundary value problems of the form $Ly = f$, or

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = f(x) \quad ...(4)$$

we are therefore considering all second order, non-singular, linear differential operators. For example, consider Hermite's equations.

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0, \quad ...(5)$$

for $-\infty < x < \infty$. This is not in self-adjoint form, but, if we follow the above procedure, the self-adjoint form of the equation is

$$\frac{d}{dx}\left(e^{-x^2}\frac{dy}{dx}\right) + \lambda e^{-x^2}y = 0$$

This can be simplified, and kept in self-adjoint form, by writing $u = e^{(-x^2/2)}y$ to obtain

$$\frac{d^2u}{dx^2} - (x^2 - 1)u = -\lambda u \quad ...(6)$$

9.2 Boundary Conditions

To complete the definition of a boundary value problem associated with (4), we need to know the boundary conditions. In general these will be of the form

$$\alpha_1 y(a) + \alpha_2 y(b) + \alpha_3 y'(a) + \alpha_4 y'(b) = 0,$$
$$\beta_1 y(a) + \beta_2 y(b) + \beta_3 y'(a) + \beta_4 y'(b) = 0. \quad ...(7)$$

Since each of these is dependent on the values of y and y' at each end of [a, b], we refer to these as mixed or coupled boundary conditions. It is unnecessarily complicated to work with the boundary conditions in this form, and we can start to simplify matters by deriving Lagrange's identity.

Lagrange's Identity: If L is the linear differential operator given by (1) on [a, b] and if $y_1, y_2 \in C^2$ [a, b], then

$$y_1(Ly_2) - y_2(Ly_1) = [p(y_1y'_2 - y'_1y_2)]'. \quad ...(8)$$

Proof: From the definition of L,

Notes

$$y_1(Ly_2) - y_2(Ly_1) = y_1\left[(py_2')' + qy_2\right] - y_2\left[(py_1')' + qy_1\right]$$

$$= y_1(py_2')' - y_2(py_1')' = y_1\left[py_2'' + p'y_2'\right] - y_2\left[py_1'' + p'y_1'\right]$$

$$= p'(y_1y_2' - y_1'y_2) + p(y_1y_2'' - y_1''y_2) = [p(y_1y_2' - y_1'y_2)]'$$

Now recall that the space $C[a, b]$ is a real inner product space with a standard inner product defined by

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx$$

If we now integrate (8) over $[a, b]$ then

$$\langle y_1, Ly_2\rangle - \langle Ly_1, y_2\rangle \;=\; [p(y_1y_2' - y_1'y_2)]_a^b \qquad ...(9)$$

This result can be used to motivate the following definitions. The adjoint operator of T, written $\overline{T}$, satisfies $\langle y_1, Ty_2\rangle - \langle \overline{T}y_1, y_2\rangle$ for all y_1 and y_2. For example, let us see if we can construct the adjoint to the operator

$$\mathcal{D} \equiv \frac{d^2}{dx^2} + \gamma\frac{d}{dx} + \delta,$$

with $\gamma, \delta \in R$, on the interval [0, 1], when the functions on which $\mathcal{D}$ operates are zero at $x = 0$ and $x = 1$. After integrating by parts and applying these boundary conditions, we find that

$$\langle \phi_1, \mathcal{D}\phi_2\rangle = \int_0^1 \phi_1(\phi_2'' + \gamma\phi_2' + \delta\phi_2)dx = \left[\phi_1\phi_2'\right]_0^1 - \int_0^1 \phi_1'\phi_2'dx + \left[\gamma\phi_1\phi_2\right]_0^1 - \int_0^1 \gamma\phi_1'\phi_2 dx + \int_0^1 \delta\phi_1\phi_2 dx$$

$$= -\left[\phi_1'\phi_2\right]_0^1 + \int_0^1 \phi_1''\phi_2 dx - \int_0^1 \gamma\phi_1'\phi_2 dx + \int_0^1 \delta\phi_1\phi_2 = (\overline{\mathcal{D}}\phi_1, \phi_2),$$

where

$$\overline{D} \equiv \frac{d_2}{dx^2} - \gamma\frac{d}{dx} + \delta$$

A linear operator is said to be Hermitian, or self-adjoint. If $\langle y_1, Ty_2\rangle = \langle Ty_1, y_2\rangle$ for all y_1 and y_2. It is clear from (9) that L is a Hermitian, or self-adjoint, operator if and only if

$$\left[p(y_1y_2' - y_1'y_2\right]_a^b = 0$$

and hence

$$p(b)\{y_1(b)y_2'(b) - y_1'(b)y_2(b)\} - p(a)\{y_1(a)y_2'(a) - y_1'(a)y_2(a)\} = 0 \qquad ...(10)$$

In other words, whether or not L is Hermitian depends only upon the boundary values of the functions in the space upon which it operates.

There are three different ways in which (10) can occur.

(i) $p(a) = p(b) = 0$. Note that this doesn't violate our definition of p as strictly non-zero on the open interval (a, b). This is the case of singular boundary conditions.

(ii) $p(a) = p(b) \neq 0$, $y_i(a) = y_i(b)$ and $y_i'(a) = y_i'(b)$. This is the case of periodic boundary conditions.

(iii) $\alpha_1 y_i(a) + \alpha_2 y_1'(a) = 0$ and $\beta_1 y_i(b) + \beta_2 y_1'(b) = 0$, with at least one of the α_i and one of the β_i non-zero. These conditions then have non-trivial solutions if and only if

$$y_1(a)y_2'(a) - y_1'(a)y_2(a) = 0, \quad y_1(b)y_2'(b) - y_1'(b)y_2(b) = 0,$$

and hence (10) is satisfied.

Conditions (iii), each of which involves y and y' at a single endpoint, are called unmixed or separated. We have therefore shown that our linear differential operator is Hermitian with respect to a pair of unmixed boundary conditions. The significance of this result becomes apparent when we examine the eigenvalues and eigenfunctions of Hermitian linear operators.

As an example of how such boundary conditions arise when we model physical systems, consider a string that is rotating or vibrating with its ends fixed. This leads to boundary conditions $y(0) = y(a) = 0$ - separated boundary conditions. In the study of the motion of electrons in a crystal lattice, the periodic conditions $p(0) = p(l)$, $y(0) = y(l)$ are frequently used to represent the repeating structure of the lattice.

9.3 Eigenvalues and Eigenfunctions of Hermitian Linear Operators

The eigenvalues and eigenfunctions of a Hermitian linear operator L are the non-trivial solutions of $Ly = \lambda y$ subject to appropriate boundary conditions.

Theorem 1. Eigenfunctions belonging to distinct eigenvalues of a Hermitian linear operator are orthogonal.

Proof: Let y_1 and y_2 be eigenfunctions that correspond to the distinct eigenvalues λ_1 and λ_2. Then

$$\langle Ly_1, y_2 \rangle = \langle \lambda_1 y_1, y_2 \rangle = \lambda_1 \langle y_1, y_2 \rangle$$

and

$$\langle y_1, Ly_2 \rangle = \langle y_1, \lambda_2 y_2 \rangle = \lambda_2 \langle y_1, y_2 \rangle$$

so that the Hermitian property $\langle Ly_1, y_2 \rangle = \langle y_1, Ly_2 \rangle$ gives

$$(\lambda_1 - \lambda_2)(y_1, y_2) = 0$$

Since $\lambda_1 \neq \lambda_2$, $(y_1, y_2) = 0$, and y_1 and y_2 are orthogonal.

As we shall see in the next section, all of the eigenvalues of a Hermitian linear operator are real, a result that we will prove once we have defined the notion of a complex inner product.

If the space of functions $C^2[a, b]$ were of finite dimension, we would now argue that the orthogonal eigenfunctions generated by a Hermitian operator are linearly independent and can be used as a basis (or in the case of repeated eigenvalues, extended into a basis). Unfortunately, $C^2[a, b]$ is not finite dimensional, and we cannot use this argument. We will have to content ourselves with presenting a credible method for solving inhomogeneous boundary value problems based upon the ideas we have developed, and simply state a theorem that guarantees that the method will work in certain circumstances.

9.4 Eigenfunction Expansions

In order to solve the inhomogeneous boundary value problem given by (4) with $f \in C[a, b]$ and unmixed boundary conditions, we begin by finding the eigenvalues and eigenfunctions of L.

Notes

We denote these eigenvalues by $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, and the eigenfunctions by $\phi_1(x)$, $\phi_2(x) \ldots$, $\phi_n(x), \ldots$ Next, we expand $f(x)$ in terms of these eigenfunctions, as

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \qquad \text{...(11)}$$

By making use of the orthogonality of the eigenfunctions, after taking the inner product of (11) with ϕ_n, we find that the expansion coefficients are

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \qquad \text{...(12)}$$

Next, we expand the solution of the boundary value problem in terms of the eigenfunctions, as

$$y(x) = \sum_{n=1}^{\infty} d_n \phi_n(x), \qquad \text{...(13)}$$

and substitute (12) and (13) into (4) to obtain

$$L\left[\sum_{n=1}^{\infty} d_n \phi_n(x)\right] = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

From the linearity of L and the definition of ϕ_n this becomes

$$\sum_{n=1}^{\infty} d_n \ \lambda_n \ \phi_n(x) = \sum_{n=1}^{\infty} c_n \ \phi_n(x).$$

We have therefore constructed a solution of the boundary value problem with $d_n = c_n/\lambda_n$, if the series (13) converges and defines a function in $C^2(a, b)$. This process will work correctly and give a unique solution provided that none of the eigenvalues λ_n is zero. When $\lambda_m = 0$, there is no solution if $c_m \neq 0$ and an infinite number of solutions if $c_m = 0$.

Example 1: Consider the boundary value problem

$$-y'' = f(x) \qquad \text{subject to } y(0) = y(\pi) = 0 \qquad \text{...(14)}$$

In this case, the eigenfunctions are solutions of

$$y'' + \lambda y = 0 \quad \text{subject to } y(0) = y(\pi) = 0,$$

which we already know to be $\lambda_n = n^2$, $\phi_n(x) = \sin nx$. We therefore write

$$f(x) = \sum_{n=1}^{\infty} c_n \sin nx,$$

and the solution of the inhomogeneous problem (14) is

$$y(x) = \sum_{n=1}^{\infty} \frac{c_n}{n^2} \sin nx,$$

In the case $f(x) = x$,

$$c_n = \frac{\int_0^{\pi} x \sin nx \, dx}{\int_0^{\pi} \sin^2 nx \, dx} = \frac{2(-1)^{n+1}}{n},$$

so that

$$y(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nx$$

This type of series is known as a Fourier series.

This example is, of course, rather artificial, and we could have integrated (14) directly. There are, however, many boundary value problems for which this eigenfunction expansion method is the only way to proceed analytically.

Example 2: Consider the inhomogeneous equation

$$(1 - x^2)y'' - 2xy' + 2y = f(x) \qquad \text{on } -1 < x < 1, \qquad ...(15)$$

with $f \in C[-1, 1]$, subject to the condition that y should be bounded on $[-1, 1]$. We begin by noting that there is a solubility condition associated with this problem. If $u(x)$ is a solution of the homogeneous problem, then, after multiplying through by u and integrating over $[-1, 1]$, we find that

$$\left[u(1-x^2)y'\right]_{-1}^{1} - \left[u'(1-x^2)y\right]_{-1}^{1} = \int_{-1}^{1} u(x)f(x)dx$$

If u and y are bounded on $[-1, 1]$, the left hand side of this equation vanishes, so that $\int_{-1}^{1} u(x)f(x)dx = 0.$ Since the Legendre polynomial, $u = P_1(x) = x$, is the bounded solution of the homogeneous problem, we have

$$\int_{-1}^{1} P_1(x)f(x)dx = 0$$

Now, to solve the boundary value problem, we first construct the eigenfunction solutions by solving $Ly = \lambda y$, which is

$$(1 - x^2)y'' - 2xy' + (2 - \lambda)y = 0$$

The choice $2 - \lambda = n(n + 1)$, with n a positive integer, gives us Legendre's equation of integer order, which has bounded solutions $y_n(x) = P_n(x)$. These Legendre polynomials are orthogonal over $[-1, 1]$. If we now write

$$f(x) = \sum_{m=0}^{\infty} A_m P_m(x),$$

where $A_1 = 0$ by the solubility condition, and then expand $y(x) = \sum_{m=0}^{\infty} B_m P_m(x)$

we find that

$$\{2 - m(m + 1)\}B_m = A_m \text{ for } m \geq 0$$

The required solution is therefore

$$y(x) = \frac{1}{2}A_0 + B_1 P_1(x) + \sum_{m=2}^{\infty} \frac{A_m}{2 - m(m+1)} P_m(x)$$

with B_1 an arbitrary constant.

Notes

Having seen that this method works, we can now state a theorem that gives the method a rigorous foundation.

Theorem: If L is a non-singular, linear differential operator defined on a closed interval $[a, b]$ and subject to unmixed boundary conditions at both endpoints, then

(i) L has an infinite sequence of real eigenvalues $\lambda_0, \lambda_1, \ldots$, which can be ordered so that

$$|\lambda_0\} < |\lambda_1| < \ldots < |\lambda_n| < \ldots$$

and

$$\lim_{n \to \infty} |\lambda_n| = \infty$$

(ii) The eigenfunctions that correspond to these eigenvalues form a basis for $C[a, b]$, and the series expansion relative to this basis of a piecewise continuous function y with piecewise continuous derivative on $[a, b]$ converges uniformly to y on any subinterval of $[a, b]$ in which y is continuous.

We will not prove this result here. Instead, we return to the equation, $Ly = \lambda y$, which defines the eigenfunctions and eigenvalues. For a self-adjoint, second order. Linear differential operator, this is

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = \lambda y, \qquad \text{...(16)}$$

which, in its simplest form, is subject to the unmixed boundary conditions

$$\alpha_1 y(\mathrm{a}) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0, \qquad \text{...(17)}$$

with $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$ to avoid a trivial condition. This is an example of a Sturm–Liouville system, and we will devote the unit II for study of the properties of the solutions of such systems.

Self Assessment

1. Consider the linear second order differential equation

$$x\frac{d^2y}{dx^2} + (1 - x)\frac{dy}{dx} + \lambda y = 0$$

Show that the Sturm–Liouville form of the above equation is

$$(xe^{-x}y')' + \lambda e^{-x}y = 0, \text{ for } x > 0$$

2. Show that the equation

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + [\lambda B(x) - C(x)]y = 0$$

can be written in self-adjoint form by defining

$$p(x) = \exp\left(\int A(x)dx\right)$$

what are $q(x)$, $r(x)$ in terms of A, B, C?

9.5 Summary

- In this unit we rearrange certain linear equations of the second order in a way in which the differential operator is self-adjoint.
- Examples of self-adjoint equations are Legendre equation, Bessel's equations, Hermite equations and many more.
- Putting these equations into self-adjoint form enables us to study certain properties known as eigenvalue and eigenfunction expansions and completeness etc.

9.6 Keywords

Eigenfunctions are a set of solutions of the self-adjoint equations that form an orthonormal set of complete system.

The real symmetric matrix is self-adjoint or an ***Hermitian operator***.

9.7 Review Question

1. Show that

   $(xy'(x))' = -\lambda xy(x)$

   is self-adjoint on the interval (0, 1), with $x = 0$ a singular end point and $x = 1$ a regular end point with the condition $y(1) = 0$.

9.8 Further Readings

Books

King A.C., Billingham and Otto S.R., Differential Equations.

Pipes L.A. and Harrill L.R., Applied Mathematics for Engineers and Physicists

Yosida K., Lectures on Differential and Integral Equations.

Notes

Unit 10: Green's Function Method

CONTENTS

Objectives

After studying this unit, you should be able to see that:

- Green's function plays an important part in the solution of the differential equations.
- It finds its applications in most of the boundary value problems.
- Green's function is quite helpful in converting a differential equation into an integral equation.

Introduction

Green's function method helps in solving most of the boundary value problems. It is quite useful in reducing a differential equation to an integral equation. With the help of the Green's function method the problem of solution of differential equations becomes simpler.

10.1 Boundary Value Problem of Sturm–Liouville Type

We consider a differential equation of the second order

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0 \qquad ...(1)$$

where $p_1(x)$, $p_2(x)$ are real-valued continuous function on a closed interval $a \le x \le b$. The equation (1) can be put into the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) = q(x)y \qquad ...(2)$$

by multiplying equation (1) with

$$\exp\left(\int_a^x p_1(x)dx\right) = p(x) \qquad ...(3)$$

and putting

$$q(x) = -p_2(x)p\ (x) \qquad ...(4)$$

The coefficients $p(x)$ and $q(x)$ satisfy the following conditions:

$p(x)$ and $q(x)$ are real-valued continuous functions on the interval $a \le x \le b$ and $p(x) > 0$ there.

Putting $z = p(x)\dfrac{dy}{dx}$ in (2) we have

$$\frac{dy}{dx} = \frac{z}{p(x)} \qquad ...(5)$$

$$\frac{dz}{dx} = q(x)\ y \qquad ...(6)$$

If a pair of functions $y(x)$ and $z(x)$ is a solution of the equations (5) and (6) and if $y(x) \ne 0$, then $y(x)$, and $z(x)$ do not vanish at any point in the interval $a \le x \le b$. So due to $y(x) \ne 0$, we may seek a solution.

$$y(x) = \rho(x)\sin\theta(x)$$

$$z(x) = \rho(x)\cos\theta(x)$$

$$\text{with } p(x) = (y^2(x) + z^2(x))^{1/2} > 0 \qquad ...(7)$$

Substituting in (5) and (6) we have

$$\frac{d\rho}{dx}\sin\theta(x) + \rho(x)\cos\theta(x) \qquad \frac{d\theta}{dx} = \frac{\rho(x)\cos\theta(x)}{p(x)}$$

and $\dfrac{d\rho}{dx}\cos\theta(x) - \rho(x)\sin\theta(x)\dfrac{d\theta}{dx} = q(x)\,\rho(x)\sin\theta(x)$

Simplifying the above equations, we have

$$\frac{d\rho(x)}{dx} = \left(\frac{1}{p(x)} + q(x)\right)P\sin\theta(x)\cos\theta(x) \qquad ...(8)$$

$$\frac{d\theta}{dx} = \frac{\cos^2\theta(x)}{p(x)} - q(x)\sin^2\theta(x), \qquad p(x) > 0$$

The second equation of (8) does not contain the unknown ρ, hence we can find a solution $\theta(x)$. Then substituting this solution in the first equation, we can obtain the general solution $p(x)$

$$\rho(x) = \rho(\alpha)\exp\left(\int_a^x \left\{\frac{1}{p(x)} + q(x)\right\}\sin\theta(x)\cos\theta(x)dx\right) \qquad ...(9)$$

Notes

Since $p(x) > 0$ or < 0 or every point $a \leq x \leq b$, according as $p(a) > 0$ or < 0, we can find a positive solution $p(x)$ from which, along with $\theta(x)$, we can obtain a solution $y(x) = p(x) \sin \theta(x)$, not identically zero, of the original equation (2).

Now for an integer n, $\theta(x) + 2\pi\ n$ is also a solution of the second equation of (8). Thus the solutions $y_1(x)$ and $y_2(x)$ obtained from $\theta(x)$ and $\theta(x) + 2n\pi$ are linearly dependent. So if the two solutions $y_1(x)$ and $y_2(x)$ given by

$$y_1(x) = \rho_1(x) \sin \theta_1(x)$$

$$y_2(x) = \rho_2(x) \sin \theta_2(x)$$

are linearly dependent, then for some integer n

$$\theta_1(x) = \theta_2(x) + 2\pi.$$

Now, an initial condition for q(x),

$$\theta(a) = \alpha \qquad ...(10)$$

gives a relation between $y(a)$ and $y_1(a)$ as follows

At $x = a$ from (5) and (7) we have

$$z(a) = p(a)\ y'(a) = \rho(a) \cos \theta(a)$$

So $\quad p(a)\ y'(a) \sin \theta(a) = \rho(a) \cos \theta(a) \sin \theta(a)$

or $\quad p(a)\ y'(a) \sin \theta(a) = y(a) \cos \theta(a)$

or $\quad p(a)\ y'(a) \sin \theta(a) - y(a) \cos \theta(a) = 0 \qquad ...(11)$

In this section we shall be concerned with the problem of finding the solution $y(x)$ corresponding to the solution $\theta(x)$ satisfying the boundary conditions

$$\theta(a) = \alpha,\ \theta(b) = \beta \qquad ...(12)$$

at both ends of the interval $a \leq x \leq b$.

Condition (12) corresponds to the conditions

$$p(a)\ y'(a) \sin \alpha - y(a) \cos \alpha = 0$$

$$p(b)\ y'(b) \sin \beta - y(b) \cos \beta = 0 \qquad ...(13)$$

for $y(x)$. It should be noted that the boundary value problem of finding the solution of (2) satisfying the boundary conditions (13) between y and y' is essentially different from the initial value problem.

10.2 Green's Function for One Dimensional Problem

Let us denote $L_x(y)$, a differential operator

$$L_x(y) = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y \qquad ...(1)$$

which is defined for every function $y(x)$ such that $\frac{dy}{dx}$ and $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]$ are defined and continuous on the interval $\alpha \leq x \leq b$. Let us define Lagrange's identity

$$y\ L_x(z) - z\ L_x(y) = \frac{d}{dx}\left[p(x)\frac{dz}{dx}\right]y - z\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]$$

Notes

$$= \frac{d}{dx}\left\{p(x)\left[y(x)\frac{dz}{dx} - z\frac{dy}{dx}\right]\right\} \qquad ...(2)$$

Integrating both sides of equation (2) we obtain

$$\left\{p(x)\left[y(x)\frac{dz}{dx} - z(x)\frac{dy}{dx}\right]\right\}_{a'}^{b'} = \int_{a'}^{b'} [y L_x(z) - Z L_x(y)]dx,\ a < a' < b' < b \qquad ...(3)$$

Equation (3) is known as Green's theorem in one dimension. If $y(x)$ and $z(x)$ both satisfy the boundary conditions

$p(a)\ y,(a) \sin \alpha - y(a) \cos \alpha = 0$

$p(b)\ y,(b) \sin \beta - y(b) \cos \beta = 0$

$p(a)\ z,(a) \sin \alpha - z(a) \cos \alpha = 0$

$p(b)\ y,(b) \sin \beta - z(b) \cos \alpha = 0$ ...(4)

Then for $a, = a$ and $b, = b$, L.H.S. is zero and we get

$$\int_a^b [y(x)L_x(x) - z(x)L_x(y)]dx = 0 \qquad ...(5)$$

Suppose that two functions $y_1(x) \neq 0$ and $y_2(x) \neq 0$ satisfy

$L_x(y_1) = 0$

$p(a)\ y_1,(a) \sin \alpha - y_1(a) \cos \alpha = 0$ ...(6)

and

$L_x(y_2) = 0$

$p(b)\ y_2,\ (b) \sin \beta - y_2(b) \cos \beta = 0$ ...(7)

respectively, and suppose that these two functions $y_1(x)$ and $y_2(x)$ are linearly independent. Write

$C = p(\xi)\ [y_1(\xi)\ y,_2(\xi) - y,_1(\xi)\ y_2(\xi)]$.

Differentiating C with respect to ξ and making use of (2), we see, by virtue of (6) and (7), that C must be constant. Moreover, the linear independence of $y_1(x)$ and $y_2(x)$ implies that C is not zero. Now we define a function $G(x, \xi)$ of two variables x and ξ by

$$G(x, \xi) = -\frac{1}{C} y_1(\xi) y_2(x) \qquad (x \geqq \xi)$$

$$= \frac{1}{C} y_1(x) y_2(\xi) \qquad (x < \xi)$$

$$C = p(\xi)\left[y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi)\right] = \text{Constant}$$

The function $G(x, \xi)$ is called *Green's Function* for the equation $L_x(y) = 0$ subject to the boundary conditions (4). Obviously Green function $G(x, \xi)$ has the following properties:

$G(x, \xi)$ is continuous at any point (x, ξ) in the domain $a \leqq x, \xi \leqq b$.

As a function of x, $G(x, \xi)$ satisfies the given boundary conditions for every ξ. ...(9)

If $x \neq \xi$, $G(x, \xi)$ satisfies the equation $L_x(G) = 0$ as a function of x.

Notes

Both $G_x(x, \xi)$ and $\{p(x)G_x(x, \xi)\}_x$ are bounded in the region $x \neq \xi$, $a \leqq x, \xi \leqq b$. ...(10)

If $a < x_0 < b$ then as $x \to x_0$, keeping the relation $x < \xi$ and as $x \to x_0$, $\xi \to x_0$, keeping the relation $x < \xi$, $G(x, \xi)$ tends to finite values $G_x(x_0 + 0, x_0)$ and $G(x_0 - 0, x_0)$ respectively, and ...(11)

$$G_x(x_0 + 0, x_0) - G_x(x_0 \to 0, x_0) = -\frac{1}{p(x_0)} \quad ...(12)$$

$$G(x, \xi) = G(\xi, x) \quad ...(13)$$

Example: On the basis of equation (8), we have

$$L_x = \frac{d^2}{dx^2}, \quad y(0) = y(1) = 0$$

$$x = 0, x = 1$$

Now solutions of

$$L_x(y) = 0$$

or
$$\frac{d^2y}{dx^2} = 0 \quad ...(14)$$

Suppose that a Green's function $G(x, \xi)$ exists. Then since

$L_x(G(x, \xi)) = 0$ for $x \neq \xi$,

$G(x, \xi)$ must be represented, by means of a fundamental system $y_1(x)$, $y_2(x)$ of the solutions of $L_x(y) = 0$, as follows:

The general solution of $\frac{d^2y}{dx^2} = 0$.

So the solution of (14) is

$$y = c_1 x + c_2 \quad ...(15)$$

Let the two solutions be $y_1(x)$ and $y_2(x)$. Thus

if $\quad y_1(0) = 0$ then $c_2 = 0$

so $\quad y_1(x) = x$, ...(16)

$\quad y_2(1) = 0 = c_1 \quad 1 + c_2 = 0$

$\therefore \quad c_1 = -c_2 = 1$

$\quad y_2 = (1 - x)$, ...(17)

Thus

$$C = 1 \cdot \{x \cdot (-1) - 1 \cdot (1 - x)\} = 1$$

$$G(x, \xi) = 1 \cdot (1 - \xi)x \qquad (x \leqq \xi)$$

$$= (1 - x)\xi \qquad (x > \xi). \quad ...(18)$$

Self Assessment

Notes

1. Find the Green function for the equation

$$L_x y = \frac{d^2}{dx^2} y = 0$$

with the conditions

$y(0) = 0, y'(1) = 0$

10.3 Periodic Solutions Generalized Green's Function

A system of important boundary conditions not included earlier is

$$y(a) = y(b),\ y'(a) = y'(b) \quad ...(1)$$

If the coefficients $p(x)$, $g(x)$, $r(x)$ are periodic functions with period $b - a$, that is

$$p(x + b - a) = p(x),\ q\ (x + b - a) = q(x),\ r(x + b - a) = r(x)$$

Then the conditions (1) are just the conditions that the solution $y(x)$ of the equation

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0 \quad ...(A)$$

is periodic with the same period $b - a$, that is

$$y(x + b - a) = y(x)$$

For in each case, $y(x)$, $y_{a,\,b}\ (x + b - a)$ both satisfy the equation (A) together with the same initial conditions

$$y(a) = y_{a,b}\ (a),\ y'(a) = y_{a,b}\ (a)$$

Hence by the uniqueness of the solutions, we must have

$$y(x) = y_{a,b}\ (x)$$

In the following we shall be concerned with more general conditions, which include the conditions (1), of the form

$$y(a) = \gamma y(b),\ p(a)\ y'(\mathrm{a}) = \frac{p(b)}{\gamma}\ y'(b) \quad ...(2)$$

or

$$y(a) = \gamma p(b),\ y'(b),\ p(a)\ y'(\mathrm{a}) = -\ \frac{1}{\gamma} y'(b) \quad ...(3)$$

where γ is a non-zero constant. It is easily seen that if $y(x)$ and $z(x)$ both satisfy either (2) or (3), then the relation

$$p(x)\ (y(x)z'(x)\ - y'(x)\ z(x))\Big|_a^b = 0 \quad ...(4)$$

holds.

10.3.1 Construction of Green's Function

Suppose that a Green's function exists. Then since $L_x(\mathrm{G}(x, \xi)) = 0$ for $x \neq \xi$, $y(x, \xi)$ must be represented by means of a fundamental system $y_1(x)$, $y_2(x)$ of the solution of $L_x(y) = 0$ as follows:

$$G(x, \xi) = \begin{cases} c_1 y_1(x) + c_2 y_2(x) & (a \leqq x < \xi) \\ c_3 y_1(x) + c_4 y_2(x) & (\xi < x \leq b) \end{cases} \quad ...(5)$$

Notes

where every C_i is a function of ξ. We shall determine the relations between C_i so that $G(x, \xi)$ satisfies the required properties for the Green's function pertaining to the boundary condition (2). Since $G(x, \xi)$ is continuous at $x = \xi$, we obtain

$$c_1 y_1(\xi) + c_2 y_2(\xi) = c_3 y_3(\xi) + c_4 y_4(\xi) \quad \text{...(6)}$$

By equation (12) of section (10.2), we obtain

$$c_1 y_1'(\xi) + c_2 y_2'(\xi) - c_3 y_3'(\xi) - c_4 y_4'(\xi) = \frac{1}{p(\xi)} \quad \text{...(7)}$$

Finally from the boundary conditions (2) we obtain

$$c_1 y_1(a) + c_2 y_2(a) = \gamma\,(c_3 y_3(b) + c_4 y_4(b))$$

$$\gamma\, p(a)\,(c_1 y_1'(a) + c_2 y_2'(a) = p(b)\,(c_3 y_3'(b) + c_4 y_4'(b)) \quad \text{...(8)}$$

Also Green's function should be symmetric i.e.

$$G(x, \xi) = G(\xi, x) \quad \text{...(8a)}$$

Only the last relation of (8) must be changed according as the corresponding boundary conditions, if we are concerned with Green's function under the boundary conditions (3).

Example: Find the Green's function for $L_x\, y = 0$ with the boundary conditions

$$y(0) = -\,y(1), \quad y'(0) = -\,y'(1).$$

Solution:

The general solution of $L_x y = 0$ is of the form $c_1 x + c_2$. Now taking as a fundamental system of the solutions of $y'' = 0$, as

$$y_1(x) = (x),\ y_2(x) = 1,\ p(x) = 1,\ \gamma = 1$$

Let $G(x, \xi)$ be given by the relation (5) where $a = 0$, $b = 1$ from the equations (6), (7) and (8) we have

$$c_1\xi + c_2 = c_3\,\xi + c_4,\ c_1 - c_3 = 1,\ c_2 = -\,(c_3 + c_4),\ c_1 = -\,c_3$$

Solving these equations, we obtain

$$2\,c_1 = 1,\ c_1 = \frac{1}{2} = -\,c_3,\ (c_1 - c_3)\xi + c_2 = c_4$$

$$c_2 - \frac{1}{2} = -\,c_4$$

$$2\,c_2 - \frac{1}{2} + \xi = 0$$

$$c_2 = \frac{1}{4} - \xi/2,\ c_4 = \frac{1}{4} + \xi/2$$

Therefore

$$G(x, \xi) = \frac{1}{2}x + \left(\frac{1}{4} - \xi/2\right)\cdot 1 \qquad \text{for } 0 \le x < \xi$$

$$= -\frac{1}{2}x + \left(\frac{1}{4} + \xi/2\right)\cdot 1 \qquad \text{for } \xi < x \le 1$$

or $$G(x, \xi) = -\frac{1}{2}|x - \xi| + \frac{1}{4} = G(\xi, x).$$

Generalized Green's Function

Notes

Let us consider the inhomogeneous equation

$L_x y = \varphi(x)$

whose solution $y(x)$ satisfies the boundary conditions. Let us assume that there exists a non-trivial solution $y_0(x) \neq 0$ of the equation $L_x\, y(x) = 0$. We can show that the function $\varphi(x)$ must satisfy

$$\int_a^b \varphi(x)\, y_0(x) dx = 0 \quad \text{...(9)}$$

where $y_0(x)$ also satisfying the boundary conditions. To see this we have

$$-\int_a^b \varphi(x) y_0(x) dx = \int_a^b [y_0(x) L_x(y) - y(x) L_x(y_0)] dx$$

$$= [p(x)\,(y_0(x) y'(x) - y'_0(x) y(x)]_a^b = 0$$

On the other hand the solution $y(x)$ may be written in the form

$y(x) = z(x) + c\, y_0(x)$

where $z(x)$ is a solution of $L_x(z) = \varphi(x)$, satisfying the boundary conditions. Since $y_0(x) \neq 0$ we can choose the constant C so that

$$\int_a^b y(x) y_0(x) dx = 0 \quad \text{...(10)}$$

Now it can be proved that such a function $y(x)$ of the boundary value problem satisfying (10) can be written as

$$y(x) = \int_a^b G(x, \xi) \varphi(\xi) d\xi \quad \text{...(11)}$$

by means of the generalized Green's function $G(x, \xi)$.

By a generalized Green's function, we mean a such $G(x, \xi)$ satisfying the following five conditions:

1. Continuity of $G(x, \xi)$ at any point (x, ξ) in the domain $a \leq x \leq \xi < b$. As a function of x, $G(x, \xi)$ satisfies the given boundary conditions.
2. If $x \neq \xi$, $G(x, \xi)$ satisfies the equation

   $G(x, \xi) = y_0(x)\, y_0(\xi)$

   as a function of x. $G_x(x, \xi)$ is bounded in the region $x \neq \xi$.
3. If $a < x_0 < b$ then as $x \to x_0$, $\xi \to x$, keeping the relation $x > \xi$ and as $x \to x_0$, $\xi \to x_0$ keeping the relation $x < \xi$, $G_x(x, \xi)$ tends to finite values $G_x(x_0 + 0, x_0)$ and $G_x(x_0 - 0, x_0)$, respectively, and

   $$G_x(x_0 + 0, x_0) - G_x(x_0 - 0, x_0) = \left(-\frac{1}{p(x_0)}\right)$$
4. $G(x, \xi) = G(\xi, x)$
5. $\int_a^b G(x, \xi) y_0(x) dx = 0$

Notes

Example: Find generalized Green's function for $L_x = \frac{d^2}{dx^2}$, with the boundary conditions $y'(0) = y'(1) = 0$.

Solution:

The general solution of $y''(x) = 0$ is a polynomial of degree 1. Hence there exists a non-trivial solution $y_0(x) = 1$ of the boundary value problem. So from the condition (2) we have

$L_x\, G(x, \xi) = 1$, that is, $G_{xx}(x, \xi) = 1$.

Hence we have

$$G(x, \xi) = A_1 + A_2 x + \frac{x^2}{2} \qquad x \leqq \xi$$

$$= B_1 + B_2 x + \frac{x^2}{2} \qquad x > \xi$$

By the boundary conditions $G_x(0, x) = 0$, $G_x(1, \xi) = 0$, we obtain

$A_2 = 0$, $B_2 = -1$. So the condition

$$G_x(\xi+0,\xi) - G_x(\xi-0,\xi) = -1$$

holds automatically. By the continuity at $x = \xi$, that is $G(\xi + 0, \xi) - G(\xi - 0, \xi) = 0$, we obtain $B_1 - \xi - A_1 = 0$. Hence we obtain

$$G(x, \xi) = A_1 + \frac{x^2}{2} \qquad x \leqq \xi$$

$$= A_1 + \xi - x + \frac{x^2}{2} \qquad x > \xi.$$

Finally, from the relation

$$\int_0^1 G(x, \xi) y_0(\xi) d\xi = 0,$$

we obtain $A_1 = 0$. Thus the generalized Green's function is given by

$$G(x, \xi) = \frac{x^2}{2} \qquad x \leqq \xi$$

$$= \xi - x + \frac{x^2}{2} \qquad x > \xi.$$

Self Assessment

2. Find the generalized Green's function for $L_x = \frac{d^2}{dx^2}$, with the boundary conditions

   $y(-1) = y(1)$, $y'(-1) = y'(1)$. (*Hint:* take $y_0(x) = \frac{1}{\sqrt{2}}$)

10.4 Green's Function for Two Independent Variables

Notes

Let us assume that a function z of x and y satisfies the differential equation

$$L(z) = f(x, y) \quad \text{...(1)}$$

Where L denotes the linear operator

$$\frac{\partial^2}{\partial x \partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c \quad \text{...(2)}$$

Now let w be another function with continuous derivatives of the first order. We may write

$$w\frac{\partial^2 z}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial y}\left(w\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z\frac{\partial w}{\partial y}\right)$$

$$wa\frac{\partial z}{\partial x} + z\frac{\partial (aw)}{\partial x} = \frac{\partial}{\partial x}(awz)$$

$$wb\frac{\partial z}{\partial y} + z\frac{\partial (aw)}{\partial y} = \frac{\partial}{\partial y}(bwz)$$

Defining the M operator by the relation

$$Mw = \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial (aw)}{\partial x} - \frac{\partial (bw)}{\partial y} + cw \quad \text{...(3)}$$

we find that

$$wLz - z\,Mw = w\left(\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz\right)$$

$$- \; z\left(\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial (aw)}{\partial x} - \frac{\partial (bw)}{\partial y} + cw\right)$$

$$= \frac{\partial}{\partial x}(awz) - \frac{\partial}{\partial x}\left(z\frac{\partial w}{\partial y} + \frac{\partial}{\partial y}(bwz) + \frac{\partial}{\partial y}\left(w\frac{\partial z}{\partial x}\right)\right)$$

or

$$wLz - zMw = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad \text{...(4)}$$

where $\quad u = awz - z\dfrac{\partial w}{\partial y}, \quad v = bwz + w\dfrac{\partial z}{\partial x}$ ...(5)

The operator M defined by equation (3) is called the *adjoint* operator. If $M = L$, we say the operator L is *self-adjoint*.

Now if Γ is a closed curve enclosing an area Σ, then it follows from equation (4) and a straight forward use of Green's theorem that

$$\iint_{\Sigma} (wLz - zLw)\,dx\,dy = \iint \left(\frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial y}\right) dx\,dy$$

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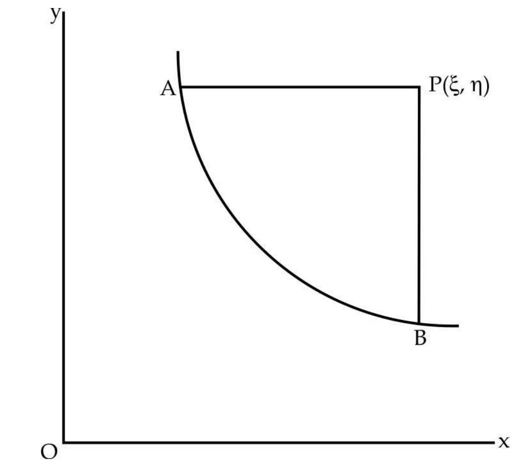
$$= \int_{\Gamma} (u\,dy - \nu\,dx)$$

$$= \int_{\Gamma} [u\cos(n, x) - \nu\cos(x, y)]\,ds \qquad ...(6)$$

where n denotes the direction of the inward drawn normal to the curve Γ.

Figure 10.1



Suppose now that the values of z, $\frac{dz}{dx}$ or $\frac{dz}{dy}$ are prescribed along a curve C in the xy plane (see Figure 10.1) and that we wish to find the solution of the equation (1) at the point $p(\xi, n)$ agreeing with boundary conditions. Through P we draw PA parallel to the x-axis and cutting the curve in the point A and PB parallel to the y-axis and cutting *curve* in B. We then take the curve to be the closed curve $PABPA$ since $dx = 0$ on PB and $dy = 0$ on PA, we have immediately from (6)

$$\iint (wLz - zMw)\,dxdy = \int_{AB} (u\,dy - v\,dx) + \int_{BP} (u\,dy - \int_{PA} v\,dx$$

Now $\quad \int v\,dx = \int (bwz + w\frac{\partial z}{\partial x})dx = \{bw\}^P + \int z(bw - \frac{\partial w}{\partial x})dx$.

So $\quad [z\,w]^P + \int z(bw - \frac{\partial w}{\partial x})dx - \int (u\,dy - v\,dx) - \int z(aw - \frac{\partial w}{\partial x})dy$

$$+ \iint (wLz - zMw)\,dx\,dy \qquad ...(7)$$

Here the function w has been arbitrary. Suppose now that we choose function $w(x, y, \xi, \eta)$ which has the properties

$$Mw = 0$$

$$\frac{\partial w}{\partial x} = b\,(x, y)w \qquad \text{when } y = \eta$$

$$\frac{\partial w}{\partial y} = a\,(x, y)w \qquad \text{when } x = \xi$$

$$w = 1 \qquad \text{when } x = \xi, y = \eta \qquad ...(8)$$

Here w function is called Green's function for the problem. Since also $Lz = f$, we find that

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$$[z\,w] = \int_{AB} wz(a\,dy - b\,dx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_{\Sigma} w\,f\,dxdy \qquad ...(9)$$

Equation (7) enables us to find the value of z at the point P when $\frac{dz}{dx}$ is prescribed along the curve C. When $\frac{dz}{dx}$ is prescribed, we make use of the following calculation

$$[z\,w]_B - [z\,w]_A = \iint_{AB}\left[\frac{\partial(zw)}{\partial x}dx - \frac{\partial(zw)}{\partial y}dy\right]$$

to show that we can write equation (7) in the form

$$[z]_P - [zw]_B - \int_{AB} wz(a\,dy - b\,dx) - \iint_{AB}\left[z\frac{\partial(w)}{\partial x}dx - \frac{\partial(z)}{\partial y}w\,dy\right] + \iint_{\Sigma}(wf)dx\,dy \qquad ...(10)$$

Finally adding (9) and (10), we obtain the symmetrical results

$$[z]_P = \frac{1}{2}[[zw]_A - [zw]_B] - \int wz(a\,dy - b\,dx) - \frac{1}{2}\int_{AB} w\left(\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right)$$

$$-\frac{1}{2}\int_{AB} z\left(\frac{\partial w}{\partial x}dx - \frac{\partial w}{\partial y}dy\right) + \iint_{\Sigma}(wf)dxdy \qquad ...(11)$$

So we can find z at any point in terms of prescribed values of $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$, along a given curve.

Self Assessment

3. If L denotes the operator

$$R\frac{\partial^2}{\partial x^2} - S\frac{\partial^2}{\partial x \partial y} - T\frac{\partial^2}{\partial y^2} - P\frac{\partial}{\partial x} - Q\frac{\partial}{\partial y} = Z$$

and M is the adjoint operator defined by

$$Mw = \frac{\partial^2(Rw)}{\partial x^2} - \frac{\partial^2(Sw)}{\partial x \partial y} - \frac{\partial^2(Tw)}{\partial y^2} - \frac{\partial(Pw)}{\partial x} - \frac{\partial(Qw)}{\partial y} = zw$$

show that

$$\iint_{\Sigma}(wLZ - ZMw)dx\,dy = \int_{\Gamma}\left[U\cos(n,x) - V\cos(n,y)\right]ds$$

where Γ is a closed curve enclosing an area Σ and

$$U = Rw\frac{\partial z}{\partial x} - z\frac{\partial(Rw)}{\partial x} - z\frac{\partial(Sw)}{\partial y} - Pzw$$

$$V = Sw\frac{\partial z}{\partial x} - Tw\frac{\partial z}{\partial y} - z\frac{\partial(Tw)}{\partial y} - Qzw.$$

Notes

10.5 Green's Function for Two Dimensional Problem

The theory of the Green function for the two dimensional Laplace equation may be developed as follows. It is well known that if $P(x, y)$ and $Q(x, y)$ are functions defined inside and on the boundary C of the closed area Σ, then

$$\int_{\Sigma}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dS = \int_{C}(Pdx+Qdy) \qquad ...(1)$$

If we put

$$P=-\psi\frac{\partial\psi}{\partial y}, Q=\psi\frac{\partial\psi'}{\partial x}, \text{ in equation (1) we find that}$$

$$\int_{\Sigma}\psi\nabla^2\psi' ds+\int_{\Sigma}\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi'}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial\psi'}{\partial y}\right)ds = \int_{C}\left(-\psi\frac{\partial\psi'}{\partial y}dx+\psi\frac{\partial\psi'}{\partial x}dy\right)$$

$$= +\int_{C}\psi\frac{\partial\psi'}{\partial n}ds \qquad ...(2)$$

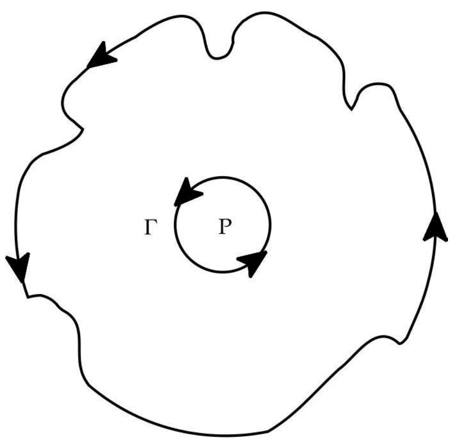
where $\frac{\partial\psi'}{\partial n}$ denotes the derivative of ψ in the direction of the outward normal to C and we have used the relation

$$\frac{\partial\psi'}{\partial x}dy-\frac{\partial\psi'}{\partial y}dx = \frac{\partial\psi'}{\partial n} \qquad ...(3)$$

If we interchange ψ and ψ′ in (2) and subtract the two equations, we find that

$$\int_{\Sigma}\left(\psi\nabla^2\psi'-\psi'\nabla^2\psi'\right)ds = \int_{C}\left(\psi\frac{\partial\psi'}{\partial n}-\psi'\frac{\partial\psi}{\partial n}\right)ds \qquad ...(4)$$

Figure 10.2



Suppose that P with co-ordinates (x, y) is a point in the interior of the region S in which the function ψ is assumed to be harmonic. Draw a small circle Γ with center P and small radius ε (see

Notes

figure) and apply the result (4) to the region k bounded by the curves C and Γ with $\psi' = \log \frac{1}{|\vec{r} - \vec{r^1}|}$.

Since both ψ and ψ' are harmonic, it follows that if S is measured in the direction shown in the fig.,

$$\left(\int_\Gamma + \int_C\right)\left[\psi(x', y')\frac{\partial}{\partial n}\log\frac{1}{|\vec{r} - \vec{r^1}|} - \log\frac{1}{|\vec{r} - \vec{r^1}|}\frac{\partial \psi}{\partial n}\right] = 0 \qquad ...(5)$$

we can show that

$$\int_\Gamma \psi \frac{\partial}{\partial n}\log\frac{1}{|\vec{r} - \vec{r^1}|}ds' = 2\pi\psi(x, y) + 0(\varepsilon)$$

and that

$$\left|\int_\Gamma \log\frac{1}{|\vec{r} - \vec{r^1}|}\frac{\partial \psi}{\partial n}ds'\right| < 2\pi M\,\varepsilon \log \varepsilon,$$

where M is an upper bound of $\frac{\partial \psi}{\partial r}$. Inserting these results into equation (5), we find that

$$\psi(x, y) = \frac{1}{2\pi}\int_C\left[\log\frac{1}{|\vec{r} - \vec{r^1}|}\frac{\partial \psi(x', y')}{\partial n} - \psi(x', y')\frac{\partial}{\partial n}\log\frac{1}{|\vec{r} - \vec{r^1}|}\right]ds' \qquad ...(6)$$

we now introduce a Green's function $G(x, y, x', y')$ defined by the equations

$$G(x, y, x', y') = W(x, y, x', y') + \log\frac{1}{|\vec{r} - \vec{r^1}|} \qquad ...(7)$$

where the function $W(x, y, x', y')$ satisfies the relations

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right)W(x, y, x', y') = 0 \qquad ...(8)$$

$$W(x, y, x', y') = \log|\vec{r} - \vec{r^1}| \quad \text{on } C \qquad ...(9)$$

then for ψ satisfying equations

$$\nabla^2 \psi = 0 \quad \text{within } \Sigma,$$

and

$$\psi = f(x, y) \quad \text{on } C \qquad ...(10)$$

is given by the expression

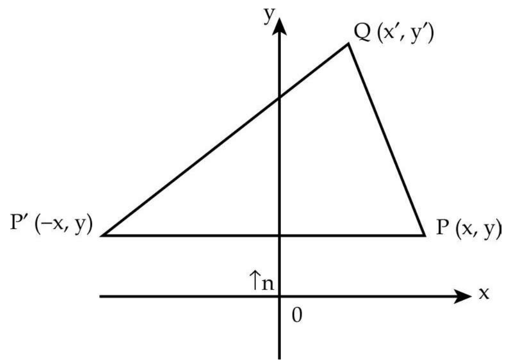
$$\psi(x, y) = -\frac{1}{2\pi}\int \psi(x', y')\frac{\partial G}{\partial n}G(x, y, x', y')ds' \qquad ...(11)$$

Notes

Where $\hat{n}$ is the outward drown normal to the boundary curve C.

Dirichlet's Problem for a Half Plane Suppose that we wish to solve the boundary value problem $\nabla^2 \psi = 0$ for $x \geq 0$, $\psi = f(y)$ on $x = 0$, and $\psi = 0$ as $x \to \infty$. If $P(x, y)$ is a point ($x > 0$), and P' is $(-x, y)$, then $G(x,y,x',y') = \log\left(\frac{QP'}{QP}\right)$, satisfies both equations (8) and (9) since $P'Q = PQ$. on $x = 0$.

Figure 10.3



The required Green's function is therefore

$$G(x,y,x',y') = \frac{1}{2}\log\left[\frac{(x+x')^2+(y-y')^2}{(x-x')+(y-y')^2}\right] \qquad ...(12)$$

Now on C

$$\frac{\partial G}{\partial x} = -\left.\frac{\partial G}{\partial x'}\right|_{x'=0} = \frac{2x}{x^2+(y-y')^2},$$ so substituting in (11), we find that

$$\psi(x, y) = \frac{\pi}{x}\int_{-\infty}^{+\infty}\frac{f(y')dy'}{\left[x^2+(y-y')^2\right]} \qquad ...(13)$$

10.6 Summary

- Green's functions and its properties are described for one and two dimensional problems.
- It is seen that depending upon the boundary conditions the structure of the Green's functions is established.
- It also gives a link to reduce a differential equation into an integral equation.

10.7 Keywords

We can have an ***initial value problem*** where the values of the dependent function and its derivatives are given.

In a ***boundary value problem*** the values of the dependent function and its derivatives are given at both the ends of the interval of the independent variable.

10.8 Review Questions

Notes

1. Find the Green's function for the one dimensional case given by

$$L_x y = \frac{d^2}{dx^2} y = 0$$

with $y(0) = y'(0),\ y(1) = -y'(1)$

2. Find the Green's function for the boundary value problem $\nabla^2\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = 0$, for $r < 0$, given that $\psi = f(0)$ for $r = a$

3. Prove that for the equation

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{2}{x-y}\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 0$$

the Green's function is

$$G(x, y, \xi, \eta) = \frac{(x-y)\left[2xy - (\xi - \eta)(x-y) - 2\xi\eta\right]}{(\xi - \eta)^3}.$$

Answers: Self Assessment

1. $G(x,\xi) = \begin{cases} x & (x \leq \xi) \\ \xi & (x > \xi) \end{cases}$

2. $G(x,\xi) = -\frac{1}{2}|x-\xi| + \frac{1}{4}(x-\xi)^2 + \frac{1}{6}.$

10.9 Further Readings

Books

K. Yosida, Lectures in Differential and Integral Equations

Sneddon L.N., Elements of Partial Differential Equations

King A.C, Billingham J. and S.R. Otto, Differential Equations

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Unit 11: Sturm–Liouville's Boundary Value Problems

CONTENTS

Objectives

After studying this unit, you should be able to:

- Understand the structure of self-adjoint equations. If we are dealing with only second order differential equations, we see that under what conditions we can put them in self-adjoint form.
- Know that Sturm-Liouville boundary value problem is a method of dealing with equations which can be put into Sturm-Liouville form.
- Find the solutions for some values of the parameters. The solutions are known as eigenfunctions and the values of the parameter are known as eigenvalues.
- Know that important examples of Sturm-Liouville boundary value problems are Legendre equation, Bessel's equations and many more.

Introduction

This method helps us in finding certain sets of functions which are orthogonal and we can express any function in terms of these eigenfunctions on the interval $a \le x \le b$ where a and b may be finite or one of them finite and the other infinite or both a and b to be infinite.

These methods are known as Fourier Legendre expansion if we use Legendre polynomials and so on.

11.1 Sturm-Liouville's Equation

In the first four units we have studied linear second order differential equations. After examining some solutions techniques that are applicable to such equations in general we studied the particular cases of Legendre's equation, Bessel's equations, the Hermite equations and Laguerre's equations, as they frequently arise in models of physical systems in spherical, cylindrical geometries and in Quantum mechanics. In each case we saw that we can construct a set of

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solutions that can be used as the basis for series expansion of the solution of the physical problem in question, namely the Fourier-Legendre's and Fourier-Bessel series. In this unit we will see that Legendre's, Bessel's, Hermite and Laguerre's equations are examples of Sturm-Liouville's equations which are also in self-adjoint form. Some of the properties of Sturm-Liouville's equations are examined in the previous unit also. In this unit we deduce some more properties of such equations independent of the function form of the coefficients.

Sturm-Liouville equations are of the form

$$(p(x)y'(x))' + q(x)y(x) = -\lambda r(x)y(x) \quad ...(1)$$

which can be written more concisely as

$$Sy(x, \lambda) = -\lambda r(x)y(x, \lambda) \quad ...(2)$$

where the differential operator S is defined as

$$S\phi \equiv \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi. \quad ...(3)$$

This is a slightly more general equation. In (1) the number λ is the eigenvalue, whose possible values, which may be complex, are critically dependent upon the given boundary conditions. It is often more important to know the properties of λ than it is to construct the actual solutions of (1).

We seek to solve the Sturm-Liouville equation (1) on an open interval, (a, b) of the real line. We will also make some assumptions about the behaviour of the coefficients of (1) for $x \in (a, b)$, namely that

(i) $p(x)$, $q(x)$ and $r(z)$ are real-valued and continuous

(ii) $p(x)$ is differentiable, ...(4)

(iii) $p(x) > 0$ and $r(z) > 0$.

Some Example of Sturm-Liouville Equations

Perhaps the simplest example of a Sturm-Liouville equation is Fourier's equations,

$$y''(x, \lambda) = -\lambda y(x, \lambda) \quad ...(5)$$

which has solutions $\cos(x\sqrt{\lambda})$ and $\sin(x\sqrt{\lambda})$. We discussed a physical problem that leads naturally to Fourier's equation at the start of least unit.

We can write Legendre's equation and Bessel's equation as Sturm-Liouville problems. Recall that Legendre's equation is

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{\lambda}{1-x^2}y = 0$$

and we are usually interested in solving this for $-1 < x < 1$. This can be written as

$$[(1 - x^2)y']' = -\lambda y.$$

If $\lambda = n(n + 1)$, we showed in unit 2 that this has solutions $P_n(x)$ and $Q_n(x)$. Similarly, Bessel's equation, which is usually solved for $0 < x < a$, is

$$x^2y'' + xy' + (\lambda x^2 - \nu^2)\phi = 0.$$

This can be rearranged into the form

$$(xy')' - \frac{\nu^2}{x}y = -\lambda xy.$$

Again, from the results of unit 1, we know that this has solutions of the form $J_v(x\sqrt{\lambda})$ and $Y_v(x\sqrt{\lambda})$.

Although the Sturm-Liouville forms of these equations may look more cumbersome than the original forms, we will see that they are very convenient for the analysis that follows. This is because of the self-adjoint nature of the differential operator.

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11.2 Boundary Conditions

We begin with a couple of definitions. The endpoint, $x = a$, of the interval (a, b) is a regular endpoint if a is finite and the conditions (4) hold on the closed interval $[a, c]$ for each $c \in (a, b)$. The endpoint $x = a$ is a singular endpoint if $a = -\infty$ or if a is finite but the conditions (4) do not hold on the closed interval $[a, c]$ for some $c \in (a, b)$. Similar definitions hold for the other endpoint, $x = b$. For example, Fourier's equation has regular endpoints if a and b are finite. Legendre's equation has regular endpoints if $-1 < a < b > 1$, but singular endpoints if $a = -1$ or $b = 1$, since $p(x) = 1 - x^2 = 0$ when $x = \pm 1$. Bessel's equation has regular endpoints for $0 < a < b < \infty$, but singular endpoints if $a = 0$ or $b = \infty$, since $q(x) = -v^2/x$ is unbounded at $x = 0$.

We can now define the types of boundary conditions that can be applied to a Sturm-Liouville equation.

(i) On a finite interval, $[a, b]$, with regular endpoints, we prescribe unmixed, or separated, boundary conditions, of the form

$$\alpha_0 y(a, \lambda) + \alpha_1 y'(a, \lambda) = 0, \ \beta_0 y(b, \lambda) + \beta_1 y'(v, \lambda) = 0. \qquad ...(6)$$

These boundary conditions are said to be real if the constants $\alpha_0, \alpha_1, \beta_0$ and β_1 are real, with $\alpha_0^2 + \alpha_1^2 > 0$ and $\beta_0^2 + \beta_1^2 > 0$.

(ii) On an interval with one or two singular endpoints, the boundary conditions that arise in models of physical problems are usually boundedness conditions. In many problems, these are equivalent to Friedrich's boundary conditions, that for some c ε (a, b) there exists $A \ \varepsilon \ \mathbb{R}^+$ such that

$$|y(x, \lambda)| \leq A \text{ for all } x \in (a, c)$$

and similarly if the other endpoint, $x = b$, is singular there exists $B \subseteq \mathbb{R}^+$ such that $|y(x, \lambda)| \leq B$ for all $x \ \varepsilon \ (a, b)$

We can now define the Sturm-Liouville boundary value problem to be the Sturm-Liouville equation,

$$(p(x)y'(x))' + q(x)y(x) = -\lambda r(x)y(x) \qquad \text{for } x \in (a, b)$$

where the coefficient functions satisfy the conditions (4), to be solved subject to a separated boundary condition at each regular endpoint and a Friedrich's boundary condition at each singular endpoint. Note that this boundary value problem is homogeneous and therefore always has the trivial solution, $y = 0$. A non-trivial solution, $y(x, \lambda) \not\equiv 0$, is an eigenfunction, and λ is the corresponding eigenvalue.

Some Examples of Sturm-Liouville Boundary Value Problems.

Consider Fourier's equation.

$$y''(x, \lambda) = -\lambda x(x, \lambda) \qquad \text{for } x \in (0, 1)$$

subject to the boundary conditions $y(0, \lambda) = y(1, \lambda) = 0$, which are appropriate since both endpoints are regular. The eigenfunctions of this system are $\sin\sqrt{\lambda_n x}$ for $x = 1, 2, ...$, with corresponding eigenvalues $\lambda = \lambda_\eta = n^2 \pi^2$.

Legendre's equation is

$$\{(1 - x^2)y'(x, \lambda)\}' = -\lambda y(x, \lambda) \text{ for } x \in (-1, 1).$$

Note that this is singular at both endpoints, since $p(\pm 1) = 0$. We therefore apply Friedrich's boundary conditions, for example with $c = 0$, in the form

$$|y(x, \lambda)| \leq A \text{ for } x \in (-1, 0), \ |y(x, \lambda)| \leq B \text{ for } x \in (0, 1),$$

for some $A, B \in \mathbb{R}^+$. In unit 2 we used the method of Frobenius to construct the solutions of Legendre's equation, and we know that the only eigenfunctions bounded at both the endpoints are the Legendre polynomials, $P_n(x)$ for $n = 0, 1, 2, \ldots$, with corresponding eigenvalues $\lambda = \lambda_n = n(n + 1)$.

Let's now consider Bessel's equation with $v = 1$, over the interval (0, 1),

$$(xy')' - \frac{y}{x} = -\lambda xy.$$

Because of the form of $q(x)$, $x = 0$ is a singular endpoint, whilst $x = 1$ is a regular endpoint. Suitable boundary conditions are therefore

$$|y(x, \lambda)| \leq A \text{ for } x \in \left(0, \frac{1}{2}\right), \; y(1, \lambda) = 0$$

for some $A \in \mathbb{R}^+$. In unit 1 we constructed the solutions of this equation using the method of Frobenius. The solution that is bounded at $x = 0$ is $J_1\left(x, \sqrt{\lambda}\right)$. The eigenvalues are solutions of

$$J_1\left(\sqrt{\lambda_n}\right) = 0,$$

which we write as $\lambda = \lambda_1^2, \lambda_2^2, \ldots$, where $J_1(\lambda_n) = 0$.

Finally, let's examine Bessel's equation with $v = 1$, but now for $x \in (0, \infty)$. Since both endpoints are now singular, appropriate boundary conditions are

$$|y(x, \lambda)| \leq A \text{ for } x \in \left(0, \frac{1}{2}\right), \; |y(x, \lambda)| \leq B \text{ for } x \in \left(\frac{1}{2}, \infty\right),$$

for some $A, B \in \mathbb{R}^+$. The eigenfunctions are again $J_1\left(x, \sqrt{\lambda}\right)$, but now the eigenvalues lie on the half-line $[0, \infty)$. In other words, the eigenfunctions exist for all real, positive λ. The set of eigenvalues for a Sturm-Liouville system is often called the spectrum. In the first of the Bessel function examples above, we have a discrete spectrum, whereas for the second there is a continuous spectrum. We will focus our attention on problems that have a discrete spectrum only.

Self Assessment

1. Put the equation

   $x^2y'' + xy' + (\lambda^2x^2 - 4)\, y = 0$

   in Sturm-Liouville's form

2. Put the equation

   $$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\lambda y = 0$$

   into Sturm-Liouville's form

11.3 Properties of the Eigenvalues and Eigenfunctions

In order to study further the properties of the eigenfunctions and eigenvalues, we begin by defining the inner product of two complex-valued functions over an interval I to be

$$\langle \phi_1(x), \phi_2(x) \rangle = \int_I \phi_1^*(x)\phi_2(x)\,dx,$$

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where a superscript asterisk denotes the complex conjugate. This means that the inner product has the properties

(i) $\langle \phi_1, \phi_2 \rangle = \langle \phi_2, \phi_1 \rangle^*$,

(ii) $\langle a_1\phi_1, a_2\phi_2 \rangle = a_1^* a_2 \langle \phi_1, \phi_2 \rangle$,

(iii) $\langle \phi_1, \phi_2 + \phi_3 \rangle = \langle \phi_1, \phi_2 \rangle + \langle \phi_1, \phi_3 \rangle$, $\langle \phi_1 + \phi_2, \phi_3 \rangle = \langle \phi_1, \phi_3 \rangle + \langle \phi_2, \phi_3 \rangle$

(iv) $\langle \phi, \phi \rangle = \int_I |\phi|^2 \, dx \geq 0$, with equality if and only if $\phi(x) \equiv 0$ in I.

Note that this reduces to the definition of a real inner product if ϕ_1 and ϕ_2 are real. If $\langle \phi_1, \phi_2 \rangle = 0$ with $\phi_1 \not\equiv 0$ and $\phi_2 \not\equiv 0$, we say that ϕ_1 and ϕ_2 are orthogonal.

Let $y_1(x)$, $y_2(x) \in C^2[a, b]$ be twice-differentiable complex-valued functions. By integrating by parts, it is straightforward to show that

$$\langle y_2 S y_1 \rangle - \langle S y_2 \; y_1 \rangle = \left[p(x)\{y_1(x)(y_2^*(x))' = y_1(x) y_2^*(x)\} \right]_\alpha^\beta \qquad ...(7)$$

which is known as Green's formula. The inner products are defined over a sub-interval $[\alpha, \beta] \subset (a, b)$, so that we can take the limits $\alpha \to a^+$ and $\beta \to b^-$ when the endpoints are singular, and the Sturm-Liouville operator, S, is given by (3). Now if $x = a$ is a regular endpoint and the function y_1 and y_2 satisfy a separated boundary condition at a, then

$$p(a)\{y_1(a)(y_2^*(a))' - y_1'(a) y_2^*(a)\} = 0. \qquad ...(8)$$

If a is a finite singular endpoint and the functions y_1 and y_2 satisfy the Friedrich's boundary condition at a,

$$\lim_{x \to a^+} [p(x)\{y_1(x) y_2^*(x))' - y_1'(x) y_2^*(x)\}] = 0 \qquad ...(9)$$

Similar results hold at $x = b$.

We can now derive several results concerning the eigenvalues and eigenfunctions of a Sturm-Liouville boundary value problem.

Theorem 1: The eigenvalues of a Sturm-Liouville boundary value problem are real.

$$\langle y^*(x,\lambda) S y(x,\lambda) \rangle - \langle S y^*(x,\lambda), y(x,\lambda) \rangle$$

$$= [p(x)\{y(x,\lambda)(y^*(x,\lambda))' - y'(x,\lambda) y^*(x,\lambda)\}]_a^b = 0$$

Proof: If we substitute $y_1(x) = y(x, \lambda)$ and $y_2(x) = y^*(x, \lambda)$ into Green's formula over the entire interval, $[a, b]$, we have $\langle y^*(x,\lambda), Sy(x,\lambda) \rangle - \langle Sy^*(x,\lambda), y(x,\lambda) \rangle$

$$= \left[p(x)\left\{ y(x,\lambda)(y^*(x,\lambda)' - y'(x,\lambda) y^*(x,\lambda) \right\} \right]_a^b = 0$$

making use of (8) and (9). Now, using the fact that the function $y(x, \lambda)$ and $y^*(x, \lambda)$ are solutions of (1) and its complex conjugate, we find that

$$\int_a^b r(x) y(x,\lambda) y^*(x,\lambda)(\lambda - \lambda^*) \, dx = (\lambda - \lambda^*) \int_a^b r(x)[y(x,\lambda)]^2 \, dx = 0$$

Since $r(x) > 0$ and $y(x, \lambda)$ is nontrivial, we must have $\lambda = \lambda^*$ and hence $\lambda \in \mathbb{R}$ i.e. the eigenvalues are real.

Theorem 2: If $y(x, \lambda)$ and $y(x, \bar{\lambda})$ are eigenfunctions of the Sturm-Liouville boundary value problem, with $\lambda \neq \bar{\lambda}$), then these eigenfunctions are orthogonal over $C^p[a, b]$ with respect to the weighing function $r(x)$, so that

$$\int_a^b r(x) y(x,\lambda) y(x,\bar{\lambda}) \, dx = 0 \qquad ...(10)$$

Proof: Firstly, notice that the separated boundary condition (6) at $x = a$ takes the form

$$\alpha_0 y_1(a) + \alpha_1 y'_1(a) = 0, \; \alpha_0 y_2(a) + \alpha_1 y'_2(a) = 0. \quad ...(11)$$

Taking the complex conjugate of the second of these gives

$$\alpha_0 y^*_2(a) + \alpha_1 (y'_2(a))^* = 0. \quad ...(12)$$

since α_0 and α_1 are real. For the pair of equations (11) and (12) to have a nontrivial solution, we need

$$y_1(a)(y'_2(a))^* - y'_1(a)y'_2(a) = 0.$$

A similar result holds at the other endpoint, $x = b$. This clearly shows that

$$p(x)\{y(x,\lambda)\,(y'(x,\bar{\lambda}))^* - y'(x,\lambda)(y(x,\bar{\lambda}))^*\} = 0$$

as $x \to a$ and $x \to b$, so that, from Green's formula (7),

$$\langle y(x,\bar{\lambda})\; Sy(x,\lambda)\rangle = \langle Sy(x,\bar{\lambda}),\, y(x,\lambda)\rangle$$

If we evaluate this formula, we find that

$$\int_a^b r(x) y(x,\lambda) y(x,\bar{\lambda}) dx = 0$$

so that the eigenfunctions associated with the distinct eigenvalues λ and $\bar{\lambda}$ are orthogonal with respect to the weighting function $r(x)$.

Example: Consider Hermite's equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0 \quad ...(i)$$

for $-\infty < x < \infty$. This is not in self-adjoint form. To do that let us define

$$\begin{aligned} p(x) &= \exp\left[\int^x (-2x)dx\right] \\ &= \exp(-x^2) \end{aligned} \quad ...(ii)$$

Thus the equation (i) becomes

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \lambda e^{-x^2} y = 0 \quad ...(iii)$$

By using the method of Frobenius, we showed in unit (3) that the solutions of equation (i) are polynomials defined by $H_n(x)$ when $\lambda = 2n$ for $n = 0, 1, 2, ...$. The solutions of equation (iii), the self-adjoint form of the equation, that are bounded at infinity for $\lambda = 2n$, then take the form

$$u_n = e^{-\frac{x^2}{2}} H_n(x) \quad ...(iv)$$

and from theorem (2) satisfy the orthogonality condition

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) dx = 0 \; \textit{for } n \neq m$$

Self Assessment

3. Put the Laguerre's equation

   $xy'' + (1 - x) + \lambda y = 0$, for $0 < x < \infty$

   into self-adjoint form and deduce orthogonality condition for Laguerre's polynomials.

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11.4 Bessel's Inequality, Approximation in the Mean and Completeness

We can now define a sequence of orthonormal eigenfunctions

$$\phi_n(x) = \frac{\sqrt{r(x)}y(x, \lambda_n)}{\left\langle \sqrt{r(x)}y(x, \lambda_n), \sqrt{r(x)}y(x, \lambda_n)\right\rangle},$$

which satisfy

$$\langle \phi_n(x), \phi_m(x) \rangle = \delta_{nm}, \qquad \text{...(13)}$$

where δ_{nm} is the Kronecker delta. We will try to establish when we can write a piecewise continuous function $f(x)$ in the form

$$f(x) = \sum_{i=0}^{\infty} a_i \phi_i(x) \qquad \text{...(14)}$$

Taking the inner product of both sides of this series with $\phi_j(x)$ shows that

$$a_j = \langle f(x), \phi_j(x) \rangle, \qquad \text{...(15)}$$

using the orthonormality condition (13). The quantities a_i are known as the expansion coefficients, or generalized Fourier coefficients. In order to motivate the infinite series expansion (14), we start by approximating $f(x)$ by a finite sum,

$$f_N(x) = \sum_{i=0}^{N} A_i \phi(x, \lambda_i)$$

for some finite N, where the A_i are to be determined so that this provides the most accurate approximation to $f(x)$. The error in this approximation is

$$R_N(x) = f(x) - \sum_{i=0}^{N} A_i \phi(x, \lambda_i)$$

We now try to minimize this error by minimizing its norm

$$\|R_N\|^2 = \langle R_N(x), R_N(x) \rangle = \int_a^b \left[f(x) - \sum_{i=0}^{N} A_i \phi_i(x) \right]^2 dx,$$

which is the mean square error in the approximation. Now

$$\|R_N\|^2 = \left\langle f(x) - \sum_{i=0}^{N} A_i \phi_i(x), f(x) - \sum_{i=0}^{N} A_i \phi_i(x) \right\rangle$$

$$= \|f(x)\|^2 - \left\langle f(x), \sum_{i=0}^{N} A_i \phi_i(x) \right\rangle$$

$$-\left\langle \sum_{i=0}^{N} A_i \phi_i(x), f(x) \right\rangle + \left\langle \sum_{i=0}^{N} A_i \phi_i(x), \sum_{i=0}^{N} A_i \phi_i(x) \right\rangle$$

We can now use the orthonormality of the eigenfunctions (13) and the expression (15), which determines the coefficients a_i, to obtain

Notes

$$\|R_N(x)\|^2 = \|f(x)\|^2 - \sum_{i=0}^{N} A_i \langle f(x), \phi_i(x)\rangle$$

$$-\sum_{i=0}^{N} A_i^* \langle \phi_i(x), f(x)\rangle, \sum_{i=0}^{N} A_i^* A_i \langle \phi_i(x), \phi_i(x)\rangle$$

$$= \|f(x)\|^2 + \sum_{i=0}^{N} \{-A_i a_i - A_i^* a_i^* + A_i^* A_i\}$$

$$= \|f(x)\|^2 + \sum_{i=0}^{N} \{|A_i - a_i|^2 - |a_i|^2\}$$

The error is therefore smallest when $A_i = a_i$ for $i = 0, 1, ..., N$, so the most accurate approximation is formed by simply truncating the series (14) after N terms. In addition, since the norm of $R_N(x)$ is positive,

$$\sum_{i=0}^{N} |a_i|^2 \le \int_a^b |f(x)|^2\, dx$$

As the right side of this is independent of N_1 if follows that

$$\sum_{i=0}^{\infty} |a_i|^2 \le \int_a^b |f(x)|^2\, dx \qquad ...(16)$$

which is Bessel's inequality. This shows that the sum of the squares of the expansion coefficients converges. Approximations by the method of least squares are often referred to as approximations in the mean, because of the way the error is minimized.

If, for a given orthonormal system, $\phi_1(x)$, $\phi_2(x)$..., any piecewise continuous function can be approximated in the mean to any desired degree of accuracy by choosing N large enough, then the orthonormal system is said to be complete. For complete orthonormal systems, $R_N(x) \to 0$ as $N \to \infty$, so that Bessel's inequality becomes an *equality,*

$$\sum_{i=0}^{\infty} |a_i|^2 = \int_a^b |f(x)|^2\, dx \qquad ...(17)$$

for every function $f(x)$.

The completeness of orthonormal systems as expressed by

$$\lim_{N\to\infty} \int_a^b \left[f(x) - \sum_{i=0}^{N} a_i \phi_i(x) \right]^2 dx = 0$$

does not necessarily imply that $f(x) = \sum_{i=0}^{\infty} a_i \phi_i(x)$, in other words that $f(x)$ has an expansion in terms of the $\phi_i(x)$. If however, the series $\sum_{i=0}^{\infty} a_i \phi_i(x)$, is uniformly convergent, then the limit and the integral can be interchanged, the expansion is valid, and we say that $\sum_{i=0}^{\infty} a_i \phi_i(x)$, converges in the mean to $f(x)$. The completeness of the systems $\phi_1(x)$, $\phi_2(x)$.... , should be seen as a necessary condition for the validity of the expansion, but, for an arbitrary function $f(x)$, the question of convergence requires a more detailed investigation.

The Legendre polynomials $P_0(x)$, $P_1(x)$,... on the interval (–1, 1) and the Bessel functions $J_\nu(\lambda_t x)$, $J_\nu(\lambda_2 x)$,... on the interval [0, a] are both examples of complete orthogonal systems (they can easily be made orthonormal), and the expansions of unit 1 to 5 are special cases of the more general

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results of this chapter. For example, the Bessel functions $J_v\left(\sqrt{\lambda}x\right)$ satisfy the Sturm-Liouville equation, with $p(x) = x$, $q(x) = -v^2/x$ and $r(x) = x$. They satisfy the orthogonality relation

$$\int_0^a xJ_v\left(\sqrt{\mu}x\right)J_v\left(\sqrt{\lambda}x\right)dx = 0$$

if λ and μ are distinct eigenvalues. Using the regular endpoint condition $J_v\left(\sqrt{\lambda}a\right) = 0$ and the singular endpoint condition at $x = 0$, the eigenvalues, that is the zeros of $J_v(x)$, can be written as $\sqrt{\lambda}\, a = \lambda_1 a_1, \lambda_2 a\ldots$, so that $\sqrt{\lambda} = \lambda_i$ for $i = 1, 2, \ldots$, and we can write

$$f(x) = \sum_{i=1}^{\infty} a_i J_v(\lambda_i x),$$

with

$$a_i = \frac{2}{a^2\{J_v'(\lambda_i a)\}^2}\int_0^a xJ_v(\lambda_i x)f(x)dx$$

Example: Show that the functions $g_m = \cos mx$, $m = 0, 1, 2, \ldots$ form orthogonal set of functions on the interval $-\pi < x > \pi$ and determine the corresponding orthonormal set of functions.

Solution: We have, for $m \neq n$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx$$

$$= 2\int_0^{\pi} \cos mx \cos nx\, dx$$

$$= \int_0^{\pi}\left\{\cos[(m+n)x] - \cos[(m-n)x]\right\}dx$$

$$= \left[\frac{\sin[(m+n)x]}{(m+n)} - \frac{\sin[(m-n)x]}{m-n}\right]_0^{\pi} = 0$$

Hence the given functions $g_m = \cos mx$, $m = 0, 1, 2, \ldots$. are orthogonal set of functions.

Now the norm of g_m is

$$\|g_m\| = \|\cos mx\| = \left|\int_{-\pi}^{\pi}\cos^2 mx\, dx\right|^{1/2}$$

$$= \left|2\int_0^{\pi}\cos^2 mx\, dx\right|^{1/2}$$

$$= \sqrt{2\pi} \quad \text{when } m = 0$$

and

$$= \sqrt{\pi} \quad \text{when } m = 1, 2, 3, \ldots.$$

Hence the orthonormal set is

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \ldots$$

Self Assessment

4. Show that the functions 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, ... form an orthogonal set on an interval $-\pi \leq x \leq \pi$ and obtain the orthonormal set.

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11.5 Summary

- The Sturm-Liouville's boundary value problems leads us to eigenvalues and eigenfunctions of certain second order differential equations.
- It is seen that the eigenfunctions form a set of orthonormal set and as so form a complete set.
- This helps us in expanding a certain function in terms of eigenfunctions on an interval (a, b).

11.6 Keywords

Bessel's differential equations, Legendre differential equations and many more equations can be written in the Sturm-Liouville equation.

Depending upon certain boundary conditions the solutions known as ***eigenfunctions*** can be found that form orthogonal set.

11.7 Review Questions

1. Find all eigenvalues and eigenfunctions of the Sturm-Liouville problem

   $$y'' + \lambda y = 0, \text{ with } y(0) = y'\left(\frac{\pi}{2}\right) = 0$$

2. Find all the eigenvalues and eigenfunctions of the Sturm-Liouville problem

   $y'' + \lambda y = 0$, with $y'(0) = 3$, $y'(c) = 0$

Answers: Self Assessment

1. $(xy')' - \dfrac{4}{x}y = -\lambda xy$
2. $(e^{-x^2}y')' + 2\lambda e^{-x^2}y = 0$
3. $(x\,e^{-x}y')' + \lambda e^{-x}y = 0$
4. $\dfrac{1}{\sqrt{2\pi}}, \dfrac{\cos x}{\sqrt{\pi}}, \dfrac{\sin x}{\sqrt{\pi}}, \dfrac{\cos 2x}{\sqrt{\pi}}, \dfrac{\sin 2x}{\sqrt{\pi}}, \ldots\ldots$

11.8 Further Readings

Books

K. Yosida, Lectures in Differential and Integral Equations

Sneddon L.N., Elements of Partial Differential Equations

King A.C, Billingham J. and S.R. Otto, Differential Equations

Notes

Unit 12: Sturm Comparison and Separation Theorems

CONTENTS

Objectives

After studying this unit, you should be able to:

- Deal with a linear second order differential equation with ease, there are a number of important processes by which the solutions are found easily.
- Know that in certain important cases the method of reduction of order helps in solving the differential equation.
- Discuss another method called the method of variation of parameters which helps in solving non-homogeneous differentiation equation.

Introduction

Sturm comparison and separation theorems help us in understanding the nature of solutions of certain differential equation where the solutions are periodic.

This process helps us in setting up the equation for Wronskian involving the solutions of the differential equation.

12.1 Linear Ordinary Second Order Differential Equation

We here consider linear, second order ordinary differential equation of the form

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=F(x)$$

Notes

where $P(x)$, $Q(x)$ and $R(x)$ are finite polynomials that contain no common factor. This equation is inhomogeneous and has variable coefficients. After dividing through $P(x)$, we obtain the more concurrent, equivalent form,

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \qquad ...(1)$$

Provided $p \neq 0$. If $p(x) = 0$ at some point $x = x_0$, we call $x = x_0$ a singular point of the equation. If $P(x) \neq 0$, x_0 is a regular or ordinary point of the equation. If $P(x) \neq 0$ for all points x in the interval where we want to solve the equation, we say the equation is non-singular or regular in the interval.

If $a_1(x)$, $a_0(x)$ and $f(x)$ are continuous on some open interval $a < x < b$ that contains the initial point, then a unique solution of the form

$$y = Au_1(x) + Bu_2(x) + G(x)$$

where A, B are constants and are fixed by initial conditions. Before we try to construct the general solution of equation (1), we will outline a series of sub-problems that are more tractable.

12.2 The Method of Reduction of Order

As a first simplification we discuss the solution of the homogeneous differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad ...(2)$$

on the assumption that we know one solution, say $y(x) = u_1(x)$, and only need to find the second solution. We will look for a solution of the form $y(x) = U(x)u_1(x)$. Differentiating $y(x)$ using the product rule gives

$$\frac{dy}{dx} = \frac{dU}{dx}u_1 + U\frac{du_1}{dx},$$

$$\frac{d^2y}{dx^2} = \frac{d^2U}{dx^2}u_1 + 2\frac{dU}{dx}\frac{du_1}{dx} + U\frac{d^2u_1}{dx^2}$$

If we substitute these expressions into (2) we obtain

$$\frac{d^2U}{dx^2}u_1 + 2\frac{dU}{dx}\frac{du_1}{dx} + U\frac{d^2u_1}{dx^2} + a_1(x)\left(\frac{dU}{dx}u_1 + U\frac{du_1}{dx}\right) + a_0(x)Uu_1 = 0$$

We can now collect terms to get

$$U\left(\frac{d^2u_1}{dx^2} + a_1(x)\frac{du_1}{dx} + a_0(x)u_1\right) + u_1\frac{d^2U}{dx^2} + \frac{dU}{dx}\left(2\frac{du_1}{dx} + a_1u_1\right) = 0$$

Now, since $u_1(x)$ is a solution of (2), the term multiplying U is zero. We have therefore obtained a differential equation for dU/dx, and, by defining $Z = dU/dx$, we have

$$u_1\frac{dZ}{dx} + Z\left(2\frac{du_1}{dx} + a_1u_1\right) = 0$$

Dividing through by Zu_1 we have

$$\frac{1}{Z}\frac{dZ}{dx} + \frac{2}{u_1}\frac{du_1}{dx} + a_1 = 0,$$

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which can be integrated directly to yield

$$\log|Z| + 2\log|u_1| + \int^x a_1(s)ds = C,$$

where s is a dummy variable, for some constant C. Thus

$$Z = \frac{c}{u_1^2}\exp\left\{-\int^x a_1(s)ds\right\} = \frac{dU}{dx}$$

where $c = e^C$. This can then be integrated to give

$$U(x) = \int^z \frac{c}{u_1^2(t)}\exp\left\{-\int^t a_1(s)ds\right\}dt + \bar{c},$$

for some constant $\bar{c}$. The solution is therefore

$$y(x) = u_1(x)\int^x \frac{c}{u_1^2(t)}\exp\left\{-\int^t a_1(s)ds\right\}dt + \in u_1(x).$$

We can recognize $\in u_1(x)$ as the part of the complementary function that we knew to start with, and

$$u_2(x) = u_1(x)\int^x \frac{1}{u_1^2(t)}\exp\left\{-\int^t a_1(s)ds\right\}dt \qquad ...(3)$$

as the second part of the complementary function. This result is called the reduction of order formula.

Example: Let us try to determine the full solution of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

given that $y = u_1(x) = x$ is a solution. We firstly write the equation in standard form as

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{2}{1-x^2}y = 0$$

Comparing this with (2), we have $a_1(x) = -2x/(1 - x^2)$. After noting that

$$\int^t a_1(s)ds = \int^t -\frac{2s}{1-s^2}ds = \log(1-t^2),$$

the reduction of order formula gives

$$u_2(x) = x\int^x \frac{1}{t^2}\exp\{-\log(1-t^2)\}dt = x\int^x \frac{dt}{t^2(1-t^2)}$$

We can express the integrand in terms of its partial fractions as

$$\frac{1}{t^2(1-t^2)} = \frac{1}{t^2} + \frac{1}{1-t^2} = \frac{1}{t^2} + \frac{1}{2(1+t)} + \frac{1}{2(1-t)}$$

This gives the second solution of (2) as

Notes

$$u_2(x) = x\int^x \left\{\frac{1}{t^2} + \frac{1}{2(1+t)} + \frac{1}{2(1-t)}\right\}dt$$

$$= x\left[-\frac{1}{t} + \frac{1}{2}\log\left(\frac{1+t}{1-t}\right)\right]^x = \frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1,$$

and hence the general solution is

$$y = Ax + B\left\{\frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1\right\}.$$

Self Assessment

1. Use the reduction of order method to find the second independent solution of the equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 0$$

with the solution $u_1(x) = x^{-1} \sin x$

12.3 The Method of Variation of Parameters

Let's now consider how to find the particular integral given the complementary function, comprising $u_1(x)$ and $u_2(x)$. As the name of this technique suggests, we take the constants in the complementary function to be variable, and assume that

$$y = c_1(x)u_1(x) + c_2(x)u_2(x)$$

Differentiating, we find that

$$\frac{dy}{dx} = c_1\frac{du_1}{dx} + u_1\frac{dc_1}{dx} + c_2\frac{du_2}{dx} + u_2\frac{dc_2}{dx}$$

We will choose to impose the condition

$$u_1\frac{dc_1}{dx} + u_2\frac{dc_2}{dx} = 0, \qquad ...(4)$$

and thus have

$$\frac{dy}{dx} = c_1\frac{du_1}{dx} + c_2\frac{du_2}{dx},$$

which, when differentiated again, yields

$$\frac{d^2y}{dx^2} = c_1\frac{d^2u_1}{dx^2} + \frac{du_1}{dx}\frac{dc_1}{dx} + c_2\frac{d^2u_2}{dx^2} + \frac{du_2}{dx}\frac{dc_2}{dx}$$

This form can then be substituted into the original differential equation to give

$$c_1\frac{d^2u_1}{dx^2} + \frac{du_1}{dx}\frac{dc_1}{dx} + c_2\frac{d^2u_2}{dx^2} + \frac{du_2}{dx}\frac{dc_2}{dx} + a_1\left(c_1\frac{du_1}{dx} + c_2\frac{du_2}{dx}\right) + a_0(c_1u_1 + c_2u_2) = f.$$

Notes

This can be rearranged to show that

$$c_1\left(\frac{d^2u_1}{dx^2}+a_1\frac{du_1}{dx}+a_0u_1\right)+c_2\left(\frac{d^2u_2}{dx^2}+a_1\frac{du_2}{dx}+a_0u_2\right)+\frac{du_1}{dx}\frac{dc_1}{dx}+\frac{du_2}{dx}\frac{dc_2}{dx}=f$$

Since u_1 and u_2 are solutions of the homogeneous equation, the first two terms are zero, which gives us

$$\frac{du_1}{dx}\frac{dc_1}{dx}+\frac{du_2}{dx}\frac{dc_2}{dx}=f \quad ...(5)$$

We now have two simultaneous equations (4) and (5), for $c_1 = dc_1/dx$ and $c_2' = dc_2/dx$, which can be written in matrix form as

$$\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}\begin{pmatrix} c_1' \\ c_2' \end{pmatrix}=\begin{pmatrix} 0 \\ f \end{pmatrix}$$

These can easily be solved to give

$$c_1' = -\frac{fu_2}{W}, c_2' = \frac{fu_1}{W},$$

where

$$W = u_1u_2' - u_2u_1' = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

is called the Wronskian. These expansions can be integrated to give

$$c_1 = \int^x -\frac{f(s)u_2(s)}{W(s)}d\dot{s}+A,\ \ c_2 = \int^x \frac{f(s)u_1(s)}{W(s)}ds+B.$$

We can now write down the solution of the entire problem as

$$y(x) = u_1(x)\int^x -\frac{f(s)u_2(s)}{W(s)}ds+u_2(x)\int^x \frac{f(s)u_1(s)}{W(s)}d\dot{s}+Au_1(x)+Bu_2(x)$$

The particular integral is therefore

$$y(x) = \int^x f(s)\left\{\frac{u_1(s)u_2(x)-u_1(x)u_2(s)}{W(s)}\right\}ds \quad ...(6)$$

This is called the variation of parameters formula.

Example: Consider the equation

$$\frac{d^2y}{dx^2}+y = x\sin x$$

The homogeneous form of this equation has constant coefficients, with solutions

$$u_1(x) = \cos x,\ u_2(x) = \sin x$$

Notes

The variation of parameters formula then gives the particular integral as

$$y = \int^x s\sin s\left\{\frac{\cos s\sin x - \cos x\sin s}{1}\right\}ds,$$

since

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

We can split the particular integral into two integrals as

$$y(x) = \sin x\int^x s\sin s\cos s\, ds - \cos x\int^x s\sin^2 s\, ds$$

$$= \frac{1}{2}\sin x\int^x s\sin 2s\, ds - \frac{1}{2}\cos x\int^x s(1-\cos 2s)\, ds$$

Using integration by parts, we can evaluate this, and find that

$$y(x) = -\frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x + \frac{1}{8}\cos x$$

is the required particular integral. The general solution is therefore

$$y = c_1\cos x + c_2\sin x - \frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x$$

Self Assessment

2. Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 4y = 2\sec 2x$$

12.4 The Wronskian

Before we carry on, let's pause to discuss some further properties of the Wronskian. Recall that if V is a vector space over $\mathbb{R}$, then two elements $v_1, v_2 \in V$ are linearly dependent if $\exists\ \alpha_1, \alpha_2 \in \mathbb{R}$, with α_1 and α_2 not both zero, such that $\alpha_1 v_1 + \alpha_2 v_2 = 0$.

Now let $V = C^1(a, b)$ be the set of once-differentiable functions over the interval $a < x < b$. If $u_1, u_2 \in C^1(a, b)$ are linearly dependent, $\exists\ \alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 u_1(x) + \alpha_2 u_2(x) = 0\ \forall x \in (a, b)$. Notice that, by direct differentiation, this also gives $\alpha_1 u_1'(x) + \alpha_2 u_2'(x) = 0$ or, in matrix form.

$$\begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

These are homogeneous equations of the form

$$Ax = 0$$

which only have nontrivial solutions if det(A) = 0, that is

$$W = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = u_1u_2' - u_1'u_2 = 0.$$

Notes

In other words, the Wronskian of two linearly dependent functions is identically zero on (a, b). The contrapositive of this result is that if $W \not\equiv 0$ on (a, b), then u_1 and u_2 are linearly independent on (a, b).

Example 1: The functions $u_1(x) = x^2$ and $u_2(x) = x^3$ are linearly independent on the interval $(-1, 1)$. To see this, note that, since $u_1(x) = x^2$, $u_2(x) = x^3$, $u_1'(x) = 2x$, and $u_2'(x) = 3x^2$, the Wronskian of these two functions is

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$$

This quantity is not identically zero, and hence x^2 and x^3 are linearly independent on $(-1, 1)$

Example 2: The functions $u_1(x) = f(x)$ and $u_2(x) = kf(x)$, with k a constant, are linearly dependent on any interval, since their Wronskian is

$$W = \begin{vmatrix} f & kf \\ f' & kf' \end{vmatrix} = 0$$

If the functions u_1 and u_2 are solutions of (2), we can show by differentiating $W = u_1u_2' - u_1'u_2$ directly that

$$\frac{dW}{dx} + a_1(x)W = 0.$$

This first order differential equation has solution

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} a_1(t)dt\right\} \quad ...(7)$$

which is known as Abel's formula. This gives us an easy way of finding the Wronskian of the solutions of any second order differential equation without having to construct the solutions themselves.

Example 3: Consider the equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{x^2}\right)y = 0$$

Using Abel's formula, this has Wronskian

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} \frac{dt}{t}\right\} = \frac{x_0W(x_0)}{x} = \frac{A}{x}$$

for some constant A.

We end this section with a useful theorem.

Theorem. If u_1 and u_2 are linearly independent solutions of the homogeneous, non-singular ordinary differential equation (2), then the Wronskian is either strictly positive or strictly negative.

Proof: From Abel's formula, and since the exponential function does not change sign, the Wronskian is identically positive, identically negative or identically zero. We just need to

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exclude the possibility that W is ever zero. Suppose that $W(x_1) = 0$. The vectors $\begin{pmatrix} u_1(x_1) \\ u_1'(x_1) \end{pmatrix}$ and $\begin{pmatrix} u_2(x_1) \\ u_2'(x_1) \end{pmatrix}$ are then linearly dependent, and hence $u_1(x_1) = ku_2(x_1)$ and $u_1'(x) = ku_2'(x)$ for some constant k. The function $u(x) = u_1(x) - ku_2(x)$ is also a solution of (2) by linearity, and satisfies the initial conditions $u(x_1) = 0$, $u'(x_1) = 0$. Since (2) has a unique solution, the obvious solution, $u \equiv 0$, is the only solution. This means that $u_1 \equiv ku_2$. Hence u_1 and u_2 are linearly dependent – a contradiction.

The non-singularity of the differential equation is crucial here. If we consider the equation $x^2y'' - 2xy' + 2y = 0$, which has $u_1(x) = x^2$ and $u_2(x) = x$ as its linearly independent solutions, the Wronskian is $-x^2$, which vanishes at $x = 0$. This is because the coefficient of y'' also vanishes at $x = 0$.

Self Assessment

3. Find the Wronskian of x, x^2 on the interval (–1, 1).

12.5 The Sturm Comparison Theorem

The theorem states that if $f(x)$ and $g(x)$ are nontrivial solutions of the differential equations

$$u'' + p(x)u = 0 \quad ...(1)$$

and

$$v'' + q(x)v = 0 \quad ...(2)$$

and $p(x) \geqq q(x)$, $f(x)$ vanishes at least once between any two zeros of $g(x)$ unless $p \equiv q$ and $f = \mu g$ where μ is a real number.

Proof: As $p(s) \geqq q(x)$ for all values of x within the interval of interest. For example consider the equation

$$w'' + a^2w = 0, a^2 > 0 \quad ...(3)$$

This equation has an oscillatory behaviour and the solution is of the form

$$w(x) = c_1 \sin ax + c_2 \cos ax \quad ...(4)$$

since

$$p(x) \geqq a^2 > 0$$

then (1) will have an oscillatory solution and so will have zeros. As (1) is more oscillatory then (2) it will have zeros also more frequently and hence in between zeros of (2) it have at least one zero.

12.6 The Sturm Separation Theorem

If $u_1(x)$ and $u_2(x)$ are the linearly independent solutions of a non-singular homogeneous equation (1), then the zeros of $u_1(x)$ and $u_2(x)$ occur alternately. In other words, successive zeros of $u_1(x)$ are separated by successive zeros of $u_2(x)$ and vice versa.

Proof: Suppose that x_1 and x_2 are successive zeros of $u_2(x)$; as the Wronskian W is given by

$$W(x) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1(x)u_2'(x) - u_2(x)u_1'(x)$$

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so that

$$W(x_i) = u_1(x_i)u_2'(x_i) \qquad \text{for } i = 1,2$$

We also know from Abel's formula that $W(x)$ is of one sign on $x_1 < x < x_2$, since $u_1(x)$ and $u_2(x)$ are linearly independent. This means that $u_1(x_i)$ and $u_2'(x_i)$ are nonzero. Now if $u_2'(x_1)$ is positive then $u_2'(x_2)$ is negative or vice versa, since $u_2(x_2) = 0$. Since the Wronskian cannot change sign between x_1 and x_2, so $u_1(x)$ must change sign and hence u_1 has a zero in between x_1 and x_2 as we claimed.

Self Assessment

4. Consider the equation

$$\frac{d^2y}{dx^2} + w^2y = 0$$

It has the solution

$$y = \text{A} \sin wx + \text{B} \cos wx$$

If we consider any two of the zeros of sin wx, it is immediately clear that cos wx has a zero between them.

Compare its solutions with respect to those of

$$\frac{d^2w}{dx^2} + 4w^2w = 0$$

12.7 Summary

- The comparison and separation theorems of Sturm are useful in the periodic solutions of the second order linear equation.
- These theorems are understood in a better way once the reduction method of order is set up.
- The variation of parameters help us in finding the particular integral of the non-homogeneous differential equation.

12.8 Keywords

Sturm comparison theorem helps us in telling when the solution of a differential equation has at least one zero in between the two zeros of the solution of another differential equation simply by studying their coefficients in the equation.

Whereas, the ***Sturm separation theorem*** helps us in predicting that one independent solution of the equation has at least one zero in between the two zeros of the other independent solution. This happens in the case of periodic solutions.

12.9 Review Questions

1. Find the Wronskian of e^x, e^{-x}

2. Find the general solution of $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = x$

3. If u_1, u_2 are linearly independent solution of $y'' + p(x)y' + q(x)y = 0$ and y is any other solution, show that Wronskian of (y, u_1, u_2)

$$W(x) = \begin{vmatrix} y & u_1 & u_2 \\ y' & u_1' & u_2' \\ y'' & u_1'' & u_2'' \end{vmatrix}$$

is zero.

Answers: Self Assessment

1. $\dfrac{\cos x}{x}$

   complete solution is $(A \sin x + B \cos x)/x$

2. $y = A \sin 2x + B \cos 2x + x \cos 2x - \sin 2x \log (\cos 2x)$

3. $-3x^2$

12.10 Further Readings

Books

Pipes, Louis A. & Lawrence R. Harvill, Applied Mathematics for Engineers & Physicists

King A.C., Billingham, J. Otto S.R., Differential Equations.

Yosida, K., Lectures on Differential and Integral Equations

Sneddon, L.N., Elements of partial differential equations

Notes

Unit 13: Orthogonality of Solutions

CONTENTS

Objectives

After studying this unit, you should be able to:

- Understand better the solutions of Bessel equations, Legendre equations, Hermite equations and Laguerre differential equations.
- See that there are solutions which are obtained for some values of the parameters known as eigenvalues. These solutions are known as eigenfunctions.
- Reduce these equations and many more differential equations of second order to Sturm-Liouville boundary value problem. Hence the solutions can be shown to be orthogonal, orthonormal and the set of various solutions of the equations form a complete set.

Introduction

Knowledge of Sturm–Liouville problem and certain methods are prerequisite to the ideas of orthogonality of the solutions of certain differential equations.

Also the solutions of these equations can be used to expand any function on an interval in terms of them in a systematic manner.

13.1 Review of Some Basic Definitions

In the last four units we had studied the properties of linear second order differential equations. By now you must have got enough inside into the solutions of the equations. It is seen that the form of self-adjoint equations as well as Sturm–Liouville's boundary value problems led to the kind of solutions of certain linear second order differential equations the orthogonal set of functions which are solutions of these equations. The most important of these solutions are the Fourier sine and cosine series, the Legendre polynomials, the Bessel functions; the Hermite polynomials and Laguerre's polynomials. In the last four chapters we had already seen that the solutions do resemble the eigenfunctions of a self-adjoint operator and also form an orthogonal set with respect to a weight factor. So it is advisable to introduce the inner product of two functions. The concept of an orthogonal set of functions arises in a natural way from an analogy with vectors in a vector space. This is a natural generalization of the concept of an orthogonal set

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of vectors, i.e. a set of mutually perpendicular vectors. In fact, a function can be considered as a generalized vector so that fundamental properties of the set of functions are suggested by an analogous properties of the set of vectors.

Some Basic Definitions

Inner Product: The inner product of two functions $f(x)$ and $g(x)$ is a number defined by the equation

$$(f, g) = \int_a^b f(x)\, g(x)\, dx$$

on the interval $a \le x \le b$.

Norm of the function: The norm of the function $f(x)$ is defined as the non-negative number

$$\|f\| = \left\{\int_a^b |f(x)|^2 dx\right\}^{1/2}$$

Orthogonal functions: The condition that the two functions be orthogonal is written as

$$(f, g) = \int_a^b f(x)\, g(x) dx = 0.$$

Orthogonality with respect to a weight (or density) function: The concept of orthogonality can be extended as follows. Let $p(x) \ge 0$. Then the condition that the two functions $f(x)$ and $g(x)$ be orthogonal with respect to the weight function $p(x)$ is written as

$$\int_a^b p(x)\, f(x)\, g(x)\, dx = 0$$

Further the norm of the function is defined as

$$\|f\|_p = \left\{\int_a^b p(x) f^2(x) dx\right\}^{1/2}$$

Again $f(x)$ is said to be normalized when

$$\int_a^b p(x)\, f^2(x) dx = 1$$

The orthogonality with respect to weight function $p(x)$ can be reduced to the ordinary type by using the product $\sqrt{p(x)}\ f(x)$ and $\sqrt{p(x)}\ g(x)$ as two functions.

Orthogonal Set of Functions:

If we have a set $\{f_n(x)\}$, (n = 1, 2, 3, ...) of real functions defined on an interval $a \le x \le b$, then the $\{f_n(x)\}$ is said to be an orthogonal set of functions on the interval $a \le x \le b$ if

$$\int_a^b f_m(x) f_n(x) dx = \}0 \text{ when } m \ne n$$

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The set $\{f_n(x)\}$ is said to be orthonormal set if

$$\int_a^b f_m(x)\, f_n(x)\, dx = \delta_{mn},$$

Where the Kronecker delta,

$$\delta_{mn} = \begin{cases} 0 & \text{if} \quad m \neq n \\ 1 & \text{if} \quad m = n \end{cases}$$

Orthonormal Set of Functions with Respect to a Weight Function

Let $\{\phi_n(x)\}$ $(n = 1, 2, 3, ...)$ be a set of real functions defined on the interval $a \le x \le b$ and $p(x) \geqq 0$. Then the set $\{\phi_n(x)\}$ is said to be orthonormal set of functions on the interval $a \le x \le b$ if

$$\int_a^b p(x)\, \phi_m(x)\phi_n(x)\, dx = \begin{cases} 0 & \text{when} \quad m \neq n \\ 1 & \text{when} \quad m = n \end{cases}$$

i.e.,

$$\int_a^b p(x)\, \phi_m(x)\phi_n(x)\, dx = \delta_{mn}.$$

Self Assessment

1. Show that the function $f_1(x) = 1, f_2(x) = x$ are orthogonal on the interval $(-1, 1)$ and determine the constants A and B so that the function $f_3(x) = 1 + Ax + Bx^2$ is orthogonal to both $f_1(x)$ and $f_2(x)$ on the interval $(-1, 1)$.

13.2 Review of Sturm-Liouville Problem - Eigenvalues and Eigenfunctions

Various important orthogonal sets of functions arise in the solution of second-order differential equation

$$[R(x)y']' + [Q(x) + \lambda P(x)]y = 0 \qquad ...(i)$$

on some interval $0 \le x \le b$ satisfying boundary conditions of the form

$$\left.\begin{aligned} &\text{(a)} \quad a_1 y + a_2 y' = 0 \quad \text{at} \quad x = a \\ &\text{(b)} \quad b_1 y + b_2 y' = 0 \quad \text{at} \quad x = b \end{aligned}\right\} \qquad ...(ii)$$

The boundary value problem given by (i), (ii) is called a Sturm−Liouville problem. Here λ is a parameter and a_1, a_2, b_1, b_2 are given real constants at least one in each of conditions (ii) being different from zero. The equation (i) is known as the Sturm−Liouville equation.

You may recall that Bessel's differential equation, Legendre's equation, Hermite equation and other important equations can be written in the form (i).

The solution $y = 0$ is the trivial solution. The solution $y \neq 0$ are called the characteristic functions or eigenfunctions and λ are called λ characteristic values or eigenvalues of the problem.

There are a few theorems about the eigenvalues and eigenfunctions as follows: **Notes**

Theorem 1: Let the functions P, Q, R in the Sturm–Liouville equation be real and continuous on the interval $a \le x \le b$. Let $y_m(x)$ and $y_n(x)$ be given functions of the Sturm–Liouville problem corresponding to different eigenvalues λ_m and λ_n respectively, and let the derivatives $y'_m(x)$, $y'_n(x)$ be also continuous on the interval. Then y_m and y_n are orthogonal on that interval with respect to the weight function P i.e.,

$$\int_a^b P(x) y_m(x) y_n(x)\, dx = 0 \qquad \text{for} \qquad \lambda_m \ne \lambda_n$$

Theorem 2: The eigenvalues of the Sturm–Liouville problem are all real.

Theorem 3: If $R(a) > 0$ or $R(b) > 0$, the Sturm–Liouville problem cannot have two linearly independent eigen functions corresponding to the same eigenvalue.

Example: The simpler example of a Sturm–Liouville equation is the Fourier's equation

$$y''(x,\lambda) + \lambda y(x,\lambda) = 0 \text{ subject to } y(0) = y(l) = 0$$

which has solutions cos $(x\sqrt{\lambda})$ and sin $(x\sqrt{\lambda})$. Using the boundary conditions, we have for y (0) = 0, only sin $(x\sqrt{\lambda})$ term is present. From the second consideration we have

$$l\sqrt{\lambda} = n\pi, \qquad n = 0,1,2,...$$

So the eigenfunctions are given by

$$y_n(x) = A_n \sin\left(\frac{n\pi x}{l}\right), \text{ for } n = 1, 2, 3,...$$

The eigenvalues are given by

$$\lambda_n = \frac{n^2\pi^2}{l^2}, n = 0,1,2,3,....$$

Self Assessment

2. Find the eigenvalues ad eigenfunctions of the equation

$$y''(x) + k^2 y(x) = 0$$

with the boundary conditions

$$y(0) = 0 \text{ and } y'(1) = 0$$

13.3 Review of Bessel's Inequality and Completeness Relation

Let $\{\Psi_n(x)$, [n = 1, 2, 3, ...]} be an orthonormal set of functions on an interval (a, b) and let an arbitrary function on the same interval be a linear combination of these functions, in the form

$$f(x) = \sum_{n=1}^{\infty} C_n \Psi_n(x) \qquad a \le x \le b$$

If the series converges and represents $f(x)$, it is called a generalized Fourier series of $f(x)$. The coefficient C_ν, ν = 1, 2, ... given by

$$C_\nu = \left(f, \Psi_\nu(x)\right) = \int_a^b f(x)\, \Psi_\nu(x)\, dx \qquad ...(i)$$

Notes

are called the expansion coefficients of $f(x)$ with respect to the given orthonormal system.

Obviously $$\int\left(f-\sum_{\nu=1}^{n}C_\nu\Psi_\nu\right)^2 dx \geq 0 \qquad ...(ii)$$

By writing out the square and integrating term by term, we get

$$0 \leq \int f^2\,dx - 2\sum_{\nu=1}^{n}C_\nu\int f.\Psi_\nu dx + \sum_{\nu=1}^{n}C_\nu^{\,2}$$

or $$0 \leq (Nf)^2 - 2\sum_{\nu=1}^{n}C_\nu^{\,2} + \sum_{\nu=1}^{n}C_\nu^{\,2} \qquad [Nf \text{ means norm of } f]$$

or $$0 \leq (Nf)^2 - \sum_{\nu=1}^{n}C_\nu^{\,2}$$

or $$\sum_{\nu=1}^{n}C_\nu^{\,2} \leq (Nf)^2 \qquad ...(iii)$$

Since the number on right is Independent of n, it follows that

$$\sum_{\nu=1}^{n}C_\nu^{\,2} < (Nf)^{2.}$$

This fundamental inequality is known as *Bessel's inequality* and is true for every orthonormal system. It proves that the *sum of the squares of the expansion coefficients always converges*.

For systems of functions with complex values the corresponding relation is

$$\sum_{\nu=1}^{n}|C_\nu|^2 \leq (Nf)^2 = (f, \bar{f}) \qquad ...(iv)$$

holds, where C_ν is the expansion coefficient $C_\nu = (\bar{f}, \Psi_\nu)$.

This relation may be obtained from the inequality

$$\int\left|f(x)-\sum_{\nu=1}^{n}C_\nu\Psi_\nu\right|^2 dx = (Nf)^2 - \sum_{\nu=1}^{n}|C_\nu|^2 \geq 0$$

The significance of the integral in (ii) is that it occurs in the problem of approximating the given function $f(x)$ by a linear combination $\sum_{\nu=1}^{n}\lambda_\nu\Psi_\nu$ with λ_ν as constant coefficient and fixed n, in such a way that the *mean square error*

$$M = \int\left(f-\sum_{\nu=1}^{n}\lambda_\nu\Psi_\nu\right)^2 dx$$

is as small as possible.

An approximation of this type is known as an approximation by the method of least squares, or an *approximation in the mean*.

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If, for a given orthonormal system $\Psi_1, \Psi_2 ...$, any piecewise continuous function f, can be approximated in the mean to any desired degree of accuracy by choosing n large enough, i.e., if n may be so chosen that the mean square error.

$$\int\left(f - \sum_{\nu=1}^{n} C_\nu \Psi_\nu\right)^2 dx$$

is less than a given arbitrary small positive number, then the system of functions $\Psi_1, \Psi_2 ...$, is said to be *complete*.

For a complete or orthonormal system of functions Bessel's inequality becomes an equality for every function f

i.e. $$\sum_{\nu=1}^{n} C_\nu^2 = (Nf)^2$$

or $$\sum_{\nu=1}^{n} (f, \Psi_\nu)^2 = \|f\|^2$$

The relation is known as the *completeness relation* or ***Parseval's equation.***

Definitions

Closed Set: The set $\{\phi_n\}$ is closed in the sense of mean convergence if for each function f of the function space

$$\sum_{n=1}^{\infty} (f, \phi_n)^2 = \|f\|^2$$

Complete Set: An orthonormal set $\{\phi_n\}$ is complete in the function space if there is no function in that space, with positive norm which is to orthogonal to each of the functions.

Theorem: If an orthonormal set $\{\phi_n(x)\}$ is closed it is complete.

If an orthonormal set is closed then for each function f of the function space

$$\sum_{n=1}^{\infty} (f, \phi_n)^2 = \|f\|^2 \qquad ...(i)$$

Now, let us suppose a function $\Psi(x)$ in the space which is orthogonal to each function $\{\phi_n(x)\}$ of the closed orthonormal set such that

$$\|\phi\| \neq 0$$

$$(f, \phi_n) \neq 0,$$

Therefore from (i), we have $\|f\| = 0$, which is a contradiction.

Therefore there is no function in space, with positive norm which is orthogonal to each of the functions $\phi_n(x)$.

Hence the closed orthonormal set $\{\phi_n(x)\}$ is complete also.

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13.4 Orthogonality of Solutions of Some Equations

(a) Orthogonality of Bessel's Functions

We know that $J_n(x')$ is the solution of Bessel's equation

$$x'^2 \frac{d^2 J_n(x')}{dx'^2} + x' \frac{d J_n(x')}{dx'} + (x'^2 - n^2)\, J_n(x') = 0$$

where n is a positive integer. Putting $x' = \lambda x$, we have

$$\frac{d J_n}{dx'} = \frac{1}{\lambda} \frac{d J_n}{dx}$$

and

$$\frac{d^2 J_n}{dx'^2} = \frac{1}{\lambda^2} \frac{d^2 J_n}{dx^2},$$

where λ is a constant,

$$x^2 \frac{d^2 J_n(\lambda x)}{dx^2} + x \frac{d J_n(\lambda x)}{dx} + (\lambda^2 x^2 - n^2) J_n(\lambda x) = 0 \qquad \text{...(i)}$$

which may be rewritten as

$$\frac{d}{dx}\left[x \frac{d J_n(x\lambda)}{dx}\right] + \left[\lambda^2 x - \frac{n^2}{x}\right] J_n(\lambda x) = 0$$

which is Sturm–Liouville equation for each fixed n i.e.

$$\frac{d}{dx}\left[p(x) \frac{d}{dx} J_n(\lambda x)\right] + [q(x) + \lambda_1 r(x)]y = 0$$

with

$$p(x) = x,\ q(x) = -\frac{n^2}{x} \text{ and } r(x) = x \text{ and } \lambda_1 = \lambda^2 .$$

Since $p(x) = 0$ for $x = 0$, it follows that the solution of (i) on an interval $0 \le x \le a$ satisfying the boundary conditions

$$J_n(\lambda a) = 0 \qquad \text{...(ii)}$$

form an orthogonal set with respect to the weight $p(x) = x$.

Let $\alpha_{1n} < \alpha_{2n} < \alpha_{3n}$ ... denote the positive zeros of $J_n(x_1)$, therefore (ii) holds for

$$\lambda a = \lambda_{mn} \text{ or } \lambda = \lambda_{mn} = \frac{\alpha_{mn}}{a} \qquad (m = 1, 2, \ldots n \text{ fixed})$$

and since $\frac{d}{dx} J_n(x)$ is continuous also at $x = 0$, therefore for each fixed $n = 0, 1, 2, \ldots$, the Bessel's function $J_n(\lambda_{mn} x)$ ($m = 1, 2, \ldots$) with $\lambda_{mn} = \frac{\alpha_{mn}}{a}$, form a orthogonal set on an interval $0 \le x \le a$ with respect to weight function $p(x) = x$,

$$\int_0^a x\, J_n\left(\lambda_{mn} x\right) J_n\left(\lambda_{pn} x\right) = 0 \qquad \text{if } p \ne m$$

Thus we have obtained infinity many orthogonal sets corresponding to each fixed value of n. **Notes**

If a function is represented by generalized Fourier Bessel series

$$f(x) = \sum_{m=1}^{\infty} C_m J_n(\lambda_{mn}x), \text{ for } n \text{ fixed} \qquad ...(iii)$$

then

$$C_m = \frac{1}{\|J_n(\lambda_{mn}x)\|^2} \int_a^b x f(x) J_n(\lambda_{mn}x)\,dx,\ m = 1, 2...$$

Since

$$p(x) = x, \qquad \lambda_{mn} = \frac{\alpha_{mn}}{a}$$

where

$$\|J_n(\lambda_{mn}x)\|^2 = \int_0^a x\, J_n^2(\lambda_{mn}x)\,dx \qquad ...(iv)$$

To bind $\|J_n(\lambda_{mn}x)\|^2$,

let us proceed as follows:

Multiplying (i) by $2x\, J_n'(\lambda x)$, we have

$$2x\, J_n^1(\lambda x)\left[x\, J_n^1(\lambda x)\right]' + \left(\lambda^2 x - \frac{n^2}{x^2}\right) 2x\, J_n(\lambda x) J_n^1(\lambda x) = 0$$

or

$$\left\{\left[x\, J_n^1(x)\right]^2\right\}' + \left(\lambda^2 x^2 - n^2\right)[J_n(\lambda x)]' = 0$$

Integrating over the limits 0 to a, we have

$$\left\{\left[x\, J_n^1(\lambda x)\right]^2\right\}_0^a = -\int \left(\lambda^2 x^2 - n^2\right)\left[J_n^2(\lambda x)\right]' dx$$

Integrating R.H.S. by parts, we have

$$\left\{\left[x\, J_n^1(\lambda x)\right]^2\right\}_0^a = -\left[\left(\lambda^2 x^2 - n^2\right) J_n^2(\lambda x)\right]_0^a + 2\lambda^2 \int_0^a x\, J_n^2(\lambda x)\,dx \qquad ...(v)$$

From the following recurrence formulas for $J_n(\mu)$, we have

$$\frac{d}{d\mu}\left[\mu^{-n} J_n(\mu)\right] = -\mu^{-n} J_{n+1}(\mu)$$

or

$$\mu^{-n} \frac{d}{d\mu} J_n(\mu) - n\, \mu^{-n-1} J_n(\mu) = -\mu^{-n} J_{n+1}(\mu)$$

Multiplying both sides by μ^{n+1}

$$\mu \frac{d}{d\mu} J_n(\mu) - n J_n(\mu) = -\mu\, J_{n+1}(\mu)$$

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Putting $\mu = \lambda x$,

$$\lambda x \frac{d}{d(\lambda x)} J_n(\lambda x) - n\, J_n(\lambda x) = -\lambda x\, J_{n+1}(\lambda x)$$

or
$$x\, J_n^1(\lambda x) - n\, J_n(\lambda x) = -\lambda x\, J_{n+1}(\lambda x)$$

Substituting in (v), we have

$$\left[\left[n\, J_n(\lambda x) - \lambda x\, J_{n+1}(\lambda x)\right]^2\right]_0^a = -\left[\left(\lambda^2 x^2 - n^2\right) J_n^2(\lambda x)\right]_0^a + 2\,\lambda^2 \int_0^a x\, J_n^2(\lambda x)\, dx$$

If $\lambda = \lambda_{mn}$, then $J_n(\lambda a) = J_n\left(\lambda_{mn} a\right) = 0$, and

Since $J_n(0) = 0$, for $n = 1, 2, ...,$

then we have

$$\lambda_{mn}^2\; a^2\; J_{n+1}^2\left(\lambda_{mn}\, a\right) = 2\,\lambda_{mn}^2 \int_0^a x\; J_n^2\left(\lambda_{mx} x\right) dx$$

$$= 2\,\lambda_{mn}^2 \left\| J_n(\lambda_{mn} x) \right\|^2 \qquad \{\text{since weight} = x\}$$

Thus

$$\left\| J_n(\lambda_{mn} x) \right\|^2 = \frac{a^2}{2} J_{n+1}^2\; (\lambda_{mn} a)$$

$$= \frac{a^2}{2} J_{n+1}^2\; (\alpha_{mn})$$

where
$$\alpha_{mn} = \lambda_{mn} a$$

So
$$C_n = \frac{2}{a^2\, J_{n+1}^2(\alpha_{mn})} \int_0^a x\, J_n(\lambda_{mn}\, x) f(x)\, dx \qquad ...(vi)$$

and
$$\lambda_{mn} = \frac{\alpha_{mn}}{a}, \text{ for } m = 1, 2, 3....$$

Thus generalized Fourier Bessel series is given by (iii) with the coefficient C_n given by (vi).

(b) Orthogonality of Legendre Polynomials

The Legendre's differential equation

$$(1 - x^2) y'' - 2xy' + n(n+1) y = 0$$

may be written as

$$[(1 - x^2) y']' + \lambda y = 0 \qquad ...(i)$$

where $\lambda = n\,(n + 1)$,

and is therefore a Sturm-Liouville equation with

$$R(x) = 1-x^2,\ P(x)=1 \text{ and } Q(x)=0$$

Here no boundary conditions are needed to form a Sturm-Liouville problem on the internal (−1, 1) since $R = 0$ when $x = \pm 1$.

Further we know that Legendre Polynomials

$$P_n(x),\ (n=0,1,2,...)$$

are the solutions of the problem, hence they are the eigenfunctions and since they have continuous derivatives, therefore it follows that $\{P_n(x)\}$, $n = 0, 1, 2, ...$ are orthogonal on the interval −1, $\leq x \leq 1$ with respect to the weight function

$$p = 1, \text{ i.e., } \int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0 \text{ if } (m \neq n)$$

and

$$\|P_m\|^2 = \int_{-1}^{1} P_m{}^2(x)\, dx = \frac{1}{2m+1},\ m = 0,1,2,...$$

If $g_0(x)$, $g_1(x)$, ... are eigenfunctions which are orthogonal on the interval $a \leq x \leq c$ with respect to the weight function $p(x)$, and if a given function $f(x)$ can be represented by a generalised Fourier series

$$f(x) = \sum_{n=1}^{\infty} C_n\, g_n(x)$$

then,

$$c_n = \frac{1}{\|g_n\|^2} \int_a^b p(x) f(x) g_m(x) dx \qquad (m = 0,1,2,...)$$

where

$$\|g_m\|^2 = \int_a^b p(x)\, g_m^2(x) dx$$

(c) Orthogonality of Hermite Polynomials

The Hermite polynomials $H_n(x)$, given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}$$

are orthogonal with respect to the weight function $p(x) = e^{-x^2}$ on the interval $-\infty \leq x \leq \infty$.

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = (-1)^n \int_{-\infty}^{\infty} H_m(x) \frac{d^n e^{-x^2}}{dx^n} dx$$

$$= (-1)^n \left[H_m(x) \frac{d^{n-1} e^{-x^2}}{dx^{n-1}} \right]_{-\infty}^{+\infty}$$

Notes

$$-(-1)^n \int_{-\infty}^{\infty} H_m'(x) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} dx$$

$$= -(-1) \int_{-\infty}^{\infty} 2m H_{m-1}(x) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} dx$$

[since e^{-x^2} and all its derivatives vanish for infinite x and $H_n' = 2n\, H_{n-1}$]

$$= (-1)^{n-1} 2m \int_{-\infty}^{\infty} H_{m-1}(x) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} dx \qquad n > m$$

proceeding similarly again and again

$$= (-1)^{n-m} 2^m m! \int_{-\infty}^{\infty} H_0(x) \frac{d^{n-m}}{dx^{n-m}} e^{-x^2} dx \qquad n > m$$

$$= (-1)^{n-m} 2^m m! \int_{-\infty}^{\infty} \frac{d^{n-m}}{dx^{n-m}} e^{-x^2} dx \qquad [\because H_0(x) = 1]$$

$$= (-1)^{n-m} 2^m m! \int_{-\infty}^{\infty} \left[\frac{d^{n-m-1}}{dx^{n-m-1}} e^{-x^2} \right]_{-\infty}^{\infty}$$

$$= 0$$

Now $$\int_{-\infty}^{\infty} H_n^2(x) e^{-x^2} dx = \int_{-\infty}^{\infty} H_n(x) \frac{d^n}{dx^n} e^{-x^2} dx$$ integrating as above n times

$$= 2^n n \int_{-\infty}^{\infty} H_0(x) e^{-x^2} dx$$

$$= 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$= 2^n n! 2 \int_{0}^{\infty} e^{-x^2} dx$$

$$= 2^n n! \sqrt{\pi}.$$

The functions of the orthogonal system are

$$\Psi_n(x) = \frac{H_n(x) e^{-x^2/2}}{\sqrt{\{2^n n! \sqrt{\pi}\}}}, (n = 0, 1, 2, ...)$$

(d) Orthogonality of Laguerre Polynomials

The Laguerre Polynomials $L_n(x)$ given by

$$L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x})$$

are orthogonal w.r.t. the weight function $p(x) = e^{-x}$ on the interval $0 \le x \le \infty$

$$\int_0^\infty L_m(x).L_n(x)\, e^{-x} dx$$

$$= \int_0^\infty L_m(x) \frac{d^n}{dx^n}(x^n e^{-x})\, dx$$

$$= \left[L_m(x) \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) \right]_0^\infty - \int_0^\infty L'_m(x) \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) dx$$

$$= \int_0^\infty L_m'(x) \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) dx$$

proceeding similarly

$$= (-1)^m \int_0^\infty L_m^{m'}(x) \frac{d^{n-m}}{dx^{n-m}}(x^n \; e^{-x})\; dx \;\; n \le m$$

$$= (-1)^m \int_0^\infty (-1)^m \; m! \frac{d^{n-m}}{dx^{n-m}}(x^n \; e^{-x})\; dx \;\; n \le m$$

$$= m! \left[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^n e^{-x}) \right]_0^\infty = 0$$

Now,

$$\int_0^\infty L_n^2(x).\; e^{-x} dx$$

$$= \int_0^\infty L_n(x) \frac{d^n}{dx^n}(x^n \; e^{-x})\; dx$$

$$= (-1)^n \int_0^\infty L_n^{n!}(x)(x^n \; e^{-x})\; dx$$

$$= (-1)^n (-1)^n n! \int_0^\infty x^n e^{-x} dx = (n!)^2$$

Thus the functions of the orthogonal system are

$$\Psi_v(x) = \frac{e^{-x/2} L_n(x)}{n!} \qquad (n = 0, 1, 2, \ldots)$$

Notes

Self Assessment

3. Find the eigenvalues and eigenfunctions of the equation

$$\frac{d^2y}{dx^2} + \lambda y \quad = \quad 0$$

when $y\ (0) = 0$, $y\ (\pi) = 0$

Show that the eigenfunctions are orthogonal to each other.

13.5 Summary

- In this unit we have review some of the properties of the solutions of equations like Bessel equations, Legendre equations, Hermite equations and Laguerre equations which are of Sturm-Liouville's form.
- This way we can construct the eigenfunctions for certain eigenvalues of other equations which resemble Sturm-Liouville problem with certain boundary conditions.

13.6 Keywords

Eigenfunctions are solutions of Sturm-Liouville problem corresponding to certain values of the parameter called the eigenvalues.

Sturm-Liouville boundary value problem helps us to find eigenvalues and eigenfunctions in a systematic way and their properties are well understood.

13.7 Review Questions

1. Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \qquad y(0) = y'\left(\frac{\pi}{2}\right) = 0$$

2. Show that the given set is orthogonal on the given interval and determine the corresponding orthonormal set

$1, \cos x, \cos 2x, \cos 3x, \ldots, 0 \le x \le \pi$

Answers: Self Assessment

1. A = 0, B = – 3

2. $K = \left(n + \frac{1}{2}\right)\pi,\ y_n(x) = A_n \sin\left[\left(n + \frac{1}{2}\right)\pi x\right],\ n = 0, 1, 2, \ldots.$

3. $\lambda = n^2,\ y_n(x) = \sin nx,\ n = 1, 2, 3, \ldots.$

13.8 Further Readings

Books

Yosida, K., Lectures in Differential and Integral Equations

King A.C., Billingham, J. and Otto S.R., Differential Equations

Unit 14: Classification of Partial Differential Equations

Notes

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Objectives

After studying this unit, you should be able to:

- Know before hand the type of the equation to be solved.
- Know that there are various methods based on the structure of the partial differential equations.
- See that the partial differential equations of the first order are generally solved by methods to get either complete solution or general solution.
- See that in the case of second order partial differential equations there are three types of equations, i.e. hyperbolic type, parabolic type or elliptic type.
- Deal with the methods of dealing with various partial differential equations.

Introduction

The classification of the partial differential equations is quite different than those of ordinary differential equations.

Some of the most important partial differential equations fall into one of the three categories i.e., the hyperbolic type, the parabolic type or elliptic type.

14.1 Types of Differential Equations

In dealing with any differential equation involving a number of variables, we first of all classify the variables into two categories. A variable may be such that it depends upon a number of other variables. Such a variable is called dependent variable and the other variables on which it is dependent are termed as independent variables.

In the case of ordinary differential equations we have to deal with one dependent and one independent variable. So the derivative of dependent variable is denoted as $\frac{dy}{dx}$, where y is a dependent variable and x is an independent variable. So the differential equation may be of the form

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$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0 \quad ...(1)$$

involving up to nth derivative of y.

In contrast to the above we may sometimes have to deal with a dependent variable and more than one independent variables. Thus we may have partial derivatives of the dependent variable u with respect to independent variable x, y, z,.... So we have partial derivatives of u in the differential equation like $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ etc. We may have a higher partial derivatives also present in the differential equations i.e. $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial z^2}, ...$. Such a differential equations involving one dependent variable u and a number of independent variables x, y, z, ... along with the partial derivatives of u with respect to x, y, z, .. is known as partial differential equation i.e.

$$f\left(x, y, z, ...u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, ... \frac{\partial^2 u}{\partial x^2}, ...\right) = 0 \quad ...(2)$$

We may have a situation in which the partial differential equation involves only first derivatives only. Such an equation is known as first order partial differential equation i.e.

$$f_1\left(x, y, z, ...u, \frac{\partial u}{\partial x}, ... \frac{\partial u}{\partial x_n}\right) = 0 \quad ...(3)$$

Here the order of the equation is one and it is known as first order partial differential equation. Let us denote independent variables, as x, y and z as dependent variable. Also let us put

$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} \\ q &= \frac{\partial z}{\partial y} \end{aligned}\right\} \quad ...(4)$$

So the partial differential equation involving x, y, z, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ will be of the form

$$f_2(x, y, z, p, q) = 0 \quad ...(4)$$

Example: The equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$$

is a partial differential equation of *second order*. The equation

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial y} = 0$$

is a first order partial differential equation and of second degree involving two independent variables x and y. The equation

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

is a first order partial differential equation involving three variables. So in these units involving partial differential equations we may have to deal with first order, second order or higher order partial differential equations.

14.2 Derivation of Partial Differential Equations

Example 1: Let us form the differential equation from the relation

$$lx + my + nz = \phi\,(x^2 + y^2 + z^2) \qquad ...(5)$$

Differentiating equation partially with respect to x and y

$$l + n\frac{\partial z}{\partial x} = \phi'(x^2 + y^2 + z^2)\left(2x + 2z\frac{dz}{dx}\right) \qquad ...(6)$$

and

$$m + n\frac{\partial z}{\partial y} = \phi'(x^2 + y^2 + z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right) \qquad ...(7)$$

Eliminating ϕ'

$$\frac{\left(l + n\frac{\partial z}{\partial x}\right)}{\left(m + n\frac{\partial z}{\partial y}\right)} = \frac{x + z\frac{\partial z}{\partial x}}{y + z\frac{\partial z}{\partial y}}$$

or

$$(l + np)y - \left(m + n\frac{\partial z}{\partial y}\right)x + z\frac{\partial z}{\partial y}\left(l + n\frac{\partial z}{\partial x}\right) - z\frac{\partial z}{\partial x}\left(m + n\frac{\partial z}{\partial y}\right) = 0$$

or

$$(l + np)y - (m + nq)x + z(lq - mp) = 0 \qquad ...(8)$$

Notes When the relation like (6) contains more than one function partial differential equations of the higher order will be obtained.

Example 2: Find the partial differential equation from the relation

$$\frac{x}{z} = \phi\left(\frac{y}{z}\right) \qquad ...(9)$$

by treating z as dependent variable and x, y as independent variables.

Solution: Differentiating (9) with respect to x, we have

$$\frac{1}{z} - \frac{x}{z^2}p = \phi'\left[-\frac{y}{z^2}p\right] \qquad ...(10)$$

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Again differentiating with respect to y, we obtain

$$-\frac{x}{z^2}q = \phi'(1/z - yq/z^2) \quad ...(11)$$

Eliminating ϕ' from (10) and (11) we have

$$\frac{z-xp}{(-xq)} = \frac{(-yp)}{z-yq}$$

or $$z^2 - zxp - zyq\ xypq = xypq$$

or $$z^2 - z\,(px + qy) = 0$$

or $$z = px + qy \quad ...(12)$$

Example 3: Find the partial differential equation from the relation

$$x^2 - z^2 = \phi\,(x^2 - y^2) \quad ...(13)$$

Solution: Differentiate (13) partially with respect to x keeping y fixed we have

$$2x - 2z\frac{\partial z}{\partial x} = 2x\phi' \quad ...(14)$$

Again differentiate (13) partially with respect to y keeping x fixed.

$$-2z\frac{\partial z}{\partial y} = -2y\phi' \quad ...(15)$$

Eliminating ϕ' from (14) and (15) we have

$$\frac{2(x-zp)}{(-2zq)} = \frac{2x}{(-2y)}$$

or $$-xy + zpy = xzq$$

or $$xzy + zpy = xy \qquad \text{Ans} \quad ...(16)$$

Example 4: Find the partial differential equation from the relation

$$z = \phi_1(y-2x) + \phi_2(2y-x) \quad ...(17)$$

Solution:

Differentiating (17) partially with respect to x keeping y fixed and z a dependent variable.

$$\frac{\partial z}{\partial x} = \phi_1'(-2) + \phi_2'(-1) \quad ...(18)$$

Now differentiate (17) with respect to y,

$$\frac{\partial z}{\partial y} = \phi_1' + 2\phi_2' \quad ...(19)$$

Eliminating ϕ_2' from (18) and (19) we have

Notes

$$2\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y} = -3\phi_1' \qquad ...(20)$$

Now differentiating (20) by x

$$2\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x \partial y} = -3\phi_1''(-2)=6\phi_1'' \qquad ...(21)$$

And differentiating (20) by y

$$2\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2} = -3\phi_1''\,(1) \qquad ...(22)$$

Now eliminating ϕ_1'' from (21) and (22) we have

$$2\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x \partial y}+4\frac{\partial^2 z}{\partial x \partial y}+2\frac{\partial^2 z}{\partial y^2} = 0$$

or

$$2\frac{\partial^2 z}{\partial x^2}+5\frac{\partial^2 z}{\partial x \partial y}+2\frac{\partial^2 z}{\partial y^2} = 0 \qquad ...(23)$$

Notes One can see that if there are two unknown functions in the relation between x, y and z then we obtain second order partial differential equation.

Self Assessment

1. Set up the partial differential equation by treating z as dependent variable and x, y as independent variables from the following relation

$$z = f_1(y+x)+f_2(y-x)$$

2. Set up the partial differential equation from the following relation by treating z as dependent variable and x, y as independent variable

$$\phi\left[e^{-5x}\left\{5z+\tan(y-3x)\right\},(y-3x)\right]=0$$

14.3 Various Classes of Partial Differential Equations

In this section we shall discuss some partial differential equations that occur in problems or propagation of waves in metals or strings, in electrostatics and gravitation, conduction of heat and diffusion of things in certain media. The partial differential equations discussed in the last two sections are generally partial differential equations. There are certain partial differential equations which are of second order in nature or of higher order. Let us define the partial derivatives of the dependent variable z of two independent variables x and y as

$$\frac{\partial z}{\partial x}=p,\ \frac{\partial z}{\partial y}-q,\ \frac{\partial^2 z}{\partial x^2}=r,\ \frac{\partial^2 z}{\partial x \partial y}=s \text{ and } \frac{\partial^2 z}{\partial y^2}=t.$$

Notes

up to second order partial differential equations i.e.

$$a_1\frac{\partial^2 z}{\partial x^2}+a_2\frac{\partial^2 z}{\partial x\partial y}+a_3\frac{\partial^2 z}{\partial y^2}+a_4\frac{\partial z}{\partial x}+a_5\frac{\partial z}{\partial y}+z = f(x,y)$$

or

$$a_1 r+a_2 s+a_3 t+a_4 p+a_5 q+z = f(x,y)$$

(a) Depending upon the values of a_1, a_2 and a_3 we can have:

1. ***Hyperbolic type*** of partial differential equations in which $4a_1 a_3 < a_2^2$.

   Such equations are found in wave motion as well as in vibration of strings etc.

   The example is wave motion

   $$\frac{\partial^2 V}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},$$ here y is replaced by time variable

2. ***Parabolic type:*** Partial differential equations in which

   $$a_2^2-4a_1\ a_3=0$$

   Examples of such type of equations are diffusion problems as well as conduction of heat problems i.e.

   $$K\frac{\partial^2 V}{\partial x^2}=\frac{\partial V}{\partial t},$$ here y is replaced by time t.

3. ***Elliptic type*** partial differential equation in which

   $$a_2^2-4a_1\ a_3<0.$$

   We come across such differential equations in electrostatics or gravitational potential problems. Such equations are Laplace equations i.e.

   $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

   The signification of these equations is that if we transform from x, y co-ordinate to another co-ordinate system by canonical transformation these three properties do not change.

(b) Homogeneous Partial Differential Equations

In these equations the coefficients of differential equations of any order is a constant multiple of the variables of the same degree i.e.

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=0$$

(c) Linear Partial Differential Equations with Constant Coefficients

In these equations the coefficients of the partial derivatives are constant i.e.

$$c_1 r+c_2 s+c_3 t+c_4 p+c_5 q+c_6 z=f(x,y)$$

where c_1, c_2, ... c_6 are constant of x and y.

By means of transformations we can reduce the homogeneous partial differential equations into those with constant coefficients.

Self Assessment

3. Classify the equation

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

into one of the categories i.e. elliptical, hyperbolic or parabolic type.

4. Reduce the equation

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

to equation with constant coefficients.

14.4 Summary

- Like ordinary differential equations partial differential equations play an important part in understanding certain processes.
- There are various types of partial equations like partial differential equations of first order. It involves only first partial derivatives of the dependent variable.
- Then there are partial differential equations of second or higher order and involve higher order than the first one, derivatives of the dependent variables.
- The most important second order partial differential equations can be either elliptic or parabolic or hyperbolic and play important role in most physical problems.
- In the subsequent units various methods will be given to tackle these types of equations.

14.5 Keyword

The classification of ***partial differential equations*** help us to choose appropriate method for solving these partial differential equations.

14.6 Review Questions

1. Set up partial differential equations by eliminating the constants a and b:

$$y^2\left\{(x-a)^2 + y^2 + 2z\right\} = b$$

2. Set up partial differential equation by eliminating b and a from the following equation

$$z = ax + 3a^2 y + b$$

3. Reduce the following equation to an equation having constant coefficients of its derivatives

$$x^2 \frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3 y^4$$

Notes

Answers: Self Assessment

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

2. $p + 3q = 5z + \tan(y - 3x)$

3. Hyperbola

4. $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 0$

   where $u = \log x$, $v = \log v$

14.7 Further Readings

Books

Piaggio, H.T.H., Differential Equations

Sneddon, L.N., Elements of Partial Differential Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 15: Cauchy's Problem and Characteristics for First Order Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- See that in the differential equation p and q may be of any degree also.
- Understand whether the solution exists for certain types of conditions or not.
- Understand that the partial differential equations can be solved by introducing certain characteristic curves.

Introduction

The method of solution involves the ideas of integral surfaces or curves through which the solution passes.

Thus one can introduce certain parameters and set up the characteristic equations for x, y, z, p and q in terms of these parameters. After solving these equations and eliminating the parameters we can get the solutions.

15.1 Cauchy's Problem for First Order Equations

We know that z is a dependent variable and x, y being independent variables. So the first order partial differential equation can be put into the form

$$\phi(x, y, z, p, q) = 0 \qquad ...(1)$$

Here $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ are partial derivatives. We are interested in seeking the solution of the partial differential equation (1). Before we attempt to find a solution we want to understand whether the solution exists or not. What is meant by the existence theorem which establishes conditions under which we can assert whether or not a given partial differential equation has a solution at all. Also further whether the solution if it exists is unique or not. The conditions to be satisfied in the case of first order partial differential equation are boiled down to the

Notes

classic problem of Cauchy, which in the case of two independent variables may be stated as follows:

Cauchy's Problem

Cauchy's problem is stated as follows:

(a) $x(t)$, $y(t)$, and $z(t)$ are functions which together with their first derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are continuous in the interval M defined by $t_1 < t < t_2$,

(b) And if $\phi(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$ is continuous function of $x, y, z, p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ in a certain region U of the *xyz pq* space, then it is required to establish the existence of the function $z = f(x, y)$ with the following properties:

(1) $f(x, y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.

(2) For all values of x and y lying in R the point $\{x, y, f(x, y), f_x(x, y), f_y(x, y)\}$ lies in U and

$$\phi[x, y, f(x, y), f_x(x, y), f_y(x, y)] = 0$$

(3) For all t belonging to the interval M, the point $\{x_0(t), y_0(t)\}$ belongs to the region R and

$$f\{x_0(t), y_0(t)\} = z_0$$

Geometrically stated, what we wish to prove is that there exists a surface $z = f(x, y)$ which passes through the curve Γ whose parametric equations are

$$x = x_0(t),\ y = y_0(t) \text{ and } z = z_0(t) \qquad ...(1)$$

and at every point of which the direction $(p, q, -1)$ of the normal is such that

$$\phi\{x, y, z, p, q\} = 0 \qquad ...(2)$$

The Cauchy's problem stated above can be formulated in seven other ways. For details you are referred to D. Berstein. To prove the existence of a solution it is necessary to make some more assumptions about the form of the functions and the curve. There are a whole class of existence theorems depending on the nature of these assumptions. However we shall be contented ourselves by quoting one of them as follows.

Theorem: If $g(y)$ and all its derivatives are continuous for $|y - y_0| < \delta$, if x_0 is a given number and $z_0 = g(y_0)$, $q_0 = g'(y_0)$ and if (x, y, z, q) and all its partial derivatives are continuous in a region S defined by

$$|x - x_0| < \delta,\ |y - y_0| < \delta,\ |q - q_0| < \delta$$

then there exists a unique function $\phi(x, y)$ such that:

(a) $\phi(x, y)$ and all its partial derivatives are continuous in a region R defined by $|x - x_0| < \delta_1$, $|y - y_0| < \delta_2$.

(b) For all (x, y) in R, $z = \phi(x, y)$ is a solution of the equation

$$\frac{\partial z}{\partial x} = f(x, y, z, \frac{\partial z}{\partial y})$$

(c) For all values of y in the interval $|y - y_0| < \delta_1$, $\phi(x_0, y) = g(y)$.

At this point we want to say a few words about different kinds of solutions. We may get a relation of the type

$$F(x, y, z, a, b) = 0$$

for the solution of the first order partial differential equation.

Any such relation containing two arbitrary constants a and b and a solution of the partial differential equation of the first order is said to be a complete solution or a complete integral of that equation.

On the other hand any relation of the type

$$F(u, v) = 0$$

involving an arbitrary function F connecting two known functions u and v of x, y and z and providing a solution of the first order partial differential equation is called a general solution or a general integral of that equation.

We shall be dealing with the classifications of the integrals of the first order partial differential equations in the unit 16 in more details.

Self Assessment

1. Eliminate constants a and b from the equation

   $$z = (x + a)(y + b)$$

2. Eliminate the arbitrary function f from the equation

   $$z = xy + f(x^2 + y^2)$$

15.2 Cauchy's Method of Characteristics

We should now consider a method due to Cauchy for solving the non-linear partial differential equation

$$F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0 \qquad ...(1)$$

The method is based on geometrical ideas. Equation (1) can be theoretically solved to obtain an expression.

$$q = G(x, y, z, p) \qquad ...(2)$$

from which q is calculated in terms of x, y, z and p. Before proceeding further let us consider a plane passing through a point $P(x_0, y_0, z_0)$ with its normal parallel to the direction n defined by the direction cosines $(p_0, q_0, -1)$. This plane is uniquely specified by the set of numbers $D(x_0, y_0, z_0, p_0, q_0)$. Conversely any such set of five numbers defines a plane in three dimensional space. We now define

A plane element: A set of five numbers $D(x, y, z, p, q)$ is called a plane element of the space.

An integral element: If the plane element (x, y, z, p, q) satisfies an equation

$$F(x, y, z, p, q) = 0 \qquad ...(3)$$

it is called an integral element of the equation (3) at the point (x_0, y_0, z_0).

Thus keeping x_0, y_0 and z_0 fixed and varying p, we obtain a set of plane elements $\{x_0, y_0, z_0, p, G(x_0, y_0, z_0, p)\}$ which depend on the single parameter p. As p varies we obtain a set of plane

Notes

elements, all of which pass through the point P and which therefore envelope a Cone with vertex P; the cone so generated is called **elementary Cone** of equation (3) at the point P (Figure 15.1). Consider now a surface S whose equation is

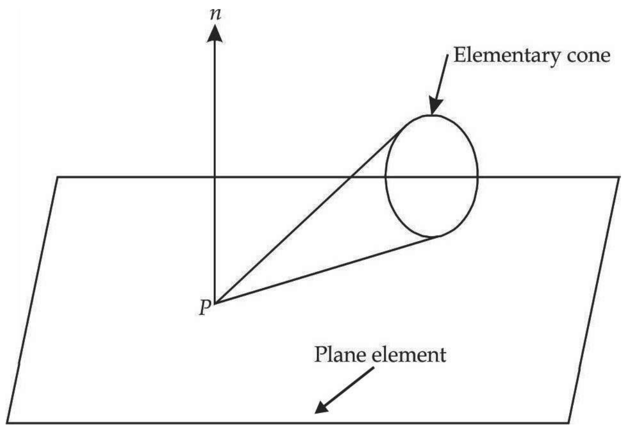
$$z = g(x, y) \qquad ...(4)$$

If the function $g(x, y)$ and its first partial derivatives $g_x(x, y)$, $g_y(x, y)$ are continuous in a certain region R of the xy plane, then the **tangent plane** at each point of S determines a plane element of the type

$$\{x_0, y_0, g(x_0, y_0), g_x(x_0, y_0), g_y(x_0, y_0)\} \qquad ...(5)$$

which we shall call the **tangent element** of the surface S at the point $(x_0, y_0, g(x_0, y_0))$.

Figure 15.1



We now state the following theorem on geometrical ground.

Theorem 1: A necessary and sufficient condition that a surface be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.

A curve C with parametric equation

$$x = x(t), y = y(t), z = z(t) \qquad ...(6)$$

lies on the surface (4) if

$$z(t) = g(x(t), y(t));$$

for all values of t in the appropriate interval l. If P_0 is a point on this curve determined by the parameter t_0, then the direction ratios of the tangent line $P_0 P_1$ (See Figure 15.2) are $(x'(t_0), y'(t_0), z'(t_0))$, where $x'(t_0)$ denotes the values of $\frac{dx}{dt}$ when $t = t_0$, etc. This direction will be perpendicular to the direction $(p_0, q_0, -1)$ if

$$z'(t_0) = p_0 x'_0(t_0) + q_0 y'_0(t_0).$$

For this reason we say that any set

$$\{x(t), y(t), z(t), p(t), q(t)\} \qquad ...(7)$$

of five real functions satisfying the conditions

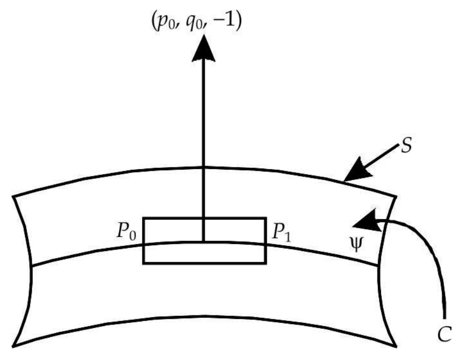
$$z'(t) = p(t)\, x'(t) + q(t)\, y'(t) \qquad ...(8)$$

defines a strip at the point (x, y, z) of the curve C. If such a strip is also an integral element of equation (3), we say that it is an integral strip of equation (3) i.e., the set of functions (7) is an integral strip of equation (3) provided they satisfy condition (8) and the condition

$$F(x(t), y(t), z(t), p(t), q(t)) = 0 \qquad ...(9)$$

for all t in l.

Figure 15.2



If at each point, the curve (6) touches a generator of the elementary cone, we say that the corresponding strip is a characteristic strip. We shall now derive the equations determining a characteristic strip for the point $(x + dx, y + dy, z + dz)$ that lies in the tangent plane to the elementary cone at P.

If $\qquad dz = p\,dx + q\,dy \qquad ...(10)$

where p and q satisfy (3). Differentiating (10) with respect to p we obtain

$$0 = dx + \frac{dq}{dp}\,dy. \qquad ...(11)$$

Also from (3)

$$\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\frac{dq}{dp} = 0 \qquad ...(12)$$

solving the equations (10), (11) and (12) for the ratios of dx, dy, dz and by putting the values of $\frac{dq}{\partial p}$ from (10) into (11), we have

$$\frac{dq}{dp} = -\frac{dx}{dy} = -\frac{\frac{\partial F}{\partial p}}{\frac{\partial F}{\partial q}}$$

or $\qquad \dfrac{dx}{\frac{\partial F}{\partial p}} = \dfrac{dy}{\frac{\partial F}{\partial q}}$

Notes

Also $$\frac{p\,dx}{p\frac{\partial F}{\partial p}} = \frac{q\,dq}{q\frac{\partial F}{\partial p}} = \frac{p\,dx+q\,dy}{p\frac{\partial F}{\partial p}+q\frac{\partial F}{\partial p}} = \frac{dz}{p\frac{\partial F}{\partial p}+q\frac{\partial F}{\partial p}}$$

Hence $$\frac{dx}{\frac{\partial F}{\partial p}} = \frac{dy}{\frac{\partial F}{\partial p}} = \frac{dz}{p\frac{\partial F}{\partial p}+q\frac{\partial F}{\partial p}} \qquad ...(13)$$

that means that along a characteristic strip, $x'(t)$, $y'(t)$, $z'(t)$ must be proportional to F_p, F_q, $p\,F_p + q\,F_q$ respectively. If we choose the parameter t in such a way that

$$x'(t) = F_p, \quad y'(t) = F_q \qquad ...(14)$$

then $$z'(t) = p\,F_p + q\,F_q$$

along a characteristic strip p is a function of t so that

$$p'(t) = \frac{\partial p}{\partial x}x'(t) + \frac{\partial p}{\partial y}y'(t)$$

$$= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q}$$

$$= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial F}{\partial q}. \qquad \left(\text{Since } \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}\right)$$

Differentiating equation (3) with respect to x, we find that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$$

so that on a characteristic strip

$$p'(t) = -(F_x + p\,F_z) \qquad ...(16)$$

and it can be shown similarly that

$$q'(t) = -(F_y + q\,F_z) \qquad ...(17)$$

Collecting equations (14) to (17), we see that we have the following system of five ordinary differential equations for the determination of the characteristic strip

$$x'(t) = F_p,\ y'(t) = F_q,\ z'(t) = p\,F_p + q\,F\,q'$$

$$p'(t) = -(F_x + p\,F_z),\ q'(t) = -(F_y + q\,F_z) \qquad ...(18)$$

These equations are known as the characteristic equations of the differential equation (3).

The main theorem about characteristic strip is:

Theorem 2: Along every characteristic strip of the equation $F(x, y, z, p, q) = 0$, the function $F(x, y, z, p, q)$ is a constant.

The proof is a matter simply of calculation. Along a characteristic strip we have

$$\frac{d}{dt}F(x(t), y(t), z(t), p(t), q(t)) = F_x x' + F_y y' + F_z z' + F_p p' + F_q q'$$

$$= F_x F_p + F_y F_q - F_z(pF_p + qF_q) - F_p(F_x + pF_z) - F_q(F_y + qF_z) = 0$$

Notes

So that $F(x, y, z, p, q) = k$, is a constant along the strip.

Theorem 3: If a characteristic strip contains at least one integral element of $F(x, y, z, p, q) = 0$, it is an integral strip of the equation $F(x, y, z, p, q) = 0$.

We are now in a position to solve Cauchy's problem. Suppose we want to find the solution of the partial differential equation (1) which passes through a curve Γ whose freedom equations are

$$x = \theta(\nu),\ y = \phi(\nu),\ z = \chi(\nu) \quad \text{...(19)}$$

then in the solution

$$x = x(p_0, q_0, x_0, y_0, z_0, t_0, t) \text{ etc.,} \quad \text{...(20)}$$

and in the characteristic equations (18) we may take

$$x_0 = \theta(\nu),\ y_0 = \phi(\nu),\ z_0 = \chi(\nu)$$

as the initial values of x, y, z. The corresponding initial values of θ, ϕ, χ are determined by the relations

$$\chi' = p_0\, \theta'(\nu) + q_0\, \phi'(\nu)$$

$$F(\theta(\nu), \phi(\nu), \chi(\nu), p_0, q_0) = 0$$

We substitute these values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_o in equation (20), and find that x, y, z can be expressed in terms of two parameters t, ν to give

$$x = X(\nu, t),\ y = Y(\nu, t),\ z = Z(\nu, t) \quad \text{...(21)}$$

Eliminating ν, t from these equations, we get a relation

$$\psi(x, y, z) = 0$$

which is the equation of the integral surface of equation (1) through the curve Γ. We shall illustrate this procedure by an example.

Example: Find the solution of the equation

$$F = \frac{1}{2}(p^2 - q^2) + (p - x)(q - y) - z \quad \text{...(1)}$$

that passes through the x-axis.

It is readily shown that the initial values are

$$x_0 = \nu,\ y_0 = 0,\ z_0 = 0,\ p_0 = 0,\ q_0 = 2\nu,\ t_0 = 0, \quad \text{...(2)}$$

The characteristic equations of this partial differential equations are

$$x'(t) = F_p,\ \ y'(t)\ F_q,\ \ z'(t) = p\,F_p + q\,F_q$$

$$p'(t) = -F_x - p\,F_z,\ \ q'(t) = -F_y - q\,F_z \quad \text{...(3)}$$

$$F_p = \frac{\partial F}{\partial p} = p + q - y,\ \ F_q = \frac{\partial F}{\partial q} = -q + p - x$$

$$F_x = \frac{\partial F}{\partial x} = -q + y,\ \ F_y = \frac{\partial F}{\partial y} = -p + x,\ F_z = -1 \quad \text{...(4)}$$

Substituting these values of partial derivatives of F in equations (3) we have

Notes

$$x'(t) = p + q - y,\ y'(t) = p - q - x,\ z'(t) = p(p + q - y) + q(p - q - x)$$

$$p'(t) = q - y + p,\ \ q'(t) = p - x + q \qquad \text{...(5)}$$

Now $x'(t) = p'(t)$, which gives $x = p + \alpha$, so that $t = 0$

$$x = \nu,\ p = 0, \text{ so } x = \nu + p \qquad \text{...(6)}$$

similarly $y = q - 2\nu$ ...(7)

Also, it is readily shown that

$$\frac{d}{dt}(p + q - x) = q - y + p + p - x + q - p - q + y$$

$$= p + q - x$$

So $\dfrac{d(p+q-x)}{p+q-x} = dt$

On integrating we get

$$\log(p + q - x) = t + \log c_1$$

or $p + q - x = c_1 e^t$ ...(8)

At $t = 0, p = 0, q = 0, x = \nu$ we get $c_1 = +\nu$

therefore $p + q - x = +\nu e^t$ ...(9)

Similarly

$$\frac{d}{dt}(p+q-y) = p + q - y + p + q - p - x - p - q + x = p + q - y$$

or $\dfrac{d}{dt}(p+q-y) = p + q - y$ ...(10)

On integrating (10) we get

$$p + q - y = 2\nu e^t \qquad \text{...(11)}$$

the constant of integration being 2ν.

From (6) and (9) we have

$$q = \nu e^t - p + x$$

or $q = \nu e^t + \nu = \nu (e^t + 1)$ ...(12)

From (7) we have

$$y = q - 2\nu = \nu (e^t - 1) \qquad \text{...(13)}$$

From (11) we have

$$p = 2\nu e^t - q + y$$

$$= 2\nu e^t - \nu (e^t + 1) + \nu (e^t - 1)$$

or $p = 2\nu (e^t - 1)$ ...(14)

Finally from (6)

$$x = p + \nu = 2\nu\,(e^t - 1) + \nu$$

or $$x = \nu\,(2e^t - 1) \quad ...(15)$$

Substituting these values of x, y, p, q in the equation for $z'(t)$, we have

$$\frac{dz}{dt} = 2\nu\,(e^t - 1)\,(2\nu\,e^t) + \nu\,(e^t + 1)\,(ve^t)$$

or $$\frac{dz}{dt} = 5\nu^2\,e^{2t} - 3\nu^2\,e^t \quad ...(16)$$

on Integration of (16) we have

$$z = \frac{5\nu^2}{2}\,(e^{2t} - 1) - 3\nu^2\,(e^t - 1) \quad ...(17)$$

From (13) and (15)

$$x - 2y = \nu\,(2e^t - 1) - 2\nu\,(e^t - 1)$$

or $$x - 2y = \nu, \quad ...(18)$$

and $$y - x = \nu\,(e^t - 1) - \nu\,(2e^t - 1)$$

$$y - x = -\,\nu\,e^t$$

so using (18) we have by eliminating ν, we get

$$e^t = \frac{y - x}{2y - x} \quad ...(19)$$

Substituting these values of e^t and ν into equation (17) we have

$$z = \frac{5}{2}(x - 2y)^2\left(\left(\frac{y - x}{2y - x}\right)^2 - 1\right) - 3(x - 2y)^2\left(\frac{y - x}{2y - x} - 1\right)$$

$$= \frac{5}{2}(y - x)^2 - \frac{5}{2}(x - 2y)^2 + 3(y - x)(x - 2y) + 3(x - 2y)^2$$

$$= \frac{5}{2}(y - x)^2 + \frac{1}{2}(x - 2y)^2 - 3(y - x)^2 - 3y\,(y - x)$$

$$= -\frac{1}{2}(y^2 - 2yx + x^2) + \frac{1}{2}(x^2 - 4xy + 4y^2) - 3y^2 + 3xy$$

$$= -\frac{3}{2}y^2 + 2xy = \frac{1}{2}y\,(4x - 3y)$$

or $$z = \frac{y}{2}(4x - 3y) \quad ...(20)$$

is the solution of the equation (1).

Notes

Self Assessment

3. Find the characteristics of the equation

   $pq = z,$

   and determine the integral surface which passes through the parabola $x = 0$, $y^2 = z$.

15.3 Summary

- Cauchy's problem is the question to be asked, if the given differential equation solution exists.
- The conditions are given in which the solution does exist.
- Cauchy's characteristics equations are set up which help in the solution of the partial differential equations.

15.4 Keywords

Depending upon the values of the parameters the solution of a particular ***partial differential equation*** represents various integral surfaces as well as certain curves.

The characteristic method of Cauchy helps in finding a particular solution passing through certain curves or surfaces.

15.5 Review Questions

1. Eliminate b and c from the equation

   $z = b^2(x + y) + b\,xy + c$

2. Eliminate the function ϕ from the equation

   $\phi(x^2 - y^2, x^2 - z^2) = 0$

Answers: Self Assessment

1. $pq = z$
2. $yp - xq + x^2 - y^2 = 0$
3. $x = 2\nu\,(e^t - 1)$, $y = 1/2\,\nu\,(e^t + 1)$, $z = \nu^2 e^{2t}$, $16z = (4y + x)^2$

15.6 Further Readings

Books

Piaggio H.T.H., Differential Equations

Sneddon L.N., Elements of Partial Differential Equations

Unit 16: Classifications of Integrals of the First Order Partial Differential Equations

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know various methods of finding the solution of the first order partial differential equation.
- See that the solution may consists of two arbitrary constants and this type of solution is called complete integral of the solution.
- Come to know that there are solutions which can be written in terms of an arbitrary function. Such a solution is called a general integral. There is a typical solution also that is called a singular solution.

Introduction

The types of integrals can be complete integrals that depend upon two arbitrary constants.

There is a general integral of the solution of partial differential equation that is expressed in terms of one arbitrary constant or function.

Then there is a singular integral which is an other solution of the partial differential equation.

16.1 Geometrical Theorems

In this unit we shall be concerned mainly with equations of geometrical interest and seek the solutions of various partial differential equations as integrals of various forms, general integrals, complete integrals, particular integrals and singular integrals and their geometrical interpretation.

For this purpose it is advisable to revise the following two geometrical theorems.

Theorem 1: The direction-cosines of the normal to the surface $f(x, y, z) = 0$ at the point (x, y, z) are in the ratio

Notes

$$\frac{\partial f}{\partial x}:\frac{\partial f}{\partial y}:\frac{\partial f}{\partial z}$$

Also $$\frac{\partial f}{\partial x}\bigg|\frac{\partial f}{\partial z}=\frac{\partial z}{\partial x}=p$$

and $$-\frac{\partial f}{\partial y}\bigg|\frac{\partial f}{\partial z}=\frac{\partial z}{\partial y}=q$$

The symbols p, q are to be understood as here defined.

Theorem 2: The envelope of the system of surfaces

$$f(x, y, z, a, b) = 0,$$

where a, b are variable parameters, is found by eliminating a and b by using the given relation and $\frac{\partial f}{\partial a}=0, \frac{\partial f}{\partial b}=0.$

Example 1: Let us consider the equation

$$x^2 + y^2 + (z - c)^2 = a^2 \quad ...(1)$$

which contains two constants a and c. This equation represents the set of all spheres whose centers lie along the z-axis. If we differentiate the equation (1) with respect to x, we obtain the relation

$$2x+2(z-c)\frac{\partial z}{\partial x} = 0 \quad ...(2)$$

And if we differentiate the equation (1) with respect to y. We obtain the relation

$$2y+2(z-c)\frac{\partial z}{\partial y} = 0 \quad ...(3)$$

Eliminating (c) from equations (2) and (3) we have

$$2x\frac{\partial z}{\partial y}-2y\frac{\partial z}{\partial x} = 0$$

or $$xq - yp = 0 \quad ...(4)$$

where $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$. The equation (4) is a first order partial differential equation and is linear.

We can show that there are other geometrical entities other than the set of all spheres with centers along the z-axis which can be described by the equation (4).

Let us consider the equation

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha \quad ...(5)$$

in which the constants c and α are arbitrary. Differentiating (5) with respect to x and y, we get the relations

$$p(z - c)\tan^2 \alpha = x, \; q(z - c)\tan^2 \alpha = y \quad ...(6)$$

Notes

Eliminating the constant c and α we get the equation (4).

We see that the common things among these two surfaces of revolution (1) and (5) is that they have the line OZ as the axis of symmetry. So if we simply take the equation

$$z = f(x^2 + y^2) \qquad \text{...(7)}$$

where the function f is arbitrary and again differentiate (7) with respect to x and y separately we get

$$\frac{\partial z}{\partial x} = p = 2xf', \frac{\partial z}{\partial y} = 2yf' \qquad \text{...(8)}$$

where $f' = \frac{\partial f}{\partial u}$ and $u = x^2 + y^2$. So after eliminating f from (8)

we get $$py - qx = 0 \qquad \text{...(4)}$$

Thus we see that the function z defined by each of the equations (1), (5) and (7), is in some sense a solution of the equation.

We now interpret the argument slightly. The relation (1) and (5) are both of the type

$$F(x, y, z, a, b) = 0 \qquad \text{...(9)}$$

where a and b denote arbitrary constants. If we differentiate this equation with respect to x and y respectively. We obtain the relations

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \qquad \text{...(10)}$$

The set of equations (9) and (10) constitute three equations involving two arbitrary constants a and b. It will be possible to eliminate a and b from these equations to obtain a relation of the kind

$$f(x, y, z, p, q) = 0 \qquad \text{...(11)}$$

showing that the system of surfaces gives rise to a partial differential equation (11) of the first order.

The obvious generalization of the equation (7) is a relation between x, y, z of the type

$$F(u, v) = 0 \qquad \text{...(12)}$$

where u and v are functions of x, y and z and F is an arbitrary function of u and v. If we differentiate (12) with respect to x and y respectively, we obtain the relations

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial F}{dv}\left(\frac{\partial \text{v}}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial F}{dv}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

and if we eliminate $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{dv}$ from these equations, we obtain the equation

$$\frac{\partial F}{\partial u}\left\{\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial y}\right)\right\} = 0$$

Notes

or $$p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial z}\frac{\partial u}{\partial y}\right)+q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right)+\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}=0$$

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \qquad ...(13)$$

which is partial differential equation of the type (11). It should be noted that equation (13) is a linear partial differential equation i.e. the powers of p and q are both unity. Whereas the partial differentiation equation (11) need not be linear. To see that consider the equation

$$(x-a)^2+(y-b)^2+z^2 = 1 \qquad ...(14)$$

Differentiating (14) with respect to x and y separately, we have

$$2(x-a)+2zp=0, \quad 2(y-b)+2zq=0$$

Substituting the values of $(x-a)$ and $(y-b)$ in equation (14) we have

$$z^2p^2+z^2q^2+z^2=1 \text{ or } z^2(p^2+q^2+1)=1. \qquad ...(15)$$

So powers of p and q are not one.

Example 2: Eliminate the constants a and b from

$$2z = (ax+y)^2+b \qquad ...(1)$$

Solution: Differentiate with respect to x we have

$$2\frac{\partial z}{\partial x} = 2p=2a(ax+y)$$

Differentiating (1) with respect to y we have

$$2\frac{\partial z}{\partial y} = 2q=2(ax+y)$$

or $$p = a(ax+y) \qquad ...(2)$$

$$q = (ax+y) \qquad ...(3)$$

$$px+qy = ax(ax+y)+y(ax+y)$$

$$= (ax+y)^2=q^2$$

or $$px+qy = q^2$$

is the answer.

Example 3: Eliminate the arbitrary function f from the equation

$$z = f\left(\frac{xy}{z}\right) \qquad ...(4)$$

Differentiating with respect to x and y respectively we have

$$\frac{\partial z}{\partial x}=p = f'\left(\frac{y}{z}-\frac{xy}{z^2}p\right) \qquad ...(15)$$

Notes

and $$q = \frac{dz}{dy} = f'\left(\frac{x}{z} - \frac{xy}{z^2}q\right) \quad ...(16)$$

so $$\frac{p}{q} = \frac{yz - xyp}{xz - xyq}$$

or $$pxz - xypq = yzq - xypq$$

or $$z(px - qy) = 0$$

is the answer.

Self Assessment

1. Eliminate the constants a and b from the equation

   $ax^2 + by^2 + z^2 = 1$

2. Eliminate the arbitrary function from the equation

   $F(x^2 + y^2 + z^2, z^2 - 2xy) = 0$

16.2 Classes of Integrals of a Partial Differential Equation

Let us consider the partial differential equation of the form

$$F(x, y, z, p, q) = 0 \quad ...(1)$$

in which the function F is not necessarily linear in p and q. We saw earlier that the solution involving two parameter system of equation can be of the form

$$f(x, y, z, a, b) = 0 \quad ...(2)$$

Any envelope of the system (2) must also be a solution of the differential equation (1). In this way we are led to three classes of integrals of a partial differential equation of type (1):

(a) Two parameter systems of surfaces $f(x, y, z, a, b) = 0$.

Such an integral is called **complete integral**.

(b) If we take any one parameter subsystem

$$f(x, y, z, a, \phi(a)) = 0$$

of the system (2) and form its envelope, we obtain a solution of equation (1). When the function $\phi(a)$ which defines the subsystem is arbitrary, the solution obtained is called general integral of (1) corresponding to the complete integral (2).

When a definite function $\phi(a)$ is used we obtain a particular case of the general integral.

(c) If the envelope of the two parameter system (2) exists, it is also a solution of the equation (1), it is called the singular integral of the equation.

Example 1: Show that

$$z = ax + by + a^2 + b^2 \quad ...(1)$$

is the complete integral of partial differential equation

$$z = px + qy + p^2 + q^2 \quad ...(2)$$

Notes

Differentiate (1) with respect to x we have

$$p = a \qquad ...(3)$$

Also differentiate (1) with respect to y we have

$$\frac{\partial z}{\partial y} = q = b \qquad ...(4)$$

Substituting the values of a and b from (3) and (4) into the equation (1) we have

$$z = px + qy + p^2 + q^2 \qquad ...(2)$$

so equation (1) having two arbitrary constants a and b is the complete integral of partial differential equation (2).

Differentiating (1) with respect to a and b respectively,

we get

and $$\left.\begin{array}{l} 0 = x + 2a \\ 0 = y + 2b \end{array}\right\} \qquad ...(5)$$

Substituting the values of a and b in (1) we have

$$Z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$4Z = -(x^2 + y^2) \qquad ...(6)$$

To see whether equation (6) satisfies (2) we have

$$\left.\begin{array}{l} 4p = -2x \\ 4q = -2y \end{array}\right\}$$

Substituting in R.H.S. of (2) we have

$$-\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4} = -\frac{(x^2 + y^2)}{4} = z = \text{L.H.S.}$$

So equation (6) satisfies equation (2).

Equation (6) represents a paraboloid of revolution, the envelops of all the planes represented by the complete integral. Equation (6) represents singular integral.

Example 2: Show that

$$Z = be^{ax + a^2 y} \qquad ...(1)$$

is the complete integral of partial differential equation

$$p^2 = zy \qquad ...(2)$$

Differentiating (1) w.r.t. x, y respectively

$$\frac{\partial z}{\partial x} = p = bae^{ax+a^2 y} \qquad ...(3)$$

$$\frac{\partial z}{\partial y} = q = ba^2 e^{ax+a^2 y} \qquad ...(4)$$

$$p^2 = b^2a^2e^{2ax+2a^2y}$$

$$qz = b^2a^2e^{2ax+2a^2y}$$

Thus $$p^2 = qz \quad ...(2)$$

So (1) is the complete integral of partial differential equation (2) since it has two arbitrary constants.

Differentiating (2) w.r.t. p and q, we get

$$2p = 0 \quad ...(5)$$

and $$z = 0 \quad ...(6)$$

Eliminating p, q from (2), (5) and (6) we have

$$z = 0$$

It satisfies equation (2). So it is a singular integral. Also if we put $b = 0$ in (1) we get

$$z = 0$$

So $z = 0$ is both a singular as well as a particular solution.

Self Assessment

3. Show that $F = ax + by + a^2 + ab + b^2 - z = 0$

   is the complete integral of the partial differential equation

   $Z = px + qy + p^2 + pq + q^2$

   and find the singular integral

4. Show that

   $$F = ax + by + \frac{1}{2}a^2b^2 - Z = 0$$

   is the complete integral of the partial differential equation

   $$Z = px + qy + \frac{1}{2}p^2q^2$$

   Find the singular integral of this partial differential equation.

16.3 General Integrals

Consider the partial differential equation of the first order

$$F(x, y, z, p, q) = 0 \quad ...(1)$$

If on integration we get a solution of the form

$$f(u, v) = 0 \quad ...(2)$$

where u and v are functions of x, y, z we call it a general integral. This will be illustrated by means of the following example.

Notes

Example: Find the partial differential equation for the general integral

$$f(x^2 + y^2, z) = 0 \quad ...(3)$$

Let $u = x^2 + y^2 = \text{constant}$

$v = z = \text{constant}$

Now differentiating (3) with respect to x

We have
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial x}$$

$$= \frac{\partial f}{\partial u}.(2x) + \frac{\partial f}{\partial v}\left(\frac{\partial z}{\partial x}\right)$$

or
$$\frac{\partial f}{\partial x} = 2x\frac{\partial f}{\partial u} + p\frac{\partial f}{\partial v} = 0 \qquad \text{(where } p = \frac{\partial z}{\partial x}\text{)} \quad ...(4)$$

Again differentiating (3) with respect to y, we have

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial y} = 0$$

or
$$\frac{\partial f}{\partial y} = 2y\frac{\partial f}{\partial u} + q\frac{\partial f}{\partial v} = 0 \qquad \text{(where } q = \frac{\partial z}{\partial y}\text{)} \quad ...(5)$$

To solve (4) and (5) we get a condition on the coefficients of the partial derivatives $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$, as

$$2xq - 2yp = 0$$

or
$$xq - yp = 0 \quad ...(6)$$

which is the required partial differential equation.

Now from (3) we can write the

$$z = \alpha(x^2 + y^2 + \beta) \quad ...(7)$$

We now show that (7) is also the solution of (3). To show this let us eliminate α and β from (7). Now

$$\frac{\partial z}{\partial x} = p = 2\alpha x$$

$$\frac{\partial z}{\partial y} = q = 2\alpha y$$

$\therefore$
$$\frac{p}{q} = \frac{x}{y}$$

or
$$xq - yp = 0$$

The solution (7) of (6) has two unknown constants and so (7) is the complete solution of the equation (6).

Equation (7) denotes the surfaces all of whose normals intersect the axis of z.

To find singular solution let us put $\beta = \alpha^2$ in equation (7) and put

$$Z = a(x^2 + y^2) + \alpha^2 \quad ...(8)$$

To find α differentiate (8) with respect to α, i.e.

$$0 = (x^2 + y^2) + 2\alpha$$

or $$\alpha = -\frac{(x^2+y^2)}{2} \quad ...(9)$$

Eliminating α from (8) we have

$$4Z = -(x^2 + y^2)^2 \quad ...(10)$$

Self Assessment

5. Eliminate the arbitrary function ϕ from the equation

$$\phi\left(\frac{y}{2},(x^2+y^2+z^2)/z\right)=0$$

16.4 Singular Integrals

The complete integral of a partial differential equation represents a family of surfaces. If these surfaces have an envelope, its equation is called a singular integral. To see that this is really an integral we have merely to notice that at any point of the envelope there is a surface of the family touching it. Therefore the normals to the envelope and this surface coincide, so the values of p and q at any point of the envelope are the same as that of some surface of the family and therefore it satisfies the same equation.

The working rule for finding out the singular integral is to start with the complete integral of the form

$$f(x, y, z, p, q, a, b) = 0 \quad ...(1)$$

Differentiate (1) with respect to a and b i.e.

$$\frac{\partial f}{\partial a} = 0 \quad ...(2)$$

$$\frac{\partial f}{\partial b} = 0 \quad ...(3)$$

and eliminate a, b, from (1), (2) and (3) to get the envelope.

or by eliminating p and q from the differential equation.

$$F(x, y, z, p, q) = 0 \quad ...(4)$$

And two derived equations

$$\frac{\partial F}{\partial p} = 0 \quad ...(5)$$

$$\frac{\partial F}{\partial q} = 0 \quad ...(6)$$

One should test whether the singular integral obtained really satisfies the differential equation.

Example: Verify that

$$Z = ax + by + a - b - ab \quad ...(7)$$

Notes

is a complete integral of the partial differential equation

$$Z = px + qy + p - q - pq \quad ...(8)$$

Also find the singular integral.

Solution: Differentiate (7) with respect to a and b respectively, i.e.,

$$0 = x + 1 - b \quad ...(9)$$

$$0 = y - 1 - a \quad ...(10)$$

So $a = y - 1, b = x + 1$

Substituting values of a and b in (7) we have

$$z = x(y-1) + y(x+1) + y - 1 - x - 1 - (y-1)(x+1)$$

Simplifying, we have

$$z = xy - x + y - 1$$

as singular integral. Differentiating (7) with respect to x and y separately we have $\frac{\partial Z}{\partial x} = p = a, \frac{\partial Z}{\partial y} = q = b,$ substituting in (7)

we have

$$z = px + qy + p - q - pq$$

which is just equation (8). So (7) is the complete integral of (8).

Self Assessment

6. Find the singular integral for the differential equation

   $Z = px + qy + p/q$

16.5 Summary

- The partial differential equation of the first order can be a function of x, y, z and the partial derivatives of z i.e., $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$.
- The differential equation can have a solution depending upon two unknown constants. Such a solution is called complete integral.
- If we substitute some fixed values for the constants we get particular integral.
- On the other hand if we get the solution of the equation in the form

  $$\phi(u, v) = 0$$

  where u, v are known functions of x, y, z then we get a general solution.

16.6 Keyword

By varying the two ***arbitrary constants*** we can get various integrals or solutions of the partial differential equations. It is advisable to visualize geometrically the integral surfaces or integral curves.

Notes

16.7 Review Questions

1. Eliminate the arbitrary constants *a*, *b* from the equation

   $zx = ax + by - a^2b$

2. Show that

   $z^2 = ax^2 + by^2 - 3a^2 + b^2$

   is the complete integral of the equation

   $(z - px - qy)x^3y^2 = q^2zx^3 - 3p^2z^2y^2$

   Find the singular integral.

Answers: Self Assessment

1. $z(px - qy) - z^2 + 1 = 0$

2. $z(q - p) + y - x = 0$

5. $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$

6. $zx = -y$

16.8 Further Readings

Books

Piaggio, H.T.H., Differential Equations

Sneddon, L.N., Elements of Partial Differential Equations

Yosida, K., Lectures in Differential and Integral Equations

Notes

Unit 17: Lagrange's Methods for Solving Partial Differential Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Understand that Lagrange's method involves one dependent variable and two or more independent variables in the differential equation.
- See that in the method the technique involved is similar to that which occurs in total differential equation.
- Know how to study some special methods of solving non-linear partial differential equations.

Introduction

Lagrange's method is quite suitable to linear differential equations involving more than two independent variables.

Four different methods are also listed to deal with special types of differential equations.

17.1 Linear Partial Differential Equations of the First Order

Let $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

Then the linear partial differential equations involving z as dependent and x, y as independent variables are of the form

$$Pp + Qq = R \qquad \text{... (1)}$$

where P, Q and R are given functions of x, y and z and they do not involve p and q. The first systematic theory of equations of this type was given by Lagrange. Equation (1) is frequently referred to as *Lagrange's equation*.

Note: If generalised to n independent variables, obviously the equation is

$$P_1p_2 + P_2p_2 + P_3p_3 + ... + P_np_n = R \qquad ...(2)$$

where $P_1, P_2, ... P_n, R$ are functions of n independent variables $x_1, x_2, ... x_n$ and a dependent variable f; $p_i = \frac{\partial f}{\partial x_i}$, $(i = 1, 2, ... n)$.

It should be noted that the term 'linear' in the section means that p and q (or, in general case p_1, ... p_n) appear to the first degree only, but P, Q and R may be any functions of x, y and z.

17.2 Lagrange's Method of Solutions

The Lagrange's equation is

$$Pp + Qq = R \qquad ...(1)$$

where P, Q, R are functions of x, y, z. Suppose

$$u = f(x, y, z) = a \qquad ...(2)$$

is a relation that satisfies (1). Differentiating (2) with respect to x, y,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0,$$

And
$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0$$

or
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p = 0$$

and
$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q = 0$$

Hence
$$p = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}} \text{ and } q = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}}$$

Substituting these values of p and q in (1) changes it to

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0 \qquad ...(2)$$

Therefore, if $u = a$ be an integral of (1), $u = a$ also satisfies (2). Conversely if $u = a$ be an integral of (2), it is also an integral of (1). This can be seen by dividing by $\frac{\partial u}{\partial z}$ and substituting p and q for the values above. Therefore equation (2) can be taken as equivalent to equation (1).

We have shown in unit (8) that $u = a$ and $v = b$ are independent solution of the system of equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad ...(3)$$

Notes

then $\phi(u, v) = 0$ is a general integral.

Hence we have the following rule:

To obtain an integral of the linear equation of the form (1), find two independent integrals of equation (3). Let they be denoted by $u = a$ and $v = b$, then $\phi(u, v) = 0$, where ϕ is an arbitrary function, is an integral of the partial differential equation. Equations (3) are called subsidiary equations.

The solution may also be written in the form

$$u = f(\nu) \qquad \text{... (4)}$$

where f denotes an arbitrary function of v.

This is known as Lagrange's solution of the linear equation.

The method given above can be extended to the general equation of the form

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \ldots\ldots + P_n\frac{\partial_z}{\partial x_n} = R \qquad \text{... (5)}$$

where $P_1, P_2, \ldots P_n, R$ are functions of $(x_1, x_2, \ldots x_n, z)$. To solve equation (5) we write the subsidiary equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots\ldots = \frac{dx_n}{P_n} \qquad \text{... (6)}$$

and find n independent integrals of this system of these subsidiary equations, in the form

$$u_1 = c_1, u_2 = c_2, u_3 = c_3, \ldots u_n = c_n \qquad \text{... (7)}$$

then the integral of the given equation (5) is

$$\phi\,(u_1, u_2, \ldots u_n) = 0 \qquad \text{... (8)}$$

17.3 Illustrative Examples

Example 1: Solve

$$(mz - ny)\,p + (nx - lz)\,q = ly - mx \qquad \text{... (1)}$$

Solution:

Here $P = mz - ny$

$Q = nx - lz$

$R = ly - mx$

The subsidiary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \qquad \text{... (2)}$$

or $$\frac{\ell\, dx}{\ell(mz - ny)} = \frac{mdy}{m(nx - \ell z)} = \frac{ndz}{n(\ell y - mx)}$$

or $$\frac{\ell dx + mdy + ndz}{\ell mz - \ell ny + mnx - m\ell z + n\ell y - nmx} = \frac{\ell dx + mdy + ndz}{O}$$

Notes

So $$\ell\, dx + m\, dy + n\, dz = 0 \qquad \text{... (3)}$$

On integrating (3) we have

$$\ell x + my + nz = a = u \text{ (say)} \qquad \text{... (4)}$$

Again from (2)

$$\frac{xdx}{x(mz-ny)} = \frac{ydy}{y(nx-\ell z)} = \frac{zdz}{z(\ell y - mx)}$$

or $$\frac{xdx+ydy+zdz}{mxz-nxy+nxy-\ell yz+\ell zy-mxz} = \frac{xdx+ydy+zdz}{O}$$

So $$xdx + ydy + zdz = 0$$

or $$x^2 + y^2 + z^2 = b = \nu \text{ (say)} \qquad \text{... (5)}$$

Hence the integral of (1) is

$$\phi(u, \nu) = 0 \qquad \text{... (6)}$$

Example 2: Solve

$$\frac{p}{x^2} + \frac{q}{y^2} = \frac{1}{zx}$$

Solution:

The subsidiary equations are

$$\frac{dx}{(1/x^2)} = \frac{dy}{(1/y^2)} = \frac{dz}{(1/zx)}$$

or $x^2dx = y^2dy = zxdz$

From the first two equations we have on integration

$$x^3 = y^3 + a$$

or $$x^3 - y^3 = a \text{ (say } u)$$

From the first and third equations

$$x^2dx = xzdz$$

or $$xdx = zdz$$

On integrating it

$$x^2 = z^2 + b$$

or $$x^2 - z^2 = b = \nu \text{ (say } b = \nu)$$

So the solution of the above equation is

$$\phi(u, \nu) = 0$$

$$\phi(x^3 - y^3, x^2 - z^2) = 0$$

Notes

Example 3: Solve: $(z^2 - 2yz - y^2)\, p + (xy + zx)\, q = xy - zx$.

Solution:

The auxiliary equations are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

or $$\frac{xdx}{xz^2 - 2xyz - xy^2} = \frac{ydy}{xy^2 + xyz} = \frac{zdz}{xyz - z^2x}$$

∴ $x\,dx + y\,dy + z\,dz = 0$.

∴ $x^2 + y^2 + z^2 = c_1$.

Also from second and third terms,

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

or $y\,dy - z\,dy - y\,dz - z\,dz = 0$

or $y\,dy - z\,dz - (z\,dy + y\,dz) = 0$

or $y^2/2 - z^2/2 - yz = c_2$.

∴ The general solution is

$\phi\,(x^2 + y^2 + z^2,\, y^2 - z^2 - 2yz) = 0$.

Example 4: Solve: $(y^2 + z^2 - x^2)\, p - 2xyq + 2zx = 0$.

Solution:

The auxiliary equations are

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2zx}.$$

From second and third terms,

$$\frac{dy}{y} = \frac{dz}{z},\ \ i.e., \frac{y}{z} = c_1.$$

Also $$\frac{2x\,dx}{2xy^2 + 2xz^2 - 2x^3} = \frac{2y\,dy}{-4xy^2} = \frac{2z\,dz}{-4xz^2}.$$

∴ $$\frac{2x\,dx + 2y\,dy + 2z\,dz}{-2x(x^2 + y^2 + z^2)} = \frac{dz}{-2zx}.$$

∴ $$\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2} = \frac{dz}{z}.$$

∴ $\log (x^2 + y^2 + z^2) = \log z + \log c_2$

∴ $(x^2 + y^2 + z^2) = c_2 z.$

∴ The solution is

$$x^2 + y^2 + z^2 = z\phi\left(\frac{y}{z}\right).$$

Example 5: Solve: $(y + z)\, p + (z + x)\, q = (x + y)$.

Solution:

The auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}.$$

∴ $$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)}$$

or $$\frac{1}{2}\log(x + y + z) = -\log c_1 (x - y)$$

and $\log (x - y) = \log c_2 (y - z)$

Hence the solution is

$$(x - y)\sqrt{(x + y + z)} = f\left(\frac{x-y}{y-z}\right).$$

Example 6: Solve: $(y^3 x - 2x^4)\, p + (2y^4 - x^3 y) q = 9z(x^3 - y^3)$.

Solution:

The auxiliary equations are

$$\frac{dx}{y^3x-2x^4} = \frac{dy}{2y^4-x^3y} = \frac{dz}{9z\left(x^3-y^3\right)}.$$

∴ $$\frac{dy}{dx} = \frac{2y^4-x^3y}{y^3x-2x^4}.$$

Put $y = \nu x, \dfrac{dy}{dx} = \nu + x\dfrac{dv}{dx}$, $\qquad v + x\dfrac{dv}{dx} = \dfrac{2v^4-v}{v^3-2}.$

∴ $$x\frac{dv}{dx} = \frac{2v^4-v-v^4+2v}{v^3-2}$$

or $$\frac{v^2-2}{v^4+v} = \frac{dx}{x}$$

Notes

or $$\frac{v^3-2}{v(v+1)(v^2-v+1)}dv = \frac{dx}{x}$$

or $$\int\left[-\frac{2}{v}+\frac{1}{v+1}+\frac{2v-1}{v^2-v+1}\right]dv = \log cx$$

or $$\log\frac{(v+1)(v^2-v+1)}{v^2} = \log cx$$

or $$\frac{(y+x)\left(y^2-xy+x^2\right)}{x^3\,\frac{y^2}{x^2}} = cx$$

or $$\frac{x^2y^2}{x^3+y^3} = k.$$

Also $$\frac{dx/x}{y^3-2x^3} = \frac{dy/y}{2y^3-x^3} = \frac{dz}{9z\left(x^3-y^3\right)}.$$

$\therefore$ $$\frac{dx/x+dy/y}{1} = \frac{dz}{-3z}.$$

$\therefore$ $3\log x + 3\log y = -\log cz$

or $x^3y^3 = 1/cz.$

$\therefore$ $$z = \frac{1}{x^3y^3}\phi\left(\frac{x}{y^2}+\frac{y}{x^2}\right).$$

Example 7: Solve: $\dfrac{(y-z)p}{yz}+\dfrac{(z-x)q}{zx} = \dfrac{x-y}{xy}.$

Solution:

$(xy - zx)\,p + (yz - yx)\,q = zx - zy.$

$\therefore$ $$\frac{dx}{y-zx} = \frac{dy}{yz-yx} = \frac{dz}{zx-zy}.$$

$\therefore$ $dx + dy + dz = 0$

or $x + y + z = c_1.$

Also $yz\,dx + zx\,dy + xy\,dz = 0.$

or $$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0.$$

$\therefore$ $\log x + \log y + \log z = \log c_2.$

$\therefore$ $xyz = c_2.$

Notes

∴ The general solution is

$$(x + y + z) = f(xyz)$$

Example 8: Solve: $p \cos(x + y) + q \sin(x + y) = z$.

Solution:

The auxiliary equations are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}.$$

From first two terms,

$$\frac{dy}{dx} = \frac{\sin(x+y)}{\cos(x+y)}.$$

Put $x + y = t$,

$$1 + \frac{dy}{dx} = \frac{dt}{dx},$$

∴ $$\frac{dt}{dx} - 1 = \tan t$$

or $$\frac{dt}{1+\tan t} = dx$$

or $$\frac{\cos t}{\sin t + \cos t} dt = dx$$

or $$\frac{1}{2}\left[\frac{(\cos t + \sin t) + (\cos t - \sin t)}{\sin t + \cos t}\right] dt = dx$$

or $$\frac{1}{2}\int \frac{\cos t + \sin t}{\cos t + \sin t} dt + \frac{1}{2}\int \frac{\cos t - \sin t}{\sin t + \cos t} dt = x + c_1$$

or $$t/2 + \frac{1}{2}\log(\sin t + \cos t) = x + c_1$$

or $(x + y) + \log[\sin(x + y) + \cos(x + y)] = 2x + \log k_1$.

∴ $[\sin(x + y) + \cos(x + y)] = ae^{x - y}$

Again $$\frac{dx + dy}{\sin(x+y) + \cos(x+y)} = \frac{dz}{z}.$$

or $$\frac{dt}{\sin t + \cos t} = \frac{dz}{z}.$$

or $$\frac{dt}{\sqrt{2}\sin\left(\frac{3\pi}{4} - t\right)} = \frac{dz}{z}$$

Notes

or $$-\log \tan\left(\frac{3\pi}{8}-\frac{t}{2}\right)=\sqrt{2}\log c_2 z.$$

$\therefore$ $$z^{\sqrt{2}}\tan\left(\frac{3\pi}{8}-\frac{x+y}{2}\right)=b.$$

Hence the general solution is

$$[\sin(x+y)+\cos(x+y)]e^{x-y}=\phi\left[z^{\sqrt{2}}\tan\left(\frac{3\pi}{8}-\frac{x+y}{2}\right)\right]$$

Example 9: Solve:

$$(t+y+z)\frac{\partial t}{\partial x}+(t+z+x)\frac{\partial t}{\partial y}+(t+x+y)\frac{\partial t}{\partial z}=x+y+z.$$

Solution:

The auxiliary equations are

$$\frac{dx}{t+y+z}=\frac{dy}{t+z+x}=\frac{dz}{t+x+y}=\frac{dt}{x+y+z}$$

or $$\frac{dx+dy+dz+dt}{3(x+y+z+t)}=\frac{dx-dt}{-(x-t)}=\frac{(dt-dt)}{-(y-t)}=\frac{dz-dt}{-(z-t)}$$

$\therefore$ $\log (x+y+z+t)^{1/3} = -\log c_1 (x-t)$

$\log (x+y+z+t)^{1/3} = -\log c_2 (y-t)$

and $\log (x+y+z+t)^{1/3} = -\log c_3 (z-t)$

Hence the solution is

$$\phi\,[x+y+z+t]^{1/3}(x-t),\ (x+y+z+t)^{1/3}(y-t),\ (x+y+z+t)^{1/3}(z-t)] = 0$$

Example 10: Solve:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=az+\frac{xy}{t}.$$

Solution:

The auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{t}=\frac{dz}{az+\frac{xy}{t}}.$$

From (1) and (2),

$\log c_1 x = \log y$, *i.e.*, $y = c_1 x$.

From (1) and (3), $t = c_2 x$

Now from (1) and (4),

$$\frac{dx}{x} = \frac{dz}{az + \frac{x.c_1 x}{c_2 x}} = \frac{dz}{az + \frac{c_1}{c_2} x}$$

or $$\frac{az + \frac{c_1}{c_2} x}{x} = \frac{dz}{dx} \quad or \quad \frac{dz}{dx} = \frac{az}{x} + \frac{c_1}{c_2}$$

which is linear in z.

$\therefore$ I.F. = $\exp.\left(-\int \frac{a}{x} dx\right) = \exp.(-a \log x) = \frac{1}{x^a}.$

$\therefore$ The solution is

$$z \times \frac{1}{x^a} = \frac{c_1}{c_2} \int \frac{dx}{x^a} = \frac{c_1}{c_2} \frac{x^{1-a}}{(1-a)} + c_3$$

or $$\frac{z}{x^a} = \frac{y}{t} \frac{x^{1-a}}{(1-a)} + c_3 \text{ since } \frac{c_1}{c_2} = \frac{y}{t}$$

Thus the solution is

$$\frac{z}{x^a} = \frac{x^{1-a}}{(1-a)} \times \frac{y}{t} = c_3 = \phi\left(\frac{y}{t}, \frac{t}{x}\right)$$

Self Assessment

1. Solve

   $x(y - z)p + (y)(z - x)q = z(x - y)$

2. $x^2 p + y^2 q = z^2$
3. $p + q = z/a$
4. $zp - zq = z^2 + (x + y)^2$
5. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = xyz$
6. $\tan x\, p + \tan y\, q = \tan z$

17.4 Some Special Types of Equations

We have so far studied the method of solving the equations of the type

Pp + Qq = R.

Now, before we take up the general method of Charpit to solve the partial differential equations of the first order but of any degree, we will deal with some special types of equations which can be solved by methods other than the general method. We give here four simple standard forms for which "complete Integral" can be obtained.

Standard I

In this form of the equation only p and q are present. The partial differential equation will be of the form

$$f(p, q) = 0 \qquad \text{... (1)}$$

in which x, y, z do not appear. The complete integral is

$$z = ax + by + c \qquad \text{... (2)}$$

where a and b are connected by the relation

$$f(a, b) = 0 \qquad \text{... (3)}$$

Since $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$, which on substitution becomes the given equation (1).

To find the general solution, let from (3) put $b = \phi(a)$ and replacing c by $\Psi(a)$, we have

$$z = ax + \phi(a)y + \Psi(a) \qquad \text{... (4)}$$

Differentiating (4) with respect to a,

$$0 = x + y\,\phi'(a) + \Psi'(a) \qquad \text{... (5)}$$

The general solution is obtained by eliminating a between (4) and (5).

Suppose from (2), $b = \phi(a)$ and replacing c by $\Psi(a)$ the general solution is obtained by eliminating 'a' between the following equations:

$$z = ax + \phi(a)\,y + \Psi(a). \qquad \text{... (6)}$$

Differentiating (3) with respect to a,

$$0 = x + y\phi'(a) + \Psi'(a) \qquad \text{... (7)}$$

Now to find the singular integral, differentiate

$$z = ax + \phi(a)\,y + c$$

with respect to a and c,

$$0 = x + y\phi'(a)$$

and

$$0 = 1.$$

Now the last equation shows that there is no singular integral.

Illustrative Examples

Example 1: Solve: $q = \exp.(-p/a)$.

Solution:

The complete integral is

$$z = \alpha x + \beta y + \gamma$$

where $\beta = \exp.(-\alpha/a)$

i.e., the complete integral is

$$z = \alpha x + \{\exp.(-\alpha/a\}\,y + \gamma$$

The general integral is obtained by eliminating α between **Notes**

$z = \alpha x + \{\exp. (-\alpha/a)\}\, y + f(\alpha)$

and $0 = x - \{\exp. (-\alpha/a)\} \dfrac{y}{a} + f(a)$

Example 2: Find the complete integral of

$x^2p^2 + y^2q^2 = z^2$

Solution:

Now put $z = e^Z$, $x = e^X$, $y = e^Y$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial x} = \frac{1}{x}.\frac{\partial z}{\partial X}$$

$\therefore$ $xp = \dfrac{\partial z}{\partial X}$.

and now $\dfrac{\partial Z}{\partial X} = \dfrac{\partial Z}{\partial z}.\dfrac{\partial z}{\partial X} = \dfrac{1}{z}.\dfrac{\partial z}{\partial X}$

$\therefore$ $xp = z\dfrac{\partial Z}{\partial X}$.

Similarly,

$$yq = z\frac{\partial Z}{\partial Y}$$

$\therefore$ The equation becomes

$$z^2\left(\frac{\partial Z}{\partial X}\right)^2 + z^2\left(\frac{\partial Z}{\partial Y}\right)^2 = z^2$$

or $\left(\dfrac{\partial Z}{\partial X}\right)^2 + \left(\dfrac{\partial Z}{\partial Y}\right)^2 = 1.$

The complete integral is

$Z = aX + bY + c$

where $a^2 + b^2 = 1$

i.e., $\log z = a \log x - \sqrt{(1-a^2)} \log y + c$.

Example 3: $p^m \sec^{2m} x + z^l q^n \operatorname{cosec}^{2n} y = z^{lm/(m-n)}$.

Solution:

Put $\cos^2 x\, dx = dX$, $\sin^2 y\, dy = dY$ and $z^{-1/(m-n)}\, dz = dZ$.

Write the given equation as

$$\left(\frac{z^{-1/(m-n)}}{\cos^2 x}.\frac{dz}{dx}\right)^m + \left(\frac{z^{-1/(m-n)}}{\sin x}.\frac{\partial z}{\partial y}\right)^n = 1$$

Notes

which on substitution becomes

$$\left(\frac{\partial Z}{\partial X}\right)^m + \left(\frac{\partial Z}{\partial X}\right)^n = 1.$$

$\therefore$ The complete integral is

Z = aX + bY + c

where

$a^m + b^n = 1$

and $$Z = \frac{m-n}{m-n-a}.z^{(m-n-l)/(m-n)}$$

$$X = \frac{1}{2}(x + \frac{1}{2}\sin 2x).$$

$$Y = \frac{1}{2}(y - \frac{1}{2}\sin 2y).$$

Example 4: Solve: $(y - x)(qy - px) = (p - q)^2$.

Solution:

Put $x + y = X$, $xy = Y$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial x}$$

$$= \frac{\partial z}{\partial X}.1 + \frac{\partial z}{\partial Y}.y;$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y}$$

$$= \frac{\partial z}{\partial X}.1 + \frac{\partial z}{\partial Y}.x.$$

The given equation by this substitution becomes

$$(y-x)\left[\left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right)y - \left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right)x\right]$$

$$= \left[\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y} - \frac{\partial z}{\partial X} - x\frac{\partial z}{\partial Y}\right]^2.$$

$\therefore$ $$(y-x)^2\left(\frac{\partial z}{\partial X}\right)^2 = (y-x)^2\left(\frac{\partial z}{\partial Y}\right)^2$$

or $$\frac{\partial z}{\partial X} = \left(\frac{\partial z}{\partial Y}\right)^2$$

which is of the form F $(p, q) = 0$, [Standard I]

$\therefore$ Solution is $z + aX + bY + c$

where $a = b^2$.

$\therefore$ $z = b^2 (x + y) + bxy + c$.

Self Assessment

Find the complete integrals of:

7. $p^2 + q^2 = m^2$.
8. $pq = k$.
9. $p^2 + q^2 = npq$.
10. $\sqrt{p} + \sqrt{q} = 1$.

Standard II

The equation

$$z = px + qy + f(p, q),$$

which is analogous to Clairaut's form, has for its complete integral

$$z = ax + by + f(a, b) \quad \text{... (1)}$$

for $\frac{\partial z}{\partial x} = p = a$ and $\frac{\partial z}{\partial y} = q = b$

In order to obtain the general integral put $b = \phi(a)$.

$\therefore$ $z = ax + y\phi(a) + f\{a, \phi(a)\}$.

Differentiating with respect to a,

$0 = x + y\phi'(a) + f'(a)$

and eliminate a between these equations.

In order to obtain the singular integral, differentiate (1) with respect to a and b, *i.e.*,

$$0 = x + \partial f/\partial a, \quad \text{... (2)}$$

$$0 = y + \partial f/\partial b \quad \text{... (3)}$$

and eliminate a and b between the equations (1), (2) and (3).

Illustrative Examples

Example 1: Solve $z = px + qy - 2\sqrt{(pq)}$.

Solution:

The complete integral is

$$z = ax + by - 2\sqrt{(ab)} \quad \text{... (1)}$$

Notes

Differentiating with respect to a and b,

$$0 = x - 2\sqrt{b}.\frac{1}{2\sqrt{a}},$$

$$0 = y - \frac{2\sqrt{a}}{2\sqrt{b}},$$

$$\frac{\sqrt{b}}{\sqrt{a}} = x \text{ and } \sqrt{\left(\frac{a}{b}\right)} = y$$

Eliminating a and b, the singular integral is

$xy = 1.$

Example 2: Solve $z - px - qy = c\sqrt{(1 + p^2 + q^2)}$.

Solution:

The complete integral is

$$z = ax + by + c\sqrt{(1 + a^2 + b^2)} \quad \text{... (1)}$$

Differentiating with respect to a and b,

$$0 = x + \frac{ca}{\sqrt{(1+a^2+b^2)}}, \quad \text{... (2)}$$

$$0 = y + \frac{bc}{\sqrt{(1+a^2+b^2)}}. \quad \text{... (3)}$$

$$\therefore \quad x^2 + y^2 = \frac{c^2(a^2+b^2)}{1+a^2+b^2}.$$

$$\therefore \quad c^2 - x^2 - y^2 = c^2 - \frac{c^2(a^2+b^2)}{1+a^2+b^2}$$

$$= \frac{c^2}{1+a^2+b^2}.$$

$$\therefore \quad 1 + a^2 + b^2 = \frac{c^2}{c^2-x^2-y^2}.$$

Putting in (2), (3),

$$\therefore \quad a = -\frac{x\sqrt{(1+a^2+b^2)}}{c} = \frac{-x}{\sqrt{(c^2-x^2-y^2)}}$$

and

$$b = \frac{-y}{\sqrt{(c^2-x^2-y^2)}}.$$

Notes

Put the values of a and b, the singular integral is

$$z = -\frac{x^2}{\sqrt{(c^2 - x^2 - y^2)}} - \frac{y^2}{\sqrt{(c^2 - x^2 - y^2)}} + \frac{c^2}{\sqrt{(c^2 - x^2 - y^2)}},$$

or $$z^2 (c^2 - x^2 - y^2) = (c^2 - x^2 - y^2)^2$$

or $$x^2 + y^2 + z^2 = c^2.$$

Self Assessment

Find a complete integral of following equations:

11. $z = px + qy + pq.$

12. $z = px + qy + p^2 + q^2.$

13. $z = px + qy + \sqrt{(\alpha p^2 + \beta q^2 + \gamma)}.$

Standard III

The equations which do not contain x and y, *i.e.*, which are of the form

$$F(z, p, q) = 0 \qquad \text{... (1)}$$

can be solved in the following way.

Write $x + ay = X$ where 'a' is an arbitrary constant and assume z to be a function of $(x + ay)$ *i.e.* of X alone.

$\therefore \quad z = f(X) \quad$ when $X = (x + ay)$;

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{dz}{dX}\frac{\partial X}{\partial x} = \frac{dz}{dX}.1,$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{dX}.\frac{\partial X}{\partial y} = a.\frac{dz}{dX}.$$

Now the equation (1) becomes

$$F\left(z, \frac{dz}{dX}, a\frac{dx}{dX}\right) = 0$$

which is an ordinary differential equation of the first order and can be integrated. So the complete integral will be known.

The general and singular integrals can be found as in first two cases.

Illustrative Examples

Example 1: Find a complete integral of: $9(p^2z + q^2) = 4$.

Solution:

Put $z = f(x + ay) = f(X)$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{dz}{dX}.\frac{\partial X}{\partial x} = \frac{dz}{dX}$$

Notes

$$q = \frac{\partial z}{\partial y} = \frac{dz}{dX}.\frac{\partial X}{\partial y} = \frac{dz}{dX}a.$$

Therefore the equation becomes

$$9\left[\left(\frac{dz}{dX}\right)^2 z + a^2\left(\frac{dz}{dX}\right)^2\right] = 4$$

$$\left(\frac{dz}{dX}\right)^2 \{9z + 9a^2\} = 4$$

$$\frac{dz}{dX} = \frac{2}{3\sqrt{(z+a^2)}}$$

or $$\int \sqrt{(z+a^2)}dz = \int \frac{2}{3}dY$$

or $$\frac{(z+a^2)^{3/2}}{(3/2)} = \frac{2}{3}X + C$$

or $(z + a^2)^3 = (X + k)^2$

or $(z + a^2)^3 = (x + ay + k)^2$.

Example 2: Find a complete integral of: $p^3 + q^3 - 3pqz = 0$.

Solution:

Put $z = f(x + ay) = f(X)$

$$\left(\frac{dz}{dX}\right)^2 + a^3\left(\frac{dz}{dX}\right)^3 - 3a\frac{dz}{dX}\left(\frac{dz}{dX}\right)z = 0$$

$$\frac{dz}{dX}(1 + a^3) = az$$

or $$\frac{dz}{3az} = \frac{dx}{1+a^3}$$

∴ $$\frac{1}{3a}\log z = \frac{X}{1+a^3} + c$$

or $3a\,(x + ay) + k = (1 + a^3)\log z$.

Example 3: Find a complete integral of: $q^2y^2 = z(z - px)$.

Solution:

Put $dY = \dfrac{dy}{y}$, *i.e.* $y = e^Y$

and $dX = \dfrac{dx}{x}$, *i.e.* $x = e^X$,

The equation becomes

$$\left(\frac{\partial z}{\partial Y}\right)^2 = z\left(z - \frac{\partial z}{\partial X}\right),$$

$$z = f(X + aY) = f(\xi).$$

$$\therefore \quad a^2\left(\frac{dz}{d\xi}\right)^2 = z\left(z - \frac{dz}{d\xi}\right)$$

$$\therefore \quad a^2\left(\frac{dz}{d\xi}\right)^2 + z\frac{dz}{d\xi} + z^2 = 0.$$

$$\therefore \quad \frac{dz}{d\xi} = \frac{-z \pm \sqrt{(z^2 + 4a^2z^2)}}{2a^2}$$

or

$$\frac{dz}{-z[1 \pm \sqrt{(1+4a^2)}]} = \frac{1}{2a^2}d\xi.$$

$$\therefore \quad \log z = \frac{[1 \pm \sqrt{(1+4a^2)} - 1]}{2a^2}\xi + c_1$$

$$\therefore \quad 2a^2 \log z = [\pm \sqrt{(1 + 4a^2)} - 1]\,[X + aY] + k$$

$$= [\pm \sqrt{(1 + 4a^2)} - 1]\,(\log x + a \log y) + k.$$

Example 4: Find complete integral of: $pq = x^m y^n z^l$.

Solution:

Put $\dfrac{x^{m+1}}{m+1} = X, \dfrac{y^{n+1}}{n+1} = Y,$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}, \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y},$$

$$p = \frac{\partial z}{\partial x} = x^m\frac{\partial z}{\partial X}, q = \frac{\partial z}{\partial Y}y^n.$$

$\therefore$ The given equation becomes $\dfrac{\partial z}{\partial X}\cdot\dfrac{\partial z}{\partial Y} = z^l,$

which is of the form $f(p, q, z) = 0$.

Putting $\dfrac{\partial z}{\partial X} = \dfrac{dz}{d\xi}, \dfrac{dz}{dy} = a\dfrac{\partial z}{d\xi},$

$$\frac{dz}{d\xi}a\frac{dz}{d\xi} = z^l;$$

Notes

$\therefore \quad \left(\frac{dz}{d\xi}\right)^2 = \frac{z^l}{a}.$

$\therefore \quad \frac{z^{-(l/2+1)}}{1-(l/2)} = \frac{\xi}{\sqrt{a}} + c,$

$$\frac{1}{2-l} z^{1-(l/2)} = \frac{aY+X}{\sqrt{a}} + c = -\frac{x^{m+1}}{\sqrt{a(m+1)}} + \sqrt{a}\frac{y^{n+1}}{n+1} + c.$$

Example 5: Solve: $z^2 (p^2 + q^2 + 1) = c^2$

Solution:

Put $z\,dz = dZ$ *i.e.* $Z = \frac{z^2}{2}$

$$\frac{\partial Z}{\partial x} = \frac{dZ}{dz} \cdot \frac{\partial Z}{\partial x} = zp = P \text{ (say)}$$

$$\frac{\partial z}{\partial Y} = \frac{dZ}{\partial z} \times \frac{\partial z}{\partial Y} = zq = Q \text{ (say)}$$

$\therefore$ The given equation becomes

$2Z + P^2 + Q^2 = c^2$

now let $Z = f(x + ay) + f(X)$

$$P = \frac{\partial Z}{\partial x} = \frac{dZ}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{dP}{dx}$$

$$Q = \frac{\partial Z}{\partial Y} = \frac{dZ}{\partial X} \cdot \frac{\partial X}{\partial y} = a\frac{dZ}{dX}$$

$\therefore \quad \left(\frac{dZ}{dx}\right)^2 (1 + a^2) = c^2 - 2Z$

or $\quad \frac{dZ\sqrt{(1+a^2)}}{\sqrt{(c^2 - a^2 z)}} = dx$

or $\quad -\sqrt{[(1 + a^2)]}\sqrt{[(c^2 - 2Z)]} = X + c$

or $\quad -\sqrt{(1 + a^2)}\sqrt{(c^2 - z^2)} = (x + ay) + c$

or $\quad (1 + a^2)(c^2 - z^2) = (x + ay + c)^2.$

Self Assessment

Solve

14. $p(1 + q^2) = q(z - a)$
15. $p^2 = z^2(1 - pq)$

Notes

16. $p^2 - q^2 = pz$.
17. $pz = 1 + q^2$
18. $p\,(1 + q) = qz$.

Standard IV

If the equation is of the type

$$f_1\,(x, p) = f_2\,(y, q), \qquad \text{... (1)}$$

write $$f_1\,(x, p) = f_2\,(y, q) = c_1 \qquad \text{... (2)}$$

Solving equations (2) for q and p, we have

$$\partial z/\partial x = p = \Psi_1\,(x, c_1)$$

and $$\partial z/\partial y = q = \Psi_2\,(y, c_1).$$

Now $$dz = p\,dx + q\,dy$$

$$= \Psi_1\,(x, c_1)\,dx + \Psi_2\,(y, c_1)\,dy,$$

$$\therefore \quad z = \int \Psi_1(x, c_1)dx + \int \Psi(y, c_1)dy + b.$$

The general integral may be obtained from the above complete integral and as in Standard I, there is no singular integral.

Illustrative Examples

Example 1: Find complete integral of:

$\sqrt{p} + \sqrt{q} = 2x.$

Solution:

$\sqrt{p} - 2x = -\sqrt{q} = a$ (say),

$p = (2x + a)^2$ and $q = a^2$,

$dz = p\,dx + q\,dy$

$= (2x + a)^2\,dx + a^2\,dy$

$\therefore \quad z = \dfrac{(2x + a)^3}{3.2} + a^2 y + b$

$\therefore$ the complete integral is

$6z - 6b = (2x + a)^3 + 6a^2 y.$

Example 2: Solve: $z^2\,(p^2 + q^2) = x^2 + y^2$.

Solution:

Put $z\,dz = dZ$; *i.e.* $Z = z^2/2$.

Notes

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}.\frac{\partial z}{\partial x} = zp = P \text{ (say)}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}.\frac{\partial z}{\partial y} = zq = Q \text{ (say)}$$

∴ The given equation becomes

$P^2 + Q^2 = x^2 + y^2$.

∴ $P^2 - x^2 = y^2 - Q^2$.

Let $P^2 - x^2 = y^2 - Q^2 = a^2$

or $P = \sqrt{(a^2 + x^2)}$ and $Q = \sqrt{(y^2 - a^2)}$.

∴ $dZ = P\,dx + Q\,dy = \sqrt{(x^2 + a^2)}\,dx + \sqrt{(y^2 - a^2)}\,dy$

$$Z = \frac{x}{2}\sqrt{(x^2+a^2)} + \frac{a^2}{2}\log[x + \sqrt{(x^2+a^2)}] + \frac{y}{2}\sqrt{(y^2-a^2)} - \frac{a^2}{2}\log[y + \sqrt{(y^2-a^2)}] + c.$$

∴ Complete integral is

$z^2 = x\sqrt{(x^2 + a^2)} + a^2 \log[x + \sqrt{(x^2 + a^2)}] + y\sqrt{(y^2 - a^2)} - a^2 \log[y + \sqrt{(y^2 - a^2)}] + k$.

Example 3: Solve: $(x^2 + y^2)(p^2 + q^2) = 1$.

Solution:

Put $x = r\cos\theta$, $y = r\sin\theta$,

i.e. $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\frac{y}{x}$.

∴ $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}.\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}.\frac{\partial z}{\partial \theta},$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}.\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}.\frac{\partial z}{\partial \theta}.$$

On substitution the equation becomes

$$r^2\left[\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2\right] = 1$$

or $$r^2\left[\left(\frac{\partial z}{\partial r}\right)^2 = 1 - \left(\frac{\partial z}{\partial \theta}\right)^2\right]$$

which is of the form $f_1(q, x) = f_2(p, y)$.

Putting

$$r^2\left(\frac{\partial z}{\partial r}\right)^2 = a^2 = 1 - \left(\frac{\partial z}{\partial \theta}\right)^2,$$

Notes

$$\frac{\partial z}{\partial r} = \frac{a}{r}, \frac{\partial z}{\partial \theta} = \sqrt{(1-a^2)}.$$

$z = a \log r + a$ quantity independent of r

and $z = \sqrt{(1 - a^2)}\, \theta + a$ quantity independent of θ.

∴ General solution is

$z = a \log r + \sqrt{(1 - a^2)}\, \theta + c$

$$= a \log (x^2 + y^2) + \sqrt{(1-a^2)} \tan^{-1} \frac{y}{x} + c.$$

Example 4: Solve: $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.

Solution:

Put $(x + y) = X$, $(x - y) = Y$,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} + \frac{\partial z}{\partial Y}.$$

$$q = \frac{\partial z}{\partial y} + \frac{\partial z}{\partial X}.\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X} + \frac{\partial z}{\partial Y}(-1).$$

On substitution the given equation becomes

$$X\left(\frac{\partial z}{\partial X}\right)^2 + Y\left(\frac{\partial z}{\partial Y}\right)^2 = \frac{1}{4}$$

or $$X\left(\frac{\partial z}{\partial X}\right)^2 = \frac{1}{4} - Y\left(\frac{\partial z}{\partial Y}\right)^2,$$

which is of the form $f_1(x, p) = f_2(q, y)$.

Putting $X\left(\frac{\partial z}{\partial X}\right)^2 = a$ and $\frac{1}{4} - Y\left(\frac{\partial z}{\partial Y}\right)^2 = a$, we get

$\partial z/\partial X = \sqrt{(a/X)}$

and $(\partial z/\partial Y) = \sqrt{\left[\left(\frac{1}{4} - a\right)/X\right]}$.

$z = 2\sqrt{(aX)}$ + a quantity independent of x

and $z = 2\sqrt{\left[\left(\frac{1}{4} - a\right)Y\right]}$ + a quantity independent of y.

∴ Complete integral is

$$z = 2\sqrt{(aX)} + 2\sqrt{\left[\left(\frac{1}{4} - a\right)Y\right]} + b$$

Notes

$$= 2\sqrt{\left[a(x+y)\right] + 2 + \sqrt{\left(\frac{1}{4} - a\right)(x-y)}\right]} + b.$$

Example 5: Solve: $z(p^2 - q^2) = x - y$.

Solution:

Putting $Z = \frac{2}{3} z^{3/2}$

$$\frac{\partial Z}{\partial x} = \frac{2}{3} \times \frac{3}{2} z^{1/2} \frac{\partial z}{\partial x}.$$

$$\therefore \qquad z\left(\frac{\partial z}{\partial x}\right)^2 = \left(\frac{\partial Z}{\partial x}\right)^2 = P^2 \quad \text{(say)}$$

Similarly,

$$z\left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial Z}{\partial y}\right)^2 = Q^2 \quad \text{(say)}$$

$\therefore \qquad P^2 - Q^2 = x - y.$

Let $\qquad P - x = Q^2 - y = c.$

$\therefore \qquad P = \sqrt{(c + x)}$ and $Q = \sqrt{(c + y)}$.

$\therefore \qquad dZ = P\,dx + Q\,dy$

$\qquad = \sqrt{(c + x)}\,dx + \sqrt{(c + y)}\,dy.$

$$Z = \frac{(c+x)^{3/2}}{\frac{3}{2}} + \frac{(c+y)^{3/2}}{\frac{3}{2}} + k_1$$

or $\qquad z^{3/2} = (c + x)^{3/2} + (c + y)^{3/2} + k.$

is the required solution.

Self Assessment

Solve the following:

19. $q = 2yp^2$.

20. $x^2p^2 = yq^2$.

17.5 Summary

- Lagrange method is quite famous. It is used also in the theory of total differential equations as well as simultaneous differential equations.
- It can be easily extended to the theory of partial differential equations involving more than two independent variables.

Notes

17.6 Keywords

The geometrical interpretation of the ***Lagrange's equation***

$$Pp + Qq = R$$

where P, Q and R are functions of Z, is that the normal to a certain surface is perpendicular to a line whose direction cosines are in the ratio P : Q : R.

The ***subsidiary equations*** help us in finding the solution of Lagrange's equation. If $u = a$, $v = b$ where u, v are functions of x, y, z and a, b being arbitrary constants but the statement that $\Psi(u, v)$ are solutions of the Lagrange equations.

17.7 Review Questions

1. Solve the following $x(y - z)p + y(z - x)q - (x - y)z = 0$
2. Solve the following $p + q = z/a$
3. Solve the following by Lagrange's method $xzp - yzq = xy$
4. $p^2 + q^2 = x + y$
5. $zp = -x$
6. $p^2q^3 = 1$

Answers: Self Assessment

1. $(x + y + z) = \phi(xyz)$
2. $\left(\frac{1}{x} - \frac{1}{y}\right) = \phi\left(\frac{1}{x} - \frac{1}{z}\right)$
3. $z = e^{y/a} f(x - y)$
4. $\phi[y + x, \log(x^2 + y^2 + 2xy + z^2) - 2x] = 0$
5. $xyz - 3u = \phi\left(\frac{y}{x}, \frac{x}{z}\right)$
6. $\frac{\sin z}{\sin y} = f\left(\frac{\sin x}{\sin y}\right)$
7. $z = ax + \sqrt{(m^2 - a^2)}y + c$
8. $z = ax + \frac{k}{a}y + c$
9. $z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]y + c$
10. $z = ax + \left(1 - \sqrt{a}\right)^2 y + c$
11. $z = ax + by + ab$

Notes

12. $z = ax + by + a^2 + b^2$

13. z = ax + by + $\sqrt{\alpha a^2 + \beta b^2 + Y}$

14. $4c\,(z - a) = (x + cy + b)^2 + 4$

15. $\frac{1}{\sqrt{a}}\log\left[z\sqrt{a} + (1 + az^2)^{1/2}\right] + (1 + az^2)^{1/2} = z + ay + b$

16. $(z - c)\,[z - c\exp\{x + ay/(1 - a^2)\}] = 0$

17. $z^2 \pm \left[z\sqrt{(z^2 - 4a^2)} - 4a^2\log\left[z + (z^2 + 4a^2)^{1/2}\right]\right] = 4x + 4ay + k$

18. $\log\,(az - 1) = x + ay + c$

19. $z = ax + a^2y^2 + b$

20. $(z - a\log x - b)^2 = 4a^2y$

17.8 Further Readings

Books

Piaggio H.T.H., Differential Equations

Sneddon L.N., Elements of Partial Differential Equations

Notes

Unit 18: Charpit's Method for Solving Partial Differential Equations

CONTENTS

Objectives

After studying this unit, you should be able to see that:

- Charpit's method is used to find the general integral of the partial differential equation.
- This method introduces a second partial differential equation of the first order that contains an arbitrary constant.
- With the help of this second equation and the original equation the partial derivatives $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$, can be found.
- After finding these p and q, the solution can be found involving two arbitrary constants.

Introduction

With the help of the second equation and the original equation Charpit's subsidiary equations are setup. Only those equations are to be solved that involve p or q.

Charpit's method helps in finding the general solution of the partial differential equations with two arbitrary constants.

18.1 General Method of Solution

After discussing Lagrange's method and some special methods of solving partial differential equation we now turn to an other general method due to Charpit in dealing with non-linear partial differential equations involving two independent variables x and y. Here again we denote $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$. Let the given equation be of the first order only. So the equation to be sold will be of the form

$$F(x, y, z, p, q) = 0 \quad \ldots (1)$$

Notes

The Charpit method of solving this equation is as follows:

Charpit's Method

Here in addition to equation (1), another equation involving the same variables, is sought i.e.

$$f(x, y, z, p, q) = 0 \qquad \text{... (2)}$$

With the help of equations (2) and (1), we solve for p and q and then substitute p and q in the equation

$$dz = p\,dx + q\,dy \qquad \text{... (3)}$$

Clearly the integral of (3) will satisfy the given equation for the values of p and q derived from it are the same as the values of p and q in (1). Now differentiating (1) and (2) w.r.t. x and y, we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}.\frac{\partial z}{\partial y} + \frac{\partial F}{\partial p}.\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}.\frac{\partial q}{\partial y} = 0$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial y} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial y} = 0$$

Eliminating $\partial p/\partial x$ from the first pair and $\partial q/\partial y$ from the second pair, we have

$$\left(\frac{\partial F}{\partial x}.\frac{\partial f}{\partial p} - \frac{\partial F}{\partial p}.\frac{\partial f}{\partial x}\right) + \frac{\partial z}{\partial x}\left(\frac{\partial F}{\partial z}.\frac{\partial f}{\partial p} - \frac{\partial F}{\partial p}.\frac{\partial f}{\partial z}\right) + \frac{\partial q}{\partial x}\left(\frac{\partial F}{\partial q}.\frac{\partial f}{\partial p} - \frac{\partial F}{\partial p}.\frac{\partial f}{\partial q}\right) = 0 \qquad \text{... (4)}$$

$$\left(\frac{\partial F}{\partial y}.\frac{\partial f}{\partial q} - \frac{\partial F}{\partial q}.\frac{\partial f}{\partial y}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial F}{\partial z}.\frac{\partial f}{\partial q} - \frac{\partial F}{\partial q}.\frac{\partial f}{\partial p}\right) + \frac{\partial p}{\partial y}\left(\frac{\partial F}{\partial p}.\frac{\partial f}{\partial q} - \frac{\partial F}{\partial q}.\frac{\partial f}{\partial p}\right) = 0 \qquad \text{... (5)}$$

Now since $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial p}{\partial y}$

and $\partial z/\partial x = p$, $\partial z/\partial y = q$,

adding (4) and (5) and rearranging,

$$\frac{\partial f}{\partial p}\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial F}{\partial q} + q\frac{\partial F}{\partial z}\right) + \frac{\partial f}{\partial z}\left(-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}\right) + \left(-\frac{\partial F}{\partial p}\right)\frac{\partial f}{\partial x} + \left(-\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial y} = 0 \qquad \text{... (6)}$$

The terms involving $\frac{\partial p}{\partial y}$ and $\frac{\partial q}{\partial x}$ cancel.

Now (6) is a linear equation of the first order, which the function f must satisfy and its integrals are integrals of

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\partial F/\partial p} = \frac{dy}{-\partial F/\partial q} = \frac{df}{0}. \qquad \text{... (7)}$$

Notes

Any of the integrals of (7) will satisfy (6). The simplest relation involving p or q or both should be taken and that will be the required relation.

18.2 Illustrative Examples

Example 1: Solve by Charpit's method $z = pq$.

Solution:

Applying Charpit's method,

$$\frac{dp}{p\,.\,1} = \frac{dp}{q} = \frac{dz}{(-p)(-q)+(-q)(-p)} = \frac{dx}{q} = \frac{dy}{p} = \frac{df}{0}$$

From first two terms,

$$\frac{p}{q} = c.$$

$\therefore$ $z = cq^2$ or $q = \sqrt{(z/c)}$ and $p = \sqrt{(cz)}$.

Now $dz = p\,dx + q\,dy$

$= \sqrt{(cz)}\,dx + \sqrt{(z/c)}\,dy$

$z^{-1/2}dz = \sqrt{c}\,dx + (1/\sqrt{c})\,dy$, on integration, we have

$2z^{1/2} = \sqrt{c}x + (y/\sqrt{c}) + b$

Example 2: Solve by Charpit's method $(p^2 + q^2)\,y = qz$.

Solution:

$$\frac{dp}{0+p(-q)} = \frac{dq}{(p^2+q^2)+q(-q)} = \frac{dz}{-p(2py)-q(2qy-z)} = \frac{dx}{-2py} = \frac{dy}{-2py+z} = \frac{df}{0}$$

From first two terms,

$$\frac{dp}{-qp} = \frac{dq}{p^2}$$

or $p\,dp = -\,q\,dq$ *i.e.* $p^2 + q^2 = c$

$\because$ $q = cy/z$ and $p = \sqrt{(c - c^2y^2/z^2)}$

$\therefore$ $dz = pd\,x + q\,dy$

$= \sqrt{(c - c^2y^2/z^2)}\,dx + cy/z\,dy$

or $z\,dz = (cz^2 - c^2y^2)^{1/2}\,dx + cy\,dy$

or $$\frac{2(z\,dz - cy\,dy)}{\sqrt{(z^2 - cy^2)}} = 2\sqrt{c}\,.\,dx,$$

$\therefore$ $(z^2 - cy^2)^{1/2} = \sqrt{c}.x + b$

Notes

$\therefore$ The complete integral is

$$(z^2 - cy^2) = (\sqrt{c}x + b)^2$$

Example 3: Solve by Charpit's method:

$q = xp + p^2$.

Solution:

Charpit's auxiliary equations are

$$\frac{dp}{p+0} = \frac{dq}{0} = \frac{dz}{-p(x+2p) - q(-1)} = \frac{dx}{-(x+2p)} = \frac{dy}{+1} = \frac{\partial f}{0}$$

i.e. $q = c$ from second term.

$\therefore$ $px + p^2 = c$

$$p = \frac{-x \pm \sqrt{(x^2+4c)}}{2}.$$

$\therefore$ $$dz = \frac{-x \pm \sqrt{(x^2+4c)}}{2}dx + c\,dy.$$

$$z = -\frac{x^2}{4} \pm \left[\frac{1}{2}.\frac{x}{2}\sqrt{(x^2+4c)} + \frac{4c}{4}\log\{x + \sqrt{(x^2+4c)}\}\right] + cy + b.$$

Aliter. Also $\frac{dp}{p} = \frac{dy}{1}$, *i.e.*, $p = ae^y$

$\therefore$ $q = axe^y + a^2e^{2y}$

$\therefore$ $dz = ae^y\,dx + axe^y\,dy + a^2e^{2y}\,dy$.

$\therefore$ $$z = axe^y + \frac{a^2}{2}e^{2y} + b.$$

Example 4: Solve by Charpit's method:

$(p + q)(px + qy) - 1 = 0$.

Solution:

By Charpit's method, auxiliary equations are

$$\frac{dp}{p(p+q)+0} = \frac{dq}{(p+q)q} = \ldots$$

$\therefore$ $$\frac{dp}{p} = \frac{dq}{q} \text{ or } \frac{p}{q} = c$$

$q^2(1 + c)(cx + y) - 1 = 0$

or $$q = \sqrt{\left[\frac{1}{(1+c)(cx+y)}\right]}$$

$\therefore \quad dz = p\,dx + q\,dy$

$$= \frac{c\,dx + dy}{\sqrt{[(1+c)(cx+y)]}}$$

$\therefore \quad z\sqrt{(1 + c)} = 2\,(cx + y)^{1/2} + b.$

Example 5: Solve by Charpit's method:

$$pq = px + qy.$$

Solution:

The auxiliary equations are

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-q)}.$$

From first two ratios,

$$p/q = a \quad i.e., p = aq.$$

Putting the value of p in the given equation,

$$aq^2 = aqx + qy$$

or $\quad q = (y + ax)/a.$

Therefore

$$p = (y + ax).$$

Now $\quad dz = p\,dx + q\,dy$

$$= (y + ax)\,dx + \frac{y + ax}{a}\,dy.$$

$\therefore \quad adz = (y + ax)\,(dy + a\,dx).$

$\therefore \quad az = (y + ax)^2/2 + c.$

Writing c as $f(a)$,

$$az = (y + ax)^2/2 + f(a). \qquad \text{... (1)}$$

Differentiating with respect to a,

$$z = x\,(y + ax) + f'(a). \qquad \text{... (2)}$$

Eliminating a between (1) and (2) the general integral will be obtained.

Example 6: Solve by Charpit's method:

$$2zx - px^2 - 2qxy + pq = 0.$$

Solution:

Applying Charpit's method,

$$\frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dz}{px^2+2xyq} = \frac{dp}{2z-2qy} = \frac{dq}{0} = \frac{df}{0}.$$

$\therefore \quad q = a.$

Notes

Putting this value in the given equation,

$$2zx - px^2 - 2axy + ap = 0.$$

$$\therefore \quad p = 2x(z - ay) / (x^2 - a).$$

Also

$$dz = p\,dx + q\,dy$$

$$= \frac{2x(z-ay)}{(x^2-a)}dx + a\,dy$$

or

$$\frac{dz - a\,dy}{z - ay} = \frac{2x}{x^2 - a}dx$$

or

$$\log(z - ay) = \log c(x^2 - a).$$

$$\therefore \quad (z - ay) = c(x^2 - a).$$

$$\therefore \quad z = ay + c(x^2 - a)$$ is the general solution.

Example 7: Solve by Charpit's method:

$$p^2 + q^2 - 2px - 2qy + 1 = 0.$$

Solution:

Applying Charpit's method,

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}}$$

i.e. $\frac{dp}{-2p} = \frac{dq}{-2q}$ *i.e.* $p = qa.$

Substituting in the given equation,

$$q^2(a^2 + 1) - 2q(ax + y) + 1 = 0.$$

$$\therefore \quad q = \frac{2(ax+y) + \sqrt{[4(ax+y)^2 - 4(a^2+1)]}}{2(a^2+1)}$$ [taking +ve sign with the radical].

$$\therefore \quad q = \frac{(ax+y) + \sqrt{[(ax+y)^2 - (a^2+1)]}}{(a^2+1)}$$

Now $dz = p\,dx + q\,dy$

$$= \frac{1}{(a+1)}(ax + y)(a\,dx + dy) + \frac{1}{(a+1)}\sqrt{[(ax+y)^2 - (a^2+1)]}\,(a\,dx + dy).$$

Now putting $ax + y = t$

$$a\,dx + dy = dt$$

$$\therefore \quad (a^2 + 1)\,dz = dt + \sqrt{[t^2 - (a^2 + 1)]}\,dt.$$

Notes

$$(a^2 + 1)\, z = t + \frac{1}{2}\ \sqrt{[t^2 - (a^2+1)]} - \frac{a^2+1}{2}\ \log\,[t + \sqrt{(t^2 - (a^2 + 1)\}}] + b$$

which is the required solution where $t = ax + y$.

Example 8: Solve by Charpit's method:

$$q = (z + px)^2.$$

Solution:

Applying Charpit's method,

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial y}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}} = \frac{df}{0}\ .$$

We have

$$\frac{dp}{2p(z+px) + p \times 2(z+px)} = \frac{dq}{2q(z+px)} = \frac{dx}{-2x(z+px)}$$

or $\quad \dfrac{dq}{q} = \dfrac{dx}{-x}$

or $\quad qx = a$

Putting this value of q in the given equation $\dfrac{a}{x} = (z+px)^2$

or $\quad p = \dfrac{1}{x}\left[\sqrt{\dfrac{a}{x}} - z\right].$

Now $dz = p\ dx + q\ dy$

$$= \frac{1}{x}\left(\sqrt{\frac{a}{x}} - z\right)dx + \frac{a}{x}dy$$

or $\quad (x\ dz + z\ dx) = \sqrt{\dfrac{a}{x}}dx + a\ dy$

or $\quad zx = 2\sqrt{(ax)} + ay + b.$

Example 9: Solve $p^2 + q^2 - 2px - 2qy + 2xy = 0$.

Solution:

Applying Charpit's method,

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

Notes

or $$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}$$

or $$\frac{dp+dq}{-2(p+q-x-y)} = \frac{dx+dy}{-2(p+q-x-y)}$$

or $p + q = x + y + c$

or $(p - x) + (q - y) = c$ ...(1)

Also the given equation can be written as

$$(p - x)^2 + (q - y)^2 = (x - y)^2 \qquad ...(2)$$

Putting the value of $(p - x)$ from (1) in (2)

$$\{c - (q - y)\}^2 + (q - y)^2 = (x - y)^2$$

or $$2(q - y)^2 - 2c\,(q - y) + c^2 - (x - y)^2 = 0$$

$$\therefore \quad q - y = \frac{2c \pm \sqrt{[4c^2 - 8\{c^2 - (x-y)^2\}]}}{2 \times 2}$$

$$= \frac{c}{2} \pm \frac{1}{2}\sqrt{\{2(x-y)^2 - c^2\}},$$

$$\therefore \quad q = y + \frac{1}{2}[c + \sqrt{\{2(x-y)^2 - c^2\}}]$$

$$\therefore \quad p - x = c - (q - y)$$

$$= c - \frac{1}{2}[c + \sqrt{2\{(x-y)^2 - c^2\}}]$$

$$\therefore \quad p = x + \frac{1}{2}\{c - \sqrt{\{2(x-y)^2 - c^2\}}]$$

Also we know that $dz = p\,dx + q\,dy$.

$$= [x + \frac{1}{2}\{c - \sqrt{\{(2(x-y)^2 - c^2\}}]dx + [y + \frac{1}{2}\{c + \sqrt{\{2(x-y)^2 - c^2]\}}]dy$$

$$= x\,dx + y\,dy + \frac{c\,dx}{2} + \frac{c\,dy}{2} - \frac{1}{2}[\sqrt{2(x-y)^2 - c^2}\}\,\{dx - dy\}$$

$$\therefore \quad Z = \frac{x^2}{2} + \frac{y^2}{2} + \frac{cx}{2} + \frac{cy}{2} - \frac{1}{2}\int (t^2 - c^2)\frac{dt}{\sqrt{2}} \qquad \text{if } 2(x-y)^2 = t^2$$

or $$2Z = x^2 + y^2 + cx + cy - \frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{(t^2c^2)} - \frac{c^2}{2}\log\{t + \sqrt{t} + \sqrt{(t^2 - c^2)}\}\,k\right]$$

Example 10: Solve by Charpit's method:

$pxy + pq + qy = yz$.

Solution: **Notes**

Here $$f = pxy + pq + qy - yz = 0 \qquad \dots (1)$$

Charpit's auxiliary equations are

$$\frac{dp}{py + p(-y)} = \frac{dq}{(px+q) - qy} = \dots$$

or $$dp = 0 \quad \text{or} \quad p = a \qquad \dots (2)$$

From (1) and (2), we get

$$p = a,\ q = \frac{y(z-ax)}{a+y}$$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = a\,dx + \frac{y(z-ax)}{a+y}\,dy$$

or $$\frac{dz - a\,dx}{z - ax} = \frac{y\,dy}{a+y} = \left(1 - \frac{a}{a+y}\right)dy$$

Integrating, $$\log(z - ax) = y - a\log(a+y) + \log b$$

or $$(z - ax)(y + a)^a = be^y.$$

Example 11: Solve by Charpit's method:

$px + qy = z(1 + pq)^{1/2}$.

Solution:

$$f = px + qy - z(1 + pq)^{1/2} = 0 \qquad \dots (1)$$

Charpit's auxiliary equations are

$$\frac{dp}{p - p(1+pq)^{1/2}} = \frac{dq}{q - q(1+pq)^{1/2}} = \dots$$

or $$\frac{dp}{p} = \frac{dq}{q} \quad \therefore p = aq \qquad \dots (2)$$

Putting in (1), we get

$$q(ax + y) = z(1 + aq^2)^{1/2}$$

or $$q^2[(ax + y^2) - az^2] = z^2$$

$$\therefore \quad q = \frac{z}{[(ax+y^2) - az^2)]^{1/2}} \text{ and } p = aq = \frac{az}{[(ax+y^2) - az^2]^{1/2}}$$

putting these values of p and q in $dz = p\,dx + q\,dy$,

$$dz = \frac{z(a\,dx + dy)}{\sqrt{\{(ax+y)^2 - az^2\}}} \text{ or } \frac{dz}{z} = \frac{a\,dx + dy}{\sqrt{\{(ax+y)^2 - az^2\}}}$$

Notes

Let $ax + y = \sqrt{(a)}u \therefore a\, dx + dy = \sqrt{(a)}\,.\,du$

$$\therefore \quad \frac{dz}{z} = \frac{\sqrt{a}\,du}{\sqrt{(au^2 - az^2)}} \text{ or } \frac{du}{dz} = \frac{\sqrt{(u^2 - z^2)}}{z}$$

This is homogeneous equation. To solve it put $u = vz$, then

$$v + z\frac{dv}{dz} = \frac{1}{z}\sqrt{(v^2z^2 - z^2)}$$

$$\text{or} \quad z\frac{dv}{dz} = \{\sqrt{(v^2-1)} - v\}$$

$$\text{or} \quad \frac{dz}{z} = \frac{dv}{\sqrt{(v^2-1)} - v}$$

$$\text{or} \quad \frac{dz}{z} = -\{\sqrt{(v^2-1)} + v\}dv$$

$$\therefore \quad \log z = -\left[\frac{v}{2}\sqrt{[(v^2-1)]} - \frac{1}{2}\log\{v + \sqrt{(v^2-1)}\}\right] - \frac{v^2}{2} + b$$

$$\text{or} \quad \log z + \frac{v^2}{2} + \frac{v}{2}\sqrt{(v^2-1)} - \frac{1}{2}\log\{v + \sqrt{(v^2-1)}\} = b.$$

This is a complete integral, where $v = \dfrac{u}{z} = \dfrac{ax+y}{z\sqrt{a}}$

Example 12: Solve by Charpit's method:

$$(x^2 - y^2)\,pq - xy\,(p^2 - q^2) - 1 = 0. \qquad \text{... (1)}$$

Solution:

$$f = (x^2 - y^2)\,pq - xy\,(p^2 - q^2) - 1 = 0$$

Charpit's auxiliary equations are

$$\frac{dp}{2pqx - z(p^2-q^2)} = \frac{dq}{-2ypq - x(p^2-q^2)} = \frac{dx}{-(x^2-y^2)y + 2pxy} = \frac{dy}{-(x^2-y^2)p - 2pxy} = \dots$$

from which it follows that each fraction

$$= \frac{x\,dp + y\,dq + p\,dx + q\,dy}{0}$$

$$\therefore \quad (x\,dp + p\,dx) + (q\,dy + y\,dq) = 0$$

Integrating, $px + qy = a$

$$\therefore \quad p = \frac{a - qy}{x} \qquad \text{... (2)}$$

Putting this value of p in (1),

$$(x^2-y^2)\left(\frac{a-qy}{x}\right)q - xy\left\{\frac{(a-qy)^2}{x^2}-q^2\right\}-1 = 0$$

or $$\frac{a-qy}{x}\{(x^2-y^2)q-(a-qy)y\}+xyq^2-1 = 0$$

or $$\frac{a-qy}{x}(x^2q-ay)+xyq^2-1 = 0$$

or $$(a-qy)\ (x^2q-ay)+x^2yq^2-x = 0$$

or $$aq\ (x^2+y^2) = a^2y+x.$$

$\therefore$ $$q = \frac{a^2y+x}{a(x^2+y^2)}$$

and $$p=\frac{1}{x}\left[a-\frac{(a^2y+x)y}{a(x^2+y^2)}\right]=\frac{a^2x-y}{a(x^2+y^2)}$$

Putting values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{(a^2x-y)dx+(a^2y+x).dy}{a(x^2+y^2)}$$

or $$dz = a\frac{(x\,dx+y\,dy)}{x^2+y^2}+\frac{x\,dy-y\,dx}{a(x^2+y^2)}$$

Integrating,

$$z=\frac{a}{2}\log(x^2+y^2)+\frac{1}{a}\tan^{-1}\frac{y}{x}+b.$$

Self Assessment

Apply Charpit's method to find the complete integrals of:

1. $pxy + qp + qy = y^2$.
2. $q = 3p^2$.
3. $p - 3x^2 = q^2 - y$.
4. $z = px + qy + p^2 + q^2$.
5. $2\ (pq + py + qx) + x^2 + y^2 = 0$.
6. $Zxp^2 - q = 0$

18.3 Special Types of First Order Equations

In the section we shall consider some special types of first-order partial differential equations whose solutions may be obtained easily by Charpit's Method.

Notes

(a) The equations involving only $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$. In this case the equation to be solved will be of the type

$$f(p, q) = 0 \quad \text{... (1)}$$

From the subsidiary equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial z} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - p\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{df}{0} \quad \text{... (2)}$$

or

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial q}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad \text{... (3)}$$

Now from first equation

$$dp = 0$$

or

$$p = a = \text{constant} \quad \text{... (4)}$$

Substituting this value of p in (1) we have

$$f(a, q) = 0 \quad \text{... (5)}$$

Solving for q from (5) we have

$$q = \phi(a) \quad \text{... (6)}$$

So from the equation

$$dz = p\,dx + q\,dy = a\,dx + \phi(a)\,dy \quad \text{... (7)}$$

We have on integration

$$z = ax + \phi(a)\,y + b$$

which is the general solution.

Example 1: Solve:

$pq = 1$

Solution:

Here again $p = a$ so $q = \frac{1}{a}$

Thus on integrating

$$dz = pdx + q\,dy$$

$$= a\,dx + \frac{1}{a}\,dy$$

$$z = ax + \frac{1}{a}y + b \qquad \text{where } a, b \text{ are constants}$$

Example 2: Solve:

$$p + q = pq \quad \text{... (1)}$$

Solution:

$$p = a \text{ (constant)}$$

so from (1)

$$a + q = aq$$

or $$q = \frac{a}{a-1}$$

Thus $$dz = a\,dx + \frac{a}{a-1}dy$$

given $$z = ax + \frac{a}{a-1}y + b$$

which is the general solution.

(b) Equations not involving independent variables consider the partial equation of the following type

$$f(z, p, q) = 0 \quad \text{... (1)}$$

which does not involve independent variables x, y.

From the subsidiary equations:

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0} \quad \text{... (2)}$$

Here the symbols used are

$$f_x = \frac{\partial f}{\partial x}, f_p = \frac{\partial f}{\partial p}, f_z = \frac{\partial f}{\partial z}, f_q = \frac{\partial f}{\partial q}, f_y = \frac{\partial f}{\partial y} \quad \text{... (3)}$$

So from the first two fractions of (2) we have

$$\frac{dp}{pf_z} = \frac{dq}{qf_z}$$

Integrating, we have

$$p = aq \quad \text{... (4)}$$

From equations (1) and (4) we can find p and q and the complete integral follows from the relation.

$$dz = pdx + q\,dy \quad \text{... (5)}$$

Example 3: Find the complete integral of the equation

$$p^2z^2 + q^2 = 1 \quad \text{... (6)}$$

As (6) does not involve x, y. So from the above method

$$q = pa_1 \quad \text{... (7)}$$

Notes

Substituting in (6) we have

$$p^2z^2 + a^2, q^2 = 1$$

$$p^2 = \frac{1}{z^2 + a_1^2}$$

or $$p = \pm\left(z^2 + a_1^2\right)^{-1/2} \quad \text{... (8)}$$

Substituting in

$$dz = p\,dx + q\,dy$$

$$dz = \pm\frac{dx}{\left(z^2 + a_1^2\right)^{-1/2}} \pm \frac{a_1 dy}{\left(z^2 + a_1^2\right)^{1/2}}$$

we have

$$\left(z^2 + a_1^2\right)^{1/2} dz = dx + a_1\,dy$$

so $$\int\left(z^2 + a_1^2\right)^{1/2} dz = x + a_1 y + a_2 \quad \text{... (9)}$$

It can be shown that

$$\int\left(z^2 + a_1^2\right)^{1/2} dz = \frac{z}{2}\left(z^2 + a_1^2\right)^{1/2} + \frac{a_1^2}{2}\log\left(\frac{z + \sqrt{z_1^2 + a_1^2}}{a_1}\right) \quad \text{... (10)}$$

So the solution is (9) with integral (10).

(c) Separable equation

Let the equation be of the form

$$f(x, p) = g(y, q) \quad \text{... (11)}$$

instead of

$$\mathrm{F}(x, y, z, p, q) = 0 \quad \text{... (12)}$$

Then from the subsidiary equations, we have

$$\frac{dp}{f_x} = \frac{dq}{-g_y} = \frac{dx}{-f_p} = \frac{dy}{+g_q} = \frac{dz}{-(pf_p + qg_q)}$$

So $$\frac{dp}{dx} - \frac{f_x}{f_p} = 0$$

or $$fp\,dp - fx\,dx = 0 \quad \text{... (13)}$$

which can be solved for p. Similarly we can solve for q and the complete integral is obtained.

Example 4: Solve

$$p^2y\,(1 + x^2) = q\,x^2 \quad \text{... (14)}$$

On rearranging we have

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a^2 \text{ (say)} \qquad \text{... (15)}$$

Then $\qquad q = a^2 y$ and $p = \dfrac{ax}{(1+x^2)^{1/2}}$

Thus $\qquad dz = pdx + qdy,$

On integration gives

$$z = \int \frac{ax\,dx}{(1+x^2)^{1/2}} + a^2 . \frac{y^2}{2} + b$$

$$z = a(1+x^2)^{1/2} + \frac{a^2}{2} y^2 + b \qquad \text{... (16)}$$

is the complete integral.

(d) Clairaut's Equations

A first order partial differential equation of the form

$$z = px + qy + f(p, q) \qquad \text{... (17)}$$

is of Clairaut type of the equation. Here

$$F = px + qy + f(p, q) - z = 0 \qquad \text{... (18)}$$

So from the corresponding Charpit's equations, we have

$$\frac{dp}{p-p} = \frac{dq}{q-q} = \frac{dz}{-p(x+f_p)-q(y+f_q)} = \frac{dx}{-x-f_p} = \frac{dy}{-y-f_q}, \qquad \text{... (19)}$$

We have

$p = a$ (say a constant)

$q = b$ (a constant).

So from (17)

$$z = ax + by + f(a, b) \qquad \text{... (20)}$$

is the complete solution of (17).

Example 5: Solve:

$$pqz = p^2(xq + p^2) + q^2(yp + q^2) \qquad \text{... (21)}$$

Solution:

From (21)

$$z = px + qy + \frac{p^3}{q} + \frac{q^3}{p}$$

Notes

So we have Clairaut equation type

$$p = a, q = b,$$

so $$z = ax + by + \frac{a^4 + b^4}{ab} \quad \text{... (22)}$$

is the complete solution.

Self Assessment

7. Find the complete integral of

   $z = px + qy + p^4 + q^4 + p^2q^2$

8. Find the solution of

   $p\,(q^2 + 1) = q\,(z - b)$

18.4 Summary

- Charpit method is quite useful in finding the complete integral of the first order partial differential equation.
- Here we are interested in setting up auxiliary equations with the help of which the values of p and q are obtained.
- Knowledge of the first derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ or p and q respectively help in finding the complete integral involving two arbitrary constants.

18.5 Keywords

Charpit's method helps in finding the complete integral of the first order partial differential equation.

Jacobi's method: It deals with two independent variables and so to solve partial differential equation having more than two independent variables we have to take the help of Jacobi's method.

18.6 Review Questions

Solve by Charpit's method:

1. $p^2x + q^2y = z$
2. $p^2 - y^2q = y^2 - x^2$
3. $yp = 2yx + \log q$
4. $z^2\,(p^2z^2 + q^2) = 1$

Answers: Self Assessment

1. $z = c_1x + c_2e^y\,(y + c_1) - c_1$
2. $z = ax + 3a^2y + b$

Notes

3. $z = x^3 + ax + \frac{2}{3}(y+a)^{3/2} + b$

4. $z = ax + by + a^2 + b^2$

5. $2z = ax - x^2 + ay - y^2 + \frac{1}{2}(x-y)\sqrt{\left[2(x-y)^2 + a^2\right]}$

6. $z^2 = 2ax + a^2y^2 + b$

7. $z = ax + by + a^4 + b^4 + a^2b^2$

8. $2\sqrt{\left[a(z-b-a)\right]} = ax + y + c$

18.7 Further Readings

Books

Piaggio H.T.H., Differential Equations

Sneddon L.N., Elements of Partial Differential Equations

Notes

Unit 19: Jacobi's Method for Solving Partial Differential Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know that Jacobi's method for solving partial differential equation is similar to that of Charpit's method.
- See that two additional equations are to be found through which the first order derivatives $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \frac{\partial z}{\partial x_3}$ can be found that help in finding the solution of the first order partial differential equations.

Introduction

Jacobi's method consists of setting up the subsidiary equations.

Through the solution of subsidiary equations two independent integrals will be found and the method uses techniques to solve the first order partial differential equation.

19.1 Jacobi's Method of Solution of Partial Differential Equations

In Jacobi's method we have to deal with three or more independent variables and one dependent variable. Consider the equation

$$F(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \qquad (1)$$

Where the dependent variable z does not occur except by its partial differential coefficients p_1, p_2, p_3 with respect to the three independent variables x_1, x_2, x_3. The basic idea of Jacobi's method is very similar to that of Charpit's.

So we try to find two additional equations

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = \alpha_1 \qquad ...(2)$$

Notes

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = \alpha_2 \qquad ...(3)$$

Here α_1 and α_2 are arbitrary constants. These equations are such that p_1, p_2, p_3 can be found from (1), (2), (3) as functions of x_1, x_2, x_3 that make the equation

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 \qquad ...(4)$$

integrable, for which the conditions are

$$\frac{\partial p_2}{\partial x_1} = \frac{\partial^2 z}{\partial x_1 \partial x_2} = \frac{\partial p_1}{\partial x_2}, \frac{\partial p_3}{\partial x_1} = \frac{\partial^2 z}{\partial x_1 \partial x_3} = \frac{\partial p_1}{\partial x_1}, \frac{\partial p_3}{\partial x_2} = \frac{\partial p_2}{\partial x_3} \qquad ...(5)$$

Now by differentiating (1) partially with respect to x_1, keeping x_2, x_3 constant, but regarding p_1, p_2, p_3, as dependent functions of x_1, x_2, x_3, we get

$$\frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial F}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial F}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0 \qquad ...(6)$$

Similarly

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial F_1}{\partial p_2}\frac{\partial \text{p}_2}{\partial x_1} + \frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0 \qquad ...(7)$$

Multiplying equation (6) by $\frac{\partial F_1}{\partial p_1}$ and equation (7) by $\frac{\partial F}{\partial p_1}$, and subtracting we get

$$\frac{\partial(F, F_1)}{\partial(x_1, p_1)} + \frac{\partial(F, F_1)}{\partial(p_2, p_1)}\frac{\partial p_2}{\partial x_1} + \frac{\partial(F, F_1)}{\partial(p_3, p_1)}\frac{\partial p_3}{\partial x_1} = 0 \qquad ...(8)$$

where

$\frac{\partial(F, F_1)}{\partial(x_1, p_1)}$ denotes "Jacobian" $\frac{\partial F}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial F}{\partial p_1}\frac{\partial F_1}{\partial x_1}$.

Similarly, like (8) we get

$$\frac{\partial(F, F_1)}{\partial(x_2, p_2)} + \frac{\partial(F, F_1)}{\partial(p_1, p_2)}\frac{\partial p_1}{\partial x_2} + \frac{\partial(F, F_1)}{\partial(p_3, p_2)}\frac{\partial p_3}{\partial x_2} = 0 \qquad ...(9)$$

and

$$\frac{\partial(F, F_1)}{\partial(x_3, p_3)} + \frac{\partial(F, F_1)}{\partial(p_1, p_3)}\frac{\partial p_1}{\partial x_3} + \frac{\partial(F, F_1)}{\partial(p_2, p_3)}\frac{\partial p_2}{\partial x_3} = 0 \qquad ...(10)$$

Add equation (8), (9) and (10) and noting that two pairs of terms are:

$$\frac{\partial(F, F_1)}{\partial(p_2, p_1)}\frac{\partial p_2}{\partial x_1} + \frac{\partial(F, F_1)}{\partial(p_1, p_2)}\frac{\partial p_1}{\partial x_2} = \frac{\partial^2 z}{\partial x_1 \partial x_2}\left[\frac{\partial(F, F_1)}{\partial(p_2, p_1)} + \frac{\partial(F, F_1)}{\partial(p_1, p_2)}\right] = 0$$

Similarly two other pairs of terms also vanish, leaving

$$\frac{\partial(F, F_1)}{\partial(x_1, p_1)} + \frac{\partial(F, F_1)}{\partial(x_2, p_2)} + \frac{\partial(F, F_1)}{\partial(x_3, p_3)} = 0 \qquad ...(11)$$

Notes

i.e. on expansion

$$\frac{\partial F}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial F}{\partial p_1}\frac{\partial F_1}{\partial x_1}+\frac{\partial F}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial F}{\partial p_2}\frac{\partial F_1}{\partial x_2}+\frac{\partial F}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial F}{\partial p_3}\frac{\partial F_1}{\partial x_3} = 0 \quad ...(12)$$

The equation (12) is generally written as $(F, F_1) = 0$.

Similarly

$$(F, F_2) = 0 \text{ and } (F_1, F_2) = 0.$$

But these are linear equations having more than two independent variables. Here we have the following rule.

Try to find two independent integrals, $F_1 = a_1$ and $F_2 = a_2$, of the subsidiary equations

$$\frac{dx_1}{-\frac{\partial F}{\partial p_1}}=\frac{dp_1}{\frac{\partial F}{\partial x_1}}=\frac{dx_2}{-\frac{\partial F}{\partial p_2}}=\frac{dp_2}{\frac{\partial F}{\partial x_2}}=\frac{\partial x_3}{-\frac{\partial F}{\partial p_3}} = \frac{dp_3}{\frac{\partial F}{\partial x_3}} \quad ...(13)$$

If F_1, F_2 satisfy the conditions

$$(F_1, F_2) = \sum_{r=1,2,3}\left[\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right]=0,$$

and if the p's can be found as functions of the x's from

$$F = F_1 - a_1 = F_2 - a_2 = 0,$$

then integrate the equation formed by substituting these functions in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3.$$

Examples of Jacobi Method

1. Solve

$$2p_1x_1x_3 + 3p_2x_3^2 + p_2^2p_3 = 0$$

Solution:

Let $F = 2p_1x_1x_3 + 3p_2x_3^2 + p_2^2p_3 = 0$ ...(1)

The subsidiary equations are

$$\frac{dx_1}{-\frac{\partial F}{\partial p_1}}=\frac{dp_1}{\frac{\partial F}{\partial x_1}}=\frac{dx_2}{-\frac{\partial F}{\partial p_2}}=\frac{dp_2}{\frac{\partial F}{\partial x_2}}=\frac{dx_3}{-\frac{\partial F}{\partial p_3}}=\frac{dp_3}{\frac{\partial F}{\partial x_3}} \quad (2)$$

Now

$$-\frac{\partial F}{\partial p_1}=-2x_1x_3, \frac{\partial F}{\partial x_1}=2p_1x_3, -\frac{\partial F}{\partial p_2}=-3x_3^2-2p_2p_3, \frac{\partial F}{\partial x_2}=0,$$

$$\frac{-\partial F}{\partial p_3}=-p_2^2, \frac{\partial F}{\partial x_3}=2p_1x_1+6p_2x_3$$

Notes

So the auxiliary equations are

$$\frac{dx_1}{-2x_1x_3} = \frac{dp_1}{2p_1x_3} = \frac{dx_2}{-3x_2^2 - 2p_2p_3} = \frac{dp_2}{0} = \frac{dx_3}{-p_2^2} = \frac{dp_3}{2p_1x_1 + 6p_2x_3^2} \quad ...(3)$$

of which integrals are obtained by integrating the equations

$$-\frac{dx_1}{x_1} = \frac{dp_1}{p_1}$$

$$dp_2 = 0$$

or

$$F_1 = x_1p_1 = a_1 \quad ...(4)$$

$$F_2 = p_2 = a_2 \quad ...(5)$$

Now consider

$$(F_1, F_2) = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

$$= p_1(0) - x_1(0) + 0 + 0 + 0 + 0 = 0$$

So equations (4) and (5) can be taken as the two additional equations required. So

$$p_1 = \frac{a_1}{x_1}, \; p_2 = a_2$$

And from equation (1) we have

$$p_3 = \left(-2x_3a_1 - 3a_2x_3^2\right)\Big| a_2^2 = -\left(2a_1x_3 + 3a_2x_3^2\right)\Big| a_2^2$$

Hence

$$dz = p_1dx_1 + p_2dx_2 + p_3dx_3$$

$$= \frac{a_1dx_1}{x_1} + a_2dx_2 - \left(2a_1x_3 + 3a_2x_3^2\right)\frac{dx_3}{a_2^2}$$

So on integration we get

$$z = a_1 \log x_1 + a_2x_2 - \frac{1}{a_2^2}\left(a_1x_3^2 + a_2x_3^3\right) + a_3$$

as the complete integral.

2. Solve

$$(x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0 \quad ...(1)$$

Solution:

This equation is not of Jacobi's type as it involves z. But put

$$z = x_4$$

so $$p_1 = \frac{\partial z}{\partial x_1} = \frac{\partial x_4}{\partial x_1} = -\frac{\partial u}{\partial x_1}\Big|\frac{\partial u}{\partial x_4} = -p_1/p_4 \; \text{...(say)}$$

Notes

where $u = 0$ is an integral of (1).

Similarly

$$p_2 = \frac{\partial z}{\partial x_2} = \frac{\partial x_4}{\partial x_2} = -\frac{\partial u}{\partial x_2}\bigg|\frac{\partial u}{\partial x_4} = -P_2 / P_4$$

$$p_3 = \frac{\partial z}{\partial x_3} = \frac{\partial x_4}{\partial x_3} = -\frac{\partial u}{\partial x_3}\bigg|\frac{\partial u}{\partial x_4} = -P_3 / P_4$$

So equation (1) becomes

$$F = (x_2 + x_3)(P_2 + P_3)^2 - x_4 p_1 p_4 = 0 \qquad ...(2)$$

So equation (2) involves four variables, but not involving the dependent variable u. Now

$$-\frac{\partial F}{\partial P_1} = x_4 P_4, \frac{\partial F}{\partial x_1} = 0, -\frac{\partial F}{\partial P_2} = -2(x_2 + x_3)(P_2 + P_3)$$

$$\frac{\partial F}{\partial x_2} = (P_2 + P_3)^2, -\frac{\partial F}{\partial P_3} = -2(x_2 + x_3)(P_2 + P_3), \frac{\partial F}{\partial x_3} = (P_2 + P_3)^2$$

$$-\frac{\partial F}{\partial P_4} = x_4 P_1; \frac{\partial F}{\partial x_4} = -P_1 P_4.$$

The subsidiary equations are

$$\frac{dx_1}{x_4 P_4} = \frac{dP_1}{0} = \frac{dx_2}{-2(x_2 + x_3)(P_2 + P_3)} = \frac{dP_2}{(P_2 + P_3)^2} = \frac{dx_3}{-2(x_2 + x_3)(P_2 + P_3)}$$

$$= \frac{dP_3}{(P_2 + P_3)^2} = \frac{dx_4}{x_4 P_1} = \frac{dP_4}{-P_1 P_4}$$

of which integrals are

$$F_1 = P_1 = a_1, \quad dp_2 = dp_3, \text{ so } P_2 - P_3 = a_2 = F_2$$

$$\frac{dx_4}{x_4 P_1} = \frac{dP_4}{-P_1 P_4}, \text{ so } x_4 P_4 = a_3 = F_3$$

so

$$F_1 = P_1 = a_1 \qquad ...(3)$$

$$F_2 = P_2 - P_3 = a_2 \qquad ...(4)$$

$$P_3 = x_4 P_4 = a_3 \qquad ...(5)$$

We have to ensure that $(F_r, F_s) = 0$, where r and s are any two of the indices 1, 2, 3. To see $(F_1, F_2) = 0$, we have

$$\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial P_1} - \frac{\partial F_1}{\partial P_1}\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial P_2} - \frac{\partial F_1}{\partial P_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial P_3} - \frac{\partial F_1}{\partial P_3}\frac{\partial F_2}{\partial x_3}$$

$$+\frac{\partial F_1}{\partial x_4}\frac{\partial F_2}{\partial P_4} - \frac{\partial F_1}{\partial P_4}\cdot\frac{\partial F_2}{\partial x_4} = 0 \qquad ...(6)$$

as F_1, F_2 do not contain x_1, x_2, x_3 and x_4.

From (3) and (5) we have

$$P_1 = a_1,\ P_4 = \frac{a_3}{x_4}$$

From (4) we have

$$P_2 = P_3 + a_2 \qquad ...(7)$$

Substituting in (2) we have

$$(x_2 + x_3)(2P_3 + a_2)^2 - a_1 a_3 = 0$$

$$P_2 + P_3 = (2P_3 + a_2) = \pm\sqrt{\frac{a_1 a_3}{(x_2 + x_3)}} \qquad ...(8)$$

$$\therefore \qquad 2P_2 = a_2 \pm \sqrt{\frac{a_1 a_3}{(x_2 + x_3)}} \qquad ...(9)$$

$$2P_3 = -a_2 \pm \sqrt{\frac{a_1 a_3}{(x_2 + x_3)}} \qquad ...(10)$$

$$du = P_1 dx_1 + P_2 dx_2 + P_3 dx_3 + P_4 dx_4$$

$$= a_1 dx_1 + \frac{a_3 dx_4}{x_4} + \frac{a_2}{2}(dx_2 - dx_3) \pm \frac{1}{2}\sqrt{\frac{a_1 a_3}{(x_2 + x_3)}}(dx_2 + dx_3)$$

on integration we get

$$u = a_1 x_1 + a_3 \log(x_4) + \frac{a_2}{2}(x_2 - x_3) \pm \frac{1}{2}\sqrt{a_1 a_3}\,(x_2 + x_3)^{1/2} + a_4$$

so $u = 0$ gives, replacing x_4 by z, and dividing by a_3 we have

$$\frac{a_1}{a_3} x_1 + \log z + \frac{a_2}{2a_3}(x_2 - x_3) \pm \sqrt{\frac{a_1}{a_3}}(x_2 + x_3)^{1/2} + \frac{a_4}{a_3} = 0$$

Let $\frac{a_1}{a_3} = A_1, \frac{a_2}{2a_3} = A_2, \frac{a_4}{a_3} = A_3$ we have the required equation:

$$\log z + A_1 x + A_2(x_2 - x_3) \pm \sqrt{A_1}(x_2 + a_3)^{1/2} + A_3 = 0 \qquad ...(11)$$

3. Solve

$$p^2 x_1 + q^2 x_2 = z \qquad ...(1)$$

Solution:

Let $z = x_3$; let $u(x_1, x_2, x_3) = 0$ be the solution.

$$p = \frac{\partial z}{\partial x_1} = \frac{\partial x_3}{\partial x_1} = \frac{p_1}{P_3}, \quad \text{where } P_1 = \frac{\partial u}{\partial x_1}, P_3 = \frac{\partial u}{\partial x_3}$$

$$q = \frac{\partial z}{\partial x_2} = \frac{\partial x_3}{\partial x_2} = \frac{P_2}{P_3} \quad \text{where } P_2 = \frac{\partial u}{\partial x_2}$$

Notes

Substituting in (1)

$$F = P_1^2 x_1 + P_2^2 x_2 - P_3^2 x_3 = 0 \qquad ...(2)$$

The subsidiary equations are

$$\frac{dx_1}{-\frac{\partial F}{\partial P_1}} = \frac{dP_1}{\frac{\partial F}{\partial x_1}} = \frac{dx_2}{-\frac{\partial F}{\partial P_2}} = \frac{dP_2}{\frac{\partial F}{\partial x_2}} = \frac{dx_3}{-\frac{\partial F}{\partial P_3}} = \frac{dP_3}{\frac{\partial F}{\partial x_3}} \qquad ...(3)$$

or

$$\frac{dx_1}{-2P_1x_1} = \frac{dP_1}{P_1^2} = \frac{dx_2}{-2P_2x_2} = \frac{dP_2}{P_2^2} = \frac{dx_3}{2P_3x_3} = \frac{dP_3}{-P_3^2} \qquad ...(4)$$

From first two terms

$$P_1^2 x_1 = c_1, P_2^2 x_2 = c_2,$$

From (2) $\qquad P_3^2 = \frac{c_1 + c_2}{x_3}$

Thus $\qquad du = P_1 dx_1 + P_2 dx_2 + P_3 dx_3 \qquad ...(5)$

Substituting the values of P_1, P_2 and P_3 we have

$$du = \sqrt{\frac{c_1}{x_1}}\,dx_1 + \sqrt{\frac{c_2}{x_2}}\,dx_2 + \sqrt{\frac{c_1+c_2}{x_3}}\,dx_3$$

On integrating we have

$$u = 2(c_1x_1)^{1/2} + 2(c_2x_2)^{1/2} + 2[(c_1+c_2)z]^{1/2} + c_3 \text{ Q.E.D.}$$

4. Solve

$$F = p_1^2 + p_2^2 + p_3 - 1 = 0$$

$$-\frac{\partial F}{\partial p} = -2p_1, \frac{\partial F}{\partial x_1} = 0, \frac{\partial F}{\partial x_2} = 0, \frac{\partial F}{\partial x_3} = 0, -\frac{\partial F}{\partial p_2} = -2p_2, -\frac{\partial F}{\partial p_3} = -1$$

Solution:

The subsidiary equations are

$$\frac{dx_1}{-2p_1} = \frac{dp_1}{0} = \frac{dx_2}{-2p_2} = \frac{dp_2}{0} = \frac{dx_3}{-1} = \frac{dp_3}{0}$$

$$p_1 = a, p_2 = b, p_3 = 1 - a^2 - b^2$$

$$F_1 = p_1 = a, F_2 = p_2 = b$$

$$(F_1, F_2) = 0$$

$$dz = a\,dx_1 + b\,dx_2 + (1 - a^2 - b^2)dx_3$$

$$z = a\,x_1 + b\,x_2 + (1 - a^2 - b^2)x_3 + a_3 \qquad \text{Q.E.D.}$$

Self Assessment

1. Apply Jacobi's method to find complete integral of the following:

$$x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2 - p_3^2 = 0$$

2. Find the complete integral for

Notes

$$p_3x_3(p_1+p_2)+x_1+x_2=0$$

19.2 Simultaneous Partial Differential Equations

In Jacobi's method two additional equations are needed to solve the partial differential equation by Jacobi's method.

In this section the problem of finding the solution of the partial differential equation $F = 0$ with some work of finding F_1 is already done. The method can be illustrated by the following examples:

Example 1: Find the complete integral for the partial differential equations.

$$F = p_1x_1+p_2x_2-p_3^2=0 \quad ...(1)$$

$$F_1 = p_1-p_2+p_3-1=0 \quad ...(2)$$

Here

$$(F,F_1) = \frac{\partial F}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial F}{\partial p_1}\cdot\frac{\partial F_1}{\partial x_1}+\frac{\partial F}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial F}{\partial p_2}\frac{\partial F_1}{\partial x_2}+\frac{\partial F}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial F}{\partial p_3}\cdot\frac{\partial F_1}{\partial x_3}$$

$$= p_1\cdot 1-x_1(0)+p_2(-1)-x_2(0)+0\cdot(1)+2p_3(0)=p_1-p_2 \quad ...(3)$$

Now $(F,F_1)\neq 0$, now to make

$$(F,F_1) = 0, \text{ we have } p_1=p_2 \quad ...(4)$$

From equation (2) $\quad p_3 = 1 \quad ...(5)$

So From (1), $p_1(x_1+x_2)-1 = 0$, so $p_1=\dfrac{1}{(x_1+x_2)}$ ...(6)

$$dz = p_1dx_1+p_2dx_2+p_3dx_3$$

$$= \frac{dx_1}{x_1+x_2}+\frac{dx_2}{x_1+x_2}+1dx_3$$

or

$$dz = \frac{dx_1+dx_2}{x_1+x_2}+dx_3 \quad ...(7)$$

on integrating (7) we have

$$z = \log(x_1+x_2)+x_3+a \quad ...(8)$$

which is the complete integral of (1).

Example 2: Find the complete integral for

$$F = 2x_3p_1p_3-x_4p_4=0 \quad ...(1)$$

$$F1 = 2p_1-p_2=0 \quad ...(2)$$

Now

$$(F,F_1) = \frac{\partial F}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial F}{\partial p_1}\cdot\frac{\partial F_1}{\partial x_1}+\frac{\partial F}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial F}{\partial p_2}\cdot\frac{\partial F_1}{\partial x_2}$$

Notes

$$+\frac{\partial F}{\partial x_3}\cdot\frac{\partial F_1}{\partial p_3}-\frac{\partial F}{\partial p_3}\cdot\frac{\partial F_1}{\partial x_3}+\frac{\partial F}{\partial x_4}\cdot\frac{\partial F_1}{\partial p_4}-\frac{\partial F}{\partial p_4}\cdot\frac{\partial F_1}{\partial x_4}$$

$$= (2p_1p_3)(0)-2x_3p_1\cdot(0)-p_4\cdot(0)+0=0 \qquad ...(3)$$

The next step is to find F_2 and F_3 such that

$$(F,F_2) = 0 = (F_1,F_2) \qquad ...(4)$$

Now

$$-\frac{\partial F}{\partial p_1} = -2x_3p_3,-\frac{\partial F}{\partial p_2}=0,-\frac{\partial F}{\partial p_3}=-2x_3p_1,-\frac{\partial F}{\partial p_4}=x_4$$

$$\frac{\partial F}{\partial x_1} = 0,\frac{\partial F}{\partial x_2}=0,\frac{\partial F}{\partial x_3}=2p_1p_3,\frac{\partial F}{\partial x_4}=-p_4$$

$$\frac{dx_1}{-2x_3p_3} = \frac{dx_2}{0}=\frac{dx_3}{-2x_3p_1}=\frac{dx_4}{-p_4}=\frac{dp_1}{0}=\frac{dp_2}{0}=\frac{dp_3}{2p_1p_3}=\frac{dp_4}{-p_4}$$

$$p_2 = a_2 \qquad ...(4)$$

so $F_2 = p_2 = a_2$, so $(F,F_2)=0=(F_1,F_2)$

Also from $\frac{dx_3}{-2x_3p_1} = \frac{dp_3}{2p_1p_3}$, on integration

$$F_3 = x_3p_3 = a_3 \qquad ...(5)$$

Again $(F_1,F_3) = 0=(F,F_3)=0=(F_2,F_3)=0$

$$p_1 = \frac{a_2}{2},\ p_2=a_2,\ p_3=\frac{a_3}{x_3},\ p_4=\frac{2x_3p_3p_1}{x_4}=\frac{a_3a_2}{2x_4}$$

so from the relation

$$du = p_1dx_1+p_2dx_2+p_3dx_3+p_4dx_4$$

$$= \frac{a_2}{2}dx_1+a_2dx_2+\frac{dx_3a_3}{x_3}+\frac{a_3a_2dx_4}{2x_4} \qquad ...(6)$$

On integrating (6) we have the complete integral

$$u = \frac{a_2}{2}x_1+a_2x_2+a_3\log x_3+\frac{a_2a_3}{2}\log x_4+a_4 \qquad ...(7)$$

Self Assessment

3. Solve for complete integral of

$$F = p_1^2+p_2p_3x_2x_3^2=0$$

$$F_1 = p_1+p_2x_2=0$$

4. Find the complete integral of

$$F = x_1p_1-x_2p_2+p_3-p_4=0$$

$$F_1 = p_1+p_2-x_1-x_2=0$$

19.3 Summary

- Jacobi's method of solution of the partial differential equation of the first order is very similar to that of Charpit's method.
- The method consists in setting up subsidiary equations through which two integrals are found that help in finding the solution.

19.4 Keywords

The ***subsidiary equations*** help us in finding the two independent integrals.

Independent integrals help in finding the partial derivatives $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}$ and so the solution can be found.

19.5 Review Questions

1. Find the solution of

$$F = p_1 + p_2 + p_2^2 - 3x_1 - 3x_2 - 4x_3^2 = 0$$

with additional equations

$$F_1 = x_1p_1 - x_2p_2 - 2x_1^2 + 2x_2^2 = 0$$

$$F_2 = p_3 - 2x_3 = 0$$

2. Find complete integral of

$$p_1x_3^2 + p_3 = 0$$

$$p_2x_3^2 + p_3x_2^2 = 0$$

3. Find the complete integral of

$$2x_1x_3p_1p_3z + x_2p_2 = 0$$

Answers: Self Assessment

1. $z = a_1x_1 + a_2x_2 \pm \sin^{-1}(a_1a_2x_3) + a_3$
2. $4a_1z = 4a_1^2 \log x_3 + 2a_1a_2(x_1 - x_2) - (x_1 + x_2)^2 + 4a_1a_3$
3. $z = a(x_1 - \log x_2 - 1/x_3) + b$
4. $z = x_1x_2 + a(x_3 + x_4) + b$

19.6 Further Readings

Books

Piaggio H.T.H., Differential Equations

Sneddon L.W., Elements of Partial Differential equations

Notes

Unit 20: Higher Order Equations with Constant Coefficients and Monge's Method

CONTENTS

Objectives

After studying this unit, you should be able to:

- Set up partial differential equations having higher order than that of first order.
- Know that various methods are employed depending upon the structure of the partial differential equation.
- See that each section is followed by a set of self assessment problems related to that section. By solving these problems the method can be understood.

Introduction

This section of the unit needs more practise for solving the various types of partial differential equations.

The problems are classified according to the method used in solving them. It is therefore essential to understand the method and its subsequent steps of solving the problem.

20.1 Linear Partial Differential Equations of Order *n* with Constant Coefficients; Complementary Functions

Notes

So far we have been dealing with partial differential equations of first order with first degree as well as with any degree. In this unit we shall introduce higher derivatives than the usual first order derivatives $\frac{\partial z}{\partial x}, \frac{\partial y}{\partial z}$. So we may have $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ and so on and so forth. If we are dealing with only second order equations we denote $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$. In dealing with higher derivatives let us denote $\frac{\partial}{\partial x}$ by D and $\frac{\partial}{\partial y}$ by D′, then

$$\frac{\partial^2}{\partial x^2} = D^2, \frac{\partial^2}{\partial x \partial y} = DD' = D'D, \frac{\partial^2}{\partial y^2} = D'^2, \ldots$$

$\ldots \frac{\partial^n}{\partial x^n} = D^n, \frac{\partial^{n-1}}{\partial x^{n-1}} \frac{\partial}{\partial y} = D^{n-1} D'$ and so on. So we have to deal with a general equation of the form

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \ldots \frac{\partial^n z}{\partial x^n}, \ldots\right) = f(x, y) \quad \ldots(1)$$

or

$$\left(A_0 D^n z + A_1 D^{n-1} D' z + A_2 D^{n-2} D'^2 + \ldots + A_n D'^n z\right)$$

$$+\left(B_0 D^{n-1} z + B_1 D^{n-2} D' Z + B_2 D^{n-3} D'^2 z + \ldots + B_{n-1} D'^{n-1} z\right)$$

$$+ \ldots + \left[M_0 Dz + M_1 D' z\right] + N_0 z = f(x, y) \quad \ldots(2)$$

Thus equation (1) may be written as

$$F(D, D')z = f(x, y) \quad \ldots(3)$$

Just as in the case of ordinary differential equations it can be shown that the complete solution of linear partial differential equation will consist of two parts, namely:

(i) The complementary function (C.F.), and

(ii) The particular integral (P.I.)

The complementary function is the general solution of the equation

$$F(D, D')z = 0 \quad \ldots(4)$$

The particular integral is that value of z in terms of x, y which satisfies the equation (3) that contains no arbitrary constants.

A Linear Homogeneous partial differential equation of order n with constant coefficients is that in which $F(D, D')$ is a homogeneous function i.e. $f(D, D')$ and is of the form

$$f(D, D')z = (A_0 D^n + A_1 D^{n-1} D' + \ldots + A_n D^n)z = f(x, y) \quad \ldots(5)$$

Non-homogeneous differential equation is not homogeneous i.e. if all terms of D, D' in the function $F(D, D')$ are not of the same degree.

Just as we deal with ordinary differential equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n)y = f(x)$$

Where $D = \frac{d}{dx}$, we shall deal briefly with the corresponding equation in two independent variables,

$$(D^n + a_1 D^{n-1} D + a_2 D^{n-2} D'^2 + ... + a_n D'^n)z = f(x, y) \quad ...(6)$$

where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$.

The simplest case is

$$(D - mD')z = 0$$

i.e

$$\left(\frac{\partial}{\partial x} - m\frac{\partial}{\partial y}\right)z = 0$$

or

$$(p - mq) = 0$$

where

$$p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial x}$$

or

$$z = \phi(y + mx)$$

This suggests what is easily verified, that the solution of (6) if $f(x, y) = 0$ is

$$Z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + ...\phi_n(y + m_n x) \quad ...(7)$$

where the constants $m_1, m_2, m_3, ..., m_n$ are the roots (supposed all different)

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n = 0 \quad ...(8)$$

Example: Solve

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial x \partial y^2} = 0$$

or

$$(D^3 - 3D^2 D' + 2D D'^2)z = 0$$

Now the roots of

$$m^3 - 3m^2 + 2m = 0$$

or 0, 1 and 2. So the solution is

$$z = F_1(y) + F_2(y + x) + F_3(y + 2x)$$

Self Assessment

Notes

1. Solve

$$(D^3 - 6D^2D' + 11D\,D'^2 - 6D'^3)z = 0$$

2. Solve

$$2r + 5s + 2t = 0$$

where $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

20.2 Case when the Auxiliary Equation has Equal Roots

Consider the equation

$$(D - mD')^2 z = 0 \quad \text{...(9)}$$

Put $(D - mD')z = u.$

Equation (9) becomes

$$(D - mD')u = 0$$

The solution is

$$u = F(y + mx)$$

Therefore

$$(D - mD')z = F(y + mx)$$

or

$$\frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = F(y + mx)$$

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{F(y + mx)}$$

From the first two terms we get

$$y + mx = a$$

and from first and last term we have

$$dz - F(y + mx)dx = 0$$

or

$$dz - F(a)dx = 0$$

So the solution is

$$z = xF(a) + b$$

Notes

Hence the solution is

$$\phi(z - x\,F(y+mx),\, y+mx) = 0$$

or $$z = x\,F(y+mx) = F_1(y+mx)$$

so $$z = x\,F(y+mx) + F_1(y+mx) \quad ...(10)$$

In general, the solution of

$$(D - mD')^r z = 0$$

is $$z = F_1(y+mx) + x\,F_2(y+mx) + ... + x^{r-1}F_r(y+mx) \quad ...(11)$$

Example 1: Solve

$$\frac{\partial^4 z}{\partial x^4} - 2\frac{\partial^4 z}{\partial x^3 \partial y} + 2\frac{\partial^4 z}{\partial x \partial y^3} - \frac{\partial^4 z}{\partial y^2} = 0$$

The auxiliary equation is

$$m^4 - 2m^3 + 2m - 1 = 0$$

$$m^4 - 1 - 2m(m^2 - 1) = 0$$

$$(m^2 - 1)(m^2 + 1) - 2m(m^2 - 1) = 0$$

$$(m^2 - 1)(m - 1)^2 = 0 = (m+1)(m-1)^3$$

So the roots are 1, 1, 1, –1

Hence the solution is

$$z = F_1(y+x) + xF_2(y+x) + x^2F_3(y+x) + F_4(y-x)$$

Example 2: Solve

$$(25D^2 - 40DD' + 16D'^2)z = 0$$

The auxiliary equation is

$$25m^2 - 40m + 16 = 0$$

$$(5m - 4)^2 = 0$$

The roots are $m = \frac{4}{5},\ 4/5$ are repeated roots so the solution is

$$z = F_1(5y + 4x) + x\,F_2(5y + 4x)$$

Self Assessment

3. Solve

$$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 0$$

4. Solve **Notes**

$$\frac{\partial^2 z}{\partial x^2} - 6\frac{\partial^2 z}{\partial x \partial y} + 9z = 0$$

20.3 The Particular Integral (P.I.)

We now return to the equation (3) *i.e.*

$$F(D, D')z = f(x, y) \quad ...(1)$$

Now the most general solution of equation (1) can be written as

$$z = \text{complementary function + Particular function}$$

or $$z = \text{C.F + P.I} \quad ...(2)$$

In the above we have found C.F. for the homogeneous equation and now in the following find the P.I. We can write

$$\text{The particular integral} = \frac{1}{F(D, D')} f(x, y) \quad ...(12)$$

Here we treat the symbolic function of D and D' as we do D alone. We can factor. $F(D, D')$, resolve $\frac{1}{F(D, D')}$ into partial fractions on expanding in power series.

(a) **On Expansion**

Example 1: Solve

$$(D^2 - 4DD' + 4D'^2)z = 0$$

The complementary function is given by

$$(D^2 - 4DD' + 4D'^2)z = 0$$

$$\text{C.F.} = F_1(y + 2x) + x\, F_2(y + 2x)$$

The particular integral is

$$\text{P.I.} = \frac{1}{D^2 - 4DD' + D'^2(4)}(x^2 + xy)$$

or $$\text{P.I.} = (D^2 - 4DD' + 4D'^2)^{-1}(x^2 + xy)$$

$$= \frac{1}{D^2}\left(1 - \frac{4D'}{D} + 4\frac{D'^2}{D^2}\right)^{-1}(x^2 + xy)$$

$$= \frac{1}{D^2}\left(1 + \frac{4D'}{D} - \frac{4D'^2}{D^2} + \frac{16D'^2}{D^2} + ...\right)(x^2 + xy)$$

$$= \frac{1}{D^2}\left(x^2 + xy + \frac{4}{D}(x) + 0\right)$$

$$= \frac{x^4}{12} + \frac{x^3 y}{6} + \frac{4}{D^3}(x)$$

$$= \frac{x^4}{12} + \frac{x^3 y}{6} + \frac{x^4}{24} = \frac{x^4}{8} + \frac{x^3 y}{6}$$

Thus the complete solution is

$$z = F_1(y+2x) + x\, F_2(y+2x) + \frac{x^4}{8} + \frac{x^3 y}{6}$$

Example 2: Solve

$$(D^2 - a^2 D'^2)z = x^2$$

Solution: The complementary function is given by the equation

$$(D^2 - a^2 D'^2)z = 0$$

The auxiliary equation is

$$m^2 - a^2 = 0$$

with roots $m = a$ and $m = -a$.

So C.F. $= F_1(y-ax) + F_2(y+ax)$

The particular integral is given by

$$\text{P.I.} = \frac{1}{(D^2 - a^2 D'^2)}(x^2)$$

$$= \frac{1}{D^2}\left(1 - \frac{a^2 D'^2}{D^2}\right)^{-1}(x^2)$$

$$= \frac{1}{D^2}\left(1 + \frac{a^2 D'^2}{D} + \ldots\right)x^2 = \frac{1}{D^2}(x^2) = \frac{x^4}{12}$$

So the complete solution is

$$z = F_1(y-ax) + F_2(y+ax) + \frac{x^4}{12}.$$

Self Assessment

5. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = xy$

6. Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

20.4 Shorter Method for Finding Particular Integral

Notes

When dealing with the equation

$$F(D,D')z = f(x,y)$$

We consider a special function of the form

$$f(x,y) = \phi(ax+by),$$

then a shorter method may be used. Now

$$D\phi(ax+by) = a\,\phi'(ax+by);\ D'\phi\ (ax+by) = b\phi'(ax+by)$$

So $$D^r\phi(ax+by) = a^r\phi^r(ax+bx)$$

$$D'^r\phi(ax+by) = b^r\phi^r(ax+bx)$$

and $$D^p\,D'q\,\phi(ax+by) = a^p\ b^q\ \phi^{p+q}(ax+by)$$

Here ϕ^n is the nth derivative of ϕ with respect to '$ax + by$' as a whole and n is the degree of $F(D,D')$.

Hence we will have

$$F(D,D')\phi(ax+by) = F(a,b)\ \phi^n(ax+by) \quad ...(13)$$

when ϕ^n is the nth derivative of ϕ with respect to '$ax + by$' as a whole and n is the degree of $F(D,D')$.

Operating by $\frac{1}{F(D,D')}$ on both sides of (13) and dividing by $F(a, b)$, we get

$$\frac{1}{F(D,D')}\phi^n(ax+by) = \frac{1}{F(a,b)}\phi(ax+by) \quad ...(14)$$

provided

$$F(a,b) \neq 0.$$

Therefore $$\frac{1}{F(D,D')}\phi_1(ax+b) = \frac{1}{F(a,b)}\int\int\int\phi_1(u)\,du...du$$

$$= \frac{1}{F(a,b)}\ n\text{th integral of } \phi_1 \text{ where } u = ax + by \quad ...(15)$$

Example 1: Solve

$$(r-2s+t) = \sin(2x+3y)$$

Solution:

Here $$r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2}{\partial y^2}$$

Notes

So $$(D^2 - 2DD' + D'^2)z = \sin(2x+3y)$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

having roots m = 1, 1, so that

$$\text{C.F.} = F_1(y+x) + x\,F_2(y+x)$$

and $$\text{P.I.} = \frac{1}{(D-D')^2}\sin(2x+3y)$$

Putting $2x+3y = u,$ so we have

$$\text{P.I.} = \frac{1}{(2-3)^2}\int\int \sin u\, du, du \qquad \text{(integrating twice)}$$

$$= 1\int(-\cos u)du$$

$$= -\sin u = -\sin(2x+3y)$$

Thus the solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$= F_1(y+x) + xF_2(y+x) - \sin(2x+3y)$$

Example 2: Solve

$$(D^2 - D'^2)z = 30(2x+y)$$

The auxiliary equation is

$$m^2 - 1 = 0$$

so, $$m = +1, -1$$

and $$\text{C.F.} = F_1(y+x) + F_2(y-x)$$

$$\text{P.I.} = \frac{1}{(D^2 - D'^2)}30(2x+y)$$

Let $u = 2x + y,$

$$\text{P.I.} = \frac{1}{(4-1)}(30)\int(u\,du)du$$

$$= \frac{1}{3}(30)\int\frac{u^2}{2}du$$

$$= 10\,\frac{u^3}{6} = \frac{5}{6}(2x+y)^3$$

So the solution is

Notes

$$z = F_1(y+x) + F_2(y-x) + \frac{5}{6}(2x+y)^3$$

Self Assessment

7. Solve

$(D^2 + 3DD' + D'^2)z = (x+y)$

8. Solve

$(D^2 + D'^2)z = \cos(mx+ny)$

Particular case when $F(a, b) = 0$

As $$\frac{1}{F(D,D')}\phi^n(ax+by) = \frac{1}{F(a,b)}\phi(ax+by)$$

but if $F(a,b)=0$ then R.H.S. becomes infinite and the above method fails.

Now consider the case

$$(bD - aD')z = x^r\phi(ax+by)$$

or $$bp - aq = x^r\phi(ax+by), \text{ where} \qquad ...(16)$$

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}.$$

Applying Lagrange's method to (1) we get

$$\frac{dx}{b} = \frac{dy}{-a} = \frac{dz}{x^r\phi(ax+by)}$$

So one solution is

$$ax+by = c, \text{ and the other solution is given by}$$

$$\frac{dx}{b} = \frac{dz}{x^r\phi(c)}$$

$$\therefore \quad z = \frac{x^{r+1}}{(r+1)b}\phi(ax+by)$$

This is the solution of the given differential equation (16).

Thus $$\frac{1}{(bD-aD')}x^r\phi(ax+by) = \frac{x^{r+1}}{b(r+1)}\phi\,(ax+by) \qquad ...(17)$$

Next consider

$$z = \frac{1}{(bD-aD')^n}\phi(ax+by)$$

$$= \frac{1}{(bD-aD')^{n-1}}, \frac{1}{(bD-aD')}\phi(ax+by)$$

$$= \frac{1}{(bD-aD')^{n-1}}.\frac{x}{b}\phi(ax+by)$$

$$= \frac{1}{(bD-aD')^{n-2}}\frac{1}{(bD-aD')}\frac{x}{b}\phi(ax+by)$$

$$= \frac{1}{(6D-aD')^{n-2}}.\frac{x}{2b^2}\phi(ax+by)$$

$$= \frac{1}{\underline{|2}b^2}\frac{1}{(bD-aD')^{n-3}}\frac{1}{(bD-aD')}x^2\phi(ax+b)$$

$$= \frac{1}{\underline{|3}b^3}.\frac{1}{(bD-aD')^{n-3}}.x^3\phi(ax+b)$$

....

....

....

$$= \frac{1}{b^{n-1}(n-1)!}\frac{1}{(bD-aD')^{n-x}}.\frac{x^n}{nb}\phi(ax+b)$$

$$= \frac{x^n}{b^n\underline{|n}}\phi(ax+by)$$

Thus $$\frac{1}{(bD-aD')^n}\phi(ax+by) = \frac{x^n}{b^n\underline{|n}}\phi(ax+b) \quad ...(18)$$

When $$F(a,b) = 0$$

Example 1: Solve

$$(D^2-2aDD'+a^2D'^2)z = f(y+ax)$$

Solution: The auxiliary equation is

$$m^2-2am+a^2 = 0$$

$$(m-a)^2 = 0$$

$$m = a, a$$

The complimentary function is

$$\text{C.F.} = F_1(y+ax)+x\,F_2(y+ax)$$

$$\text{P.I.} = \frac{1}{D^2-2aDD'+a^2D'^2}f(y+ax)$$

$$= \frac{1}{(D-aD')^2}f(y+ax) = \frac{x^2}{\underline{|2}}f(y+ax)$$

Notes

So the complete solution is

$$z = F_1(y+ax) + x\,F_2(y+ax) + \frac{x^2}{\underline{|2}} f(y+ax)$$

Example 2: Solve

$$(4D^2 - 4DD' + D'^2)z = e^{x+2y} + x^3$$

Solution: The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$m = 1/2, 1/2$$

$$\text{C.F.} = F_1(2y+x) + x\,F_1(2y+x)$$

$$\text{P.I.} = \frac{1}{(2D-D')^2}\left\{e^{x+2y} + x^3\right\}$$

$$= \frac{1}{(2D-D')^2} e^{x+2y} + \frac{1}{(2D-D')^2} x^3$$

$$= \frac{x^2}{2.4} e^{x+2y} + \frac{1}{4D^2}\left(1 - \frac{D'}{2D}\right)^{-2} x^3$$

$$= \frac{x^2}{8} e^{x+2y} + \frac{1}{4D^2}\left(1 + \frac{D'}{D} + \ldots\right) x^3$$

So $$\text{P.I.} = \frac{x^2}{8} e^{x+2y} + \frac{1}{4}\cdot\frac{x^5}{4.5} = \frac{x^2}{8} e^{x+2y} + \frac{x^5}{80}$$

Thus the solution is

$$z = F_1(2y+x) + x\,F_1(2y+x) + \frac{x^2}{8} e^{x+2y} + \frac{x^5}{80}$$

Self Assessment

9. Solve

   $(D - D')^2 = x + \phi(x+y)$

10. Solve

   $(D^3 - 4D^2D' + 4DD'^2)z = \cos(y+2x)$

20.5 General Method for Finding Particular Integral (P.I.)

Consider the equation

$$(D - mD')z = f(x, y)$$

Notes

i.e $$\frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = f(x,y)$$

or $$p - mq = f(x,y) \qquad ...(1)$$

where $$p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}.$$

So Lagrange's auxiliary equations (A.E.) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(x,y)}$$

From the first two fractions, we have

$$y = -mx + c \qquad ...(2)$$

From the first and last fractions

$$dz = f(x,y)dx = f(x, c - mx)dx$$

$$\therefore \qquad z = f(x, c - mx)dx$$

and after integration ($c - mx$) is replaced by y because the P.I. does not contain any arbitrary constant.

Now, the particular integral of

$$\frac{1}{f(D,D')}f(x,y) = \frac{1}{D - m_1D'}\cdot\frac{1}{(D - m_2D')}\cdots\frac{1}{D - m_nD'}f(x,y)$$

can be determined by the repeated application of the method given above.

Illustrative Examples

Example 1: Solve: $r + s - 6t = y\cos x$

Solution: The given equation can be written as

$$(D^2 + DD' - 6D'^2)z = y\cos x$$

A.E. is $m^2 + m - 6 = 0$, i.e., $m = 2, -3$

$$\therefore \qquad \text{C.F.} = \phi_1(y + 2x) + \phi_2(y - 3x)$$

Now, $$\text{P.I.} = \frac{1}{(D - 2D')(D + 3D')}y\cos x$$

$$= \frac{1}{(D - 2D')}\cdot\int(c + 3x)\cos x dx \qquad [\because y = c + 3x]$$

$$= \frac{1}{(D - 2D')}[c\sin x + 3x\sin x + 3\cos x]$$

Notes

$$= \frac{1}{(D-2D')}[(y-3x)\sin x + 3x\sin x + 3\cos x]$$

[putting back $c = -3x + y$]

$$= \frac{1}{(D-2D')}[y\sin x + 3\cos x]$$

$$= \int \{(k-2x)\sin x + 3\cos x\}dx \qquad [\because -2x + k = y]$$

$$= -k\cos x - 2(-x\cos x + \sin x) + 3\sin x$$

$$= -(y+2x)\cos x + 2x\cos x + \sin x \qquad [\because y = k + 2x]$$

$$= -y\cos x + \sin x.$$

Hence the complete solution is

$$z = C.F. + P.I. = \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + \sin x.$$

Example 2: Solve: $(D^2 - 4D'^2)z = \frac{4x}{y^2} - \frac{y}{x^2}$

Solution: The C.F. = $\phi_1(y+2x) + \phi_2(y-2x)$

Now, P.I. $= \frac{1}{(D+2D')(D-2D')}\left\{\frac{4x}{y^2} - \frac{y}{x^2}\right\}$

$$= \frac{1}{D+2D'}\int\left\{\frac{4x}{(c-2x)^2} - \frac{c-2x}{x^2}\right\}dx \qquad (\because c - 2x = y)$$

$$= \frac{1}{D+2D'}\int\left[-2\left\{\frac{-2x+c-c}{(c-2x)^2}\right\} - \frac{c}{x^2} + \frac{2}{x}\right]dx$$

$$= \frac{1}{D+2D'}\int\left[-\frac{1}{c-2x} + \frac{2c}{(c-2x)^2} - \frac{c}{x^2} + \frac{2}{x}\right]dx$$

$$= \frac{1}{D+2D'}\left[\log(c-2x) + \frac{c}{c-2x} + \frac{c}{x} + 2\log x\right]$$

$$= \frac{1}{D+2D'}\left[\log y + \frac{y+2x}{y} + \frac{y+2x}{x} + 2\log x\right] \qquad [\text{putting } c = y + 2x]$$

$$= \int\left[\log(k+2x) + \frac{k+4x}{k+2x} + \frac{k+4x}{x} + 2\log x\right]dx \qquad \text{where } y = k + 2x$$

$$= \int\left[\log(k+2x) + 1 + \frac{2x+k-k}{k+2x} + \frac{k}{x} + 4 + 2\log x\right]dx$$

Notes

$$= \int\left[\log(k+2x)+1+1-\frac{k}{k+2x}+\frac{k}{x}+4+2\log x\right]dx$$

$$= \int\left[\log(k+2x)+6-\frac{k}{k+2x}+\frac{k}{x}+2\log x\right]dx$$

$$= \left[\log(k+2x).x-\int\frac{2}{k+2x}.x\,dx+6x\ -\frac{k}{2}\log(k+2x)+k\log x+2\left\{\log x.x-\int\frac{1}{x}.xdx\right\}\right]$$

$$= x\log(k+2x)-\int\frac{k+2x-k}{k+2x}dx+6x-\frac{k}{2}\log(k+2x)+k\log x+2x\log x-2x$$

$$= x\log(k+2x)-x+\frac{k}{2}\log(k+2x)+6x-\frac{k}{2}\log(k+2x)+k\log x+2x\log x-2x$$

$$= x\log y-x+\frac{k}{2}\log y+6x-\frac{k}{2}\log y+k\log x+2x\log x-2x \text{ (putting back } y = k+2x)$$

$$= x\log y-x+6x+k\log x+2x\log x-2x$$

$$= x\log y+3x+(y-2x)\log x+2x\log x$$

$$= x\log y+3x+y\log x.$$

Hence the complete solution is

$$z = \phi_1(y+2x)+\phi_2(y-2x)+x\log y+3x+y\log x.$$

Example 3: Solve: $r-t = \tan^3 x\tan y-\tan x\tan^3 y$

Solution: The given equation is

$$(D^2-D'^2)z = \tan x\tan y\ (\tan^2 x-\tan^2 y).$$

$$= \tan x\tan y\ (\sec^2 x-\sec^2 y)$$

$$\therefore \quad \text{C.F.} = \phi_1(y-x)+\phi_2(y+x).$$

$$\text{P.I.} = \frac{1}{(D+D')(D-D')}\tan x\tan y(\sec^2 x-\sec^2 y)$$

$$= \frac{1}{D+D'}\int\tan x\tan(c-x)\{\sec^2 x-\sec^2(c-x)\}dx \qquad \text{[where } c-x=y]$$

$$= \frac{1}{D+D'}\left[\int\tan x\tan(c-x)\sec^2 x\ dx-\int\tan x\tan(c-x)\sec^2(c-x)dx\right]$$

$$= \frac{1}{D+D'}\left[\frac{1}{2}\tan^2 x\tan(c-x)+\frac{1}{2}\int\tan^2 x\sec^2(c-x)dx\right.$$

$$\left.+\frac{1}{2}\tan x\tan^2(c-x)-\frac{1}{2}\int\tan^2(c-x)\sec^2 x\,dx\right]$$

Notes

$$= \frac{1}{2(D+D')}\left[\tan^2 x\tan(c-x)+\tan x\tan^2(c-x)+\int\{\sec^2 x-\sec^2(c-x)\}dx\right]$$

$$= \frac{1}{2(D+D')}\left[\tan^2 x\tan(c-x)+\tan x\tan^2(c-x)+\tan x+\tan(c-x)\right]$$

$$= \frac{1}{2(D+D')}\left[\tan^2 x\tan y+\tan x\tan^2 y+\tan x+\tan y\right] \quad \text{[By putting back } y = c-x\text{]}$$

$$= \frac{1}{2(D+D')}[\tan y\sec^2 x+\tan x\sec^2 y]$$

$$= \frac{1}{2}\int[\tan(k+x)\sec^2 x+\tan x\sec^2(k+x)]dx \qquad \text{where } k+x=y$$

$$= \frac{1}{2}\int\left[\frac{d}{dx}\{\tan x\tan(k+x)\}\right]dx$$

$$= \frac{1}{2}\tan x\tan(k+x) = \frac{1}{2}\tan x\tan y \qquad \text{[putting } k+x=y\text{]}$$

Hence the complete solution is

$$z = \phi_1(y-x)+\phi_2(y+x)+\frac{1}{2}\tan x\tan y$$

Example 4: Find the particular integral with the help of general method for

$$(D^2-2DD'-15D'^2)z = 12xy$$

Solution: We have

$$\text{P.I.} = \frac{1}{(D^2-2DD'-15D'^2)}12xy$$

$$= \frac{1}{(D+3D')(D-5D')}12xy$$

$$= \frac{12}{(D+3D')}\int x(c-5x)\,dx, \qquad \text{where } y = c-5x$$

$$= \frac{12}{D+3D'}\left(\frac{cx^2}{2}-\frac{5x^3}{3}\right)$$

$$= \frac{2}{D+3D'}(3cx^2-10x^3)$$

$$= \frac{2}{D+3D'}x^2(3y+15x-10x), \qquad \text{(putting back } c = y+5x\text{)}$$

Notes

$$= \frac{2}{(D+3D')}x^2(3y+5x)$$

$$= 2\int x^2\{3(k+3x)+5x\}\,dx, \qquad \text{where } k + 3x = y$$

$$= 2\int x^2(3k+14x)\,dx$$

$$= 2kx^3+7x^4 = 2x^3(y-3x)+7x^4$$

$$= x^3(2y+x).$$

Self Assessment

11. Solve

$$(D+D')^2 z = 2\cos y - x\sin y$$

12. Solve

$$(D^2 - DD' - 2D'^2)z = (y-1)e^x$$

20.6 The Non-homogeneous Equation with Constant Coefficients

The simplest case is

$$(D - mD' - \alpha)z = 0$$

or

$$z = e^{(mD'+a)x}\phi(y)$$

where D' has been considered algebraic and ϕ is arbitrary.

$$= e^{ax}\phi(y+mx).$$

Note. Also

$$(D - mD' - \alpha)z = 0.$$

or

$$p - mq = \alpha z.$$

∴ The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\alpha z}.$$

∴

$$z = e^{\alpha x}\phi(y+mx).$$

Similarly the integral of

$$(D - m_1D' - \alpha_1)(D - m_2D' - \alpha_2)(D - m_3D' - \alpha_3)... = 0$$

is

$$z = e^{\alpha_1 n}\phi_1(y+m_1x) + e^{\alpha_2 n}\phi_2(y+m_2x) + e^{\alpha_3 n}\phi_3(y+m_2x) + ...$$

In case of repeated factors **Notes**

$$(D - mD' - \alpha)^2 z = 0 \quad ...(1)$$

or $$(D - mD' - \alpha)(D - mD' - \alpha)z = 0$$

let $$(D - mD' - \alpha)z = \nu,$$

Then, $(D - mD' - \alpha)\nu$ [from (1)]

or $$\nu = e^{ax}\phi_1(y + mx)$$

or $$(D - mD' - \alpha)z = e^{ax}\phi_1(y + mx);$$

$$\therefore \quad z = e^{(mD'+\alpha)x}\left[\int \{e^{-(mD'-\alpha)x} + e^{ax}\phi_1(y + mx)\}dx + \phi_2(y)\right]$$

$$= e^{(\alpha+mD')}x\int \phi_1(y)\,dx + e^{ax}e^{(mxD')}\phi_2(y)$$

$$= e^{\alpha x}.x\phi_1(y + mx) + e^{\alpha x}\phi_2(y + mx)$$

Similarly proceeding in the case of $(D - mD' - \alpha)^r z = 0,$ we have

$$z = e^{\alpha x}\phi_1(y + mx) + e^{\alpha x}x\phi_2(y + mx) + e^{\alpha x}x^2\phi_3(y + mx) + ... + e^{\alpha x}x^{r-1}\phi_r(y + mx)$$

The Particular Integral

The methods for obtaining particular integrals of non-homogeneous partial differential equations are very similar to those used in solving linear equation with constant coefficients.

Note: It can be easily shown that

I. $$\frac{1}{F(D, D')}e^{ax+by} = \frac{e^{ax+by}}{F(a, b)}$$

provided $F(a, b) \neq 0.$

II. $$\frac{1}{F(D, D')}\sin(ax + by) \text{ or } \cos(ax + by)$$

is obtained by putting $D^2 = -a^2$, $DD' = -ab$ and $D'^2 = -b^2$, provided the denominator is not zero.

III. $$\frac{1}{F(D, D')}x^m y^n = [F(D, D')]^{-1} x^m y^n$$

which can be evaluated after expanding $[F(D, D')]^{-1}$ in ascending powers of D or D′.

IV. $$\frac{1}{F(D, D')}(e^{ax+by}.V)$$

$$e^{ax+by}\frac{1}{F\{(D + a).(D' + b)\}}.V$$

Notes

Illustrative Examples

Example 1: Solve: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = xy + e^{x+2y}$.

Solution: Here, $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

or, $(D - D')(D + D' - 3)z = xy + e^{x+2y}$

$\therefore$ The complementary function is

$$\phi(x+y) + e^{3x}\,\Psi\,(y-x).$$

$$\text{Now P.I.} = \frac{xy}{(D-D')(D+D'-3)} + \frac{e^{x+2y}}{(D-D')(D+D'-3)}$$

$$= \frac{1}{3(D'-D)\left[1 - \frac{D'+D}{3}\right]}xy + \frac{e^{x+2y}}{(1-D')(1+D'-3)}$$

$$= \frac{1}{3(D'-D)}\left[1 + \frac{(D'+D)}{3} + \frac{(D'+D)^2}{9} + \ldots\right]xy + \frac{e^x.e^{2y}}{(-1)(D'-2)}$$

$$= \frac{1}{3(D'-D)}\left[xy + \frac{x}{3} + \frac{y}{3} + \frac{2}{9}\right] - e^x.e^{2y} \times \frac{1}{D'}.1$$

$$= \frac{1}{-3D\left(1 - \frac{D'}{D}\right)}\left(xy + \frac{x}{3} + \frac{y}{3} + \frac{2}{9}\right) - ye^{x+2y}$$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \ldots\right)\left(xy + \frac{x}{3} + \frac{y}{3} + \frac{2}{9}\right) - ye^{x+2y}$$

$$= \frac{1}{3D}\left[xy + \frac{x}{3} + \frac{y}{3} + \frac{2}{9} + \frac{x^2}{2} + \frac{x}{3}\right] - ye^{x+2y}$$

$$= -\frac{x^2 y}{3.2} - \frac{1}{9} - \frac{x^2}{2} - \frac{x}{9} - \frac{2x}{27} - \frac{x^3}{18} - \frac{x^2}{18} - ye^{x+2y}$$

$\therefore$ The solution is

$$z = \phi(x+y) + e^{3x}\Psi(y-x) - \frac{x^2 y}{6} - \frac{x^2}{9} - \frac{xy}{9} - \frac{2x}{27} - \frac{x^2}{18} - ye^{x+2y}$$

Example 2: Solve: $(D - D' - 1)(D - D' - 2)z = e^{2x-y} + x$.

Solution: The complementary function is

$$e^x\phi_1(y+x) + e^{2x}\phi_2(y+x)$$

Notes

$$\text{P.I.} = \frac{1}{(D-D'-1)(D-D'-2)}[e^{2x-y}+x]$$

$$= \frac{1}{(D-D'-1)(D-D'-2)}e^{2x-y} + \frac{1}{(D-D'-1)(D-D'-2)}x$$

Now, $\frac{1}{(D-D'-1)(D-D'-2)}e^{2x-y}$

$$= e^{2x}\frac{1}{(D+2-D'-1)(D+2-D'-2)}e^{-y}$$

$$= e^{2x}\frac{1}{(D-D'+1)(D-D')}e^{-y}$$

$$= e^{2x}\frac{1}{[0-(-1)+1][0-(-1)]}e^{-y}$$

$$= e^{2x}.\frac{1}{2}e^{-y} - \frac{1}{2}e^{2x-y}$$

Also, $\frac{1}{(D-D'-1)(D-D'-2)}x$

$$= \frac{1}{2}[1-(D-D')]^{-1}\left[1-\frac{1}{2}(D-D')\right]^{-1}x$$

$$= \frac{1}{2}[1+D-D'+...]\left[1+\frac{1}{2}(D-D')+...\right]x$$

$$= \frac{1}{2}\left[1+\frac{3}{2}D-\frac{3}{2}D'\right]x$$

$$= \frac{1}{2}\left[x+\frac{3}{2}-\frac{3}{2}\times 0\right] = \frac{1}{2}x+\frac{3}{4}$$

∴ The solution is

$$z = e^{x}\phi_1(y+x)+e^{2x}\phi_2(y+x)+\frac{1}{2}e^{2x-y}+\frac{1}{2}x+\frac{3}{4}.$$

Example 3: Solve: $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}-2z = e^{x-y}-x^2y.$

Solution: $[(D-D')(D+D')+2(D+D')-(D-D')-2]z = e^{x-y}-x^2y$

or $[(D-D'+2)(D+D'-1)]z = e^{x-y}-x^2y$

∴ The complementary function is

$$z = e^{-2x}\phi(y+x)+e^{x}\,\Psi\,(y-x)$$

Notes

Now, $$\text{P.I.} = \frac{e^{x-y}}{(D-D'+2)(D+D'-1)} - \frac{x^2 y}{D^2 - D'^2 + 3D' - 2}$$

$$= \frac{e^{x-y}}{(1-D'+2)D'} - \frac{x^2 y}{-2\left[1-\left\{\frac{D}{2}+\frac{3D'}{2}-\frac{D'2}{2}+\frac{D^2}{2}\right\}\right]}$$

$$= \frac{e^{x-y}}{4} + \frac{1}{2}\left[1+0\left(\frac{D}{2}+\frac{3D'}{2}-\frac{D'^2}{2}+\frac{D^2}{2}\right)+\left\{\frac{D}{2}+\frac{3D'}{2}-\frac{D'^2}{2}+\frac{D^2}{2}\right)^2\right.$$

$$\left.+\left(\frac{D}{2}+\frac{3D'}{2}-\frac{D'^2}{2}+\frac{D^2}{2}\right)^3+\ldots\right]x^2$$

$$= -\frac{e^{x-y}}{4} + \frac{1}{2}\left[1+\frac{D}{2}+\frac{3D'}{2}+\frac{D^2}{2}+\frac{D^2}{4}+\frac{3DD'}{2}+\frac{3D^2D'}{2}+\frac{3D^2D'}{4}+\frac{3D^2D'}{8}+\ldots\right]x^2.$$

$$= -\frac{e^{x-y}}{4} + \frac{1}{2}\left[x^2y + xy + \frac{3x^2}{2} + y + \frac{y}{2} + 3x + 3 + \frac{3}{2} + \frac{3}{4}\right]$$

$$= -\frac{e^{x-y}}{4} + \left(\frac{x^2y}{2} + \frac{3x^2}{4} + \frac{3y}{4} + \frac{xy}{2} + \frac{3x}{2} + \frac{21}{8}\right)$$

$\therefore$ The solution is

$$z = e^{-2x}\phi(y+x) + e^x\,\Psi(y-x) - \frac{e^{x-y}}{4} + \left(\frac{x^2y}{2} + \frac{3x^2}{4} + \frac{3y}{4} + \frac{xy}{2} + \frac{3x}{2} + \frac{21}{8}\right)$$

Example 4: Solve the equation:

$$(D^3 - 4D^2D' + 4DD^2)u = \cos(y+2x)$$

or $$D(D-2D')^2 u = \cos(\nu + 2x)$$

Solution: C.F. is $\phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$

$$\text{P.I.} = \frac{1}{(D-2D')^2 D}\cos(y+2x) = \frac{1}{(D-2D')^2}\left\{\sin\frac{(y+2x)}{2}\right\},$$

Now since $$\frac{1}{(bD-aD')}\phi(ax+by) = \frac{x}{b}\phi(ax+by),$$

$$\text{P.I.} = \frac{1}{(D-2D')}\left\{\frac{1}{D-2D'}\frac{\sin(y+2x)}{2}\right\} = \frac{1}{(D-2D')}\left\{\frac{x\sin(y+2x)}{2}\right\}$$

$$= \frac{x^2}{4}\sin(y+2x)$$

∴ The solution is

$$u = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{x^2}{4}\sin(y+2x).$$

Self Assessment

13. Solve $\frac{\partial^2 z}{\partial x^2} - a\frac{\partial^2 z}{\partial y^2} + 2ab\frac{\partial z}{\partial x} + 2a^2 b\frac{\partial z}{\partial y} = 0$

14. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} - z = \cos(x+2y)$

20.7 Equation Reducible to Homogeneous Linear Form

An equation in which the coefficient of a differential coefficient of any order is a constant multiple of the variables of the same degree may be transformed into one having constant coefficients. The method is explained with the help of the following equations.

Example 1: Solve

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial x} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

Solution: Assume, $u = \log x,\ v = \log y,$ also denoting $\frac{\partial}{\partial u}$ by D and $\frac{\partial}{\partial V}$ by D', the given equation reduces to

$$[D(D-1) + 2DD' + D'(D'-1)]z = 0$$

or

$$(D+D')(D+D'-1)z = 0$$

Hence the solution is

$$\begin{aligned} z &= \phi_1(\nu - u) + e^u\,\phi_2(\nu - u) \\ &= \phi_1(\log y - \log x) + \phi_2(\log y - \log x) \\ &= \phi_1\left(\log\frac{y}{x}\right) + x\phi_2\left(\log\frac{y}{x}\right) \\ &= \psi_1\left\{\frac{y}{x}\right) + x\,\psi_2\left(\frac{y}{x}\right) \end{aligned}$$

Example 2: Solve: $yt - q = xy$.

Solution: The equation can be written as

$$y^2\frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial y} = xy^2 \qquad ...(1)$$

Notes

Put $x = e^u$, $y = e^v$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\times\frac{1}{x}, \frac{\partial z}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial v}$$

$$\left(x\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x}\right)z = \frac{\partial^2 z}{\partial x^2}$$

or $$x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial u^2}$$

and $$y^2\frac{\partial^2 z}{\partial y^2} + y\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial v^2}$$

$\therefore$ The equation (1) becomes

$$\frac{\partial^2 z}{\partial v^2} - 2\frac{\partial z}{\partial v} = e^{u+2v}$$

$\therefore$ The complementary function is

$$= \phi_1(u) + e^{2v}\phi_2(u)$$

$$= \phi_1(\log x) + y^2\phi_2(\log x)$$

$$= \psi_1(x) + y^2\psi_2(x)$$

$$\text{P.I.} = \frac{1}{D'(D'-2)}\times e^{u+2v}$$

$$= \frac{1}{D'(D'-2)}\times e^{u+2v}$$

$$= \frac{e^{u+2v}}{2}\times\frac{1}{(D'-2+2)}(1) = \frac{e^{u+2v}}{2}.v$$

$$= \frac{1}{2}xy^2\log y$$

$\therefore$ The solution is $z = \phi_1(x) + y^2\phi_2(x) + \frac{xy^2}{2}\log y$

Aliter. $yt - q = xy$

The equation can be written as

$$\frac{\partial q}{\partial y} - \frac{1}{y}q = x$$

Solving,

$$q.e^{\int -\frac{1}{y}dy} = \int xe^{-\int\frac{1}{y}dy}\,dy + \phi_1(y)$$

Notes

$$\therefore \qquad \frac{q}{y} = \int \frac{x}{y} dy + \phi_1(x)$$

$$\therefore \qquad q = xy \log y + y\phi_1(x)$$

$$\text{or} \qquad \frac{\partial z}{\partial y} = xy \log y + y\phi_1(x)$$

$$\therefore \qquad z = x\int y \log y \, dy + \phi_1(x).\frac{y^2}{2} + \phi_2(x)$$

$$= x\left[\frac{y^2}{2}\log y - \int \frac{y^2}{2} \times \frac{1}{y} dy\right] + y^2 f(x) + F(x)$$

$$\therefore \qquad z = \frac{xy^2}{2}\log y - \frac{xy^2}{4} + y^2 f(x) + F(x)$$

is the required solution.

Example 3: Solve: $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial x} = 0$

Solution: Assume $u = \log x$, $\nu = \log y$. Then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \times \frac{1}{x}$$

$$\text{or} \qquad x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}, \text{ so that } x\frac{\partial}{\partial x} = \frac{\partial}{\partial x} \qquad \text{...(1)}$$

$$\therefore \qquad x\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial x}\right) = x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial u^2} \qquad \text{[from (1)]}$$

Similarly

$$y^2\frac{\partial^2 z}{\partial y^2} + y\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}.$$

$\therefore$ The given equation reduces to

$$\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial \nu^2} = 0,$$

for which

$$z = \phi(u+\nu) + \psi(\nu - u)$$

$$= \phi[\log x + \log y] + \psi[\log y - \log x]$$

$$= \phi(\log xy) + \psi\left[\log\left(\frac{y}{x}\right)\right]$$

$$= f_1(xy) + f_2\left(\frac{y}{x}\right)$$

Notes

Example 4: Solve: $x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x\partial y} + 4y^3\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4.$

Solution: As shown in the last example, if $u = \log x, \nu = \log y$,

$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u},\ y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \nu}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial u^2} \text{ and } y^2\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}$$

Now
$$y\frac{\partial}{\partial t}\left(x\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial \nu}\left(\frac{\partial}{\partial u}\right)$$

or
$$yx\frac{\partial^3 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial \nu\partial u}.$$

With these substitution the equation takes the form

$$\frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u} - 4\frac{\partial^2 z}{\partial u\partial \nu} + 4\frac{\partial^2}{\partial \nu^2} - 4\frac{\partial z}{\partial \nu} + 6\frac{\partial z}{\partial \nu} = e^{3u}.e^{4\nu}$$

or
$$\frac{\partial^2 z}{\partial u^2} - 4\frac{\partial^2 z}{\partial u\partial \nu} + 4\frac{\partial^2 z}{\partial \nu^2} - \frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial \nu} = e^{3u+4\nu} \quad ...(1)$$

Denoting $\frac{\partial}{\partial u}$ by D and $\frac{\partial}{\partial \nu}$ by D' in (1).

$$(D^2 - 4DD' + 4D^2 - D + 2D')z = e^{2u+4\nu}.$$

$$[(D-2D')(D-2D'-1)]z = e^{2u+4\nu}$$

$\therefore$ The complementary function is

$$= \phi_1(\nu + 2u + e^u\phi_2(\nu + 2u).$$

$$= \phi_1(\log x^2y) + x\phi_2(\log x^2y)$$

$$= \phi(x^2y) + x\ \psi(x^2y)$$

$$\text{P.I.} = \frac{1}{(D-2D')(D-2D'-1)}e^{3u+4\nu}$$

$$= \frac{1}{(-5)(-6)}e^{3u+4\nu} = \frac{x^3y^4}{30}$$

$\therefore$ The solution is

$$z = \phi(x^2y) + x\psi(x^2y) + \frac{x^3y^4}{30}.$$

Self Assessment

Notes

15. Solve

$$\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}-\frac{1}{x^3}\frac{\partial z}{\partial x}-\frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}+\frac{1}{y^3}\frac{\partial z}{\partial y}=0$$

16. Solve

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=xy$$

20.8 Monge's Method

We shall usually take z as dependent and x, y as independent variables and throughout this chapter we shall denote

$\frac{\partial z}{\partial x}$ by p, $\frac{\partial z}{\partial y}$ by q, $\frac{\partial^2 z}{\partial x^2}$ by r, $\frac{\partial^2 z}{\partial x\partial y}$ by s, and $\frac{\partial^2 z}{\partial y^2}$ by t.

Monge's Method of Solving the Equation

$$Rr+Ss+Tt = V \qquad ...(1)$$

where r, s, t have their usual meanings and R, S, T and V are functions of x, y, z, p and q.

We know

$$dp = \frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy$$

$$= r\,dx+s\,dy$$

and

$$dq = \frac{\partial q}{\partial x}dx+\frac{\partial q}{\partial y}dy$$

$$= s\,dx + t\,dy.$$

Putting the values of r and t in (1),

$$R\left(\frac{dp-s\,dy}{dx}\right)+S.\,s+T.\left(\frac{dq-s\,dx}{dy}\right) = V$$

or $$R\,dp\,dy+T\,dq\,dx+Ss\,dx\,dy-Rs\,dy^2-Ts\,dx^2 = V\,dx\,dy$$

or $$(R\,dp\,dy+T\,dq\,dx-V\,dx\,dy) = s(R\,dy^2-S\,dx\,dy+T\,dx^2) \qquad ...(2)$$

If some relation between x, y, z, p, q makes each of the bracketed expressions vanish, the relation will satisfy (2); therefore

$$R\,dy^2-S\,dx\,dy+T\,dx^2 = 0 \qquad ...(3)$$

$$R\,dp\,dy+T\,dq\,dx-V\,dx\,dy = 0 \qquad ...(4)$$

Notes

Now it may be possible to get one or two relations between x, y, z, p, q called intermediate integrals, and then to find the general solution of (1).

If (3) resolves into two linear equations in dx and dy such as

$$dy - m_1\ dx = 0, \text{ and } dy - m_2\ dx = 0, \quad ...(5)$$

from one of the equations (5) combined with (4) and if necessary with $dz = p\,dx + q\,dy$, we may obtain two integrals $u_1 = a$ and $v_1 = b$; then $u_1 = f_1(v_1)$,

where f_1 is an arbitrary function, is an intermediate integral.

Proceeding similarly from the second equation, we may get another intermediate integral $u_2 = f_2(v_2)$.

From these two integrals we may find the values of p and q and putting these values in $dz = p\,dx + q\,dy$ and integrating it we get the complete integral of the original equation.

Illustrative Examples

Example 1: Solve by Monge's method $r = a^2 t$.

Solution: (This can be easily solved by the method discussed in the last section. Here we solve it by Monge's Method).

Putting $r = \dfrac{dp - s\ dy}{dx}$ and $t = \dfrac{dq - s\ dx}{dy}$ in the given equation, $dp\,dy - a^2\ dx\ dq = s(dy^2 - a^2\,dx^2)$.

So the subsidiary equations are

$$dy^2 - a^2\ dx^2 = 0 \quad ...(1)$$

and

$$dp\ dy - a^2\ dx\ dq = 0. \quad ...(2)$$

From (1)

$$dy + a\ dx = 0 \quad ...(3)$$

$$dy - a\ dx = 0. \quad ...(4)$$

Taking (3) and combining with (2), we get

$$dp + a\,dq = 0.$$

$$p + qa = A.$$

Also

$$y + ax = B.$$

$\therefore \quad p + aq = \phi_1(y + ax)$ is an intermediate integral.

Similarly $p - aq = \phi_2(y - ax)$ is the second intermediate integral.

From these,

$$p = \frac{1}{2}[\phi_1(y + ax) + \phi_2(y + ax)]$$

Notes

and $$q = \frac{1}{2a}[\phi_1(y+ax)-\phi_2(y-ax)]$$

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{1}{2}[\phi_1(y+ax)+\phi_2(y-ax)]\,dx + \frac{1}{2a}[\phi_1(y+ax)-\phi_2(y-dx)]dy$$

or $$dz = \frac{1}{2a}(dy + a\,dx)\phi_1(y+ax) - \frac{dy - a\,dx}{2a}\phi_2(y-ax),$$

or $$z = f_1(y+ax)+f_2(y-ax).$$

Example 2: Solve by Monge's method:

$$(b+cq)^2 r - 2(b+cq)(a+cp)s + (a+cp)^2 t = 0.$$

Solution. Putting

$$r = \frac{dp - s\,dy}{dx}, \qquad t = \frac{dq - s\,dx}{dy},$$

$$(b+cq)^2 \frac{dp - s\,dy}{dx} - 2(b+cq)(a+cp)s + (a+cp)^2 \frac{dq - s\,dx}{dy} = 0.$$

$\therefore$ The subsidiary equations are,

$$(b+cq)^2\,dy^2 + 2(b+cq)(a+cp)dx\,dy + (a+cp)^2\,dx^2 = 0, \qquad ...(1)$$

$$(b+cq)^2\,dp\,dy + (a+cp)^2\,dx = 0, \qquad ...(2)$$

From (1),

$$(p+cq)\,dy + (a+cp)\,dx = 0 \qquad ...(3)$$

Combining it with (2),

$$(b+cq)\,dp - (a+cp)\,dq = 0$$

From which $$\frac{dp}{a+cp} = \frac{dq}{b+cq}$$

and therefore, $$(a+cp) = A(b+cq). \qquad ...(4)$$

Also from (3) and $dz = p\,dx + q\,dy$, we get

$$a\,dx + b\,dy + c\,dz = 0$$

or $$ax + by + az = B. \qquad ...(5)$$

$\therefore$ From (4) and (5),

$$a + cp = (b+cq)\,\phi(ax+by+cz)$$

Notes

$$\therefore \quad \frac{dx}{c} = \frac{dy}{-c\phi} = \frac{dz}{-a + b\phi} = \frac{a\,dx + b\,dy + c\,dz}{0} \quad ...(6)$$

where ϕ stands for $\phi\,(ax + by + cz)$,

so that

$$ax + by + dz = K_1$$

and
$$\frac{dx}{c} = \frac{dy}{-c\phi(K_1)}$$

Integrating

$$x\phi(K_1) = -y + K_2.$$

$$\therefore \quad y + x\phi(ax + by + cz) = \psi(ax + by + cz). \qquad [\text{as } K_2 = \Psi\ (K_1)]$$

Example 3: Solve by Monge's method $r + (a+b)s + abt = xy$.

Solution: Putting

$$r = \frac{dp - s\,dy}{dx}, \text{ and } r = \frac{dq - s\,dx}{dy},$$

$$\frac{dp - s\,dy}{dx} + (a+b)s + ab\frac{dq - s\,dx}{dy} = xy$$

or $\quad dp\ dy + ab\ dq\ dx - xy\ dx\,dy = s[dy^2 - (a+b)dx\ dy + ab\,dx^2]$

The subsidiary equations are

$$dy^2 - (a+b)dx\ dy + ab\ dx^2 = 0 \quad ...(1)$$

and
$$dp\ dy + ab\ dq\ dx - xy\,dx\ dy = 0. \quad ...(2)$$

From (1)

$$dy - a\ dx = 0, \quad ...(3)$$

$$dy - b\ dx = 0, \quad ...(4)$$

Whence $y - ax = c_1$, and $y - bx = c_2$.

Combining these with (2), we get

$$a\,dp + ab\ dq - ax(c_1 + ax)\,dx = 0$$

and
$$b\,dp + abq - bx(c_2 + bx)\,dx = 0$$

or
$$p + bq - c_1\frac{x^2}{2} - \frac{ax^3}{3} = A,$$

$$\therefore \quad p + aq - c_2\frac{x^2}{2} - \frac{bx^3}{3} = B$$

Notes

or $$p + bq - (y - ax)\frac{x^2}{2} - \frac{ax^3}{3} = \phi_1(c_1) + \phi(y - ax)$$

$$p + aq - (y - bx)\frac{x^2}{2} - \frac{bx^3}{3} = \phi_2(c_2) = \phi_2(y - bx).$$

Solving,

$$p = \frac{1}{a-b}\left[\frac{yx^2}{2}(a-b) - (a^2 - b^2)\frac{x^2}{6} + a\phi_1(y - ax) - b\phi_2(y - bx)\right],$$

$$q = \frac{1}{b-a}\left[-\frac{x^3}{6}(a-b) + \phi_1(y - ax) - \phi_2(y - bx)\right]$$

Putting these values in $dz = p\,dx + q\,dy$,

$$dz = \left[\frac{yx^2}{2} - (a+b)\frac{x^3}{6} + \frac{a\phi_1(y-ax)}{a-b}dx - \frac{a\phi_2(y-bx)}{a-b}\right] + \left[\frac{x^3}{6} - \frac{\phi_1(y-ax)}{a-b} + \frac{\phi_2(y-bx)}{a-b}\right]dy$$

$$= -\frac{(a+b)x^3}{6}dx + \frac{3x^2y\,dx + x^3\,dy}{6} - \frac{1}{a-b}[\phi_1(y-ax)(dy - a\,dx)] + \frac{1}{a-b}[\phi_2(y-bx)(dy - b\,dx)]$$

$$\therefore\ z = -\frac{(a+b)x^3}{24} + \frac{yx^3}{6} + \Psi_1(y-ax) + \Psi_2(y-bx).$$

Note: This question could be solved by the method of Ist chapter also.

Example 4: Solve by Monge's method

$$q(1+q)r - (p+q+2pq)s + p(1+p)t = 0.$$

Solution: Putting

$$r = \frac{dp - s\,dy}{dx},\ t = \frac{dq - s\,dx}{dy}.$$

$$(q+q^2)\frac{dp - s\,dy}{dx} - (p+q+2pq)s + p(1+p)\frac{dq - s\,dx}{dy} = 0$$

or $$[(q+q^2)dp\,dy + (p+p^2)dq\,dx]$$

$$= s[(q+q^2)dy^2 + (q+q+2pq)dx\,dy + (p+p^2)dx^2]$$

$\therefore$ The subsidiary equations are

$$(q+q^2)dp\,dy + p(1+p)dq\,dx = 0 \quad ...(1)$$

and $$[(q+q^2)dy^2 + (p+q+2pq)dx\,dy + (p+p^2)dx^2] = 0 \quad ...(2)$$

From (2), $$q\,dy + p\,dx = 0 \quad ...(3)$$

Notes

and $(1+q)\,dy+(1+p)\,dx = 0$ ...(4)

From (3), and

$$dz = p\,dx+q\,dy \text{, we have}$$

$$dz = 0, \text{ or } z = C_1 \quad ...(5)$$

and from (4), and

$$dz = p\,dx + q\,dy, \text{ we have}$$

$$dx + dy + dz = 0,$$

or, $x + y + z = C_2$ ...(6)

Now combining (3) with (1)

$$(q-1)\,dp-(p+1)\,dq = 0 \quad ...(7)$$

and combining (4) with (1),

$$q\,dp - p\,dq = 0 \quad ...(8)$$

i.e., $dp - dq = 0$ [from (7) and (8)]

or $p - q = k_1 = \phi_1(C_1) = \phi_1(z)$

$\therefore$ $$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\phi_1(z)}$$

or $x = F_1(z)+k_2 = F_1(z)+F_2(C_2)$

$$= F_1(z)+F_2(x+y+z)$$

Example 5: Solve : $q^2r-2pqs+p^2t=0$ and show that the integral represents a surface generated by straight lines which are parallel to a fixed plane.

Solution: Putting

$$r = \frac{dp-s\,dy}{dx}, \text{ and } t = \frac{dq-s\,dx}{dy},$$

$$(q^2\,dp\,dy+p^2dq\,dx) = s(q^2\,dy^2+2pq\,dx\,dy+p^2\,dx^2)$$

$\therefore$ The subsidiary equations are

$$q^2dp\,dy+p^2dq\,dy = 0 \quad ...(1)$$

$$q\,dy+p\,dx = 0 \quad ...(2)$$

Also $dz = p\,dx+q\,dy=0.$

$\therefore$ $z = c.$

From (1) and (2),

or $q\,dp-p\,dq = 0$

or $$p/q = k = f(c)$$

$$p - qf(c) = 0.$$

$\therefore$ $$\frac{dx}{1} = \frac{dy}{-f(c)} = \frac{dz}{0},$$

$$y + xf(c) = K = F(c)$$

$$y + xf(z) = F(z). \quad ...(3)$$

The integral of the differential equation is the surface (3) which is the locus of the straight lines given by the intersections of planes $y + xf(c) = F(c)$, and $z = c$. These lines are all parallel to the plane $z = 0$ as they lie on the plane $z = c$ for varying values of c.

Example 6: Solve by Monge's method

$$r - a^2t + 2ab(p + qa) = 0.$$

Solution: Putting

$$r = \frac{dp - s\,dy}{dx} \text{ and } t = \frac{dq - s\,dx}{dy}, \text{ we get}$$

$$dp\,dy - a^2\,dq\,dx + 2ab(p + aq)dx\,dy = s(dy^2 - a^2\,dx^2)$$

$\therefore$ The subsidiary equations are

$$dy^2 - a^2dx^2 = 0 \quad ...(1)$$

$$dp\,dy - a^2\,dq\,dx + 2ab(p + qa)dx\,dy = 0 \quad ...(2)$$

From (1),

$$y + ax = \alpha, \quad ...(3)$$

$$y - ax = \beta. \quad ...(4)$$

From (3) and (2)

$$dp + a\,dq + 2ab(p + qa)dx = 0$$

or $$\frac{dp + a\,dq}{p + aq} = -2ab\,dx$$

$\therefore$ $$\log(p + qa) = -2abx + \log c,$$

$\therefore$ $$\frac{p + aq}{c} = \frac{(p + aq)}{f(\alpha)} = e^{-2abx}$$

or $$p + qa = f(\alpha)\,e^{-2abx} \quad ...(5)$$

$\therefore$ $$\frac{dx}{1} = \frac{dy}{a} = \frac{dz}{f(\alpha)e^{-2abx}}$$

Notes

Integrating,

$$\frac{f(\alpha)e^{-2abx}}{-2ab} = z+k = z+\phi(\beta)$$

$$z = f_1(y+ax)e^{-2abx} + f_2(y-ax)$$

Example 7: Solve by Monge's method

$$r - t\cos^2 x + p\tan x = 0.$$

Solution: Putting

$$r = \frac{dp - s\,dy}{dx},\ t = \frac{dq - s\,dx}{dy},\ \text{we get}$$

$$dp\,dy - \cos^2 x\,dx\,dq + q\tan x\,dx\,dy = s(dy^2 - \cos^2 x\,dx^2).$$

∴ The subsidiary equations are

$$dy^2 - \cos^2 x\,dx^2 = 0, \quad ...(1)$$

$$dp\,dy - \cos^2 x\,dx\,pq + p\tan x\,dx\,dy = 0. \quad ...(2)$$

From (1), $y = \sin x + \alpha$, ...(3)

$$y = -\sin x + \beta. \quad ...(4)$$

From (2) and (3),

$$\cos x\,dp - \cos^2 x\,dq + p\sin x\,dx = 0$$

or $$\sec x\,dp - dq + p\tan x\sec x\,dx = 0$$

or $$p\sec x - q = c_1 = f(a) = f(y-\sin x).$$

∴ $$\frac{dx}{\sec x} = \frac{dy}{-1} = \frac{dz}{(y-\sin x)}$$

and hence,

$$f(y-\sin x)\frac{(dy-\cos x\,dx)}{2} = -dz.$$

∴ $$F(y-\sin x)+2z = c_2G(\beta).$$

∴ $$F(y-\sin x)+2z = G(y+\sin x). \quad \text{[From (4)]}$$

Example 8: Solve the equation by Monge's method:

$$t - r\sec^4 y = 2q\tan y.$$

Solution: Putting

$$r = \frac{dp - s\,dy}{dx},\ t = \frac{dq - s\,dx}{dy}$$

$$\frac{dq - s\,dx}{dy} - \frac{dp - s\,dy}{dx}\sec^4 y = 2q \tan y$$

or $$dq\,dx - \sec^4 y\,dp\,dy - 2q \tan y\,dx\,dy = s(dx^2 - \sec^4 y\,dy^2)$$

∴ Subsidiary equations are

$$dx^2 - \sec^4 y\,dy^2 = 0 \quad ...(1)$$

$$dq\,dx - \sin^4 y\,dp\,dy - 2q \tan y\,dx\,dy = 0 \quad ...(2)$$

From (1) $x = \tan y + \alpha$. ...(3)

$$x = -\tan y + \beta. \quad ...(4)$$

From (2) and (3)

$$\sec^2 y\,dq\,dy - \sec^4 y\,dp\,dy - 2q \tan y \sec^2 y\,dy^2 = 0$$

or $$dq - \sec^2 y\,dp - 2q \tan y\,dy = 0$$

or $$\cos^2 y\,dq - dp - 2q \sin y \cos y\,dy = 0$$

or $$q\cos^2 y - p = C = f(x - \tan y)$$

∴ $$\frac{dx}{-1} = \frac{dy}{\cos^2 y} = \frac{dz}{f(x - \tan y)}$$

or $$\frac{dx - \sec^2 y\,dy}{2} = \frac{-dz}{f(x - \tan y)}$$

∴ $$\frac{1}{2} f(x - \tan y)(dx - \sec^2 y\,dy) = -dz$$

∴ $$F(x - \tan y) + 2z = K.$$

or $$F(x - \tan y) + 2z = \phi(x + \tan y) \quad \text{from (4)}$$

∴ The solution is

$$z = \phi_1(x - \tan y) + \phi_2(x + \tan y).$$

Self Assessment

Solve the following differential equations by Monge's method

17. $2x^2 r - 5xys + 2y^2 t + 2(px + qy) = 0$

18. $pt - qs = q^3$

Notes

20.9 Monge's Method of Integrating $Rr + Ss + Tt + U(rt - s^2) = V$

R, S, T, U are functions of x, y, z, p, q.

As before put

$$r = (dp - s\,dy)/dx$$

and $$t = (dq - s\,dx)/dy.$$

The equation reduces to

$$R\,dp\,dy + T\,dq\,dx + U\,dp\,dq - V\,dx\,dy - s(R\,dy^2 - S\,dx\,dy + T\,dx^2 + U\,dp\,dx + V\,dp\,dy) = 0$$

or $\quad N - Ms = 0.$

So far, we used to factorise M, but on account of the presence of $U\,dx\,dp + V\,dq\,dy$, the factors are not possible; so let us try to factorise $M + \lambda N$, where λ is some multiplier to be determined later.

$$\text{Now } \lambda N + M = \lambda(R\,dp\,dy + T\,dq\,dx + U\,dp\,dq - V\,dx\,dy)$$

$$+(R\,dy^2 - S\,dx\,dy + T\,dx^2 + U\,dp\,dx + V\,dq\,dy)$$

$$= R\,dy^2 + T\,dx^2 - (S + \lambda V)\,dx\,dy + U\,dp\,dx + U\,dq\,dy + \lambda R\,dp\,dy + \lambda T\,dq\,dx + \lambda U\,dp\,dq.$$

Let the factors of the above be

$$\alpha\,dy + \beta\,dx + \gamma\,dp \text{ and } \alpha'\,dy + \beta'\,dx + \gamma'\,dq.$$

Equating coefficient of dy^2, dx^2, $dp\,dq$ in the product,

$$\alpha\alpha' = R, \beta\beta' = T, \gamma\gamma' = \lambda U.$$

Now if we take

$$\alpha = R, \alpha' = 1, \beta = kT, \beta' = (1/k), \gamma = mU, \gamma' = \lambda/m$$

equating the coefficients of the other five terms.

$$kT + R/k = -(S + \lambda V). \quad ...(1)$$

$$\lambda R/m = U, \quad ...(2)$$

$$kT\lambda/m = \lambda T, \quad ...(3)$$

$$mU = \lambda R, \quad ...(4)$$

$$mU/k = U. \quad ...(5)$$

From (5), $m = k$ and this satisfies (3).

From (2) and (3), $m = \lambda R/U = k.\left(\text{on putting } k = \dfrac{\lambda R}{U}\right)$

$\therefore$ From (1),

$$\lambda^2(RT + UV) + \lambda US + U^2 = 0 \quad ...(6)$$

The first step in practical working is to form the equation (6) in λ and to determine the two roots λ_1 and λ_2 of this equation.

So if λ_1 is a root of (6), factorised $M + \lambda N$ is

$$\left(R\,dy + \lambda_1 \frac{RT}{U} dx + \lambda_1 R\,dp\right)\left(dy + \frac{U}{\lambda_1 R} dx + \frac{U}{R} dq\right)$$

Or $\quad \frac{R}{U}(U\,dy + T\lambda_1 dx + \lambda_1 U\,dp)\frac{1}{\lambda_1 R}(\lambda_1 R\,dy + U\,dp + \lambda_1 U\,dq).$

Similarly if λ_2 is a root of (6), the same is,

$$\frac{R}{U}(U\,dy + T\lambda_2 dx + \lambda_2 U\,dp)\times\frac{1}{\lambda_2 R}(\lambda_2 R\,dy + U\,dx + \lambda_2 U\,dq).$$

Now we may obtain two integrals $u_1 = a_1,\ v_1 = b_1$ of the equations

and $$\left.\begin{aligned} U\,dy + \lambda_1 T\,dx + \lambda_1 U\,dp = 0 \\ U\,dx + \lambda_2 R\,dy + \lambda_2 U\,dq = 0 \end{aligned}\right\} \quad ...(7)$$

or we may obtain two integrals $u_2 = a_2,\ v_2 = b_2$ of the equations

$$\left.\begin{aligned} U\,dy + \lambda_2 T\,dx + \lambda_2 U\,dp = 0 \\ U\,dx + \lambda_1 R\,dy + \lambda_1 U\,dq = 0 \end{aligned}\right\} \quad ...(8)$$

Sets of equations (7) and (8), when written down, constitute the second important step in the solution of the given equation.

Thus we get two intermediate integrals $u_1 = f_1(v_1)$ and $u_2 = f_2(v_2)$ and substituting in $dz = p\,dz + q\,dy$, the values of p and q obtained from the two intermediate integrals, and we get the solution after integrating.

In case the two roots of the equation (6) are equal, we shall get only intermediate integral $u_1 = f_1(v_1)$ which together with one of the integrals $u_1 = a_1$ and $v_1 = b_1$ will give values of p and q suitable to solve $dz = p\,dx + q\,dy$.

If it is not possible to obtain the values of p and q from the two intermediate integrals $u_1 = f_1(v_1)$ and $u_2 = f_2(v_2)$, suitable for integration in $dz = p\,dx + q\,dy$, we may take one of the intermediate integrals say $u_1 = f_1(v_1)$ and one of the integrals from $u_2 = a_2$ and $v_2 = b_2$.

The values of p and q obtained from these and substituted in $dz = p\,dx + q\,dy$ will give the solution of the given equation.

Illustrative Examples

Example 1: Solve:

$ar + bs + ct + e(rt - s^2) = h$ where a, b, c, e and h are constants.

Solution: Here $R = a, S = b, T = c, U = e, V = h$

The equation in λ is

$$\lambda^2(ac + eh) + \lambda be + e^2 = 0. \quad ...(1)$$

Putting $$\lambda = -e/m, \quad ...(2)$$

(1) becomes

$$\frac{e^2}{m^2}(ac + eh) - \frac{e^2 b}{m} + e^2 = 0$$

Notes

or $$m^2 - bm + (ac + eh) = 0 \qquad ...(3)$$

If m_1, m_2 are the roots of (3), the first system of intermediate integrals is given by

$$U\,dy + \lambda_1 T\,dx + \lambda_1 U\,dp = 0,$$

$$U\,dx + \lambda_2 R\,dy + \lambda_2 U\,dq = 0,$$

i.e., by $$e\,dy + \left(-\frac{e}{m_1}\right)c\,dx + \left(-\frac{e}{m_1}\right)e\,dp = 0.$$

$$e\,dx + \left(-\frac{e}{m_2}\right)a\,dy + \left(-\frac{e}{m_2}\right)e\,dq = 0.$$

or by $$c\,dx + e\,dp - m_1\,dy = 0,$$

$$a\,dy + e\,dq - m_2\,dx = 0;$$

so one of the intermediate integrals is

$$cx + ep - m_1 y = f(ay + eq - m_2 x). \qquad ...(4)$$

Similarly the second intermediate integral is

$$(cx + ep - m_1 y) = F(ay + ap - m_1 x), \qquad ...(5)$$

It is not possible to get the values of p and q from (4), (5); so we combine (4) with $cx + ep - m_2 y = A$,

Thus we have

$$(m_2 - m_1)y + A = f(ay + eq - m_2 x)$$

or $$ay + eq = m_2 x + \phi[(m_2 - m_1)y + A]$$

where ϕ is inverse function of f.

This gives q, and $cx + ep - m_2 y = A$ gives p.

Substituting these values in $dz = p\,dx + q\,dy$,

$$e\,dz = (A - cx + m_2 y)dx + [-ay + m_2 x + \phi\{(m_2 - m_1)y + A\}]dy.$$

Integrating,

$$ez + \frac{cx^2}{2} + \frac{ay^2}{2} = m_2 xy + Ax + \{\Psi(m_2 - m_1)y + A\} + B$$

where $$\Psi(t) = \frac{f\phi(t)\,dt}{m_2 - m_3}$$

Example 2: Solve:

$$z(1+q^2)r - 2pqzs + z(1+p^2)t - z^2(s^2 - rt) + 1 + p^2 + q^2 = 0.$$

Notes

Solution: Here

$$R = z(1+q^2),\ S = -2pqz,\ T = (1+p^2)z$$

$$U = z^2, V = -(1+p^2+q^2).$$

The equation in λ is

$$(RT+UV)\lambda^2 + \lambda US + U^2 = 0$$

or $$z^2\lambda^2 p^2 q^2 - 2\lambda z^3 pq + z^4 = 0$$

or $$p^2 q^2 \lambda^2 - 2z\lambda pq + z^2 = 0$$

or $$\lambda = z/pq. \qquad \text{(roots are equal).}$$

$\therefore$ The system of intermediate integrals is given by

$$U\,dy + \lambda T\ dx + \lambda U\ dp = 0$$

$$U\,dx + \lambda R\,dy + \lambda U\,dq = 0.$$

i.e., by $$pq\ dy + (1+p^2)dx + zdp = 0$$

$$pq\,dx + (1+q^2)dy + zdq = 0.$$

Also $$dz = p\ dx + q\ dy.$$

We write (1) as

$$dx + p(p\,dx + q\,dy) + z\,dp = 0,$$

With the help of (3), it reduces to

$$dx + p\ dz + z\ dp = 0$$

or $$x + pz = \alpha.$$

Similarly from (2) and (3), $y + zq = \beta$.

Putting the values of p and q in $dz = p\ dx + q\ dy$,

$$dz = \frac{\alpha - x}{z}dx + \frac{\beta - y}{z}dy$$

or $$-z\,dz = (\alpha - x)(-dx) + (\beta - y)(-dy)$$

or $$-\frac{z^2}{2} = \frac{(\alpha - x)^2}{2} + \frac{(\beta - y)^2}{2} + k$$

or $$z^2 + (x-\alpha)^2 + (y-\beta)^2 = \lambda^2$$

Where α, β, λ are constants.

Example 3: Solve: $(1+q^2)r - 2pqs + (1+p^2)t$

$$+(1+p^2+q^2)^{-1/2}(rt - s^2) = -(1+p^2+q^2)^{3/2}.$$

Notes

Solution: Here $R = (1+q^2),\ S = -2pq, T = (1+p^2),$

$$U = (1+p^2+q^2)^{-1/2}, V = -(1+p^2+q^2)^{3/2}.$$

The equation in λ is $(RT+UV)\lambda^2 + \lambda US + U^2 = 0$

or $$[(1+p^2)(1+q^2)-(1+p^2+q^2)]\lambda^2 + \lambda\frac{(-2pq)}{\sqrt{1+p^2+q^2}} + \frac{1}{1+p^2+q^2} = 0$$

or $$\lambda^2 p^2 q^2 (1+p^2+q^2) - 2pq\sqrt{(1+p^2+q^2)}\,\lambda + 1 = 0$$

or $$\lambda = \frac{1}{pq\sqrt{(1+p^2+q^2)}}$$ (roots being equal).

We get only one system which will give only one intermediate integral.

The system is $U\,dy + \lambda T\ dx + \lambda U\ dp = 0,$

$$U\ dx + \lambda R\,dy + \lambda U\,dq = 0,$$

$$\frac{1}{\sqrt{(1+p^2+q^2)}}dy + \frac{(1+p^2)}{pq\sqrt{(1+p^2+q^2)}}dx + \frac{dp}{dq(1+p^2+q^2)} = 0$$

$$\frac{1}{\sqrt{(1+p^2+q^2)}}dx + \frac{(1+q^2)}{pq\sqrt{(1+p^2+q^2)}}dy + \frac{dq}{pq(1+p^2+q^2)} = 0$$

or $$pq\ dy + (1+p^2)dx + \frac{dp}{\sqrt{(1+p^2+q^2)}} = 0,$$

$$pq\,dx + (1+q^2)dy + \frac{dq}{\sqrt{(1+p^2+q^2)}} = 0.$$

Eliminating

$$dy, [(1+p^2)(1+q^2) - p^2q^2]dx + [(1+q^2)dp - pq\,dq]/\sqrt{(1+p^2+q^2)}$$

or $$dx + \frac{(1+q^2)\,dp - pq\ dq}{(1+p^2+q^2)^{3/2}} = 0$$

or $$dx + \frac{(1+p^2+q^2)\,dp}{(1+p^2-q^2)^{3/2}} - \frac{(p^2dp + pq\ dq)}{(1+p^2+q^2)^{3/2}} = 0$$

or $$dx + (1+p^2+q^2)^{-1/2}\ dp - \frac{\frac{1}{2}p(2p\ dp + 2q\ dq)}{(1+p^2+q^2)^{3/2}} = 0$$

or $$x + p\,(1+p^2+q^2)^{-1/2} = \alpha. \qquad ...(1)$$

Similarly eliminating $dx, y + q(1+p^2+q^2)^{-1/2} = \beta$ ...(2)

From (1) and (2),

$$\frac{(x-\alpha)}{(y-\beta)} = \frac{p}{q}. \qquad ...(3)$$

Substituting in (1) the value of p as found from (3),

$$q = \frac{y-\beta}{\sqrt{[1-\{(x-\alpha)^2+(y-\beta)^2\}]}}$$

Similarly from (3) and (2),

$$p = \frac{x-\alpha}{\sqrt{[1-\{(x-\alpha)^2+(y-\beta)^2\}]}}$$

Now, $$dz = p\,dx + q\,dy$$

or $$dz = \frac{(x-\alpha)\,dx+(y-\beta)dy}{\sqrt{[1-\{(x-\alpha)^2.+(y+\beta)^2]}}$$

Integrating,

$$(z-\gamma) = -[1-\{(x-\alpha)^2+(y-\beta)^2\}]^{1/2}$$

or $$(z-\gamma)^2 = 1-[(x-\alpha)^2+(y-\beta)^2]$$

or $$(x-\alpha)^2+(y-\beta)^2+(z-y)^2 = 1.$$

Example 4: Solve $s^2-rt=a^2$

or $$rt-s^2 = -a^2.$$

Solution: Here $R=0, S=0, T=0, U=1, V=-a^2$.

∴ The equation in λ is

$$\lambda^2(-a^2)+\lambda.\,0+1 = 0$$

or $$\lambda = \pm 1/a.$$

The two intermediate integrals are given by

$$\left.\begin{aligned} -dy-\frac{1}{a}dp &= 0, \\ -dx+\frac{1}{a}dq &= 0. \end{aligned}\right\} \qquad ...(a)$$

$$\left.\begin{aligned} -dy+\frac{1}{a}dp &= 0, \\ -dx-\frac{1}{a}dq &= 0. \end{aligned}\right\} \qquad ...(b)$$

Notes

From (a),

$$\left.\begin{aligned} p + ay &= F(\alpha) \\ q - ax &= \alpha \end{aligned}\right\} \quad ...(c)$$

and from (b),

$$\left.\begin{aligned} p - ay &= F(\beta) \\ q + ax &= \beta \end{aligned}\right\} \quad ...(d)$$

i.e., the two intermediate integrals are

$$p + ay = f(q - ax) \quad ...(1)$$

and $$p - ay = F(q + ax) \quad ...(2)$$

Now since it is not possible to find the values of p and q from (1) and (2), we proceed as follows. Suppose α, β are not constants, but parameters.

Solving (*c*) and (*d*),

$$x = \frac{\beta - \alpha}{2a}, q = \frac{\alpha + b}{2}. \quad ...(3)$$

$$p = \frac{1}{2}[F(\alpha) + f(\beta)], \quad ...(4)$$

$$y = \frac{1}{2a}[F(\alpha) - f(\beta)]. \quad ...(5)$$

Substituting these values in $dz = p\, dx + q\, dy$,

$$dz = \frac{1}{4a}[F(\alpha) + f(\beta)]\,(d\beta - dx) + \frac{\alpha + \beta}{4a}[F'(\alpha)\,d\alpha - f'(\beta)\,d\beta]$$

$$= \frac{1}{4a}[\{F(\alpha)\,d\beta + \beta F'(\alpha)\,d\alpha\} - \{f(\beta)\,d\alpha + \alpha F'(\beta)\,d\beta\}]$$

$$+ \frac{1}{4a}[\{F(\alpha)d\alpha + \alpha F'(\alpha)d\alpha\} - \{f(\beta)d\beta + \beta f'(\beta)dB\}] + \frac{1}{4a}[2f(\beta)\; d\beta - 2F(\alpha)\,d\alpha].$$

$$\therefore \quad z = \frac{1}{4a}[\beta F(\alpha) - \alpha f(\beta) - \beta f(\beta) + \alpha F(\alpha)] + \frac{2}{4a}\int f(\alpha)\; d\beta - \frac{2}{4a}\int F(\beta)\,d\alpha$$

$$= \frac{1}{4a}[F(\alpha))(\alpha + \beta) - f(\beta)(\alpha + \beta)] + \frac{2}{4a}G(\beta) - \frac{2}{4a}\phi(\alpha)$$

$$= \frac{\alpha + \beta}{2}\left[\frac{F(\alpha) - f(\beta)}{2a}\right] + \frac{1}{2a}G(\beta) - \frac{1}{2a}\phi\,(\alpha)$$

or $$z - qy = \Psi_1(q + ax) + \phi_2(q - ax) \quad \text{[from (3) and (5)]}$$

where

Notes

$$\Psi_1(t) = \int \frac{f(t)}{2a} dt. \qquad ...(6)$$

and $$\Psi_2(t) = -\int \frac{F(t)}{2a} dt. \qquad ...(7)$$

Hence the primitive is

$$z - qy = \Psi_1(q + ax) + \Psi_2(q - ax)$$

$$-y = \phi_1'(q + ax) + \Psi_2'(q - ax) \qquad \text{[from (5), (6) and (7)].}$$

Example 5: Solve:

$$rq + (p + x)s + yt + y(rt - s^2) + q = 0$$

Solution: Here $R = q, S = (p + x), T = y, U = y, V = -q.$

The equation in λ is

$$\lambda^2[qy - qy] + \lambda.y(p + x) + y^2 = 0$$

or $$\lambda = \infty, \text{ or } \lambda = -y/(p + x).$$

$\therefore$ The intermediate integrals are given by

$$\left.\begin{aligned} y\,dy - \frac{y^2}{p + x}dx - \frac{y}{p + x}dp &= 0 \\ \frac{y}{\infty}dx + q\,dy + y\,dq &= 0 \end{aligned}\right\} \qquad ...(a)$$

$$\left.\begin{aligned} y\,dx - \frac{qy}{p + x}dy - \frac{y^2\,dq}{p + x} &= 0 \\ \frac{y}{\infty}dy + y\,dx + y\,dp &= 0 \end{aligned}\right\} \qquad ...(b)$$

From (a)

$$[(p + x)/y] = \alpha \qquad ...(1)$$

$$qy = F(\alpha) \qquad ...(2)$$

or one of the integrals is

$$qy = F[(p + x)/y].$$

From second equation of (*b*),

$$p + x = \beta, \frac{p + x}{y} = \frac{\beta}{y} = \alpha \qquad \text{[from (1)]...(2')}$$

$$p = \beta - x. \qquad ...(3)$$

Notes

and from (2) and (1),

$$q = \frac{1}{y}F\left(\frac{p+x}{y}\right) = \frac{1}{y}F\left(\frac{\beta}{y}\right) = \frac{1}{y}F(\alpha) \qquad \text{[from (2′)]}$$

$$= \frac{\alpha}{\beta}.F(\alpha) \quad \left[\because \text{ From (1) and (3)}, \frac{1}{y} = \frac{\alpha}{\beta}\right] \qquad \text{...(4)}$$

Now

$$dz = p\,dx + q\,dy$$

$$= (\beta - x)\,dx + \frac{\alpha}{\beta}F(\alpha)\,dy \qquad \text{[from (3) and (4)]}$$

$$\therefore \quad z = \beta x - \frac{x^2}{2} + \frac{\alpha}{\beta}F(\alpha)\,y + k$$

$$= \beta x - \frac{x^2}{2} + \frac{1}{y}F\left(\frac{\beta}{y}\right)y + \phi(\beta)$$

or

$$z = \beta x - \frac{x^2}{2} + F\left(\frac{\beta}{y}\right) + \phi(\beta)$$

Example 6: Solve:

$$5r + 6s + 3t + 2(rt - s^2) + 3 = 0 \qquad \text{...(1)}$$

Solution: Comparing it with

$$Rr + Ss + Tt + U(rt - s^2) = V$$

We have $R = 5, S = 6, T = 3, U = 2, V = -3$

The λ-quadratic will be

$$\lambda^2(UV + RT) + \lambda\, SU + U^2 = 0$$

or

$$9\lambda^2 + 12\lambda + 4 = 0$$

or

$$(3\lambda + 2)^2 = 0$$

$$\therefore \quad \lambda_2 = -\frac{2}{3}, \qquad \lambda = -\frac{2}{3}.$$

The intermediate integral will be

$$U\,dy + \lambda_1 T\,dx + \lambda_1 U.\,dp = 0$$

and

$$\lambda_2 R\,dy + U\,dx + \lambda_2 U.\,dq = 0$$

or

$$3\,dy - 3\,dx - 2\,dp = 0 \quad \text{and} \quad -5\,dy + 3\,dx - 2\,dq = 0.$$

Notes

Integrating,

$$3y - 3x - 2p = a, \qquad -5y + 3x - 2q = b \qquad ...(2)$$

∴ The intermediate integral is

$$3y - 3x - 2p = f(-5y + 3x - 2q) \qquad ...(3)$$

From (2),

$$p = \frac{1}{2}(3y - 3x - a), \qquad q = \frac{1}{2}(-5y + 3x - b)$$

Putting these values of p and q in

$$dz = p\,dx + q\,dy$$

$$dz = \frac{1}{2}(3y - 3x - a)\,dx + \frac{1}{2}(-5y + 3x - b)\,dy$$

or

$$2\,dz = 3(y\,dx + x\,dy) - 3x\,dx - 5y\,dy - a\,dx - b\,dy$$

Integrating

$$2z = 3xy - \frac{3}{2}x^2 - \frac{5}{2}y^2 - ax - by + c$$

This is the required complete integral of (1).

Self Assessment

19. Solve

$2s + (rt - s^2) = 1$

20. Solve

$3r + 4s + t + (rt - s^2) = 1$

20.10 Summary

- The partial differential equations are classified according to their structure.
- Similar method as used in ordinary differential equations is adopted for partial differential equations with constant coefficients.
- The methods, adopted in solving various equations are given in details. It is advisable to understand the partial differential equations and apply the appropriate methods.

20.11 Keywords

C.F. or Complimentary Function is the solution of the partial differential equations containing a number of arbitrary constants.

P.I. or Particular Integral is the particular solution of the partial differential equation containing any arbitrary constants.

Notes

20.12 Review Questions

1. Solve

$$\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$$

2. Solve

$$(D^3 - 3D^2D' + 2DD'^2)z = 0$$

3. Solve

$$\frac{\partial^2 z}{\partial x^2} - 2a\frac{\partial^2 z}{\partial x\partial y} + a^2\frac{\partial^2 z}{\partial y^2} = 0$$

4. Solve

$$\frac{\partial^4 z}{\partial x^4} - 2\frac{\partial^4 z}{\partial x^3\partial y} - 3\frac{\partial^4 z}{\partial x^2\partial y^2} + 8\frac{\partial^4 z}{\partial x\partial y^3} - 4\frac{\partial^4 z}{\partial y^4} = 0$$

5. Solve

$$\frac{\partial^2 z}{\partial x^2} + (a+b)\frac{\partial^2 z}{\partial x\partial y} + ab\frac{\partial^2 z}{\partial y^2} = xy$$

6. Solve

$$\left(\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2}\right) = e^{x+2y}$$

7. Solve

$$\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\partial y} + \frac{\partial z}{\partial y} - z\right) = \cos(x+2y) + e^y$$

8. Solve

$$(DD' + D - D - 1)z = xy$$

9. Solve

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = x^2 y$$

10. Solve

$$r + t - (rt - s^2) = 1$$

Answers: Self Assessment

1. $Z = F_1(y+mx) + F_2(y+3x) + F_3(y+2x)$

2. $Z = F_1\left(y - \frac{x}{2}\right) + F_2(y-2x)$

3. $Z = F_1(y) + F_2(y+2x) + xF_3(y+2x)$

4. $Z = F_1(y+3x) + x\,F_2(y+3x)$

5. $Z = F_1(y-2x) + F_2(y+3x) + \frac{x^3}{6}y + \frac{x^4}{24}$

6. $Z = F_1(y-2x) + F_2(y-x) + \frac{x^3}{6} + \frac{y^3}{12}$

7. $Z = F_1(y-2x) + F_2(y-x) + \frac{1}{36}(x+y)^3$

8. $Z = F_1(y-ix) + F_2(y+ix) - \frac{1}{(m^2+n^2)}\cos(mx+xy)$

9. $Z = F_1(y+x) + x\,F_2(x+y) + \frac{x^3}{6} + \frac{x^2}{2}\phi(x+y)$

10. $Z = F_1(y) + F_2(y+2x) + x\,F_3(y+2x) + \frac{x^2}{4}\sin(2x+y)$

11. $Z = F_1(y-x) + x\,F_2(y-x) + x\sin y$

12. $Z = F_1(y+2x) + F_2(y-x) + y\,e^x$

13. $Z = F_1(y-ax) + e^{2abx}F_2(y+ax)$

14. $Z = e^x F_1(y) + e^{-x}F_2(y-x) + \frac{1}{2}\sin(x+2y)$

15. $Z = F_1(y^2+x^2) + F_2(y^2-x^2)$

16. $Z = F_1(xy) + x\,F_2\left(\frac{y}{x}\right) + xy\log x$

17. $Z = F_1(x^2y) + F_2(xy^2)$

18. $y = zx + F_1(z) + F_2(x)$

19. $Z = xy + C_1x + C_2y + C_3$

20. $Z = 2xy - \frac{1}{2}(x^2+3y^2) + C_1x + \psi(y+mx)$

20.13 Further Readings

Books

Piaggio, H.T.H., Differential Equations

Sneddon L.N., Elements of Partial Differential Equations.

Notes

Unit 21: Classifications of Second Order Partial Differential Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Observe that the partial differential equations of the second order can be of linear type or non-linear type.
- Understand that linear partial differential equations can be classified into three categories, namely hyperbolic, parabolic and elliptic type.
- Know that we have equations having variable coefficients there are some cases where the equations involve variable coefficients but they can be transformed into equations with constant coefficients.

Introduction

Classification of the partial differential equations help us in solving them in a systematic way. It is advisable to understand the type of the partial differential equation before trying to solve it.

The methods of solving various classes of differential equations are also different.

21.1 Classification of Linear, Second Order Partial Differential Equations in two Independent Variables

Consider a second order linear partial differential equation in two independent variables x and y which can be written as

$$a(x, y)\frac{\partial^2 \phi}{\partial x^2} + 2b(x, y)\frac{\partial^2 \phi}{\partial x \partial y} + c(x, y)\frac{\partial^2 \phi}{\partial y^2} + d_1(x, y)\frac{\partial \phi}{\partial x} + d_2(x, y)\frac{\partial \phi}{\partial x} + d_3(x, y)\phi = f(x, y) \quad ...(1)$$

Notes

It will be seen that the first three terms of equation (1) allow us to classify the equation into one of three distinct types: *Elliptic*, for example Laplace's equation, *Parabolic*, for example the diffusion equation or *Hyperbolic*, for example the wave equation as follows:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad \text{(Laplace equations for two variables } x, y\text{)}$$

$$K\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t} \qquad \text{(Diffusion equation)}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 V}{\partial t^2} \qquad \text{(Wave equation)}$$

Each of these types of equation has distinctive properties. We would like to know about those properties of equation (1) that are unchanged by any change of co-ordinates since these must be of fundamental significance and not just a result of our choice of co-ordinate system. We can write this change of co-ordinates as

$$(x, y) \to \{\varepsilon(x, y), \eta(x, y)\}$$

with

$$\frac{\partial(\varepsilon, \eta)}{\partial(x, y)} \neq 0 \qquad ...(2)$$

If equation represents a model physical system, a change of co-ordinates should not affect its qualitative behaviour. Writing $\phi(x, y) \equiv \psi(\varepsilon, \eta)$ and using subscripts to denote partial derivatives, we find that

$$\phi_x = \varepsilon_x\psi_\varepsilon + \eta_x\psi_\eta,\ \phi_{xx} = \varepsilon^2{}_x\psi_{\varepsilon\varepsilon} + 2\varepsilon_x\eta_x\psi_{\varepsilon x} + \eta^2{}_x\psi_{\eta\eta} + \varepsilon_{xx}\psi_\varepsilon + \eta_{xx}\psi_\eta$$

and similarly for the other derivatives. Substituting these into equation (1) gives us

$$A\psi_{\varepsilon\varepsilon} + 2B\psi_{\varepsilon\eta} + C\psi_{\eta\eta} + b_1(\varepsilon, \eta)\psi_\eta + b_2(\varepsilon, \eta)\psi_\eta + b_3(\varepsilon, \eta)\psi = g(\varepsilon, \eta) \qquad ...(3)$$

where

$$\begin{aligned} &A + a\varepsilon_x^2 + 2b\varepsilon_x\eta_x + c\eta_y^2, \\ &B + a\varepsilon_x\eta_x + b(\eta_x\varepsilon_y + \eta_y\varepsilon_x) + c\varepsilon_y\eta_y \\ &C + a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2, \end{aligned} \qquad ....(4)$$

We do not need to consider other co-efficient functions b_1, (ε, η), $b_2(\varepsilon, \eta)$, $b_3(\varepsilon, \eta)$.

We can express (4) in a concise matrix form as

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} + \begin{pmatrix} \varepsilon_x & \eta_x \\ \varepsilon_y & \eta_y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \varepsilon_x & \varepsilon_y \\ \eta_x & \eta_y \end{pmatrix} \qquad ...(5)$$

which shows that

$$\det\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \det\begin{pmatrix} a & b \\ b & c \end{pmatrix}\left(\frac{\partial(\varepsilon, \eta)}{\partial(x, y)}\right)^2 \qquad ...(6)$$

In (6) $\left(\frac{\partial(\varepsilon, \eta)}{\partial(x, y)}\right)$ = Jacobian of transformation.

This shows that the sign of a $c - b^2$ is independent of the choice of co-ordinate system which allows us to classify the equation.

An *Elliptic* equation has $ac < b^2$, for example Laplace equation

Notes

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}.$$

A *Parabolic* equation has $ac = b^2$, for example the diffusion equation

$$K\frac{\partial^2\phi}{\partial x^2}-\frac{\partial\phi}{\partial y}=0 \quad \text{...(here } y = t\text{)}$$

A *hyperbolic* equation has $ac < b^2$, for example the wave equation

$$\frac{\partial^2\phi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{...(here } y \text{ is time)}$$

21.2 Canonical Form

Any equation of the form (1) can be written in Canonical form by choosing the canonical co-ordinate system in terms of which the second derivative appear in the simplest possible way.

Hyperbolic Equation $ac < b^2$

In this case we can factorize A and C to give

$$A = a\varepsilon_x^2+2b\varepsilon_x\varepsilon_y+c\varepsilon_y^2=(p_1\varepsilon_x+q_1\varepsilon_y)(p_2\varepsilon_x+q_2\varepsilon_y)$$

$$C = a\eta_x^2+2b\eta_x\eta_y+c\eta_y^2=(p_1\eta_x+q_1\eta_y)(p_2\eta_x+q_2\eta_y)$$

with the two factors not multiples of each other. We can then choose ε and η so that

$$p_1\varepsilon_x + q_1\varepsilon_y = p_2\eta_x + q_2\eta_y = 0$$

and hence $A = C = 0$. This means that

ε is constant on curves with $\frac{dy}{dx}=\frac{q_1}{p_1}$, η is constant

on curves with $\frac{dy}{dx}=\frac{q_2}{p_2}$

we can therefore write

$$p_1dy - q_1dx\ p_2dy - q_2dx = 0$$

and hence

$$(p_1dy - q_1dx)(p_2dy - q_2dx) = 0$$

which gives

$$ad^2y - 2b\,dxdy + cdx^2 = 0 \quad \text{...(7)}$$

As we shall see, this is the easiest equation to use to determine (ε, η). We call (ε, η) the characteristic co-ordinate system in terms of which (1) takes its Canonical form

$$\psi_{\varepsilon\eta} + b_1(\varepsilon,\eta)\psi_\varepsilon + b_2(\varepsilon,\eta)\psi_\eta + b_3\psi = g(\varepsilon,\eta) \quad \text{...(8)}$$

The curves where ε is constant and the curves where η is constant are called characteristic curves or simply characteristics. As we shall see it is the existence or non-existence of characteristic curves for the three types of equations that determines the distinctive properties of their solutions.

Notes

As a less trivial example, consider the hyperbolic equation

$$\phi_{xx} - \text{sech}^4 x\, \phi_{yy} = 0 \qquad ...(9)$$

Equation (7) shows that the characteristics are given by

$$dy^2 - \text{sech}^4 x\, dx^2 = (dy + \text{sech}^2 x\, dx)(dy - \text{sech}^2 x\, dx) = 0$$

and hence

$$\frac{dy}{dx} = \pm\, \text{sech}^2 x$$

The characteristics are therefore

$$y \pm \text{tanb } x = \text{constant},$$

and the characteristic co-ordinates are

$\varepsilon = y + \text{tanb } x$, $\eta = y - \text{tanb } x$. On writing (9) in terms of these variables with $\phi = (x, y) = \psi(\varepsilon, \eta)$, we find that its canonical form is

$$\psi_{\varepsilon\eta} = \frac{(\eta - \varepsilon)(\psi_\varepsilon - \psi_\eta)}{[4 - (\varepsilon - \eta)^2]} \qquad ...(10)$$

in the domain $(\eta - \varepsilon)^2 < 4$.

Parabolic Equation $ac = b^2$

In this case

$$A = a\varepsilon_x^2 + 2b\varepsilon_x\varepsilon_y + c\varepsilon_y^2 = (p\varepsilon_x + q\varepsilon_y)^2$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = (p\eta_x + q\eta_y)^2$$

so we can construct one set of characteristic curves. We therefore take ε to be constant on the curves $pdy - qdx = 0$. This gives us $A = 0$ and since $AC + B^2$, $B = 0$. For any set of curves where η is constant that is never parallel to the characteristics, C does not vanish, and the canonical form is

$$\psi_{\eta\eta} + b_1(\varepsilon, \eta)\psi_\varepsilon + b(\varepsilon, \eta)\psi_\eta + b_3(\varepsilon, \eta)\psi = g(\varepsilon, \eta) \qquad ...(11)$$

We can now see that the diffusion equation is in canonical form.

As a further example, consider the parabolic equation

$$\phi_{xx} + 2\text{cosec } y\, \phi_{xy} + \text{cosec}^2 y\, \phi_{yy} = 0 \qquad ...(12)$$

The characteristic curves satisfy

$$dy^2 - 2\text{ cosec } y\, dxdy + \text{cosec}^2 y\, dx^2 = (dy - \text{cosec } dx)^2 = 0,$$

and hence

$$\frac{dy}{dx} = \text{cosec } y$$

The characteristic curves are therefore given by $x + \cos y = \text{constant}$, and we can take $\varepsilon = x + \cos y$ as the characteristic. A suitable choice for the other co-ordinate is $\eta = y$. On writing (12) in terms of these variables, with $\phi(x, y) = \psi(\varepsilon, \eta)$, we find that its canonical form is

$$\psi_{\eta\eta} = \sin^2\eta \cos \eta\psi\varepsilon, \qquad ...(13)$$

in the whole (ε, η) plane.

Elliptic Equations: $ac > b^2$

In this case we can make neither A nor C zero, since no real characteristic curves exist. Instead we can simplify by making $A = C$ and $B = 0$, so that the second derivative form the Laplacian $\Delta^2\psi$ and the canonical form is

$$\psi_{\varepsilon\varepsilon} + \psi_{\eta\eta} + b_1(\varepsilon, \eta)\psi_\varepsilon + b_2(\varepsilon, \eta)\psi_\eta + b_3\psi = g(\varepsilon, \eta) \qquad ...(14)$$

Clearly Laplace's equation is in canonical form

In order to proceed, we must solve

$$A - C = a(\varepsilon_x^2 - \eta_y^2) + 2b(\varepsilon_x\varepsilon_y - \eta_x\eta_y) + c(\varepsilon_y^2 - \eta_y^2) = 0$$

$$B = a\varepsilon_x\eta_x + b(\eta_x\varepsilon_y + \varepsilon_x\eta_y) + c\varepsilon_y\varepsilon_y = 0.$$

We can do this by defining $x = \varepsilon + i\eta$, and noting that these two equations form the real and imaginary parts of

$$a\chi_x^2 + 2b\chi_x\chi_y + c\chi_x^2 = 0$$

and hence

$$\frac{\chi_x}{\chi_y} = \frac{-b \pm \sqrt{ac - b^2}}{a} \qquad ...(15)$$

Now χ is constant on curves given by $\chi_y dy + \chi_x dx = 0$, and hence from (15) on

$$\frac{dy}{dx} = \frac{b \pm \sqrt{ac - b^2}}{a} \qquad ...(16)$$

By solving (16) we can deduce ε, η. For example consider elliptic equation

$$\phi_{xx} + \text{sech}^4 x\, \phi_{yy} = 0 \qquad ...(17)$$

In this case $\chi = \varepsilon + i\eta$ is constant on the curves given by

$$\frac{dy}{dx} = \pm i\,\text{sech}^2 x,$$

and hence $y \pm i$ tanb x = constant. We can therefore take $\chi = y + i$ tanb x, and hence $\varepsilon = y$, η = tanb x. On writing (17) in terms of these variables, with $\phi(x, y) = \psi(\varepsilon, \eta)$, we find that the canonical form is

$$\psi_{\varepsilon\varepsilon} + \psi_{\eta\eta} = \frac{2n}{(1 - \eta^2)}\psi_\eta, \qquad ...(18)$$

in the domain $|\eta| < 1$.

21.3 Classification of Second order Partial Differential Equations

Let us consider a function z of two independent variables x and y. Writing various partial derivatives as

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x dy}, t = \frac{\partial^2 z}{\partial y^2} \qquad ...(1)$$

We find that the most general form of the partial differential equation of the second order will be of the form

$$F(x, y, z, p, q, r, s, t) = 0 \qquad ...(2)$$

Example: Consider z as a function of x, y through two functions f and g as follows

$$z = f(x^2 - y) + g(x^2 + y) = 0 \qquad ...(3)$$

Find the differential equation by eliminating f and g

Solution:

$$p = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial q}{\partial x}$$

Let $u = x^2 - y$ and $v = x^2 + y$, so that

$$z = f(u) + g(v)$$

then
$$p = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial q}{\partial v}.\frac{\partial v}{\partial x}$$

$$= \frac{\partial f}{\partial u}.(2x) + (2x)\frac{\partial q}{\partial v} = 2x\left(\frac{\partial f}{\partial u} + \frac{\partial q}{\partial v}\right) \qquad ...(4)$$

$$q = \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}.(1)$$

$$= -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \qquad ...(5)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}p = 2\left(\frac{\partial f}{\partial u} + \frac{\partial q}{\partial v}\right) + 2x\left(2x\frac{\partial^2 f}{\partial u^2} + 2x\frac{\partial^2 f}{\partial v^2}\right)$$

$$= 2\left(\frac{\partial f}{\partial u} + \frac{\partial q}{\partial v}\right) + 4x^2\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) \qquad ...(6)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}q = -\frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)$$

$$= -2x\frac{\partial^2 f}{\partial x^2} + 2x\frac{\partial^2 f}{\partial v^2} \qquad ...(7)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}q = -\frac{\partial^2 f}{\partial u^2}.\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)$$

$$= +\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \qquad ...(8)$$

Now using equations (4), (6) and (8) we have

$$r = \frac{\partial^2 z}{\partial x^2} = 2\left(\frac{\partial f}{\partial u} + \frac{\partial q}{\partial v}\right) + 4x^2\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

or
$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x}\frac{\partial z}{\partial x} + 4x^2\frac{\partial^2 z}{\partial y^2} \qquad ...(9)$$

Notes

We can have various types of partial differential equations.

1. ***Linear partial differential equations with constant coefficients***

   We may have equations of the type

   $C_1r + C_2s + C_3t + C_4p + C_5q + C_6z = f(x, y)$

   where C_1, C_2, C_3, C_4, C_5 are constants. We have already given the methods of solving these types of equations in the earlier unit no. 20.

   The examples are $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x, y)$

   $$\frac{\partial^2 z}{\partial x \partial y} = f(x, y)$$

   $$\frac{\partial^2 z}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 z}{\partial y^2}$$

   $K\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$ (here K is a constant)

2. ***Equations with Variable Coefficients***

   In this type of partial differential equations we will have a structure as follows

   $$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \text{...(1a)}$$

   where R, S, T are functions of x, y, z.

   As suggested in the section (21.1) we classify this equation into three classes

   (a) Hyperbolic if $s^2 - 4rt > 0$

   (b) Parabolic if $s^2 - 4rt = 0$ and

   (c) Elliptic if $s^2 - 4rt < 0$

   In dealing with equations of the above types first we reduce them to canonical form. The solution of Laplace equation, Wave equation and conduction of heat or diffusion we defer cases to next two units.

3. ***Equations reducible to homogeneous linear form***

   An equation in which the coefficient of a differential coefficient of any order is a constant multiple of the variables of the same degree, may be transformed into one having constant coefficients.

Example: Transform the equation

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x} = 0 \qquad \text{...(1)}$$

into a form with constant coefficients.

Solution: Put $u = \log x$, $v = \log y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{1}{x}$$

or $$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}$$

So operator

$$x\frac{\partial}{\partial x} = \frac{\partial}{\partial u}$$

$$\therefore \qquad x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)z = x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial u^2}$$

Similarly

$$y^2\frac{\partial^2 z}{\partial y^2} + y\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial v^2}$$

So the equation reduces to

$$\frac{\partial^2 z_1}{\partial u^2} - \frac{\partial^2 z_1}{\partial v^2} = 0$$

where $z_1(u, v) = z(x, y)$.

Self Assessment

1. Reduce the equation

   $$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}$$

   to canonical form.

2. Reduce the equation

   $$\frac{\partial^2 z}{\partial x^2} - x^2\frac{\partial^2 z}{\partial y^2} = 0$$

   to canonical form

3. Transpose the partial differential equation into one having constant coefficients

   $$y\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial q} = 0$$

21.4 Summary

- In units 17 to 20 we studied and solved various types of partial differential equations both first order and higher orders as well as linear and non-linear equations.
- There are three main classes of partial differential equations i.e. hyperbolic type, parabolic type and elliptic type.
- The wave equation is of hyperbolic type, diffusion equation is of parabolic type and Laplace equation is of elliptic type.

21.5 Keywords

An *Elliptic* equation has $ac < b^2$, for example Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}.$$

Notes

A *Parabolic* equation has $ac = b^2$, for example the diffusion equation

$$K\frac{\partial^2\phi}{\partial x^2} - \frac{\partial\phi}{\partial y} = 0 \quad \text{...(here } y = t\text{)}$$

A *hyperbolic* equation has $ac < b^2$, for example the wave equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{...(here } y \text{ is time)}$$

21.6 Review Questions

1. Reduce the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

to canonical form

2. Transform the partial differential equation into the form having constant coefficients

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

Answers: Self Assessment

1. $\frac{\partial^2\psi}{\partial\eta^2} = 0$ where $\psi(\varepsilon, \eta) = z\,(x, y)$

and $\varepsilon = x - y$, $\eta = x + y$.

2. $$\frac{\partial^2\psi}{\partial\varepsilon\partial\eta} = \frac{1}{\Psi(\varepsilon+\eta)}\left(\frac{\partial\psi}{\partial\varepsilon} - \frac{\partial\psi}{\partial\eta}\right)$$

3. $$\frac{\partial^2\psi}{\partial v2} - 2\frac{\partial\psi}{\partial v} = 0$$

where $\psi(\upsilon, v) = z(x, y)$

21.7 Further Readings

Books

Piaggio H.T.H, Differential Equations

Yosida K., Lectures in Differential and Integral Equations

Unit 22: Solution of Laplace Differential Equation

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know that Laplace equation is a partial differential equation involving one dependent variable and three independent variables.
- See that it has a vast number of applications in gravitational potential process in electrostatic potential distributions, in the propagation of waves, in diffusion process or heat conductions.
- Note that three major co-ordinate systems namely the Cartesian co-ordinate system the spherical polar co-ordinate system or the cylindrical co-ordinate systems are used to express Laplacian operator.

Introduction

This Laplace equation is seen to be written in such a way that the dependence of dependent variable on three independent variables can be separated.

Both spherical polar co-ordinates and cylindrical co-ordinates are used to find the solution of Laplace equation.

22.1 Solution of Laplace Differential Equation – Cylindrical Co-ordinates

The most important partial differential equation of applied mathematics is the differential equation of Laplace i.e.

$$\nabla^2 V = 0 \qquad ...(1)$$

The Laplace operator is expressed in general curvilinear co-ordinates u_1, u_2, u_3 in the following manner,

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right\} \qquad ...(2)$$

Notes

If we use cylindrical co-ordinates (r, θ, z) given by

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \qquad ...(3)$$

Then $\nabla^2 V$ in this co-ordinate system is given by

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} \qquad ...(4)$$

So Laplace differential equation in cylindrical co-ordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

or, $$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(5)$$

Here V is a function of r, θ and z. Let us suppose the solution of (5) as

$$V = R(r)\,\Theta\,(r)\,Z(r) \qquad ...(6)$$

Where $R(r)$ is a function of r, Θ is a function of θ and Z is a function of z only. This method is known as method of separation of variable. Substituting in (6) and dividing by $R\,\Theta\,Z$, we have

$$\frac{1}{R^2}\frac{d^2R}{dr^2} + \frac{1}{Rr}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} \qquad ...(7)$$

Now the right hand side is only a function of z whereas L.H.S. is function of r and θ, so each side must be constant i.e.

$$\frac{1}{R^2}\frac{d^2R}{dr} + \frac{1}{Rr}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} = -\frac{1}{z}\frac{d^2z}{dz^2} = -\lambda^2 \qquad ...(8)$$

Where λ^2 is a negative constant. This gives us

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} = -\lambda^2 \qquad ...(9)$$

and

$$\frac{d^2Z}{dz^2} - \lambda^2 Z = 0 \qquad ...(10)$$

The equation (9) can be rewritten as

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + \lambda^2 r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} \qquad ...(11)$$

Keeping in view the same argument, we have from (11)

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+\lambda^2 r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=\mu^2 \quad ...(12)$$

which gives

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(\lambda^2 r^2-\mu^2)R = 0 \quad ...(13)$$

and

$$\frac{d^2\Theta}{d\theta^2}+\mu^2\Theta = 0 \quad ...(14)$$

In equation (13) if we use the substitution $r=\frac{x}{\lambda}$, it reduces to

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{\mu^2}{x^2}\right)R = 0 \quad ...(15)$$

Equation (15) is Bessel's differential equation and so the solution is given by

$$R = A\,J_\mu(x)+B\,J_{-\mu}(x)$$

or

$$R = A\,J_\mu(\lambda r)+B\,J_{-\mu}(\lambda r) \quad ...(16)$$

where μ is not an integer and

$$R = A_1\,J_\mu(\lambda r)+B_1\,Yy_\mu(\lambda r) \quad ...(17)$$

when μ is an integer. The solutions of equations (10), (14) are given by

$$Z = A_2\,e^{\lambda.z}+B_2\,e^{-\lambda z} \quad ...(18)$$

and

$$\Theta = A_3\cos(\mu\,\theta)+B_3\sin(\mu\,\theta) \quad ...(19)$$

Hence the total solution is

$$V = R\,\Theta\,Z=\left[A\,J_\mu(\lambda r)+B\,J_{-\mu}(\lambda r)\right]\left[A_2\,e^{\lambda z}+B_2\,e^{-\lambda z}\right]\left[A_3\cos(\mu\theta)+B_3\sin(\mu\theta)\right] \quad ...(20)$$

where μ is a fraction and λ = 1, 2, 3... and

$$V = R\Theta Z=\left[A_1\,J_\mu(\lambda r)+B\,Y_\mu(\lambda r)\right]\left[A_3\cos(\mu\theta)+B_3\sin(\mu\theta)\right]\left[A_2e^{\lambda z}+B_2e^{-\lambda z}\right] \quad ...(21)$$

When μ is an integer and λ = 1, 2,...

The solutions (20) and (21) depend upon the parameters μ, λ. If we see a solution that is finite at $r = 0$ and also be single valued in θ then μ be *a* positive integer and taking all values from 0 to ∞. Thus for a fixed λ,

$$V = \sum_{\mu=0}^{\infty}A_1\,J_\mu(\lambda r)\left[A_3\cos\mu\theta+A_4\sin\mu\theta\right]\left[A_2e^{\lambda z}+A_2e^{-\lambda z}\right] \quad ...(22)$$

Thus the above solution is known as cylindrical Harmonics and will be useful for certain physical problems.

The solution (22) V for a single value of μ is called general cylindrical Harmonics.

Notes

22.2 Circular Harmonics

Laplace equation in cylindrical co-ordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(1)$$

Assume that V is independent of co-ordinates *z*, we then have

$$\frac{1}{r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0 \qquad ...(2)$$

We now attempt to find a solution of this equation of the form.

$$V = F_1(\theta)F_2(r) \qquad ...(3)$$

Substituting this in (2), we have

$$\frac{F_1(\theta)}{r}\frac{d}{dr}\left(r\frac{dF_2}{dr}\right)+\frac{F_2(r)}{r^2}\frac{d^2F_1(\theta)}{d\theta^2} = 0 \qquad ...(4)$$

Multiplying by r^2 and dividing by $F_1\,F_2$, we have

$$\frac{1}{F_1}\left(r^2\frac{d^2F_2}{dr^2}+r\frac{dF_2}{dr}\right) = -\frac{1}{F_1}\frac{d^2F_1}{d\theta^2}=n^2 \qquad ...(5)$$

Since L.H.S. is a function of *r* and the R.H.S. is a function of θ, so each one of them is a constant. We thus have the two solutions.

$$\frac{d^2F_1}{d\theta^2}+n^2F_1 = 0 \qquad ...(6)$$

and

$$r^2\frac{d^2F_1}{dr^2}+r\frac{dF_2}{dr}-n^2F_2 = 0 \qquad ...(7)$$

The solutions are separable. The solution of (6) is given by

$$F_1 = A\cos n\theta + B\sin n\theta \qquad ...(8)$$

Also it is easily verified that the solution of (7) is

$$F_2 = Cr^n + Dr^{-n},\ if\ n \neq 0 \qquad ...(9)$$

If $n = 0$, we have the solution

$$F_2 = C_0 \log r + D_0 \qquad ...(10)$$

Where *A*, *B*, *C* and *D* are arbitrary constants. The solution of Laplace equation in cylindrical co-ordinates when *V* is independent of the co-ordinate *z* are called circular harmonics. The circular harmonics are then

$$\left.\begin{aligned} V_0 &= (A_0\theta + B_0)(C_0\log r + D) \quad \text{degree zero} \\ V &= (A_n\cos n\theta + B_n\sin\theta)(C_n r^n + D_n r^{-n}) \quad \text{degree } n \end{aligned}\right\} \qquad ...(11)$$

Notes

In most applications of circular harmonics, V is usually single-valued function of θ. So if we change θ by 2 π, we reach the conclusion

$$V(r, \theta+2\pi) = V(r,\theta) \quad ...(12)$$

It is necessary that n take integer values. So a general single valued solution of Laplace equation is obtained by summing over n i.e.

$$V = a_0 \log r + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)r^n + \sum_{n=1}^{\infty}\frac{1}{r^n}(q_n \cos n\theta + p_n \sin n\theta) + c_0 \quad ...(13)$$

where a_0, a_n, b_n, q_n and p_n and c_0 are constants.

Example: Find the steady state temperature in the region inside a cylinder, the two halves of the cylinder are 'thermally insulated from each other, and the upper half of it is kept at temperature ν_1, while the lower half is kept at temperature ν_2. It is assumed that cylinder is so long in the z-direction that the temperate is independent of z.

Solution: To solve this problem, let $\nu(r,\theta,z,t)$ be the temperature that satisfies heat equation

$$\frac{\partial \nu}{\partial t} = \nabla^2 \nu \quad ...(1)$$

In the steady state ν is independent of t so that we have to solve Laplace equation

$$\nabla^2 \nu = 0 \quad ...(2)$$

in the region inside the cylinder and satisfy the boundary conditions

$$\nu = \nu_1 \quad \text{at} \quad r = R \quad 0 < \theta < \pi \quad ...(3)$$

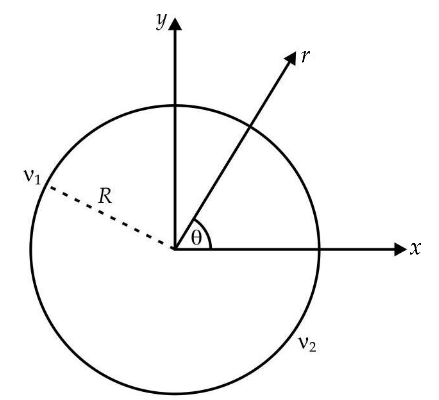
$$\nu = \nu_2 \quad \text{at} \quad r = R \quad \pi < \theta < 2\pi$$

we do this by taking the general solution independent of z as, we have

$$\nu = a_0 \log r + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + B_n \sin n\theta) + c_0 + \sum_{n=1}^{\infty} r^{-n-1}(q_n \cos n\theta + f_n \sin n\theta) \quad ...(4)$$

and use the boundary conditions (3). We first see that the temperature must be finite

Figure 22.1



Notes

at the origin $r = 0$. so a_0, q_x and f_x must be equal to zero. Therefore the solution (4) reduces to

$$v = \sum_{n=1}^{\infty} r^n (a_x \cos(n\theta) + b_x \sin(n\theta)) + c_0 \quad ...(5)$$

As a first step let us assume that the temperature on the circumference of the cylinder $r = a$ is specified as

$$v = F(\theta) \quad \text{at} \quad r = R$$

Then placing $r = R$ in (5) we have

$$F(\theta) = \sum_{n=1}^{\infty} r^n (a_x \cos(n\theta) + b_x \sin(n\theta)) + c_0 \quad ...(6)$$

Now c_0, a_x and b_x are Fourier coefficients and so are given by the relations

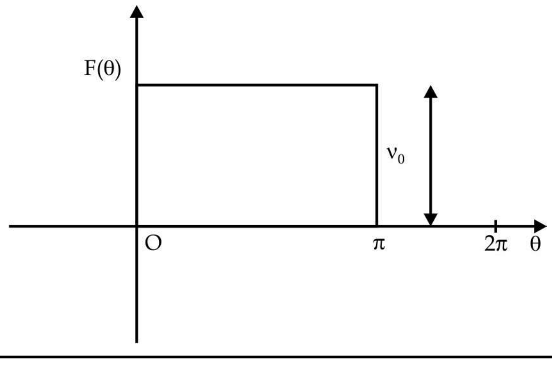
$$\left.\begin{aligned} a_x &= \frac{1}{R^n \pi} \int_0^{2\pi} F(\theta) \cos n\theta \, d\theta \\ b_x &= \frac{1}{R^n \pi} \int_0^{2\pi} F(\theta) \sin n\theta \, d\theta \\ \text{and} \quad c_0 &= \frac{1}{R\pi} \int_0^{2\pi} F(\theta) \, d\theta \end{aligned}\right\} \quad ...(7)$$

An interesting special case arises when the temperature of the upper half of the cylinder is kept at v_0 and the lower half is kept at zero degree. The function then is given geographically by figure 22.2. We have

$$a_x = \frac{v_0}{R^n \pi} \int_0^{\pi} \cos n\,\theta \, d\theta = 0$$

$$b_x = \frac{v_0}{R^n \pi} \int_0^{\pi} \sin n\,\theta \, d\theta = \frac{2v_0}{R^n \pi n}, n \text{ odd}$$

Figure 22.2



Notes

and

$$C_0 = \frac{1}{2\pi}\int_0^{\pi} \nu_0 \, d\theta = \frac{\nu_0}{2} \quad \text{...(8)}$$

substituting into (6), we obtain

$$\nu(r, \theta) = \frac{2C_0}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \frac{\sin n\theta}{n} + \frac{\nu_0}{2}.... \qquad \text{for } n \text{ odd} \quad \text{...(9)}$$

Self Assessment

1. Find the potential $u(r,\theta)$ in the exterior of a unit sphere satisfying the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}u\right) = 0$$

under the conditions

$$u(1,0) = \cos 2\theta$$

and

$$\lim_{r\to\infty} u(r, \theta) = 0$$

22.2.1 Solution of Laplace's Equation in Spherical Polar Co-ordinates

The Laplace equation in spherical polar co-ordinates is given by

$$r^2 \frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \quad \text{...(1)}$$

we apply here a separation of variable's method and write the solution of (1) in the form

$$V(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi) \quad \text{...(2)}$$

where R is a function of r only, Θ that of θ and ϕ that of ϕ only. Substituting in (1) we get

$$\left\{\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + \frac{1}{\Theta\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right\}\sin^2\theta = \frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2} \quad \text{...(3)}$$

Since both sides are functions of different independent variables hence each side should be equal to some constant. Let this constant be λ^2. Then equation (3) gives

$$\frac{d^2\Phi}{d\phi^2} + \lambda\Phi = 0 \quad \text{...(4)}$$

and

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{R} = \frac{1}{\Theta\sin q}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\lambda^2}{\sin^2\theta} \quad \text{...(5)}$$

Again in (5) both sides are functions of different variables and hence both will be equal to a constant say $n(n+1)$. This gives us from (5)

Notes

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-n(n+1)R = 0 \quad ...(6)$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[n(n+1)-\frac{\lambda^2}{\sin^2\theta}\right]\Theta = 0 \quad ...(7)$$

To solve (6), let

$$r = e^p,$$

so that

$$\frac{dr}{dp} = e^p = r$$

Therefore

$$\frac{dR}{dr} = \frac{dR}{dr}.\frac{dp}{dr}=\frac{1}{r}\frac{dR}{dp}$$

or

$$r\frac{d}{dr} = \frac{d}{dp}$$

Let us denote the operator $\frac{d}{dp}$ by D, then

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}$$

So

$$r^2\frac{d^2R}{dr^2} = r\frac{d}{dr}\left(r\frac{dR}{dr}\right)-r\frac{dR}{dr}$$

$$= r\frac{d}{dr}\left\{\left(r\frac{d}{dr}-1\right)R\right\}$$

$$= D(D-1)R$$

Using these values in (6), we get

$$\left[D(D-1)+2D-n(n+1)\right]R = 0$$

or

$$(D-n)(D+n+1)R = 0 \quad ...(6a)$$

The solution of (6a) is

$$R = A'e^{np}+B'e^{-(n+1)p}$$

or

$$R = A'r^n+B'r^{-(n+1)} \quad ...(5)$$

To solve (7) put $\cos\theta=\mu$

so that

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\mu}\frac{d\mu}{d\theta}=-\sin\theta\frac{d\Theta}{d\mu}$$

Substituting these values in (7) we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[n(n+1)-\frac{\lambda^2}{\sin^2\theta}\right]\Theta = 0$$

$$= \frac{1}{\sin\theta}\frac{d}{d\theta}\left\{-\sin^2\theta\frac{d\Theta}{d\mu}\right\}+\left[n(n+1)-\frac{\lambda^2}{1-\mu^2}\right]=0$$

$$= \frac{-2\sin\theta\cos\theta}{\sin\theta}\frac{d\Theta}{d\mu}-\sin\theta\frac{d}{d\theta}\left(\frac{d\Theta}{d\mu}\right)+\left\{n(n+1)-\frac{\lambda^2}{1-\mu^2}\right\}\Theta=0$$

$$= \sin^2\theta\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left[n(n+1)-\frac{\lambda^2}{(1-\mu^2)}\right]\Theta=0$$

or

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left[n(n+1)-\frac{\lambda^2}{(1-\mu^2)}\right]\Theta=0 \quad ...(9)$$

It is clear that Θ will be a function of μ i.e.

$$\Theta(z) \text{ or } \Theta(\cos\theta)$$

Hence the solution of Laplace equation is

$$V = (A'r^n+B'r^{-(n+1)})\Theta(\cos\theta)\left[A''e^{i\lambda\phi}+B''e^{-i\lambda\phi}\right] \quad ...(10)$$

where the solution of (μ) is

$$\Phi = A''e^{a\lambda\phi}+B''e^{-i\lambda\phi} \quad ...(11)$$

For $\lambda^2 = m^2$, integer m, the solution is satisfied by associated Legendre polynomial $P_n^m(x)$ as shown below:

Consider the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0 \quad ...(12)$$

Differentiating it m times and putting

$$v = \frac{d^my}{dx^m} \quad ...(13)$$

We have

$$\frac{d^m}{dx^m}\left[(1-x^2)\frac{d^2y}{dx^2}\right]-2\frac{d^m}{dx^m}\left[x\frac{dy}{dx}\right]+n(n+1)\frac{d^my}{dx^m} = 0$$

or

$$(1-x^2)\frac{d^{m+2}y}{dx^{m+2}}-2.mx\frac{d^{m+1}}{dx^m}y-m(m-1)x\frac{dmy}{dx^m}-\ 2x\frac{d^{m+1}y}{dx^{m+1}}-2\frac{d^my}{dx^m}(m)+n(n+1)\frac{d^my}{dx^m}=0$$

Notes

or from (13)

$$(1-x^2)\frac{d^2\nu}{dx^2}-2x(m+1)\frac{d\nu}{dx}+\left[n(n+1)-m(m+1)\right]\nu = 0 \quad ...(14)$$

Let us put

$$w = (1-x^2)^{m/2}\ \nu = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \quad ...(15)$$

then $\nu = (1-x^2)^{\frac{-m}{2}}w$

$$\frac{d\nu}{dx} = -\frac{m}{2}(-2x)(1-x^2)^{\frac{-m}{2}-1}w+(1-x^2)^{\frac{-m}{2}}\frac{dw}{dx}$$

$$\frac{d^2\nu}{dx^2} = m(1-x^2)^{\frac{-m}{2}-1}w+mx(-2x)\left(\frac{-m}{2}-1\right)(1-x^2)^{\frac{-m}{2}-2}w+2mx(1-x^2)^{\frac{-m}{2}-1}\frac{dw}{dx}+(1-x^2)^{\frac{-m}{2}}\frac{d^2w}{dx^2}$$

$$= (1-x^2)^{\frac{-m}{2}}\frac{d^2w}{dx^2}+2mx(1-x^2)^{\frac{-m}{2}-1}\frac{dw}{dx}+(1-x^2)^{\frac{-m}{2}-2}w\left\{m(1-x^2)+mx^2(m+2)\right\}$$

Substituting in equation (14) we have

$$(1-x^2)^{\frac{-m}{2}+1}\frac{d^2w}{dx^2}+2mx(1-x^2)^{\frac{-m}{2}}\frac{dw}{dx}+(1-x^2)^{\frac{-m}{2}-1}\left\{m+mx^2(m+1)\right\}w-$$

$$-2x(m+1)mx(1-x^2)^{\frac{-m}{2}-1}w-2x(m+1)(1-x^2)^{\frac{-m}{2}}\frac{dw}{dx}+$$

$$+\left[n(n+1)-m(m+1)\right](1-x^2)^{\frac{-m}{2}}w = 0$$

Dividing by $(1-x^2)^{\frac{-m}{2}}$ we have

$$(1-x^2)\frac{d^2w}{dx^2}-2x\frac{dw}{dx}+w\left\{n(n+1)-m(m+1)-\frac{2x^2m(m+1)}{(1-x^2)}+\frac{m+mx^2(m+1)}{(1-x^2)}\right\} = 0$$

$$(1-x^2)\frac{d^2w}{dx^2}-2x\frac{dw}{dx}+\left[n(n+1)-\frac{m^2}{(1-x^2)}\right]w = 0 \quad ...(16)$$

The equation (16) is same as equation (9) where

$\Theta = w$ and $\mu = x$

Thus the solution of equation (9) is given by

$$\Theta = w = (1-\mu^2)^{\frac{m}{2}}\nu = (1-\mu^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(\mu)\equiv P_n^m(\mu) \quad ...(17)$$

Where $P_n^m(\mu)$ is known as associated Legendre polynomial. Hence the solution of Laplace differential equation is given by (for $\lambda = m$)

$$V = \left[A'r^n+B'r^{-n-1}\right]\left[A''e^{im\phi}+B''e^{-im\phi}\right]P_n^m(\mu) \quad ...(18)$$

Notes

For solution which exist for $r = 0$, then $B' = 0$.

The complete solution is given by summing over m or

$$V = \sum_{\substack{m=0,1,2,\ldots \\ n=0,1,2,\ldots}}^{\substack{n=\infty \\ m=\lambda}} A' r^n \left[A'' e^{im\phi} + B''\, e^{-im\phi}\right] P_n^m(\mu) \qquad \text{...(19)}$$

Since $P_n^m(x)$ involves mth derivative of $P_n(x)$ which is polynomial of degree n, so for $m > n$

$$P_n^m(m) = 0 \qquad \text{...(20)}$$

for $m > n$. Defining S, the surface Harmonic by

$$S_n = \left[A'' e^{im\phi} + B''\, e^{-im\phi}\right] P_n^m(\mu) \qquad \text{...(21)}$$

If S_n is independent of ϕ, then

$$\frac{dS_n}{d\phi} = 0$$

So S_n has only $m = 0$ value hence

$S_n = P_n(\mu)$. In the case V becomes

$$V = \sum_n (A' r^n + B' r^{-n-1}) P_n(\mu) \qquad \text{For } m = \cos\theta \qquad \text{...(22)}$$

Example 1: Gravitational Potential Due to Uniform Circular Ring

Let us consider a particle of mass m situated at a point (x_1, y_1, z_1) of a reference Cartesian coordinate system, then the gravitational potential θ due to this mass at the point with coordinate (x, y, z) is given by

$$V = \frac{\text{mass}}{\text{distance}} = \frac{m}{\sqrt{\{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\}}} \qquad \text{...(i)}$$

We know that potential V, satisfies Laplace equation

$$\nabla^2 V = 0 \qquad \text{...(ii)}$$

in matter free space.

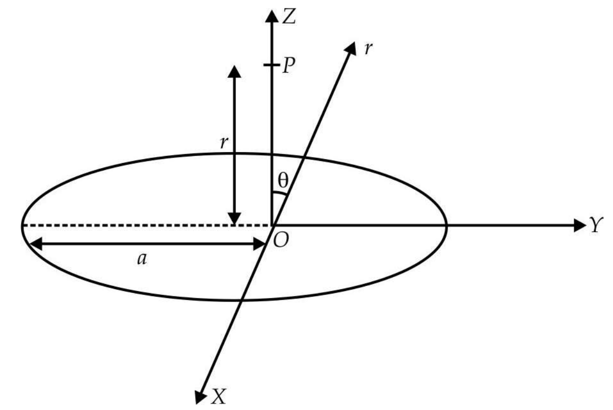
Now, we have to calculate the gravitational potential at any point due to a uniform circular ring of small cross-section, lying in the $x - y$ plane and with its centre situated at the point O, (Figure 22.3).

Obviously, the gravitational potential is symmetric about the z-axis and so it should be independent of the angle θ. The potential V, therefore may be written with following form:

$$V = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta) \qquad \text{...(iii)}$$

Notes

Figure 22.3



where A_n and B_n are constant coefficients and are to be evaluated. To evaluate these coefficients, we know that the gravitational potential is symmetric about the z-axis and therefore any point P on the same distance $\sqrt{(a^2 + r^2)}$ from all the points of the ring, where a is the radius of the ring and distance $OP = r$.

Let M denote the total mass of the ring, then the gravitational potential at P due to the ring will be

$$V = \frac{\text{mass}}{\text{distance}} = \frac{M}{\sqrt{(a^2 + r^2)}} \quad \text{...(iv)}$$

but
$$\frac{M}{\sqrt{(a^2 + r^2)}} = M(a^2 + r^2)^{-1/2} = \frac{M}{a}\left(1 + \frac{r^2}{a^2}\right)^{-1/2}$$

or
$$V = \frac{M}{a}\left[1 - \frac{r^2}{2a^2} + \frac{1}{2}.\frac{3}{4}.\frac{r^4}{a^4}...\right] \quad \text{...(v)}$$

by Binomial theorem for $r < a$

However in case $r > a$, we can write

$$\frac{M}{\sqrt{(a^2 + r^2)}} = M(a^2 + r^2)^{-1/2} = \frac{M}{r}\left(1 + \frac{a^2}{r^2}\right)^{-1/2}$$

$$= \frac{M}{r}\left(1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{1}{2}.\frac{3}{4}\frac{a^4}{r^4}...\right)$$

or
$$V = \frac{M}{a}\left\{\frac{a}{r} - \frac{1}{2}\frac{a^3}{r^3} + \frac{1}{2}.\frac{3}{4}\frac{a^5}{r^5}...\right\} \quad \text{...(vi)}$$

Now, for point situated on the z-axis, $\theta = 0$ and the general solution as contained in equation (iii) must reduce either to equation (v) or equation (vi). Now the Legendre polynomials $P_n(\cos\theta)$ for a point on the z-axis ($\cos 0°$) become

$$P_n(\cos 0°) = P_n(1) = 1$$

Notes

Therefore for all points situated on the z-axis, the general form of the potential as contained in (iii), reduces to

$$V = \sum_{n=0}^{\infty}\left[A_n r^n + \frac{B_n}{r^{n+1}}\right] \qquad ...(vii)$$

Comparing this equation with equation (vi) we see that for $r > a$, the coefficients $A_n = 0$ and B_n are the coefficients of equation (vi).

Again comparing equation (vii) with (v), we see that for $r < a$, the coefficients $B_n = 0$ and A_n are the coefficients of equation (v).

Hence the solution for the case $r > a$ may be written as

$$V = \frac{M}{a}\left[\frac{a}{r}P_0(\cos\theta) - \frac{1}{2}\frac{a^3}{r^3}P_2(\cos\theta) + \frac{1}{2}.\frac{3}{4}.\frac{a^5}{r^5}P_4(\cos\theta)...\right] \qquad ...(viii)$$

and that for $r < a$ is

$$V = \frac{M}{a}\left[P_0(\cos\theta) - \frac{1}{2}\frac{r^2}{a^2}P_2(\cos\theta) + \frac{1}{2}.\frac{3}{4}.P_4(\cos\theta)...\right] \qquad ...(ix)$$

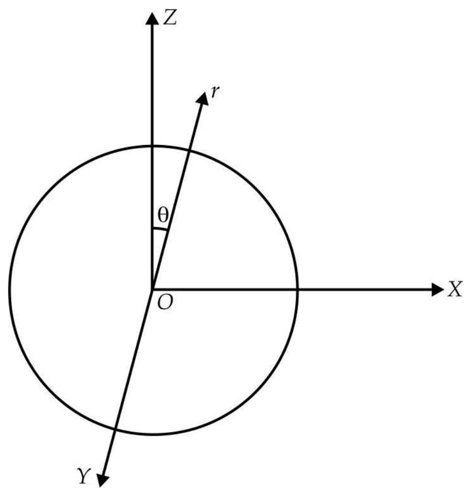
Example 2: Electrical Potential about a Spherical Surface

Let us consider a spherical surface which is being kept at a fixed distribution of the electrical potential of the form

$$V = f(\theta) \qquad ...(i)$$

On the surface of the sphere.

Figure 22.4



Let us assume that the space both *inside* and *outside* the surface is free of electrical charge and we will determine the potential at points within and outside the spherical surface under consideration.

Obviously, the potential V is quite symmetric around the z-axis and as such it shall be independent of angle Φ.

Notes

Therefore we have

$$\frac{\partial^2 V}{\partial \phi^2} = 0 \qquad ...(ii)$$

So Laplace equation expressed in spherical polar co-ordinates reduces to

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \left(\frac{1}{r^2 \tan\theta}\right)\frac{\partial V}{\partial r} = 0 \qquad ...(iii)$$

The general solution of this equation can be written in the form

$$V = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta) \qquad ...(iv)$$

The potential satisfies the boundary conditions

$$V = f(\theta) \text{ when } r = 0 \text{ and } \underset{r\to\infty}{Lt}\ V = 0 \qquad ...(v)$$

Potential in the Region outside the spherical surface

According to the second boundary condition of equation (v), the potential may not be zero at $r = \infty$. Therefore in the region outside the spherical surface no positive powers of r are admissible in the solution of Laplace's equation. Thus in the general solution we should have $A_n = 0$ and so

$$V = \sum_{n=0}^{\infty}\frac{B_n}{r^{n+1}}P_n(\cos\theta) \qquad \text{for } r > a \qquad ...(vi)$$

The coefficients B_n are to be determined. This can be done by making use of the first boundary of equation (v). Hence from (vi) we get

$$V = F(\theta) = f(\cos\theta) = \sum_{n=0}^{\infty}\frac{B_n}{a^{n+1}}P_n(\cos\theta) \qquad ...(vii)$$

Let cos θ = u then

$$V = f(u) = \sum_{n=0}^{\infty}\frac{B_n}{a^{n+1}}P_n(u) \qquad ...(viii)$$

To obtain the value of the general coefficient B_n, we multiply both sides of equation (viii) with $P_n(u)$ and integrate with respect to u in between the limit −1 to +1 we obtain

$$\int_{-1}^{+1} f(u)P_n(u)du = \int_{-1}^{+1}\frac{B_n}{a^{n+1}}[P_n(u)^2]du$$

All other integrals vanish because of the orthogonal property of $P_n(u)$.

$$\therefore \qquad \int_{-1}^{+1} f(u)P_n(u)du = \frac{1}{a^{n+1}}\frac{2B_n}{(2n+1)}$$

or

$$B_n = \frac{(2n+1)}{2}.a^{n+1}\int_0^{\pi} f(\theta)\,P_n(\cos\theta)\sin\theta\,d\theta \qquad ...(ix)$$

This gives us the value of the coefficient B_n. Hence the potential outside the spherical surface is given by equation (viii) with B_n given by equation (ix).

Potential in Region within the Spherical Surface

Notes

The potential within the spherical surface cannot be infinite and therefore negative powers of r are inadmissible in the general solution as contained in equation (iv). This means that potential inside spherical surface will be

$$V = \sum_{n=0}^{\infty} A_n r^n \, P_n(\cos\theta) \qquad \text{for } r < a \qquad ...(x)$$

Again the coefficients A_n are determined by the boundary condition at the surface, viz., $V = f(\theta)$ at $r = a$

$$\therefore \quad V = F(\theta) = f(\cos\theta)$$

$$= \sum_{n=0}^{\infty} A_n a^n \, P_n(\cos\theta) \qquad ...(xi)$$

Let $u = \cos\theta$, then

$$V = F(u) = \sum_{n=0}^{\infty} A_n a^n \, P_n(u) \qquad ...(xii)$$

multiplying both sides by $P_n(u)$ and integrating within the limits −1 to +1, we get

$$\int_{-1}^{+1} F(u)\, P_n(u)\, du = \int_{-1}^{1} A_n \, a^n [P_n(u)]^2 \, du$$

All other coefficients vanish on account of the orthogonal property of $P_n(u)$

$$\therefore \quad \int_{-1}^{+1} F(u) P_n(u)\, du = A_n a^n \frac{2}{(2n+1)}$$

or

$$A_n = \frac{(2n+1)}{2a^n} \int_{-1}^{+1} F(u)\, P_n(u)\, du$$

or

$$A_n = \frac{(2n+1)}{2a^n} \int_{-1}^{+1} F(\theta)\, P_n(\cos\theta) \sin\theta \, d\theta \qquad ...(xiii)$$

So the potential within the spherical surface is given by equation (xi) or (xii) with values of A_n given by the equation (xiii).

Self Assessment

2. Solve

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right) = 0$$

subject to the boundary conditions

Notes

$$u(r) = u_{10} \quad at \quad r = a$$

and $$u(r) = u_{20} \quad at \quad r = b$$

22.2.2 Steady Flow of Heat in Rectangular Plate

We now consider the steady state temperature distribution in a rectangular metallic sheet. In this case temperature is every where independent of time, and hence the equation governing the temperature distribution is given by

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad ...(i)$$

This equation is called Laplace's equation of two Dimensions. We shall now solve this equation under various boundary conditions.

Case I: Let there is a thin plate bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = \infty$, the sides $x = 0$ and $x = a$ being kept at temperature zero. The lower edge y = 0 is kept at $f(x)$ and the edge $y = \infty$ at temperature zero.

In this case the boundary conditions are:

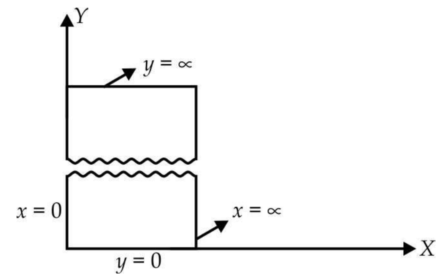
$$V(0, y) = 0 \qquad ...(ii)$$

$$V(a, y) = 0 \qquad ...(iii)$$

$$V(x, 0) = f(x) \qquad ...(iv)$$

$$V(x, \infty) = 0 \qquad ...(v)$$

Figure 22.5



Let the solution of (i) be in the following form

$$V(x, y) = X(x)Y(y) = X\ Y \text{ (say)} \qquad ...(vi)$$

where X and Y are the functions of x and y respectively. Substituting this solution in (i). We have

$$\frac{1}{X}\frac{\partial^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

Since L.H.S. is the function of x only and R.H.S. is the function of y only, both sides will be equal only when both reduce to a constant,

Notes

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dY^2} = -\lambda^2.$$

Here we have taken the negative constant because it suits the boundary conditions.

Therefore the corresponding differential equations are

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \text{ and } \frac{d^2X}{dx^2} + \lambda^2 Y = 0$$

whose general solutions are

$$X = A\cos\lambda x + B\sin\lambda x$$

and $$Y = Ce^{\lambda y} + De^{-\lambda y}$$

Hence

$$V(x,y) = XY = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \quad \text{...(vii)}$$

using boundary condition (v), we get C = 0

Otherwise $V \to \infty$ as $y \to \infty$ and hence

$$V(x,y) = (A\cos\lambda x + B\sin\lambda x)e^{-\lambda y}. \quad \text{(we have put } D = 1\text{)}$$

and using boundary condition (iii), we have

$$\sin\lambda a = 0$$

or $$\lambda = \frac{n\pi}{a}(n = 1, 2, 3,...)$$

Thus for each value of n, we have

$$V_n(x,y) = B_n \sin\frac{n\pi}{a}xe^{-n\pi y/a} \qquad (n = 1,2,3,....) \quad \text{...(viii)}$$

and therefore for different values of n, the solution may be taken as

$$V(x,y) = \sum_{n=1}^{\infty} V_n(x,y)$$

or $$V(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}xe^{-n\pi y/a} \quad \text{...(ix)}$$

Using boundary condition (iv), we have

$$V(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x = f(x)$$

which gives

$$B_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx \quad \text{...(x)}$$

Notes

Hence (ix) with the coefficient (x) is the solution of Laplace's equation (i), which satisfy all the given boundary conditions.

Case II: Let there be a thin rectangular metallic plate bounded by the lines $x=0,\ x=a,\ y=0$ and $y=b,$ the edges $x=0, x=a, y=0$ are kept at temperature zero while the edge $y=b$ is kept at temperature $f(x)$.

Here the boundary conditions are given by

$$V(0,y) = 0 \quad ...(xi)$$

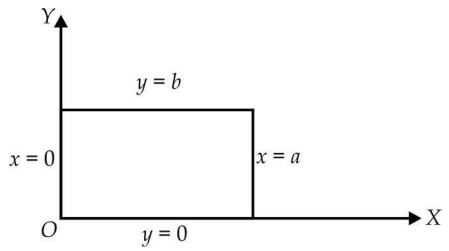
$$V(a,y) = 0 \quad ...(xii)$$

$$V(x,0) = 0 \quad ...(xiii)$$

$$V(x,b) = f(x) \quad ...(xiv)$$

Proceeding as in Case I and using (xi) and (xii), we get

Figure 22.6



$$A = 0 \text{ and } \lambda = \frac{n\pi}{a} \qquad (n=1, 2, 3, ...)$$

Therefore for each value of n, we have

$$V_n(x,y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a} \sin\frac{n\pi}{a}x... \quad (n=1, 2, 3,...)$$

Hence for different values of n, the solution of (i) is

$$V(x,y) = \sum_{n=1}^{\infty}\left(C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}\right)\sin\frac{n\pi}{a}x$$

In this result using (xiii), we get

$$D_n = -C_n.$$

Therefore

$$V(x,y) = \sum_{n=1}^{\infty} C_n\left(e^{n\pi y/a} - e^{-n\pi y/a}\right)\sin\frac{n\pi}{a}x$$

or

$$V(x,y) = \sum_{n=1}^{\infty} C'_n \sin h\frac{n\,\pi\,y}{a}\sin\frac{n\,\pi\,x}{a} \text{ where } C'_n = 2\,C_n \quad ...(xv)$$

Now using (xiv), we get

$$V(x,b) = \sum_{n=1}^{\infty} C_n' \sin h \frac{n\pi b}{a} \sin \frac{n\pi}{a} x = f(x)$$

or $$C_n' \sin \text{h} \frac{n\pi b}{a} = \frac{2}{a}\int_b^a f(x) \sin \frac{n\pi x}{a} dx$$

or $$C_n' = \frac{2}{a \sin h \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \quad \text{...(xvi)}$$

Hence (xv) with coefficient (xvi) in the solution of (i) satisfying the given boundary conditions.

Case III: Let there be a rectangular plate of length *a* and width *b*, the sides of which are kept at temperature zero, the lower end is kept at temperature *f*(*x*) and the upper edge is kept insulated.

Boundary conditions are:

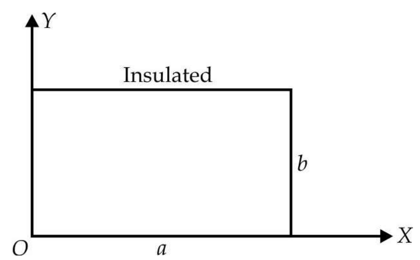
$$V(0,y) = 0 \quad \text{...(xvii)}$$

$$V(a,y) = 0 \quad \text{...(xviii)}$$

$$V(x,0) = f(x) \quad \text{...(xix)}$$

$$\left(\frac{\partial V}{\partial y}\right)_{Y=b} = 0 \quad \text{...(xx)}$$

Figure 22.7



Proceeding as in Case I, assuming the solution of equation (i) as $V(x,y) = X(x)Y(y)$ and substituting this in equation (i) itself. We get two differential equations.

$$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0 \text{ and } \frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0$$

whose general solutions are

$$X = A\cos \lambda x + B \sin \lambda x$$

and $$Y = C \cos h\, \lambda y + D \sin h\, \lambda y$$

Notes

respectively. Therefore

$$V(x,y) = (A\cos\lambda x + B\sin\lambda x)(C\cos h\,\lambda y + D\sin h\,\lambda y) \quad \text{...(xxi)}$$

Using boundary conditions (xvii) and (xviii) in (xxi), we get

$$A = 0 \text{ and } \lambda = \frac{n\pi}{a} \qquad (n = 1, 2, 3,...)$$

Hence for each value of n, we have

$$V(x,y) = \sum_{n=1}^{\infty}\left(C_n \cos h\frac{n\,\pi\,y}{a} + D_n \sin h\frac{n\,\pi\,y}{a}\right)\sin\frac{n\,\pi}{a}x \quad \text{...(xxii)}$$

Using (xix) in (xxii) we have

$$V(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\,\pi}{a}x = f(x)$$

Therefore

$$C_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\,\pi}{a}x\,dx \quad \text{...(xxiii)}$$

Again using (xx) in (xxii), we have

$$\left(\frac{\partial V}{\partial y}\right)_{y=b} \sum_{n=1}^{\infty}\left(C_m \sin h\frac{n\,\pi\,b}{a} + D_n \cos h\frac{n\pi b}{a}\right)\sin\frac{n\,\pi}{a} = 0$$

This will be true for all values of x, if

$$C_n \sin h\frac{n\pi b}{a} + D_n \cos h\frac{n\pi}{a}b = 0$$

or

$$D_n = -C_n \tan h\frac{n\pi b}{a}s \quad \text{...(xxiv)}$$

Therefore (xxii) with coefficients given by (xxiii) and (xxiv) is the solution of the equation (i) satisfying all the given boundary conditions.

Self Assessment

3. Solve

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

subject to the conditions

$U(0,y) = 0$

$U(l,y) = 0$

and $U(x,a) = \sin\frac{n\,\pi\,x}{l}$ and $U(x,0) = 0$ for $n = 1, 2, 3, ...$

Notes

22.3 Summary

- Laplacian operator is expressed in Cartesian spherical polar co-ordinates and cylindrical co-ordinates.
- The solution of Laplace equation in these co-ordinate systems is solved.
- Laplace differential equations finds its applications in potential problems, in wave propagation and diffusion and heat conduction processes.

22.4 Keywords

Method of Separation of Variables helps in finding the solution of Laplace differential equation in all the three co-ordinate systems.

Partial Differential Equation involve one dependent variable which is a function of more than one independent variable.

22.5 Review Questions

1. Solve Laplace's equation in cylindrical co-ordinates and independent of Z.
2. Solve

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 0$$

subject to the boundary conditions

$$u(r) = 0 \text{ at } r = a$$

and $$r(u) = u_0 \text{ at } r = 2a$$

3. Solve for $U(x,y)$ distribution

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

subject to the conditions

$$U(0,y) = U(l,y) = 0, U(x,0) = x^2$$

and $$\left(\frac{\partial U}{\partial y}\right)_{y=b} = 0$$

4. Find the potential $U(r, \theta)$ inside the spherical surface of radius R when its spherical surface is kept at fixed distribution

$$U(R,\theta) = U_0 \cos\theta$$

Answers: Self Assessment

1. $U(r,\theta) = \dfrac{2(3\cos^2\theta - 1) - r^2}{3r^3}$

Notes

2. $U(r) = \frac{(a\,u_{10} - b\,u_{20})}{(a-b)} - \frac{ab(u_{10} - u_{20})}{(a-b)r}$

3. $U(r,y) = \sin h \frac{n\,\pi y}{l} \sin \frac{n\,\pi\,x}{l} \Big/ \sin h\left(\frac{n\,\pi\,a}{l}\right)$

22.6 Further Readings

Books

K. Yosida, Lectures in Differential and Integral Equations

L.N. Sneddon, Elements of Partial Differential Equations

Louis A. Pipes and L.R. Harnvill, Applied Mathematics for Engineers and Physicists

Unit 23: Wave and Diffusion Equations by Separation of Variable

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Note that it finds its applications in almost all branches of applied sciences.
- Understand how heat flows in solids
- See how the electrical current and potentials are distributed in certain medias.
- Know how the diffusion problem is tackled by means of diffusion equation.

Introduction

It is seen that Laplace equation plays an important role in the solution of wave equation as well as conduction of heat.

The problems occurring in this unit are based on boundary values of the waves as well as the temperature distribution of the substance.

Depending upon the symmetry of the problem the Laplace equation is solved in Cartesian or spherical polar co-ordinates or cylindrical co-ordinates.

23.1 On Solution of Wave Equation

When a stone is dropped into a pond, the surface of the water is disturbed and waves of displacement travel radially outward, when a tuning fork or a bill is struck, sound waves are

Notes

propagated from the source of the sound. The electrical oscillations of a radio antenna generate electromagnetic waves that are propagated through space. All these entities are governed by a certain differential equation, called a wave equation. This equation has the form

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...(1)}$$

Where c is a constant having dimension of velocity, t is the time, x, y, z are the co-ordinates of a certain reference frame and u is the entity under consideration, whether it be a mechanical displacement of components of electromagnetic wave or currents or potentials of an electrical transmission line.

In finding the solution of equation (1) we some times also employ cylindrical co-ordinate system or spherical polar co-ordinate system.

In cylindrical co-ordinate system, wave equation is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...(A)}$$

where as in cylindrical co-ordinate system r, θ, z the wave equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...(B)}$$

Example: Solution of wave equation symmetric in all directions about the origin, i.e. independent of θ and ϕ.

In this case u is independent of θ and ϕ. So from equation (A) we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...(C)}$$

Putting

$$v = ru$$

$$\frac{\partial v}{\partial r} = r\frac{\partial u}{\partial r} + u$$

$$\frac{\partial v}{\partial r} = r\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial r}$$

so from (C)

$$\frac{\partial^2 v}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...(D)}$$

Putting

$$R = r - ct$$

$$T = r + ct$$

gives

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial R}\frac{\partial R}{\partial r} + \frac{\partial v}{\partial T}\frac{\partial T}{\partial r}$$

Notes

$$= \frac{\partial v}{\partial R} + \frac{\partial v}{\partial T}$$

$$\frac{\partial^2 v}{\partial r^2} = \frac{\partial^2 v}{\partial R^2}\frac{\partial R}{\partial r} + 2\frac{\partial^2 v}{\partial R \partial T}.\frac{\partial T}{\partial r} + \frac{\partial^2 v}{\partial T^2}.\frac{\partial T}{\partial r}$$

$$= \frac{\partial^2 v}{\partial R^2} + 2\frac{\partial^2 v}{\partial R \partial T} + \frac{\partial^2 v}{\partial T^2}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial R}\frac{\partial R}{\partial t} + \frac{\partial v}{\partial T}.\frac{\partial T}{\partial t}$$

$$= \frac{\partial r}{\partial R}(-e) + e\frac{\partial v}{\partial T}$$

$$\frac{\partial^2 v}{\partial r^2} = (-e)\frac{\partial^2 v}{\partial R^2}\frac{\partial R}{\partial t} - 2e^2\frac{\partial^2 v}{\partial R \partial T} + e^2\frac{\partial^2 v}{\partial T^2}$$

$$= e^2\left(\frac{\partial^2 v}{\partial R^2} - 2\frac{\partial^2 v}{\partial R \partial T} + \frac{\partial^2 v}{\partial T^2}\right)$$

Substituting in (D) we have

$$\frac{\partial^2 v}{\partial R^2} + 2\frac{\partial^2 v}{\partial R \partial T} + \frac{\partial^2 v}{\partial T^2} = \frac{c^2}{a^2}\left(\frac{\partial^2 v}{\partial R^2} - 2\frac{\partial^2 v}{\partial R \partial T} + \frac{\partial^2 v}{\partial T^2}\right)$$

or $\quad \frac{\partial^2 v}{\partial R \partial T} = 0$ ...(E)

Integrating with respect to T we have

$$\frac{\partial v}{\partial R} = F(R) \qquad \text{...(F)}$$

where F(R) is a constant as far as T is concerned.

Integrating (F) we have

$$v = \int F(R)dR + G(T)$$

$$= H(R) + G(T)$$

or $\quad v = H(r - ct) + G(r + ct)$

This is known as D, Alemberts, solution of the wave equation.

The Transverse Vibrations of a Stretched String

Consider a perfectly flexible string that is stretched between two points having a constant tension *T* which is large enough so that the gravity may be neglected. Let the string be uniform and have a mass per unit length equal to *m*.

Notes

Let us take the initial i.e. undisturbed position of the string to be the axis of x and suppose that the motion is confined to the xy plane. Consider the motion of an element PQ of length as shown in the Figure 23.1.

The net force in the y direction, Fy, is given by

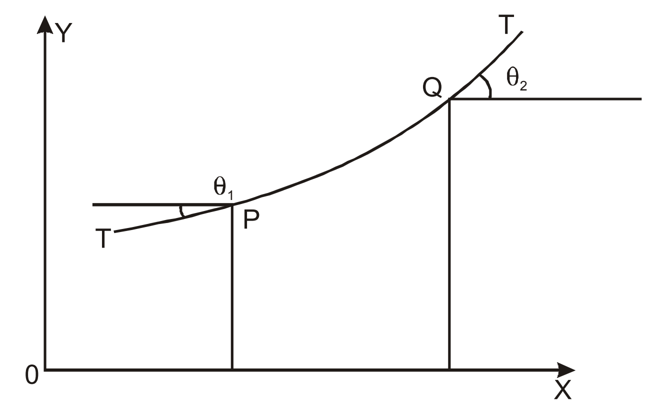
$$Fy = T\sin\theta_2 - T\sin\theta_1 \quad \text{...(i)}$$

Now, for small oscillations, we may write

$$\sin\theta_2 = \tan\theta_2 = \left(\frac{\partial y}{\partial x}\right)_{x+dx} \quad \text{...(ii)}$$

$$\sin\theta_1 = \tan\theta_1 = \left(\frac{\partial y}{\partial x}\right)_{x} \quad \text{...(iii)}$$

Figure 23.1



Therefore, we have

$$F_y = \left(T\frac{\partial y}{\partial x}\right)_{x+dx} - \left(T\frac{\partial y}{\partial x}\right)_{x} \quad \text{...(iv)}$$

Using Taylor's expansion and neglecting terms of order dx^2 and higher, we have

$$F_y = \left(T\frac{\partial y}{\partial x}\right)_{x} + \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)_{x} dx - \left(T\frac{\partial y}{\partial x}\right)_{x}$$

or $$F_y = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)_{x} dx \quad \text{...(v)}$$

By Newton's Law of motion, we have

$$F_y = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)dx = mdx\left(\frac{\partial^2 y}{\partial x^2}\right) \quad \text{...(vi)}$$

where mdx represents the mass of the section of string under consideration and where we have written dx for ds since the placement is small $\frac{\partial^2 y}{\partial x^2}$ is the acceleration of the section of string in the y direction, we thus have

Notes

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = m\frac{\partial^2 y}{\partial t^2} \quad \text{...(vii)}$$

Now if the stretching force is constant throughout the string then we can write

$$T\frac{\partial^2 y}{\partial x^2} = m\frac{\partial^2 y}{\partial t^2} \quad \text{...(viii)}$$

or $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{...(ix)}$$

where $c = \sqrt{\frac{T}{m}}$ ...(x)

This equation (ix) is known as one dimensional wave equation and is a special case of the general wave equation.

The Oscillations of a Hanging Chain

Let us consider the small coplanar oscillations of a uniform flexible string or chain hanging from a support under the action of gravity as shown in Figure 23.2. We consider only small deviations y from the equilibrium position; x is measured from the free end of the chain. Let it be required to determine the position of the chain

$$y = y(x,t) \quad \text{...(1)}$$

where at $t = 0$ we give the chain an arbitrary displacement

$$y = y_0(x) \quad \text{...(2)}$$

In this case the tension T of the chain is variable, and hence eq. governing the displacement of the chain at any instant is given by

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = m\frac{\partial^2 y}{\partial t^2} \quad \text{...(3)}$$

where m is the mass per unit length of the chain. In this case the tension T is given by

$$T = mgx \quad \text{...(4)}$$

Hence we have

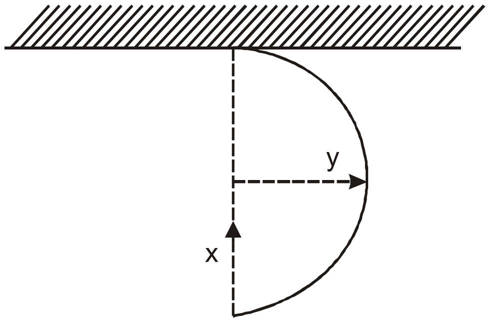
$$\frac{\partial}{\partial x}\left(mgx\frac{\partial y}{\partial x}\right) = m\frac{\partial^2 y}{\partial t^2} \quad \text{...(5)}$$

Or, differentiating and dividing both members by the common factor m, we have

$$x\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} = \frac{1\partial^2 y}{g\partial t^2} \quad \text{...(6)}$$

Notes

Figure 23.2



As in the case of the tightly stretched string, let us assume

$$y(x,t) = e^{jwt} v(x) \quad ...(7)$$

Substituting this into (6), we obtain

$$x\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + \frac{\omega^2}{g} v = 0 \quad ...(8)$$

This equation resembles Bessel's differential equation. Changing the variable x to Z by the relation:

$$Z^2 = \frac{4\omega^2 x}{g} \quad ...(9)$$

reduces (8) to

$$Z^2 \frac{\partial^2 v}{\partial Z^2} + Z\frac{\partial v}{\partial Z} + Z^2 v = 0 \quad ...(10)$$

whose general solution is

$$v = AJ_0(Z) + BY_0(Z) \quad ...(11)$$

where $J_0(Z), Y_0(Z)$ are Bessel functions of first and second kind.

In order to satisfy the condition that the displacement of the string y remain finite when $x = 0$, we must place

$$B = 0 \quad ...(12)$$

Accordingly, in terms of the original variable x, we have the solution

$$v = AJ_0\left(2\omega\sqrt{\frac{x}{g}}\right) \quad ...(13)$$

for the function v.

So far, the value of ω is undetermined. In order to determine it, we make use of the boundary condition

$$v = 0 : \text{at } x = s \quad ...(14)$$

This leads to the equation

$$0 = AJ_0\left(2\omega\sqrt{\frac{s}{g}}\right) \quad ...(15)$$

Now, for a non-trivial solution, A cannot be equal to zero, and hence we have

$$J_0\left(2\omega\sqrt{\frac{s}{g}}\right) = 0 \quad ...(16)$$

If we let

$$u = 2\omega\sqrt{\frac{s}{g}} \quad ...(17)$$

we must find the roots of the equation

$$J_0(u) = 0 \quad ...(18)$$

If we consult a table of Bessel functions, we find that the first three zeros of the Bessel function $J_0(u)$ are given by the values

2.405, 5.52, 8.654

Accordingly the various possible values of ω are given by

$$\omega_1 = \frac{2.405}{2}\sqrt{\frac{g}{s}} \qquad \omega_2 = \frac{5.52}{2}\sqrt{\frac{g}{s}} \qquad \omega_3 = \frac{8.654}{2}\sqrt{\frac{g}{s}} \text{ etc.} \quad ...(19)$$

To each value of ω we associate a characteristic function or eigenfunction v_n of the form

$$v_n = A_n J_0\left(2\omega_n\sqrt{\frac{x}{g}}\right) \quad ...(20)$$

Since the real and imaginary parts of the assumed solution (7) are solutions of the original differential equation, we can construct a general solution of (6) satisfying the boundary conditions by summing the particular solutions corresponding to the various possible values of *n* in the manner

$$y(x,t) = \sum_{n=1}^{n=\infty} J_0\left(2\omega_n\sqrt{\frac{x}{g}}\right)(A_n\cos\omega_n t + B_n\sin\omega_n t) \quad ...(21)$$

where the quantities A_n and B_n are arbitrary constants to be determined from the boundary conditions of the problem. In the case under consideration there is no initial velocity imparted to the chain; hence

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(22)$$

This leads to the condition

$$B_n = 0 \quad ...(23)$$

At t = 0 we have

$$y_0(x) = \sum_{n=1}^{n=\infty} A_n J_0\left(2\omega_n\sqrt{\frac{x}{g}}\right) \quad ...(24)$$

Notes

That is, we must expand the arbitrary displacement $y_0(x)$ into a series of Bessel functions to zeroth order. To do this, we can make use of the results of unit 13. It is shown there that an arbitrary function of $F(x)$ may be expanded in a series of the form

$$F(x) = \sum_{n=1}^{n=\infty} A_n J_0(u_n x) \quad \text{...(25)}$$

where the quantities u_n are successive positive roots of the equation

$$J_n(u) = 0 \quad \text{...(26)}$$

The coefficient A_n are then given by the equation

$$A_n = \frac{2}{J_1^2(u_n)} \int_0^1 z J_0(u_n z) F(z) dz \quad \text{...(27)}$$

To make use of this result to obtain the coefficients of the expansion (24), it is necessary to introduce the variable

$$z = \sqrt{\frac{x}{s}} \quad \text{...(28)}$$

In view of (17) and (18), eq. (24) becomes

$$y_0(x) = y_0(sz^2) = F(z) = \sum_{n=1}^{n=\infty} A_n J_0(u_n z) \quad \text{...(29)}$$

This is the form (25), and the arbitrary constants are determined by (27).

The determination of the possible frequencies and modes of oscillation of a hanging chain is of historical interest. It appears to have been the first instance where the various normal modes of a continuous system were determined by Daniel Bernoulli (1732).

Self Assessment

1. Find the relations between l, m , n and k so that

$$V(x, y, z, t) = A\exp\left[i(lx + my + nz + kct)\right] + B\exp\left[-i(lx + my + nz + kct)\right]$$

is the solution of wave equation

$$\nabla^2 V = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

23.1.1 Solution of One Dimensional Wave Equation

We shall now solve one dimensional wave equation under some boundary conditions. Let $f(x)$ and $g(x)$ be the initial deflection and initial velocity of the string and the string is stretched between two points (0, 0), (L, 0). Hence for the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \text{...(i)}$$

$u(0, t) = 0$,

and $u(L, t) = 0$, for all t, and initial conditions ...(ii)

Notes

$u(x, 0) = f(x)$ ...(iii)

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$ ...(iv)

It is obvious from the equation (i), that u is a function of x and t. Therefore we suppose that the solution of equation is of the form by

$u(x, t) = X(x)T(t)$

or $u(x, t) = XT$(say) ...(v)

where X is a function of x only and T is that of t only.

Substituting this solution in (i), we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2}.\frac{1}{T}.\frac{d^2T}{dt^2}$$

Now L.H.S. is a function of the independent variable x, while R.H.S. is a function of independent variable t. Therefore both sides cannot be equal unless both reduce to a constant value. Hence

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2}.\frac{1}{T}.\frac{d^2T}{dt^2} = 0 \text{ or } \lambda^2 \text{or } -\lambda^2$$

Therefore in the three cases, we have

$$\frac{d^2X}{dx^2} = 0, \qquad \frac{d^2T}{dt^2} = 0,$$

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0, \qquad \frac{d^2X}{dt^2} - \lambda^2 c^2 T = 0,$$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \qquad \frac{d^2X}{dt^2} + \lambda^2 c^2 T = 0$$

The general solutions in the above three cases are

(a) $X = Ax + B$, $T = Ct + D$

(b) $X = Ae^{\lambda x} + Be^{-\lambda x}$, $T = Ce^{\lambda ct} + De^{-\lambda ct}$

(c) $X = A\cos\lambda x + B\sin\lambda x$, $T = \cos\lambda ct + D\sin\lambda ct$

Using boundary conditions and the solution (a), we have

$u(0, t) = X(0)\ T(t) = 0$

and $u(L, t) = X(l)\ T(t) = 0$

which gives either $T(t) = 0$ or $X(0) = X(L)= 0$

But $T(t) \neq 0$ otherwise we get

$u(x, t) = 0$

Therefore $X(0) = X(L) = 0$

Using this in solution (a), we have

$X(0) = B = 0$

and $X(L) = AL + B = 0$

Giving $A = B = 0$. Hence $X(x) = 0$ and therefore $u\ (x, t) = 0$ which is absurd. This proves that (a) cannot be solution of the wave equation (i).

Notes

Now from solution (b) using boundary conditions

$X(0) = A + B = 0$

and $X(L) = Ae^{\lambda x} + Be^{-\lambda x} = 0$

Giving $A - B = 0$, so that $X(x) = 0$ therefore 0 which is absurd.

Hence (a) and (b) are not the solutions of wave equation (i). The third solution (c) is periodic (in time). Therefore the solution is $u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda ct + D\sin\lambda ct) = 0$. Using the boundary conditions (i) and (ii), we have

$$u(0,t) = A(C\cos\lambda ct + D\sin\lambda ct) = 0.$$

Hence $A = 0$

and $u(L,t) = B\sin\lambda L(C\cos\lambda ct + D\sin\lambda ct) = 0.$

this gives $\sin\lambda_L = 0$

or $\lambda L = n\pi$

or $\lambda = \frac{n\pi}{L}$

where n = 1, 2, 3......, (i.e. a + ive integer).

Hence the solution of equation (i) satisfying boundary conditions is

$$u_n(x,t) = \left(C_n\cos\frac{n\pi ct}{L} + D_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L} \qquad ...(vii)$$

Now using initial conditions (iii) and (iv), we have

$$u_n(x,0) = C_n\sin\frac{n\pi x}{L} = f(x)$$

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \left[\frac{-n\pi c}{L}C_n\sin\frac{n\pi ct}{L} + \frac{n\pi c}{L}D_n\cos\frac{n\pi ct}{L}\right]\frac{\sin n\pi x}{L}$$

$$= \frac{n\pi c}{L}D_n\sin\frac{n\pi x}{L} = g(x).$$

Clearly these will not be satisfied if we take only a single term as our solution. The equation (i) is a linear and homogeneous therefore the sum of different solutions will still be a solution.

This instead of (vii), the solution may be taken as

$$u(x,t) = \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi ct}{L} + D_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L} \qquad ...(viii)$$

Therefore using initial conditions

$$u(x,0) = \sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{L} = f(x)$$

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty}\frac{n\pi c}{L}D_n\sin\frac{n\pi x}{L} = g(x)$$

Notes

L.H.S. can be considered as the Fourier since expansion of the R.H.S. Hence

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad \text{...(ix)}$$

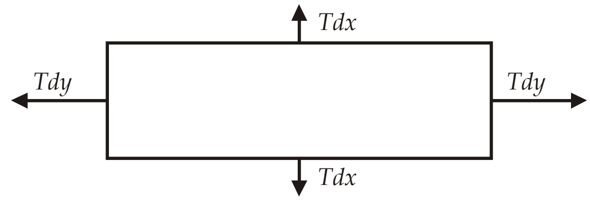
and $$\frac{n\pi c}{L}D_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx \qquad \text{...(x)}$$

These values completely satisfy the solution (viii). Thus u (x, t) given by (viii) with the coefficients (iv) and (x) is the solution of the above equation that satisfies the conditions (i), (ii), (iii) and (iv).

23.1.2 Two Dimensional Wave Equation

As another example leading to the solution of the wave equation, let us consider the oscillations of a flexible membrane. Let us suppose that the membrane has a density of m gms. per cm^2 and that it is pulled evenly around its edge with a tension of T dynes per cm. length of edge. If the membrane is perfectly flexible, this tension will be distributed evenly throughout its area, that is, the material on opposite sides of any line segment dx is pulled apart with a force of $T\,dx$ dynes.

Figure 23.3



Let u is the displacement of the membrane from its equilibrium position. u is then clearly a function of time and of the position on the membrane of the point in question.

If we use rectangular co-ordinates to locate the point, u will be a function of x, y and t. Let us consider an element $dx\,dy$ of the membrane shown in the figure 23.3.

If we refer to the analogous argument for the string, we see that the new force normal to the surface of the membrane due to the pair of tensions Tdy is given by

$$Tdy\left[\left(\frac{\partial u}{\partial x}\right)_{x+dx} - \left(\frac{\partial u}{\partial x}\right)_x\right] = T\frac{\partial^2 u}{\partial x^2}dxdy \qquad \text{...(i)}$$

The net normal force due to the pair Tdx by the same reasoning is

$$Tdx\left[\left(\frac{\partial u}{\partial y}\right)_{y+dy} - \left(\frac{\partial u}{\partial y}\right)_y\right] = T\frac{\partial^2 u}{\partial y^2}dxdy \qquad \text{...(ii)}$$

The sum of these forces is the net force on the element and is equal to the mass of the element times its acceleration. That is, we have

$$Tdy\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]dxdy = m\frac{\partial^2 u}{\partial t^2}dxdy \qquad \text{...(iii)}$$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...(iv)}$$

Notes

where $c = \sqrt{\frac{T}{m}}$

Equation (iv) is the wave equation for membrane.

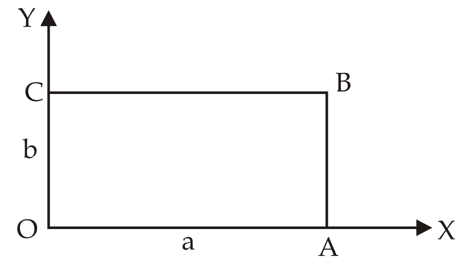
Solution of Two Dimensional Wave Equation

Let us now obtain the solution of the two dimensional wave equation. In the last section we have derived that the oscillations of a perfectly flexible membrane stretched to a uniform tension T are governed by the two dimensional wave equation. Here in this equation $u(x, y, t)$ is the deflection of the membrane.

Let $f(x, y)$ be the initial deflection and $g(x, y)$ be the initial velocity of the membrane.

Therefore the boundary conditions and initial conditions are

Figure 23.4



$$\left.\begin{aligned} u(0,y,t) &= 0 \\ u(a,y,t) &= 0 \\ u(x,0,t) &= 0 \\ u(x,b,t) &= 0 \end{aligned}\right\} \text{ for all t,} \qquad \text{...(i)}$$

and $\quad u(x,y,0) = f(x,y)$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x,y) \text{ respectively.} \qquad \text{...(ii)}$$

It is obvious that u is a function of x, y and t. Hence we suppose that the solution of the equation is of the form

$$u(x,y,t) = X(x)Y(y)T(t)$$

or $\quad u(x,y,t) = XYT(say) \qquad$ ...(iii)

where X is a function of x only, Y is that of y only and T is that of t only.

Substituting this solution in wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

we have

$$\frac{1}{c^2}\cdot\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

Notes

L.H.S. is purely a function of t and R.H.S. is a function of x and y. Hence both sides will be equal only when both reduce to some constant value. Again in R.H.S. the sum of two terms $\frac{1}{X}\frac{\partial^2 Y}{\partial x^2}$ and $\frac{1}{X}\frac{\partial^2 X}{\partial y^2}$ cannot be equal to a constant unless each of these is constant.

Thus we have following three possibilities

(a) $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = 0, \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0, \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0, z$

(b) $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = \lambda^2, \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \lambda_1^2, \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \lambda_2^2,$

where $\lambda^2 = \lambda_1^2 + \lambda_2^2$ and

(c) $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -\lambda^2, \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\lambda_1^2, \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\lambda_2^2,$

where again $\lambda^2 = \lambda_1^2 + \lambda_2^2$

The general solution in above three cases are

$$X = A_1 x + B_1,\ Y = A_2 y + B_2,\ T = A_3 t + B_3, \qquad \text{...(iv)}$$

$$X = A_1 e^{\lambda 1 x} + B_1 e^{-\lambda 1 x},\ Y = A_2 2 e^{\lambda 2 y} + B_2 2 e^{-\lambda 2 y} \text{ and } T = A_3 e^{\lambda c t} + B_3 e^{-\lambda c t} \qquad \text{...(v)}$$

$$X = A_1 \cos\lambda_1 x + B_1 \sin\lambda_1 x$$

$$Y = A_2 \cos\lambda_2 x + B_2 \sin\lambda_2 x$$

$$T = A_3 \cos(C\lambda t) + B_3 \sin(C\lambda t) \qquad \text{...(vi)}$$

From the boundary conditions (i) it is clear that (iv) and (v) are not the solution of the wave equation. Therefore (vi) must be required solution which is periodic in time. Hence we have

$$u(x, y, t) = (A_1 \cos\lambda_1 x + B_1 \sin\lambda_1 x)(A_2 \cos\lambda_2 y + B_2 \sin\lambda_2 y)(A_3 \cos c\lambda t + B_3 \sin c\lambda t) \qquad \text{...(vii)}$$

Using the boundary condition (i), we get

$$u(0, y, t) = A_1 (A_2 \cos\lambda_2 y + B_2 \sin\lambda_2 y)(A_3 \cos c\lambda t + B_3 \sin c\lambda t) = 0$$

$\therefore \quad A_1 = 0$

$$u(a, y, t) = B_1 \sin\lambda_1 a (A_2 \cos\lambda_2 y + B_2 \sin\lambda_2 y)(A_3 \cos c\lambda t + B_3 \sin c\lambda t) = 0;$$

$\therefore \quad \text{Sin}\lambda_1 a = 0$

or $\quad \lambda_1 a = m\pi$

$$\lambda_1 = \frac{m\pi}{a} \qquad (m = 1, 2, 3, ...)$$

Notes

Similarly using other boundary condition, we get

$$A_2 = 0 \text{ and } \lambda_2 = \frac{n\pi}{b} \qquad (n = 1, 2, 3, ...)$$

Now (vii) becomes

$$u_{mn}(x,y,t) = (A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t)x\ x\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad ...(viii)$$

where $\lambda = \lambda_{mn} = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$.

Since the wave equation is linear and homogeneous, therefore sums of any number of different solution will still be a solution.

Thus instead of (viii) an appropriate solution of $u(x,y,t)$ is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{mn}\cos\lambda_{mn}t + B_{mn}\sin(\lambda_{mn}t))\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad ...(ix)$$

where $\lambda^2 = \lambda_{mn}^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$

Now using the initial conditions (ii), we have

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b} = f(x,y).$$

This series is called the double Fourier series of $f(x,y)$ therefore.

$$A_{mn} = \frac{2}{a}.\frac{2}{b}\int_{x=0}^{a}\int_{y=0}^{b}f(x,y)\sin\frac{m\pi x}{x}\sin\frac{n\pi y}{b}dxdy$$

and $$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C\lambda_{mn}B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = g(x,y). \qquad ...(x)$$

Therefore,

$$c\lambda_{mn}\text{B}_{mn} = \frac{2}{a},\frac{2}{b},\int_{x=0}^{a}\int_{y=0}^{b}g(x,y)\sin\frac{m\pi x}{x}\sin\frac{n\pi y}{b}dxdy$$

or $$\text{B}_{mn} = \frac{4}{abc\lambda_{mn}},\int_{x=0}^{a}\int_{y=0}^{b}g(x,y)\sin\frac{m\pi x}{x}\sin\frac{n\pi y}{b}dxdy \qquad ...(xi)$$

Hence the solution of two dimensional wave equation is given by (ix) with the coefficients (x) and (xi) satisfying all the conditions (i) and (ii).

23.1.3 The Vibrations of a Circular Membrane

In the case of the circular membrane we naturally have recourse to polar co-ordinates with the origin at the centre. In this case the equation of motion obtained in Cartesian co-ordinates must

Notes

be transformed to polar co-ordinates, we may write the basic equation of motion of the membrane in the form.

$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ where ∇^2 is Laplacian operator in two dimensions. ...(i)

Transforming this equation to polar co-ordinates, we have

$$c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2} \quad ...(ii)$$

Let $f(r,\theta)$ be the initial displacement and $g(r,\theta)$ the initial velocity of the membrane. Therefore the function $u(r,\theta,t)$ is required to satisfy (ii) and all the boundary and initial conditions, i.e.

Boundary Condition

$$u(a,\theta,t) = 0 \qquad (-\pi < \theta \le \pi; l \ge 0) \quad ...(iii)$$

Initial Condition

$$u(r,\theta,0) = f(r,\theta) \quad ...(iv)$$

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(r,\theta) \qquad 0 \le r \le a, \qquad -\pi \le \theta \le \pi$ ...(v)

since u is a function of r, θ and t, we suppose the solution of equation (ii) as

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

or $u(r,\theta,t) = R\Theta(T)$say ...(vi)

Using the solution (ii) we have

$$\frac{1}{T}\frac{1}{c^2}\frac{d^2T}{dt^2} = \frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{r}.\frac{1}{R}.\frac{dR}{dr} + \frac{1}{r^2}\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}$$

L.H.S. is a function of t and R.H.S. is a function of r and θ, hence both sides will be equal only when both reduce to a constant.

Hence

$$\frac{1}{c^2T}\frac{dT}{dt^2} = \frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{Rr}.\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\theta}{d\theta^2} = -\lambda^2 \quad ...(vii)$$

where $-\lambda^2$ is any constant. We separate the variable in equation (vii) and write

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\Theta^2} = -\mu^2$$

thus we get

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{\mu^2}{r^2}\right)R = 0 \quad ...(viii)$$

Notes

$$\frac{d^2\Theta}{d\theta^2} + \mu^2\theta = 0. \qquad \text{...(ix)}$$

$$\frac{d^2T}{dt^2} + c^2\lambda^2 T = 0. \qquad \text{...(x)}$$

Equation (ix) has the solution of the form

$$\Theta = Ae^{\pm i\mu\theta} \qquad \text{...(xi)}$$

Substituting new variable $s = \lambda r$ in equation (vii), we have

$$\frac{d^2R}{ds^2} + \frac{1}{s}\frac{dR}{ds} + \left(1 - \frac{\mu^2}{s^2}\right)R = 0$$

which is Bessel's equation whose general solution is

$$R = C_1 J_\mu(s) + C_2 Y_\mu(s)$$

or $\quad R = C_1 J_\mu(\lambda r) + C_2 Y_\mu(\lambda r)$

But since the deflection of the membrane is always finite while Y_μ becomes infinite as $r \to 0$ hence we cannot use Y_μ and must choose $C_2 = 0$.

Now using boundary condition (iii)

$$u(a, \theta, t) = R(a)\Theta(\theta)T(t)$$

$\therefore \quad \mathrm{R}(a) = 0$

Otherwise if $\Theta(\theta) = 0$ or $T(t) = 0,\ u = 0$

$$\mathrm{R}(a) = GJ\mu(\lambda\ a) = 0$$

or $\quad J_\mu(\lambda\ a) = 0 \qquad$ ...(xii)

Let $\lambda\mu_1,\ \lambda\mu_2$ be the positive root of (xii),

The corresponding solution of (viii)

$$T = A\mu n \cos e\lambda\mu nt + B\mu n \sin C\lambda\mu nt$$

Thus we get the general solution as

$$u(r, \theta, t) = \sum_{\mu=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{\mu n}\cos C\lambda_{\mu n}t + B_{\mu n}\sin\lambda_{\mu n}t\right)e^{\pm i\mu\theta}J_\mu\left(\lambda_{\mu n}r\right) \qquad \text{...(xiii)}$$

which satisfies the boundary condition (iii).

Considering the solution of the wave equation (ii) which are radially symmetric i.e. when the solution is independent of θ, we get the general solution as

$$u(r, t) = \sum_{n=1}^{\infty}\left(A_n\cos C\lambda_n t + B_n\sin C\lambda_n t\right)J_0\left(\lambda_n r\right) \qquad \text{...(xiv)}$$

Notes

when $\lambda_1, \lambda_2 ...$ are the positive roots of the equation

$$J_0(\lambda a) = 0$$

From (xii) and initial condition (iv) when $t = 0$, we have

$$u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = f(r)$$

$u(r,0)$ becomes $f(r)$ when independent of θ.

Hence A_n must be the coefficients of Fourier Bessel series which represent $f(r)$ in terms of $J_0(\lambda_n r)$ i.e.

$$A_n = \frac{2}{a^2 J_0^2(\lambda_n a)} \int_0^a r f(r) J_0(\lambda_n r) dr, \qquad r = 1, 2, ... \qquad ...(xv)$$

The initial condition (v) gives

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} C\lambda_n B_n J_0(\lambda_n r) = g(r)$$

$[g(r,\theta)$ becomes $g(r)$ when independent of $\theta]$

Again using Fourier Bessel series, we get

$$c\lambda_n B_n = \frac{2}{a^2 J_0^2(\lambda_n a)} \int_0^a r g(r) J_0(\lambda_n r) dr$$

$$B_n = \frac{2}{a^2 J_0^2(\lambda_n a) c\lambda_n} \int_0^a r g(r) J_0(\lambda_n r) dr \qquad ...(xvi)$$

$n = 1, 2, 3.....$

Hence (xiv) is the solution of the wave equation with the coefficients given by the equations (xv) and (xvi) which is radially symmetric.

D, Alembert's Solution of Wave Equation

Given wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(i)$$

Let us introduce two independent variables v and w given by

and $\begin{Bmatrix} v = x + ct \\ w = x - ct \end{Bmatrix}$ ...(ii)

$\therefore \quad \frac{\partial v}{\partial x} = 1$ and $\frac{\partial w}{\partial x} = 1$

Therefore, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial u}{\partial x}$

Notes

$$= \frac{\partial u}{\partial v} + \frac{\partial w}{\partial w}$$

i.e., $\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial v} + \frac{\partial}{\partial w}$

Now $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial w}\right)\left(\frac{\partial u}{\partial v} + \frac{\partial w}{\partial w}\right)$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \qquad \text{...(iii)}$$

Again $\frac{\partial v}{\partial t} = c$ and $\frac{\partial w}{\partial t} = -c$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}.\frac{\partial u}{\partial t} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial t} = c\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\right)\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)$$

$$= c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}\right) \qquad \text{...(iv)}$$

Substituting from (iii) and (iv) in (i), we get

$$= c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}\right)$$

$$= c^2\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}\right)$$

or $\quad \frac{\partial^2 u}{\partial v \partial w} = 0$

Integrating with respect to w, we get

$$\frac{\partial u}{\partial v} = F(v)$$

where $F(v)$ is an arbitrary function of v.

Integrating this with respect to v, we get

$$u = \Phi(v) + \Psi(w).$$

where $\int f(v)\,dv = \Phi(v)$

and $\Psi(w)$ is an arbitrary function of w.

$$\therefore \quad u(x,t) = \phi(x + ct) + \Psi(x - ct) \qquad \text{...(v)}$$

This is known as D, Alembert's Solution of the wave equation (i).

Notes

Example 1: A string is stretched between the fixed points (0, 0) and (1, 0) and released at rest from the positions $u = A \sin \pi x$. Find the formula for its subsequent displacement $u(x, t)$.

Solution: Here the variation of the string is governed by one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions are $u(0,t) = 0$

and $u(1,t) = 0$

Initial conditions are $u(x,0) = A\sin \pi x$

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$

Hence, we have

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos n\pi ct \sin n\pi x$$

where $C_n = 2\int_0^1 A\sin \pi x \sin n\pi x \, dx$

$C_1, C_2, C_3, \ldots$ are all zero, since R.H.S. vanish for all these values

and $C_1 = 2\int_0^1 A\sin \pi x \sin \pi x \, dx$

$$= A\int_0^1 (1 - \cos 2\pi x) dx$$

$$= A$$

Hence $u(x,t) = c_1 \cos(c\pi t) \sin \pi x$

$$= A\cos c\pi t \sin \pi x$$

Example 2: Find the deflection $u(x, y, t)$ of a square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is

$$f(x,y) = A\sin \pi x \sin^2 \pi y$$

Solution: Equation governing the deflection of the membrane is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

Notes

Boundary Conditions

$$u(0,y,t)=0$$
$$u(1,y,t)=0$$
$$u(x,0,t)=0$$
$$u(x,1,t)=0$$

Initial Conditions

$$u(x,y,0)=f(x,y)=A\sin\pi x\sin\pi^2 y$$

and $\left(\dfrac{\partial u}{\partial t}\right)_{t=0}=0$

Now $u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos\lambda_{mn}t\sin m\pi x\sin n\pi y$

Since C = 1, a = 1, b = 1

and $\lambda_{mn}^2=\lambda^2\left(m^2+n^2\right)$

where $A_{mn}=4\int_0^1\int_0^1 f(x,y)\sin mn\pi.\sin n\pi y dxdy$

$$=4A\int_0^1\int_0^1\sin\pi x\sin mn\pi x.\sin^2\pi y\sin n\pi y dxdy.$$

clearly $A_{m1}=A_{m3}=A_{m4}=A_{m5}=...0$

and $\quad A_{m2}=4A\int_0^1\int_0^1\sin\pi x\sin m\pi x.\sin^2 2\pi y dxdy.$

$$=2A\int_0^1\sin\pi x\sin m\pi x dx.$$

Now $A_{22}=A_{32}=A_{42}=...=0$

and $\quad A_{12}=2A\int_0^1\sin^2\pi x dx=A$

Hence we have

$$u(x,y,t)=A_{12}\cos\lambda_{12}t\sin\pi x 2\pi y$$

$=A\cos\sqrt{5}\pi t\sin\pi x\sin 2\pi y,$ as all coefficients

Vanish except $\lambda_{12}^2=\pi^2\left(1^2+2^2\right).$

or $\qquad \lambda_{12}=\sqrt{5}\,\pi$

Self Assessment

2. Solve one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0$$

with the boundary equations

Notes

$u(0,t) = 0$
$u(L,t) = 0$
$u(x,0) = 0$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$$

23.2 Boundary Value Problems (Heat Conduction or Diffusion)

Derivation of the Equation of Heat Conduction

In applied mathematics the partial differential equation

$$\frac{\partial V}{\partial t} = h^2 \nabla^2 V$$

where h^2 is a constant and ∇^2 is the Laplacian operator governs the temperature distribution V in homogeneous solids.

To prove this, we know that the role of flow of heat in a homogeneous solid across the surface is $-K\frac{\partial V}{\partial n}$ per unit area, where V is the temperature and K a constant called the thermal conductivity, $\frac{\partial}{\partial n}$ denotes the differentiation along the normal. Taking an element of the solid at the point P (x, y, z) as a rectangular parallelepiped with P centre and edges parallel to the co-ordinate axes, of lengths dx, dy and dz, we find that the rate of flow of heat into the element is

$$K\nabla^2 V dxdydz$$

But the element is gaining heat at the rate

$$\rho C \frac{\partial V}{\partial t} dxdydz$$

where ρ is the density and C the specific heat. Thus, if there is no gain of heat in the element other than by conduction, we have

$$\frac{\partial V}{\partial t} = C^2 \nabla^2 V$$

where $C^2 = \frac{K}{C\rho}$. ...(i)

If heat is being produced at (x, y, z) in any other way, a term must be added to the right hand side of (i).

23.2.1 Variable Heat Flow in One Dimension

If we consider the heat flow in a long thin bar or wire of constant cross-section and homogeneous material which is along x-axis λ and is perfectly insulated, so that the heat flows in the x-direction only, V depends only on x and t and therefore the heat equation becomes.

Notes

$$\frac{\partial V}{\partial t} = c^2 \frac{\partial^2 V}{\partial x^2} \quad \text{...(i)}$$

Equation (i) is known as one dimensional heat equation.

Now we shall find out the solution of equation (i) under different initial and boundary conditions.

Case I: Let L is length of the rod whose ends are kept at zero temperature and whose initial temperature is *f(x)*.

The boundary conditions are

$$V(0,t) = 0 \quad \text{...(ii)}$$

$$V(L,t) = 0 \text{ for all } t \quad \text{...(iii)}$$

The initial conditions are

$$V(x,0) = f(x) \qquad 0 < x < L \quad \text{...(iv)}$$

Let the solution of equation (i) is of the form

$$V(x,t) = X(x)T(t)$$

$$V = XT(say) \quad \text{...(v)}$$

where X is a function of x only and T is that of t only.

Substituting this solution in equation (i), we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{dT}{dt}$$

since L.H.S. is a function of x and R.H.S. is a function of t, hence both sides will be equal only when both reduces to same constant. Therefore

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2t}\frac{dT}{dt} = 0 \text{ or } \lambda^2 \text{or} - \lambda^2$$

and hence in these three cases, we have

(a) $\frac{d^2X}{dx^2} = 0$ and $\frac{dT}{dt} = 0$,

(b) $\frac{d^2X}{dx^2} - \lambda^2 X = 0$ and $\frac{dT}{dt} - \lambda^2 c^2 t = 0$,

(c) $\frac{d^2X}{dx^2} + \lambda^2 X = 0$ and $\frac{dT}{dt} + \lambda^2 c^2 t = 0$

The general solution in these three cases are

(i) $X = Ax + B$ $\qquad T = c$

(ii) $X = Ae^{\lambda x} + Be^{-\lambda x}$ $\qquad T = c^{e^{x^2}c^2t}$

(iii) $X = A\cos\lambda x + B\sin\lambda x,$ $\qquad T = Ce^{-\lambda^2c^2t}$

If we use the boundary conditions (ii) and (iii) we observe that (i) and (ii) do not constitute the solution as they give $A = B = 0$ i.e. $X = 0$ and hence $V(x, t) = 0$, which is absurd.

Using boundary conditions (ii) and (iii) the solution (iii) gives.

$X(0) = A = 0$ and $X(L) = 0 + B\sin\lambda L = 0.$

Now $B \neq 0$ otherwise $X = 0$ and hence $V(x,t) = 0.$

Therefore

$$\sin\lambda L = 0$$

or $$\lambda L = n\pi$$

or $$\lambda = \frac{n\pi}{L}, n = 1,2,3,.....$$

Hence for each value of n.

$$V_n(x,t) = B_n \sin\frac{n\pi}{L} x e^{-n^2\pi^2c^2t/L^2}$$

are solution of (i) satisfying the given boundary condition. Therefore for each value of n, we take the solution as

$$V_n(x,t) = \sum_{n=1}^{\infty} V_n(x,t)$$

or $$V_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x.e^{-n^2\pi^2c^2t/L^2} \quad ...(vi)$$

Using initial condition, we have

$$V(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x = f(x)$$

which gives

$$B_n = \frac{2}{L}\int_0^L F(x)\sin\frac{n\pi}{L} x.dx \quad ...(vii)$$

Thus (vi) with coefficient (vii) is the solution of one dimensional heat equation in (i).

Case II: Let L be the length of a uniform wire whose end $x = 0$ is kept at 0 temperature and other end $x = L$ is kept at constant temperature t_0 and we have to obtain the temperature function of the wire as t increases, the initial temperature being t_1.

Hence boundary conditions are

$$V(0,t) = 0 \quad ...(viii)$$

$$V(L,t) = t_0 \text{ for all } t \quad ...(ix)$$

and initial condition is

$$V(x,0) = t_i \quad ...(x)$$

Notes

Let the solution of heat equation be

$$V(x,t) = XT \quad \text{...(xi)}$$

where X is a function of x only and T that of t only.

Substituting this solution in (i) as we have done in Case I, we get the following three solutions:

(i) $X = Ax + B$ $\quad T = C$

(ii) $X = Ae^{\lambda x} + Be^{\lambda x}$ $\quad T = Ce^{\lambda^2 c^2 t}$

(iii) $X = A\cos\lambda x + B\sin\lambda x$ $\quad T = Ce^{-\lambda^2 c^2 t}$

Hence (ii) does not constitute the solution of (i), since in this case $V(x,t) = XT$ increase indefinitely with time, which is not the case. (iii) is also inadequate to give complete solution since in this case temps tends to zero as t tends to infinity. Hence the complete solution must be a compilation of (i) and (iii) Therefore

$$V(x,t) = V_3(x) + V_t(x,t) \quad \text{...(xii)}$$

where $V_s(x)$ denotes the temperature distribution after a long period of time when the rod has reached a steady state of temperature distribution, $V_t(x, t)$ denotes the transient effects which die down with the passage of time. These two must be the solutions of the types (i) and (iii) respectively.

It is obvious that when the end $x = 0$ is maintained at temperature $V = 0$ and the end $x = L$ at $V = t_0$ ultimately there will be uniform gradation of temperature.

Therefore $V_s(x) = \frac{t_0}{L}x.$

(xii) then becomes

$$V(x,t) = \frac{t_0}{L}x + V_t(x,t)$$

with the help of (viii), (ix) and (x) the boundary and initial conditions for $V_t(x,t)$ are as follows:

$$V(0,t) = V_t(0,t) = 0 \quad \text{...(xiii)}$$

$$V(L,t) = t_0 + V_t(L,t) = t_0$$

or $\quad V_t(L,t) = 0$

and $\quad V(x,0) = \frac{t_0}{L}x + V_t(x,0) = t_i \quad \text{...(xiv)}$

or $\quad V_t(x,0) = t_i - \frac{t_0}{L}x. \quad \text{...(xv)}$

Therefore let us take

$$V_t(x,t) = (A'\cos\lambda x + B'\sin\lambda x)e^{-\lambda^2 c^2 t} \quad \text{...(xvi)}$$

Notes

In this result by making use of (xiii), we get

$$V_t(0,t) = A'e^{-\lambda^2 c^2 t} = 0$$

$\therefore \quad A' = 0$

Then making use of (xiv) in (xvi), we get

$$V_t(L,t) = B'\sin\lambda L = 0$$

$\therefore \quad \sin\lambda L = 0$

or $\quad \lambda L = n\pi$

or $\quad \lambda = \dfrac{n\pi}{L} \qquad (n = 1, 2, 3, ...)$

Therefore a solution for $V_t(x,t)$ is

$$B_n \sin\frac{n\pi}{L}x.e^{-x^2\pi^2c^2t/L2} \qquad (n = 1,2,3,...)$$

Now adding the solutions for different n the general solution may be written as

$$V_t(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x.e^{-x^2\pi^2 t/L2} \qquad ...(xvii)$$

In this result if we use (*xv*), we get

$$V_t(x,0) = t_i - \frac{t_0}{L}x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$$

which gives $B_n = \dfrac{2}{L}\displaystyle\int_0^L \left(t_i - \frac{t_0}{L}x\right)\sin\frac{n\pi}{L}x dx$

Integrating by parts, we get

$$B_n = \frac{2}{n\pi}\left[t_i - (-1)n(t_i - t_0)\right]$$

Therefore

$$V_t(x,t) = \frac{t_0}{L}x + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[t_i - (-1)^n(t_i - t_0)e^{-x^2\pi^2c^2t/L}\sin\frac{n\pi x}{L}\right] \qquad ...(xviii)$$

Here if the initial temperature of the wire is zero then, we get

$$V_t(x,t) = \sqrt{\frac{t_0}{L}}\left[x + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n e^{-x^2\pi^2c^2t/L}\sin\frac{n\pi x}{L}.\right] \qquad ...(xix)$$

Case III: Let there is a bar of infinite length (i.e. extending up to infinity on both sides) which is insulated laterally. Then we have to find out the solution of heat equation (1) if the initial temperature of the bar is *f(x)*.

Notes

In this case there is no boundary condition and the initial condition is

$$V(x,0) = f(x) \qquad (-\infty < x < \infty) \qquad ...(xx)$$

Again we assume the solution of equation (xi) as

$$V(x,t) = X.T.$$

Proceeding as in the last two cases, we get the three solutions and here we find that (i) and (ii) do not constitute the solution. Hence we take here the third solution (iii), i.e.

$$X = A\cos px + B\sin px \text{ and } T = C_0 e^{-c^2 p^2 t}$$

Here we have taken the constant as $-p^2$ instead of $-\lambda^2$.

Hence $V(x,t,p) = XT = (C\cos px + D\sin px)e^{-c^2 p^2 t}$ ...(xxi)

Since $f(x)$ is not periodic here, therefore we will use Fourier integrals and not Fourier series. Also, we may consider C and D as functions of p

write $C = C(p)$, $D = D(p)$.

Now since the heat equation is linear and homogeneous, we have

$$V(x,t) = \int_0^\infty V(x,t,p)dp$$

or $$V(x,t) = \int_0^\infty \left[C(p)\cos px + D(p)\sin px\right]e^{-c^2 p^2 t}dp \qquad ...(xxii)$$

(xxiii) is the solution of (i) provided this integral exists and can be differentiated w.r.t. ,x, and w.r.t. ,t,.

Using the initial condition (xx), we get

$$V(x,0) = \int_0^\infty \left[C(p)\cos px + D(p)\sin px\right]dp = X(x)$$

$$\therefore \quad C(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\lambda)\sin p\lambda d\lambda$$

and $$D(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\lambda)\sin p\lambda d\lambda;$$

$$\therefore \quad V(x,t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(\lambda)\cos(px - p\lambda)e^{-c^2 p^2 t}d\lambda\right]dp$$

$$= \frac{1}{\pi}\int_0^\infty f(\lambda)\left[\int_0^\infty e^{-c^2 p^2 t}\cos(x-\lambda)p.dp\right]d\lambda$$

The change of the order of integration is justified, since inner integral exists and after changing the order of integration resulting integral also exists.

Solving the inner integral by using the substitution $cp\sqrt{t} = s$ and using the well known integral

$$\int_0^\infty e^{-s^2}\cos 2bs ds = \frac{\sqrt{\pi e^{-b^2}}}{2}$$

Notes

we get $V(x,t) = \frac{1}{2^6 \sqrt{\pi} t^{-\infty}} \int_{-\infty}^{\infty} f(\lambda) e^{-(x-\lambda)^2/4c^2 t d\lambda}$

Putting $\frac{\lambda - x}{2c\sqrt{t}} = w$, so that $dx = -2c\sqrt{t}dw$, we have

$$V(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f\left(x + 2cw\sqrt{t}\right) e^{-w^2 dw} \qquad \text{...(xxiii)}$$

which is the required solution.

Case IV: Let there be a bar of length L which is perfectly insulated. Both ends i.e. $x = 0$ and $x = L$ are also perfectly insulated and the initial temperature of the bar is

$$V(x,0) = f(x)$$

The flux of heat across the faces $x = 0$ and $x = L$ is proportional $t_0 \frac{\partial V}{\partial x}$ at the end, since these ends are insulated. In this case the boundary conditions are

$$\frac{\partial}{\partial x} V(0,t) = 0 \qquad \text{...(xxiv)}$$

$$\frac{\partial}{\partial x} V(L,t) = 0 \qquad \text{...(xxv)}$$

and the initial condition is

$$V(x,0) = f(x) \qquad (0 < x < L) \qquad \text{...(xxvi)}$$

Proceeding as in Case I, here also we get three solutions. Solution (ii) is inadmissible as in this $V = XT$ increases indefinitely with time. The solution (iii) by itself is inadequate since in this case the temperature will tend to zero as t tends to infinity. Therefore general solution will consist of the solution of (i) and (iii).

Using boundary condition (xxiv) in solution (i), i.e.

$$X = Ax + B \text{ and } T = C$$

or $\quad V = A'x + B'$

we get $A' = 0.$

Therefore $V = B'$ is one of the solution of (i). Considering solution (iii) i.e.

$$X = A\cos\lambda x + B\sin\lambda x,\ T = Ce^{-\lambda^2 c^2 t}$$

or $\quad V(x,t) = (C'\cos\lambda x + D'\sin\lambda x)e^{-\lambda^2 c^2 t}$

Using boundary condition (xxiv) and (xxv), we get

$$D' = 0$$

and $\quad \lambda = \frac{n\pi}{L} \qquad (n = 1,2,3,.....)$

Notes

Therefore for each value of n, we have a solution of (i) of the type

$$V(x,t) = A_n \cos\frac{n\pi}{L} x e^{-n^2\lambda^2 c^2 t/L^2}$$

Hence the complete solution of (i) is

$$V(x,t) = B' + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L} x e^{-n^2\lambda^2 c^2 t/L^2} \qquad ...(xxvii)$$

Using the initial condition (xxvi), we have

$$V(x,0) = f(x) = B' + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} dx \qquad ...(xxviii)$$

If we integrate both sides w.r.t. x between the limits 0 to L, we have

$$B' = \frac{1}{L}\int_0^L f(x)dx \qquad ...(xxix)$$

Also if we multiply both sides of (xxviii) by $\cos\frac{n\pi x}{L}$ and then integrate w.r.t. x between 0 to L, we have

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L} dx \qquad ...(xxx)$$

B, can also be written in a better way as

$$B' = \frac{1}{L}\int_0^L f(x)dx$$

$$= \frac{1}{2}.\frac{2}{L}\int_0^L f(x)\cos\frac{\pi x}{L} 0dx$$

$$= \frac{1}{2}A_0$$

Hence complete solution of (i) to be given by

$$V(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-n^2 r^2 c^2 t/L} \qquad ...(xxxi)$$

where $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L} dx$ ...(xxxii)

Self Assessment

3. The heat equation is given by

$$K\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial u}{\partial t}$$

show that the function

Notes

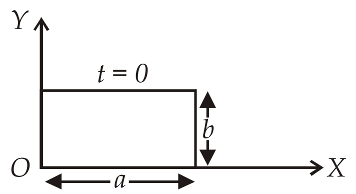
$$U(x,t) = \frac{1}{\sqrt{t}} \exp\left(\frac{-x^2}{4xt}\right)$$

is also the solution of heat equation.

23.2.2 Heat Flow in Two Dimensional Rectangular System

To illustrate the solution of the two dimensional diffusion equation, let us consider the following problem.

Figure 23.5



A thin rectangular plate whose surface is impervious to heat flow has at t = 0 an arbitrary distribution of temperature. Its four edges are kept at zero temperature. It is required to determine the subsequent temperature of the plate as t increases.

Let the plate extend from x = 0 to x = a and from y = 0 to y = b. Expressing the problem Mathematically, we must solve the equation

$$c^2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) = \frac{\partial V}{\partial t} . \qquad \text{...(i)}$$

Subject to the boundary conditions

$$\left.\begin{matrix} V(0,y,t)=0 \\ V(a,y,t)=0 \\ V(x,0,t)=0 \\ V(x,b,t)=0 \end{matrix}\right\} \text{ for all } t. \qquad \text{...(ii)}$$

The initial conditions are

$$V(x,y,0) = F(x,y) \text{ for } 0 \le x \le a, 0 \le y \le b$$

$$V(x,y,\infty) = 0 \qquad \text{...(iii)}$$

To solve equation (i) assume a solution of the form

$$V(x,y,t) = e^{-\theta t} X(x) Y(y) = e^{-\theta t} XY(say) . \qquad \text{...(iv)}$$

where X is a function of x only and Y is function of y only. Substituting (iv) in (i) we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{\theta}{c^2}$$

or $$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{c^2}\theta = \frac{-1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 . \qquad \text{...(v)}$$

Notes

We have now succeeded in separating the variables since the left hand member of (v) is a function of Y only and hence both members of (v) are equal to a constant which we have called λ^2.

Let $\frac{\theta}{C^2} - \lambda^2 = \mu^2$ then ...(vi)

the solutions are

$$X = A_1 \sin \mu x + B_1 \cos \mu x$$

$$X = A_2 \sin \lambda x + B_2 \cos \lambda x \quad ...(vii)$$

And A's and B's are arbitrary constants. Now, to satisfy the boundary conditions (ii), it is obvious that there cannot be any cosine forms present so that we must have

$$B_1 = B_2 = 0$$

Also we must have

$$\sin \mu a = 0$$

and $\sin \lambda b = 0$

which gives $\mu = \frac{m\pi}{a}$ $m = 0,1,2,.....$

and $\lambda = \frac{n\pi}{b}$ $n = 0,1,2,.....$

From (vi) we find that

$$\theta_{mn} = c^2\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \quad ...(viii)$$

Hence for all value of *m* and *n* we find a particular solution of (i) that satisfies the boundary conditions (ii) of the form

$$V = B_{mn} e^{-\theta mnt} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

If we sum over all possible values of *m* and *n* construct the general solution

$$V = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} e^{-\theta mnt} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad ...(ix)$$

Using initial conditions (iii), we get

$$F(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad ...(x)$$

Multiplying both sides of (x) by

$$\sin \frac{r\pi x}{a} \sin \frac{s\pi y}{b} \quad ...(x)$$

and integrating w.r.t. x and y from $x = 0$ and $y = 0$ to $y = b$, because of the orthogonality properties of the sin θ all the terms in the summation vanish except the term for which $m = r$ and $n = s$ and we obtain the result.

$$B_{r\lambda} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} F(x,y)\sin\frac{r\pi x}{a}\sin\frac{\lambda\pi y}{b}dxdy \qquad \text{...(xi)}$$

This determines the arbitrary constants of the general solution (ix)

Three Dimensional Heat Flow

The heat equation in three dimensions is given by

$$\frac{\partial V}{\partial t} = c^2\nabla^2 V = c^2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) \qquad \text{...(i)}$$

where $c^2 = \dfrac{k}{cp}$

Consider now a slab of dimensions a, b, c, the boundary conditions are

$$\text{and} \quad \left.\begin{aligned} V(0,y,z,t) &= 0, \\ V(a,y,z,t) &= 0, \\ V(x,0,z,t) &= 0, \\ V(x,b,z,t) &= 0, \\ V(x,y,0,t) &= 0, \\ V(x,y,c,t) &= 0, \end{aligned}\right\} \qquad \text{...(ii)}$$

for all t.

$$V(x,y,z,0) = F(x,y,z) \text{ for } 0 \le x \le a, \quad 0 \le y \le b \text{ and } 0 \le z \le c. \qquad \text{...(iii)}$$

To solve equation (i) we assume as usually a solution of the form

$$V(x,y,z,t) = e^{-\theta t}X(x)Y(y)Z(z) \qquad \text{...(iv)}$$

and then find the solutions similar to the case of two dimensions.

23.2.3 Temperature Inside a Circular Plate

Consider a thin circular plate whose faces are impervious to heat flow and whose circular edge is kept at zero temperature. At $t = 0$ the initial temperature of the plate is a function *f(r)* of the distance r from the center of the plate only. It is required to find the temperature u (r, t). Let the radius of the plate be a.

The equation of heat conduction is

$$\frac{\partial u}{\partial t} = h^2\nabla^2 u \qquad \text{...(i)}$$

Notes

It is clear that the temperature u must be a function of r and t only (due to symmetry). So using cylindrical co-ordinates, we have

$$\frac{\partial u}{\partial t} = h^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial t}\right), \qquad 0 < r < a \qquad \text{...(ii)}$$

The boundary condition is

$$u = 0 \text{ at } r = a \qquad \text{...(iii)}$$

The initial condition is

$$u(r,0) = f(r) \qquad \text{...(iv)}$$

To solve eq. (ii), let us assume

$$u = e^{-mt}v(r) \qquad \text{...(v)}$$

Substituting in eq. (ii), we obtain

$$-mv(r) = h^2\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \qquad \text{...(vi)}$$

Rewriting (vi) in the form

$$r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} + \frac{mr}{h^2}v = 0 \qquad \text{...(vii)}$$

Let $k^2 = m/h^2$ ...(viii)

and $t = kr$, we have from (vii)

$$t\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} + tv = 0 \qquad \text{...(ix)}$$

which has the same form as Bessel's differential equation for $n = 0$. Hence the general solution of (ix) is

$$v = AJ_0(kr) + BY_0(kr) \qquad \text{...(x)}$$

where A and B are arbitrary constants. Now since the temperature must remain finite at $r = 0$, the arbitrary constant B in *(X)* must be equal to zero. We thus have

$$v = AJ_0(kr) \qquad \text{...(xi)}$$

Since the boundary $r = a$, of the plate is maintained at zero temperature for all values of t, we must have

$$J_0(ka) = 0 \qquad \text{...(xii)}$$

Thus only those values of k are allowed that satisfy equation (xii). Let these values be $k_i\ (i = 1,2,3,...)$. Equation (viii) gives the following values for m:

$$m_i = (k_i h)^2 \qquad \text{...(xiii)}$$

A particular solution of (v) that satisfies the boundary condition is

$$u_i = A_i e^{-k_i^2 th^2} J_0(k_i r)$$

Notes

The general solution is obtained by summing over all values of i.e.

$$u = \sum_{i=1}^{\infty} A_i e^{-k_i^2 th^2} J_0\left(k_i r\right) \quad \text{...(xiv)}$$

where the arbitrary constants A_i must be obtained from the initial conditions i.e. at $t = 0$, $u = f(r)$. Putting $t = 0$ in (xiv), we have

$$f(r) = \sum_{i=1}^{\infty} A_i J_0\left(k_i r\right) \quad \text{...(xv)}$$

Here A_i are now obtained as

$$A_i = \frac{2}{a^2 \left|J_1\left(k_i a\right)\right|^2} \int_0^a r f(r) J_0\left(k_i r\right) dr,\ i = 1, 2, ... \quad \text{...(xvi)}$$

Example 1: Determine the solution of one dimensional heat equation under the following boundary and initial conditions:

$V(0,t) = V(L,t) = 0 \qquad t > 0$

and $V(x,0) = x \qquad 0 < x < L$ where L is the length of the bar.

Solution: Proceeding as before for Case I; we have

$$V(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.e^{-n^2\pi^2 a^2 t/L^2}$$

where $B_n = \frac{2}{L}\int_0^L x.\sin\frac{n\pi}{L} x dx$

Integrating by parts, we get

$$B_n = \frac{2}{n}\frac{L}{\pi}\cos n\pi$$

Therefore $V(x,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi \sin\frac{n\pi x}{L}.e^{-n^2\pi^2 a^2 t/L^2}$

Example 2: A rectangular plate bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$ has an initial distribution of temperature given by.

$$V(x,y,0) = A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

The edges are kept at zero temperature and the plane faces are impervious to heat. Find V at any point and at a time.

Solution: We have the heat equation as

$$\frac{\partial 2V}{\partial x^2} + \frac{\partial 2V}{\partial y^2} = \frac{1}{c^2}\frac{2V}{\partial t}$$

Notes

Let us put the solution as

$$V(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, e^{-c^2\lambda_{mnl}} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

where

$$\lambda^2{}_{mn} = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

and $$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$= \frac{4A}{ab}\int_{x=0}^{a}\sin\frac{\pi x}{a}\sin\frac{m\pi x}{a}\left[\int_{y=0}^{b}\sin\frac{\pi y}{b}\sin\frac{n\pi}{b}dy\right]dx$$

for n = 2, 3, 4, ... the inner integral vanishes and for $n = 1$, the value of the integral is $\frac{1}{2}a,$ we have

$$A_{11} = A$$

and $$\lambda_{11} = \pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$$

Therefore $V(x,y,t) = Ae^{-c^2\lambda_{11}t}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}.$

This give the temperature of the plate at any point and time.

Example 3: Find the temperature $u(x,t)$ of a slab whose ends $x = 0$ and $x = L$ are kept at temperature zero and whose initial temperature $f(x)$ is given by

$$f(x) = A \quad \text{when } 0 < x < \frac{L}{2}$$

$$f(x) = 0 \quad \text{when } \frac{L}{2} < x < L$$

Solution: Let L be the length of the slab whose ends are kept at zero temperature and whose initial temperature is $f(x)$.

The boundary conditions are

$$u(0,t) = 0$$

$$u(L,t) = 0 \text{ for all } t. \qquad \ldots(A_1)$$

The initial conditions are

$$u(x,0) = f(x) = A \quad \text{when } 0 < x < \frac{L}{2}$$

$$= f(x) = 0 \quad \text{when } \frac{L}{2} < x < L \qquad \ldots(A_2)$$

Let the solution of the heat equation

Notes

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{dx^2}, \quad \ldots(1)$$

is of the form

$$u(x,t) = X(x)T(t) \quad \ldots(2)$$

where X is a function of x only and T is that of t only.

Substituting in (1), we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} \quad \ldots(3)$$

Since L.H.S. is a function of x only and R.H.S. is a function of t only, both sides will be equal if they are constant i.e. equal to $-\lambda^2$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{dT}{dt} = -\lambda^2$$

Thus

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

and

$$\frac{dT}{dt} + c^2\lambda^2 T = 0 \quad \ldots(4)$$

The solutions of equations (4) are

$$X = A\cos\lambda x + B\sin\lambda x\ ;\ T = Ce^{-e^2\lambda^2 t} \quad \ldots(5)$$

using boundary conditions (A_1), the solution (5) gives

$$X(0) = 0 = A \text{ and } X(L) = 0 + B\sin\lambda L = 0 \quad \ldots(6)$$

Now $B \neq 0$ hence

$$\sin\lambda L = 0$$

or $\lambda L = n\pi,$ for n = 1, 2, 3, … …(7)

$$i.e.\ \lambda = n\pi/L$$

Hence for each value of n

$$u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right)e^{-c^2n^2\pi^2 t/L^2} \quad \ldots(8)$$

are solution of equation (i) satisfying the given boundary conditions (A_1). So the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\frac{\pi^2c^2n^2t}{L^2}} \quad \ldots(9)$$

Notes

The coefficients B_n are given by

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$= \frac{2A}{L}\int_0^{L/2}\sin\frac{n\pi x}{L}dx = \frac{2A}{L}\left.\frac{\left[-\cos\frac{n\pi x}{L}\right]}{(n\pi/L)}\right|_0^{L/2}$$

$$= \frac{2A}{n\pi}\left[-\cos\frac{(n\pi)}{2}\right]$$

or

$$B_n = \frac{2A}{n\pi}\left[1-\cos\frac{n\pi}{2L}\right]$$

$$= \frac{2A}{n\pi}\left(2\sin^2\frac{n\pi}{4L}\right)$$

$$= \frac{4A}{n\pi}\sin^2\left(\frac{n\pi}{4L}\right) \qquad \ldots(10)$$

Thus the solution (9) becomes

$$u(x,t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{n\pi}{4L}\right)}{n}\sin\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2c^2t}{L^2}} \qquad ...(11)$$

So the solution of equation (i) subject to the conditions (A_1) and (A_2) is given by equation (11).

Self Assessment

4. Find the solution of heat equation.

$$\frac{\partial^2 V}{dx^2}+\frac{\partial^2 V}{dy^2}=\frac{\partial V}{dt}$$

Subject to the boundary conditions

$V = 0$ when $t = +\infty$, when $x = 0$ or $x = \ell$ and when $y = 0$ or ℓ.

Also initially

$V(x, y, 0) = f(x, y)$

23.3 Summary

- Wave equation is written in Cartesian co-ordinates, cylindrical co-ordinates and spherical polar co-ordinates.

- It is shown that depending upon the nature of the process the suitable wave equation can be set up and solved.
- One dimensional wave equation suits in most problems. So the solution of wave equation in one dimension is solved.
- Two dimensional wave equation depending upon the symmetry of the problem is solved both in rectangular and circular cases. Also heat conduction is studied.

23.4 Keywords

Heat Conduction: It is an other process that occurs in so many processes. Diffusion process is very very similar to conduction process.

Wave Motion: It can be obtained in mechanical vibrations, electrical vibrations and other processes.

23.5 Review Questions

1. Show that the solution of the wave equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2}$$

can be of the form

$$u(r,t) = \frac{1}{r}\left[t(r-at) + F(r+at)\right]$$

where f and F are arbitrary functions.

2. Solve the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{dt^2} = 0$$

with the boundary conditions

$$\left.\begin{array}{l} u(0,t) = 0 \\ u(l,t) = 0 \end{array}\right] \text{for all } t$$

and

$$u(x, 0) = \text{A} \sin 2\pi x$$

$$\left.\frac{\partial u}{\partial t}\right]_{t=0} = 0$$

3. Solve the heat equation in one dimension:

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 V}{\partial x^2} = 0$$

subject to the conditions

$$u(0, t) = u(\pi, t) = 0$$

$$\text{and } V(x, 0) = \sin 3x$$

Notes

4. Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = L$ are kept at temperature zero and whose initial temperature $F(x)$ is given by

$$f(x) = A \qquad \text{when } 0 < x < \frac{L}{2}$$

$$= 0 \qquad \text{when } \frac{1}{2}L < x < L.$$

Answers: Self Assessment

1. $l^2 + m^2 + n^2 = k^2$

2. $$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

   where

   $$D_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

4. $$V(x,y,t) = \sum A_{mn} \sin\frac{m\pi x}{l}\sin\frac{n\pi y}{l}e^{-rt}$$

   where

   $rl^2 = \pi^2\left(m^2 + n^2\right)$ and

   $$A_{mn} = \frac{4}{l^2}\int_{x=0}^{l}\int_{y=0}^{l} f(x,y)\sin\frac{m\pi x}{\ell}\sin\frac{n\pi y}{\ell}dxdy$$

23.6 Further Readings

Books

H.T.H. Piaggio, Differential Equation

L.N. Sneddon, Elements of Partial Differential Equations

Louis A. Pipes, and L. R. Harnvill, Applied Mathematics for Engineers and Physicists

Unit 24: Integral Equations and Algebraic System of Linear Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Remind ourselves that in unit six we studied Picard's method of showing the existence of the solution of first order differential equations which let us to integral equations.
- Study how to express a differential equation with boundary conditions or initial conditions into an integral equation.
- See the connection between an integral equation and an algebraic system of linear equations.

Introduction

In the next few units we are interested in studying various types of integral equations and see how to solve them.

You will learn how to express a differential equation with initial conditions into an integral equation.

In the case of boundary value problem of a differential equation we are let to Fredholm type of integral equations.

By dividing the interval into segments we will see how the solution of an integral equation reduced to an algebraic system to equations.

24.1 Connection between a First Order Differential Equation and Integral Equation

In unit 6 we studied the existence and uniqueness of the solution of the first order differential equation of the type

$$\frac{dy}{dx} = f(x, y) \qquad ...(1)$$

Notes

with the initial conditions that at $x = x_0$, $y = y_0$. We assume that

1. The function $f(x, y)$ is real valued and continuous on a domain D of the xy plane given by

$$x_0 - a \leqq x \leqq x_0 + a,\ y_0 - b \leqq y \leqq y_0 + b \qquad \text{...(2)}$$

where a and b are positive numbers

2. $f(x, y)$ satisfies the Lipschitz condition with respect to y in D, that is, there exists a positive constant k such that

$$|f(x,y_1) - f(x,y_2)| \leqq K|y_1 - y_2| \qquad \text{...(3)}$$

for every pair of points $(x, y_2), (x, y_2)$ of D,

with the help of Picard's method of successive approximation, it is then seen that $y(x)$ satisfies the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \qquad \text{...(4)}$$

The integrand $f(t, y(t))$ on the right hand side of (4) is a continuous function, hence $y(x)$ is differentiable with respect to x, and its derivatives is equal to $f(x, y(x))$. Here the integral equation (4) can be solved by the method of successive approximation.

Uniqueness of the Solution: We have obtained the integral equations (4) for the solution $y(x)$ of (1) satisfying the initial conditions $x = x_0$, $y = y_0$. There remains an other important problem, the problem of uniqueness. Is there any other solution satisfying the same initial condition. Fortunately under our two assumptions, we can prove the uniqueness of the solution. To see this let $z(x)$ be another solution of (1) such that $x = x_0$, $z(x_0) = y_0$. Then

$$z(x) = y_0 + \int_{x_0}^{x} f(t, z(t))\,dt.$$

By the assumption 2, we obtain for $|x - x_0| \leqq b$

$$|y(x) - z(x)| \leqq K\left|\int_{x_0}^{x} |y(t) - z(t)|\,dt\right| \qquad \text{...(5)}$$

Therefore, we also obtain $|x - x_0| \leqq h$

$$|y(x) - z(x)| \leqq KN|x - x_0|$$

where

$$N = Sup_{|x-x_0| \leqq h} |y(x) - z(x)|$$

Substituting the above estimate for $|y(t) - z(t)|$ on the right side of (5), we obtain further

$$|y(x) - z(x)| \leqq NK|x - x_0|^2 / \underline{|2},$$

for $(x-x_0) \leqq h$. Substituting this estimate for $|y(t)-z(t)|$ once more on the right side of (5) we have

$$|y(x)-z(x)| \leqq KN|x-x_0|^3/\underline{|3}$$

for $|x-x_0| \leqq h$. Repeating this substitution we obtain

$$|y(x)-z(x)| \leqq NK|x-x_0|^m/m!, \quad m=1,2,...$$

for $|x-x_0| \leqq h$. The right side of the above inequality tends to zero as $m \to \infty$. This means that

$$N = \sup_{|x-x_0| \leqq h} |y(x)-z(x)|$$

is equal to zero. So the solution of $y(x)$ by the integral equal is unique also.

24.2 Conversion of a Differential Equation of Second Order to an Integral Equation

Example: Convert the differential equation

$$\frac{d^2y}{dx}+2\frac{dy}{dx}-8y = 5x^2-3x \quad ...(1)$$

with the initial conditions

$$x = 0,\ y(x_0)=-2,\ \left.\frac{dy}{dx}\right|_{x=0}=3. \quad ...(2)$$

Solution 1: Let

$$y'' = \frac{d^2y}{dx^2}=G(x) \quad ...(3)$$

Integrating (3) once yields the result

$$y'(x) = \int_0^x G(t)\,dt+C_1$$

For $x = 0$, this gives

$$y'(0) = 0 + C_1 = 3$$

therefore

$$y'(x) = \int_0^x G(t)\,dt+3 \quad ...(4)$$

Again integrating (4),

$$y(x) = \int_0^x\int_0^t G(t')\,dt'\,dt+3\int_0^x dx+C_2$$

Notes

Integrating the first term on the right by parts we have

$$y(x) = \left[t\int_0^t G(t')dt'\right]\Bigg|_0^x - \int_0^x tG(t)dt + 3x + C_2$$

$$= x\int_0^x G(t')dt' - \int_0^x t'G(t')dt' + 3x + c_2$$

or

$$y(x) = \int_0^x (x-t')G(t')dt' + 3x + c_2$$

Subjecting this to the condition

$$y(x) = -2 \text{ at } x = 0$$

we get

$$-2 = 0 + 0 + c_2 \text{ or } c_2 = -2$$

so

$$y(x) = \int_0^x (x-t)G(t)dt + 3t - 2 \quad ...(5)$$

Writing (1) with the help of (3), (4) and (5), we have

$$G(x) + 2\int_0^x G(t)dt + 6 - 8\int_0^x (x-t)G(t)dt - 24t + 16 = 5x^2 - 3x$$

or

$$G(x) + \int_0^x (2 - 8x + 8t)G(t)dt - 5x^2 - 21t + 22 = 0 \quad ...(6)$$

Where

$$G(x) = \frac{d^2y}{dx^2} \quad ...(7)$$

Solution 2: We follow an other method. In this method we integrate equation (1) from 0 to x,

$$[y'(t)]\Big|_0^x + 2[y(t)]\Big|_0^x - 8\int_a^x y(t)dt = \frac{5}{3}x^3 - \frac{3}{2}x^2$$

or

$$y'(x) - y'(0) + 2y(x) - 2y(0) - 8\int_0^x y(t)dt = \frac{5}{3}x^3 - \frac{3}{2}x^2$$

but

$$y'(0) = 3,\ y(0) = -2$$

$$-y'(0) - 2y(0) = 1$$

$\therefore$

$$y'(x) + 2y(x) - 8\int_0^x y(t)dt = \frac{5}{3}x^3 - \frac{3}{2}x^2 - 1$$

Again integrating, we get

$$[y(t)]\Big|_0^x + 2\int_0^x y(t)dt - 8\int_0^x (x-t)y(t)dt = \frac{5}{12}x^4 - \frac{x^3}{2} - x$$

Notes

or $$y(x) - y(0) + \int_0^x (-8x + 8t + 2)\, y(t)\, dt = \frac{5}{12}x^4 - \frac{x^3}{2} - x$$

or $$y(x) + \int_0^x (-8x + 8t + 2)\, y(t)\, dt = \frac{5}{12}x^4 - \frac{x^3}{2} - x - 2 \quad \text{...(8)}$$

Note: In this problem we have two answers, i.e. one for $\frac{d^2y}{dx^2}$ another for y for the same problem, but they lead to the same conclusion.

Self Assessment

1. Express the differential equation

$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + x^2 y(x) = 1 + x$ with the condition at $x = 0$, $y(0) = 4$, $\left.\frac{dy}{dx}\right|_{x=0} = 2$, into integral equation.

24.3 Fredholm Integral Equations and Boundary Value Problem

Let us consider the following example of a second order differential equation with the given boundary conditions and establish the integral equation

Example 1: Express the differential equation

$$\frac{d^2y}{dx^2} + a\,y(x) = 0,$$

with the boundary conditions

$$x = 0,\ y(0) = 0,\ x = 1,\ y(1) = 0,$$

as an integral equation

Solution: We have

$$\frac{d^2y}{dx^2} + a\,y(x) = 0 \quad \text{...(i)}$$

with $$y(0) = 0 = y(1) \quad \text{...(ii)}$$

Method 1: Let

$$\frac{d^2y}{dx^2} = G(x)$$

Integrating, we get

$$\frac{dy}{dx} = \int_0^x G(t)\, dt + c_1 \quad \text{...(iii)}$$

Notes

Again Integrating

$$y(x) = \int_0^x dt' \int_0^{t'} G(t)\,dt + c_1 + c_2$$

$$= \left[t' \int_0^{t'} G(t)\,dt \right]_0^x - \int_0^x t' G(t')\,dt' + c_1 x + c_2$$

$$= x \int_0^x G(t)\,dt - \int_0^x t G(t)\,dt + c_1 x + c_2$$

or

$$y(x) = \int_0^x (x - t) G(t)\,dt + c_1 x + c_2 \qquad \text{...(iv)}$$

For $x = 0$, equation (iv) gives

$$0 = y(0) = 0 + c_1 0 + c_2 \text{ or } c_2 = 0$$

Now (iv) becomes

$$y(x) = \int_0^x (x - t) G(t)\,dt + c_1 x \qquad \text{...(v)}$$

For $t = 1$

$$y(1) = 0 = \int_0^1 (1 - t) G(t)\,dt + c_1 .1$$

or

$$c_1 = -\int_0^1 (1 - t) G(t)\,dt$$

Now equation (v) becomes

$$y(x) = \int_0^x (x - t) G(t)\,dt - \int_0^1 x(1 - t) G(t)\,dt$$

$$= \int_0^x (x - t) G(t)\,dt + \int_0^1 (xt - x) G(t)\,dt$$

$$= \int_0^x (x - t) G(t)\,dt + \int_0^x (xt - x) G(t)\,dt + \int_x^1 (xt - x) G(t)\,dt$$

$$= \int_0^x (x - 1) + G(t)\,dt + \int_x^1 x(t - 1) G(t)\,dt$$

or

$$y(x) = \int_0^1 K(x, t) G(t)\,dt \qquad \text{...(vi)}$$

Notes

with $$K(x, t) = \begin{Bmatrix} (x-1)t & \text{if} & t < x \\ x(t-1) & \text{if} & t > x \end{Bmatrix} \qquad ...(vii)$$

Using this in (i),

$$G(x) + a\int_0^1 K(x,t)G(t)\,dt = 0$$

where $$G(x) = \frac{d^2y}{dx^2},\ K(x,t) = \begin{Bmatrix} (x-1)t, & t < x \\ (t-1)x, & t > x \end{Bmatrix}$$

Method 2:

Integrating (i) from 0 to x

$$\int_0^x y''(t)\,dt + a\int_0^x y(t)\,dt = 0$$

$$\left[y'(t)\right]\Big|_0^x + a\int_0^x y(t)\,dt = 0$$

or $$y'(x) - y'(0) + a\int_0^t y(t)\,dt = 0$$

Again integrating,

$$\left[y(t)\right]\Big|_0^x - y'(0)[t]\Big|_0^x + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or $$y(x) - y(0) - y'(0)x + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or $$y(x) - y'(0)x + a\int_0^x (x-t)\,y(t)\,dt = 0 \qquad ...(viii)$$

Putting $x = 1$, this gives

$$y(1) - y'(0) + a\int_0^1 (1-t)\,y(t)\,dt = 0$$

or as $y(1) = 0$, we have

$$y'(0) = a\int_0^1 (1-t)\,y(t)\,dt$$

Notes

Substituting this in (viii) we have

$$y(x) - xa\int_0^1 (1-t)\,y(t)\,dt + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or
$$y(x) + a\int_0^1 x(t-1)\,y(t)\,dt + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or
$$y(x) + a\int_0^x x(t-1)\,y(t)\,dt + a\int_x^1 x(t-1)\,y(t)\,dt + a\int_0^x (x-t)\,y(t)\,dt = 0$$

or
$$y(x) + a\int_0^x t(x-1)\,y(t)dt + a\int_x^1 x(t-1)\,y(t)\,dt = 0$$

Taking

$$K(t, x) = \begin{cases} t(x-1) & , \quad t < x \\ x(t-1) & , \quad t > x \end{cases}$$

So we get

$$y(x) + a\int_0^1 K(t,x)\,y(t)\,dt = 0 \qquad \text{...(ix)}$$

Example 2: Express the differential equation

$$\frac{d^2y(x)}{dx^2} + \lambda\, y(x) = f(x) \qquad \text{...(1)}$$

into an integral equation. Here y, y' and f are continuous differentiable on the interval $0 < x < 1$ with the boundary conditions.

$$y(0) = 0 = y(1)$$

Following the method 2, let us integrate (1) from 0 to x, we have

$$\int_0^x y''(u)\,du + \lambda\int_0^x y(u)\,du - \int_0^x f(u)\,du = 0$$

or
$$y'(x) - y'(0) + \lambda\int_0^x y(u)\,du - \int_0^x f(u)\,du = 0 \qquad \text{...(2)}$$

Integrating once again, we have

$$\int_0^x y'(x)\,dx - y'(0)\,x + \lambda\int_0^x (x-u)\,y(u)\,du - \int_0^x (x-u)\,f(u)\,du = 0$$

or $$y(x) - y(0) - y'(0)x + \lambda\int_0^x (x-u)y(u)\,du - \int_0^x (x-u)f(u)\,du = 0$$

or $$y(x) - y'(0)x + \lambda\int_0^x (x-u)y(u)\,du - \int_0^x (x-u)f(u)\,du = 0 \quad ...(3)$$

To find the value of $y'(0)$, put $x = 1$ in equation (3), we get

$$0 - y'(0).1 + \lambda\int_0^1 (1-u)\,y(u)\,du - \int_0^1 (1-u)f(u)\,du = 0$$

so $y'(0)$ is given by

$$y'(0) = \lambda\int_0^1 (1-u)y(u)\,du - \int_0^1 (1-u)f(u)\,du \quad ...(4)$$

Substituting this value of $y'(0)$ in equation (3) and rearranging terms we get

$$y(x) = \lambda\int_0^1 x(1-u)y(u)\,du - \lambda\int_0^x (x-u)y(u)\,du - \int_0^1 (1-u)f(u)\,du + \int_0^x (x-u)f(u)\,du$$

or $$y(x) = \lambda\int_0^x x(1-u)y(u)\,du - \lambda\int_0^x (x-u)y(u)\,du + \lambda\int_x^1 x(1-u)y(u)\,du + \int_0^x (x-u)f(u)\,du -$$

$$\int_0^x x(1-u)f(u)\,du - \int_x^1 (1-u)f(u)x\,du = 0$$

Simplifying the above equation we have

$$y(x) = \lambda\int_0^x u(1-x)y(u)\,du + \lambda\int_x^1 x(1-u)y(u)du + \int_0^x u(x-1)f(u)\,du - \int_x^1 x(1-u)f(u)\,du = 0 \quad ...(5)$$

Defining

$$K(u, x) = \begin{cases} u(x-1) & u < x \\ x(u-1) & u > x \end{cases}$$

We write equation (4) as

$$y(x) = -\lambda\int_0^1 K(u,x)y(u)\,du + \int_0^1 K(u,x)f(u)\,dx$$

Knowing $K(u, x)$ and $f(u)$, we know the second integral on the right hand side. Let us put

$$\int_0^1 K(u,x)f(u)\,du = \phi(x) \quad ...(6)$$

Notes

Thus $y(x)$ is given by the integral equation

$$y(x) = -\lambda\int_0^1 K(u,x)\,y(u)\,du + \phi(x) \qquad ...(7)$$

Self Assessment

2. Express

   $2y''(x) - 3y'(x) - 2y(x) = 4e^{-t} + 2\cos t$ with initial conditions $y(0) = 4, y'(0) = -1$, into integral equation.

   (**Hint:** Integrate the differential equation twice and use initial conditions.)

24.4 Relation between Integral Equations and Algebraic System of Linear Equations

Consider the general linear Fredholm integral of the second kind for a function $\phi(x)$ of the type

$$\phi(x) - \lambda\int_0^1 K(x,y)\,dy = f(x) \qquad (0 \le x \le 1) \qquad ...(1)$$

and the linear Fredholm equation of the first kind is given by

$$\int_0^1 K(x,y)\,\phi(y)\,dy = f(x) \qquad (0 \le x \le 1) \qquad ...(2)$$

The problem of solving (1) and (2) can be considered as a generalization of the problem of solving a set of n linear algebraic equations in n unknown:

$$\sum_{s=1}^{n} a_{rs}x_s = b_r. \qquad (r = 1, 2, ...n) \qquad ...(3)$$

For this purpose we divide the interval $(0 \le x \le 1)$ into n segments and define

$$K(x, y) = K_{rs} \qquad (r, s = 1, 2, ...\ n)$$

and $$f(x) = f_r$$

Here, x, y are divided into strips as ...(4)

$$\frac{r-1}{n} < x \le \frac{r}{n} \qquad (r = 1, 2, ...\ n)$$

$$\frac{s-1}{n} < y \le \frac{s}{n} \qquad (s = 1, 2, 3, ...\ n)$$

Then equation (1) becomes

$$\phi(x) = f_r + \lambda\sum_{s=1}^{n}\int_{(s-1)/n}^{s/n} K_{rs}\,\phi(y)\,dy \qquad \left(\frac{r-1}{n} < x \le \frac{r}{n}\right) \qquad ...(5)$$

Equation (5) shows that if a function $\phi(x)$ exists it must be a step function, i.e.

$$\phi(x) = \phi_r \qquad \left(\frac{r-1}{n} < x \leq \frac{r}{n}\right), \quad (r = 1, 2, \ldots n)$$

then equation (1) can be written in the form

$$\phi_r - \lambda \sum_{s=1}^{n} K_{rs}\phi_s = f_r \qquad (r = 1, 2, 3, \ldots. n) \qquad \ldots(6)$$

Define the determinant A with elements

$$\delta_{rs} - \frac{\lambda}{n} K_{rs} \qquad \text{for } r, s = 1, 2, \ldots \text{n}$$

and

$$\delta_{rs} = \begin{cases} 0 & , \quad r \neq s \\ 1 & , \quad r = s \end{cases}$$

If the determinant A does not vanish, then (6), and therefore (5) has a unique solution for any given step function $f(x)$.

In the same way if we take up equation (2) and use equations (3) and (4) then equation (2) takes up the form

$$f_r = \sum_{s=1}^{n} K_{rs} \int_{(s-1)/n}^{s/m} \phi(y)\, dy \qquad \left(\frac{r-1}{n} < x \leq \frac{r}{n}\right), \qquad \ldots(7)$$

This case of (7) is different than that of (5) as here one cannot conclude that $\phi(x)$ is necessarily a step function. All that can be said is that if we set

$$n \int_{(s-1)/n}^{s/n} \phi(y)\, dy = x_s,$$

then (7) becomes

$$F_r = \frac{\lambda}{n} \sum_{s=1}^{n} K_{rs}\, x_s \qquad (r = 1, 2, \ldots n) \qquad \ldots(8)$$

Here $x_1, x_2, \ldots x_n$ give the mean values of $\phi(x)$ is the successive intervals $\left(0, \frac{1}{n}\right), \left(\frac{1}{n}, \frac{2}{n}\right) \ldots \left(\frac{n-1}{n}, \frac{n}{n}\right)$,

So there are infinitely many solutions of $\phi(x)$.

24.5 Summary

- In this unit we have seen how to convert a differential equation with conditions into an integral equation.
- The existence and uniqueness of the solution of the integral equation is based on Picard's method which puts some conditions on the Kernel as well on the function.
- It is seen that the integral equation is reduced to a algebraic system of equations if we divide the interval into segment.

Notes

24.6 Keywords

Integral equation is an equation in which the unknown variable appears under the integral sign.

The conversion of a ***differential equation*** into an integral equation is possible if the function and its first derivatives are continuous in the interval.

24.7 Review Questions

1. Express the differential equation

$$y'' + 2y' - 8y = 0$$

with boundary conditions $y(0) = 0 = y(1)$ as in integral equation.

2. Convert the differential equation

$$y'' + 2y' - 8y = 5x^2 - 3x,$$

with $y(0) = -2, y'(0) = 3$ into integral equation.

Answers: Self Assessment

1. $$G(x) + \int_0^x (x^3 - x - 4x^2)\, G(t)dt = 1 + 3x - 4x^2 - 2, \text{ with } G(x) = \frac{d^2y}{dx^2}$$

2. $$2y(x) + \int_0^t (-2t + 2u - 3)\, y(u)\, du = 4\, e^{-x} - 2\cos x - 10t + 6$$

24.8 Further Readings

Books

Louis A. Pipes and Lawrence R. Harnvill, Applied Mathematics for Engineers and Physicists

Tricomi, P.G., Integral Equation

Yosida, K., Lectures in Differential and Integral Equations

Unit 25: Volterra Equations and L_2 Kernels and Functions

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know that integral equations can be of Volterra type equations of first or second kind or they can be Fredholm type of first or second kind.
- See that in the case of Volterra integral equations the upper limit depends upon the independent variable while in the case of Fredholm integral equations the limits are fixed.
- Understand that there are certain conditions on the Kernels as well on the functions for the existence of the solution. Here it is seen that the Kernels as well as the functions are L_2 class and so the solution does exist.

Introduction

L_2 class Kernels as well as functions are square integrable. So if the iteration procedure is applied one can see that product of two L_2 class Kernels is also L_2-class.

This method enables us to find the resolvent Kernels by L_2-class method and the solution of the integral equation is obtainable.

25.1 Classification of Integral Equations

In the last unit we studied the integral equations by converting a differential equation with boundary conditions or initial conditions. We see that the boundary conditions lead us to integral equations of the type

$$y(x) = f(x) + \int_a^b K(x, u)\, y(u)\, du \qquad ...(1)$$

Notes

or $$\int_a^b K(x,u)\,y(u) = f(x) \qquad ...(2)$$

In these cases the limits of integrations are fixed by some constants and the unknown variable appears inside the integral sign. These equations are known as Fredholm integral equations of the second kind (1) and the first kind (2) respectively.

We can also have integral equations of the following type.

$$y(x) = f(x) + \int_a^x K(x,u)\,y(u)du \qquad ...(3)$$

or $$\int_a^x K(x,u)\,y(u)du = f(x) \qquad ...(4)$$

In the equations (3) and (4) the limits of integration depends on the independent variable. Equations (3) and (4) are known as Volterra integral equations of the second kind and the first kind respectively.

We can take up the various types of integral equations and study them and devise methods of solving them. The solution of the integral equation is based on the properties of the Kernels $K(u, x)$ as well as the function $f(x)$.

In this unit we concentrate on the Volterra integral equations and in particular see how the solution of the Volterra integral equations are carried out along with the discussion of the L_2-Kernel.

25.2 Volterra Integral Equations

In the previous unit we had seen some difficulties in the solutions of the integral equation by converting them into an algebraic system of equations. It is seem there that when dealing with integral equation of the first kind we find the mean values of the function in the successive intervals $\left(0, \frac{1}{n}\right)\left(\frac{1}{n}, \frac{2}{n}\right)$, ... and so therefore the equation (2) of that section will possess infinite many solutions.

To avoid these difficulties, Vito Volterra investigated the solution of the integral equations in which the Kernel satisfies the conditions

$$K(x, y) = 0 \qquad \text{if } u > x \qquad ...(1)$$

This corresponds (in the sense of the previous unit) to the simple case of a system of algebraic linear equations where the elements of the determinant above the main diagonal are all zero.

We rewrite the integral equations of Volterra type of the second kind and first kind as follows:

$$y(x) - \lambda \int_0^t K(x,u)\,y(u)du = f(x) \qquad ...(2)$$

and $$\int_0^t K(x,u)\,y(u)du = f(x) \qquad ...(3)$$

In this section we shall study the Volterra integral equation of the second kind (2) that we can readily solve by Picard's process of successive approximation as discussed in unit 6. We state by setting $y_0(x) = f(x)$ and then determine $y_1(x)$:

Notes

$$y_1(x) = f(x) + \lambda \int_0^k K(x, u)\, f(u) du$$

Continuing in this manner we obtain an infinite sequence of functions

$$y_0(x), y_1(x), y_2(x)......y_n(x)...... \quad ...(4)$$

satisfying the recurrence relations

$$y_n(x) = f(x) + \lambda \int_0^k K(x, u)\, y_{n-1}(u) du, \quad (n = 1, 2, 3....) \quad ...(5)$$

Setting

$$y_n(x) - y_{n-1}(x) = \lambda^n \, \Psi_n(x) \quad (n = 1, 2, 3....) \quad ...(6)$$

and putting

$\Psi_0(x) = f(x)$, we get

$$y_n(x) = \sum_{v=0}^{n} \lambda^v \psi_2 \,(x) \quad ...(7)$$

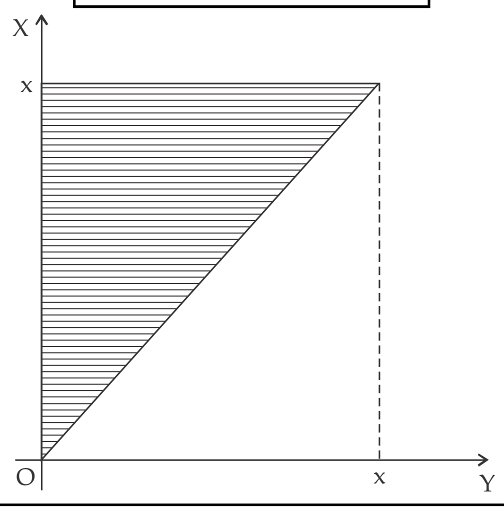
Also $\quad \psi_n(x) = \int_0^x K(x, u)\, \psi_{n-1}(u) du \quad (n= 1, 2, 3, ...)$

Hence $\quad \psi_1(x) = \int_0^t K(x, u)\, f(u) du$

and $\quad \psi_2(x) = \int_0^t K(x, u_1)\, du_1 \int_0^{u_1} K(u_1, u) f(u) du$

This repeated integral be considered as a double integral over the triangular region indicated in the figure 25.1 thus interchanging the order of integration, we obtain

Figure 25.1



$$\psi_2(x) = \int_0^x f(u)\, du \int_u^x K(x, u_1)\, K(u_1, u) du_1$$

or $\quad \psi_2(x) = \int_0^x K_2(x, u) f(u) du$

Notes

where $K_2(x, u) = \int_u^x K(x, u_1)\, K(u_1, u) du_1$

Similarly, we find in general

$$\psi_n(x) = \int_0^x K_n(x, u)\, f(u) du \qquad \text{(n = 1, 2, 3, ...)} \qquad ...(8)$$

Where the integrated Kernels are defined as

$K_1(x, u) \equiv K(x, u)$, $K_2(x, u)$, $K_3(x, u)$......

are defined by the recurrence formula

$$K_{n+1}(x, u) = \int_0^x K(x, u_1)\, K_n(u_1, u) du_1 \qquad (n = 1, 2, 3, ...) \quad ...(9)$$

Moreover, it is easily seen that we also have

$$\begin{aligned} K_{n+1}(x, u) &= \int_0^x K_1(x, u_1)\, K_n(u_1, u) du_1 \\ &= \int_0^x K_{r_0}(x, u_1)\, K_{s_0}(u_1, u) du_1 \qquad r_0 + s_0 = n + 1 \end{aligned} \qquad ...(9)$$

where $r_0 = 1$, $s_0 = n$.

Now $K_{n+1}(x, u) = \int_0^x K_1(x, u_1) \int_{u_1}^x K_1(u_1, u_2)\, K_{n-1}(u_2, u) du_2 du_1$

Interchanging the integrals we have

$$\begin{aligned} K_{n+1}(x, u) &= \int_0^x K_{n-1}(u_2, u) du_2 \int_{u_2}^x K_1(x, u_1)\, K_1(u_1, u_2) du_1 \\ &= \int_0^x K_{n-1}(u_2, u)\, K_2(x, u_2) du_2 \\ &= \int_0^x K_2(x, u_2)\, K_{n-1}(u_2, u) du_2 \end{aligned}$$

In the same way we get

$$K_{n+1}(x, u) = \int_0^x K_3(x, u_2)\, K_{n-2}(u_2, u) du_2$$

and so on. So we may write

$$K_{n+1}(x, u) = \int_0^x K_r(x, u_2)\, K_s(u_2, u) du_2 \qquad \text{where } (r = 1, 2, ...n,\ s = n - r + 1) \quad ...(10)$$

Now from equation (7)

$$\begin{aligned} y_n(x) &= \sum_{v=0}^{n} \lambda^v \psi_v(x) \\ &= \sum_{v=0}^{n} \lambda^v \int_0^x K_v(x, u) f(u) du \\ &= f(x) + \sum_{v=1}^{n} \lambda^v \int_0^x K_v(x, u) f(u) du \end{aligned}$$

or $$y_n(x) = f(x) + \int_0^x \left[\sum_{v=1}^{n} K_v(x, u) \right] f(u) du$$

Notes

Hence if the solution exists, it should be given by letting $n \to \infty$ and given by

$$f(x) = f(x) - \lambda \int_0^x H(x, u, \lambda)\, f(u) du \qquad ...(11)$$

where $H(x, u, \lambda)$ is the resolvent Kernel given by the series

$$H(x, u, \lambda) = -\sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, u) \qquad ...(12)$$

This method of successive approximation cannot only be applied to those of Volterra type integral equations but a whole lot of other equations including the Fredholm integral equations.

Example: Let the Volterra integral equation be given by

$$y(x) = \overset{x}{X} - \int_0^x (x-t)\, y(t) dt \qquad \text{for } 0 \le x \le 1$$

The interacted Kernels are

$$K_1(x, t) = x - t$$

$$K_2(x, t) = \int_t^x (x-r)(r-t) dr = \frac{(x-t)^3}{1.2.3.}$$

$$K_3(x, t) = \int \frac{(x-r)^3 (r-t) dr}{1.2.3} = \frac{(s-t)^5}{\underline{|5}}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$K_n(x, t) = \frac{(x-t)^{2n-1}}{(2n-1)!}$$

Hence

$$y(x) = x - \int_0^s \left[(x-t) + \frac{(x-t)^3}{3!} + \frac{(x-t)^5}{5!} + ... \right] t.dt$$

$$= x + \left[\frac{(x-t)^3}{3!} - \frac{(x-t)^5}{5!} + \frac{(x-t)^7}{7!} .... \right]_0^x = \sin x$$

So the answer is

$y(x) = \sin x$

25.3 L_2-Kernels and Functions

In the case of Volterra integral equation

$$y(x) = f(x) + \lambda \int_0^x K(x, u)\, f(u) du \qquad ...(1)$$

The Kernel $K(x, u)$ and the $f(x)$ are supposed to be continuous and differentiable in the double interval $0 \le x \le h$ and $0 \le u \le h$. They are consequently bounded in the L_2-space. Namely the Kernel and the function $f(x)$ are quadratically integrable in the L_2-space i.e. $0 \le x \le h$ and $0 \le u \le h$ where h is constant i.e. the integrals

$$\|K\| = \int_0^h \int_0^h K^2(x, u) dx du \le N^2 \qquad ...(2)$$

$$\|t\| = \int_0^h f^2(x) dx \qquad ...(3)$$

Notes

exist and are finite in the Lebesgue sense while N is finite. Such a Kernel as well as the function will be called L_2 Kernel and *L2-function,* respectively.

The consequences of the Kernel being L_2-Kernel are many. One of them is as follows: The functions

$$A(x) = \left[\int_0^h K^2(x, u)du\right]^{1/2}, B(u) = \left[\int_0^h K^2(x, u)dx\right]^{1/2} \quad ...(4)$$

exist almost everywhere for $0 \leq x \leq h$ an $0 \leq u \leq h$ respectively. Also $A(x)$, $B(u)$ belong to L_2 class. and finally that

$$\|K\|^2 = \int_0^h A^2(x)dx = \int_0^h B^2(u)du \quad ...(5)$$

Secondly, if $\phi(x)$ is any L_2-function in (0, h) then the two functions

$$\psi(x) = \int_0^h K(x, u)\phi(u)du, \psi(u) = \int_0^h K(x, u)\phi(x)dx \quad ...(6)$$

are also L_2-functions. This is an immediate consequence of the Schwarz inequality

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

From (6) it follows that

$$\|\psi\| \leq \|K\| \|\phi\|, \|\chi\| \leq \|K\| \|\phi\| \quad ...(7)$$

In the same way, it is easy to show that the composition of two L_2 Kernels $K(x, u)$ and $H(u, t)$ i.e. the formation of two new Kernels

$$G_1(x, u) = \int_0^h K(x, u_1)\, H(u_1\ u)du_1$$
$$G_2(x, u) = \int_0^h H(x, u_1)\, K(u_1\ u)du_1 \quad ...(8)$$

yields two new L_2-Kernels, such that

$$\|G_1\| \leqq \|K\| \|H\|, \ \|G_2\| \leq \|H\| \|K\| \quad ...(9)$$

and so on. In fact this last formula give us useful bounds for the norms of the iterated Kernels

$$\|K_n\| \leqq \|K\|^n \quad ...(10)$$

Self Assessment

1. Show that the nth iterated Kernel $K_n(x, u)$ satisfies the bound

$$\|K_n\| \leqq \|K\|^n$$

25.4 Solution of Volterra Integral Equation of Second Kind

In the section we want to prove the existence and uniqueness of the solution of the Volterra integral equation of the second kind

$$y(x) - \lambda\int_0^x K(x, u)\, y(u)du = f(x) \quad (0 \leq x \leq h) \quad ...(1)$$

Notes

where the Kernel $K(x, u)$ and the function $f(x)$ belong to the class L_2. In the last sections we had seen that the solution is given by the formula

$$y(x) = f(x) - \lambda \int_0^x H(x, u, \lambda)\, f(u) du \qquad ...(2)$$

where the resolvent Kernel $H(x, u, \lambda)$ is given by the series of iterated Kernels

$$-H(x, y, \lambda) = \sum_{v=0}^{\infty} \lambda^v\, K_{v+1}\,(x, u) \qquad ...(3)$$

The series (3) converges almost everywhere. $H(x, u, \lambda)$ satisfies the integral equation

$$\begin{aligned} K(x, u) + H(x, u, \lambda) &= \lambda \int_y^x K(x, z)\, H(z, u, \lambda) dz \\ &= \lambda \int_y^x H(x, z, \lambda)\, K(z, u) dz \end{aligned} \qquad ..(4)$$

Proof: with the help of the Schwarz inequality, we first find

$$\begin{aligned} K_2^2(x, u) &= \left[\int_y^x K(x, u_1)\, K(u_1, u) du_1 \right]^2 \\ &\le \int_y^x K^2(x, u_1)\, du_1 \int_y^x k^2(u_1, y) du_1 \\ &\le \int_0^h K^2(x, u_1) du_1 \int_0^h K^2(u_1, u)\, du_1 = A^2(x)\, B^2(y), \end{aligned}$$

and successively

$$\begin{aligned} K_3^2(x, u) &\le \int_y^x K^2(x, u_1) du_1 \int_y^x K_2^2(u_1, u) du_1 \\ &\le \int_0^h K^2(x, u_1) du_1 \int_y^x A^2(u_1)\, B^2(u) du_1 \\ &= A^2(x) B^2(u) \int_y^x A^2(u_1) du_1 \end{aligned}$$

$$\begin{aligned} K_4^2(x, u) &\le \int_y^x K^2(x, u_1) du_1 \int_y^x K_3^2(u_1, u) du_1 \\ &\le \int_0^h K^2(x, u_1) du_1 \int_y^x A^2(u_1)\, B^2(u) du_1 \int_u^x A^2(u_2) du_2 \\ &= A^2(x) B^2(u) \int_y^x A^2(u_1) du_1 \int_u^x A^2(u_2) du_2 \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{aligned}$$

In general, we can write

$$K_{n+2}^2(x, u) \le A^2(x) B^2(u)\, F_n(x, u) \qquad (n = 1, 2, 3, ...) \quad ...(5)$$

where

$$F_1(x, u) = \int_y^x A^2(u_1) du_1,\ F_2(x, u) = \int_u^x A^2(u_1) F_1(u_1, u) dz, ...$$

or generally

$$F_n(x, u) = \int_y^x A^2(z)\, F_{n-1}(z, u) dz, \qquad (n = 2, 3, ...) \quad ...(6)$$

Notes

Now we state that

$$F_n(x, u) = \frac{1}{n!} F_1^n(x, u) \qquad (n = 1, 2, 3, ...) \quad ...(7)$$

This formula is obviously valid for n = 1. If it is assumed true for n = 1, it also remains valid for n, since it follows from (6) that

$$F_n(x, u) = \frac{1}{(n-1)!} \int_y^x A^2(z)\, F_1^{n-1}(z, u) dz$$
$$= \frac{1}{(n-1)!} \int_y^x F_1^{n-1}(z, u) \frac{\partial F_1(z, u)}{\partial z} dz$$
$$= \frac{1}{(n-1)!} \left[\frac{1}{n} F_1^n(z, u) \right]_{z=u}^{z=x} = \frac{1}{n!} F_1^n(x, u)$$

On the other hand from equation (2) of the section it follows that

$$0 \le F_1(x, u) \le \int_0^h A^2(z) dz \le N^2$$

hence

$$0 \le F_n(x, u) \le \frac{1}{n!} N^{2n}$$

and by substituting into (5) we obtain

$$\left| K_{n+2}(x, u) \right| \le A(x)\, B(u) \frac{N^n}{\sqrt{n!}}, (n = 0, 1, 2, ...)$$

Neglecting the first term, this shows that the infinite series (3) or that of equation (12) of section (25.2) which gives the resolvent Kernel H, has the majorant

$$M(x, u) = |\lambda|\, A(x)\, B(u) \sum_{n=0}^{\infty} \frac{(N|\lambda|)^n}{\sqrt{n!}},$$

where the last series always converges because the power series

$$\sum_{n=0}^{\infty} \frac{Z^n}{\sqrt{n!}}$$

has an infinite radius of convergence. This is not sufficient to insure that the series (3) be uniformly and absolutely convergent everywhere, but it is sufficient to ensure its uniform convergence almost every where, because the functions $A(x)$ and $B(u)$ may become infinite in a subset of (0, h) of measure zero. However a fundamental theorem of Lebesgue allows the integration of the series term-by-term, because $M(x, u)$ is a L_2 function. In such a case, we will say that the series is almost uniformly convergent.

If follows that term-by-term integration can be used to evaluate

$$\int_u^x K(x, t)\, H(t, u, \lambda) dt, \int_u^x H(x, u_1, \lambda)\, K(u_1, u) du_1$$

Remembering that

$$K_n(x, u) = \int_y^x K_v(x, z) K_{n-v}(z, u) dz, \qquad (h = 1, 2, ...n - 1) \quad ...(8)$$

we obtain the basic equation (4). Here the interchange of order improving (8) is allowed under our hypothesis that K and hence K_n and H belong to L_2-class.

Notes

With the help of (4), it is easy to prove that the function $y(x)$ given by (2) satisfies (1). Also

$$y_0(x) = f(x) - \lambda\int_0^x H(x, u, \lambda) f(u) du \qquad ...(9)$$

certainly belongs to L_2 proved that $f(x)$ belongs to the same class. But then we have

$$y_0(x) - \lambda\int_0^x K(x, u) y_0(u)\, du$$

$$= f(x) - \lambda\int_0^x H(x, u, \lambda) f(u) du - \lambda\int_0^x K(x, u) f(u) du + \lambda^r \int_0^x K(x, z) dz \int_0^z H(z, u, \lambda) f(u) du$$

$$= f(x) - \lambda\int_0^x \left[K(x, u) + H(x, u, \lambda) - \lambda\int_0^x K(x, z) H(z, u, \lambda) dz\right] f(u) du$$

$$= f(x) + 0 = f(x)$$

So the function $y_0(x)$ from (9) is the only function of class L_2 of the given equation, neglecting the function $y(x)$ given by

$$y(x) - \lambda \int_0^x K(x, u)\phi(u) du \qquad ...(10)$$

known as a zero function in L_2-space. For this we observe that let v be the norm of $y(x)$ in the basic interval $(0, h)$

$$v^2 = \int_0^h y^2(x) dx$$

then from (10) using Schwarz inequality, it follows that

$$y^2(x) \le |\lambda|^2 \int_0^x K^2(x, u) du \int_0^x y^2(z) dz \le |\lambda|^2\ A^2(x) v^2$$

and successively

$$y^2(x) \le |\lambda|^4\ v^2 \int_0^x K^2(x, u) du \int_0^x A^2(z) dz = |\lambda|^4\ v^2 A^2(x) \int_0^x A^2(u) du$$

$$y^2(x) \le |\lambda|^6\ v^2 \int_0^x K^2(x, u) du \int_0^x A^2(u) du \int_0^u A^2(z) dz$$

$$= |\lambda|^6\ v^2 A^2(x) \int_0^x A^2(u) du \int_0^y A^2(z) dz$$

By analogy to (7) we have

$$\int_0^x A^2(u_1) du_1 \int_0^{u_1} ...... \int_0^{u_{n-1}} A^2(u_n) du_n = \frac{1}{n!}\left[\int_0^x A^2(u) du\right]^n \le \frac{N^{2n}}{n!} \qquad ...(11)$$

hence we can write

$$\phi^2(x) \le |\lambda|^2\ v^2 A^2(x) \frac{(|\lambda|^2 N^2)^n}{n!}, (n = 0, 1, 2, ...)$$

and this shows that $y(x) = 0$ at any point where $A(x)$ is finite. So we have shown that $y_0(x)$ is a unique solution of (1).

An alternative approx of proving the existence and uniqueness of the solution of the Volterra integral equation is by Picard's process of successive approximation method. It is advisable to try it as an alternative as given in Yosida book.

Notes

Self Assessment

2. Show that for L_2 Kernel $K(x, t)$ the nth iterated Kernel of Volterra integral equation $K_n(x, t)$ is also L_2 class.

25.5 Summary

- Volterra integral equations are obtained by converting a differential equation with initial conditions.
- For L_2-Kernels the resolvent Kernel can be found by iterated Kernel in the limit of $n \to \infty$.
- For degenerate type of Kernels the resolvent Kernel can be obtained in a simpler way.

25.6 Keywords

Kernel that is L_2 class has the same properties as a square integrable integral.

The ***L_2 class*** nature of the Kernel as well as the function of L_2 class helps finding the solution by iteration.

25.7 Review Questions

1. What ae integral equation. Give examples.
2. How will you classify integral equations?
3. Account for volterra integral equations.
4. What are L_2 Kernel and functions? Explain with suitable examples.
5. Consider the volterra equation with Kernel function

$$\hat{K}(t) = K_k(t) + \theta$$

where $k = 2$, $\theta = 10^{-3}$ and k_k indefined by

$$k_k(t) = \frac{1}{2t^{3/2}\sqrt{k\pi}} \exp\left[\frac{-1}{4kt}\right]$$

construct a solution function.

25.8 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 26: Volterra Integral Equation of the First Kind

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know various types of Kernels and how they help in solving the integral equations.
- Understand that it is difficult to solve Volterra integral equation of the first kind. It can first be converted to Volterra integral equation of the second kind and the methods discussed earlier in units can be employed to solve it.

Introduction

Volterra integral equations of the first kind is by suitable method converted into Volterra integral equations to solve it by suitable method.

The resolvent Kernel can be found easily in the case of Volterra integral equation of the second kind.

26.1 Volterra Equations of First Kind, function and Kernel Classes

In this unit we present methods for solving Volterra linear equations of the first kind which have the form

$$\int_0^x K(x,u)\,y(u)\,du = f(x) \qquad \text{...(1)}$$

Here $y(x)$ is unknown function on the interval $a \le x \le b$ $K(x, u)$ is the Kernel of the equation and $f(x)$ is a given known function. The functions $y(x)$, $f(x)$ are usually assumed to be continuous or square integrable on (a, b). The Kernel $K(x, u)$ is assumed to either continuous or the square $a \le x \le b$, $a \le u \le b$ or it satisfies the condition

$$\int_a^b\int_a^b K^2(x,u)\,dx\,du = N^2 < \infty \qquad \text{...(2)}$$

i.e. $K(x, u)$ is of class L_2. Also $K(x, u) = 0$ for $u > x$.

Notes

We now classify some of the Kernels as follows:

1. ***Degenerate Kernels or Poincere Goursat Kernels***

   The Kernel $K(x, u)$ of the integral equation is said to be degenerate if it can be represented in the form

$$K(x, u) = g_1(x)h_1(u) + g_2(x)h_2(u) + ... \qquad ...(3)$$

2. ***Difference Kernel***

   The Kernel of the integral equation is said to be difference Kernel if it depends upon the difference of the arguments,

$$K(x, u) = K(x - u).$$

3. ***Polar Kernels***

   They are of the form

$$K(x, u) = \frac{L(x,u)}{(x-u)^{\beta}} + M(x,u) \qquad 0 < \beta < 1 \qquad ...(4)$$

   where $L(x, u)$ and $M(x, u)$ are continuous on the square

   $a \le x \le b$, $a \le u \le b$ and $L(x, x) \ne 0$

4. ***Logarithmic Kernels***

   They are of the form

$$K(x, u) = L(x, u) \log (x - u) + M(x - u) \qquad ...(5)$$

   The following generalized Abel equation is a special case of equation (1) with the Kernel of the form (4)

$$\int_0^x \frac{y(u)\,du}{(x-u)^{\beta}} = f(x) \qquad 0 \le \beta \le 1 \qquad ...(6)$$

Example: In case the Kernel $K(x, u)$ and $f(x)$ are continuous then $f(x)$ must satisfy the following conditions:

(i) If $K(a,a) \ne 0$, then $f(\alpha) = 0$

(ii) If $K(a,a) = K_x^1(a,a) = K_x^2(a,a) ... = K_x^{n-1}(a,a) = 0$, and

$$0 \le K_x^n(a,a) \le \infty,$$

then $$f(a) = f^1(a) = ... = f^n(a) = 0.$$

26.2 Reduction of Volterra Equations of the First Kind to Volterra Equations of the Second Kind

Consider Volterra integral equation of the first kind

$$\int_0^x K(x,u)\, y(u)\,du = f(x) \qquad ...(1)$$

Notes

Also suppose that the Kernel $K(x, u)$ and the function $f(x)$ have continuous derivatives on the interval $a \le x \le b$ and $a \le u \le b$ i.e.

$$\frac{d}{dx}f(x), \frac{\partial K(x,u)}{\partial x}, \frac{\partial K(x,u)}{\partial u}$$

exist and continuous, the equation (1) can be reduced to that of second kind provided $k(x, x) \neq 0$.

To see that differentiate (1) with respect to x,

$$K(x,x)y(x) + \int_0^x \frac{\partial K(x,u)}{\partial x}.y(u)\,du = \frac{df}{dx}$$

or
$$y(x) + \int_0^a \frac{\frac{\partial}{\partial x}K(x,u)}{K(x,x)} y(u)\,du = \frac{\frac{dt}{dx}}{K(x,x)} \quad ...(2)$$

which is the Volterra equation of the second kind with Kernel

$$\frac{\partial}{\partial x}[K(x,u)] \Big/ K(x,x)$$

and the function

$$\frac{df}{dx} \Big/ K(x,x).$$

If $K(x, x) = 0$ then we have to differentiate twice to reduce the equation to that of second kind.

There is a second method of reducing the Volterra equation of the first kind to Volterra equation of the second kind. For this consider the equation (1)

$$\int_0^x K(x,u)\, y(u)\,du = f(x) \quad ...(1)$$

If we set
$$\int_0^x y(u)\,du = Z(x) \quad ...(2)$$

Clearly
$$Z(0) = 0$$

Now integrate by parts of L.H.S. of the integral i.e.

$$\int_0^x K(x,u)\frac{dz}{du}(u)\,du = f(x)$$

or
$$K(x,u)Z(u)\Big|_{u=0}^{u=x} - \int_0^x \frac{\partial K(r,u)}{\partial u} Z(u)\,du = f(x)$$

or
$$K(x,x)Z(x) - \int_0^x \frac{\partial K(x,u)}{\partial u} Z(u)\,du = f(x)$$

Notes

or
$$Z(x) - \int_0^x \frac{\frac{\partial K(x,u)}{\partial u}}{K(x,x)} Z(u)\,du = \frac{f(x)}{K(x,x)} \quad ...(3)$$

which is Volterra equation of second kind with Kernel

$$\frac{\frac{\partial K(x,u)}{\partial u}}{K(x,x)}$$

and the function $f(x)/K(x,x)$. Here it is assumed that $K(x, x) \neq 0$. Applying the techniques of last unit we can write the solution of $Z(x)$ as

$$Z(x) = \frac{f(x)}{K(x,x)} - \int_0^x H^{\alpha}(x,u,1)\frac{f(u)}{K(u,u)}du \quad ...(4)$$

where H^{α} $(x, u, 1)$ is the resolvent Kernel corresponding to the Kernel $\frac{d}{du}K(x,u)/K(x,x)$.

Example 1: Consider the Volterra integral equation of the first kind

$$\int_0^x K(x,u)y(u)\,du = f(x) \quad ...(1)$$

with the Kernel $K(x, u)$ given by

$$K(x, u) = e^{x-u}$$

So the equation (1) becomes

$$\int_0^x e^{x-u}y(u)\,du = f(x) \quad ...(2)$$

Let us put

$$\int_0^x y(u)\,du = Z(x) \quad ...(3)$$

So that $Z(0) = 0$ and $y(x) = \frac{dz(x)}{dx}$.

Substituting this value of y in (2) we have

$$\int_0^x e^{x-u}\frac{dz}{du}(u)\,du = f(x)$$

Integrating L.H.S. by parts once we have

$$e^{x-u}Z(u)\Big|_0^x + \int_0^x e^{x-u}Z(u)\,du = f(x)$$

Notes

or $$z(x)+\int_0^x e^{x-u}Z(u)\,du = f(x) \quad ...(4)$$

Which is the Volterra integral equation of the second kind. The equation (4) can be solved by the method developed in the last unit. Here

$$K(x,u) = e^x.e^{-u} \quad ...(5)$$
$$= K_1(x,u)$$

Example 2: Consider the Volterra equation of the first kind

$$\int_0^x K(x,u)y(u) = f(x) \quad ...(1)$$

Where $K(x, u)$ is a degenerate Kernel of the form

$$K(x,u) = g_1(x)g_2(u)+h_1(x)h_2(u) \quad ...(2)$$

Substituting in (1) we get

$$\int_0^x g_1(x)g_2(u)y(u)\,du+\int_0^x h_1(x)h_2(x)y(u)\,du = f(x)$$

or

$$g_1(x)\int_0^x g_2(u)y(u)du+h_1(x)\int_0^x h_2(u)y(u)du = f(x) \quad ...(3)$$

Let us introduce an other variable $Z(x)$ by the relation

$$Z(x) = \int_0^x g_2(u)y(u)\,du \quad ...(4)$$

where $$Z(0) = 0$$

and $$\frac{dz(x)}{dx} = g_2(x)y(x) \quad ...(5)$$

So equation (3) becomes

$$g_1(x)Z(x)+h_1(x)\int_0^x \frac{h_2\frac{dz}{dx}}{g_z(u)}du = f(x)$$

Now integrating by parts the integral on L.H.S. we have

$$g_1(x)Z(x)+h_1(x)\frac{h_2(u)}{g_2(u)}Z(u)\Big|_0^x - h_1(x)\int_0^x \frac{d}{du}\left[\frac{h_2(u)}{g_2(u)}\right]Z(u)\,du = f(x)$$

Notes

or

$$[g_1(x)g_2(x)+h_1(x)h_2(x)]Z(x)-h_1(x)g_2(x)\int_0^x \frac{d}{dx}\left[\frac{h_2(u)}{g_2(u)}\right]Z(u)\,du \quad = \quad f(x)g_2(x).$$

Simplifying equation (6) we have

$$Z(x)-\int_0^x \frac{h_1(x)g_2(x)}{[g_1(x)g_2(x)+h_1(x)h_2(x)]}\frac{d}{du}\left[\frac{h_2(u)}{g_2(u)}\right]Z(u)\,du \quad = \quad \frac{f(x)g_2(x)}{[g_1(x)g_2(x)+h_1(x)h_2(x)]} \quad \text{...(7)}$$

So equation (7) is Volterra equation of the second kind. Putting

$$K^N(x,u) \quad = \quad \frac{h_1(x)g_2(x)}{[g_1(x)g_2(x)+h_1(x)h_2(a)]}\frac{d}{du}\left[\frac{h_2(u)}{g_2(u)}\right]$$

and

$$f^N(x) \quad = \quad \frac{f(x)g_2(x)}{[g_1(x)g_2(x)+h_1(x)h_2(x)]}$$

equation (7) can be put into the form

$$Z(x)-\int_0^x K^N(x,u)\,Z(u)\,du \quad = \quad f^N(x) \quad \text{...(8)}$$

Knowing $g_1(x), g_2(x), h_1(x), h_2(x)$ and $f(x)$ we can then solve equation (8) by the methods of the last unit.

Self Assessment

1. Solve the integral equation

$$y(x) = f(x)+\lambda\int_0^x e^{2(x-u)}\,y(u)\,du$$

26.3 Summary

- Volterra integral equation of the first kind may have a number of different kinds of Kernels.
- It is sometimes useful to convert Volterra integral equation into Volterra integral equation of the second kind.
- By converting Volterra integral equation into that of second order the method of solving the Volterra integral equation of second kind may be employed.

26.4 Keyword

Volterra integral equation of the first kind is related to Volterra integral equation of the second kind and the solution of Volterra integral equation of the first kind can be found by the methods already used.

26.5 Review Question

Convert the Volterra integral equation of the first kind

$$\int_0^x K(x,t)\, y(t)\, dt = x^2$$

where $K(x, t)$ is a degenerate Kernel of the form

$$K(x,t) = xt + (x+1)(t+1).$$

into integral equation of the second kind.

Answer: Self Assessment

1. $y(x) = f(x) + \lambda \int_0^x e^{(2+\lambda)(x-u)} f(u)\, du$

26.6 Further Readings

Books

Tricomi, F.G., Integral Equation

Yosida, K., Lectures in Differential and Integral Equation

Notes

Unit 27: Volterra Integral Equations and Linear Differential Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know that the existence and uniqueness of the solution of differential equations leads us to the integral equations
- See the relation between the integral equations and the linear differential equations with initial conditions.
- Understand that the solution of the integral equation also satisfies a certain differential equation with boundary conditions.

Introduction

The connection between a differential equation and integral equation should be seen clearly.

This connection helps us to solve certain differential equations by converting it into an integral equation and vice versa.

27.1 Relation between Linear Differential Equations and Volterra Integral Equations

In the unit 24 we had seen that a differential equation of first order or second order under certain conditions is converted into an integral equation. This idea can be further explained in details in this unit. Let us consider an nth order linear differential equation as follows:

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_n y = f(x) \quad \ldots(1)$$

It is assumed that the unknown functions $y(x)$, $f(x)$, $a_1(x)$ $a_2(x)$,...$a_n(x)$ are continuous and differentiable on the interval (a, b). The function $y(x)$ satisfies the following initial conditions:

$$y(0) = y_0, y'(0) = y'_0, y''(0) = y''_0 \ldots\ldots \overset{(n-1)}{y}(0) = \overset{(n-1)}{y_0} \ldots \quad \ldots(2)$$

Notes

To convert the linear differential equation (1) into an integral equation we introduce a function $\phi(x)$ by the relation

$$\frac{d^n y}{dx^n} = \phi(x) \qquad ...(3)$$

Integrating once we have by taking into account (2),

$$\left.\frac{d^{n-1}y}{dx^{n-1}}\right]_0^x = \int_0^x \phi(u)du$$

$$\frac{d^{n-1}y(x)}{dx^{n-1}} = y_0^{n-1} + \int_0^x \phi(u)du$$

Integrating once more we have

$$\frac{d^{n-2}}{dx^{n-2}}y(x) = y_0^{n-2} + y_0^{n-1}x + \int_0^x du \int_0^u \phi(u_1)du_1$$

$$= y_0^{n-2} + y_0^{n-1}x + \int_0^x (x-u)\phi(u)du \qquad ...(4)$$

In general integrating up to n times we have

$$y(x) = y_0 + y_0'x + y_0''\frac{x^2}{\lfloor 2} + y_0'''\frac{x^3}{\lfloor 3} + ... + y_0^{n-2}\frac{x^{n-2}}{\lfloor n-2} + y_0^{n-1}\frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1}\phi(u)du... \qquad ...(5)$$

Writing (1) with the help of (3), (4) and (5) we have

$$\phi(x) + a_1(x)\left[y_0^{n-1} + \int_0^x \phi(u)du\right] + a_2(x)\left[y_0^{n-2} + xy_0^{n-1} + \int_0^x (x-u)\phi(u)du\right]$$

$$+ a_3(x)\left[y_0^{n-3} + xy_0^{n-2} + \frac{x^2}{\lfloor 2}y_0^{n-1} + \frac{1}{\lfloor 2}\int_0^x (x-u)^2\,\phi(u)du\right] + ... + ...$$

$$+ a_n\left[y_0 + y_0'\,x + y_0''\frac{x^2}{\lfloor 2} + ... + y_0^{n-1}\frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_0^x (x-n)^{n-1}\phi(u)du\right] = f(x) \qquad ...(6)$$

Defining

$$F(x) = f(x) - a_1(x)y_0^{n-1} - a_2(x)\left[y_0^{n-2} + xy_0^{n-1}\right] - a_3\left[y_0^{n-3} + xy_0^{n-2} + \frac{x^2}{2}y_0^{n-2}\right]$$

$$+ ...... - a_n\left[y_0 + xy_0' + \frac{x^2}{\lfloor 2}y_0'' + ... + y_0^{n-1}\frac{x^{n-1}}{(n-1)!}\right] \qquad ...(7)$$

and

$$K(x,u) = a_1(x) + a_2(x)\,(x-u) + \frac{a_3(x)}{2!}(x-u)^2 + ... + a_n\frac{(x-u)^{n-1}}{(x-1)!}$$

or

$$K(x,u) = \sum_{k=1}^{n} a_k \frac{(x-u)^{k-1}}{(k-1)!} \qquad ...(8)$$

Substituting (7) and (8) into (6) we get

$$\phi(x) + \int_0^x K(x,u)\,\phi(u)du = F(x) \qquad ...(9)$$

Notes

So we have converted a differential equation (1) into the Volterra integral equation of the second kind with Kernel given by (8) and the function given by (7). The unknown function being given by (3).

Example: Convert

$$2y''(x) - 3y'(x) - 2y(x) = 4e^{-x} + 2\cos x \qquad ...(1)$$

with $y(0) = 4$, $y'(0) = -1$, into integral equation

Let us put

$$y''(x) = G(x) \qquad ...(2)$$

Integrating (1) with respect to x, we have

$$y'(x)\Big|_0^x = \int_0^x G(u)du$$

or $$\frac{dy}{dx} - y'(0) = \int_0^x G(u)du$$

$$\frac{dy}{dx} = -1 + \int_0^x G(u)du \qquad ...(3)$$

Integrating with respect to x again we have

$$y(x)\Big|_0^x = -x + \int_0^x du_1 \int_0^{u1} G(u)du$$

or $$y(x) = y(0) - x + \int_0^x (x-u)G(u)du$$

$$y(x) = 4 - x + \int_0^x (x-u)\, G(u)du \qquad ...(4)$$

Substituting from equations (2), (3) and (4) into (1) we have

$$2G(x) - 3\left[-1 + \int_0^x G(u)du\right] - 2\left[4 - x + \int_0^x (x-u)\, G(u)du\right] = 4e^{-x} + 2\cos x$$

Rearranging we have

$$2G(x) + \int_0^x G(u)du\left[-3 - 2(x-u)\right] = 4e^{-x} + 2\cos x - 3 + 2\,(4-x) \qquad ...(5)$$

$$2G(x) + \int_0^x K(x, u)\, G(u)du = F(x) \qquad ...(6)$$

where

$$K(x, u) = -3 - 2(x-u)$$
$$F(x) = 4e^{-x} + 2\cos x + 5 - 2x \qquad ...(7)$$

So we get Volterra integral equation of the second kind.

Self Assessment

1. Convert the linear differential equation

$$\frac{d^3y}{dx^3} + 6y(x) = 0 \text{ with } y(0) = 4, y'(0) = -3, y''(0) = 2$$

Notes

27.2 Conversion of Volterra Integral Equation of Second Kind into a Differential Equation

We have seen that a linear differential equation with initial conditions can be expressed into a Volterra integral equation. In this section we can show that an integral equation can also be converted into a linear differential equation. To see that we take up the following example.

Example: Convert the integral equation

$$y(x) = 3x - 4 - 2\sin x + \int_0^x \left[(x-u)^2 - 3(x-u) + 2\right] y(u)du \quad ...(1)$$

into the linear differential equation.

Before attempting the problem we know that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} K(t,u)y(u)du = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}K(t,u)y(u)du + K[t,b(t)]\frac{db}{dt} - K[t,a(t)]\frac{da}{dt} \quad ...(2)$$

using equation (2), differentiate (1) with respect to x, we have

$$y'(x) = 3 - 2\cos x + \left[(x-x)^2 - 3(x-x) + 2\right]y(x) + \int_0^x \left[2(x-u) - 3\right]y(u)du$$

or $$y'(x) = 3 - 2\cos x + 2y(x) + \int_0^x \left[2(x-u) - 3\right]y(u)du \quad ...(3)$$

Differentiating (3) again, we have

$$y''(x) = 2\sin x + 2y'(x) + \left[2(x-x) - 3\right]y(x) + \int_0^x (2)y(u)du$$

or $$y''(x) = 2\sin x + 2y'(x) - 3y(x) + 2\int_0^x y(u)du \quad ...(4)$$

Differentiating equation (4) again, we have

$$y'''(x) = 2\cos x + 2y''(x) - 3y'(x) + 2y(x)$$

or $$y'''(x) - 2y''(x) + 3y'(x) - 2y(x) = 2\cos x \quad ...(5)$$

Self Assessment

2. Convert the integral equation

$$y(x) = 2x^2 - 3x + 3\cos x + \int_0^x \left[2(x-u)^3 + 3(x-u)^2 + 6\right]y(u)du$$

27.3 Summary

- We have taken up the case of nth order differential equation and have seen how an integral equation can be established.
- There is a strong connection between the initial value differential equation and the Volterra integral equation of the second type or of first type.

Notes

27.4 Keywords

The relation between the ***Volterra integral equation*** and ***linear differential equation*** with initial condition has to be understood.

Method of conversion of differential equation to integral equation shows that the solution is unique as we show that the new integral equation satisfies the original differential equation.

27.5 Review Questions

1. Convert the differential equation

$$y''(x) - xy'(x) + x^2 y(x) = 1 + x$$

with $y(0) = 4$, $y'(0) = 2$, into integral equation.

2. Convert the differential equation

$$y''(x) - 3y'(x) + 2y(x) = 4\left(x - \frac{x^3}{6}\right)$$

with $y(0) = 1$, $y'(0) = -2$

Answers: Self Assessment

1. $G(x) + 3\int_0^x (x-u)^2 G(u)du = 18x - 24 - 3x^2$ with $G(x) = \frac{d^3y}{dx^3}$

2. $y^{iv}(x) - 6y'''(x) - 6y'(x) - 12y(x) = 3\cos x$

27.6 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 28: Integral Equations

CONTENTS

Objectives

After studying this unit, you should be able to:

- Classify the type of Fredholm integral equations.
- Classify the Kernel of any integral equation i.e. is it symmetric or Poincere Goursat type or of different type?
- Choose the right method of solving the integral equation.

Introduction

You have learnt in the previous few units the Volterra integral equation of the second and first kind.

You will find similarities and differences in approach between the two types of integral equations.

28.1 Fredholm Equations

In the last three units we studied one type of integral equation known as Volterra integral equation. In the next few units we are interested in studying an other integral equation known as Fredholm integral equation.

In the case of Volterra integral equation we saw that linear differential equations with initial condition lead us to Volterra integral equation. In the case of boundary value problem, the differential equations can be converted into Fredholm integral equation.

Now the Fredholm equations can be of the form

$$Q(x) = f(x) + \lambda \int_a^b K(x,t)\, Q(t)\, dt \qquad ...(1)$$

or

$$f(x) = \lambda \int_a^b K(x,t) Q(t)\, dt \qquad ...(2)$$

Notes

Here $K(x, t)$ the Kernel and $f(x)$ the function are known and $Q(x)$ is an unknown function on the interval $a \le x \le b$.

Let $\Psi(x)$ be a function which satisfies the Fredholm integral equation

$$\Psi(t) = g(t) + \lambda \int_a^b \bar{K}(t,x) Q(x)\, dx \qquad ...(3)$$

Here $\bar{K}(t,x) = K(x, t)$

28.2 Types of Kernels

Just like in Volterra integral equation in the case of Fredholm integral equations are a variety of Kernels as follows:

1. ***Symmetric Kernels:*** Kernels having properties

   as $K(x, t) = K(t, x)$

   are called symmetric Kernels.

2. ***Degenerate Kernels or Poincere Goursat type of Kernels.*** The Kernels of the type

   $$K(x, t) = \sum_{i=1}^{n} g_i(x)\, h_i(t)$$

   These Kernels play an important part in the development of Fredholm theory of integral equation like the eigenvalue and eigenfunction problems.

3. ***Difference Kernels:*** The Kernels of the type

   $$K(x, t) = K(x - t)$$

   are known as difference Kernels. These types of Kernels do arise while converting a differential equation with boundary conditions.

The conditions on Kernels are that they should be continuous and its partial derivatives should be continuous. Also they should be square integrable.

28.3 Methods of Solving Fredholm Integral Equations

There are various methods of solving integral equations which can briefly summarized as follows:

(a) We can reduce integral equation to a differential equation which can be solved easily.

(b) The Fredholm integral equations can be solved by transform method. In this method the Laplace transformation helps in writing an integral equation into an algebraic equation and then by inverse Laplace transformation get the final solution.

(c) ***The Iteration Method:*** The most important method of solving the Fredholm integral equation is the iterative method. In this method the unknown function is expanded in powers of the iterated parameter. This series is known as Neumann series. There is an other alternate approach in which the Kernels are iterated up to nth times and then solved the integral equations. The famous iterative method are that of Picard's methods or by using the idea of L_2 class Kernels in the iterative approaches.

(d) ***Numerical Methods:*** Sometimes the Kernel of the Fredholm equations is approximated by a suitable Poincere Goursat Kernel on step functions, then the integral equations can be reduced to an algebraic system of linear equations. If the integral of the given integral equation is replaced by a suitable sum then instead of dividing the basic integral into sub-intervals of the same size, it may be useful to divide it according to the zeros of a certain polynomial of Legendre. This method is developed in most books on numerical methods.

28.4 Description of Some Methods used in the solution of Fredholm Integral Equation

In the next few units we are interested in studying the Fredholm integral equations. In the unit 29, we study the Fredholm equations by the method of *successive approximation*. In this iteration method either the unknown function or the Kernel is iterated into a series known as Neumann's series. The convergence of the series depends upon the iterative parameter and the nature of the Kernel as well as the function in the domain $a \le x \le b$, $a \le t \le b$ when the Kernel $K(x, t)$ and the function $f(x)$ are square integrable. For this purpose the function as well as Kernel has continuous derivations. Then in the unit 31 we will study the solution of Fredholm equations with a special type of Kernels known as Poincere Goursat Kernels (P.G.). In the light of P.G. Kernels the existence and uniqueness of the solution of Fredholm equations of both kinds. In the unit 32 the final unit the famous Fredholm theorem on the existence and uniqueness of the solutions is described along with the conditions put on the functions.

Example: Express the differential equation

$$\frac{d^2y}{dx^2} + 9y = 18x \qquad y(0) = y\left(\frac{\pi}{2}\right) = 0 \qquad ...(1)$$

as an integral equation.

Solution: Integrating (1) from 0 to x,

$$\int_0^x \frac{d^2y}{dx^2}dx + 9\int_0^x y(u)\,du = \int_0^x 18x\,dx$$

or $$\frac{dy}{dx} - \left.\frac{dy}{dx}\right|_{x=0} + 9\int_0^x y(u)\,du = 9x^2$$

or $$y'(x) - y'(0) + 9\int_0^x y(u)\,du = 9x^2$$

Again integrating

$$y(x)\big|_0^x - y'(0)x + 9\int_0^x dt\int_0^t y(u)\,du = \frac{9x^3}{3}$$

or

$$y(x) - y(0) - y'(0)x + 9\int_0^x y(u)\,du\int_0^x dt = 3x^3$$

Notes

or $$y(x) - y'(0)x + 9\int_0^x y(u)\,du\,(x-u) = 3x^3 \qquad ...(2)$$

Putting $x = \pi/2$ in (2) and using boundary condition we have

$$-y'(0)\frac{\pi}{2} + 9\int_0^{\pi/2}\left(\frac{\pi}{2} - u\right)y(u)\,du = 3\pi^3$$

Solving for $y'(0)$, we have

$$y'(0) = \frac{18}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2} - u\right)y(u)\,du - 6\pi^2 \qquad ...(3)$$

Substituting in equation (3) we have

$$y(x) - \frac{18x}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2} - u\right)y(u)\,du - 6\pi^2 x + 9\int_0^x (x-u)y(u) = 3x^3$$

or $$y(x) - \frac{18x}{\pi}\int_0^x\left(\frac{\pi}{2} - u\right)y(u)\,du - \frac{18x}{\pi}\int_x^{\pi/2}\left(\frac{\pi}{2} - u\right)y(u) + 9\int_0^x (x-u)y(u) = 3x^3 + 6\pi^2 x \qquad ...(4)$$

or letting

$$F(x) = 3x\,(x^2 + 2\pi^2) \qquad ...(5)$$

and $$K(x, u) = \begin{cases} \dfrac{18x}{\pi}\left(\dfrac{\pi}{2} - u\right) - 9(x-u) & \text{for} \quad u < x \\ \dfrac{18x}{\pi}\left(\dfrac{\pi}{2} - u\right) & \text{for} \quad u \geq x \end{cases} \qquad ...(6)$$

Equation (4) then becomes

$$y(x) - \int_0^{\pi/2} K(x,u)\,y(u) = F(x) \qquad ...(7)$$

which is the required integral equation of the second kind.

Self Assessment

1. Convert the differential equation

   $y'' + 4y = \sin 3x$ with $y(0) = y(1) = 0$

   into integral equation.

28.5 Summary

- In Fredholm integral equations of first kind and second kind the upper limit of integration is fixed.
- Fredholm Integral equation can be obtained from linear differential equations by applying certain boundary conditions.
- Types of Kernels appearing in Fredholm equations are of the type; symmetric Kernels, difference Kernels, Poincere Goursat Kernels.

28.6 Keywords

Degenerate Kernels or Poincere Goursat Kernels are of the type $K(x,t) = \sum_{i=1}^{n} g_i(x)h_i(t)$ where $g_i(x)$, $h_i(t)$ are known functions.

Symmetric Kernels: The Kernels $K(x, t)$ having the property $K(x, t) = K(t, x)$ are known as symmetric Kernels.

28.7 Review Question

1. Express the differential equation

   $$y''(x) - y'(x) - 6y = x^2 + 1$$

   with $y(0) = y(1) = 0$ into Fredholm integral equation.

Answer: Self Assessment

1. $$G(x) - 4\int_0^1 K(x,t)G(t)\,dt = \sin 3x$$

   where $G(x) = 4''(x)$,

   $$K(x,t) = \begin{cases} t(1-x) & t < x \\ x(1-t) & t > x \end{cases}$$

28.8 Further Readings

Books

Erwin Kreyzig, Introductory Functional Analysis with Application

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equation

Notes

Unit 29: Fredholm Equations Solution by the Method of Successive Approximation

CONTENTS

Objectives

After studying this unit, you should be able to:

- Realize that when the expansion parameter is small the unknown function is iterated in powers of this parameter.
- Describe the Kernel iteratively in powers of the expansion parameter.
- Explain and calculate the iterated function $\psi_n(x)$ or iterated Kernel $K_n(x, t)$.
- Estimate the lower bound for the radius of convergence of Neumann series.

Introduction

You have learnt the method of successive approximation in the case of Volterra integral equations.

The method of successive approximation becomes all the more easy as upper limit of integration is fixed.

29.1 The Method of Successive Approximation

The method of successive approximation in the earlier unit has been applied to the solution of Volterra integral equation. This method can be applied even more easily to the basic Fredholm equation of the second kind. Let us consider the Fredholm integral equation of the second kind.

$$y(x) - \lambda \int_0^1 K(x, u)\, y(u)du = f(x) \qquad ...(1)$$

However, the solution obtained in this way has some difficulty in case $|\lambda|$ is not small and hence may no longer converge. The method of successive approximation can be used more easily because now all integrations are to be performed between the limits 0 and 1.

Now let

$$y(x) = f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + ... \qquad ...(2)$$

This is called Neumann Series. Substituting (2) into (1) we obtain

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$$f(x)+\lambda\psi_1(x)+\lambda^2\psi_2(x)+\ldots.-\lambda\int_0^1 K(x,u)\left[f(u)+\lambda\psi_1(u)+\lambda^2\psi_2(x)+\ldots.\right]du \equiv f(x) \quad \ldots(3)$$

Comparing the powers of λ on both sides we have

$$\psi_1(x)-\int_0^1 K(x,u)f(u)du=0$$

$$\psi_1(x) = \int K(x,u)f(u)du$$

$$\psi_2(x) = \int_0^1 K(x,u)\psi_1(u)du=\int_0^1 K(x,u)\int_0^1 K(u,u_1)f(u_1)du_1$$

$$= \int_0^1 K_2(x,y_1)\,f(u_1)\,du$$

$$\psi_3 = \int_0^1 K(x,u)\psi_2(u)du=\int_0^1 K_3(x,u)f(u)du$$

……………………………………………………………………

$$\psi_n = \int_0^1 K(x,u)\psi_{n-1}(u)du=\int_0^1 K_n(x,u)du \qquad (\text{for } n = 1, 2, \ldots)$$

In the above we have

$$\left.\begin{array}{l} K_2(x,u)=\int K(x,u_1)\,K(u_1,u)du_1 \\ K_3(x,u)=\int K(x,u_1)\,K_2(u_1,u)du_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}. \quad \ldots(4)$$

and so on.

More generally

$$K_n(x,u)=\int K_r(x,u_1)K_{n-r}(u_1,u)du_1 \qquad [n = 2, 3, 4, \ldots; r = 1, 2, \ldots, n-1; K_1 = K \quad \ldots(5)$$

Thus the series for the resolvent Kernel $H(x, u, \lambda)$ is given by

$$-H(x,u,\lambda)=K(x,u)+\lambda K_2(x,u)+\lambda^2 K_3(x,u)+\ldots+\lambda^n K_n(x,u) \quad \ldots(6)$$

The solution then is given by

$$y(x)=f(x)=-\lambda\int H(x,u,\lambda)f(u)du \quad \ldots(7)$$

The main difference from the Volterra case is that the series for the resolvent Kernel (6) now converges only for sufficiently small values of $|\lambda|$. In other words, although $H(x, u, \lambda)$ is still analytic function of λ it is no longer an entire function of λ.

29.2 Lower Bound for the Radius of Convergence

We shall now determine a lower bound for the radius of convergence of the power series (6). We observe that if we preserve the basic hypothesis i.e. that the Kernel $K(x, y)$ is an L_2 Kernel,

i.e. $$\|K\|^2=\iint K^2(x,u)dxdu=\int A^2(x)dx=\int B^2(u)du \le N^2 \quad \ldots(8)$$

where

$$A(x)=\left[\int_0^1 K^2(x,u)du\right]^{1/2},\ B(u)=\left[\int_0^1 K^2(x,u)dx\right]^{1/2} \quad \ldots(9)$$

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we have successively

$$K_2^2(x,u) = \left[\int K(x,u_1)\,K(u_1,u)du_1\right]^2 \le A^2(x)\,B^2(u)$$

$$K_3^2(x,u) = \int K^2(x,u_1)du_1 \int K_2^2(u_1,u)du_1 \le A^2(x)\,B^2(u)\int A^2(u_1)du_1 \le A^2(x)\,B^2(u)N^2$$

$$K_4^2(x,u) \le \int K^2(x,z)dz \int K_3^2(z,u)dz \le A^2(x)\,B^2(u)\,N^2\int A^2(z)dz \le A^2(x)\,B^2(u)N^4$$

and hence in general

$$\left|K_{n+2}(x,u)\right| \le A(x)\,B(x)N^n \qquad (n = 0, 1, 2, ...) \quad ...(10)$$

If we neglect the first term of (6), this process that (6) has the majorant

$$A(x)\,B(x)\,|\,\lambda\,|\sum_{n=0}^{\infty}(\,|\,\lambda\,|\,N)^n;$$

This is a geometric series with the common ratio $|\,\lambda\,|\,N$, hence it converges for $|\,\lambda\,|\,N < 1$, i.e. for

$$|\lambda| < \|K\|^{-1} \qquad ...(11)$$

We thus see that under the condition (11), the partial sums of (6) have a majorant of the type

$$C\,A(x)\,B(x)$$

where C is a constant i.e., a majorant which is L_2 function of both x and u. In other words (6) is an almost uniformly convergent series, hence a series which can be integrated term-by-term in either x or u (by Lebesgue fundamental theorem). Now the resolvent Kernel is an analytic function whose singular points are outside or on the boundary of the circle (11).

Since term-by-term integration is permitted, we see that by using (5) under condition (11) we have

$$-\int_0^1 K(x,u_1)H(u_1,u,\lambda)du_1 = -\int H(x,u_1,\lambda)\,K(u_1,u)du_1$$

$$= K_2(x,u) + \lambda K_3(x,u) + ...$$

$$= \lambda^{-1}[H(x,u,\lambda) + K(x,u)]$$

that is,

$$\left.\begin{aligned} K(x,u) + H(x,u,\lambda) &= \lambda\int K(x,u_1)\,H(u_1,u,\lambda)du_1 \\ &= \lambda\int H(x,u_1,\lambda)\,K(u_1,u)du_1 \end{aligned}\right\} \qquad ...(12)$$

Now considering that all the terms of this double equality are analytic functions of λ, we can thus assert that the basic equation (12) for the resolvent Kernel are valid not only in the circle (11), but in the whole domain of existence of the resolvent Kernel H in the λ plane. If now $f(x)$ belongs to the class L_2, then the given equation (1) has at least one solution of the same class L_2, this solution is

$$y(x) = f(x) - \lambda\int H(x,u,\lambda)f(u)du \qquad ...(13)$$

in the domain of existence H, of H.

Moreover, it is easy to see that the solution (13) is the unique L_2-solution of our equation, not only inside the circle $|\lambda| < \|K\|^{-1}$ but also in the whole domain of existence H.

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Example: Let us consider the following integral equation

$$y(x) - \lambda \int_0^1 e^{x-u} y(u) du = f(x) \qquad ...(1)$$

we now have

$$K_2(x, u) = \int_0^1 e^{x - u_1 + u_1 - u} du_1 = e^{x-u} \int_0^1 du_1 = e^{x-u} = K(x, u)$$

with this consequence that all the iterated Kernels K_n coincide with the given Kernel $K(x, u)$ and the series (6) becomes

$$-H(x, u, \lambda) = K(x, u)\,(1 + \lambda + \lambda^2 + ...) \qquad ...(2)$$

Hence we have

$$H(x, u, \lambda) = \frac{K(x, u)}{(\lambda - 1)} \qquad ...(3)$$

and we see that the resolvent Kernel is analytic function of λ. So we have one and only one solution for $\lambda \neq 1$.

$$y(x) = f(x) - \frac{\lambda e^x}{(\lambda - 1)} \int_0^1 e^{-u} f(u) du \qquad ...(4)$$

We started with the integral equation (1)

$$y(x) - \lambda \int_0^1 K(x, u)\, y(u) = f(x) \qquad ...(1)$$

and arrived at the equation (13)

$$y(x) = f(x) - \lambda \int_0^1 H(x, u, \lambda)\, f(u) du \qquad ...(13)$$

where the resolvent Kernel satisfies the equation (12)

$$K(x, u) + H(x, u, \lambda) = \lambda \int H(x, u_1, \lambda)\, (u_1, u) du \qquad ...(11)$$

Substituting (13) into L.H.S. of (1)

$$f(x) - \lambda \int_0^1 H(x, u, \lambda)\, f(u) du - \lambda \int_0^1 K(x, u) \left[f(u) - \lambda \int_0^1 H(u, u_1, \lambda)\, f(u_1) du_1 \right] du$$

$$= f(x) - \lambda \int_0^1 du\ f(u) \left[K(x, u) + H(x, u, \lambda) \right] - \lambda \int_0^1 \int_0^1 H(u, u_1, \lambda) f(u_1) du_1 K(x,u) du$$

$$= f(x) - \lambda \int_0^1 du \left[K(x, u) + H(x, u, \lambda) - \lambda \int_0^1 H(x, u_1, \lambda) K(u_1, u) \right] f(u)$$

$= f(x) + 0 =$ R.H.S.

Notes

Self Assessment

1. Solve the following integral equation

$$y(x) = m\int_0^1 y(t)dt = 1.$$

Also find the Neumann series for $y(x)$

$$\left[\textbf{Hint}: \int_0^1 y(t)dt = \text{constant}.\right]$$

29.3 Summary

- The iterative method gives the solution of the function in terms of the powers of the parameter of the equation.
- We can either get an iterative power series in the wave function or the iterated Kernel.
- After iterating it nth times we get the solution as limiting as n tends to ∞.
- In this way we get the Resolvent Kernel in the nth iteration when n is very large.

29.4 Keyword

The successive method helps in getting the solution of the problem as a power series in terms of powers of the parameter known as ***Neumann series***. The estimate of the radius of convergence of the Neumann series gives an estimate of the accuracy of the solution.

29.5 Review Question

The Fredholm integral equation is

$$y(x) - \lambda\int_0^{2\pi} K(x, t)\, y(t)\, dt = f(x)$$

where $K(x, t) = \sum_{v=1}^{\infty} v^{-2} \sin(vs) \sin[(v+1)t]$

find $K_3(x, t)$, the third iterative Kernel.

Answer: Self Assessment

1. $y(x) = \dfrac{1}{1-u}$, the Neumann series is $y(x) = 1 + \mu^2 + \mu^3 + \ldots$.

29.6 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Unit 30: Neumann's Series

Notes

CONTENTS

Objectives

After studying this unit, you should be able to:

- Find lots of similarities of the description of the successive approximation approach in regard to getting Neumann Series.
- Observe that the unknown function can either be expanded in power series of λ or the resolvent Kernel is expanded in power series in λ.
- Understand the convergence of the Neumann Series as given in unit 29.

Introduction

For small values of λ the solution of the Fredholm equation can be determined as power series known as Neumann's Series.

The resolvent kernel is an analytic function of the parameter λ but it is not an entire function of the whole complex plane.

30.1 Fredholm Integral Equations, Successive Approximation Neumann's Series

Consider the Fredholm integral equations of the first kind and second kind:

$$f(x) = \lambda \int_a^b K(x, t)\, y(t) dt \qquad ...(1)$$

and

$$y(x) = f(x) + \lambda \int_a^b K(x, t)\, y(t) dt \qquad ...(2)$$

In these equations $y(x)$ is an unknown function that has to be found and $f(x)$ and $K(x, t)$ are given as function and the Kernel of the integral equations. Unless in the case of Volterra integral equation, here the limits of the integral are fixed as constants a and b. The range of x and t are given as $a \le x \le b$ and $a \le t \le b$. Depending upon the nature of Kernel $K(x, t)$ a suitable method of

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solving the integral equation is to be chosen. Here the parameter λ also plays an important part. So if λ is small as well as the Kernel $K(x, t)$ is continuous along with its partial derivatives, we can use the method of successive approximation.

Let us consider first the equation (2) of Fredholm integral equation of the second kind. To a zero approximation

$$y(x) \simeq f(x).$$

If we substitute this value of $y(x)$ in the integral (2) we get

$$f(x) \simeq f(x) + \lambda\int_a^b K(x, t)f(t)dt \quad ...(3)$$

or $$y(x) \cong f(x) + \lambda\psi_1(x)$$

where $$\psi_1(x) = \int_a^b K(x, t)f(t)dt \quad ...(4)$$

So to a first approximation $y(x)$ is given by (2). To get an improvement over the above approximation we put this new value of $y(x)$ given by (3) into (2) to improve the solution as follows:

$$y(x) \cong f(x) + \lambda\int_a^b K(x, t)\left[f(t) + \lambda\int_a^b K(t, u)f(u)du\right]dt$$

$$= f(x) + \lambda\int_a^b K(x, t)f(t)dt + \lambda^2\int_a^b K(x, t)dt\int_a^b K(t, u)f(u)du$$

or $$y(x) \cong f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) \quad ...(5)$$

where

$$\psi_2(x) = \int_a^b K(x, t)dt\int_a^b K(t, u)\, f(u)du$$

$$= \int_a^b du\, f(u)\int_a^b K(x, t)\, K(t, u)dt$$

or $$\psi_2(x) = \int_a^b du\, K_2(x, u)f(u) \quad ...(6a)$$

where $$K_2(x, u) = \int_a^b K(x, t)\, K(t, u)dt \quad ...(6b)$$

We can improve the accuracy by taking more powers of λ in $y(x)$ i.e. we may write

$$y(x) \cong f(x) + \lambda\psi_1 + \lambda^2\psi_2 + \lambda^3\psi_3 + ... + \lambda^n\psi_n + ... \quad ...(7)$$

where ψ_1, ψ_2 are given by (4) and (6a) and other ψ's are given by

$$\psi_n(x) = \int_a^b du\, K_n(x, u)\, f(u) \qquad \text{for } n = 1, 2, ... \quad ...(8)$$

and the n^{th} Kernel $K_n(x, u)$ given by

$$K_n(x, u) = \int_a^b K_r(x, u_1)K_{n-r}(u_1, u)du_1 \qquad [n = 2, 3, 4,...; r = 1, 2, ...n - 1 \quad ...(9)$$

while $K_1(x, u) = K(x, u)$

Thus $$y(x) \cong f(x) + \sum_{i=1}^{n}\lambda^i\psi_i(x) \text{ .... for any } n \quad ...(10)$$

Notes

This series for the solution of the Fredholm integral equation of the second kind is known as Neumann Series.

30.2 Successive Approximation for the Resolvent Kernel

Writing in full the expression for the function $y(x)$, we have

$$y(x) \cong f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + \lambda^3\ \Psi_3(x) + \ldots \qquad \ldots(10)$$

Making use of (4) (6a) and (8) for ψ_1, ψ_2, ψ_3,...... into (10) we get

$$y(x) = f(x) + \lambda\int_a^b K(x,t)\ f(t)dt + \lambda^2\int_a^b K_2(x,t)f(t)dt + \lambda^3\int_a^b K_3(x,t)\ f(t)dt + \ldots$$

$$= f(x) + \int\left[\lambda K_1(x,t) + \lambda^2 K_2(x,t) + \lambda^3 K_3(x,t) + \ldots\right]f(t)dt$$

$$y(x) = f(x) + \lambda\int_a^b H(x,t,\lambda)f(t)\ dt \qquad ..(11)$$

where the resolvent Kernel $H(x, t, \lambda)$ is given by the series

$$-H(x,t,\lambda) = K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \ldots. \qquad \ldots(12)$$

Equation (12) is now the power series known again as Neumann Series.

As discussed in unit 29, we see that the resolvent Kernel is still analytic function of λ but is no longer an entire function of λ. Also the resolvent Kernel satisfies the integral equation

$$-H(x,u,\lambda) = K(x,u) - \lambda\int H(x,u_1,\lambda)\ K(u_1,u)du_1 \qquad \ldots(13)$$

Now the solution (11) is the unique L_2-solution of the equation (2), as $f(x)$ and $K(x, t)$ are L_2-class and it exists in the whole domain of $C(a, b)$. We now show that if the homogeneous equation (1) for $\lambda = \lambda_0$ has a certain non-trivial solution then with the help of equation (13) we obtain

$$\phi_0(x) = \lambda_0\int K(x,t)\phi_0(t)dt$$

$$= -\lambda_0\int H(x,t,\lambda_0)\phi_0(t)dt + \lambda_0^2\int\phi_0(t)dt\int H(x,z,\lambda_0)\ K(z,t)dz$$

$$= -\lambda_0\int H(x,t,\lambda_0)\phi_0(t)dt + \lambda_0^2\int H(x,z,\lambda_0)dz\int K(z,t)\phi_0(t)dt$$

$$= -\lambda_0\int H(x,t,\lambda_0)\phi_0(t)dt + \lambda_0\int H(x,z,\lambda_0)dz\ \phi_0(z)$$

$$\equiv 0$$

This shows that if equation (2) has a unique non-trivial solution of the form (12) then the non-trivial solution of the homogeneous equation (1) is $\phi_0(x)$, vanishes almost everywhere.

The above analysis process the following theorem to each quadratically integrable Kernel $K(x, t)$ there corresponds a resolvent Kernel $H(x, t, \lambda)$ which is analytic function of λ, regular at least inside the circle $|\lambda| < \|K\|^{-1}$ and represented these by the power series (12). Let the domain of existence of the resolvent Kernel in the complex plane λ be H. Then if $f(x)$ also belongs to the class L_2, the unique quadratically integrable solution of Fredholm's equation (2) valid in H is given by (11).

For the proof of this theorem please refer to the treatment in the unit 29.

Notes

Example: Consider the integral equation

$$y(x) = f(x) + \lambda \int_a^b K(x, t)\, y(t) dt \qquad ...(1)$$

Find the solution when $f(x) = e^x$, $K(x, t) = 2e^{x+t}$, $a = 0$, $b = 1$.

Substitute the value of $f(x)$ and $K(x, t)$ in (1) we have

$$y(x) = e^x + 2\lambda e^x \int_0^1 e^t y(t) dt$$

$$= e^x \left[1 + 2\lambda \int_0^1 e^t y(t) dt\right]$$

Let $\quad C = \int_0^1 e^t y(t) dt = \text{constant} \qquad ...(2)$

then $\quad y(x) = e^x (1 + 2\lambda C) \qquad ...(3)$

Substituting this value of y in (2) we have

$$C = (1 + 2\lambda C) \int_0^1 e^t . e^t dt = (1 + 2\lambda C) \frac{(e^2 - 1)}{2}$$

Solving for C i.e.

$$2C - 2\lambda C(e^2 - 1) = (e^2 - 1)$$

$$C = \frac{(e^2 - 1)}{2\left[1 - \lambda(e^2 - 1)\right]} \qquad ....(4)$$

Substituting in (3) we have

$$y(x) = e^x / [1 - \lambda(e^2 - 1)] \qquad ...(5)$$

The denominator is non-zero.

Self Assessment

1. Solve the Fredholm integral equation

$$y(x) = f(x) + \lambda \int_a^b K_0 y(t) dt$$

where K_0 is a constant and show that for $|\lambda| < 1/K_0(b - a)$ the corresponding Neumann Series is convergent.

30.3 Summary

- In case the parameter λ is small one gets the solution of Fredholm equation of the second kind as a power series in λ called Neumann series.
- The Resolvent Kernel can also be expanded in powers of λ provided the Kernel $K(x, t)$ is of L_2-class. The resolvent Kernel is though an analytic function of λ but is not an entire function in whole of complex λ-plane.

30.4 Keywords

The $C(a, b)$ is a space of all ***continuous functions*** defined on the interval (a, b).

The ***unknown functions*** $y(x)$ and $f(x)$ are of $C(a, b)$ type while $K(x, t)$ is of $C^2(a, b) \to C(a, b)$ type on the square $a \le x \le b$ and $a \le t \le b$.

30.5 Review Question

Solve the Fredholm integral equation of the second kind

$$Y(x) = f(x) + \lambda \int_0^1 x(1+t)y(t)dt$$

when λ is not an eigenvalue.

Answer: Self Assessment

1. $$y(x) = f(x) + \frac{\lambda K_o C_o}{1 - \lambda K_0(b-a)]}, C_0 = \int_a^b f(x)dx$$

   Expand $\dfrac{1}{1-\lambda K_0(b-a)}$ in powers of λ to get Neumann Series.

30.6 Further Readings

Books

Erwin Kreyzig, Introductory Functional Analysis with Applications

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Notes

Unit 31: Fredholm Equations with Poincere Goursat Kernels

CONTENTS

Objectives

After studying this unit, you should be able to:

- Know that Fredholm equations may have varieties of Kernels. Among them the Poincere-Goursat Kernel also plays an important part.
- Observe that in this type of Fredholm equation the resolvent Kernel is a quotient of two polynomials of the *n*th degree in λ and the denominator is independent of the variables of the Kernel.
- Understand the nature of singular points of resolvent Kernel in terms of zeros of the denominator polynomial $D(\lambda)$.

Introduction

In this unit we saw that resolvent Kernel has a structure that helps in understanding the nature of the solution of non-homogeneous as well as homogeneous equations.

Fredholm integral equation as well as its conjugate equation can be studied together to understand the structure of the solutions.

31.1 The Poincere Goursat Kernels

In the unit we consider again the Fredholm integral equation of the second kind i.e.

$$y(x) - \lambda\int_0^1 K(x, u)y(u)du = f(x) \quad ...(1)$$

Here we take the structure of the Kernel to be of the form

$$K(x, u) = \sum_{i=i}^{n} g_i(x)h_i(u) \quad ...(2)$$

Notes

Such a Kernel has been mentioned in the case of Volterra integral equations and are called degenerate Kernels or Poincere-Goursat Kernels i.e. P-G Kernels. In equation (2) the functions $g_i(x)$ and $h_i(u)$, for $i = 1, 2, ...n$, are L_2-class. We will see that in this case the Fredholm integral equation can be reduced to an algebraic system of a linear equation in n unknown. Here $g_i(x)$ and $h_i(x)$ $i = 1, 2, ...n$ are independent in the basic interval (0, 1).

Substituting (2) in one we have

$$y(x) - \lambda\int_0^1 \sum_{i=1}^{n} g_i(x)\, h_i(u) y(u) du = f(x)$$

or $$y(x) - \sum_{i=1}^{n} (\lambda)\; g_i(x) \int_0^1 h_i(u) y(u) du = f(x)$$

If we put

$$\varepsilon_k = \int_0^1 h_k(u) g(u) du \qquad (x = 1, 2, ...n) \qquad ...(3)$$

then equation (1) becomes

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} \varepsilon_i g_i(x) \qquad ...(4)$$

From equation (4) it is already seen that the difference $y(x)$ -$f(x)$ must necessarily coincide with a suitable linear combination of the functions $g_i(x)$. Now multiply equation (4) by $h_k(x)$ $i = 1, 2,...n$ and integrate between 0 and 1 we have

$$\int_0^1 h_k(x) y(x) dx = \int_0^1 h_k(x) f(x) dx + \lambda \sum_{i=1}^{n} \varepsilon_i \int_0^1 g_i(x) h_k(x) dx$$

Defining

$$\left.\begin{aligned} &\int g_i(x) h_k(x) dx = a_{ik} \\ &\int h_i(x) f(x) dx = b_i \end{aligned}\right\} \qquad ...(5)$$

We have

$$\varepsilon_k - \lambda \sum_{i=1}^{n} a_{ik} \varepsilon_i = b_k \qquad ...(6)$$

We thus see that the unknowns ε_1, ...ε_n must satisfy the following system of linear equations

$$\left.\begin{aligned} &(1 - a_{11})\varepsilon_1 - \lambda a_{12}\varepsilon_2 - \lambda a_{13}\varepsilon_3 ...... - \lambda a_{1n}\varepsilon_n = b_1 \\ &-\lambda a_{21}\varepsilon_1 + (1 - a_{22})\varepsilon_2 - \lambda a_{23}\varepsilon_3 .... - \lambda a_{2n}\varepsilon_n = b_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &-\lambda a_{n1}\varepsilon_1 - \lambda a_{n2}\varepsilon_2 ...... + (1 - \lambda a_{nn})\varepsilon_n = b_n \end{aligned}\right\} \qquad ...(7)$$

To each set of solution $\varepsilon^\circ_1, \varepsilon^\circ_2 ... \varepsilon^\circ_n$ of this system there corresponds a solution of equation (1) given by (4). Now the solution of equation exist if the determinant formed by the coefficients ε_i in equation (7) defined by

$$D(\lambda) = \begin{vmatrix} 1 - \lambda a_{11} - \lambda a_{12} - \lambda a_{13} ...... - \lambda a_{1n} \\ -\lambda a_{21} (1 - \lambda a_{22}) .... - \lambda a_{2n} \\ -\lambda a_{31} .... \\ .... \\ -\lambda a_{n1} - \lambda a_{n2} ....(1 - \lambda a_{nn}) \end{vmatrix} \qquad ...(8)$$

Notes

is non-zero. So there is one and only one solution of the system of n simultaneous equations ε_1, ε_2, ...ε_n. Thus if $D(\lambda) \neq 0$, then the system (7) has one and only one solution given by Gamer's rule i.e.

$$\varepsilon_k = \frac{1}{D(\lambda)}\left(D_{1k}b_1 + D_{2k}b_2 + ... + D_{nk}b_n\right) \quad (k = 1, 2, 3, ...n)$$

where D_{hk} denotes co-factor of (h, k) the elements of the determinant (8), correspondingly, the solution (1) has the unique solution

$$y(x) = f(x) + \frac{\lambda \sum_{k=1}^{n}}{D(\lambda)}\left[D_{1k}b_1 + D_{2k}b_2 + ... D_{nk}b_n\right]g_k(x) \qquad ...(9)$$

As $D(\lambda) \neq 0$, the corresponding Fredholm equation of the first kind

$$y(x) - \lambda\int_0^1 K(x, u)y(u)du \qquad ...(10)$$

has only the trivial solution $y(x) \equiv 0$ as $D(\lambda) \neq 0$.

31.2 Resolvent Kernel $H(x, u, \lambda)$

If we now substitute the expression of b_i in (5), the solution (9) can also be written as

$$y(x) = f(x) + \frac{\lambda}{D(\lambda)}\int_0^1\left[D_{1k}h_1(u) + D_{2k}h_2(u) + D_{3k}h_3(u) + ... + D_{nk}h_n(u)\right]f(u)g_k(x)du$$

but the sum under the integral sign can be considered as the expansion of the negative of a determinant of the (n + 1) order i.e.

$$-\left[(D_{ik}h_1(u) + D_{2k}h_2(u) + D_{3k}h_3(4) + ... + D_{nk}b_n(u)g_k(x)\right]$$

$$= D(x,u,\lambda) = \begin{vmatrix} 0 \; g_1(x)g_2(x) \; ....g_n(x) \\ h_1(u) \; 1 - \lambda a_{11} \; - \lambda a_{12} .... \lambda a_{1n} \\ \vdots .... - \lambda a_{2n} \\ .... \\ h_n(u) \qquad\qquad (1 - \lambda a_{nn}) \end{vmatrix} \qquad ...(11)$$

Hence we can write equation (9) as

$$y(x) = f(x) - \frac{\lambda}{D(\lambda)}\int_0^1 D(x, u, \lambda)f(u)du \qquad ...(12)$$

Defining the resolvent Kernel $H(x, u, \lambda)$ by

$$H(x, u, \lambda) = \frac{D(x, u, \lambda)}{D(\lambda)} \qquad ...(13)$$

so equation (12) becomes

$$y(x) = f(x) - \lambda\int_0^1 H(x, u, \lambda)f(u)du \qquad ...(14)$$

In the equation (13) the resolvent Kernel $H(x, u, \lambda)$ is the quotient of two polynomials of the nth degree in λ and the denominator is independent of x and u and this has important consequences.

At this point it is to be noticed that the only singular points of $H(x, u, \lambda)$ in the λ-plane are the roots of the equation

$$D(\lambda) = 0 \qquad ...(15)$$

which will be called the eigenvalues of our Kernel $K(x, u)$

31.3 Eigenvalues and Eigenvectors

If $D(\lambda) = 0$ the non-homogeneous equation (1) has no solution in general, because an algebraic linear system with vanishing determinant can only be solved for certain values of the quantities on the right hand side of equation (7).

Furthermore, from each non-trivial solution $\varepsilon_1^0, \varepsilon_2^0, ..., \varepsilon_n^0$ of the homogeneous algebraic system we obtain a non-trivial solution of the homogeneous equation (10), which we call an eigenfunction and vice versa.

To be precise, from the theory of algebraic systems of linear equations. We infer that, if λ coincides with a certain eigenvalue λ_0 for which the determinant $D(\lambda_0)$ has the characteristic $P(1 \le p \le n - 1)$, and we put $n - p = r$, then there are ∞^r solutions of the homogeneous system (7). Furthermore, these solutions can be represented by formulae of the type

$$\xi_k = B_{1k}C_1 + B_{2k}C_2 + ... + B_{rk}C_r \quad (k = 1, 2, ..., n) \qquad ...(16)$$

where $C_1, C_2, ..., C_r$ denote r arbitrary constants and

$$\left.\begin{matrix} B_{11}, B_{12}, ... B_{1n} \\ .... \\ B_{r1}, B_{r2} ... B_{rn} \end{matrix}\right\} \qquad ...(17)$$

are r arbitrarily fixed but linearly independent solutions of the system in question.

This shows that to each eigenvalue λ_0 of index $r = n - p$ there corresponds a solution of the homogeneous equation (10) of the form

$$\phi_0(x) = C_1\phi_{01}(x)\, C_2\phi_{02}(x) + ... + C_r\phi_{0r}(x) \qquad ...(18)$$

where $C_1, C_2, ... C_r$ are r arbitrary constants and

$$\phi_{01}(x), \phi_{02}(x), ..., \phi_{0r}(x)$$

are r linearly independent functions, which can be expressed in terms of the B_{hk} as follows:

$$\phi_{0h}(x) = \sum_{k=1}^{n} B_{hk}\, g_k(x) \qquad (h = 1, 2, ..., r) \qquad ...(19)$$

Moreover, we can assume that these functions are normalized, i.e., that their norms are all equal to unity,

$$\int \phi_{0h}^2(x)dx = 1 \quad (h = 1, 2, ..., r) \qquad ...(20)$$

Al these eigenfunctions are annihilated by the Fredholm operator

$$F_s[\phi_{oh}(y)] \equiv 0. \qquad ...(21)$$

Using elementary transformations on the determinant (7), we can see that the index $r = n - p$ of an eigenvalue is never larger than its multiplicity m as a root of the equation $D(\lambda) = 0$. Moreover, in the important case $a_{hk} = a_{kh}$ we have

$$r = m$$

Notes

Another important fact is that to the given Kernel (1) and to the associated one

$$K(y, x) = \sum_{k=1}^{n} g_k(y)\, h_k(x) \qquad ...(21)$$

there corresponds the same function $D(\lambda)$ and consequently the same eigenvalues. This is because the interchange of g_k and h_k carries a_{hk} into a_{kh} and hence only interchanges the rows and columns of determinant (8).

However, the eigenfunctions of the associated Kernel, i.e. the non-trivial solutions of the associated homogeneous equation

$$\psi(x) - \lambda \int K(y, x)\psi(y)dy = 0 \qquad ...(22)$$

for $\lambda = \lambda_0$ are not the previous function (16) but other ones,

$$\psi_{0h}(x) - \sum_{k=1}^{n} B^*_{hk} h_k(x) \quad (h = 1, 2, ..., r), \qquad ...(23)$$

where

$$\left.\begin{matrix} B^*_{11}, B^*_{12}, ... B^*_{1n} \\ .... \\ B^*_{r1}, B^*_{r2} .... B^*_{rn} \end{matrix}\right\} \qquad ...(24)$$

are any r linearly independent solutions of the associated homogeneous system

$$\left.\begin{matrix} (1-\lambda a_{11})\xi_1 - \lambda a_{21}\xi_2 - ... - \lambda a_{n1}\xi_n = 0 \\ -\lambda a_{21}\xi_1 + (1-\lambda a_{22})\xi_2 - ... - \lambda a_{n2}\xi_n = 0, \\ .... \\ -\lambda a_{1n}\xi_1 - \lambda a_{2n}\xi_2 - ... + (1-\lambda a_{nn})\xi_n = 0 \end{matrix}\right\} \qquad ...(25)$$

Any eigenfunction $\phi_{0h}(x)$ corresponding to the eigenvalue λ_0 and any associated eigenfunction $\psi_{1k}(x)$ corresponding to a different eigenvalue λ_1 are always orthogonal in the basic interval (0, 1).

In fact we have

$$I = \int \phi_{oh}(x)\psi_{1k}(x)dx = \lambda_0 \int \psi_{1k}(x)\,dx \int K(x, y)\phi_{oh}(y)dy$$
$$= \lambda_0 \int \phi_{0h}(y)dy \int K(x, y)\psi_{1k}(x)dx = \frac{\lambda_0}{\lambda_1} \int \phi_{oh}(y)\psi_{1k}(y)dy = \frac{\lambda_0}{\lambda_1} I,$$

and this equality can be true only if $\lambda_0 = \lambda_1$ or if $I = 0$.

We now return to the non-homogeneous equation (1) for the case $D(\lambda) = 0$. We prove that for $\lambda = \lambda_0$ the non-homogeneous equation can be solved if and only if the r orthogonality conditions

$$(f, \psi_{oh}) \equiv \int f(x)\psi_{oh}(x)dx = 0 \qquad (h = 1, 2, ..., r) \qquad ...(26)$$

are satisfied. In this case the non-homogeneous equation has ∞^r solutions of the form

$$\phi(x) = \Phi(x) + C_1\phi_{01}(x) + C_2\phi_{02}(x) + ... + C_r\phi_{or}(x), \qquad ...(27)$$

where $\Phi(x)$ is a suitable linear combination of $g_1(x)$, $g_2(x)$,...$g_n(x)$.

Notes

In fact, conditions (26) are necessary because if equation (1) for $\lambda = \lambda_0$ admits a certain solution $\Phi(x)$, then from the equation itself, it follows that

$$\int f(x)\psi_{0h}(x)dx = \int \Phi(x)\psi_{0h}(x)dx - \lambda_0 \int \psi_{0h}(x)dx \int K(x, y)\Phi(y)dy$$
$$= \int \Phi(x)\psi_{0h}(x)dx - \lambda_0 \int \Phi(y)dy \int K(x, y)\psi_{0h}(x)dx.$$

But, since λ_0 and $\psi_{0h}(x)$ are eigenvalue and corresponding eigenfunction of the associated Kernel, we have

$$\lambda_0 \int K(x, y)\psi_{0h}(x)dx = \psi_{0h}(y);$$

hence

$$\int f(x)\psi_{0h}(x)dx = 0$$

Furthermore, conditions (26) are also sufficient, since from them it can be easily deduced that the non-homogeneous system (7), which we shall write briefly as

$$\Xi_1 = b_1\ \Xi_2 = b_2\ ...,\ \Xi_n = b_n,$$

reduces to only n - r independent equations. Consequently we can now solve it readily (carrying r unknowns on the right hand side), since the characteristic of matrix of the coefficients is exactly $p = n - r$.

We can reduce the system for the following reason: Let us multiply the previous equations by B^*_{h1}, B^*_{h2}, ...B^*_{hr}, respectively and add. Bearing in mind equations (25), we have

$$\sum_{k=1}^{n} B^*_{hk}\Xi_k = [(1 - \lambda a_{11})B^*_{h1} - \lambda a_{21}B^*_{h2} - ... - \lambda a_{n1}B^*_{hn}]\ \xi_1$$
$$+ [-\lambda a_{12}B^*_{h1} + (1 - \lambda a_{22})B^*_{h2} - ... - \lambda a_{n2}B^*_{hn}]\xi_2$$
$$+ ....$$
$$+ [-\lambda a_{1n}B^*_{h1} - \lambda a_{2n}B^*_{h2} - ... + (1 - \lambda a_{nn})B^*_{hn}]\xi_n \equiv 0,$$

while on the other side, by virtue of (26), we also have

$$\sum_{k=1}^{n} B^*_{hk}b_k = \int \left[\sum_{k=1}^{n} B^*_{hk}Y_k(x)\right] f(x)dx = \int \psi_{oh}(x)f(x)dx = 0.$$

Among other things, form (27) of the solution demonstrates the following obvious fact: the general solution of equation (1) when $D(\lambda) = 0$ can be considered as the sum of any particular solution $\Phi(x)$ and of the general solution (18) of the homogeneous equation.

Thus we have proved for PG Kernels the following basic Fredholm theorem, which will be extended to general Kernels in the next section:

Fredholm's integral equation of the second kind

$$\phi(x) - \lambda \int K(x, y)\phi(y)dy = f(x)$$

has, in general, one and only one solution of the class L_2 given by the formula

$$\phi(x) = f(x) - \lambda \int H(x, y; \lambda)f(y)dy,$$

Notes

where $H(x, y; \lambda)$ is the resolvent Kernel. $H(x, y; \lambda)$ is an analytic function λ, and if $|\lambda| < ||K||^{-1}$ it is given by the Neumann series

$$-H(x, y; \lambda) = K(x, y) + \lambda K_2(x, y) + \lambda^2 K_3(x, y) + ...,$$

where K_2, K_3,... are the iterated Kernels. The only exceptions are the singular points of $H(x, y; \lambda)$ which coincide with the zeros (called eigenvalues) of an analytic function $D(\lambda)$ of λ. In the case of a PG Kernel, $D(\lambda)$ is a polynomial.

If $\lambda = \lambda_0$ is a root of multiplicity $m \geq 1$ of the equation $D(\lambda) = 0$, then the homogeneous equation

$$\phi(x) - \lambda \int K(x, y)\phi(y)dy = 0$$

has r linearly independent non-trivial solutions, called eigenfunctions, where r, the index of the eigenvalue, satisfies the condition $1 \leq r \leq m$. The same is true of the associated homogeneous equation.

$$\psi(x) - \lambda \int K(x, y)\psi(y)dy = 0.$$

However, if $\lambda = \lambda_0$ the non-homogeneous equation has solutions (exactly ∞^r solutions) if and only if the given function $f(x)$ is orthogonal to all the eigenfunctions of the associated homogeneous equation.

A very important alternative theorem can immediately be deduced as a corollary:

Alternative Theorem: If the homogeneous Fredholm integral equation has only the trivial solution, then the corresponding non-homogeneous equation always has one and only one solution. On the contrary, if the homogeneous equation has some non-trivial solutions, then the non-homogeneous integral equation has either no solution or an infinity of solutions, depending on the given function $f(x)$.

But even this corollary has been proved only for PG Kernels.

Self Assessment

1. The Kernel of Fredholm integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} K(x, t)y(t)dt$$

is given by

$$k(x, t) = \sum_{v=1}^{\infty} \frac{1}{V^2} \sin(vx) \sin[(v+1)t]$$

Find the iterated Kernel.

$K_2(x,t)$

$$\left[\textbf{Hint :} \text{Use the relation } \lim_{\alpha \to 0} \frac{\sin \alpha\, u}{\alpha} = u\right].$$

31.4 Summary

- Fredholm integral equation of the second kind is studied with the help of Poincere Goursat Kernels.

- It is seen that the resolvent Kernel can be expressed in terms of quotient of two polynomials of the nth degree in λ and denominator is independent of the independent variables.
- Also conditions are discussed when λ is an eigenvalue and the corresponding eigenfunctions are discussed with respect to P.G. Kernel only.

31.5 Keywords

In this unit the resolvent Kernel of the ***Fredholm integral equation*** of the second kind as well as corresponding conjugate equation is discussed.

In the next unit we shall be studying ***Fredholm theorem*** for the existence and uniqueness of the eigenvalue solution of the problem with only general Kernel.

31.6 Review Question

The Kernel of Fredholm integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} K(x, t)\, y(t) dt$$

is given by

$$K(x, t) = \sum_{v=1}^{\infty} \frac{1}{v^2} \sin(vx) \sin[(v+1)t]$$

Find the iterated Kernel

$K_3\,(x, t)$

$$\left[\textbf{Hint :} \text{ Use the relation } \lim_{\alpha \to 0} \frac{\sin \alpha\, u}{\alpha} = u \right].$$

Answer: Self Assessment

1. $$K_2(x, t) = \sum_{v=1}^{\infty} \pi \frac{\sin(vx) \sin[(v+2)t]}{v^2 (v+1)^2}$$

31.7 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

Notes

Unit 32: The Fredholm Theorem

CONTENTS

Objectives

After studying this unit, you should be able to:

- Learn that Fredholm integral equations are of two types – of first kind and of second kind
- Prove that if λ is not an eigenvalue then the Fredholm Integral equation has a solution for the second kind and the solution for the homogeneous equation is zero.
- Show that for an eigenvalue problem the Fredholm integral equation of second kind has a solution which also contains a set of *r*-constants in addition to one of its solution.

Introduction

The proof of the Fredholm theorem consists of two parts. In the first part the solution is unique and λ is not an eigenvalue.

The second part explains the eigenvalue problem of the homogeneous Fredholm integral equation and explains the structure of the main integral equation and the conjugate one.

32.1 Fredholm Alternate Theorem

The theorem states that:

Either the integral equation of the second kind

$$f(s) = Q(s) - \lambda \int_a^b K(s, t) Q(t) dt \qquad ...(1)$$

with fixed λ, admits a unique continuous solutions $Q(s)$ for any continuous function $f(s)$, in particular $Q(s) = 0$ for $f(s) \equiv 0$, or the associated homogeneous equation

$$\bar{Q}(s) - \lambda \int_a^b K(s, t)\, \bar{Q}(t) dt = 0 \qquad ...(2)$$

admits a number $r(r \geqq 1)$ of linearly independent continuous solutions $\bar{Q}_1(s), \bar{Q}_2(s)....\bar{Q}_n(s)$. In the first case, the conjugate equation

Notes

$$g(s) = \psi(s) - \lambda\int_a^b K(t, s)\psi(t)dt \quad ...(3)$$

also admits a unique continuous solution $\psi(s)$ for any continuous function $g(s)$. In the second case the associated homogeneous equation

$$\bar{\psi}(s) = \lambda\int_a^b K(t, s)\bar{\psi}(t)dt \quad ...(4)$$

admits a number r of linearly independent continuous solutions $\bar{\psi}_1(s), \bar{\psi}_2(s), \bar{\psi}_3(s)...\bar{\psi}_r(s)$. In the second case, the equation (1) admits a solution if and only if

$$\int_a^b t(s)\bar{\psi}_i(s)ds = 0 \quad (i = 1, 2, ...r) \quad ...(5)$$

If condition (5) is satisfied, the general solution of (1) is written as

$$Q(s) = Q^{(1)}(s) + \sum_{j=1}^{r} C_j\bar{Q}_j(s) \quad ...(6)$$

by means of a particular solution $Q^{(1)}(s)$ of (1) and r arbitrary constants $C_1, C_2, ..., C_r$. Similarly, the conjugate equation (3) admits a solution if and only if

$$\int_a^b g(s)\bar{Q}_j(s) = 0 \qquad (j = 1, 2, 3....r) \quad ...(7)$$

If condition (7) is satisfied, the general solution of (3) is written as

$$\psi(s) = \psi^{(1)}(s) + \sum_{j=1}^{r} C_j\bar{\psi};(s)$$

by means of a particular solution $\psi^{(1)}(s)$ of (3) and r arbitrary constants $C_1, C_2, ...C_r$.

The theorem also shows that the unique solution of (1) exists for any continuous function $f(x)$ if and only if λ is not an eigenvalue.

The proof of the Fredholm's alternative theorem is given in two parts for continuous Kernel $K(s, t)$ on the domain $a \le s \le b, \; a \le t \le b$. We shall start proving the theorem by Schmidt's method instead of L_2-class method. Of course both the methods had to the same conclusion.

32.2 Proof of Fredholm Theorem

The case when $\int_a^b\int_a^b |K(s, t)|^2 \, ds \, dt < 1$

For the sake of simplicity, we take $\lambda = 1$ and consider the equation

$$\varphi(s) - \int_a^b K(s, t)\varphi(t)dt = f(s) \quad ...(1)$$

An equation in the unknown $\phi(t)$, of the form

$$\phi(t) - \int_a^b K(s, t)\phi(s)ds = g(t) \quad ...(2)$$

$g(t)$ being a given continuous function on the interval $a \leqq t \leqq b$, is called the conjugate equation of (1).

Notes

Theorem 1: Under the assumption

$$\int_a^b\int_a^b |K(s,t)|^2\, ds\, dt < 1 \qquad ...(3)$$

the equation (1) [(2)] admits one and only one solution $\varphi(s)[\phi(t)]$ for any $f(s)[g(t)]$; in particular $\varphi(s) \equiv 0[\varphi(t) \equiv 0]$ for the homogeneous equation

$$\varphi(s) - \int_a^b K(s,t)\varphi(t)dt = 0 \qquad ...(4)$$

$$\phi(t) - \int_a^b K(s,t)\phi(s)ds = 0 \qquad ...(5)$$

Proof: Starting with the Kernel $K(s, t)$, we define the iterated Kernels $K^{(1)}(s, t)$, $K^{(2)}(s, t)$, ..., $K^{(n)}(s, t)$, ... as follows:

$$\begin{aligned} K^{(1)}(s,t) &= K(s,t) \\ K^{(2)}(s,t) &= \int_a^b K(s,r)K(r,t)dr \\ &\cdots\ \cdots\ \cdots \\ K^{(n)}(s,t) &= \int_a^b K(s,r)K^{(n-1)}(r,t)dr \end{aligned} \qquad ...(6)$$

The following relation clearly holds for the iterated Kernels

$$K^{(n+m)}(s,t) = \int_a^b K^{(n)}(s,r)K^{(m)}(r,t)dr \qquad ...(7)$$

By (6) and the Schwarz inequality, we have

$$|K^{(n)}(s,t)|^2 \leqq \int_a^b |K(s,r)|^2\, dr \int_a^b |K^{(n-1)}(r,t)|^2\, dr$$

hence

$$\int_a^b\int_a^b |K^{(n)}(s,t)|^2\, ds\, dt$$

$$\leqq \int_a^b\int_a^b |K(s,r)|^2\, ds\, dt \int_a^b\int_a^b |K^{(n-1)}(r,t)|^2\, dr\, dt$$

Repeating this procedure, we finally obtain

$$\int_a^b\int_a^b |K^{(n)}(s,t)|^2\, ds\, dt \leqq \left[\int_a^b\int_a^b |K(s,t)|^2\, ds\, dt\right]^n \qquad ...(8)$$

On the other hand, according to (6) and (7), we see that for $n \geqq 3$.

$$K^{(n)}(s,t) = \int_a^b\int_a^b K(s,r)K^{(n-2)}(r,r_1)K(r_1,t)dr\, dr_1$$

Hence by the Schwarz inequality we have

$$|K^{(n)}(s,t)|^2 \leqq \int_a^b\int_a^b |K^{(n-2)}(r,r_1)|^2\, dr\, dr_1 \int_a^b\int_a^b |K(s,r)K(r_1,t)|^2\, dr\, dr_1$$

Accordingly, by making use of (8), we obtain

$$|K^{(n)}(s,t)|^2 \leqq \left[\int_a^b\int_a^b |K(s,t)|^2\, ds\, dt\right]^{n-2} \left\{\int_a^b |K(s,r)|^2 dr \int_a^b |K(r_1,t)|^2\, dr_1\right\}$$

Notes

The term in braces on the right side is continuous on the domain $a \leqq s \leqq b$, $a \leqq t \leqq b$; hence bounded. Therefore, according to the assumption (3), the series.

$$\Gamma'(s, t) = \sum_{n-1}^{\infty} K^{(n)}(s, t) \qquad ...(9)$$

converge uniformly on the domain $a \leqq s \leqq b$, $a \leqq t \leqq b$. Hence by term-by-term integration and by using (7) we obtain

$$\Gamma(s, t) = K(s, t) + \int_a^b K(s, r)\Gamma(r, t)dr \qquad ...(10)$$

$$\Gamma(s, t) = K(s, t) + \int_a^b \Gamma(s, r)K(r, t)dr \qquad ...(11)$$

The series (9) is known as the Neumann series for the Kernel $K(s, t)$.

Now, by making use of (10), we can prove that

$$\varphi(s) = f(s) + \int_a^b \Gamma(s, t)f(t)dt \qquad ...(12)$$

satisfies the equation (1). In fact, substituting (12) in (1) and using (10), we have

$$\begin{aligned}
&\varphi(s) - \int_a^b K(s, t)\varphi(t)dt \\
&= f(s) + \int_a^b \Gamma(s, t)f(t)dt - \int_a^b K(s, t)\left\{f(t) + \int_a^b \Gamma(t, r)f(r)dr\right\}dt \\
&= f(s) + \int_a^b \left\{\Gamma(s, t) - K(s, t) - \int_a^b K(s, r)\Gamma(r, t)dr\right\}f(t)dt \\
&= f(s)
\end{aligned}$$

Conversely, we can prove that if $\varphi(s)$ satisfies the equation (1), then $\varphi(s)$ satisfies (12). In fact, substituting $f(s) = \varphi(s) - \int_a^b K(s, t)\varphi(t)dt$ in (12) and using (11), we see that

$$\begin{aligned}
&\varphi(s) - \int_a^b K(s, t)\varphi(t)dt \\
&+ \int_a^b \Gamma(s, t)\left\{\varphi(t) - \int_a^b K(t, r)\varphi(r)dr\right\}dt \\
&= \varphi(s) + \int_a^b \left\{\Gamma(s, t) - K(s, t) - \int_a^b \Gamma(s, r)K(r, t)dr\right\}\varphi(t)dt \\
&= \varphi(s)
\end{aligned}$$

Accordingly, we see that the equation (1) is equivalent to the equation (12). Similarly, we can prove that conjugate equation (2) is equivalent to the equation

$$\phi(t) = g(t) + \int_a^b \Gamma(s, t)g(s)ds \qquad ...(13)$$

Example: Under the assumption (3), every solution $\varphi(s)$ of the equation (1) is given by (12) by means of the Kernel $\Gamma(s, t)$ and every solution $\phi(t)$ of the conjugate equation (2) is given by (13) by means of the conjugate Kernel $\Gamma(s, t)$ of $\Gamma(s, t)$, defined by

$$\Gamma(s, t) = \Gamma(t, s) \qquad ...(14)$$

Notes

For this reason, the Kernels $\Gamma(s, t)$ and $\Gamma'(s, t)$ are called the resolvent Kernels of the equation (1) and (2) respectively.

The foregoing theorem shows that λ is not an eigenvalue of either the Kernel $K(s, t)$ or its conjugate Kernel $K'(s, t)$,

$$K'(s, t) = K(t, s) \qquad ...(15)$$

The General Case

We shall prove that there exist two sets of linearly independent continuous functions

$$\begin{aligned} &\alpha_1(s), \alpha_2(s), ..., \alpha_m(s) \\ &B_1(t), \beta_2(t), ..., \beta_m(t) \end{aligned} \qquad ...(15)$$

defined on the interval $[a, b]$, such that

$$\int_a^b \int_a^b \left| K(s, t) - \sum_{v=1}^{m} \alpha_v(s)\beta_v(t) \right|^2 ds\, dt < 1 \qquad ...(16)$$

To prove this, let ε be an arbitrary positive number. Then we divide the interval (a, b) into a finite number of sub-intervals $I_1, I_2, ..., I_n$, such that

$$\sup_{a \leqq s \leqq v} |K(s, t') - K(s, t'')| \leqq \varepsilon$$

for any pair of points t', t'' in each I_v. This is possible, because of the uniform continuity of $K(s, t)$ on the domain $a \leqq s \leqq b$, $a \leqq t \leqq b$. Let t_v be an inner point of I_v. Let I'_v be an interval contained in the interior of I_v and containing the point t_v. Then we define $\beta_v(t)$ as follows:

$$\beta_v(t) \equiv \begin{cases} 0 & \text{outside of } I_v \\ 1 & \text{on } I'_v \end{cases}$$

such that the function $\beta_v(t)$ is continuous and $0 \leqq \beta_v(t) \leqq 1$ on the interval $[a, b]$. We now set $\alpha_r(s) \equiv K(s,t)$ and

$$N(s, t) = \left| K(s, t) - \sum_{v=1}^{n} \alpha_v(s)\beta_v(t) \right|$$

Then we see that

$$|N(s, t)| = |K(s, t) - K(s, t_v)| \leqq \varepsilon$$

for t in I'_v, and

$$|N(s, t)| = |K(s, t) - K(s, t_v)\beta_v(t)| \leqq 2M$$

for t in $I_v - I'_v$ where

$$M = \sup_{a \leqq s \leqq b \leqq a \leqq t \leqq b \leqq} |K(s, t)|$$

Since ε and the sum of lengths of $I_v - I'_v$ are both arbitrary, we can choose the values of them so small that

$$\int_a^b \int_a^b \left| K(s, t) - \sum_{v=1}^{n} \alpha_v(s)\beta_v(t) \right|^2 ds\, dt < 1$$

Clearly, the function $\beta_1(t)$, $\beta_2(t)$, ..., $\beta_n(t)$ are linearly independent. Hence, if $\alpha_1(s)$, $\beta_2(s)$, ..., $\alpha_n(s)$ are linearly independent, then our proof is completed. If otherwise, say, $\alpha_n(s)$ is written as a linear combination of $\alpha_1(s)$, $\alpha_2(s)$, ..., $\alpha_n-_1(s)$, then

Notes

$$R(s, t) \equiv \sum_{v=1}^{n} \alpha_v(s)\beta_v(t)$$

is also written in the form

$$R(s, t) \equiv \sum_{v=1}^{n-1} \alpha_v(s)\beta_v^{(1)}(t)$$

If $\beta_1^{(1)}(t), \beta_2^{(1)}(t), ..., \beta_{n-1}^{(1)}(t)$ are linearly independent, then, by setting $\beta_v^{(1)}(t) = \beta_v(t)$, the number n is diminished. If otherwise, say, $\beta_{n-1}^{(1)}(t)$ is written as a linear combination of $\beta_1^{(1)}(t), \beta_2^{(2)}(t), ..., \beta_{n-2}^{(1)}(t)$ then $R(s, t)$ is also written as

$$R(s, t) \equiv \sum_{v=1}^{n-2} \alpha_v^{(1)}(s)\beta_v^{(1)}(t)$$

Repeating this argument alternatively for α and β, we finally obtain two sets of linearly independent functions

$$\gamma_1(s), \gamma_2(s), ..., \gamma_m(s) \text{ and } \delta_1(t), \delta_2(t), ..., \ \delta_m(t)$$

in terms of which $R(s, t)$ is written as

$$R(s, t) \equiv \sum_{v=1}^{m} \gamma_v(s)\delta_v(t)$$

provided that $K(s, t) \not\equiv 0$ and $R(s, t) \not\equiv 0$. Then by setting $\gamma_v(s) = \alpha_v(s)$, and $\delta_v(t) = \beta_v(t)$, the proof is completed

we now set

$$K_1(s, t) = K(s, t) - \sum_{v=1}^{m} \alpha_v(s)\beta_v(t)$$

and denote the resolvent Kernel of $K_1(s, t)$ by

$$\Gamma_1(s, t) = \sum_{n=1}^{\infty} K_1^{(n)}(s, t)$$

Then, the equation (1) is written as

$$\varphi(s) - \int_a^b K_1(s, t)\varphi(t)dt$$

$$= f(s) + \int_a^b \left(\sum_{v=1}^{m} \alpha_v(s)\beta_v(t) \right)\varphi(t)dt \qquad ...(17)$$

and we can prove in the same way as in last that φ(s) is determined by

$$\varphi(s) - \int_a^b \left[\sum_{v=1}^{m} \left(\alpha_v(s) + \int_a^b \Gamma_1(s, r)\alpha_v(r)dr \right) \beta_v(t) \right] \varphi(t)dt$$
$$= f(s) + \int_a^b \Gamma_1(s, r) f(r)\, dr \qquad ...(18)$$

Notes

From this follows the fact that to solve the equation (1) is equivalent to finding a solution $\varphi(s)$ of the equation (18) with the term in brackets as the Kernel and for the right side the given function

$$f(s)+\int_a^b \Gamma_1(s,r)f(r)\,dr$$

We shall prove incidentally that

$$\alpha_v(s)+\int_a^b \Gamma_1(s,r)\alpha_v(r)dr \qquad (v=1,2,\ldots,m) \quad \ldots(19)$$

are linearly independent. To prove this, suppose

$$0\equiv\sum_{v=1}^{m} c_v\left(\alpha_v(s)+\int_a^b \Gamma_1(s,r)\,\alpha_v(r)dr\right)$$

$$=\sum_{v=1}^{m} c_v\alpha_v(s)+\int_a^b \Gamma_1(s,r)\left(\sum_{v=1}^{m} c_v\alpha_v(r)\right)dr$$

and $\sum_{v=1}^{m}|c_v|\neq 0.$ Then, by the properties of the resolvent Kernel $\Gamma_1(s, t)$, we have

$$\sum_{v=1}^{m} c_v\alpha_v(s)\equiv 0-\int_a^b K_1(s,r)0.dr\equiv 0$$

This contradicts the linear independence of $\alpha_v(s)$.

The equation (18) is reduced to the system of equations

$$\varphi(s)=f(s)+\int_a^b \Gamma_1(s,r)f(r)+\sum_{v=1}^{m}\rho_v\left(\alpha_v(s)+\int_a^b \Gamma_1(s,r)\alpha_v(r)dr\right) \quad \ldots(20)$$

$$\rho_\mu=\int_a^b \beta_\mu(t)\varphi(t)dt \qquad (\mu=1,2,\ldots,m) \quad \ldots(21)$$

Hence, substituting (20) in (21), we have a system of linear equations in unknowns, $\rho_1, \rho_2, \ldots \rho_m$,

$$\rho_\mu-\sum_{v=1}^{m}\rho_v\left[\int_a^b \alpha_v(s)\beta_\mu(s)ds+\int_a^b \Gamma_1(s,r)\,\alpha_v(r)\beta_\mu(s)dr\,ds\right]$$

$$=\int_a^b\left(\beta_\mu(t)+\int_a^b \Gamma_1(r,t)\,\beta_\mu(r)dr\right)f(t)dt \qquad (\mu=1,2,\ldots,m) \quad \ldots(22)$$

Accordingly to solve the equation (1) is equivalent to finding the solutions ρ_μ of (22); indeed, substituting the solution ρ_μ in (20), we obtain the solution of (1).

Similarly, we see that to solve the equation (2) is equivalent to solving the following system of linear equations in the unknowns

$\rho'_1, \rho'_2, \ldots, \rho'_m$,

$$\rho'_\mu-\sum_{v=1}^{m}\rho'_v\left[\int_a^b \alpha_\mu(t)\beta_v(t)dt+\int_a^b \Gamma_1(r,t)\alpha_\mu(t)\beta_v(r)dr\,dt\right] \quad \ldots(23)$$

$$=\int_a^b\left(a_\mu(s)+\int_a^b \Gamma_1(s,r)\,\alpha_\mu(r)dr\right)g(s)ds \qquad (\mu=1,2,\ldots,m)$$

and the solution $\phi(t)$ of (2) is given by

$$\phi(t)=g(t)+\int_a^b \Gamma_1(r,t)g(r)dr+\sum_{v=1}^{m}\rho'_v\left(\beta_v(t)+\int_a^b \Gamma_1(r,t)\beta_v(r)dr\right) \quad \ldots(24)$$

Notes

where the ρ'_v are the solution of (23).

Let Δ be the matrix of the equations (22), in the unknowns ρ, and Δ' that of the equations (23), in the unknowns ρ'. Then

Δ' is the transposed matrix of Δ ...(25)

Hence det $\Delta' \neq 0$ if and only if det $\Delta \neq 0$.

We first consider the case when det $\Delta \neq 0$, and hence, det $\Delta' \neq 0$. In this case, the equation (22)[(23)] for any function $f(s)[g(t)]$, admits a unique solution

$$\rho = (\rho_1, \rho_2, ..., \rho_m)\ [\rho' = (\rho'_1, \rho'_2, ..., \rho'_m)]$$

Therefore, for the given function $f(s)[g(t)]$, the equation (1) [2] admits a unique solution $\varphi(s)[\phi(t)]$. In particular, if $f(s) \equiv 0$ $[g(t) \equiv 0]$, then

$$\rho = (\rho_1, \rho_2, ..., \rho_m) = (0, 0, ..., 0)$$
$$[\rho' = (\rho'_1, \rho'_2, ..., \rho'_m) = (0, 0, ..., 0)]$$

hence, $\varphi(s) \equiv 0[\phi(t) \equiv 0]$

We next consider the case when det $\Delta = 0$, and hence det $\Delta' = 0$. For the sake of simplicity we write (22), (23) as

$$\rho_\mu - \sum_{v=1}^{m} c_{\mu v}\rho_v = f_\mu \qquad (\mu = 1, 2, ..., m) \quad ...(23')$$

$$\rho'_\mu - \sum_{v=1}^{m} c_{v\mu}\rho'_v = g_\mu \qquad (\mu = 1, 2, ..., m) \quad ...(24')$$

respectively. The matrices Δ, Δ' are of course written as

$$\Delta = (\delta_{\mu\nu} - c_{\mu\nu}),\ \Delta' = (\delta_{\nu\mu} - c_{\nu\mu})$$

where $\delta_{\mu\nu} = 0$ for $\mu \neq \nu$, and $\delta_{\mu\nu} = 1$ for $\mu = \nu$. For the case when det Δ = det $\Delta' = 0$, the following facts are known:

The associated systems of linear homogeneous equations

$$\rho_\mu - \sum_{v=1}^{m} c_{\mu v}\rho_v = 0 \qquad (\mu = 1, 2, ..., m) \quad ...(22'')$$

and

$$\rho'_\mu - \sum_{v=1}^{m} c_{v\mu}\rho'_v = 0 \qquad (\mu = 1, 2, ..., m) \quad ...(23'')$$

admit a number $r(r \geqq 1)$ of linearly independent solutions

$$\rho(1) = (\rho_{11}, \rho_{12}, ..., \rho_{1m}), ...,$$
$$\rho(r) = (\rho_{r1}, \rho_{r2}, ..., \rho_{rm})$$

and

$$\rho'(1) = (\rho'_{11}, \rho'_{12}, ..., \rho'_{1m}), ...,$$
$$\rho'(r) = (\rho'_{r1}, \rho'_{r2}, ..., \rho'_{rm})$$

respectively. The inhomogeneous system (22′) admits a solution for given $f_1, f_2, ..., f_m$ if and only if

$$\sum_{\mu=1}^{m} f_\mu \rho'_{j\mu} = 0 \qquad (j = 1, 2, ..., m)$$

Notes

in other words, for the general solution of (23′′).

$$\sum_{j=1}^{r} c_j \rho'(j) = \left(\sum_{j=1}^{r} c_j \rho'_{j1}, \sum_{j=1}^{r} c_j \rho'_{j2}, ..., \sum_{j=1}^{r} c_j \rho'_{jm} \right)$$

which contains a number r of arbitrary constants c_1, c_2, ..., c_r, there hold the following relations

$$\sum_{\mu=1}^{m} f_\mu = \left(\sum_{j=1}^{r} c_j \rho'_{j\mu} \right) = 0 \qquad ...(26')$$

If the condition (26′) is satisfied, then the general solution of (22′) is given by the sum of a particular solution $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2, ..., \bar{\rho}_m)$ of (22′) and the general solution $\sum_{j=1}^{r} c_j \rho(j)$ of (22′), that is, by the following expression containing r arbitrary constants c_1, c_2, ..., c_r.

$$\rho = \bar{\rho} + \sum_{j=1}^{r} c_j \rho(j) \qquad ...(27)$$

$$= \left(\bar{\rho}_1 + \sum_{j=1}^{r} c_j \rho_{j1}, \bar{\rho}_2 + \sum_{j=1}^{r} c_j \rho_{j2}, ..., \bar{\rho}_m + \sum_{j=1}^{r} c_j \rho_{jm} \right)$$

Similarly, the equations (23′) admit a solution for given g_1, g_2, ...g_m if and only if the following relations

$$\sum_{\mu=1}^{m} g_\mu \left(\sum_{j=1}^{r} c_j \rho_{j\mu} \right) = 0 \qquad ...(28)$$

hold; and under the condition (28), the general solution of (23′) is given by the sum of a particular solution $\bar{\rho}' = (\bar{\rho}'_1, \bar{\rho}'_2, ..., \bar{\rho}'_m)$ of (23′) and the general solution $\sum_{j=1}^{r} c_j p'(j)$ of (23′), that is, by the following expression containing r arbitrary constants c_1, c_2,.... c_r.

$$\rho' = \bar{\rho}' + \sum_{j=1}^{r} c_j \rho'(j)$$

$$= \left(\bar{\rho}'_1 + \sum_{j=1}^{r} c_j \rho'_{j1}, \bar{\rho}'_2 + \sum_{j=1}^{r} c_j \rho'_{j2}, ..., \bar{\rho}'_m + \sum_{j=1}^{r} c_j \rho'_{jm} \right) \qquad ...(29)$$

Accordingly, substituting the solution-ρ given by (27), if any, in (20), we obtain the general solution φ(s) of the equation (1). The function φ(s) contains r arbitrary constants. In fact, if

$$0 = \sum_{v=1}^{m} \left(\alpha_v(s) + \int_a^b \Gamma_1(s, r) \alpha_v(r) dr \right) \left(\sum_{j=1}^{r} c_j \rho_{jv} \right)$$

then, by the linear independence of (19)

$$0 = \sum_{j=1}^{r} c_j \rho_{jv} \qquad (v = 1, 2, ..., m,)$$

This contradicts the fact that ρ(1), ρ(2), ... ρ(r) are linearly independent solution of (22′′). We can also obtain, substituting (29) in (24), the general solution ϕ(t) of (2) which contains a number r of arbitrary constants.

Notes

Finally, we shall reduce the solvability condition (26′) to a more readable and usual form as follows:

The term on the left side of (26) is, by (22) and (22′)

$$\sum_{\mu=1}^{m} f_\mu \rho'_{j\mu} = \int_a^b \left[\sum_{\mu=1}^{m} \rho'_{j\mu} \left(\beta_m(t) + \int_a^b \Gamma_1(r,t)\beta_\mu(r)dr \right) \right] f(t)dt$$

From (24), it is easily seen that the function in brackets on the right side is a solution of (2) with $g(t) \equiv 0$, that is, of

$$\phi(t) - \int_a^b K(s,t)\phi(s)ds = 0 \qquad ...(30)$$

On the other hand, the general solution of (30) is given by linear combinations of the functions in brackets. Therefore (26) is equivalent to the following for every solution $\phi(t)$ of (30).

$$\int_a^b f(t)\phi(t)dt = 0 \qquad ...(31)$$

Similarly, we see that the condition (28) is equivalent to the following: for every solution $\varphi(s)$ of the equation

$$\varphi(s) - \int_a^b K(s,t)\varphi(t)dt = 0$$
$$\int_a^b g(s)\varphi(s)ds = 0 \qquad ...(32)$$

Self Assessment

1. The Fredholm equation is given by

$$y(x) = f(x) + \lambda \int_0^1 (xt - x^2t^2)y(t)dt$$

solve for $y(x)$ when $f(x) = x^3$.

32.3 Summary

- We have seen that Fredholm integral equation has solutions that depend on the nature of the resolvent Kernel as well on the function $f(s)$.
- If the parameter λ is not an eigenvalue then the non-homogeneous equation has one and only one solution and the homogeneous equation has a solution $Q(x) = 0$.
- For λ to be one of the eigenvalues, the homogeneous equation admits a number of independent solutions.

32.4 Keywords

The nature of the solution of the ***Fredholm integral equation*** of the second kind as well as on the first kind depends upon the constant parameter λ as well as on the function $f(x)$.

The ***eigenvalue problem*** puts certain conditions on the function $f(s)$ for the solutions to exist. Fredholm theorem elaborates on these points.

Notes

32.5 Review Question

Show that for the unsymmetric Kernel

$$K(x, t) = \sum_{v=1}^{\infty} v^{-2} \sin(vx) \sin[(v+1)t]$$

defined on the domain $0 \le x \le 2\pi, 0 \le t \le 2\pi$ has the iterated Kernel given by

$$K_n(x, t) = \sum_{v=1}^{\infty} \pi^{n-1}[v^2(v+1)^2(v+2)^2 \ldots (v+n-1)^2]^{-1} \sin xv \sin[(n+v)t]$$

[**Hint:** Integrate term-by-term and use the relation $\lim_{\alpha \to 0} \frac{\sin \alpha\mu}{\alpha} = u.$]

Answer: Self Assessment

1. $y(x) = x^3 + \lambda x c_1 - \lambda x^2 c_2$

   where $C_1 = \dfrac{(120 - \lambda)}{(600 - 80\lambda - \frac{5}{2}\lambda^2)}$,

   $C_2 = \dfrac{4 - 5C_1(1 - \lambda/3)}{5\lambda}$.

32.6 Further Readings

Books

Tricomi, F.G., Integral Equations

Yosida, K., Lectures in Differential and Integral Equations

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