A STUDY OF BICOMPLEX SPACE WITH A TOPOLOGICAL VIEW POINT

A

THESIS

Submitted to

For the award of

DOCTOR OF PHILOSOPHY (Ph.D)

in

MATHEMATICS

by

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2018

Dedicated to

(Late) Professor Rajiv Kumar Srivastava

always inspiring!

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I, **Sukhdev Singh**, Department of Mathematics, Lovely Professional University, Punjab certify that the work embodied in this Ph.D thesis titled, "**A Study of Bicomplex Space With a Topological View Point**" is my own bonafide work carried out by me under the supervision of **Dr. Rajesh Kumar Gupta** and the co-supervision of **Dr. Sanjeev Kumar**. I confirm that:

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Abstract

This thesis entitled **"A Study of Bicomplex Space with a Topological View Point"** is being submitted in partial fulfillment for the award of degree of Doctor of Philosophy in Mathematics to Lovely Professional University, Phagwara, Punjab. The work done is divided into five chapters.

Chapter 1 provided a brief literature of several aspects of bicomplex algebra, nets and filters, which will be required to understand and apprehend the work done in the remaining parts of the thesis.

In Chapter 2, three types of order relations on the bicomplex space are defined and studied them. Further, three order topologies on the bicomplex space namely, $\mathbb{C}_0$(o)-topology, $\mathbb{C}_1$(o)-topology and Id(o)-topology are defined by using the order relations defined and compared. Also, metric on the bicomplex space is defined.

In Chapter 3, deals with the study of bicomplex nets called as $\mathbb{C}_2$-nets. Due to the multi-dimensionality of the bicomplex space $\mathbb{C}_2$, there arise different types of tendencies called confluences. The bicomplex space, $\mathbb{C}_2$ equipped with $\mathbb{C}_0$(o)-topology as well as Id(o)-topology exhibits interesting and challenging behaviour of $\mathbb{C}_2$-nets. Different types of confluences have been characterized in terms of convergence of the component nets.

Chapter 4, initiates the study of clustering of $\mathbb{C}_2$-nets. Clustering of $\mathbb{C}_2$-nets on different types of zones in $\mathbb{C}_2$ have been defined. Clustering in Id(o)-topology and Id(p)-topology have been compared. Relation between clustering of $\mathbb{C}_2$-nets and the clustering of its component nets have been defined. Finally, investigations have been made connecting clustering of a $\mathbb{C}_2$-nets and confluence of its subnets and studied the compactness of some subsets of the bicomplex space in the Id(o)-topology. Also given a result regarding homeomorphism in Id(o)-topology and $\mathbb{C}_1$(o)-topology on $\mathbb{C}_2$ are given. Further, the compatibility of the $\mathbb{C}_2$-nets with the filters on the bicomplex space is tested and discussed the confluence of the filters on $\mathbb{C}_2$ with respect to the different types of order topologies as defined earlier. The relations between the $\mathbb{C}_2$-nets and their corresponding filters on $\mathbb{C}_2$ and vice-versa are also established.

In Chapter 5, the compatibility of the algebraic and topological structures have been discussed. For this purpose, the $\mathbb{C}_2$-sequence space and discussed their algebraic properties using the Orlicz functions and paranorm are considered. Further, the properties of the sets of $\mathbb{C}_2$-nets with respect to the Id(o)-topology are also discussed. As the last objective was on the orderability problem, therefore, the orderability of some of the topological structures was tested using the convex subsets of them.

At last, we gave the conclusion of Thesis and further scope of study.

Acknowledgements

First of all, I am grateful to God who kept on showering His blessings upon me throughout this task and blessed me with success. It is a joyful, pleasant and enjoying moment to express my deep gratitude to all people who helped me to make this thesis possible.

I express a deep gratitude to my respected supervisor, Dr. Rajesh Kumar Gupta, Department of Mathematics, Lovely Professional University, Punjab and the co-supervisor Dr. Sanjeev Kumar, Department of Mathematics, Institute of Basic Science, Khandari Campus, Dr. B. R. Ambedkar University Agra, for their guidance and support throughout my Ph.D. work. I am greatly indebted to them for their inspirations, constant supervision and the great patience, which they showed to me while answering my queries without which it was just impossible for me even to imagine the successful completion of this work. The great efforts of my supervisor Dr. Rajesh Kumar Gupta to explain things clearly and simply. His influences as an advisor, mentor and as a coach on the lacrosse field have an extremely significant and positive impact on my life.

The proper guidance of my co-supervisor, Dr. Sanjeev Kumar made the work easy for me and helped me to establish useful findings through my research pursuit. Whenever I needed his guidance, he helped and encouraged me during the work. He helped to make Mathematics fun for me. Throughout my thesis-writing period, he provided encouragement, sound advice, good teaching, good company and great ideas. I would have been lost in this critical world without him.

I am heartily thankful to my teachers during M.Phil. programme, Prof. (Late) Rajiv K. Srivastava, Prof. Sunder Lal, Dr. Sanjay Chaudhary, Department of Mathematics, Institute of Basic Science, Khandari Campus, Dr. B. R. Ambedkar University Agra, for their constant encouragement.

I wish to thank my family for providing a loving environment for me. My wife Mrs. Mandeep Rani, my brothers Mr. Pritpal Singh, Mr. Gurmail Singh, my sister Ms. Gurpreet Kaur, my uncles and my grand mother were particularly supportive. Lastly and most importantly, I feel highly grateful to my parents, Smt. Kamlesh Rani and Shri Avtar Singh. They encouraged me, raised me, supported me, taught me and loved me.

I wish to thank my friends Mr. Pranav Sharma, Mr. Kuldeep Kumar, Mr. Prince, Dr. Sachin Kaushal, Mr. Tarun Kumar, Mr. Ankit Kumar Singh, Mr. Munendra Agrawal, Mr. Raman Kumar, Mr. Rahul Sharma, Ms. Sapna, Dr. Rinku Mathur and many others, for helping me get through the difficult times, and for all the emotional support, camaraderie, entertainment and caring they provided.

I am sincerely thankful to my dear and best friend Ms. Alpana Vasudeva for her kind support, helpful nature and care for me. I am also thankful to my senior research scholar Ms. Mamta Nigam for her support and helpful attitude. Most importantly, I wish to thank my senior Dr. Jogendra Kumar for his help and corporation throughout.

I would like to thank all the staff members of the Department of Mathematics, Lovely Professional University, Punjab especially Dr. Dilbaj Singh for his support during my Ph.D. course.

Last but not the least, I am grateful to management of Lovely Professional University, for allowing me to the research work and for providing me with the working facilities.

Sukhdev Singh

Contents

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List of Symbols

$\mathbb{N}$	:	Set of natural numbers
$\mathbb{Q}$	:	Set of rational numbers
$\mathbb{Q}^+$	:	Set of positive rational numbers
$\mathbb{C}_0$	:	Set of real numbers
$\mathbb{C}_1$	:	Set of complex numbers
$\mathbb{C}_0^3$	:	Three dimensional space
$\mathbb{C}_2$	:	Set of bicomplex numbers
$\mathcal{H}$	:	Set of all hyperbolic numbers
$\mathbb{O}_0$	:	Set of singular elements in real space
$\mathbb{O}_1$	:	Set of singular elements in complex space
$\mathbb{O}_2$	:	Set of singular elements in bicomplex space
$\mathbb{O}_h$	:	Set of all hyperbolic singular elements
τ_1	:	Norm topology on $\mathbb{C}_2$
τ_2	:	Complex topology on $\mathbb{C}_2$
τ_3	:	Idempotent topology on $\mathbb{C}_2$
τ_4	:	$\mathbb{C}_0$(o)-topology on $\mathbb{C}_2$
τ_5	:	$\mathbb{C}_1$(o)-topology on $\mathbb{C}_2$
τ_6	:	Id(o)-topology on $\mathbb{C}_2$
τ_7	:	Id(p)-topology on $\mathbb{C}_2$
G_1	:	Family of R-frame segments
G_2	:	Family of R-plane segments
G_3	:	Family of R-line segments
G_4	:	Family of R-open intervals
M_1	:	Family of C-frame segments
M_2	:	Family of C-plane segments
M_3	:	Family of C-line segments
M_4	:	Family of C-open intervals

N_1	:	Family of Id-frame segments
N_2	:	Family of Id-plane segments
N_3	:	Family of Id-line segments
N_4	:	Family of Id-open intervals
$(\xi, \eta)_{\mathbb{C}_0}$	:	Basis element of the $\mathbb{C}_0$(o)-topology τ_4
$(\xi, \eta)_{\mathbb{C}_1}$	:	Basis element of the $\mathbb{C}_1$(o)-topology τ_5
$(\xi, \eta)_{Id}$	:	Basis element of the Id(o)topology τ_6
$C(\xi;\, r_1,\, r_2)$	:	C-discus (basis elements of complex topology τ_2)
$D(\xi;\, r_1,\, r_2)$	:	D-discus (basis elements of idempotent topology τ_3)
$\xi,\ \eta,\ \zeta$	:	Notations for bicomplex numbers
e_1 and e_2	:	Non-trivial idempotent elements in $\mathbb{C}_2$
${}^1\xi$ and ${}^2\xi$	:	Idempotent components of a bicomplex number
$\mathbb{A}_1$ and $\mathbb{A}_1$	:	Auxiliary complex spaces
$\mathbb{I}_1$ and $\mathbb{I}_2$	:	Principal ideals in bicomplex space
$\{\xi_\alpha\}$	:	$\mathbb{C}_2$-net
$\mathcal{F}$	:	$\mathbb{C}_2$-filter
$\mathcal{U}$	:	$\mathbb{C}_2$-ultrafilter

Chapter 1

Introduction

In this chapter, a brief literature of several aspects of Bicomplex Analysis, nets and filters, paranorm and the orderability problem which is required to understand work done in the remaining part of thesis has been presented.

Section 1.1 contains definitions, and results related to the algebraic structure of the bicomplex numbers. This section is divided into four subsections explaining the historical background, algebraic structure of the bicomplex space, different types of the conjugations of the bicomplex numbers and their properties are discussed. Algebraic structures of the complex and bicomplex spaces are compared and the differences between them are given in details. For details of the theory, we refer to the monograph by Price [52] and an article by Srivastava [71] and Rochon [58].

Section 1.2 is given for the literature review of the research work done on the algebraic and the topological structures on the bicomplex space by the researchers.

Section 1.3 contains the definitions and results related to the different types of order relations on a non-empty set and the order topology.

In Section 1.4, we discuss about the basics of the orderability problem. The conditions for a subspace to be orderable are discussed in the section

Section 1.5 contains the theory of the nets structure on a non - empty set. Several results of the theory of nets and sequences are compared in the general setup.

In Section 1.6, the basic concepts of the filters is discussed. The structures of the and nets and filters on a set are compared.

In Section 1.7, the concept of Orlicz functions and the paranorm structures on the set of sequence spaces have been discussed.

In Section 1.8, some applications of the bicomplex numbers are given in physics like signal systems, dynamics of spiral waves are discussed in details. Some of the work done by Srivsatava and Naveen [70] on the structures of sequence spaces are also explained in brief.

1.1 Introduction to Bicomplex Numbers

1.1.1 Historical Background

A complex number may be viewed as a pair of real numbers. With the importance and applicability of the complex analysis, naturally attempts were made to generalize the theory in different direction.

Euler (1707-1783) was the mathematician who introduced the symbol i with property $i^2 = -1$ and accordingly as a root of the equation $x^2 + 1 = 0$. He also called the symbol i imaginary. Also a number of the form $a + ib$ where a, b are real numbers is called a complex number.

Complex Analysis, is undoubtedly the most developed and most applicable branch of Pure Mathematics. But, during the period of its infancy, the progress of this subject was very slow. Karl Friedrich Gauss (1777-1855) had worked on this new concept and had actually obtained very good results. Augustin-Louis Cauchy (1789-1857) presented the theory of complex numbers and functions of complex variable in such an organized and lucid manner that he is regarded as the effective founder of complex analysis. In 1843, William Rowan Hamilton (1805-1865), an Irish mathematician and astronomer-better known for his work in vector analysis and in optics-developed an algebra of real numbers which is more or less is the present day algebra of complex numbers. It was beginning of theory of algebra different from the algebra of real numbers. Ten years later, Hamilton conceived the term Quaternion.

Extensions of basic complex numbers to higher dimensions have a renewed interest in mathematics, physics, and engineering because of fruitful applications. Quaternions is one of the most popular sets of tetra-numbers, however, they form a noncommutative algebra. A couple of interesting commutative algebras of tetranumbers are defined by bicomplex and bihyperbolic numbers. Bicomplex numbers are a natural extension of complex numbers, whereas bihyperbolic numbers are a natural extension of hyperbolic numbers to four dimensions. A quaternion is a hypercomplex number that can be presented as linear combination

$$X = x_0 + ix_1 + jx_2 + kx_3,$$

where $x_p \in \mathbb{C}_0; 0 \leq x \leq 3$ and i, j, k are units, such that $i^2 = j^2 = k^2 = -1$. Also, $i.j = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Given two quaternions $X = x_0 + ix_1 + jx_2 + kx_3$ and $Y = y_0 + iy_1 + jy_2 + ky_3$ of this kind we may add them by applying component-wise addition. Formally,

$$X + Y = (x_0 + y_0) + i(x_1 + y_1) + j(x_2 + y_2) + k(x_3 + y_3).$$

Now, definition of multiplication of the quaternions is given in the following table:

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

TABLE 1.1: Multiplication of Quaternions

This multiplication yields the following expression for non-commutative product:

$$\begin{aligned} X \times Y &= (x_0 + ix_1 + jx_2 + kx_3) \times (y_0 + iy_1 + jy_2 + ky_3) \\ &= (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) + i(x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2) \\ &+ j(x_0y_2 + x_2y_0 + x_3y_1 - x_1y_3) + k(x_0y_3 + x_3y_0 + x_1y_2 - x_2y_1). \end{aligned}$$

There are very cumbersome equations mean, besides other things, that multiplication of quaternions loses commutativity. Thus, $q_1q_2 \neq q_2q_1$. However, it remains associative, i.e.,

$$(q_1q_2)q_3 = q_1(q_2q_3),$$

where q_1, q_2 and q_3 are quaternions.

Somehow, the researchers of his time simply could not accept the notion that there could be "fourth dimension", especially if it was claimed to be time. They clung tenaciously to be a primitive combination of component by component calculation and extensive use of geometry. By late 1800's, their calculations were accompanied by elaborated geometrical figure that look alike the 2×4 framing of a house.

Hamilton suggested that in the quaternion one should distinguish the scalar part x_0 from the vector part $V_x = ix_1 + jx_2 + kx_3$. In this case as it is easy to check the product of two quaternion vectors $V_x = ix_1 + jx_2 + kx_3$ and $V_y = iy_1 + jy_2 + ky_3$, is a common quaternion, viz.

$$V_x \cdot V_y = (-x_1y_1 - x_2y_2 - x_3y_3) + [i(x_2y_3 - x_3y_2) + j(x_3y_1 - x_1y_3) + k(x_1y_2 - x_2y_1)],$$

whose scalar part has a symmetric bilinear form and the vector part looks like a conversions vector multiplication. As a matter of fact the terms "scalar product" and "vector product" appeared right from here and for the first time were introduced by Hamilton (Pavlov [49]). Gibbs in America and Heaviside in Britain reformulated quaternion analysis so that all expressions would be constrained to three dimensions or less. In particular, they defined the cross product of two, 3-dimensional vectors as

$$\begin{array}{ccc} i \times i = -1, & j \times j = -1, & k \times k = -1, \\ i \times j = k, & j \times k = i, & k \times i = j, \\ i \times k = -j, & j \times i = -k, & k \times j = -i, \end{array}$$

so that result would come out as another 3-dimensional vector. The quaternion product of two, 3-dimensional vectors a and b is

$$ab = a \cdot b \, + \, a \times b$$

which has a scalar part and 3-D vector part. Therefore, Gibbs and Heaviside avoided quaternion product notation and used only dot (modified), and cross component in what they cleverly renamed as vector analysis. Scientists and engineers accepted this subterfuge because it met their prejudice about 3-D being invalidate and it did not have the word "quaternion" mentioned anywhere. Nevertheless, vector analysis is form of quaternion analysis (cf. Davenport [16]).

Every beginner of algebra has learnt about quaternion as a counter example of division ring which is not a field, simply because multiplication of quaternion is not commutative. This was a very big drawback in the theory and probably the biggest hurdle in the development of the subject.

1.1.2 Certain Basics of Bicomplex Numbers

The bicomplex numbers were introduced by Segre [62] in 1892. Here some of the basic results of the theory of bicomplex numbers are reproduced. The set of bicomplex numbers is denoted by $\mathbb{C}_2$ and the sets of real and complex numbers are denoted as $\mathbb{C}_0$ and $\mathbb{C}_1$, respectively.

A *bicomplex number* is defined as

$$\begin{aligned} \xi &= x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \\ &= z_1 + i_2 z_2 \end{aligned} \tag{1.1.1}$$

where $x_p \in \mathbb{C}_0; 1 \leq p \leq 4$ and $z_1, z_2 \in \mathbb{C}_1$. Also, $i_1 i_2 = i_2 i_1$ and $i_1^2 = i_2^2 = -1$.

Study of the bicomplex numbers had been started with the work of the Italian school of Segre [62], Spampinato [64, 65] and Scorza [66]. Their interest arose from the fact that such numbers offer a commutative alternative to the skew field of quaternion (both sets are real four dimensional spaces), and that in many ways they generalize complex numbers more closely and more accurately than quaternions do.

Corrado Segre read the work of Hamilton [24] on quaternions. Segre used some of the Hamilton's notation to develop his system of bicomplex numbers: Let i_1 and i_2 be square root of -1 that commute with each other. Then, presuming associativity of multiplication, the product $i_1 i_2$ must have $+1$ for its square. The algebra constructed on the basis $\{1, i_1, i_2, i_1 i_2\}$ is then nearly the same as James' tessarines [29].

The University of Kansas has contributed to the development of bicomplex analysis. In 1953, a Ph.D. student Riley had his thesis "*Contributions to the theory of functions of a bicomplex variable*" published in the Tohoku Mathematical Journal (2nd Ser., 5:132-165). Then, in 1991, emeritus professor Price [52] published his book on bicomplex numbers, multicomplex numbers, and their function theory. Another book developing bicomplex numbers and their applications is by Catoni, Bocaletti, Cannata, Nichelatti and Zampetti (2008).

Definition 1.1.1. A bicomplex number $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ is said to be *hyperbolic number* if $x_2 = 0$ and $x_3 = 0$. The set of all hyperbolic numbers is denoted by $\mathcal{H}$ and the plane of all hyperbolic numbers is called as the $\mathcal{H}$-Plane.

Algebra of Bicomplex Numbers

Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ and $\eta = y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4$ be any two bicomplex numbers. Define

1. **Addition**

$$\begin{aligned}
\xi + \eta &= (x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4) + (y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4) \\
&= (x_1 + y_1) + i_1(x_2 + y_2) + i_2(x_3 + y_3) + i_1 i_2(x_4 + y_4) \\
\xi + \eta &= (z_1 + i_2 z_2) + (w_1 + i_2 w_2) \\
&= (z_1 + w_1) + i_2(z_2 + w_2).
\end{aligned}$$

2. **Scalar Multiplication:** Let $\alpha \in \mathbb{C}_0$ be an arbitrary scalar, then

$$\begin{aligned}
\alpha\,\xi &= \alpha\,(x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4) \\
&= \alpha\, x_1 + i_1 \alpha\, x_2 + i_2 \alpha\, x_3 + i_1 i_2 \alpha\, x_4 \\
\alpha\,\xi &= \alpha\,(z_1 + i_2 z_2) \\
&= \alpha\, z_1 + i_2 \alpha\, z_2.
\end{aligned}$$

3. **Multiplication**

$$\begin{aligned}
\xi \times \eta &= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) \times (y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4) \\
&= (x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4) + i_1(x_1y_2 + x_2y_1 - x_3y_4 + x_4y_3) \\
&+ i_2(x_1y_3 + x_3y_1 - x_2y_4 - x_4y_2) + i_1i_2(x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1) \\
\xi \times \eta &= (z_1 + i_2z_2) \times (w_1 + i_2w_2) \\
&= (z_1w_1 - z_2w_2) + i_2(z_1w_2 + z_2w_1).
\end{aligned}$$

With these binary compositions, the bicomplex space $\mathbb{C}_2$ becomes a *commutative algebra with identity*.

1.1.3 Conjugations and Moduli of Bicomplex Numbers

We shall use the notation $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$ for the following sets:

$$\begin{aligned}
\mathbb{C}(i_1) &= \{x_1 + i_1x_2 \,:\, x_1,\, x_2 \in \mathbb{C}_0\}, \\
\mathbb{C}(i_2) &= \{x_1 + i_2x_2 \,:\, x_1,\, x_2 \in \mathbb{C}_0\}.
\end{aligned}$$

Every bicomplex number can be represented in six different form as given below:

$$\begin{aligned}
\xi &= (x_1 + i_1x_2) + i_2(x_3 + i_1x_4) = z_1 + i_2z_2 \\
&= (x_1 + i_2x_3) + i_1(x_2 + i_2x_4) = w_1 + i_1w_2 \\
&= (x_1 + i_1i_2x_4) + i_2(x_3 - i_1i_2x_2) = \mu_1 + i_2\mu_2 \\
&= (x_1 + i_1i_2x_4) + i_1(x_2 - i_1i_2x_3) = \kappa_1 + i_1\kappa_2 \\
&= (x_1 + i_1x_2) + i_1i_2(x_4 - i_1x_3) = \alpha_1 + i_1i_2\alpha_2 \\
&= (x_1 + i_2x_3) + i_1i_2(x_4 - i_2x_2) = \nu_1 + i_1i_2\nu_2
\end{aligned}$$

where $z_1, z_2, \alpha_1, \alpha_2 \in \mathbb{C}(i_1)$; $w_1, w_2, \nu_1, \nu_2 \in \mathbb{C}(i_2)$ and $\mu_1, \mu_2, \kappa_1, \kappa_2 \in \mathcal{H}$.

As the set of bicomplex numbers, $\mathbb{C}_2$ contains two imaginary units whose square is -1 and one hyperbolic unit whose square is 1, we can define three types of conjugations for the bicomplex numbers as analogous to the usual conjugation of the complex numbers:

Definition 1.1.2. The i_1-*conjugation* of a bicomplex number $\xi = z_1 + i_2 z_2$ is denoted by $\xi^\star$ and is defined as $\xi^\star = \overline{z_1} + i_2\,\overline{z_2}$, $\forall\, z_1, z_2 \in \mathbb{C}(i_1)$, $\overline{z_1}$ and $\overline{z_2}$ being complex conjugate of z_1 and z_2, respectively.

Definition 1.1.3. The i_2-*conjugation* of a bicomplex number $\xi = z_1 + i_2 z_2$ is denoted by $\widetilde{\xi}$ and is defined as $\widetilde{\xi} = z_1 - i_2\, z_2$, $\forall\, z_1, z_2 \in \mathbb{C}(i_1)$.

Definition 1.1.4. The j-*conjugation* of a bicomplex number $\xi = z_1 + i_2 z_2$ is denoted by ξ' and is defined as $\xi' = \overline{z_1} - i_2\,\overline{z_2}$, $\forall\, z_1, z_2 \in \mathbb{C}(i_1)$.

Here, some relations between the different types of conjugations of the bicomplex numbers are given as follows:

(i) If $\xi \in \mathbb{C}(i_1)$, i.e., $\xi = z_1$ and $z_2 = 0$, then $\xi = z_1 = x_1 + i_1 x_2$ and $\xi^\star = \overline{z_1} = x_1 - i_1 x_2 = \xi'$ and $\widetilde{\xi} = z_1 = \xi$.
Here we observe that the i_1-conjugation and j-conjugation, restricted to $\mathbb{C}(i_1)$, coincide with the usual conjugation of the complex numbers in $\mathbb{C}_1$.

(ii) If $\xi = w_1 \in \mathbb{C}(i_2)$, i.e. $w_1 = x_1 + i_2 x_2$, then $\xi^\star = w_1 = \xi$ and $\xi' = x_1 - i_2 x_2 = \widetilde{\xi}$.
In this case, both the j-conjugation and i_2-conjugation, restricted to $\mathbb{C}(i_2)$, coincide with the conjugation of the complex numbers in $\mathbb{C}_1$.

(iii) If $\xi = x_1 + i_1 i_2 x_2 \in \mathcal{H}$, then $\xi^\star = x_1 - i_1 i_2 x_2 = \widetilde{\xi}$ and $\xi' = \xi$.
Thus, the i_1-conjugation and the i_2-conjugation restricted to $\mathcal{H}$ coincide with the intrinsic conjugation in $\mathcal{H}$. Further, every hyperbolic number is fixed to the j-conjugation.

Properties of i_1-conjugation

Some of the properties of i_1 -conjugation, which are obtained by Rochon and Shapiro [58], are listed as follows:

(i) $(\xi + \eta)^\star = \xi^\star + \eta^\star$ and $(\xi - \eta)^\star = \xi^\star - \eta^\star$

(ii) $(\alpha\,\xi)^\star = \alpha\,\xi^\star$

(iii) $(\xi^\star)^\star = \xi$

(iv) $(\xi\,\eta)^\star = \xi^\star\eta^\star$

(v) $(\xi^{-1})^\star = (\xi^\star)^{-1}$, if ξ^{-1} exists

(vi) $\left(\dfrac{\xi}{\eta}\right)^\star = \dfrac{\xi^\star}{\eta^\star}$, if η^{-1} exists

(vii) $\xi^\star + \xi \in \mathbb{C}(i_2)$

(viii) $\widetilde{\xi} + \xi \in \mathbb{C}(i_1)$

(ix) $\xi' + \xi \in \mathcal{H}$.

They obtained analogous properties of i_2 -conjugation and j -conjugation also. Some particular bicomplex modulus can be defined with the help of each conjugate of a bicomplex number. We explain them as follows:

$$\begin{aligned}
|\xi|^2_{i_1} &= \xi \cdot \widetilde{\xi} = z_1^2 + z_2^2 \in \mathbb{C}(i_1) \\
|\xi|^2_{i_2} &= \xi \cdot \xi^\star = (|z_1|^2 - |z_2|^2) + 2Re(z_1\overline{z_2})i_2 \in \mathbb{C}(i_2) \\
|\xi|^2_{j} &= \xi \cdot \xi^\star = (|z_1|^2 + |z_2|^2) - 2Im(z_1\overline{z_2})j \in \mathcal{H}
\end{aligned}$$

Further, $|\xi \cdot \eta|^2_k = |\xi|^2_k \cdot |\eta|^2_k$, for all $k = i_1, i_2, j$.

1.1.4 Differences between Algebraic Structures of $\mathbb{C}_1$ and $\mathbb{C}_2$

Algebraic structure of $\mathbb{C}_2$ differs from that of $\mathbb{C}_1$ in many aspects. Few of such differences, which are concerned with our work, are mentioned below. For details, we refer to Price [52] and Srivastava [72].

(a) Non-invertible Elements in $\mathbb{C}_2$

An element which has an inverse is said to be non-singular (regular) and an element which does not have inverse is said to be singular element. There is unique element in $\mathbb{C}_0$ which does not have a multiplicative inverse, viz. the zero. Again there is only one element in $\mathbb{C}_1$ which does not have an inverse, viz., the element $0 + i_1 0$. However there are many singular elements in $\mathbb{C}_2$.

An element $\xi = z_1 + i_2 z_2$ is singular if and only if $|z_1^2 + z_2^2| = 0$ and it is non-singular if and only if $|z_1^2 + z_2^2| \neq 0$. Hence, there are uncountable singular elements in $\mathbb{C}_2$.

Theorem 1.1.1 ([52]). *A bicomplex number $\xi = x_1+i_1x_2+i_2x_3+i_1i_2x_4$ is singular if and only if either $(x_1 = -x_4\ ;\ x_2 = x_3)$ or $(x_1 = x_4\ ;\ x_2 = -x_3)$.*

Denote the set of all singular elements in $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ by $\mathbb{O}_0$, $\mathbb{O}_1$ and $\mathbb{O}_2$, respectively. Since $\mathbb{C}_0$ is isomorphic to a subset of $\mathbb{C}_1$ and $\mathbb{C}_1$ is isomorphic to subset of $\mathbb{C}_2$, it is customary to say simply that $\mathbb{C}_0$ is a subset of $\mathbb{C}_1$ and $\mathbb{C}_1$ is a subset of $\mathbb{C}_2$. Then, 0 is the common element of $\mathbb{O}_0$, $\mathbb{O}_1$ and $\mathbb{O}_2$. Thus, $\mathbb{O}_0 = \mathbb{O}_1 \subseteq \mathbb{O}_2$.

Due to the existence of singular elements, the division by a bicomplex number and the cancellation laws are restricted to non-singular bicomplex numbers.

(b) Hyperbolic Singular Elements

The set $\mathcal{H} \cap \mathbb{O}_2$ is the collection of all singular numbers which are hyperbolic numbers. These elements are called as *hyperbolic singular elements* and are denoted by $\mathbb{O}_h$. Therefore, $\mathbb{O}_h = \mathcal{H} \cap \mathbb{O}_2$.

(c) Non-trivial Idempotent Elements in $\mathbb{C}_2$

Besides the additive and multiplicative identities 0 and 1 there are exactly two non-trivial idempotent elements denoted by e_1 and e_2 defined as

$$e_1 = \frac{1+i_1i_2}{2} \quad \text{and} \quad e_2 = \frac{1-i_1i_2}{2}.$$

Note that $e_1 + e_2 = 0$ and $e_1e_2 = e_2e_1 = 0$. Obviously, $e_1^n = e_1$ and $e_2^n = e_2$, where n is a positive integer.

The last property says that e_1 and e_2 are zero divisors. Sometimes e_1 and e_2 called as orthogonal idempotents because their product is zero. Moreover, a bicomplex number $\xi = z_1 + i_2z_2$ can be *uniquely* expressed as a complex combination of e_1 and e_2 as follows:

$$\begin{aligned} \xi &= z_1 + i_2z_2 \\ &= (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 \\ &= {}^1\xi e_1 + {}^2\xi e_2, \end{aligned} \tag{1.1.2}$$

where ${}^1\xi = z_1 - i_1 z_2$ and ${}^2\xi = z_1 + i_1 z_2$ (for details cf. [72]). The coefficients ${}^1\xi$ and ${}^2\xi$ are called *idempotent components* and Equation (1.1.2) is known as *idempotent representation* of ξ.

Definition 1.1.5. The auxiliary spaces $\mathbb{A}_1$ (or first nil-plane) and $\mathbb{A}_2$ (or second nil-plane) are defined as follows:

$$\begin{aligned} \mathbb{A}_1 &= \{z_1 - i_1 z_2 \,:\, z_1, z_2 \in \mathbb{C}_1\} = \left\{{}^1\xi \,:\, \xi \in \mathbb{C}_2\right\}, \\ \mathbb{A}_2 &= \{z_1 + i_1 z_2 \,:\, z_1, z_2 \in \mathbb{C}_1\} = \left\{{}^2\xi \,:\, \xi \in \mathbb{C}_2\right\}. \end{aligned}$$

Both $\mathbb{A}_1$ and $\mathbb{A}_2$ are homeomorphic to $\mathbb{C}_1$. With the complex coefficients, e_1 and e_2 form a basis for $\mathbb{C}_2$. Thus, any bicomplex number $\xi \in \mathbb{C}_2$ can be represented as

$$\xi = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 \;=\; {}^1\xi e_1 + {}^2\xi e_2 \tag{1.1.3}$$

where ${}^1\xi = z_1 - i_1 z_2 \in \mathbb{A}_1$ and ${}^2\xi = z_1 + i_1 z_2 \in \mathbb{A}_1$ and to each pair of points $({}^1\xi, {}^2\xi) \in \mathbb{A}_1 \times_e \mathbb{A}_2$, there corresponds a unique point in $\mathbb{C}_2$. The elements ${}^1\xi$ and ${}^2\xi$, respectively called as first and second nil-factors (also known as idempotent components). The representation $\xi = {}^1\xi e_1 + {}^2\xi e_2$ is called as the idempotent representation of ξ.

Theorem 1.1.2 ([72]). *Let ξ and η be two arbitrary bicomplex numbers, p and q are real scalars and m and n are integers. Then*

(i) $p.\xi^m + q.\eta^n = [p.\,({}^1\xi)^m + q.\,({}^1\eta)^n]e_1 + [p.\,({}^2\xi)^m + q.\,({}^2\eta)^n]e_2$

(ii) $\xi^m \times \eta^n = [({}^1\xi)^m \times ({}^1\eta)^n]\, e_1 + [({}^2\xi)^m \times ({}^2\eta)^n]\, e_2.$

Proof. As e_1 and e_2 are two non-trivial idempotent elements of $\mathbb{C}_2$ and $e_1^n = e_1$ and $e_2^n = e_2$. So, $\xi^m = ({}^1\xi)^m e_1 + ({}^2\xi)^m e_2$ and $\eta^n = ({}^1\eta)^n\, e_1 + ({}^2\eta)^n\, e_2$.

Then,

$$\begin{aligned} (i) \qquad p\,\xi^m + q\,\eta^n &= p\,[({}^1\xi)^m e_1 + ({}^2\xi)^m e_2] + q\,[({}^1\eta)^n\, e_1 + ({}^2\eta)^n\, e_2] \\ &= [p({}^1\xi)^m e_1 + p({}^2\xi)^m e_2] + [q({}^1\eta)^n e_1 + q({}^2\eta)^n e_2] \\ &= [p({}^1\xi)^m + q({}^1\eta)^n]e_1 + [p({}^2\xi)^m + q({}^2\eta)^n]e_2 \end{aligned}$$

$$
\begin{aligned}
(ii) \qquad \xi^m \times \eta^n &= [({}^1\xi)^m e_1 + ({}^2\xi)^m e_2] \times [({}^1\eta)^n e_1 + ({}^2\eta)^n e_2] \\
&= [({}^1\xi)^m \times ({}^1\eta)^n] e_1 + [({}^2\xi)^m \times ({}^2\eta)^n] e_1
\end{aligned}
$$

Hence proved. $\square$

Remark 1.1.1 ([72])**.** In particular, we have

(i) ${}^1(\xi \pm \eta) = {}^1\xi \pm {}^1\eta$ and ${}^2(\xi \pm \eta) = {}^2\xi \pm {}^2\eta$
$\Rightarrow \xi \pm \eta = ({}^1\xi \pm {}^1\eta)e_1 + ({}^2\xi \pm {}^2\eta)e_2$

(ii) ${}^1(\xi \times \eta) = {}^1\xi \times {}^1\eta$ and ${}^2(\xi \times \eta) = {}^2\xi \times {}^2\eta$
$\Rightarrow \xi \times \eta = ({}^1\xi \times {}^1\eta)e_1 + ({}^2\xi \times {}^2\eta)e_2$

(iii) ${}^1\xi^m = ({}^1\xi)^m$ and ${}^2\xi^m = ({}^2\xi)^m$
$\Rightarrow \xi^m = ({}^1\xi)^m e_1 + ({}^2\xi)^m e_2$

(iv) If η is a non-singular bicomplex number, then

${}^1[\xi/\eta] = {}^1\xi/{}^1\eta$ and ${}^2[\xi/\eta] = {}^2\xi/{}^2\eta$
$\Rightarrow \xi/\eta = ({}^1\xi/{}^1\eta)e_1 + ({}^2\xi/{}^2\eta)e_2.$

Note that η is non-singular element if and only if both ${}^1\eta$ and ${}^2\eta$ are non–zero. i.e., ξ/η exists if and only if ${}^1\xi/{}^1\eta$ and ${}^2\xi/{}^2\eta$ exists.

(d) Principal Ideals

The principal ideals in $\mathbb{C}_2$ are determined by the idempotent elements e_1 and e_2 are denoted by $\mathbb{I}_1$ and $\mathbb{I}_2$, respectively and are defined as

$$
\begin{aligned}
\mathbb{I}_1 &= \{(z_1 - i_1 z_2)e_1 : z_1, z_2 \in \mathbb{C}_1\} = \left\{{}^1\xi e_1 : {}^1\xi \in \mathbb{A}_1\right\} = \{\xi e_1 : \xi \in \mathbb{C}_2\} \\
\mathbb{I}_2 &= \{(z_1 + i_1 z_2)e_2 : z_1, z_2 \in \mathbb{C}_1\}, = \left\{{}^2\xi e_2 : {}^2\xi \in \mathbb{A}_2\right\} = \{\xi e_2 : \xi \in \mathbb{C}_2\}.
\end{aligned}
$$

Both $\mathbb{I}_1$ and $\mathbb{I}_2$ are uniquely determined but their elements admits different representations. Also, note that $\mathbb{I}_1 \oplus \mathbb{I}_2 = \mathbb{C}_2$, we denote the set of all singular elements in $\mathbb{C}_2$ by $\mathbb{O}_2$.

(e) Non-trivial Zero Divisors in $\mathbb{C}_2$

The set of all complex numbers $\mathbb{C}_1$ forms a field but the set of all bicomplex numbers $\mathbb{C}_2$ does not form a field because $\mathbb{C}_2$ contains non-trivial divisors of zero. The existence of non-trivial zero divisors in the $\mathbb{C}_2$ is evident by the fact that $e_1 e_2 = 0$.

In fact, two bicomplex numbers are divisors of zero if and only if one of them is complex multiple of e_1 and other is the complex multiple of e_2. In other words, two elements of $\mathbb{C}_2$ are non-trivial divisors of zero if and only if one of them belongs to $\mathbb{I}_1 - \{0\}$ and other element belongs to $\mathbb{I}_2 - \{0\}$. An element $\xi \in \mathbb{C}_2$ will be a singular element if and only if $\xi \in \mathbb{I}_1 \cup \mathbb{I}_2$ and it will be non-singular iff $\xi \notin \mathbb{I}_1 \cup \mathbb{I}_2$. Hence, we can say that $\mathbb{O}_2 = \mathbb{I}_1 \cup \mathbb{I}_2$.

The norm in $\mathbb{C}_2$ is defined as follows:

$$\begin{aligned} \|\xi\| &= \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \\ &= \sqrt{|z_1|^2 + |z_2|^2} \\ &= \sqrt{\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}} \end{aligned} \tag{1.1.4}$$

It can be verified that the functional $||.||$ defined as above satisfies the four required postulates, viz.

(i) $\|\xi\| \geq 0, \ \forall \, \xi \in \mathbb{C}_2,$

(ii) $\|\xi\| = 0 \iff \xi = 0,$

(iii) $\|\alpha \, \xi\| = |\alpha| \, \|\xi\|, \ \forall \xi \in \mathbb{C}_2, \ \alpha \in \mathbb{C}_0,$

(iv) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|, \ \forall \, \xi, \, \eta \in \mathbb{C}_2.$

Equipped with this norm, $\mathbb{C}_2$ is a Banach space. Also, $\mathbb{C}_2$ is a commutative algebra.

Further, the norm of the product of two bicomplex numbers and the product of their norms are connected by means of the following inequality:

$$\|\xi \, . \, \eta\| \leq \sqrt{2} \, \|\xi\| \, . \, \|\eta\| \tag{1.1.5}$$

Definition 1.1.6 ([72]). An inequality (and inclusion relation) is said to be *best possible* if the slightest reduction of any parameter of the greater value (superset) disturbs the inequality (inclusion).

The inequality given in (1.1.5) is the best possible. For this reason, we call $\mathbb{C}_2$ as *modified complex Banach algebra* [52].

Remark 1.1.2. From the above inequality we obtain the following identities:

$$\|e_p \cdot e_p\| = \|e_p\| \leq \frac{\sqrt{2}}{2} = \sqrt{2}\,\|e_p\|\,\|e_p\|, \qquad p = 1, 2. \tag{1.1.6}$$

Corollary 1.1.1. If $\xi \in \mathbb{C}_2$, $z \in \mathbb{C}(i_1)$ or $\mathbb{C}(i_2)$. Then

$$\|z\,\xi\| = |z|\,\|\xi\|.$$

Also, note that if ξ is in a nil-plane ($\mathbb{A}_1$ or $\mathbb{A}_2$), then $\|\xi\| = (1/\sqrt{2})\,|^k\xi|$, $k = 1, 2$. Thus, the distance in a nil-plane, measured by the absolute value of the complex number $^k\xi$, ($k = 1, 2$), differs from the distance in the bicomplex space by a constant value of $1/\sqrt{2}$. If ξ is a complex number, then $\|\xi\| = |^k\xi|$.

1.2 Literature Review

In this section, a brief literature of the analysis of the set of bicomplex numbers done by the researchers from different aspects are given.

Alpay et al. [1] and Luna-Elizarraras et al. [42] provide some elementary functions, viz., polynomials, exponentials and trigonometric functions of bicomplex numbers in algebra. They also defined the inverse functions for these function which are not possible in quaternions. They have shown that any two holomorphic functions admit derivative in the form of the bicomplex numbers. The analysis of bicomplex holomorphic functions were developed with bicomplex scalars and to compare with the quaternionic scalars which were studied by Teichmuller [75].

The theory of bicomplex numbers is relatively very much young and in particular in the field of bicomplex functional analysis. The concept of bicomplex functional analysis was given by Gervais et. al. [21, 22]. Finite-dimensional modules on bicomplex space was studied by them and proved some results on this concept by using square matrices with bicomplex entries, linear operators, self-adjoint operators and orthogonal bases, based on the spectral decomposition theorem.

In recent study of the bicomplex space, people are working in many areas of analysis and applications, viz., complex dynamics, functional analysis, polynomials with bicomplex variables, bicomplex sequence spaces, fractals, matrices, bicomplex Riemann zeta function, bicomplex hyperfunctions, bicomplex gamma and beta functions, etc.

In 2006, Goyal et. al. [23] introduced the bicomplex version of the gamma and beta functions. They studied the properties of the gamma and beta functions for the bicomplex variable. They studied the Legendre duplication formula, Gauss multiplication theorem and Binomial theorem along with the holomorphicity of these functions. In the same year, Javtokas [30] have also studied the Hurwitz zeta functions satisfying the complexified Cauchy-Riemann equations.

In 2007, Srivastava and Srivastava [70] have started the work on the sequences spaces in the bicomplex numbers. Nigam [47] investigated a particular class denoted as B' of bicomplex holomorphic functions. The class of B' is the subclass of the class defined by Srivastava and Srivastava [70]. This class of functions have been shown as an Gelfand algebra and also studied the invertible and quasi-invertible elements. Kumar and Srivastava [38] have studied the poles of Riemann zeta function. They investigate the properties of the poles of the Riemann zeta function in the bicomplex variable using its idempotent components.

In 2011, Kumar et. al. [36] had generalize the fundamental theorems of functional analysis in the framework of bicomplex modules. Colombo et. al. [12, 13] had studied the singularities of the holomorphic functions of the bicomplex space. Junliang and Pingping [32] introduced a series of bicomplex representation methods of quaternion division algebra. They gave a new multiplication method of quaternion matrices. Under this new concept, many quaternion algebra problems can be solved using complex algebra. Kumar and Srivastava [40] investigated the entireness of the Dirichlet series and obtained the conditions for the representation of the Dirichlet series by the entire function of bicomplex variable.

In 2012, Shapiro et. al. [42] have discussed about the elementary functions of the bicomplex variables and their properties. They have studied the function f from $\mathbb{C}^2$ to $\mathbb{C}^2$ and the complexified Cauchy-Riemann system of equations. Vajiac and Vajiac [76] have extended their previous work on the hyperfunction theory. They worked upon space of analytic functions of one or several bicomplex variables to multicomplex scene of $\mathbb{C}_n$.

The study of multicomplex dynamics was started by Pelletier and Rochon [50] in 2009. They have developed the hypercomplex 3D fractals generated from multicomplex dynamics. They gave the generalization of the Mandelbrot and Julia sets for the multicomplex numbers, particulary for bicomplex numbers. Further they studied the multicomplex version of the so-called Fatou-Julia theorem. In 2013, Wang and Song [77] had studied the generalized Mandlebrot-Julia sets (M-J sets) for the bicomplex space. They studied the connectedness of the generalized Mandlebrot-Julia sets, the properties of the generalized Tetrabrot, and the connection between the generalized Mandlebrot sets and its corresponding generalized Julia sets for bicomplex numbers. Campos and Kravchenko [6] introduced the bicomplex analogue for the pseudo-analytic (or generalized analytic) functions. They developed the theory of bicomplex pseudoanalytic formal powers. Further. they provide the fundamental solutions for the Darboux Schrodinger operators.

In 2014, Struppa et. al. [73] had interpreted the different aspects of derivative and holomorphy of the bicomplex function. Charak and Sharma [10] studied the Zalcman lemma for the bicomplex numbers. They focused on the dynamics of the bicomplex meromorphic functions. They explained the theory to see whether the results from one variable theory: normal families of meromorphic functions on plane-hold for the families of bicomplex meromorphic functions. They obtained the bicomplex analogue of Zalcman principle and proved the Lappan's five-point theorem in bicomplex variable.

Kumar [39] investigated a class T of entire functions which are represented by the Dirichlet series on the modified Banach algebra structure on the bicomplex space. He has proved that T is neither a division algebra nor a B^*-algebra and found that invertible and quasi-invertible elements of class T. Colombo et. al. [14] have introduced the functional calculus for bicomplex linear bounded operators. It is based on the decomposition of bicomplex numbers and of the linear operators by using the idempotent elements.

In 2015, Kumar and Singh [41] have studied the hyperbolic norm ($\mathbb{D}$-valued) norm on bicomplex module ($\mathbb{BC}$-module). They have studied the homomorphism in the ring $\mathbb{BC}$ and hence the maximal ideals in $\mathbb{BC}$ by using the fundamental theorem of ring homomorphism. They have also studied the spectrum of an element of bicomplex C^*-algebra. They have proved that the spectrum of a bicomplex bounded linear operator T on a bicomplex Banach module X is unbounded.

In 2016, Kumar et. al. [37] have studied bicomplex version of weighted Hardy spaces. They have generalized the results which holds for the classical weighted Hardy spaces. Kim and Shon [35] used the different forms of conjugations of the bicomplex numbers to study the properties of them. Saini and Kumar [61] have studied some of the fundamental theorem on the functional analysis with bicomplex and hyperbolic scalars. They verified some properties of linear functionals on topological hyperbolic and topological bicomplex modules. The hyperbolic and bicomplex analogues of the uniform boundedness principle, the open mapping theorem, the closed graph theorem and the Hahn Banach separation theorem are also discussed in detail. In 2017, Choi et.al. [11] elaborated the fixed point theorems with the weakly compatible mappings in the bicomplex valued metric spaces.

1.3 Order Relations and Order Topology

Definition 1.3.1 (**Partially Ordered Set**). Let P be a non–empty set. A partial order relation $\leq$ in P is a relation which satisfies the following properties:

(i) $x \leq x, \quad \forall \in P$ (Reflexive)

(ii) $x \leq y$ and $y \leq x \;\Rightarrow\; x = y$ (Anti-symmetry)

(iii) $x \leq y$ and $y \leq z \;\Rightarrow\; x \leq z.$ (Transitivity)

The non-empty set P with partial order relation is called as a partially order set, (or *poset*).

Example 1.3.1. The set $\mathbb{N}$ of all positive integers is partially ordered with respect to ordering ($<$) defined as "$m < n$, if m divides n".

Example 1.3.2. Set inclusion is a partial ordering in any class of sets.

Definition 1.3.2. Two elements x and y of a poset are called *comparable* if one of them is less than or equal to the other with respect to the partial ordering. In other words, x and y are comparable if either $x \leq y$ or $y \leq x$.

Definition 1.3.3. A partially ordered set P is said to be *totally ordered set* if any two elements of P are comparable. Total ordered set is also known as *linearly ordered set* or a *chain*.

Example 1.3.3. Consider the relation $\leq$ on the real line defined as "$x \leq y$ if x is less than or equal to y". It is a total order, known as usual ordering on $\mathbb{R}$.

Remark 1.3.1. The ordering in Example 1.3.1 is in fact total ordering, where as the ordering in Example 1.3.2 is not a total ordering. If every chain in a poset has an upper bound then the poset contains a maximal element. Also, the restriction on the total order relation is a total order relation.

Definition 1.3.4 (**Dictionary Order Relation**). Assume that A and B are two sets with order relations $\leq_A$ and $\leq_B$ respectively. Define an order relation $\preceq$ on $A \times B$ as: $a_1 \times b_1 \preceq a_2 \times b_2$.

if either (i) $a_1 <_A a_2$

or (ii) $a_1 =_A a_2$ and $b_1 \leq_B b_2$.

It is called *dictionary order relation* or *lexicographical order relation* on $A \times B$.

Definition 1.3.5 (The Order Topology). Let X be a set with suitably defined ordering. Suppose that X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(i) All open intervals $(a,\, b)$ in X.

(ii) All intervals of type $[a_0,\, b)$, where a_0 is the smallest element (if exists) of X.

(iii) All intervals of type $(a,\, b_0]$, where b_0 is the largest element (if exists) of X.

The collection $\mathcal{B}$ is a basis for the *order topology* on X.

Example 1.3.4. The order topology on $\mathbb{N}$ is the discrete topology, for which $\{1\} = \{n \in \mathbb{N} : n < 2\} = (-\infty, 2)$ and for $n > 1$ $\{n\} = (n-1, n+1)$.

Example 1.3.5. Let X_1 denote the topological space $\mathbb{C}_0$ with discrete topology and let X_2 be $\mathbb{C}_0$ with usual topology. Then the product topology on $X_1 \times X_2$ is same as the lexicographic order topology on $\mathbb{C}_0^2$.

Definition 1.3.6. A space X is *first countable* at $x \in X$ if there is a countable base at x. A space is *first countable* if it is first countable at each point of X.

Theorem 1.3.1 ([45])**.** *Metric spaces are first countable.*

Definition 1.3.7 (Hausdorff Space). A non empty space X is said to be Hausdorff space if for any two elements $x,\, y \in X$, there exists two disjoint open subsets U and V such that $x \in U$ and $y \in V$.

Theorem 1.3.2. *A first countable space in which each sequence converges to at most one point is Hausdorff.*

Theorem 1.3.3. *Let $\{x_n\}$ be a sequence in a first countable space. Then*

(i) x is a limit point of the sequence.

(ii) there exists a subsequence converging to x.

Theorem 1.3.4 ([45])**.** *Every order topology is Hausdorff.*

Remark 1.3.2. $\mathbb{A}_1$ and $\mathbb{A}_2$ are Hausdorff space with respect to the order topology generated by lexicographic order relation $\prec$.

The topological study of the bicomplex space was initiated by Srivastava [72] and developed three topologies on the bicomplex space with the help of certain known topologies on the component spaces. He has also compared these topologies. here we are discussing the work done by Srivastava in detail.

Definition 1.3.8 (**Topological Space**)**.** Let X be a non-empty set. A class τ of subsets of X is called a *topology* on X if it satisfies the following conditions:

(i) $X, \emptyset \in \tau$.
(ii) The union of every class of sets in τ is a set in τ.
(iii) The intersection of every finite class of sets in τ is a set in τ.

A *topological space* consists of two objects: a non-empty set X and a topology τ on X.

Definition 1.3.9. The *idempotent parts* of a set S of bicomplex numbers are denoted as ${}^1S$ and ${}^2S$, and are defined as

$$ {}^1S = \{ {}^1\xi \, : \, \xi \in \mathbb{C}_2 \} \ \text{ and } \ {}^2S = \{ {}^2\xi \, : \, \xi \in \mathbb{C}_2 \}. $$

The idempotent parts are nothing but the auxiliary complex spaces $\mathbb{A}_1$ and $\mathbb{A}_2$, (for details cf. [52]) which are defined by in the Equation (1.1.3).

Definition 1.3.10. *Cartesian idempotent set* D (say) determined by the sets S and T of complex numbers is the set of bicomplex numbers denoted by $S \times_e T$ and is defined as

$$ S \times_e T \;\; = \;\; \{\xi = w_1 e_1 + w_2 e_2 \, : \, w_1 \in \mathbb{A}_1, \, w_2 \in \mathbb{A}_2\}. $$

For convenience, we can denoted the element $w_1e_1+w_2e_2$ by $w_1 \times_e w_2$, the Cartesian product of the elements of the complex sets of the Cartesian idempotent set.

Definition 1.3.11. *Cartesian complex set* of bicomplex numbers, determined by the sets S and T of complex numbers is denoted by $S \times T$ and is defined as

$$S \times T \;=\; \{\xi = z_1 + i_2 z_2 \,:\, z_1 \in S,\, z_2 \in T\}.$$

In particular, $\mathbb{C}_1 \times \mathbb{C}_1 = \mathbb{C}_2$.

Definition 1.3.12. An *open circular disc* in $\mathbb{C}_1$ with centre at z and the radius r is denoted by $S(z\,;\,r)$ and is defined as

$$S(z\,;\,r) \;=\; \{w \,:\, w \in \mathbb{C}_1,\, |\,z - w\,| < r\}.$$

Definition 1.3.13 (Norm Topology on $\mathbb{C}_2$). The topology generated by the norm defined in the Equation (1.1.4) is called as *norm topology* and is denoted by τ_1 (cf. [72]).

A basis element of the norm topology is the set $\mathbb{B}_1$ of all open balls, where an open ball in $\mathbb{C}_2$ with centre at ξ and radius r is denoted by $B(\xi;\, r)$ and is defined as

$$B(\xi;\, r) \;=\; \{\eta \,:\, \eta \in \mathbb{C}_2,\, ||\,\xi - \eta\,|| < r\}. \tag{1.3.1}$$

Definition 1.3.14 ([72]). An *open complex discus* (or C-discus) with centre at $\xi = z_1 + i_2 z_2$ and associated radii $r_1,\, r_2$ is denoted as $C(\xi\,;\, r_1,\, r_2)$ and is defined as

$$C(\xi;\, r_1,\, r_2) \;=\; \{\eta = w_1 + i_2 w_2 \,:\, \eta \in \mathbb{C}_2\,;\, |\,z_1 - w_1\,| < r_1\,,\, |\,z_2 - w_2\,| < r_2\}$$

It may be noted $C(\xi\,;\, r_1,\, r_2)$ is the Cartesian complex set by the open circular discs $S(z_1;\, r_1)$ and $S(z_2;\, r_2)$. The family of C–discuses is denoted by $\mathbb{B}_2$.

Definition 1.3.15 (Complex Topology on $\mathbb{C}_2$). The topology generated by $\mathbb{B}_2$ on $\mathbb{C}_2$ is called as *complex topology* and is denoted by τ_2 (cf. [72]).

Definition 1.3.16. An *open idempotent discus* (or D-discus) with centre at ξ and associated radii r_1 and r_2 is denoted by $D(\xi;\, r_1,\, r_2)$ and is denoted as the

Cartesian idempotent set determined by open circular discs $S({}^1\xi;\, r_1)$ in $\mathbb{A}_1$ and $S({}^2\xi;\, r_2)$ in $\mathbb{A}_2$. Thus,

$$D(\xi;\, r_1,\, r_2) \;=\; \{\eta \,:\, \eta \in \mathbb{C}_2,\, |\,{}^1\xi - {}^1\eta\,| < r_1,\, |\,{}^2\xi - {}^2\eta\,| < r_2\}$$

Denote the family of all D–discuses in $\mathbb{C}_2$ by $\mathbb{B}_3$.

Definition 1.3.17 (**Idempotent Topology on** $\mathbb{C}_2$)**.** The topology generated by $\mathbb{B}_3$ on $\mathbb{C}_2$ is called as *idempotent topology* and is denoted by τ_2 (cf. [72]).

Definition 1.3.18. A proper inclusion relation is said to be *best possible* if the slightest reduction of any parameter of the superset disturb the inclusion.

Remark 1.3.3. For some statements and theorems, we shall denote root square mean and arithmetic mean of non-negative numbers r_1 and r_2 by $r^\star$ and r'', wherever it will be required. Further, the case $0 < r_1 \leq r_2$ will be discussed (the result can be proved for other possible cases).

Theorem 1.3.5 ([72])**.** *For given* $\xi \in \mathbb{C}_2$

(i) $B(\xi\,;\, r) \subset C(\xi\,;\, r,\, r)$*,*

(ii) $C(\xi;\, r_1,\, r_2) \subset B(\xi;\, \sqrt{2}\,r^\star)$*.*

The inclusion relations are best possible.

Remark 1.3.4. Note that if η is a point in the discus $C(\xi;\, r_1,\, r_2)$ (or is the ball $B(\xi;\, r)$) one can always find a C–discus $C(\xi;\, s_1,\, s_2)$ (or ball $B(\eta;\, s)$) centered at η and sufficiently small size so as to contain original discus (or original ball).

Therefore, from the Theorem 1.3.5, we can conclude that for every point η of a C–discus, there corresponds a ball containing η and contained in the C–discus and vice-versa. In other words, For every point belong to the basis element B of τ_1 such that B is containing that point and contained in $\mathbb{C}_2$. This implies that

Theorem 1.3.6 ([72])**.** *Norm topology and complex topology on* $\mathbb{C}_2$ *are equivalent.*

Theorem 1.3.7 ([72])**.** *For every* $\xi \in \mathbb{C}_2$

(i) $D(\xi;\, r_1,\, r_2) \subset C(\xi;\, r^\star,\, r^\star)$*,*

(ii) $C(\xi;\, r_1,\, r_2) \subset D(\xi;\, 2r^\star,\, 2r^\star)$*.*

These inclusions are proper.

Theorem 1.3.8 ([72]). *For every $\xi \in \mathbb{C}_2$*

(i) $D(\xi;\, r_1,\, r_2) \subset C(\xi;\, r'',\, r'')$,

(ii) $C(\xi;\, r_1,\, r_2) \subset D(\xi;\, 2r'',\, 2r'')$.

These inclusions are proper.

Remark 1.3.5. The inclusion relations obtained above are important at their places. However, as $r'' < r^\star$, so it can be directly obtained that

$$C(\xi;\, r'',\, r'') \subset C(\xi;\, r^\star,\, r^\star) \text{ and } D(\xi;\, r'',\, r'') \subset D(\xi;\, r^\star,\, r^\star). \tag{1.3.2}$$

Further, note that these two inclusions are best possible.

Theorem 1.3.9 ([72]). *The inclusions*

(i) $D(\xi;\, r_1,\, r_2) \subset C(\xi;\, r'',\, r'')$,

(ii) $C(\xi;\, r_1,\, r_2) \subset D(\xi;\, 2r'',\, 2r'')$

are best possible.

Theorem 1.3.10 ([72]). *The complex topology and idempotent topology are same.*

Remark 1.3.6 ([52]). In the bicomplex space we know that

$$B\left(\xi;\, \frac{r}{\sqrt{2}}\right) \subset D(\xi;\, r_1,\, r_2) \subset B(\xi;\, r^\star). \tag{1.3.3}$$

In particular, if $r_1 = r_2$, the inclusion relation (1.3.3) becomes

$$B\left(\xi;\, \frac{r}{\sqrt{2}}\right) \subset D(\xi;\, r,\, r) \subset B(\xi;\, r). \tag{1.3.4}$$

Remark 1.3.7. The first part of the inclusion (1.3.3), $B(\xi;\, r_1/\sqrt{2}) \subset D(\xi;\, r_1,\, r_2)$ or equivalently $B(\xi;\, r_1) \subset D(\xi;\, \sqrt{2}\, r_1,\, \sqrt{2}\, r_2)$, $0 < r_1 \leq r_2$ is not the best possible inclusion of a given ball in a D-discus, but if $r_1 - r_2 = 2k$. Then

$$B(\xi;\, r_1) \subset D(\xi;\, \sqrt{2}\, r_1,\, \sqrt{2}(r_2 - k)) \subset D(\xi;\, \sqrt{2}\, r_1,\, \sqrt{2}\, r_2) \tag{1.3.5}$$

is a better inclusion.

Note that, $\eta \in B(\xi;\, r_1)$

$\Rightarrow ||\,\xi - \eta\,|| < r_1$

$\Rightarrow |\,{}^1\xi - {}^1\eta\,|^2 + |\,{}^2\xi - {}^2\eta\,|^2 < 2\,r_1^2$ (from the Equation (1.1.4))

Therefore, $B(\xi;\, r_1) \subset D(\xi;\, \sqrt{2}\,r_1,\, \sqrt{2}\,r_1)$ is much better result. In fact, this is a best possible inclusion. Again as

$$D(\xi;\, r_1,\, r_2) \subset C(\xi;\, r'',\, r'') \subset B(\xi;\, \sqrt{2}\,r'')$$

Note that these inclusions are best possible, yet $D(\xi;\, r_1,\, r_2) \subset B(\xi;\, \sqrt{2}\,r'')$ is not best possible inclusion.

1.4 Orderability of Topological Spaces

Order is a concept as old as the idea of numbers, and much of early mathematics was devoted to construct and study various subsets of the real line.

The class of linearly ordered topological spaces, (i.e. spaces equipped with a topology generated by a linear order) contains many important spaces, like the set of real numbers, the set of rational numbers, the set of bicomplex numbers, the ordinals, etc. The orderability of topological spaces is a very important topic, as defined on whether a topological space admits a linear order which generates a topology equal to the topology of the space. A general solution for this problem was first given by Dalen and Wattel [15] in 1973.

For a subset A of a set X, it is possible that the relative topology $\tau(<)|_A$ on A is not coinciding with the open interval topology $\tau(<|_A)$ induced on A by restricted ordering. Cech [9] had introduced the concept of *generalized ordered spaces* or GO-spaces. He also introduced the study of subspaces of LOTS.

In the survey paper by Purisch [53], one can find the problems on topological spaces which deal, with orderability problem i.e., orderable spaces and the problems dealing with the sub-orderability problem, i.e., the GO-spaces. The concept of orderability theorem for compact connected spaces in [27] explains that any compact connected space is orderable iff it has exactly two no-cut points.

The metrization theory for the linearly ordered topological spaces is simple by some means: Lutzer [43] have shown that a linearly ordered topological space is metrizable if and only if it has a G_δ-diagonal. This theorem does not hold for GO-spaces, as the examples of the Sorhenfrey and Micheal lines.

1.5 Nets and Subnets

In metric spaces, properties such as continuity, closure, and compactness can be stated completely in terms of sequences. This breaks down in general topological spaces, where sequences can't even tell whether a set is closed. Sequences suffice to handle all convergence problems in space that satisfies first axiom of countability, in particular all metric spaces.

Certain spaces (e.g. Hilbert spaces in the weak topology) require the more general notions of nets, and some complicated convergence arguments (refinement of sequences by Cantor's diagonal principle) are effectively trivialized by the use of universal nets. Nets are also called as *generalized sequences* in the literature.

Modern general topology, and in particular its manifestations in the so-called weak topologies of certain function spaces, taught us that limits of sequences are no longer a strong enough tool for analysis. In classical analysis we learn that a set is closed if and only if it contains the limits of all convergent sequences in it, but there are many important and useful topological spaces for which that statement is not true.

If, however, sequences are replaced by "generalized sequences" (the streamlined word is "nets"), and correspondingly, ordinary limits of sequences are replaced by Moore -Smith limits of nets, the classical proofs work again, often with no changes except terminological ones, and they yield results just as useful as the classical ones. (Example: a set is closed if and only if it contains the Moore-Smith limits of all convergent nets in it.).

The Moore of the seminal 1922 Moore-Smith paper is the great Moore [46] (one of the teachers of Moore) and the Smith is the otherwise largely forgotten H. L. Smith. The theory of nets was further developed by Birkhoff [4]. Hence nets at first were called Moore-Smith sequences. The word "net" was first used by Kelley [33] in 1950. This was followed by Cartan's discovery of filters in 1937 [7], which, while they look nothing similar to sequences, can also be used to describe those concepts.

Definition 1.5.1 (Directed Set). A directed set is a pair $(D, \geq)$ where D is a non - empty set and $\geq$ a binary relation on D satisfying (for details cf. [3]):

(i) $m \geq n$ and $n \geq p \Rightarrow m \geq p$, $\forall\, m, n, p \in D$,

(ii) $n \geq n$, $\forall\, n \in D$,

(iii) $\forall\, m,\, n \in D$, $\exists\, p \in D$ such that $p \geq m$ and $p \geq n$.

In other words, we can say that A directed set is a set D with a pre-order relation (i.e. a reflexive and transitive binary relation) such that pair of every two elements have an upper bound.

Remark 1.5.1. The composition of finite number of directed sets is a directed set.

Remark 1.5.2. We don't require that a pair of elements has a least upper bound, we just require that some upper bound exists.

Example 1.5.1. Every linearly ordered set (such as the $\mathbb{N}$, the set of natural numbers with the usual order) is a directed set.

Example 1.5.2. Any collection of sets that is closed under binary intersections is a directed set when ordered by reverse inclusion, i.e., $X \leq Y$ iff $Y \subseteq X$.

In particular, given any point of a topological space, the collection of all neighbourhoods of x ordered by reverse inclusion is a directed set, which we write as $\mathcal{N}(x)$.

Example 1.5.3. If $(D, \trianglelefteq)$ and $(E, \propto)$ are directed sets, then so is their product $D \times E$ ordered by $\preccurlyeq$ as $(d_1, c_1) \preccurlyeq (d_2, c_2)$ if and only if $d_1 \trianglelefteq d_2$ in D and $c_1 \propto c_2$ in E.

Definition 1.5.2 (Net). A net in a set X is a function $S : D \longrightarrow X$, where D is a directed set.

Theorem 1.5.1. *A net in a Hausdorff topological space has at most one limit.*

Definition 1.5.3 (Subnet). The subnet of a net $S : D \to X$ is the composition $S \cdot \phi : M \to D$ is an increasing cofinal function from some directed set M to D. That is

(i) $\phi(a_1) \leq_D \phi(a_2)$ when $a_1 \leq_M a_2$

(ii) for each $k \in D$, there is some $p \in M$ such that $\phi(p) \leq_D k$.

Theorem 1.5.2. *Let $(X, \mathcal{T})$ be a topological space and suppose that $M \subseteq X$. Then M is closed if and only if $\lim_{x \in A} x_\alpha \in M$ for all the convergent nets $\{x_\alpha\} \subseteq M$.*

Theorem 1.5.3. *A topological space X is compact if and only if every net in X has a convergent subnet.*

1.6 Filters and Ultrafilters

In order to study the convergence in general topological spaces, the concept of sequences (i.e., the functions defined on the natural numbers $\mathbb{N}$), are too restrictive. There are two generalizations, one is the concept of a *filter* introduced by Cartan [7, 8] and the other is the concept of *net* introduced by Moore and Smith [46]. In this section, we have given some introduction to the concept of filters.

Filters have applications beyond just generalizing the notion of convergent sequences: in completions and compactifications, in Boolean algebra and in mathematical logic, where ultrafilters are arguably the single most important (and certainly the most elegant) single technical tool.

Definition 1.6.1 (Filter). A filter $\mathcal{F}$ on a set X is the subset of the power set $P(X)$ satisfying the conditions:

(i) $\emptyset \notin \mathcal{F}$ and $\emptyset \neq \mathcal{F}$,

(ii) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$,

(iii) If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Remark 1.6.1. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on a set X, we say that $\mathcal{F}_2$ is finer than $\mathcal{F}_1$ if $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

The set of all filters on a set X with inclusion relation is a partially ordered set.

Definition 1.6.2. For any non-empty subset Y of the set X, the collection $\mathcal{F}_Y = \{A \,|\, Y \subset A\}$ of all subsets containing Y is a filter on X. This type of filter is called as *principal filter*.

Example 1.6.1. Every filter on a finite set is principal filter.

Definition 1.6.3 (Filter Basis). A basis $\mathbb{B}$ for a filter on a set X is a subset of $P(X)$, satisfying the conditions:

(i) $\emptyset \notin \mathbb{B}$ and $\emptyset \neq \mathbb{B}$

(ii) If $A \in \mathbb{B}$ and $B \in \mathbb{B}$, there is a set $C \in \mathbb{B}$ such that $C \subseteq A \cap B$.

Definition 1.6.4 (Ultrafilter). A filter $\mathcal{F}$ on a set X is an ultrafilter if for every $S \subset X$, either $S \in \mathcal{F}$ or $X \setminus S \in \mathcal{F}$. Further, A filter $\mathcal{F}$ on X is *maximal* if there is no filter $\mathcal{F}'$ on X such that $\mathcal{F} \subset \mathcal{F}'$ and $\mathcal{F} \neq \mathcal{F}'$.

Remark 1.6.2. Let x a point in X. Consider a collection as follows:

$$\mathcal{U}_x = \{U \setminus X \,:\, x \in U\}. \tag{1.6.1}$$

Clearly, $\mathcal{U}_x$ is an ultrafilter in X. This is called as *constant ultrafilter* at x. If $(X, \mathcal{T})$ is a topological space, then $\mathcal{U}_x$ is convergent to x.

Remark 1.6.3. The ultrafilters are not always convergent.

Example 1.6.2. Let $\mathbb{N}$ be the set of non-negative integers. Suppose that $\mathbb{N}$ is equipped with discrete topology (in which every subset is open). Consider the collection $\mathcal{F}$ as follows:

$$\mathcal{F} \; = \; \{F : F \subset \mathbb{N}, \;\; \mathbb{N} \setminus F \;\; \text{is finite}\} \tag{1.6.2}$$

Obviously, $\mathcal{F}$ is a filter.

Let $\mathcal{U}$ be any ultrafilter such that $\mathcal{U} \supset \mathcal{F}$. Since $\mathbb{N}$ is equipped with discrete topology. Therefore, the only convergent ultrafilters are the constant ultrafilters on $\mathbb{N}$.

Now as $\mathbb{N} \setminus \{n\} \in \mathcal{F}$, $n \in \mathbb{N}$ (from Equation (1.6.2))

$\Rightarrow \;\; \mathbb{N} \setminus \{n\} \in \mathcal{U}$

$\Rightarrow \;\; \{n\} \notin \mathcal{U}$.

Therefore, $\mathcal{U}$ cannot be a constant ultrafilter.

Definition 1.6.5 (Fréchet Filter). For any infinite set X, the family of all cofinite subsets of X is a filter on the finite set X and is called as *Fréchet Filter*.

Definition 1.6.6. A filter $\mathcal{F}$ on a non-empty set is said to be *free* if

$$\bigcap_{A \in \mathcal{F}} A = \emptyset.$$

Lemma 1.6.1. Every filter is contained in an ultrafilter.

Theorem 1.6.1. *For a filter $\mathcal{F}$ of X, the following are equivalent:*
(i) For every subset Y on X, $\mathcal{F}$ contains exactly one of Y or $X \setminus Y$.
(ii) $\mathcal{F}$ is an ultrafilter.

Remark 1.6.4. Condition (ii), of Theorem 1.6.1 says that a filter basis with the relation $\supseteq$ is a directed set.

Lemma 1.6.2. If $f : X \to Y$ is a function and $\mathcal{F}$ is a filter on X, then $A \in f(\mathcal{F})$ if and only if $f^{-1}(A) \in \mathcal{F}$

Definition 1.6.7 (**Accumulation Point of a Filter**)**.** A point x is said to be accumulation point of filter $\mathcal{F}$ if every neighborhood of x meets every set in $\mathcal{F}$. The set of all accumulation points of the filter $\mathcal{F}$ is denoted by $\overline{\mathcal{F}}$

Definition 1.6.8 (**Limit Point**)**.** A point x is said to be limit point of filter $\mathcal{F}$ if every neighborhood of x is member of $\mathcal{F}$.

Theorem 1.6.2. *In a Hausdorff space, every convergent filter has exactly one limit point.*

Theorem 1.6.3. *A function $f : X \to Y$ is continuous if and only if for every filter $\mathcal{F}$ and every limit point $x \in X$, $f(x)$ is a limit point of $f(\mathcal{F})$.*

Theorem 1.6.4. *A filter $\mathcal{F}$ on a product space $Y = \prod_{i \in I} X_i$ converges to x, if and only if each filter $\mathcal{F}_i = \pi_i(\mathcal{F})$ converges to $x_i = \pi_i(x)$.*

1.7 Paranorm and Orlicz Function

The concept of paranormed sequences was studied in detail by Maddox [44]. The notion of difference sequence spaces was introduced by Kizmaz [34] as follows:

$$X_\Delta \;=\; \{x = (x_n) : (\Delta x_n) \in X\}, \tag{1.7.1}$$

where $X = c, c_0, \ell^\infty$ and $\Delta x_n = x_n - x_{n+1}$.

The *Orlicz function* $\mathcal{M}$ is defined as $\mathcal{M} : [0, \infty) \to [0, \infty)$. It is continuous, non-decreasing and $\mathcal{M}(0) = 0, \mathcal{M}(x) > 0$ for $x > 0$. Also, for $\lambda \in (0, 1)$ it satisfies the condition

$$\mathcal{M}(\lambda x + (1 - \lambda)y) \;\leq\; \lambda\mathcal{M}(x) + (1 - \lambda)\mathcal{M}(y), \tag{1.7.2}$$

and if the condition of convexity of Orlicz function $\mathcal{M}$ is replaced by the condition $\mathcal{M}(x + y) \leq \mathcal{M}(x) + \mathcal{M}(y)$, then the function $\mathcal{M}$ is called *modulus function*.

The Orlicz function $\mathcal{M}$ satisfies the Δ_2-condition for all values of $x \geq 0$ if there exists a constant $P > 0$ such that

$$\mathcal{M}(2x) \leq P\mathcal{M}(x). \tag{1.7.3}$$

Definition 1.7.1. A sequence space S is said to be *solid* if $\{\alpha_n \xi_n\} \in S$, for some $\{\xi_n\} \in S$ and for all $\{\alpha_n\}$ sequences of complex numbers such that $|\alpha_n| \leq 1$, $\forall n \in \mathbb{N}$. The sequence space S is said to be *symmetric* if every rearrangement of every sequence in S is in S.

Suppose that $h = \sup\{p_n\}$ and $k = \max\{1, 2^{h-1}\}$, then

$$\|\xi_n + \eta_n\|^{p_n} \leq k(\|\xi_n\|^{p_n} + \|\eta_n\|^{p_n})$$

Remark 1.7.1. Let $\mathcal{M}$ be an Orlicz function and $\mu \in (0,1)$, then $\mathcal{M}(\mu x) \leq \mu\,\mathcal{M}(x)$, $\forall\, x > 0$.

1.8 Applications of Theory of Bicomplex Numbers

In this section, we have given some brief review of the applications of the bicomplex numbers.

1.8.1 Several Complex Variables

The best beneficiary of the development of the theory of bicomplex numbers and functions of a bicomplex variable is the theory of functions of several complex variables. The theory of functions of a complex variable is, in many ways, advantageous over the theory of functions of several real variables. In the similar fashion, bicomplex variable techniques have come to the rescue of many problems in the theory of functions of several complex variables.

(a) Complex Signal Systems

These systems have been found useful in manipulating analytic and complex signals. In 1998, Toyoshima [74] and Hashimoto [26] employed bicomplex numbers and proposed an all - pass filter structure with bicomplex coefficients. With the help of this structure a power complementary filter pair with complex coefficients can be realized with a single bicomplex filter.

(b) Sequences Spaces and Series of Bicomplex Numbers

The study of certain sequences and series of complex numbers has played an important role in the development of complex analysis. In analogy, the study of bicomplex certain sequence and series was initiated.

In 2002, Srivastava and Srivastava [70] have defined a class $\mathcal{B}$ of bicomplex sequences as

$$\mathcal{B} = \left\{f : f = \{\xi_n\}, \xi_n \in \mathbb{C}_2, \ \sup n! \, ||\xi_n|| < \infty\right\}. \qquad (1.8.1)$$

They have furnished $\mathcal{B}$ with a modified Banach algebraic structure. They have succeeded in defining an involution on $\mathcal{B}$ equipped with which, $\mathcal{B}$ becomes (modified) Banach *-algebra, which is not a $\mathcal{B}^\star$-algebra. The class $\mathcal{P}$ of bicomplex series, defined as

$$\mathcal{P} = \left\{f : f = \{\xi_n\}, \xi_n \in \mathbb{C}_2, \sum n! \, ||\xi_n|| < \infty\right\}. \qquad (1.8.2)$$

By defining suitable binary compositions and norms, $\mathcal{P}$ has been provided a *two norm space* structure, which is γ-complete. The closed unit ball of $\mathcal{P}$ shown to be a Saks space.

In 2002, Srivastava & Srivastava [70] have studied the class of entire bicomplex sequences defined as

$$\mathbf{B} = \Big\{ f : f = \{\xi_k\}, \xi_k = (a_k - i_1 b_k)e_1 + (a_k + i_1 b_k)e_2; \sup\ k^k|a_k - i_1 b_k| < \infty, \\ \sup\ k^k|a_k + i_1 b_k| < \infty \Big\} \tag{1.8.3}$$

After providing a suitable functional analytic structure to **B**, invertible elements and zero divisors in **B** have been characterized. Topological zero divisors and Quasi invertible elements in **B** have also been investigated.

1.8.2 Signal Processing

Blind source separation is a signal-processing problem concerned with the recovery of a set of unobservable source signals or random variables from the only observation. In 2001, the Signal Processing and Communication group at the Department of Electrical Engineering and Electronics, University of Liverpool, headed by Zarzoso and Nandi [79] focused their research on Blind Source Separation.

In 2002, they defined a bicomplex formalism, which enables an elegant extension of *analytical Blind Source Separation solutions* from the real valued signal case to the complex valued signal case. A number of closed - form estimated ideas in real mixture scenario have been extended to the complex mixture case, including the concept of scatter diagram and centroids.

1.8.3 Electromagnetic Fields Appearing in Scattering and Diffraction Problems

In 1997, Hashimoto [25] has used a bicomplex representation for time harmonic electromagnetic fields appearing in Scattering and Diffraction problems and has been able to obtain new relations between high frequency diffraction.

1.8.4 Dynamics of Spiral Waves

Spiral waves are observed in various physical, chemical and biological systems. In 2001, Biktasheva and Biktashev [2] studied the response functions of Spiral wave solutions of the *Complex Ginzburg – Landau Equation*, using bicomplex numbers and functions of bicomplex variable. They have concluded that the response functions may be used to predict new qualitative features in behavior of spiral waves.

□ □ □

Chapter 2

Certain Bicomplex Dictionary Order Topologies

In this chapter, we introduce central notion of this thesis, order topologies and initiated the study of certain order topologies on bicomplex space $\mathbb{C}_2$. Srivastava [72] has defined three topologies on $\mathbb{C}_2$ such as norm topology τ_1, complex topology τ_2 and idempotent topology τ_3, and shown that all three topologies are equivalent.

In Section 2.1, three types of order relations on $\mathbb{C}_2$, viz. $\ell(\mathbb{C}_0)$-order, $\ell(\mathbb{C}_1)$-order and ℓ_{Id}-order using the concept of dictionary order relation are defined and have shown that $\ell(\mathbb{C}_0)$-order and $\ell(\mathbb{C}_1)$-order are equivalent in some sense, whereas the third ordering ℓ_{Id}-order is different from the other two.

Section 2.2 deals with the topologies, viz. $\mathbb{C}_0$(o)-topology, $\mathbb{C}_1$(o)-topology and Id(o)-topology generated by these three order relations. All these topologies are compared in this section.

Section 2.3, discusses two interesting topologies namely, Id(p)-topology and Id(m)-topology on $\mathbb{C}_2$ which are different from the earlier topologies.

2.1 Order Topology on $\mathbb{C}_0^3$

We have extended the topological study of the bicomplex space which was initiated by Srivastava [71]. Firstly, we have defined the dictionary order relation on $\mathbb{C}_0^3$. With the help of this order relation, a dictionary order topology on $\mathbb{C}_0^3$.

Consider that $\mathbb{C}_0^3$ is dictionary ordered topological space. We denote the element (x, y, z) of $\mathbb{C}_0^3$ as $x \times y \times z$. The order topology on $\mathbb{C}_0^3$ is generated by a basis B as the collection of all open intervals of the forms:

(i) $(a \times b \times c, r \times s \times t)$ when $a < r$.

(ii) $(a \times b \times c, a \times s \times t)$ when $b < s$.

(iii) $(a \times b \times c, a \times b \times t)$ when $c < t$.

Since bicomplex space, $\mathbb{C}_2$ is investigated, so we constructed only the shapes of the basis elements of order topology on $\mathbb{C}_0^3$ was constructed. For all three possible types of open intervals, three types of figures of the open sets in the order topology on $\mathbb{C}_0^3$ are defined.

In Figure 2.1, an arrow drawn between the two planes represents an open interval which contains all the points on the other planes that lies between these two planes.

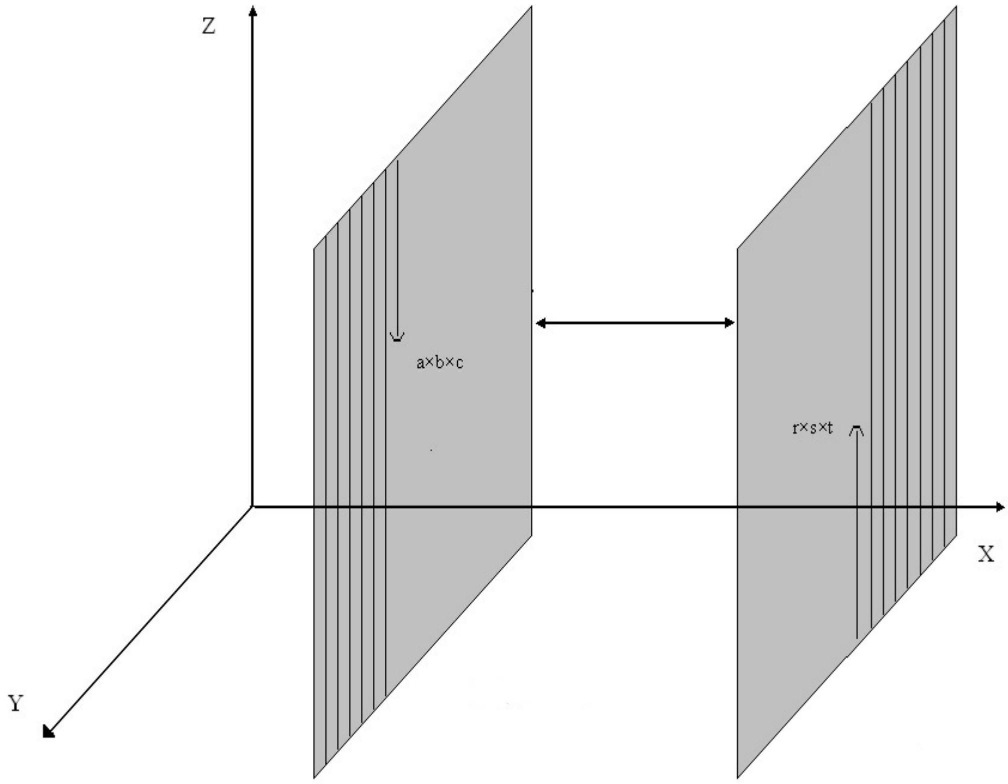

FIGURE 2.1: Collection of Planes

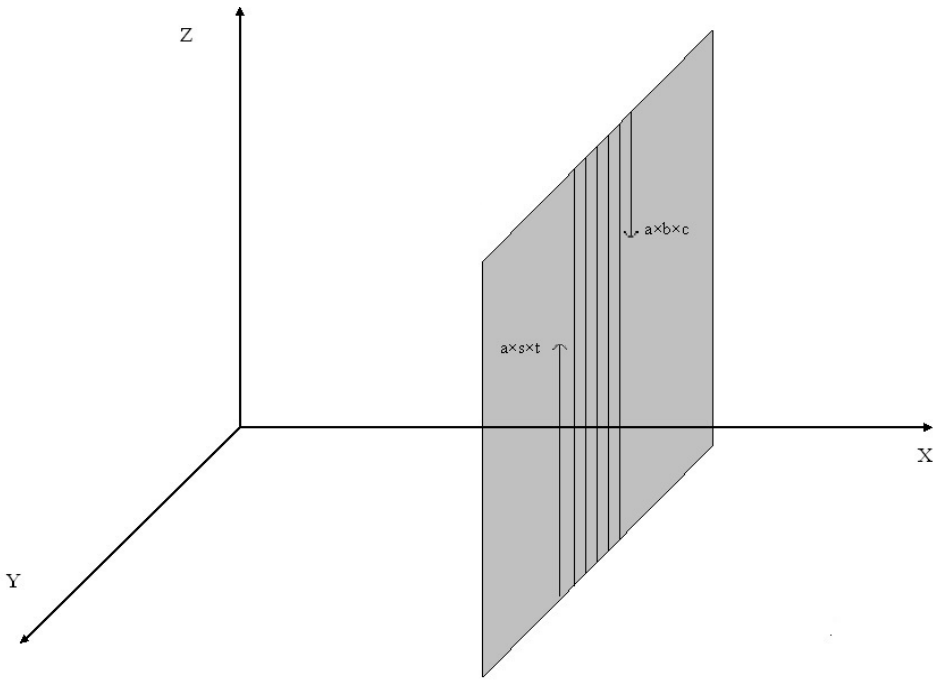

FIGURE 2.2: Collection of Infinite Strips

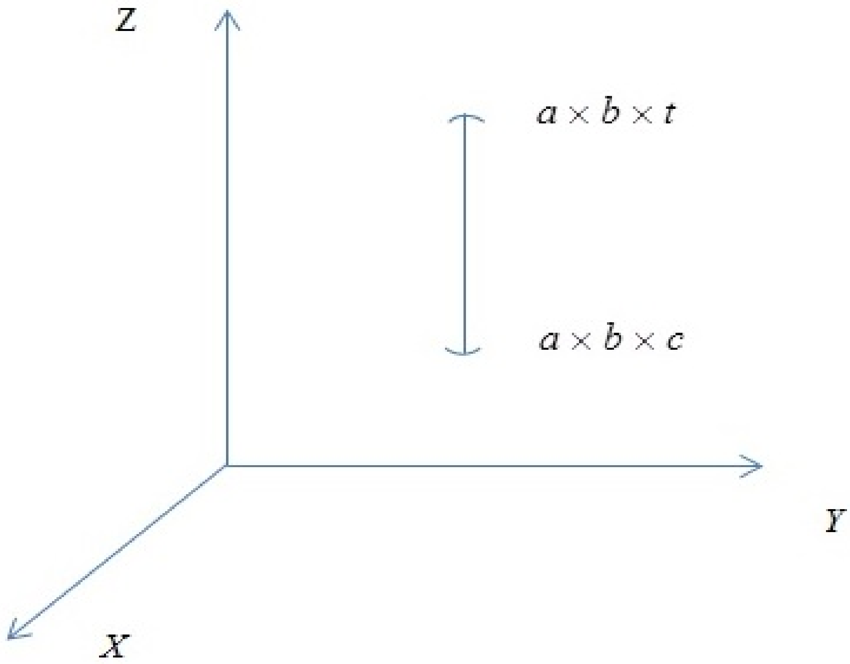

FIGURE 2.3: Vertical Interval

2.2 Certain Types of Order Relations in $\mathbb{C}_2$

In this section, on the bicomplex space $\mathbb{C}_2$, three types of dictionary order relation on $\mathbb{C}_2$, namely $\ell(\mathbb{C}_0)$-order, $\ell(\mathbb{C}_1)$-order and ℓ_{Id}-order relation are defined. Some new topological structures on $\mathbb{C}_2$ are also developed and compared these order relations.

Throughout, ξ and η will denote bicomplex numbers defined by

$\xi = a_1 + i_1 a_2 + i_2 a_3 + i_1 i_2 a_4 = z_1 + i_2 z_2 = {}^1\xi e_1 + {}^2\xi e_2$ and

$\eta = b_1 + i_1 b_2 + i_2 b_3 + i_1 i_2 b_4 = w_1 + i_2 w_2 = {}^1\eta e_1 + {}^2\eta e_2$.

Definition 2.2.1 ($\ell(\mathbb{C}_0)$**-Order**)**.** We say $\xi \prec_{\mathbb{C}_0} \eta$, if $a_k \leq b_k$, for some $k \in \mathbb{N}$, $1 \leq k \leq 4$ and $x_p = y_p$, for $p \in \mathbb{N}$, $1 \leq p < k$.

Definition 2.2.2 ($\ell(\mathbb{C}_1)$**-Order**)**.** We say $\xi \prec_{\mathbb{C}_1} \eta$. if $z_1 \prec w_1$ or $z_1 = w_1$, $z_2 \prec w_2$.

Where the symbol "$\prec$" denotes the lexicographic order relation in the complex space.

Definition 2.2.3 (ℓ_{Id}**-Order**)**.** We say $\xi \prec_{Id} \eta$, if ${}^1\xi \prec {}^1\eta$ or ${}^1\xi = {}^1\eta$, ${}^2\xi \prec {}^2\eta$.

Remark 2.2.1. The $\ell(\mathbb{C}_0)$-order and $\ell(\mathbb{C}_1)$-order are equivalent, i.e.,

$$\begin{aligned}
\xi \prec_{\mathbb{C}_0} \eta \iff & x_k < y_k \quad \text{and} \quad x_p = y_p,\ p < k,\ 1 \leq k \leq 4 \\
\iff & z_1 \prec w_1 \quad \text{or if} \quad z_1 = w_1,\ z_2 \prec w_2 \\
\iff & \xi \prec_{\mathbb{C}} \eta
\end{aligned}$$

Remark 2.2.2. The $\ell(\mathbb{C}_1)$-order is different from ℓ_{Id}-order, which can be proved by means of two examples.

Example 2.2.1. Let $\xi = (x_1 + i_1 x_2) + i_2(x_3 + i_1 x_4) = (2 + 2i_1) + i_2(3 + 4i_1)$

and $\eta = (y_1 + i_1 y_2) + i_2(y_3 + i_1 y_4) = (1 + 2i_1) + i_2(4 + 7i_1)$.

As $(1 + 2i_1) \prec (2 + 2i_1)$, by Definition 2.2.2, $\eta \prec_{\mathbb{C}_1} \xi$.

Now, we have

$$\begin{aligned}
\xi = {}^1\xi e_1 + {}^2\xi e_2 &= [(2+2i_1) - i_1(3+4i_1)]e_1 + [(2+2i_1) + i_1(3+4i_1)]e_2 \\
&= (6 - 1_1)e_1 + (-2+5i_1)e_2 \\
\eta = {}^1\eta e_1 + {}^2\eta e_2 &= [(1+2i_1) - i_1(4+7i_1)]e_1 + [(1+2i_1) + i_1(4+7i_1)]e_2 \\
&= (8-2i_1)e_1 + (-6+6i_1)e_2
\end{aligned}$$

Since, ${}^1\xi \prec {}^1\eta$, so by Definition 2.2.3, we have $\xi \prec_{Id} \eta$.

Thus, there exists ξ, $\eta \in \mathbb{C}_2$ for which $\eta \prec_C \xi$ although $\xi \prec_{ID} \eta$.

Example 2.2.2. Let $\xi = {}^1\xi e_1 + {}^2\xi e_2$, where ${}^1\xi = 7 + 5i_1$ and ${}^2\xi = 1 + 3i_1$ and $\eta = {}^1\eta e_1 + {}^2\eta e_2$, where ${}^1\eta = 6 + 4i_1$ and ${}^2\eta = 4 + 8i_1$. Now as, ${}^1\eta \prec {}^1\xi$, so by Definition 2.2.3, $\eta \prec_{Id} \xi$. However,

$$\begin{aligned}
\xi = {}^1\xi e_1 + {}^2\xi e_2 &= (7+5i_1)e_1 + (1+3i_1)e_2 \\
&= (7+5i_1)\left(\frac{1+i_1i_2}{2}\right) + (1+3i_1)\left(\frac{1-i_1i_2}{2}\right) \\
&= (4+4i_1) + i_2(-1+3i_1), \\
\eta = {}^1\eta e_1 + {}^2\eta e_2 &= (6+4i_1)e_1 + (4+8i_1)e_2 \\
&= (6+4i_1)\left(\frac{1+i_1i_2}{2}\right) + (4+8i_1)\left(\frac{1-i_1i_2}{2}\right) \\
&= (5+6i_1) + i_2(2+i_1)
\end{aligned}$$

Therefore, by the Definition 2.2.2, we have $\xi \prec_{\mathbb{C}_1} \eta$.

Thus, there exist ξ, $\eta \in \mathbb{C}_2$ for which $\xi \prec_{\mathbb{C}_1} \eta$ although $\eta \prec_{Id} \xi$.

2.3 Certain Order Topologies on $\mathbb{C}_2$

In this section, with the help of the order relations defined in the last section, some order topologies have been developed on $\mathbb{C}_2$. These order topologies are also compared with each other.

The reverse order of $\ell(\mathbb{C}_0)$-order is also a linear order. The notation of open intervals $(\xi, \eta)_{\mathbb{C}_0}$ denote a basis element of order topology on $\mathbb{C}_2$ w.r.t the $\ell(\mathbb{C}_0)$-order. Now we have

$$\begin{aligned}
(\xi, \rightarrow)_{\mathbb{C}_0} &= \{\zeta \in \mathbb{C}_2 : \xi \prec_{\mathbb{C}_0} \zeta\}, \\
(\leftarrow, \xi)_{\mathbb{C}_0} &= \{\zeta \in \mathbb{C}_2 : \zeta \prec_{\mathbb{C}_0} \xi\}, \\
(\xi, \eta)_{\mathbb{C}_0} &= \{\zeta \in \mathbb{C}_2 : \xi \prec_{\mathbb{C}_0} \zeta \prec_{\mathbb{C}_0} \eta\}.
\end{aligned}$$

In the similar manner, the intervals $[\xi, \rightarrow)_{\mathbb{C}_0}$, $(\leftarrow, \xi]_{\mathbb{C}_0}$, $[\xi, \eta)_{\mathbb{C}_0}$, $(\xi, \eta]_{\mathbb{C}_0}$ and $[\xi, \eta]_{\mathbb{C}_0}$ can be defined.

Definition 2.3.1 ($\mathbb{C}_0$**(o)-topology).** Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4, \eta = y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4$ be two bicomplex numbers such that $\xi \prec_{\mathbb{C}_0} \eta$. The families of open intervals in $\mathbb{C}_2$ for $\ell(\mathbb{C}_0)$-order type are defined as follows:

(i) $G_1 = \{(a_1 + i_1a_2 + i_2a_3 + i_1i_2a_4,\, b_1 + i_1b_2 + i_2b_3 + i_1i_2b_4)_{\mathbb{C}_0} \,:\, a_1 < b_1\}$,

(ii) $G_2 = \{(a_1 + i_1a_2 + i_2a_3 + i_1i_2a_4,\, a_1 + i_1b_2 + i_2b_3 + i_1bi_2b_4)_{\mathbb{C}_0} \,:\, a_2 < b_2\}$,

(iii) $G_3 = \{(a_1 + i_1a_2 + i_2a_3 + i_1i_2a_4,\, a_1 + i_1a_2 + i_2b_3 + i_1i_2b_4)_{\mathbb{C}_0} \,:\, a_3 < b_3\}$,

(iv) $G_4 = \{(a_1 + i_1a_2 + i_2a_3 + i_1i_2a_4,\, a_1 + i_1a_2 + i_2a_3 + i_1i_2b_4)_{\mathbb{C}_0} \,:\, a_4 < b_4\}$.

Lemma 2.3.1. The collection $\mathbb{B}_1 = \bigcup_{k=1}^{4} G_k$ construct a basis for a topology on $\mathbb{C}_2$.

Proof. Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4$ be an arbitrary element of $\mathbb{C}_2$.

Then there exist a member $(y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4,\, r_1 + i_1r_2 + i_2r_3 + i_1i_2r_4)_{\mathbb{C}_0}$ in the collection $\mathbb{B}_4$ such that

$x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 \in (y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4,\, r_1 + i_1r_2 + i_2r_3 + i_1i_2r_4)_{\mathbb{C}_0}$.

Hence, $\mathbb{B}_4$ covers $\mathbb{C}_2$. Now, Suppose that

$S_1 = (y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4,\, r_1 + i_1r_2 + i_2r_3 + i_1i_2r_4)_{\mathbb{C}_0} = (\xi_1, \eta_1)_{\mathbb{C}_0}$ and

$S_2 = (c_1 + i_1c_2 + i_2c_3 + i_1i_2c_4,\, d_1 + i_1d_2 + i_2d_3 + i_1i_2d_4)_{\mathbb{C}_0} = (\xi_2, \eta_2)_{\mathbb{C}_0}$

are any two elements from the collection $\mathbb{B}_4$.

Let $\zeta = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4$ be a arbitrary bicomplex number such that $\zeta \in S_1 \cap S_2$.

$\Rightarrow \quad \zeta \in S_1 \quad$ as well as $\quad \zeta \in S_2$.

$\Rightarrow \quad \xi_1 \prec_{\mathbb{C}_0} \zeta \prec_{\mathbb{C}_0} \eta_1 \quad$ as well as $\quad \xi_2 \prec_{\mathbb{C}_0} \zeta \prec_{\mathbb{C}_0} \eta_2$.

Define $\xi = \max\,\{\xi_1,\, \xi_2\}$, $\eta = \min\,\{\eta_1,\, \eta_2\}$ and $S = (\xi,\, \eta)_{\mathbb{C}_0}$.

Obviously, $S \in \mathbb{B}_4$ such that $\zeta \in (\xi,\, \eta)_{\mathbb{C}_0}$.

We claim that $S \subset S_1 \cap S_2$. Let $\alpha = k_1 + i_1k_2 + i_2k_3 + i_1i_2k_4$ be an arbitrary element of S, then $\xi \prec_{\mathbb{C}_0} \alpha$ and $\alpha \prec_{\mathbb{C}_0} \eta$.

Now, if $\xi \prec_{\mathbb{C}_0} \alpha$. Then $\xi_1 \prec_{\mathbb{C}_0} \alpha$ as well as $\xi_2 \prec_{\mathbb{C}_0} \alpha$

and $\alpha \prec_{\mathbb{C}_0} \eta$. Then $\alpha \prec_{\mathbb{C}_0} \eta_1$ as well as $\alpha \prec_{\mathbb{C}_0} \eta_2$

$\Rightarrow \quad \xi_1 \prec_{\mathbb{C}_0} \alpha \prec_{\mathbb{C}_0} \eta_1$ and $\xi_2 \prec_{\mathbb{C}_0} \alpha \prec_{\mathbb{C}_0} \eta_2$

$\Rightarrow \quad \alpha \in S_1$ and $\alpha \in S_2$.

Thus, we have shown that all pairs S_1 and S_2 of members of $\mathbb{B}_4$ and for every element ζ in their intersection, we have member S of $\mathbb{B}_4$ such that $\zeta \in S \subset S_1 \cap S_2$. So that $\mathbb{B}_4$ forms a basis for some topology on $\mathbb{C}_2$. □

Remark 2.3.1. The topology generated by $\mathbb{B}_4$ as $\mathbb{C}_0$*(o)-topology* are defined and it is denoted as τ_4. Further, the collection $\{(\xi, \rightarrow)_{\mathbb{C}_0} : \xi \in \mathbb{C}_2\} \cup \{(\leftarrow, \xi)_{\mathbb{C}_0} : \xi \in \mathbb{C}_2\}$ forms a sub-basis for this topology. The interval $(\xi, \eta)_{\mathbb{C}_0} = (a_1 + i_1a_2 + i_2a_3 + i_1i_2a_4,\, b_1 + i_1b_2 + i_2b_3 + i_1i_2b_4)_{\mathbb{C}_0}$ is called as $\mathbb{C}_0$-open set in τ_4. It may be a $\mathbb{C}_0$-frame, $\mathbb{C}_0$-plane, $\mathbb{C}_0$-line segment or $\mathbb{C}_0$-interval depends upon the order of the elements of the intervals.

Definition 2.3.2 ($\mathbb{C}_1$**(o)-topology**)**.** Let $\xi \prec_C \eta$. Two types of open intervals in $\mathbb{C}_2$ can be defined, viz.,

(i) $K_1 = \{(z_1 + i_2z_2,\, w_1 + i_2w_2)_{\mathbb{C}_1} \,:\, z_1 \prec w_1\}$,

(ii) $K_2 = \{(z_1 + i_2z_2,\, z_1 + i_2w_2)_{\mathbb{C}_1} \,:\, z_2 \prec w_2\}$.

Define $\mathbb{B}_5 = K_1 \cup K_2$. In the similar lines of Lemma 2.3.1, it can be prove that $\mathbb{B}_5$ forms a basis for some topology on $\mathbb{C}_2$, we shall call this topology as $\mathbb{C}_1$*(o)-topology* and denote it as τ_5.

Remark 2.3.2. Now for $\alpha = z_1 + i_2 z_2$ and $\beta = w_1 + i_2 w_2$, the following families are constructed as:

(i) $M_1 = \{(\alpha,\, \beta)_{\mathbb{C}_1} \,:\, Re\, z_1 < Re\, w_1\}$,

(ii) $M_2 = \{(\alpha,\, \beta)_{\mathbb{C}_1} \,:\, Re\, z_1 = Re\, w_1,\, Im\, z_1 < Im\, w_1\}$,

(iii) $M_3 = \{(\alpha,\, \beta)_{\mathbb{C}_1} \,:\, Re\, z_2 < Re\, w_2\}$,

(iv) $M_4 = \{(\alpha,\, \beta)_{\mathbb{C}_1} \,:\, Re\, z_2 = Re\, w_2,\, Im\, z_2 < Im\, w_2\}$.

Note that, for $z_1 \prec w_1$, $z_2 \prec w_2$, K_1 and K_2 can are described as $K_1 = M_1 \cup M_2$ and $K_2 = M_3 \cup M_4$. Therefore, $\mathbb{B}_5 = K_1 \cup K_2 = \bigcup_{p=1}^{4} M_p$. Note further M_1, M_2, M_3 and M_4 are in fact, families of $\mathbb{C}_1$-space-segments, $\mathbb{C}_1$-frame-segments, $\mathbb{C}_1$-plane-segments and $\mathbb{C}_1$-line segments, respectively.

Definition 2.3.3 (Id(o)-topology). Let $\xi \prec_{Id} \eta$. Two families of open intervals in $\mathbb{C}_2$ with respect to ℓ_{Id}-order are defined as

(i) $L_1 = \{({}^1\xi e_1 + {}^2\xi e_2,\, {}^1\eta e_1 + {}^2\eta e_2)_{Id} \,:\, {}^1\xi \prec {}^1\eta\}$.

(ii) $L_1 = \{({}^1\xi e_1 + {}^2\xi e_2,\, {}^1\xi e_1 + {}^2\eta e_2)_{Id} \,:\, {}^2\xi \prec {}^2\eta\}$.

Define $\mathbb{B}_6 = L_1 \cap L_2$. It can be proved that collection $\mathbb{B}_6$ forms a basis for some topology on $\mathbb{C}_2$. This topology is known as *Id(o)-topology* and denoted by τ_6.

Remark 2.3.3. So far the basis elements of the idempotent order topology classified as the classes L_1 and L_2. As a matter of fact, the idempotent components of a bicomplex number are themselves complex numbers, the ordering of ${}^1\xi$ and ${}^1\eta$ can be further split into orderings of the real parts or - in the case real parts are same their imaginary parts. In this sense, L_1 can be considered as the union of two classes, say N_1 and N_2, and L_2 can be considered as the union of two classes, say N_3 and N_4 defined as

(i) $N_1 = \{(\zeta, \theta)_{Id} \,:\, Re\, {}^1\zeta \,<\, Re\, {}^1\theta\}$

(ii) $N_2 = \{(\zeta, \theta)_{Id} \,:\, Re\, {}^1\zeta = Re\, {}^1\theta,\, Im\, {}^1\zeta \,<\, Im\, {}^1\theta\}$

(iii) $N_3 = \{(\zeta,\theta)_{Id} : Re^2\zeta < Re^2\theta\}$

(iv) $N_4 = \{(\zeta,\theta)_{Id} : Re^2\zeta = Re^2\theta,\ Im^2\zeta < Im^2\theta\}$.

Theorem 2.3.1. *$\mathbb{C}_0(o)$-topology and $\mathbb{C}_1(o)$-topology are same.*

Proof. Since the $\mathbb{C}_0$-order and $\mathbb{C}_1$-order are equivalent, so any basis element of the $\mathbb{C}_0(o)$-topology is equal to some basis element of the $\mathbb{C}_1$(o)-topology. Therefore, the $\mathbb{C}_0(o)$-topology and the $\mathbb{C}_1$(o)-topology are same. □

Theorem 2.3.2. *$\mathbb{C}_1$(o)-topology and Id(o)-topology are not comparable.*

To prove this assertion, we gave two examples are given as follows:

Example 2.3.1. Let $(\xi,\,\eta)_{\mathbb{C}_1}$ be a basis element of the complex order topology, where $\xi = (1+2i_1)+i_2(4+3i_1)$ and $\eta = (7+4i_1)+i_2(2+5i_1)$.

Let $\zeta = (1+2i_1)+i_2(4+4i_1)$. Clearly, $\zeta \in (\xi,\,\eta)_{\mathbb{C}_1}$. Firstly, define

$$\begin{aligned}
\xi &= (1+2i_1)+i_2(4+3i_1)\\
&= [(1+2i_1)-i_1(4+3i_1)]e_1+[(1+2i_1)+i_1(4+3i_1)]e_2\\
&= (4-2i_1)e_1+(-2+6i_1)e_2,\\
\eta &= (7+4i_1)+i_2(2+5i_1)\\
&= [(7+4i_1)-i_1(2+5i_1)]e_1+[(7+4i_1)+i_1(2+5i_1)]e_2\\
&= (12+2i_1)e_1+(2+6i_1)e_2
\end{aligned}$$

and

$$\begin{aligned}
\zeta &= (1+2i_1)+i_2(4+4i_1)\\
&= [(1+2i_1)-i_1(4+4i_1)]e_1+[(1+2i_1)+i_1(4+4i_1)]e_2\\
&= (5-2i_1)e_1+(-3+6i_1)e_2
\end{aligned}$$

Now, given $\epsilon > 0$, an interval $(\alpha, \beta)_{Id}$, where $\alpha = (5-2i_1)e_1 + (-3+(6-\epsilon)i_1)e_2$ and $\beta = (5-2i_1)e_1 + (-3+(6+\epsilon)i_1)e_2$. Then

$$\begin{aligned} \alpha &= (5-2i_1)e_1 + (-3+(6-\epsilon)i_1)e_2 \\ &= (5-2i_1)\left(\frac{1+i_1i_2}{2}\right) + (-3+(6-\epsilon)i_1)\left(\frac{1-i_1i_2}{2}\right) \\ &= \left(1+\left(2-\frac{\epsilon}{2}\right)i_1\right) + i_2\left(\left(4-\frac{\epsilon}{2}\right)+4i_1\right) \end{aligned}$$

and

$$\begin{aligned} \beta &= (5-2i_1)e_1 + (-3+(6+\epsilon)i_1)e_2 \\ &= (5-2i_1)\left(\frac{1+i_1i_2}{2}\right) + (-3+(6-\epsilon)i_1)\left(\frac{1-i_1i_2}{2}\right) \\ &= \left(1+\left(2+\frac{\epsilon}{2}\right)i_1\right) + i_2\left(\left(4+\frac{\epsilon}{2}\right)+4i_1\right) \end{aligned}$$

Clearly, $\zeta \in (\alpha, \beta)_{Id}$ but $\alpha \prec_{\mathbb{C}_1} \xi$, which implies that $\alpha \in (\alpha, \beta)_{Id} \nsubseteq (\xi, \eta)_{\mathbb{C}_1}$.

Thus, no basis element around ζ in Id(o)-topology is contained in the basis element $(\xi, \eta)_{\mathbb{C}_1}$ around ζ in $\mathbb{C}_1$(o)-topology.

Example 2.3.2. Let $(\xi, \eta)_{Id}$ be an arbitrary element of B_6, where

$\xi = (6+4i_1)e_1 + (2+6i_1)e_2$ and $\eta = (7+5i_1)e_1 + (3+4i_1)e_2$.

Suppose that $\zeta = (6+4i_1)e_1 + (2+7i_1)e_2$. Obviously, $\zeta \in (\xi, \eta)_{Id}$.

Now, $\xi = (6+4i_1)e_1 + (2+6i_1)e_2 = (4+5i_1) + i_2(1+2i_1)$,
$\eta = (7+5i_1)e_1 + (3+4i_1)e_2 = (5+\frac{9}{2}i_1) + i_2(-\frac{1}{2}+2i_1)$

and $\zeta = (6+4i_1)e_1 + (2+7i_1)e_2 = (4+\frac{11}{2}i_1) + i_2(\frac{3}{2}+2i_1)$.

Given any $\epsilon > 0$, consider an element $(\alpha, \beta)_{\mathbb{C}_1} \in B_6$ containing ζ, where

$\alpha = 4 + \frac{11}{2}i_1 + i_2(\frac{3}{2}+(2-\epsilon)i_1)$ and $\beta = 4 + \frac{11}{2}i_1 + i_2(\frac{3}{2}+(2+\epsilon)i_1)$.

Then, $\alpha = 4 + \frac{11}{2}i_1 + i_2(\frac{3}{2}+(2-\epsilon)i_1) = ((6-\epsilon)+4i_1)e_1 + ((2+\epsilon)+7i_1)e_2$

and $\beta = 4 + \frac{11}{2}i_1 + i_2(\frac{3}{2}+(2+\epsilon)i_1) = ((6+\epsilon)+4i_1)e_1 + ((2-\epsilon)+7i_1)e_2$.

Clearly, $\zeta \in (\alpha, \beta)_{\mathbb{C}_1}$ but $\alpha \prec_{Id} \xi$. Hence, $\zeta \in (\alpha, \beta)_{\mathbb{C}_1} \nsubseteq (\xi, \eta)_{Id}$.

From Example 2.3.1 and Example 2.3.2, we conclude that $\mathbb{C}_1$(o)-topology and Id(o)-topology are not comparable.

Corollary 2.3.1. $\mathbb{C}_0$(o)-topology and the Id(o)-topology are not comparable.

2.4 Some Other Topologies on $\mathbb{C}_2$

In this section, we have defined a product topology and a metric topology on the bicomplex space are defined and studied.

Definition 2.4.1 (Id(p)-topology). A topology on $\mathbb{C}_2$ is constructed by treating it as $\mathbb{C}_2 = \mathbb{A}_1 \times_e \mathbb{A}_2$. This topology is called as Id(p)-topology and denoted as τ_4.

The Id(p)-topology have basis B_4 as the collection of the basis elements of the type $({}^1\alpha,\, {}^1\beta) \times_e ({}^2\alpha,\, {}^2\beta)$, where $({}^1\xi,\, {}^1\eta)$ and $({}^2\xi,\, {}^2\eta)$ are the basis elements of the order topologies on auxiliary complex spaces $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively.

Theorem 2.4.1. *The Id(o)-topology is strictly finer than the Id(p)-topology.*

Proof. Let $({}^1\xi,\, {}^1\eta) \times_e ({}^2\xi,\, {}^2\eta)$ be an arbitrary basis element of the Id(p)-topology and $\zeta = {}^1\zeta e_1 + {}^2\zeta e_2$ be any bicomplex number such that $\zeta \in ({}^1\xi,\, {}^1\eta) \times_e ({}^2\xi,\, {}^2\eta)$. So ${}^1\zeta \in ({}^1\xi,\, {}^1\eta)$ and ${}^2\zeta \in ({}^2\xi,\, {}^2\eta)$. For ${}^2\zeta \in ({}^2\xi,\, {}^2\eta)$, $\exists\, a,\, b \in \mathbb{A}_2$ such that ${}^1\zeta = (a,\, b) \subset ({}^2\xi,\, {}^2\eta)$. In particular, consider $({}^1\zeta e_1 + ae_2,\, {}^1\zeta e_1 + be_2)_{Id} \in B_6$. Note that $\zeta \in ({}^1\zeta e_1 + ae_2,\, {}^1\zeta e_1 + be_2)_{Id} \subset ({}^1\xi,\, {}^1\eta) \times_e ({}^2\xi,\, {}^2\eta)$. Hence, Id(o)-topology is finer than Id(p)-topology.

Conversely, to show that Id(p)-topology is not finer than Id(o)-topology, consider an element $(\xi,\, \eta)_{Id} \in B_6$ such that ${}^1\xi = {}^1\eta$. Note that, if ζ is any point belonging to $(\xi,\, \eta)_{Id}$, then ${}^1\zeta = {}^1\xi = {}^1\eta$. Let $(z_1,\, w_1) \times (z_2,\, w_2)$ be any basis element of the Id(p)-topology containing. Note that $z_1 \neq w_1$ and $z_2 \neq w_2$, since singleton sets are not treated as open sets.

Now, $\zeta \in (z_1,\, w_1) \times_e (z_2,\, w_2) \quad \Rightarrow \quad {}^1\zeta \in (z_1,\, w_1)$ and ${}^2\zeta \in (z_2,\, w_2)$.

Let u be any point in the interval (z_1, w_1), $u \neq {}^1\zeta$. We find that the bicomplex number $\alpha = ue_1 + {}^2\zeta e_2$ belongs to $\zeta \in (z_1, w_1) \times_e (z_2, w_2)$. However,

${}^1\alpha = u \neq {}^1\zeta(= {}^1\xi = {}^1\eta)$. So, $\alpha \in (\xi, \eta)_{Id}$. Hence proved. $\square$

The mechanism behind the failure of the equivalence of two topologies is elaborated in the following remark.

Remark 2.4.1. Let $({}^1\xi e_1 + {}^2\xi e_2,\, {}^1\eta e_1 + {}^2\eta e_2)_{ID}$ be an arbitrary basis element of the Id(o)-topology and γ be any bicomplex number such that

$$\gamma = {}^1\gamma e_1 + {}^2\gamma e_2 \in ({}^1\xi e_1 + {}^2\xi e_2,\, {}^1\eta e_1 + {}^2\eta e_2)_{Id}. \tag{2.4.1}$$

If ${}^1\xi \neq {}^1\eta$, three cases arise as follows:

(i) ${}^1\xi = {}^1\gamma$ and ${}^1\gamma \prec {}^1\eta$.

(ii) ${}^1\xi \prec {}^1\gamma$ and ${}^1\gamma \prec {}^1\eta$.

(iii) ${}^1\gamma = {}^1\eta$ and ${}^2\gamma \prec {}^2\eta$.

Case (i): When ${}^1\xi = {}^1\gamma$ and ${}^1\gamma \prec {}^1\eta$. Then, ${}^1\gamma \in ({}^1\xi, {}^1\eta)$.

Since, $({}^1\xi, {}^1\eta)$ is open in the order topology on $\mathbb{A}_1$ space, therefore there exists a basis element $({}^1\alpha, {}^1\beta)$ of the basis element of the order topology on A_1 such that ${}^1\gamma \in ({}^1\alpha, {}^1\beta) \subset ({}^1\xi, {}^1\eta)$.

Also, ${}^2\gamma \in A_2$ and A_2 is order topological (auxiliary complex) space, so that there must exist a basis element $({}^2\alpha, {}^2\beta)$ of the order topology on A_2 such that ${}^1\gamma \in ({}^1\alpha, {}^1\beta) \subset ({}^1\xi, {}^1\eta)$. Therefore,

$${}^1\gamma e_1 + {}^2\gamma e_2 \in ({}^1\alpha, {}^1\beta) \times_e ({}^2\alpha, {}^2\beta) \subset ({}^1\xi e_1 + {}^2\xi e_2,\, {}^1\eta e_1 + {}^2\eta e_2)_{Id}$$

Case (ii): When ${}^1\xi \prec {}^1\gamma$ and ${}^1\gamma \prec {}^1\eta$.

If there exists a basis element $({}^1\alpha, {}^1\beta)$ of the order topology on $\mathbb{A}_1$ such that ${}^1\gamma \in ({}^1\alpha, {}^1\beta)$, then ${}^1\alpha \prec {}^1\gamma(= {}^1\xi)$ and ${}^1\gamma \prec {}^1\beta$.

Therefore,

$$^1\alpha e_1 + ze_2 \prec_{ID} {}^1\xi e_1 + {}^2\xi e_2, \ \forall z \in \mathbb{A}_2 \tag{2.4.2}$$

$\Rightarrow \ {}^1\alpha e_1 + ze_2 \notin ({}^1\xi e_1 + {}^2\xi e_2, {}^1\eta e_1 + {}^2\eta e_2)_{Id}$.

By adopting the same method for ${}^2\gamma$, one can find a basis element $({}^2\alpha, {}^2\beta)$ of the order topology on $\mathbb{A}_2$ such that $\gamma = {}^1\gamma e_1 + {}^2\gamma e_2 \in ({}^1\alpha, {}^1\beta) \times_e ({}^2\alpha, {}^2\beta)$.

But from the Equation (2.4.2), we have

$$\gamma = {}^1\gamma e_1 + {}^2\gamma e_2 \in ({}^1\alpha, {}^1\beta) \times_e ({}^2\alpha, {}^2\beta) \nsubseteq ({}^1\xi e_1 + {}^2\xi e_2, {}^1\eta e_1 + {}^2\eta e_2)_{Id}.$$

Case (iii): Similar situation arises. Therefore, one cannot necessarily find a basis element of the idempotent product topology, which contains the given bicomplex number and contained in some basis element of the Id(o)-topology. Hence, the Id(o)-topology is strictly finer than the Id(p)-topology.

Definition 2.4.2 (Idempotent Metric on $\mathbb{C}_2$). Let us define a metric on $\mathbb{C}_2$ as follows:

$$d_{Id}(\xi, \eta) = \max\left\{|{}^1\xi - {}^1\eta|, |{}^2\xi - {}^2\eta|\right\}.$$

That d_{Id} is actually a metric which can be verified easily. We shall call this metric as the *idempotent metric* on $\mathbb{C}_2$.

Definition 2.4.3 (Id(m)-topology). The topology generated by the idempotent metric will be called as the Id(m)-topology on $\mathbb{C}_2$ and will be denoted by τ_5. The basis elements of this topology will be denoted by $B_{Id}(\xi; r)$, where

$$B_{Id}(\xi; r) \ = \ \{\eta : \eta \in \mathbb{C}_2, \ d_{Id}(\xi, \eta) < r\} \tag{2.4.3}$$

where r is a real number.

Remark 2.4.2. The set $B_{Id}(\xi; r)$ is same as the set $D(\xi; r, r)$.

We know that $B_{Id}(\xi; r) = \{\eta : \eta \in \mathbb{C}_2, \ d_{Id}(\xi, \eta) < r\}$. Let $\eta \in B_{Id}(\xi; r)$.

Then $d_{Id}(\xi,\, \eta) < r \quad \Leftrightarrow \ max\ \{|^1\xi - {}^1\eta|,\ |^2\xi - {}^2\eta|\} < r$

$\Leftrightarrow \ |^1\xi - {}^1\eta| < r$ as well as $|^2\xi - {}^2\eta| < r \quad \Leftrightarrow \quad \eta \in D(\xi;\, r,\, r)$.

Remark 2.4.3. The idempotent metric d_{Id} on $\mathbb{C}_2$ satisfies the condition of homogeneity, i.e., $d_{Id}(\alpha\xi,\, \alpha\eta) = |\alpha|\, d_{Id}(\xi,\, \eta), \ \ \forall\, \alpha \in \mathbb{C}_1$.

Let $\xi,\, \eta \in \mathbb{C}_2$ and $\alpha \in \mathbb{C}_1$.

$$\begin{aligned} d_{Id}(\alpha\xi,\, \alpha\eta) &= \max\left\{|\alpha\ {}^1\xi - \alpha\ {}^1\eta\,|,\, |\alpha\ {}^2\xi - \alpha\ {}^2\eta\,|\right\} \\ &= \max\left\{|\alpha\,|\,|^1\xi - {}^1\eta\,|,\, |\alpha\,|\,|^2\xi - {}^2\eta\,|\right\} \\ &= |\alpha\,|\ \max\left\{|\,{}^1\xi - {}^1\eta\,|,\, |\,{}^2\xi - {}^2\eta\,|\right\} \\ &= |\alpha|\, d_{ID}(\xi,\, \eta) \end{aligned}$$

So that the idempotent metric satisfies the condition of homogeneity and hence this metric defines a norm on $\mathbb{C}_2$ defined as

$$||\xi\,||^\star \ = \ \max\left\{|\,{}^1\xi|,\, |\,{}^2\xi|\right\}.$$

Also, the norm $||\,.\,||^\star$ is equivalent to the norm defined in the Equation (1.1.4).

Theorem 2.4.2. *Id(p)-topology and Id(m)-topology are same.*

Proof. Let $({}^1\xi,\, {}^1\eta) \times_e ({}^2\xi,\, {}^2\eta)$ be any basis elements of the Id(p)-topology and ζ be an arbitrary bicomplex number such that

$$\zeta \in ({}^1\xi,\, {}^1\eta) \times_e ({}^2\xi,\, {}^2\eta),$$
$${}^1\zeta \in ({}^1\xi,\, {}^1\eta) \quad \text{and} \quad {}^2\zeta \in ({}^2\xi,\, {}^2\eta).$$

Note that $({}^1\xi,\, {}^1\eta)$ is a basis element of the order topology on $\mathbb{A}_1$. Then there exists a basis element $B(\,{}^1\alpha;\, r);\, r > 0$ of the metric topology on $\mathbb{A}_1$ such that

$${}^1\zeta \in B(\,{}^1\alpha;\, r) \subset (\,{}^1\xi,\, {}^1\eta).$$

Similarly, there exists a basis element $B({}^2\alpha;\, r);\, r > 0$ of the metric topology on $\mathbb{A}_2$ such that ${}^2\zeta \in B({}^2\alpha;\, r) \subset ({}^2\xi,\, {}^2\eta)$. Then

$$\zeta \;=\; {}^1\zeta e_1 + {}^2\zeta e_2 \in \mathbb{D}(\,\alpha;\, r,\, r) \subset ({}^1\xi,\, {}^1\eta) \times_e ({}^2\xi,\, {}^2\eta) \tag{2.4.4}$$

$$(\because B({}^1\alpha;\, r) \times_e B({}^2\alpha;\, r) = \mathbb{D}(\,\alpha;\, r,\, r))$$

By Remark 2.4.2, the Equation (2.4.4) implies that

$$\zeta \in B_{Id}(\alpha;\, r) \subset ({}^1\xi,\, {}^1\eta) \times_e ({}^2\xi,\, {}^2\eta).$$

Therefore, Id(m)-topology is finer than Id(p)-topology.

Conversely, let $\eta \in B_{Id}(\xi;\, r)$. By Remark 2.4.2, we have $\eta \in \mathbb{D}(\xi;\, r,\, r)$

$\Rightarrow \; |{}^1\xi - {}^1\eta| < r$ and $|{}^2\xi - {}^2\eta| < r$

$\Rightarrow \; {}^1\eta \in B({}^1\xi;\, r) = {}^1B$ (say) and ${}^2\eta \in B({}^2\xi;\, r) = {}^2B$ (say)

Let ${}^1\eta = x_1 + i_1 y_1$ and ${}^2\eta = x_2 + i_1 y_2$. Then there exists ${}^1D = (x_1 + i_1 c_1,\, x_1 + i_1 d_1)$, where $c_1 < y_1 < d_1$ of order topology on $\mathbb{C}_1$ such that

$${}^1\eta \in {}^1D \subset {}^1B. \tag{2.4.5}$$

Also, we can find a basis element ${}^2D = (x_2 + i_1 c_2,\, x_2 + i_1 d_2)$, where $c_2 < y_2 < d_2$ of order topology corresponding to ${}^2B$ such that

$${}^2\eta \in {}^2D \subset {}^2B. \tag{2.4.6}$$

From the Equation (2.4.5) and Equation (2.4.6), we have

$$\eta \in {}^1D \times_e {}^2D \subset {}^1B \times_e {}^2B$$
$$\Rightarrow \eta \in {}^1D \times_e {}^2D \subset D(\xi;\, r,\, r)$$

Therefore, Id(p)-topology is finer than Id(m)-topology. So that Id(p)-topology and Id(m)-topology are same. $\square$

Theorem 2.4.1 and Theorem 2.4.2 together gives the following Corollary.

Corollary 2.4.1. Id(o)-topology is strictly finer than Id(m)-topology.

Conclusion

In this chapter, some topological structures on the bicomplex space and their properties are developed. Also, the basis elements of the order topology on the three dimensional space, $\mathbb{C}_0^3$ are constructed with the concept of lexicographic order. Three ordered topological structures, namely $\mathbb{C}_0$(o)-topology, $\mathbb{C}_1$(o)-topology and Id(o)-topology are framed and compared on the bicomplex space.

□ □ □

Chapter 3

$\mathbb{C}_2$-Nets and their Confluences

In this chapter, the properties of the $\mathbb{C}_2$-nets are defined and discussed. Also, the convergence (or confluence) of the $\mathbb{C}_2$-nets in the $\mathbb{C}_0$(o)-topology as well as Id(o)-topology is defined. Due to distinct forms of the bicomplex numbers their are different types of different types of tendencies called confluences. In this chapter, we concentrated on the confluence of the $\mathbb{C}_2$-nets in the $\mathbb{C}_0$(o)-topology and Id(o)-topology.

In section 3.1, the $\mathbb{C}_2$-net and $\mathbb{C}_2$-subnet are defined. The concept of different types of amplitudes in terms of convergence called $\mathbb{C}_0$-confluences are given and studied in the different forms of confluences which have been described in terms of $\mathbb{C}_0$(o)-topology.

In section 3.2, the confluences of the $\mathbb{C}_2$-nets have been discussed with respect to the Id(o)-topology. The confluences of the $\mathbb{C}_2$-nets along with their component nets are also explained in detail.

In section 3.3, the confluence of the $\mathbb{C}_2$-net is discussed in the Id(p)-topology. The ID(p)-topology is defined on the $\mathbb{C}_2$ space as the product of two lexicographical order topological spaces as $\mathbb{A}_1$ and $\mathbb{A}_2$. The convergence of the components nets in the $\mathbb{C}_2$-net is also studied in this section.

3.1 Confluences of $\mathbb{C}_2$-nets in $\mathbb{C}_0$(o)-topology

In this section, we constructed the $\mathbb{C}_2$-nets and also defined the various types of confluences of these nets w.r.t the $\mathbb{C}_0$(o)-topology, τ_4. In this section some results on the $\mathbb{C}_0$-confluences of the $\mathbb{C}_2$-nets have been discussed.The results of $\mathbb{C}_2$-subnets and Cauchy $\mathbb{C}_2$-net have been defined and studied thoroughly.

Definition 3.1.1 ($\mathbb{C}_2$**-Net**)**.** Let D be any arbitrary directed set, then a $\mathbb{C}_2$-net $\Phi : D \to \mathbb{C}_2$ is defined as

$$\begin{aligned}\Phi(\alpha) &= x_{1\alpha} + i_1x_{2\alpha} + i_2x_{3\alpha} + i_1i_2x_{4\alpha}\\ &= z_{1\alpha} + i_2z_{2\alpha}\\ &= {}^1\xi_\alpha e_1 + {}^2\xi_\alpha e_2 \quad \forall\, \alpha \in D. \qquad (3.1.1)\end{aligned}$$

We denote the $\mathbb{C}_2$-net $\Phi(\alpha)$ as $\{\xi_\alpha\}_{\alpha\in D}$ or $\{\xi_\alpha\}$ where D denotes the directed set. Also, a *tail* in the directed set $(D, \geq)$ is the set $T_\alpha = \{\beta : \beta \geq \alpha\}$.

Definition 3.1.2 ($\mathbb{C}_2$**-Subnet**)**.** A $\mathbb{C}_2$-net $\{\eta_\beta\}_{\beta\in E}$ is a $\mathbb{C}_2$-*subnet* of a $\mathbb{C}_2$-net $\{\xi_\alpha\}_{\alpha\in D}$ if for each tail T_α of elements of D, there is a tail T_β of elements of E such that $\{\eta_\delta : \delta \in T_\beta\} \subset \{\xi_\gamma : \gamma \in T_\alpha\}$.

We have studied the concept of $\mathbb{C}_2$-nets in the order topologies defined on $\mathbb{C}_2$. In this section, the $\mathbb{C}_2$-net will be considered as $\{\xi_\alpha\} = \{x_{1\alpha} + i_1x_{2\alpha} + i_2x_{3\alpha} + i_1i_2x_{4\alpha}\}$. In this context, we gave concept of convergence of $\mathbb{C}_2$-nets in the sense of confluence. Certain types of confluences in $\mathbb{C}_0$(o)-topology, τ_4 are defined as follows:

Definition 3.1.3 ($\mathbb{C}_0$($\boldsymbol{F}$)**-Confluence**)**.** A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is said to be $\mathbb{C}_0$*(F)-confluence* to $[x_1 = a]_{\mathbb{C}_0}$, if for every $\beta \in D$, $\exists\, N \in G_1$ such that $\xi_\alpha \in N$, $\forall\alpha \geq \beta$ and $[x_1 = a]_{\mathbb{C}_0} \subset N$.

Definition 3.1.4 ($\mathbb{C}_0$($\boldsymbol{P}$)**-Confluence**)**.** A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is said to be $\mathbb{C}_0$*(P)-confluence* to $[x_1 = a,\, x_2 = b]_{\mathbb{C}_0}$, if for every $\beta \in D$, $\exists\, N \in G_2$ such that $\xi_\alpha \in N$, $\forall\alpha \geq \beta$ and $[x_1 = a,\, x_2 = b]_{\mathbb{C}_0} \subset N$.

Definition 3.1.5 ($\mathbb{C}_0(\boldsymbol{L})$**-Confluence**)**.** A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is said to be $\mathbb{C}_0$*(L)-confluence* to $[x_1 = a,\, x_2 = b,\, x_3 = c]_{\mathbb{C}_0}$ of for every $\beta \in D$, $\exists N \in G_3$ such that $\xi_\alpha \in N$, $\forall \alpha \geq \beta$ and $[x_1 = a,\, x_2 = b,\, x_3 = c]_{\mathbb{C}_0} \subset N$.

Definition 3.1.6 ($\mathbb{C}_0$**-Point Confluence**)**.** A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is said to be $\mathbb{C}_0$-*Point confluence* to ξ, if for every $\beta \in D$, $\exists N \in G_4$ such that $\xi_\alpha \in N$, $\alpha \geq \beta$. This type of confluence is denoted as $\mathbb{C}_0\text{-}\lim\limits_{\alpha \in D} \xi_\alpha = \xi$

Remark 3.1.1. Let $\{\xi_\alpha\}$ be a $\mathbb{C}_2$-net on the directed set D and A be a subset of $\mathbb{C}_2$. If there is a $\beta \in D$ such that $\xi_\alpha \in A$, $\alpha > \beta$, $\forall \alpha \in D$ then $\{\xi_\alpha\}$ is called *finally in* A. If $\{\xi_\alpha\}$ is finally in every neighbourhood of ξ, then we say $\{\xi_\alpha\}$ is converges to ξ. Further, $\{\xi_\alpha\}$ is *stable* on ξ if $\xi_\alpha = \xi$, $\forall\, \alpha \in D$. It is *finally stable* on ξ if $\exists\, \beta \in D$ such that $\xi_\alpha = \xi$, $\forall\, \alpha \geq \beta$.

Remark 3.1.2. Note that if a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is $\mathbb{C}_0(F)$-confluence to $[x_1 = a]_{\mathbb{C}_0}$ it will not be finally in any $K \in G_2$ unless $\{x_{1\alpha}\}$ is finally stable at 'a'. Analogous cases will be explored with the other types of $\mathbb{C}_2$-nets in the real form.

Remark 3.1.3. The $\mathbb{C}_0$-Point confluence of a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is a necessary but not sufficient condition for the usual convergence of $\{\xi_\alpha\}$ in norm topology τ_1 induced by the Euclidean norm. Clearly, every finally stable net $\{x_{k\alpha}\}$, $1 \leq k \leq 4$, converges and thus, $\mathbb{C}_0$-Point confluence of $\{\xi_\alpha\}$ to ξ implies convergence of $\{\xi_\alpha\}$ to ξ in τ_1.

Example 3.1.1. For the verification of the insufficiency condition, consider the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ on the directed set $(\mathbb{Q}^+, \geq)$ defined as $\xi_\alpha = x_{1\alpha} + i_1 x_{2\alpha} + i_2 x_{3\alpha} + i_1 i_2 x_{4\alpha}$, $\forall \alpha \in D$, where $x_{k\alpha} = 1 + 1/(\alpha^2 + k^2)$, $1 \leq k \leq 4$. The net converges to $\xi = 1 + i_1 + i_2 + i_1 i_2$ in τ_1 but not $\mathbb{C}_0$-Point confluence to ξ. Because for any $\epsilon > 0$,

$$\xi_\alpha \notin (1 + i_1 + i_2 + (1 - \epsilon) i_1 i_2,\, 1 + i_1 + i_2 + (1 + \epsilon) i_1 i_2)_{\mathbb{C}_0}, \quad \forall a \in D.$$

Theorem 3.1.1. *A* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *is* $\mathbb{C}_0$*(F)-confluence to* $[x_1 = k]_{\mathbb{C}_0}$ *if and only if* $\{Re\, z_{1\alpha}\}$ *converges to* k *in* $\mathbb{C}_1$.

Proof. Consider that $\{Re\, z_{1\alpha}\}$ converges to k. For some given $\epsilon > 0$, consider a set as follows:

$$B_\epsilon = \{\xi \,:\, \xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4\,;\, k - \epsilon < x_1 < k + \epsilon\}\,. \tag{3.1.2}$$

Obviously, $B_\delta \in G_1$ and $[x_1 = k]_{\mathbb{C}_0} \subset B_\delta$. Since, $\{Re\, z_{1\alpha}\}$ converges to k, then $\exists\, \gamma \in D$ such that

$Re\, z_{1\alpha} \in (k - \epsilon,\, k + \epsilon) \quad \Rightarrow \quad k - \epsilon < x_{1\alpha} < k + \epsilon,\, \forall\, \alpha \geq \gamma$

$\Rightarrow \quad (k - \epsilon) + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \;\prec_{\mathbb{C}_0}\; Re\, z_{1\alpha} + i_1 x_{2\alpha} + i_2 x_{3\alpha} + i_1 i_2 x_{4\alpha}$

and $\;Re\, z_{1\alpha} + i_1 x_{2\alpha} + i_2 x_{3\alpha} + i_1 i_2 x_{4\alpha} \prec_{\mathbb{C}_0} \;(k + \epsilon) + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4,$

$\forall\, x_p,\, y_p \in \mathbb{C}_0,\, 2 \leq p \leq 4$ and $\forall\, \alpha \geq \gamma$. Therefore,

$$\xi_\alpha \in ((k - \epsilon) + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\, (k + \epsilon) + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4)_{\mathbb{C}_0},$$

$$\forall\, x_p,\, y_p \in \mathbb{C}_0,\, 2 \leq p \leq 4 \text{ and } \forall \alpha \geq \gamma.$$

Hence, $\{\xi_\alpha\}$ is finally in B_ϵ. Since, $\epsilon > 0$ is arbitrary and each element of G_1 contains a set B_ϵ (for some $\epsilon > 0$), then $\{\xi_\alpha\}$ is $\mathbb{C}_0(F)$-confluence to $[x_1 = k]_{\mathbb{C}_0}$. Conversely, let the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ be $\mathbb{C}_0(F)$-confluence to $[x_1 = k]_{\mathbb{C}_0}$. Then $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is finally in every member of G_1 containing the R-frame $[x_1 = k]_{\mathbb{C}_0}$. In particular, $\{\xi_\alpha\}$ is finally in B_ϵ $(\epsilon > 0)$ defined by the Equation (3.1.2). Thus, $\exists\, \beta \in D$ such that $\xi_\alpha \in B_\epsilon\,,\;\; \forall\, \alpha \geq \beta$

$\Rightarrow \quad a - \epsilon < Re\, z_{1\alpha} < a + \epsilon,\;\; \forall\, \alpha \geq \beta$

$\Rightarrow \quad Re\, z_{1\alpha} \to a.$ □

Theorem 3.1.2. *A* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *is* $\mathbb{C}_0(F)$*-confluence to* $[x_1 = k,\, x_2 = \ell]_{\mathbb{C}_0}$ *if and only if* $\{Re\, z_{1\alpha}\}$ *is finally stable at* k *and* $\{Im\, z_{1\alpha}\}$ *converges to* ℓ.

Proof. Suppose that $\{Re\, z_{1\alpha}\}$ be finally stable on k and $\{Im\, z_{1\alpha}\}$ converge to ℓ. Since $\{Re\, z_{1\alpha}\}$ is finally stable on k, $\exists\, \beta \in D$ such that $\forall\, \alpha \,\geq \beta$, $x_{1\alpha} = k$. For a given $\epsilon > 0$, construct a set as follows:

$$U_\epsilon = \{\xi \,:\, \xi = k + i_1x_2 + i_2x_3 + i_1i_2x_4\,;\, \ell - \epsilon < x_2 < \ell + \epsilon\}. \tag{3.1.3}$$

Then, $U_\epsilon \in G_2$ and $[x_1 = k,\, x_2 = \ell]_{\mathbb{C}_0} \subset U_\epsilon$. Since, the net $\{Im\, z_{1\alpha}\}$ converges to ℓ, then there exists some $\gamma \in D$ such that

$$Im\, z_{1\alpha} \in (\ell - \epsilon,\, \ell + \epsilon),\ \forall\, \alpha \geq \gamma$$

Also, for $\beta,\, \gamma \in D$, there exists some $\delta \in D$ such that $\delta \geq \beta$ and $\delta \geq \gamma$.

Therefore, $Re\, z_{1\alpha} = k$ and $Im\, z_{1\alpha} \in (\ell - \epsilon,\, \ell + \epsilon),\ \forall\, \alpha \geq \delta$.

$\Rightarrow\ \ell - \epsilon < Im\, z_{1\alpha} < \ell + \epsilon, \quad \forall\, \alpha \geq \delta$

$$\Rightarrow \xi_\alpha \in (k + i_1(\ell - \epsilon) + i_2x_3 + i_1i_2x_4,\, k + i_1(\ell + \epsilon) + i_2y_3 + i_1i_2y_4)_{\mathbb{C}_0},$$

$$\forall x_3,\, x_4,\, y_3,\, y_4 \in \mathbb{C}_0,\ \forall\, \alpha \geq \delta.$$

So that the net $\{\xi_\alpha\}$ is finally in $U_\epsilon(\epsilon > 0)$. Since every member of G_2 contains an U_ϵ, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is $\mathbb{C}_0$(P)-confluence to $[x_1 = k,\, x_2 = \ell]_{\mathbb{C}_0}$.

Conversely, suppose that the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is $\mathbb{C}_0$(P)-confluence to the R-plane $[x_1 = k,\, x_2 = \ell]_{\mathbb{C}_0}$. Then, it is finally in every member of the type $U_\epsilon > 0$ defined by the Equation (3.1.3) of the family G_2 containing the plane $[x_1 = k,\, x_2 = \ell]_{\mathbb{C}_0}$. So for given $\epsilon > 0$, there exists some $\beta \in D$ such that $\xi_\alpha \in U_\epsilon, \quad \forall\, \alpha \geq \beta$.

$\Rightarrow\ Re\, z_{1\alpha} = k$ and $Im\, z_{1\alpha} \in (\ell - \epsilon,\, \ell + \epsilon), \quad \forall\, \alpha \geq \beta$.

Hence, the net $\{Re\, z_{1\alpha}\}$ is finally stable on k and $\{Im\, z_{1\alpha}\}$ converges to ℓ. □

On the similar lines, the following theorems can be proved.

Theorem 3.1.3. *A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is $\mathbb{C}_0$(L)-confluence to $[x_1 = k, x_2 = \ell, x_3 = m]_{\mathbb{C}_2}$ if and only if the net $\{Re\, z_{1\alpha}\}$ is finally stable at k, $\{Im\, z_{1\alpha}\}$ is finally stable at ℓ and $\{Re\, z_{2\alpha}\}$ converges to m.*

Theorem 3.1.4. *A* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *is* $\mathbb{C}_0$*-Point confluence to* $a + bi_1 + ci_2 + di_1i_2$ *if and only if the net* $\{Re\, z_{1\alpha}\}$ *is finally stable at a,* $\{Im\, z_{1\alpha}\}$ *is finally stable at b,* $\{Re\, z_{2\alpha}\}$ *is finally stable at c and* $\{Im\, z_{2\alpha}\}$ *is converges to d.*

Theorem 3.1.5. *The following inferences can be proved:*

(i) Every $\mathbb{C}_0$*-Point confluence* $\mathbb{C}_2$*-net is* $\mathbb{C}_0(L)$*-confluence.*

(ii) Every $\mathbb{C}_0(L)$*-confluence* $\mathbb{C}_2$*-net is* $\mathbb{C}_0(P)$*-confluence.*

(iii) Every $\mathbb{C}_0(P)$*-confluence* $\mathbb{C}_2$*-net is* $\mathbb{C}_0(F)$*-confluence.*

The converses of these implications are not true, in general.

Proof. In fact, if a $\mathbb{C}_2$-net, we conclude that $\{\xi_\alpha\}$ is $\mathbb{C}_0$-Point confluence to $a + bi_1 + c\, i_2 + di_1i_2$, then it is $\mathbb{C}_0(L)$-confluence to R-line $[x_1 = a, , x_2 = b,\ x_3 = c]_{\mathbb{C}_0}$. Similarly, a $\mathbb{C}_2$-net which is $\mathbb{C}_0(L)$-confluence to R-line $[x_1 = a, , x_2 = b,\ x_3 = c]_{\mathbb{C}_0}$ is $\mathbb{C}_0(P)$-confluence to R-plane $[x_1 = a, , x_2 = b]_{\mathbb{C}_0}$ and a $\mathbb{C}_2$-net which is $\mathbb{C}_0(P)$-confluence to R-plane $[x_1 = a, , x_2 = b]_{\mathbb{C}_0}$ is $\mathbb{C}_0(F)$-confluence to $[x_1 = a]_{\mathbb{C}_0}$. $\square$

The converse is not true, in general.

Example 3.1.2. Consider a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ on the directed set $(\mathbb{Q}^+, \geq)$ as follows:

$$\{\xi_\alpha\} = (k - x_\alpha) + (1/\alpha)i_1 + (\alpha + 1)i_2 + \alpha i_1 i_2,\ \ \forall\, \alpha \in \mathbb{Q}^+, \tag{3.1.4}$$

where the net $\{x_\alpha\}$ is finally stable at 0. Then, the net $\{x_{1\alpha}\}$ is finally stable at 'k' and the net $\{x_{2\alpha}\}$ converges on 0. So that the bicomplex net $\{\xi_\alpha\}$ is $\mathbb{C}_0(P)$-confluence to the R-plane $[x_1 = k,\ x_2 = \ell]_{\mathbb{C}_0}$. Since, the net $\{x_{1\alpha}\}$ is finally stable on 'k'. Then, for each $M \in G_1$, $\exists\, \beta \in D$ such that $\xi_\alpha \in M$, $\forall\, \alpha \geq \beta$ and $[x_1 = k]_{\mathbb{C}_0} \subset M$. Hence, the net $\{\xi_\alpha\}$ is $\mathbb{C}_0(F)$-confluence to the frame $[x_1 = k]_{\mathbb{C}_0}$.

Example 3.1.3. Consider a $\mathbb{C}_2$-net $\{\eta_\alpha\}$ on the directed set $(\mathbb{Q}^+, \geq)$ as follows:

$$\{\eta_\alpha\} = k - (1/\alpha) + (\alpha + 1)i_1 + (\alpha + 2)i_2 + (\alpha + 3)i_1i_2,\ \ \forall\, \alpha \in \mathbb{Q}^+.$$

This $\mathbb{C}_2$-net is $\mathbb{C}_0$(F)-confluence to the R-frame $[x_1 = k]_{\mathbb{C}_0}$. Then the component net $\{x_{1\alpha}\}$ converges to 'k' but $\{x_{2\alpha}\}$ is not convergent. Therefore, the $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is not $\mathbb{C}_0$(P)-confluence to any R-plane contained in the R-frame $[x_1 = k]_{\mathbb{C}_0}$.

3.2 Confluences of $\mathbb{C}_2$-Nets in Id(o)-topology

We assume the bicomplex space, $\mathbb{C}_2$ to be furnished with the Id(o)-topology, τ_6 (cf. [72]). Now, rewriting the bicomplex net $\{\xi_\alpha\}$ as $\{{}^1\xi_\alpha e_1 + {}^2\xi_\alpha e_2\}$, where

$$ {}^1\xi_\alpha e_1 + {}^2\xi_\alpha e_2 = [(x_{1\alpha} + x_{4\alpha}) + i_1(x_{2\alpha} - x_{3\alpha})]e_1 + [(x_{1\alpha} - x_{4\alpha}) + i_1(x_{2\alpha} + x_{3\alpha})]e_2. \tag{3.2.1} $$

For the convenience, we represent the numbers $x_1 + x_4$ and $x_2 - x_3$ as $Re\,{}^1\xi$ and $Im\,{}^1\xi$, respectively. Thus $[Re\,{}^1\xi = a]_{Id}$ denote the frame $x_1 + x_4 = a$, whereas $\{Re\,{}^1\xi_\alpha\}$ denote the net $\{x_{1\alpha}+x_{4\alpha}\}$ and so on. Under these notations, the Equation (3.2.1) can be rewritten as

$$ {}^1\xi_\alpha e_1 + {}^2\xi_\alpha e_2 = (Re\ {}^1\xi_\alpha + i_1\, Im\ {}^1\xi_\alpha)e_1 + (Re\ {}^2\xi_\alpha + i_1\, Im\ {}^2\xi_\alpha)e_2. \tag{3.2.2} $$

Note that for the net $\{Re\ {}^1\xi_\alpha\}$ to be convergent, $\{x_{1\alpha} + x_{4\alpha}\}$ must be convergent. The conditions for the convergence of the nets $\{Im\ {}^1\xi_\alpha\}$, $\{Re\ {}^2\xi_\alpha\}$ and $\{Im\ {}^2\xi_\alpha\}$ can be similarly interpreted.

Definition 3.2.1 (**Id(F)-Confluence**)**.** The $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is said to be Id(F)-confluence to $[Re\,{}^1\xi = a]_{Id}$ if for every $K \in N_1$, $\exists\, \alpha \in D$ such that $\xi_\beta \in K$, $\forall \beta \geq \alpha$ and $[Re\,{}^1\xi = a]_{Id} \subset K$.

Definition 3.2.2 (**Id(P)-Confluence**)**.** The $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is said to be Id(P)-confluence to $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b]_{Id}$ if for every $K \in N_2$, $\exists\, \alpha \in D$ such that $\xi_\beta \in K$, $\forall \beta \geq \alpha$ and $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b]_{Id} \subset K$.

Definition 3.2.3 (**Id(L)-Confluence**)**.** The $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is said to be Id(L)-confluence to $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b,\ Re\ {}^1\xi = c]_{Id}$ if for every $K \in N_3$, $\exists\, \alpha \in D$ such that $\xi_\beta \in K$, $\forall \beta \geq \alpha$ and $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b,\ Re\ {}^1\xi = c]_{Id} \subset K$.

Definition 3.2.4 (**Id-Point Confluence**). The $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is said to be Id-Point confluence to the point $\xi = {}^1\xi e_1 + {}^2\xi e_2$ if it is finally in each $K \in N_4$ containing ξ.

Remark 3.2.1. If a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ defined by the Equation (3.2.2) is Id(F)-confluence to $[Re\ {}^1\xi = a]_{Id}$, it cannot be finally in any member of the family N_2 unless $\{Re\ {}^1\xi_\alpha\}$ is finally static on 'a'. Similar cases will arise with the other types of the confluences of the $\mathbb{C}_2$-nets with respect to the Id(o)-topology.

Theorem 3.2.1. *A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(F)-confluence to $[Re\ {}^1\xi = a]_{Id}$ if and only if the net $\{Re\ {}^1\xi_\alpha\}$ converges to 'a'.*

Proof. Assume that the net $\{Re\ {}^1\xi_\alpha\}$ converges to 'a'. Given $\epsilon > 0$, let

$$S_\epsilon \;=\; \{\xi \,:\, a - \epsilon < Re\ {}^1\xi < a + \epsilon\}, \tag{3.2.3}$$

be a member of N_1 such that $[Re\ {}^1\xi = a]_{Id} \subset S_\epsilon$. Since, the net $\{Re\ {}^1\xi_\alpha\}$ is converging to a, then there exists a $\beta \in D$ such that

$$a - \epsilon < Re\ {}^1\xi_\alpha < a + \epsilon,\ \ \forall\, \alpha \geq \beta$$

Hence, by the definition of $\prec$, we have $\forall\, \alpha \geq \beta$ and $\forall\, x_2,\, y_2 \in \mathbb{C}_2$

$$a - \epsilon + i_1 x_2 \prec (Re\ {}^1\xi_\alpha) + i_1(Im\ {}^1\xi_\alpha),$$

and

$$(Re\ {}^1\xi_\alpha) + i_1(Im\ {}^1\xi_\alpha) \prec a + \epsilon + i_1 y_2.$$

Therefore,

$${}^1\xi_\alpha e_1 + {}^2\xi_\alpha e_2 \in \,((a - \epsilon + i_1 x_2)e_1 + (x_3 + i_1 x_4)e_2,\, (a + \epsilon + i_1 x_2)e_1 + (y_3 + i_1 y_4)e_2)_{Id},$$

$\forall\, x_3,\, x_4,\, y_3,\, y_4 \in \mathbb{C}_0$ and $\forall\, \alpha \geq \beta$.

So the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is finally in S_ϵ. Since, $\epsilon > 0$ is arbitrary and every S in N_1 contains S_ϵ for some $\epsilon > 0$, by Definition (3.2.1), $\{\xi_\alpha\}$ is Id(F)-confluence to the ID-frame $[Re\ {}^1\xi = a]_{Id}$.

Conversely, let the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ be Id(F)-confluence to the ID-frame $[Re\ ^1\xi = a]_{Id}$. By definition, it is finally in every member S_ϵ (for $\epsilon > 0$, by Equation (3.2.3)) containing $[Re\ ^1\xi = a]_{Id}$. Then, $\exists\, \beta \in D$ such that

$\xi_\alpha \in S_\epsilon,\ \forall\, \alpha \geq \beta$

$\Rightarrow\ \ \xi_\alpha \in ((a - \epsilon + i_1 x_2)e_1 + (x_3 + i_1 x_4)e_2,\ (a + \epsilon + i_1 y_2)e_1 + (y_3 + i_1 y_4)e_2)_{ID}$

$$\forall\, x_p,\, y_p \in \mathbb{C}_0,\ 2 \leq p \leq 4 \ \text{ and } \ \forall\, \alpha \geq \beta.$$

By definition of N_1 and S_ϵ, we infer

$a - \epsilon < Re\ ^1\xi_\alpha < a + \epsilon$

$\Rightarrow\ \ Re\ ^1\xi_\alpha \in (a - \epsilon,\, a + \epsilon),\ \ \forall\, \alpha \geq \beta$

$\Rightarrow\ \ \{Re\ \xi_\alpha\}\ \to\ a.$

Hence proved. $\square$

Theorem 3.2.2. *The $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(P)-confluence to $[Re\ ^1\xi = a,\ Im\ ^1\xi = b]_{Id}$ if and only if $\{Re\ ^1\xi_\alpha\}$ is finally stable on 'a' and $\{Im\ ^1\xi_\alpha\}$ converges to 'b'.*

Proof. Suppose that the net $\{Re\ ^1\xi_\alpha\}$ is finally stable at a and $\{Im\ ^1\xi_\alpha\}$ converges to b. Since $\{Re\ ^1\xi_\alpha\}$ is finally stable on a. Then, $\exists\, \beta \in D$ such that

$$Re\ ^1\xi_\alpha = a,\ \ \forall\, \alpha \geq \beta.$$

Let $\epsilon > 0$ be given, suppose that

$$F_\epsilon \ = \ \{\xi\ :\ Re\ ^1\xi = a,\ b - \epsilon < Im\ ^1\xi < b + \epsilon\}, \tag{3.2.4}$$

is a member of N_2 containing the ID-plane $[Re\ ^1\xi = a,\ Im\ ^1\xi = b]_{Id}$.

Since, the net $\{Im\ \xi_\alpha\}$ converges to b, $\exists\, \gamma \in D$ such that

$$Im\ ^1\xi_\alpha \in (b - \epsilon,\, b + \epsilon),\ \forall\, \alpha \geq \gamma.$$

As β, $\gamma \in D$. Then, there exists some $\delta \in D$ such that $\delta \geq \beta$ and $\delta \geq \gamma$.

Therefore,

$$Re\ {}^1\xi_\alpha = a \quad \text{and} \quad Im\ {}^1\xi_\alpha \in (b - \epsilon,\, b + \epsilon),\ \forall\, \alpha \geq \delta \tag{3.2.5}$$

$$\Rightarrow\ \xi_\alpha \in ((a + i_1(b - \epsilon))e_1 + (x_3 + i_1 x_4)e_2,\, (a + i_1(b + \epsilon))e_1 + (y_3 + i_1 y_4)e_2)_{Id},$$

$$\forall\, x_3,\, x_4,\, y_3,\, y_4 \in C_0,\ \forall\, \alpha \geq \delta.$$

Thus, $\{\xi_\alpha\}$ is finally in F_ϵ. Since $\epsilon > 0$ is arbitrary and every member of N_2 contains an F_ϵ (for some $\epsilon > 0$), by the definition, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(P)-confluence to the ID-plane $[Re\ {}^1\xi = a,\, Im\ {}^1\xi = b]_{Id}$.

Conversely, suppose that the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(P)-confluence to the ID-plane $[Re\ {}^1\xi = a,\, Im\ {}^1\xi = b]_{Id}$. Therefore, $\{\xi_\alpha\}$ is finally in every member of the family N_2 containing the ID-plane $[Re\ {}^1\xi = a,\, Im\ {}^1\xi = b]_{Id}$. In particular, the net is finally in every open ID-frame segment F_ϵ ($\epsilon > 0$) defined by the Equation (3.2.4) containing the ID-plane $[Re\ {}^1\xi = a,\, Im\ {}^1\xi = b]_{Id}$. For given $\epsilon > 0$, there exists some $\beta \in D$ such that $\xi_\alpha \in F_\epsilon$, $\alpha \geq \beta$. Then, by the Equation (3.2.4)

$$x_{1\alpha} = a \text{ and } x_{2\alpha} \in (b - \epsilon,\, b + \epsilon),\ \ \forall\, \alpha \geq \beta$$

Hence the theorem. $\square$

On similar lines, the following theorems can be proved.

Theorem 3.2.3. *A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(L)confluence to the ID-line $[Re\ {}^1\xi = a,\, Im\ {}^1\xi = b,\, Re\ {}^2\xi = c]_{Id}$ if the net $\{Re\ {}^1\xi_\alpha\}$ is finally stable on 'a', the net $\{Im\ {}^1\xi_\alpha\}$ is finally stable on 'b' and $\{Re\ {}^2\xi_\alpha\}$ converges to 'c'.*

Theorem 3.2.4. *A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id-Point confluence to the point $\xi = (a + i_1 b)e_1 + (c + i_1 d)e_2$, if the net $\{Re\ {}^1\xi_\alpha\}$ is finally stable on 'a', the net $\{Im\ {}^1\xi_\alpha\}$ is finally stable on 'b', the net $\{Re\ {}^2\xi_\alpha\}$ is finally stable on 'c' and the net $\{Im\ {}^2\xi_\alpha\}$ converges to 'd'.*

Theorem 3.2.5. *The following implications can be proved:*

(i) Every Id-Point confluence $\mathbb{C}_2$-net is Id(L)-confluence.

(ii) Every Id(L)-confluence $\mathbb{C}_2$-net is Id(P)-confluence.

(iii) Every Id(P)-confluence $\mathbb{C}_2$-net is Id(F)-confluence.

The converses of these implications are not true, in general.

Proof. In fact, if a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id-Point confluence to the point $\xi = (a + i_1 b)e_1 + (c + i_1 d)e_2$, say, then it is Id(L)-confluence to the ID-line $[Re\ ^1\xi = a,\ Im\ ^1\xi = b,\ Re\ ^2\xi = c]_{Id}$. Further, a $\mathbb{C}_2$-net which is Id(L)-confluence to $[Re\ ^1\xi = a,\ Im\ ^1\xi = b,\ Re\ ^2\xi = c]_{Id}$ is Id(P)-confluence to the ID-plane $[Re\ ^1\xi = a,\ Im\ ^1\xi = b]_{Id}$. Furthermore, a $\mathbb{C}_2$-net which is Id(P)-confluence to the ID-plane $[Re\ ^1\xi = a,\ Im\ ^1\xi = b]_{Id}$ is Id(F)-confluence to ID-frame $[Re\ ^1\xi = a]_{Id}$. $\square$

an example is discussed below for which Id(F)-confluence $\mathbb{C}_2$-net is also Id(P)-confluence and an example for Id(F)-confluence $\mathbb{C}_2$-net which is not Id(P)-confluence.

Example 3.2.1. Consider the directed set $(\mathbb{Q}^+, \geq)$. Define the $\mathbb{C}_2$-net

$$\{\xi_\alpha\} = \{(a - x_\alpha + \alpha^2) + i_1(a - (1/\alpha) + \alpha^3) + i_2(\alpha^3 - (1/\alpha) - a) + i_1 i_2(a - x_\alpha - \alpha^2)\},$$

where $\{x_\alpha\}$ is finally stable on 0, $\forall\, \alpha \in \mathbb{Q}^+$.

By the Equation (3.2.1), the net $\{Re\ ^1\xi_\alpha\}$ is finally stable on $2a$ and then converging to $2a$, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is finally in every element of N_1 containing the ID-frame $[Re\ ^1\xi = 2a]_{Id}$. Thus the net is Id(F)-confluence to the ID-frame $[Re\ ^1\xi = 2a]_{Id}$. Also the net $\{Re\ ^1\xi_\alpha\}$ is finally stable on $2a$ and the net $\{Im\ ^1\xi_\alpha\}$ is converging to $2a$. So, $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(P)-confluence to $[Re\ ^1\xi = 2a,\ Im\ ^1\xi = 2b]_{Id}$.

Example 3.2.2. Consider the $\mathbb{C}_2$-net

$$\{\xi_\alpha\} = \{(a - x_\alpha + \alpha) + i_1((1/\alpha) + \alpha^2) + i_2(-(1/\alpha) + \alpha^2) + i_1 i_2(a - x_\alpha - \alpha)\},$$

where $\forall\, \alpha \in \mathbb{Q}^+$. By Equation (3.2.1), the net $\{Re\ ^1\xi_\alpha\}$ is finally stable on $2a$ and $\{Im\ ^1\xi_\alpha\}$ converges to 0. Therefore, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(F)-confluence to the ID-frame $[Re\ ^1\xi = 2a]_{Id}$. Note that whilst $\{Re\ ^1\xi_\alpha\}$ is converging to $2a$, it is not finally stable on $2a$. Thus, $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is not Id(P)-confluence to any of the ID-plane contained in the ID-frame $[Re\ ^1\xi = 2a]_{Id}$.

Definition 3.2.5 (Cauchy $\mathbb{C}_2$-Net). A $\mathbb{C}_2$-net $\{\xi_\alpha\}_{\alpha\in D}$ is said to be a Cauchy $\mathbb{C}_2$-net if the $\mathbb{C}_2$-net $\{\xi_\alpha - \xi_\beta\}_{(\alpha,\beta)\in D\times D}$ is finally in every neighbourhood $U(0)$ of 0.

Remark 3.2.2. As for Cauchy $\mathbb{C}_2$-net $\{\xi_\alpha\}_{\alpha\in D}$, the net $\{\xi_\alpha - \xi_\beta\}_{(\alpha,\beta)\in D\times D}$ is finally stable in every neighbourhood of zero, hence ID-point confluence to zero. Then the nets $\{^1\xi_\alpha - {}^1\xi_\beta\}_{(\alpha,\beta)\in D\times D}$ and $\{Re\,(^2\xi_\alpha - {}^2\xi_\beta)\}_{(\alpha,\beta)\in D\times D}$ are both finally stable at 0 and the net $\{Im\,(^2\xi_\alpha - {}^2\xi_\beta)\}_{(\alpha,\beta)\in D\times D}$ is converging to zero.

Theorem 3.2.6. *A $\mathbb{C}_2$-net $\{\xi_\alpha\}_{\alpha\in D}$ is a Cauchy $\mathbb{C}_2$-net if and only if the net $\{\xi_\alpha - \xi_\beta\}_{(\alpha,\beta)\in D\times D}$ is ID-point confluence at 0.*

Proof. Let the $\mathbb{C}_2$-net $\{\xi_\alpha - \xi_\beta\}_{(\alpha,\beta)\in D\times D}$ be ID-point confluence at 0. Then the nets $\{Re\ (^1\xi_\alpha - {}^1\xi_\beta)\}$, $\{Im\ (^1\xi_\alpha - {}^1\xi_\beta)\}$, $\{Re\ (^2\xi_\alpha - {}^2\xi_\beta)\}$ are finally stable at 0 and the net $\{Im\ (^2\xi_\alpha - {}^2\xi_\beta)\}$ converges to 0. Since the net $\{Im\ (^2\xi_\alpha - {}^2\xi_\beta)\}$ converges to 0 in $\mathbb{C}_0$ and $\mathbb{C}_0$ is a Banach space. Thus, the net $\{Im\ (^2\xi_\alpha - {}^2\xi_\beta)\}$ is a Cauchy net in $\mathbb{C}_0$. Therefore, for each neighbourhood $U(0)$, there exists $(\gamma,\delta) \in D\times D$ such that $\xi_\alpha - \xi_\beta \in U(0)$, $\forall(\alpha,\beta) > (\gamma,\delta)$. The converse is straight forward. □

Corollary 3.2.1. Every ID-point confluence $\mathbb{C}_2$-net is a Cauchy $\mathbb{C}_2$-net.

3.3 $\mathbb{C}_2$-Nets and their Projection Nets

This section is dedicated to the study of correspondence between confluence of $\mathbb{C}_2$-nets and the convergence of their projection nets (cf. [78]) in Id(p)-topology.

Theorem 3.3.1. *A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ converges to ξ in Id(p)-topology iff $\{^k\xi_\alpha\}$ is confluence to $^k\xi$ in $\mathbb{A}_k$, for $k = 1, 2$.*

Proof. If the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ converges to ξ, then it is finally in every neighbourhood of ξ with respect to the Id(p)-topology. Note that $\pi_k(\xi_\alpha) = \{^k\xi_\alpha\}$ in A_k is finally in every neighbourhood of $\pi_k(\xi)$, k = 1, 2. Hence the net $\{\xi_\alpha\}$ in $\mathbb{A}_k$ is confluence to $\mathbb{A}_k$, $k = 1, 2$.

The converse is straight forward. □

Remark 3.3.1. The related results are not true for any type of ID-confluence (except Id-Point confluence) of the $\mathbb{C}_2$-nets w.r.t. Id(o)-topology on $\mathbb{C}_2$. Moreover, there is a characteristic difference between the confluence of the $\mathbb{C}_2$-nets in the Id(p)-topology and the confluence in the Id(o)-topology in the sense that for any type of confluence (except Id-Point confluence) of a $\mathbb{C}_2$-net with respect to the Id(o)-topology is not necessarily to have all component nets to be convergent. We are giving the following results in this frame of reference.

Theorem 3.3.2. *If the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(F)-confluence to $[Re\ {}^1\xi = a]_{Id}$, then the net $\{{}^1\xi_\alpha\}$ is confluence to the line $x = a$ in $\mathbb{A}_1$.*

Proof. Assume that the $\mathbb{C}_2$-net defined by the Equation (3.2.2) is Id(F)-confluence to $[Re\ {}^1\xi = a]_{Id}$. Then, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is finally in each element of N_1 containing $[Re\ {}^1\xi = a]_{Id}$. Now, the projection of every $K \in N_1$ on $\mathbb{A}_1$ is a plane segment in $\mathbb{A}_1$ and then it is a basis element of the lexicographical order topology on $\mathbb{A}_1$. So, $\{{}^1\xi_\alpha\}$ is finally in every basis element of lexicographical order topology on $\mathbb{A}_1$ containing the line $x = a$ in $\mathbb{A}_1$. Hence $\{{}^1\xi_\alpha\}$ is confluence to line $x = a$ in $\mathbb{A}_1$. □

Theorem 3.3.3. *If $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(P)-confluence to the ID-plane $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b]_{Id}$, then $\{{}^1\xi_\alpha\}$ is confluence to $a + i_1 b$ in $\mathbb{A}_1$.*

Proof. Suppose that the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ defined by the Equation (3.2.2), is Id(P) confluence to $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b]_{Id}$. Therefore, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is finally in every member of the family N_2 containing $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b]_{Id}$. Note that the projection of every member of the family N_2 on the auxiliary space $\mathbb{A}_1$ is a basis element of lexicographical order topology on $\mathbb{A}_1$ containing the point $a + i_1 b$. So the projection net $\{{}^1\xi_\alpha\}$ is finally in every basis element of lexicographical order topology on the auxiliary complex space $\mathbb{A}_1$ containing $a + i_1 b$. Thus, the net $\{{}^1\xi_\alpha\}$ confluence to $a + i_1 b$ in $\mathbb{A}_1$. □

Theorem 3.3.4. *If the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id(L) confluence to the ID-line $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b,\ Re\ {}^2\xi = c]_{Id}$, then the projection net $\{{}^1\xi_\alpha\}$ converges to $a + i_1 b$ in $\mathbb{A}_1$ and $\{{}^2\xi_\alpha\}$ is confluence to the line $x = c$ in $\mathbb{A}_2$.*

Proof. Let the $\mathbb{C}_2$-net $\{\xi_\alpha\}$, given by the Equation (3.2.2) be Id(L) confluence to $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b,\ Re\ {}^2\xi = c]_{Id}$. Then, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is finally in every member of the family N_3 containing $[Re\ {}^1\xi = a,\ Im\ {}^1\xi = b,\ Re\ {}^2\xi = c]_{Id}$. Since, the projection of every member of the family N_3 on $\mathbb{A}_1$ is a point $a + i_1\, b$. Then, the projection net $\{{}^1\xi_\alpha\}$ in $\mathbb{A}_1$ is finally stable at $a + i_1\, b$ in $\mathbb{A}_1$, so it converges to the point $a + i_1\, b$ in $\mathbb{A}_1$. The projection on $\mathbb{A}_2$ of every member of N_3 is a plane segment in $\mathbb{A}_2$, which is a basis element of the lexicographical order topology on $\mathbb{A}_2$. Therefore, the projection net $\{{}^2\xi_\alpha\}$ in $\mathbb{A}_2$ is finally in every basis element of the lexicographical order topology on $\mathbb{A}_2$ containing the line $x = c$ in $\mathbb{A}_2$. Thus, the component net $\{{}^2\xi_\alpha\}$ in $\mathbb{A}_2$ is confluence to line $x = c$ in $\mathbb{A}_2$. $\square$

Theorem 3.3.5. *If the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is Id-Point confluence to $(a + i_1 b)e_1 + (c + i_1 d)e_2$, then the net $\{{}^1\xi_\alpha\}$ converge to $a + i_1\, b$ in $\mathbb{A}_1$ and the net $\{{}^2\xi_\alpha\}$ confluence to $c + i_1\, d$ in $\mathbb{A}_2$.*

Proof. Let the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ be Id-Point confluence to the point $(a+i_1 b)e_1(c+i_1 d)e_2$. Then, it is finally in every member of the family N_4 containing $(a+i_1 b)e_1(c+i_1 d)e_2$. So that the projection net $\{{}^1\xi_\alpha\}$ in $\mathbb{A}_1$ is confluence to the point $a + i_1\, b$ in $\mathbb{A}_1$ and the projection net $\{{}^2\xi_\alpha\}$ in $\mathbb{A}_2$ is confluence to the point $c + i_1\, d$. $\square$

Example 3.3.1. Consider the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ as defined in the Example 3.2.1.

The net $\{Re\ {}^1\xi_\alpha\}$ converges to 2a and the net $\{Im\ {}^1\xi_\alpha\}$ converges to 2a but the nets $\{Re\ {}^2\xi_\alpha\}$ and $\{Im\ {}^2\xi_\alpha\}$ are not convergent. Since, the component nets $\{Re\ {}^2\xi_\alpha\}$ and $\{Im\ {}^2\xi_\alpha\}$ are not convergent. Therefore, the projection net $\{{}^2\xi_\alpha\}$ in $\mathbb{A}_2$ is not confluence in $\mathbb{A}_2$. So the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ does not converge in the Id(p)-topology. But the projection net $\{{}^1\xi_\alpha\}$ in $\mathbb{A}_1$ is confluence to the point $2a + 2a\, i_1$ in $\mathbb{A}_2$. Hence the $\mathbb{C}_2$-net is I(P) confluence to $[Re\ {}^1\xi = 2a,\ Im\ {}^1\xi = 2a]_{Id}$.

Example 3.3.2. Define a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ on the directed set $(\mathbb{Q}^+, \geq)$ as follows:

$$\xi_\alpha = (a - k_\alpha) + i_1(b - h_\alpha + (1/\alpha)) + i_2(-b + h_\alpha + (1/\alpha)) + i_1 i_2 k_\alpha, \forall\, \alpha \in \mathbb{Q}^+,$$

where $\{h_\alpha\}$ and $\{k_\alpha\}$ are finally stable at 0, $\forall\, \alpha \in \mathbb{Q}^+$. The projection net $\{{}^1\xi_\alpha\}$ in $\mathbb{A}_1$ is confluence to the point $a + 2b\, i_1$ in $\mathbb{A}_1$ and the projection net $\{{}^2\xi_\alpha\}$ in $\mathbb{A}_2$ is confluence to the point $a + 0\, i_1$ in $\mathbb{A}_2$. Therefore, the $\mathbb{C}_2$-net is Id-Point confluence to the point $(a + 2b\, i_1)e_1 + (a + 0\, i_1)e_2$. As, all of the component nets of the $\mathbb{C}_2$-net are convergent. Thus, the $\mathbb{C}_2$-net converges to the point $(a + 2b\, i_1)e_1 + (a + 0\, i_1)e_2$ in the norm topology, τ_1.

Conclusion

The concept of nets is considered as generalization of sequences. By using the different representations of the bicomplex numbers, three distinct types of nets known as $\mathbb{C}_2$-nets are developed on the bicomplex numbers. The convergence called confluence in the $\mathbb{C}_0$(o)-topology and Id(o)-topology is studied. The product topological space with the Id(p)topology is constructed as the product of the two order topological spaces as $\mathbb{C}_2 = \mathbb{A}_1 \times_e \mathbb{A}_2$. The convergence of the $\mathbb{C}_2$-nets in the Id(p)-topology has been studied using the convergence of the components in $\mathbb{C}_0$ space.

□ □ □

Chapter 4

$\mathbb{C}_2$-Nets, $\mathbb{C}_2$-Filters and their Zones of Clustering

This chapter emphasis on the concept of clustering of $\mathbb{C}_2$-nets. The clustering on different types of zones in the bicomplex space have been defined. The main focus is on $\mathbb{C}_2$ equipped with Id(o)-topology, τ_6. Further the chapter is divided into four sections. In section 4.1, the concept of clustering of the $\mathbb{C}_2$-nets in the Id(o)-topology. The conditions for the clustering of $\mathbb{C}_2$-nets in different Id-zones in the Id(o)-topology are discussed in detail. In section 4.2, the Id-confluence and the clustering of $\mathbb{C}_2$-nets, and their $\mathbb{C}_2$-subnets is discussed. In this section, the $\mathbb{C}_2$-subnets are defined on the cofinal subsets of the directed sets of $\mathbb{C}_2$-nets.

In section 4.3, the topological properties such as compactness, countability and the homeomorphism of some subsets of the bicomplex space are discussed. It have been shown that the principal ideals $\mathbb{I}_1$ and $\mathbb{I}_2$ are nowhere dense subsets of $\mathbb{C}_2$ in the Id(o)-topology. The investigations have been made connecting the clustering and Id-confluence of the $\mathbb{C}_2$-nets, and their $\mathbb{C}_2$-subnets. In section 4.4, the concept of $\mathbb{C}_2$-filter is discussed in brief. The Id-confluence of the $\mathbb{C}_2$-filters is analogous to the concept of Id-confluence of $\mathbb{C}_2$-nets. Some properties of the $\mathbb{C}_2$-filters are also discussed in detail.

4.1 Clustering of $\mathbb{C}_2$-nets in Id(o)-topology

In this section, the concept of clustering of $\mathbb{C}_2$-nets and their confluences on various types of zones in $\mathbb{C}_2$ are studied. In this section, the clustering of a $\mathbb{C}_2$-net in different types of zones is defined and investigate the conditions required for the clustering of the $\mathbb{C}_2$-net. One can refer the Definition 3.1.1 for $\mathbb{C}_2$-net. Some definitions about clustering of the $\mathbb{C}_2$-nets are as follows:

Definition 4.1.1 (Cofinal Set). A subset K of directed set P is said to be *cofinal* in P, if for each $\alpha \in P$, there exists some $\gamma \in K$ such that $\gamma \geq \alpha$ (for details cf. [78]).

Example 4.1.1. The set of integers, $\mathbb{Z}$ is a cofinal subset of the set of rational numbers, $\mathbb{Q}$.

Definition 4.1.2. Let $\{\xi_\alpha\}$ be a $\mathbb{C}_2$-net. Then

(i) $\{\xi_\alpha\}$ cluster on $[Re^1\xi_\alpha = a]_{Id}$ if for every $K \in N_1$ and $\alpha \in D$, $\exists\, \beta \in D$ with $\beta \geq \alpha$ such that $\xi_\beta \in K$, and $[Re^1\xi_\alpha = a]_{Id} \subset K$.

(ii) $\{\xi_\alpha\}$ cluster on $[Re^1\xi = a,\, Im^1\xi = b]_{Id}$ if for every $K \in N_2$ and $\alpha \in D$, $\exists\, \beta \in D$ with $\beta \geq \alpha$ such that $\xi_\beta \in K$ and $[Re^1\xi = a,\, Im^1\xi = b]_{Id} \subset K$.

(iii) $\{\xi_\alpha\}$ cluster on $[Re^1\xi = a,\, Im^1\xi = b,\, Re^2\xi = c]_{Id}$ if for every $K \in N_3$ and $\alpha \in D$, $\exists\, \beta \in D$ with $\beta \geq \alpha$ such that $\xi_\beta \in K$ and $[Re^1\xi = a,\, Im^1\xi = b,\, Re^2\xi = c]_{Id} \subset K$.

(iv) $\{\xi_\alpha\}$ cluster on the point ξ if for every $K \in N_4$ and $\alpha \in D$, $\exists\, \beta \in D$ with $\beta \geq \alpha$ such that $\xi_\beta \in K$ and $\xi \in K$.

Remark 4.1.1. If any $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is frequently in every member of the family N_4 containing ξ, then it is frequently in every member of the basis $\mathbb{B}_6$ of the Id(o)-topology containing ξ.

Theorem 4.1.1. *If the* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *clusters on* ξ *in the Id(o)-topology, then it clusters on* ξ *in the Id(p)-topology.*

Proof. Consider that the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ cluster on the point $\xi = (a+i_1b)e_1+(c+i_1d)e_2$ in the Id(o)-topology. Let $P = ({}^1\zeta, {}^1\eta) \times_e ({}^2\zeta, {}^2\eta)$ be an arbitrary basis element of the Id(p)-topology containing the bicomplex point $\xi = (a+i_1b)e_1+(c+i_1d)e_2$. Thus, $a+i_1b \in ({}^1\zeta,\, {}^1\eta)$ and $c+i_1d \in ({}^2\zeta,\, {}^2\eta)$. Then for some $\epsilon > 0$, we have

$$(c+i_1d-\epsilon,\, c+i_1d+\epsilon) \subset ({}^2\zeta,\, {}^2\eta)$$

Hence,

$$K = ((a+i_1b)e_1+(c+i_1(d-\epsilon))e_2, (a+i_1b)e_1+(c+i_1(d+\epsilon))e_2)_{ID} \subset P$$

Clearly, $K \in N_4$, $\xi \in K$ and $K \subset P$. Now as $\{\xi_\alpha\}$ clusters on $\xi = (a+i_1b)e_1+(c+i_1d)e_2$ in the Id(o)-topology, it is frequently in every member of N_4 containing ξ. In particular, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ frequently in every open ID-line segment containing ξ. Hence it is frequently in K. So it is frequently in P. Since P is an arbitrary basis element of the Id(p)- topology containing ξ, then $\{\xi_\alpha\}$ is frequently in every basis element of the Id(p)-topology containing ξ. Therefore, $\{\xi_\alpha\}$ clusters on ξ in the Id(p)-topology. □

Remark 4.1.2. The converse for this theorem is not true, in general, i.e. if $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on a point ξ in the Id(p)-topology, then it may or may not cluster on ξ in the Id(o)-topology.

Example 4.1.2. Consider the directed set $(\mathbb{Q}^+, \geq)$. Define the $\mathbb{C}_2$-net

$$\{\xi_\alpha\} = \left\{a + i_1(1/\alpha^2) - i_2(2/\alpha^2) + i_1i_2 0\right\}, \forall \alpha \in \mathbb{Q}^+.$$

Since, both the nets $\{{}^1\xi_\alpha\}$ and $\{{}^2\xi_\alpha\}$ cluster on a. Then, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on the point $a \in \mathbb{C}_0$ with respect to the Id(p)-topology. However, although the net $\{Re\,{}^1\xi_\alpha\}$ is stable at a, the net $\{Im\,{}^1\xi_\alpha\}$ does not attain the value '0', frequently. Therefore, $\{\xi_\alpha\}$ is not frequently in any member of the family N_4 containing the point 'a'. Hence, it does not cluster on a in the Id(o)-topology.

Theorem 4.1.2. *A* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *clusters on an ID- frame* $[Re\,{}^1\xi = k]_{Id}$ *if and only if* k *is a cluster point of* $\{Re\,{}^1\xi_\alpha\}$.

Proof. Consider that the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on an ID-frame $[Re\,^1\xi = k]_{Id}$. Then it is frequently in each $P \in N_1$ such that $[Re\,^1\xi = k]_{Id} \subset P$. Therefore, for given $\epsilon > 0$ and $\forall\, \beta \in D$, $\exists\, \alpha \in D$, $\alpha \geq \beta$ such that $\forall\, x_p,\, y_p \in \mathbb{C}_0$, $p = 2, 3, 4$, we have

$$\xi_\alpha \in ((k - \epsilon + i_1 x_2)e_1 + (x_3 + i_1 x_4)e_2, (k + \epsilon + i_1 y_2)e_1 + (y_3 + i_1 y_4)e_2)_{Id},$$

$\Rightarrow k - \epsilon < Re\,^1\xi_\alpha < k + \epsilon.$

Then, for each $\beta \in D$, there is some $\alpha \in D$ such that

$$k - \epsilon < Re\,^1\xi_\alpha < k + \epsilon.$$

Hence, a is the cluster point of $\{Re\,^1\xi_\alpha\}$.

Conversely, let a be a cluster point of the net $\{Re\,^1\xi_\alpha\}$. Let $\epsilon > 0$ be given. Then for each $\beta \in D$, there exists some $\alpha \in D$, $\alpha \geq \beta$ such that

$$Re\,^1\xi_\alpha \in (k - \epsilon,\, k + \epsilon) \tag{4.1.1}$$

$\Rightarrow\;\; k - \epsilon < Re\,^1\xi_\alpha < k + \epsilon$

$\Rightarrow\;\; (k - \epsilon + i_1 x_2)e_1 + (x_3 + i_1 x_4)e_2 \;\prec_{Id}\; {}^1\xi_\alpha\, e_1 + {}^2\xi_\alpha\, e_2$

and $\quad {}^1\xi_\alpha\, e_1 + {}^2\xi_\alpha\, e_2 \;\prec_{ID}\; (k + \epsilon + i_1 y_2)\, e_1 + (y_3 + i_1 y_4)\, e_2$

$\Rightarrow\;\; \xi_\alpha \in ((k - \epsilon + i_1 x_2)e_1 + (x_3 + i_1 x_4)e_2, (k + \epsilon + i_1 y_2)e_1 + (y_3 + i_1 y_4)e_2)_{ID},$

$$\forall\, x_p,\, y_p \in \mathbb{C}_0,\; 2 \leq p \leq 4.$$

Therefore, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on $[Re\,^1\xi = k]_{Id}$. □

Remark 4.1.3. In view of the above theorem one may intuitively infer its analogue for clustering on an ID-plane viz., a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on the ID-plane $[Re\,^1\xi = k,\, Im\,^1\xi = \ell]_{Id}$ if the net $\{Re\,^1\xi_\alpha\}$ attains the value k frequently and ℓ is the cluster point of $\{Im\,^1\xi_\alpha\}$.

However, this result does not hold good in the general setup.

Example 4.1.3. Let D' and D'' be two infinite and disjoint cofinal subsets of D. Define $\{\xi_\alpha\}$ as follows:

(i) $Re\,{}^1\xi_\alpha = k, \quad \forall \alpha \in D'$ and nowhere else.

(ii) Given $\epsilon > 0$ and given $\beta \in D$, $\exists \gamma \in D''$, $\gamma \geq \beta$ such that $Im\,{}^1\xi_\gamma \in (\ell - \epsilon,\ \ell + \epsilon)$.

Note that D' and D'' are disjoint. In particular, this implies there is no member of D for which both the conditions (i) and (ii) are attained, simultaneously. Therefore, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ cannot cluster on the ID-plane $[Re\,{}^1\xi = k,\ Im\,{}^1\xi = \ell]_{Id}$.

Theorem 4.1.3. *The* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *clusters on* $[Re\,{}^1\xi = k,\ Im\,{}^1\xi = \ell]_{Id}$ *only if for a cofinal subset* D' *of* D *such that* $\{Re\,{}^1\xi_\alpha\}_{D'}$ *is stable at* k *and the net* $\{Im\,{}^1\xi_\alpha\}_{\alpha \in D'}$ *clusters on the point* ℓ*.*

Proof. Suppose that $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on an ID-plane $[Re\,{}^1\xi = k,\ Im\,{}^1\xi = \ell]_{Id}$. Then, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is frequently in every member of N_2 containing $[Re\,{}^1\xi = k,\ Im\,{}^1\xi = \ell]_{Id}$.

Let $\epsilon > 0$ be given. Consider a set $F_\epsilon \in N_2$ defined as

$$F_\epsilon = \left\{\xi\,:\, Re\,{}^1\xi = k,\, \ell - \epsilon < Im\,{}^1\xi\ <\ \ell + \epsilon\right\}.$$

Since $\{\xi_\alpha\}$ is frequently in F_ϵ, given $\epsilon > 0$ and given $\beta \in D$, there $\exists\,\alpha \in D$, $\alpha \geq \beta$ such that $\forall x_3, x_4, y_3, y_4 \in \mathbb{C}_0$

$$\xi_\alpha \in ((k + i_1(\ell - \epsilon))e_1 + (x_3 + i_1 x_4)e_2,\ (k + i_1(\ell + \epsilon))e_1 + (y_3 + i_1 y_4)e_2)_{Id},$$

$$\Rightarrow Re\,{}^1\xi_\alpha = k \quad \text{and} \quad \ell - \epsilon < Im\,{}^1\xi\ <\ \ell + \epsilon.$$

Define a subset D' of D as follows:

$$D' = \left\{\alpha\,:\, \alpha \in D, Re\,{}^1\xi_\alpha = k\right\}.$$

Then, D' is the desired cofinal subset of D. $\square$

If we define a $\mathbb{C}_2$-net $\{\eta_\beta\}$ on D' as $\eta_\beta = \xi_\beta$, $\forall\, \beta \in D'$, we see that $\{\eta_\beta\}$ is a $\mathbb{C}_2$-subnet of $\{\xi_\alpha\}_{\alpha \in D}$. Now, if $\{\xi_\alpha\}_{\alpha \in D}$ clusters on $[Re^{\,1}\xi = k,\, Im^{\,1}\xi = \ell]_{Id}$, then $\{Re^{\,1}\eta_\beta\}_{D'}$ is stable at k and the net $\{Im^{\,1}\eta_\beta\}_{D'}$ clusters on ℓ.

Hence, the above theorem can be reworded as:

Theorem 4.1.4. *A* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *clusters on* $[Re^{\,1}\xi = k,\, Im^{\,1}\xi = \ell]_{Id}$ *only if there exists a* $\mathbb{C}_2$*-subnet* $\{\eta_\beta\}$ *of* $\{\xi_\alpha\}$ *such that* $\{Re^{\,1}\eta_\beta\}$ *is stable at* k *and* $\{Im^{\,1}\eta_\beta\}$ *clusters on* ℓ*.*

On the similar lines, we can prove the following theorems:

Theorem 4.1.5. *A* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *cluster on* $[Re^{\,1}\xi = k,\, Im^{\,1}\xi = \ell,\, Re^{\,2}\xi = m]_{Id}$ *only if there is a* $\mathbb{C}_2$*-subnet* $\{\eta_\beta\}_{\beta \in D'}$ *of* $\{\xi_\alpha\}$ *such that* $\{Re^{\,1}\eta_\beta\}$ *is stable at* k *and there is a* $\mathbb{C}_2$*-subnet* $\{\zeta_\gamma\}_{\gamma \in D''}$ *of* $\{\eta_\beta\}_{\beta \in D'}$*, for which* $\{Im^{\,1}\zeta_\gamma\}$ *is stable at* ℓ *and* $\{Im^{\,2}\zeta_\gamma\}$ *clusters on* m*.*

Theorem 4.1.6. *A* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *cluster on* $\xi = (a + i_1 b)e_1 + (c + i_1 d)e_2$ *only if there is a* $\mathbb{C}_2$*-subnet* $\{\eta_\beta\}_{\beta \in D'}$ *of* $\{\xi_\alpha\}_{\alpha \in D}$ *a* $\mathbb{C}_2$*-subnet* $\{\zeta_\gamma\}_{\gamma \in D''}$ *of* $\{\eta_\beta\}_{\beta \in D'}$ *and a* $\mathbb{C}_2$*-subnet* $\{\psi_\delta\}_{\delta \in D'''}$ *such that* $\{Re^{\,1}\eta_\beta\}$ *is stable at* a*,* $\{Im^{\,1}\zeta_\gamma\}$ *is stable at* b*,* $\{Re^{\,2}\psi_\delta\}$ *is stable at* c *and* $\{Im^{\,2}\psi_\delta\}$ *clusters on* d*.*

Remark 4.1.4. If a $\mathbb{C}_2$-net is confluence to a particular ID-zone (ID-frame, ID-plane, ID-line or ID-point), then it clusters on that particular ID-zone. The converse of this is not true in the general set up.

Example 4.1.4. Define a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ on the directed set $(\mathbb{Q}^+, \geq)$ as follows:

$$\{\xi_\alpha\} = \left\{(a + i_1 x_\alpha)e_1 + (b + i_1(1/\alpha^2))e_2\right\}, \quad \forall \alpha \in \mathbb{Q}^+,$$

where the net $\{Im^{\,1}\xi_\alpha\} = \{x_\alpha\}$ attains the value 0, frequently. Therefore, the net $\{\xi_\alpha\}$ clusters on the bicomplex point $\xi = a\, e_1 + b\, e_2$ with respect to the Id(o)-topology. But as the net $\{Im^{\,1}\xi_\alpha\}$ is not finally stable at 0, so the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is not ID-Point confluence to ξ .

Remark 4.1.5. For the clustering of the $\mathbb{C}_2$-net on different Id-zones, one can have

(i) If a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on a point $(a+i_1b)e_1+(c+i_1d)e_2$, then it clusters on the ID-line $[Re\,^1\xi=k,\ Im\,^1\xi=\ell,\ Re\,^2\xi=m]_{Id}$.

(ii) If a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on an ID-line $(Re\,^1\xi=k,\ Im\,^1\xi=\ell,\ Re\,^2\xi=c)$, then it clusters on the ID-plane $[Re\,^1\xi=k,\ Im\,^1\xi=\ell]_{Id}$.

(iii) If a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on an ID-plane $[Re\,^1\xi=k,\ Im\,^1\xi=\ell]_{Id}$, then it clusters on the ID-frame $(Re\,^1\xi=k)$.

Converses of these implications are not true, in general, for obvious reasons.

For any $\mathbb{C}_2$-net $\{\eta_\alpha\}$, let $L(\eta_\alpha)$, $\Delta(\eta_\alpha)$ and $\nabla(\eta_\alpha)$ denote the set

$$\{\zeta\in\mathbb{C}_2 : \text{for each neighbourhood V of } \zeta, \text{ the set } \{\alpha\in D:\eta_\alpha\in V\} \text{ is infinite}\},$$

all confluence zones and the set of all cluster zones of $\{\eta_\alpha\}$, respectively. Then $\Delta(\eta_\alpha)\subset L(\eta_\alpha)$ and $\nabla(\eta_\alpha)\subset L(\eta_\alpha)$.

Remark 4.1.6. For any $\mathbb{C}_2$-net $\{\eta_\alpha\}$, we have $\Delta(\eta_\alpha)\subset\nabla(\eta_\alpha)$

Theorem 4.1.7. *For any* $\mathbb{C}_2$*-net* $\{\eta_\alpha\}$*, the set* $\nabla(\eta_\alpha)$ *is closed.*

Proof. Let $\xi\in\overline{\nabla(\eta_\alpha)}$, for any neighbourhood U of ξ, we have $U\cap\nabla(\eta_\alpha)\neq\emptyset$. Assume that $\zeta\in U\cap\nabla(\eta_\alpha)$. Select a neighbourhood V of ζ. Then $\{\alpha\in D:\eta_\alpha InU\}\supset\{\alpha:\eta_\alpha\in V\}$. Therefore, $\xi\in\nabla(\eta_\alpha)$. Hence proved. □

4.2 Confluence and Clustering of $\mathbb{C}_2$-nets and their $\mathbb{C}_2$-subnets

In this section, the ID-confluence and clustering of subnets of $\mathbb{C}_2$-nets is investigated. Note that various types of $\mathbb{C}_2$-subnets may be formed depending upon its domain. The domains of the $\mathbb{C}_2$-net and $\mathbb{C}_2$-subnet may be disjoint directed sets or domain of the $\mathbb{C}_2$-subnet may be subset of the domain of the $\mathbb{C}_2$-net. In the later case, there are two possibilities. Either domain of the $\mathbb{C}_2$-subnet is cofinal subset of domain of the $\mathbb{C}_2$-net or a proper subset of domain of the $\mathbb{C}_2$-net.

Theorem 4.2.1. *If a* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}$ *is ID-Point confluence to* ξ*, then every cofinal* $\mathbb{C}_2$*-subnet of* $\{\xi_\alpha\}$ *is also ID-Point confluence to* ξ*.*

Proof. Let $\{\xi_\alpha\}$ be a $\mathbb{C}_2$-net on the directed set $(D, \geq)$ which is ID-Point confluence to point ξ. Now, suppose that D' is a cofinal subset of D. Thus, for every $\alpha \in D$ there exists some $\lambda \in D'$ such that $\lambda \geq \alpha$. Define a $\mathbb{C}_2$-net $\{\eta_\lambda\}$ on the directed set D'. Clearly, $\{\eta_\lambda\}$ is a cofinal subnet of $\{\xi_\alpha\}$. Therefore, for each tail T_α of D there is a T_λ of D' such that for each $\gamma \in T_\lambda$ there is a $\delta \in T_\alpha$ such that

$$\{\eta_\lambda \,:\, \lambda \geq \gamma\} \subset \{\xi_\alpha \,:\, \alpha \geq \delta\}. \tag{4.2.1}$$

Now as the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is ID-Point confluence to ξ, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is eventually in every member of N_4 containing ξ. From Equation (4.2.1) we have obtained that every tail of points of the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ contains some tail of the points of the $\mathbb{C}_2$-subnet $\{\eta_\lambda\}$ and also D' is a cofinal subset of D. Therefore, it can be concluded that the $\mathbb{C}_2$-subnet $\{\eta_\lambda\}$ lies eventually in every member of N_4 containing the point ξ. Hence, $\{\eta_\lambda\}$ is ID-Point confluence to ξ. □

Theorem 4.2.2. *A* $\mathbb{C}_2$*-net* $\{\xi_\alpha\}_{\alpha\in D}$ *clusters on a bicomplex point* ξ *if there exists a* $\mathbb{C}_2$*-subnet of* $\{\xi_\alpha\}_{\alpha\in D}$ *which is Id-point confluence to* ξ*.*

Proof. Let $\{\eta_\beta\}_{\beta\in E}$ be a $\mathbb{C}_2$-subnet of the $\mathbb{C}_2$-net $\{\xi_\alpha\}_{\alpha\in D}$, which is Id-point confluence to $\xi = (a + i_1 b)e_1 + (c + i_1 d)e_2$. We have to show that the net $\{\xi_\alpha\}_{\alpha\in D}$ clusters on ξ. Let U be an arbitrary neighbourhood of ξ and $\alpha \in D$ be given. Now for $\{\eta_\beta\}_{\beta\in E}$ as a $\mathbb{C}_2$-subnet of $\{\xi_\alpha\}_{\alpha\in D}$, there exists a tail T_β of E such that (cf. Definition 3.1.1)

$$\{\eta_\delta : \delta \in T_\beta \subset E\} \subset \{\xi_\gamma : \gamma \in T_\alpha \subset D\} \tag{4.2.2}$$

Since the net $\{\eta_\beta\}$ is Id-point confluence to the point ξ, there exists a $\lambda \in E$ such that $\forall\, \delta \geq \lambda$, $\eta_\delta \in U$. Now consider some $\mu \in E$ where $\mu \geq \beta$ and $\mu \geq \lambda$. Then clearly, $\eta_\mu \in U$. Due to Equation (4.2.2), there exists $\nu \in T_\alpha$ such that $\xi_\nu = \eta_\mu$.

Thus there corresponds some $\nu \in D$ such that $\xi_\nu \in U$. Since $\alpha \in D$ and U are arbitrary. Then, $\mathbb{C}_2$-net $\{\xi_\alpha\}$ clusters on the point ξ. $\square$

Remark 4.2.1. The converse of this theorem is not true, in general. An example to explain it as follows.

Example 4.2.1. Let S be a non-empty subset of $\mathbb{Q}^+$. Define a set D as follows:

$$D := \{(\alpha, \beta) \,:\, \alpha, \beta \in S\} \subseteq \mathbb{Q}^+ \times_c \mathbb{Q}^+.$$

Define an order relation as: $(\alpha,\, \beta) \lhd (\gamma,\, \delta)$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$. Then, D is a directed set. Now, define a $\mathbb{C}_2$-net $\Phi : D \longrightarrow \mathbb{C}_2$ as follows:

$$\Phi(\alpha, \beta) = \frac{1}{2}\left(\left[\frac{1}{\alpha}\right] + \left[\frac{1}{\beta}\right]\right) + \frac{i_1 i_2}{2}\left(\left[\frac{1}{\alpha}\right] - \left[\frac{1}{\beta}\right]\right), \forall\, (\alpha, \beta) \in D,$$

where $[\frac{1}{\alpha}]$ and $[\frac{1}{\beta}]$ are integral parts of $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, respectively. This $\mathbb{C}_2$-net Φ is ID-point confluence to '0'. Further, define a subset D' as follows:

$$D' = \left\{(\alpha, 1) \,:\, \alpha \in \mathbb{Q}^+\right\} \subset D$$

Clearly, D' is a directed set under the order relation of D. Now, consider $\delta > 1$. Then for any $(\lambda,\, \delta) \in D$, there does not exist any element $(\gamma,\, 1) \in D'$ such that $(\lambda,\, \delta) \lhd (\gamma,\, 1)$. Therefore, D' is not a cofinal subset of D. Now, define a $\mathbb{C}_2$-subnet $\Phi' = \Phi|_{D'}$ of the net Φ as follows:

$$\Phi'(\lambda, 1) = \frac{1}{2}\left(\left[\frac{1}{\lambda}\right] + 1\right) + \frac{i_1 i_2}{2}, \forall\, (\lambda, 1) \in D'$$

Since the net $\{Re\,^1\xi_\theta\}_{\theta \in D'}$ is not finally stable on 0. Then, Φ' is not Id-point confluence to 0. Also as the $\mathbb{C}_2$-net $\{Re\,^1\xi_\theta\}_{\theta \in D'}$ does not attain the value 0 frequently, $\mathbb{C}_2$-subnet Φ' of given net Φ does not cluster on 0.

Remark 4.2.2. If the domain of the $\mathbb{C}_2$-subnet of given $\mathbb{C}_2$-net is not a cofinal subset of domain of the $\mathbb{C}_2$-net, then it is possible that the $\mathbb{C}_2$-subnet does not cluster on the Id-zone even when the $\mathbb{C}_2$-net is Id-confluence to that Id-zone.

Remark 4.2.3. It can be easily verified that the Remark 4.2.2 holds good for every type of Id-confinement of the $\mathbb{C}_2$-net.

4.3 Topological Aspects of Some Subsets of the Bicomplex Space

In this section, the compactness, countability and the homeomorphism of some subsets of the bicomplex space is discussed with respect to the Id(o)-topology. We have also given a result regarding homeomorphism in the Id(o)-topology and the $\mathbb{C}_1$(o)-topology on the bicomplex space. It is proved that the principal ideals $\mathbb{I}_1$ and $\mathbb{I}_2$ of the bicomplex algebra are nowhere dense and $\mathbb{O}_2$ is uncountable subset of $\mathbb{C}_2$ of first category.

Theorem 4.3.1. *The principal ideals $\mathbb{I}_1$ and $\mathbb{I}_2$ of $\mathbb{C}_2$ are nowhere dense in the Id(o)-topology.*

Proof. Let ξ be an arbitrary limit point of $\mathbb{I}_1$. Then, there exists a $\mathbb{C}_2$-net $\{\xi_\alpha\}_{\alpha\in D}$ in $\mathbb{I}_1$ which is Id-point confluence to ξ, where D is a directed set. Since, $\xi_\alpha \in \mathbb{I}_1, \forall\, \alpha \in D$, then ${}^2\xi_\alpha = 0, \forall \alpha \in D$. Now, by the Id-point confluence of $\{\xi_\alpha\}$, the net $\{Re\ {}^1\xi_\alpha\}$ is finally stable at $Re\ {}^1\xi$, $\{Im\ {}^1\xi_\alpha\}$ is finally stable at $Im\ {}^1\xi$ and $\{{}^2\xi_\alpha\}$ is stable at 0. Then, $\{\xi_\alpha\}$ is Id-point confluence to $\xi = ((Re\ {}^1\xi) + i_1(Im\ {}^1\xi))e_1 \in \mathbb{I}_1$. Hence,

$$\xi \in \mathbb{I}_1.$$

Therefore, we have

$$\overline{\mathbb{I}_1} = \mathbb{I}_1. \tag{4.3.1}$$

Now, let $\xi = \xi e_1$ be an arbitrary element of $\mathbb{I}_1$ and $({}^1\zeta e_1 + {}^2\zeta e_2, {}^1\eta e_1 + {}^2\eta e_2)_{Id}$ be an arbitrary basis element around ξ. So ${}^1\zeta \prec {}^1\xi \prec {}^1\eta$. If ${}^2\zeta$ and/or ${}^2\eta$ are non zero.

Then, the neighbourhood $({}^1\zeta e_1 + {}^2\zeta e_2, {}^1\eta e_1 + {}^2\eta e_2)_{Id}$ contains some non-singular bicomplex numbers. But all elements of $\mathbb{I}_1$ are singular elements. Therefore,

$$\xi \in ({}^1\zeta e_1 + {}^2\zeta e_2, {}^1\eta e_1 + {}^2\eta e_2)_{ID} \not\subset \mathbb{I}_1. \tag{4.3.2}$$

Thus,

$$(\mathbb{I}_1)^o = \emptyset. \tag{4.3.3}$$

The Equation (4.3.2) and Equation (4.3.3) together implies that

$$(\overline{\mathbb{I}_1})^o = \emptyset. \tag{4.3.4}$$

Hence, $\mathbb{I}_1$ is a nowhere dense subset of $\mathbb{C}_2$ in Id(o)-topology. Similarly, it can be proved that $\mathbb{I}_2$ is a nowhere dense subset of $\mathbb{C}_2$ in the Id(o)-topology. □

Corollary 4.3.1. From the above theorem, the following results are established in the Id(o)-topology, τ_6:

(i) $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{O}_2$ are closed but not open subsets of $\mathbb{C}_2$.
(ii) $\mathbb{O}_2$ is uncountable set of first category.
(iii) The set $\mathbb{C}_2 \setminus \mathbb{O}_2$ of regular elements is uncountable dense subset of $\mathbb{C}_2$.

Proof. Following assertions are proved .

(i) The Equation (4.3.1) guarantees the closedness of $\mathbb{I}_1$. Equation (4.3.3) assures that the interior of $\mathbb{I}_1$ is empty, so is not equal to $\mathbb{I}_1$. Hence, $\mathbb{I}_1$ is not an open set. Similarly, $\mathbb{I}_2$ and $\mathbb{O}_2$ can be shown to be closed but not open sets.

(ii) As $\mathbb{O}_2$ is the union of two disjoint nowhere dense sets, so it is of first category.

(iii) Since, no point of $\mathbb{O}_2$ is an interior point of $\mathbb{O}_2$. Then every point of $\mathbb{O}_2$ is a frontier point of $\mathbb{O}_2$ and also frontier a frontier point of $\mathbb{C}_2 \setminus \mathbb{O}_2$. This makes $\mathbb{C}_2 \setminus \mathbb{O}_2$ an uncountable dense subset of $\mathbb{C}_2$ in the Id(o)-topology.

Hence, the proof. □

Lemma 4.3.1. Suppose that $S = \{\zeta = ze_1 + we_2 \,:\, z \in \mathbb{A}_1, w = z^{-1},\, 0 \prec z \prec 1\}$. Any subset of the type $(z_0e_1 + w_1e_2,\, z_0e_1 + w_2e_2)_{Id}$, for $0 \prec z_0 \prec 1$ in $\mathbb{C}_2$ contains at most one point of S.

Proof. We have $S = \{\zeta = ze_1 + we_2 \,:\, z \in \mathbb{A}_1, w = z^{-1},\, 0 \prec z \prec 1\}$. Let $z_0 \in \mathbb{C}_1$, with $0 \prec z_0 \prec 1$. As $z_0 \neq 0$, there exists a $w \in \mathbb{C}_1$ such that $z_0^{-1} = w$. Thus, there exists some $\zeta = z_0e_1 + we_2 \in \mathbb{C}_2$ such that

$$\zeta \in (z_0e_1 + w_1e_2,\, z_0e_1 + w_2e_2)_{Id}, \tag{4.3.5}$$

where $w_1,\, w_2 \in \mathbb{A}_2$ and $w_1 \prec w \prec w_2$. Further, if $(z_0e_1 + w_1e_2,\, z_0e_1 + w_2e_2)_{Id}$ is a subset of $\mathbb{C}_2$ such that $w \prec w_1$, where $w = z_0^{-1}$, then

$$(z_0e_1 + w_1e_2,\, z_0e_1 + w_2e_2)_{Id} \cap S = \phi.$$

Hence proved. □

Theorem 4.3.2. *The set* $S = \{\zeta = ze_1 + we_2 \,:\, z \in \mathbb{A}_1, w = z^{-1},\, 0 \prec z \prec 1\}$ *is a compact subset of* $\mathbb{C}_2$ *in Id(o)-topology.*

Proof. It is sufficient to prove that the set S is closed and bounded subset of on $\mathbb{C}_2$ in the Id(o)-topology. Let

$$S_1 = \{z : \zeta = ze_1 + we_2 \in S\}$$

Clearly, S is bounded by 0 and 1. Thus, for each w_1 in $\mathbb{A}_2$, the set S is bounded by w_1e_2 and $e_1 + w_1e_2$ as lower and upper bounds, respectively, i.e., $\zeta \in (w_1e_2,\, e_1 + w_1e_2)_{Id}$, where $0 \prec Re^1\zeta \prec 1$. Hence, S is bounded. Now to prove that S is closed, it is sufficient to prove that S^c is open in $\mathbb{C}_2$. For any $\xi = z_0e_1 + w_0e_2 \in S^c$, there are three options given as follows:

(1) $z_0^{-1} \neq w_0$ and $z_0 \in (0, 1)$.
(2) $z_0^{-1} = w_0$ and $z_0 \notin (0, 1)$.
(3) $z_0^{-1} \neq w_0$ and $z_0 \notin (0, 1)$.

Case (1): $z_0^{-1} \neq w_0$ and $z_0 \in (0, 1)$.

Since both z_0^{-1} and w_0 are elements of $\mathbb{A}_2$ (where $\mathbb{A}_2$ is a Hausdorff space in the lexicographic order topology). Then, $\exists\, u_1, u_2 \in \mathbb{A}_2$ such that $z_0^{-1} \in (u_1, u_2)$. Similarly, $\exists\, v_1, v_2 \in \mathbb{A}_2$ such that $w_0 \in (v_1, v_2)$ and

$$(u_1, u_2) \cap (v_1, v_2) = \phi. \tag{4.3.6}$$

So, we have

$$z_0e_1 + w_0e_2 \in (z_0e_1 + v_1e_2,\ z_0e_1 + v_2e_2)_{Id}.$$

Now if possible, suppose that $(z_0e_1 + v_1e_2,\ z_0e_1 + v_2e_2)_{Id} \cap S \neq \emptyset$, then their exists some $\xi = z_0e_1 + z_0^{-1}e_2 \in S$ such that ξ is contained in $(z_0e_1 + v_1e_2,\ z_0e_1 + v_2e_2)_{Id}$. But for $z_0^{-1} \neq w_0$ and $z_0 \in (0, 1)$, we have

$$\xi = z_0e_1 + z_0^{-1}e_2 \notin (z_0e_1 + v_1e_2,\ z_0e_1 + v_2e_2)_{Id}.$$

So, there is a contradiction, hence

$$(z_0e_1 + v_1e_2,\ z_0e_1 + v_2e_2)_{Id} \cap S = \phi.$$

$\Rightarrow\ z_0e_1 + w_0e_2 \in (z_0e_1 + v_1e_2,\ z_0e_1 + v_2e_2)_{Id} \subset S^c.$

So that S^c is open subset of $\mathbb{C}_2$. Therefore, S is closed subset of $\mathbb{C}_2$.

Case (2): When $z_0^{-1} = w_0$ and $z_0 \notin (0, 1)$. Two sub cases arise as follows:

Either $z_0 \preceq 0$ or $z_0 \succeq 1$.

Sub-case (2.1): Let $z_0 \preceq 0$. If $z_0 = 0$. Then $\xi = z_0e_1 + w_0e_2$ does not exist. So, $z_0 \neq 0$. Now assume that $z_0 \prec 0$, then for $z_0(\neq 0) \in \mathbb{A}_1$, there exist $a_1,\, a_2 \in \mathbb{A}_1$ such that $0 \in (a_1,\, a_2)$ and similarly, there exists some $w_1,\, w_2 \in \mathbb{A}_1$ such that $z_0 \in (w_1,\, w_2)$ and $(a_1,\, a_2) \cap (w_1,\, w_2) = \phi$. Also, $(0,\, 1) \cap (w_1,\, w_2) = \phi$. Therefore,

$w_1 \prec 0$ as well as $w_2 \prec 0$. Hence, for $b_1,\, b_2 \in \mathbb{A}_2$, we have

$$z_0e_1 + w_0e_2 \in (w_1e_1 + b_1e_2,\, w_2e_1 + b_2e_2)_{Id} \subset S^c.$$

Sub-case (2.2): Let $z_0 \succeq 1$, then either $z_0 = 1$ or $z_0 \succ 1$. If $z_0 = 1$, then $\xi = 1$. So there exists $\eta = e_1 + (1/2)e_2$ and $\zeta = e_1 + (3/2)e_2$ such that $\xi \in (\eta,\, \zeta)_{Id} \subset S^c$. If $z_0 \succ 1$, then $z_0^{-1}(= w_0) \prec 1$. Since $z_0(\neq 1) \in \mathbb{A}_1$ and as w_0 and $1 \in \mathbb{A}_2$. Also, $\mathbb{A}_2$ is Hausdorff under dictionary order topology. So, there exists, $u_1,\, u_2 \in \mathbb{A}_2$ such that $w_0 \in (u_1,\, u_2)$ and similarly, $\exists\, v_1,\, v_2 \in \mathbb{A}_2$ such that

$$1 \in (v_1,\, v_2) \qquad \text{and} \qquad (u_1,\, u_2) \cap (v_1, v_2) = \phi.$$

Therefore, we get

$$z_0e_1 + w_0e_2 \in (z_0e_1 + u_1e_2,\, z_0e_1 + u_2e_2)_{Id},$$

and

$$(z_0e_1 + u_1e_2,\, z_0e_1 + u_2e_2)_{Id} \cap S = \phi.$$

This implies that

$$(z_0e_1 + u_1e_2,\, z_0e_1 + u_2e_2)_{Id} \subset S^c.$$

Case (3): When $z_0^{-1} \neq w_0$ and $z_0 \notin (0,\, 1)$. As $z_0^{-1} \neq w_0$ and $z_0^{-1}, w_0 \in \mathbb{A}_2$. Then, there exists some $c_1,\, c_2 \in \mathbb{A}_2$ such that $z_0^{-1} \in (c_1,\, c_2)$. Similarly $\exists\, d_1,\, d_2 \in \mathbb{A}_2$ such that $w_0 \in (d_1,\, d_2)$ and $(c_1,\, c_2) \cap (d_1,\, d_2) = \phi$. Therefore one obtains an interval $(z_0e_1 + d_1e_2,\, z_0e_1 + d_2e_2)_{ID}$ such that $z_0e_1 + w_0e_2 \in (z_0e_1 + d_1e_2,\, z_0e_1 + d_2e_2)_{ID}$ and $(z_0e_1 + d_1e_2,\, z_0e_1 + d_2e_2)_{Id} \subset S^c$.

Thus, S is a closed and bounded set. Hence, S is a compact subset of $\mathbb{C}_2$. $\square$

Theorem 4.3.3. *The set* $S = \{(z,\, \sin z^{-1}) : z \in (0,\, 1)\}$ *is a compact subset of* $\mathbb{C}_2$ *in Id(o)-topology.*

Proof. We have $S = \{(z, \sin z^{-1}) \, : \, z \in (0, 1)\}$.

$\Rightarrow S = \{(z, \sin w) \, : \, z \in (0, 1), \, w = z^{-1}\}$

$\Rightarrow S = \{ze_1 + (\sin w)e_2 \, : \, z \in (0, 1), \, w = z^{-1}\}$

Since, $z \in (0, 1)$. Hence, S is bounded subset of $\mathbb{C}_2$ with respect to the ℓ_{Id}-order relation with lower and upper bounds as $w_1 e_1$ and $e_1 + w_2 e_2$, respectively . Now to show S is a closed set. Let $\xi = u\, e_1 + v\, e_2 \in S^c$. Then there are three different possibilities as follows:

(i) $u \notin (0, 1)$ and $v = \sin u^{-1}$,
(ii) $u \in (0, 1)$ and $v \neq \sin u^{-1}$,
(iii) $u \notin (0, 1)$ and $v \neq \sin u^{-1}$.

Case (i): $u \notin (0, 1)$ and $v = \sin u^{-1}$. Now as $u \notin (0, 1)$. Therefore, either $u \preceq 0$ or $1 \preceq u$. If $u = 0$, then $\sin u^{-1}$ is not defined. So that $u \neq 0$. Now consider that $u <$ 0. Since $u \neq 0$ in $\mathbb{A}_1$. Then, $\exists \; u_1, \, u_2 \in \mathbb{A}_1$ such that $u \in (u_1, \, u_2)$. Similarly, there exists $v_1, \, v_2 \in A_1$ such that $0 \in (v_1, \, v_2)$ and also $(u_1, \, u_2) \cap (v_1, \, v_2) = \phi$. Thus,

$$u\, e_1 + v\, e_2 \in (u_1\, e_1 + v_1\, e_2, \, u_2\, e_1 + v_2\, e_2)_{Id}, \tag{4.3.7}$$

and

$$1 + v_1\, e_2, \, u_2\, e_1 + v_2\, e_2)_{Id} \, \cap \, S = \phi. \tag{4.3.8}$$

Therefore, from Equation (4.3.7) and Equation (4.3.8), we obtain that

$$ue_1 + ve_2 \in (u_1e_1 + v_1e_2, \, u_2e_1 + v_2e_2)_{Id} \subset S^c.$$

Thus, S^c is an open set in $\mathbb{C}_2$. So, S is a closed set.

Similarly, we can prove that S is closed if $1 \prec u$.

Case (ii): If $u \in (0, 1)$ and $v \neq sin\, u^{-1}$. Then, there exists $v_1, \, v_2 \in \mathbb{A}_2$ such that $v \in (v_1, \, v_2)$. Similarly, $\exists \, w_1, \, w_2 \in \mathbb{A}_2$ such that $sin\, u^{-1} \in (w_1, \, w_2)$ and

$(v_1,\, v_2) \cap (w_1,\, w_2) = \phi$. Therefore,

$$u\,e_1 + v\,e_2 \in (u_1\,e_1 + v_1\,e_2,\; u_2\,e_1 + v_2\,e_2)_{Id}. \tag{4.3.9}$$

and

$$(u_1 e_1 + v_1 e_2,\; u_2 e_1 + v_2 e_2)_{Id} \cap S = \phi. \tag{4.3.10}$$

Therefore, from Equation (4.3.9) and Equation (4.3.10), we obtain that

$$u\,e_1 + v\,e_2 \in (u_1\,e_1 + v_1\,e_2,\; u_2\,e_1 + v_2\,e_2)_{Id} \subset S^c.$$

So that S^c is open subset of $\mathbb{C}_2$. Hence, S is a closed subset of $\mathbb{C}_2$.

Case (iii): If $u \notin (0,\, 1)$ and $v \neq sin\, u^{-1}$. By the method of case (i), S is a closed subset of $\mathbb{C}_2$. Hence we can conclude that S is a closed and bounded subset of $\mathbb{C}_2$. So, S is a compact subset of $\mathbb{C}_2$. □

Theorem 4.3.4. *The space* $(\mathbb{C}_2;\, \tau_5)$ *is not homeomorphic to space* $(\mathbb{C}_2;\, \tau_6)$.

Proof. If possible, suppose that f is homeomorphism between $(\mathbb{C}_2,\, \tau_5)$ and $(\mathbb{C}_2,\, \tau_6)$.
Case (1): If f is an identity function. Then topologies τ_5 and τ_6 on $\mathbb{C}_2$ are same. This is a contradiction, as $\mathbb{C}_1$(o)-topology and Id(o)-topology are not comparable.

Case (2): If f is a non-identity homeomorphism between $(\mathbb{C}_2,\, \tau_5)$ and $(\mathbb{C}_2,\, \tau_6)$. Therefore, f is bijective and order preserving functions, i.e., f is one-one, onto and $\xi \prec_{\mathbb{C}_1} \eta$. This implies that $f(\xi) \preceq_{Id} f(\eta)$. This means the space $(\mathbb{C}_2,\, \tau_5)$ and $(\mathbb{C}_2,\, \tau_6)$ have same topological structures, which is not possible because the topologies τ_5 and τ_6 on $\mathbb{C}_2$ are not same (cf. [**?**]). Hence, the spaces $(\mathbb{C}_2,\, \tau_5)$ and $(\mathbb{C}_2,\, \tau_6)$ cannot be homeomorphic. □

Example 4.3.1. Define a function $f : (\mathbb{C}_2, \tau_5) \to (\mathbb{C}_2, \tau_6)$ as $f(\xi) = \xi,\ \forall\, \xi \in \mathbb{C}_2$. This is an onto function for obvious reasons. Let $\xi = (3 + 2i_1) + i_2(3 + 4i_1)$ and $\eta = (1 + 2i_1) + i_2(9 + 7i_1)$. Now, $f(\xi) = (7 - i_1)\,e_1 + (-1 + 5i_1)\,e_2$ and $f(\eta) = (8 - 7i_1)\,e_1 + (-6 + 11i_1)\,e_2$. Then, $\eta \prec_{\mathbb{C}_1} \xi$ but $f(\xi) \prec_{Id} f(\eta)$. So, this is not an order preserving map. Hence, f is not a homeomorphism.

4.4 $\mathbb{C}_2$-filters and Order Topologies on $\mathbb{C}_2$

In this section, the analogous concept $\mathbb{C}_2$-filters of $\mathbb{C}_2$-nets is studied. The results of confluence and clustering of the $\mathbb{C}_2$-nets and $\mathbb{C}_2$-filters are similar. We studied some new members of the filters and developed the algebraic structure on the set of all filters on $\mathbb{C}_2$.

Definition 4.4.1 (**Convergence of a** $\mathbb{C}_2$**-filter**)**.** A filter $\wp$ is said to converge to a point x if every neighborhood of x contains some member of the filter.

As in Id(o)-topology, we have studied the convergence of the $\mathbb{C}_2$-nets with the concept of ID-confinements. Therefore, in this topology, we shall define the convergence as follows:

Definition 4.4.2 (**Id(F)-confinement of** $\mathbb{C}_2$**-filter**)**.** A $\mathbb{C}_2$-filter $\wp$ is said to ID-frame confined to $[Re\,^1\xi = a]_{Id}$ if every ID-space segment containing the ID-frame $[Re\,^1\xi = a]_{Id}$ contains some member of $\mathbb{C}_2$-filter $\wp$.

In the similar manner, we can define the ID-confinement of the $\mathbb{C}_2$-filters to the ID-plane, ID-line and to a point.

Remark 4.4.1. If $\mathcal{F}$ and $\mathcal{F}'$ are two $\mathbb{C}_2$-filters on the same set M, and if $\mathcal{F}$ is a subclass of $\mathcal{F}'$, then $\mathcal{F}$ is said to be coarser than $\mathcal{F}'$. If τ' is finer topology on M than τ, then neighborhood filter of a point ξ_0 relative to τ' is finer than then neighborhood filter of the point ξ_0 relative to topology τ.

Definition 4.4.3 (**Cluster Point of a** $\mathbb{C}_2$**-filter**)**.** A point ξ is said to be cluster point of a filter $\wp$ if it is contained in the closure of every member of the filter.

Theorem 4.4.1. *A bicomplex point* $\xi = (a+i_1b)e_1 + (c+i_1d)e_2$ *is a cluster point of the filter* $\mathcal{F}$ *if* $\mathcal{F}$ *is Id-point confluence to the point* ξ*.*

Proof. Consider that the filter $\mathcal{F}$ is Id-point confined to the bicomplex number $\xi = (a+i_1b)e_1 + (c+i_1d)e_2$.

Therefore, every member of the filter containing ξ is contained in some member of N_4 containing the point ξ. So one can say that the point ξ is contained in every member of the filter $\mathcal{F}$. Therefore, the point ξ is contained in every closure of every member of filter $\mathcal{F}$. Hence, the point ξ is the cluster point of filter $\mathcal{F}$. □

Remark 4.4.2. The converse of the above theorem is not true, in general.

Remark 4.4.3. Let S be the set of all cluster points of the $\mathbb{C}_2$-net $\{\xi_\alpha\}$. Then the collection of all supersets of the set S forms a filter on $\mathbb{C}_2$, i.e., the set of all cluster points of a $\mathbb{C}_2$-net is a basis of some $\mathbb{C}_2$-filter as the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is frequently in every member of the family containing the elements of the set S.

Remark 4.4.4. Since $\mathbb{C}_0$(o)-topology and $\mathbb{C}_1$(o)-topology are same. Thus, the neighbourhood filters of any bicomplex point with respect to these two topologies are same. As the $\mathbb{C}_0$(o)-topology is not comparable to the Id(o)-topology on $\mathbb{C}_2$. Then, the neighbourhood filters of any bicomplex point with respect to these two topologies are not comparable.

Further, the Id(o)-topology is strictly finer than the Id(p)-topology on $\mathbb{C}_2$. So the neighbourhood filter of any bicomplex point, ξ with respect to the Id(o)- topology is strictly finer than the neighbourhood filter of the same bicomplex point with respect to Id(p)-topology.

We define some particular type of subsets of the $\mathbb{C}_2$-filters and also give their properties. We define the product of $\mathbb{C}_2$-filters using these subsets.

Definition 4.4.4. Let S be a non-empty subset of $\mathbb{C}_2$ and $\mathcal{F}$ be a filter on $\mathbb{C}_2$. Then we define some sets as follows:

(i) $\overline{\xi}S := \{\eta \in \mathbb{C}_2 \,:\, \xi\eta \in S\}$

(ii) $S^{\overline{\mathcal{F}}} := \{\xi \in \mathbb{C}_2 : \overline{\xi}S \in \mathcal{F}\}$

(iii) $S^\star(\mathcal{F}) := S \cap S^{\overline{\mathcal{F}}}$.

Note $\xi \in S$ if and only if $\eta = 1$. Then, $S \subseteq \overline{\xi}S$.

Lemma 4.4.1. Let S be a non-empty subset of the set of bicomplex numbers, $\mathbb{C}_2$ and $\mathcal{F}$, $\mathcal{G}$ be any two filters on $\mathbb{C}_2$. Then

(i) $\overline{\xi}\,\overline{\eta}\,S = \overline{\eta\,\xi}\,S$

(ii) $\overline{\xi}\,S^{\overline{\mathcal{F}}} = (\overline{\xi}\,A)^{\overline{\mathcal{F}}}$

(iii) $(\overline{\xi}S)^{\star}(\mathcal{F}) = \overline{\xi}S^{\star}(\mathcal{F})$

(iv) $(S^{\overline{\mathcal{G}}})^{\overline{\mathcal{F}}} = S^{\overline{\mathcal{G}\mathcal{F}}}$

Proof. (i) Let $S \subseteq \mathbb{C}_2$ and $\mathcal{F}$ be a filter on $\mathbb{C}_2$.

$$\begin{aligned} \overline{\xi}\,\overline{\eta}\,S &= \{\zeta \in \mathbb{C}_2 \,:\, \xi\,\zeta \in \overline{\eta}S\} \\ &= \{\zeta \in \mathbb{C}_2 \,:\, \eta(\xi\,\zeta) \in S\} \\ &= \{\zeta \in \mathbb{C}_2 \,:\, (\eta\,\xi)\,\zeta \in S\} \\ &= (\overline{\eta\,\xi})\,S. \end{aligned}$$

(ii) One can calculate

$$\begin{aligned} \overline{\xi}\,S^{\overline{\mathcal{F}}} &= \{\zeta \,:\, \xi\,\zeta \in S^{\overline{\mathcal{F}}}\} \\ &= \{\zeta \,:\, (\overline{\xi\,\zeta})S \in \mathcal{F}\} \\ &= \{\zeta \,:\, \overline{\xi}\,(\overline{\zeta}\,S) \in \mathcal{F}\} \\ &= \{\zeta \,:\, \zeta \in (\overline{\xi}\,S)^{\overline{\mathcal{F}}}\}\} \\ &= (\overline{\xi}\,S)^{\overline{\mathcal{F}}} \end{aligned}$$

(iii) One can calculate

$$\begin{aligned} (\overline{\xi}S)^{\star}(\mathcal{F}) &= \overline{\xi}S \cap (\overline{\xi}S)^{\overline{\mathcal{F}}} \\ &= \overline{\xi}S \cap (\overline{\xi}S^{\overline{\mathcal{F}}}) \\ &= \overline{\xi}(S \cap S^{\overline{\mathcal{F}}}) \\ &= \overline{\xi}S^{\star}(\mathcal{F}) \end{aligned}$$

(iv) One can calculate

$$\begin{aligned}(S^{\overline{\mathcal{G}}})^{\overline{\mathcal{F}}} &= \{\xi : \overline{\xi} S^{\overline{\mathcal{G}}} \in \mathcal{F}\} \\ &= \{\xi : (\overline{\xi}\, S)^{\overline{\mathcal{G}}} \in \mathcal{F}\} \\ &= \{\xi : \overline{\xi}\, S \in \mathcal{F}\mathcal{G}\} \\ &= S^{\overline{\mathcal{F}\mathcal{G}}}.\end{aligned}$$

Hence the results. □

Example 4.4.1. There are some particular examples for the sets given as follows:

(i) $\overline{e_1}\,\mathbb{I}_1 = \{\eta \in \mathbb{C}_2 : e_1\eta \in \mathbb{I}_1\} = \mathbb{C}_2$

(ii) $\overline{e_2}\,\mathbb{I}_2 = \{\eta \in \mathbb{C}_2 : e_2\eta \in \mathbb{I}_2\} = \mathbb{C}_2$

(iii) $\overline{e_1}\,\mathbb{I}_2 = \{\eta \in \mathbb{C}_2 : e_1\eta \in \mathbb{I}_2\} = \{0\}$

(iv) $\overline{e_2}\,\mathbb{I}_1 = \{\eta \in \mathbb{C}_2 : e_2\eta \in \mathbb{I}_1\} = \{0\}$

(v) $\overline{e_1}\,\mathbb{A}_1 = \{\eta \in \mathbb{C}_2 : e_1\eta \in \mathbb{A}_1\} = \{0\}$

(vi) $\overline{e_2}\,\mathbb{A}_2 = \{\eta \in \mathbb{C}_2 : e_2\eta \in \mathbb{A}_2\} = \{0\}$

(vii) $\overline{e_1}\,\mathbb{A}_2 = \{\eta \in \mathbb{C}_2 : e_1\eta \in \mathbb{A}_2\} = \{0\}$

(viii) $\overline{e_2}\,\mathbb{A}_1 = \{\eta \in \mathbb{C}_2 : e_2\eta \in \mathbb{A}_1\} = \{0\}$.

Example 4.4.2. Let $\mathcal{F} = \{N_4\}$, be a filter generated by the family N_1. Let $S \subseteq \mathbb{I}_1$, then $\overline{\xi} S \notin S^{\mathcal{F}}$, $\forall\, \xi(\neq 0) \in \mathbb{C}_2$ and $S^{\mathcal{F}} = \{0\}$ if for some $\xi, \eta \in \mathbb{C}_2$, with $\xi \prec_{Id} 0 \prec_{Id} \eta$ such that $(\xi, \eta)_{Id} \in N_4$.

Conclusion

In this chapter, the concept of clustering of the $\mathbb{C}_2$-nets have been developed for the different Id-zones with respect to the Id(o)-topology. The $\mathbb{C}_2$-subnets are defined on cofinal subsets of the directed sets. The confluence of these $\mathbb{C}_2$-subnets is used to study the clustering of the $\mathbb{C}_2$-nets. The topological properties, particularly the countability, compactness, denseness and the homeomorphism is studies and found that the principal ideals $\mathbb{I}_1$ and $\mathbb{I}_1$ are uncountable nowhere dense subsets of the bicomplex space in the Id(o)-topology. Further, it has been proved that the set of singular elements in $\mathbb{C}_2$ is a uncountable subset of $\mathbb{C}_2$ of the first category. Also, it is proved that the topological space $(\mathbb{C}_2; \tau_5)$ is not homeomorphic to the topological space $(\mathbb{C}_2; \tau_6)$. In the last part of the chapter, the concept of $\mathbb{C}_2$-filter is developed and discussed in detail. Some particular type of subsets of the bicomplex space are defined using the $\mathbb{C}_2$-filters and are explored with some examples.

□ □ □

Chapter 5

Topological-Algebraic Structures on the Bicomplex Space

In this chapter, the compatibility of the algebraic and topological structures on the bicomplex space as well as the orderability problem are studied. This chapter is mainly divided into three sections.

In section 5.1, a space of all bounded $\mathbb{C}_2$-sequences, $\ell^M_{\mathbb{C}_2}$ is defined by using the Orlicz function. It is shown that $\ell^M_{\mathbb{C}_2}$ is a Banach space. Further, four other $\mathbb{C}_2$-sequence spaces are defined by using the concept of Orlicz function and the paranorm. The completeness, symmetry and the solidness of these spaces are discussed in detail. In section 5.2, the concept of summability via $\mathbb{C}_2$-nets have been explored. Some sets of $\mathbb{C}_2$-nets are defined, viz., $\mathbb{F}$ as set of all bounded $\mathbb{C}_2$-nets, $\mathbb{F}_1$ as the set of all convergent $\mathbb{C}_2$-nets in the norm topology, $\mathbb{F}_2$ as the set of all Id-confluence $\mathbb{C}_2$-nets in the Id(o)-topology, $\mathbb{F}_3$ as the set of all Id-point confluence $\mathbb{C}_2$-nets and $\mathbb{F}_4$ be the set of all null $\mathbb{C}_2$-nets. In particular, we focused on properties of $\mathbb{F}$. In section 5.3, we studied the orderability condition for convex subspaces of the topological spaces on $\mathbb{C}_2$. Properties for the equivalence of subspace topology and the induced order topology are also verified in this section.

5.1 Paranormed $\mathbb{C}_2$-Sequence Spaces

Throughout the chapter the notations ω_4, c, c_0 and $\ell^{\infty}_{\mathbb{C}_2}$ are denoting the spaces of all $\mathbb{C}_2$-sequences, convergent $\mathbb{C}_2$-sequences, null $\mathbb{C}_2$-sequences and all bounded $\mathbb{C}_2$-sequences, respectively. We denote the zero sequence $(0, 0, 0, \dots, 0, \dots)$ by π and $s = \{s_k\}$ is a sequence of strictly positive real numbers, $\{s_k^{-1}\} = \{t_k\}$. By using the concept of Orlicz function $\mathcal{M}$, we define the sequence spaces on the bicomplex space as follows:

$$\ell^{M}_{\mathbb{C}_2} = \left\{\{\xi_n\} \in \omega_4 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{K}\right) < \infty, \text{for some}\, K > 0\right\}. \qquad (5.1.1)$$

Lemma 5.1.1. The set $\ell^{M}_{\mathbb{C}_2}$ is a linear space over $\mathbb{C}_1$.

Proof. Let $\{\xi_\alpha\}, \{\eta_\alpha\}$ be two arbitrary $\mathbb{C}_2$-sequences in $\ell^{M}_{\mathbb{C}_2}$. Then for some $K_1 > 0$ and $K_2 > 0$, we have

$$\sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{K_1}\right) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\eta_n\|}{K_2}\right) < \infty. \qquad (5.1.2)$$

Let $a, b \in \ell^{M}_{\mathbb{C}_2}$ and $K = \max\{2|a|K_1, 2|b|K_2\}$. Then,

$$\sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|a\,\xi_n + b\,\eta_n\|}{K}\right) \leq K \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{K_1}\right) + K \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\eta_n\|}{K_2}\right).$$

Thus, $\{a\,\xi_n + b\,\eta_n\} \in \ell^{M}_{\mathbb{C}_2}$. Hence, $\ell^{M}_{\mathbb{C}_2}$ is linear over $\mathbb{C}_1$. □

Lemma 5.1.2. The function $\|.\|_{\mathcal{M}}$ defined as

$$\|\xi_n\|_{\mathcal{M}} = \inf\left\{K > 0 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{K}\right) \leq 1\right\} \qquad (5.1.3)$$

is the norm on $\ell^{M}_{\mathbb{C}_2}$.

Proof. Let θ be the null sequence in $\mathbb{C}_2$, then $\mathcal{M}\left(\frac{\|\theta_n\|}{K}\right) = 0$, for any $K > 0$. Thus, $\|\theta_n\|_{\mathcal{M}} = 0$. Since, $\|\xi_n\| = \|-\xi_n\|$, then $\|\xi_n\|_{\mathcal{M}} = \|-\xi_n\|_{\mathcal{M}}$.

Let $\xi = \{\xi_n\}, \eta = \{\eta_n\} \in \ell^M_{\mathbb{C}_2}$. Then there exists $K_1 > 0$, $K_2 > 0$ such that

$$\sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{K_1}\right) \leq 1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{K_2}\right) \leq 1$$

Assume that $K = K_1 + K_2$. Then,

$$\sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n + \eta_n\|}{K}\right) \leq \frac{K_1}{K} \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{K}\right) + \frac{K_2}{K} \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\eta_n\|}{K}\right) \leq 1$$

Then

$$\begin{aligned}
\|\xi + \eta\|_{\mathcal{M}} &= \inf\left\{K > 0 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n + \eta_n\|}{K}\right) \leq 1\right\} \\
&\leq \inf\left\{K_1 > 0 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{K_1}\right) \leq 1\right\} \\
&+ \inf\left\{K_2 > 0 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\eta_n\|}{K_2}\right) \leq 1\right\} \\
&\leq \|\xi_n\|_{\mathcal{M}} + \|\eta_n\|_{\mathcal{M}}
\end{aligned}$$

Now, let $\alpha \in \mathbb{C}_2 \setminus \mathbb{O}_2$. Then

$$\begin{aligned}
\|\alpha\, \xi_n\|_{\mathcal{M}} &= \inf\left\{K > 0 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\alpha\, \xi_n\|}{K}\right) \leq 1\right\} \\
&= \inf\left\{K > 0 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\sqrt{2}\, \|\alpha\|\, \|\xi_n\|}{K}\right) \leq 1\right\} \\
&= \inf\left\{\|\alpha\| H > 0 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{H}\right) \leq 1\right\} \\
&= \|\alpha\| \;\inf\left\{H > 0 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n\|}{H}\right) \leq 1\right\} \\
&= \|\alpha\|\, \|\xi\|_{\mathcal{M}}
\end{aligned}$$

where $H = K/(\|\alpha\|\sqrt{2})$. Hence proved $\square$

Theorem 5.1.1. *The space $\ell^M_{\mathbb{C}_2}$ is Banach space in the norm $\|.\|_{\mathcal{M}}$.*

Proof. Let $\{\xi_n^m\}$ be a Cauchy sequence in $\ell_{\mathbb{C}_2}^M$. Then for all $n \in \mathbb{N}$,

$$\|\xi_n^i - \xi_n^j\|_{\mathcal{M}} \to 0, \tag{5.1.4}$$

as $i, j \to \infty$. Let for given $\epsilon > 0$, there exists $H > 0$ and some $k > 0$ such that $\dfrac{\epsilon}{kH} > 0$. Now, by the Equation (5.1.4), there exists some $n_0 \in \mathbb{N}$ such that

$$\|\xi_n^i - \xi_n^j\|_{\mathcal{M}} < \frac{\epsilon}{kH}, \ \forall\, i, j \geq n_0, \ \forall\, n \in \mathbb{N}.$$
$$\inf\left\{K > 0 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n^i - \xi_n^j\|}{K}\right) \leq 1\right\} < \frac{\epsilon}{kH}.$$

Therefore, we have

$$\mathcal{M}\left(\frac{\|\xi_n^i - \xi_n^j\|}{K}\right) \leq 1.$$

This implies that

$$\mathcal{M}\left(\frac{\|\xi_n^i - \xi_n^j\|}{K}\right) < \mathcal{M}\left(\frac{kH}{2}\right)$$

Thus,

$$\|\xi_n^i - \xi_n^j\| < \frac{kH}{2} \cdot \frac{\epsilon}{kH} < \frac{\epsilon}{2}.$$

Therefore, the sequence $\{\xi_n^m\}$ is a Cauchy sequence in $\mathbb{C}_2$ for all $m \in \mathbb{N}$. As we know that $\mathbb{C}_2$ is a modified Banach space, thus the sequence $\{\xi_n^m\}$ converges in $\mathbb{C}_2$. Suppose that $\lim\limits_{i \to \infty} \xi_n^i = \xi_n$. Thus, by continuity of the Orlicz function, we have

$$\lim_{j \to \infty} \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n^i - \xi_n^j\|}{K}\right) \ \leq \ 1.$$

This implies that

$$\sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n^i - \xi_n\|}{K}\right) \ \leq \ 1. \tag{5.1.5}$$

Then for all $i > n_0$, we have

$$\inf\left\{K > 0 : \sum_{n=1}^{\infty}\mathcal{M}\left(\frac{\|\xi_n^i - \xi_n^j\|}{K}\right) \leq 1\right\} < \epsilon.$$

Thus, $\|\xi_n^i - \xi_n\|_{\mathcal{M}} < \epsilon$. So, $\{\xi_n^i - \xi_n\} \in \ell_{\mathbb{C}_2}^M$. Hence $\{\xi_n\} \in \ell_{\mathbb{C}_2}^M$. This implies that $\ell_{\mathbb{C}_2}^M$ is a complete space. So it is a Banach space. $\square$

Note that $(\mathbb{C}_2, \|.\|)$ is a normed space by the norm $\|.\|$ defined in Equation (1.1.4). Let $\mathcal{M}$ be an Orlicz functions. We define the following Orlicz difference $\mathbb{C}_2$-sequence spaces:

$$\begin{aligned}
\ell(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|) &= \left\{\{\xi_n\} \in \omega_4 : \sum_{n=1}^{\infty}\left[\mathcal{M}\left(\frac{\|\Delta\xi_n\|}{K}\right)\right]^{s_n} < \infty,\ K > 0\right\},\\
c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|) &= \left\{\{\xi_n\} \in \omega_4 : \left[\mathcal{M}\left(\frac{\|\Delta\xi_n - L\|}{K}\right)\right]^{s_n} \to 0, L \in \mathbb{C}_2,\ K > 0\right\},\\
c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|) &= \left\{\{\xi_n\} \in \omega_4 : \left[\mathcal{M}\left(\frac{\|\Delta\xi_n\|}{K}\right)\right]^{s_n} \to 0, K > 0\right\},\\
\ell^{\infty}(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|) &= \left\{\{\xi_n\} \in \omega_4 : \sup_n\left[\mathcal{M}\left(\frac{\|\Delta\xi_n\|}{K}\right)\right]^{s_n} < \infty,\ K > 0\right\}.
\end{aligned}$$

Theorem 5.1.2. *The set $\ell^{\infty}(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ is linear space over $\mathbb{C}_1$ for some sequence $s = \{s_n\}$ of positive numbers.*

Proof. Let $\{\xi_n\}, \{\eta_n\} \in \ell^{\infty}(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Then there exists $K_1 > 0$, $K_2 > 0$ such that

$$\sup_n\left[\mathcal{M}\left(\frac{\|\Delta\xi_n\|}{K_1}\right)\right]^{s_n} < \infty \quad \text{and} \quad \sup_n\left[\mathcal{M}\left(\frac{\|\Delta\eta_n\|}{K_2}\right)\right]^{s_n} < \infty$$

Let $a, b \in (\mathbb{C}_2 \setminus \mathbb{O}_2)$ and $K = \max\{2\|a\|K_1, 2\|b\|K_2\}$. Then

$$\begin{aligned}
\sup_n\left[\mathcal{M}\left(\frac{\|a(\Delta\xi_n) + b(\Delta\eta_n)\|}{K}\right)\right]^{s_n} &\leq k\sup_n\left[\mathcal{M}\left(\frac{\|\Delta\xi_n\|}{K_1}\right)\right]^{s_n}\\
&+ k\sup_n\left[\mathcal{M}\left(\frac{\|\Delta\eta_n\|}{K_2}\right)\right]^{s_n}\\
&< \infty
\end{aligned}$$

for some $k > |a|$ and $k > |b|$. Therefore, $\{a\xi_n + b\eta_n\} \in \ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, q, \|.\|)$. Hence, $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ is a linear space over $\mathbb{C}_2 \setminus \mathbb{O}_2$. □

Theorem 5.1.3. *For every sequence $s = \{s_n\}$ of strictly positive numbers, the sets $\ell(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$, $c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ and $c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ are linear spaces.*

Proof. The proof is analogous to the previous theorem, hence omitted. □

Theorem 5.1.4. *The space $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ is paranormed by*

$$p(\xi) = \|\xi_1\| + \inf\left\{K^{\frac{s_n}{P}} : \sup_n \left\{\mathcal{M}\left(\frac{\|\Delta\xi_n\|}{K}\right)(t_n)^{1/s_n}\right\} \leq 1,\ K > 0\right\},$$

where $P = \max\{1, \sup s_n\}$.

Proof. For the null $\mathbb{C}_2$-sequence θ, we have $q(\theta_1) = 0$ and $\mathcal{M}\left(\frac{\|\Delta\theta_n\|}{K}\right) = 0$ for any $K > 0$. Therefore, $p(\theta) = 0$.

Also, $p(-\xi) = p(\xi)$, $\forall\, \xi \in \ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$.

Now, let $\xi, \eta \in \ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Then there exist $K_1 > 0$, $K_2 > 0$ such that

$$\mathcal{M}\left(\frac{\|\Delta\xi_n\|}{K_1}\right)(t_n)^{1/s_n} \leq 1 \quad \text{and} \quad \mathcal{M}\left(\frac{\|\Delta\eta_n\|}{K_2}\right)(t_n)^{1/s_n} \leq 1$$

Suppose $K = K_1 + K_2$. Then

$$\begin{aligned}
&\sup_n \left\{\mathcal{M}\left(\frac{\|\Delta\xi_n + \Delta\eta_n\|}{K}\right)(t_n)^{1/s_n}\right\} \\
&\leq \frac{K_1}{K} \sup_n \left\{\mathcal{M}\left(\frac{\|\Delta\xi_n\|}{K}\right)(t_n)^{1/s_n}\right\} + \frac{K_2}{K} \sup_n \left\{\mathcal{M}\left(\frac{\|\Delta\eta_n\|}{K}\right)(t_n)^{1/s_n}\right\} \\
&\leq 1
\end{aligned}$$

Further,

$$\begin{aligned}
p(\xi+\eta) &= \|\xi_1+\eta_1\| + \inf\left\{(K)^{\frac{s_n}{P}} : \sup_n\left\{\mathcal{M}\left(\frac{\|\Delta\xi_n+\Delta\eta_n\|}{K}\right)(t_n)^{1/s_n}\right\}\leq 1\right\}\\
&\leq \|\xi_1\| + \inf\left\{K_1^{\frac{s_n}{P}} : \sup_n\left\{\mathcal{M}\left(\frac{\|\Delta\xi_n\|}{K_1}\right)(t_n)^{1/s_n}\right\}\leq 1\right\}\\
&+ \|\eta_1\| + \inf\left\{K_2^{\frac{s_n}{P}} : \sup_n\left\{\mathcal{M}\left(\frac{\|\Delta\eta_n\|}{K_2}\right)(t_n)^{1/s_n}\right\}\leq 1\right\}\\
&\leq p(\xi)+p(\eta)
\end{aligned}$$

Let $a \in (\mathbb{C}_2 \setminus \mathbb{O}_2)$. Then

$$\begin{aligned}
p(a\,\xi) &= \|a\,\xi_1\| + \inf\left\{K^{\frac{s_n}{P}} : \sup_n\left\{\mathcal{M}\left(\frac{a\,\|\Delta\xi_n\|}{K}\right)t_n^{\frac{1}{s_n}}\right\}\leq 1,\ K>0\right\}\\
&\leq \sqrt{2}\,\|a\|\,\|\xi_1\| + \inf\left\{\|a\|(\sqrt{2}\,H)^{\frac{s_n}{P}} : \sup_n\left\{\mathcal{M}\left(\frac{a\,\|\Delta\xi_n\|}{K}\right)t_n^{\frac{1}{s_n}}\right\}\leq 1,\ H>0\right\}
\end{aligned}$$

where $H = K/\|a\|$. □

Theorem 5.1.5. *$\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ is a complete paranormed space for $s \in \ell^\infty(\mathbb{C}_0)$.*

Proof. Let $\{\xi_n^m\}$ be a Cauchy sequence in $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Then for all $n \in \mathbb{N}$,

$p(\xi_n^i - \xi_n^j) \to 0$ as $i, j \to \infty$.

Let for given $\epsilon > 0$, there exists $H > 0$ and some $x > 0$ such that $\frac{\epsilon}{xH} > 0$ and

$\sup_n (s_n)^{t_n} \leq \mathcal{M}(\frac{xH}{2})$.

Now for $p(\xi_n^i - \xi_n^j) \to 0$ as $i, j \to \infty$, there exists some $k_0 \in \mathbb{N}$ such that

$$p(\xi_n^i - \xi_n^j) < \frac{\epsilon}{x\,H}, \quad \forall\, i, j \geq k_0,\ \forall\, n \in \mathbb{N}.$$

Therefore,

$$\|\xi_1^i - \xi_1^j\| + \inf\left\{K^{\frac{s_n}{P}} : \sup_n\left\{\mathcal{M}\left(\frac{\|\Delta\xi_n^i - \Delta\xi_n^j\|}{K}\right)(t_n)^{1/s_n}\right\}\leq 1, K>0\right\} < \frac{\epsilon}{x\,H},$$

$\Rightarrow \|\xi_1^i - \xi_1^j\| < \frac{\epsilon}{x\,H}$ and

$$\inf\left\{K^{\frac{s_n}{P}} : \sup_n \left\{\mathcal{M}\left(\frac{\|\Delta\xi_n^i - \Delta\xi_n^j\|}{K}\right)(t_n)^{1/s_n}\right\} \leq 1, K > 0\right\} < \frac{\epsilon}{x\,H}. \quad (5.1.6)$$

This implies that $\{\xi_1^i\}$ is a Cauchy sequence in $\mathbb{C}_2$. Since $\mathbb{C}_2$ is a modified Banach Algebra, then $\{\xi_1^i\}$ converges in $\mathbb{C}_2$. Let $\lim\limits_{n\to\infty} \xi_1^i = \xi_1$. Thus

$$\|\xi_1^i - \xi_1\| < \frac{\epsilon}{x\,H} \quad \text{as} \quad j \to \infty.$$

Now, from the Equation (5.1.6), we have

$$\mathcal{M}\left(\frac{\|\Delta\xi_n^i - \Delta\xi_n^j\|}{p(\xi_n^i - \xi_n^j)}\right)(t_n)^{1/s_n} \leq 1.$$

This implies that

$$\mathcal{M}\left(\frac{\|\Delta\xi_n^i - \Delta\xi_n^j\|}{p(\xi_n^i - \xi_n^j)}\right) \leq (p_n)^{1/t_n} \leq \mathcal{M}\left(\frac{xH}{2}\right).$$

Therefore,

$$\|\Delta\xi_n^i - \Delta\xi_n^j\| < \frac{xH}{2} \cdot \frac{\epsilon}{x\,H} < \frac{\epsilon}{2}.$$

Thus, $\{\Delta\xi_n^i\}$ is a Cauchy sequence in $\mathbb{C}_2$, $\forall k \in \mathbb{N}$. Hence, $\{\Delta\xi_n^i\}$ converges in $\mathbb{C}_2$. Suppose $\lim\limits_{i\to\infty} \Delta\xi_n^i = \Delta\eta_n$, $\forall\, n \in \mathbb{N}$. So, $\lim\limits_{i\to\infty} \Delta\xi_2^i = \Delta\eta_1 - \xi_1$ and in general, we have $\lim\limits_{i\to\infty} \Delta\xi_{n+1}^i = \eta_n - \xi_n$, $\forall\, n \in \mathbb{N}$. Hence, by the continuity of the Orlicz function $\mathcal{M}$, we have

$$\lim_{j\to\infty} \sup_n \left\{\mathcal{M}\left(\frac{\|\Delta\xi_n^i - \Delta\xi_n^j\|}{K}\right)(t_n)^{1/s_n}\right\} \leq 1,$$

this implies that

$$\sup_n \left\{\mathcal{M}\left(\frac{\|\Delta\xi_n^i - \Delta\xi_n\|}{K}\right)(t_n)^{1/s_n}\right\} \leq 1,$$

Let $i \geq k_0$ and by taking infimum for the values of $K > 0$, we obtain

$p(\xi^i - \xi) < \epsilon$. So, $\{\xi^i - \xi\} \in \ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Hence, $\xi \in \ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Therefore, $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ is complete. $\square$

Corollary 5.1.1. $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ is a Banach space for $s \in \ell^\infty(\mathbb{C}_0)$.

In the similar manner, we can prove the following theorem:

Theorem 5.1.6. *The sequence spaces* $\ell(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$, $c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ *and* $c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ *are Banach Spaces.*

Theorem 5.1.7. *The Banach spaces* $c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$, $c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ *and* $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ *are not solid.*

An example for this theorem is as follows:

Example 5.1.1. Suppose that the Orlicz function $\mathcal{M}$ is an identity function and $s_n = 1, \forall\, n \in \mathbb{N}$, $K = 1$. Consider a sequence $\{\xi_n^{(m)}\} \in \omega_4$ given as

$$\xi_n = \{\xi_n^{(m)}\} = \{2, 2, 2, \ldots\ldots\}.$$

Then $\{\alpha_n \xi_n^{(m)}\} \in c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Now let $\{\alpha_n\} = \{(-1)^n\}$, $\forall\, n \in \mathbb{N}$. Then $\{\alpha_n \xi_n^{(m)}\} \notin c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Thus, $c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ is not solid. Let $\{\xi_n\} \in \omega_4$, defined as $\xi_n = \{\xi_n^{(m)}\} = \{n^2, n^2+1, n^2+2, \ldots,\}$, $\forall\, n \in \mathbb{N}$. Then $\{\xi_n^{(m)}\} \in c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ as well as $\{\xi_n^{(m)}\} \in \ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Now let $\alpha_n = (-1)^n, \forall\, n \in \mathbb{N}$. Then $\{\alpha_n\, \xi_n^{(m)}\} \notin c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ as well as $\{\alpha_n\, \xi_n^{(m)}\} \notin \ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Also, it is clear that $\{\alpha_n\, \xi_n^{(m)}\} \notin c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Hence, spaces $c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|), c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ and $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ are not solid.

Theorem 5.1.8. *The Banach spaces* $c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$, $c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ *and* $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ *are not symmetric, in general.*

We gave an example for this theorem as follows:

Example 5.1.2. Let $\mathcal{M}(x) = x$ and $s_n = 2$, $\forall n \in \mathbb{N}$. Suppose $\{\xi_n\} = \{\xi_n^k\}$ defined as $\xi_n^k = \{n, n+1, n+2,\}$, for all $n \in \mathbb{N}$. Then $\{\xi_n\}$ is in $c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ and $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Consider an rearrangement $\{\eta_n\}$ of $\{\xi_n\}$ defined as

$$\{\xi_n\} = \{\xi_1^k, \xi_8^k, \xi_2^k, \xi_{27}^k, \xi_3^k, \xi_{64}^k, \xi_4^k, \ldots, \ldots\}$$

Then $\{\xi_n\} \notin c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ as well as $\{\xi_n\} \notin \ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$. Hence $c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ and $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ are not symmetric space. Similarly, the space $c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ is not symmetric.

Theorem 5.1.9. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Orlicz functions with Δ_2-condition and $s = \{s_n\}$ be a bounded sequence of real numbers, then*

$$c_0(\mathbb{C}_2, \mathcal{M}_2, \Delta, s, \|.\|) \subset c_0(\mathbb{C}_2, \mathcal{M}_1 * \mathcal{M}_2, \Delta, s, \|.\|)$$

Proof. Let $\{\xi_n\} \in c(\mathbb{C}_2, \mathcal{M}_2, \Delta, s, \|.\|)$. Then there exists some $K > 0$ such that

$$\left[\mathcal{M}_2\left(\frac{\|\Delta\xi_n\|}{K}\right)\right]^{s_n} t_n \to 0, \quad \text{as} \quad k \to \infty$$

Suppose that $\eta_n = \mathcal{M}_2\left(\frac{\|\Delta\xi_n\|}{K}\right)$, $\forall n \in \mathbb{N}$. Also, $\mathcal{M}_1$ satisfies the Δ_2-condition, Then for some $0 < \delta < 1$, there exists some $P \geq 1$ such that

$$\mathcal{M}_1(\eta_n) \leq P\frac{\eta_n}{\delta}\mathcal{M}_1(2). \tag{5.1.7}$$

Therefore,

$$\begin{aligned}
\left[(\mathcal{M}_1 * \mathcal{M}_2)\left(\frac{\|\Delta\xi_n\|}{K}\right)\right]^{s_n} t_n &= \left[\mathcal{M}_1\left(\mathcal{M}_2\left(\frac{\|\Delta\xi_n\|}{K}\right)\right)\right]^{s_n} t_n \\
&= [\mathcal{M}_1(\eta_n)]^{s_n} t_n \\
&= \max\left\{\sup_n([\mathcal{M}_1(1)]^{s_n}), \sup_n([P\mathcal{M}_1(2)(1/\delta)]^{s_n})\right\}[\eta_n]^{s_n} t_n \\
&= \to 0, \quad \text{as} \quad n \to \infty
\end{aligned}$$

Hence, $\{\xi_n\} \in c_0(\mathbb{C}_2, \mathcal{M}_1 * \mathcal{M}_2, \Delta, s, \|.\|)$. □

Theorem 5.1.10. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Orlicz functions with Δ_2-condition and $s = \{s_n\} \in \ell^\infty$, then*

$$c(\mathbb{C}_2, \mathcal{M}_1, \Delta, s, \|.\|) \cap c(\mathbb{C}_2, \mathcal{M}_2, \Delta, s, \|.\|) \subset c(\mathbb{C}_2, \mathcal{M}_1 + \mathcal{M}_2, \Delta, s, \|.\|)$$

Proof. Let $\{\xi_n\} \in c(\mathbb{C}_2, \mathcal{M}_1, \Delta, s, \|.\|) \cap c(\mathbb{C}_2, \mathcal{M}_2, \Delta, s, \|.\|)$. Then there exists some $L \in \mathbb{C}_2$, $K_1 > 0$ and $K_1 > 0$ such that

$$\left[\mathcal{M}_1\left(\frac{\|\Delta\xi_n - L\|}{K_1}\right)\right]^{s_n} t_n \to 0 \quad \text{and} \quad \left[\mathcal{M}_2\left(\frac{\|\Delta\xi_n - L\|}{K_1}\right)\right]^{s_n} t_n \to 0 \quad (5.1.8)$$

Let $K = \max\{K_1, K_2\}$. Then

$$\begin{aligned}\left\{\left[(\mathcal{M}_1 + \mathcal{M}_2)\left(\frac{\|\Delta\xi_n - L\|}{K_1}\right)\right]^{s_n} t_n\right\} &\leq D\left[\mathcal{M}_1\left(\frac{\|\Delta\xi_n - L\|}{K_1}\right)\right]^{s_n} t_n \\ &+ D\left[\mathcal{M}_2\left(\frac{\|\Delta\xi_n - L\|}{K_1}\right)\right]^{s_n} t_n\end{aligned}$$

From the Equation (5.1.8), we obtain, $\{\xi_n\} \in c(\mathbb{C}_2, \mathcal{M}_1 + \mathcal{M}_2, \Delta, s, \|.\|)$. Hence,

$$c(\mathbb{C}_2, \mathcal{M}_1, \Delta, s, \|.\|) \cap c(\mathbb{C}_2, \mathcal{M}_2, \Delta, s, \|.\|) \subset c(\mathbb{C}_2, \mathcal{M}_1 + \mathcal{M}_2, \Delta, s, \|.\|)$$

Analogously, result can be proved for the $\mathbb{C}_2$-sequence spaces $c_0(\mathbb{C}_2, \mathcal{M}_1, \Delta, s, \|.\|)$ and $\ell^\infty(\mathbb{C}_2, \mathcal{M}_1, \Delta, s, \|.\|)$. □

5.2 Some Classes of $\mathbb{C}_2$-nets and Id(o)-topology

In this section, some classes of $\mathbb{C}_2$-nets on the bicomplex space are constructed and discuss about the summability by the $\mathbb{C}_2$-nets in the Id(o)-topology.

Definition 5.2.1 (Bounded $\mathbb{C}_2$-net). A $\mathbb{C}_2$-net $\{\xi_\alpha\}_D$ is said to be bounded in Id(o)-topology, τ_7, if there exist some η and ζ such that $\eta \prec_{Id} \xi_\alpha \prec_{Id} \zeta$, $\forall \alpha \in D$.

Let us assume that Δ be a family of directed sets. Let $D \in \Delta$ with $Card(D) < Card(\mathbb{C}_0)$ be a given directed set. Define some sets of $\mathbb{C}_2$-net as follows:

(i) $\mathbb{F}$ be the set of all bounded $\mathbb{C}_2$-nets.

(ii) $\mathbb{F}_1$ be the set of all convergent $\mathbb{C}_2$-nets.

(iii) $\mathbb{F}_2$ be the set of all Id-confluence $\mathbb{C}_2$-nets.

(iv) $\mathbb{F}_3$ be the set of all Id-point confluence $\mathbb{C}_2$-nets.

(v) $\mathbb{F}_4$ be the set of all null $\mathbb{C}_2$-nets.

Remark 5.2.1. The set $\mathbb{F}_2$ contains all the Id(F)-confluence, Id(P)-confluence, Id(L)-confluence an Id-point confluence $\mathbb{C}_2$-nets. The $\mathbb{C}_2$-nets which are Id(F)-confluence, Id(P)-confluence and Id(L)-confluence may or may not be bounded.

Example 5.2.1. Consider a $\mathbb{C}_2$-net $\{\xi_\alpha\}$ on $(\mathbb{Q}^+, \geq)$ as $\xi_\alpha = 2e_1 + (1/\alpha + \alpha)e_2$, $\forall\, \alpha \in \mathbb{Q}^+$. Then, $\mathbb{C}_2$ is Id(L)-confluence to $[Re\,^1\xi = 2, Im\,^1\xi = 2, Re\,^2\xi = 0]_{Id}$ but is unbounded as the net $\{Im\,^2\xi_\alpha\}$ is unbounded.

Remark 5.2.2. Every convergent $\mathbb{C}_2$-net in norm topology (cf. [72]) and as well as Id-point confluence $\mathbb{C}_2$-net in Id(o)-topology is bounded. Thus, $\mathbb{F}_4 \subset \mathbb{F}_1 \subset \mathbb{F}$ and $\mathbb{F}_3 \subset \mathbb{F}$.

Theorem 5.2.1. *The set* $\mathbb{F}$ *is linear space with the algebraic operations:*

(i) $\{\xi_\alpha\} + \{\eta_\alpha\} = \{\xi_\alpha + \eta_\alpha\}$, $\forall \alpha \in D$.

(ii) $k\{\xi_\alpha\} = \{k\,\xi_\alpha\}$, $\forall \alpha \in D$ *and* $k \in \mathbb{C}_1$.

Proof. The proof is straight forward, hence omitted. □

Definition 5.2.2. Define a function $||\,.\,||_{\mathbb{F}}$ on $\mathbb{F}$ as follows:

$$||\xi_\alpha||_{\mathbb{F}} = \max\{\sup\{|^1\xi_\alpha| : \alpha \in D\}, \sup\{|^2\xi_\alpha| : \alpha \in D\}\}$$

Lemma 5.2.1. The function $||\,.\,||_{\mathbb{F}}$ on $\mathbb{F}$ is a norm.

Proof. We shall verify all the properties of a norm on $\mathbb{F}$. Let us assume that the $\mathbb{C}_2$-net $\{\xi_\alpha\} = \{^1\xi_\alpha e_1 + {}^2\xi_\alpha e_2\} \in \mathbb{F}$ be a bounded $\mathbb{C}_2$-net.

(1) Suppose $|^1\xi_\alpha| \geq 0 \quad \text{and} \quad |^2\xi_\alpha| \geq 0$.

$\Rightarrow \sup\{|^1\xi_\alpha| : \alpha \in D\} \geq 0 \quad \text{and} \quad \sup\{|^2\xi_\alpha| : \alpha \in D\} \geq 0$

$\Rightarrow \max\{\sup\{|^1\xi_\alpha| : \alpha \in D\}, \sup\{|^2\xi_\alpha| : \alpha \in D\}\} \geq 0$

$\Rightarrow ||\xi_\alpha||_{\mathbb{F}} \geq 0$.

(2) Let $\max\{\sup\{|^1\xi_\alpha| : \alpha \in D\}, \sup\{|^2\xi_\alpha| : \alpha \in D\}\} = 0$

$\Leftrightarrow \sup\{|{}^1\xi_\alpha| : \alpha \in D\} = 0$ and $\sup\{|{}^2\xi_\alpha| : \alpha \in D\} = 0$

$\Leftrightarrow |{}^1\xi_\alpha| = 0 \quad$ and $\quad |{}^2\xi_\alpha| = 0, \forall \alpha \in D$

$\Leftrightarrow {}^1\xi_\alpha = 0 \quad$ and $\quad {}^2\xi_\alpha = 0, \forall \alpha \in D$

$\Leftrightarrow \xi_\alpha = 0, \forall \alpha \in D.$

(3) Let $\{\xi_\alpha\}_{\alpha \in D} \in \mathbb{F}$ be a bounded $\mathbb{C}_2$-net and $k \in \mathbb{C}_1$. Then

$$\begin{aligned}
||k\,\xi_\alpha||_{\mathbb{F}} &= \max\{\sup\{|k\ {}^1\xi_\alpha| : \alpha \in D\}, \sup\{|k\ {}^2\xi_\alpha| : \alpha \in D\}\} \\
&= \max\{\sup\{|k|\,|{}^1\xi_\alpha| : \alpha \in D\}, \sup\{|k|\,|{}^2\xi_\alpha| : \alpha \in D\}\} \\
&= |k|\,\max\{\sup\{|{}^1\xi_\alpha| : \alpha \in D\}, \sup\{|{}^2\xi_\alpha| : \alpha \in D\}\} \\
&= |k|\,||\xi_\alpha||_{\mathbb{F}}
\end{aligned}$$

Hence the proof. □

Theorem 5.2.2. *The space* $(\mathbb{F}, ||\,.\,||_{\mathbb{F}})$ *is a Banach space.*

Proof. Let D be a directed set with $card(D) < card(\mathbb{C}_0)$. Consider $\mathbb{C}_2$-net $\{\xi_\alpha^\beta\} \in \mathbb{F}$ be an arbitrary Cauchy $\mathbb{C}_2$-net. Then, there exists some $\alpha' \in D$ such that

$$\begin{aligned}
&||\xi_\alpha^\beta - \xi_\alpha^\gamma||_{\mathbb{F}} \to 0, \quad \forall\,\beta,\gamma \geq \alpha', \forall\,\alpha \in D, \\
\Rightarrow \quad & \max\{\sup\{|{}^1\xi_\alpha - {}^1\xi_\alpha^\gamma| : \alpha \in D\}, \sup\{|{}^2\xi_\alpha - {}^2\xi_\alpha^\gamma| : \alpha \in D\}\} \to 0, \\
\Rightarrow \quad & \sup\{|{}^1\xi_\alpha - {}^1\xi_\alpha^\gamma| : \alpha \in D\} \to 0 \quad \text{and} \quad \sup\{|{}^2\xi_\alpha - {}^2\xi_\alpha^\gamma| : \alpha \in D\} \to 0,
\end{aligned}$$

Thus, the nets $\{{}^1\xi_\alpha^\beta\}$ and $\{{}^2\xi_\alpha^\beta\}$ are Cauchy nets in $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively. The spaces $\mathbb{A}_1$ and $\mathbb{A}_2$ are Banach spaces. Therefore, the nets $\{{}^1\xi_\alpha^\beta\}$ and $\{{}^2\xi_\alpha^\beta\}$ converges in $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively. Assume ${}^1\xi_\alpha \in \mathbb{A}_1$ and ${}^2\xi_\alpha \in \mathbb{A}_2$, $\forall \alpha \in D$ such that $\{{}^1\xi_\alpha^\beta\}$ converges to ${}^1\xi_\alpha$ and $\{{}^2\xi_\alpha^\beta\}$ converges to ${}^2\xi_\alpha$, for $\beta \geq \alpha'$ and $\forall\,\alpha \in D$. Therefore, the $\mathbb{C}_2$-net $\{\xi_\alpha^\beta\}$ converges to ${}^1\xi_\alpha e_1 + {}^2\xi_\alpha e_2$. Hence, $(\mathbb{F}, ||\,.\,||_{\mathbb{F}})$ is a Banach space. □

Definition 5.2.3. Let D be a directed set. A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ on D is said to be *monotonically increasing*, if $\alpha \preceq \beta$ implies $\xi_\alpha \preceq \xi_\beta$. Similarly, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ on D is said to be *monotonically decreasing*, if $\xi_\alpha \succeq \xi_\beta$ when $\alpha \preceq \beta$.

Definition 5.2.4 (Summable $\mathbb{C}_2$-net). A $\mathbb{C}_2$-net $\{\xi_\alpha\}$ in $\mathbb{F}$ is said to be summable if and only if the $\mathbb{C}_2$-net $\{\eta_\alpha\}$ defined as $\eta_\alpha = \sum_{\beta \geq \alpha} \xi_\beta$ is convergent.

Theorem 5.2.3. *Let $\{\xi_\alpha\}$ be a $\mathbb{C}_2$-net. The following statements are equivalent:*

(a) $\{\xi_\alpha\}$ is summable.

(b) $\{{}^1\xi_\alpha\}$ and $\{{}^2\xi_\alpha\}$ are summable.

Proof. Let $\{\xi_\alpha\}$ be a summable $\mathbb{C}_2$-net. Thus, the net $\eta_\alpha = \sum_{\beta \geq \alpha} \xi_\beta$ is convergent to η (say) in $\mathbb{C}_2$. Then, for given $\epsilon > 0$, there exists $\gamma \in D$ such that

$$\|\eta_\alpha - \eta\|_{\mathbb{F}} < \epsilon$$

Thus,

$$\max\{\sup\{|{}^1\eta_\alpha - {}^1\eta|\}, \sup\{|{}^2\eta_\alpha - {}^2\eta|\}\} < \epsilon, \ \forall \alpha \geq \gamma$$

This implies that

$$\sup\{|{}^1\eta_\alpha - {}^1\eta| : \alpha \in D\} < \epsilon \quad \text{as well as} \quad \sup\{|{}^2\eta_\alpha - {}^2\eta| : \alpha \in D\} < \epsilon, \ \forall \alpha \geq \gamma$$

This implies that

$$\{|{}^1\eta_\alpha - {}^1\eta|\} < \epsilon \quad \text{as well as} \quad \{|{}^2\eta_\alpha - {}^2\eta|\} < \epsilon, \ \forall \alpha \geq \gamma$$

Hence the nets $\{{}^1\eta_\alpha\}$ and $\{{}^2\eta_\alpha\}$ are converging to ${}^1\eta$ and ${}^2\eta$, respectively. Also,

$$\begin{aligned} \eta_\alpha = {}^1\eta_\alpha e_1 + {}^2\eta_\alpha e_2 &= \sum_{\beta \geq \alpha} \xi_\beta \\ &= \sum_{\beta \geq \alpha} {}^1\xi_\beta \, e_1 + \sum_{\beta \geq \alpha} {}^2\xi_\beta \, e_2 \end{aligned}$$

Therefore, the nets $\{{}^1\xi_\alpha\}$ and $\{{}^2\xi_\alpha\}$ are summable in $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively. $\square$

Theorem 5.2.4. *If $\{\xi_\alpha\}$ and $\{\eta_\alpha\}$ are summable $\mathbb{C}_2$-nets. Then $\{\xi_\alpha + \eta_\alpha\}$ is summable $\mathbb{C}_2$-net.*

Proof. Let $\{\xi_\alpha\}$ and $\{\eta_\alpha\}$ be two summable $\mathbb{C}_2$-nets, then the $\mathbb{C}_2$-nets given by

$$\xi_\alpha = \sum_{\beta \geq \alpha} \gamma_\beta \quad \text{and} \quad \eta_\alpha = \sum_{\beta \geq \alpha} \delta_\beta, \quad \forall\, \alpha \in D,$$

are convergent. Since $\xi_\alpha = {}^1\xi_\alpha\, e_1 + {}^2\xi_\alpha\, e_2$ and $\eta_\alpha = {}^1\eta_\alpha\, e_1 + {}^2\eta_\alpha\, e_2$, $\forall\, \alpha \in D$. Thus,

$${}^1\xi_\alpha\, e_1 + {}^2\xi_\alpha\, e_2 = \sum_{\beta \geq \alpha} {}^1\gamma_\beta\, e_1 + \sum_{\beta \geq \alpha} {}^2\gamma_\beta\, e_2,$$

and

$${}^1\eta_\alpha\, e_1 + {}^2\eta_\alpha\, e_2 = \sum_{\beta \geq \alpha} {}^1\delta_\beta\, e_1 + \sum_{\beta \geq \alpha} {}^2\delta_\beta\, e_2.$$

Therefore,

$$({}^1\xi_\alpha + {}^1\eta_\alpha)\, e_1 + ({}^2\xi_\alpha + {}^2\eta_\alpha)\, e_2 = \sum_{\beta \geq \alpha} ({}^1\gamma_\beta + {}^1\delta_\beta)\, e_1 + \sum_{\beta \geq \alpha} ({}^2\gamma_\beta + {}^2\delta_\beta)\, e_2.$$

Since, the sum of two convergent nets in $\mathbb{A}_i$, $(i = 1, 2)$ is a convergent net. Hence by the Theorem 5.2.3, the result can be obtained. □

Theorem 5.2.5. *If $\{\xi_\alpha\}$ is summable $\mathbb{C}_2$-net and $\lambda \in \mathbb{C}_1$, then $\{\lambda\, \xi_\alpha\}$ is summable $\mathbb{C}_2$-net.*

Proof. Assume that the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ in $\mathbb{F}$ is summable. Then, the $\mathbb{C}_2$-net $\{\eta_\alpha\}$, with $\eta_\alpha = \sum_{\beta \geq \alpha} \xi_\beta$ is convergent. Now,

$$\sum_{\beta \geq \alpha} \lambda\, \xi_\beta = \lambda\, \eta_\alpha = \lambda\, {}^1\eta_\alpha\, e_1 + \lambda\, {}^2\eta_\alpha\, e_2. \tag{5.2.1}$$

Since the nets $\{{}^1\eta_\alpha\}$ and $\{{}^1\eta_\alpha\}$ are convergent, then the nets $\{\lambda\, {}^1\eta_\alpha\}$ and $\{\lambda\, {}^1\eta_\alpha\}$ are convergent in $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively. Thus, the nets $\{\lambda\, {}^1\xi_\alpha\}$ and $\{\lambda\, {}^2\xi_\alpha\}$ are summable in $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively. Hence, $\{\lambda\, \xi_\alpha\}$ is summable in $\mathbb{C}_2$. □

5.3 Orderability of Topological Structures on $\mathbb{C}_2$

In this section, the orderability problem on the bicomplex space is obtained. To study the orderability problem in the bicomplex space with respect to the topologies defined by in Section 2.3.

Remark 5.3.1. Since the Id(o)-topology is strictly coarser than the Id(p)-topology on $\mathbb{C}_2$. Then, the Id(p)-topology is not orderable in the interval topology (Id(o)-topology) on $\mathbb{C}_2$.

Remark 5.3.2. The norm topology, complex topology and the idempotent topology are all coarser that the Id(o)-topology, so none of these topologies is orderable in the interval topology (Id(o)-topology) on $\mathbb{C}_2$.

We shall now focus on the convex subsets of the bicomplex space for the orderability of the subspace space topologies.

Definition 5.3.1 (Convex set). Let $<$ be the linear order on set X. The subset P is called a *convex set* in X if $(a, b) \subseteq P$ for every $a, b \in P$ with $a < b$.

Definition 5.3.2. Let Y be a subset of the space X, Then $Lim(Y)$ denotes the set of all cluster points of Y.

Also, for $G = Lim(Y) \setminus Y$, we define a set $I(\xi) = X \cap [\sup(G \cap (\leftarrow, \xi), \xi)]$. Note that for each $\xi \in X$, the set $I(\xi)$ is a convex set in $(X, <)$.

Remark 5.3.3. If P is a convex set in X, and r be any point in X such that $P \cap (\leftarrow, r) \neq \emptyset$ and $P \cap (r, \rightarrow) \neq \emptyset$, then $r \in P$.

Lemma 5.3.1. Let Y be a subset of $\mathbb{C}_2$. Suppose that for every $\xi \in \mathbb{C}_2 \setminus Y$ such that $Y \cap (\leftarrow, \xi)_{Id} \neq \emptyset$, $Y \cap (\xi, \rightarrow)_{Id} \neq \emptyset$ and $Y \cap (\leftarrow, \xi)_{Id}$ has maximum element if and only if $Y \cap (\xi, \rightarrow)_{Id}$ has minimum element. Then the subspace topology $\tau(\mathbb{C}_2, \prec_{Id})|Y$ coincides with the order topology $\tau(Y, \prec_{Id})$.

Lemma 5.3.2. Let Y be a convex subset of $\mathbb{C}_2$, then the subspace topology $\tau(\mathbb{C}_2, \prec_{Id})|Y$ coincides with order topology $\tau(Y, \prec_{Id})$.

Proof. Given that Y is convex in $\mathbb{C}_2$. Then for every pair of elements $\xi, \eta \in \mathbb{C}_2$, $(\xi, \eta)_{Id} \subseteq Y$. And for every $\zeta \in \mathbb{C}_2$ with $Y \cap (\leftarrow, \zeta)_{Id} \neq \emptyset$ and $Y \cap (\zeta, \rightarrow)_{Id} \neq \emptyset$, $\zeta \in Y$. Then $Y \cap (\zeta, \rightarrow)_{Id}$ has minimal element if and only if $Y \cap (\rightarrow, \zeta)_{Id}$ has maximal element. Therefore, the space $\tau(\mathbb{C}_2, \prec_{Id})|Y$ coincides with $\tau(Y, \prec_{Id})$. □

Example 5.3.1. Since the set $\mathbb{I}_2$ is a convex subset of $\mathbb{C}_2$ under $\prec_{Id}$. Thus, the Lemma 5.3.2 holds good for $\mathbb{I}_2$ in Id(o)-topology.

Lemma 5.3.3. Let Y be a subset of $\mathbb{C}_2$ such that $Y \subset \lim(Y)$ and $\lim(Y) \setminus Y \subset (\leftarrow, \sup Y)$. Then $\tau(\mathbb{C}_2, \prec_{Id})|Y$ coincides with the order topology $\tau(Y, \prec_{Id})$.

Proof. Let $\xi \in \mathbb{C}_2 \setminus Y$. Then $Y \cap (\leftarrow, \xi)_{Id} \neq \emptyset$ and $Y \cap (\xi, \rightarrow)_{Id} \neq \emptyset$. As $Y \cap (\xi, \rightarrow)_{Id}$ has a minimal element. Then we claim that $Y \cap (\leftarrow, \xi)_{Id}$ has a maximal element. If possible, suppose that $Y \cap (\leftarrow, \xi)_{Id}$ does not have a maximal element. Therefore, $\sup\{Y \cap (\leftarrow, \xi)_{Id}\} \notin Y \cap (\leftarrow, \xi)_{Id}$. Let $\alpha = \sup\{Y \cap (\leftarrow, \xi)_{Id}\}$. Then $\alpha \notin Y \cap (\leftarrow, \xi)_{Id}$ and $\alpha \in Lim(Y)$. Also $\alpha \prec_{Id} \xi$. Therefore, $\alpha \notin Y \cap (\leftarrow, \xi)_{Id}$ and $\alpha \prec_{Id} \xi$, where $\xi \notin Y$. So $\alpha \notin Y$. This is a contradiction. Hence the proof. □

Theorem 5.3.1. *Let τ and τ' be topologies on $\mathbb{C}_2$. Then following properties hold:*

(i) If $K \subseteq \mathbb{C}_2$, $\tau|K = \tau'|K$ and $\xi \in int_{(\mathbb{C}_2,\tau)}(K) \cap int_{(\mathbb{C}_2,\tau')}(K)$, then for every subset V of $\mathbb{C}_2$, V is a neighbourhood of ξ in $(\mathbb{C}_2, \tau)$ if and only if it is a neighbourhood of ξ in $(\mathbb{C}_2, \tau')$.

(ii) If there a set $S \subset \mathbb{C}_2$ and a cover $\{I(\eta) : \eta \in S\}$ such that for each $\eta \in S, \tau|I(\eta) = \tau'|I(\eta)$ and $int_{(\mathbb{C}_2,\tau)}(K) = int_{(\mathbb{C}_2,\tau')}(K)$, for every $\xi \in I(\eta) int_{\mathbb{C}_2,\tau}$ and for every subset V of $\mathbb{C}_2$, V is a neighbourhood of ξ in $(\mathbb{C}_2, \tau)$ if and only if it is a neighbourhood of ξ in $(\mathbb{C}_2, \tau')$.

Proof. (i) Suppose that N is a neighbourhood of ξ in $(\mathbb{C}_2, \tau)$. Let $K_1 = int_{(\mathbb{C}_2,\tau)}(K)$. Then $\xi \in K_1 \subset K$ and $N \cap K_1$ is a neighbourhood of ξ in $\tau|K_1$. As $\tau|K_1 = (\tau|K)|K_1 = (\tau'|K)|K_1 = \tau'|K_1$ holds and K_1 is open in $(\mathbb{C}_2, \tau')$ and $N \cap K_1$ is a neighbourhood of ξ in $(\mathbb{C}_2, \tau')$.

(ii) It is sufficient to show that both topologies τ and τ' are same. For $\tau \subseteq \tau'$, let V be a neighbourhood of η in τ. Select an element $\xi \in S$ and $\eta \in I(\xi)$. By the given conditions, we suppose that $\eta \in I(\xi) \setminus int_{(\mathbb{C}_2,\tau)}(I(\xi))$. Then by the the assumption, V is a neighbourhood of η in τ'. Similarly, we have $\tau' \subseteq \tau$.

Hence the proof. $\square$

Conclusion

In this chapter, some $\mathbb{C}_2$-sequence spaces are defined using the Orlicz functions. The properties as linear structures, completeness, solidness of these $\mathbb{C}_2$-sequence spaces are discussed. In particular, the $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ space is analyzed under the paranorm defined on it. It has been proved that $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ is Banach space, not solid and not symmetric space. The condition of containment between different $\mathbb{C}_2$-sequence space have also been discussed. The problem of summability is discussed for different spaces of $\mathbb{C}_2$-nets. Further, the concept of orderability is also analyzed for norm topology, real topology, complex topology and the idempotent topology. The orderability condition for some convex subsets of $\mathbb{C}_2$ has been discussed in the subspace topology and the induced order topology.

$\square$ $\square$ $\square$

Conclusion

In this thesis, the main focus is to study the order topological structures and their compatibility with algebraic structures on the bicomplex space.

In Chapter 1, the introduction to the quaternions and the bicomplex numbers is given. The literature of bicomplex numbers have been given in brief. The basic concepts of the nets, filters and orderability problem are also discussed in this chapter. Work done by various authors is also reviewed in this chapter.

In Chapter 2, the order topological structures are defined and studied. Firstly, an order topology on the three dimensional space, $\mathbb{C}_0^3$ is defined by using the lexicographical order relation on it and three order relation namely $\ell(\mathbb{C}_0)$-order, $\ell(\mathbb{C}_1)$-order and ℓ_{Id}-order are defined using the lexicographical order relation on the bicomplex numbers in the real, complex and idempotent forms, which are defined on $\mathbb{C}_2$. The order topologies, namely $\mathbb{C}_0$(o)-topology, $\mathbb{C}_1$(o)-topology and Id(o)-topology are defined on the bicomplex space by using the order relations. It is also proved that the $\mathbb{C}_0$(o)-topology and the $\mathbb{C}_1$(o)-topology are same but the $\mathbb{C}_0$(o)-topology and the Id(o)-topology are not comparable. Two more topologies namely Id(p)topology and Id(m)-topology are defined and compared with order topologies and concluded that Id(o)topology is strictly finer than the Id(p)-topology.

In Chapter 3, the net of bicomplex numbers called as $\mathbb{C}_2$-net is defined and the concept of convergence called as $\mathbb{C}_0$(F)-confluence, $\mathbb{C}_0$(P)-confluence, $\mathbb{C}_0$(L)-confluence and $\mathbb{C}_0$-Point confluence are defined. It has been proved that every $\mathbb{C}_2$-net, which is $\mathbb{C}_0$-Point confluence is also $\mathbb{C}_0$(L)-confluence and every $\mathbb{C}_0$(L)-confluence $\mathbb{C}_2$-net is $\mathbb{C}_0$(P)-confluence. Also every $\mathbb{C}_0$(P)-confluence $\mathbb{C}_2$-net is $\mathbb{C}_0$(F)-confluence, but the converse is not true. Id(F)-confluence, Id(P)-confluence, Id(L)-confluence and the Id-Point confluence are defined in the Id(o)-topology. It is also proved that any $\mathbb{C}_2$-net $\{\xi_\alpha\}_D$ is a Cauchy $\mathbb{C}_2$-net iff the net $\{\xi_\alpha - \xi_\beta\}_{D\times D}$ is Id-point confluence to zero.

In Chapter 4, the different possibilities of the clustering of the $\mathbb{C}_2$-nets has been given with respect to the Id(o)-topology. It is proved that any $\mathbb{C}_2$-net cluster on a point with respect to the Id(o)-topology is also cluster with respect to Id(p)-topology. But the converse of this implication is not true in some cases. It has been proved that if any $\mathbb{C}_2$-net is Id-point confluence to a point ξ, then every cofinal $\mathbb{C}_2$-subnet of the $\mathbb{C}_2$-net is Id-point confluence to ξ. Also, it is proved that the principal ideals $\mathbb{I}_1$ and $\mathbb{I}_2$ are nowhere dense subsets of $\mathbb{C}_2$ and the set of all bicomplex singular elements is uncountable set of first category.

In Chapter 5, the compatibility of the algebraic and topological structures have been discussed. The paranormed $\mathbb{C}_2$-sequence spaces such as $\ell(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$, $c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$, $c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ and the space $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ are defined on the bicomplex space. It has been shown that all these spaces are Banach spaces. It is also proved that the spaces $c(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$, $c_0(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ and $\ell^\infty(\mathbb{C}_2, \mathcal{M}, \Delta, s, \|.\|)$ are not solid as well as not symmetric. Some classes such as $\mathbb{F}$, $\mathbb{F}_1$, $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$ of $\mathbb{C}_2$-nets are defined and their respective results are proved for $\mathbb{F}$. It is proved that $\mathbb{C}_2$-net is summable iff both of its idempotent component nets are summable. In last, orderability problem on $\mathbb{C}_2$ was discussed and proved that real topology, complex and idempotent topology are not orderable in ℓ_{Id}-order.

Future Work

For future work, one can study about the analysis of order topologies, summability theory via $\mathbb{C}_2$-nets as well as the orderability problem on the bicomplex space. The summability theory is a very intensive area for research. An idea about the concept of summability of the $\mathbb{C}_2$-nets in the Id(o)-topology was derived. Therefore, different classes of $\mathbb{C}_2$-nets can be studied in the future. The conditions for the equivalence of different topologies with the lexicographical order topologies on $\mathbb{C}_2$ can be developed. Also, the concept of convex subsets for the orderability condition in the subspace topologies on subsets of $\mathbb{C}_2$ is studied. Therefore, the general conditions for the orderability of subsets can be developed.

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Appendix

List of Publications

Published/Communicated Papers

1. Cluster Sets of $\mathbb{C}_2$-nets, *Global J. of Pure and Appl. Math.*, **13**(6), (2017), 2589-2599. (UGC Approved)
2. On Certain Topological Aspects of $\mathbb{C}_0^3$ and $\mathbb{C}_2$ with a Special Emphasis on Lexicographical Order, *AIP Conf. Proc. RAFAS-2016*, **1860**, (20029-1), (2017), 1-9. (Scopus Indexed)
3. $\mathbf{C}_2$-nets and $\mathbf{C}_0$(o)-topology, *Inter. J. Math. Sci. & Engg. Appls.,* **11** (III), (2017), 89-99. (UGC Approved)
4. Orlicz Function and Some Paranormed Difference $\mathbb{C}_2$-Sequence Spaces, (Communicated with a Scopus Indexed Journal)

National and International Conferences/Workshops/Seminars

1. Paper entitled "Order Topologies and Compactness in the Bicomplex Space" presented in the National Conference on *Recent Advancements in Mathematics* held at Beant College of Engineering and Technology, Gurdaspur, (Punjab), February, 2014.

2. Participated in the International Conference on *Advances in Pure and Applied Mathematics*, held at Jawaharlal Nehru Government Degree College, Haripur (Manali), (Himachal Pradesh), 7-9 March, 2014.

3. Participated in the Teacher's Enrichment Workshop (ATM School) on *Finite Group Theory* organized by National Centre for Mathematics at Department of Mathematics, Punjabi University, Patiala, (Punjab), 27 May-2 June, 2014.

4. Paper entitled "Product of Filters on the Bicomplex Space" presented in the conference on *Emerging Basic and Applied Sciences* by School of Chemical Engineering & Physical Sciences, Lovely Professional University, Phagwara, (Punjab), 14-15 November, 2014.

5. Paper entitled "ID-Confinement Theory of Filters in Bicomplex Space" presented in the 80^{th} annual conference of *Indian Mathematical Society* organized by Department of Applied Mathematics, Indian School of Mines, Dhanbad, (Jharkhand), 27-30 December, 2014.

6. Paper entitled "A Study of Order Topologies and Nets in the Bicomplex Space" presented in the 81^{st} annual conference of *Indian Mathematical Society* organized by Department of Mathematics, Visvesvaraya National Institute of Technology (NIT), Nagpur, (Maharashtra), 27-30 December, 2015.

7. Paper entitled "On Topological Aspects of $\mathbb{C}_0^3$ and $\mathbb{C}_2$ with Special Emphasis on Lexicographic Order" presented in the International Conference *Recent Advances on Fundamental and Applied Sciences* organized by the School of Chemical Engineering & Physical Sciences, Lovely Professional University, Phagwara (Punjab), 25-26 November, 2016.

8. Paper entitled "On Problems Related to Summability of $\mathbb{C}_2$-nets" presented in the International Conference of The Indian Mathematical Consortium (TIMC) in cooperation with American Mathematical Society (AMS) in the Banaras Hindu University, Varanasi (Uttar Pradesh), 14-17 December, 2016.

9. Participated in the Teacher's Enrichment Workshop (ATM School) on the *"Connections Across Disciplines in Mathematics"* organized by National Centre for Mathematics at Department of Mathematics, Indian Institute of Science Education and Research, Mohali (Punjab), 18-23 December, 2017.

10. Paper entitled "On Problems Related to Summability of $\mathbb{C}_2$-sequences" presented in the National Conference on *Recent Trends in Pure and Applied*

Mathematics organized by Post Graduate Department of Mathematics, DAV College, Jalandhar (Punjab), 23-24 February, 2018.

11. Participated in the Teacher's Enrichment Workshop (ATM School) on *"Algebra and Multivariable Calculus"* organized by the National Centre for Mathematics at DAV University, Jalandhar (Punjab), 28 May-2 June, 2018.

Memberships

1. Indian Mathematical Society *(Life member)*
2. Ramanujan Mathematical Society *(Life member)*
3. Indian Science Congress Association *(Life member)*
4. Indian Society of Mathematics & Mathematical Sciences *(Life member)*
5. American Mathematical Society *(Affiliate Member)*

Other Achievements

1. Awarded with *best paper presentation award* in the national conference on *Recent Advancements in Mathematics* held at Beant College of Engineering and Technology, Gurdaspur, (Punjab); February, 2014.
2. An article on the *Bicomplex Topology* is published in the annual magazine of Deen Dayal Upadhyaya College, University of Delhi, Delhi, 2014.
3. Our work on Order Topologies has been cited in the book "*Bicomplex Holomorphic Functions: Algebra, Geometry and Analysis of Bicomplex Numbers*" by the authors M. Shapiro et. al. published by Birkhauser, USA in 2015.

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