A STUDY OF COUPLED FIXED POINT THEOREMS

A

Thesis

Submitted to

LOVELY PROFESSIONAL UNIVERSITY

Transforming Education Transforming India

For the award of

DOCTOR OF PHILOSOPHY

in

Mathematics

By

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December, 2018

DECLARATION

I hereby declare that the work being presented in the thesis entitled "**A Study of Coupled Fixed Point Theorems**" in fulfillment of the requirements for the award of the degree of Doctor of Philosophy, submitted in the Lovely Faculty of Technology and Sciences of Lovely Professional University, Phagwara, is an authentic record of my own work under the supervision of Dr. Virendra Singh Chouhan and Dr. Sanjay Mishra.

The matter embodied in the thesis has not been submitted by me for the award of any other degree.

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CERTIFICATE

This is to certify that Richa Sharma, has completed the thesis titled "**A Study of Coupled Fixed Point Theorems**" under my guidance and supervision. To the best of my knowledge, the present work is the result of his original investigation and study. No part of this thesis has ever been submitted for any other degree at any University.

The thesis is fit for the submission and the partial fulfilment of the conditions for the award of Doctor of Philosophy in Mathematics.

Signed:

Supervisor: **Dr. Virendra Singh Chouhan**

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Abstract

Fixed point theory is a mesmerizing subject, with an immense number of applications in various fields of Mathematics. In maximum cases, I noticed that fixed points pop up when they are needed. Bhaskar and Laksmikantham, extend this theory to partially ordered metric spaces and introduce the concept of coupled fixed point for mixed-monotone operators. There is a vast literature on this topic, which has influenced me in many ways. This is the reason that made me decide to work on this.

The present thesis entitled "A study of coupled fixed point theorems" contained coupled fixed point and coupled coincidence point theorems associated with metric spaces. We have established certain coupled fixed point results using altering distances and rational expressions. Also, new results for various types of mapping such as mixed monotone, mixed g-monotone and mixed weakly monotone have been proved in abstract spaces such as G-metric spaces, cone metric spaces, ordered metric spaces. We also introduced Y-cone metric space and study some topological properties of Y-cone metric space.

The aim of this work is to establish the existence of coupled fixed points for mixed monotone operators, coupled coincidence points and coupled common fixed points for mixed g-monotone operators and mixed weakly monotone operators satisfying different contractive conditions in abstract metric spaces.

The thesis is divided into eight chapters.

Chapter 1, is introductory in nature. This chapter divulges the general review of the literature on the fixed point theorems and coupled fixed point theorems and it has given a brief summary of each chapter towards the end of this chapter.

In chapter 2, the concepts of coupled fixed points have been discussed. The conception of the property (mixed monotone) has been used to demonstrate the coupled fixed point results in ordered metric spaces for the non-linear contractive condition of altering distance functions. To illustrate these outcomes, an example has been given. The results presented in this chapter have applied to achieve the solution of an integral equation.

The aim of chapter 3, is to show some unique coupled fixed point theorems involving rational expression in an ordered cone metric spaces. Examples are provided to support our results. Further, some unique coupled fixed point theorems satisfying certain rational contractive condition have proved in an ordered cone metric space. We support these results by giving an example.

The motivation behind this chapter 4, is to build up some coupled coincidence point results for a mixed g-monotone mappings satisfying a non-linear contractive condition in the framework of partially ordered G-metric spaces. Also, we present a result on the existence and uniqueness of coupled common fixed points. The results presented in this chapter generalize and extend several well-known results in the literature. To illustrate these results, an example and a solution of integral equations have also been given.

Chapter 5, is devoted to establish coupled coincidence point results for non-linear contractive maps using mixed g-monotone property in ordered metric spaces by altering distances. We provide examples and an application of integral equations to support the usability of our results.

In chapter 6, we introduce the idea of Y-cone metric space and to study some topological properties of Y-cone metric space. Then, some coupled common fixed point results have been established using the property of mixed weakly monotone in ordered Y-cone metric spaces. Finally, we give an example, which supports the main theorem we develop in this chapter.

The objective of chapter 7, is to prove some unique coupled fixed point theorem in ordered metric space. Also, an example in support to illustrate the effectiveness of these results has been given. Further, some unique coupled fixed point results along with rational contractive condition in a partially ordered metric space have been proved. We support these results by giving some examples.

In last, we provides the conclusion which is based on the present study, also relevant topics for future research have been suggested.

Acknowledgements

First of all, I would like to pay high regards to Supreme GOD, who encourages me to do this tough errand.

I express my sincere gratitude and indebtedness to my Supervisor Dr. Virendra Singh Chouhan, Department of Mathematics, for introducing me to an extremely rich and beautiful area of research, for his competent guidance throughout my work, as well as for his encouragement and strong support on numerous occasions. I express my heartfelt thanks and gratitude to my Co-Supervisor Dr. Sanjay Mishra, Associate Professor, Department of Mathematics, Lovely Professional University, Punjab, for his helpful suggestions, co-operation, encouragement and guidance during the research work.

I am deeply thankful to Dr. Ramesh Thakur, Head of School (HOS), Lovely Professional University, Punjab, for their support and needful help during the various stages of my research work. I am also grateful to Dr. Rekha, Associate Professor, School of Research Degree Programme, Lovely Professional University, Punjab, for the learned advice and constant encouragement during my research work.

I am grateful to the Department of Mathematics and Lovely Professional University for offering me a thesis work, without which not have been able to complete this research. I would also like to thank the members of staff at the Department who helped me on various occasions. I would also like to thank the experts who were involved in the evaluation of thesis work. Without their passionate participation and input, the thesis could not have been successfully completed.

Finally, I must express my very profound gratitude to my parents for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them. I would also like to thank my brother, sister and their families for all their support.

Contents

Chapter 1

Introduction

The field of fixed point theory is critical and beneficial in Mathematics and may be recognized as one of the thrust areas of exploration in nonlinear analysis. Fixed point theory has shown a pivotal aspect in the dilemma-solving approaches of nonlinear functional analysis.

A point is usually called fixed point when it remains invariant. For a function $f\colon U \to U$, fixed point is a point $s \in U$, for which $f(s) = s$.

The study of existence and the uniqueness of coincidence points and common fixed point of maps fulfilling some contractive conditions has been an impressive field of Mathematics since 1922 when Banach [19] stated and demonstrated his great result. The Banach contraction principle [19] is uncomplicated and most adaptable elementary results in fixed point theory. According to this principle, if S is a contraction on a complete metric space U, thus S has a unique fixed point in U. This principle provides useful applications which occur in diverse places in the mathematical literature. An impressive application is a differentiable function, the system of linear algebraic equations, ordinary differential equations and implicit functions. Fractals and holomorphic mappings are also the applications of Banach contraction principle. Various researchers enhance the approach of Banach's contraction principle (see [23, 28, 29, 38–40, 53, 64, 83, 98, 131, 139–143]). Jungck [79] investigated common fixed point results in metric spaces by introducing commuting maps that have occurred in the literature.

The first time conception of provided by Opoitsev [110–112], and then Guo and Lakshmikantham [62] gave the perception of coupled fixed point in relation with coupled quasi-solutions of an initial value problem for ordinary differential equations.

Several years later, an unusual research direction for the theory of coupled fixed points in the setting of an ordered metric space was initiated by Bhaskar and Lakshmikantham.

Bhaskar and Lakshmikantham [21] presented the interpretation of mixed monotone maps and obtained certain coupled fixed point results. Also, they gave an application in which they have deliberated the existence and uniqueness of solution for a boundary value problem. Several researchers ([41, 42, 74, 93, 127, 137]) have obtained coupled fixed point, and coupled coincidence results in ordered metric spaces, fuzzy metric spaces, and other spaces.

In this thesis, coupled fixed point outcomes in abstract spaces like ordered metric spaces, G-metric spaces, and cone metric spaces have been presented and many existing results in the literature generalized.

To provide adequate background for consequent chapters. We present some basic definitions, notations and some classical and recent results connected to this work in this introductory chapter. However, some basic definitions and notations will be rehashed at times in numerous chapters for the purpose of convenience.

1.1 Preliminaries Related to Metric Fixed Point Theory

Metric spaces play a substantial role in the study of functional analysis and topology. The term 'metric' is consequent from the word 'metor' (measure). Frechet [55] acquainted with the conception of a metric space in 1906. Though, the definition presented by the Hausdorff [70] in 1914 is use and expressed as:

Definition 1.1. [70] Suppose U be any nonempty set. A metric on U is a real-valued function $d\colon U \times U \to \mathbb{R}$ which satisfies the succeeding conditions for all $u_1, u_2, u_3 \in U$

1. $d(u_1, u_2) \geq 0,$ (Positivity)
2. $d(u_1, u_2) = 0 \iff u_1 = u_2,$ (Identity)
3. $d(u_1, u_2) = d(u_2, u_1),$ (Symmetry)
4. $d(u_1, u_2) \leq d(u_1, u_3) + d(u_3, u_2).$ (Triangle inequality)

A metric space is a non-empty set U equipped with a metric d on U and is denoted as (U, d). Geometrically, $d(u_1, u_2)$ represents distance between two points u_1 and u_2 on the real line.

Example 1.1. *[70]*

1. *The set $\mathbb{R}$ of real numbers with the distance function $d(u_1, u_2) = |u_1 - u_2|$ is a metric space.*

2. *Every normed linear space is a metric space underneath the metric* $d(u_1, u_2) = \parallel u_1 - u_2 \parallel$.

Definition 1.2. [70] A sequence $\{u_t\}$ of a metric space (U, d) it calls

1. convergent to a point $u \in U$ if $\lim_{t\to\infty} d(u_t, u) = 0$.
2. Cauchy sequence if $\lim_{t,s\to\infty} d(u_t, u_s) = 0$.

Definition 1.3. [70] If each Cauchy sequence in (U, d) is convergent in U then a metric space (U, d) is called complete.

In a metric space, each convergent sequence is a Cauchy sequence, but the converse is not true. For example, Euclidean n-space with the Euclidean distance is the complete metric space whereas the set of rational numbers with metric $d(u_1, u_2) = |u_1 - u_2|$ is not a complete metric space.

Famous mathematician Brouwer [24] investigated the study of the fixed point theory. The recognition of creating the fixed point theory is pragmatic and popularity extends to mathematician Stefan Banach. In 1922, Banach proved a fixed point theorem, which ensures the existence and uniqueness of a fixed point under suitable conditions. This result [19] is known as the Banach contraction principle expressed as:

Theorem 1.4. *[19] Suppose (U, d) is a complete metric space. Also, M be a self-map on U and $0 \le k < 1$ such that*

$$d(Mu, Mv) \le kd(u, v), \;\; \forall \;\; u, v \in U.$$

Then M has a unique fixed point in U. Further for some $u_0 \in U$, the sequence of iterates $\{M^n u_0\}$ is Cauchy and its limit is the unique fixed point of M.

Banach contraction principle has many applications, but it suffers from one drawback, it requires the mapping to be continuous at all points of its domain. In 1968, Kannan [83] showed the existence of a fixed point for a map that can have a discontinuity in a domain, however, the maps involved in every case were continuous at the fixed point. Many authors started working along this direction and proved related fixed point theorems.

Theorem 1.5. *[83] Suppose (U, d) be a complete metric space and M is a self-map of U satisfy $d(Mu, Mv) \le k[d(u, Mu) + d(v, Mv)], \; \forall \, u, v \in U$ and k is any real number such that $k \in \left[0, \frac{1}{2}\right)$. Then M has a unique fixed point in U.*

Dass and Gupta [46] built up an augmentation of Banach contraction principle through rational articulation.

Theorem 1.6. *[46] Let g be a mapping of U into itself, thus*

$$d(g(c), g(c')) \leq \alpha \frac{d(c', g(c'))[1 + d(c, g(c))]}{1 + d(c, c')} + \beta\, d(c, c'),\ \forall\, c', c \in U,$$

$\alpha > 0$, $\beta > 0$, $(\beta + \alpha) < 1$ for any $c_0 \in U$, the sequence of iterates $\{g^n(c_0)\}$ has a subsequence $\{g^n{}_k(c_0)\}$ with $\zeta = g^n{}_k(c_0)$

Then, Jaggi [73] generalized certain unique fixed point hypothesis which fulfil a contractive condition of rational expression. Afterwards many authors ([13, 26, 56, 67, 94, 105]) generalized this concept.

Theorem 1.7. *[73] Suppose M be a continuous self-map on a complete metric space (U, d). Assume that M gratifies:*

$$d(Mz, Mx) \leq \alpha \frac{d(z, Mz)\, d(x, Mx)}{d(z, x)} + \beta\, d(z, x),$$

each $z, x \in U, z \neq x$, and for any $\beta, \alpha \in [0, 1)$ with $(\beta + \alpha) < 1$, then M has a unique fixed point in U.

Khan [89] acquaint with the usage of a control function in metric fixed point problems. This function was referred to as altering distance function, this function and its extension have utilized in several problems of a fixed point theory, some of these are noted in ([18, 49, 52, 66, 104, 116, 136]).

Definition 1.8. [89] If it satisfies the subsequent properties, then $\phi\colon [0, \infty) \to [0, \infty)$ is termed as an altering distance function.

1. The function ϕ is monotone increasing and continuous,
2. $\phi(z) = 0$ iff $z = 0$.

Weak contraction principle is a generality of Banach's contraction principle which was first specified by Alber and Gurre-Delabriere [8] in Hilbert spaces and evidenced the existence of fixed points.

Definition 1.9. [8] Suppose (U, d) be a metric space. It terms a mapping $P\colon U \to U$ as weakly contractive provided that

$$d(Ps, Pu) \leq d(s, u) - \phi(d(s, u))$$

where $s, u \in U$, $\phi\colon [0,\infty) \to [0,\infty)$ is a non-decreasing and continuous function such that $\phi(z) = 0$ iff $z = 0$.

Rhoades [125] has shown that the outcome which Alber demonstrated in [8] is also substantial in complete metric spaces. Rhoades [125] also established the subsequent remarkable fixed point theorem which is one of the generalizations of the Banach contraction principle because it contains contractions as special cases $\phi(z) = (1-k)z$ for some $0 \leq k < 1$.

Theorem 1.10. *[125] Suppose (U, d) be a complete metric space. Presume $M\colon U \to U$ be a weakly contractive mapping. Thus M has a fixed point.*

Dutta and Choudhury [52] proved the succeeding theorem which generalized the Rhoades theorem.

Theorem 1.11. *[52] Suppose (U, d) be a complete metric space and $M\colon U \to U$ be a self-mapping satisfy*

$$\psi(d(Mu, Mv)) \leq \psi(d(u, v)) - \phi(d(u, v))$$

where $\phi, \psi\colon [0,\infty) \to [0,\infty)$ are two continuous and monotone non-decreasing functions with $\phi(z) = 0 = \psi(z)$ iff $z = 0$. Then M has a fixed point.

For the duration of the last few spans, several works on weakly contractions have been published, some of these are noted in ([35, 37, 60, 65, 76, 121, 123]).

1.2 Abstract Spaces for Fixed Point

Numerous generalities of a metric space have been discussed by several prominent researcher. Now, we present preliminaries related to G-metric spaces, Cone metric spaces, partial metric spaces and partially ordered metric spaces which will be used in later chapters.

1.2.1 G-Metric Space

Gahler [57] presented the different approach of 2-metric spaces in 1963, later many researchers demonstrated that there is no relationship among these two functions. Then, Dhage [47], declared a distinctive theory of generalized metric space called the D-metric spaces. But fundamental topological arrangements of such spaces were irrelevant.

Mustafa in association with Sims [101] presented another structure of generalized metric spaces, are described as G-metric spaces, to establish and displayed another fixed point hypothesis for many mappings in this new arrangement.

Definition 1.12. [101] Suppose $G\colon U \times U \times U \to \mathbb{R}^+$ be a function and assume U be a non-empty set satisfies the following axioms for all $u_1, u_2, u_3, a \in U$,

1. $G(u_1, u_2, u_3) = 0$ if $u_1 = u_2 = u_3$,
2. $0 < G(u_1, u_1, u_2)$ with $u_1 \neq u_2$,
3. $G(u_1, u_1, u_2) \leq G(u_1, u_2, u_3)$ with $u_3 \neq u_2$,
4. $G(u_1, u_2, u_3) = G(u_1, u_3, u_2) = G(u_2, u_3, u_1) = \dots$,
5. $G(u_1, u_2, u_3) \leq G(u_1, a, a) + G(a, u_2, u_3)$,

then G termed as Generalized metric or G-metric on U and (U, G) termed as G-metric space.

Example 1.2. *[101] Suppose (U, d) be a metric space. Define $G\colon U \times U \times U \to \mathbb{R}^+$ by*

$$G(u_1, u_2, u_3) = \frac{d(u_1, u_2) + d(u_2, u_3) + d(u_3, u_1)}{3}.$$

Then (U, d) is a G-metric space.

They have obtained lateral, several fixed point theorems on G-metric space. For more results, we allude the peruser ([2, 78, 102, 103, 117, 126, 132, 138, 146, 147]).

1.2.2 Cone Metric Space

Huang and Zhang [69] presented the perception of cone metric spaces, supplanting the set of the real numbers by an ordered Banach space. In that paper, they studied the convergence in cone metric spaces, completeness, and demonstrated certain fixed point hypotheses for contractive mappings on these spaces.

Let P a subset of E and E be a real Banach space. They call P a cone iff

1. $P \neq \{\theta\}$, P is non-empty and closed set,
2. $p, q \in R,\ p, q \geq 0$ and $l, m \in P \implies pl_1 + ql_2 \in P$,
3. $P \cap (-P) = \{\theta\}$.

Given cone $P \subset E$, we represent a partial ordering $\leq$ with respect to P by $l_1 \leq l_2$ iff $l_2 - l_1 \in P$. We shall write $l_1 < l_2$ to show that $l_1 \leq l_2$ but $l_1 \neq l_2$, while $l_1 \ll l_2$ will hold for $l_2 - l_1 \in intP$, $intP$ describes the interior of P.

The cone P is called normal if there is a constant $M > 0$, for each $l_1, l_2 \in E$,

$$\theta \leq l_1 \leq l_2 \text{ then } \parallel l_1 \parallel \leq M \parallel l_2 \parallel .$$

The least positive number M satisfies this inequality is known as the normal constant of P. Rezapour [124] demonstrated that there are no normal cones with normal constants $M < 1$, for every $k > 1$ there are normal cones with normal constants $M > k$.

The cone P is termed as regular if each non-increasing (non-decreasing) sequence which is bounded below (above) is convergent respectively. It is observed that each regular cone is normal.

Definition 1.13. [69] Suppose U be a non-empty set. Presume the mapping $d\colon U \times U \to E$ gratifies the following axioms for all $u_1, u_2, u_3 \in U$,

1. $\theta \leq d(u_1, u_2)$ and $d(u_1, u_2) = \theta \iff u_1 = u_2$,
2. $d(u_1, u_2) = d(u_2, u_1)$,
3. $d(u_1, u_2) \leq d(u_1, u_3) + d(u_3, u_1)$.

Then d termed as a cone metric on U and (U, d) termed as a cone metric space.

Example 1.3. *[69] Suppose* $E = \mathbb{R}^2, P = \{(s, v) \in E \mid u_1, u_2 \geq 0\} \subseteq \mathbb{R}^2$, $U = \mathbb{R}$ *and* $d\colon U \times U \to E$ *such that* $d(u_1, u_2) = (|u_1 - u_2|, \alpha|u_1 - u_2|)$, *where* $\alpha \geq 0$ *is a constant. Then* (U, d) *is a cone metric space.*

In recent years, several authors have studied, some of references are noted in ([1, 11, 77, 80–82, 88, 91, 120, 134, 144]).

1.2.3 Partial Metric Space

One of the most important concept is partial metric spaces, which is given by Matthews [96]. In which the distance of a point from itself may not be zero. This concept has a wide formation of applications not only in many branches of Mathematics, but also in the field of computer domain and semantics.

Definition 1.14. [96, 97] A partial metric on a non-empty set U is a function $p\colon U \times U \to \mathbb{R}_n$ for each $u_1, u_2, u_3 \in U$,

1. $u_1 = u_2 \iff p(u_1, u_1) = p(u_1, u_2) = p(u_2, u_2)$,
2. $p(u_1, u_1) \leq p(u_1, u_2)$,
3. $p(u_1, u_2) = p(u_2, u_1)$,
4. $p(u_1, u_2) \leq p(u_1, u_3) + p(u_3, u_2) - p(u_3, u_3)$.

The pair (U, p) is known as partial metric space (pms).
If a partial metric p on U, then the function $p^s : U \times U \to \mathbb{R}^+$ given by

$$p^s(u_1, u_2) = 2p(u_1, u_2) - p(u_1, u_1) - p(u_2, u_2)$$

is a metric on U.

Remark 1.15. If $u_1 = u_2$, $p(u_1, u_2)$ may not be '0'.

Example 1.4. *[97] Take into account* $U = [0, \infty)$ *with* $p(u_1, u_2) = max\{u_1, u_2\}$. *Then* (U, p) *be a partial metric space. It is clear that* p *is not a (usual) metric. Note that in this case* $d_p(u_1, u_2) = |u_1 - u_2|$

In topical years there has been a growing interest in studying the existence of fixed points for contractive maps satisfying monotone properties in partial metric spaces, for more details, we refer to the reader ([6, 12, 15, 72, 108, 109, 114, 145]).

1.2.4 Partially Ordered Metric Spaces

Ran and Reurings [122] have investigated another paramount direction in generalizing the Banach contraction principle by considering a partially ordered on the metric space and presented some applications to matrix equations. Since then several researchers have studied the problem of the existence and uniqueness of a fixed point for contraction type mapping on partially ordered sets (see [7, 106, 107]).

Definition 1.16. [90] A partially ordered set is a set U taken together with a binary relation $\leq$, represented by $(U, \leq)$ such that for all $u_1, u_2, u_3 \in U$,

1. $u_1 \leq u_1$ (reflexivity),
2. $u_1 \leq u_2$ and $u_2 \leq u_1$ implies $u_1 = u_2$ (anti-symmetry),
3. $u_1 \leq u_2$ and $u_2 \leq u_3$ implies $u_1 \leq u_3$ (transitivity).

Definition 1.17. Suppose $(U, \leq)$ be a partially ordered set. Presuppose there is a metric d on U then we call $(U, \leq, d)$ a partially ordered metric space.

Definition 1.18. [44] Suppose $(U, \leq)$ be partially ordered set. Any elements $u_1, u_2 \in U$, is known to be comparable if either $u_1 \leq u_2$ or $u_2 \leq u_1$.

Other topical works in this area are noted in ([10, 20, 43, 50, 54, 63, 65, 68, 71, 119]). Also, these results have been studied in partially ordered cone metric spaces [9, 11, 17] and ordered G-metric spaces [16, 25, 27, 115].

1.2.5 Coupled Fixed Point Theory

One of the significant and imperative ideas, a coupled fixed point result, was present by Lakshmikantham and Bhaskar [21] and considered some results in ordered metric spaces. This concept is fundamental for many scientific works.

Definition 1.19. [21] An element $(u, v) \in U \times U$ is called coupled fixed point of the mapping $T : U \times U \to U$ if $T(u, v) = u,\ T(v, u) = v$.

Definition 1.20. [21] Suppose $(U, \leq)$ be a partially ordered set and a mapping $T : U \times U \to U$. Then the map T has the property of mixed monotone if $T(u, v)$ is monotone non-decreasing in u and is monotone non-increasing in v, that is, for some $u, v \in U$,

$$u_1, u_2 \in U, u_1 \leq u_2 \implies T(u_1, v) \leq T(u_2, v),$$
$$v_1, v_2 \in U, v_1 \leq v_2 \implies T(u, v_1) \geq T(u, v_2).$$

Theorem 1.21. *[21] Suppose $(U, \leq)$ be a partially ordered set. Presuppose there exists a metric d on U, (U, d) is a complete metric space. Presume $T : U \times U \to U$ be a mapping having the mixed monotone property on U. Suppose that there exists, $k \in [0, 1)$ with*

$$d(T(s, v), T(u, t)) \leq \frac{k}{2}[d(s, u) + d(v, t)]$$

for each $s, v, u, t \in U$ with $u \leq s,\ v \leq t$. Presume either T is continuous or U has the following properties:

1. *if a non-decreasing sequence $s_n \to s$, then $s_n \leq s$, for all n,*
2. *if a non-increasing sequence $v_n \to v$, then $v_n \geq v$, for all n.*

If there exist $s_0, v_0 \in U$ with $s_0 \leq T(s_0, v_0)$ and $T(v_0, s_0) \leq v_0$, then T has a coupled fixed point.

Lakshmikantham and Ciric [42] established a new conception of mixed g-monotone property and coupled coincidence point.

Definition 1.22. [42] An element $(s,v) \in U \times U$ is called coupled coincidence point of the mappings $g\colon U \to U$ and $T\colon U \times U \to U$ if $T(s,v) = gs,\ T(v,s) = gv$.

Definition 1.23. [42] Suppose U be a non-empty set. Then the mappings $g\colon U \to U$, $T\colon U \times U \to U$ are commutative if $T(gs, gv) = gT(s,v)$, for all $s, v \in U$.

Definition 1.24. [42] Suppose $(U, \leq)$ be a partially ordered set. The mappings $g\colon U \to U$, $T\colon U \times U \to U$ is said to have mixed g-monotone property if $T(u,v)$ is monotone g-non-decreasing in u and is monotone g-non-increasing in v, for some $u, v \in U$,

$$u_1, u_2 \in U, gu_1 \leq gu_2 \implies T(u_1, v) \leq T(u_2, v),$$
$$v_1, v_2 \in U, gv_1 \leq gv_2 \implies T(u, v_2) \leq T(u, v_1).$$

Sabetghadam [127] presented this concept in cone metric spaces. Abbas [3] presented a new concept of w-compatible mapping and employed this conception to get a uniqueness result of coupled coincidence point in G-metric space.

Definition 1.25. [3] The mappings $g\colon U \to U$ and $T\colon U \times U \to U$ are called w-compatible if $g(T(s,v)) = T(gs, gv)$ whenever $gs = T(s,v)$ and $gv = T(v,s)$.

Aydi [14] proved results on partial metric spaces in 2011.

Theorem 1.26. *[14] Suppose (U, p) be a complete partial metric space. Presuppose that the mapping $H\colon U \times U \to U$ satisfy,*

$$p(H(s,v), H(u,t)) \leq k\, p(s,u) + l\, p(v,t)$$

for each $s, u, t, v \in U$, where l, k are positive constants with $(l + k) < 1$. Thus H has a unique coupled fixed point.

In 2012, Gordji et.al [59] established a new notion of mixed weakly monotone property as follows.

Definition 1.27. [59] Suppose $(U, \leq)$ be a partially ordered set and $F_1, F_2\colon U \times U \to U$ be mappings. Then a pair (F_1, F_2) has the mixed weakly monotone property on U, for some $u_1, u_2 \in U$,

$$u_1 \leq F_1(u_1, u_2),\ u_2 \geq F_1(u_2, u_1)$$
$$\implies F_1(u_1, u_2) \leq F_2(F_1(u_1, u_2), F_!(u_2, u_1)),\ \ F_1(u_2, u_1) \geq F_2(F_1(u_2, u_1), F_1(u_1, u_2)),$$

and

$$u_1 \leq F_2(u_1, u_2),\ u_2 \geq F_2(u_2, u_1)$$
$$\implies F_2(u_1, u_2) \leq F_1(F_2(u_1, u_2), F_2(u_2, u_1)),\ \ F_2(u_2, u_1) \geq F_1(F_2(u_2, u_1), F_2(u_1, u_2)).$$

After that several authors extended these concepts for different contractive conditions in abstract spaces. For this direction, we refer works of [3–5, 14, 16, 22, 25, 27, 30–32, 34, 36, 48, 51, 58, 59, 61, 75, 84–87, 92, 95, 99, 100, 113, 118, 128, 133–135, 148].

1.3 Objectives of The Study

In precise terms, the objectives of this study are as follows:

1. To improve and extend the known results considering new contractive conditions.
2. To obtain coupled fixed point theorems for an operator with the property of mixed monotone.
3. To obtain coupled coincidence point theorems with the property of mixed g-monotone.
4. To study related coupled fixed point theorems in abstract spaces.
5. To study the applications of coupled fixed points theorems.

We have established some results in abstract spaces such as partially ordered G-metric space, partially ordered cone metric spaces and ordered metric spaces by using rational expression and altering distances.

1.4 Thesis Organization

The thesis is divided into eight chapters.

Chapter 1 is introductory in nature. It reveals the general review of the literature on the fixed point and coupled fixed point and also the summary of the subsequent chapters.

In chapter 2, the concepts of coupled fixed points have been discussed. The conception of the property (mixed monotone) has been used to demonstrate the coupled fixed point results in ordered metric spaces for the non-linear contractive condition of altering distance functions. To illustrate these outcomes, an example has been given. The results presented in this chapter have been applied to achieve the solution of an integral equation.

The aim of chapter 3, is to show some unique coupled fixed point results involving rational expression in an ordered cone metric spaces. Illustrations are presented to support our results. Further, some unique coupled fixed point theorems satisfying certain rational contractive condition have been proved in an ordered cone metric space. We support these results by giving an example.

The purpose of chapter 4, is to establish some coupled coincidence point results for maps possesses the property (mixed g-monotone) adequate a non-linear contractive condition in the structure of ordered G-metric spaces. Also, we show a result on the existence and uniqueness of coupled common fixed points. The results produced in this chapter generalize and expand many acclaimed results in the literature. To illustrate these results, it has also given an example and a solution of integral equations.

Chapter 5, is devoted to establishing coupled coincidence point theorems for non-linear contractive maps using mixed g-monotone property in ordered metric spaces by altering distances. We provide illustrations and an application to integral equations to help the ease of use of these outcomes.

In chapter 6, we introduce the concept of Y-cone metric space and study some topological properties of Y-cone metric space. Then, some coupled common fixed point theorems have been established utilizing the property of mixed weakly monotone map in ordered Y-cone metric spaces. Finally, we gave an illustration, which constitutes the main theorem we develop in this chapter.

The objective of chapter 7, is to demonstrate a certain unique coupled fixed point result in ordered metric space. Also, examples in support to interpret the efficacy of these results have been given. Further, some unique coupled fixed point results along with rational contractive condition in a partially ordered metric space have proved. We support these results by giving examples.

In the last, we provides the conclusion which is based on the present study, also relevant topics for future research have been suggested.

Chapter 2

Coupled Fixed Point Results Associated with Altering Distances in Partially Ordered Metric Spaces

In this chapter, some unique coupled fixed point theorems along with the property of mixed monotone including altering distance functions have been proved. This chapter has been divided into various sections. In section 2.1, the concept of coupled fixed point and altering distance function have been discussed. In section 2.2, some coupled fixed point theorem has been proved by using the concept of altering distance function. In section 2.3, applications to the solution of integral equation have been given by using the results proved in the section 2.2.

2.1 Introduction

Khan [89] started the use of a control function that alters distance among two points in a metric space. They call such mappings an altering distances. Altering distance has been applied in metric fixed point theory by many researchers (perceive [49, 52, 104]). It has also been expanded in fuzzy and multivalued maps. In Menger spaces [33], the perception of altering distance function has also been presented. Recently, utilizing these functions Harjani and Sadarangani [66] demonstrated certain fixed point hypotheses in partially ordered metric spaces.

Bhaskar and Lakshmikantham [21] started an investigation of a coupled fixed point result in ordered metric spaces and connected the outcomes to demonstrate the existence and uniqueness of results for a boundary value problem. Using this concept many researchers have obtained their results for maps underneath numerous contractive conditions [34, 41, 42, 128, 135].

At first we need the following definitions and results.

Definition 2.1. [89]. If it satisfy subsequent axioms, then $\phi\colon [0,\infty) \to [0,\infty)$ is termed as altering distance function:

1. The function ϕ is monotone increasing and continuous.
2. $\phi(z) = 0$ iff $z = 0$.

Definition 2.2. [21]. An element $(s, v) \in U \times U$ is termed as coupled fixed point of the mapping $F\colon U \times U \to U$ if $F(s, v) = s, F(v, s) = v$.

Definition 2.3. [21]. Suppose $(U, \leq)$ be a partially ordered set and a mapping $F\colon U \times U \to U$. Then the map F has the property of mixed monotone if $F(s, v)$ is monotone non decreasing in s and is monotone non-increasing in v, for some $s, v \in U$,

$$s_1, s_2 \in U, s_1 \leq s_2 \implies F(s_1, v) \leq F(s_2, v),$$
$$v_1, v_2 \in U, v_1 \leq v_2 \implies F(s, v_1) \geq F(s, v_2).$$

2.2 Coupled Fixed Point Theorems Including Altering Distance Functions

In this section, certain coupled fixed point theorem has been established by utilizing the concept of altering distance function in complete metric space.

Theorem 2.4. *Suppose $(U, \leq)$ be a partially ordered set equipped with a metric d in U so that (U, d) be a complete space. Presume $F\colon U \times U \to U$ be a continuous mapping on U possesses the property of mixed monotone satisfy*

$$\varphi(d(F(r, s), F(w, v))) \leq \varphi(M((r, s),(w, v))) - \phi(M((r, s),(w, v))) \tag{2.1}$$

where,

$$M((r, s),(w, v)) = max\{d(r, w), d(s, v), d(F(r, s), r), d(F(w, v), w)\}$$

$\forall\ r, w, v, s \in U$ with $r \geq w$ and $v \geq s$, here φ and ϕ are altering distance functions. Suppose that there exists $r_0, s_0 \in U$ such that $r_0 \leq F(r_0, s_0)$, $s_0 \geq F(s_0, r_0)$, then F has a coupled fixed point.

Proof. Take $r_0, s_0 \in U$. Set $r_1 = F(r_0, s_0)$ and $s_1 = F(s_0, r_0)$. Repeating this process, set $r_{t+1} = F(r_t, s_t)$ and $s_{t+1} = F(s_t, r_t)$. Then by (2.1), we get

$$\begin{aligned}\varphi(d(r_{t+1}, r_t)) &= \varphi(d(F(r_t, s_t), F(r_{t-1}, s_{t-1}))) \\ &\leq \varphi(M((r_t, s_t), (r_{t-1}, s_{t-1}))) - \phi(M((r_t, s_t), (r_{t-1}, s_{t-1}))),\end{aligned}$$

and

$$\begin{aligned}\varphi(d(s_{t+1}, s_t)) &= \varphi(d(F(s_t, r_t), F(s_{t-1}, r_{t-1}))) \\ &\leq \varphi(M((s_t, r_t), (s_{t-1}, r_{t-1}))) - \phi(M((s_t, r_t), (s_{t-1}, r_{t-1}))),\end{aligned}$$

where,

$$\begin{aligned}M((r_t, s_t), (r_{t-1}, s_{t-1})) &= max\{d(r_t, r_{t-1}), d(s_t, s_{t-1}), \\ &\qquad d(F(r_t, s_t), r_t), d(F(r_{t-1}, s_{t-1}), r_{t-1})\} \\ &= max\{d(r_t, r_{t-1}), d(s_t, s_{t-1}), d(r_{t+1}, r_t), d(r_t, r_{t-1})\} \\ &= max\{d(r_t, r_{t-1}), d(s_t, s_{t-1}), d(r_{t+1}, r_t)\}.\end{aligned}$$

Now, let us consider two cases.

Case 1: If $M((r_t, s_t), (r_{t-1}, s_{t-1})) = max\{d(r_t, r_{t-1}), d(s_t, s_{t-1})\}$.
We get

$$\begin{aligned}\varphi(d(r_{t+1}, r_t)) \leq\ & \varphi(max\{d(r_t, r_{t-1}), d(s_t, s_{t-1})\}) \\ & - \phi(max\{d(r_t, r_{t-1}), d(s_t, s_{t-1})\}),\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\varphi(d(s_{t+1}, s_t)) \leq\ & \varphi(max\{d(s_t, s_{t-1}), d(r_t, r_{t-1})\}) \\ & - \phi(max\{d(s_t, s_{t-1}), d(r_t, r_{t-1})\}).\end{aligned} \tag{2.3}$$

Case 2: If $M((r_t, s_t), (r_{t-1}, s_{t-1})) = d(r_{t+1}, r_t)$.
We claim that $M((r_t, s_t), (r_{t-1}, s_{t-1})) = d(r_{t+1}, r_t) = 0$.

In fact if $d(r_{t+1}, r_t) \neq 0$, then

$$\varphi(d(r_{t+1}, r_t)) \leq \varphi(d(r_{t+1}, r_t)) - \phi(d(r_{t+1}, r_t)) < \varphi(d(r_{t+1}, r_t)) \text{ as } \phi \geq 0.$$

This implies $d(r_{t+1}, r_t) < d(r_{t+1}, r_t)$, which is a contradiction.
Since $M((r_t, s_t), (r_{t-1}, s_{t-1})) = 0$. Then it is obvious that (2.2) and (2.3) hold.

Now, by (2.2) and (2.3), we obtain

$$\varphi(d(r_{t+1},r_t)) \leq \varphi(max\{d(r_t,r_{t-1}),d(s_t,s_{t-1})\}) - \phi(max\{d(r_t,r_{t-1}),d(s_t,s_{t-1})\}). \tag{2.4}$$

As $0 \leq \phi$. $\varphi(d(r_{t+1},r_t)) \leq \varphi(max\{d(r_t,r_{t-1}),d(s_t,s_{t-1})\})$ and utilizing the way that φ is non decreasing, we get

$$d(r_{t+1},r_t) \leq max\{d(r_t,r_{t-1}),d(s_t,s_{t-1})\}. \tag{2.5}$$

Similarly,

$$\begin{aligned}\varphi(d(s_{t+1},s_t)) &\leq \varphi(max\{d(s_t,s_{t-1}),d(r_t,r_{t-1})\}) - \phi(max\{d(s_t,s_{t-1}),d(r_t,r_{t-1})\}) \\ &\leq \varphi(max\{d(s_t,s_{t-1}),d(r_t,r_{t-1})\}),\end{aligned} \tag{2.6}$$

and consequently

$$d(s_{t+1},s_t) \leq max\{d(s_t,s_{t-1}),d(r_t,r_{t-1})\}, \tag{2.7}$$

by (2.5) and (2.7), we have

$$max\{d(r_{t+1},r_t),d(s_{t+1},s_t)\} \leq max\{d(r_t,r_{t-1}),d(s_t,s_{t-1})\},$$

thus, $max\{d(r_{t+1},r_t),d(s_{t+1},s_t)\}$ is non-negative non-increasing sequence. This indicates that there is $k \geq 0$,

$$\lim_{t\to\infty} max\{d(r_{t+1},r_t),d(s_{t+1},s_t)\} = k. \tag{2.8}$$

It is effortlessly observed that if $\varphi : [0,\infty) \to [0,\infty)$ is non-decreasing, $\varphi(max(p,q)) = max(\varphi(p),\varphi(q))$ for $p,q \in [0,\infty)$. Considering this and (2.4) and (2.6), we have

$$\begin{aligned}max\{\varphi(d(r_{t+1},r_t)),\varphi(d(s_{t+1},s_t))\} &= \varphi(max\{d(r_{t+1},r_t),d(s_{t+1},s_t)\}) \\ &\leq \varphi(max\{d(r_t,r_{t-1}),d(s_t,s_{t-1})\}) \\ &\quad - \phi(max\{d(r_t,r_{t-1}),d(s_t,s_{t-1})\}).\end{aligned} \tag{2.9}$$

Letting $t \to \infty$ in (2.9) and using (2.8), we have

$$\varphi(k) \leq \varphi(k) - \phi(k) \leq \varphi(k) \implies \phi(k) = 0.$$

As, function ϕ is an altering distance, $k = 0$, this implies

$$\begin{aligned}&\lim_{t\to\infty} max\{d(r_{t+1},r_t),d(s_{t+1},s_t)\} = 0. \\ \implies &\lim_{t\to\infty} d(r_{t+1},r_t) = \lim_{t\to\infty} d(s_{t+1},s_t) = 0.\end{aligned} \tag{2.10}$$

Next, we claim that $\{r_t\}, \{s_t\}$ are Cauchy sequences.

We will establish that for each $0 < \varepsilon$, there is a natural number c, if $t, m \geq c$,

$$max\{d(r_{m(c)}, r_{t(c)}), d(s_{m(c)}, s_{t(c)})\} < \varepsilon.$$

Presuppose the above statement is not true.

At that point, there exists a $\varepsilon > 0$ for which we can discover sequence $\{r_{m(c)}\}, \{r_{t(c)}\}$ with $c < m(c) < t(c)$ such that

$$max\{d(r_{m(c)}, r_{t(c)}), d(s_{m(c)}, s_{t(c)})\} \geq \varepsilon. \tag{2.11}$$

Furthermore, we can take $t(c)$ comparing to $m(c)$ in such a manner that it should be least integer with $t(c) > m(c)$ and satisfy (2.11). Thus

$$\varepsilon > max\{d(r_{m(c)}, r_{t(c)-1}), d(s_{m(c)}, s_{t(c)-1})\}. \tag{2.12}$$

Using triangle inequality

$$d(r_{t(c)}, r_{m(c)}) \leq d(r_{t(c)}, r_{t(c)-1}) + d(r_{t(c)-1}, r_{m(c)}). \tag{2.13}$$

Similarly,

$$d(s_{t(c)}, s_{m(c)}) \leq d(s_{t(c)}, s_{t(c)-1}) + d(s_{t(c)-1}, s_{m(c)}). \tag{2.14}$$

From (2.13) and (2.14), we have

$$\begin{aligned} max\{d(r_{t(c)}, r_{m(c)}), d(s_{t(c)}, s_{m(c)})\} \leq\ & max\{d(r_{t(c)}, r_{t(c)-1}), d(s_{t(c)}, s_{t(c)-1})\} \\ & + max\{d(r_{t(c)-1}, r_{m(c)}), d(s_{t(c)-1}, s_{m(c)})\}. \end{aligned} \tag{2.15}$$

From (2.11), (2.12) and (2.15), we get

$$\varepsilon \leq max\{d(r_{t(c)}, r_{m(c)}), d(s_{t(c)}, s_{m(c)})\} \leq max\{d(r_{t(c)}, r_{t(c)-1}), d(s_{t(c)}, s_{t(c)-1})\} + \varepsilon. \tag{2.16}$$

Letting $c \to \infty$ in (2.16) and using (2.10), we have

$$\lim_{c\to\infty} max\{d(r_{t(c)}, r_{m(c)}), d(s_{t(c)}, s_{m(c)})\} = \varepsilon. \tag{2.17}$$

Again, the triangle inequality, we obtain

$$d(r_{t(c)-1}, r_{m(c)-1}) \leq d(r_{t(c)-1}, r_{m(c)}) + d(r_{m(c)}, r_{m(c)-1}), \tag{2.18}$$

and

$$d(s_{t(c)-1}, s_{m(c)-1}) \leq d(s_{t(c)-1}, s_{m(c)}) + d(s_{m(c)}, s_{m(c)-1}). \tag{2.19}$$

From, (2.18) and (2.19), we get

$$\begin{aligned} max\{d(r_{t(c)-1}, r_{m(c)-1}), &d(s_{t(c)-1}, s_{m(c)-1})\} \\ &\leq max\{d(r_{t(c)-1}, r_{m(c)}), d(s_{t(c)-1}, s_{m(c)})\} \\ &\quad + max\{d(r_{m(c)}, r_{m(c)-1}), d(s_{m(c)}, s_{m(c)-1})\}. \end{aligned} \tag{2.20}$$

From, (2.12), we have

$$\begin{aligned} max\{d(r_{t(c)-1}, r_{m(c)-1}), &d(s_{t(c)-1}, s_{m(c)-1})\} \\ &\leq max\{d(r_{m(c)}, r_{m(c)-1}), d(s_{m(c)}, s_{m(c)-1})\} + \varepsilon. \end{aligned} \tag{2.21}$$

Applying the triangle inequality, we obtain

$$d(r_{t(c)}, r_{m(c)}) \leq d(r_{t(c)}, r_{t(c)-1}) + d(r_{t(c)-1}, r_{m(c)-1}) + d(r_{m(c)-1}, r_{m(c)}), \tag{2.22}$$

and

$$d(s_{t(c)}, s_{m(c)}) \leq d(s_{t(c)}, s_{t(c)-1}) + d(s_{t(c)-1}, s_{m(c)-1}) + d(s_{m(c)-1}, s_{m(c)}). \tag{2.23}$$

From (2.11), (2.22), and (2.23), we get

$$\begin{aligned} \varepsilon \leq\ & max\{d(r_{t(c)}, r_{t(c)-1}), d(s_{t(c)}, s_{t(c)-1})\} \\ &+ max\{d(r_{t(c)-1}, r_{m(c)-1}), d(s_{t(c)-1}, s_{m(c)-1})\} \\ &+ max\{d(r_{m(c)-1}, r_{m(c)}), d(s_{m(c)-1}, s_{m(c)})\}. \end{aligned} \tag{2.24}$$

From, (2.21) and (2.24), we have

$$\begin{aligned} &\varepsilon - max\{d(r_{t(c)}, r_{t(c)-1}), d(s_{t(c)}, s_{t(c)-1})\} \\ &\quad - max\{d(r_{m(c)-1}, r_{m(c)}), d(s_{m(c)-1}, s_{m(c)})\} \\ &\quad \leq max\{d(r_{t(c)-1}, r_{m(c)-1}), d(s_{t(c)-1}, s_{m(c)-1})\} \\ &\quad < max\{d(r_{m(c)-1}, r_{m(c)}), d(s_{m(c)-1}, s_{m(c)})\} + \varepsilon. \end{aligned} \tag{2.25}$$

Letting $c \to \infty$ in (2.25) and using (2.10), we get

$$\lim_{c \to \infty} max\{d(r_{t(c)-1}, r_{m(c)-1}), d(s_{t(c)-1}, s_{m(c)-1})\} = \varepsilon. \tag{2.26}$$

Since $r_{t(c)-1} \geq r_{m(c)-1}$ and $s_{t(c)-1} \leq s_{m(c)-1}$, utilize the contractive condition (2.1), we can acquire

$$\begin{aligned}\varphi(d(r_{t(c)}, r_{m(c)})) &= \varphi(d(F(r_{t(c)-1}, s_{t(c)-1}), F(r_{m(c)-1}, s_{m(c)-1}))) \\ &\leq \varphi(M((r_{t(c)-1}, s_{t(c)-1}), (r_{m(c)-1}, s_{m(c)-1}))) \\ &\quad - \phi(M((r_{t(c)-1}, s_{t(c)-1}), (r_{m(c)-1}, s_{m(c)-1}))),\end{aligned} \tag{2.27}$$

where,

$$\begin{aligned}M((r_{t(c)-1}, s_{t(c)-1}), (r_{m(c)-1}, s_{m(c)-1})) &= max\{d(r_{t(c)-1}, r_{m(c)-1}), d(s_{t(c)-1}, s_{m(c)-1}), \\ &\quad d(F(r_{t(c)-1}, s_{t(c)-1}), r_{t(c)-1}), \\ &\quad d(F(r_{m(c)-1}, s_{m(c)-1}), r_{m(c)-1})\} \\ &= max\{d(r_{t(c)-1}, r_{m(c)-1}), d(s_{t(c)-1}, s_{m(c)-1}), \\ &\quad d(r_{t(c)}, r_{t(c)-1}), d(r_{m(c)}, r_{m(c)-1})\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\varphi(d(s_{t(c)}, s_{m(c)})) &= \varphi(d(F(s_{t(c)-1}, r_{t(c)-1}), d(F(s_{m(c)-1}, r_{m(c)-1}))) \\ &\leq \varphi(M((s_{t(c)-1}, r_{t(c)-1}), (s_{m(c)-1}, r_{m(c)-1}))) \\ &\quad - \phi(M((y_{t(c)-1}, r_{t(c)-1}), (s_{m(c)-1}, r_{m(c)-1}))),\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}M((s_{t(c)-1}, r_{t(c)-1}), (s_{m(c)-1}, r_{m(c)-1}) &= max\{d(s_{t(c)-1}, s_{m(c)-1}), d(r_{t(c)-1}, r_{m(c)-1}), \\ &\quad d(F(s_{t(c)-1}, r_{t(c)-1}), s_{t(c)-1}), \\ &\quad d(F(s_{m(c)-1}, r_{m(c)-1}), s_{m(c)-1})\} \\ &= max\{d(r_{t(c)-1}, r_{m(c)-1}), d(s_{t(c)-1}, s_{m(c)-1}), \\ &\quad d(s_{t(c)}, s_{t(c)-1}), d(s_{m(c)}, s_{m(c)-1})\}.\end{aligned}$$

From (2.27) and (2.28), we have

$$max\{\varphi(d(r_{t(c)}, r_{m(c)}), d(s_{t(c)}, s_{m(c)}))\} \leq \varphi(z_t) - \phi(z_t),$$

where

$$\begin{aligned}z_t = max\{&d(r_{t(c)-1}, r_{m(c)-1}), d(s_{t(c)-1}, s_{m(c)-1}), \\ &d(r_{t(c)}, r_{t(c)-1}), d(s_{t(c)}, s_{t(c)-1}), \\ &d(r_{m(c)}, r_{m(c)-1}), d(s_{m(c)}, s_{m(c)-1})\}.\end{aligned}$$

Finally letting $c \to \infty$ in last two inequalities and using (2.10), (2.17) and (2.26). Also the continuity of φ and ϕ, we have

$$\varphi(\varepsilon) \leq \varphi(max(\varepsilon, 0, 0)) - \phi(max(\varepsilon, 0, 0)) < \varphi(\varepsilon)$$

as a result, $\phi(\varepsilon) = 0$. As ϕ be an altering distance function, then $\varepsilon = 0$, a contradiction. This verifies our claim.

As U is complete, we find $r, s \in U$ then

$$\lim_{t\to\infty} r_t = r \text{ and } \lim_{t\to\infty} s_t = s.$$

We now establish that F possesses a coupled fixed point (r, s). Since, we have

$$\begin{aligned} r &= \lim_{t\to\infty} r_{t+1} = \lim_{t\to\infty} F(r_t, s_t) = F\Big(\lim_{t\to\infty} r_t, \lim_{t\to\infty} s_t\Big) = F(r, s), \\ s &= \lim_{t\to\infty} s_{t+1} = \lim_{t\to\infty} F(s_t, r_t) = F\Big(\lim_{t\to\infty} s_t, \lim_{t\to\infty} r_t\Big) = F(s, r). \end{aligned}$$

Then, F has a coupled fixed point (r, s).

Theorem 2.5. *Presume all the presumptions of Theorem 2.4 are satisfied . Furthermore, assume that U has the accompanying properties*

(a) if an increasing sequence $\{r_t\} \to r$, then $r_t \le r$, $\forall\, t$,

(b) if a decreasing sequence $\{s_t\} \to s$, then $s_t \ge s$, $\forall\, t$.

Thus the Theorem 2.4 hold the same result.

Proof. Succeeding the evidence of Theorem 2.4, we demonstrate that (r, s) is a coupled fixed point of F.

In fact, since $\{r_t\}$ is increasing and $r_t \to r$ and $\{s_t\}$ is decreasing and $s_t \to s$, by our hypothesis, $r_t \le r$ and $s_t \ge s$, $\forall\, t$.

Utilizing the condition (2.1), we get

$$\begin{aligned} \varphi(d(F(r, s), F(r_t, s_t))) &\le \varphi(M((r, s), (r_t, s_t))) - \phi(M((r, s), (r_t, s_t))) \\ &\le \varphi(M((r, s), (r_t, s_t))), \end{aligned} \tag{2.29}$$

and as φ is non-decreasing, we have

$$d(F(r, s), F(r_t, s_t)) \le M((r, s), (r_t, s_t)),$$

where

$$M((r, s), (r_t, s_t)) = max\{d(r, r_t), d(s, s_t), d(F(r, s), r), d(F(r_t, s_t), r_t)\}. \tag{2.30}$$

Letting $t \to \infty$ in (2.29) and (2.30), we obtain $d(r, F(r, s)) = 0$, and consequently $F(r, s) = r$.

Utilizing a comparative contention it can be demonstrated that $s = F(s, r)$ and this completes the verification.

To guarantee the uniqueness of the coupled fixed point in Theorem 2.4 and 2.5, we give the condition. That is

$$\text{for } (r, s), (t, v) \in U \times U \text{ we can find } (z, u) \in U \times U \text{ comparable to } (r, s) \text{ and } (t, v). \tag{2.31}$$

In $U \times U$ consider the partial order relation as follow

$$(r, s) \leq (t, v) \iff r \leq t \text{ and } s \geq v.$$

Theorem 2.6. *Including the condition (2.31) to the assumptions of Theorem 2.4 (respectively Theorem 2.5) we acquire the unique coupled fixed point of F.*

Proof. Let (r, s) and $(\acute{r}, \acute{s})$ are coupled fixed points of F, then, $F(r, s) = r,\ F(s, r) = s,\ F(\acute{r}, \acute{s}) = \acute{r}$ and $F(\acute{s}, \acute{r}) = \acute{s}$. We shall prove that $r = \acute{r}, s = \acute{s}$.

Let (r, s) and $(\acute{r}, \acute{s})$ are not comparable. By assumption there exist $(z, u) \in U \times U$ comparable with both of them. Suppose that $(r, s) \geq (z, u)$.

We define sequences $\{z_t\}, \{u_t\}$ as follows

$$z_0 = z,\ u_0 = u,\ z_{t+1} = F(z_t, u_t) \text{ and } u_{t+1} = F(u_t, z_t)\ \forall\ t.$$

Since (z, u) is comparable with (r, s). We claim that $(r, s) \geq (z_t, u_t)$ for every $t \in N$.

Now, by utilizing the induction.

For $n = 0$, $(r, s) \geq (z, u)$, therefore $z_0 = z \leq r$ and $s \geq u = u_0$ and consequently, $(r, s) \geq (z_0, u_0)$.

Suppose that $(r, s) \geq (z_t, u_t)$; by applying the property of mixed monotone of F, we get

$$\begin{aligned} z_{t+1} &= F(z_t, u_t) \leq F(r, u_t) \leq F(r, s) = r, \\ u_{t+1} &= F(u_t, z_t) \geq F(s, z_t) \geq F(s, r) = s, \end{aligned}$$

and this confirms our claim.

Now, since $z_t \leq r$ and $u_t \geq s$, using (2.1), we have

$$\begin{aligned} \varphi(d(r, z_{t+1})) &= \varphi(d(F(r, s), F(z_t, u_t))) \\ &\leq \varphi(M((r, s), (z_t, u_t))) - \phi(M((r, s), (z_t, u_t))), \end{aligned} \tag{2.32}$$

where

$$\begin{aligned} M((r,s),(z_t,u_t)) &= max\{d(r,z_t), d(s,u_t), d(F(r,s),r), d(F(z_t,u_t),z_t)\} \\ &= max\{d(r,z_t), d(s,u_t)\}. \end{aligned}$$

Therefore

$$\begin{aligned} \varphi(d(r,z_{t+1})) &\leq \varphi(max\{d(r,z_t), d(s,u_t)\}) - \phi(max\{d(r,z_t), d(s,u_t)\}) \\ &\leq \varphi(max\{d(r,z_t), d(s,u_t)\}), \end{aligned} \tag{2.33}$$

and analogously

$$\varphi(d(s,u_{t+1})) \leq \varphi(max\{d(s,u_t), d(r,z_t)\}). \tag{2.34}$$

From (2.33) and (2.34) and utilizing the way that φ is non decreasing, we attain

$$\begin{aligned} \varphi(max\{d(r,z_{t+1}), d(s,u_{t+1})\}) &= max\{\varphi(d(r,z_{t+1}), \varphi(d(s,u_{t+1}))\} \\ &\leq \varphi(max\{d(r,z_t), d(s,u_t)\}) - \phi(max\{d(r,z_t), d(s,u_t)\}) \\ &\leq \varphi(max\{d(r,z_t), d(s,u_t)\}). \end{aligned} \tag{2.35}$$

This implies that

$$max\{d(r,z_{t+1}), d(s,u_{t+1})\} \leq max\{d(r,z_t), d(s,u_t)\},$$

and consequently the sequence $max\{d(r,z_{t+1}), d(s,u_{t+1})\}$ is non-negative and decreasing and thus,

$$\lim_{t\to\infty} max\{d(r,z_{t+1}), d(s,u_{t+1})\} = a, \tag{2.36}$$

for certain $a \geq 0$. Using (2.36) and letting $t \to \infty$ in (2.35), we attain

$$\varphi(a) \leq \varphi(a) - \phi(a) \leq \varphi(a),$$

and as a result $\phi(a) = 0$. Therefore $a = 0$.

Finally, as

$$\lim_{t\to\infty} max\{d(r,z_{t+1}), d(s,u_{t+1})\} = 0. \tag{2.37}$$

This implies

$$\lim_{t\to\infty} d(r,z_{t+1}) = \lim_{t\to\infty} d(s,u_{t+1}) = 0. \tag{2.38}$$

Similarly

$$\lim_{t\to\infty} d(\acute{r}, z_{t+1}) = \lim_{t\to\infty} d(\acute{s}, u_{t+1}) = 0. \tag{2.39}$$

From (2.38) and (2.39), we have $r = \acute{r}, s = \acute{s}$. The proof is complete.

Theorem 2.7. *In extension to the assumptions of Theorem 2.4 (respectively Theorem 2.5), suppose that r_0 and s_0 in U are comparable, then $r = s$.*

Proof. Suppose that $r_0 \leq s_0$. We claim that

$$r_t \leq s_t, \forall\, t \in \mathbb{N}. \tag{2.40}$$

As F possesses the property of mixed monotone, we get

$$r_1 = F(r_0, s_0) \leq F(s_0, s_0) \leq F(s_0, r_0) = s_1.$$

Assume that $r_t \leq s_t$, for some t. Now,

$$r_{t+1} = F(r_t, s_t) \leq F(s_t, s_t) \leq F(s_t, r_t) = s_{t+1}.$$

Hence, this confirms our claim.

Now, applying (2.40) and (2.1), we have

$$\begin{aligned} \varphi(d(r_{t+1}, s_{t+1})) &= \varphi(d(s_{t+1}, r_{t+1})) = \varphi(d(F(s_t, r_t), F(r_t, s_t))) \\ &\leq \varphi(M((s_t, r_t), (r_t, s_t))) - \phi(M((s_t, r_t), (r_t, s_t))) \\ &\leq \varphi(M((s_t, r_t), (r_t, s_t))), \end{aligned} \tag{2.41}$$

and as φ is nondecreasing,

$$d(r_{t+1}, s_{t+1}) \leq M((s_t, r_t), (r_t, s_t)),$$

where

$$\begin{aligned} M((s_t, r_t), F(r_t, s_t)) &= max\{d(s_t, r_t), d(r_t, s_t), d(F(s_t, r_t), s_t), d(F(r_t, s_t), r_t)\} \\ &= max\{d(s_t, r_t), d(s_{t+1}, s_t), d(r_{t+1}, r_t)\}. \end{aligned} \tag{2.42}$$

Thus, $\lim\limits_{t\to\infty} d(r_t, s_t) = a$ for certain $a \geq 0$.

Taking $t \to \infty$ in (2.41) and hence (2.42), and utilizing the concept of continuity of ϕ and φ, we attain

$$\varphi(a) \leq \varphi(a) - \phi(a) \leq \varphi(a),$$

this provides $a = 0$.

As $r_t \to r$, $s_t \to s$ and $\lim_{t\to\infty} d(r_t, s_t) = 0$.

We have, $\lim_{t\to\infty} d(r_t, s_t) = d(\lim_{t\to\infty} r_t, \lim_{t\to\infty} s_t) = d(r, s) = 0$ and thus $r = s$.

This finishes the proof.

Example 2.1. *Presume $U = \mathbb{R}$ with usual metric and order. Define $F\colon U \times U \to U$ as $F(r, s) = \frac{1}{4}(r^2 - 3s^2)$ for all $r, s \in U$.*

Let φ, ϕ: $[0, \infty) \to [0, \infty)$ be defined by $\varphi(z) = z$ and $\phi(z) = \frac{1}{3}(z)$. Clearly, φ, ϕ are altering distance functions.

Now, suppose $r \le w$ and $s \ge v$. Thus, we obtain

$$
\begin{aligned}
\varphi(d(F(r,s), F(w,v))) &= d(F(r,s), F(w,v)) \\
&= \left|\frac{1}{4}(r^2 - 3s^2) - \frac{1}{4}(w^2 - 3v^2)\right| \\
&= \frac{1}{4}|(r^2 - w^2) - 3(s^2 - v^2)| \\
&\le \frac{1}{4}[d(r,w) + 3d(s,v)] \\
&\le \frac{2}{3} max\{d(r,w), d(s,v), d(F(r,s),r), d(F(w,v),w)\} \\
&= max\{d(r,w), d(s,v), d(F(r,s),r), d(F(w,v),w)\} \\
&\quad - \frac{1}{3} max\{d(r,w), d(s,v), d(F(r,s),r), d(F(w,v),w)\} \\
&= \varphi(max\{d(r,w), d(s,v), d(F(r,s),r), d(F(w,v),w)\}) \\
&\quad - \phi(max\{d(r,w), d(s,v), d(F(r,s),r), d(F(w,v),w)\}).
\end{aligned}
$$

Thus, all the assumptions of Theorem 2.4 hold. Moreover, $(0, 0)$ is the coupled fixed point of F.

Corollary 2.8. *Suppose $(U, \le)$ be a partially ordered set equipped with a metric d in U such that (U, d) be a complete space. Presume $F\colon U \times U \to U$ is a continuous mapping on U possesses the property of mixed monotone, there exists $l \in [0, 1)$ satisfy*

$$d(F(r,s), F(w,v))) \le l \; max\{d(r,w), d(s,v), d(F(r,s),r), d(F(w,v),w)\}$$

$\forall\ r, s, w, v \in U$ with $r \ge w$ and $s \le v$. Assume either F is continuous or U has the subsequent properties

(a) if an increasing sequence $\{r_t\} \to U$, then $r_t \le r$, $\forall\, t$,

(b) if a decreasing sequence $\{s_t\} \to U$, then $s_t \ge s$, $\forall\, t$.

Furthermore for each $r_0, s_0 \in U$ with $r_0 \leq F(r_0, s_0)$ and $s_0 \geq F(s_0, r_0)$, then F has a coupled fixed point.

Proof. Take $\varphi = I$ (identity) and $\phi = (1-l)\varphi$, we acquire the result.

2.3 Application to Integral Equations

In, this segment we discuss the existence and the uniqueness of solutions of a non-linear integral equation by utilizing the outcome demonstrated in section 2.2.

Consider an equation of an integral of the following type:

$$r(q) = \int_0^1 (k_1(q,a) + k_2(q,a))(f(a,r(a)) + g(a,r(a)))da + c(q), \ q \in [0,1]. \tag{2.43}$$

We will analyze (2.43) under the subsequent hypothesis:

(1) k_j: $[0,1] \times [0,1] \to \mathbb{R}(j=1,2)$ be continuous and $k_1(q,w) \geq 0$ and $k_2(q,w) \leq 0$.

(2) $c \in C[0,1]$.

(3) g, f: $[0,1] \times \mathbb{R} \to \mathbb{R}$ be two continuous functions.

(4) There are constants $\mu, \lambda > 0$, for every $r, s \in \mathbb{R}$, $r \geq s$

$$\lambda\sqrt{ln[(s-r)^2+1]} \geq f(q,r) - f(q,s) \geq 0,$$
$$0 \geq g(q,r) - g(q,s) \geq -\mu\sqrt{ln[(s-r)^2+1]}.$$

(5) There are $\beta, \alpha \in C[0,1]$ satisfy

$$\begin{aligned}\alpha(q) \leq &\int_0^1 k_1(q,a)(f(a,\alpha(a)) + g(a,\beta(a)))da \\ &+ \int_0^1 k_2(q,a)(f(a,\beta(a)) + g(a,\alpha(a)))da + c(q) \\ \leq &\int_0^1 k_1(q,a)(f(a,\beta(a)) + g(a,\alpha(a)))da \\ &+ \int_0^1 k_2(q,a)(f(a,\alpha(a))) + g(a,\beta(a)))da + c(q) \leq \beta(q).\end{aligned}$$

(6) $2.max(\lambda,\mu) \parallel k_1 - k_2 \parallel_\infty \leq 1$, where

$$\parallel k_1 - k_2 \parallel_\infty = sup\{(k_1(q,a) - k_2(q,a)): \ q, a \in [0,1]\}.$$

Let $U = C[0,1]$ be the space of continuous functions defined on [0, 1] with the usual metric provided by

$$d(r,s) = \sup_{q\in[0,1]} \mid r(q) - s(q) \mid, \text{ for } r,s \in C[0,1].$$

It can endow this space with a partial order as follows

$$r,s \in C[0,1],\ r \leq s \iff r(q) \leq s(q), \text{ for some } q \in [0,1].$$

If $U \times U$ consider the order as follow

$$(r,s),(w,v) \in U \times U,\ (r,s) \leq (w,v) \iff r \leq w, s \geq v,$$

for some $r,s \in U$ we have that $max(r,s), min(r,s) \in U$, condition (2.31) is fulfilled.

In [106] it is demonstrated that $(C[0,1], \leq)$ fulfills hypothesis (1).

Now, we formulate our outcome.

Theorem 2.9. *Under hypothesis (1)-(6), equation* (2.43) *possesses a unique solution in* $C[0,1]$.

Proof. Consider the mapping $F\colon U \times U \to U$ given by

$$\begin{aligned} F(r,s)(q) = &\int_0^1 k_1(q,a)(f(a,r(a)) + g(a,s(a)))da \\ &+ \int_0^1 k_2(q,a)(f(a,s(a)) + g(a,r(a)))da + c(q), \text{ for } q \in [0,1]. \end{aligned}$$

By virtuousness of our hypothesis, F is a well-defined (that for $r,s \in U$ then $F(r,s) \in U$).

We show that F satisfies the property of mixed monotone.

For $r_1 \leq r_2$ and $q \in [0,1]$, we have

$$\begin{aligned} F(r_1,s)(q) - F(r_2,s)(q) = &\int_0^1 k_1(q,a)(f(a,r_1(a)) + g(a,s(a)))da \\ &+ \int_0^1 k_2(q,a)(f(a,s(a)) + g(a,r_1(a)))da + c(q) \\ &- \int_0^1 k_1(q,a)(f(a,r_2(a)) + g(a,s(a)))da \\ &- \int_0^1 k_2(q,a)(f(a,s(a)) + g(a,r_2(a)))da - c(q) \\ = &\int_0^1 k_1(q,a)(f(a,r_1(a)) - f(a,r_2(a)))da \end{aligned}$$

$$+\int_0^1 k_2(q,a)(g(a,r_1(a))-g(a,r_2(a)))da. \tag{2.44}$$

Consider that $r_1 \le r_2$ and our hypothesis,

$$f(a,r_1(a))-f(a,r_2(a)) \le 0,$$
$$g(a,r_1(a))-g(a,r_2(a)) \ge 0,$$

and from (2.44) we obtain $F(r_1,s)(q)-F(r_2,s)(q) \le 0$

and this shows that $F(r_1,s) \le F(r_2,s)$.

Similarly, if $s_1 \ge s_2$ and $q \in [0,1]$, we have

$$\begin{aligned} F(r,s_1)(q)-F(r,s_2)(q) &= \int_0^1 k_1(q,a)(f(a,r(a))+g(a,s_1(a)))da \\ &\quad + \int_0^1 k_2(q,a)(f(a,s_1(a))+g(a,r(a)))da + c(q) \\ &\quad - \int_0^1 k_1(q,a)(f(a,r(a))+g(a,s_2(a)))da \\ &\quad - \int_0^1 k_2(q,a)(f(a,s_2(a))+g(a,r(a)))da - c(q) \\ &= \int_0^1 k_1(q,a)(g(a,s_1(a))-g(a,s_2(a)))da \\ &\quad + \int_0^1 k_2(q,a)(f(a,s_1(a))-f(a,s_2(a)))da, \end{aligned}$$

and by our assumptions, as $s_1 \ge s_2$,

$$g(a,s_1(a))-g(a,s_2(a)) \le 0, \; f(a,s_1(a))-f(a,s_2(a)) \ge 0,$$

and thus,

$$F(r,s_1)(q)-F(r,s_2)(q) \le 0,$$

or, equivalently,

$$F(r,s_1) \le F(r,s_2).$$

Thus, F possesses the property of mixed monotone.

In what follows, we estimate $d(F(r,s),F(w,v))$ for $r \ge w$, $s \le v$.

Certainly, as F possesses the property of mixed monotone, $F(r,s) \ge F(w,v)$. We have

$$\begin{aligned} d(F(r,s),F(w,v)) &= \sup_{q\in[0,1]} \mid F(r,s)(q)-F(w,v)(q) \mid = \sup_{q\in[0,1]} (F(r,s)(q)-F(w,v)(q)) \\ &= \sup_{q\in[0,1]} \Big[\int_0^1 k_1(q,a)(f(a,r(a))+g(a,s(a)))da \end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 k_2(q,a)(f(a,s(a))+g(a,r(a)))da+c(q)\\
&-\int_0^1 k_1(q,a)(f(a,w(a))+g(a,v(a)))da\\
&-\int_0^1 k_2(q,a)(f(a,v(a))+g(a,w(a)))da-c(q)\Bigg]\\
=&\sup_{q\in[0,1]}\Bigg[\int_0^1 k_1(q,a)[(f(a,r(a))-f(a,w(a)))-(g(a,v(a))-g(a,s(a)))]\\
&-\int_0^1 k_2(q,a)[(f(a,v(a))-f(a,s(a)))-(g(a,r(a))-g(a,w(a)))]da\Bigg].
\end{aligned}
\tag{2.45}
$$

Using our hypothesis (notice that $r \geq w$, $s \leq v$)

$$
\begin{aligned}
f(a,r(a))-f(a,w(a)) &\leq \lambda\sqrt{ln[(r(a)-w(a))^2+1]}\\
g(a,v(a))-g(a,s(a)) &\geq -\mu\sqrt{ln[(s(a)-v(a))^2+1]}\\
f(a,v(a))-f(a,s(a)) &\leq \lambda\sqrt{ln[(v(a)-s(a))^2+1]}\\
g(a,r(a))-g(a,w(a)) &\geq -\mu\sqrt{ln[(r(a)-w(a))^2+1]}.
\end{aligned}
$$

Consider these last inequalities, $k_2 \leq 0$ and (2.45), we have

$$
\begin{aligned}
&d(F(r,s),F(w,v))\\
&\leq \sup_{q\in[0,1]}\Bigg[\int_0^1 k_1(q,a)[\lambda\sqrt{ln[(r(a)-w(a))^2+1]}+\mu\sqrt{ln[(s(a)-v(a))^2+1]}]ds\\
&\quad+\int_0^1(-k_2(q,a))[\lambda\sqrt{ln[(v(a)-s(a))^2+1]}+\mu\sqrt{ln[(r(a)-w(a))^2+1]}]ds\Bigg]\\
&=max(\lambda,\mu)\sup_{q\in[0,1]}\Bigg[\int_0^1(k_1(q,a)-k_2(q,a))\sqrt{ln[(r(a)-w(a))^2+1]}ds\\
&\quad+\int_0^1(k_1(q,a)-k_2(q,a))\sqrt{ln[(s(a)-v(a))^2+1]}ds\Bigg].
\end{aligned}
\tag{2.46}
$$

Defining

$$
\begin{aligned}
(A)\ &= \textstyle\int_0^1(k_1(q,a)-k_2(q,a))\sqrt{ln[(r(a)-w(a))^2+1]}da.\\
(B)\ &= \textstyle\int_0^1(k_1(q,a)-k_2(q,a))\sqrt{ln[(s(a)-v(a))^2+1]}da.
\end{aligned}
$$

and utilizing the inequality of Cauchy-Schwartz in (A), we get

$$
\begin{aligned}
(A) &\leq \left(\int_0^1(k_1(q,a)-k_2(q,a))^2da\right)^{\frac{1}{2}}.\left(\int_0^1 ln[(r(a)-w(a))^2+1]da\right)^{\frac{1}{2}}\\
&\leq \| k_1-k_2\|_\infty .(ln\| r-w\|^2+1))^{\frac{1}{2}} = \| k_1-k_2\|_\infty .(ln(d(r,w)^2+1))^{\frac{1}{2}}.
\end{aligned}
\tag{2.47}
$$

In similar way, we can attain the subsequent estimate for (B):

$$(B) \leq \| k_1 - k_2 \|_\infty .(ln(d(s,v)^2+1))^{\frac{1}{2}}. \tag{2.48}$$

From (2.46)-(2.48), we get

$$\begin{aligned} d(F(r,s),F(w,v)) &\leq max(\lambda,\mu) \| k_1 - k_2 \|_\infty \left[(ln(d(r,w)^2+1))^{\frac{1}{2}} + (ln(d(s,v)^2+1))^{\frac{1}{2}}\right] \\ &\leq max(\lambda,\mu) \| k_1 - k_2 \|_\infty \Big[(ln(max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w))^2+1))^{\frac{1}{2}} \\ &\quad + (ln(max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w))^2+1))^{\frac{1}{2}}\Big] \\ &= 2max(\lambda,\mu) \| k_1 - k_2 \|_\infty \left[(ln(max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w))^2+1))^{\frac{1}{2}}\right]. \end{aligned} \tag{2.49}$$

From (2.49) and hypothesis (6) provide us

$$d(F(r,s),F(w,v)) \leq (ln(max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w))^2+1))^{\frac{1}{2}},$$

$$\implies d(F(r,s),F(w,v))^2 \leq (ln(max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w))^2+1)),$$

or,

$$\begin{aligned} d(F(r,s),F(w,v))^2 \leq\ & (max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w))^2 \\ & - [(max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w))^2 \\ & - ln(max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w))^2+1))]. \end{aligned} \tag{2.50}$$

Put $\varphi(r) = r^2$ and $\phi(r) = r^2 - ln(r^2+1)$. Clearly, ϕ and φ are altering distance functions. From (2.50) we have

$$\begin{aligned} \varphi(d(F(r,s),F(w,v)) \leq\ & \varphi(max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w)) \\ & - \phi(max(d(r,w),d(s,v),d(F(r,s),r),d(F(w,v),w))) \end{aligned}$$

This demonstrates that it fulfills the contractive condition (2.1).

Lastly, suppose β, α be the functions appeared in hypothesis (5); then, by (5), we have

$$\beta \geq F(\beta,\alpha) \geq F(\alpha,\beta) \geq \alpha.$$

Theorem 2.6 provides us that F has a unique coupled fixed point $(r,s) \in U \times U$. As $\beta \geq \alpha$, Theorem 2.7 states us that $r = s$ and this indicates $r = F(r,r)$ and r is the unique solution of equation (2.43).

This completes the proof.

Chapter 3

Coupled Fixed Point Theorems Involving Rational Expression in Partially Ordered Cone Metric Spaces

In this chapter, certain unique coupled fixed point theorems using the property of mixed monotone involving rational expression have been proved in an ordered cone metric space. In section 3.1, the conception of cone metric space have been discussed. In section 3.2, a coupled fixed point theorem has been established by using rational expression in a partially ordered cone metric space. Section 3.3, provides a necessary and sufficient condition for the existence of coupled fixed points of contractive condition with rational expression in complete cone metric space. Also, an example is provided to illustrate our results. Coupled fixed point results for rational contraction have been given in section 3.4. The results presented in this chapter have improved and generalize many known coupled fixed point theorems in the existing literature.

3.1 Introduction

Dass and Gupta [46] introduced the new idea of rational expression in complete metric spaces. Later on, Jaggi [73] generalized certain unique fixed point hypothesis which fulfil a contractive condition of a rational sort. Afterwards many authors [67, 94] generalized this concept.

Huang [69] introduced cone metric spaces by supplanting an ordered Banach space for the real numbers. After that, many researchers generalized the fixed point theorems in different ways (perceive [84, 85, 127, 133]).

Let P a subset of E and E be a real Banach space. P is called a cone iff

(a) $P \neq \{\theta\}$, P is non-empty and closed set ,

(b) $P \cap (-P) = \{\theta\}$,

(c) $pt + qv \in P$ for every $p, q \in \mathbb{R}$, p, $q \geq \theta$ and t, $v \in P$.

Given cone $P \subseteq E$, the partial ordering $\leq$ with respect to P by $t \leq v$ iff $v - t \in P$. The notation $t \ll v$ will stand for $v - t \in intP$ where $intP$ denotes the interior of P. Also, we will use $t < v$ to indicate that $t \leq v$ and $t \neq v$.

Definition 3.1. [69] Suppose U be a non empty set. Presuppose the mapping $d\colon U \times U \to E$ gratifies the subsequent conditions for all $u_1, u_2, u_3 \in U$

(d_1) $\theta \leq d(u_1, u_2)$ and $d(u_1, u_2) = \theta \iff u_1 = u_2$,

(d_2) $d(u_1, u_2) = d(u_2, u_1)$,

(d_3) $d(u_1, u_2) \leq d(u_1, u_3) + d(u_3, u_2)$.

Then d is termed as a cone metric on U and (U, d) is said to be a cone metric space.

3.2 Coupled Fixed Point Theorem Involving Rational Expression

In this section, the following result on complete cone metric space has been established:

Theorem 3.2. *Suppose $(U, \leq)$ be a partially ordered set endowed with a cone metric d in U then cone metric (U, d) is complete. Presume $H\colon U \times U \to U$ possesses the property of mixed monotone on U satisfy*

$$\begin{aligned} d(H(r,s), H(w,v)) \leq {} & \alpha \left(max \left\{ \frac{d(r, H(r,s))\, d(w, H(w,v))}{d(r,w)}, d(r,w) \right\} \right) \\ & + \beta \left(max \left\{ \frac{d(r, H(r,s))\, d(w, H(w,v))}{d(r,w)}, d(r,w) \right\} \right) \\ & + \gamma \left(M((r,s),(w,v)) \right) \end{aligned} \tag{3.1}$$

where,

$$M((r,s),(w,v)) = min\Bigg(d(r,H(r,s))\,\frac{2+d(w,H(w,v))+d(v,H(v,w))}{2+d(r,w)+d(s,v)},\\ d(w,H(w,v))\,\frac{2+d(r,H(r,s))+d(s,H(s,r))}{2+d(r,w)+d(s,v)}\Bigg)$$

$\forall\ r,s,w,v \in X$ with $r \geq w, s \leq v$ and there exist positive real numbers $\alpha,\beta,\gamma \in [0,1)$ and $\alpha+\beta+\gamma < 1$. Presume either H is continuous or U has the subsequent properties,

(a) if an increasing sequence $\{r_p\}$ in U converges to some point $r \in U$, then $r_p \leq r$, $\forall\ p$,

(b) if a decreasing sequence $\{s_p\}$ in U converges to some point $s \in U$, then $s_p \geq s$, $\forall\ p$.

Then H has a coupled fixed point.

Proof. Take $r_0, s_0 \in U$, set $r_1 = H(r_0,s_0)$ and $s_1 = H(s_0,r_0)$. Repeating this process, set $r_{p+1} = H(r_p,s_p)$ and $s_{p+1} = H(s_p,r_p)$. Therefore, from (3.1), we get

$$\begin{aligned} d(r_p,r_{p+1}) &= d(H(r_{p-1},s_{p-1}),H(r_p,s_p))\\ &\leq \alpha\left(max\left\{\frac{d(r_{p-1},H(r_{p-1},s_{p-1}))\,d(r_p,H(r_p,s_p))}{d(r_{p-1},r_p)},d(r_{p-1},r_p)\right\}\right)\\ &\quad+\beta\left(max\left\{\frac{d(r_{p-1},H(r_{p-1},s_{p-1}))\,d(r_p,H(r_p,s_p))}{d(r_{p-1},r_p)},d(r_{p-1},r_p)\right\}\right)\\ &\quad+\gamma\,(M((r_{p-1},s_{p-1})),(r_p,s_p)))\\ &= \alpha\,(max\{d(r_p,r_{p+1}),\ d(r_{p-1},r_p)\}) + \beta\,(max\{d(r_p,r_{p+1}),\ d(r_{p-1},r_p)\})\\ &\quad+\gamma\left(d(r_p,H(r_p,s_p))\,\frac{2+d(r_{p-1},H(r_{p-1},s_{p-1})+d(s_{p-1},H(s_{p-1},r_{p-1})}{2+d(r_{p-1},r_p)+d(s_p,s_{p-1})}\right)\\ &= \alpha\,(max\{d(r_p,r_{p+1}),d(r_{p-1},r_p)\}) + \beta\,(max\{d(r_p,r_{p+1}),d(r_{p-1},r_p)\})\\ &\quad+\gamma\,(d(r_p,r_{p+1})). \end{aligned} \tag{3.2}$$

Similarly, by (3.1), also we attain

$$\begin{aligned} d(s_p,s_{p+1}) &= d(H(s_{p-1},r_{p-1}),H(s_p,r_p))\\ &= \alpha\,(max\{d(s_p,s_{p+1}),d(s_{p-1},s_p)\}) + \beta\,(max\{d(s_p,s_{p+1}),d(s_{p-1},s_p)\})\\ &\quad+\gamma\,(d(s_p,s_{p+1})). \end{aligned} \tag{3.3}$$

Suppose that $max\{d(r_p,r_{p+1}),d(r_{p-1},r_p)\} = d(r_p,r_{p+1})$ for some $n \geq 1$. Then the inequality turns into $d(r_p,r_{p+1}) \leq \alpha\,(d(r_p,r_{p+1})) + \beta\,(d(r_p,r_{p+1})) + \gamma\,(d(r_p,r_{p+1}))$ which is a contradiction. Thus $max\{d(r_p,r_{p+1}),d(r_{p-1},r_p)\} = d(r_{p-1},r_p)$ for some $p \geq 1$.

Therefore, the inequality yields

$$d(r_p, r_{p+1}) \leq \alpha\, d(r_{p-1}, r_p) + \beta\, d(r_{p-1}, r_p) + \gamma\, d(r_p, r_{p+1}) \tag{3.4}$$

$$d(s_p, s_{p+1}) \leq \alpha\, d(s_{p-1}, s_p) + \beta\, d(s_{p-1}, s_p) + \gamma\, d(s_p, s_{p+1}) \tag{3.5}$$

Which implies that

$$(1-\gamma)\, d(r_p, r_{p+1}) \leq (\alpha + \beta)\, d(r_{p-1}, r_p) \tag{3.6}$$

$$(1-\gamma)\, d(s_p, s_{p+1}) \leq (\alpha + \beta)\, d(s_{p-1}, s_p). \tag{3.7}$$

By adding (3.6) and (3.7), we have

$$d_p \leq \frac{(\alpha+\beta)}{(1-\gamma)}\, d_{p-1}. \tag{3.8}$$

Let $d_p = d(r_p, r_{p+1}) + d(s_p, s_{p+1})$. Subsequently, if we set $\lambda = \dfrac{(\alpha+\beta)}{(1-\gamma)}$, then we have

$$d_p \leq \lambda\, d_{p-1} \leq \cdots \leq \lambda^p\, d_0. \tag{3.9}$$

If $d_0 = 0$, then H possesses a coupled fixed point (r_0, s_0).

Presume that $d_0 \geq 0$. Then, for each $a \in \mathbb{N}$, the repeated application of triangle inequality, we acquire

$$\begin{aligned} d(r_p, r_{p+a}) + d(s_p, s_{p+a}) &\leq [d(r_p, r_{p+1}) + d(r_{p+1}, r_{p+2}) + \ldots + d(r_{p+a-1}, r_{p+a})] \\ &\quad + [d(s_p, s_{p+1}) + d(s_{p+1}, s_{p+2}) + \ldots + d(s_{p+a-1}, s_{p+a})] \\ &= [d(r_p, r_{p+1}) + d(s_p, s_{p+1})] + [d(r_{p+1}, r_{p+2}) \\ &\quad + d(s_{p+1}, s_{p+2})] + \ldots + [d(r_{p+a-1}, r_{p+a}) \\ &\quad + d(s_{p+a-1}, s_{p+a})] \\ &\leq d_p + d_{p+1} + \ldots + d_{p+a-1} \\ &\leq \frac{\lambda^p(1-\lambda^a)}{1-\lambda}\, d_0. \end{aligned} \tag{3.10}$$

Let $0 \ll c$ be given. Choose a natural number K such that $\dfrac{\lambda^p(1-\lambda^a)}{1-\lambda}\, d_0 \ll c$ for all $m > K$. Thus $d(r_p, r_{p+r}) + d(s_p, s_{p+r}) \ll c$. Therefore $\{r_p\}$ and $\{s_p\}$ are Cauchy sequences.

As U is complete, $\exists\ r, s \in U$ then $\lim\limits_{p\to\infty} r_p = r$, $\lim\limits_{p\to\infty} s_p = s$. Now, we show that if H is continuous, then (r, s) is coupled fixed point of H.

As, we have

$$r = \lim_{p\to\infty} r_{p+1} = \lim_{p\to\infty} H(r_p, s_p) = H\left(\lim_{p\to\infty} r_p, \lim_{p\to\infty} s_p\right) = H(r, s),$$

$$s = \lim_{p\to\infty} s_{p+1} = \lim_{p\to\infty} H(s_p, r_p) = H\left(\lim_{p\to\infty} s_p, \lim_{p\to\infty} r_p\right) = H(s, r).$$

Therefore, (r, s) is coupled fixed point of H.

Now, presume that the assumption (a) and (b) of the theorem satisfy.

The sequence $\{r_p\} \to r$, $\{s_p\} \to s$

$$\begin{aligned} d(H(r,s), H(r_p, s_p)) \leq \alpha &\left(max\left\{ \frac{d(r, H(r,s))d(r_p, H(r_p, s_p))}{d(r, r_p)}, d(r, r_p) \right\} \right) \\ &+ \beta \left(max\left\{ \frac{d(r, H(r,s))d(r_p, H(r_p, s_p))}{d(r, r_p)}, d(r, r_p) \right\} \right) \\ &+ \gamma \left(d(r, H(r,s)) \frac{2 + d(r_p, H(r_p, s_p)) + d(s_p, H(s_p, r_p))}{2 + d(r, r_p) + d(s, s_p)} \right). \end{aligned}$$

Letting $p\to \infty$, we have $d(H(r,s), r) \leq 0$. Thus, $H(r,s) = r$. In similar way, we can prove that $H(s,r) = s$. This completes the theorem.

Theorem 3.3. *Let the presumptions of Theorem 3.2 hold. We acquire the uniqueness of the coupled fixed point of H.*

Proof. Presume (r, s) and (r^*, s^*) are coupled fixed points of H, thus, $H(r,s) = r, H(s,r) = s, H(r^*, s^*) = r^*$ and $H(s^*, r^*) = s^*$. We shall prove that $r = r^*, s = s^*$. Consider the following cases:

Case 1: If (r, s) and (r^*, s^*) are comparable. We get

$$\begin{aligned} d(r, r^*) &= d(H(r,s), H(r^*, s^*)) \\ &\leq \alpha \left(max\left\{ \frac{d(r, H(r,s))d(r^*, H(r^*, s^*))}{d(r, r^*)}, d(r, r^*) \right\} \right) \\ &\quad + \beta \left(max\left\{ \frac{d(r, H(r,s))d(r^*, H(r^*, s^*))}{d(r, r^*)}, d(r, r^*) \right\} \right) \\ &\quad + \gamma \left(d(r, H(r,s)) \frac{2 + d(r^*, H(r^*, s^*)) + d(s^*, H(s^*, r^*))}{2 + d(r, r^*) + d(s, s^*)} \right) \end{aligned}$$

Which gives $d(r, r^*) \leq 0$, $(\alpha + \beta + \gamma) < 1$(a contradiction). Thus $r = r^*$. Similarly, $d(s, s^*) = d(H(s,r), H(s^*, r^*)) \leq 0$. Hence, $s = s^*$. Therefore, H acquire a unique coupled fixed point (r, s).

Case 2: Presume (r, s) and (r^*, s^*) are not comparable.

Presume that there exist $(z, u) \in U \times U$, comparable with both of them.

We define sequences $\{z_p\}$, $\{u_p\}$ as follows

$$z_0 = z,\ u_0 = u,\ z_{p+1} = H(z_p, u_p)\ and\ u_{p+1} = H(u_p, z_p)\ \forall\ p.$$

Since (z, u) is comparable with (r, s), we may presume that $(r, s) \geq (z, u) = (z_0, u_0)$.

Using mathematical induction, we can easily prove that

$$(r, s) \geq (z_p, t_p) \; \forall \; p. \tag{3.11}$$

From (3.1) and (3.11), we have

$$\begin{aligned} d(H(r,s), H(z_p,u_p)) \leq \alpha &\left(max \left\{ \frac{d(r, H(r,s)) d(z_p, H(z_p,u_p))}{d(r,z_p)}, d(r,z_p) \right\} \right) \\ &+ \beta \left(max \left\{ \frac{d(r, H(r,s)) d(z_p, H(z_p,u_p))}{d(r,z_p)}, d(r,z_p) \right\} \right) \\ &+ \gamma \left(d(r, H(r,s)) \, \frac{2 + d(z_p, H(z_p,u_p)) + d(u_p, H(u_p,z_p))}{2 + d(r,z_p) + d(s,u_p)} \right), \end{aligned}$$

or

$$d(r, z_{p+1}) \leq (\alpha + \beta) \, d(r, z_p). \tag{3.12}$$

Similarly, we also have

$$d(u_{p+1}, s) \leq (\alpha + \beta) \, d(u_p, s). \tag{3.13}$$

Adding (3.12) and (3.13), we get

$$\begin{aligned} d(r, z_{p+1}) + d(u_{p+1}, s) &\leq (\alpha + \beta) \left[d(r, z_p) + d(u_p, s) \right] \\ &\leq (\alpha + \beta)^2 \left[d(r, z_{p-1}) + d(u_{p-1}, s) \right] \\ &\quad \vdots \\ &\leq (\alpha + \beta)^{p+1} \left[d(r, z_0) + d(u_0, s) \right] \to 0 \text{ as } p \to \infty. \end{aligned}$$

Thus,

$$\lim_{p \to \infty} d(r, z_{p+1}) = \lim_{p \to \infty} d(u_{p+1}, s) = 0. \tag{3.14}$$

In similar way, we can show that

$$\lim_{p \to \infty} d(r^*, z_{p+1}) = \lim_{p \to \infty} d(u_{p+1}, s^*) = 0. \tag{3.15}$$

From (3.14) and (3.15), we obtain $r = r^*$ and $s = s^*$.

3.3 A Necessary and Sufficient Condition For the Existence of Coupled Fixed Points

Theorem 3.4. *Suppose $(U, \leq)$ be a partially ordered set endowed with a cone metric d in U then cone metric (U, d) be complete. Presume a mapping $H\colon U \times U \to U$ possess* *the property of mixed monotone on U satisfy*

$$
\begin{aligned}
d(H(r,s),H(w,v)) \leq \alpha &\left(max\left\{ \frac{d(r,H(r,s))d(w,H(w,v))}{d(r,w)}, d(r,w) \right\}\right) \\
&+ \beta \left(max\left\{ \frac{d(r,H(r,s))d(w,H(w,v))}{d(r,w)}, d(r,w) \right\}\right)
\end{aligned}
\tag{3.16}
$$

$\forall\ r,s,w,v \in X$ with $r \geq w, s \leq v$ and there exist positive real numbers $\beta, \alpha \in [0,1)$ with $\beta + \alpha < 1$. Presume either H is continuous or U has the subsequent properties,

(a) if an increasing sequence $\{r_p\} \to r$, then $r_p \leq r$, $\forall\, p$,

(b) if a decreasing sequence $\{s_p\} \to s$, then $s_p \geq s$, $\forall\, p$.

Then H has a coupled fixed point.

Proof. Take $r_0,\ s_0 \in U$. Set $r_1 = H(r_0,s_0)$ and $s_1 = H(s_0,r_0)$. Repeating this process, set $r_{p+1} = H(r_p,s_p)$ and $s_{p+1} = H(s_p,r_p)$. Then by (3.16), we have

$$
\begin{aligned}
d(r_p,r_{p+1}) &= d(H(r_{p-1},s_{p-1}),H(r_p,s_p)) \\
&\leq \alpha \left(max\left\{ \frac{d(r_{p-1},H(r_{p-1},s_{p-1}))d(r_p,H(r_p,s_p))}{d(r_{p-1},r_p)}, d(r_{p-1},r_p) \right\}\right) \\
&\quad + \beta \left(max\left\{ \frac{d(r_{p-1},H(r_{p-1},s_{p-1}))d(r_p,H(r_p,s_p))}{d(r_{p-1},r_p)}, d(r_{p-1},r_p) \right\}\right) \\
&= \alpha \left(max\{d(r_p,r_{p+1}), d(r_{p-1},r_p)\}\right) + \beta \left(max\{d(r_p,r_{p+1}), d(r_{p-1},r_p)\}\right).
\end{aligned}
\tag{3.17}
$$

Similarly, from (3.16), we obtain

$$
\begin{aligned}
d(s_p,s_{p+1}) &= d(H(s_{p-1},r_{p-1}),H(s_p,r_p)) \\
&= \alpha \left(max\{d(s_p,s_{p+1}), d(s_{p-1},s_p)\}\right) \\
&\quad + \beta \left(max\{d(s_p,s_{p+1}), d(s_{p-1},s_p)\}\right).
\end{aligned}
\tag{3.18}
$$

Suppose that $max\{d(r_p,r_{p+1}), d(r_{p-1},r_p)\} = d(r_p,r_{p+1})$ for some $p \geq 1$. Then the inequality turns into $d(r_p,r_{p+1}) \leq \alpha\,(d(r_p,r_{p+1})) + \beta\,(d(r_p,r_{p+1}))$, which is a contradiction. Thus $max\{d(r_p,r_{p+1}), d(r_{p-1},r_p)\} = d(r_{p-1},r_p)$ for some $p \geq 1$. Therefore, the inequality yields

$$
d(r_p,r_{p+1}) \leq (\alpha + \beta)\, d(r_{p-1},r_p), \tag{3.19}
$$

$$
d(s_p,s_{p+1}) \leq (\alpha + \beta)\, d(s_{p-1},s_p). \tag{3.20}
$$

By adding (3.19) and (3.20), we have

$$d_p \leq (\alpha+\beta)\, d_{p-1}. \tag{3.21}$$

Let $d_p = d(r_p, r_{p+1}) + d(s_p, s_{p+1})$. Subsequently, if we set $\lambda = (\alpha+\beta)$, then we have

$$d_p \leq \lambda d_{p-1} \leq \ldots \leq \lambda^p d_0. \tag{3.22}$$

If $d_0 = 0$, then H acquire a coupled fixed point (r_0, s_0).

Presume $d_0 \geq 0$. Then, for each $a \in \mathbb{N}$, by the repeated application of triangle inequality, we acquire

$$\begin{aligned}
d(r_p, r_{p+a}) + d(s_p, s_{p+a}) &\leq [d(r_p, r_{p+1}) + d(r_{p+1}, r_{p+2}) + \ldots + d(r_{p+a-1}, x_{p+a})] \\
&\quad + [d(s_p, s_{p+1}) + d(s_{p+1}, s_{p+2}) + \ldots + d(s_{p+a-1}, s_{p+a})] \\
&= [d(r_p, r_{p+1}) + d(s_p, s_{p+1})] + [d(r_{p+1}, r_{p+2}) \\
&\quad + d(s_{p+1}, s_{p+2})] + \ldots + [d(r_{p+a-1}, r_{p+a}) + d(s_{p+a-1}, s_{p+a})] \\
&\leq d_p + d_{p+1} + \ldots + d_{p+a-1} \\
&\leq \frac{\lambda^p(1-\lambda^a)}{1-\lambda}\, d_0.
\end{aligned} \tag{3.23}$$

Let $0 \ll c$ be given. Choose a natural number M such that $\dfrac{\lambda^p(1-\lambda^a)}{1-\lambda}\, d_0 \ll c$ for all $m > M$. Thus $d(r_p, r_{p+a}) + d(s_p, s_{p+a}) \ll c$. Therefore $\{r_p\}$ and $\{s_p\}$ are Cauchy sequences.

As U is complete, $\exists\ r, s \in U$ then $\lim\limits_{p\to\infty} r_p = r$, $\lim\limits_{p\to\infty} s_p = s$. Now, we show that if H is continuous, then (r, s) is coupled fixed point of H.

As, we have

$$\begin{aligned}
r &= \lim_{p\to\infty} r_{p+1} = \lim_{p\to\infty} H(r_p, s_p) = H\left(\lim_{p\to\infty} r_p, \lim_{p\to\infty} s_p\right) = H(r, s), \\
s &= \lim_{p\to\infty} s_{p+1} = \lim_{p\to\infty} H(s_p, r_p) = H\left(\lim_{p\to\infty} s_p, \lim_{p\to\infty} r_p\right) = H(s, r).
\end{aligned}$$

Therefore, (r, s) is coupled fixed point of H.

Now, presume that the assumption (a) and (b) of the theorem holds.

The sequence $\{r_p\} \to r$, $\{s_p\} \to s$

$$\begin{aligned}
d(H(r,s), H(r_p, s_p)) \leq \alpha &\left(max\left\{\frac{d(r, H(r,s))d(r_p, H(r_p, s_p))}{d(r, r_p)}, d(r, r_p)\right\}\right) \\
&+ \beta\left(max\left\{\frac{d(r, H(r,s))d(r_p, H(r_p, s_p))}{d(r, r_p)}, d(r, r_p)\right\}\right).
\end{aligned}$$

Letting $p\to\infty$, we have $d(H(r,s), r) \leq 0$. Thus, $H(r,s) = r$. Similarly, we can produce

$H(s,r) = s$. This concludes the theorem.

Theorem 3.5. *Let the assumptions of Theorem 3.4 hold. We acquire the uniqueness of the coupled fixed point of H.*

Proof. Presume (r,s) and (r^*,s^*) are coupled fixed points of H, thus, $H(r,s) = r,\ H(s,r) = s,\ H(r^*,s^*) = r^*$ and $H(s^*,r^*) = s^*$. We shall prove that $r = r^*,\ s = s^*$. Examine the succeeding cases:

Case 1: If (r,s) and (r^*,s^*) are comparable. We have

$$\begin{aligned} d(r,r^*) &= d(H(r,s),H(r^*,s^*)) \\ &\leq \alpha\left(max\left\{\frac{d(r,H(r,s))d(r^*,H(r^*,s^*))}{d(r,r^*)}, d(r,r^*)\right\}\right) \\ &\quad + \beta\left(max\left\{\frac{d(r,H(r,s))d(r^*,H(r^*,s^*))}{d(r,r^*)}, d(r,r^*)\right\}\right), \end{aligned}$$

which gives $d(r,r^*) \leq 0$, $(\alpha+\beta) < 1$(a contradiction). Thus $r = r^*$.

Similarly,

$$d(s,s^*) = d(H(s,r),H(s^*,r^*)) \leq 0.$$

Hence, $s = s^*$. Therefore, (r,s) is a unique coupled fixed point of H.

Case 2: Presume (r,s) and (r^*,s^*) are not comparable.

Presume that there exist $(z,u) \in X \times X$, comparable with both of them.
We define sequences $\{z_p\}, \{u_p\}$ as follows

$$z_0 = z,\ u_0 = u,\ z_{p+1} = H(z_p,u_p)\ and\ u_{p+1} = H(u_p,z_p)\ \forall\ p.$$

Since (z,u) is comparable with (r,s), we may presume that $(r,s) \geq (z,u) = (z_0,u_0)$.

Using mathematical induction, we can easily prove that

$$(r,s) \geq (z_p,u_p)\ \forall\ p. \tag{3.24}$$

From (3.16) and (3.24), we have

$$\begin{aligned} d(H(r,s),H(z_p,u_p)) &\leq \alpha\left(max\left\{\frac{d(r,H(r,s))d(z_p,H(z_p,u_p))}{d(r,z_p)}, d(r,z_p)\right\}\right) \\ &\quad + \beta\left(max\left\{\frac{d(r,H(r,s))d(z_p,H(z_p,u_p))}{d(r,z_p)}, d(r,z_p)\right\}\right) \end{aligned}$$

or

$$d(r,z_{p+1}) \leq (\alpha+\beta)d(r,z_p). \tag{3.25}$$

Similarly, we also have

$$d(u_{p+1}, s) \leq (\alpha + \beta) d(u_p, s). \tag{3.26}$$

Adding (3.25) and (3.26), we get

$$\begin{aligned} d(r, z_{p+1}) + d(u_{p+1}, s) &\leq (\alpha + \beta)[d(r, z_p) + d(u_p, s)] \\ &\leq (\alpha + \beta)^2 [d(r, z_{p-1}) + d(u_{p-1}, s)] \\ &\quad \vdots \\ &\leq (\alpha + \beta)^{p+1} [d(r, z_0) + d(u_0, s)] \to 0 \ as \ p \to \infty. \end{aligned}$$

Thus,

$$\lim_{p\to\infty} d(r, z_{p+1}) = \lim_{p\to\infty} d(u_{p+1}, s) = 0. \tag{3.27}$$

In similar way, we can prove that

$$\lim_{p\to\infty} d(r^*, z_{p+1}) = \lim_{p\to\infty} d(u_{p+1}, s^*) = 0. \tag{3.28}$$

From (3.27) and (3.28), we obtain $r = r^*$ and $s = s^*$.

Example 3.1. *Let $E = \mathbb{R}^2$, the Euclidean plane and $P = \{(r, s) \in \mathbb{R}^2 \colon r, s \geq 0\}$ a normal cone in P. Let $r = \{(r, 0) \in \mathbb{R}^2 \colon 0 \leq r \leq 1\} \cup \{(0, r) \in \mathbb{R}^2 \colon 0 \leq r \leq 1\}$. The mapping $d \colon U \times U \to E$ is defined by*

$$\begin{aligned} d((r, 0), (s, 0)) &= \left(\frac{5}{3}|r - s|, |r - s|\right), \\ d((0, r), (0, s)) &= \left(|r - s|, \frac{2}{3}|r - s|\right), \\ d((r, 0), (0, s)) = d((0, s), (r, 0)) &= \left(\frac{5}{3}r + s, r + \frac{2}{3}s\right). \end{aligned}$$

Thus (U, d) is complete cone metric space.

Consider the operator $H \colon U \times U \to U$ given by

$$H((r, 0), (0, r)) = \left(\frac{r}{4}, 0\right), \ H((0, r), (r, 0)) = \left(0, \frac{r}{2}\right).$$

U satisfies the properties (i) and (ii) of Theorem 3.4. Clearly H is continuous and possesses the property of mixed monotone. There are $r_0 = 0; s_0 = 0$ in U, $r_0 = 0 \leq H(0, 0) = H(r_0, s_0)$ and $s_0 = 0 \geq H(0, 0) = H(s_0, r_0)$.

We claim that (3.16) holds for each $r \geq w$, $s \leq v$.
Case 1:

$$d(H((r, 0), (0, r)), H((0, s), (s, 0)) = d\left(\left(\frac{r}{4}, 0\right), \left(0, \frac{s}{2}\right)\right)$$

$$= \left(\frac{5}{3}\left(\frac{r}{4}\right) + \frac{s}{2}, \frac{r}{4} + \frac{2}{6}s\right) \leq \left(\frac{5}{12}r + s, \frac{r}{2} + \frac{2}{3}s\right)$$
$$\leq \frac{2}{3}\left(\frac{5}{3}r + s, r + \frac{2}{3}s\right)$$

Hence, inequality (3.16) *holds.*

Case 2:

$$d(H((0,r),(r,0), H((0,s),(s,0)) = d\left(\left(0, \frac{r}{2}\right), \left(0, \frac{s}{2}\right)\right)$$
$$= \left(\left|\frac{r}{2} - \frac{s}{2}\right|, \frac{2}{3}\left|\frac{r}{2} - \frac{s}{2}\right|\right) \leq \frac{2}{3}\left(|r-s|, \frac{2}{3}|r-s|\right)$$

Hence, inequality (3.16) *holds.*

Case 3:

$$d(H((r,0),(0,r)), H((s,0),(0,s)) = d\left(\left(\frac{r}{4}, 0\right), \left(\frac{s}{4}, 0\right)\right)$$
$$= \left(\frac{5}{3}\left|\frac{r}{4} - \frac{s}{4}\right|, \left|\frac{r}{4} - \frac{s}{4}\right|\right) \leq \frac{2}{3}\left(\frac{5}{3}|r-s|, |r-s|\right)$$

Hence, inequality (3.16) *holds.*

Case 4:

$$d(H((0,r),(r,0), H((s,0),(0,s)) = d\left(\left(0, \frac{r}{2}\right), \left(\frac{s}{4}, 0\right)\right)$$
$$= \left(\left|\frac{r}{2} - \frac{s}{4}\right|, \frac{2}{3}\left|\frac{r}{2} - \frac{s}{4}\right|\right) \leq \frac{2}{3}\left(\frac{5}{3}r + s, r + \frac{2}{3}s\right)$$

Hence, inequality (3.16) *holds.*

We deduce that all the assumptions of Theorem 3.4 are fulfilled with $(\alpha + \beta) < 1$ *where* α, β *are such that* $\alpha = \beta = \frac{1}{3}$ *. Here,* H *possesses the unique coupled fixed point* $(0,0)$.

3.4 Coupled Fixed Point Theorems for Rational Contractions

Theorem 3.6. *Suppose* $(U, \leq, d)$ *be a partially ordered complete cone metric space. Presume a mapping* $H\colon U \times U \to U$ *possesses the property of mixed monotone on* U *satisfy*

$$\varphi(d(H(r,s), H(w,v))) \leq \varphi\bigg[\alpha\, \frac{d(r, H(r,s))d(w, H(w,v))}{d(r,w)} + \beta\left(d(r,w) + d(s,v)\right)$$
$$+ \gamma\left(d(r, H(r,s)) + d(s, H(s,r))\right)\bigg]$$

$$
\begin{aligned}
&- \phi\bigg[\alpha\, \frac{d(r,H(r,s))d(w,H(w,v))}{d(r,w)} + \beta\,(d(r,w)+d(s,v)) \\
&\qquad + \gamma\,(d(r,H(r,s))+d(s,H(s,r)))\bigg] \\
&+ L\,min\,\{d(r,H(r,s)), d(w,H(w,v)), \\
&\qquad\qquad d(w,H(r,s)), d(r,H(w,v))\}. \qquad (3.29)
\end{aligned}
$$

$\forall\ r,s,w,v \in U$ with $r \geq w, s \leq v$ and where $\varphi, \phi : [0,\infty) \to [0,\infty)$ are continuous and nondecreasing function. Also, there exist positive real numbers $\beta, \alpha, \gamma \in [0,1)$ and $L \geq 0$ with $(\alpha + 2\beta + 2\gamma) < 1$. Presuppose either H is continuous or U has the subsequent properties,

(a) if an increasing sequence $\{r_t\}$ in U converges to some point $\varrho \in U$, then $r_t \leq \varrho$, $\forall\ t$,

(b) if a decreasing sequence $\{s_t\}$ in U converges to some point $\varrho' \in U$, then $s_t \geq \varrho'$, $\forall\ t$.

Then H has a coupled fixed point.

Proof. Take $r_0, s_0 \in U$. Set $r_1 = H(r_0, s_0)$ and $s_1 = H(s_0, r_0)$. Repeating this process, set $r_{t+1} = H(r_t, s_t)$ and $s_{t+1} = H(s_t, r_t)$.

Then by (3.29), we have

$$
\begin{aligned}
\varphi(d(r_t, r_{t+1})) &= \varphi(d(H(r_{t-1}, s_{t-1}), H(r_t, s_t))) \\
&\leq \varphi\bigg[\alpha\, \frac{d(r_{t-1}, H(r_{t-1}, s_{t-1}))d(r_t, H(r_t, s_t))}{d(r_{t-1}, r_t)} + \beta\,(d(r_{t-1}, r_t) + d(s_{t-1}, s_t)) \\
&\qquad + \gamma\,(d(r_{t-1}, H(r_{t-1}, s_{t-1})) + d(s_{t-1}, H(s_{t-1}, r_{t-1})))\bigg] \\
&\quad - \phi\bigg[\alpha\, \frac{d(r_{t-1}, H(r_{t-1}, s_{t-1}))d(r_t, H(r_t, s_t))}{d(r_{t-1}, r_t)} + \beta\, d(r_{t-1}, r_t) + d(s_{t-1}, s_t)) \\
&\qquad + \gamma\,(d(r_{t-1}, H(r_{t-1}, s_{t-1}))) + d(s_{t-1}, H(s_{t-1}, r_{t-1})))\bigg] \\
&\quad + L\,min\,\{d(r_{t-1}, H(r_{t-1}, s_{t-1})), d(r_t, H(r_t, s_t)), \\
&\qquad\qquad d(r_t, H(r_{t-1}, s_{t-1})), d(r_{t-1}, H(r_t, s_t))\} \\
&= \varphi[\alpha\, d(r_t, r_{t+1}) + \beta\,(d(r_{t-1}, r_t) + d(s_{t-1}, s_t)) + \gamma\,(d(r_{t-1}, r_t) + d(s_{t-1}, s_t)))] \\
&\quad - \phi[\alpha\, d(r_t, r_{t+1}) + \beta\,(d(r_{t-1}, r_t) + d(s_{t-1}, s_t)) + \gamma\,(d(r_{t-1}, r_t) + d(s_{t-1}, s_t)))] \\
&\leq \varphi[\alpha\, d(r_t, r_{t+1}) + \beta\,(d(r_{t-1}, r_t) + d(s_{t-1}, s_t)) + \gamma\,(d(r_{t-1}, r_t) + d(s_{t-1}, s_t)))]. \quad (3.30)
\end{aligned}
$$

Similarly, from (3.29), we also have

$$\begin{aligned}\varphi(d(s_t,s_{t+1})) &= \varphi(d(H(s_{t-1},r_{t-1}),H(s_t,r_t)))\\ &\le \varphi[\alpha\, d(s_t,s_{t+1})+\beta\,(d(s_{t-1},s_t)+d(r_{t-1},r_t))+\gamma\,(d(s_{t-1},s_t)+d(r_{t-1},r_t))]. \end{aligned} \tag{3.31}$$

Consequently, since φ is non-decreasing, using (3.30) and (3.31), we get

$$d(r_t,r_{t+1}) \le \frac{\beta+\gamma}{1-\alpha}\, d(r_{t-1},r_t), \tag{3.32}$$

$$d(s_t,s_{t+1}) \le \frac{\beta+\gamma}{1-\alpha}\, d(s_{t-1},s_t). \tag{3.33}$$

By adding (3.32) and (3.33), we have

$$d_t \le \frac{2\beta+2\gamma}{1-\alpha}\, d_{t-1} \tag{3.34}$$

Let $d_t = d(r_t,r_{t+1})+d(s_t,s_{t+1})$. Consequently, if we set $\lambda = \dfrac{2\beta+2\gamma}{1-\alpha}$, then we have

$$d_t \le \lambda\, d_{t-1} \le \ldots \le \lambda^t\, d_0. \tag{3.35}$$

If $d_0 = 0$, then (r_0, s_0) is a coupled fixed point of H.

Presuppose that $d_0 \ge 0$. Then, for each $k \in \mathbb{N}$, by the repeated application of triangle inequality, we acquire

$$\begin{aligned} d(r_t,r_{t+k})+d(s_t,s_{t+k}) &\le [d(r_t,r_{t+1})+d(r_{t+1},r_{t+2})+\ldots+d(r_{t+k-1},r_{t+k})]\\ &\quad+[d(s_t,s_{t+1})+d(s_{t+1},s_{t+2})+\ldots+d(s_{t+k-1},s_{t+k})]\\ &= [d(r_t,r_{t+1})+d(s_t,s_{t+1})]+[d(r_{t+1},r_{t+2})\\ &\quad+d(s_{t+1},s_{t+2})]+\ldots+[d(r_{t+k-1},r_{t+k})+d(s_{t+k-1},s_{t+k})]\\ &\le d_t+d_{t+1}+\ldots+d_{t+k-1}\\ &\le \frac{\lambda^t(1-\lambda^k)}{1-\lambda}\, d_0. \end{aligned} \tag{3.36}$$

Let $0 \ll c$ be given. Choose a natural number M such that $\dfrac{\lambda^t(1-\lambda^k)}{1-\lambda}\, d_0 \ll c$ for all $t > M$. Thus $d(r_t,r_{t+k})+d(s_t,s_{t+k}) \ll c$. Therefore $\{r_t\}$ and $\{s_t\}$ are Cauchy sequences.

As U is complete, $\exists\ \varrho,\ \varrho' \in U,\ \lim_{t\to\infty} r_t = \varrho,\ \lim_{t\to\infty} s_t = \varrho'$. Now, we show that if H is continuous, then (ϱ,ϱ') is coupled fixed point of H.

As, we have

$$\begin{aligned} \varrho &= \lim_{t\to\infty} r_{t+1} = \lim_{t\to\infty} F(r_t,s_t) = F\left(\lim_{t\to\infty} r_t, \lim_{t\to\infty} s_t\right) = F(\varrho,\varrho'),\\ \varrho' &= \lim_{t\to\infty} s_{t+1} = \lim_{t\to\infty} F(s_t,r_t) = F\left(\lim_{t\to\infty} s_t, \lim_{t\to\infty} r_t\right) = F(\varrho',\varrho). \end{aligned}$$

Therefore, (ϱ, ϱ') is coupled fixed point of H.

Now, presuppose that the assumptions (I) and (II) of the theorem holds.
The sequence $\{r_t\} \to \varrho$, $\{s_t\} \to \varrho'$

$$\begin{aligned}\varphi(d(H(\varrho,\varrho'),H(r_t,s_t))) \leq \varphi\Big[&\alpha\,\frac{d(\varrho,H(\varrho,\varrho'))d(r_t,H(r_t,s_t))}{d(\varrho,r_t)}+\beta\,(d(\varrho,r_t))+d(\varrho',s_t))\\&+\gamma\,(d(\varrho,H(\varrho,\varrho'))+d(\varrho',H(\varrho',\varrho)))\Big]\\&-\phi\Big[\alpha\,\frac{d(\varrho,H(\varrho,\varrho'))d(r_t,H(r_t,s_t))}{d(\varrho,r_t)}+\beta\,(d(\varrho,r_t)+d(\varrho',s_t))\\&+\gamma\,(d(\varrho,H(\varrho,\varrho'))+d(\varrho',H(\varrho',\varrho)))\Big]\\&+L\,min\,\{d(\varrho,H(\varrho,\varrho')),d(r_t,H(r_t,s_t)),\\&\qquad d(r_t,H(\varrho,\varrho')),d(\varrho,H(r_t,s_t))\}.\end{aligned}$$

Letting $t\to\infty$, we have

$$\begin{aligned}\varphi(d(H(\varrho,\varrho'),\varrho)) &\leq \varphi[\gamma\,(d(\varrho,H(\varrho,\varrho'))+d(\varrho',H(\varrho',\varrho)))]\\&\quad-\phi[\gamma\,(d(\varrho,H(\varrho,\varrho'))+d(\varrho',H(\varrho',\varrho)))]\\&\leq \varphi\,[\gamma\,(d(\varrho,H(\varrho,\varrho'))+d(\varrho',H(\varrho',\varrho)))].\end{aligned} \tag{3.37}$$

In similar way, we have

$$\begin{aligned}\varphi(d(H(\varrho',\varrho),\varrho')) &\leq \varphi[\gamma\,(d(\varrho',H(\varrho',\varrho))+d(\varrho,H(\varrho,\varrho')))]\\&\quad-\phi[\gamma\,(d(\varrho',H(\varrho',\varrho))+d(\varrho,H(\varrho,\varrho')))]\\&\leq \varphi(\gamma\,(d(\varrho',H(\varrho',\varrho))+d(\varrho,H(\varrho,\varrho')))\end{aligned} \tag{3.38}$$

As φ is non-decreasing, utilizing (3.37) and (3.38), we get

$$d(H(\varrho,\varrho'),\varrho) \leq \gamma\,(d(\varrho,H(\varrho,\varrho'))+d(\varrho',H(\varrho',\varrho))) \tag{3.39}$$

$$d(H(\varrho',\varrho),\varrho') \leq \gamma\,(d(\varrho',H(\varrho',\varrho))+d(\varrho,H(\varrho,\varrho'))) \tag{3.40}$$

Adding (3.39) and (3.40), we have

$$d(H(\varrho,\varrho'),\varrho)+d(H(\varrho',\varrho),\varrho') \leq 2\,\gamma\,(d(\varrho,H(\varrho,\varrho'))+d(\varrho',H(\varrho',\varrho)))$$

a contraction, we acquire $H(\varrho,\varrho')=\varrho$ and $H(\varrho',\varrho)=\varrho'$. This completes the theorem.

Corollary 3.7. *Suppose $(U,\leq,d)$ be a partially ordered complete cone metric space. Presume $H\colon U\times U\to U$ be a mapping possesses the property of mixed monotone on U satisfy*

$$d(H(r,s),H(w,v)) \leq \alpha\frac{d(r,H(r,s))d(w,H(w,v))}{d(r,w)} + \beta(d(r,w)+d(s,v)) \\ + \gamma(d(r,H(r,s))+d(s,H(s,r))).$$

$\forall\ r,s,w,v \in U$ with $r \geq w,\ s \leq v$ and there exist positive real numbers $\alpha,\beta,\gamma \in [0,1)$ with $1 > \alpha+2\beta+2\gamma$. Presuppose either H is continuous or U has the subsequent properties,

(a) if an increasing sequence $\{r_t\}$ in U converges to some point $r \in U$, then $r_t \leq \varrho$, $\forall\, t$,

(b) if a decreasing sequence $\{s_t\}$ in U converges to some point $s \in U$, then $s_t \geq \varrho'$, $\forall\, t$.

Then H has a coupled fixed point.

Proof. For $\alpha+2\beta+2\gamma < 1$, taking $\varphi(z) = z$ and $\phi(z) = 0 = L$ in Theorem 3.6, we acquire Corollary 3.7.

Theorem 3.8. *Let the hypotheses of Theorem 3.6 hold. We acquire the uniqueness of the coupled fixed point of H.*

Proof. Suppose (ϱ,ϱ') and (ζ,ζ') are coupled fixed points of H, that is, $H(\varrho,\varrho') = \varrho, H(\varrho',\varrho) = \varrho', H(\zeta,\zeta') = \zeta$ and $H(\zeta',\zeta) = \zeta'$. We shall prove that $\varrho = \zeta, \varrho' = \zeta'$.

Consider the subsequent cases:

Case 1: If (ϱ,ϱ') and (ζ,ζ') are comparable. We have

$$\begin{aligned}
\varphi(d(\varrho,\zeta)) &= \varphi[d(H(\varrho,\varrho'),H(\zeta,\zeta'))] \\
&\leq \varphi\Big[\alpha\,\frac{d(\varrho,H(\varrho,\varrho'))d(\zeta,H(\zeta,\zeta'))}{d(\varrho,\zeta)} + \beta\,(d(\varrho,\zeta)) + d(\varrho',\zeta')) \\
&\qquad + \gamma\,(d(\varrho,H(\varrho,\varrho')) + d(\varrho',H(\varrho',\varrho)))\Big] \\
&\quad - \phi\Big[\alpha\,\frac{d(\varrho,H(\varrho,\varrho'))d(\zeta,H(\zeta,\zeta'))}{d(\varrho,\zeta)} + \beta\,d(\varrho,\zeta) + d(\varrho',\zeta')) \\
&\qquad + \gamma\,(d(\varrho,H(\varrho,\varrho')) + d(\varrho',H(\varrho',\varrho)))\Big] \\
&\quad + L\,min\,\{d(\varrho,H(\varrho,\varrho')), d(\zeta,H(\zeta,\zeta')), d(\zeta,H(\varrho,\varrho')), d(\varrho,H(\zeta,\zeta'))\} \\
&\leq \varphi(\beta\,(d(\varrho,\zeta) + d(\varrho',\zeta'))) - \phi(\beta\,d(\varrho,\zeta) + d(\varrho',\zeta'))) \\
&\leq \varphi(\beta\,(d(\varrho,\zeta) + d(\varrho',\zeta'))).
\end{aligned}$$

As φ is non-decreasing, therefore we obtain

$$d(\varrho, \zeta) \leq \beta\,(d(\varrho, \zeta) + d(\varrho', \zeta')) \tag{3.41}$$

$$d(\varrho', \zeta') \leq \beta\,(d(\varrho', \zeta') + d(\varrho, \zeta)) \tag{3.42}$$

Adding up (3.41) and (3.42), we get

$$d(\varrho, \zeta) + d(\varrho', \zeta') \leq 2\beta\,(d(\varrho, \zeta) + d(\varrho', \zeta'))$$

a contradiction. Thus $\varrho = \zeta$ and $\varrho' = \zeta'$. Therefore, (ϱ, ϱ') is a unique coupled fixed point of H.

Case 2: Presuppose (ϱ, ϱ') and (ζ, ζ') are not comparable. Assume that there exist $(z, u) \in U \times U$, comparable with both of them.

We define sequences $\{z_t\}, \{u_t\}$ as follows

$$z_0 = z,\; u_0 = u,\; z_{t+1} = H(z_t, u_t) \;\textit{and}\; u_{t+1} = H(u_t, z_t)\;\forall\, t.$$

Since (z, u) is comparable with (ϱ, ϱ'), we may presume that $(\varrho, \varrho') \geq (z, u) = (z_0, u_0)$.

Using mathematical induction, we can easily prove that

$$(\varrho, \varrho') \geq (z_t, u_t)\;\forall\, t. \tag{3.43}$$

From (3.29) and (3.43), we have

$$\begin{aligned}
\varphi(d(H(\varrho, \varrho'), H(z_t, u_t))) \leq\;& \varphi\Big[\alpha\,\frac{d(\varrho, H(\varrho, \varrho'))d(z_t, H(z_t, u_t))}{d(r, z_t)} + \beta\,d(\varrho, z_t)) + d(\varrho', u_t)) \\
&+ \gamma\,(d(\varrho, H(\varrho, \varrho')) + d(\varrho', H(\varrho', \varrho)))\Big] \\
&- \phi\Big[\alpha\,\frac{d(\varrho, H(\varrho, \varrho'))d(z_t, H(z_t, u_t))}{d(\varrho, z_t)} + \beta\,d(\varrho, z_t)) + d(\varrho', u_t)) \\
&+ \gamma\,(d(\varrho, H(\varrho, \varrho')) + d(\varrho', H(\varrho', \varrho)))\Big] \\
&+ L\,min\,\{d(\varrho, H(\varrho, \varrho')), d(z_t, H(z_t, u_t)), \\
&d(z_t, H(\varrho, \varrho')), d(\varrho, H(z_t, u_t))\} \\
\implies \varphi(d(\varrho, z_{t+1})) \leq\;& \varphi\Big[\alpha\,\frac{d(\varrho, H(\varrho, \varrho'))d(z_t, H(z_t, u_t))}{d(\varrho, z_t)} + \beta\,d(\varrho, z_t)) + d(\varrho', u_t)) \\
&+ \gamma\,(d(\varrho, H(\varrho, \varrho')) + d(\varrho', H(\varrho', \varrho)))\Big].
\end{aligned} \tag{3.44}$$

Similarly, we have

$$\varphi(d(\varrho', u_{t+1})) \leq \varphi\Big[\alpha\,\frac{d(u_t, H(u_t, z_t))d(\varrho', H(\varrho', \varrho))}{d(u_t, \varrho')} + \beta\,d(u_t, \varrho')) + d(\varrho, z_t))$$

$$+\,\gamma\left(d(\varrho', H(\varrho',\varrho)) + d(\varrho, H(\varrho,\varrho'))\right)\Big]. \tag{3.45}$$

Since φ is non-decreasing, from (3.44) and (3.45), we have

$$d(\varrho, z_{t+1}) \leq \beta\left(d(\varrho, z_t) + d(\varrho', u_t)\right), \tag{3.46}$$

$$d(\varrho', u_{t+1}) \leq \beta\left(d(\varrho, z_t) + d(\varrho', u_t)\right). \tag{3.47}$$

Adding (3.46) and (3.47), we get

$$\begin{aligned} d(\varrho, z_{t+1}) + d(u_{t+1}, \varrho') &\leq 2\,\beta\,[d(\varrho, z_t) + d(\varrho', u_t)] \\ &\leq (2\beta)^2\,[d(\varrho, z_{t-1}) + d(u_{t-1}, \varrho')] \\ &\;\;\vdots \\ &\leq (2\beta)^{t+1}\,[d(\varrho, z_0) + d(u_0, \varrho')] \to 0 \; as \; t \to \infty. \end{aligned}$$

Thus,

$$\lim_{t\to\infty} d(\varrho, z_{t+1}) = \lim_{t\to\infty} d(u_{t+1}, \varrho') = 0. \tag{3.48}$$

Correspondingly, we can show that

$$\lim_{t\to\infty} d(\zeta, z_{t+1}) = \lim_{t\to\infty} d(u_{t+1}, \zeta') = 0. \tag{3.49}$$

From (3.48) and (3.49), we obtain $\varrho = \zeta$ and $\varrho' = \zeta'$.

Example 3.2. *Let $E = \mathbb{R}^2, P = \{(t,v) \in \mathbb{R}^2\colon t, v \geq 0\} \subseteq \mathbb{R}^2$, and $U = [0,1]$. Define $d\colon U \times U \to U$ by $d(s,v) = (|s-v|, |s-v|), \forall\, s, v \in U$. Then (U,d) be a complete cone metric space.*

Consider the operator $H\colon U \times U \to U$ given by $H(r,s) = \dfrac{r}{9}$.

U satisfies the properties (i) and (ii) in Theorem 3.6. Take $\beta = \dfrac{1}{7}$. Then, for any $\alpha, \gamma \in [0,1)$ with $\alpha + 2\beta + 2\gamma < 1$. We claim that (3.29) holds for each $t \leq r,\ v \geq s$.

$$\begin{aligned} d(H(r,s), H(w,v)) &= d\left(\frac{r}{9}, \frac{w}{9}\right) = \left(\frac{|r-w|}{9}, \frac{|r-w|}{9}\right) \\ &\leq \frac{1}{7}(|r-w|, |r-w|) = \frac{1}{7} d(r,w) \leq \frac{1}{7}(d(r,w) + d(s,v)) \\ &\leq \alpha \frac{d(r, H(r,s)) d(w, H(w,v))}{d(r,w)} + \beta(d(r,w) + d(s,v)) \\ &\quad + \gamma(d(r, H(r,s)) + d(d(s, H(s,r)))). \end{aligned}$$

We deduce that all the assumptions of Theorem 3.6 are satisfied with taking $\varphi(z) = z$ and $\phi(z) = 0 = L$. Here, H possesses the unique coupled fixed point $(0,0)$.

Chapter 4

Coupled Coincidence Point Results in Ordered G-Metric Spaces

The persistence of this chapter is to demonstrate certain coupled coincidence point outcomes for non-linear contraction maps in ordered G-metric spaces with a property of mixed g-monotone. This chapter has been divided into three sections. In section 4.1, the concepts of G-metric space has been discussed. In section 4.2, some new coupled fixed point theorems for weak contractions have been proved. Also, an example is given to illustrate our result. Section 4.3, application to the solution of the integral equation have been given by using the result proved in the section 4.2.

4.1 Introduction

Mustafa and Sims [101] presented another structure of generalized metric spaces, which are termed as G-metric spaces, to create and present another fixed point theory for numerous mappings in this new structure. Later, several fixed point theorems on G-metric spaces have been obtained. For more results, we refer to the reader [102, 103, 126]. In recent times, coupled fixed point and coupled coincidence point theory has been established in partially ordered G-metric space. Many researchers have studied, coupled fixed point idea in ordered G-metric space(perceive [16, 18, 32, 36, 61, 95, 135]).

The notion of w and w^*-compatible mappings was initially introduced Abbas et.al [3]. Abbas et.al [4] used this idea to prove uniqueness theorem of coupled fixed point in G-metric spaces.

For more information on the subsequent definitions and results, see Mustafa and Sims [101].

Definition 4.1. [101] Suppose $G\colon U \times U \times U \to \mathbb{R}^+$ be a function and suppose U be a nonempty set persuade the following axioms for all $u_1, u_2, u_3, a \in U$,

(G_1) $G(u_1, u_2, u_3) = 0$ if $u_1 = u_2 = u_3$,

(G_2) $0 < G(u_1, u_1, u_2)$ with $u_1 \neq u_2$,

(G_3) $G(u_1, u_1, u_2) \leq G(u_1, u_2, u_3)$ with $u_3 \neq u_2$,

(G_4) $G(u_1, u_2, u_3) = G(u_1, u_3, u_2) = G(u_2, u_3, u_1) = ...$,

(G_5) $G(u_1, u_2, u_3) \leq G(u_1, a, a) + G(a, u_2, u_3)$,

then G is termed as a Generalized metric, or a G-metric on U and (U, G) is said to be G-metric space.

The concept of an altering distance function has been presented by Khan [89].

Definition 4.2. [89]. If it satisfies the subsequent properties, then $\phi\colon [0, \infty) \to [0, \infty)$ is termed as an altering distance function.

1. The function ϕ is monotone increasing and continuous.
2. $\phi(z) = 0 \iff z = 0$.

Abbas [3] initiated the innovative idea of w and w^*-compatible mappings and employed this conception to get a uniqueness theorem of coupled coincidence point for mapping g and F in G-metric space.

Definition 4.3. [3] Mappings $g\colon U \to U$ and $F\colon U \times U \to U$ are called

(1) w-compatible if $F(gs, gv) = g(F(s, v))$, whenever $gs = F(s, v)$ and $gv = F(v, s)$;

(2) w^*-compatible if $F(gs, gs) = g(F(s, s))$, whenever $gs = F(s, s)$.

4.2 Coupled Fixed Point Theorems for (ψ, α, β)-Weak Contractions

Now we establish the following theorem concerning (ψ, α, β)-weak contractions in ordered G-metric spaces.

Theorem 4.4. *Suppose $(U, \leq, G)$ be a partially ordered complete G-metric space. Presuppose g: $U \to U$ and F: $U \times U \to U$ be continuous mappings and F possess the property of mixed g-monotone and g commutes with F, satisfy*

$$\psi\big(G(F(r,s),F(t,v),F(z,w))\big) \leq \alpha\left(M((r,s),(t,v),(z,w))\right) - \beta\left(M((r,s),(t,v),(z,w))\right), \tag{4.1}$$

where,

$$\begin{aligned} M((r,s),(t,v),(z,w)) = max\{&G(gr,gt,gw), G(gs,gv,gz), G(gr,F(r,s),F(r,s)),\\ &G(gt,F(t,v),F(t,v)), G(gz,F(z,w),F(z,w)),\\ &G(gs,F(s,r),F(s,r)), G(gv,F(v,t),F(v,t)),\\ &G(gw,F(w,z),F(w,z))\},\end{aligned}$$

for all $r,s,t,v,w,z \in U$ with $gz \leq gt \leq gr$ and $gs \leq gv \leq gw$, where ψ, α, β: $[0,\infty) \to [0,\infty)$ and ψ is an altering distance function, β is lower semi-continuous, α is continuous,

$$\beta\,(0) = \alpha\,(0) = 0, \tag{4.2}$$

$$\psi(u) - \alpha\,(u) + \beta\,(u) > 0 \text{ for each } 0 < u. \tag{4.3}$$

Assume that $F(U \times U) \subseteq g(U)$. Furthermore for each $r_0, s_0 \in U$ with $gr_0 \leq F(r_0,s_0)$ and $gs_0 \geq F(s_0,r_0)$, then g and F have a coupled coincidence point, there exist $r,s \in U$, $g(r) = F(r,s)$ and $g(s) = F(s,r)$.

Proof. Assume $r_0, s_0 \in U$ then $gr_0 \leq F(r_0,s_0)$ and $gs_0 \geq F(s_0,r_0)$. Utilizing the way that $F(U \times U) \subseteq g(U)$, take $r_1, s_1 \in U$ then $gr_1 = F(r_0,s_0)$ and $gs_1 = F(s_0,r_0)$.

By similar contentions, $F(U \times U) \subseteq g(U)$, take $r_2, s_2 \in U$ such that $gr_2 = F(r_1,s_1)$, $gs_2 = F(s_1,r_1)$. As F possesses the property of mixed g-monotone, we get $gr_0 \leq gr_1 \leq gr_2$ and $gs_2 \leq gs_1 \leq gs_0$. Proceeding with this procedure, we can create two sequences $\{r_n\}$ and $\{s_n\}$ in U, such that

$$\begin{aligned} gr_n &= F(r_{n-1},s_{n-1}) \leq gr_{n+1} = F(r_n,s_n),\\ gs_{n+1} &= F(s_n,r_n) \leq gs_n = F(s_{n-1},r_{n-1}).\end{aligned}$$

If, for some integer n, we have $(gr_{n+1}, gs_{n+1}) = (gr_n, gs_n)$, then $F(r_n,s_n) = gr_n$ and $F(s_n,r_n) = gs_n$, that is, (r_n,s_n) is a coincidence point of g and F. Thus, we presume that $(gr_{n+1}, gs_{n+1}) \neq (gr_n, gs_n)$ for all $n \in \mathbb{N}$ All the more accurately, we expect that either $gr_{n+1} \neq gr_n$ or $gs_{n+1} \neq gs_n$.

For every $n \in \mathbb{N}$, utilizing the inequality (4.1), we have

$$\begin{aligned}\psi(G(gr_n, gr_{n+1}, gr_{n+1}) &= \psi(G(F(r_{n-1}, s_{n-1}), F(r_n, s_n), F(r_n, s_n))) \\ &\leq \alpha\,(M((r_{n-1}, s_{n-1}), (r_n, s_n), (r_n, s_n))) \\ &\quad - \beta\,(M((r_{n-1}, s_{n-1}), (r_n, s_n), (r_n, s_n))),\end{aligned}$$

and

$$\begin{aligned}\psi(G(gs_n, gs_{n+1}, gs_{n+1}) &= \psi(G(F(s_{n-1}, r_{n-1}), F(s_n, r_n), F(s_n, r_n))) \\ &\leq \alpha\,(M((s_{n-1}, r_{n-1}), (s_n, r_n), (s_n, r_n))) \\ &\quad - \beta\,(M((s_{n-1}, r_{n-1}), (s_n, r_n), (s_n, r_n))),\end{aligned}$$

where,

$$\begin{aligned}M((r_{n-1}, s_{n-1}), (r_n, s_n), (r_n, s_n)) &= M((s_{n-1}, r_{n-1}), (s_n, r_n), (s_n, r_n)) \\ &= max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n), \\ &\qquad G(gr_{n-1}, F(r_{n-1}, s_{n-1}), F(r_{n-1}, s_{n-1})), \\ &\qquad G(gs_{n-1}, F(s_{n-1}, r_{n-1}), F(s_{n-1}, r_{n-1})), \\ &\qquad G(gr_n, F(r_n, s_n), F(r_n, s_n)), \\ &\qquad G(gs_n, F(s_n, r_n), F(s_n, r_n)), \\ &\qquad G(gr_n, F(r_n, s_n), F(r_n, s_n)), \\ &\qquad G(gs_n, F(s_n, r_n), F(s_n, r_n))\} \\ &= max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n), \\ &\qquad G(gr_n, gr_{n+1}, gr_{n+1}), G(gs_n, gs_{n+1}, gs_{n+1})\}.\end{aligned}$$

Now, let us consider three cases.
Case 1: $M((r_{n-1}, s_{n-1}), (r_n, s_n), (r_n, s_n)) = max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\}$.
We have

$$\begin{aligned}\psi(G(gr_n, gr_{n+1}, gr_{n+1}) &\leq \alpha\,(max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\}) \\ &\quad - \beta\,(max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\}), \qquad (4.4)\end{aligned}$$

and

$$\begin{aligned}\psi(G(gs_n, gs_{n+1}, gs_{n+1}) &\leq \alpha\,(max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\}) \\ &\quad - \beta\,(max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\}). \qquad (4.5)\end{aligned}$$

Case 2: $M(r_{n-1}, s_{n-1}, r_n, s_n, r_n, s_n) = G(gr_n, gr_{n+1}, gr_{n+1})$.
We claim that

$$M(r_{n-1}, s_{n-1}, r_n, s_n, r_n, s_n) = G(gr_n, gr_{n+1}, gr_{n+1}) = 0.$$

In fact, if $G(gr_n, gr_{n+1}, gr_{n+1}) \neq 0$, then

$$\psi(G(gr_n, gr_{n+1}, gr_{n+1}) \leq \alpha\,(G(gr_n, gr_{n+1}, gr_{n+1})) - \beta\,(G(gr_n, gr_{n+1}, gr_{n+1})),$$

By (4.2), which is contradiction. Thus, we have $G(gr_n, gr_{n+1}, gr_{n+1}) = 0$. Then, it is obvious that (4.4) and (4.5) hold.

Case 3: $M(r_{n-1}, s_{n-1}, r_n, s_n, r_n, s_n) = G(gs_n, gs_{n+1}, gs_{n+1})$.

Similar to the proof of Case 2, one can also show that (4.4) and (4.5) hold.

Let $\delta_n = max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\}$. So for $n \geq 1$, $\delta_n = 0$, then the deduction of the theorem follows. Thus, we presume that

$$\delta_n \neq 0,\ \forall\, n \geq 1. \tag{4.6}$$

Suppose, for any n, $\delta_{n-1} < \delta_n$. Thus, from (4.4) and (4.5), as ψ is non decreasing, we obtain

$$\begin{aligned}
&\psi(max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\}) \\
&\qquad < \psi(max\{G(gr_n, gr_{n+1}, gr_{n+1}), G(gs_n, gs_{n+1}, gs_{n+1})\}) \\
&\qquad = max\{\psi(G(gr_n, gr_{n+1}, gr_{n+1}), \psi(G(gs_n, gs_{n+1}, gs_{n+1}))\} \\
&\qquad \leq \alpha\,(max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\}) \\
&\qquad\quad - \beta\,(max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\}),
\end{aligned} \tag{4.7}$$

that is, $\psi(\delta_n) - \alpha\,(\delta_n) + \beta\,(\delta_n) \leq 0$. By our hypothesis, we have $\delta_n = 0$, which is a contradiction (4.6). So, for every $n \geq 1$ we conclude that

$$\delta_{n+1} \leq \delta_n, \tag{4.8}$$

thus, $\{\delta_n\}$ is a non increasing sequence of non-negative real numbers. Then, we can find $k \geq 0$ such that $\lim_{n\to\infty} \delta_n = k$.

Taking $n \to \infty$ in equation (4.7) and using the the continuity of ψ and α and lower semi-continuity of β, we acquire $\psi(k) \leq \alpha\,(k) - \beta\,(k)$, which implies $k = 0$, as of our hypothesis about ψ, β, α. Thus,

$$\lim_{n\to\infty} max\{G(gr_{n-1}, gr_n, gr_n), G(gs_{n-1}, gs_n, gs_n)\} = 0. \tag{4.9}$$

Next, we claim that $\{gr_n\}$ and $\{gs_n\}$ are G-Cauchy sequences.

We will prove that for each $\varepsilon > 0$, we can find $a \in \mathbb{N}$, if $n, m \geq a$,

$$max\{G(gr_{m(a)}, gr_{n(a)-1}, gr_{n(a)-1}), G(gs_{m(a)}, gs_{n(a)-1}, gs_{n(a)-1})\} < \varepsilon. \tag{4.10}$$

Presuppose the above statement is not true.

Then, for some $\varepsilon > 0$ there exists subsequences of integers $\{m(a)\}$ and $\{n(a)\}$ with $n(a) > m(a) > a$ such that

$$max\{G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)}), G(gs_{m(a)}, gs_{n(a)}, gs_{n(a)})\} \geq \varepsilon. \tag{4.11}$$

Now, from (G_5), we have

$$G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)}) \leq G(gr_{m(a)}, gr_{n(a)-1}, gr_{n(a)-1}) + G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)}).$$

Then, from (4.10), we get

$$G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)}) < G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)}) + \varepsilon. \tag{4.12}$$

Similarly, from (G_5) and (4.10), we have

$$G(gs_{m(a)}, gs_{n(a)}, gs_{n(a)}) < G(gs_{n(a)-1}, gs_{n(a)}, gs_{n(a)}) + \varepsilon. \tag{4.13}$$

From (4.11), (4.12) and (4.13), we get

$$\begin{aligned} \varepsilon &\leq max\{G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)}), G(gs_{m(a)}, gs_{n(a)}, gs_{n(a)})\} \\ &< max\{G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)}), G(gs_{n(a)-1}, gs_{n(a)}, gs_{n(a)})\} + \varepsilon. \end{aligned} \tag{4.14}$$

Letting, $a \to \infty$ in (4.14) and using (4.9), we get

$$\lim_{a\to\infty} max\{G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)}), G(gs_{m(a)}, gs_{n(a)}, gs_{n(a)})\} = \varepsilon. \tag{4.15}$$

Again, from (G_5) and (4.10), we get

$$\begin{aligned} G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}) &\leq G(gr_{m(a)-1}, gr_{m(a)}, gr_{m(a)}) + G(gr_{m(a)}, gr_{n(a)-1}, gr_{n(a)-1}) \\ &< G(gr_{m(a)-1}, gr_{m(a)}, gr_{m(a)}) + \varepsilon, \end{aligned} \tag{4.16}$$

$$\begin{aligned} G(gs_{m(a)-1}, gs_{n(a)-1}, gs_{n(a)-1}) &\leq G(gs_{m(a)-1}, gs_{m(a)}, gs_{m(a)}) + G(gs_{m(a)}, gs_{n(a)-1}, gs_{n(a)-1}) \\ &< G(gs_{m(a)-1}, gs_{m(a)}, gs_{m(a)}) + \varepsilon. \end{aligned} \tag{4.17}$$

From (4.16) and (4.17), we have

$$\begin{aligned} &max\{G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}), G(gs_{m(a)-1}, gs_{n(a)-1}, gs_{n(a)-1})\} \\ &\quad < max\{G(gr_{m(a)-1}, gr_{m(a)}, gr_{m(a)}), G(gs_{m(a)-1}, gs_{m(a)}, gs_{m(a)})\} + \varepsilon. \end{aligned} \tag{4.18}$$

Using (G_5), we get

$$G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)}) \leq G(gr_{m(a)}, gr_{n(a)-1}, gr_{n(a)-1}) + G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)})$$

$$\leq G(gr_{m(a)}, gr_{m(a)-1}, gr_{m(a)-1}) + G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}) + G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)}).$$

From Proposition, $G(u, v, v) \leq 2\ G(v, u, u)$, we have

$$G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)}) \leq 2\ G(gr_{m(a)-1}, gr_{m(a)}, gr_{m(a)}) + G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}) + G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)}). \tag{4.19}$$

Similarly, we get

$$G(gs_{m(a)}, gs_{n(a)}, gs_{n(a)}) \leq 2\ G(gs_{m(a)-1}, gs_{m(a)}, gs_{m(a)}) + G(gs_{m(a)-1}, gs_{n(a)-1}, gs_{n(a)-1}) + G(gs_{n(a)-1}, gs_{n(a)}, gs_{n(a)}). \tag{4.20}$$

So, from (4.11), (4.19) and (4.20), we obtain

$$\begin{aligned} \varepsilon &\leq max\{G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)}), G(gs_{m(a)}, gs_{n(a)}, gs_{n(a)})\} \\ &\leq 2\ max\{G(gr_{m(a)-1}, gr_{m(a)}, gr_{m(a)}), G(gs_{m(a)-1}, gs_{m(a)}, gs_{m(a)})\} \\ &\quad + max\{G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}), G(gs_{m(a)-1}, gs_{n(a)-1}, gs_{n(a)-1})\} \\ &\quad + max\{G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)})), G(gs_{n(a)-1}, gs_{n(a)}, gs_{n(a)}))\}. \end{aligned} \tag{4.21}$$

From (4.18) and (4.21), we get

$$\begin{aligned} &\varepsilon - 2\ max\{G(gr_{m(a)-1}, gr_{m(a)}, gr_{m(a)}), G(gs_{m(a)-1}, gs_{m(a)}, gs_{m(a)})\} \\ &- max\{G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)})), G(gs_{n(a)-1}, gs_{n(a)}, gs_{n(a)}))\} \\ &\leq max\{G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}), G(gs_{m(a)-1}, gs_{n(a)-1}, gs_{n(a)-1})\} \\ &< max\{G(gr_{m(a)-1}, gr_{m(a)}, gr_{m(a)}), G(gs_{m(a)-1}, gs_{m(a)}, gs_{m(a)}) + \varepsilon. \end{aligned} \tag{4.22}$$

Letting, $a \to \infty$ in (4.22) and using (4.9), we get

$$\lim_{a\to\infty} max\{G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}), G(gs_{m(a)-1}, gs_{n(a)-1}, gs_{n(a)-1})\} = \varepsilon. \tag{4.23}$$

From using the inequality (4.1), we get

$$\begin{aligned} &\psi\big(G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)})\big) \\ &\quad = \psi\big(G(F(r_{m(a)-1}, s_{m(a)-1}), F(r_{n(a)-1}, s_{n(a)-1}), F(r_{n(a)-1}, s_{n(a)-1}))\big) \\ &\quad \leq \alpha\left(M((r_{m(a)-1}, s_{m(a)-1}), (r_{n(a)-1}, s_{n(a)-1}), (r_{n(a)-1}, s_{n(a)-1}))\right) \\ &\qquad - \beta\left(M((r_{m(a)-1}, s_{m(a)-1}), (r_{n(a)-1}, s_{n(a)-1}), (r_{n(a)-1}, s_{n(a)-1}))\right), \end{aligned} \tag{4.24}$$

$$\begin{aligned} &\psi\big(G(gs_{m(a)}, gs_{n(a)}, gs_{n(a)})\big) \\ &\quad = \psi\big(G(F(s_{m(a)-1}, r_{m(a)-1}), F(s_{n(a)-1}, r_{n(a)-1}), F(s_{n(a)-1}, r_{n(a)-1}))\big) \end{aligned}$$

$$\begin{aligned} &\leq \alpha\left(M((s_{m(a)-1}, r_{m(a)-1}), (s_{n(a)-1}, r_{n(a)-1}), (s_{n(a)-1}, r_{n(a)-1}))\right) \\ &\quad - \beta\left(M((s_{m(a)-1}, r_{m(a)-1}), (s_{n(a)-1}, r_{n(a)-1}), (s_{n(a)-1}, r_{n(a)-1}))\right). \end{aligned} \tag{4.25}$$

Where,

$$\begin{aligned} &M((s_{m(a)-1}, r_{m(a)-1}), (s_{n(a)-1}, r_{n(a)-1}), (s_{n(a)-1}, r_{n(a)-1})) \\ &\quad = M((r_{m(a)-1}, s_{m(a)-1}), (r_{n(a)-1}, s_{n(a)-1}), (r_{n(a)-1}, s_{n(a)-1})) \\ &\quad = max\{G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}), G(gs_{m(a)-1}, gs_{n(a)-1}, gs_{n(a)-1}), \\ &\qquad\qquad G(gr_{m(a)-1}, F(r_{m(a)-1}, s_{m(a)-1}), F(r_{m(a)-1}, s_{m(a)-1})), \\ &\qquad\qquad G(gs_{m(a)-1}, F(s_{m(a)-1}, r_{m(a)-1}), F(s_{m(a)-1}, r_{m(a)-1})), \\ &\qquad\qquad G(gr_{n(a)-1}, F(r_{n(a)-1}, s_{n(a)-1}), F(r_{n(a)-1}, s_{n(a)-1})), \\ &\qquad\qquad G(gs_{n(a)-1}, F(s_{n(a)-1}, r_{n(a)-1}), F(s_{n(a)-1}, r_{n(a)-1})), \\ &\qquad\qquad G(gr_{n(a)-1}, F(r_{n(a)-1}, s_{n(a)-1}), F(r_{n(a)-1}, s_{n(a)-1})), \\ &\qquad\qquad G(gs_{n(a)-1}, F(s_{n(a)-1}, r_{n(a)-1}), F(s_{n(a)-1}, r_{n(a)-1}))\} \\ &\quad = max\{G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}), G(gs_{m(a)-1}, gs_{n(a)-1}, gs_{n(a)-1}), \\ &\qquad\qquad G(gr_{m(a)-1}, gr_{m(a)}, gr_{m(a)}), G(gs_{m(a)-1}, gs_{m(a)}, gs_{m(a)}), \\ &\qquad\qquad G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)}), G(gs_{n(a)-1}, gs_{n(a)}, gs_{n(a)})\}. \end{aligned}$$

Now, by (4.24) and (4.25), we have

$$\begin{aligned} &\psi(max\{G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)}), G(gs_{m(a)}, gs_{n(a)}, gs_{n(a)})\}) \\ &\qquad = max\{\psi(G(gr_{m(a)}, gr_{n(a)}, gr_{n(a)})), \psi(G(gs_{m(a)}, gs_{n(a)}, gs_{n(a)}))\} \\ &\qquad \leq \alpha\,(Z_n) - \beta\,(Z_n), \end{aligned} \tag{4.26}$$

where,

$$\begin{aligned} Z_n = &max\{G(gr_{m(a)-1}, gr_{n(a)-1}, gr_{n(a)-1}), G(gs_{m(a)-1}, gs_{n(a)-1}, gs_{n(a)-1}), \\ &\qquad G(gr_{m(a)-1}, gr_{m(a)}, gr_{m(a)}), G(gs_{m(a)-1}, gs_{m(a)}, gs_{m(a)}), \\ &\qquad G(gr_{n(a)-1}, gr_{n(a)}, gr_{n(a)}), G(gs_{n(a)-1}, gs_{n(a)}, gs_{n(a)}). \end{aligned} \tag{4.27}$$

Finally, Letting $a \to \infty$ in (4.26) ((4.27))and using (4.9), (4.15) and (4.23), we get

$$\psi(\varepsilon) \leq \alpha\,(max(\varepsilon, 0, 0)) - \beta\,(max(\varepsilon, 0, 0)). \tag{4.28}$$

Therefore, $\psi(\varepsilon) - \alpha\,(\varepsilon) + \beta\,(\varepsilon) \leq 0$ and therefore $\varepsilon = 0$, a contradiction. As a result, $\{gr_n\}$ and $\{gs_n\}$ are G-Cauchy sequences in the G-metric space (U, G), which is complete. Then, we can find $r, s \in U$, $\{gr_n\}$ and $\{gs_n\}$ are respectively G-convergent to r and s.

From Proposition , we have

$$\lim_{n\to\infty} G(gr_n, gr_n, r) = \lim_{n\to\infty} G(gr_n, r, r) = 0, \tag{4.29}$$

$$\lim_{n\to\infty} G(gs_n, gs_n, s) = \lim_{n\to\infty} G(gs_n, s, s) = 0. \tag{4.30}$$

From (4.29) and (4.30), the continuity of g, we obtain

$$\lim_{n\to\infty} G(g(gr_n), g(gr_n), gr) = \lim_{n\to\infty} G(g(gr_n), gr, gr) = 0, \tag{4.31}$$

$$\lim_{n\to\infty} G(g(gs_n), g(gs_n), gs) = \lim_{n\to\infty} G(g(gs_n), gs, gs) = 0. \tag{4.32}$$

Since $gr_{n+1} = F(r_n, s_n)$ and $gs_{n+1} = F(s_n, r_n)$, the commutativity of g and F provides that

$$g(gr_{n+1}) = g(F(r_n, s_n)) = F(gr_n, gs_n), \tag{4.33}$$

$$g(gs_{n+1}) = g(F(s_n, r_n)) = F(gs_n, gr_n). \tag{4.34}$$

By using (4.33) and (4.34) and the continuity of F, we get $\{g(gr_{n+1})\}$ is G-convergent to $F(r,s)$ and $\{g(gs_{n+1})\}$ is G-convergent to $F(s,r)$. By the uniqueness of limit, we get $F(r,s) = gr$ and $F(s,r) = gs$, and this ends the proof.

Theorem 4.5. *Suppose all the assumptions of Theorem 4.4 are satisfied. Moreover, presume that U has the subsequent properties*

(a) if an increasing sequence $\{r_n\} \to r$, then $r_n \leq r$, $\forall\, n$,

(b) if a decreasing sequence $\{s_n\} \to s$, then $s_n \geq s$, $\forall\, n$.

Then the conclusion of Theorem 4.4 also hold.

Proof. Succeeding the proof of Theorem 4.4, we have that $\{gr_n\}$ and $\{gs_n\}$ are Cauchy sequences in the complete G-metric space $(g(U), G)$. Then, we can find $r, s \in U$ such that $gr_n \to gr$ and $gs_n \to gs$.

Since $\{gr_n\}$ is increasing and $\{gs_n\}$ is decreasing, using the regularity of $(U, G, \leq)$. We have $gr_n \leq gr$ and $gs_n \geq gs$ for all $n \geq 0$. If $gr_n = gr$ and $gs_n = gs$ for some $n \geq 0$. Then $gr = gr_n \leq gr_{n+1} \leq gr = gr_n$ and $gs \leq gs_{n+1} \leq gs_n = gs$. Which implies that $gr_n = gr_{n+1} = F(r_n, s_n)$ and $gr_n = gr_{n+1} = F(r_n, s_n)$, that is (r_n, s_n) is a coupled coincidence point of g and F. Then, we presuppose that $(gr_n, gs_n) \neq (gr, gs)$ for all $n \geq 0$. By inequality (4.1), we have

$$\psi\big(G(gr_{n+1}, F(r,s), F(r,s))\big) = \psi\big(G(F(r_n, s_n), F(r,s), F(r,s))\big)$$

$$\leq \alpha\left(M((r_n,s_n),(r,s),(r,s))\right)-\beta\left(M((r_n,s_n),(r,s),(r,s))\right), \tag{4.35}$$

$$\begin{aligned}\psi\big(G(gs_{n+1},F(s,r),F(s,r))\big) &= \psi(G(F(s_n,r_n),F(s,r),F(s,r)))\\ &\leq \alpha\left(M((s_n,r_n),(s,r),(s,r))\right)-\beta\left(M((s_n,r_n),(s,r),(s,r))\right),\end{aligned} \tag{4.36}$$

where,

$$\begin{aligned}M((r_n,s_n),(r,s),(r,s)) &= M((s_n,r_n),(s,r),(s,r))\\ &= max\{G(gr_n,gr,gr),G(gr_n,F(r_n,s_n),F(r_n,s_n)),\\ &\qquad G(gr,F(r,s),F(r,s)),G(gr,F(r,s),F(r,s)),\\ &\qquad G(gs_n,gs,gs),G(gs_n,F(s_n,r_n),F(s_n,r_n)),\\ &\qquad G(gs,F(s,r),F(s,r)),G(gs,F(s,r),F(s,r))\}\\ &= max\{G(gr_n,gr,gr),G(gs_n,gs,gs),G(gr_n,gr_{n+1},gr_{n+1}),\\ &\qquad G(gr,F(r,s),F(r,s)),G(gs_n,gs_{n+1},gs_{n+1}),\\ &\qquad G(gs,F(s,r),F(s,r))\}.\end{aligned}$$

Now, we claim that

$$max\{G(gr,F(r,s),F(r,s)),G(gs,F(s,r),F(s,r))\}=0. \tag{4.37}$$

If this not true, then $max\{G(gr,F(r,s),F(r,s)),G(gs,F(s,r),F(s,r))\} > 0$. Since $\lim_{n\to\infty} gr_n = gr$, $\lim_{n\to\infty} gs_n = gs$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\begin{aligned}M((r_n,s_n),(r,s),(r,s)) &= M((s_n,r_n),(s,r),(s,r))\\ &= max\{G(gr,F(r,s),F(r,s)),G(gs,F(s,r),F(s,r))\}.\end{aligned}$$

Combining this with (4.35) and (4.36), we get for all $n > N$,

$$\begin{aligned}&\psi\big(max\{G(gr_{n+1},F(r,s),F(r,s)),G(gs_{n+1},F(s,r),F(s,r)))\}\big)\\ &\quad = max\{\psi(G(gr_{n+1},F(r,s),F(r,s))),\psi(G(gs_{n+1},F(s,r),F(s,r))))\}\\ &\quad \leq \alpha\left(max\{G(gr,F(r,s),F(r,s)),G(gs,F(s,r),F(s,r))\}\right)\\ &\qquad -\beta\left(max\{G(gr,F(r,s),F(r,s)),G(gs,F(s,r),F(s,r))\}\right).\end{aligned}$$

Letting $n \to \infty$ it follows that

$$\begin{aligned}&\psi\big(max\{G(gr,F(r,s),F(r,s)),G(gs,F(s,r),F(s,r)))\}\big)\\ &\quad \leq \alpha\left(max\{G(gr,F(r,s),F(r,s)),G(gs,F(s,r),F(s,r))\}\right)\\ &\qquad -\beta\left(max\{G(gr,F(r,s),F(r,s)),G(gs,F(s,r),F(s,r))\}\right).\end{aligned}$$

From our assumptions about ψ, α, β, which is a contradiction. So (4.37) hold. Then, it follows that $gr = F(r, s)$ and $gs = F(s, r)$.

Example 4.1. *Presume $U = [0, 1]$. Define G: $U \times U \times U \to \mathbb{R}^+$ by*

$$G(s, v, w) = |s - v| + |v - w| + |w - s|, \forall\, s, v, w \in U.$$

Then (U, G) is a complete G-metric space.
Presuppose the mapping F: $U \times U \to U$ defined by

$$F(r, s) = \frac{1}{5}r - \frac{1}{3}s^2 \;\; if\, r \geq s$$

for all $r, s \in U$. Also define g: $U \to U$ by $gr = r$ for $r \in U$.

Let ψ, α, β: $[0, \infty) \to [0, \infty)$ be given by $\psi(u) = \alpha(u) = u$ and $\beta(u) = \frac{1}{5}u$. Clearly, ψ be an altering distance function, β is lower semicontinuous, α is continuous, $\beta(0) = \alpha(0) = 0$, $\psi(u) - \alpha(u) + \beta(u) = \frac{u}{5} > 0$ for each $u > 0$.
Now, we have following possibility for value of (r, s), (t, v) and (z, w) such that $r \geq t \geq z$, $s \leq v \leq w$.

$$\begin{aligned} G(F(r,s), F(t,v), F(z,w)) &= \left|\left(\frac{1}{5}r - \frac{1}{3}s^2\right) - \left(\frac{1}{5}t - \frac{1}{3}v^2\right)\right| + \left|\left(\frac{1}{5}t - \frac{1}{3}v^2\right) - \left(\frac{1}{5}z - \frac{1}{3}w^2\right)\right| \\ &+ \left|\left(\frac{1}{5}z - \frac{1}{3}w^2\right) - \left(\frac{1}{5}r - \frac{1}{3}s^2\right)\right| \\ &\leq \frac{1}{5}(|r - t| + |t - z| + |z - r|) \\ &+ \frac{1}{3}\left(\left|s^2 - v^2\right| + \left|v^2 - w^2\right| + \left|w^2 - s^2\right|\right) \end{aligned}$$

Since, $\left(s^2 - v^2\right) \leq (s - v)$. Similarly, $\left(v^2 - w^2\right) \leq (v - w)$ and $\left(w^2 - s^2\right) \leq (w - s)$. Thus, we obtain

$$\begin{aligned} G(F(r,s), F(t,v), F(z,w)) &\leq \frac{1}{5}(|(r - t)| + |t - z| + |z - r|) \\ &+ \frac{1}{3}(|s - v| + |v - w| + |w - s|) \\ &\leq \frac{1}{5}(G(gr, gt, gz)) + \frac{1}{3}(G(gs, gv, gw)) \\ &\leq \frac{8}{15}\, max\, \{G(gr, gt, gz), G(gs, gv, gw)\} \\ &\leq \frac{4}{5}M((r,s),(t,v),(z,w)) \\ &= M((r,s),(t,v),(z,w)) - \frac{1}{5}M((r,s),(t,v),(z,w)) \\ &= \alpha\left(M((r,s),(t,v),(z,w))\right) - \beta\left(M((r,s),(t,v),(z,w))\right), \end{aligned}$$

where,

$$M((r,s),(t,v),(z,w)) = max\{G(gr,gt,gz), G(gs,gv,gw), G(gr,F(r,s),F(r,s)), \\ G(gt,F(t,v),F(t,v)), G(gz,F(z,w),F(z,w)), \\ G(gs,F(s,r),F(s,r)), G(gv,F(v,t),F(v,t)), \\ G(gw,F(w,z),F(w,z))\}.$$

Therefore, all the assumptions of Theorem 4.4 hold. Furthermore, $(0,0)$ *is the unique coupled coincidence point of* g *and* F*.*

Now, putting $g = I_U$ (the identity map of U) in the equation (4.1), we obtain

Corollary 4.6. *Suppose* $(U,\leq)$ *be a partially ordered set and* G *be a* G*-metric on* U*. Presume* $F\colon U\times U\to U$ *be a function satisfy* (4.1) *(with* $g = I_U$*) for all* $r,s,t,v,z,w\in U$ *with* $r\geq t\geq z$ *and* $s\leq v\leq w$*. Assume that* (U,G) *is complete and* F *possesses the property of mixed monotone. Presuppose either* F *is continuous or* U *has the subsequent:*

(a) if an increasing sequence $\{r_n\}\to r$*, then* $r_n\leq r$*,* $\forall\, n$*,*

(b) if a decreasing sequence $\{s_n\}\to s$*, then* $s_n\geq s$*,* $\forall\, n$*.*

Furthermore for each $r_0, s_0\in U$ *with* $r_0\leq F(r_0,s_0)$*,* $s_0\geq F(s_0,r_0)$*, then* F *has a coupled fixed point.*

Corollary 4.7. *Suppose* $(U,\leq)$ *be a partially ordered set and* G *be a* G*-metric on* U*. Presuppose* $F\colon U\times U\to U$ *and* $g\colon U\to U$ *be two mappings such that* $F(U\times U)\subseteq g(U)$ *and* F *has the property of mixed* g*-monotone. Presume there exists* $l\in[0,1)$ *satisfy*

$$G(F(r,s),F(t,v),F(z,w)) \leq l\, max\{G(gr,gt,gz), G(gs,gv,gw), \\ G(gr,F(r,s),F(r,s)), G(gt,F(t,v),F(t,v)), \\ G(gz,F(z,w),F(z,w)), G(gs,F(s,r),F(s,r)), \\ G(gv,F(v,t),F(v,t)), G(gw,F(w,z),F(w,z))\} \tag{4.38}$$

for all $r,s,t,v,w,z\in U$ *with* $gr\geq gt\geq gz$ *and* $gs\leq gv\leq gw$*. Presume either*

1. F *and* g *are continuous,* (U,G) *is complete and* g *commutes with* F*, or*
2. $(g(U),G)$ *is complete and* U *has the subsequent:*
   (a) if an increasing sequence $\{r_n\}\to r$*, then* $r_n\leq r$*,* $\forall\, n$*,*
   (b) if a decreasing sequence $\{s_n\}\to s$*, then* $s_n\geq s$*,* $\forall\, n$*.*

Furthermore for each $r_0, s_0 \in U$ with $gr_0 \leq F(r_0, s_0)$, $gs_0 \geq F(s_0, r_0)$, then g and F have a coupled coincidence point.

Proof. Define ψ, α, β: $[0, \infty) \to [0, \infty)$, $\psi(z) = z = \alpha(z)$, $\beta(z) = (1-l)z$ where $l \in [0, 1)$. Then (4.38) holds. Hence the result follows from Theorem 4.4 or Theorem 4.5.

Theorem 4.8. *Under the hypothesis of Theorem 4.5, suppose that $gs_0 \leq gr_0$. Then, it follows $gr = F(r, s) = F(s, r) = gs$. Furthermore, if g and F are w- compatible, then g and F have a coupled coincidence point of the form (u, u).*

Proof. If $gs_0 \leq gr_0$, then $gs \leq gs_n \leq gs_0 \leq gr_0 \leq gr_n \leq gr$ for all $n \in \mathbb{N}$. Thus, if $gr \neq gs$ (and then $G(gr, gr, gs) \neq 0$ and $G(gs, gs, gr) \neq 0$), hence by inequality (4.1), we have

$$\begin{aligned}\psi(G(gs, gr, gr)) &= \psi(G(F(s, r), F(r, s), F(r, s))) \\ &\leq \alpha(M((s, r), (r, s), (r, s))) - \beta(M((s, r), (r, s), (r, s))),\end{aligned}$$

where,

$$\begin{aligned}M((s, r), (r, s), (r, s)) = max\{&G(gs, gr, gr), G(gs, F(s, r), F(s, r)), \\ &G(gr, F(r, s), F(r, s)), G(gr, F(r, s), F(r, s)), \\ &G(gs, F(s, r), F(s, r)), G(gs, F(s, r), F(s, r)) \\ &G(gr, gs, gs), G(gr, F(r, s), F(r, s))\} \\ = max\{&G(gs, gr, gr), G(gr, gs, gs)\}.\end{aligned}$$

Hence

$$\begin{aligned}\psi(G(gs, gr, gr)) \leq\ &\alpha(max\{G(gs, gr, gr), G(gr, gs, gs)\}) \\ &- \beta(max\{G(gs, gr, gr), G(gr, gs, gs)\}).\end{aligned} \tag{4.39}$$

Since $gs \leq gr$, hence using the same idea we have

$$\begin{aligned}\psi(G(gr, gs, gs)) \leq\ &\alpha(max\{G(gs, gr, gr), G(gr, gs, gs)\}) \\ &- \beta(max\{G(gs, gr, gr), G(gr, gs, gs)\}).\end{aligned} \tag{4.40}$$

From (4.39) and (4.40), we have

$$\begin{aligned}\psi(max\{G(gs, gr, gr), G(gr, gs, gs)) &= max\{\psi(G(gs, gr, gr)), \psi(G(gr, gs, gs))\} \\ &\leq \alpha(max\{G(gs, gr, gr), G(gr, gs, gs)\}) \\ &\quad - \beta(max\{G(gs, gr, gr), G(gr, gs, gs)\}).\end{aligned}$$

Therefore, from properties of functions ψ, β, α, a contradiction. Thus, we acquire $G(gs, gr, gr) = 0$ and $G(gr, gs, gs) = 0$. Hence $gr = gs$, that is, $gr = F(r, s) = F(s, r) = gs$. Now, let $u = gr = gs$. Since g and F are w-compatible, then

$$gu = g(gr) = g(F(r, s)) = F(gr, gs) = F(u, u).$$

Thus, g and F have a coupled coincidence point of the form (u, u).

To guarantee the unique ness of coupled coincidence point. Let $(U, \leq)$ is a partially ordered set, we endow the product $U \times U$ with we need the subsequent idea of the partial order relation:

$$\text{for each } (t, v), (w, z) \in U \times U,\ (t, v) \leq (w, z) \iff t \leq w,\ v \geq z. \tag{4.41}$$

Theorem 4.9. *Under the hypothesis of Theorem 4.4, assume that, for each $(r, s), (\acute{r}, \acute{s}) \in U \times U$, we can find $(t, v) \in U \times U$ that is comparable to (r, s) and $(\acute{r}, \acute{s})$. Then g and F have a unique common coupled fixed point.*

Proof. From Theorem 4.4, the set of coupled coincidence point is non empty. We will prove that if (r, s) and $(\acute{r}, \acute{s})$ are coupled coincidence points, that is,

$$\begin{aligned} g(r) &= F(r, s), \quad g(s) = F(s, r) \\ \text{and } g(\acute{r}) &= F(\acute{r}, \acute{s}), \quad g(\acute{s}) = F(\acute{s}, \acute{r}), \end{aligned}$$

then

$$gr = g\acute{r} \text{ and } gs = g\acute{s}. \tag{4.42}$$

Choose an element $(t, v) \in U \times U$ comparable with both of them.

Let $t_0 = t, v_0 = v$ and choose $t_1, v_1 \in U$ so that $gt_1 = F(t_0, v_0)$ and $gv_1 = F(v_0, t_0)$. Then, in similar manner as in the proof of Theorem 4.4, we can define sequences $\{gt_n\}$ and $\{gv_n\}$ as follows

$$gt_{n+1} = F(t_n, v_n) \textit{ and } gv_{n+1} = F(v_n, t_n).$$

Further, set $r_0 = r$, $s_0 = s$, $\acute{r}_0 = \acute{r}$, $\acute{s}_0 = \acute{s}$ and on the same way, define the sequences $\{gr_n\}$, $\{gs_n\}$ and $\{g\acute{r}_n\}$, $\{g\acute{s}_n\}$.

Since $(gr, gs) = (F(r, s), F(s, r)) = (gr_1, gs_1)$ and $(F(t, v), F(v, t)) = (gt_1, gv_1)$ are comparable, then $gr \leq gt_1$ and $gs \geq gv_1$. Using mathematical induction, we can easily prove this $gr \leq gt_n,\ gy \geq gv_n,\ \forall\, n \in \mathbb{N}$.

Let $\gamma_n = max\{G(gr, gr, gt_n), G(gs, gs, gv_n)\}$. We will show that $\lim_{n\to\infty} \gamma_n = 0$. First, assume that $\gamma_n = 0$, for an $n \geq 1$.

By inequality (4.1), we have

$$\begin{aligned}\psi(G(gr, gr, gt_{n+1})) =& \psi(G(F(r, s), F(r, s), F(t_n, v_n))) \\ & \leq \alpha\left(M((r, s), (r, s), (t_n, v_n))\right) - \beta\left(M((r, s), (r, s), (t_n, v_n))\right),\end{aligned}$$

where,

$$\begin{aligned}M((r, s), (r, s), (t_n, v_n)) = & max\{G(gr, gr, gt_n), G(gs, gs, gv_n), \\ & G(gr, F(r, s), F(r, s)), G(gr, F(r, s), F(r, s)) \\ & G(gt_n, F(t_n, v_n), F(t_n, v_n)), G(gs, F(s, r), F(s, r)) \\ & G(gs, F(s, r), F(s, r)), G(gv_n, F(v_n, t_n), F(v_n, t_n))\} \\ = & max\{G(gr, gr, gt_n), G(gs, gs, gv_n)\}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\psi(G(gr, gr, gt_{n+1})) \leq & \alpha\left(max\{G(gr, gr, gt_n), G(gs, gs, gv_n)\}\right) \\ & - \beta\left(max\{G(gr, gr, gt_n), G(gs, gs, gv_n)\}\right). \qquad (4.43)\end{aligned}$$

Similarly, we have

$$\begin{aligned}\psi(G(gs, gs, gv_{n+1})) \leq & \alpha\left(max\{G(gr, gr, gt_n), G(gs, gs, gv_n)\}\right) \\ & - \beta\left(max\{G(gr, gr, gt_n), G(gs, gs, gv_n)\}\right). \qquad (4.44)\end{aligned}$$

Therefore, from (4.43) and (4.44), we get

$$\begin{aligned}\psi(\gamma_{n+1}) &= \psi(max\{G(gr, gr, gt_{n+1})), G(gs, gs, gv_{n+1})\} \\ &= max\{\psi(G(gr, gr, gt_{n+1}))), \psi(G(gs, gs, gv_{n+1}))\} \\ &\leq \alpha\left(max\{G(gr, gr, gt_n), G(gs, gs, gv_n)\}\right) \\ &\quad - \beta\left(max\{G(gr, gr, gt_n), G(gs, gs, gv_n)\}\right) \\ &= \alpha\left(\gamma_n\right) - \beta\left(\gamma_n\right) = \alpha\left(0\right) - \beta\left(0\right). \qquad (4.45)\end{aligned}$$

As, from the assumptions of ψ, β and α, we can deduce $\gamma_{n+1} = 0$. Reiterating this procedure, we can prove that $\gamma_m = 0$, for each $n \leq m$. So, $\lim_{n\to\infty} \gamma_n = 0$.

Now, let $\gamma_n \neq 0$, for all n and let $\gamma_n < \gamma_{n+1}$, for any n.

Since ψ is an altering distance function, from (4.45)

$$
\begin{aligned}
\psi(\gamma_n) &= \psi(max\{G(gr, gr, gu_n), G(gs, gs, gv_n)\}) \\
&< \psi(\gamma_{n+1}) \\
&= \psi(max\{G(gr, gr, gu_{n+1})), G(gs, gs, gv_{n+1})\} \\
&= max\{\psi(G(gr, gr, gu_{n+1}))), \psi(G(gs, gs, gv_{n+1}))\} \\
&\leq \alpha\,(max\{G(gr, gr, gu_n), G(gs, gs, gv_n)\}) \\
&\quad - \beta\,(max\{G(gr, gr, gu_n), G(gs, gs, gv_n)\}) \\
&= \alpha\,(\gamma_n) - \beta\,(\gamma_n)
\end{aligned}
$$

Thus, $\gamma_n = 0$, a contraction.

Therefore, $\gamma_{n+1} \leq \gamma_n$, for each $n \geq 1$. Now, if we continue as in Theorem 4.4, we can prove that

$$\lim_{n\to\infty} max\{G(gr, gr, gu_n), G(gs, gs, gv_n)\} = 0. \tag{4.46}$$

So, $\{gu_n\} \to gr$ and $\{gv_n\} \to gs$.

In similar argument, we can prove that

$$\lim_{n\to\infty} max\{G(g\acute{r}, g\acute{r}, gu_n), G(g\acute{s}, g\acute{s}, gv_n)\} = 0 \tag{4.47}$$

That is, $\{gu_n\} \to g\acute{r}$ and $\{gv_n\} \to g\acute{s}$. Finally, as the limit is unique, $gr = g\acute{r}$ and $gs = g\acute{s}$.

As $gr = F(r, s)$ and $gs = F(s, r)$, by commutativity of g and F, we get

$$g(gr) = g(F(r, s)) = F(gr, gs) \text{ and } g(gs) = g(F(s, r)) = F(gs, gr) \tag{4.48}$$

Denote $gr = x$ and $gs = w$. Then, from(4.48), it follows that

$$gx = F(x, w) \text{ and } gw = F(w, x). \tag{4.49}$$

Therefore, (x, w) is a coupled coincidence point of F and g. Then, from (4.42) with $\acute{r} = x$ and $\acute{s} = w$, it follows that $gx = gr$ and $gw = gs$, that is,

$$gx = x \text{ and } gw = w \tag{4.50}$$

Thus, from (4.49) and (4.50), we get $x = gx = F(x, w)$ and $w = gw = F(w, x)$. Therefore, (x, w) is a coupled common fixed point of g and F.

To show the uniqueness, presume that (h, t) is another coupled common fixed point of g and F. Then $h = gh = F(h, t)$ and $t = gt = F(t, h)$. As (h, t) is a coupled coincidence point of F and g, we get $gh = gr = x$ and $gt = gs = w$. Thus, $h = gh = gz = z$ and $t = gt = gw = w$. Therefore, the coupled fixed point is unique, this completes the proof.

4.3 Application to Integral Equations

In this segment, a useful application of Corollary 4.6 is presented. By utilizing the Corollary 4.6, the existence and uniqueness of solutions of a non-linear integral equation has been presented.

Consider an integral equation of the following type:

$$r(m) = \int_0^1 (k_1(m,b) + k_2(m,b))(f(b,r(b)) + g(b,r(b)))db + a(m),\ m \in [0,1]. \tag{4.51}$$

We will analyze equation (4.51) under the following accompanying presumptions:

(1) k_i: $[0,1] \times [0,1] \to \mathbb{R}(i = 1,2)$ are continuous and $k_1(m,b) \geq 0$ and $k_2(m,b) \leq 0$.

(2) $a \in C[0,1]$.

(3) g, f: $[0,1] \times \mathbb{R} \to \mathbb{R}$ be two continuous functions.

(4) There are constants $\lambda, \mu > 0$, for all $m \in [0,1]$, $r, s \in \mathbb{R}$ and $r \geq s$,

$$0 \leq f(m,r) - f(m,s) \leq \lambda(r-s),\ -\mu(r-s) \leq g(m,r) - g(m,s) \leq 0.$$

(5) There exist $\gamma, \delta \in C[0,1]$, then

$$\begin{aligned}\gamma(m) \leq & \int_0^1 k_1(m,b)(f(b,\gamma(b)) + g(b,\delta(b)))db \\ & + \int_0^1 k_2(m,b)(f(b,\delta(b)) + g(b,\gamma(b)))db + a(m), \\ \delta(m) \geq & \int_0^1 k_1(m,b)(f(b,\delta(b)) + g(b,\gamma(b)))db \\ & + \int_0^1 k_2(m,b)(f(b,\gamma(b))) + g(b,\delta(b))db + a(m).\end{aligned}$$

(6) $3.max(\lambda,\mu) \parallel k_1 - k_2 \parallel_\infty \leq \frac{1}{2}$, we have

$$\parallel k_1 - k_2 \parallel_\infty = sup\{(k_1(m,b) - k_2(m,b)) \colon m, b \in [0,1]\}.$$

Let $U = C[0,1]$ be a space of continuous functions defined on [0, 1]. Define G: $U \times U \times U \to \mathbb{R}^+$ by

$$G(u_1,u_2,u_3) = \sup_{m\in[0,1]} \mid u_1(m) - u_2(m) \mid + \sup_{m\in[0,1]} \mid u_2(m) - u_3(m) \mid$$
$$+ \sup_{m\in[0,1]} \mid u_3(m) - u_1(m) \mid,$$

for each $u_1, u_2, u_3 \in U$. Then (U, G) is a G-complete metric space.

This space can be endowed with a partial order as follow

$$r, s \in C[0,1],\ r \leq s \iff r(m) \leq s(m),\ \text{for any } m \in [0,1].$$

Evidently, if $U \times U$ we consider the order as follow

$$(r,s),(w,z) \in U \times U,\ (r,s) \leq (w,z) \iff r \leq w \text{ and } s \geq z,$$

Therefore, for some $r, s \in U$ we have that $max(r,s), min(r,s) \in U$, condition (4.41) is fulfilled.

Furthermore, in [106] it is demonstrated that $(C[0,1], \leq)$ fulfills hypothesis (1).

We now formulate our outcome.

Theorem 4.10. *Under hypothesis (1)-(6), equation* (4.51) *has a unique solution in* $C[0,1]$.

Proof. Presume $F\colon U \times U \to U$ be a mapping given by

$$F(r,s)(m) = \int_0^1 k_1(m,b)(f(b,r(b)) + g(b,s(b)))db$$
$$+ \int_0^1 k_2(m,b)(f(b,s(b)) + g(b,r(b)))db + a(m),\ \text{for } m \in [0,1].$$

By virtuousness of our hypothesis, F is well defined (for $r, s \in U$ then $F(r,s) \in U$).

Initially, we show that F possesses the property of mixed monotone.

For $r_1 \leq r_2$ and $m \in [0,1]$, we get

$$\begin{aligned} F(r_1,s)(m) - F(r_2,s)(m) &= \int_0^1 k_1(m,b)(f(b,r_1(b)) + g(b,s(b)))db \\ &+ \int_0^1 k_2(m,b)(f(b,s(b)) + g(b,r_1(b)))db + a(m) \\ &- \int_0^1 k_1(m,b)(f(b,r_2(b)) + g(b,s(b)))db \\ &- \int_0^1 k_2(m,b)(f(b,s(b)) + g(b,r_2(b)))db - a(m) \\ &= \int_0^1 k_1(m,b)(f(b,r_1(b)) - f(b,r_2(b)))db \end{aligned}$$

$$+\int_0^1 k_2(m,b)(g(b,r_1(b))-g(b,r_2(b)))db. \tag{4.52}$$

Consider $r_1 \leq r_2$ and our hypothesis,

$$f(b,r_1(b))-f(b,r_2(b)) \leq 0,$$
$$g(b,r_1(b))-g(b,r_2(b)) \geq 0,$$

and from (4.52) we obtain

$$F(r_1,s)(m)-F(r_2,s)(m) \leq 0$$

and this shows that $F(r_1,s) \leq F(r_2,s)$.

Similarly, if $s_1 \geq s_2$ and $m \in [0,1]$, we have $F(r,s_1) \leq F(r,s_2)$. Thus, F possesses the property of mixed monotone.

In what follows, we estimate the quantity $G(F(r,s),F(t,v),F(z,w))$ for all $r,s,t,v,z,w \in U$, with $r \geq t \geq z$ and $s \leq v \leq w$.

Indeed, as F possesses the property of mixed monotone, $F(r,s) \geq F(t,v) \geq F(z,w)$ and we can attain

$$\begin{aligned}
G(F(r,s),F(t,v),F(z,w)) &= \sup_{m\in[0,1]} \mid F(r,s)(m)-F(t,v)(m) \mid \\
&\quad + \sup_{m\in[0,1]} \mid F(t,v)(m)-F(z,w)(m) \mid \\
&\quad + \sup_{m\in[0,1]} \mid F(z,w)(m)-F(r,s)(m) \mid \\
&= \sup_{m\in[0,1]} (F(r,s)(m)-F(t,v)(m)) \\
&\quad + \sup_{m\in[0,1]} (F(t,v)(m)-F(z,w)(m)) \\
&\quad + \sup_{m\in[0,1]} (F(r,s)(m)-F(z,w)(m)) \\
&= \sup_{m\in[0,1]} \Big[\int_0^1 k_1(m,b)(f(b,r(b))+g(b,s(b)))db \\
&\qquad + \int_0^1 k_2(m,b)(f(b,s(b))+g(b,r(b)))db + a(m) \\
&\qquad - \int_0^1 k_1(m,b)(f(b,t(b))+g(b,v(b)))db \\
&\qquad - \int_0^1 k_2(m,b)(f(b,v(b))+g(b,t(b)))db - a(m)\Big] \\
&\quad + \sup_{m\in[0,1]} \Big[\int_0^1 k_1(m,b)(f(b,t(b))+g(b,v(b)))db
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^1 k_2(m,b)(f(b,v(b)) + g(b,t(b)))db + a(m) \\
&\quad - \int_0^1 k_1(m,b)(f(b,z(b)) + g(b,w(b)))db \\
&\quad - \int_0^1 k_2(m,b)(f(b,w(b)) + g(b,z(b)))db - a(m)\Big] \\
&+ \sup_{m\in[0,1]} \Big[\int_0^1 k_1(m,b)(f(b,r(b)) + g(b,s(b)))db \\
&\quad + \int_0^1 k_2(m,b)(f(b,s(b)) + g(b,r(b)))db + a(m) \\
&\quad - \int_0^1 k_1(m,b)(f(b,z(b)) + g(b,w(b)))db \\
&\quad - \int_0^1 k_2(m,b)(f(b,w(b)) + g(b,z(b)))db - a(m)\Big] \\
&= \sup_{m\in[0,1]} \Big[\int_0^1 k_1(m,b)[(f(b,r(b)) - f(b,t(b))) - (g(b,v(b)) - g(b,s(b)))]db \\
&\quad - \int_0^1 k_2(m,b)[(f(b,v(b)) - f(b,s(b))) - (g(b,r(b)) - g(b,t(b)))]db\Big] \\
&\quad + \sup_{m\in[0,1]} \Big[\int_0^1 k_1(m,b)[(f(b,t(b)) - f(b,z(b))) - (g(b,w(b)) - g(b,v(b)))]db \\
&\quad - \int_0^1 k_2(m,b)[(f(b,w(b)) - f(b,v(b))) - (g(b,t(b)) - g(b,z(b)))]db\Big] \\
&\quad + \sup_{m\in[0,1]} \Big[\int_0^1 k_1(m,b)[(f(b,r(b)) - f(b,z(b))) - (g(b,w(b)) - g(b,s(b)))]db \\
&\quad - \int_0^1 k_2(m,b)[(f(b,w(b)) - f(b,s(b))) - (g(b,r(b)) - g(b,z(b)))]db\Big].
\end{aligned} \tag{4.53}
$$

By our hypothesis (that $r \geq t \geq z$ and $s \leq v \leq w$)

$$
\begin{aligned}
&f(b,r(b)) - f(b,t(b)) \leq \lambda\,(r(b) - t(b)), \quad g(b,v(b)) - g(b,s(b)) \geq -\mu\,(v(b) - s(b)), \\
&f(b,v(b)) - f(b,s(b)) \leq \lambda\,(v(b) - s(b)), \quad g(b,r(b)) - g(b,t(b)) \geq -\mu\,(r(b) - t(b)), \\
&f(b,t(b)) - f(b,z(b)) \leq \lambda\,(t(b) - z(b)), \quad g(b,w(b)) - g(b,v(b)) \geq -\mu\,(w(b) - v(b)), \\
&f(b,z(b)) - f(b,v(b)) \leq \lambda\,(z(b) - v(b)), \quad g(b,t(b)) - g(b,z(b)) \geq -\mu\,(t(b) - z(b)), \\
&f(b,r(b)) - f(b,z(b)) \leq \lambda\,(r(b) - z(b)), \quad g(b,w(b)) - g(b,s(b)) \geq -\mu\,(w(b) - s(b)), \\
&f(b,w(b)) - f(b,s(b)) \leq \lambda\,(w(b) - s(b)), \quad g(b,r(b)) - g(b,z(b)) \geq -\mu\,(r(b) - z(b)).
\end{aligned}
$$

Consider these last inequalities and (4.53), we have

$$
\begin{aligned}
G(F(r,s),F(t,v),F(z,w)) \leq \sup_{m\in[0,1]} \Big[&\int_0^1 k_1(m,b)[\lambda(r(b) - t(b)) + \mu(v(b) - s(b))]db \\
&+ \int_0^1 (-k_2(m,b))[\lambda(v(b) - s(b)) + \mu(r(b) - t(b))]db\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{m\in[0,1]} \Big[\int_0^1 k_1(m,b)[\lambda(t(b)-z(b)) + \mu(w(b)-v(b))]db \\
&\qquad + \int_0^1 (-k_2(m,b))[\lambda(w(b)-v(b)) + \mu(t(b)-z(b))]db \Big] \\
&+ \sup_{m\in[0,1]} \Big[\int_0^1 k_1(m,b)[\lambda(r(b)-z(b)) + \mu(w(b)-s(b))]db \\
&\qquad + \int_0^1 (-k_2(m,b))[\lambda(w(b)-s(b)) + \mu(r(b)-z(b))]db \Big] \\
&= max(\lambda,\mu) \sup_{m\in[0,1]} \Big[\int_0^1 (k_1(m,b)-k_2(t,b))(r(b)-t(b))db \\
&\qquad + \int_0^1 (k_1(m,b)-k_2(m,b))(v(b)-s(b))db \\
&\qquad + \int_0^1 (k_1(m,b)-k_2(m,b))(t(b)-z(b))db \\
&\qquad + \int_0^1 (k_1(m,b)-k_2(m,b))(w(b)-v(b))db \\
&\qquad + \int_0^1 (k_1(m,b)-k_2(m,b))(r(b)-z(b))db \\
&\qquad + \int_0^1 (k_1(m,b)-k_2(m,b))(w(b)-s(b))db \Big].
\end{aligned} \tag{4.54}
$$

Defining

$$
\begin{aligned}
&(1) = \int_0^1 (k_1(m,b)-k_2(m,b))(r(b)-t(b))db, \quad (2) = \int_0^1 (k_1(m,b)-k_2(m,b))(v(b)-s(b))db, \\
&(3) = \int_0^1 (k_1(m,b)-k_2(m,b))(t(b)-z(b))db, \quad (4) = \int_0^1 (k_1(m,b)-k_2(m,b))(w(b)-v(b))db, \\
&(5) = \int_0^1 (k_1(m,b)-k_2(m,b))(r(b)-z(b))db, \quad (6) = \int_0^1 (k_1(m,b)-k_2(m,b))(w(b)-s(b))db.
\end{aligned}
$$

and applying the Cauchy-Schwartz inequality in (1), we acquire

$$
(1) \leq \Big(\int_0^1 (k_1(m,b)-k_2(m,b))^2 db \Big)^{\frac{1}{2}} . \Big(\int_0^1 (r(b)-t(b))^2 db \Big)^{\frac{1}{2}} \leq \parallel k_1 - k_2 \parallel_\infty .G(r,t,z). \tag{4.55}
$$

In similar way, we can attain the subsequent estimate for:

$$
(2) \leq \parallel k_1 - k_2 \parallel_\infty . G(s,v,w), \ (3) \leq \parallel k_1 - k_2 \parallel_\infty . G(r,t,z), \tag{4.56}
$$

$$
(4) \leq \parallel k_1 - k_2 \parallel_\infty . G(s,v,w), \ (5) \leq \parallel k_1 - k_2 \parallel_\infty . G(r,t,z), \tag{4.57}
$$

$$
(6) \leq \parallel k_1 - k_2 \parallel_\infty . G(s,v,w). \tag{4.58}
$$

from (4.54)-(4.58), we have

$$
G(F(r,s), F(t,v), F(z,w)) \leq max(\lambda,\mu) \parallel k_1 - k_2 \parallel_\infty [G(r,t,z) + G(s,v,w)
$$

$$
\begin{aligned}
&+ G(r,t,z) + G(s,v,w) + G(r,t,z) + G(s,v,w)] \\
&= 3.max(\lambda,\mu) \parallel k_1 - k_2 \parallel_\infty [G(r,t,z) + G(s,v,w)] \\
&\leq 3.max(\lambda,\mu) \parallel k_1 - k_2 \parallel_\infty [(M((r,s),(t,v),(z,w))]. \qquad (4.59)
\end{aligned}
$$

from (4.59) and hypothesis (6.), we get

$$
G(F(r,s),F(t,v),F(z,w)) \leq \frac{1}{2}(M((r,s),(t,v),(z,w))),
$$

where,

$$
\begin{aligned}
M((r,s),(t,v),(z,w)) = max\{&G(r,t,z), G(s,v,w), G(r,F(r,s),F(r,s)), \\
&G(t,F(t,v),F(t,v)), G(w,F(w,z),F(w,z)), \\
&G(s,F(s,r),F(s,r)), G(v,F(v,t),F(v,t)), \\
&G(z,F(z,w),F(z,w))\}
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
G(F(r,s),F(t,v),F(z,w)) \leq\ &(M((r,s),(t,v),(z,w))^2 \\
&- [(M((r,s),(t,v),(z,w))^2 \\
&- \frac{1}{2}(M((r,s),(t,v),(z,w)))]. \qquad (4.60)
\end{aligned}
$$

Put $\psi(r) = r$, $\alpha\,(r) = r^2$ and $\beta\,(r) = r^2 - \frac{1}{2}r$. Obviously, ψ, α and β are satisfying conditions of Corollary 4.6 and from (4.60) we get

$$
\begin{aligned}
\psi(G(F(r,s),F(t,v,F(z,w))) \leq\ &\alpha\,(M((r,s),(t,v),(z,w))) \\
&- \beta\,(M((r,s),(t,v),(z,w)))),
\end{aligned}
$$

where,

$$
\begin{aligned}
M((r,s),(t,v),(z,w)) = max\{&G(r,t,z), G(s,v,w), G(r,F(r,s),F(r,s)), \\
&G(t,F(t,v),F(t,v)), G(w,F(w,z),F(w,z)), \\
&G(s,F(s,r),F(s,r)), G(v,F(v,t),F(v,t)), \\
&G(z,F(z,w),F(z,w))\}
\end{aligned}
$$

This shows that the mapping F fulfill the contractive condition arising in Corollary 4.6.

Lastly, let γ, δ be the functions appearing in hypothesis (5); then, by (5), we have $\gamma \leq F(\gamma,\delta), \delta \geq F(\delta,\gamma)$.

Applying Corollary 4.6, we deduce the existence of $(r,s) \in U \times U$ such that $r = F(r,s)$ and $s = F(s,r)$, that is, (r,s) is a solution of the system (4.51).

Chapter 5

Coupled Coincidence Point Results in Partially Ordered Metric Spaces by Altering Distances

In this chapter, certain coupled coincidence point theorems have been obtained for mappings possesses the property of mixed g-monotone in partially ordered metric spaces involving altering distance functions. This chapter has been divided into various sections. Section 5.1 deals with the preliminaries related to coupled coincidence points. In section 5.2, some coupled coincidence results have been established by using altering distance function. In section 5.3 we present an application to integral equations.

5.1 Introduction

Bhaskar and Lakshmikantham [21] presented an idea of mixed monotone mappings and coupled fixed points and showed the certain coupled fixed point and fixed point results. Further, they studied their results on a first-order differential equation with periodic boundary value problems. Later on, Ciric and Lakshmikantham [42] presented a new notion of a mixed g-monotone mapping and coupled coincidence point. They established certain coupled coincidence results by utilizing the property of a mixed g-monotone in a partially ordered complete metric space. In recent years, numerous researchers have attained coupled fixed point and coupled coincidence point results for different classes of mappings on abstract metric spaces such as complete metric spaces, partially ordered metric spaces, cone metric spaces and G-metric spaces (see [18, 34, 41, 93, 127, 135]).

In 2010, Abbas [3] presented a new perception of w and w^*-compatible mappings. Abbas [4] used this idea to show an uniqueness theorem of coupled fixed point for contractive maps in G-metric spaces.

Ciric and Lakshmikantham gave the succeeding definition in [42].

Definition 5.1. [42] Suppose $(U, \leq)$ be a partially ordered set and a mapping $F\colon U \times U \to U$ is said to have the property of mixed g-monotone if $F(s, v)$ is monotone g-nondecreasing in s and is monotone g-nonincreasing in v, that is, for any $s, v \in U$,

$$s_1, s_2 \in U, gs_1 \leq gs_2 \implies F(s_1, v) \leq F(s_2, v),$$
$$v_1, v_2 \in U, gv_1 \leq gv_2 \implies F(s, v_2) \leq F(s, v_1).$$

Definition 5.2. [42] An element $(s, v) \in U \times U$ is said to be coupled coincidence point of the mappings $F\colon U \times U \to U$ and $g\colon U \to U$ if $F(s, v) = gs,\ F(v, s) = gv$.

Definition 5.3. [42] Suppose U be a non-empty set. We say that the mappings $g\colon U \to U$ and $F\colon U \times U \to U$ are commutative if $gF(s, v) = F(gs, gv)$, for all $s, v \in U$.

Khan [89] presented the conception of an altering distance function as follows.

Definition 5.4. [89]. A function $\phi\colon [0, \infty) \to [0, \infty)$ is termed as an altering distance function if it satisfy following axioms:

1. ϕ is nondecreasing and continuous.
2. $\phi(z) = 0$ iff $z = 0$.

Abbas [3] acquaint with perception of w and w^*-compatible mappings as follows.

Definition 5.5. [3] Mappings $g\colon U \to U$ and $F\colon U \times U \to U$ are called

(1) w-compatible if $F(gs, gv) = g(F(s, v))$ whenever $gs = F(s, v)$ and $gv = F(v, s)$;

(2) w^*-compatible if $F(gs, gs) = g(F(s, s))$ whenever $gs = F(s, s)$.

5.2 Coupled Coincidence Point Theorems for Mappings Having the Mixed g-Monotone Property

Theorem 5.6. *Suppose $(U, \leq, d)$ be an ordered complete metric space. Consider that $g\colon U \to U$ and $F\colon U \times U \to U$ be continuous mappings and F acquires the property of a* *mixed g-monotone and g commutes with F, satisfy*

$$\varphi(d(F(r,s),F(t,v))) \leq \varphi(M(r,s,t,v)) - \phi(M(r,s,t,v)) + \theta(N(r,s,t,v)) \tag{5.1}$$

where,

$$\begin{aligned} M(r,s,t,v) = max\{&d(gr,gt), d(gs,gv), d(F(r,s),gr), d(F(t,v),gt), \\ &d(F(s,r),gs), d(F(v,t),gv)\}, \end{aligned}$$

and

$$N(r,s,t,v) = min\{d(F(r,s),gt), d(F(t,v),gr), d(F(r,s),gr), d(F(t,v),gt)\},$$

for each $r,t,v,s \in U$ with $gr \geq gt$ and $gv \geq gs$, here ϕ and φ are altering distance functions and θ: $[0,\infty) \to [0,\infty)$ is a continuous function, $\theta(z) = 0$ iff $z = 0$. Assume that $F(U \times U) \subseteq g(U)$ and furthermore for each $r_0, s_0 \in U$ with $F(r_0,s_0) \geq gr_0$ and $F(s_0,r_0) \leq gs_0$, then g and F possesses a coupled coincidence point in U.

Proof. Suppose that $r_0, s_0 \in U$ with $F(r_0,s_0) \geq gr_0$, $F(s_0,r_0) \leq gs_0$. As $F(U \times U) \subseteq g(U)$, we can take $r_1, s_1 \in U$ then $gr_1 = F(r_0,s_0)$ and $gs_1 = F(s_0,r_0)$.

Again, we can take $r_2, s_2 \in U$ then $gr_2 = F(r_1,s_1)$ and $gs_2 = F(s_1,r_1)$. As F acquires the property of mixed g-monotone, then we have $gr_0 \leq gr_1 \leq gr_2$ and $gs_2 \leq gs_1 \leq gs_0$. Persistent this procedure, we can create two sequences $\{r_n\}$ and $\{s_n\}$ in U such that

$$\begin{aligned} gr_n &= F(r_{n-1},s_{n-1}) \leq gr_{n+1} = F(r_n,s_n), \\ gs_{n+1} &= F(s_n,r_n) \leq gs_n = F(s_{n-1},r_{n-1}). \end{aligned}$$

If, for some integer n, we have $(gr_{n+1}, gs_{n+1}) = (gr_n, gs_n)$, thus $F(r_n,s_n) = gr_n$ and $F(s_n,r_n) = gs_n$, (r_n,s_n) is a coincidence point of g and F. So, now we take $(gr_{n+1}, gs_{n+1}) \neq (gr_n, gs_n)\ \forall\ n \in \mathbb{N}$, we accept that one of $gr_{n+1} \neq gr_n$ or $gs_{n+1} \neq gs_n$.

For each $n \in \mathbb{N}$, employing the inequality (5.1), we have

$$\begin{aligned} \varphi(d(gr_{n+1},gr_n)) &= \varphi(d(F(r_n,s_n),F(r_{n-1},s_{n-1}))) \\ &\leq \varphi(M(r_n,s_n,r_{n-1},s_{n-1})) - \phi(M(r_n,s_n,r_{n-1},s_{n-1})) \\ &\quad + \theta(N(r_n,s_n,r_{n-1},s_{n-1})), \\ \varphi(d(gs_{n+1},gs_n)) &= \varphi(d(F(s_n,r_n),F(s_{n-1},r_{n-1}))) \\ &\leq \varphi(M(s_n,r_n,s_{n-1},r_{n-1})) - \phi(M(s_n,r_n,s_{n-1},r_{n-1})) \\ &\quad + \theta(N(s_n,r_n,s_{n-1},r_{n-1})), \end{aligned}$$

where,

$$\begin{aligned}
M(r_n, s_n, r_{n-1}, s_{n-1}) &= M(s_n, r_n, s_{n-1}, r_{n-1}) \\
&= max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1}), d(F(r_{n-1}, s_{n-1}), gr_{n-1}), \\
&\qquad d(F(r_n, s_n), gr_n), d(F(s_n, r_n), gs_n), d(F(s_{n-1}, r_{n-1}), gs_{n-1})\} \\
&= max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1}), d(gr_{n+1}, gr_n), d(gs_{n+1}, gs_n)\}.
\end{aligned}$$

and

$$N(r_n, s_n, r_{n-1}, s_{n-1}) = min\{d(gr_{n+1}, gr_{n-1}), d(gr_n, gr_n), d(gr_{n+1}, gr_n), d(gr_n, gr_{n-1})\} = 0.$$

Also,

$$N(s_n, r_n, s_{n-1}, r_{n-1}) = min\{d(gs_{n+1}, gs_{n-1}), d(gs_n, gs_n), d(gs_{n+1}, gs_n), d(gs_n, gs_{n-1})\} = 0.$$

Now, let us consider three cases.
Case 1: $M(r_n, s_n, r_{n-1}, s_{n-1}) = max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\}$.
We attain

$$\begin{aligned}
\varphi(d(gr_{n+1}, gr_n)) &\le \varphi(max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\}) \\
&\qquad - \phi(max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\}), \qquad (5.2) \\
\varphi(d(gs_{n+1}, s_n)) &\le \varphi(max\{d(gs_n, gs_{n-1}), d(gr_n, gr_{n-1})\}) \\
&\qquad - \phi(max\{d(gs_n, gs_{n-1}), d(gr_n, gr_{n-1})\}). \qquad (5.3)
\end{aligned}$$

Case 2: $M(r_n, s_n, r_{n-1}, s_{n-1}) = d(gr_{n+1}, gr_n)$.
We claim that $M(r_n, s_n, r_{n-1}, s_{n-1}) = d(gr_{n+1}, gr_n) = 0$.
In fact if $d(gr_{n+1}, gr_n) \neq 0$, then

$$\varphi(d(gr_{n+1}, gr_n)) \le \varphi(d(gr_{n+1}, gr_n)) - \phi(d(gr_{n+1}, gr_n)) < \varphi(d(gr_{n+1}, gr_n)) \text{ as } \phi \ge 0.$$

which is a contradiction. Since $M(r_n, s_n, r_{n-1}, s_{n-1}) = 0$. Then it is obvious that (5.2) and (5.3) hold.

Case 3: $M(r_n, s_n, r_{n-1}, s_{n-1})) = d(gs_{n+1}, gs_n)$.
Similar to the proof of Case 2, one can also show that (5.2) and (5.3) hold.
Now, by (5.2) and (5.3), $\forall\, n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\varphi(d(gr_{n+1}, gr_n)) \le &\varphi(max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\}) \\
&- \phi(max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\}).
\end{aligned}$$

As $\phi \ge 0$.

$$\varphi(d(gr_{n+1}, gr_n)) \le \varphi(max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\}),$$

and utilizing the way that φ is non-decreasing, we obtain

$$d(gr_{n+1}, gr_n) \leq max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\}. \tag{5.4}$$

Similarly, we get

$$\begin{aligned}\varphi(d(gs_{n+1}, gs_n)) &\leq \varphi(max\{d(gs_n, gs_{n-1}), d(gr_n, gr_{n-1})\}) \\ &\quad - \phi(max\{d(gs_n, gs_{n-1}), d(gr_n, gr_{n-1})\}) \\ &\leq \varphi(max\{d(gs_n, gs_{n-1}), d(gr_n, gr_{n-1})\}),\end{aligned}$$

and consequently

$$d(gs_{n+1}, gs_n) \leq max\{d(gs_n, gs_{n-1}), d(gr_n, gr_{n-1})\}, \tag{5.5}$$

by (5.4) and (5.5), we have

$$max\{d(gr_{n+1}, gr_n), d(gs_{n+1}, gs_n)\} \leq max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\},$$

and thus, the sequence $max\{d(gr_{n+1}, gr_n), d(gs_{n+1}, gs_n)\}$ is non-negative decreasing. Thus we can find $a \geq 0$, then

$$\lim_{n\to\infty} max\{d(gr_{n+1}, gr_n), d(gs_{n+1}, gs_n)\} = a. \tag{5.6}$$

It is effortlessly observed that if φ: $[0, \infty) \to [0, \infty)$ is increasing, $\varphi(max(p, q)) = max(\varphi(p), \varphi(q))$ for $p, q \in [0, \infty)$. Consider this and (5.2) and (5.3), we have

$$\begin{aligned}max\{\varphi(d(gr_{n+1}, gr_n)), \varphi(d(gs_{n+1}, gs_n))\} &= \varphi(max\{d(gr_{n+1}, gr_n), d(gs_{n+1}, gs_n)\}) \\ \leq \varphi(max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\}) &- \phi(max\{d(gr_n, gr_{n-1}), d(gs_n, gs_{n-1})\}).\end{aligned} \tag{5.7}$$

Letting $n \to \infty$ in (5.7) and consider (5.6), we have $\varphi(a) \leq \varphi(a) - \phi(a) \leq \varphi(a)$, implies $\phi(a) = 0$. As ϕ is an altering distance function, $a = 0$ thus

$$\lim_{n\to\infty} max\{d(gr_{n+1}, gr_n), d(gs_{n+1}, gs_n)\} = 0. \tag{5.8}$$

Thus $\lim_{n\to\infty} d(gr_{n+1}, gr_n) = \lim_{n\to\infty} d(gs_{n+1}, gs_n) = 0$.

Next, we claim that $\{gr_n\}, \{gs_n\}$ are Cauchy sequences.

We will prove that for each $\varepsilon > 0$, we can find $\lambda \in \mathbb{N}$ if $n, m \geq \lambda$,

$$max\{d(gr_{m(\lambda)}, gr_{n(\lambda)}), d(gs_{m(\lambda)}, gs_{n(\lambda)})\} < \varepsilon.$$

Presuppose the above statement is not true.

At that point, there exists an $\varepsilon > 0$ and sequences $\{gr_{m(\lambda)}\}, \{gr_{n(\lambda)}\}$ with $n(\lambda) > m(\lambda) > \lambda$ such that

$$max\{d(gr_{m(\lambda)}, gr_{n(\lambda)}), d(gs_{m(\lambda)}, gs_{n(\lambda)})\} \geq \varepsilon. \tag{5.9}$$

Furthermore, we can take $n(\lambda)$ corresponding to $m(\lambda)$ in a manner, it is least integer with $n(\lambda) > m(\lambda)$ and satisfying (5.9). Then

$$max\{d(gr_{m(\lambda)}, gr_{n(\lambda)-1}), d(gs_{m(\lambda)}, gs_{n(\lambda)-1})\} < \varepsilon. \tag{5.10}$$

From triangle inequality

$$d(gr_{n(\lambda)}, gr_{m(\lambda)}) \leq d(gr_{n(\lambda)}, gr_{n(\lambda)-1}) + d(gr_{n(\lambda)-1}, gr_{m(\lambda)}). \tag{5.11}$$

$$d(gs_{n(\lambda)}, gs_{m(\lambda)}) \leq d(gs_{n(\lambda)}, gs_{n(\lambda)-1}) + d(gs_{n(\lambda)-1}, gs_{m(\lambda)}). \tag{5.12}$$

From (5.11) and (5.12), we have

$$\begin{aligned} &max\{d(gr_{n(\lambda)}, gr_{m(\lambda)}), d(gs_{n(\lambda)}, gs_{m(\lambda)})\} \\ &\quad \leq max\{d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gs_{n(\lambda)}, gs_{n(\lambda)-1})\} \\ &\qquad + max\{d(gr_{n(\lambda)-1}, gr_{m(\lambda)}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)})\}. \end{aligned} \tag{5.13}$$

From (5.9), (5.10) and (5.13), we get

$$\begin{aligned} \varepsilon &\leq max\{d(gr_{n(\lambda)}, gr_{m(\lambda)}), d(gs_{n(\lambda)}, gs_{m(\lambda)})\} \\ &\leq max\{d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gs_{n(\lambda)}, gs_{n(\lambda)-1})\} + \varepsilon. \end{aligned} \tag{5.14}$$

Letting $\lambda \to \infty$ in (5.14) and taking into account (5.8) we have

$$\lim_{\lambda\to\infty} max\{d(gr_{n(\lambda)}, gr_{m(\lambda)}), d(gs_{n(\lambda)}, gs_{m(\lambda)})\} = \varepsilon. \tag{5.15}$$

Again, from the triangle inequality, we have

$$d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}) \leq d(gr_{n(\lambda)-1}, gr_{m(\lambda)}) + d(gr_{m(\lambda)}, gr_{m(\lambda)-1}), \tag{5.16}$$

$$d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1}) \leq d(gs_{n(\lambda)-1}, gs_{m(\lambda)}) + d(gs_{m(\lambda)}, s_{m(\lambda)-1}). \tag{5.17}$$

From (5.16) and (5.17), we have

$$\begin{aligned} &max\{d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1})\} \\ &\quad \leq max\{d(gr_{n(\lambda)-1}, gr_{m(\lambda)}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)})\} \\ &\qquad + max\{d(gr_{m(\lambda)}, gr_{m(\lambda)-1}), d(gs_{m(\lambda)}, gs_{m(\lambda)-1})\} \end{aligned} \tag{5.18}$$

From (5.10), we have

$$\begin{aligned} &max\{d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1})\} \\ &\quad \leq max\{d(gr_{m(\lambda)}, gr_{m(\lambda)-1}), d(gs_{m(\lambda)}, gs_{m(\lambda)-1})\} + \varepsilon \end{aligned} \tag{5.19}$$

Using the triangle inequality, we have

$$d(gr_{n(\lambda)}, gr_{m(\lambda)}) \leq d(gr_{n(\lambda)}, gr_{n(\lambda)-1}) + d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}) + d(gr_{m(\lambda)-1}, gr_{m(\lambda)}), \tag{5.20}$$

and

$$d(gs_{n(\lambda)}, gs_{m(\lambda)}) \leq d(gs_{n(\lambda)}, gs_{n(\lambda)-1}) + d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1}) + d(gs_{m(\lambda)-1}), gs_{m(\lambda)}). \tag{5.21}$$

From (5.9), (5.20) and (5.21), we get

$$\begin{aligned} \varepsilon &\leq max\{d(gr_{n(\lambda)}, gr_{m(\lambda)}), d(gs_{n(\lambda)}, gs_{m(\lambda)})\} \\ &\leq max\{d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gs_{n(\lambda)}, gs_{n(\lambda)-1})\} \\ &\quad + max\{d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1})\} \\ &\quad\quad + max\{d(gr_{m(\lambda)-1}, gr_{m(\lambda)}), d(gs_{m(\lambda)-1}), gs_{m(\lambda)})\}. \end{aligned} \tag{5.22}$$

From (5.19) and (5.22), we have

$$\begin{aligned} &\varepsilon - max\{d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gs_{n(\lambda)}, gs_{n(\lambda)-1})\} \\ &\quad - max\{d(gr_{m(\lambda)-1}, gr_{m(\lambda)}), d(gs_{m(\lambda)-1}), gs_{m(\lambda)})\} \\ &\quad \leq max\{d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1})\} \\ &\quad < max\{d(gr_{m(\lambda)-1}, gr_{m(\lambda)}), d(gs_{m(\lambda)-1}, gs_{m(\lambda)})\} + \varepsilon. \end{aligned} \tag{5.23}$$

Letting $\lambda \to \infty$ in (5.23) and using (5.8), we get

$$\lim_{\lambda \to \infty} max\{d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1})\} = \varepsilon. \tag{5.24}$$

Now, applying the inequality (5.1), we have

$$\begin{aligned} \varphi(d(gr_{n(\lambda)}, gr_{m(\lambda)})) &= \varphi(d(F(r_{n(\lambda)-1}, s_{n(\lambda)-1}), d(F(r_{m(\lambda)-1}, s_{m(\lambda)-1}))) \\ &\leq \varphi(M(r_{n(\lambda)-1}, s_{n(\lambda)-1}, r_{m(\lambda)-1}, s_{m(\lambda)-1})) \\ &\quad - \phi(M(r_{n(\lambda)-1}, s_{n(\lambda)-1}, r_{m(\lambda)-1}, s_{m(\lambda)-1})) \\ &\quad + \theta(N(r_{n(\lambda)-1}, s_{n(\lambda)-1}, r_{m(\lambda)-1}, s_{m(\lambda)-1})), \end{aligned} \tag{5.25}$$

where,

$$\begin{aligned}&M(r_{n(\lambda)-1}, s_{n(\lambda)-1}, r_{m(\lambda)-1}, s_{m(\lambda)-1})\\&\quad= max\{d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}), d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1}),\\&\qquad d(gr_{m(\lambda)}, gr_{m(\lambda)-1}), d(gs_{n(\lambda)}, gs_{n(\lambda)-1}), d(gs_{m(\lambda)}, gs_{m(\lambda)-1})\}\end{aligned}$$

and

$$\begin{aligned}N(r_{n(\lambda)-1}, s_{n(\lambda)-1}, r_{m(\lambda)-1}, s_{m(\lambda)-1}) = min\{&d(gr_{n(\lambda)}, gr_{m(\lambda)-1}), d(gr_{m(\lambda)}, gr_{n(\lambda)-1}),\\&d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gr_{m(\lambda)}, gr_{m(\lambda)-1})\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\varphi(d(gs_{n(\lambda)}, gs_{m(\lambda)})) &= \varphi(d(F(gs_{n(\lambda)-1}, gr_{n(\lambda)-1}), d(F(gs_{m(\lambda)-1}, gr_{m(\lambda)-1})))\\&\leq \varphi(M(s_{n(\lambda)-1}, r_{n(\lambda)-1}, s_{m(\lambda)-1}, r_{m(\lambda)-1}))\\&\quad - \phi(M(s_{n(\lambda)-1}, r_{n(\lambda)-1}, s_{m(\lambda)-1}, r_{m(\lambda)-1}))\\&\quad + \theta(N(s_{n(\lambda)-1}, r_{n(\lambda)-1}, s_{m(\lambda)-1}, r_{m(\lambda)-1})),\end{aligned} \tag{5.26}$$

where,

$$\begin{aligned}&M(s_{n(\lambda)-1}, r_{n(\lambda)-1}, s_{m(\lambda)-1}, r_{m(\lambda)-1})\\&\quad= max\{d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}), d(gs_{n(\lambda)}, gs_{n(\lambda)-1}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1}),\\&\qquad d(gs_{m(\lambda)}, gs_{m(\lambda)-1}), d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gr_{m(\lambda)}, gr_{m(\lambda)-1})\}\end{aligned}$$

and

$$\begin{aligned}N(s_{n(\lambda)-1}, r_{n(\lambda)-1}, s_{m(\lambda)-1}, r_{m(\lambda)-1}) = min\{&d(gs_{n(\lambda)}, gs_{m(\lambda)-1}), d(gs_{m(\lambda)}, gs_{n(\lambda)-1}),\\&d(gs_{n(\lambda)}, gs_{n(\lambda)-1}), d(gs_{m(\lambda)}, gs_{m(\lambda)-1})\}.\end{aligned}$$

From (5.25) and (5.26), we have

$$max\{\varphi(d(gr_{n(\lambda)}, gr_{m(\lambda)}), d(gs_{n(\lambda)}, gs_{m(\lambda)}))\} \leq \varphi(z_n) - \phi(z_n) + \theta(u_n),$$

where,

$$\begin{aligned}z_n = max\{&d(gr_{n(\lambda)-1}, gr_{m(\lambda)-1}), d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gs_{n(\lambda)-1}, gs_{m(\lambda)-1}),\\&d(gr_{m(\lambda)}, gr_{m(\lambda)-1}), d(gs_{n(\lambda)}, gs_{n(\lambda)-1}), d(gs_{m(\lambda)}, gs_{m(\lambda)-1})\},\end{aligned}$$

and

$$\begin{aligned}u_n &= min\{d(gr_{n(\lambda)}, gr_{m(\lambda)-1}), d(gs_{n(\lambda)}, gs_{m(\lambda)-1}), d(gr_{m(\lambda)}, gr_{n(\lambda)-1}), d(gs_{m(\lambda)}, gs_{n(\lambda)-1}),\\&\qquad d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gs_{n(\lambda)}, gs_{n(\lambda)-1}), d(gr_{m(\lambda)}, gr_{m(\lambda)-1}), d(gs_{m(\lambda)}, gs_{m(\lambda)-1})\}\\&\leq min\{d(gr_{n(\lambda)}, gr_{n(\lambda)-1}), d(gs_{n(\lambda)}, gs_{n(\lambda)-1}), d(gr_{m(\lambda)}, gr_{m(\lambda)-1}), d(gs_{m(\lambda)}, gs_{m(\lambda)-1})\}.\end{aligned}$$

Finally letting $\lambda \to \infty$ in last two inequalities and using (5.8), (5.15) and (5.24) and the continuity of ϕ, φ and θ, we acquire

$$\varphi(\varepsilon) \leq \varphi(max(\varepsilon, 0, 0)) - \phi(max(\varepsilon, 0, 0)) + \theta(min(0, 0)) \leq \varphi(\varepsilon)$$

and as a result, $\phi(\varepsilon) = 0$. As ϕ is an altering distance function, $\varepsilon = 0$ and a contradiction.

This verifies our claim.

As U is complete, $\exists\ r, s \in U$ then

$$\begin{aligned} r &= \lim_{n\to\infty} gr_n = \lim_{n\to\infty} F(r_n, s_n) = F\left(\lim_{n\to\infty} r_n, \lim_{n\to\infty} s_n\right), \\ s &= \lim_{n\to\infty} gs_n = \lim_{n\to\infty} F(s_n, r_n) = F\left(\lim_{n\to\infty} s_n, \lim_{n\to\infty} r_n\right). \end{aligned} \tag{5.27}$$

As g is continuous, then from (5.27), we obtain

$$\lim_{n\to\infty} g(gr_n) = gr, \ \lim_{n\to\infty} g(gs_n) = gs. \tag{5.28}$$

F and g have the property of commutativity yields that

$$\begin{aligned} g(gr_{n+1}) &= g(F(r_n, s_n)) = F(gr_n, gs_n) \\ g(gs_{n+1}) &= g(F(s_n, r_n)) = F(gs_n, gr_n). \end{aligned} \tag{5.29}$$

As, F is continuous, then $\{g(gr_{n+1})\}$ is convergent to $F(r, s)$ and $\{g(gs_{n+1})\}$ convergent to $F(s, r)$. By (5.28) and uniqueness of the limit, we get $F(r, s) = gr$ and $F(s, r) = gs$, that is, g and F possesses a coupled coincidence point.

This concludes the theorem. In the subsequent theorem we neglect the hypothesis that F is continuous.

Theorem 5.7. *Presuppose all the presumptions of Theorem 5.6 are fulfilled . Moreover, assume that g is monotone under the partial order $\leq$ and U has the subsequent properties*

(a) *if an increasing sequence $\{r_n\}$ in U converges to some point $r \in U$, then $r_n \leq r$, for each n,*

(b) *if a decreasing sequence $\{s_n\}$ in U converges to some point $s \in U$, then $s_n \geq s$, for each n.*

Then the conclusion of Theorem 5.6 also hold.

Proof. Following the proof of Theorem 5.6. Then $\lim_{n\to\infty} gr_n = r$ and $\lim_{n\to\infty} gs_n = s$.

To prove $F(r,s) = gr$, $F(s,r) = gs$.

$\{gr_n\}$ is increasing and $gr_n \to r$ and $\{gs_n\}$ is decreasing and $gs_n \to s$, from the assumptions (a) and (b) that $gr_n \leq r$ and $gs_n \geq s$, $\forall\, n \in \mathbb{N}$, In addition, without loss of generality, one can presume that g is nondecreasing about the partial order $\leq$. Then $g^2 r_n \leq gr$ and $g^2 s_n \geq gs$, $\forall\, n \in \mathbb{N}$, where $g^2 u = g(gu)$, for all $u \in U$.

Utilizing the inequality (5.1) we get

$$\begin{aligned} \varphi(d(F(r,s), g^2 r_{n+1}) &= \varphi(d(F(r,s), F(gr_n, gs_n))) \\ &\leq \varphi(M(r,s,gr_n,gs_n)) - \phi(M(r,s,gr_n,gs_n))) \\ &\quad + \theta(N(r,s,gr_n,gs_n)) \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} \varphi(d(F(s,r), g^2 s_{n+1}) &= \varphi(d(F(s,r), F(gs_n, gr_n))) \\ &\leq \varphi(M(s,r,gs_n,gr_n)) - \phi(M(s,r,gs_n,gr_n)) \\ &\quad + \theta(N(s,r,gs_n,gr_n)) \end{aligned} \tag{5.31}$$

where,

$$\begin{aligned} M((r,s,gr_n,gs_n) &= M(s,r,gs_n,gr_n) \\ &= max\{d(gr,gr_n), d(gs,gs_n), d(F(r,s),gr), \\ &\qquad d(F(gr_n,gs_n),ggr_n), d(F(s,r),gs), d(F(gs_n,gr_n),ggs_n)\}, \end{aligned}$$

and

$$N(r,s,gr_n,gs_n) = min\{d(F(r,s),g^2 r_n), d(g^2 r_{n+1},gr), d(F(r,s),gr), d(g^2 r_{n+1}, g^2 r_n)\} \tag{5.32}$$

and

$$N(s,r,gs_n,gr_n) = min\{d(F(s,r),g^2 s_n), d(g^2 s_{n+1},gs), d(F(s,r),gs), d(g^2 s_{n+1}, g^2 s_n)\}.$$

Now, we claim that

$$mar\{d(F(r,s),gr), d(F(s,r),gs)\} = 0. \tag{5.33}$$

If this not true, then $max\{d(F(r,s),gr), d(F(s,r),gs)\} > 0$. Since $\lim_{n\to\infty} gr_n = r$, $\lim_{n\to\infty} gs_n = s$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$M(r,s,gr_n,gs_n) = M(s,r,gs_n,gr_n) = max\{d(F(r,s),gr), d(F(s,r),gs)\}.$$

Combining this with (5.30), (5.31) and (5.32), we get for all $n > N$,

$$\begin{aligned}
&\varphi(max\{d(F(r,s), g^2 r_{n+1}), d(F(s,r), g^2 s_{n+1})\}) \\
&\quad = max\{\varphi(d(F(r,s), g^2 r_{n+1}), \varphi(d(F(s,r), g^2 s_{n+1}))\} \\
&\quad \leq \varphi(max\{d(F(r,s), gr), d(F(s,r), gs)\} - \phi(max\{d(F(r,s), gr), d(F(s,r), gs)\} \\
&\qquad + \theta(min\{d(F(r,s), g^2 r_n), d(g^2 r_{n+1}, gr), d(F(r,s), gr), d(g^2 r_{n+1}, g^2 r_n), \\
&\qquad\qquad d(F(s,r), g^2 s_n), d(g^2 s_{n+1}, gs), d(F(s,r), gs), d(g^2 s_{n+1}, g^2 s_n)\})
\end{aligned}$$

Letting $n \to \infty$, it follows that

$$\begin{aligned}
&\varphi(max\{d(F(r,s), gr), d(F(s,r), gs)\}) \\
&\quad \leq \varphi(max\{d(F(r,s), gr), d(F(s,r), gs)\}) \\
&\qquad - \phi(max\{d(F(r,s), gr), d(F(s,r), gs)\} \\
&\qquad + \theta(min\{d(F(r,s), gr), d(gr, gr), d(F(r,s), gr), d(gr, gr) \\
&\qquad\qquad d(F(s,r), gs), d(gs, gs), d(F(s,r), gs), d(gs, gs)\})
\end{aligned}$$

As the property of θ, we get

$$\begin{aligned}
&\varphi(max\{d(F(r,s), gr), d(F(s,r), gs)\}) \\
&\quad \leq \varphi(max\{d(F(r,s), gr), d(F(s,r), gs)\}) \\
&\qquad - \phi(max\{d(F(r,s), gr), d(F(s,r), gs)\} \\
&\quad \leq \varphi(max\{d(F(r,s), gr), d(F(s,r), gs)\})
\end{aligned}$$

and consequently, $\phi(max\{d(F(r,s), gr), d(F(s,r), gs)\} = 0$. As ϕ is an altering distance function, a contradiction. So, (5.33) holds Thus, $F(r,s) = gr$ and $F(s,r) = gs$.

Theorem 5.8. *Under the hypothesis of Theorem 5.7, suppose that $gs_0 \leq gr_0$. Then, it follows $gr = F(r,s) = F(s,r) = gs$. Furthermore, if g and F are w-compatible, then g and F possesses a coupled coincidence point of the form (u, u).*

Proof. If $gs_0 \leq gr_0$, then $gs \leq gs_n \leq gs_0 \leq gr_0 \leq gr_n \leq gr$ for all $n \in \mathbb{N}$. Thus, if $gr \neq gs$ (and then $d(gr, gs) \neq 0$ and $d(gs, gr) \neq 0$), hence by inequality (5.1), we acquire

$$\begin{aligned}
\varphi(d(gs, gr)) &= \varphi(d(F(s,r), F(r,s))) \\
&\leq \varphi(M(s,r,r,s)) - \phi(M(s,r,r,s)) + \theta(N(s,r,r,s)),
\end{aligned}$$

where,

$$\begin{aligned}
N(s,r,r,s) = min\{&d(F(s,r), gs), d(F(r,s), gr), \\
&d(F(s,r), gr), d(F(r,s), gs)\} = 0.
\end{aligned}$$

and

$$\begin{aligned}M(s,r,r,s) &= max\{d(gs,gr), d(gr,gs), d(F(s,r),gs), d(F(r,s),gr),\\ &\qquad d(F(s,r),gr), d(F(r,s),gs)\}\\ &= max\{d(gs,gr), d(gr,gs)\},\end{aligned}$$

Hence

$$\varphi(d(gs,gr)) \leq \varphi(max\{d(gs,gr), d(gr,gs)\}) - \phi(max\{d(gs,gr), d(gr,gs)\}). \tag{5.34}$$

Since $gs \leq gr$, hence using the same idea we have

$$\varphi(d(gr,gs)) \leq \phi(max\{d(gs,gr), d(gr,gs)\}) - \beta(max\{d(gs,gr), d(gr,gs)\}). \tag{5.35}$$

From (5.34) and (5.35), we have

$$\begin{aligned}\varphi(max\{d(gs,gr), d(gr,gs)\}) &= max\{\varphi(d(gs,gr)), \varphi(d(gr,gs))\}\\ &\leq \varphi(max\{d(gs,gr), d(gr,gs)\})\\ &\quad - \phi(max\{d(gs,gr), d(gr,gs)\})\\ &\leq \varphi(max\{d(gs,gr), d(gr,gs)\}).\end{aligned}$$

and consequently, $\phi(max\{d(gs,gr), d(gr,gs)\}) = 0$. Since ϕ is an altering distance function, we obtain $d(gs,gr) = 0, d(gr,gs) = 0$, a contradiction. Hence $gr = gs$, that is, $gr = F(r,s) = F(s,r) = gs$.

Now, let $u = gr = gs$. Since g and F are w-compatible, thus $gu = g(gr) = g(F(r,s)) = F(gr,gs) = F(u,u)$. Thus, g and F possesses a coupled coincidence point of the form (u,u).

To guarantee the uniqueness of the common coupled fixed point. If $(U, \leq)$ is a partially ordered set, we endow the product $U \times U$ with we need the subsequent idea for the partial order relation:

$$\text{for each } (\acute{r},\acute{s}), (r,s) \in U \times U \ (r,s) \leq (\acute{r},\acute{s}) \iff r \leq \acute{r} \text{ and } s \geq \acute{s}. \tag{5.36}$$

Theorem 5.9. *Including the condition (5.36) to the assumption of Theorem 5.6 (respectively Theorem 5.7), assume that, for each (r,s), $(\acute{r},\acute{s}) \in U \times U$, there exists $(z,t) \in U \times U$ that is comparable to (r,s) and $(\acute{r},\acute{s})$. Then g and F acquire a unique common coupled fixed point.*

Proof: From Theorem 5.6, the set of coupled coincidence points of g and F is non-empty. We will demonstrate that if (r,s) and $(\acute{r},\acute{s})$ are coupled coincidence points,

$g(r) = F(r, s),\ g(s) = F(s, r)$ and $g(\acute{r}) = F(\acute{r}, \acute{s}),\ g(\acute{s}) = F(\acute{s}, \acute{r})$,

then

$$gr = g\acute{r} \text{ and } gs = g\acute{s}. \tag{5.37}$$

Choose an element $(z, t) \in U \times U$ comparable with (r, s) and $(\acute{r}, \acute{s})$.

Let $z_0 = z, t_0 = t$ and choose $z_1, t_1 \in U$ so that $gz_1 = F(z_0, t_0)$ and $gt_1 = F(t_0, z_0)$. Then, in the same way as in the proof of Theorem 5.6, we can define sequences $\{gz_n\}$ and $\{gt_n\}$ as follows

$$gz_{n+1} = F(z_n, t_n) \text{ and } gt_{n+1} = F(t_n, z_n).$$

Since $(gr, gs) = (F(r, s), F(s, r))$ and $(F(z, t), F(t, z)) = (gz_1, gt_1)$ are comparable, then $gr \leq gz_1$ and $gs \geq gt_1$. It is simple to prove by using mathematical induction,

$$gr \leq gz_n,\ gs \geq gt_n,\ \forall\, n \in \mathbb{N}.$$

Now, from the contractive condition (5.1)

$$\begin{aligned} \varphi(d(gr, gz_{n+1})) &= \varphi(d(F(r, s), F(z_n, t_n))) \\ &\leq \varphi(M(r, s, z_n, t_n)) - \phi(M(r, s, z_n, t_n)) \\ &\quad + \theta(N(r, s, z_n, t_n)), \end{aligned} \tag{5.38}$$

where,

$$\begin{aligned} M(r, s, z_n, t_n) = max\{&d(gr, gz_n), d(gs, gt_n), d(F(r, s), gr), \\ &d(F(z_n, t_n), gz_n), d(F(s, r), gs), d(F(t_n, z_n), gt_n)\} \\ = max\{&d(gr, gz_n), d(gs, gt_n)\}. \end{aligned}$$

and

$$\begin{aligned} N(r, s, z_n, t_n) &= min\{d(F(r, s), gz_n), d(F(z_n, t_n), gr), d(F(r, s), gr), d(F(z_n, t_n), gz_n)\} \\ &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} \varphi(d(gr, gz_{n+1})) &\leq \varphi(max\{d(gr, gz_n), d(gs, gt_n)\}) - \phi(max\{d(gr, gz_n), d(gs, gt_n)\}) \\ &\leq \varphi(max\{d(gr, gz_n), d(gs, gt_n)\}), \end{aligned} \tag{5.39}$$

and analogously

$$\begin{aligned}\varphi(d(gs, gt_{n+1})) &\leq \varphi(max\{d(gs, gt_n), d(gr, gz_n\}) - \phi(max\{d(gs, gt_n), d(gr, gz_n\}) \\ &\leq \varphi(max\{d(gs, gt_n), d(gr, gz_n)\}).\end{aligned} \tag{5.40}$$

From (5.39) and (5.40) and utilized the fact that φ is non-decreasing, we have

$$\begin{aligned}\varphi(max\{d(gr, gz_{n+1}), d(gs, gt_{n+1})\}) &= max\{\varphi(d(gr, gz_{n+1}), \varphi(d(gs, gt_{n+1}))\} \\ &\leq \varphi(max\{d(gr, gz_n), d(gs, gt_n)\}) \\ &\quad - \phi(max\{d(gr, gz_n), d(gs, gt_n)\} \\ &\leq \varphi(max\{d(gr, gz_n), d(gs, gt_n)\}).\end{aligned} \tag{5.41}$$

This implies that

$$max\{d(gr, gz_{n+1}), d(gs, gt_{n+1})\} \leq max\{d(gr, gz_n), d(gs, gt_n)\},$$

and consequently the sequence $max\{d(gr, gz_{n+1}), d(gs, gt_{n+1})\}$ is decreasing and non-negative and so,

$$\lim_{n\to\infty} max\{d(gr, gz_{n+1}), d(gs, gt_{n+1})\} = j, \tag{5.42}$$

for certain $j \geq 0$. Using (5.42) and letting $n \to \infty$ in (5.41), we get

$$\varphi(j) \leq \varphi(j) - \phi(j) \leq \varphi(j),$$

and as a result $\phi(j) = 0$ and thus $j = 0$.

Lastly, as

$$\lim_{n\to\infty} max\{d(gr, gz_{n+1}), d(gs, gt_{n+1})\} = 0. \tag{5.43}$$

This implies

$$\lim_{n\to\infty} d(gr, gz_{n+1}) = \lim_{n\to\infty} d(gs, gt_{n+1}) = 0. \tag{5.44}$$

Similarly

$$\lim_{n\to\infty} d(g\acute{r}, gz_{n+1}) = \lim_{n\to\infty} d(g\acute{s}, gt_{n+1}) = 0. \tag{5.45}$$

From (5.44) and (5.45), we have $gr = g\acute{r}, gs = g\acute{s}$.

Since $F(r, s) = gr$ and $F(s, r) = gs$, by commutativity of g and F, we get

$$g(gr) = g(F(r, s)) = F(gr, gs) \text{ and } g(gs) = g(F(s, r)) = F(gs, gr). \tag{5.46}$$

Denote $gr = m$ and $gs = n$. Then, from(5.46), it follows that

$$gm = F(m, n) \text{ and } gn = F(n, m). \tag{5.47}$$

Thus (m, n) is an other coupled coincidence point of g and F. Then, $m = gr = gm$ and $n = gs = gn$. So, (m, n) is a coupled common fixed point of g and F.

To show the uniqueness, presume that there exists another coupled common fixed point of g and F is (h, j). Then $h = gh = F(h, j)$ and $j = gj = F(j, h)$. Since the pair (h, j) is a coupled coincidence point of g and F, we have $gh = gm$ and $gj = gn$. Thus, $h = gh = gm = m$ and $j = gj = gn = n$. Therefore, the coupled fixed point is unique.

Theorem 5.10. *Under the hypothesis of Theorem 5.7, presuppose in addition that for each (r, s) and $(\acute{r}, \acute{s})$ in U, there is $(z, t) \in U \times U$, $(F(z, t), F(t, z))$ is comparable to $(F(r, s), F(s, r))$ and $(F(\acute{r}, \acute{s}), F(\acute{s}, \acute{r}))$. If F and g are w-compatible, then g and F acquire a unique common coupled fixed point of the form (t, t).*

Proof: Using Theorem 5.7, the set of coupled fixed points of g and F is non empty. Presume (r, s) and $(\acute{r}, \acute{s})$ be coupled coincidence points of F and g. Succeeding the proof of Theorem 5.9, we can show that

$$g\acute{r} = gr \text{ and } g\acute{s} = gs. \tag{5.48}$$

Note that if (r, s) is a coupled coincidence point of g and F, then (s, r) is also a coupled coincidence point of g and F. Thus, by (5.48) we have $gr = gs$. Put $t = gr = gs$. As, $F(r, s) = gr$, $F(s, r) = gs$ and g and F are w-compatible, we obtain $gt = g(gr) = g(F(r, s)) = F(gr, gs) = F(t, t)$. Therefore, (t, t) is a common coupled fixed point of g and F. Thus, $gt = gr = gs = t$ and hence we have $t = gt = F(t, t)$. Therefore, (t, t) is a common coupled fixed point of g and F.

Example 5.1. *Let $U = \{0, 1, 2\}$ and define d: $U \times U \to \mathbb{R}^+$ as $d(r, s) = max\{r, s\}$. Let F: $U \times U \to U$ as $r = F(r, s)$ for all $s, r \in U$ and g: $U \to U$ with $g(0) = 1, g(1) = 2, g(2) = 2$ for all $r \in U$.*
Let φ, ϕ: $[0, \infty) \to [0, \infty)$ and θ: $[0, \infty) \to [0, \infty)$ be defined by $\varphi(z) = z$ and $\phi(z) = \frac{1}{2}(z)$, $\theta(z) = z$. Then, φ, ϕ, θ have the properties mentioned in Theorem 5.6.
First, we verify that g commutes with F, that is, $F(gr, gs) = g(F(r, s))$.
Case 1: If $r = 0, s = 0$, then $gF(0, 0) = g(0) = 1$ and $F(g(0), g(0)) = 1$.
Case 2: If $r = 0, s = 1$, then $gF(0, 1) = g(0) = 1$ and $F(g(0), g(1)) = 1$.
Case 3: If $r = 1, s = 0$, then $gF(1, 0) = g(1) = 2$ and $F(g(1), g(0)) = 2$.
Case 4: If $r = 1, s = 1$, then $gF(1, 1) = g(1) = 2$ and $F(g(1), g(1)) = 2$.
Case 5: If $r = 0, s = 2$, then $gF(0, 2) = g(0) = 1$ and $F(g(0), g(2)) = 1$.

Case 6: If $r = 2, s = 0$, *then* $gF(2,0) = g(2) = 2$ *and* $F(g(2), g(0)) = 2$.
Case 7: If $r = 2, s = 1$, *then,* $gF(2,1) = g(2) = 2$ *and* $F(g(2), g(1)) = 2$.
Case 8: If $r = 1, s = 2$, *then,* $gF(1,2) = g(1) = 2$ *and* $F(g(1), g(2)) = 2$.
Case 9: If $r = 2, s = 2$, *then,* $gF(2,2) = g(2) = 2$ *and* $F(g(2), g(2)) = 2$.
In all above cases g *commutes with* F.
Now, we verify that the function F *and* g *satisfies the inequality* (5.1) *and let* $gr \geq gu$ *and* $gy \leq gv$. *Then, we have the succeeding cases.*
Case 1: If $s = r = t = v = 0$, *then* $d(F(0,0), F(0,0)) = d(0,0) = 0$,
and

$$\begin{aligned}
M(r,s,t,v) &= M(0,0,0,0) \\
&= max\{d(g(0), g(0)), d(g(0), g(0)), d(F(0,0), g(0)), d(F(0,0), g(0)), \\
&\quad d(F(0,0), g(0)), d(F(0,0), g(0))\} \\
&= max\{d(1,1), d(1,1), d(0,1), d(0,1), d(0,1), d(0,1)\} = max\{1,1,1,1,1,1\} = 1,
\end{aligned}$$

also

$$\begin{aligned}
N(r,s,t,v) = N(0,0,0,0) &= min\{d(F(0,0), g(0)), d(F(0,0), g(0)), \\
&\quad d(F(0,0), g(0)), d(F(0,0), g(0))\} \\
&= min\{d(0,1), d(0,1), d(0,1), d(0,1)\} = min\{1,1,1,1\} = 1.
\end{aligned}$$

As, $\varphi(0) = 0 < \varphi(1) - \phi(1) + \theta(1) = \frac{3}{2}$, *the inequality* (5.1) *is satisfied in this case.*
Case 2: If $r = 1, t = 1, s = 0, v = 0$, *then* $d(F(1,0), F(1,0)) = d(1,1) = 1$,
and

$$\begin{aligned}
M(r,s,t,v) &= M(1,0,1,0) \\
&= max\{d(g(1), g(1)), d(g(0), g(0)), d(F(1,0), g(1)), d(F(1,0), g(1)), \\
&\quad d(F(0,1), g(0)), d(F(0,1), g(0))\} \\
&= max\{2,1,2,2,1,1\} = 2,
\end{aligned}$$

also

$$\begin{aligned}
N(r,s,t,v) &= N(1,0,1,0) \\
&= min\{d(F(1,0), g(1)), d(F(1,0), g(1)), d(F(1,0), g(1)), d(F(1,0), g(1))\} \\
&= min\{d(1,2), d(1,2), d(1,2), d(1,2)\} = min\{2,2,2,2\} = 2.
\end{aligned}$$

As, $\varphi(1) = 1 < \varphi(2) - \phi(2) + \theta(2) = 3$, *the inequality* (5.1) *is satisfied in this case.*
Case 3: If $r = 1, t = 1, v = 1, s = 0$, *then as* $d(F(1,0), F(1,1)) = 1$, $M(1,0,1,1) = 2$ *and* $N(1,0,1,1) = 2$, *the inequality* (5.1) *is satisfied in this case.*
Case 4: If $r = 1, s = 1, t = 1, v = 1$, *then as* $d(F(1,0), F(1,1)) = 1$, $M(1,0,1,1) = 2$

and $N(1,0,1,1) = 2$*, the inequality* (5.1) *is satisfied in this case.*
Case 5: If $r = s = t = v = 2$*, then as* $d(F(1,0), F(1,1)) = 1$*,* $M(1,0,1,1) = 2$ *and* $N(1,0,1,1) = 2$*, the inequality* (5.1) *is satisfied in this case.*
Case 6: If $r = 2, s = 0, t = 2, v = 0$*, then as* $d(F(1,0), F(1,1)) = 1$*,* $M(1,0,1,1) = 2$ *and* $N(1,0,1,1) = 2$*, the inequality* (5.1) *is satisfied in this case.*
Case 7: If $r = 2, s = 1, t = 2, v = 1$*, then as* $d(F(1,0), F(1,1)) = 1$*,* $M(1,0,1,1) = 2$ *and* $N(1,0,1,1) = 2$*, the inequality* (5.1) *is satisfied in this case.*
Case 8: If $r = t = v = 2, s = 0$*, then as* $d(F(1,0), F(1,1)) = 1$*,* $M(1,0,1,1) = 2$ *and* $N(1,0,1,1) = 2$*, the inequality* (5.1) *is satisfied in this case.*
Case 9: If $r = 2, s = t = v = 1$*, then as* $d(F(1,0), F(1,1)) = 1$*,* $M(1,0,1,1) = 2$ *and* $N(1,0,1,1) = 2$*, the inequality* (5.1) *is satisfied in this case.*
So, φ, ϕ *and* θ *satisfy all the hypothesis of Theorem 6.19. Moreover,*$(2,2)$ *is the coupled coincidence point of* g *and* F*.*

Example 5.2. *Suppose* $U = \mathbb{R}$ *with usual metric and order. Define* $F\colon U \times U \to U$ *as* $F(r,s) = \frac{1}{5}(r^2 + s^2 + rs)$ *for all* $r, s \in U$*. Also* $g\colon U \to U$ *with* $g(r) = r$ *for each* $r \in U$*.*

Presume $\varphi, \phi\colon [0,\infty) \to [0,\infty)$ *and* $\theta\colon [0,\infty) \to [0,\infty)$ *be defined by* $\varphi(z) = z$ *and* $\phi(z) = \frac{1}{4}(z)$*,* $\theta(z) = z$*. Then,* φ, ϕ, θ *have the assumptions mentioned in Theorem 5.6.*

Now, let $gt \leq gr$ *and* $gv \geq gs$*. So, we obtain*

$$\begin{aligned}
&\varphi(d(F(r,s), F(t,v))) = d(F(r,s), F(t,v)) \\
&\quad = |\frac{1}{5}(r^2 + s^2 + rs) - \frac{1}{5}(t^2 + v^2 + tv)| \\
&\quad \leq \frac{1}{5}|(r^2 - t^2)| + |(s^2 - v^2)| + |(rs - tv)| \\
&\quad \leq \frac{1}{5}|(r - t)| + |(s - v)| + |r - t| + |s - v| \\
&\quad = \frac{1}{5}[2d(gr, gt) + 2d(gs, gv)] \\
&\quad \leq \frac{3}{4}M(r,s,t,v) \\
&\quad = M(r,s,t,v) - \frac{1}{4}M(r,s,t,v) \\
&\quad \leq \varphi(M(r,s,t,v) - \phi(M(r,s,t,v)) + \theta(N(r,s,t,v))
\end{aligned}$$

where,

$$\begin{aligned}
M(r,s,t,v) = max\{&d(gr,gt), d(gs,gv), d(F(r,s), gr), \\
&d(F(t,v), gt), d(F(r,s), gt), d(F(t,v), gr)\}
\end{aligned}$$

and

$$N(r,s,t,v) = min\{d(F(r,s), gr), d(F(t,v), gt), d(F(r,s), gt), d(F(t,v), gr)\}$$

Thus, all of the assumptions of Theorem 5.6 are fulfilled. Furthermore, g and F possesses the unique coupled coincidence point $(0,0)$.

Theorem 5.11. *Suppose (U,d) be a complete metric space. Suppose $g\colon U \to U$ and $F\colon U \times U \to U$ be continuous mappings also F has the property of mixed g-monotone and g commutes with F, satisfy*

$$\varphi(d(F(r,s),F(t,v))) \leq \varphi(M(r,s,t,v)) - \phi((r,s,t,v)) \tag{5.49}$$

where,

$$M(r,s,t,v) = max\{d(gr,gt), d(gs,gv), d(F(r,s),gr), d(F(t,v),gt), d(F(v,t),gv)\}$$

for each $r,v,t,s \in U$ with $gr \geq gt$ and $gv \geq gs$, here ϕ and φ are altering distance functions. Assume that $F(U \times U) \subseteq g(U)$ and furthermore for each $r_0, s_0 \in U$ such that $F(r_0,s_0) \geq gr_0$ and $F(s_0,r_0) \leq gs_0$. Presuppose that U has the subsequent properties

- *(a) if an increasing sequence $\{r_n\}$ in U converges to some point $r \in U$, then $r_n \leq r$, for each n,*
- *(b) if a decreasing sequence $\{s_n\}$ in U converges to some point $s \in U$, then $s_n \geq s$, for each n.*

Then g and F possesses a coupled coincidence point in U.

Corollary 5.12. *Suppose (U,d) be a complete metric space. Presume $F\colon U \times U \to U$ be a function satisfying (5.49) (with $g = I_U$) $\forall\, r,t,v,s \in U$ with $t \leq r$ and $v \geq s$. Let F acquire property of mixed monotone . Presuppose*

1. *F is continuous or*
2. *U has the subsequent:*
   - *(a) if an increasing sequence $\{r_n\} \to r$, then $r_n \leq r$, for each n,*
   - *(b) if a decreasing sequence $\{s_n\} \to s$, then $s_n \geq s$, for each n.*

furthermore for each $r_0, s_0 \in U$ with $r_0 \leq F(r_0,s_0)$ and $s_0 \geq F(s_0,r_0)$, then F has a coupled fixed point.

Corollary 5.13. *Suppose $(U,\leq,d)$ be an ordered complete metric space. Suppose $F\colon U \times U \to U$ is a continuous mapping on U possesses the property of mixed monotone, that there is $l \in [0,1)$ satisfying*

$$d(F(r,s),F(t,v))) \leq l\ max\{d(gr,gt),d(gs,gv),d(F(r,s),gr),$$
$$d(F(t,v),gt),d(F(s,r),gs),d(F(v,t),gv)\}$$

$\forall\ r,s,v,t \in U$ with $t \leq r$ and $v \geq s$. Assume either F is continuous or U has the subsequent properties:

(a) if an increasing sequence $\{r_n\}$ in U converges to some point $r \in U$, then $r_n \leq r$, for each n,

(b) if a decreasing sequence $\{s_n\}$ in U converges to some point $s \in U$, then $s_n \geq s$, for each n.

If there exists $r_0, s_0 \in U$ such that $r_0 \leq F(r_0,s_0)$ and $s_0 \geq F(s_0,r_0)$, thus F possesses a coupled fixed point.

Proof. Using Theorem 5.11 and choosing as φ= identity and $\phi = (1-l)\varphi$, we attain the result.

5.3 Application to Integral Equations

In this segment, a useful application is presented. By utilizing the Corollary 5.12, the existence and uniqueness of solutions of a non-linear integral equation has been showed.

Consider an integral equation of the following type:

$$r(b) = \int_0^1 (k_1(b,L)+k_2(b,L))(f(L,r(L))+g(L,r(L)))dL + a(b),\ b \in [0,1]. \tag{5.50}$$

We will analyze (5.50) under the accompanying presumptions:

(1) k_i: $[0,1]\times[0,1] \to \mathbb{R}(i=1,2)$ be continuous and $0 \leq k_1(b,L)$ and $0 \geq k_2(b,L)$.

(2) $a \in C[0,1]$.

(3) g, f: $[0,1]\times\mathbb{R} \to \mathbb{R}$ be two continuous functions.

(4) There exist constants $\mu, \nu > 0$, for each $s, r \in \mathbb{R}$ with $s \leq r$

$$0 \leq f(b,r) - f(b,s) \leq \nu\big[ln[(r-s)+1]\big],$$
$$-\mu\big[ln[(r-s)+1]\big] \leq g(b,r) - g(b,s) \leq 0.$$

(5) There exist $\gamma, \delta \in C[0,1]$ satisfy

$$\gamma(b) \leq \int_0^1 k_1(b,L)(f(L,\gamma(L)) + g(L,\delta(L)))dL + \int_0^1 k_2(b,L)(f(L,\delta(L)) + g(L,\gamma(L))dL + a(b),$$
$$\delta(b) \geq \int_0^1 k_1(b,L)(f(L,\delta(L)) + g(L,\gamma(L)))dL + \int_0^1 k_2(b,L)(f(L,\gamma(L))) + g(L,\delta(L))dL + a(b).$$

(6) $2.max(\nu,\mu) \parallel k_1 - k_2 \parallel_\infty \leq 1$, where

$$\parallel k_1 - k_2 \parallel_\infty = sup\{(k_1(b,L) - k_2(b,L)) \colon b, L \in [0,1]\}.$$

Let $U = C[0,1]$ be the space of continuous functions defined on [0, 1] with the usual metric as follow

$$d(u,v) = \sup_{b\in[0,1]} \mid u(b) - v(b) \mid, \text{ for } u, v \in C[0,1].$$

This space endowed with a partial order specified as

$$u, v \in C[0,1], u \leq v \iff u(b) \leq v(b), \text{ for some } b \in [0,1].$$

If in $U \times U$ we contemplate the order as follow

$$(u,z),(w,v) \in U \times U,\ (w,v) \geq (u,z) \iff u \leq w \text{ and } z \geq v,$$

and for some $u, z \in U$ we obtain $max(u,z), min(u,z) \in U$, condition (6.12) is satisfied.

Furthermore, in [106] showed that $(C[0,1], \leq)$ satisfy the presumption (1).

We now formulate our outcome.

Theorem 5.14. *Under presumptions (1)-(6), equation* (5.50) *possesses a unique solution in* $C[0,1]$.

Proof: We contemplate the mapping $F \colon U \times U \to U$ given by:

$$F(r,s)(b) = \int_0^1 k_1(b,L)(f(L,r(L)) + g(L,s(L)))dL + \int_0^1 k_2(b,L)(f(L,s(L)) + g(L,r(L)))dL + a(b), \text{ for } b \in [0,1].$$

By virtuousness of our presumptions, F is well defined (for $r, s \in U$ then $F(r,s) \in U$).

Initially, we show that F possesses the property of mixed monotone.

For $r_1 \leq r_2$ and $b \in [0,1]$, we get

$$\begin{aligned} F(r_1,s)(b) - F(r_2,s)(b) &= \int_0^1 k_1(b,L)(f(L,r_1(L)) + g(L,s(L)))dL \\ &+ \int_0^1 k_2(b,L)(f(L,s(L)) + g(L,r_1(L)))dL + a(b) \\ &- \int_0^1 k_1(b,L)(f(L,r_2(L)) + g(L,s(L)))dL \\ &- \int_0^1 k_2(b,L)(f(L,s(L)) + g(L,r_2(L)))dL - a(b) \\ &= \int_0^1 k_1(b,L)(f(L,r_1(L)) - f(L,r_2(L)))dL \\ &+ \int_0^1 k_2(b,L)(g(L,r_1(L)) - g(L,r_2(L)))dL. \end{aligned} \tag{5.51}$$

Consider $r_1 \leq r_2$ and our presumptions,

$$f(L,r_1(L)) - f(L,r_2(L)) \leq 0,\ g(L,r_1(L) - g(L,r_2(L) \geq 0,$$

and from (5.51) we obtain $F(r_1,s)(b) - F(r_2,s)(b) \leq 0$ and this proves that $F(r_1,s) \leq F(r_2,s)$.

Similarly, if $s_1 \geq s_2$ and $b \in [0,1]$, we obtain $F(r,s_1)(b) - F(r,s_2)(b) \leq 0$, or, equivalently, $F(r,s_1) \leq F(r,s_2)$.

Thus, F possesses the property of mixed monotone.

In the following, we estimate $d(F(r,s),F(t,v))$ for $r \geq t,\ s \leq v$.

As F possesses the property of mixed monotone, $F(r,s) \geq F(t,v)$, we have

$$\begin{aligned} d(F(r,s),F(t,v)) &= \sup_{b\in[0,1]} \mid F(r,s)(b) - F(t,v)(b) \mid = \sup_{b\in[0,1]} (F(r,s)(b) - F(t,v)(b)) \\ &= \sup_{b\in[0,1]} \Big[\int_0^1 k_1(b,L)(f(L,r(L)) + g(L,s(L)))dL \\ &+ \int_0^1 k_2(b,L)(f(L,s(L)) + g(L,r(L)))dL + a(b) \\ &- \int_0^1 k_1(b,L)(f(L,t(L)) + g(L,v(L)))dL \\ &- \int_0^1 k_2(b,L)(f(L,v(L)) + g(L,t(L)))dL - a(b) \Big] \\ &= \sup_{b\in[0,1]} \Big[\int_0^1 k_1(b,L)[(f(L,r(L)) - f(L,t(L))) - (g(L,v(L)) - g(L,s(L)))]dL \end{aligned}$$

$$- \int_0^1 k_2(b,e)[(f(L,v(L)) - f(L,s(L))) - (g(L,r(L)) - g(L,t(L)))]dL\Big]. \qquad (5.52)$$

By our presumptions (that $r \geq t$, $s \leq v$)

$$\begin{aligned} f(L,r(L)) - f(L,t(L)) &\leq \nu\big[ln[(r(L) - t(L)) + 1]\big], \\ g(L,v(L)) - g(L,s(L)) &\geq -\mu\big[ln[(v(L) - s(L)) + 1]\big], \\ f(L,v(L)) - f(L,s(L)) &\leq \nu\big[ln[(v(L) - s(L)) + 1]\big], \\ g(L,r(L)) - g(L,t(L)) &\geq -\mu\big[ln[(r(L) - t(L)) + 1]\big]. \end{aligned}$$

Consider these last inequalities, $k_2 \leq 0$ and (5.52), we have

$$\begin{aligned} d(F(r,s),F(t,v)) &\leq \sup_{b\in[0,1]} \Big[\int_0^1 k_1(b,L)[\nu[ln[(r(L) - t(L)) + 1]] + \mu[ln[(v(L) - s(L)) + 1]]]de \\ &\quad + \int_0^1 (-k_2(b,L))[\nu[ln[(v(L) - s(L)) + 1]] + \mu[ln[(r(L) - t(L)) + 1]]]de\Big] \\ &= max(\nu,\mu) \sup_{b\in[0,1]} \Big[\int_0^1 (k_1(b,L) - k_2(b,L))ln[(r(L) - t(L)) + 1]de \\ &\quad + \int_0^1 (k_1(b,L) - k_2(b,L))ln[(v(L) - s(L)) + 1]dL\Big] \qquad (5.53) \end{aligned}$$

Defining

$$\begin{aligned} \text{I} &= \textstyle\int_0^1 (k_1(b,L) - k_2(b,L))\big[ln[(r(L) - t(L)) + 1]\big]dL \\ \text{II} &= \textstyle\int_0^1 (k_1(b,L) - k_2(b,L))\big[ln[(v(L) - s(L)) + 1]\big]dL \end{aligned}$$

and applying the Cauchy-Schwartz inequality in (I), we have

$$\begin{aligned} (I) &\leq \Big(\int_0^1 (k_1(b,e) - k_2(b,e))^2 de\Big)^{\frac{1}{2}} . \Big(\int_0^1 \big[ln[(r(e) - t(e)) + 1]\big]^2 de\Big)^{\frac{1}{2}} \\ &\leq \| k_1 - k_2 \|_\infty .(ln \| r - t \| + 1)) = \| k_1 - k_2 \|_\infty .(ln(d(r,t) + 1)) \end{aligned} \qquad (5.54)$$

In similar way, we can obtain the subsequent estimate for (II):

$$(II) \leq \| k_1 - k_2 \|_\infty .(ln(d(s,v) + 1)) \qquad (5.55)$$

from (5.53)-(5.55), we get

$$\begin{aligned} d(F(r,s),F(t,v)) &\leq max(\nu,\mu) \| k_1 - k_2 \|_\infty [(ln(d(r,t) + 1)) + (ln(d(s,v) + 1))] \\ &\leq max(\nu,\mu) \| k_1 - k_2 \|_\infty [(ln(M(r,s,t,v)) + 1)) + (ln(M(r,s,t,v)) + 1))] \\ &= 2max(\nu,\mu) \| k_1 - k_2 \|_\infty \Big[\Big(ln\big(M(r,s,t,v)) + 1\big)\Big)\Big] \qquad (5.56) \end{aligned}$$

where,

$$M(r,s,t,v) = max\{d(gr,gt), d(gs,gv), d(F(r,s),gr),$$
$$d(F(t,v),gt), d(F(s,r),gs), d(F(v,t),gv)\}$$

from (5.56) and presumption (6) give us

$$d(F(r,s),F(t,v)) \le (ln(M(r,s,t,v))+1)),$$

or, equivalently,

$$\begin{aligned} d(F(r,s),F(t,v)) \le\ & (M(r,s,t,v)) \\ & -[(M(r,s,t,v)) - ln(M(r,s,t,v))+1))]. \end{aligned} \tag{5.57}$$

Put $\varphi(r) = r$ and $\phi(r) = r - ln(r+1)$. Clearly, ϕ and φ are altering distance functions and from (5.57) we have

$$\varphi(d(F(r,s),F(t,v)) \le \varphi(M(r,s,t,v)) - \phi(M(r,s,t,v))$$

where,

$$M(r,s,t,v) = max\{d(gr,gt), d(gs,gv), d(F(r,s),gr),$$
$$d(F(t,v),gt), d(F(s,r),gs), d(F(v,t),gv)\}.$$

This shows that the mapping F fulfill the contractive condition appeared in Corollary 5.12.

Finally, let γ, δ be the functions appeared in presumption (5); then, from (5), we obtain

$$\gamma \le F(\gamma,\delta), \delta \ge F(\delta,\gamma).$$

Applying Corollary 5.12, we deduce the existence of $(r,s) \in U \times U$, $F(r,s) = r$ and $F(s,r) = s$, that is, (r,s) is a solution of equation (5.50).

This finishes the proof.

Chapter 6

Y-Cone Metric Spaces and Coupled Common Fixed Point Results

In this chapter, a new generalization of cone metric space, which is termed as Y-cone metric space have been introduced. In section 6.1, the concepts of Y-cone metric space have been introduced. Section 6.2, deals with some topological properties of Y-cone metric space. In section 6.3 certain coupled common fixed point results have been proved in partially ordered Y-cone metric spaces.

6.1 Introduction

Metric spaces play a significant role in the study of Functional Analysis and Topology. A metric space is a set in which we can talk about the distance between any two of its elements. To discover a proper concept of a metric space, diverse concepts exist in this sphere. So, different notions of distance lead to new notions of convergence and continuity. Several generalizations of the metric space have then developed in many papers (see [5, 45, 69, 101, 129, 130]). Recently, cone metric spaces were presented by Huang and Zhang [69]. There they defined convergence in cone metric spaces and presented the completeness. Again they showed some fixed point results of contractive mappings on cone metric spaces. To apply this approach in Topology, the theory of cone metric spaces have acquired by distinct researchers (see [1, 3, 124, 127, 144]).

First, let us start by making some basic definitions.

Let E be a real Banach space and P is a subset of E. By θ we denote the zero element of E. The subset P is said to be a cone if and only if

(1) $P \neq \{\theta\}$, P is non empty and closed set;

(2) if $p, q \in R, p, q \geq 0$ and $u, v \in P$, then $pu + qv \in P$;

(3) $P \cap (-P) = \{\theta\}$.

Given cone $P \subset E$, we Characterize a partial ordering $\leq$ with respect to P by $s \leq v$ iff $v - s \in P$. We shall write $s < v$ to indicate that $s \leq v$ but $s \neq v$, although $s \ll v$ will stand for $v - s \in intP$ where $intP$ represents the interior of P.

Definition 6.1. [69] Suppose U be a non empty set. Presume the mapping $d\colon U \times U \to E$ satisfies the subsequent axioms for all $u_1, u_2, u_3 \in U$,

(1) $\theta \leq d(u_1, u_2)$ and $d(u_1, u_2) = \theta \iff s = v$,

(2) $d(u_1, u_2) = d(u_2, u_1)$,

(3) $d(u_1, u_2) \leq d(u_1, u_3) + d(u_3, u_2)$.

Then d is termed as a cone metric on U, and (U, d) is said to be a cone metric space.

The notion of b-metric space was introduced by Czerwik in [45]. For more insights about the accompanying definitions, we allude the peruser to [45].

Definition 6.2. [45] Suppose U be a non-empty set and $t \geq 1$ be a given real number. A function $d\colon U \times U \to \mathbb{R}^+$ is a b-metric on U if, for all $u_1, u_2, u_3 \in U$, the subsequent conditions hold:

(1) $d(u_1, u_2) = 0 \iff u_1 = u_2$,

(2) $d(u_1, u_2) = d(u_2, u_1)$,

(3) $d(u_1, u_2) \leq t[d(u_1, u_3) + d(u_3, u_2)]$.

Then, (U, d) is termed as a b-metric space.

For more insights about the accompanying definitions, we allude the peruser to [5].

Definition 6.3. [5] Suppose U be a non-empty set. The function $A\colon U^n \to [0, \infty)$ is said to be an A-metric on U if, for some $x_i, a \in U, i = 1, 2, \ldots, n$, the subsequent conditions hold:

(A_1) $A(u_1, u_2, u_3, \ldots, u_{n-1}, u_n) \geq 0$,

(A_2) $A(u_1, u_2, u_3, \ldots, u_{n-1}, u_n) = 0 \iff u_1 = u_2 = u_3 = \ldots = u_{n-1} = u_n$,

(A_3)

$$\begin{aligned} A(u_1, u_2, u_3, \ldots, u_{n-1}, u_n) \leq\ & A(u_1, u_1, u_1, \ldots, (u_1)_{n-1}, a) \\ & + A(u_2, u_2, u_2, \ldots, (u_2)_{n-1}, a) \\ & \vdots \\ & + A(u_{n-1}, u_{n-1}, u_{n-1}, \ldots, (u_{(n-1)})_{n-1}, a) \\ & + A(u_n, u_n, u_n, \ldots, (u_n)_{n-1}, a). \end{aligned}$$

Then (U, A) is called an A-metric space.

In the accompanying we generally assume E is a Banach space, P is a cone in E with $intP \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 6.4. Suppose U be a non-empty set and $k \geq 1$ be a given real number. Suppose a mapping $Y \colon U^t \to E$ is called a Y-cone metric on U if, for any $r_i, a \in U, i = 1, 2, \ldots, t$, the following conditions hold:

(Y_1) $Y(r_1, r_2, r_3, \ldots, r_{t-1}, r_t) \geq \theta$,

(Y_2) $Y(r_1, r_2, r_3, \ldots, r_{t-1}, r_t) = \theta \iff r_1 = r_2 = r_3 = \ldots = r_{t-1} = r_t$,

(Y_3)

$$\begin{aligned} Y(r_1, r_2, r_3, \ldots, r_{t-1}, r_t) \leq\ & k[Y(r_1, r_1, r_1, \ldots, (r_1)_{t-1}, a) \\ & + Y(r_2, r_2, r_2, \ldots, (r_2)_{t-1}, a) + \ldots \\ & + Y(r_{t-1}, r_{t-1}, r_{t-1}, \ldots, (r_{(t-1)})_{t-1}, a) \\ & + Y(r_t, r_t, r_t, \ldots, (r_t)_{t-1}, a)]. \end{aligned}$$

The (U, Y) is called an Y-cone metric space.

Note that cone b-metric space is a special case of Y-cone metric space with $t = 2$.

Proposition 6.5. *If (U, Y) is Y-cone metric space, then for all $r_1, r_2 \in U$, we have*

$$Y(r_1, r_1, \ldots, r_1, r_2) = Y(r_2, r_2, \ldots, r_2, r_1).$$

Example 6.1. *Let $U = [0, 1]$ and $E = C^1_{\mathbb{R}}$ with $\| h \| = \| h \|_\infty + \| h' \|_\infty$, $r \in E$ and let $P = \{h \in E \colon h(t) \geq 0 \text{ on } [0, 1]\}$. It is well-known that this cone is solid but it is not normal. Define a Y-cone metric $Y \colon U^t \to E$ by*

$$\begin{aligned}Y(r_1, r_2, r_3, \ldots, r_{t-1}, r_t) =& [|r_1 - r_2|^2 + |r_1 - r_3| + \ldots + |r_1 - r_t|^2 \\ &+ |r_2 - r_3|^2 + |r_2 - r_4|^2 + \ldots + |r_2 - r_t|^2 \\ &+ \ldots + |r_{t-1} - r_t|^2]e^w \\ &= \sum_{i=1}^{t}\sum_{i<j} |r_i - r_j|^2 e^w\end{aligned}$$

Then (U, Y) is a complete Y-cone metric space with the coefficient $k = 2$.

Lemma 6.6. *Suppose U be an Y-cone metric space, for all $r_1, r_2, r_3 \in U$ we have,*

$Y(r_1, r_1, \ldots, r_1, r_3) \leq k[(t-1)Y(r_1, r_1, \ldots, r_1, r_2) + Y(r_3, r_3, \ldots, r_3, r_2)]$

and $Y(r_1, r_1, \ldots, r_1, r_3) \leq k[(t-1)Y(r_1, r_1, \ldots, r_1, r_2) + Y(r_2, r_2, \ldots, r_2, r_3)]$.

Definition 6.7. Let (U, Y) is an Y-cone metric space. Then, for $r \in U$ and $\theta \ll e$, the Y-balls with center r and radius $\theta \ll e$ are

$$B_Y(r, e) = \{s \in U \colon Y(r, r, \ldots, r, s) \ll e\}.$$

6.2 Topological Y-cone Metric Spaces

In this section, we define the Topology of Y-cone metric space and study its Topological properties.

Definition 6.8. Presuppose (U, Y) be a Y-cone metric space with coefficient $k \geq 1$. For each $r \in U$ and each $\theta \ll e$, put $B_Y(r, e) = \{s \in U \colon Y(r, r, r, \ldots, r, s) \ll e\}$ and put $B = \{B_Y(r, e) : r \in U \text{ and } \theta \ll e\}$. Then, B is a subbase for some topology τ on U.

Remark 6.9. Presuppose (U, Y) be an Y-cone metric space. In this chapter, τ denotes the topology on U, B denotes a subbase for the Topology on τ and $B_Y(r, e)$ denotes the Y- ball in (U, Y), which are described in Definition (6.8). In addition, U denotes the base generated by subbase B.

Definition 6.10. Presuppose (U, Y) be a Y-cone metric space, a sequence $\{r_p\}$ in U converges to r if for each $c \in E$ with $\theta \ll c$, there is a natural number $\mathbb{N}$ such that for all $p > \mathbb{N}$, $Y(r_p, r_p, r_p, \ldots, r_p, r) \ll c$ for certain fixed r in U. Hence r is known as the limit of a sequence $\{r_p\}$ and it is represented by $\lim_{p\to\infty} r_p = r$ or $r_p \to r$ as $p \to \infty$.

Definition 6.11. Presuppose (U, Y) be a Y-cone metric space, a sequence $\{r_p\}$ in U. If for each $c \in E$ with $\theta \ll c$, there is a natural number $\mathbb{N}$ such that for all $p, m > \mathbb{N}$, we have $Y(r_p, r_p, r_p, \ldots, r_p, r_m) \ll c$, then $\{r_p\}$ is called a Cauchy sequence in U.

Definition 6.12. If each Cauchy sequence in U is convergent in U, then the Y-cone metric space U is said to be complete.

Lemma 6.13. *Suppose (U, Y) be a Y-cone metric space. If $\{r_n\}$ be a sequence in U converges to point r, then r is unique.*

Lemma 6.14. *Suppose (U, Y) be an Y-cone metric space. If $\{r_n\}$ be a sequence in U converges to r, then $\{r_n\}$ is a Cauchy sequence.*

Lemma 6.15. *Let (U, Y) be a Y-cone metric space. If there exist sequences $\{r_p\}, \{s_p\}$ such that $r_p \to r, s_p \to s$, then $\lim_{p\to\infty} Y(r_p, r_p, r_p, \ldots, r_p, s_p) = Y(r, r, r, \ldots, r, s)$.*

Remark 6.16. Suppose (U, Y) be a Y-cone metric space over the ordered real Banach space E with a cone P. Then the subsequent properties are used:

(1) If $p_1 \leq p_2$, $p_2 \ll p_3$, thus $p_1 \ll p_3$.

(2) If $p_1 \ll p_2$ and $p_2 \ll p_3$, then $p_1 \ll p_3$.

(3) If $\theta \leq v \ll c$ for every $c \in intP$, then $v = \theta$.

(4) If $c \in intP$, $\theta \leq a_n$, $a_n \to \theta$, then we can find n_0 so that for each $n > n_0$ we get $a_n \ll c$.

(5) If $\theta \leq a_n \leq b_n$ and $a_n \to a$, $b_n \to b$, then$a \leq b$, for each cone P.

(6) If E is a real Banach space with cone P and if $e \leq \lambda a$ here $e \in P$, $\theta \leq \lambda < 1$, then $e = \theta$.

(7) $\alpha\, intP \subseteq intP$ for $\alpha > 0$.

(8) For each $\delta > 0$ and $x \in intP$ there is $0 < \gamma < 1$, such that $\| \gamma x \| < \delta$.

(9) For each $\theta \ll c_1$ and $c_2 \in P$, there is an element $\theta \ll d$ such that $c_1 \ll d$, $c_2 \ll d$.

(10) For each $\theta \ll c_1$ and $\theta \ll c_2$, there is an element $\theta \ll e$ such that $e \ll c_1$, $e \ll c_2$.

6.3 Coupled Common Fixed Point Results

Now, we obtain common coupled fixed point results of maps satisfying more general contractive conditions in the framework of partially ordered Y-cone metric spaces. we start with the following result.

Remark 6.17. [59] Suppose $(U, \leq)$ be a partially ordered set, F: $U \times U \to U$ be an operator with the property of mixed monotone on U. Then for all $p \in \mathbb{N}$, the pair (F^p, F^p) possesses the property of mixed weakly monotone on U.

Lemma 6.18. *Let (U, Y) be an Y-cone metric space, then $U \times U$ is an Y-cone metric space with the Y-cone metric D given by*

$$D((r_1, s_1), (r_2, s_2), \ldots, (r_t, s_t)) = Y(r_1, r_2, \ldots, r_t) + Y(s_1, s_2, \ldots, s_t)$$

for all $r_i, s_j \in U,\ i, j = 1, 2, \ldots, t$.

Proof: For all $r_i, s_j \in U,\ i, j = 1, 2, 3, \ldots, t$, we have $D((r_1, s_1), (r_2, s_2), (r_3, s_3), \ldots, (r_t, s_t)) \geq 0$.

Note that

$$\begin{aligned} & D((r_1, s_1), (r_2, s_2), \ldots, (r_t, s_t)) = 0 \\ \iff & Y(r_1, r_2, \ldots, r_t) + Y(s_1, s_2, \ldots, s_t) = 0 \\ \iff & Y(r_1, r_2, \ldots, r_t) = 0, Y(s_1, s_2, \ldots, s_t) = 0 \\ \iff & r_1 = r_2 = \ldots = r_t, s_1 = s_2 = \ldots = s_t \\ \iff & (r_1, s_1) = (r_2, s_2) = \ldots = (r_t, s_t). \end{aligned}$$

Consider

$$\begin{aligned} D((r_1, s_1), (r_2, s_2), \ldots, (r_t, s_t)) &= Y(r_1, r_2, \ldots, r_t) + Y(s_1, s_2, \ldots, s_t) \\ &\leq k[Y(r_1, r_1, \ldots, r_1, a) + Y(r_2, r_2, \ldots, r_2, a) \\ &\quad + Y(r, r_3, \ldots, r_3, a) + \ldots + Y(r_t, r_t, \ldots, r_t, a) \\ &\quad + Y(s_1, s_1, \ldots, s_1, b) + Y(s_2, s_2, \ldots, s_2, b) \\ &\quad + Y(s_3, s_3, \ldots, s_3, b) + \ldots + Y(s_t, s_t, \ldots, s_t, b)] \\ &= k[D((r_1, s_1), (r_1, s_1), \ldots, (r_1, s_1), (a, b)) \\ &\quad + D((r_2, s_2), (r_2, s_2), \ldots, (r_2, s_2), (a, b)) \\ &\quad + \ldots + D((r_t, s_t), (r_t, s_t), \ldots, (r_t, s_t), (a, b))]. \end{aligned}$$

By the above D is an Y-cone metric on $U \times U$.

Theorem 6.19. *Suppose $(U, \leq, Y)$ be a partially ordered complete Y-cone metric space with the coefficient $k \geq 1$ relative to a solid cone P. Presume $F, G \colon U \times U \to U$ be the mappings such that (F, G) has the property of mixed weakly monotone on U. Presuppose that there exist $a_i \geq 0, i = 1, 2, \ldots, 6$ with $a_1 + a_2 + a_3 + 2ka_4 < 1$ and $\sum_{i=1}^{6} a_i < 1$ such that*

$$\begin{aligned} & Y(F(r, s), F(r, s), \ldots, F(r, s), G(w, v)) \\ & \quad + Y(F(s, r), F(s, r), \ldots, F(s, r), G(v, w)) \\ & \quad \leq a_1 D((r, s), (r, s), \ldots, (r, s), (w, v)) \end{aligned}$$

$$\begin{aligned}
&+ a_2 D((r,s),(r,s),\ldots,(r,s),(F(r,s),F(s,r))) \\
&+ a_3 D((w,v),(w,v),\ldots,(w,v),(G(w,v),G(v,w))) \\
&+ a_4 D((r,s),(r,s),\ldots,(r,s),(G(w,v),G(w,v))) \\
&+ a_5 D((w,v),(w,v),\ldots,(w,v),(F(r,s),F(s,r))) \\
&+ a_6 (min\{D((w,v),(w,v),\ldots,(w,v),(G(w,v),G(v,w))), \\
&\qquad D((r,s),(r,s),\ldots,(r,s),(G(w,v),G(w,v))), \\
&\qquad D((w,v),(w,v),\ldots,(w,v),(F(r,s),F(s,r)))\}) \qquad (6.1)
\end{aligned}$$

$\forall\ r,v,w,s \in U$ *with* $r \le w$ *and* $s \ge v$*, where* D *is defined as in Lemma* (6.18)*. Suppose that there exists* $r_0, s_0 \in U$ *such that* $r_0 \le F(r_0,s_0)$*,* $s_0 \ge F(s_0,r_0)$ *or* $r_0 \le G(r_0,s_0)$*,* $s_0 \ge G(s_0,r_0)$*, then* G *and* F *possesses a coupled common fixed point in* U*.*

Proof. Take $r_0, s_0 \in U$. Set $r_1 = F(r_0,s_0), s_1 = F(s_0,r_0), r_2 = G(r_1,s_1)$ and $s_2 = G(s_1,r_1)$.
From the condition $r_0 \le F(r_0,s_0)$, $s_0 \ge F(s_0,r_0)$ and (F,G) has the property of mixed weakly monotone, we have

$$\begin{aligned}
r_1 = F(r_0,s_0) \le G(F(r_0,s_0),F(s_0,r_0)) = G(r_1,s_1) &\implies r_1 \le r_2, \\
r_2 = G(r_1,s_1) \le F(G(r_1,s_1),G(s_1,r_1)) = F(r_2,s_2) &\implies r_2 \le r_3.
\end{aligned}$$

Thus,

$$\begin{aligned}
s_1 = F(s_0,r_0) \ge G(F(s_0,r_0),F(r_0,s_0)) = G(s_1,r_1) &\implies s_1 \ge s_2, \\
s_2 = G(s_1,r_1) \ge F(G(s_1,r_1),G(r_1,s_1)) = F(s_2,r_2) &\implies s_2 \ge s_3.
\end{aligned}$$

Repeating this process, we obtain

$$\begin{aligned}
&r_{2p+1} = F(r_{2p},s_{2p}),\ \ s_{2p+1} = F(s_2p,r_2p), \\
&r_{2p+2} = G(r_{2p+1},s_{2p+1}),\ \text{ and }\ s_{2p+2} = G(s_{2p+1},r_{2p+1}) \text{ for all } p \in \mathbb{N}.
\end{aligned}$$

Therefore the sequences $\{r_p\}$ and $\{s_p\}$ are monotone:

$$\begin{aligned}
r_0 \le r_1 \le \ldots \le r_p \le r_{p+1} \le \ldots, \\
s_0 \ge s_1 \ge \ldots \ge s_p \ge s_{p+1} \ge \ldots.
\end{aligned}$$

Similarly, from the condition $r_0 \le G(r_0,s_0)$, $s_0 \ge G(s_0,r_0)$. Then by (6.1), we have

$$\begin{aligned}
&Y(r_{2p+1},r_{2p+1},\ldots,r_{2p+1},r_{2p+2}) + Y(s_{2p+1},s_{2p+1},\ldots,s_{2p+1},s_{2p+2}) \\
&= Y\,(F(r_{2p},s_{2p}),F(r_{2p},s_{2p}),\ldots,F(r_{2p},s_{2p}),G(r_{2p+1},s_{2p+1})) \\
&\quad + Y\,(F(s_{2p},r_{2p}),F(s_{2p},r_{2p}),\ldots,F(s_{2p},r_{2p}),G(s_{2p+1},r_{2p+1}))
\end{aligned}$$

$$\begin{aligned}
&\leq a_1 D\left((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+1}, s_{2p+1})\right) \\
&\quad + a_2 D\left((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (F(r_{2p}, s_{2p}), F(s_{2p}, r_{2p}))\right) \\
&\quad + a_3 D\left((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (G(r_{2p+1}, s_{2p+1}), G(s_{2p+1}, r_{2p+1}))\right) \\
&\quad + a_4 D\left((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (G(r_{2p+1}, s_{2p+1}), G(s_{2p+1}, r_{2p+1}))\right) \\
&\quad + a_5 D\left((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (F(r_{2p}, s_{2p}), F(s_{2p}, r_{2p}))\right) \\
&\quad + a_6 [min\{D\left((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (G(r_{2p+1}, s_{2p+1}), G(s_{2p+1}, r_{2p+1}))\right), \\
&\qquad\qquad D\left((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (G(r_{2p+1}, s_{2p+1}), G(s_{2p+1}, r_{2p+1}))\right), \\
&\qquad\qquad D\left((r_{2p+1}, s_{2p+1})), (r_{2p+1}, s_{2p+1})), \ldots, (r_{2p+1}, s_{2p+1}), (F(r_{2p}, s_{2p}), F(s_{2p}, r_{2p}))\right)\}].
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2}) \\
&\leq a_1 D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+1}, s_{2p+1})) \\
&\quad + a_2 D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+1}, s_{2p+1})) \\
&\quad + a_3 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+2}, s_{2p+2})) \\
&\quad + a_4 D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+2}, s_{2p+2})) \\
&\quad + a_5 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1})) \\
&\quad + a_6 [min\{D\big((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+2}, s_{2p+2})\big), \\
&\qquad\quad D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+2}, s_{2p+2})), \\
&\qquad\quad D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}))\}] \qquad (6.2) \\
&= a_1 D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+1}, s_{2p+1})) \\
&\quad + a_2 D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+1}, s_{2p+1})) \\
&\quad + a_3 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+2}, s_{2p+2})) \\
&\quad + a_4 D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+2}, s_{2p+2})) \\
&\leq a_1 D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+1}, s_{2p+1})) \\
&\quad + a_2 D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+1}, s_{2p+1})) \\
&\quad + a_3 D((r_{2p+1}, s_{2p+1})), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+2}, s_{2p+2})) \\
&\quad + ka_4 D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+1}, s_{2p+1})) \\
&\quad + ka_4 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+2}, s_{2p+2})) \\
&= (a_1 + a_2 + ka_4) D((r_{2p}, s_{2p}), (r_{2p}, s_{2p}), \ldots, (r_{2p}, s_{2p}), (r_{2p+1}, s_{2p+1})) \\
&\quad + (a_3 + ka_4) D((r_{2p+1}, s_{2p+1})), (r_{2p+1}, s_{2p+1}), \ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+2}, s_{2p+2})). \\
&\leq (a_1 + a_2 + ka_4)[Y(r_{2p}, r_{2p}, \ldots, r_{2p}, r_{2p+1}) + Y(s_{2p}, s_{2p}, \ldots, s_{2p}, s_{2p+1})] \\
&\quad + (a_3 + ka_4)[Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2})]. \qquad (6.3)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2}) + Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2})\\
&\quad\leq (a_1 + a_2 + ka_4)[Y(s_{2p}, s_{2p}, \ldots, s_{2p}, s_{2p+1}) + Y(r_{2p}, r_{2p}, \ldots, r_{2p}, r_{2p+1})]\\
&\qquad + (a_3 + ka_4)[Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2}) + Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2})].
\end{aligned} \tag{6.4}$$

It follows from (6.3) and (6.4) that

$$\begin{aligned}
&[Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2})]\\
&\quad\leq \frac{a_1 + a_2 + ka_4}{1 - (a_3 + ka_4)}[Y(r_{2p}, r_{2p}, \ldots, r_{2p}, r_{2p+1}) + Y(s_{2p}, s_{2p}, \ldots, s_{2p}, s_{2p+1})].
\end{aligned}$$

Let $\delta = \frac{(a_1+a_2+ka_4)}{1-(a_3+ka_4)}$, then $0 \leq \delta < 1$ and

$$\begin{aligned}
&[Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2})]\\
&\quad\leq \delta[Y(r_{2p}, r_{2p}, \ldots, r_{2p}, r_{2p+1}) + Y(s_{2p}, s_{2p}, \ldots, s_{2p}, s_{2p+1})].
\end{aligned} \tag{6.5}$$

For all $p \in \mathbb{N}$, by interchanging the roles of F and G and using (6.1), we have

$$\begin{aligned}
&Y(r_{2p+2}, r_{2p+2}, \ldots, r_{2p+2}, r_{2p+3}) + Y(s_{2p+2}, s_{2p+2}, \ldots, s_{2p+2}, s_{2p+3})\\
&\quad\leq \delta[Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2})].
\end{aligned} \tag{6.6}$$

It follows from (6.5) that

$$\begin{aligned}
&[Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2})]\\
&\quad\leq \delta[Y(r_{2p}, r_{2p}, \ldots, r_{2p}, r_{2p+1}) + Y(s_{2p}, s_{2p}, \ldots, s_{2p}, s_{2p+1})]\\
&\quad\leq \delta(\delta(Y(r_{2p-1}, r_{2p-1}, \ldots, r_{2p-1}, r_{2p}) + Y(s_{2p-1}, s_{2p-1}, \ldots, s_{2p-1}, s_{2p})))\\
&\quad\leq \delta(\delta(\delta(Y(r_{2p-2}, r_{2p-2}, \ldots, r_{2p-2}, r_{2p-1}) + Y(s_{2p-2}, s_{2p-2}, \ldots, s_{2p-2}, s_{2p-1})))).
\end{aligned}$$

This implies

$$\begin{aligned}
&[Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2})]\\
&\quad\leq \delta^3[Y(r_{2p-2}, r_{2p-2}, \ldots, r_{2p-2}, r_{2p-1}) + Y(s_{2p-2}, s_{2p-2}, \ldots, s_{2p-2}, s_{2p-1})]\\
&\qquad\vdots\\
&\quad\leq \delta^{2p+1}(Y(r_0, r_0, \ldots, r_0, r_1) + Y(s_0, s_0, \ldots, s_0, s_1).
\end{aligned} \tag{6.7}$$

and similarly, by (6.6), we get

$$\begin{aligned}
&Y(r_{2p+2}, r_{2p+2}, \ldots, r_{2p+2}, r_{2p+3}) + Y(s_{2p+2}, s_{2p+2}, \ldots, s_{2p+2}, s_{2p+3})\\
&\quad\leq \delta^{2p+2}[Y(r_0, r_0, \ldots, r_0, r_1) + Y(s_0, s_0, \ldots, s_0, s_1)].
\end{aligned} \tag{6.8}$$

By Lemma (6.6) we have for all $p, m \in \mathbb{N}$ with $p \leq m$

$$\begin{aligned}
&Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2m+1}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2m+1}) \\
&\quad \leq k[(p-1)Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + Y(r_{2p+2}, r_{2p+2}, \ldots, r_{2p+2}, r_{2m+1})] \\
&\qquad + k[(p-1)Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2})) + Y(s_{2p+2}, s_{2p+2}, \ldots, s_{2p+2}, s_{2m+1})] \\
&\quad \leq (k(p-1)Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + k(p-1)Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2})) \\
&\qquad + (k^2(p-1)Y(r_{2p+2}, r_{2p+2}, \ldots, r_{2p+2}, r_{2p+3}) + k^2(p-1)Y(s_{2p+2}, s_{2p+2}, \ldots, s_{2p+2}, s_{2p+3})) \\
&\qquad + \ldots + (p-1)(k^{2m-1}Y(r_{2m-1}, r_{2m-1}, \ldots, r_{2m-1}, r_{2m}) \\
&\qquad + k^{2m-1}Y(s_{2m-1}, s_{2m-1}, \ldots, s_{2m-1}, s_{2m})) \\
&\qquad + k^{2m-1}(Y(r_{2m}, r_{2m}, \ldots, r_{2m}, r_{2m+1}) + Y(s_{2m}, s_{2m}, \ldots, s_{2m}, s_{2m+1})) \\
&\quad \leq (k(p-1)Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2p+2}) + k(p-1)Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2p+2})) \\
&\qquad + (k^2(p-1)Y(r_{2p+2}, r_{2p+2}, \ldots, r_{2p+2}, r_{2p+3}) + k^2(p-1)Y(s_{2p+2}, s_{2p+2}, \ldots, s_{2p+2}, s_{2p+3})) \\
&\qquad + \ldots + (k^{2m}(p-1)Y(r_{2m}, r_{2m}, \ldots, r_{2m}, r_{2m+1}) + k^{2m}(p-1)Y(s_{2m}, s_{2m}, \ldots, s_{2m}, s_{2m+1})) \\
&\quad \leq k(p-1)\delta^{2p+1}(1 + k\delta + k^2\delta^2 + \ldots)(Y(r_0, r_0, \ldots, r_0, r_1) + Y(s_0, s_0, \ldots, s_0, s_1)).
\end{aligned}$$

This implies

$$\begin{aligned}
&Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2m+1}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2m+1}) \\
&\quad \leq (p-1)\frac{k\delta^{2p+1}}{1-k\delta}(Y(r_0, r_0, \ldots, r_0, r_1) + Y(s_0, s_0, \ldots, s_0, s_1)).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&Y(r_{2p}, r_{2p}, \ldots, r_{2p}, r_{2m+1}) + Y(s_{2p}, s_{2p}, \ldots, s_{2p}, s_{2m+1}) \\
&\quad \leq (p-1)\frac{k\delta^{2p}}{1-k\delta}(Y(r_0, r_0, \ldots, r_0, r_1) + Y(s_0, s_0, \ldots, s_0, s_1)),
\end{aligned}$$

and

$$\begin{aligned}
&Y(r_{2p}, r_{2p}, \ldots, r_{2p}, r_{2m}) + Y(s_{2p}, s_{2p}, \ldots, s_{2p}, s_{2m}) \\
&\quad \leq (p-1)\frac{k\delta^{2p}}{1-k\delta}(Y(r_0, r_0, \ldots, r_0, r_1) + Y(s_0, s_0, \ldots, s_0, s_1)),
\end{aligned}$$

also,

$$\begin{aligned}
&Y(r_{2p+1}, r_{2p+1}, \ldots, r_{2p+1}, r_{2m}) + Y(s_{2p+1}, s_{2p+1}, \ldots, s_{2p+1}, s_{2m}) \\
&\quad \leq (p-1)\frac{k\delta^{2p+1}}{1-k\delta}(Y(r_0, r_0, \ldots, r_0, r_1) + Y(s_0, s_0, \ldots, s_0, s_1)).
\end{aligned}$$

Hence, for each $p, m \in \mathbb{N}$ with $p \leq m$, we get

$$Y(r_p, r_p, \ldots, r_p, r_m) + Y(s_p, s_p, \ldots, s_p, s_m)$$

$$\leq (p-1)\frac{k\delta^p}{1-k\delta}(Y(r_0, r_0, \dots, r_0, r_1) + Y(s_0, s_0, \dots, s_0, s_1)).$$

According to Remark 3.12(4), and for any $c \in E$ with $\theta \ll c$, there exists t_0 such that for any

$$p > p_0,\ (p-1)\frac{k\delta^p}{1-k\delta}(Y(r_0, r_0, \dots, r_0, r_1) + Y(s_0, s_0, \dots, s_0, s_1)) \ll c.$$

Furthermore, for any $m > p > p_0$, Remark (6.16) (1) shows that $Y(r_0, r_0, \dots, r_0, r_1) + Y(s_0, s_0, \dots, s_0, s_1) \ll c$. Hence, by Definition (6.11), $\{r_p\}$ and $\{s_p\}$ are Cauchy sequences in U. By the completeness of U, $\exists\ n, \jmath \in U$ such that

$$\lim_{p\to\infty} r_p = n \text{ and } \lim_{p\to\infty} s_p = \jmath.$$

We now show that $(n, \jmath)$ is a coupled common fixed point of F and G.

Suppose F is continuous, then we have

$$\begin{aligned}
n &= \lim_{p\to\infty} r_{p+1} = \lim_{p\to\infty} F(r_p, s_p) = F\left(\lim_{p\to\infty} r_p, \lim_{p\to\infty} s_p\right) = F(n, \jmath),\\
\jmath &= \lim_{p\to\infty} s_{p+1} = \lim_{p\to\infty} F(s_p, r_p) = F\left(\lim_{p\to\infty} s_p, \lim_{p\to\infty} r_p\right) = F(\jmath, n).
\end{aligned}$$

Using (6.1), we get

$$\begin{aligned}
&Y(F(n,\jmath), F(n,\jmath), F(n,\jmath), \dots\dots, F(n,\jmath), G(n,\jmath))\\
&\quad + Y(F(\jmath,n), F(\jmath,n), F(\jmath,n), \dots\dots, F(\jmath,n), G(\jmath,n))\\
&\quad \leq a_1 D((n,\jmath), (n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (n,\jmath))\\
&\qquad + a_2 D((n,\jmath), (n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (F(n,\jmath), F(\jmath,n)))\\
&\qquad + a_3 D((n,\jmath), (n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (G(n,\jmath), G(\jmath,n)))\\
&\qquad + a_4 D((n,\jmath), (n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (G(n,\jmath), G(\jmath,n)))\\
&\qquad + a_5 D((n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (F(n,\jmath), F(\jmath,n)))\\
&\qquad + a_6 (min\{D((n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (G(n,\jmath), G(\jmath,n))),\\
&\qquad\qquad\qquad D((n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (G(n,\jmath), G(\jmath,n))),\\
&\qquad\qquad\qquad D((n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (F(n,\jmath), F(\jmath,n)))\})\\
&\quad = a_2 D((n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (n,\jmath))\\
&\qquad + a_3 D((n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (G(n,\jmath), G(\jmath,n)))\\
&\qquad + a_4 D((n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (G(n,\jmath), G(s,r)))\\
&\qquad + a_5 D((n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (n,\jmath))\\
&\quad = (a_3 + a_4) D((n,\jmath), (n,\jmath), (n,\jmath), \dots\dots, (n,\jmath), (G(n,\jmath), G(\jmath,n)))
\end{aligned}$$

$$= (a_3 + a_4)(Y(n, n, \ldots\ldots, n, G(n, \jmath)) + Y(\jmath, \jmath, \ldots\ldots, \jmath, G(\jmath, n))).$$

Therefore

$$\begin{aligned} &Y(n, n, \ldots\ldots, n, G(n, \jmath)) + Y(\jmath, \jmath, \ldots\ldots, \jmath, G(\jmath, n)) \\ &\quad \leq (a_3 + a_4)(Y(n, n, \ldots\ldots, n, G(n, \jmath)) + Y(\jmath, \jmath, \ldots\ldots, \jmath, G(\jmath, n))). \end{aligned}$$

Since $0 \leq (a_3 + a_4) < 1$, Remark 3.12(6) shows that

$Y(n, n, \ldots\ldots, n, G(n, \jmath)) + Y(\jmath, \jmath, \ldots\ldots, \jmath, G(\jmath, n)) = \theta$. Hence, $Y(n, n, \ldots\ldots, n, G(n, \jmath)) = \theta$ and $Y(\jmath, \jmath, \ldots\ldots, \jmath, G(\jmath, n)) = \theta$. That is $G(n, \jmath) = n$ and $G(\jmath, n) = \jmath$. This implies $(n, \jmath)$ is a coupled fixed point of G. Similarly, we can show that $(n, \jmath)$ is a coupled fixed point of F when G is a continuous mapping. This completes the proof.

Theorem 6.20. *Suppose all the assumptions of Theorem 6.19 are satisfied . Moreover, presume that U has the subsequent properties*

(a) *if an increasing sequence $\{r_p\}$ in U converges to some point $n \in U$, then $r_p \leq n$, $\forall\, p$,*

(b) *if a decreasing sequence $\{s_p\}$ in U converges to some point $\jmath \in U$, then $s_p \geq \jmath$, $\forall\, p$.*

Then the conclusion of Theorem 6.19 also hold.

Proof. Succeeding the proof of Theorem 6.19 just we need to show that $(n, \jmath)$ is a coupled fixed point of F.

As $\{r_p\}$ is non-decreasing and $r_p \to n$ and $\{s_p\}$ is non-increasing and $s_p \to \jmath$, by our assumption, $r_p \leq n$ and $s_p \geq \jmath$, $\forall\, p$.

Applying the contractive condition, we have

$$\begin{aligned} &Y(n, n, \ldots\ldots, n, F(n, \jmath)) + Y(\jmath, \jmath, \ldots\ldots, \jmath, F(\jmath, n)) \\ &\quad \leq k[(p-1)Y(n, n, n, \ldots\ldots, n, r_{2p+2}) + (p-1)Y(\jmath, \jmath, \ldots\ldots, \jmath, s_{2p+2}) \\ &\qquad + Y(r_{2p+2}, r_{2p+2}, \ldots\ldots, r_{2p+2}, F(n, \jmath)) + Y(s_{2p+2}, s_{2p+2}, \ldots\ldots, s_{2p+2}, F(\jmath, n))] \\ &\quad = k[(p-1)Y(n, n, \ldots\ldots, n, r_{2p+2}) + (p-1)Y(\jmath, \jmath, \ldots\ldots, s, s_{2p+2}) \\ &\qquad + Y(G(r_{2p+1}, s_{2p+1}), G(r_{2p+1}, s_{2p+1}), \ldots\ldots, G(r_{2p+1}, s_{2p+1}), F(n, \jmath)) \\ &\qquad + Y(G(s_{2p+1}, r_{2p+1}), G(s_{2p+1}, r_{2p+1}), \ldots\ldots, G(s_{2p+1}, r_{2p+1}), F(\jmath, n))]. \end{aligned} \tag{6.9}$$

By using (6.1) and interchanging the roles of F and G we obtain

$$Y(G(r_{2p+1}, s_{2p+1}), G(r_{2p+1}, s_{2p+1}), \ldots\ldots, G(r_{2p+1}, s_{2p+1}), F(n, \jmath))$$

$$
\begin{aligned}
&+ Y(G(s_{2p+1}, r_{2p+1}), G(s_{2p+1}, r_{2p+1}), \ldots\ldots, G(s_{2p+1}, r_{2p+1}), F(\jmath, n))\\
&\leq a_1 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (n, \jmath))\\
&\quad + a_2 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (G(r_{2p+1}, s_{2p+1}), G(s_{2p+1}, r_{2p+1})))\\
&\quad + a_3 D((n, \jmath), (n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (F(n, \jmath), F(\jmath, n)))\\
&\quad + a_4 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (F(n, \jmath), F(\jmath, n)))\\
&\quad + a_5 D((n, \jmath), (n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (G(r_{2p+1}, s_{2p+1}), G(s_{2p+1}, r_{2p+1})))\\
&\quad + a_6 (mit\{D((n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (F(n, \jmath), F(\jmath, n))),\\
&\qquad D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (F(n, \jmath), F(\jmath, n))),\\
&\qquad D((n, \jmath), (n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (G(r_{2p+1}, s_{2p+1}), G(s_{2p+1}, r_{2p+1})))\}\\
&= a_1 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (n, \jmath))\\
&\quad + a_2 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+2}, s_{2p+2}))\\
&\quad + a_3 D((n, \jmath), (n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (F(n, \jmath), F(\jmath, n)))\\
&\quad + a_4 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (F(n, \jmath), F(\jmath, n)))\\
&\quad + a_5 D((n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (r_{2p+2}, s_{2p+2}))\\
&\quad + a_6 (min\{D((n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (F(n, \jmath), F(\jmath, n))),\\
&\qquad\qquad D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (F(n, \jmath), F(\jmath, n))),\\
&\qquad\qquad D((n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (r_{2p+2}, s_{2p+2})).
\end{aligned} \tag{6.10}
$$

It follows (6.9) and (6.10) that

$$
\begin{aligned}
&Y(n, n, \ldots\ldots, n, F(n, \jmath)) + Y(\jmath, \jmath, \ldots\ldots, \jmath, F(\jmath, n))\\
&\quad \leq k[(p-1)Y(n, n, \ldots\ldots, n, r_{2p+2}) + (p-1)Y(\jmath, \jmath, \ldots\ldots, \jmath, s_{2p+2})\\
&\qquad + a_1 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (n, \jmath))\\
&\qquad + a_2 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (r_{2p+2}, s_{2p+2}))\\
&\qquad + a_3 D((n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (F(n, \jmath), F(\jmath, n)))\\
&\qquad + a_4 D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (F(n, \jmath), F(\jmath, n)))\\
&\qquad + a_5 D((n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (r_{2p+2}, s_{2p+2}))\\
&\qquad + a_6 (min\{D((n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (F(n, \jmath), F(\jmath, n))),\\
&\qquad\qquad D((r_{2p+1}, s_{2p+1}), (r_{2p+1}, s_{2p+1}), \ldots\ldots, (r_{2p+1}, s_{2p+1}), (F(n, \jmath), F(\jmath, n))),\\
&\qquad\qquad D((n, \jmath), (n, \jmath), \ldots\ldots, (n, \jmath), (r_{2p+2}, s_{2p+2}))].
\end{aligned} \tag{6.11}
$$

Taking the limit as $p \to \infty$ in above inequality, we have

$$
\begin{aligned}
&Y(n, n, \ldots\ldots, n, F(n, \jmath)) + Y(\jmath, \jmath, \ldots\ldots, \jmath, F(\jmath, n))\\
&\quad \leq k[(p-1)Y(n, n, \ldots\ldots, n, n) + (p-1)Y(\jmath, \jmath, \ldots\ldots, \jmath, \jmath)
\end{aligned}
$$

$$\begin{aligned}
&+ a_1 D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(n,\jmath)) \\
&+ a_2 D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(n,\jmath)) \\
&+ a_3 D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(F(n,\jmath),F(\jmath,n))) \\
&+ a_4 D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(F(n,\jmath),F(\jmath,n))) \\
&+ a_5 D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(n,\jmath)) \\
&+ a_6 (min\{D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(F(n,\jmath),F(\jmath,n))), \\
&\quad D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(F(n,\jmath),F(\jmath,n))) \\
&\quad D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(n,\jmath))\}) \\
&= k[a_3 D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(F(n,\jmath),F(\jmath,n))) \\
&\quad + a_4 D((n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(F(n,\jmath),F(\jmath,n)))].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&Y(n,n,\ldots\ldots,n,F(n,\jmath)) + Y(\jmath,\jmath,\ldots\ldots,\jmath,F(\jmath,n)) \\
&\quad \leq k(a_3+a_4) D((n,\jmath),(n,\jmath),(n,\jmath),\ldots\ldots,(n,\jmath),(F(n,\jmath),F(\jmath,n))) \\
&\quad = k(a_3+a_4)[Y(n,n,\ldots\ldots,n,F(n,\jmath)) + Y(\jmath,\jmath,\ldots\ldots,\jmath,F(\jmath,n))]
\end{aligned}$$

Since, $0 \leq k(a_3+a_4) < 1$, remark (6.16)(6) shows

$Y(n,n,\ldots\ldots,n,F(n,\jmath)) + Y(\jmath,\jmath,\ldots\ldots,\jmath,F(\jmath,n)) = \theta$,

that is, $F(n,\jmath) = n$ and $F(\jmath,n) = \jmath$. Similarly, one can show that $G(n,\jmath) = n$ and $G(\jmath,n) = \jmath$. This proves that $(n,\jmath)$ is a coupled common fixed point of F and G and this finishes the proof.

To guarantee the uniqueness of coupled fixed point in Theorem 6.19 and 6.20 we give the condition.

For each $(r,s),(w,v) \in U \times U$ there is $(z,u) \in U \times U$ that is comparable to (r,s) and (w,v). (6.12)

Theorem 6.21. *Including the condition* (6.12) *to the assumption of Theorem 6.19 (respectively Theorem 6.20) we acquire the uniqueness of the coupled common fixed point of F and G. Furthermore, any fixed point of F is a fixed point of G and conversely.*

Proof: Suppose $(n,\jmath)$ and (ζ,ζ') are coupled common fixed points of F and G, thus, $n = F(n,\jmath)$, $\zeta = G(\zeta,\zeta')$, $\jmath = F(\jmath,n)$ and $\zeta' = G(\zeta',\zeta)$. We shall prove that $n = \zeta$, $\jmath = \zeta'$.

Suppose that $(n,\jmath) \leq (\zeta,\zeta')$ without loss of generality, then it follows from Theorem 6.19.

$$
\begin{aligned}
&Y(n,n,\ldots\ldots,n,\zeta)+Y(\jmath,\jmath,\ldots\ldots,\jmath,\zeta')\\
&\quad=Y(F(n,\jmath),F(n,\jmath),\ldots,F(n,\jmath),G(\zeta,\zeta'))\\
&\quad\ +Y(F(\jmath,n),F(\jmath,n),\ldots,F(\jmath,n),G(\zeta',\zeta))\\
&\quad\le a_1D((n,\jmath),(n,\jmath),(n,\jmath),\ldots,(n,\jmath),(\zeta,\zeta'))\\
&\quad\ +a_2D((n,\jmath),(n,\jmath),\ldots,(n,\jmath),(F(n,\jmath),F(\jmath,n)))\\
&\quad\ +a_3D((\zeta,\zeta'),(\zeta,\zeta'),\ldots,(\zeta,\zeta'),(G(\zeta,\zeta'),G(\zeta',\zeta)))\\
&\quad\ +a_4D((n,\jmath),(n,\jmath),\ldots,(n,\jmath),(G(\zeta,\zeta'),G(\zeta',\zeta)))\\
&\quad\ +a_5D((\zeta,\zeta'),(\zeta,\zeta'),\ldots,(\zeta,\zeta'),(F(n,\jmath),F(\jmath,n)))\\
&\quad\ +a_6(min\{D((\zeta,\zeta'),(\zeta,\zeta'),\ldots,(\zeta,\zeta'),(G(\zeta,\zeta'),G(\zeta',\zeta))),\\
&\qquad\qquad D((n,\jmath),(n,\jmath),\ldots,(n,\jmath),(G(\zeta,\zeta'),G(\zeta',\zeta))),\\
&\qquad\qquad D((\zeta,\zeta'),(\zeta,\zeta'),\ldots,(\zeta,\zeta'),(F(n,\jmath),F(\jmath,n)))\})\\
&\quad=a_1D((n,\jmath),(n,\jmath),(n,\jmath),\ldots,(n,\jmath),(\zeta,\zeta'))\\
&\quad\ +a_2D((n,\jmath),(n,\jmath),\ldots,(n,\jmath),(n,\jmath))\\
&\quad\ +a_3D((\zeta,\zeta'),(\zeta,\zeta'),\ldots,(\zeta,\zeta'),(\zeta,\zeta'))\\
&\quad\ +a_4D((n,\jmath),(n,\jmath),\ldots,(n,\jmath),(\zeta,\zeta'))\\
&\quad\ +a_5D((\zeta,\zeta'),(\zeta,\zeta'),\ldots,(\zeta,\zeta'),(n,\jmath))\\
&\quad\ +a_6(min\{D((\zeta,\zeta'),(\zeta,\zeta'),\ldots,(\zeta,\zeta'),(\zeta,\zeta')),\\
&\qquad\qquad D((n,\jmath),(n,\jmath),\ldots,(n,\jmath),(\zeta,\zeta')),\\
&\qquad\qquad D((\zeta,\zeta'),(\zeta,\zeta'),\ldots,(\zeta,\zeta'),(n,\jmath))\})
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Y(n,n,\ldots\ldots,n,\zeta)+Y(\jmath,\jmath,\ldots,\jmath,\zeta')&\le(a_1+a_4+a_5)D((n,\jmath),(n,\jmath),\ldots,(n,\jmath),(\zeta,\zeta'))\\
&=(a_1+a_4+a_5)Y(n,n,\ldots,n,\zeta)+Y(\jmath,\jmath,\ldots,\jmath,\zeta').
\end{aligned}
$$

Since, $0\le(a_1+a_4+a_5)<1$, remark (6.16)(6) shows $Y(n,n,\ldots,n,\zeta)+Y(\jmath,\jmath,\ldots,\jmath,\zeta')=\theta$, which implies $n=\zeta$ and $\jmath=\zeta'$.

Now, we show that any fixed point of F is a fixed point of G and conversely. Applying Theorem 6.19, we get

$$
\begin{aligned}
&Y(n,n,\ldots,n,\jmath)+Y(\jmath,\jmath,\ldots,\jmath,n)\\
&\quad=Y(F(n,\jmath),F(n,\jmath),F(n,\jmath),\ldots,F(n,\jmath),G(\jmath,n))\\
&\quad\ +Y(F(\jmath,n),F(\jmath,n),F(\jmath,n),\ldots,F(\jmath,n),G(n,\jmath))\\
&\quad\le a_1D((n,\jmath),(n,\jmath),(n,\jmath),\ldots,(n,\jmath),(\jmath,n))\\
&\quad\ +a_2D((n,\jmath),(n,\jmath),\ldots,(n,\jmath),(F(n,\jmath),F(\jmath,n)))
\end{aligned}
$$

$$
\begin{aligned}
&+ a_3 D((\jmath, n), (\jmath, n), (\jmath, n), \ldots, (\jmath, n), (G(\jmath, n), G(n, \jmath))) \\
&+ a_4 D((n, \jmath), (n, \jmath), \ldots, (n, \jmath), (G(\jmath, n), G(n, \jmath))) \\
&+ a_5 D((\jmath, n), (\jmath, n), \ldots, (\jmath, n), (F(n, \jmath), F(\jmath, n))) \\
&+ a_6 (min\{D((\jmath, n), (\jmath, n), \ldots, (\jmath, n), (G(\jmath, n), G(n, \jmath))), \\
&\qquad D((n, \jmath), (n, \jmath), (n, \jmath), \ldots, (n, \jmath), (G(\jmath, n), G(n, \jmath))), \\
&\qquad D((\jmath, n), (\jmath, n), \ldots, (\jmath, n), (F(n, \jmath), F(\jmath, n)))\}) \\
= {}& a_1 D((n, \jmath), (n, \jmath), \ldots, (n, \jmath), (\jmath, n)) \\
&+ a_2 D((n, \jmath), (n, \jmath), (n, \jmath), \ldots, (n, \jmath), (n, \jmath)) \\
&+ a_3 D((\jmath, n), (\jmath, n), (\jmath, n), \ldots, (\jmath, n), (\jmath, n)) \\
&+ a_4 D((n, \jmath), (n, \jmath), (n, \jmath), \ldots, (n, \jmath), (\jmath, n)) \\
&+ a_5 D((\jmath, n), (\jmath, n), \ldots, (\jmath, n), (n, \jmath)) \\
&+ a_6 (min\{D((\jmath, n), (\jmath, n), \ldots, (\jmath, n), (\jmath, n)), \\
&\qquad D((n, \jmath), (n, \jmath), (n, \jmath), \ldots, (n, \jmath), (\jmath, n)), \\
&\qquad D((\jmath, n), (\jmath, n), \ldots, (\jmath, n), (n, \jmath))\}) \\
= {}& (a_1 + a_4 + a_5) D((n, \jmath), (n, \jmath), (n, \jmath), \ldots, (n, \jmath), (\jmath, n)) \\
= {}& (a_1 + a_4 + a_5) Y(n, n, \ldots, n, \jmath) + Y(\jmath, \jmath, \ldots, \jmath, n).
\end{aligned}
$$

Since, $0 \leq (a_1 + a_4 + a_5) < 1$, Remark 6.16(6) shows $Y(n, n, \ldots, n, \jmath) + Y(\jmath, \jmath, \ldots, \jmath, n) = \theta$, which implies $n = \jmath$. The coupled common fired point of F and G is unique. This finishes the proof.

Example 6.2. *Presume $(U, \leq, Y)$ be a totally ordered complete Y-cone metric space with Y-cone metric defined as in Example (6.1). Let $F, G \colon U \times U \to U$ as $F(r, s) = G(r, s) = \frac{(r+2s)}{7}$ for all $r, s \in U$.*

The pair (F, G) has the property of mixed weakly monotone on U.

$$
\begin{aligned}
&Y(F(r, s), F(r, s), \ldots\ldots, F(r, s), G(t, v)) + Y(F(s, r), F(s, r), \ldots\ldots, F(s, r), G(s, r)) \\
&= [(n-1)|F(r, s) - G(t, v)|^2 + (n-1)|F(s, r) - G(v, t)|^2] e^w \\
&= \left[(n-1) \left| \frac{r+2s}{7} - \frac{t+2v}{7} \right|^2 + (n-1) \left| \frac{s+2r}{7} - \frac{v+2t}{7} \right|^2 \right] e^w \\
&= \left[(n-1) \left| (\frac{r}{7} - \frac{t}{7}\right) + \left(\frac{2s}{7} - \frac{2v}{7}) \right|^2 + (n-1) \left| (\frac{s}{7} - \frac{v}{7}) + (\frac{2t}{7} - \frac{2r}{7}) \right|^2 \right] e^w \\
&\leq 2(n-1) \left[\left| \frac{r}{7} - \frac{t}{7} \right|^2 + \left| \frac{2s}{7} - \frac{2v}{7} \right|^2 + \left| \frac{s}{7} - \frac{v}{7} \right|^2 + \left| \frac{2t}{7} - \frac{2r}{7} \right|^2 \right] e^w \\
&= \frac{2(n-1)}{49} [|r-t|^2 + |2s-2v|^2 + |s-v|^2 + |2t-2r|^2] e^w
\end{aligned}
$$

$$\leq \frac{2(n-1)}{7}[|r-t|^2 + |s-v|^2]e^w$$
$$= \frac{2}{7}D((r,s),(r,s),\ldots\ldots,(r,s),(t,v))$$

where $a_1 = \frac{2}{7}, a_2 = a_3 = a_4 = a_5 = a_6 = 0$. *Hence, the conditions of Theorem 6.19 are satisfied. Moreover,* $(0,0)$ *is the unique coupled common fixed point of* F *and* G.

Corollary 6.22. *Presume* $(U, \leq, Y)$ *be a partially ordered complete* Y*-cone metric space with the coefficient* $k \geq 1$ *relative to a solid cone* P. *Presuppose* $F \colon U \times U \to U$ *be the mappings and possesses the property of mixed monotone on* U. *Suppose that there exist* $a_i \geq 0$ *with* $a_1 + a_2 + a_3 + 2ka_4 < 1$ *and* $\sum_i^6 a_i < 1$ *such that*

$$\begin{aligned}
&Y(F(r,s),F(r,s),\ldots\ldots,F(r,s),F(w,v))\\
&\quad + Y(F(s,r),F(s,r),\ldots\ldots,F(s,r),F(v,w))\\
&\leq a_1 D(((r,s),(r,s),\ldots\ldots,(r,s),(w,v)))\\
&\quad + a_2 D((r,s),(r,s),\ldots\ldots,(r,s),(F(r,s),F(s,r)))\\
&\quad + a_3 D((w,v),(w,v),\ldots\ldots,(w,v),(F(w,v),F(v,w)))\\
&\quad + a_4 D((r,s),(r,s),\ldots\ldots,(r,s),(F(w,v),F(w,v)))\\
&\quad + a_5 D((w,v),(w,v),\ldots\ldots,(w,v),(F(r,s),F(s,r)))\\
&\quad + a_6 (min\{D((w,v),(w,v),\ldots\ldots,(w,v),(F(w,v),F(v,w))),\\
&\qquad D((r,s),(r,s),\ldots\ldots,(r,s),(F(w,v),F(w,v))),\\
&\qquad D((w,v),(w,v),\ldots\ldots,(w,v),(F(r,s),F(s,r)))\}) \qquad (6.13)
\end{aligned}$$

$\forall\, r,s,v,w \in U$ *with* $r \leq w$ *and* $s \geq v$, *where* D *is defined as in Lemma 6.18.*

Presuppose either F *is continuous or* U *has the subsequent properties*

(a) *if an increasing sequence* $\{r_p\}$ *in* U *converges to some point* $n \in U$, *then* $r_p \leq n$, $\forall\, p$,

(b) *if a decreasing sequence* $\{s_p\}$ *in* U *converges to some point* $\jmath \in U$, *then* $s_p \geq \jmath$, $\forall\, p$.

Furthermore for each $r_0, s_0 \in U$ *with* $r_0 \leq F(r_0, s_0)$ *and* $s_0 \geq F(s_0, r_0)$, *then* F *has a coupled fixed point.*

Proof. Taking $G = F$ in Theorems 6.19 and 6.20 and using Remark 6.17, we acquire the corollary.

Corollary 6.23. *Presume* $(U, \leq, Y)$ *be a partially ordered complete* Y*-cone metric space with the coefficient* $k \geq 1$ *relative to a solid cone* P. *Suppose* $F \colon U \times U \to U$ *be the* *mappings and possesses the property of mixed monotone on U. Presuppose that there exist $K \in [0, 1)$ such that*

$$\begin{aligned} &Y(F(r,s), F(r,s), F(r,s), \ldots\ldots, F(r,s), F(w,v)) \\ &\quad + Y(F(s,r), F(s,r), F(s,r), \ldots\ldots, F(s,r), F(v,w)) \\ &\leq K(Y(r,r,r,\ldots\ldots,r,w) + Y(s,s,s,\ldots\ldots,s,v)) \end{aligned} \tag{6.14}$$

$\forall\ r, s, w, v \in U$ *with* $r \leq w$ *and* $s \geq v$.

Presuppose either F is continuous or U has the subsequent properties

(a) if an increasing sequence $\{r_p\}$ in U converges to some point $n \in U$, then $r_p \leq n$, $\forall\, p$,

(b) if a decreasing sequence $\{s_p\}$ in U converges to some point $\jmath \in U$, then $s_p \geq \jmath$, $\forall\, p$.

Furthermore for each $r_0, s_0 \in U$ such that $r_0 \leq F(r_0, s_0)$ and $s_0 \geq F(s_0, r_0)$, then F has a coupled fixed point.

Proof. Taking $G = F$ and $a_1 = K, a_2 = a_3 = a_4 = a_5 = a_6 = 0$ in Theorems 6.19 and 6.20 and using Remark 6.17, we obtain the corollary.

Chapter 7

Results on Coupled Fixed Point in Partially Ordered Metric Spaces

In this chapter, certain unique coupled fixed point results in ordered metric space have been proved. This chapter has been divided into two sections. In section 7.1, some coupled fixed point theorems have been established. An example is also given in order to illustrate the effectiveness of our result at the end of the Section. In section 7.2, coupled fixed point results have been proved for rational expressions in partially ordered metric spaces.

7.1 Coupled Fixed Point Results in Partially Ordered Metric Spaces

In this segment, the following result has been established in complete ordered metric spaces.

Theorem 7.1. *Suppose* $(X, \leq)$ *be a partially ordered set endowed with a metric* d *so* (X, d) *is complete. Let a map* $F\colon X \times X \to X$ *possess the property of mixed monotone on* X *and there exist* $r_0, s_0 \in X$ *with* $r_0 \leq F(r_0,\ s_0)$ *and* $s_0 \geq F(s_0,\ r_0)$*. Let there exist* $\psi\colon [0, \infty) \to [0, \infty)$ *is a continuous and non decreasing function, it is positive in* $(0, \infty), \psi(0) = 0$ *and* $\lim\limits_{t\to\infty} \psi(t) = \infty$*; so that*

$$d(F(r, s), F(t, v)) \leq d(r, t) + \psi(d(s, v)) \tag{7.1}$$

for all $r, t, s, v \in U$*, with* $t \leq r$*,* $v \geq s$*. Presuppose,* *I) F is continuous or*

II) X has the subsequent properties,

(a) if an increasing sequence $\{r_n\}$ in X converges to some point $t^* \in X$, then $r_n \leq t^*$, for each n,

(b) if a decreasing sequence $\{s_n\}$ in X converges to some point $v^* \in X$, then $s_n \geq v^*$, for each n.

Then F has a coupled fixed point $(t^*, v^*) \in X \times X$.

Proof. Take $r_0, s_0 \in X$ and set $r_1 = F(r_0, s_0)$ and $s_1 = F(s_0, r_0)$. Repeating this process, set $r_{n+1} = F(r_n, s_n)$ and $s_{n+1} = F(s_n, r_n)$. Then by (7.1), we have

$$d(r_n, r_{n+1}) = d(F(r_{n-1}, s_{n-1}), F(r_n, s_n)) \leq d(r_{n-1}, r_n) + \psi(d(s_{n-1}, s_n)), \tag{7.2}$$

and similarly,

$$d(s_n, s_{n+1}) = d(F(s_{n-1}, r_{n-1}), F(s_n, r_n)) \leq d(s_{n-1}, s_n) + \psi(d(r_{n-1}, r_n)). \tag{7.3}$$

By adding, we have

$$p_n \leq p_{n-1} + \psi(p_{n-1}). \tag{7.4}$$

Let $\quad p_n = d(r_n, r_{n+1}) + d(s_n, s_{n+1}).$

If $\exists\ n_1 \in N^*$ such that $d(r_{n_1}, r_{n_1-1}) = 0$, $d(s_{n_1}, s_{n_1-1}) = 0$, then $r_{n_1-1} = r_{n_1} = F(r_{n_1-1}, s_{n_1-1}), s_{n_1-1} = s_{n_1} = F(s_{n_1}, r_{n_1-1})$ and $r_{n_1-1}; s_{n_1-1}$ is fixed point of F and this completes the proof. In other case $d(r_{n+1}, r_n) \neq 0$; $d(s_{n+1}, s_n) \neq 0$,$\forall\, n \in \mathbb{N}$. Then by using assumption on ψ , we have,

$$p_n \leq p_{n-1} + \psi(p_{n-1}) \leq p_{n-1} \tag{7.5}$$

p_n is a non - negative sequence and posses a limit δ^*. Taking limit $n \to \infty$, we have

$$\delta^* \leq \delta^* + \psi(\delta^*)$$

and as a result $\psi(\delta^*) = 0$. Though our assumption on ψ , we conclude $\delta^* = 0$, ie. $\lim\limits_{n\to\infty} (p_n) = 0$.

$$\lim_{n\to\infty} d(r_{n+1}, r_n) + d(s_{n+1}, s_n) = 0.$$
$$\implies \lim_{n\to\infty} d(r_{n+1}, r_n) = \lim_{n\to\infty} d(s_{n+1}, s_n) = 0. \tag{7.6}$$

Next, we prove that $\{r_n\}$, $\{s_n\}$ are Cauchy sequences. Presuppose that atleast one $\{r_n\}$ or $\{s_n\}$ be not a Cauchy sequence. Then $\exists\ \varepsilon{>}0$ and subsequence of integers n_k, m_k with $n_k > m_k \geq k$, such that

$$r_k = d(r_{m_k}, r_{n_k}) + d(s_{m_k}, s_{n_k}) \geq \varepsilon,\ \forall\ k = 1, 2, 3.... \tag{7.7}$$

Further, conforming to m_k, we can take n_k in such a manner that it is least integer with $n_k > m_k \geq k$ fulfilling equation (7.7), we have

$$d(r_{m_k}, r_{n_k-1}) + d(s_{m_k}, s_{n_k-1}) < \varepsilon. \tag{7.8}$$

Using (7.7) and (7.8) and triangle inequality, we get

$$\begin{aligned}\varepsilon \leq r_k &= d(r_{m_k}, r_{n_k}) + d(s_{m_k}, s_{n_k})\\ &\leq d(r_{m_k}, r_{n_k-1}) + d(r_{n_k-1}, r_{n_k}) + d(s_{m_k}, s_{n_k-1}) + d(s_{n_k-1}, s_{n_k})\\ &= d(r_{m_k}, r_{n_k-1}) + d(s_{m_k}, s_{n_k-1}) + d(r_{n_k-1}, r_{n_k}) + d(s_{n_k-1}, s_{n_k})\\ &< \varepsilon + p_{n_k-1}.\end{aligned} \tag{7.9}$$

Letting $k \to \infty$ and applying (7.6), we get

$$\lim_{n,m\to\infty} r_k = \varepsilon > 0. \tag{7.10}$$

Now, we get

$$\begin{aligned}d(r_{m_k+1}, r_{n_k+1}) &= d(F(r_{m_k}, s_{m_k}), F(r_{n_k}, s_{n_k}))\\ &= d(F(r_{n_k}, s_{n_k}), F(r_{m_k}, s_{m_k}))\\ &\leq d(r_{n_k}, r_{m_k}) + \psi(p(s_{n_k}, s_{m_k})).\end{aligned} \tag{7.11}$$

Similarly,

$$\begin{aligned}d(s_{m_k+1}, s_{n_k+1}) &= d(F(s_{m_k}, r_{m_k}), F(s_{n_k}, r_{n_k}))\\ &= d(F(s_{n_k}, r_{n_k}), F(s_{m_k}, r_{m_k}))\\ &\leq d(s_{n_k}, s_{m_k}) + \psi(d(r_{n_k}, r_{m_k})).\end{aligned} \tag{7.12}$$

Using (7.11) and (7.12), we get

$$r_{k+1} \leq r_k + \psi(r_k) \tag{7.13}$$

$\forall\ k \in 1, 2, 3, \ldots$ taking $k \to \infty$ of both sides of equation (7.13) and from equation (7.10), it follows that $\varepsilon = \lim\limits_{k\to\infty} r_{k+1} \leq \lim\limits_{k\to\infty} r_k + \psi(r_k) < \varepsilon$ a contradiction. Therefore $\{r_n\}$ and

$\{s_n\}$ are Cauchy sequences. We now prove that $F(t^*, v^*) = t^*, F(v^*, t^*) = v^*$. We shall distinguish the cases (I), II(a) and II(b) of the Theorem 7.1.

As X is complete, $\exists\ t^*, v^* \in X$ so $\lim\limits_{n\to\infty} r_n = t^*$, $\lim\limits_{n\to\infty} s_n = v^*$.

We now show that if the assumption (1) holds, then (t^*, v^*) is coupled fixed point of F.

As, we have

$$t^* = \lim_{n\to\infty} r_{n+1} = \lim_{n\to\infty} F(r_n, s_n) = F\Big(\lim_{n\to\infty} r_n, \lim_{n\to\infty} s_n\Big) = F(t^*, v^*),$$
$$v^* = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} F(s_n, r_n) = F\Big(\lim_{n\to\infty} s_n, \lim_{n\to\infty} r_n\Big) = F(v^*, t^*)$$

Therefore, (t^*, v^*) is coupled fixed point of F.

Suppose now that the condition II(a) and II(b) of the theorem holds. The sequence $\{r_n\} \to t^*$, $\{s_n\} \to v^*$

$$\begin{aligned} d(F(t^*, v^*), t^*) &\leq d(F(t^*, v^*), r_{n+1}) + d(r_{n+1}, t^*) \\ &= d(F(t^*, v^*), F(r_n, s_n)) + d(r_{n+1}, t^*) \\ &\leq d(t^*, r_n) + \psi(d(v^*, s_n)) + d(r_{n+1}, t^*). \end{aligned}$$

Letting $n\to\infty$, we have
$$d(F(t^*, v^*), t^*) \leq 0 + \psi(0) = 0.$$

This implies that $F(t^*, v^*) = t^*$, similarly, we can show that $F(v^*, t^*) = v^*$. This completes the theorem.

Theorem 7.2. *Suppose the assumptions of Theorem 7.1 hold. In addition, suppose that $z \in X$ is comparable to t and v for all $t, v \in X$. Then F has a unique coupled fixed point.*

Proof. Presuppose $(t', v'), (t^*, v^*) \in X \times X$ are coupled fixed points of F.

Consider the subsequent two cases:

Case 1: (t', v') and (t^*, v^*) are comparable. We get

$$d(t', t^*) = d(F(t', v'), F(t^*, v^*)) \leq d(t', t^*) + \psi(d(v', v^*)),$$

similarly,
$$d(v', v^*) = d(F(v', u'), F(v^*, u^*)) \leq d(v', v^*) + \psi(d(u', u^*)).$$

It follows that

$$\begin{aligned} d(t', t^*) + d(v', v^*) &\le d(t', t^*) + d(v', v^*) + \psi[d(v', v^*) + d(t', t^*)] \\ &\implies d(t', t^*) + d(v', v^*) = 0 \end{aligned}$$

So, $t' = t^*,\ v' = v^*$. The proof is complete.

Case 2: Suppose now that (t', v') and (t^*, v^*) are not comparable. Choose an element $(w, z) \in X$ comparable with both of them.

Monotonicity $\implies (F^n(w, z), F^n(z, w))$

$$\begin{aligned} d\begin{pmatrix} (t^*, v^*) \\ (t', v') \end{pmatrix} &= d\left(\begin{pmatrix} F^n(t^*, v^*) \\ F^n(v^*, t^*) \end{pmatrix}, \begin{pmatrix} F^n(t', v') \\ F^n(v', t') \end{pmatrix}\right) \\ &\le d\left(\begin{pmatrix} F^n(t^*, v^*) \\ F^n(v^*, t^*) \end{pmatrix}, \begin{pmatrix} F^n(w, z) \\ F^n(z, w) \end{pmatrix}\right) + d\left(\begin{pmatrix} F^n(w, z) \\ F^n(z, w) \end{pmatrix}, \begin{pmatrix} F^n(t', v') \\ F^n(v', t') \end{pmatrix}\right) \\ &\le d(t^*, w) + \psi(d(v^*, z))) + (d(v^*, z) + \psi(d(t^*, w))) \\ &\quad + (d(w, t') + \psi(d(z, v'))) + (d(z, v') + \psi(d(w, t'))) = 0. \end{aligned}$$

so $t^* = t', v^* = v'$. The proof is complete.

Example 7.1. *Suppose* $X = [0, \infty)$ *be endowed with the standard metric* $d(r, s) = |r - s|, \forall\, r, s \in X$*. Then* (X, d) *is complete metric space.*

Consider the map $F\colon X \times X \to X$ *defined by*

$$F(r, s) = \frac{r - 2s}{3};\ 2s{\le}r.$$

Let us take $\psi\colon [0, \infty) \to [0, \infty)$ *such that* $\psi(t) = \frac{2t}{3}$.

Evidently F *is continuous and has the property of mixed monotone. Moreover there are* $r_0 = 0$; s_0 *=0 in* X *such that* $r_0 = 0 \le F(0, 0) = F(r_0, s_0)$ *and* $s_0 = 0 \ge F(0, 0) = F(s_0, r_0)$.

Then it is obvious $(0, 0)$ *is the unique coupled fixed point of* F*.*

Now, we have following possibility for values of (r, s) *and* (t, v) *such that* $r{\ge}t,\ v{\ge}s$

$$\begin{aligned} d(F(r, s), F(t, v)) = d\left(\frac{r - 2s}{3}, \frac{t - 2v}{3}\right) &= \frac{1}{3}\left|(r - t) - 2(s - v)\right| \\ &\le \frac{1}{3}|(r - t)| + \frac{2}{3}|(s - v)| \\ &\le |(r - t)| + \frac{2}{3}|(s - v)| = d(r, t) + \psi(d(s, v)). \end{aligned}$$

Therefore, all the assumptions of Theorem 7.1 hold.

Hence, F *has a coupled fixed point in X.*

7.2 Coupled Fixed Point Results for Rational Contractions

In the following section, some coupled fixed point theorems for rational contractions have been presented.

Theorem 7.3. *Suppose $(X, \leq)$ be a partially ordered set endowed with a metric d so (X, d) is complete. Presume a map F: $X \times X \to X$ possess the property of mixed monotone on X. Let $\alpha, \beta \in [0, 1)$ and $L \geq 0$ be positive real numbers with $(\beta + \alpha) < 1$, for all $r, s, v, t \in X$, satisfy*

$$\begin{aligned} d(F(r,s), F(t,v)) \leq \alpha \frac{d(r, F(r,s)) d(t, F(t,v))}{d(r,t)} + \beta d(r,t) \\ + L\, min\{d(r, F(r,s)), d(t, F(t,v)), \\ d(t, F(r,s)), d(r, F(t,v))\}. \end{aligned} \tag{7.14}$$

Presuppose

I) F is continuous or

II) X has the subsequent properties,

(a) if an increasing sequence $\{r_n\}$ in X converges to some point $r \in X$, then $r_n \leq r$, for each n,

(b) if a decreasing sequence $\{s_n\}$ in X converges to some point $s \in X$, then $s_n \geq s$, for each n.

Then F has a coupled fixed point.

Proof. Take $r_0,\ s_0 \in X$. Set $r_1 = F(r_0, s_0)$ and $s_1 = F(s_0, r_0)$. Repeating this process, set $r_{n+1} = F(r_n, s_n)$ and $s_{n+1} = F(s_n, r_n)$. Then by (7.14), we have

$$\begin{aligned} d(r_n, r_{n+1}) &= d(F(r_{n-1}, s_{n-1}), F(r_n, s_n)) \\ &\leq \alpha \frac{d(r_{n-1}, F(r_{n-1}, s_{n-1})) d(r_n, F(r_n, s_n))}{d(r_{n-1}, r_n)} + \beta d(r_{n-1}, r_n) \\ &\quad + L\, min\{d(r_{n-1}, F(r_{n-1}, s_{n-1})), d(r_n, F(r_n, s_n)), \\ &\qquad d(r_n, F(r_{n-1}, s_{n-1})), d(r_{n-1}, F(r_n, s_n))\} \\ &= \alpha \frac{d(r_{n-1}, r_n) d(r_n, r_{n+1})}{d(r_{n-1}, r_n)} + \beta d(r_{n-1}, r_n) \\ &\quad + L\, min\{d(r_{n-1}, r_n), d(r_n, r_{n+1}), d(r_n, r_n), d(r_{n-1}, r_{n+1})\}, \end{aligned}$$

which implies that

$$d(r_n, r_{n+1}) \leq \frac{\beta}{1-\alpha} d(r_{n-1}, r_n). \tag{7.15}$$

Similarly, we get

$$d(s_n, s_{n+1}) \leq \frac{\beta}{1-\alpha} d(s_{n-1}, s_n). \tag{7.16}$$

By adding (7.15) and (7.16), we have

$$d_n \leq \frac{\beta}{1-\alpha} d_{n-1} \tag{7.17}$$

Let $\qquad d_n = d(r_n, r_{n+1}) + d(s_n, s_{n+1}).$

Consequently, if we set $\lambda = \frac{\beta}{1-\alpha}$, then we have

$$d_n \leq \lambda d_{n-1} \leq \ldots \leq \lambda^n d_0. \tag{7.18}$$

If $d_0 = 0$, then (r_0, s_0) is a coupled fixed point of F.

Presume $d_0 \geq 0$. Thus, for every $k \in \mathbb{N}$, by repeated application of triangle inequality, we obtain

$$\begin{aligned} d(r_n, r_{n+k}) + d(s_n, s_{n+k}) &\leq [d(r_n, r_{n+1}) + d(r_{n+1}, r_{n+2}) + \ldots + d(r_{n+k-1}, r_{n+k})] \\ &\quad + [d(s_n, s_{n+1}) + d(s_{n+1}, s_{n+2}) + \ldots + d(s_{n+k-1}, s_{n+k})] \\ &\leq d_n + d_{n+1} + \ldots + d_{n+k-1} \\ &\leq \frac{\lambda^n(1-\lambda^k)}{1-\lambda} d_0 \to 0 \quad as \quad n \to \infty. \end{aligned} \tag{7.19}$$

Therefore $\{r_n\}$ and $\{s_n\}$ are Cauchy sequences.

As X is complete, $\exists\ r, s \in X$ so $\lim\limits_{n\to\infty} r_n = r$, $\lim\limits_{n\to\infty} s_n = s$. Now, we show that if the assumption (I) holds, then (r, s) is coupled fixed point of F.

As, we have

$$\begin{aligned} r &= \lim_{n\to\infty} r_{n+1} = \lim_{n\to\infty} F(r_n, s_n) = F\Big(\lim_{n\to\infty} r_n, \lim_{n\to\infty} s_n\Big) = F(r, s), \\ s &= \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} F(s_n, r_n) = F\Big(\lim_{n\to\infty} s_n, \lim_{n\to\infty} r_n\Big) = F(s, r). \end{aligned}$$

Consequently, (r, s) is coupled fixed point of F.

Now, presume that the condition II(a) and II(b) of the theorem holds.
The sequence $\{r_n\} \to r$, $\{s_n\} \to s$

$$\begin{aligned} d(F(r_n, s_n), F(r, s)) \leq \alpha &\frac{d(r_n, F(r_n, s_n))d(r, F(r, s))}{d(r_n, r)} + \beta d(r_n, r) \\ &+ L\ min\{d(r_n, F(r_n, s_n)), d(r, F(r, s)), \end{aligned}$$

$$d(r_n, F(r,s)), d(r, F(r_n, s_n))\}.$$

Letting $n \to \infty$, we have $d(F(r,s), r) \leq 0$.

This implies that $F(r,s) = r$. Similarly, we can show that $F(s,r) = s$. This completes the theorem.

Corollary 7.4. *Suppose ($X, \leq$) be a partially ordered set endowed with a metric d so (X,d) is complete. Presume a map F: $X \times X \to X$ possesses the property of mixed monotone on X. Let there be non-negative real numbers $\beta, \alpha \in [0,1)$ with $\beta + \alpha < 1$ for all $r, s, t, v \in X$, we have*

$$d(F(r,s), F(t,v)) \leq \alpha \frac{d(r, F(r,s))d(t, F(t,v))}{d(r,t)} + \beta d(r,t).$$

Suppose

I) F is continuous or

II) X has the subsequent properties,

(a) if an increasing sequence $\{r_n\}$ in X converges to some point $r \in U$, then $r_n \leq r$, for each n,

(b) if a decreasing sequence $\{s_n\}$ in X converges to some point $s \in U$, then $s_n \geq s$, for each n.

Then F has a coupled fixed point.

Proof. Take $L = 0$ in Theorem 7.3, we acquire Corollary 7.4.

Theorem 7.5. *Let the assumptions of Theorem 7.3 hold. We acquire the uniqueness of the coupled fixed point of F.*

Proof. Let (r,s) and $(\acute{r}, \acute{s})$ are coupled fixed points of F, then, $F(r,s) = r, F(s,r) = s, F(\acute{r}, \acute{s}) = \acute{r}$ and $F(\acute{s}, \acute{r}) = \acute{s}$. We shall prove that $r = \acute{r},\ s = \acute{s}$.

Consider the subsequent two cases:

Case 1: If (r,s) and $(\acute{r}, \acute{s})$ are comparable. We get

$$\begin{aligned} d(\acute{r}, r) = d(F(\acute{r}, \acute{s}), F(r,s)) \leq \alpha \frac{d(\acute{r}, F(\acute{r}, \acute{s}))d(r, F(r,s))}{d(\acute{r}, r)} + \beta d(\acute{r}, r) \\ + L\, min\{d(\acute{r}, F(\acute{r}, \acute{s})), d(r, F(r,s)), \\ d(\acute{r}, F(r,s)), d(r, F(\acute{r}, \acute{s}))\} \end{aligned}$$

$$\leq \beta\, d(\acute{r}, r),$$

which provides $d(\acute{r}, r) \leq 0, \beta < 1$ (a contradiction). Thus $r = \acute{r}$.

In similar way, $\quad d(\acute{s}, s) = d(F(\acute{s}, \acute{r}), F(s, r)) \leq 0.$

Thus, $s = \acute{s}$. Hence, (r, s) is a unique coupled fixed point of F.

Case 2: Presume (r, s) and $(\acute{r}, \acute{s})$ are not comparable. By supposition there is $(z, u) \in X \times X$ comparable with both of them.

We define sequences $\{z_n\}, \{u_n\}$ as follows

$$z_0 = z,\ u_0 = u,\ z_{n+1} = F(z_n, u_n) \text{ and } u_{n+1} = F(u_n, z_n)\ \forall\, n$$

Since (z, u) is comparable with (r, s), we may suppose that $(r, s) \geq (z, u) = (z_0, u_0)$. It is easy to prove by using the mathematical induction, that

$$(r, s) \geq (z_n, u_n),\ \forall\, n. \tag{7.20}$$

From (7.14) and (7.20), we have

$$\begin{aligned} d(F(r, s), F(z_n, u_n)) &\leq \alpha \frac{d(r, F(r, s)) d(z_n, F(z_n, u_n))}{d(r, z_n)} + \beta d(r, z_n) \\ &\quad + L\, min\{d(r, F(r, s)), d(z_n, F(z_n, u_n)), \\ &\qquad d(z_n, F(r, s)), d(r, F(z_n, u_n))\} \\ &\leq \beta d(r, z_n). \end{aligned} \tag{7.21}$$

Similarly, we also have

$$d(u_{n+1}, s) \leq \beta d(u_n, s). \tag{7.22}$$

Adding (7.21) and (7.22), we get

$$\begin{aligned} d(r, z_{n+1}) + d(u_{n+1}, s) &\leq \beta[d(r, z_n) + d(u_n, s)] \\ &\leq \beta^2[d(r, z_{n-1}) + d(u_{n-1}, s)] \\ &\ \ \vdots \\ &\leq \beta^{n+1}[d(r, z_0) + d(u_0, s)] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(u_{n+1}, s) = \lim_{n\to\infty} d(r, z_{n+1}) = 0. \tag{7.23}$$

In a similar way, we can show that

$$\lim_{n\to\infty} d(u_{n+1}, \acute{s}) = \lim_{n\to\infty} d(\acute{r}, z_{n+1}) = 0. \tag{7.24}$$

From (7.23) and (7.24), we obtain $r = \acute{r}$ and $s = \acute{s}$.

Example 7.2. *Suppose $X = [0,1]$ with metric $d(r,s) = max\{r,s\}$, for all $r, s \in X$. Then (X,d) is complete metric space.*

Consider the map $F\colon X \times X \to X$ defined by

$$F(r,s) = \begin{cases} r/(s+5) & \text{if } s \leq r, \\ 0 & \text{if not.} \end{cases}$$

Evidently F is continuous and possesses the property of mixed monotone. Furthermore there are $r_0 = 0$; s_0 =0 in X so $r_0 = 0 \leq F(0,0) = F(r_0, s_0)$ and $s_0 = 0 \geq F(0,0) = F(s_0, r_0)$.

Then it is obvious $(0,0)$ is the unique coupled fixed point of F.

Now, we have following possibilities for values of (r,s) and (t,v) so $r \geq t$, $s \leq v$.

Case 1: If $t \geq v, r \geq s$, we get

$$\begin{aligned} &F(t,v) = \frac{t}{v+5},\ F(r,s) = \frac{r}{s+5}. \\ &\implies d(F(r,s),F(t,v)) = max\left\{\frac{r}{s+5}, \frac{t}{v+5}\right\} = \frac{r}{s+5} \leq \frac{1}{3}\left(\frac{tr}{r} + r\right) \\ &\qquad\qquad \leq \alpha \frac{d(r,F(r,s))d(t,F(t,v))}{d(r,t)} + \beta d(r,t) \\ &\qquad\qquad\quad + L\ min\{d(r,F(r,s)), d(t,F(t,v)), d(t,F(r,s), d(r,F(t,v))\}. \end{aligned}$$

Case 2: If $t < v$, $r \geq s$, we get

$$\begin{aligned} &F(t,v) = 0,\ F(r,s) = \frac{r}{s+5}. \\ &\implies d(F(r,s),F(t,v)) = max\left\{\frac{r}{s+5}, 0\right\} = \frac{r}{s+5} \leq \frac{1}{3}\left(\frac{tr}{r} + r\right) \\ &\qquad\qquad \leq \alpha \frac{d(r,F(r,s))d(t,F(t,v))}{d(r,t))} + \beta d(r,t) \\ &\qquad\qquad\quad + L\ min\{d(r,F(r,s)), d(t,F(t,v)), d(t,F(r,s)), d(r,F(t,v))\}. \end{aligned}$$

Case 3: If $t \geq v$, $r < s$, we get

$$\begin{aligned} &F(t,v) = \frac{t}{v+5},\ F(r,s) = 0. \\ &\implies d(F(r,s),F(t,v)) = max\left\{0, \frac{t}{v+5}\right\} = \frac{t}{v+5} \leq \frac{1}{3}\left(\frac{tr}{r} + r\right) \\ &\qquad\qquad \leq \alpha \frac{d(r,F(r,s))d(t,F(t,v))}{d(r,t)} + \beta d(r,t) \\ &\qquad\qquad\quad + L\ min\{d(r,F(r,s)), d(t,F(t,v)), d(r,F(t,v)), d(t,F(r,s))\}. \end{aligned}$$

Case 4: If $t < v$, $r < s$, we get $F(t,v) = 0$ and $F(r,s) = 0$.

Thus all the assumptions of Theorem 7.3 hold.

Hence, F has a coupled fixed point in X.

Theorem 7.6. *Suppose $(X, \leq)$ be a partially ordered set endowed with a metric d so (X, d) is complete. Presume a map $F : X \times X \to X$ possesses the property of mixed monotone on X. Presuppose there exist non-negative real numbers $\beta, \alpha \in [0, 1)$ and $L \geq 0$ with $(\beta + \alpha) < 1$, for all $r, s, t, v \in X$, we have*

$$d(F(r,s), F(t,v)) \leq \alpha \frac{d(t, F(t,v))[1 + d(r, F(r,s))]}{[1 + d(r,t)]} + \beta d(r,t) + L\, min\{d(r, F(r,s)), d(t, F(r,s)), d(r, F(t,v))\}. \tag{7.25}$$

Suppose

I) F is continuous or

II) X has the subsequent properties,

(a) if an increasing sequence$\{r_n\}$ in X converges to some point $r \in X$, then $r_n \leq r$, for each n,

(b) if a decreasing sequence $\{s_n\}$ in X converges to some point $s \in X$, then $s_n \geq s$, for each n.

Then F has a coupled fixed point.

Proof. Take $r_0, s_0 \in X$. Set $r_1 = F(r_0, s_0)$ and $s_1 = F(s_0, r_0)$. Repeating this process, set $r_{n+1} = F(r_n, s_n)$ and $s_{n+1} = F(s_n, r_n)$. Then by (7.25), we have

$$\begin{aligned} d(r_n, r_{n+1}) &= d(F(r_{n-1}, s_{n-1}), F(r_n, s_n)) \\ &\leq \alpha \frac{d(r_n, F(r_n, s_n))[1 + d(r_{n-1}, F(r_{n-1}, s_{n-1}))]}{[1 + d(r_{n-1}, r_n)]} + \beta d(r_{n-1}, r_n) \\ &\quad + L\, min\{d(r_{n-1}, F(r_{n-1}, s_{n-1})), d(r_n, F(r_{n-1}, s_{n-1})), d(r_{n-1}, F(r_n, s_n))\} \\ &= \alpha \frac{d(r_n, r_{n+1})[1 + d(r_{n-1}, r_n)]}{[1 + d(r_{n-1}, r_n)]} + \beta d(r_{n-1}, r_n) \\ &\quad + L\, min\{d(r_{n-1}, r_n), d(r_n, r_n), d(r_{n-1}, r_{n+1})\}, \end{aligned}$$

which implies that

$$d(r_n, r_{n+1}) \leq \frac{\beta}{1 - \alpha} d(r_{n-1}, r_n). \tag{7.26}$$

Similarly, we get

$$d(y_n, y_{n+1}) \leq \frac{\beta}{1-\alpha} d(y_{n-1}, y_n). \tag{7.27}$$

By adding (7.26) and (7.27), we have

$$d_n \leq \frac{\beta}{1-\alpha} d_{n-1} \tag{7.28}$$

Let $\qquad d_n = d(r_n, r_{n+1}) + d(s_n, s_{n+1}).$

Consequently, if we set $\lambda = \frac{\beta}{1-\alpha}$, then we have

$$d_n \leq \lambda d_{n-1} \leq \ldots \leq \lambda^n d_0. \tag{7.29}$$

If $d_0 = 0$, then (r_0, s_0) is a coupled fixed point of F.

Assume that $d_0 \geq 0$. Then, for every $k \in \mathbb{N}$, by repeated application of triangle inequality, we obtain

$$\begin{aligned} d(r_n, r_{n+k}) + d(s_n, s_{n+k}) &\leq [d(r_n, r_{n+1}) + d(r_{n+1}, r_{n+2}) + \ldots + d(r_{n+k-1}, r_{n+k})] \\ &\quad + [d(s_n, s_{n+1}) + d(s_{n+1}, s_{n+2}) + \ldots + d(s_{n+k-1}, s_{n+k})] \\ &= [d(r_n, r_{n+1}) + d(s_n, s_{n+1})] + [d(r_{n+1}, r_{n+2}) \\ &\quad + d(s_{n+1}, s_{n+2})] + \ldots + [d(r_{n+k-1}, r_{n+k}) + d(s_{n+k-1}, s_{n+k})] \\ &\leq d_n + d_{n+1} + \ldots + d_{n+k-1} \\ &\leq \frac{\lambda^n(1-\lambda^k)}{1-\lambda} d_0 \to 0 \ as \ n \to \infty. \end{aligned} \tag{7.30}$$

Therefore $\{r_n\}$ and $\{s_n\}$ are Cauchy sequences.

As (X, d) is a complete metric space, $\exists\ r, s \in X$ so $\lim_{n\to\infty} r_n = r$, $\lim_{n\to\infty} s_n = s$. Now, we show that if the assumption (I) holds, then (r, s) is coupled fixed point of F.
As, we have

$$\begin{aligned} r &= \lim_{n\to\infty} r_{n+1} = \lim_{n\to\infty} F(r_n, s_n) = F\left(\lim_{n\to\infty} r_n, \lim_{n\to\infty} s_n\right) = F(r, s), \\ s &= \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} F(s_n, r_n) = F\left(\lim_{n\to\infty} s_n, \lim_{n\to\infty} r_n\right) = F(s, r). \end{aligned}$$

Thus, (r, s) is coupled fixed point of F.

Assume now that the condition II(a) and II(b) of the theorem holds.
The sequence $\{r_n\} \to r$, $\{s_n\} \to s$

$$d(F(r, s), F(r_n, s_n)) \leq \alpha \frac{d(r_n, F(r_n, s_n))[1 + d(r, F(r, s))]}{[1 + d(r, r_n)]} + \beta d(r, r_n)$$

$$+ L\, min\{d(r, F(r, s)), d(r_n, F(r, s)), d(r, F(r_n, s_n))\}.$$

Letting $n \to \infty$, we have

$$d(F(r, s), r) \leq 0.$$

This implies that $F(r, s) = r$. In the same way, we can prove that $F(s, r) = s$. This completes the theorem.

Corollary 7.7. *Suppose $(X, \leq)$ be a partially ordered set endowed with a metric d so (X, d) is complete. Presume a map $F\colon X \times X \to X$ possesses the property of mixed monotone on X. Suppose there exist non-negative real numbers $\beta, \alpha \in [0, 1)$ with $\beta + \alpha < 1$, for all $r, s, t, v \in X$, satisfy*

$$d(F(r, s), F(t, v)) \leq \alpha \frac{d(t, F(t, v))[1 + d(r, F(r, s)]}{[1 + d(r, t)]} + \beta d(r, t).$$

Suppose

I) F is continuous or

II) X has the subsequent properties,

(a) if an increasing sequence $\{r_n\}$ in X converges to some point $r \in X$, then $r_n \leq r$, for each n,

(b) if a decreasing sequence $\{s_n\}$ in X converges to some point $r \in X$, then $s_n \geq s$, for each n.

Then F has a coupled fixed point.

Proof Take $L = 0$ in Theorem 7.6, we acquire Corollary 7.7.

Theorem 7.8. *Let the assumptions of Theorem 7.6 hold. We acquire the uniqueness of the coupled fixed point of F.*

Proof. Let (r, s) and $(\acute{r}, \acute{s})$ are coupled fixed points of F, then, $F(r, s) = r,\ F(s, r) = s,\ F(\acute{r}, \acute{s}) = \acute{r}$ and $F(\acute{s}, \acute{r}) = \acute{s}$. We shall show that $r = \acute{r},\ s = \acute{s}$.

Consider the subsequent two cases.

Case 1: Let (r, s) and $(\acute{r}, \acute{s})$ are comparable. We have

$$d(\acute{r}, r) = d(F(\acute{r}, \acute{s}), F(r, s)) \leq \alpha \frac{d(\acute{r}, F(\acute{r}, \acute{s}))d(r, F(r, s))}{d(\acute{r}, r)} + \beta d(\acute{r}, r)$$
$$+ L\, min\{d(\acute{r}, F(\acute{r}, \acute{s})), d(r, F(r, s)),$$
$$d(\acute{r}, F(r, s)), d(r, F(\acute{r}, \acute{s}))\}$$

$$\leq \beta\, d(\acute{r}, r),$$

which provides $d(\acute{r}, r) \leq 0,\ \beta < 1$(a contradiction). Thus $r = \acute{r}$.
In similar way, $d(\acute{s}, s) = d(F(\acute{s}, \acute{r}), F(s, r)) \leq 0$.

Thus, $s = \acute{s}$. Hence, (r, s) is a unique coupled fixed point of F.

Case 2: Presume (r, s) and $(\acute{r}, \acute{s})$ are not comparable. By supposition there is $(z, u) \in X \times X$ comparable with both of them.

We define sequences $\{z_n\}, \{u_n\}$ as follows

$$z_0 = z,\ u_0 = u,\ z_{n+1} = F(z_n, u_n)\ and\ u_{n+1} = F(u_n, z_n)\ \forall n.$$

Since (z, u) is comparable with (r, s), we may suppose that $(r, s) \geq (z, u) = (z_0, u_0)$.
It is easy to prove by using the mathematical induction,

$$(r, s) \geq (z_n, u_n)\ \forall n. \tag{7.31}$$

From (7.25) and (7.31), we have

$$\begin{aligned} d(F(r, s), F(z_n, u_n)) \leq\ & \alpha \frac{d(z_n, F(z_n, u_n))[1 + d(r, F(r, s))]}{[1 + d(r, z_n)]} + \beta d(r, z_n) \\ & + L\, min\{d(r, F(r, s)), d(r, F(z_n, u_n)), d(z_n, F(r, s))\} \end{aligned}$$

or

$$d(r, z_{n+1}) \leq \beta d(r, z_n). \tag{7.32}$$

Similarly, also we have

$$d(u_{n+1}, s) \leq \beta d(u_n, s). \tag{7.33}$$

Adding (7.32) and (7.33), we get

$$\begin{aligned} d(r, z_{n+1}) + d(u_{n+1}, s) &\leq \beta[d(r, z_n) + d(u_n, s)] \\ &\leq \beta^2[d(r, z_{n-1}) + d(u_{n-1}, s)] \\ &\ \ \vdots \\ &\leq \beta^{n+1}[d(r, z_0) + d(u_0, s)] \to 0\ as\ n \to \infty. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(u_{n+1}, s) = \lim_{n\to\infty} d(r, z_{n+1}) = 0. \tag{7.34}$$

In similar way, we can show that

$$\lim_{n\to\infty} d(u_{n+1}, \acute{s}) = \lim_{n\to\infty} d(\acute{r}, z_{n+1}) = 0. \tag{7.35}$$

From (7.34) and (7.35), we obtain $r = \acute{r}$ and $s = \acute{s}$.

Example 7.3. *Let $X = [0, 1]$ with metric $d(r, s) = |r - s|$, for all $r, s \in X$. Then (X, d) is complete metric space.*
Consider the map $F\colon X \times X \to X$ defined by

$$F(r, s) = \frac{(r - s)}{17} \; if\, s \leq r.$$

Evidently F is continuous and possesses the property of mixed monotone. Also there are $r_0 = 0; s_0$ =0 in X such that $r_0 = 0 \leq F(0, 0) = F(r_0, s_0)$ and $s_0 = 0 \geq F(0, 0) = F(s_0, r_0)$.
Then it is obvious that $(0, 0)$ is the unique coupled fixed point of F.

Now, we have following possibility for value of (r, s) and (t, v) such that $r \geq t$, $v \geq s$.

$$\begin{aligned} d(F(r, s), F(t, v)) = \left| \frac{r - s}{17} - \frac{t - v}{17} \right| &\leq \frac{1}{17}[|(r - t)| + |(s - v)|] \leq \frac{3}{4}|r - t| \\ &\leq \alpha \frac{d(t, F(t, v)[1 + d(r, F(r, s)]}{[1 + d(r, t)]} + \beta d(r, t) \\ &\quad + L\, min\{d(r, F(r, s)), d(r, F(t, v)), d(t, F(r, s))\}. \end{aligned}$$

Thus all the conditions of Theorem 7.6 hold.

Hence F has a coupled fixed point in X.

Conclusion and Future Scope of Study

Conclusion

As presented at the beginning of this work, the role of the fixed point theory is a major one in developing the science and the technique, through the pure theoretical contributions, as well as through the applicative contributions. Bhaskar and Laksmikantham, extend this theory to partially ordered metric spaces and introduce the concept of coupled fixed point for mixed-monotone operators, obtaining results about the existence, the existence and the uniqueness of the coincidence points for mixed g-monotone operators. We consider that the results obtained in couple fixed points are of a great importance within the fixed point theory.

1. Obtaining results as concerns the existence and the uniqueness of certain coupled fixed point theorems for mixed monotone mapping with a new rational contractive condition in ordered cone metric spaces, in which the operator verifies another contraction type.

2. Obtaining results as regards the existence and the uniqueness of the couple coincidence point possess the property of mixed g-monotone operators in the framework of ordered metric spaces and G-metric spaces. We conclude some applications on integral equations by using coupled fixed point theorems in ordered metric spaces and G-metric spaces.

3. We introduce a new idea of Y-cone metric spaces and study certain topological properties of Y-cone metric spaces. Then some couple common fixed point results have been generalized using the property of mixed weakly monotone in ordered Y-cone metric spaces.

Future Scope of Study

Based on present research study, it is suggested that b-metric spaces [45], S-metric spaces [130], A-cone metric spaces [5] have sufficiently wide mathematical structures. They opens the scope of their applicability in practical situations to a large extent. The analysis of these spaces, in particular the investigation of coupled fixed point, coupled coincidence and coupled common fixed point theorems defined on such spaces may be fruitful from many angles.

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List of Research Papers

List of Research Papers Published

1. V. S. Chouhan, R. Sharma, *Coupled fixed point theorems in partially ordered metric spaces*, American Journal of Mathematical Analysis 2(2) (2014) 19-22.

2. V. S. Chouhan, R. Sharma, *Coupled fixed point theorems for rational contractions in partially ordered metric spaces*, International Journal of Modern Mathematical Sciences 12(3) (2014) 165-174.

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List of Research Papers Communicated

1. R. Sharma, V. S. Chouhan, S. Mishra, *Certain coupled fixed point results for rational contractions in partially ordered cone metric spaces.*

2. R. Sharma, V. S. Chouhan, S. Mishra, *Coupled common fixed point theorems in partially ordered metric spaces.*

3. R. Sharma, V. S. Chouhan, S. Mishra, *Coupled coincidence point results in partially ordered metric spaces by altering distances.*

4. R. Sharma, V. S. Chouhan, S. Mishra, *Coupled coincidence point theorems for* (ψ, α, β) *weak contractions in partially ordered G- metric spaces.*

5. R. Sharma, V. S. Chouhan, S. Mishra, *Coupled fixed point theorems for rational contractions in partially ordered cone metric spaces.*

6. R. Sharma, V. S. Chouhan, S. Mishra, *Y- cone metric spaces and coupled fixed point results.*

List of Research Papers Presented in National/ International Conferences

1. V. S. Chouhan, R. Sharma, *Coupled fixed point results for a contractive condition in ordered partial metric spaces*, National Conference on Advances in Engineering, Science and Technology, 17^{th}-18^{th} September, 2016 at Chandigarh.

2. V. S. Chouhan, R. Sharma, *Coupled fixed point results in partially ordered metric spaces*, International Conference on Computer Systems & Mathematical Sciences, 18^{th}-19^{th} November, 2016 at Institute of Technology & Science, Ghaziabad.

3. V. S. Chouhan, R. Sharma, S. Mishra, *Results on coupled common fixed point in partially ordered metric spaces*, National Conference on Emerging Trends in Applied Sciences, 17^{th}-18^{th} August, 2017 at School of Basic and Applied Sciences, Galgotias University, Noida.

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