EXTENSIONS OF PONTRYAGIN REFLEXIVE TOPOLOGICAL GROUPS BEYOND LOCAL COMPACTNESS

A

Thesis

Submitted to

For the award of

DOCTOR OF PHILOSOPHY (Ph.D.)

in

MATHEMATICS

By

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PUNJAB
2019

Almighty God

I owe Him everything I know about Mathematics.

Declaration

I, Pranav Sharma, declare that this thesis entitled “Extensions of Pontryagin reflexive topological groups beyond local compactness”, and the work presented in it are my own. This work is done independently while in candidature for a research degree at Lovely Professional University and no part of this work has previously formed the basis for the award of any degree. Wherever contributions of others are involved every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

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This is to certify that the thesis entitled "Extensions of Pontryagin reflexive topological groups beyond local compactness" submitted by Pranav Sharma to Lovely Professional University for the award of the degree of Doctor of Philosophy by research is a bonafide record of research work carried out by him under my supervision. The contents of this thesis, in full or in parts, have not been previously submitted to any other Institute or University for the award of any degree or diploma.

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Abstract

Pontryagin duality theorem makes sense beyond the assumption of local compactness, and hence it has been studied for more general classes of topological groups in different directions using diverse techniques and approaches, but the problem of characterisation and extension of the class of Pontryagin reflexive topological abelian groups proposed in the pioneering paper of S Kaplan published in the year 1948 is still unsolved. In this work, we address the problem of extension of the reflexive abelian groups beyond locally compact abelian (LCA) topological groups with reference to Pontryagin duality theory and following three objectives are achieved:

1. To study the reflexivity in topological vector spaces and convergence vector spaces.
2. To analyse the topological and convergence structures on the groups and the character groups with reference to Pontryagin duality theory.
3. To examine the extensions of the Pontryagin reflexive groups to the wider class of groups than locally compact abelian groups.

The inspiration to explore the extensions of the Pontryagin duality theory to the wider class of groups than LCA groups is two-fold: one aspect is the duality theory of topological vector spaces which is restricted to the class of locally convex spaces, and another aspect is the duality theory of topological abelian groups which is restricted to locally quasi-convex groups. As the roots of this research lie in the theory of locally convex vector spaces so, in order to develop this research we have focused mainly on the framework of general topology and functional analysis.

This thesis is divided into five chapters which are further divided into sections. An overview of the chapters is as follows:

The first chapter of the thesis is preliminary in nature and presents the basic terms and notations related to topological groups and Pontryagin duality theory.

The second chapter deals with extensions of Pontryagin duality theory in various classes of topological groups which are not necessarily locally compact. Special emphasis is given to present the influence of functional analysis on the development of the subject.

The third chapter is devoted to present a self-contained introduction to (filter) convergence groups. This chapter contains certain examples and properties of

convergence groups with particular reference to homeomorphism groups. Further, this chapter deals with the properties of convergence groups with boundedness. We introduce the notation of bounded convergence groups and present certain properties of bounded sets in convergence groups.

In the fourth chapter, we introduce the notation of convergence measure space which is based on the concept of topological modification of a convergence space, and we present how this kind of approach can be used to study the class of locally compact convergence groups. A general definition of a convergence measure space is still unknown and is required for analysis over convergence groups.

In the fifth chapter, we define local quasi-convexity for the class of convergence groups (convexity here is not an algebraic property of group) and prove that if a convergence group is c-reflexive then it must be locally quasi-convex and hence, we obtain that in contrast to the topological case locally compact abelian convergence groups do not lie in the class of locally quasi-convex convergence groups. Further, we prove that the condition of local quasi-convexity is sufficient for a compact (non-topological) convergence group (if it exists) to be c-reflexive. It is worth mentioning; we do not know any example of a compact convergence group which is not topological. So, the problem of existence of a Hausdorff non-topological compact convergence group is still open. Finally, this chapter ends with certain categorial aspects of the Pontryagin dual and the continuous dual.

The thesis ends with some concluding remarks and an extensive bibliography.

Acknowledgement

First and foremost, I owe my deepest gratitude to my parents for their tremendous support without which this work would not have been possible.

I am thankful to Dr. Sanjay Mishra for having agreed to be my thesis supervisor and for the trust he placed in me throughout this period.

I am thankful to Mr. Prince Singh and Mr. Sukhdev Singh for always standing beside me during the happy and hard moments.

I am indebted to Shri. Ashok Mittal, Chancellor, and Mr. Aman Mittal, Associate Director, Lovely Professional University for their support throughout this research.

Thanks are due to Shri. Sukhbir Singh Sandhu, AS, MHRD, New Delhi for his support to receive a fellowship of University Grants Commission.

I acknowledge the financial support by the University Grants Commission, New Delhi in the form of Junior Research Fellowship and Senior Research Fellowship which enabled me to work on a full-time basis.

I am thankful to Prof. Maria Jesus Chasco, Universidad de Navarra (Spain), for motivating me to work on convergence groups.

I am grateful to Prof. Szymon Dolecki, University of Burgundy (France), for having continuously been willing to talk to me and for his permanent encouragement.

Last but not the least, I am indebted to Prof. Frederic Mynard, New Jersey City University (USA) for his patience, generosity with his time, and moreover for pointing out my errors which prevented me from losing my way. Most of the results described in this thesis would not have been obtained without his help.

Contents

Symbols and Abbreviations

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{T} = \mathbb{R}/\mathbb{Z}$	Circle group
$\mathbb{T}_+$	$\{z \in \mathbb{T} : \text{ Re } (z) \geq 0\}$
$H(\mathbb{Q})$	Homeomorphism group of $\mathbb{Q}$
χ	Character of a group
$Hom(G, \mathbb{T})$	Character group
$\mathbb{C}Hom(G, \mathbb{T})$	Continuous character group
$\hat{G}$	Pontryagin dual group
$(\Gamma G, \Lambda_c)$	Convergence dual group
τ_{co}	Compact open topology
α_G or κ	Evaluation homomorphism
$O^{\perp}$ and ${}^{\perp}E$	Annihilator and inverse annihilator
$H^{\triangleright}$ and $L^{\triangleleft}$	Polar and inverse polar
$\prod_{i \in I} G_i$	Infinite products
$\bigoplus_{i \in I} G_i$	Direct sums
$\coprod\limits_{i \in I} G_i$	Disjoint union
$\lim\limits_{\leftarrow} G_n$	Direct limit
$\lim\limits_{\rightarrow} G_n$	Inverse limit
$\mathcal{F}$ or $\mathcal{G}$	Filters
$\mathcal{A} \leq \mathcal{B}$	$\mathcal{A}$ is coarser than $\mathcal{B}$
$\mathcal{A}^{\uparrow}$	Isotonization of $\mathcal{A}$
$\mathcal{A}^{\cap}$	Family of finitely many elements of $\mathcal{A}$

$\mathbb{F}X$	Set of all filters in a set X
$\mathcal{A}\#\mathcal{B}$	$\mathcal{A}$ mesh $\mathcal{B}$
(X, λ)	Convergence space
$\mathcal{F} \underset{\lambda}{\rightarrow} x$	$\mathcal{F}$ converges to x in λ
$\lim_{\lambda} \mathcal{F}$	$\{x : \mathcal{F} \rightarrow x \text{ in } \lambda\}$
$\sigma_{C_W}, \sigma_{C_S}$	Boundedness on convergence groups
$\mathcal{N}(x)$	Neighbourhood filter of x
λ_{tm}	Topological modification of the convergence space λ
$(X, \lambda, \mathcal{M}, \mu)$	Convergence measure space
$ker(\kappa_G)$	Kernel of κ_G
$Re(z)$	Real part of z
$C_c(X, \mathbb{T})$	Group of all continuous unimodular functions
CAG	Category of convergence abelian groups
LCAG	Category of locally quasi-convex Hausdorff convergence abelian groups

Abbreviations

LCA	Locally compact abelian
LQC	Locally quasi-convex
MAP	Maximal almost periodic
P-Duality	Pontryagin duality
BB-Duality	Binz and Butzmann duality

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Chapter 1

Introduction and Preliminaries

The most versatile structure based on algebraic operations is the structure of a group. In the mathematical environment, the group structure coexists overwhelmingly with an additional structure of topology, limit, measure or order. The presence of a geometric structure of topology with the algebraic structure of the group enables us to define the convergence and continuity and hence, plays a vital role in the analysis. One of the most prominent ways to obtain information about the space is to study the functions which preserve the underlying structure of the space. For instance, in functional analysis, the underlying object of study is a vector space with compatible topological or some limit-related structure, and the information about the vectors is extracted by using the continuous linear functionals. The continuous dual space and hence the reflexivity theory plays the most prominent role in the development of locally convex spaces which further has a profound impact on the progress of duality theory of groups. The primary purpose of this thesis is to contribute to the theory of reflexive abelian groups (with some limit related structures) that is, to present the local aspects of reflexivity in topological groups those are not necessarily locally compact abelian and the objectives of the research are accomplished by proving new results to widen the knowledge on reflexivity in groups with limit related structures.

Influenced from the study of the group of continuous transformations Leja, F. introduced the modern concept of the topological group as an object which is a blend of both algebraic and topological structures with group inverse continuous and group multiplication jointly continuous. The versatility of the structure makes a topological group a significant object of study in different branches of mathematics which include representation theory, topological algebra and harmonic analysis. An important field of investigation to get a better insight of topological groups is the study of their structure. This kind of investigation depends on the representation theory. The idea behind the representations is to connect the topological groups to more concrete objects like matrices and then to study these new objects to investigate the properties of the topological groups.

While working with the irreducible representations, Peter and Weyl obtained the existence of invariant integrals on compact Lie groups. Using the tools of the abstract integration theory, Haar proved the existence of a left invariant integral on locally compact groups and laid the foundations of modern harmonic analysis. Later, von-Neumann proved the existence as well as the uniqueness of an invariant integral (in general) for the compact groups. Another important discovery of that period was the Pontryagin duality theorem. Using the method of characters of Peter and Weyl on the theory of irreducible representations, Pontryagin obtained a generalisation of the fundamental structure theorem for finitely generated abelian groups to compact groups with a countable base. In most of these studies (Haar integral and Pontryagin duality theory) the topological groups were restricted to satisfy certain axiom of countability, but with the development of the subject, the countability conditions on topological groups in these results are removed. van-Kampen proves the duality theorem for all (without any countability condition) locally compact abelian (LCA) groups. With the progress of functional analysis and measure theory, several alternative treatments of the duality theory are proposed. Rudin proves the Pontryagin duality theorem using the powerful tools of functional analysis (Banach algebras) and measure theory (Haar measure) and this proof presents an elegant combination of algebra and topology (a brief outline of this proof is presented in Chapter 4, further for a proof without much use of functional analysis we refer the reader to [39]).

The striking consequences of the Pontryagin duality theorem enable us to describe the topological or algebraic property of LCA groups in terms of the respective properties of their dual groups. The theorem proves that an LCA group is canonically isomorphic to its double dual group and explains why the Pontryagin duality is satisfied in these groups. In addition to this, the theorem serves as a base for abstract harmonic analysis (analysis over topological groups) because the dual group is used as the underlying group in the abstraction of the Fourier transform and is an essential tool to study the structure of LCA groups. Several fields of research are devoted to these kinds of studies which include the study of the character of all groups, the study of the structure of topological groups, extensions of the (abelian or non-abelian) duality theorems, analysis over topological groups, study of amenable group, etc. The work of Kaplan [62] provides the first example of topological groups beyond LCA groups which satisfy Pontryagin duality and brings into the picture the problem of characterisation of the class of reflexive topological abelian groups. To date different reflexivity theories viz. Binz-Butzmann reflexivity, Chu duality, Tannaka-Krein duality, etc. are proposed and studied for different classes of (abelian and non-abelian) topological groups.

Here we deal with the problem of extension of reflexive topological abelian groups beyond the class of LCA groups with reference to Pontryagin duality theory. Most of this work is the study of abstract mathematical structures and their duals with particular emphasis on the groups and vector spaces with topological or convergence structure. In the rest of this chapter, we introduce the notations related to topological groups and Pontryagin duality theory. In this chapter, we do not attribute the specific definitions and results, but our primary sources are included in the bibliography.

1.1 Topological groups

An abstract group $(G, .)$ with a compatible topological structure τ is called topological group . The term compatible means that algebraic operation of the group is linked to the topological structure through the property that the group inverse

$$g \mapsto g^{-1}$$

is continuous and group multiplication

$$(g_1, g_2) \mapsto g_1.g_2$$

is jointly (*in product topology*) continuous, here g, g_1, g_2 represents the elements of the topological group G. When no confusion is likely to occur, we write G for $(G, \tau, .)$ and represent the group operation $g_1.g_2$ as g_1g_2. The conditions of continuity of product in the definition of the topological group asserts the existence of the neighbourhoods P of g_1 and Q of g_2 for every neighbourhood O of g_1g_2 such that $PQ \subset O$. Similarly, the continuity of the inverse operation implies the existence of a neighbourhood P of g for every neighbourhood Q of g^{-1} such that $P^{-1} \subset Q$. If $H \subset G$ is a (algebraic) subgroup of a topological group then, the group H with the subspace topology induced from the group G is called a *topological subgroup*.

The right ($g \mapsto ga$) and the left ($g \mapsto ag$) translations of a topological group can also be viewed as the action of a topological group onto itself. The very first application of the continuity of the group operations is to prove that the left translation and the right translation are topological isomorphisms (a *topological isomorphism* is a map between two topological groups which is an isomorphism of groups and homeomorphism of topological spaces).

Another consequence of the continuity of the operations allows us to define the symmetric neighbourhoods of identity, that is, every neighbourhood V of identity contains a neighbourhood U (say, $= V \cap V^{-1}$) such that $UU^{-1} \subseteq V$. Further, for each open set O and a subset E of a topological group G, the sets O^{-1}, EO and OE are open in G. So, for any two elements g_1 and g_2 of a topological group G there is a topological isomorphism (viz. left and right translations) of G onto itself which takes g_1 to g_2 and hence every topological group is *homogeneous*. It follows from the homogeneity that space behaves in the same way at all the points and it is not necessary to describe the basis of the whole space in order to describe the topology of a topological group. A complete system of the neighbourhood of identity $\mathfrak{U}_e$ (or an open basis at the identity) is sufficient to describe the topology of the topological group completely. It is a consequence of homogeneity that, to prove a group to be locally compact it is sufficient to prove that the identity has a neighbourhood with compact closure.

The very first application of the homogeneity and symmetric neighbourhoods lie in proving the separation properties for topological groups. If a topological group satisfy T_1 (point sets are closed) axiom of separation, we can find disjoint neighbourhoods for any two disjoint elements of the topological group which proves that every topological group which satisfies T_1 axiom of separation is Hausdorff. Further, in a topological group satisfying T_1 axiom we can find the open sets which separate the identity e from the closed sets not containing e, this fact along with the homogeneity of the topological group implies the regularity of the topological group. The regularity of the topological group may be used to find the continuous function which separates the identity element from the closed set not containing the identity and hence proves the complete regularity of a topological group. So, for a topological group satisfying T_1 separation property the Hausdorff separation axiom, regularity and complete regularity are equivalent conditions. In general definition of a topological group no separation axiom is involved, but for the purpose of analysis, the topological groups are assumed to satisfy the T_1 axiom of separation. A topological group is metrizable under very mild conditions of being T_0 and first countable and further, a topological group is said to be almost metrizable if the quotient group G/H is metrizable for some compact subgroup H of the group G.

Another important class of groups is the class of k-group which include the class of locally compact groups and metrizable groups. For a topological space (X, τ) the k-extension of the given topology τ of X is the strongest topology (denoted $k\tau$) on X which agrees with the topology τ on each compact set. So, the k-refinement of X denoted as kX is the same underlying set X with the topology $k\tau$ where U (called k-open set) belongs to $k\tau$ if $U \cap K$ is relatively open in K for every compact subset K of X. The space X is called k-space if $k\tau = \tau$ or $kX = X$. A group is called k-group [66] if it is a group with k-topology such that the inverse operation (of the group) is continuous and product is continuous in k-products, (k-product is the product with the k-refinement of the product topology). It is important to note that any topological group (by adding the k-refinement to the topology) can be turned into a k-group but, it is not necessary that every k-group is a refinement of k-topology. Another inequivalent definition of k-group is due to Noble [75] where all the k-groups are considered to be topological groups and the k-groups are characterised in terms of k-continuous functions (a function is k-continuous if the restriction of that function to every compact subset (of its domain) is continuous.)

Taking the quotient is an important tool to generate new topological group from a given topological group. Let H be one of the normal subgroups (the subgroup invariant under inner automorphisms) of a topological group G and G/H denotes the set of all cosets of the subgroup H in the group G. A subset O of G/H is said to be open if the projection map

$$\pi : G \to G/H$$

defined as

$$\pi(g) = \text{the coset of } H \text{ which contains } g,$$

is continuous. Equivalently, we can say that $\pi^{-1}(O)$ is open in G. With this quotient topology, the group G/H is called the *quotient space*. Another way to generate a new topological group from given topological groups is by taking the direct product of given groups and endowing the product with the product topology (product topology is compatible with group structure). As a consequence of Tychonoff's theorem a product of compact topological groups is compact, and hence, the product of finitely many locally compact groups is locally compact.

In a topological group, the left and the right translations are homeomorphisms and this fact makes it possible to introduce a notation of sufficiently near points in G and hence, makes G a uniform space. For $g_1, g_2 \in G$ we translate say g_1 to the identity and the proximity of g_1 and g_2 is evaluated by neighbourhood V of the identity (in some sense) into which g_2 is translated. Formally, the sets L_U and R_U in $G \times G$ corresponding to each neighbourhood U of identity are defined as:

$$L_U = \{(g_1, g_2) : {g_1}^{-1} g_2 \in U\};$$

$$R_U = \{(g_1, g_2) : g_2 {g_1}^{-1} \in U\}.$$

As U runs through all neighbourhoods of the identity, the family of all sets L_U (respectively L_R) is called left (respectively right) uniform structure compatible with group G. The structure of a uniform space on a group allows us to use uniformly continuous functions and Cauchy sequences to study convergence in these groups.

1.2 Pontryagin duality

Any abstract group, when equipped with a discrete topology, is a topological group. Beyond this, the non-trivial examples of the abelian topological groups include the (additive or multiplicative) group of reals or complex numbers with usual topology (of the respective topological space). The underlying additive structure of a topological vector space (in particular, of the locally convex spaces) serves as an important class of the additive abelian topological groups with reference to duality theory. Matrix groups are an example of non-abelian topological groups and include the group of all invertible $n \times n$ matrices with real (or complex) entries equipped with the topology induced from the usual topology of $\mathbb{R}^{n^2}$ (or $\mathbb{C}^{n^2}$). Some other classes of the matrix groups are the special linear group, orthogonal and unitary groups which are endowed with the relative topology as a subset of the corresponding matrix group. The set of all homeomorphisms of a topological space with the binary operation as the composition of maps becomes a group called homeomorphism group and the homeomorphism group equipped with a suitable function space topology becomes a topological group.

An important topological group from the viewpoint of the Pontryagin duality is the circle group (denoted, $\mathbb{T}$). It is a group (multiplicative) of complex numbers with unit modulus and with topology inherited as the subspace topology of $\mathbb{C}$. This group can be identified in several equivalent (isomorphic) forms viz. $\mathbb{T} \cong \mathbb{R}/\mathbb{Z} \cong U(1)$ where $\mathbb{R}/\mathbb{Z}$ and $U(1)$ denotes the quotient group of reals by integers and 1×1 unitary matrices respectively. It is important to point out that the circle group is compact and metrizable. The set $Hom(G, \mathbb{T})$ of all homomorphisms

$$\chi : G \to \mathbb{T}$$

called characters, of an abelian group G to the circle group $\mathbb{T}$ with the operation of point-wise multiplication

$$(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$$

is called the algebraic dual of G. Further, if G is a topological abelian group, then the set of all continuous characters is called character group and is denoted as $\hat{G}$ or $\mathbb{C}Hom(G, \mathbb{T})$. The weakest topology on G with respect to which all elements of $\hat{G}$ are continuous is called Bohr topology (denoted $\sigma(G, \hat{G})$) of G. A topological group is said to have sufficiently many characters if for each g_1, g_2 in G there is some $\chi \in \hat{G}$ such that $\chi(g_1) \neq \chi(g_2)$ and if a group has sufficiently many continuous characters then it is said to be maximal almost periodic (MAP). Further, a topology τ_1 on a group G is called compatible with another topology τ_2 on G if $\widehat{(G, \tau_1)} = \widehat{(G, \tau_2)}$.

For the purpose of analysis, the character group can be topologised using different function space topologies (for instance in case of topological vector spaces: weak, weak*, strong, Mackey, etc topologies are induced on the dual space). The most common is the compact-open topology τ_{co}. The character group with a compact-open topology is called a *dual group* $(\hat{G}, \tau_{co})$ of the underlying group. It is quite important to note here that this definition of the dual group fails in general for the non-abelian case [44]. For a topological abelian group G there is an evaluation homomorphism

$$\alpha_G : G \to \hat{\hat{G}} \text{ defined as } \alpha_G(g)(\chi) = \chi(g) \; \forall \; g \in G, \chi \in \hat{G}$$

(not necessarily continuous, injective or onto) from the group to its double dual (dual of a dual group). If this evaluation map is a topological isomorphism, then the group is said to satisfy the *Pontryagin duality* or is called Pontryagin reflexive.

Now we define the annihilator and the polar which is the underlying concept in defining locally quasi-convex topological groups. The *annihilator* $O^{\perp}$ of a subgroup O of a topological abelian group G and the *inverse annihilator* $^{\perp}E$ of a subgroup E of $\hat{G}$ are the subgroups of $\hat{G}$ and G respectively defined as:

$$O^{\perp} = \{\chi \in \hat{G} : \chi\{O\} = \{1\}\};$$

$$^{\perp}E = \{g \in G : \chi(g) = \{1\} \; \forall \; \chi \in E\}.$$

The more general notion for the annihilator and the inverse annihilator of a subgroup is the *polar* ($H^{\triangleright}$) and the *inverse polar* ($L^{\triangleleft}$). For any subset H of G and L of $\hat{G}$ the polar and the inverse polar of H and L respectively are subsets defined as:

$$H^{\triangleright} = \{\chi \in \hat{G} : \chi(H) \subset \mathbb{T}_+\};$$

$$L^{\triangleleft} = \{g \in G : \chi(g) \subset \mathbb{T}_+, \; \forall \; \chi \in L\}$$

here $\mathbb{T}_+ = \{z \in \mathbb{T} : \text{ Re } (z) \geq 0\}$. If for each $g \notin H$ (H is a subgroup of G) there is some $\chi \in H^{\perp}$ with $\chi(g) \neq 0$ then H is called *dually closed* in G, that is, $G \backslash H$ has sufficiently many characters. A subgroup H is said to be *dually embedded (or h-embedded)* in G if each continuous character (or any character) on H can be extended to a continuous character of the group G.

Conclusion

In this chapter, we have presented the basic facts about the topological groups and Pontryagin duality theorem. In the next chapter, we present the influence of functional analysis on the development of this subject, and the subsequent chapters are devoted to present the results obtained while investigating the extensions of Pontryagin duality theory to the class of filter convergence groups.

Chapter 2

Duality in Topological Groups

2.1 Influence of functional analysis

The first instance of extension of Pontryagin duality theorem is due to Kaplan [62] where he obtains the extension of the Pontryagin duality theorem for the infinite products ($\prod_{i\in I} G_i$) and direct sums ($\bigoplus_{i\in I} G_i$) of reflexive groups. Similar results are obtained by him in [63] for the (suitable) direct limit ($\lim\limits_{\leftarrow} G_n$) and inverse limit ($\lim\limits_{\rightarrow} G_n$) of the LCA groups. Influenced by the work of Arens [4] on the reflexivity (in functional analysis sense) in vector spaces, Smith [85], proves that the Banach spaces as topological groups are Pontryagin reflexive. Smith's work infers that the reflexivity in functional analytic sense is stronger than the reflexivity in Pontryagin sense and with this work the study of Pontryagin duality in topological vector spaces is initiated. Due to the lack of the notation of convexity similar to that of vector spaces and the absence of the theorem like Hahn-Banach theorem for the general topological groups the duality theory for topological groups cannot be deduced trivially (or analogously) from the theory of topological vector spaces. (It is worth mentioning that a form of Hahn-Banach theorem for topological abelian groups is available which is based on the theory of the additive functionals.)

Hahn-Banach theorem (extension version) is one of the fundamental theorems in functional analysis and plays a prominent role to study the linear functionals on topological vector spaces and hence, the duality (or reflexivity) theory of topological vector spaces is restricted to the class of locally convex spaces. To overcome the difficulties arising due to lack of the notation of convexity, Vilenkin [90] inspired from the Hahn Banach theorem (separation version) for topological vector space introduce the notation of quasi-convexity and defines the class of topological groups called locally quasi-convex groups which play the role similar to the class of locally convex spaces. This chapter is devoted to present the known results related to the extensions of Pontryagin duality with special emphasis to present the influence of functional analysis on the development of the subject.

After local quasi-convexity in topological groups the second instance where we find a deep impact of functional analysis on the development of the Pontryagin duality theory is the concept of group dualities. Vector space duality is a well-studied topic in functional analysis, but the translation of the concepts like duality pairs, compatible topologies, Mackey spaces, etc. from locally convex spaces to quasi-convex groups is not trivial and is an active area for research. The origin of the concept of the group dualities is due to Varopoulos [87] and this work is motivated from the concept of vector space dualities. For an abelian (topological) group G, let H be a subgroup of $\hat{G}$, then the pair (G, H) is called a group duality. In general (literature) there is no condition on H to separate the points of G, in case H separates the points of G the pair (G, H) is called separating.

Corresponding to each pair (G, H), two topologies viz $\sigma(G, H)$ and $\sigma(H, G)$ are associated. The former is the weakest topology on the group which make members of H continuous, while the latter is the topology of pointwise convergence in the sense that it is the weakest topology on H which makes the map

$$\chi \mapsto \alpha_g(\chi) = \chi(g)$$

continuous for all g in G. For a given duality pair (G, H) the topologies compatible with the duality are those topologies of G which admit H as a dual group. A duality $(G, \hat{G})$ is obtained by replacing H with $\hat{G}$ and it is interesting to note that $\sigma(G, \hat{G})$ is Bohr topology on G.

Varopoulos [87] considers only the special case of locally precompact compatible topologies and proves that the least upper bound (or supremum) of all compatible locally precompact topologies is a compatible topology. Working on the problem of Mackey topology for groups or in more general, on the problem of extension of Mackey Arens theorem from locally convex space to locally quasi-convex groups, Chasco et al. [29] prove that the least upper bound of all compatible group topologies for a topological abelian group (in general) need not always be a compatible topology. They point out that some restriction should be made on the class of (all) compatible topologies to make their least upper bound compatible and to overcome this problem they consider the class of locally quasi-convex group topologies and study the problems similar to Varopoulos for the wider classes of locally quasi-convex topologies. Based on Mackey-Arens theorem the two non-equivalent candidates for the definition of Mackey topology are proposed by Chasco et al. [29].

In [67] the author studies several strong and weak topologies on abelian topological groups and defines the Mackey topology $\nu(G, \hat{G})$ (if it exists) for a maximal almost periodic groups G as the strongest locally quasi-convex topology on the group compatible with $(G, \hat{G})$. Further, a locally quasi-convex group (G, τ) is called Mackey group if the topology τ of the group coincides with its Mackey topology $\nu(G, \hat{G})$ that is $\tau = \nu(G, \hat{G})$. Außenhofer et al. [10] studies the qualitative and quantitative aspects of the poset of locally quasi-convex compatible topologies and obtain the answers for the problem left open in [67]. Further, they propose several open problems and one of those is to find sufficient condition for a pre-compact metrizable group to be Mackey.

Nieto and Peinador [74] by characterising the locally quasi-convex topologies with reference to families of equicontinuous subsets introduce some new classes of topological groups by grading the property of being a Mackey group. Dikranjan et al. [40] give a more generalised definition of the Mackey topology which is based on the definition of the general (not necessarily locally quasi-convex) Mackey group. (Some statements of the Mackey problem for bounded groups are mentioned in [71, section 3.4].) In the same paper [40] they provide the examples of non-Mackey non-complete metrizable locally quasi-convex groups. Similar results are obtained in [8] where the authors prove that the group of all integers with a linear non-discrete Hausdorff topology is non-Mackey. Conjecture ([40, Conjecture 8.1]) states that every metrizable group in LCS is a Mackey group in locally quasi-convex topology and a negative answer to this conjecture is obtained by Gabriyelyan ([48, Theorem 3.1]) as they obtain that the group of all finite sequences $(\mathbb{R}^N, \mathfrak{p}_0)$ with topology $\mathfrak{p}_0$ induced from the product space $\mathbb{R}^N$ is not Mackey (in the category of locally quasi-convex groups) and this serves as an example of a Mackey locally convex space which is not locally quasi-convex (this is the first example in this regard). Further, the author defines a topology μ on a group G in the class of maximal almost periodic abelian group $\mathcal{G}$ to be quasi-$\mathcal{G}$-Mackey if there is no topology $\nu \in \mathfrak{T}_{\mathcal{G}}(G, \tau)$ such that $\mu < \nu$ and μ is $\mathcal{G}$-compatible with τ. In [49] the author gives an example of a locally quasi-convex group which is not pre-Mackey. In this regard, the author proves that $A_G(s)$, the (Graev) free abelian group over s,

$$s = \{0 \cup \frac{1}{j} : j \in \mathbb{N}\}$$

is a convergent sequence equipped with the topology induced from real line, is neither a quasi-Mackey group nor a pre-Mackey group. Similar results are proved in [13]. These results present how the locally quasi-convex groups differ from the locally convex vector spaces.

If any metrizable locally quasi-convex topology on an abelian group G is a Mackey topology then the group G is said to satisfy the Varopoulos paradigm [11] and it is quite noteworthy that Varopoulos paradigm is a topological property of the group and it is characterizing the fact (being of finite exponent) which is purely an algebraic feature. In [11], the author proves that an abelian topological group satisfies Varopoulos paradigm iff it is bounded. Further, Außenhofer and Dikranjan [12], proves that the Mackey topologies exist for ”linearly topologized Hausdorff abelian groups” and they describe these Mackey topologies in terms of B-embedded subgroups and hence, in the class of locally quasi-convex bounded groups.

Peinador and Tarieladze [69] makes a comparison of the Mackey theory based on the two settings viz topological vector spaces and topological groups and point out some open problems in this regard. For a categorial approach to the Mackey problem, we refer the reader to [14].

Nuclear groups

Brown et al. [23] obtain an extension of Pontryagin duality and formulate it as the functorial duality between the following two classes: (i) the class of all abelian Hausdorff groups topologically isomorphic to product of a compact group with a product (countable) of copies of $\mathbb{Z}$ and $\mathbb{R}$; (ii) the class of abelian Hausdorff topological groups isomorphic to sum of a discrete group with sum (countable) of copies of $\mathbb{T}$ and $\mathbb{R}$. They prove that for the groups from these classes the Hausdorff quotients and closed subgroups of the groups again belong to the class itself, and in this regard, the term strong duality is introduced for the first time. Further, Banaszczyk [15] defines a topological group G to be strongly reflexive if all Hausdorff quotients and closed subgroups of G and its dual group $\hat{G}$ are reflexive. The results of [23] are the first example of non locally compact strongly reflexive groups and these results are extended in [15] where the notation of nuclear groups is introduced.

In the theory of topological vector spaces, nuclear spaces were introduced by Grothendieck in connection with the nuclear products and after that several characterisation of the nuclear spaces are obtained. One of the characterisations describe the nuclear spaces in terms of Kolmogorov diameter [19]. Banaszczyk [15] defines a group to be a nuclear group if every neighbourhood of zero contains a neighbourhood which is sufficiently small with respect to the given one. This class of nuclear groups contain the nuclear spaces and LCA groups. Further, the class of nuclear groups contains the class of groups which are closed under the formation of subgroups and Hausdorff quotients. Various results related to the structure of nuclear groups, boundedness, compactness of nuclear groups are presented in [51] and the author proves that the evaluation mapping

$$\alpha_G : G \to \hat{\hat{G}}$$

is always surjective if the group G is a complete nuclear group.

Außenhofer proves in [6], that the $\check{C}$ech complete nuclear groups are strongly reflexive. Chasco and Peinador in [34] study the strong reflexivity for the class of almost metrizable reflexive topological abelian groups and they prove that the requirements for the strong reflexivity can be weakened for these classes. To this date, no example of strongly reflexive groups is known out of the class formed by nuclear groups and their dual.

Another characterisation of the nuclear spaces is due to Pietsch; they characterise nuclear spaces as locally convex spaces for which (appropriate) topologized spaces of summable sequences and absolutely summable sequences are topologically and algebraically same. Referring this characterisation as Grothendieck- Pietsch theorem [43] Domínguez and Tarieladze introduced the GP-nuclear groups as the groups for which summable sequences and absolutely summable sequences are the same topologically as well as algebraically. They prove that the nuclear groups defined in the sense of Banaszczyk (using Kolmogorov diameter) are GP-nuclear and the converse problem is still open. For a survey of GP- nuclear groups along with properties of these groups under certain operations like countable direct sums, we refer the reader to [42].

Außenhofer et al. in [9] by defining the notion of Schwartz topological groups introduce the group version of the concept of Schwartz spaces. They define a Hausdorff, abelian topological group G to be a Schwartz group if for every neighbourhood of zero U in the group G there exists another neighbourhood V of zero in G and a sequence of finite subsets (F_n) of the group G such that V is a subset of

$$F_n + U_{(n)}$$

for every n in $\mathbb{N}$. Along with the certain permanence properties of Schwartz groups, it has been proved in [9] that the nuclear groups lie in a wider class of locally quasi-convex Schwartz group. The property of being a Schwartz group in terms of the dual group is studied by Chasco et al. in [32]. Chasco et al. [31] divide the strong reflexivity in two separate properties and introduce the notation of q-reflexive groups (Hausdorff quotients of the group are reflexive) and s-reflexive groups (closed subgroups of the group are reflexive) and propose certain problems in this regard which include the problem of the study of self dual nuclear groups.

Characterisation of Locally Convex Spaces

Venkataraman [88] study the conditions under which the dually embedded and dually closed subgroups of a topological abelian group (especially the open subgroups) satisfy duality and prove that if a topological abelian group satisfies the Pontryagin duality then so does its open subgroups. In [89], he proves a characterisation for the Pontryagin reflexive groups but later it has been observed in [82] that the characterisation obtained in [89] contains a wrong statement. According to Remus and Trigos-Arrieta [82] a group is said to respect compactness if the original topology of the group and the weakest topology that makes each element of $\hat{G}$ continuous produce the same compact subspaces and based on the fact that a reflexive linear space respects compactness if and only if it is a Montel space they prove the existence of groups which do not respect compactness but satisfy Pontryagin duality and hence produced a counter-example for the characterisation obtained in [89].

Kye [64] also gives a characterisation for a class of additive topological abelian groups of locally convex topological vector spaces where the condition provides the characterisation of the continuity of the evaluation map that every convex, balanced and closed set which is a neighbourhood of zero in the k-topology is a neighbourhood of zero in the given topology. This characterisation due to Kye is based on the proof of the Venkataraman's characterisation, and hence it contains the statement similar to that present in the previous characterisations, and thus the proof contains a gap and is incomplete.

Hern$\grave{a}$ndez [58] presents the counter-example towards the characterisation of Venkataraman and Kye. Further, the conditions on a topological group that are equivalent to the Pontryagin reflexivity are also obtained in [58] but this characterisation is very technical to be considered as the intern characterisation of the reflexive groups. However, the question of intern characterisation has been obtained for some particular classes of groups which include locally convex spaces, free topological groups, etc. It has been pointed out in [58, p. 501] that "in general, if that intern characterisation does exist, it will not be an easy one with all probability since it was proved in [60] that the question of characterising for what topological spaces X, the corresponding additive groups of the rings of all continuous functions $C_p(X)$, equipped with the point-wise open topology, are P-reflexive, is undecidable in ZFC".

Working on a particular class Bonales et al. [55] offers an alternative characterisation of the duality of real locally convex spaces, and they prove in [54], that the same characterisation also holds for complex locally convex spaces. Several questions are left open in these two articles which include the problem related to the characterisation of polar reflexive spaces. Hern$\grave{a}$ndez and Javier [59] proves a new characterisation of topological abelian groups satisfying duality in terms of the precompact open (denoted τ_{pc}) topology (that is the character group is equipped with precompact open topology) and hence obtain a characterisation of polar reflexive spaces and prove that the Kye's characterisation in [65] is correct (but the proof is different). Further, they answer in negative the question asked in [55] and prove that the group duality does not imply the polar reflexivity.

In the next section, we present the duality results in some other classes of topological groups.

2.2 Duality in metrizable, bounded and other classes of topological groups

The class of metrizable groups forms an important class from the viewpoint of duality theory. Metrizable groups are contained in the class of k-groups. Noble [75] characterise the k-groups (different from that present in other literature) to be topological groups in which each k-continuous homomorphism is continuous and proves that for the k-groups the evaluation map

$$\alpha_G : G \to \hat{\hat{G}}$$

is continuous and propose the problem related to the reflexivity of complete k-groups. Nickolas [73] provides the example of free groups which fail to be reflexive and gives a negative answer to the question posed by Nobel. Lamartin [66] defines the k-group dual of the Hausdorff abelian k-group as the group of all k-group morphisms from G into the circle group and equip this group with the k-refinement of the compact-open topology. Lamartin studies the duality properties for k-groups and points out that the difference in the proofs arise because the product operation is not continuous in product topology but in k-products.

Chasco [30] studies the duality properties of the metrizable topological groups and proves that the dual group of every metrizable group is a k-space and from this result, she proves that completeness in an important condition for reflexivity in the class of metrizable groups. Further, she proves that Pontryagin duality and BB-reflexivity (discussed in chapter 4) are the equivalent conditions for the metrizable groups. Another significant result obtained by her is about the determined groups (For D a dense subgroup of a topological group G, the group D is said to determined G if as topological groups $\hat{G}$ and $\hat{D}$ are equal) where she proves that the metrizable groups are determined.

Comfort et al. [38] while considering the class of determined groups give an example of non-metrizable, non-compact determined groups and propose several questions in this regard. Studying the conditions on the sequence of metrizable abelian groups to obtain the reflexivity of inverse and direct limits of the sequence of metrizable groups Ardanza and Chasco [1] obtain an extension of the results of Kaplan [63]. The class of almost metrizable groups is a broader class of groups than the metrizable groups, and the results regarding the duality properties of almost metrizable groups are obtained in [57].

For a topological abelian group G, a subset E of G is said to be bounded if for every neighbourhood U of the identity e there is a finite subset F of the group such that

$$E \subset FU (= \bigcup_{x \in F} xU).$$

The topological abelian group is called totally bounded if the group itself is bounded. If the topology of a topological group is Hausdorff and totally bounded then (by the theorem of Weil [37]) the completion of the topological group G is compact, and this group is called precompact that is, a group is precompact if and only if the closure of the group is compact. Comfort and Ross [37] obtain an identification of the precompact group topologies on an abelian topological group by proving that precompact group topology is generated by some suitable subgroup of the characters which is point separating. More precisely Raczkowski and Trigos [81] formulate this result as: If G is a topological group, then it carries the topology of pointwise convergence on its character group if and only if the group G is totally bounded and they further study the duality (similar to Pontryagin duality) in totally bounded abelian groups by equipping the character group with the finite open topology in place of the compact-open topology. The class of precompact groups contains the class of psudocompact groups as well as the ω-bounded groups.

Chasco and Peinador [35] while studying the conditions that a dense subgroup of a locally compact abelian topological group must satisfy so that the character group with the compact-open topology coincide with the whole group, proposed the problem: Is a precompact, reflexive abelian group necessarily compact ? Working on this problem and with the results of [30] Ardanza et al. [3] point out that the reflexive precompact non-compact groups can only lie within the class of non-metrizable groups and they give the examples of non-compact, precompact abelian groups and countably compact non-compact reflexive abelian groups. After this result, the duality properties of precompact groups are studied in different directions. Tkachenko [86] studies the self-duality in different classes of precompact abelian groups. Bruguera and Tkachenko [26] present a class of reflexive precompact non-compact abelian groups. Various examples of non-discrete reflexive P-groups (groups with G_δ sets open) and non-compact reflexive ω-bounded groups (the precompact groups with the closure of every countable set compact) are obtained by Galindo et al. [52]. Ferrer and Hern$\acute{a}$ndez [46, 45] generalise several duality properties of the topological abelian groups to the non-abelian case by defining the dual object as the irreducible representations of the group using the Fell topology on the dual object. Galindo et al. [53] study some general principles of reflexivity like the behaviour of reflexivity under group extension and present some counter-examples with reference to the Pontryagin duality in the class of P-groups and precompact groups. Gabriyelyan [47] uses the method of $T-$sequences to prove the existence of non-discrete reflexive topologies on abelian groups of infinite exponents and propose a similar problem for the abelian groups of finite exponents. Außenhofer and Gabriyelyan [7] under the assumption of the continuum hypothesis provide a complete negative answer to this problem and prove that every countable reflexive group of finite exponent is discrete, and further they propose the problem of existence in ZFC of a bounded reflexive abelian group of cardinality $\aleph_1$. In view of the results presented in [7] Chasco et al. [36] points out that the class of bounded and precompact, torsion abelian topological groups acquire special features in reference to Pontryagin duality theory and they prove that if a group G is pseudocompact (or respectively Baire), then all countably compact (or respectively compact) subsets of $\hat{G}_p$ (dual of G equipped with the topology of pointwise convergence) are finite. Further, they prove that the group G is pseudo-compact if and only if all countable subgroups of the dual group $\hat{G}_p$ are closed. Finally, certain characterisations of pseudo-compactness and Baire property of $\hat{G}_p$ are obtained in terms of the properties that express the richness of continuous characters of the group G.

Conclusion

In this chapter, we have presented the influence of functional analysis on the extension of the Pontryagin duality theory beyond local compactness. We have presented the known results of the extension of duality theory for the class of topological vector spaces, nuclear spaces and groups, metrizable groups, bounded topological groups, etc.

Another important class from the viewpoint of duality theory is the class of (filter) convergence groups which contains the class of topological groups, and the corresponding extension of the Pontryagin duality is the (continuous) c-duality. In the rest of this thesis, we present the results related to the filter convergence groups and the continuous duality theory.

Chapter 3

Convergence Groups with Boundedness

The generalisation of the concept of a topological space is the convergence space which is based on the concept of (convergent) filters. This category is more suitable for analysis as we can find convergence structures satisfying certain properties for which no topology exist. Different authors use different notations to specify (filter) convergence spaces (cf. [18, 41, 76]). (For the notations related to convergence spaces via sequences see. [80].) In order to overcome the difficulty related to different notations, we give a self-contained introduction to (filter) convergence groups so that a unified notation can be used thought the thesis. This chapter is also devoted to present certain examples of convergence groups with particular emphasis on the homeomorphism groups.

A family of sets $\mathcal{A}$ is said to be coarser than the family $\mathcal{B}$, (written $\mathcal{A} \leq \mathcal{B}$), if for each $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ with $B \subset A$. Further, a family $\mathcal{A}$ of subsets of a set is said to be isotone if $A \in \mathcal{A}$ and $A \subset B$ implies $B \in \mathcal{A}$. Isotonisation (written $\mathcal{A}^{\uparrow}$) of $\mathcal{A}$ is the least isotone family that contains $\mathcal{A}$. Another operation on the family $\mathcal{A}_i$ of subsets of set X is the family of the intersection of elements (finitely many) of A and is denoted as $\mathcal{A}^{\cap}$.

Definition 3.1. A filter $\mathcal{F}$ is a family of proper ($\phi \notin \mathcal{F}$) subsets of a set X for which the following conditions hold
(i) it is closed under supersets (i.e. isotone ($F_1 \supset F_2 \in \mathcal{F} \Rightarrow F_1 \in \mathcal{F}$))
(ii) it is closed under finite intersection ($F_1 \in \mathcal{F}, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$).

Hence, we have $\mathcal{F}$ is a filter iff $\mathcal{F} = \mathcal{F}^{\uparrow\cap}$.
Following [41] we denote by $\mathbb{F}X$ the set of all filters in a set X. A family of subsets (non-empty) $\mathcal{B}$ of a set X is called filter base if isotonisation of $\mathcal{B}$ is filter on X and $\mathcal{B}^{\uparrow}$ is called filter generated by the filter base $\mathcal{B}$. We have $\mathcal{B}$ is a filter base iff for $B_1, B_2 \in \mathcal{B}$ there exists a $B \in \mathcal{B}$ with $B \subset B_1 \cap B_2$ and $\phi \notin \mathcal{B}$.

Some basic examples of filters include the collection of all supersets of a given nonempty subset; the collection of all cofinite subsets of an infinite set; the collection of all the neighbourhoods of a point of a topological space.

The intersection of all elements of a filter is called kernel of the filter. A filter is called principal if the kernel of that filter belongs to that filter and is called free if the kernel of the filter is empty. An ultrafilter or maximal filter is a filter that is not contained (properly) in any other filter. Families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X mesh (written $\mathcal{A}\#\mathcal{B}$) if A and B are not disjoint for all A in $\mathcal{A}$ and B in $\mathcal{B}$. For any two filters $\mathcal{F}$ and $\mathcal{G}$ on a set X we have, if $\mathcal{F}$ mesh with $\mathcal{G}$ then,

$$\mathcal{F} \wedge \mathcal{G} = \{A \cap B : A \in \mathcal{F} \text{ and } B \in \mathcal{G}\}$$

is a filter. Further, for a family, $\mathcal{A}$ of subsets of a set X the grill (denoted $A^{\#}$) is the family of subsets of X that intersect with every element of $\mathcal{A}$. So, a filter is an ultrafilter iff it is a filter- grill. It is quite noteworthy that an ultrafilter is either free or principal. Further, filter $\mathcal{F}$ on set X is ultrafilter iff for each $A \subset X$ only one of the conditions $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$ holds.

Let $\mathcal{F}$ be a filter on a set X and $f : X \to Y$ a map then,

$$f[\mathcal{F}] = \{f(F) : F \in \mathcal{F}\}$$

is called the image filter. For $F_1, F_2 \in \mathcal{F}$ we have

$$f(F_1 \cap F_2) \subseteq f(F_1) \cap f(F_2)$$

is not a filter on Y unless f is not surjective.

Theorem 3.2. *For $\mathcal{F}$ an ultrafilter on set X and $f : X \to Y$, we have $\mathcal{G} = f(\mathcal{F})$ is an ultrafilter on the set Y.*

Proof. Let, $V \notin f(\mathcal{F})$ for some $V \subseteq Y$. Then $f^{-1}(V) \notin U$. Now,

$$X - f^{-1}(V) \in \mathcal{F}.$$

Thus,

$$f(X - f^{-1}(V)) \in f(\mathcal{F}).$$

As,

$$f(X - f^{-1}(V)) \subseteq Y - V,$$

we have $Y - V \in f(U)$ and hence the proof. □

For $\mathcal{F} \in \mathbb{F}(X)$ and $\mathcal{G} \in \mathbb{F}(Y)$ define,

$$\mathcal{F} \times \mathcal{G} = \{F \times G : F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}.$$

$\mathcal{F} \times \mathcal{G}$ is a base for a filter on $X \times Y$ and its isotonisation (i.e. filter generated) is called the product filter. This definition can be extended for the product of infinitely many filters.

Definition 3.3. Let λ be an arbitrary relation between X and the set of all filters on X. The relation is called convergence on that set if for $\mathcal{F}_1, \mathcal{F}_2$ in $\mathbb{F}X$ and x in X the following conditions hold:
(i) **Centred:** $x^{\uparrow} \in \lambda(x)$,
(ii) **Isotone**: If $\mathcal{F}_1 \in \lambda(x)$ and $\mathcal{F}_1 \leq \mathcal{F}_2$ then $\mathcal{F}_2 \in \lambda(x)$, and
(iii) **Finitely deep**: If $\mathcal{F}_1, \mathcal{F}_2 \in \lambda(x)$ then, $\mathcal{F}_1 \cap \mathcal{F}_2 \in \lambda(x)$.
A Convergence space is a pair (X, λ).

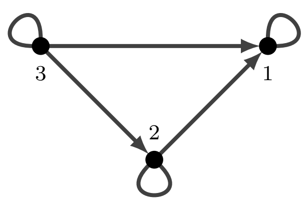
This relation is also denoted by $\mathcal{F} \underset{\lambda}{\to} x$ (or $\mathcal{F} \to x$, if no ambiguity is possible) and we say, filter $\mathcal{F}$ converges to x, or x is limit of $\mathcal{F}$, whenever $\mathcal{F} \in \lambda(x)$.

Further, we have,

$$\lim_{\lambda} \mathcal{F} = \{x : \mathcal{F} \text{ converge to } x \text{ in } \lambda\}$$

and convergence is called Hausdorff if every filter on X converges to at most one point.

Example 3.4. Consider the following reflexive directed graph,



The graph neighbourhood of the vertices [77, Definition 3.1] of this graph are:

$$\overrightarrow{1} = \{1\}, \quad \overrightarrow{2} = \{1, 2\}, \quad \overrightarrow{3} = \{1, 2, 3\};$$

and this graph can be represented by the following convergence:

$$\begin{array}{ccc} \{1\}^{\uparrow} \to \{1,2,3\} & \{1,2\}^{\uparrow} \to \{2,3\} & \{1,2,3\}^{\uparrow} \to \{3\} \\ \{2\}^{\uparrow} \to \{2,3\} & \{1,3\}^{\uparrow} \to \{3\} & \\ \{3\}^{\uparrow} \to \{3\} & \{2,3\}^{\uparrow} \to \{3\} & \end{array}$$

Remark 3.5. The following properties of the convergence spaces make them worth investigation:

1. Category of convergence spaces is

    - Cartesian closed.
    - Contains category of topological spaces.
    - Contains the category of reflexive directed graphs.

2. Convergence spaces are used in unifying (see. [21]) discrete and continuous models of computation.
3. Convergence spaces play a vital role in extending the definition of differential to the discrete structures (see. [77, 79]).

If λ_1 and λ_2 are two convergences on X then λ_1 is called finner than λ_2 (denoted as $\lambda_1 \geq \lambda_2$) iff

$$\lim_{\lambda_1} \mathcal{F} \subset \lim_{\lambda_2} \mathcal{F}$$

for each filter $\mathcal{F}$ on X.

For a topological space X, the convergence structure can be defined by associating the filter $\mathcal{F}$ to x as: x is the limit of the filter $\mathcal{F}$ if the filter $\mathcal{F}$ is finer than the neighbourhood filter of x. Hence, every topological space is a convergence space. It is quite important to see that the concepts of point-set topology can be defined using convergence but, the converse is not always true. In analysis, we can find many situations (like convergence in measure), where non-topological convergence originate.

A map $f : X \to Y$ between two convergence spaces is said to be continuous if

$$\mathcal{F} \underset{X}{\to} x \quad \Rightarrow \quad f(\mathcal{F}) \underset{Y}{\to} f(x).$$

A convergence group is an (abstract) group with a compatible convergence structure, here compatible means that the group operations (in the sense of convergence) are continuous. It is evident from the above discussion that the class of convergence groups contains the class of topological groups.

Some examples of non-topological convergence groups are as follows:

Example 3.6. Convergence groups arise in complex analysis from the theory of quasi-conformal mappings [56]. For, Γ a group acting by homeomorphisms on a compact metrizable space M, this action is called a convergence action if: for every infinite distinct sequence of elements $\gamma_n \in \Gamma$ there exist a subsequence $\gamma_{n_k}, k = 1, 2, \ldots$ and points $a, b \in M$ such that the maps $\gamma_{n_k}\big|_{M \setminus \{a\}}$ converge uniformly on compact subsets to the constant map sending $M \setminus \{a\}$ to b. Γ is called a convergence group.

Example 3.7. Convergence structures on the groups also arise from the theory of $\Sigma-$ groups, where an abelian group is assigned some infinite sums, and these unconditional sums satisfy certain properties to make them the convergence groups.

Another large class of convergence groups is the class of the underlying groups of the convergence vector spaces.

Another important class is the class of homeomorphism groups. The next section is devoted to present certain example of a homeomorphism group which is a non-topological convergence group.

3.1 Homeomorphism groups

The problem of giving an admissible topology to a homeomorphism group is quite old. Arens considered this problem for the group of homeomorphisms of locally compact spaces and proved that for a class of continuous functions defined on some locally compact Hausdorff space, the strongest admissible topology is the k-topology that can be given to the space of continuous functions (admissible here means that the evaluation map is jointly continuous). The extension of the problem to non-locally compact case was considered by Park [76] and two types of convergence structures on the homeomorphism groups were studied in this regard.

The first being the coarsest admissible convergence structure with which a homeomorphism group is convergence group and other is the continuous convergence structure which is coarsest admissible (evaluation is jointly continuous map) convergence structure.

Continuous convergence structure plays a central role throughout the thesis.

Let (X, λ) and (Y, μ) be two convergence spaces and $C(X, Y)$ denotes the set of all continuous (in the sense of convergence) maps from X to Y and the evaluation mapping

$$e : C(X, Y) \times X \to Y$$

is defined as

$$e(f, x) = f(x) \;\; \forall\, f \in C(X, Y) \text{ and } x \in X.$$

The continuous convergence structure (denoted, $(C(X, Y), \lambda_c)$)is defined as

$$\mathcal{G} \underset{\lambda_c}{\longrightarrow} f \;\text{ iff }\; e(\mathcal{G} \times \mathcal{F}) \underset{\mu}{\to} f(x),\; \forall\, x \in X, \text{ and } \mathcal{F} \underset{\lambda}{\to} x,$$

that is a filter $\mathcal{G}$ converge to f in $C(X, Y)$ iff $e(\mathcal{G} \times \mathcal{F})$ converge to $f(x)$ in Y whenever, $\mathcal{F}$ converge to x in X. It is important to see here that when X is a locally compact space and $Y = \mathbb{R}$, then the continuous convergence structure and the compact-open topology coincide.

If $\mathcal{G}$ is a filter on $C(X, Y)$ with $G \in \mathcal{G}$ and $\mathcal{F}$ is a filter on the set X with $F \in \mathcal{F}$ then we denote

$$\mathcal{G}(\mathcal{F}) = e(\mathcal{G} \times \mathcal{F}) = \{G(F)\}^{\uparrow};$$

$$G(\mathcal{F}) = G^{\uparrow}(\mathcal{F});$$

$$\mathcal{G}(F) = \mathcal{G}(F^{\uparrow}).$$

With these notations continuous convergence can be defined as

$$\mathcal{G} \underset{\lambda_c}{\rightarrow} f \text{ iff } \mathcal{G}(\mathcal{F}) \underset{\mu}{\rightarrow} f(x) \, \forall \, \mathcal{F} \underset{\lambda}{\rightarrow} x.$$

Now consider the homeomorphism group $H(\mathbb{Q})$ of the rational numbers $(\mathbb{Q}, \tau_{\mathbb{R}\cap\mathbb{Q}})$ endowed with the subspace topology of the real line. Clearly, the group $H(\mathbb{Q})$ is not locally compact. If this group is endowed with the continuous convergence structure then it is evident from [76] that the algebraic structure of $H(\mathbb{Q})$ is not compatible with the continuous convergence structure.

The coarsest convergence structure (denoted here, λ_a) is another convergence structure defined on $H(\mathbb{Q})$ which makes it a convergence group.

Let, $\mathcal{F} \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x$ a filter on $\mathbb{Q}$ and $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ filters on $H(\mathbb{Q})$. For the evaluation map

$$\omega : H(\mathbb{Q}) \times \mathbb{Q} \rightarrow \mathbb{Q}$$

we get,

$$\mathcal{G} \times \mathcal{F} = \omega(\mathcal{G} \times \mathcal{F}).$$

Now define convergence λ_a on $H(\mathbb{Q})$ as:

$$\mathcal{G} \underset{\lambda_a}{\rightarrow} f \text{ iff } \mathcal{G}(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} f(x)$$

and

$$\mathcal{G}^{-1}(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} f^{-1}(x) \, \forall \, \mathcal{F} \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x,$$

here, $\mathcal{G}^{-1}$ is the filter generated by $\{G^{-1} : G \in \mathcal{G}\}$, i.e. $\{G^{-1} : G \in \mathcal{G}\}^{\uparrow}$.

Theorem 3.8. *$(H(\mathbb{Q}), \lambda_a)$ is a convergence group.*

Proof. **Centred:**
As $H(\mathbb{Q})$ is a homeomorphism group so, both $g, g^{-1} \in H(\mathbb{Q})$ are continuous.
For $\mathcal{F} \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x$ we have,

$$\{g\}^{\uparrow}(\mathcal{F}) = g(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g(x)$$

and

$$\{g^{-1}\}^{\uparrow}(\mathcal{F}) = g^{-1}(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g^{-1}(x).$$

Hence, $\{g\}^{\uparrow} \underset{\lambda_a}{\rightarrow} g$.
Isotone:

For, $\mathcal{G}_1 \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g_1$ and $\mathcal{G}_1 \leq \mathcal{G}_2$ we have,

$$\mathcal{G}_1(\mathcal{F}) \leq \mathcal{G}_2(\mathcal{F})\ \forall\ \mathcal{F} \underset{\lambda_a}{\longrightarrow} x$$
$$\Rightarrow\ \mathcal{G}_2 \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g_2$$

and $\mathcal{G}_1 \leq \mathcal{G}_2 \Rightarrow \mathcal{G}_1^{-1} \geq \mathcal{G}_2^{-1}$ so,

$$\mathcal{G}_1^{-1}(\mathcal{F}) \geq \mathcal{G}_2^{-1}(\mathcal{F})\ \forall\ \mathcal{F} \underset{\lambda_a}{\longrightarrow} x$$
$$\Rightarrow\ \mathcal{G}_2^{-1} \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g_2$$

Hence, $\mathcal{G}_2 \geq \mathcal{G}_1 \underset{\lambda_a}{\longrightarrow} g_1 \Rightarrow \mathcal{G}_2 \underset{\lambda_a}{\longrightarrow} g_2$.

Finitely deep:

For, $\mathcal{G}_1 \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g$ and $\mathcal{G}_2 \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g$ we have,

$$\mathcal{G}_1(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x \text{ and } \mathcal{G}_2(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x$$

$$\Rightarrow \mathcal{G}_1(\mathcal{F}) \wedge \mathcal{G}_2(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x$$

as

$$\mathcal{G}_1(\mathcal{F}) \wedge \mathcal{G}_2(\mathcal{F}) = (\mathcal{G}_1 \wedge \mathcal{G}_2)\mathcal{F}$$

we have,

$$(\mathcal{G}_1 \wedge \mathcal{G}_2)\mathcal{F} \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x.$$

Similarly, for, $\mathcal{G}_1^{-1} \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g$ and $\mathcal{G}_2^{-1} \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g$ we get,

$$\mathcal{G}_1^{-1}(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x \text{ and } \mathcal{G}_2^{-1}(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x$$

$$\Rightarrow \mathcal{G}_1^{-1}(\mathcal{F}) \wedge \mathcal{F}_2^{-1}(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x$$

as

$$\mathcal{G}_1^{-1}(\mathcal{F}) \wedge \mathcal{G}_2^{-1}(\mathcal{F}) = (\mathcal{G}_1^{-1} \wedge \mathcal{G}_2^{-1})\mathcal{F}$$

we have,

$$(\mathcal{G}_1^{-1} \wedge \mathcal{G}_2^{-1})\mathcal{F} \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} x.$$

Hence, we have $\mathcal{G}_1 \underset{\lambda_a}{\longrightarrow} g$ and $\mathcal{G}_2 \underset{\lambda_a}{\longrightarrow} g \Rightarrow \mathcal{G}_1 \wedge \mathcal{G}_2 \underset{\lambda_a}{\longrightarrow} g$

Continuity of composition:

For, $\mathcal{G}_1 \underset{\tau_{\lambda_a}}{\longrightarrow} g_1$ and $\mathcal{G}_2 \underset{\tau_{\lambda_a}}{\longrightarrow} g_2$

$$(\mathcal{G}_1 \circ \mathcal{G}_2)(\mathcal{F}) = (\mathcal{G}_1(\mathcal{G}_2))(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} (g_1 \circ g_2)(x)$$
$$\Rightarrow \mathcal{G}_1 \circ \mathcal{G}_2 \underset{\tau_{\lambda_a}}{\longrightarrow} g_1 \circ g_2.$$

Similarly,

$$(\mathcal{G}_1 \circ \mathcal{G}_2)^{-1}(\mathcal{F}) = (\mathcal{G}_2^{-1}(\mathcal{G}_1^{-1}))(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} (g_2^{-1}(g_1^{-1}))(x) = (g_1 \circ g_2)^{-1}(x)$$
$$\Rightarrow (\mathcal{G}_1 \circ \mathcal{G}_2)^{-1} \underset{\tau_{\lambda_a}}{\longrightarrow} (g_1 \circ g_2)^{-1}.$$

Hence, If $\mathcal{G}_1 \underset{\tau_{\lambda_a}}{\longrightarrow} g_1$ and $\mathcal{G}_2 \underset{\tau_{\lambda_a}}{\longrightarrow} g_2$ we have, $\mathcal{G}_1 \circ \mathcal{G}_2 \underset{\tau_{\lambda_a}}{\longrightarrow} g_1 \circ g_2$.

Continuity of inversion:

For, $\mathcal{G} \underset{\tau_{\lambda_a}}{\longrightarrow} g$ we have,

$$\mathcal{G}^{-1}(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g^{-1}(x)$$

and

$$(\mathcal{G}^{-1})^{-1}(\mathcal{F}) = \mathcal{G}(\mathcal{F}) \underset{\tau_{\mathbb{R}\cap\mathbb{Q}}}{\longrightarrow} g(x) = (g^{-1^{-1}})(x).$$

Hence, $\mathcal{G} \underset{\lambda_a}{\longrightarrow} g \Rightarrow \mathcal{G}^{-1} \underset{\lambda_a}{\longrightarrow} g^{-1}$ □

3.2 Bounded convergence groups

Let G be an abelian (not necessarily topological or convergence) group (additive notation is used throughout this section). Vilenkin [90, Definition 1.1], defines the boundedness on G as the family (called bounded sets) of subsets of the group G, satisfying the following conditions:

1. B is bounded implies that $-B$ is bounded;
2. finite sets are bounded;
3. subsets of all bounded sets are bounded;
4. $B_1 \cup B_2$ and $B_1 + B_2$ are bounded whenever, B_1, B_2 are bounded.

Some examples are: the family of all finite subsets define a boundedness on any abelian group; the family of all precompact subsets define boundedness on a topological group. For a more detailed account of topological groups with boundedness we refer the reader to [50]. Here, we study the convergence groups with boundedness. Most of the results of this chapter are motivated from [15, Section 18] and [18, Section 3.7].

Consider the family of all subsets σ_{C_W}, σ_{C_S} respectively of a convergence group G which are bounded in the following sense.

Definition 3.9. A subset B of the convergence group G is said to be bounded ($B \in \sigma_{C_W}$) if there exists a filter $\mathcal{F}$ converging to 0 (additive identity) in G such that for every $F \in \mathcal{F}$ there exists $n \in \mathbb{N}$ with $B \subset nF$.

Definition 3.10. A subset B of the convergence group G is said to be bounded ($B \in \sigma_{C_S}$) if for every filter $\mathcal{F}$ converging to 0 (additive identity) in G, there is a coarser filter $\mathcal{F}_1$ converging to 0, such that for every $F_1 \in \mathcal{F}_1$ there exists $n \in \mathbb{N}$ with $B \subset nF_1$.

Theorem 3.11. *σ_{C_W} defines a boundedness on G.*

Proof. (i) Let, $B \in \sigma_{C_W}$, then

$$B \in \sigma_{C_W} \Leftrightarrow \exists \mathcal{F} \underset{G}{\rightarrow} 0 \ : \ \forall F \in \mathcal{F} \ \exists \ n \in \mathbb{N} \ , \ B \subset nF.$$

G is a convergence group so, $\mathcal{F} \underset{G}{\rightarrow} 0 \Rightarrow -\mathcal{F} \underset{G}{\rightarrow} 0$. As $B \in \sigma_{C_W}$ hence, $-B \subset n(-F), \ \forall \ -F \in -\mathcal{F} \Rightarrow -B \in \sigma_{C_W}$.
(ii) Let $g \in G$ then we first prove that $\{g\}$ is bounded. Clearly, $\{g\}^{\uparrow} \underset{G}{\rightarrow} g$ and $\{g\}^{\uparrow} - g \underset{G}{\rightarrow} 0$. As, $\{g\} \subset \{g\}^{\uparrow} - g$ hence, $\{g\} \in \sigma_{C_W}$. Extending the result for a subset of G with finite elements we get that finite subsets of G are bounded.
(iii) Let, $B \in \sigma_{C_W}$ and $B_1 \subset B$ then $B_1 \subset nF$, hence, $B_1 \in \sigma_{C_W}$.
(iv) Let, $B_1, B_2 \in \sigma_{C_W}$ then,

$$B_1 \in \sigma_{C_W} \Leftrightarrow \exists \mathcal{F}_1 \underset{G}{\rightarrow} 0 \ : \ \forall F_1 \in \mathcal{F}_1 \ \exists \ n \in \mathbb{N} \ , \ B_1 \subset nF_1;$$

$$B_2 \in \sigma_{C_W} \Leftrightarrow \exists \mathcal{F}_2 \underset{G}{\rightarrow} 0 \ : \ \forall F_2 \in \mathcal{F}_2 \ \exists \ m \in \mathbb{N} \ , \ B_2 \subset mF_2.$$

Now as G is a convergence group hence, $\mathcal{G} =< \mathcal{F}_1 + \mathcal{F}_2 > \underset{G}{\rightarrow} 0$ and $B_1 + B_2 \subset kG \ \forall \ G \in \mathcal{G}, \ k = max\{n, m\}$. Hence, $B_1 + B_2 \in \sigma_{C_W}$.
Similarly, we can prove that $B_1 \cup B_2 \in \sigma_{C_W}$. □

Theorem 3.12. *σ_{C_S} defines a boundedness on G.*

Proof. The proof is similar to Theorem 3.11, hence omitted. □

Remark 3.13. We do not know whether σ_{C_W} and σ_{C_S} induce two different boundedness on a convergence group or they coincide.

Definition 3.14. A convergence group G is called locally bounded group if every filter which converges to 0 in the group G contains a bounded set.

Proposition 3.15. *For B a bounded subset of the convergence group (G, σ_{C_W}), $\{kB : k \in \mathbb{N}\}$ is a bounded subset of (G, σ_{C_W}).*

Proof. As $B \in \sigma_{C_W}$ so, there exists a filter $\mathcal{F} \to 0$ in the convergence group G such that for every $F \in \mathcal{F}$ there exists $n \in \mathbb{N}$ with $B \subset nF$. Further, for every nB we have, $k(nB) = (nkB) \subset mF$, where $m = nk \in \mathbb{N}$. Hence $kB \in \sigma_{C_W}$. □

Theorem 3.16. *Let (G_1, σ_{C_W}) and (G_2, σ_{C_W}) be convergence groups with boundedness; B a bounded subset of G_1 and $\phi : G_1 \to G_2$ a continuous homomorphism then, $\phi(B) \subset G_2$ is bounded in G_2.*

Proof. As B is bounded subset of (G_1, σ_{C_W}) so there exists a filter $\mathcal{F} \to 0$ in the group G_1 such that for every $F \in \mathcal{F}$ there exists $n \in \mathbb{N}$ with $B \subset nF$. Now, as ϕ is continuous homomorphism so $\mathcal{G} = \phi(\mathcal{F}) \to 0$ in G_2. Further, for every $G \in \mathcal{G}$ there exist $n \in \mathbb{N}$ with $\phi(B) \subset nG = n\phi(F)$ and hence the proof. □

Theorem 3.17. *For (G, σ) be a convergence group with boundedness and H its subgroup, $\sigma_H = \{H \cap B : \forall B \in \sigma\}$, defines a boundedness on H induced by G.*

Proof. (i) For $B_1 \in \sigma_H$, there is some B in σ such that $B_1 = H \cap B$. Hence,

$$-B_1 = (-H) \cap (-B) = H \cap (-B).$$

As $B \in \sigma \implies (-B) \in \sigma$ hence, $-B_1 \in \sigma_H$.

(ii) For, B a finite subset of H we have $B \in \sigma$ and hence, $B = B \cap B \in \sigma_H$.

(iii) For $B_1 \in \sigma_H$ we have $B \in \sigma$ such that $B_1 = H \cap B$. Further for $B_2 \subset B$ we have $B_3 \subset B$ such that $B_2 = H \cap B_3$ hence, $B_2 \in \sigma_H$.

(iv) For, $B_1, B_2 \in \sigma_H$ we have $B_1 = H \cap B_3$ and $B_2 = H \cap B_4$ for some $B_3, B_4 \in \sigma$. Now,

$$B_1 \cup B_2 = (H \cap B_3) \cup (H \cap B_4) = H \cap (B_3 \cup B_4).$$

As, $B_3 \cup B_4 \in \sigma$ hence, $B_1 \cap B_2 \in \sigma_H$.
Similarly, we have

$$B_1 + B_2 = (H \cap B_3) + (H \cap B_4) = H \cap (B_3 + B_4).$$

hence, $B_1 + B_2 \in \sigma_H$. □

Theorem 3.18. *For a convergence group (G, σ) with boundedness and H its normal subgroup, canonical images of the bounded subsets of the group define a boundedness on G/H (denoted $\sigma_{G/H}$) induced by (G, σ).*

Proof. Let, the canonical mapping be $q : G \to G/H$ then we need to prove that

$$q(\sigma) = \{q(B) : \forall B \in \sigma\}$$

induce the boundedness $\sigma_{G/H}$.
The proof is similar to Theorem 3.17 and hence omitted. □

Theorem 3.19. *For $((G_i)_{i\in I}, (e_{i,j})_{i\leq j})$ a system of convergence groups with boundedness $\sigma^i_{C_W}$, if G is the reduced inductive limit of this system (i.e. all maps e_i are injective) then a $B \subset G$ is bounded iff there is an index $i \in I$ and $A \subset G_i$, a bounded subset such that $B = e_i(A)$.*

Proof. If there is index $i \in I$ and $A \subset G_i$, a bounded subset such that $B = e_i(A)$ then subset B of G is bounded is evident from Theorem 3.16.
For, $B \subset G$ a bounded set there is an index $i \in I$ and filter $\mathcal{F} \to 0$ in the convergence group G_i such that for each $F \in e_i(\mathcal{F})$ there is $n \in \mathcal{N}$ with $B \subset nF$. So, $B \subset e_i(G_i)$. Further, $e_i^{-1}(B) \subset nF$ and as e_i is injective hence, $e_i^{-1}(B) = A$ is a bounded subset of G_i and hence, $B = e_i(A)$. □

Remark 3.20. Compact subsets of bounded convergence group need not be bounded (see. [18, Proposition 3.7.4, Proposition 3.7.6]).

Theorem 3.21. *Inductive limits, coproducts and quotients of locally bounded convergence groups are locally bounded.*

Proof. For a family $(G_i)_{i\in I}$ of locally bounded convergence groups let, G be the convergence group with final convergence(with respect to $u_i : G_i \to G$) group structure. Locally boundedness of G implies the theorem.
Let, $\mathcal{F} \to g$ be filter on G then there are i_j's, $j = 1 \ldots n$ in I and for every $k \in \mathbb{N}$ a filter $\mathcal{F}_k \to 0$ in G_{i_k}. Further, we have finitely many elements $g_1, g_2 \ldots g_m$ in the group G such that

$$\mathcal{F} - x \supset u_{i_1}(\mathcal{F}_1) + u_{i_1}(\mathcal{F}_1) + \cdots + u_{i_n}(\mathcal{F}_n) + \mathcal{U}(g_1) + \cdots + \mathcal{U}(g_m)$$

where $\mathcal{U}(g_m) = \{\mathcal{U} : \mathcal{U} \to g_m \text{ in } G\}$ is the neighbourhood filter of g_m. As every summand contains a bounded set hence, the proof. □

Conclusion

This chapter begins with a self-contained introduction to the filter convergence groups. Then we present a certain (homeomorphism) group which is a non-topological convergence group. Finally, the notation of boundedness in convergence groups is introduced, and certain properties of the bounded sets are presented. It is evident that boundedness does not play the same role for convergence groups which it plays in the topological case. In this chapter, we have seen certain examples of boundedness on the convergence groups, but the inverse problem has not been explored, i.e. to define the convergence structure if a bounded group is given.

Chapter 4

Convergence Measure Space

One of the major problems in extending Pontryagin duality of topological groups beyond local compactness is the fact that the evaluation map of a topological group to its second dual group fails to be continuous. In this regard, Peinador [68] proves that the evaluation map

$$e : \hat{G} \times G \to \mathbb{T}$$

defined as

$$e(\chi, x) = \chi(x)$$

is continuous if and only if the group G is locally compact and thus the result explains why the class of locally compact abelian groups fits best for the theory of Pontryagin reflexive topological groups, but as we replace compact-open topology with continuous convergence structure on character group the evaluation map is always continuous.

To obtain an extension of Pontryagin duality theorem for the convergence group first, we look at some facts about the Rudin's proof of Pontryagin duality theorem.

Theorem 4.1 (Pontryagin duality theorem)**.** *For any LCA topological group the evaluation mapping*

$$\alpha_G : G \to \hat{\hat{G}}$$

is a topological isomorphism. ($\hat{\hat{G}}$ denotes the double dual group.)

The proof of the theorem follows in three parts
(i) α_G is isomorphism of G into $\hat{\hat{G}}$
(ii) α_G is homeomorphism of G into $\hat{\hat{G}}$.
(iii) $\alpha_G(G)$ is closed and dense in $\hat{\hat{G}}$.

The proof of (i), i.e. to prove that character group separates points of G requires hard tools of measure theory and operator algebra, so we present some basic facts about it. For a detailed account of the theorem and for the proof of (ii) and (iii) we refer the reader to [83, Theorem 1.7.2].

For a locally compact abelian group G, let, $L_1(G)$ denotes the space of all integrable functions on G ($L_1(G)$ is a commutative Banach algebra), $C_0(G)$ the space of all complex-valued, continuous functions on G which vanish at infinity and $M(G)$ the set of all complex-valued completely additive Borel measures on G.

For $f \in L_1(G), \chi \in \hat{G}$ and μ_G a Haar measures on G, we have a bounded linear operator,

$$\mathfrak{F} : L_1(G) \to C_0(\hat{G}),\ f \mapsto \mathfrak{F}(f) = \hat{f}$$

defined as

$$\hat{f}(\chi) = \int f.\bar{\chi} d\mu_G$$

called Fourier transform of f on G.

Example 4.2. [83, Example 1.2.7] For $G = \mathbb{T}, \mathbb{Z}$ and $\mathbb{R}$ we have,

1. For $G = \mathbb{T}$, $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{\iota\theta})e^{-\iota n\theta} d\theta \qquad (n \in \mathbb{Z})$.
2. For $G = \mathbb{Z}$, $\hat{f}(e^{\iota\alpha}) = \sum_{n=-\infty}^{\infty} f(n)e^{-\iota n\alpha} \qquad (e^{\iota\alpha} \in \mathbb{T}, \alpha \in \mathbb{R})$.
3. For $G = \mathbb{R}$, $\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-\iota xy} dx \qquad (x, y \in \mathbb{R})$.

Further, we have $L_1(G)$ is associative algebra with the (multiplication) convolution defined as

$$(f * g)(x) = \int f(x-z)g(z)\mu_G(z)\ , x, z \in G$$

and $C_0(\hat{G})$ is an associative algebra with the pointwise multiplication defined as

$$(\phi.\psi)(\chi) = \phi(\chi).\psi(\chi),\ \chi \in \hat{G}.$$

The image of $\mathfrak{F}$ under the map

$$\mathfrak{F} : L_1(\hat{G}) \to C_0(\hat{\hat{G}} \cong G)$$

is called Fourier algebra $A(\hat{G})$ on the group.
$\hat{G}$ can be given a weak topology induced by $A(\hat{G})$ and the Galfand theory gives us that $A(\hat{G})$ is a separating sub-algebra of $C_0(\hat{G})$. We have $\hat{G}$ is a group as well as locally compact Hausdorff space and via compact-open topology these two structures fit together to make it a locally compact abelian group.
Let, us denote by $M(G)$ the set of all complex-valued, bounded, regular, Borel measures μ on the topological group G. For $\mu, \nu \in M(G)$ the convolution is defined as

$$(\mu * \nu)E = (\mu \times \nu)(E_2)$$

here, E is the Borel set in G and $E_2 = \{(g_1, g_2) \in G \times G : g_1 + g_2 \in E\}$.
$M(G)$ with multiplication defined by convolution is a commutative Banach algebra with unit.

In view of class $M(G)$ the Fourier transform

$$\mathfrak{F} : L_1(G) \to C_0(\hat{G})$$

is extended to Fourier -Stieltjes transform given as

$$\mathfrak{F} : M(G) \to C_{bu}(\hat{G}), \quad \mu \to \mathcal{F}(\mu) = \hat{\mu},$$

defined as

$$\hat{\mu}(\chi) = \int_G \bar{\chi} d\mu, \chi \in \hat{G}$$

here $C_{bu}(\hat{G})$ is the Banach space of all bounded, uniformly continuous, complex valued functions on $\hat{G}$.

The inverse Fourier transform can be defined as

$$f = \int \hat{f}(\chi) d\hat{\mu}.$$

This inversion formula does not always hold. Let us denote by $B(G)$, the set of all functions for which the inversion formula holds (Bochner's theorem and Jordan decomposition theorem implies that this set consists of linear combinations (finite) of positive definite, continuous functions on G). Hence, we obtain the inversion theorem for a large class of functions, $L_1(G) \cap B(G)$. It is a consequence of the inversion theorem that $\hat{G}$ separates the points on G.

To obtain analogues results for the class of convergence groups, we need a theory of abstract integration on convergence groups. Some results in this regard are presented in the next section.

4.1 Duality in convergence groups

An approach towards the abstract integration theory on topological spaces is restricted exclusively to the class of locally compact Hausdorff spaces. The difficulty of the problem to develop a theory of abstract integration for convergence spaces which are not necessarily topological, lies in, defining a sigma algebra related to the convergence structure of the underlying space so that the measure is compatible with the convergence structure. Here, we use the topological modification to define the $\sigma-$ algebra compatible with the convergence structure.

In any convergence space the filter

$$\mathcal{N}(x) = \bigcap\{\mathcal{F} : \mathcal{F} \to x\}$$

is defined as the neighbourhood filter of x and elements of $\mathcal{N}(x)$ as the neighbourhoods of x. A set U in X is said to be open if it is a neighbourhood of all of its points. The collection of all open sets satisfy the axioms of topology, and hence, a topology can be associated to every convergence space (called topological modification [18, Definition 1.3.8], $o(X)$ of the convergence space).

Definition 4.3. The topological modification,(denoted λ_{tm}), is the finest topology (with respect to associated convergence structure of this topology) coarser than convergence structure λ.

We define the convergence measure space using topological modification as follows:

Definition 4.4 (Convergence Measure Space)**.** A convergence measure space is quadruple $(X, \lambda, \mathcal{M}, \mu)$, with $(X, \mathcal{M}, \mu)$ a measure space and (X, λ) a convergence space such that $\lambda_{tm} \subset \mathcal{M}$, i.e every open set (in the sense of convergence) is measurable.

Remark 4.5. This definition coincides with the definition of the topological measure space if the underlying convergence structure is topological. With this kind of approach to define the $\sigma-$algebra most of the results for the convergence measure spaces can be obtained from the theory of topological measure spaces.

The next example illustrates that there exists two different convergence spaces with same topological modification.

Definition 4.6. [18, Definition 1.3.23] A convergence space (X, λ) is pseudo-topological if $\mathcal{F} \underset{X}{\to} x$ whenever every ultrafilter finer then $\mathcal{F}$ converge to x and the pseudo-topological modification, $\chi(X)$ associated to a convergence space is defined as

$$\mathcal{F} \underset{\chi(X)}{\longrightarrow} x \text{ iff } \mathcal{G} \underset{X}{\to} x \text{ for every ultrafilter } \mathcal{G} \geq \mathcal{F}.$$

Example 4.7. Consider a non pseudo-topological convergence space X. Clearly, X and $\chi(X)$ need not be homeomorphic but the topological modification of X and $\chi(X)$ are homeomorphic.

Example 4.8. [18, Example 1.4.5][A compact convergence space which is non-topological] The finest convergence on [0, 1] that has same convergent ultrafilter as the usual topology of [0, 1] is a compact Hausdorff convergence space which is not topological.

Theorem 4.9. *The topological modification (X, λ_{tm}) of a (not necessarily topological) compact convergence space (X, λ) is always a compact topological convergence space.*

Proof. The proof is evident from the facts that, λ_{tm} has more convergent filters than λ. The identity map

$$i : (X, \lambda) \to (X, \lambda_{tm})$$

is continuous. Every ultrafilter which is convergent in λ is convergent in λ_{tm}; (X, λ_{tm}) has more compact subsets than (X, λ). □

There is no result analogous to 4.9 for locally compact convergence spaces and this brings into the picture the problem of characterisation of the class of convergence spaces whose topological modification is locally compact topological space. Further, the topological modification (G, Λ_{tm}) of a convergence group (G, Λ) not always gives rise to a group topology, i.e. the resulting topology need not be a group topology, and hence for convergence groups, the following problem arises:

Problem 4.10. To characterise those convergence groups whose topological modification is a locally compact topological group.

To best of our knowledge, no result analogues to Theorem 4.9 holds for compact convergence groups, and this gives rise to the following problem in the class of locally compact convergence groups.

Problem 4.11.

Are the convergence groups whose topological modification locally compact topological group, reflexive?

It is quite important to point out here that a general definition of a convergence measure space is still unknown and is required for analysis over convergence groups. Further, it has been obtained that introducing the theory like Haar integrals for convergence groups is not a trivial process as the analytical behaviour of locally compact convergence groups (or spaces) is not similar to locally compact topological groups (or spaces).

4.2 BB-Duality and continuous duality

The comparison of the two different notations of reflexivity for topological groups viz BB-duality (Binz and Butzmann Duality) and the Pontryagin duality has been initiated by Chasco and Peinador [33]. They prove that if the evaluation map

$$\alpha_G : G \to \hat{\hat{G}}$$

is continuous then the P-bidual is a topological subgroup of BB-bidual. Using this theorem, the example of the BB-reflexive group which is not Pontryagin reflexive is obtained. Bruguera and Chasco [25] while studying the properties of the subgroups and quotients with respect to BB-duality theory define a BB-reflexive group to be BB-strongly reflexive (convergence) group if the Hausdorff quotients of G and $\Gamma_c G$ are BB-reflexive, and they prove that this class of topological groups contains (or is larger than) the class of locally compact abelian groups.

Study of duality theory of convergence abelian groups is initiated by Butzmann [28]. Let (G, Λ) be a convergence abelian group and ΓG denotes the set of all continuous (as convergence space) homomorphisms of G into the circle group. The set of continuous homomorphisms (of convergence groups) with the structure of continuous convergence is defined as convergence dual of G and is denoted as $(\Gamma G, \Lambda_c)$. For a convergence group G, the map

$$\kappa : G \to \Gamma\Gamma G$$

defined as

$$\kappa(g)(\chi) = \chi(g) \; \forall \; g \in G, \; \chi \in \Gamma G$$

is a continuous group homomorphism and the convergence group G is called c-reflexive if this evaluation map κ is an isomorphism. The behaviour of continuous duality (in the sense of continuous convergence structure) under the operations like products and co-products is studied by Butzmann. While studying the subgroups and the quotients of the convergence groups he finds the sufficient condition for the continuous character group ($\mathbb{C}\mathrm{Hom}(H, \mathbb{T})$) of a subgroup H of the group G to be isomorphic to the quotient of a continuous-character group of given convergence group. Finally, he obtains the characterisation of relatively compact subsets of the continuous character group of a topological group and proves that if the topological group is locally quasi-convex, then the natural mapping from the given topological group into its bidual group is an embedding (the converse is also true). Ardanza-Trevijano and Chasco [2] investigate the conditions under which the continuous-dual of the limit of the inverse system is the direct limit of the continuous-dual system and the conditions for which the direct and inverse limits are related via continuous-duality. While studying the limits and co-limits of the topological groups, Beattie and Butzmann [16] point out that inductive limits and quotients of reflexive convergence groups may not be reflexive and they prove that, inductive limits of LCA groups is reflexive if this limit is separated.

Chasco and Peinador [35] point out that all the abelian groups are determined from the viewpoint of convergence, that is for a dense subgroup $H \subset G$ of a topological abelian group the continuous convergence structure in ΓG and ΓH coincide. Beattie and Butzmann [17] while extending the results of [33] point out that the problem of determined groups is simple to handle with the continuous duality than with the Pontryagin dual. Further, the difference in the behaviour of the continuous duality and Pontryagin duality is studied for the problem of distinguishing the topological groups with same dual and it has been pointed out that there might be many group topologies on underlying group of a topological group all having the same Pontryagin dual and in this case if one of these groups is P-reflexive then none of the others can be P-reflexive.

Conclusion

In this chapter, we have presented the role of measure theory in analytical proof of Pontryagin duality theorem. We have defined the convergence measure space using the topological modification but due to the lack of the notation of a general convergence measure space the extension of Pontryagin duality theorem for convergence groups is not trivial (even for some small classes). It is important to point out that no theorem analogues to Pontryagin duality theorem hold for the locally compact convergence groups. The counterexample in this regard is presented in [18, Example 8.5.14]. In this regard, the problem of characterisation of the class of c-reflexive locally compact convergence groups is still open, and we obtain certain partial answers in chapter 5.

Chapter 5

Locally Quasi-Convex Convergence Groups

We begin this chapter with an example of a non-reflexive, locally compact convergence abelian group.

Let X be a locally compact topological space, $C(X)$ and $C(X,\mathbb{T})$ respectively denote the group of all continuous, real-valued functions on X and the group of unimodular ($X \to \mathbb{T}$) continuous functions on X (for details refer [18, Section 8.5]). Define

$$\rho : C_c(X) \to C_c(X,\mathbb{T}) \ \text{ as } \ \rho(f) = \rho \circ f.$$

Clearly, ρ is continuous and a group homomorphism. Further, as X is locally compact so $C_c(X) = C_{co}(X)$ and $C_c(X,\mathbb{T}) = C_{co}(X,\mathbb{T})$. From [18, Theorem 5.8.11] we have, $C_c(X,\mathbb{T})$ is reflexive. For

$$\kappa : X \to \Gamma_c C(X,\mathbb{T})$$

defined as

$$\kappa(x)f = f(x),$$

the group generated by $\kappa(X)$, (denoted $G =< \kappa(X) >$) is a locally compact subgroup [18, Proposition 8.5.12] of $\Gamma_c C_c(X,\mathbb{T})$. Further, from [18, Example 8.5.14] we have, for X a connected, compact topological space, the group $G =< \kappa(X) >$ is not reflexive.

It is clear from the above discussion that class of locally compact convergence abelian groups is not contained in the class of c-reflexive groups. So, in this chapter, we deal with the problem of characterisation of the class of reflexive locally compact convergence groups. To obtain a class of reflexive locally compact convergence groups, we introduce the notation of local-quasi-convexity in convergence groups and prove that local quasi-convexity is a necessary condition for a convergence group to be c-reflexive.

Further, we prove that a non-topological compact convergence group (if it exists) is reflexive iff it is locally quasi-convex. This chapter ends with some results related to the duality of limits and co-limits in the class of locally quasi-convex convergence abelian groups.

5.1 Local quasi-convexity and continuous duality

The concept of polar is well know in the theory [84] of topological vector spaces and plays the central role in the development of the duality theory of locally convex spaces. Similarly the notion of annihilator and polar plays an important role in defining locally quasi-convex topological groups. The idea behind the definition of a quasi-convex subset is to separate (with continuous characters) the subset from the points in the (relative) complement of that subset. Formally, a subset $A \subset G$ of a topological abelian group is quasi-convex if for each point g in $G \backslash A$, there exists a character χ in the polar set of A such that

$$Re(\chi(g)) < 0,$$

that is $A^{\triangleright\triangleleft} = A$. Further, G is locally quasi-convex if it has a base comprising of quasi-convex neighbourhoods of identity. As any reflexive group is dual of its character group, so the dual groups are locally quasi-convex and hence, the reflexive abelian groups lie in the class of locally quasi-convex topological abelian groups but the converse may fail as additive group of rational numbers $\mathbb{Q}$ equipped with the Euclidian topology induced from the additive group $\mathbb{R}$ of real numbers is locally convex but not reflexive. Banaszczyk [15] relates the local convexity of the topological vector space to local quasi-convexity of the underlying additive abelian topological group and proves that a topological vector space is a locally convex space iff when considered as an additive group (an underlying group of topological vector space) it is a locally quasi-convex. It is quite interesting to note that convexity (of vector spaces) is purely an algebraic property while the definition of quasi-convexity involves the topology of the group and to this date, there is no physical interpretation for quasi-convexity. A study of quasi-convex subsets is conducted in [67] where the author studies the quasi-convex subsets of $\mathbb{Z}$ and presents its relation with Bohr sets.

We extend the definition of local quasi-convexity[1] to the class of convergence groups and obtain a characterisation of the c-reflexive locally compact convergence abelian groups.

For any convergence abelian group G, the annihilator and inverse annihilator are defined as:

[1]We are thankful to Prof Frédéric Mynard for introducing the current version of the definition of local quasi-convexity.

Definition 5.1 (Annihilator and inverse annihilator)**.** The *annihilator* $O^{\perp}$ of a subgroup O of convergence abelian group G and the *inverse annihilator* ${}^{\perp}E$ of a subgroup E of ΓG are the subgroups of ΓG and G respectively defined as:

$$O^{\perp} = \{\chi \in \Gamma G : \chi\{O\} = \{1\}\};$$

$${}^{\perp}E = \{g : \chi(g) = \{1\} \ \forall\, \chi \in E\} \subset G.$$

The more general notion for the annihilator and the inverse annihilator of a subgroup is the *polar* ($H^{\rhd}$) and the *inverse polar* ($L^{\lhd}$).

Definition 5.2 (Polar and inverse polar)**.** For any subset H of G and L of ΓG the polar and the inverse polar of H and L respectively are subsets defined as:

$$H^{\rhd} = \{\chi \in \Gamma G : \chi(H) \subset \mathbb{T}_{+}\};$$

$$L^{\lhd} = \{g \in G : \chi(g) \subset \mathbb{T}_{+},\ \forall\, \chi \in L\},$$

here $\mathbb{T}_{+} = \{z \in \mathbb{T} : \text{ Re } (z) \geq 0\}$.

If $\mathbb{T}$ is identified with the interval $(-\frac{1}{2}, \frac{1}{2}]$ the equivalent notations of polar and annihilator are presented in [15].

The idea behind the definition of a quasi-convex subset [90] is to separate (with continuous characters) the subset from the points in the (relative) complement of that subset. Using this idea, we define the local quasi-convexity for convergence groups as:

Definition 5.3 (Quasi-convex set)**.** A subset A of a convergence abelian groups G is quasi-convex if for each point g in $G\backslash A$, there is a character χ in the polar set of A such that $Re\chi(g) < 0$, that is

$$A^{\rhd\lhd} = A$$

.

Definition 5.4 (Locally quasi-convex convergence group)**.** A convergence group G is locally quasi-convex if for each filter $\mathcal{F} \underset{G}{\to} 0$, there exists another filter $\mathcal{G}$ coarser than $\mathcal{F}$ such that $\mathcal{G} \underset{G}{\to} 0$ and $\mathcal{G}$ has a filter base composed of quasi-convex sets.

In view of [41] local quasi-convexity can be defined in terms of regularity with respect to the family of quasi-convex sets.

Proposition 5.5. *Let G_1 and G_2 be convergence groups and*

$$f : G_1 \to G_2$$

a continuous (convergence) homomorphism and $H \subset G_2$ *a quasi-convex set then* $f^{-1}(H) \subset G_1$ *is a quasi-convex set.*

Proof. Let $g \notin f^{-1}(H)$, then $f(x) \notin H$. As H is quasi-convex then there exists,

$$\chi \in \Gamma G_2 : \chi(f(g)) \notin \mathbb{T}_+ \text{ and } \chi(H) \subset \mathbb{T}_+.$$

Now,

$$(\chi \circ f) \circ f^{-1}(H) \subset \chi(H) \subset \mathbb{T}_+$$

and hence, $\chi \circ f(g) \notin \mathbb{T}_+$. □

Theorem 5.6. *A subgroup of a locally quasi-convex convergence group is a locally quasi-convex convergence group.*

Proof. Let G be locally quasi-convex so for each filter $\mathcal{F} \to 0$ in G, there is another filter $\mathcal{G}$ coarser than $\mathcal{F}$ such that $\mathcal{G} \to 0$ and $\mathcal{G}$ has a filter base composed of quasi-convex sets. With respect to the inclusion map

$$i : A \to G$$

the subgroup convergence structure is an initial convergence structure so, if $\mathcal{G} \to 0$ in A then $i(\mathcal{G}) \to 0$ in G and hence, $\mathcal{G}$ has a filter base composed of quasi-convex sets. □

Theorem 5.7. *Direct sum of the locally quasi-convex convergence groups is a locally quasi-convex convergence group.*

Proof. The proof is similar to Theorem 5.6. □

Theorem 5.8. *For a locally quasi-convex convergence abelian group,* Γ_c *separates the points on the group.*

Proof. Let, G be locally quasi-convex convergence abelian group so it has filter base $\mathcal{B}$ composed of quasi-convex sets. Now, $A \in \mathcal{B}$ is a quasi-convex set if for every $x \in G \backslash A$ there exists a character $\chi \in \Gamma(G)$ such that

$$|\chi(g)| > \frac{1}{4}$$

and

$$|\chi(A)| = \sup\{|\chi(g)| : g \in A\} \leq \frac{1}{4},$$

that is every element of the filter base is separated and hence the proof. □

Corollary 5.9. *For every locally quasi-convex convergence abelian group,* κ_G *is injective.*

For any convergence space X let,

$$C_c(X, \mathbb{T}) = \{f : \ f : X \to \mathbb{T} \text{ and } f \text{ is continuous}\}$$

be the group of all continuous unimodular functions. Then we have:

Theorem 5.10. *If $C_c(X, \mathbb{T})$ is locally quasi-convex then it is c-reflexive.*

Proof. Let, $\kappa_x : X \to \Gamma_c C_c(X, \mathbb{T})$ be defined as

$$\kappa_x(x)(f) = f(x) \ \forall \ x \in X \ , \ f \in C_c(X, \mathbb{T}).$$

We have, κ_x is well defined and as $\Gamma_c C_c(X, \mathbb{T})$ has continuous convergence structure so κ_x is continuous. κ_x is embedded is evident from the following commutative diagram:

$$\begin{array}{ccc} C_c(X, \mathbb{T}) & \xrightarrow{\kappa_{C_c(X,\mathbb{T})}} & \Gamma_C \Gamma_C C_c(X, \mathbb{T}) \\ & \searrow^{id} & \downarrow \kappa_x^* \\ & & C_c(X, \mathbb{T}) \end{array}$$

Hence, κ_x is an embedding and for

$$\kappa_x^* : \Gamma_C \Gamma_C C_c(X, \mathbb{T}) \to C_c(X, \mathbb{T})$$

we have we have $\kappa_{C_G(X,\tau)}$ is isomorphism iff κ_x^* is injective. Hence, from Theorem 5.9 the result follows. □

The following theorem is evident as the special case of [18, Proposition 8.1.3].

Theorem 5.11. *For every locally quasi-convex convergence abelian group G, κ_G is continuous.*

Proof. The map

$$e : \Gamma_c G \times G \to \mathbb{T}$$

defined as

$$e(\chi, x) = \chi(x)$$

is a continuous. The continuity of κ_G follows from the following commutative diagram:

$$\begin{array}{ccc} G \times \Gamma_c G & \xrightarrow{\kappa_G \times id} & \Gamma_c \Gamma_c G \times \Gamma_c G \\ \downarrow v & & \downarrow \omega \\ \Gamma_c G \times G & \xrightarrow{\quad e \quad} & \mathbb{T} \end{array}$$

Here, v represents the exchange of components and ω represents the evaluation mapping. $\square$

Corollary 5.12. *κ_G is a continuous group homomorphism.*

Proof. For $x, y \in G$ and $\chi \in \Gamma_c$ we have

$$\begin{aligned} \kappa_G(x+y)(\chi) &= \chi(x+y) \\ &= \chi(x)\chi(y) \\ &= \kappa_G(x)(\chi) \\ &= \kappa_G(y)(\chi) \\ &= (\kappa_G(x) + \kappa_G(y))\,(\chi). \end{aligned}$$

So, from Theorem 5.11, we get that κ_G is a continuous group homomorphism. $\square$

Theorem 5.13. *For every locally quasi-convex convergence abelian group, κ_G is an embedding.*

Proof. Let U be a quasi-convex set contained in a filter base, the filter generated by which converge to zero in G then, $U^{\triangleright\triangleright}$ is contained in a filter base, the filter generated by which converges to zero in $\Gamma_c\Gamma_c G$. As

$$\kappa_G^{-1}(U^{\triangleright\triangleright}) = U^{\triangleright\triangleleft} = U$$

so,

$$\kappa_G(U) = \kappa_G(\kappa_G^{-1} U^{\triangleright\triangleright}) = U^{\triangleright\triangleright} \cap \kappa_G(G).$$

From Theroem 5.9 and Theorem 5.11, we get that κ_G is an embedding. $\square$

Example 5.14 (A locally quasi-convex convergence group need not be c-reflexive.). The topological group $L^2_Z[0,1]$ of almost every-where integer integrable functions ([6, Corollary 11.15.]]) is a locally quasi-convex metrizable complete group which does not satisfy P-reflexivity. Hence as a convergence group, $L^2_Z[0,1]$ is not c-reflexive but it is locally quasi-convex.

Theorem 5.15. *For a convergence abelian group G,*
(i) a filter $\Phi \xrightarrow[\Gamma_c G]{} 0$ iff $A^{\triangleright} \in \Phi$ for every finite subset A of G.
(ii) For every filter $\mathcal{F} \xrightarrow[G]{} 0$, there is $B \in \mathcal{F}$ such that $B^{\triangleright} \in \Phi$.
(iii) If $\mathcal{U} \xrightarrow[G]{} 0$ then, $U^{\triangleright\triangleright}$ is contained in a filter base, filter generated by which converge to zero in $\Gamma_c\Gamma_c G$.

Proof. (i) and (ii) are evident from [18, Proposition 8.1.8] and the proof of (iii) is evident from (ii). $\square$

Theorem 5.16. *If a convergence group is c-reflexive then it must be locally quasi-convex.*

Proof. To prove this result it is sufficient to prove that if a convergence group is embedded then it must be locally quasi-convex.
In a convergence abelian group G let,

$$\kappa_G : G \to \Gamma_c\Gamma_c G$$

be an embedding. Let, U is a quasi-convex set contained in a filter base, the filter generated by which converge to zero in G. Then, $\kappa_G(U)$ is a quasi-convex set contained in a filter base, the filter generated by which converge to zero in $\kappa_G(G)$. So, there exists a quasi-convex set W contained in a filter base, the filter generated by which converge to zero in $\Gamma_c\Gamma_c G$ such that $\kappa_G(U) = W \cap \kappa_G(G)$. Using theorem 5.15 we obtain that there is a V contained in a filter base, the filter generated by which converge to zero in G with $W \supseteq V^{\rhd\rhd}$ and hence,

$$\kappa_G(U) \supseteq V^{\rhd\rhd} \cap \kappa_G(G).$$

Further,

$$V \subseteq V^{\rhd\lhd} = \kappa_G^{-1}(V^{\rhd\rhd}) \subseteq \kappa_G^{-1}(\kappa_G(U)) = U$$

so, $U(\supseteq V^{\rhd\lhd})$ is a quasi-convex set contained in a filter base, the filter generated by which converge to zero in G. Hence, the proof. □

Remark 5.17. As there exists non reflexive locally compact convergence group so, this group serves as an example of a non locally quasi-convex, convergence abelian group and this situation is in contrast to the nature of locally quasi-convex topological abelian groups as locally compact abelian topological groups are locally quasi-convex.

Theorem 5.18. *$C_c(X, \mathbb{T})$ is c-reflexive iff it is locally quasi-convex.*

Proof. The proof follows from 5.10 and 5.16. □

Motivated from [24, Proposition 3.2.9] we obtain the following result:

Theorem 5.19. *A compact convergence abelian group which is locally quasi-convex is reflexive.*

Proof. The evaluation map κ_G is an embedding is evident from 5.13.
The evaluation map κ_G is onto:
If G is a compact group then, we have $\Gamma_c G$ is discrete, and hence, $\Gamma_c\Gamma_c G$ is compact. As $\Gamma_c G$ is locally compact, its dual is topological and has the compact-open topology, therefore $\Gamma_c\Gamma_c G = \widehat{(\Gamma_c G)}$.
As, G is compact Hausdorff space, κ_G is continuous, hence $\kappa_G(G) = \Gamma_c\Gamma_c G$. As every

subgroup of the dual of a discrete group, which separates points from the group, is dense, the subgroup $\kappa_G(G)$ of $\Gamma_c\Gamma_c G$ is dense and hence the proof. □

Remark 5.20. We do not know any example of a non-topological, Hausdorff compact convergence abelian group. The following elementary facts are known in this regard.

- A non-topological compact convergence group, must be infinite.
- Every reflexive compact convergence abelian group must be topological, and hence, a non-topological compact convergence group must not be locally quasi-convex.

Next we present certain facts obtained while investigating the examples of groups with convergence.

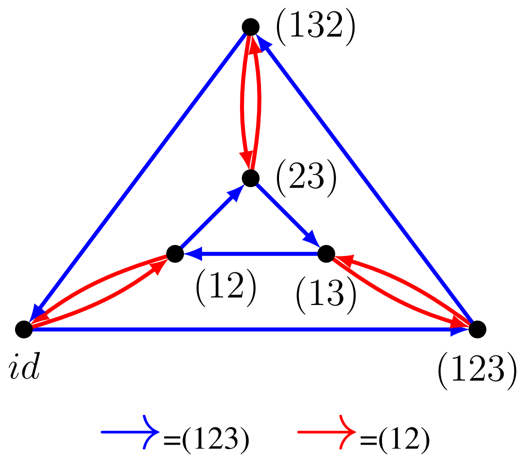
Interaction between groups and convergence spaces

Let Γ be a subset of a group G such that each element of G is a product of elements of Γ and no element of Γ is redundant, i.e. no element of Γ can be written as product of its other elements. The edge colored Cayley digraph (directed graph) for G generated by Γ is the directed graph C such that the vertex set of C is G and the edge set of C is $E = \{(g, g\gamma) : g \in G\ ,\ \gamma \in \Gamma\}$. The edges are colored by $j : E \to \Gamma$, where $j(g, h) = s$.

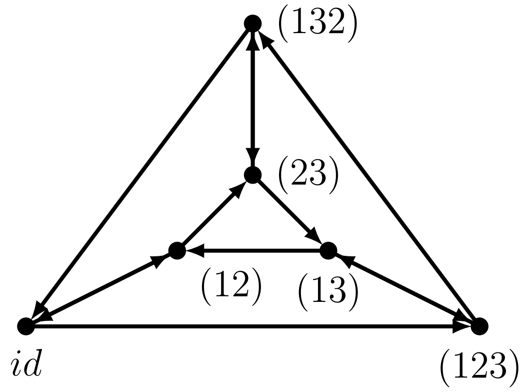
Consider the symmetric group S_3 with generators $\{(12), (123)\}$. Then the left product of the elements of the generator set with the elements of the group is given as:

Generator	id	(12)	(13)	(23)	(123)	(132)
(12)	(12)	id	(123)	(132)	(13)	(23)
(123)	(123)	(23)	(12)	(13)	(132)	id

The following edge colored Cayley digraph is obtained

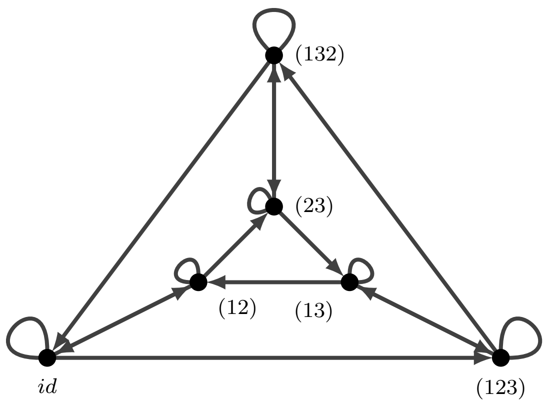


Without colors the edge colored Cayley digraph reduces to the following Cayley digraph:



Further, in addition to the condition that $\gamma \in \Gamma$ if another condition $\gamma = id$ is added to the definition of Cayley graph we obtain a reflexive Cayley digraph as follows:
The reflexive Cayley digraph for G generated by Γ is the reflexive digraph C such that the vertex set of C is G and the edge set of C is $\{(g, g\gamma) : g \in G$ and $\gamma = id$ or $\gamma \in \Gamma\}$.

With this definition the reflexive Cayley digraph for S_3 with generator set $\{(12), (123)\}$ is:



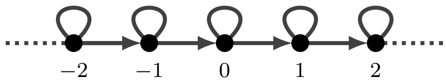
As explained in Example 3.4, a reflexive diagraph can be represented by a convergence space [21, Proposition 5]. So in view of above discussion we can see how fixing a generating set of a group generates the convergence.

Remark 5.21.

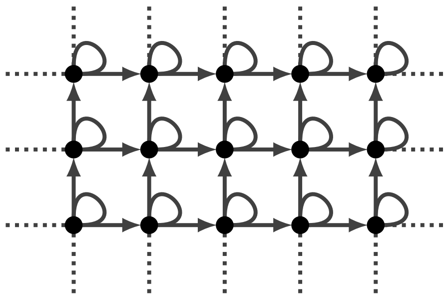
1. The convergence structure defined via Cayley graphs is not necessarily compatible with group structure and is not necessarily Hausdroff.
2. We have not explored here, different convergences a group can generate on considering different generators, but this kind of approach provides a view to analyse "groups with convergence" in place of convergence groups.

Apart from finite groups, the convergence generated by the infinite groups is also of a

great interest, an elementary example in this case includes:
Reflexive Cayley digraph of $\mathbb{Z}$ generated by $\{1\}$



Further, in view of [78, Example 4.6] a Cayley graph for $\mathbb{Z} \oplus \mathbb{Z}$ is the graph Cartesian product[2] $\mathbb{Z} \times \mathbb{Z}$ generated by $(\Gamma \times \{e\}) \cup (\{e\} \times \Gamma)$ and can be represented as follows:



Now, define a function $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as $+(a, b) = a + b$. Clearly, + is continuous but this convergence is not Hausdroff, hence the problem to obtain a non topological, Hausdroff, compact convergence group is still unsolved.

5.2 Duality of limits

In this section, we present the results obtained regarding the duality properties of limits of locally quasi-convex convergence groups. For a detailed study of the duality of projective and inductive limits of topological and convergence groups we refer the reader to [1, 2, 16, 17].

We denote by **CAG** and **LCAG** the categories of abelian convergence groups and of locally quasi-convex Hausdorff convergence abelian groups, respectively. A directed system is an indexed set I in which for each pair i, j in $(I, \leq)$ there is a k in I such that $i, j \leq k$. *Projective system* in **CAG** is family of pairs

$$\{(G_i, f_{ij})|i, j \in I, j \leq i\}$$

of a collection of convergence groups $\{G_i\}$ and continuous homomorphisms $\{f_{ij} : G_j \to G_i\}$ such that f_{ii} is identity homomorphism and $f_{ik} = f_{ij} \circ f_{jk}$ for $i \leq j \leq k$. The subset

$$\{(g_i)_{i\in I} \in \prod_{i\in I} G_i | f_{ij}(g_j) = g_i \text{ with } i \leq j\}$$

[2]We are thankful to Prof Daniel R. Patten for suggesting this example.

of the product $\prod_{i\in I} G_i$ is called *inverse limit or projective limit* $(\lim\limits_{\leftarrow} G_n)$ of projective system $\{G_i, f_{ij}\}$, here the product $\prod_{i\in I} G_i$ is equipped with initial convergence structure in respect of the projections

$$\pi_i : \prod_{i\in I} G_i \to G_i.$$

Dual (categorical) to the notion of projective (or inverse limit) is the inductive (or direct limit). An inductive system in **CAG** over a directed system I is defined as the family (of pairs)

$$\{G_i, f_{ij} | i, j \in I, i \leq j\}$$

of the collection of convergence abelian groups $\{G_i\}$ and continuous homomorphisms $\{f_{ij} : G_i \to G_j\}$ such that $f_{ii} = id_{G_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$ when $i \leq j \leq k$. The *inductive or direct limit* is defined as the quotient

$$(\lim_{\rightarrow} G_n = \coprod_{i\in I} G_i \backslash \sim)$$

of an inductive system $\{G_i, f_{ij}\}$, of the disjoint union $\coprod_{i\in I} G_i$, where two elements $g_i \in G_i$ and $g_j \in G_j$ are equivalent if there is a $k \geq i, j$ such that $f_{ik}(g_i) = f_{jk}(g_j)$.
For a family $(G_i)_{i\in I}$ of convergence abelian groups, $\sum_{i\in I} G_i$ and $\bigoplus_{i\in I} G_i$ denotes the product and the coproduct of G_i's, equipped with product convergence structure final group convergence structure with respect to the natural injections

$$\epsilon_J : G_J \to \bigoplus_{i\in I} G_i, \text{ (for all finite } J \subset I)$$

respectively.

Proposition 5.22. *For G a locally quasi-convex convergence abelian group, $G/ker(\kappa_G)$ is dually closed in G, i.e. it has enough characters.*

In order to prove that the dual group of a locally quasi-convex, compact, abelian convergence group is c-reflexive we prove the following factorization property. The proof is motivated from [18, Theorem 8.1.14].

Theorem 5.23. *For convergence abelian groups H and G. If $\phi : L \to H$ and $\psi : H \to L$ are continuous homomorphisms such that, $\phi \circ \psi = id$ or equivalently the following diagram is commutative*

$$\begin{array}{ccc} H & \xrightarrow{id} & H \\ & \searrow^{\psi} & \uparrow{\scriptstyle \phi} \\ & & L \end{array}$$

then, $L = \psi(H) \oplus ker(\phi)$.

Proof. Let, $y \in \psi(H) \cap \ker(\phi)$. Then, there is an $x \in H$ for which $y = \psi(x)$ and hence,

$$\begin{aligned} 0 = \psi(y) = id(x) = x \\ \Rightarrow y = 0. \end{aligned}$$

For, $x \in L$,

$$\phi(x - \psi(\phi(x))) = \phi(x) - \phi(\psi(\phi(x))) = 0$$

and hence,

$$x = \psi(\phi(x)) + (x - \psi(\phi(x))) \in \psi(H) \oplus ker(\phi).$$

So, L is algebraic direct sum. Clearly, this sum is topological and hence the proof. □

Proposition 5.24. *For a locally quasi-convex, compact, abelian convergence group G the convergence group $\Gamma_c G$ is c-reflexive.*

Proof. $\Gamma_c G$ is embedded is evident from the following diagram

$$\begin{array}{ccc} \Gamma_c G & \xrightarrow{\kappa \Gamma_c G} & \Gamma_C \Gamma_C \Gamma_C G \\ & \overset{id}{\searrow} & \downarrow \kappa_G^* \\ & & \Gamma_c G \end{array}$$

As G is reflexive, we have

$$\kappa_G(G) = \Gamma_C \Gamma_C G$$

and hence $\kappa_G(G)^{\triangleright} = 0$. Now, using Theorem 5.23, we have,

$$\Gamma_c \Gamma_c \Gamma_c G = \kappa_{\Gamma_c G}(\Gamma_c G) \oplus ker(\kappa_G^*)$$

and as $\kappa_G(G)^{\triangleright} = 0$ we have

$$\Gamma_c \Gamma_c \Gamma_c G = \kappa_{\Gamma_c G}(\Gamma_c G).$$

Hence, the convergence group $\Gamma_c G$ is c-reflexive. □

For a family $(G_i)_{i \in I}$ of convergence groups, the maps

$$\Lambda : \Gamma_c(\sum_{i \in I} G_i) \to \bigoplus_{i \in I} (\Gamma_c G_i)$$

and

$$\Delta : \Gamma_c(\bigoplus_{i \in I} G_i) \to \sum_{i \in I} (\Gamma_c G_i)$$

are isomorphism [28, Proposition 2.2, 2.3]. These two statements are dual to each other and hence seems analogues but the proof to the second is not straight forward. From [28, Theroem 2.4] we obtain the following:

Theorem 5.25. *For family $(G_i)_{i\in I}$ of compact, locally quasi-convex convergence abelian groups, the convergence groups $\sum_{i\in I} G_i$ and $\bigoplus_{i\in I} G_i$ are c-reflexive.*

Proof. For, compact, locally quasi-convex convergence abelian groups we have $E(\xi) = (\kappa_{G_i}(\xi(i)))_{i\in I}$ is an isomorphism which we get from theorem [28, Proposition 2.2]. Further, ω is a mapping from $\sum_{i\in I} \Gamma_c\Gamma_c G_i$ to $\Gamma_c(\bigoplus_{i\in I} \Gamma_c G_i)$ The proof is evident from the following diagram

$$\begin{array}{ccc} \sum_{i\in I} G_i & \xrightarrow{\quad E \quad} & \sum_{i\in I} \Gamma_c\Gamma_c G_i \\ \Big\downarrow \kappa & & \Big\downarrow \omega \\ \Gamma_c\Gamma_c(\sum_{i\in I} G_i) & \xrightarrow{(u^{-1*})} & \Gamma_c(\bigoplus_{i\in I} \Gamma_c G_i) \end{array}$$

□

We obtain the following results as a special case of [2, Theorem 4.1].

Theorem 5.26. *The c-dual of the direct limit of the compact, locally quasi-convex convergence abelian groups is the inverse limit of the c-dual i.e. for family $(G_i)_{i\in I}$ of compact, locally quasi-convex convergence abelian groups, $\Gamma_c(\varinjlim G_i) \cong (\varprojlim \Gamma_c G_i)$.*

Theorem 5.27. *For functor F : $\mathbf{CAG} \to \mathbf{LCAG}$ defined by $F(G) = G/ker(\kappa_G)$, $F(\varinjlim G_i) = (\varprojlim FG_i)$.*

Proof. The proof is evident from Theorem 5.22 and from fact that the functor F is left adjoint to the inclusion functor $\mathbf{LCAG} \to \mathbf{CAG}$. □

Conclusion

In this chapter, we have presented the notion of local quasi-convexity in convergence groups. We have seen that locally compact convergence groups (and, non-topological compact convergence groups, if they exist) are not contained in the class of locally quasi-convex convergence abelian groups. Finally, we have proved certain results related to the limits of local-quasi-convex convergence abelian groups.

Concluding Remarks

It is evident that the behaviour (in reference to duality theory) of non-topological locally compact convergence groups is quite different from the class of locally compact topological groups and it is not trivial to extend the implications of boundedness and compactness from topological groups to convergence groups.

In order to make an attempt to obtain an extension of the theory of integration from topological spaces to convergence spaces we have introduced the term convergence measure space whose underlying idea is to define the σ-algebra compatible with the convergence structure. Our idea of the convergence measure space depends on the topological modification of the convergence space but as compactness and the open sets do not play the same role in theory of filter convergence groups as they play in topological case, so the problem of obtaining a suitable class of measurable sets for the purpose of analysis is still unsolved, and hence the general definition of a convergence measure space is still not known.

Motivated from the convexity in topological vector spaces and quasi-convexity in topological groups we have introduced the notation of local-quasi-convexity for the class of convergence abelian groups and in contrast to topological case it has been obtained that the class of locally compact convergence groups do not lie in the class of locally quasi-convex convergence groups. We prove that the local quasi-convexity is necessary for a convergence group to be c-reflexive. Further, we obtain that locally quasi-convex, compact convergence groups are c-reflective. The problem to obtain the concrete examples of filter convergence groups beyond homeomorphism groups and the problem of existence of non-topological compact convergence groups is still unsolved. In this regard, we have obtained that if non-topological compact convergence group exists then, it cannot be reflexive.

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Publications and Presentations

Publications

- *Spaces over non-Newtonian numbers*. International Journal of Mathematics and its Applications (2016). Co-authored with S. Mishra.
- *Duality in topological and convergence groups*. Topology Proceedings. Accepted. Co-authored with S. Mishra.
- *On local quasi-convexity in convergence groups*. Submitted. Co-authored with S. Mishra.
- *Boundedness in convergence groups*. Submitted. Co-authored with S. Mishra.

Book Chapter

- *Duality in convergence groups and vector spaces*. In Research trends in Mathematics and Statistics. ISBN:9789353350499. Co-authored with S. Mishra.

Presentations and Talks

- *Interaction between convergence spaces and discrete groups*, at Advances in Group Theory and Applications held at University of Salento (Italy) from June 25-28, 2019.
- *Boundedness in convergence groups*, at Interdisciplinary Colloquium in Topology and its Applications held at University of Vigo (Spain) from June 19-22, 2019
- *Duality in topological and convergence groups,* at the International Conference on Set-Theoretic Topology and Topological Algebra in honor of Professor Alexander Arhangelskii on his 80th birthday held at Lomonosov Moscow State University (Russia) from August 23-28, 2018.
- *Duality of limits of convergence groups,* in the special session on topology in analysis and topological algebras at 33rd Summer Conference on Topology and its Applications held at Western Kentucky University (USA) from July 17-20, 2018.
- *Sequences valued in convergence groups*, at 42 nd Annual Summer Symposium in Real Analysis held at the V. A. Steklov Institute of Mathematics and Herzen University, Saint- Petersburg, Russia from June 9-15, 2018.

- *Locally quasi-convex convergence groups,* at 19th Postgraduate Group Theory Conference held at University of Cambridge, (UK) from June 27-30, 2017.
- *Continuous duality of groups,* at 82nd Annual Conference of Indian Mathematical Society held at University of Kalyani from December 27-30, 2016.
- *Convergence structures and Pontryagin duality theory,* at International conference of the Indian Mathematics Consortium in cooperation with American Mathematical Society (**AMS**) held at Banaras Hindu University from December 14 -17, 2016.
- *Convergence measure spaces*, at 12th Symposium on General Topology and its Relations to Modern Analysis and Algebra held at Charles University, Prague (Czech Republic) from July 25-29, 2016.

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