STUDY OF FOURIER SERIES AND BOUNDARY VALUE PROBLEMS INVOLVING A-FUNCTION

A

Thesis

Submitted to

For the award of

DOCTOR OF PHILOSOPHY (Ph.D.)

in

MATHEMATICS

By

KAMAL KISHORE

(41400702)

Supervised By

Dr. S.S. Shrivastava

Co-Supervised by

Dr. Rajesh Kumar Gupta

LOVELY FACULTY OF TECHNOLOGY AND SCIENCES

LOVELY PROFESSIONAL UNIVERSITY

PUNJAB

2019

DECLARATION

I, Kamal Kishore, Department of Mathematics, Lovely Professional University, Punjab certify that the work embodied in this Ph.D thesis titled, **"Study of Fourier Series and Boundary Value Problems Involving A-Function**" is my own bonafide work carried out by me under the Supervision of **Dr. S.S. Shrivastava** and the Co-supervision of **Dr. Rajesh Kumar Gupta**, I confirm that

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given with the exception of such quotations this thesis is entirely my own work.
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- I have made clear exactly what was done by others and what I have contributed myself.

Date

Place : Phagwara **Kamal Kishore**

Registration No. 41400702

CERTIFICATE BY SUPERVISOR

This is to certified that the study embodied in this thesis entitled **“Study of Fourier Series and Boundary Value Problems Involving A-Function”** being submitted by **Mr. Kamal Kishore** for the award of the degree of **Doctor of Philosophy (Ph.D.)** in **Mathematics** of **Lovely Professional University, Phagwara (Punjab)** and is the outcome of research carried out by him under my supervision and guidance. Further this work has not been submitted to any other university or institution for the award of any degree or diploma. No extensive use has been made of the work of other investigators and whereas it has been used, references have been given in the text.

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CERTIFICATE BY CO-SUPERVISOR

This is to certified that the study embodied in this thesis entitled **"Study of Fourier Series and Boundary Value Problems Involving A-Function"** being submitted by **Mr. Kamal Kishore** for the award of the degree of **Doctor of Philosophy (Ph.D.)** in **Mathematics** of **Lovely Professional University, Phagwara (Punjab)** and is the outcome of research carried out by him under my supervision and guidance. Further this work has not been submitted to any other university or institution for the award of any degree or diploma. No extensive use has been made of the work of other investigators and whereas it has been used, references have been given in the text.

Date :

Place : Phagwara

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The thesis entitled **"Study of Fourier Series and Boundary Value Problems Involving A-Function"** is being submitted in partial fulfillment for the award of degree of **Doctor of Philosophy in Mathematics** to **Lovely Professional University, Phagwara, Punjab**.

Usually we call a function 'special' when the function belongs to the toolbox of the applied mathematician, the physicist or the engineer. They have a particular notation and a number of properties. Mathematically, special functions are functions defined on R, the set of real numbers or C, the set of complex numbers and they possess not only series representations, but also integral representations. This thesis is mainly concerned with the development of special functions especially A-function. So the concept of Pochhammer notation, Mellin-Barnes integrals, convergence and residue calculus are essential for the detailed study of these functions. Recently the attention of mathematicians towards these functions has increased from both the analytical and numerical point of view due to their relation with the fractional calculus.

The whole thesis is divided into nine chapters, each divided into three to six sections. The formulae and results are numbered progressively in each chapter. For instance (3.2.5) denotes the Fifth formula of the Second section in the Third chapter. Bibliography to the literature are given in full at the end of the thesis arranged alphabetical order. In the text, they have been referred to by putting within rectangular brackets, the serial number of the references, where so ever necessary; the page of the references and the number of the result have also been given i.e. [34, p.122(ii)] means second result of page 122 of the thirty fourth reference.

The **First Chapter** deals with the historical background, development and definitions of the A-functions and polynomials in the context of the research work accomplished in the subsequent chapters of this thesis. It also provide brief literature of several aspects of special functions.

Since generating relations plays an important role in the investigation of various useful properties of the sequences, which they generate and also used as

z-transform in solving certain classes of difference equation which arise in a wide variety of problems in operation research (including, for example, queening theory and related stochastic process). Looking into the requirement and importance of various properties of generating relations in the analysis of many problems of mathematics and mathematical physics, in the **Second Chapter**, we have established some new linear and bilinear generating relations involving A-function of one variable. In section (2.2) and (2.3) by increasing the number of parameters in the definition of A-Function and by using properties of gamma function we have derived these relations.

Several authors have discussed a number of bilateral and trilateral generating relations involving generalized hypergeometric functions time to time. The A-function of one variable plays an important role in the development and study of special functions. In **Third Chapter**, the usefulness of this function has inspired us to find some new bilateral and trilateral generating relations involving A-function of one variable.

Integrals are useful in connection with the study of certain boundary value problems. It is also helpful for obtaining the expansion formula. These are also used in the study of statistical distribution, probability and integral equation. **Fourth Chapter** contains some definite and indefinite integrals involving the A-function and other commonly used functions. Some double integrals involving A-function have been also evaluated with the help of some known results. We have used the results of Bajpai, Shrivastava, Rainville and others to derive these integrals.

In **Fifth Chapter**, in the section (5.3), we have established two integrals containing the products of A-Function and other hypergeometric functions. At the end of this section we have also discussed particular cases. In section (5.4) some new integrals involving A-functions are evaluated with the help of finite difference operator [$E_a f(a) = f(a + 1)$].

Looking into the requirement and importance of various properties of expansion in several field, in **Sixth Chapter** we have established some new Expansion and Identities involving A-Function of one variable by increasing the number of parameters. In section (6.2) six new expansions and in section (6.3) nine new identities involving A-Function of one variable has been established by increasing the number of parameters.

Various problems in science and technology, when formulated mathematically, lead naturally to certain classes of partial differential equations involving one or more unknown functions together with the prescribed conditions (known as boundary conditions) which arise from the physical situation. Several researchers have obtained solutions to the differential equations related to certain problems, which satisfy the given boundary conditions. The classical method in obtaining solutions of the boundary value problems of mathematical physics can be derived from Fourier's another technique using integral transforms, which had its origin in Heaviside's work, has been developed in the past and has certain advantages over the classical method. Several authors such as Arora (1998), Chandel (2002), Chaurasia (1997), Srivastava (1998, 1999, 2000), Tiwari (1993) have used various classes of orthogonal polynomials and generalized hypergeometric functions of one or more variables in finding the solutions of the boundary value problems concerning

(a) heat conduction in

  (i) a non-homogenous finite bar
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(c) transverse vibrations in a circular membranes

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(e) angular displacement in a shaft of circular cross-section

(f) potential theory, etc.

Inspired by these authors in **Seventh Chapter,** in section (7.3) first we have evaluated an integral involving A-function of one variable and then applied it to solve two boundary value. In section (7.4) we employ the A-function of one variable in obtaining a solution of a partial differential equation related to heat conduction along with Hermite polynomials. In section (7.5) we derive the solution of special one-dimensional time dependent Schrodinger equation involving Hermite polynomials and A-function of one variable. In section (7.6) we employ the A-function of one variable in obtaining a solution of a problems on (i) heat conduction in a bar (ii) deflection of vibrating string and bounded electrostatic potential in the semi-infinite space under certain conditions.

The subject of Fourier series for generalized hypergeometric functions occupies outstanding place in the literature of special functions and boundary value problems. Certain double Fourier series of generalized hypergeometric functions play vital role in the improvement of the theories of special functions and two-dimensional boundary value problems.

In the **Eighth Chapter**, we have founded some new Fourier series involving A-function of one variable. We have taken help of the results obtained in chapter 4 to prove these Fourier series.

At the last in **Ninth Chapter** we have given the summary and conclusion of the thesis.

Dedicated to

My Parents

&

Teachers

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First of all, I am grateful to God who kept on showering His blessings upon me throughout this task and blessed me with success. It is a joyful, privilege to express my deep gratitude to all the people who helped me to make this thesis possible.

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Kamal Kishore

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LIST OF SYMBOLS AND ABBREVIATIONS

R	Set of real numbers
C	Set of complex number
Re(z)	Real part of complex number z
Im(z)	Imaginary part of complex number z
D^n	n^{th} order derivative
RHS	Right hand side
LHS	Left hand side
BVP	Boundary value problem
$(\alpha)_n$, (α, n)	Pochhammer Notation
$\Gamma(z)$	Gamma function of z
$A_{p,q}^{m,n}\left[x\Big\vert_{((b_q,\beta_q))}^{((a_p,\alpha_p))}\right]$	A-function
E	Finite difference operator
${}_2F_1$	Gauss hypergeometric function
${}_pF_q$	Generalized hypergeometric function
$P_n\ (u)$,	Legendre's polynomial
$P_n^m(u)$, $Q_n^m(u)$	Associated Legendre's polynomial
$L_n\,(x)$	Laguerre's polynomial
$L_n^{\alpha}(x)$	Generelized Laguerre's polynomial
$H_n\ (x)$	Hermite polynomial
$H_{e_n}(x)$	Chebyshev's Hermite polynomial
$P_n^{(\alpha,\beta)}(z)$	Jacobi polynomial

$y_n(x; \alpha, \beta)$	Generalized Bessel function
$D_n(x)$	Webber's parabolic cylinder function
$M_{k,m}$	Whittaker's function
$F_1, F_2, F_3, F_4,$	Appell's function
$G_1, G_2, G_3,$ H_1 to H_7	Horn's function
F_E ------- F_S	Saran's function
G_A, G_B	Panday's Hypergeometric function
G_C, G_D	Dhawan's Hypergeometric function

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LIST OF RESEARCH PAPERS

[1] Kishore, K., Srivastava, S. S. (2010). Some new Identities involving A-Function. *Vijnana Parishad Anusandhan Patrika.* 53(2): 121-124.

[2] Kishore, K., Srivastava, S.S. (2011). Bounded Electrostatic potential in the Semi-infinite Space and A-Function. *Journal of Indian Academy of Mathematics.* 33(2): 479-482.

[3] Kishore, K., Srivastava, S.S. (2012). Some Double Integrals involving A-Function. *The Mathematics Education.* 46(4): 191-194.

[4] Kishore, K., Srivastava, S.S. (2013). A-Function, Hermite Polynomials and Time Dependent Schrodinger Equations. *The Mathematics Education*. 47(2):135-147.

[5] Kishore, K., Srivastava, S.S. (2013). Some New Bilinear Generating Relations Involving A-Function. *Applied Science Periodical* 15(1): 46-49.

[6] Kishore, K., Srivastava, S.S. (2016). Expansion Formulae Involving A-Function. *International Research Journal of Mathematics, Engineering and IT*. 3(11): 8-12.

[7] Kishore, K., Srivastava, S.S. (2016). Some Integrals Involving Product of Hypergeometric Function and A-Function. *Applied Science Periodical.* 18(3): 64-74.

[8] Kishore, K., Srivastava, S.S. (2016). Some New Linear Generating Relations Involving A-Function of One Variable *IOSR Journal of Mathematics*. 12(6): 01-03.

[9] Kishore, K., Srivastava, S.S. (2017). Some Bilateral and Trilateral Generating Relations Involving A-Function. *Aryabhatta Journal of Mathematics and Informatics*. 9(1): 551-556.

[10] Kishore, K., Srivastava, S.S. (2018). Fourier Series Involving A-Function. *International Journal of Scientific Research and Reviews*. 7(4): 2694-2696.

CHAPTER–1

INTRODUCTION

1.1 HYPERGEOMETRIC FUNCTION

In the theory of special functions, the Gaussian hypergeometric function is very important. In fact nearly all the functions used in mathematical physics and applied mathematics can be expressed in term of hypergeometric function or in terms of confluent cases. This function is the extensions and generalization of the basic geometric series and simple transcendental functions.

The function

$$_2F_1[a, b; c; z] = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{1.1.1}$$

arises in the study of following second order linear differential equation having regular singular points [56]

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0, \tag{1.1.2}$$

for $c > 0$ and $c \in Z$. In (1.1.2), Pochhammer's symbol $(a)_n$ is factorial function defined as

$$(a)_n = (a, n) = (a + n - 1)\,(a + n - 2) \ldots. (a + 1)a,\ n \geq 1$$

$$= \frac{\Gamma(a+n)}{\Gamma(a)} \text{ for n positive integer}$$

and $a \neq 0$, $(a, 0) = 1$. The quantities a, b and c in (1.1.2) are independent of z and are called parameters, z is called argument.

The function $_2F_1\left[{a,b; \atop c;} z\right]$, where a, b, c are parameters and z is variable, is known as Gauss's hypergeometric function.

All four of these quantities may be any numbers, real or complex. There is one exception, namely, that the series is not defined, then numerical value of the series becomes infinite if $c \leq 0$, if one of the parameters in numerator $a \leq 0$ or $b \leq 0$, such that $-a > -c$, say. In general, if either of the numerator parameters is a negative integer the series (1.1.1) terminates to a polynomial in z. The convergence conditions of (1.1.1) are as follows:

(i) The series is convergent if $|z| < 1$ and divergent if $|z| > 1$, $\forall$ z $\in$ R or C.

(ii) For $|z| = 1$, the absolute convergence of series required Real $(- a - b + c) > 0$ and for divergence Real $(- a - b + c) \leq 0$.

most of the classical orthogonal polynomials, complete elliptic functions of first and second kinds, incomplete beta function and Legendre functions are the special cases of $_2F_1$. Coulomb wave functions, parabolic cylinder functions, Bessel functions, etc. are also the special cases of confluent hypergeometric function.

1.2 GENERALIZED HYPERGEOMETRIC FUNCTIONS

The function $_pF_q$ is the generalization of hypergeometric function $_2F_1$, where nature of p parameters is similar as of a and b, and nature of q parameters is same as of c. Thus the generalized hypergeometric series is:

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots\ldots (a_p)_n \; z^n}{(b_1)_n \ldots\ldots (b_q)_n \; n!} $$

$$ = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n \, z^n}{\prod_{j=1}^{q} (b_j)_n \; n!}, $$

where $_pF_q$ is known as generalized hypergeometric function of variable z. If for any q, $b_q = 0$ or $b_q < 0$, the function $_pF_q$ is not defined. If for any p, $a_p = 0$ or $a_p < 0$, the series will terminates. In case non terminating $_pF_q$,

(i) for $|z| < 1$ if $p = q + 1$;

(ii) for $|z| = 1$ if $p = q + 1$ and $R\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$

(iii) for all finite z if $p \leq q$;

the series converges and diverges $\forall$ $z \neq 0$, $q + 1 < p$.

Functions considered above and the class of the hypergeometric series are of single variable. Countless achievement of philosophy of hypergeometric series in one variable takes inspired the growth of equivalent theory in two and more than two variables.

It was Appell (1880), who for the first time introduced the following four series F_1, F_2, F_3, F_4 in two variables:

$$F_1\left[{}^{a,b,b';}_{c;}x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}\, m!\, n!} x^m y^n, \tag{1.2.1}$$

$\max\{|x|, |y|\} < 1;$

$$F_2\left[{}^{a,b,b';}_{c,c';}x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n\, m!\, n!} x^m y^n, \tag{1.2.2}$$

$1 > |x| + |y|;$

$$F_3\left[{}^{a,a',b,b';}_{c;}x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}\, m!\, n!} x^m y^n, \tag{1.2.3}$$

$1 > |x| + |y|;$

$$F_4\left[{}^{a,b;}_{c,c';}x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n\, m!\, n!} x^m y^n, \tag{1.2.4}$$

$1 > \sqrt{|x|} + \sqrt{|y|};$

In 1920, Humbert [27] introduced the confluent hypergeometric function of two variables

$$\phi_1(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n} m!\, n!} x^m y^n, \tag{1.2.5}$$

$|y| < \infty, |x| < 1;$

$$\phi_2(\beta, \beta'; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m\, (\beta')_n}{(\gamma)_{m+n} m!\, n!} x^m y^n, \tag{1.2.6}$$

$|y| < \infty, |x| < \infty;$

$$\phi_3(\beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n} m!\, n!} x^m y^n, \tag{1.2.7}$$

$$\Psi_1(\alpha, \beta; \gamma, \gamma'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n m!\, n!} x^m y^n, \tag{1.2.8}$$

$|x| < 1, |y| < \infty;$

$$\Psi_2(\alpha; \gamma, \gamma'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n m!\, n!} x^m y^n, \tag{1.2.9}$$

$|y| < \infty, |x| < \infty;$

Horn (1931), while giving a general definition for the double power series, constructed ten more hypergeometric functions viz. G_1 to G_3 and H_1 to H_7 and thirteen confluent out of these ten functions. Thus, there are 34 distinct convergent hypergeometric series of two variables as shown by Horn [26]. Some of them, which are useful in our research, are given as:

$$G_1(\alpha, \beta, \beta'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{n-m}(\beta')_{m-n}}{m!\, n!} x^m y^n, \tag{1.2.10}$$

$|y| < s,\ r + s = 1,\ |x| < r;$

$$G_2(\alpha, \alpha', \beta, \beta'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_{n-m}(\beta')_{m-n}}{m!\, n!} x^m y^n, \tag{1.2.11}$$

$|y| < 1,\ |x| < 1;$

$$G_3(\alpha, \alpha'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2n-m}(\alpha')_{2m-n}}{m!\, n!} x^m y^n, \tag{1.2.12}$$

$|y| < s,\ 27r^2s^2 + 18rs \pm 4(r - s) - 1 = 0,\ |x| < r;$

$$H_1(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m m!\, n!} x^m y^n, \tag{1.2.13}$$

$|y| < s,\ (s - 1)^2 = 4rs,\ |x| < r;$

$$H_2(\alpha, \beta, \gamma, \delta; \varepsilon; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\varepsilon)_m m!\, n!} x^m y^n, \tag{1.2.14}$$

$|y| < s,\ (r + s) = 1,\ |x| < r;$

$$H_3(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_m}{(\gamma)_m m!\, n!} x^m y^n, \tag{1.2.15}$$

$|x| < 1;$

$$H_4(\alpha, \gamma; \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\gamma)_n}{(\delta)_m m!\, n!} x^m y^n, \tag{1.2.16}$$

$|y| < s,\ (s - 1)^2 = 4r =,\ |x| < r;$

$$H_5(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_{n-m}}{(\gamma)_n m!\, n!} x^m y^n, \tag{1.2.17}$$

$|y| < s,\ 16r^2 - 36rs \pm (8r - s + 27rs^2) + 1 = 0,\ |x| < r;$

$$H_6(\alpha, \beta, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n}(\beta)_{n-m}(\gamma)_n}{m!\, n!} x^m y^n, \tag{1.2.18}$$

$|y| < s,\ s + rs^2 - 1,\ |x| < r;$

$$H_7(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n}(\beta)_n(\gamma)_n}{(\delta)_m m!\, n!} x^m y^n, \tag{1.2.19}$$

$|y| < s,\ (s^{-1} - 1)^2 = 4r,\ |x| < r;$

$$H_8(\alpha, \beta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n}(\beta)_{n-m}}{m!\, n!} x^m y^n, \tag{1.2.20}$$

$|x| < ¼$;

$$H_9(\alpha, \beta; \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n}(\beta)_n}{(\delta)_n m!\, n!} x^m y^n, \tag{1.2.21}$$

$$\Gamma_1(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_{n-m}(\beta')_{m-n}}{(\gamma)_n m!\, n!} x^m y^n. \tag{1.2.22}$$

In 1954, Saran [62] completed Lauricella's series of hypergeometric function of three variables by defining the functions F_E, F_F, F_G, F_K, F_M, F_H, F_P, F_R, F_S and F_T.

$$F_E[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1,m+n+p)(\beta_1,m)(\beta_2,n+p)}{(\gamma_1,m)(\gamma_2,n)(\gamma_3,p)(1,m)(1,n)(1,p)} x^m y^n z^p, \tag{1.2.23}$$

$$F_G[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1,m+n+p)(\beta_1,m)(\beta_2,n)(\beta_3,p)}{(\gamma_1,m)(\gamma_2,n+p)(1,m)(1,n)(1,p)} x^m y^n z^p, \tag{1.2.24}$$

$$F_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1,m)(\alpha_2,n+p)(\beta_1,m+p)(\beta_2,n)}{(\gamma_1,m)(\gamma_2,n)(\gamma_3,p)(1,m)(1,n)(1,p)} x^m y^n z^p, \tag{1.2.25}$$

$$F_N[\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1,m)(\alpha_2,n)(\alpha_3,p)(\beta_1,m+p)(\beta_2,n)}{(\gamma_1,m)(\gamma_2,n+p)(1,m)(1,n)(1,p)} x^m y^n z^p, \tag{1.2.26}$$

$$F_S[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1,m)(\alpha_2,n+p)(\beta_1,m)(\beta_2,n)(\beta_3,p)}{(\gamma_1,m+n+p)(1,m)(1,n)(1,p)} x^m y^n z^p. \tag{1.2.27}$$

In addition to Lauricella's and Saran's functions Pandey [52] defined G_A and G_B and Dhawan [18] considered G_C and G_D hypergeometric function of three variables, are given as follows:

$$G_A(\alpha, \beta, \beta'; \gamma; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta)_{m+p}(\beta')_n}{(\gamma)_{n+p-m} m!\, n!\, p!} x^m y^n z^p, \tag{1.2.28}$$

$|y| < 1, |z| < 1, |x| < 1$;

$$G_B(\alpha, \beta_1, \beta_2, \beta_3; \gamma; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma)_{n+p-m} m!\, n!\, p!} x^m y^n z^p, \tag{1.2.29}$$

$|y| < 1, |z| < 1, |x| < 1$;

$G_C(\alpha, \beta, \beta_1; \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+n}(\beta_1)_{n-p}}{(\gamma)_{m+n-p} m!\, n!\, p!} x^m y^n z^p, \qquad (1.2.30)$$

$|y| < 1, |z| < 1, |x| < 1;$

$G_D(\alpha, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{p-m}(\alpha_1)_n(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma)_{n+p-m} m!\, n!\, p!} x^m y^n z^p, \qquad (1.2.31)$$

$|y| < 1, |z| < 1, |x| < 1;$

Taking the limiting cases of fourteen triple hypergeometric functions due to Lauricella and Saran, Dhawan [17] defined five more confluent hypergeometric functions ${}_3G_A^{(1)}$, ${}_3G_A^{(2)}$, ${}_3G_B^{(1)}$, ${}_3H_A^{(1)}$, and ${}_3H_B^{(1)}$. Some of them, are given as follows:

${}_3G_A^{(1)}(\alpha, \beta_1; \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_{m+p}}{(\gamma)_{n+p-m} m!\, n!\, p!} x^m y^n z^p, \qquad (1.2.32)$$

$|y| < 1, |z| < 1, |x| < 1;$

${}_3G_B^{(1)}(\alpha, \beta_1, \beta_2; \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_m(\beta_2)_n}{(\gamma)_{n+p-m} m!\, n!\, p!} x^m y^n z^p, \qquad (1.2.33)$$

$|y| < 1, |z| < 1, |x| < 1;$

${}_3H_A^{(1)}(\alpha, \beta; \gamma, \gamma'; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+n}}{(\gamma)_m(\gamma')_{n+p} m!\, n!\, p!} x^m y^n z^p, \qquad (1.2.34)$$

$|y| < s, |z| < t, 1 + st = r + s + t, |x| < r;$

In recent research work, the double hypergeometric function has been generalized by taking more variables and more parameters. Moreover, G, H and A-function also have been generalized by increasing the number of variables, in terms of contour integral.

The A-function of one variable is defined by Gautam [22] and we will represent here in the following manner:

$$A_{p,q}^{m,n}\left[x\Big|_{((b_q,\beta_q))}^{((a_p,\alpha_p))}\right] = \frac{1}{2\pi i}\int_L \theta(s)\, x^s\, ds \qquad (1.2.35)$$

where $i = \sqrt{(-1)}$ and

(i)

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(a_j + s\alpha_j)\prod_{j=1}^{n}\Gamma(1 - b_j - s\beta_j)}{\prod_{j=m+1}^{p}\Gamma(1 - a_j - s\alpha_j)\prod_{j=n+1}^{q}\Gamma(b_j + s\beta_j)} \qquad (1.2.36)$$

(ii) m, n, p and q are non-negative numbers in which $m \le p$, $n \le q$.

(iii) $x \neq 0$ and parameters a_j, α_j, b_k and β_k (j = 1 to p and k = 1 to q) are all complex.

In (1.2.35), the integral is convergent if

(i) $x \neq 0$, $k = 0$, $h > 0$, $|\arg(ux)| < \pi h/2$

(ii) $x > 0$, $k = 0 = h$, $(\nu - \sigma\omega) < -1$

where

$$k = \operatorname{Im}\left(\sum_1^p \alpha_j - \sum_1^q \beta_j\right)$$

$$h = \operatorname{Re}\left(\sum_{j=1}^{n}\beta_j - \sum_{j=n+1}^{q}\beta_j + \sum_{j=1}^{m}\alpha_j - \sum_{j=n+1}^{p}\alpha_j\right) \qquad (1.2.37)$$

$$u = \prod_1^p {\alpha_j}^{\alpha_j} \prod_1^q {\beta_j}^{\beta_j} \qquad (1.2.38)$$

$$\nu = \operatorname{Re}\left(\sum_1^p a_j - \sum_1^q b_j\right) - (p - q)/2,$$

$$w = \operatorname{Re}\left(\sum_1^q \beta_j - \sum_1^p \alpha_j\right)$$

and $s = \sigma + it$ is on path L when $|t| \to \infty$.

1.3 POLYNOMIALS

1.3.1 Legendre Polynomials:

In the study of attraction of spheroids and planetary motion, Legendre was led to the consideration of the series of the function

$$1/r = (1 - 2\rho\cos\gamma + \rho^2)^{-1/2} \qquad (1.3.1)$$

The expansion of this expression in ascending powers of ρ is of the form

$$\sum_{n=0}^{\infty} \rho^h P_n(\mu), \text{ where } \mu = \cos\gamma,\ 0 < \rho < 1. \quad (1.3.2)$$

The coefficients $P_n(\mu)$ are known as Legendre polynomials and it depends on cosγ only and can be shown to be polynomials of degree n in cosγ. In term of hypergeometric function as

$$P_n(\mu) = {}_2F_1\left[{}^{-n,n+1;}_{1;}\frac{1-\mu}{2}\right] \quad (1.3.3)$$

1.3.2 Associated Legendre Polynomials:

Ferrer (1877) introduced the associated Legendre polynomial $P_n^{(m)}(\mu)$ and $Q_n^{(m)}(\mu)$ of the first and second kinds respectively of degree n and order m, as the solution of the differential equation.

$$\frac{d}{d\mu}\{(1-\mu^2)\frac{dz}{d\mu}\} + \{n(n+1) - m^2/(1-\mu^2)\}z = 0, \quad (1.3.4)$$

where (μ = cosθ).

It can be proved easily that if m is a positive integer and $-1 \leq \mu \leq 1$, then

$$P_n^{(m)}(\mu) = (1-\mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m}[P_n(\mu) \quad (1.3.5)$$

where $(1-\mu^2)^{m/2}$ indicates the numerical value of the root.

Further $P_n^{(m)}(\mu)$ and $Q_n^{(m)}(\mu)$ are surface spherical harmonics of degree n and order m where $Q_n^{(m)}(\mu) = (-1)^m \frac{(n-m)!}{n+m} P_n^{(m)}(\mu)$

Legendre polynomials have been widely used in many applied problems related to this spherical regions, steady temperatures in a solid and hemisphere, temperature in non-homogeneous insulated bar etc.

1.3.3 Laguerre Polynomials:

Simple Laguerre polynomials $L_n(x)$ were introduced by Laguerre, E. N. in (1879). These Laguerre polynomials also occur in an unedited manuscript (1881) of Able. N. H.

Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by the generating function [56]

$$\frac{1}{(1-t)^{1+\alpha}} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n \tag{1.3.6}$$

and Rodrigues formula [56].

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^{x}}{n!} D^n (x^{n+\alpha} e^{-x}) \tag{1.3.7}$$

In hypergeometric form, these polynomials $L_n^{(\alpha)}(x)$ are expressed [56] by

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \; {}_1F_1\left[\begin{matrix}-n,;\\ 1+\alpha;\end{matrix}\; x\right] \tag{1.3.8}$$

and known as generalized Laguerre or sonine polynomials. Moreover, the solution of differential equation of second order [56]

$$D^2 L_n^{(\alpha)}(x) + (1+\alpha+n) D L_n^{(\alpha)}(x) + n L_n^{(\alpha)}(x) = 0 \tag{1.3.9}$$

gives these polynomials.

For $\alpha = 0$, the polynomials $L_n^{(\alpha)}(x)$ reduces to simple Laguerre polynomials

$L_n(x)$ i.e. $L_n^{(\alpha)}(x) = \; {}_1F_1\left[\begin{matrix}-n,;\\ 1;\end{matrix}\; x\right]$

1.3.4 Hermite Polynomials:

The notation $H_n(x)$ for Hermite polynomial was introduced by Szego's in 1939. $H_n(x)$ (Hermite polynomials) are defined by generating function [56]

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x))t^n}{n!} \tag{1.3.10}$$

and Rodrigues formula [56]

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2} \tag{1.3.11}$$

The hypergeometric form, these polynomials $H_n(x)$ expressed [56] by

$$H_n(x) = (2x)^{x^2} \; {}_2F_0\left[\begin{matrix}\frac{-n}{2}, \frac{-n}{2}+1;\\ -;\end{matrix}\; \frac{-1}{x^2}\right] \tag{1.3.12}$$

Moreover, the solution of differential equation of second order [56]

$$H_n(x)'' - 2x H_n(x)' + 2n H_n(x) = 0 \tag{1.3.13}$$

gives these polynomials. Chebyshev Hermite polynomial $H_{e_n}(\mathrm{x})$, is given by the generating relation

$$e^{xt-\frac{t^2}{2}} = \sum_{n=0}^{\infty} \frac{H_{e_n}(\mathrm{x})t^n}{n!} \tag{1.3.14}$$

and related to Hermite polynomial by relation

$$\mathrm{Hn}(\mathrm{x}) = 2^{\frac{n}{2}} H_{e_n}(\sqrt{2}\mathrm{x}) \tag{1.3.15}$$

1.3.5 Jacobi Polynomials:

The orthogonal polynomials which have occupied a significant place in the recent research papers are the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, introduced by C. G. J. Jacobi (1859) and $P_n^{(\alpha,\beta)}(z)$is the solution of second order linear homogeneous differential equation namely:

$$(1 - z)^2 w'' + [\beta - \alpha - (\alpha + \beta + 2)z]\, w' + n(n + \alpha + \beta + 1)w = 0, \tag{1.3.16}$$

where n is positive integer.

The Jacobi polynomials may be expressed in the hypergeometric form as:

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[{}^{-n,n+\beta+1;}_{\alpha+1;}\frac{1-z}{2}\right] \tag{1.3.17}$$

When we substitute $\alpha = \beta$ in the Jacobi polynomial, we get ultraspherical polynomial $P_n^{(\alpha,\alpha)}(z)$and by the substitution $\alpha = \beta = 0$, these degenerate into Legendre polynomial $P_n(x)$.

1.3.6 Generalized Bessel Polynomials:

In 1949, Krall and Frink defined generalized Bessel Polynomial as follows

$$y_n(x;\alpha,\beta) = \ {}_2F_0\left[{}^{-n,\alpha+n-1;}_{-;}\frac{x}{\beta}\right] \tag{1.3.18}$$

1.3.7 Orthogonal Polynomials:

If $\{\phi_n(x)\}$ be a sequence of functions and w(x) is a non-negative weight function such that $w\phi_n^2$ is integrable in (a, b), then the scalar product is defined by

$$(\phi_n, \phi_m) = \int_a^b w(x)\, \phi_n(x)\, \phi_m(x)\, dx. \tag{1.3.19}$$

If

$(\phi_n, \phi_m) = h_n \, \delta_{mn}$,

then sequence of function $\{\phi_n(x)\}$ is said to be orthogonal, where

$$h_n = (\phi_n, \phi_n) = \int_a^b [\phi_n(x)]^2 \, w(x) \, dx, \qquad (1.3.20)$$

and

$\delta_{mn} = 1$ if $m = n$

$= 0$ if $m \neq n$.

CHAPTER–2

LINEAR AND BILINEAR GENERATING RELATIONS INVOLVING A-FUNCTION

2.1 INTRODUCTION

The sequences, which is generated by generating relations plays significant role in the study of numerous valuable properties. In solving certain classes of difference equation which arise in a wide variety of problems in operation research (for instance, queening theory and related stochastic process), the generating relations are used as z-transform. Generating relations can also be used with good effect for the determination of the asymptotic behavior of the generalized sequence $\{f_n\}_{n=0}^{\infty}$ as n $\rightarrow \infty$ by suitably adopting Darboux's method.

Shrivastava [71], Hussain [28], [29], Majumdar [46], Srivastava [78], Singh [72], Patel [53], Ming [48] and several other authors have discussed a number of linear and bilinear generating relations involving other generalized hypergeometric functions time to time.

Looking into the requirement and importance of various properties of generating relations in the analysis of many problems of mathematics and mathematical physics, in this chapter we established some new linear and bilinear generating relations involving A-function of one variable.

In section (2.3), we have established some new linear generating relations for A-function of one variable.

In section (2.4), we have discussed some bilinear generating relations involving A-function of one variable.

The content of this chapter in the form of two research papers has been published in Applied Science Periodical [37] and IOSR Journal of Mathematics [40].

2.2 LINEAR GENERATING RELATIONS

Since linear generating relation has large role in the study of hypergeometric functions. Thus in this section we have established the eight linear generating relations involving A-Function. We have used some basic results from Shrivastava and Manocha [69, p. 34, 44, 37 (10)].

$$(\alpha)_n = (\alpha, n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \tag{2.2.1}$$

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}. \tag{2.2.2}$$

$$e^x = {}_0F_0[-;-; x], \tag{2.2.3}$$

$$(1-x)^{-a} = {}_1F_0[a; -; x], \ |x| < 1, \tag{2.2.4}$$

$$(1-x)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{x^n}{n!}. \tag{2.2.5}$$

$$(1+x)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{(-x)^n}{n!}. \tag{2.2.6}$$

to prove the following results.

Theorem 2.2.1: Prove that

(i) $$\sum_{r=0}^{\infty} \frac{t^r}{r!} A_{p+1,q}^{m+1,n}\left[x\Big|_{(b_j,\beta_j)_{1,q}}^{(\lambda+r,\alpha),(a_j,\alpha_j)_{1,p}}\right]$$

$$= (1-t)^{-\lambda} A_{p+1,q}^{m+1,n}\left[x(1-t)^{-\alpha}\Big|_{(b_j,\beta_j)_{1,q}}^{(\lambda,\alpha),(a_j,\alpha_j)_{1,p}}\right]; \tag{2.2.7}$$

(ii) $$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} A_{p+1,q}^{m+1,n}\left[x\Big|_{(b_j,\beta_j)_{1,q}}^{(\lambda+r,\alpha)(a_j,\alpha_j)_{1,p}}\right]$$

$$= (1+t)^{-\lambda} A_{p+1,q}^{m+1,n}\left[x(1+t)^{-\alpha}\Big|_{(b_j,\beta_j)_{1,q}}^{(\lambda,\alpha),(a_j,\alpha_j)_{1,p}}\right]; \tag{2.2.8}$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

(i) Consider

$$\Delta = \sum_{r=0}^{\infty} \frac{t^r}{r!} A_{p+1,q}^{m+1,n}\left[x\Big|_{(b_j,\beta_j)_{1,q}}^{(\lambda+r,\alpha),(a_j,\alpha_j)_{1,p}}\right]$$

On expressing A-function in contour integral form as given in (1.2.35), we get

$$\Delta = \sum_{r=0}^{\infty} \frac{t^r}{r!} \{\frac{1}{2\pi i}\int_L \theta(s) x^s \Gamma(\lambda + r + \alpha s)\, ds\}$$

$$= \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \left\{\frac{1}{2\pi i}\int_L \theta(s) x^s (\lambda + \alpha s)_r \Gamma(\lambda + \alpha s)\, ds\right\}.$$

On altering the order of integration and summation, we get

$$\Delta = \frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma(\lambda+\alpha s)\{\sum_{r=0}^{\infty}\frac{(t)^r}{r!}(\lambda+\alpha s)_r\}ds$$

$$= (1-t)^{-\lambda}A_{p+1,q}^{m+1,n}\left[x(1-t)^{-\alpha}\Big|_{(b_j,\beta_j)_{1,q}}^{(\lambda,\alpha),(a_j,\alpha_j)_{1,p}}\right]$$

(in view of (1.2.35) and (1.2.36))

(ii) Proceed as above (i) and using (2.2.6)

Theorem 2.2.2: Prove that

(i) $$\sum_{r=0}^{\infty}\frac{t^r}{r!}A_{p,q+1}^{m,n+1}\left[x\Big|_{(\lambda-r,\alpha),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= (1-t)^{\lambda-1}A_{p,q+1}^{m,n+1}\left[x(1-t)^{\alpha}\Big|_{(\lambda,\alpha),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]; \quad (2.2.9)$$

(ii) $$\sum_{r=0}^{\infty}\frac{(-t)^r}{r!}A_{p,q+1}^{m,n+1}\left[x\Big|_{(\lambda-r,\alpha),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= (1+t)^{\lambda-1}A_{p,q+1}^{m,n+1}\left[x(1+t)^{\alpha}\Big|_{(\lambda,\alpha),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]; \quad (2.2.10)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

(i) Consider

$$\Delta = \sum_{r=0}^{\infty}\frac{t^r}{r!}A_{p,q+1}^{m,n+1}\left[x\Big|_{(\lambda-r,\alpha),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

On expressing A-function in contour integral form as given in (1.2.35), we get

$$\Delta = \sum_{r=0}^{\infty}\frac{t^r}{r!}\{\frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma(1-\lambda+r-\alpha s)\,ds\}$$

$$= \sum_{r=0}^{\infty}\frac{(t)^r}{r!}\left\{\frac{1}{2\pi i}\int_L \theta(s)x^s(1-\lambda-\alpha s)_r\Gamma(1-\lambda-\alpha s)\,ds\right\}.$$

On altering the order of integration and summation, we get

$$\Delta = \frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma(1-\lambda-\alpha s)\{\sum_{r=0}^{\infty}\frac{(t)^r}{r!}(1-\lambda-\alpha s)_r\}ds$$

$$= (1-t)^{\lambda-1}\frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma(1-\lambda-\alpha s)\,(1-t)^{\alpha s}ds$$

$$= (1-t)^{\lambda-1}A_{p,q+1}^{m,n+1}\left[x(1-t)^{\alpha}\Big|_{(\lambda,\alpha),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$ (in view of (1.2.35) and (1.2.36))

(ii) Proceed on same line as in (i) and use (2.2.6) to prove this result.

Theorem 2.2.3: Prove that

(i) $$\sum_{r=0}^{\infty}\frac{t^r}{r!}A_{p+1,q}^{m,n}\left[x\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\lambda+r,\alpha)}\right]$$

$$=(1+t)^{-\lambda}A_{p+1,q}^{m,n}\left[x(1+t)^{-\alpha}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\lambda,\alpha)}\right]; \qquad (2.2.11)$$

(ii) $$\sum_{r=0}^{\infty}\frac{(-t)^r}{r!}A_{p+1,q}^{m,n}\left[x\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\lambda+r,\alpha)}\right]$$

$$=(1-t)^{-\lambda}A_{p+1,q}^{m,n}\left[x(1-t)^{-\alpha}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\lambda,\alpha)}\right]; \qquad (2.2.12)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

(i) Consider

$$\Delta = \sum_{r=0}^{\infty}\frac{t^r}{r!}A_{p+1,q}^{m,n}\left[x\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\lambda+r,\alpha)}\right]$$

On expressing A-function in contour integral form as given in (1.2.35), we get

$$\Delta = \sum_{r=0}^{\infty}\frac{t^r}{r!}\{\frac{1}{2\pi i}\int_L \frac{\theta(s)x^s}{\Gamma(1-\lambda-r-\alpha s)}ds\}$$

$$=\sum_{r=0}^{\infty}\frac{(t)^r}{r!}\left\{\frac{1}{2\pi i}\int_L \frac{\theta(s)x^s(\lambda+\alpha s)_r}{(-1)^r\Gamma(1-\lambda-\alpha s)}ds\right\}.$$

On altering the order of integration and summation, we get

$$\Delta = \frac{1}{2\pi i}\int_L \frac{\theta(s)x^s}{\Gamma(1-\lambda-\alpha s)}\{\sum_{r=0}^{\infty}\frac{(-1)^r(t)^r}{r!}(\lambda+\alpha s)_r\}ds$$

$$=(1+t)^{-\lambda}\frac{1}{2\pi i}\int_L \frac{\theta(s)x^s(1-t)^{-\alpha s}}{\Gamma(1-\lambda-\alpha s)}ds$$

$$=(1+t)^{-\lambda}A_{p+1,q}^{m,n}\left[x(1+t)^{-\alpha}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\lambda,\alpha)}\right];$$ (in view of (1.2.35) and (1.2.36))

(ii) Proceed as above (i)

Theorem 2.2.4: Prove that

(i) $$\sum_{r=0}^{\infty}\frac{t^r}{r!}A_{p,q+1}^{m,n}\left[x\Big|_{(b_j,\beta_j)_{1,q},(1-\lambda-r,\alpha)}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= (1+t)^{-\lambda} A^{m,n}_{p,q+1}\left[x(1+t)^{\alpha}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1-\lambda,\alpha)}\right]; \qquad (2.2.13)$$

(ii) $\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} A^{m,n}_{p,q+1}\left[x\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1-\lambda-r,\alpha)}\right]$

$$= (1-t)^{-\lambda} A^{m,n}_{p,q+1}\left[x(1-t)^{\alpha}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1-\lambda,\alpha)}\right]; \qquad (2.2.14)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

(i) Consider

$$\Delta = \sum_{r=0}^{\infty} \frac{t^r}{r!} A^{m,n}_{p,q+1}\left[x\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1-\lambda-r,\alpha)}\right]$$

On expressing A-function in contour integral form as given in (1.2.35), we get

$$\Delta = \sum_{r=0}^{\infty} \frac{t^r}{r!}\left\{\frac{1}{2\pi i}\int_L \frac{\theta(s)x^s}{\Gamma(1-\lambda-r+\alpha s)} ds\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(t)^r}{r!}\left\{\frac{1}{2\pi i}\int_L \frac{\theta(s)x^s(\lambda+\alpha s)_r}{(-1)^r\Gamma(1-\lambda+\alpha s)} ds\right\}.$$

On altering the order of integration and summation, we get

$$\Delta = \frac{1}{2\pi i}\int_L \frac{\theta(s)x^s}{\Gamma(1-\lambda-\alpha s)}\left\{\sum_{r=0}^{\infty} \frac{(-1)^r(t)^r}{r!}(\lambda-\alpha s)_r\right\}ds$$

$$= (1+t)^{-\lambda}\frac{1}{2\pi i}\int_L \frac{\theta(s)x^s(1-t)^{\alpha s}}{\Gamma(1-\lambda+\alpha s)} ds$$

$$= (1+t)^{-\lambda} A^{m,n}_{p,q+1}\left[x(1+t)^{\alpha}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1-\lambda,\alpha)}\right]$$ (in view of (1.2.35) and (1.2.36))

(ii) Same as part (i)

2.3 BILINEAR GENERATING RELATIONS

In this section we establish the four bilinear generating relations involving two A-Functions. In order to prove these relation we have use the relations given in section (2.2) from Shrivastava and Manocha [69, p.37 (10), 34, 44].

Theorem 2.3.1: Prove that

(i) $\sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^l}{l!}\frac{v^r}{r!} A^{m,n}_{p+1,q}\left[x\Big|^{(a_j,\alpha_j)_{1,p},(\lambda+l,\alpha)}_{(b_j,\beta_j)_{1,q}}\right]. A^{m,n}_{p+1,q}\left[y\Big|^{(a_j,\alpha_j)_{1,p},(\mu+r,\beta)}_{(b_j,\beta_j)_{1,q}}\right]$

$$= (1+t)^{-\lambda}(1+v)^{-\mu}\, A^{m,n}_{p+1,q}\left[x(1+t)^{-\alpha}\Big|^{(a_j,\alpha_j)_{1,p},(\lambda,\alpha)}_{(b_j,\beta_j)_{1,q}}\right].$$

$$.A^{m,n}_{p+1,q}\left[y(1+v)^{-\beta}\Big|^{(a_j,\alpha_j)_{1,p},(\mu,\beta)}_{(b_j,\beta_j)_{1,q}}\right], \qquad (2.3.1)$$

(ii) $$\sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-t)^l}{l!}\frac{(-v)^r}{r!} A^{m,n}_{p+1,q}\left[x\Big|^{(a_j,\alpha_j)_{1,p},\,(\lambda+l,\alpha)}_{(b_j,\beta_j)_{1,q}}\right].$$

$$.A^{m,n}_{p+1,q}\left[y\Big|^{(a_j,\alpha_j)_{1,p},\,(\mu+r,\beta)}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= (1-t)^{-\lambda}(1-v)^{-\mu}\, A^{m,n}_{p+1,q}\left[x(1-t)^{-\alpha}\Big|^{(a_j,\alpha_j)_{1,p},\,(\lambda,\alpha)}_{(b_j,\beta_j)_{1,q}}\right].$$

$$.A^{m,n}_{p+1,q}\left[y(1-v)^{-\beta}\Big|^{(a_j,\alpha_j)_{1,p},\,(\mu,\beta)}_{(b_j,\beta_j)_{1,q}}\right], \qquad (2.3.2)$$

provided that $|\arg(ux)| < \pi h/2$ and $|\arg(uy)| < \pi h/2$, where u and h are given in (1.2.37) and (1.2.38) respectively.

Proof

(i) Consider

$$\Delta = \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^l}{l!}\frac{v^r}{r!} A^{m,n}_{p+1,q}\left[x\Big|^{(a_j,\alpha_j)_{1,p},(\lambda+l,\alpha)}_{(b_j,\beta_j)_{1,q}}\right].A^{m,n}_{p+1,q}\left[y\Big|^{(a_j,\alpha_j)_{1,p},(\mu+r,\beta)}_{(b_j,\beta_j)_{1,q}}\right]$$

On expressing A-function in contour integral form as given in (1.2.35), we get

$$\Delta = \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^l}{l!}\frac{v^r}{r!}\left\{\frac{1}{2\pi i}\int_L \theta(s)x^s \frac{1}{\Gamma(1-\lambda-l-\alpha s)}ds\right\}.$$

$$.\left\{\frac{1}{2\pi i}\int_L \emptyset(z)y^z \frac{1}{\Gamma(1-\mu-r-\beta z)}dz\right\}$$

$$= \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-t)^l}{l!}\frac{(-v)^r}{r!}\left\{\frac{1}{2\pi i}\int_L \theta(s)x^s \frac{(\lambda+\alpha s)_l}{\Gamma(1-\lambda-\alpha s)}ds\right\}.$$

$$.\left\{\frac{1}{2\pi i}\int_L \emptyset(z)y^z \frac{(\mu+\beta z)_r}{\Gamma(1-\mu-\beta z)}dz\right\}$$

On altering the order of integration and summation, we get

$$\Delta = \left[\frac{1}{2\pi i}\int_L \theta(s)x^s \frac{1}{\Gamma(1-\lambda-\alpha s)}\left\{\sum_{l=0}^{\infty}\frac{(-t)^l}{l!}(\lambda+\alpha s)_l\right\}ds\right].$$

$$.\left[\frac{1}{2\pi i}\int_L \emptyset(z)y^z \frac{1}{\Gamma(1-\mu-\beta s)}\left\{\sum_{r=0}^{\infty}\frac{(-t)^r}{r!}(\mu+\beta z)_r\right\}dz\right].$$

$$= (1+t)^{-\lambda}(1+v)^{-\mu}\left[\frac{1}{2\pi i}\int_L \theta(s)x^s \frac{(1+t)^{-\alpha s}}{\Gamma(1-\lambda-\alpha s)}ds\right].$$

$$.\left[\frac{1}{2\pi i}\int_L \emptyset(z)y^z\frac{(1+v)^{-\beta z}}{\Gamma(1-\mu-\beta z)}dz\right] \qquad \text{(in view of (1.2.36))}$$

Hence Proved.

(iii) Proceed as above (i)

Theorem 2.3.2: Prove that

(i) $$\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^l}{l!}\frac{v^r}{r!}A_{p+1,q}^{m+1,n}\left[x\middle|\begin{matrix}(\lambda+l,\alpha),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right].\,A_{p+1,q}^{m+1,n}\left[y\middle|\begin{matrix}(\mu+r,\alpha),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

$$= (1-t)^{-\lambda}(1-v)^{-\mu}A_{p+1,q}^{m+1,n}\left[x(1-t)^{-\alpha}\middle|\begin{matrix}(\lambda,\alpha),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right].$$

$$\times A_{p+1,q}^{m+1,n}\left[y(1-v)^{-\beta}\middle|\begin{matrix}(\mu,\beta),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right], \qquad (2.3.3)$$

(ii) $$\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-t)^l}{l!}\frac{(-v)^r}{r!}A_{p+1,q}^{m+1,n}\left[x\middle|\begin{matrix}(\lambda+l,\alpha),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right].$$

$$.A_{p+1,q}^{m+1,n}\left[x\middle|\begin{matrix}(\mu+r,\beta),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

$$= (1+t)^{-\lambda}(1+v)^{-\mu}A_{p+1,q}^{m+1,n}\left[x(1+t)^{-\alpha}\middle|\begin{matrix}(\lambda,\alpha),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right].$$

$$\times A_{p+1,q}^{m+1,n}\left[y(1+v)^{-\beta}\middle|\begin{matrix}(\mu,\beta),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right], \qquad (2.3.4)$$

provided that $|\arg(ux)| < \pi h/2$ and $|\arg(uy)| < \pi h/2$, where u and h are given in (1.2.37) and (1.2.38) respectively.

Proof

(i) Consider

$$\Delta = \sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^l}{l!}\frac{v^r}{r!}A_{p+1,q}^{m+1,n}\left[x\middle|\begin{matrix}(\lambda+l,\alpha),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right].\,A_{p+1,q}^{m+1,n}\left[y\middle|\begin{matrix}(\mu+r,\alpha),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

On expressing A-function in contour integral form as given in (1.2.35), we get

$$\Delta = \sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^l}{l!}\frac{v^r}{r!}\left\{\frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma(\lambda+l+\alpha s)\,ds\right\}.$$

$$.\left\{\frac{1}{2\pi i}\int_L \emptyset(z)y^z\Gamma(\mu+r+\beta z)\,dz\right\}$$

$$= \sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-t)^l}{l!}\frac{(-v)^r}{r!}\left\{\frac{1}{2\pi i}\int_L \theta(s)x^s(\lambda+\alpha s)_l\Gamma(\lambda+\alpha s)\,ds\right\}.$$

$$.\left\{\frac{1}{2\pi i}\int_L \emptyset(z)y^z(\mu+\beta z)_r\Gamma(\mu+\beta z)\,dz\right\}$$

On altering the order of integration and summation, we get

$$\Delta = \left[\frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma(\lambda+\alpha s)\left\{\sum_{l=0}^{\infty}\frac{t^l}{l!}(\lambda+\alpha s)_l\right\}ds\right].$$

$$\cdot\left[\frac{1}{2\pi i}\int_L \emptyset(z)y^z\Gamma(\mu+\beta z)\left\{\sum_{r=0}^{\infty}\frac{v^r}{r!}(\mu+\beta z)_r\right\}dz\right].$$

$$= (1-t)^{-\lambda}(1-v)^{-\mu}\left[\frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma(\lambda+\alpha s)(1-t)^{-\alpha s}\,ds\right].$$

$$\cdot\left[\frac{1}{2\pi i}\int_L \emptyset(z)y^z\Gamma(\mu+\beta z)(1-v)^{-\beta z}\,dz\right]$$

$$= (1-t)^{-\lambda}(1-v)^{-\mu}A^{m+1,n}_{p+1,q}[x(1-t)^{-\alpha}|^{(\lambda,\alpha),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}].$$

$$\times A^{m+1,n}_{p+1,q}[y(1-v)^{-\beta}|^{(\mu,\beta),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}} \qquad \text{(in view of (1.2.36))}$$

Hence proved.

(ii) Proceed as above (i)

CHAPTER–3

BILATERAL AND TRILATERAL GENERATING RELATIONS INVOLVING A-FUNCTION

3.1 INTRODUCTION

In the progress and study of special functions A-function of one variable plays a vital role. The usefulness of this function has inspired us to find some new generating relations.

Hussain [28], Majumdar [46], Shrivastava [78], Singh [72], Ming [48] and several other authors have discussed a number of bilateral and trilateral generating relations involving generalized hypergeometric functions time to time.

In this chapter some new bilateral and trilateral generating relations have been established involving A-function of one variable and other hypergeometric functions.

In section (3.3), we have discussed some new bilateral generating relations involving A-function of one variable.

In section (3.4), we find some new trilateral generating relations for A-function of one variable.

Most of the results in this chapter have been published in Arybhatta Journal of Mathematics and Informatics [41] in form of a research paper.

3.2 RESULTS AND FORMULAE USED

In the present investigation we require the following formulae:

From Shrivastava and Manocha [69, p.37 (10), 34, 44],

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} P_n^{(\alpha-n,\beta-n)}(z)t^n = F_1\left[\lambda, -\alpha, -\beta; \mu; -(z+1)\frac{t}{2}, -(z-1)\frac{t}{2}\right], \tag{3.2.1}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n(\delta)_n}{(\alpha+1)_n(\beta+1)_n} P_n^{(\alpha,\beta)}(z)t^n = F_4\left[\lambda, \delta; \alpha+1, \beta+1; (z-1)\frac{t}{2}, (z+1)\frac{t}{2}\right]. \tag{3.2.2}$$

From Rainvile [56]:

$$ {}_2F_1\left[{-n,\ a; \atop 1+a+n;} -1\right] = \frac{(1+a)_n}{(1+a/2)_n}, \tag{3.2.3}$$

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{3.2.4}$$

$$ (\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha,n)}, \tag{3.2.5}$$

$$ (\alpha', p-q) = (\alpha', -q)(\alpha'-q, p) = \frac{(-1)^q(\alpha'-q,p)}{(1-\alpha',q)}, \tag{3.2.6}$$

$$ (\mu, p)\ (\mu + x, y + z) = (\mu, x + y + z), \tag{3.2.7}$$

$$ (\mu, x + y)\ (\mu + x + y, t + z) = (\mu, x + y + t + z)$$

$$ = (\mu, y)\ (\mu + y, x + t + z), \tag{3.2.8}$$

$$ (\mu, n)\ (\mu + n, q) = (\mu, n + q) = (\mu, q)\ (\mu + q, n). \tag{3.2.9}$$

3.3 BILATERAL GENERATING RELATIONS

Since bilateral generating relations are of great importance in the study of A-Functions therefore in this section we have established the four bilateral Generating Relations involving A-Function and Gauss hypergometric function.

Theorem 3.3.1: Prove that

(i) $$\sum_{l=0}^{\infty} \frac{t^l}{l!}\ {}_2F_1\left[{-l,\ a; \atop 1+a+l;} -1\right] A_{p,q+1}^{m,n+1}\left[x\Big|{(a_j,\alpha_j)_{1,p} \atop (-a/2-l,0),(b_j,\beta_j)_{1,q}}\right]$$

$$ = (1-t)^{-(a+1)} A_{p,q+1}^{m,n+1}\left[x\Big|{(a_j,\alpha_j)_{1,p} \atop (-a/2,0),(b_j,\beta_j)_{1,q}}\right], \tag{3.3.1}$$

(ii) $$\sum_{l=0}^{\infty} \frac{t^l}{l!}\ {}_2F_1\left[{-l,\ a; \atop 1+a+l;} -1\right] A_{p+1,q}^{m+1,n}\left[x\Big|{(1+a/2+l,0),(a_j,\alpha_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right]$$

$$ = (1-t)^{-(a+1)} A_{p+1,q}^{m+1,n}\left[x\Big|{(1+a/2,0),(a_j,\alpha_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right], \tag{3.3.2}$$

(iii) $$\sum_{l=0}^{\infty} \frac{t^l}{l!}\ {}_2F_1\left[{-l,\ a; \atop 1-a-l;} -1\right] A_{p+1,q}^{m+1,n}\left[x\Big|{(a+l,0),(a_j,\alpha_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right]$$

$$= (1-t)^{-a/2} A_{p+1,q}^{m+1,n}\left[x\Big|_{(b_j,\beta_j)_{1,q}}^{(a,0),(a_j,\alpha_j)_{1,p}}\right], \quad (3.3.3)$$

(iv) $\sum_{l=0}^{\infty}\frac{t^l}{l!} \quad {}_2F_1\left[{}^{-l,-a;}_{1-a-l;}-1\right] A_{p,q+1}^{m,n+1}\left[x\Big|_{(1-a-l,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$

$$= (1-t)^{-a/2} A_{p,q+1}^{m,n+1}\left[x\Big|_{(1-a,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]; \quad (3.3.4)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively and ${}_2F_1$ is Gauss hypergeometric function.

Proof

(i) Consider

$$\Delta = \sum_{l=0}^{\infty}\frac{t^l}{l!} \quad {}_2F_1\left[{}^{-l,\ a;}_{1+a+l;}-1\right] A_{p,q+1}^{m,n+1}\left[x\Big|_{(-a/2-l,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right].$$

On expressing A-function in contour integral form as given in (1.2.35) and using (3.2.3), we get

$$\Delta = \sum_{l=0}^{\infty}\frac{t^l}{l!}\frac{(1+a)_l}{(1+a/2)_l}\left[\frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma\{1-\left(-\frac{a}{2}-l\right)-0s\}\,ds\right].$$

In the view of (3.2.4) and (2.2.5), we arrive at R.H.S. of (3.3.1) as follows:

$$\Delta = \sum_{l=0}^{\infty}\frac{t^l}{l!}\frac{(1+a)_l}{(1+a/2)_l}\left[\frac{1}{2\pi i}\int_L \theta(s)x^s\left(1+\frac{a}{2}\right)_l\Gamma(1+a/2)\,ds\right]$$

$$= \frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma\left(1+\frac{a}{2}\right)\left[\sum_{l=0}^{\infty}\frac{t^l}{l!}(1+a)_l\right]$$

$$= \frac{1}{2\pi i}\int_L \theta(s)x^s\Gamma\left(1+\frac{a}{2}\right)(1-t)^{-(a+1)}\,ds$$

$$= (1-t)^{-(a+1)} A_{p,q+1}^{m,n+1}\left[x\Big|_{(-a/2,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]. \quad \text{(in view of (1.2.35))}$$

Which we have to prove.

(ii) - (iv) Proceed as above (i) and using the results of section 3.3.

3.4 TRILATERAL GENERATING RELATIONS

Theory of trilateral generating relations for different kind of special functions is of great significance. We are going to establish the five trilateral generating relations in this section.

In first result we have established a trilateral generating relation involving Horn's hypergeometric function H_2 and hypergeometric function F_S.

Theorem 3.4.1

$$\sum_{n=0}^{\infty} H_2\,[\alpha', \beta', \gamma', \delta'; \mu+n; x, y] P_n^{(\alpha-n,\beta-n)}(z)$$

$$.\,A_{p+1,q+1}^{m+1,l}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu+n,0)}^{(\lambda+n,0),(a_j,\alpha_j)_{1,p}}\right] t^n$$

$$= \sum_{q=0}^{\infty} \frac{(\gamma', q)(\delta', q)}{(1-\alpha', q)(1, q)} (-y)^q A_{p+1,q+1}^{m+1,l}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu,0)}^{(\lambda,0),(a_j,\alpha_j)_{1,p}}\right]$$

$$.F_S[\alpha'-q, \lambda, \lambda, \beta', -\alpha, -\beta; \mu, \mu, \mu; x, -(z+1)\tfrac{t}{2}, -(z-1)\tfrac{t}{2}], \qquad (3.4.1)$$

$|x| < r$, $|y| < s$, $(r + s) = 1$, $|\arg (uv)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively;

Proof

To prove (3.4.1), consider

$$\sum_{n=0}^{\infty} H_2\,[\alpha', \beta', \gamma', \delta'; \mu+n; x, y] P_n^{(\alpha-n,\beta-n)}(z)$$

$$.\,A_{p+1,q+1}^{m+1,l}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu+n,0)}^{(\lambda+n,0),(a_j,\alpha_j)_{1,p}}\right] t^n.$$

Expressing H_2 in series form, by using (1.2.14) and A-function (1.2.35) and using (3.2.4), we get

$$\Delta = \sum_{n=0}^{\infty}\sum_{p,q=0}^{\infty} \frac{(\alpha', p-q)(\beta', p)(\gamma', q)(\delta', q)}{(\mu+n, p)(1, p)(1, q)} x^p y^q P_n^{(\alpha-n,\beta-n)}(z)$$

$$.[\frac{1}{2\pi i}\int_L \theta(s)u^s\frac{(\lambda,n)\Gamma(\lambda)}{(\mu,n)\Gamma(\mu)}ds]t^n.$$

After changing integration and summation order and using (3.2.9), we get

$$\Delta=\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(\lambda)}{\Gamma(\mu)}u^s$$

$$.\sum_{p,q=0}^{\infty}\frac{(\alpha',p-q)(\beta',p)(\gamma',q)(\delta',q)}{(\mu,p)(1,p)(1,q)}x^p y^q$$

$$.\left[\sum_{n=0}^{\infty}\frac{(\lambda,n)}{(\mu+p,n)}P_n^{(\alpha-n,\beta-n)}(z)t^n\right]ds.$$

Again applying (3.2.1), we find that

$$\Delta=\frac{1}{2\pi i}\int_L \theta(s)u^s\frac{\Gamma(\lambda)}{\Gamma(\mu)}\sum_{p,q=0}^{\infty}\frac{(\alpha',p-q)(\beta',p)(\gamma',q)(\delta',q)}{(\mu,p)(1,p)(1,q)}x^p y^q$$

$$.F_1\left[\lambda,-\alpha,-\beta;\mu+p;-(z+1)\frac{t}{2},-(z-1)\frac{t}{2}\right]ds.$$

Further writing F_1 in series form, on using (1.2.2), we find that

$$\Delta=\frac{1}{2\pi i}\int_L \theta(s)u^s\frac{\Gamma(\lambda)}{\Gamma(\mu)}\sum_{p,q=0}^{\infty}\frac{(\alpha',p-q)(\beta',p)(\gamma',q)(\delta',q)}{(\mu,p)(1,p)(1,q)}x^p y^q$$

$$.\sum_{j,k=0}^{\infty}\frac{(\lambda,j+k)(-\alpha,j)(-\beta,k)}{(\mu+p,j+k)(1,j)(1,k)}[-(z+1)\frac{t}{2}]^j[-(z-1)\frac{t}{2}]^k ds.$$

Now using relation (3.2.7) and (3.2.6), we find that

$$\Delta=\frac{1}{2\pi i}\int_L \theta(s)u^s\frac{\Gamma(\lambda)}{\Gamma(\mu)}\sum_{q=0}^{\infty}\frac{(\gamma',q)(\delta',q)}{(1-\alpha',q)(1,q)}(-y)^q$$

$$.\sum_{p,j,k=0}^{\infty}\frac{(\alpha'-q,p)(\lambda,j+k)(\beta',p)(-\alpha,j)(-\beta,k)}{(\mu,p+j+k)(1,p)(1,j)(1,k)}x^p[-(z+1)\frac{t}{2}]^j[-(z-1)\frac{t}{2}]^k ds,$$

which in the light of (1.2.27) and (1.2.35) provides (3.4.1).

In the following results we have given the trilateral generating relations involving some hypergeometric functions given in section 1.2 of chapter 1.

Theorem 3.4.2: Prove that

(i) $$\sum_{n=0}^{\infty} G_1\,[\delta+n,\beta',\beta'';x,y]P_n^{(\alpha,\beta)}(z)$$

$$.A_{p+2,q+2}^{m+2,l}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\alpha+1+n,0)\,(\beta+1+n,0)}^{(\gamma+n,0),(\delta+n,0),(a_j,\alpha_j)_{1,p}}\right]t^n$$

$$=\sum_{p=0}^{\infty}\frac{(\delta,p)(\beta'',p)}{(1-\beta',p)(1,p)}(-x)^p A_{p+2,q+2}^{m+2,l}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\alpha+1,0)\,(\beta+1,0)}^{(\gamma,0),(\delta,0),(a_j,\alpha_j)_{1,p}}\right]$$

$$.F_E[\delta+p,\delta+p,\delta+p,\beta'-p,\gamma,\gamma;1-\beta''-p,\alpha+1,\beta+1;-y,(z-1)\tfrac{t}{2},(z+1)\tfrac{t}{2}],$$

(3.4.2)

$r+s=1$, $|y|<s$, $|x|<r$, $|\arg(uv)|<\tfrac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively, G_1 is Horn's function as in (1.2.10) and F_E is Saran's function as in (1.2.23).

(ii) $$\sum_{n=0}^{\infty} H_3\,[\alpha',\lambda+n;\mu+n;x,y]P_n^{(\alpha-n,\beta-n)}(z)$$

$$.A_{p+1,q+1}^{m+1,l}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu+n,0)}^{(\lambda+n,0),(a_j,\alpha_j)_{1,p}}\right]t^n$$

$$=\sum_{p=0}^{\infty}\frac{(\alpha',2p)}{(\mu,p)(1,p)}(x)^p A_{p+1,q+1}^{m+1,l}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu,0)}^{(\lambda,0),(a_j,\alpha_j)_{1,p}}\right]$$

$$.F_N[\alpha'+2p,-\alpha,-\beta,\lambda+r,\lambda,\lambda+r;\mu,\mu+q,\mu+q;y,-(z+1)\tfrac{t}{2},-(z-1)\tfrac{t}{2}],$$

(3.4.3)

$|x|<1$, $|\arg(uv)|<\tfrac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively, H_3 is Horn's function as in (1.2.15) and F_N is Saran's function as in (1.2.26).

(iii) $$\sum_{n=0}^{\infty} H_6\,[\alpha',\lambda+n;\gamma';x,y]P_n^{(\alpha-n,\beta-n)}(z)$$

$$.A_{p+1,q+1}^{m,l+1}\left[v\Big|_{(1-\lambda-n,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}.(1-\mu-n,0)}\right]t^n$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha', 2p)}{(1-\lambda, p)(1, p)} (-x)^p A_{p+1,q+1}^{m,l+1} \left[v \Big|_{(1-\lambda,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}.(1-\mu,0)} \right]$$

$$.F_G[\lambda - p, \lambda - p, \lambda - p, \gamma, -\alpha, -\beta; 1 - \alpha' - 2p, \mu, \mu; -y, -(z+1)\tfrac{t}{2}, -(z-1)\tfrac{t}{2}],$$

(3.4.4)

$|x| < r$, $|y| < s$, $rs^2 + \tilde{s}\ 1$, $|\arg (uv)| < ½\ \pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively, H_6 is Horn's function as in (1.2.18) and F_G is Saran's function as in (1.2.24).

(iv) $\sum_{n=0}^{\infty} H_7\, [\alpha', \gamma + n, \delta + n; \delta'; x, y] P_n^{(\alpha,\beta)}(z)$

$$. A_{p+2,q+2}^{m+2,l} \left[v \Big|_{(b_j,\beta_j)_{1,q},(\alpha+1+n,0)\ (\beta+1+n,0)}^{(\gamma+n,0),(\delta+n,0),(a_j,\alpha_j)_{1,p}} \right] t^n$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha', 2p)}{(\delta', p)(1, p)} (-x)^p A_{p+2,q+2}^{m+2,l} \left[v \Big|_{(b_j,\beta_j)_{1,q},(\alpha+1,0)\ (\beta+1,0)}^{(\gamma,0),(\delta,0),(a_j,\alpha_j)_{1,p}} \right]$$

$$. F_K[\gamma, \gamma + q, \gamma + q, \delta + r, \delta, \delta + r; 1 - \alpha' - 2p, \alpha + 1, \beta + 1; -y, (z-1)\tfrac{t}{2}, (z+1)\tfrac{t}{2}],$$

(3.4.5)

$|y| < s$, $|x| < r$, $(s^{-1} - 1)^2 = 4r$, $|\arg (uv)| < ½\ \pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively, H_7 is Horn's function as in (1.2.19) and F_K is Saran's function as in (1.2.25).

Proof

(i) – (iv) Proceed as theorem 3.4.1 and using the results of section 3.2.

CHAPTER–4

DEFINITE AND INDEFINITE INTEGRALS INVOLVING A-FUNCTION

4.1 INTRODUCTION

In the study of boundary value problems Integral plays an important role. Its usefulness cannot be ignored in getting expansion formulae. These are also significant when integral equation, probability and statistical distribution are studied.

Ronghe [59], Saxena [63], Sharma [67], Goyal [24], Mohan [50], Srivastava [76], [75], Jaloree [31] and several other authors have evaluated some definite, indefinite and double integrals involving the generalized hypergeometric functions.

Looking importance and usefulness of integral in various fields we have established some new integrals of various types, which will be helpful in the study of boundary value problems, expansion formula, statistical distribution, probability and integral equation.

Most of the results in this chapter have been published in The Mathematics Education [35] in form of a research paper.

4.2 PREREQUISITE

In order to prove the results in the coming sections we shall need the following results:

From Shrivastava [70], we have

$$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)dx$$

$$=\frac{2^{\rho+\sigma+1}\Gamma(\rho+1)\Gamma(1+\sigma+n)\Gamma(-n-\sigma)}{n!\Gamma(2+n+\rho+\sigma)\Gamma(1+n+\rho)}$$

$$.\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)\Gamma(1+\beta+n+k)\Gamma(\alpha+\beta+n+k-\rho-\sigma)}{k!\,\Gamma(1+\alpha+\beta+n+k-\sigma)\Gamma(\alpha+k-\rho-\sigma)},$$

(4.2.1)

provided that $\mathrm{Re}(1 + \sigma) > 0$, $\mathrm{Re}(\rho + 1) > 0$, $\mathrm{Re}\,(1 + \alpha) > 0$, $\mathrm{Re}\,(-n - \sigma) > 0$, $\mathrm{Re}(\alpha + \beta + n + k - \rho - \sigma) > 0$.

$$\int_{-1}^{1} (1-x)^{\rho}\,(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{(-1)^n 2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(1+\rho+n)\Gamma(-n-\rho)}{n!\Gamma(2+n+\rho+\sigma)\Gamma(1+n+\sigma)}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma(1+\beta+n+k)\Gamma(1+\sigma+n+k)\Gamma(\alpha+\beta+n+k-\rho-\sigma)}{k!\,\Gamma(1+\alpha+\beta+n+k-\rho)\Gamma(\beta+k-\rho-\sigma)},$$

(4.2.2)

only if $\mathrm{Re}(\rho + 1) > 0$, $\mathrm{Re}(\sigma + 1) > 0$, $\mathrm{Re}\,(-\rho) > 0$, $\mathrm{Re}\,(-n - \rho) > 0$, $\mathrm{Re}(\alpha + \beta + n + k - \rho - \sigma) > 0$, $\mathrm{Re}\,(1 + \beta) > 0$.

$$\int_{-1}^{1} (1-x)^{\rho}\,(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(\rho+1)}{n!\Gamma(1+n+\rho)}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha+n+k)\Gamma(1+\rho+n+k)\Gamma(1-\beta+n+\sigma)}{k!\,\Gamma(2+k+\rho+\sigma)\Gamma(2+n+k+\alpha+\sigma)},$$

(4.2.3)

only if $\mathrm{Re}(1 + \rho) > 0$, $\mathrm{Re}(\sigma + 1) > 0$, $\mathrm{Re}(1 - \beta + n + \sigma) > 0$, $\mathrm{Re}\,(1 + \alpha) > 0$.

$$\int_{-1}^{1} (1-x)^{\rho}\,(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)dx =$$

$$\frac{2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(\rho+1)}{n!\Gamma(1+n+\sigma)}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma(1+\beta+n+k)\Gamma(1+\sigma+n+k)\Gamma(1-\alpha+k+\rho)}{k!\Gamma(2+k+\sigma+\rho)\Gamma(2+n+k+\beta+\rho)}, \qquad (4.2.4)$$

only if $\mathrm{Re}(1 + \sigma) > 0$, $\mathrm{Re}(\rho + 1) > 0$, $\mathrm{Re}(1 - \alpha + k + \rho) > 0$, $\mathrm{Re}\,(1 + \beta) > 0$.

$$\int_{-1}^{1} (1-x)^{\rho}\,(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(1+\rho+n)\Gamma(-n-\rho)}{n!\Gamma(2+n+\rho+\sigma)\Gamma(-\rho-\sigma-1)}$$

$$\cdot\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)\Gamma(-1-\rho-\sigma-k)\Gamma(1-\beta+\sigma+k)}{k!\,\Gamma(-\beta+k-n-\rho)\Gamma(-\alpha-\beta-n+k+\sigma)},$$

(4.2.5)

provided that Re(1 + 2n + α + β) > 0, Re(− 2n − α − β) > 0, Re(1 + σ) > 0, Re(ρ + 1) > 0, Re(− α − β − n + k) > 0, Re (− n − ρ) > 0, Re(− 1 − ρ − σ + k) > 0, Re(1 − β + σ + k) > 0.

$$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)dx$$

$$=\frac{(-1)^n 2^{\rho+\sigma+1}\Gamma(\rho+1)\Gamma(1+\sigma+n)\Gamma(-n-\sigma)}{n!\Gamma(2+n+\rho+\sigma)\Gamma(-1-\rho-\sigma)}$$

$$\cdot\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)\Gamma(-1-\rho-\sigma-k)\Gamma(1-\beta+\rho+k)}{k!\,\Gamma(-\beta-n+k-\sigma)\Gamma(-\alpha-\beta-n+k+\rho)},$$

(4.2.6)

only if Re(1 + 2n + α + β) > 0, Re(− 2n − α − β) > 0, Re(1 + σ) > 0, Re(ρ + 1) > 0, Re(− α − β − n + k) > 0, Re (n − σ) > 0, Re(1 − β + ρ + k) > 0, Re(− 1 − ρ − σ + k) > 0.

From Bajpai [8], we have

$$\int_0^{\infty} x^{\sigma-1}e^{-1/x}y_n(x;a,1)dx=\frac{(-1)^n\Gamma(-\sigma-n)\Gamma(2+\sigma-a)}{\Gamma(2+\sigma-a-n)}, \qquad (4.2.7)$$

where $y_n(x;a,1)$ is generalized Bessel function, Re(σ) < 0, Re(a − σ) < 2, $\sigma \neq -1,-2,-3,\ldots$.

From Whitaker and Watson [84], we have

$$\int_0^{\pi/2}\frac{2^{\alpha+\beta+1}}{\pi}e^{i(\alpha-\beta)\theta}(\cos\theta)^{\alpha+\beta}d\theta=\frac{\pi\,\Gamma(\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}, \qquad (4.2.8)$$

$\mathrm{Re}(\alpha+\beta) > -1$.

From MacRobert [44], we have

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}d\theta=\frac{e^{\pi i\alpha/2}\,\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad (4.2.9)$$

$\text{Re}(\alpha) > 0, Re(\beta) > 0.$

From Rainville [56], we have

$$\int_0^\infty x^{\alpha-1}e^{-x}dx = \Gamma(\alpha),\ \ \text{Re}(\alpha) > 0; \tag{4.2.10}$$

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}dx = t^{\rho+\sigma-1}\frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}, \text{Re}(\rho) > 0, Re(\sigma) > 0. \tag{4.2.11}$$

From Erdelyi [21]:

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, d\theta \ = (1/2)\frac{\Gamma(\rho+1/2)\Gamma(\sigma+1/2)}{\Gamma(\rho+\sigma+1/2)}, \tag{4.2.12}$$

provided that ρ > 0, σ > 0.

From Nielsen [51]:

$$\int_0^\pi (\sin\theta)^\rho \cos u\theta\, d\theta = \frac{\pi\Gamma(1+\rho)\cos(\frac{\pi u}{2})}{2^\rho\Gamma(1+\frac{\rho+u}{2})\Gamma(1+\frac{\rho-u}{2})} \tag{4.2.13}$$

provided that ρ > −1.

$$\int_0^\pi (\sin\theta)^\rho \sin u\theta\, d\theta = \frac{\pi\Gamma(1+\rho)\sin(\frac{\pi u}{2})}{2^\rho\Gamma(1+\frac{\rho+u}{2})\Gamma(1+\frac{\rho-u}{2})} \tag{4.2.14}$$

provided that ρ > −1.

From Mishra [49]:

$$\int_0^\pi (\sin x)^{\omega-1}\, e^{imx}\ {}_PF_Q[{}^{\alpha_P:c(\sin x)^{2h}}_{\beta_Q}]\ {}_UF_V[{}^{\gamma_U:d(\sin x)^{2k}}_{\delta_V}]\, dx$$

$$= \frac{\pi e^{im\pi}}{2^{\omega-1}}\sum_{r,t=0}^{\infty}\frac{(\alpha_P)_r c^r(\gamma_U)_t d^t\Gamma(\omega+2hr+2kt)}{2^{2(hr+kt)}(\beta_Q)_r r!\, (\delta_V)_t\, t!\, \Gamma(\frac{\omega+2hr+2kt\pm m+1}{2})}, \tag{4.2.15}$$

where h and k are positive integers, Q > P (or Q + 1 = P, |c| < 1), V < U (or V + 1 = U, |d| < 1), none of the β_Q and δ_V = 0 or < 0 and Re (ω) > 0.

From MacRobert [45]:

$$\int_0^\pi \sin(2n+1)\theta\, (\sin\theta)^{1-2u}\, d\theta = \frac{\sqrt{\pi}\Gamma(\frac{3}{2}-u)\Gamma(u+n)}{\Gamma(u)\Gamma(2-u+n)} \tag{4.2.16}$$

where Re(3 − 2u) > 0, n = 0, 1, 2, …;

$$\int_0^{\pi} \cos n\theta\,(\sin\theta/2)^{-2u}\,d\theta = \frac{\sqrt{\pi}\,\Gamma(u+n)\Gamma(\frac{1}{2}-u)}{\Gamma(u)\Gamma(1-u+n)} \tag{4.2.17}$$

where Re(1 – 2u) > 0, n = 0, 1, 2, …

4.3 DEFINITE AND INDEFINITE INTEGRALS

Following Ronghe [59], Saxena [63], Sharma [67], Goyal [24], Mohan [50], Srivastava [75, 76], Jaloree [31] and other authors, in this section we have evaluated some definite and indefinite integrals involving the A-function of one variable with the help of results given in the previous section.

Theorem 4.3.1

Prove that if Re(1 + ρ + μ) > 0, Re(1 + α) > 0, Re(σ + n + δ + 1) > 0, Re(α + β + n + k – ρ – σ – (μ + δ)) > 0, Re(1 + n + k + ρ + μ) > 0, Re(– n – σ – δ) > 0, |arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively, then

$$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\,A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dx$$

$$=\frac{2^{\rho+\sigma+1}}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)}{k!}\Gamma(1+\beta+n+k)$$

$$A_{p+4,q+4}^{m+2,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\sigma,\delta),(1-\alpha-\beta-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},(2+\rho+\sigma+n,\mu+\delta),(1+n+\rho,\mu)}^{(1+\rho,\mu),(1+\sigma+n,\delta),(a_j,\alpha_j)_{1,p},(-\alpha-\beta-n-k+\sigma,\delta),(1-\alpha-k+\rho+\sigma,\mu+\delta)}\right], \tag{4.3.1}$$

Proof

Replace the A-function by its equivalent counter integral in L.H.S. of (4.3.1) as given in (1.2.35), we get

$$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\,.[\frac{1}{2\pi i}\int_L\ \theta(s)\,z^s(1-x)^{\mu s}(1+x)^{\delta s}ds]dx.$$

Under the given condition, changing the order of integration is valid, we arrive at

$$\frac{1}{2\pi i}\int_L\ \theta(s)\,z^s\left[\int_{-1}^{1}(1-x)^{\rho+\mu s}\,(1+x)^{\sigma+\delta s}P_n^{(\alpha,\beta)}(x)dx\right]ds$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\, z^s \frac{2^{\rho+\mu s+\sigma+\delta s+1}\Gamma(\rho+\mu s+1)\Gamma(1+\sigma+\delta s+n)\Gamma(-n-\sigma-\delta s)}{n!\Gamma(2+n+\rho+\mu s+\sigma+\delta s)\Gamma(1+n+\rho+\mu s)}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha+n+k)\Gamma(1+\beta+n+k)\Gamma(\alpha+\beta+n+k-\rho+\mu s-\sigma-\delta s)}{k!\Gamma(1+\alpha+\beta+n+k-\rho-\mu s)\Gamma(\alpha+k-\rho-\rho-\mu s)} ds, \qquad \text{(By (4.2.1))}$$

$$= \frac{2^{\rho+\sigma+1}}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)}{k!}\Gamma(1+\beta+n+k)$$

$$A_{p+4,q+4}^{m+2,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\sigma,\delta),(1-\alpha-\beta-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},(2+\rho+\sigma+n,\mu+\delta),(1+n+\rho,\mu)}^{(1+\rho,\mu),(1+\sigma+n,\delta),(a_j,\alpha_j)_{1,p},(-\alpha-\beta-n-k+\sigma,\delta),(1-\alpha-k+\rho+\sigma,\mu+\delta)}\right]$$

(Interpreting with (1.2.35)).

Which we have to prove.

Theorem 4.3.2: Prove that

(i) only if Re$(1 + \sigma + \delta) > 0$, Re$(1 + \beta) > 0$, Re$(\rho + n + \mu + 1) > 0$, Re$(\alpha + \beta + n + k - \rho - \sigma - (\mu + \delta)) > 0$, Re$(1 + n + k + \sigma + \mu) > 0$, Re$(- n - \rho - \mu) > 0$, $|\arg (uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively then

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\, A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dx$$

$$= (-1)^n\frac{2^{\rho+\sigma+1}}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(1+\beta+n+k)}{k!}$$

$$A_{p+5,q+4}^{m+3,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\rho,\mu),(1-\alpha-\beta-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},(2+\rho+\sigma+n,\mu+\delta),(1+n+\sigma,\delta)}^{(1+\sigma,\delta),(1+\rho+n,\mu),(1+n+k+\sigma,\delta),(a_j,\alpha_j)_{1,p},(-\alpha-\beta-n-k+\rho,\mu),(1-\beta-k+\rho+\sigma,\mu+\delta)}\right],$$

(4.3.2)

(ii) If Re$(\alpha + 1) > 0$, Re$(- \beta + k + \sigma + \delta + 1) > 0$, Re$(\rho + \mu + 1) > 0$, Re$(\sigma + \delta + 1) > 0$, $|\arg (uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively, then

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\, A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dx$$

$$= \frac{2^{\rho+\sigma+1}}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)}{k!}$$

$$A^{m+4,l}_{p+4,q+3}\left[z2^{\mu+\delta}\Big|^{(1+\rho,\mu),(1+\sigma,\delta),(1+n+k+\rho,\mu),(1-\beta+n+\sigma,\delta),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1+n+\rho,\mu),(2+k+\rho+\sigma,\mu+\delta),(2+\alpha+n+k+\sigma,\delta)}\right], \quad (4.3.3)$$

(iii) If $\text{Re}(-\alpha+k+\rho+\mu+1)>0$, $\text{Re}(\alpha+1)>0$, $\text{Re}(\rho+\mu+1)>0$, $\text{Re}(\sigma+\delta+1)>0$, $|\arg(uz)|<\frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively, then

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\,A^{m,l}_{p,q}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dx$$

$$=\frac{2^{\rho+\sigma+1}}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(1+\beta+n+k)}{k!}$$

$$A^{m+4,l}_{p+4,q+3}\left[z2^{\mu+\delta}\Big|^{(1+\rho,\mu),(1+\sigma,\delta),(1+n+k+\sigma,\delta),(1-\alpha+k+\rho,\mu),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1+n+\sigma,\delta),(2+k+\rho+\sigma,\mu+\delta),(2+\beta+n+k+\rho,\mu)}\right], \quad (4.3.4)$$

(iv) $$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\,A^{m,l}_{p,q}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dx$$

$$=\frac{2^{\rho+\sigma+1}}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)}{k!}$$

$$A^{m+3,l+2}_{p+5,q+4}\left[z2^{\mu+\delta}(1+x)^{\delta}\Big|^{(1+\sigma,\delta),(1+\rho+n,\mu),(1-\beta+\sigma+k,\delta),(a_j,\alpha_j)_{1,p},(2+\rho+\sigma,\mu+\delta),(1+\beta+n-k+\rho,\mu)}_{(1+n+\rho,\mu),(2+k+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},(2+n+\rho+\sigma,\mu+\delta),(-\alpha-\beta-n+k+\sigma,\delta)}\right],$$

(4.3.5)

provided that $\text{Re}(1+\alpha+\beta)>0$, $\text{Re}(-\alpha-\beta-2n)>0$, $\text{Re}(-\alpha-\beta-n+k)>0$, $\text{Re}(1+\sigma+\delta)>0$, $\text{Re}(1+n+\rho+\mu)>0$, $\text{Re}(-1+k-\rho-\sigma-(\mu+\delta))>0$, $\text{Re}(1-\beta+\sigma+k+\delta)>0$, $\text{Re}(-\rho-n-\mu)>0$, $|\arg(uz)|<\frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(v) $$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\,A^{m,l}_{p,q}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dx$$

$$=\frac{(-1)^n 2^{\rho+\sigma+1}}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)}{k!}$$

$$A_{p+5,q+4}^{m+3,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\sigma,\delta),(2+k+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},(2+n+\rho+\sigma,\mu+\delta),(-\alpha-\beta-n+k+\rho,\mu)}^{(1+\rho,\mu),(1+\sigma+n,\delta),(1-\beta+\rho+k,\mu),(a_j,\alpha_j)_{1,p},(2+\rho+\sigma,\mu+\delta),(1+\beta+n-k+\sigma,\delta)}\right],$$

(4.3.6)

provided that Re(1 + α + β) > 0, Re(− α − β − 2n) > 0, Re(− α − β − n + k) > 0, Re(1 + ρ + μ) > 0, Re(1 + n + σ + δ) > 0, Re(− 1 + k − ρ − σ + (μ + δ)) > 0, Re(1 − β + ρ + k + μ) > 0, Re(− σ − n − δ) > 0, |arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

(i) Proceed as in theorem 4.3.1 and using the results (4.2.2)

(ii) It can be established using (4.2.3).

(iii) It can be established using (4.2.4).

(iv) It can be established using (4.2.5).

(v) It can be established using (4.2.6).

Theorem 4.3.3: Prove that

$$\int_0^\infty x^{\rho-1}e^{-1/x}y_m(x;a,1)\,A_{p,q}^{k,l}\left[zx^{\lambda}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dx$$

$$=(-1)^m A_{p+1,q+2}^{k+1,l+1}\left[z\Big|_{(1+\rho+m,\lambda),(b_j,\beta_j)_{1,q},(2-a-m+\rho,\lambda)}^{(2-a+\rho,\lambda),(a_j,\alpha_j)_{1,p}}\right], \quad (4.3.7)$$

where Re(λ) > 0, Re(ρ) < 0, Re(a − ρ) < 2, $\rho \neq -1,-2,-3,....$ and |arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To establish (4.3.7), replace the A-function by its equivalent counter integral as given in (1.2.35), we get

$$\int_0^\infty x^{\rho-1}e^{-1/x}y_m(x;a,1)\left[\frac{1}{2\pi i}\int_L \theta(s)\,z^s x^{\lambda s}ds\right]dx.$$

Under the given condition, changing the order of integration is valid, we arrive at

$$\frac{1}{2\pi i}\int_L \theta(s)\, z^s \left[\int_0^\infty x^{\rho+\lambda s-1} e^{-\frac{1}{x}} y_m(x;a,1)\, dx\right] ds.$$

Now evaluate the integral in the braces using (4.2.7) and finally interpret it with (1.2.35), we get (4.3.7).

The following theorem (4.3.4) can be established easily in the view of (4.2.7) exactly on the same lines as given above respectively.

Theorem 4.3.4: Prove that

$$\int_0^\infty x^{\rho-1} e^{-1/x} y_m(x;a,1)\, A_{p,q}^{k,l}\left[zx^{-\lambda}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] dx$$

$$= (-1)^m A_{p+2,q+1}^{k+1,l+1}\left[zx^{\lambda}\Big|_{(-1-\rho+a,\lambda),(b_j,\beta_j)_{1,q}}^{(-\rho-m,\lambda),(a_j,\alpha_j)_{1,p},(-1-\rho+a+m,\lambda)}\right], \tag{4.3.8}$$

where $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(\rho) < 0$, $\mathrm{Re}(a-\rho) < 2$, $\rho \neq -1,-2,-3,\ldots.$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Theorem 4.3.5: Prove that

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta A_{p,q}^{m,n}\left[x.\sin^{2h}\theta\cos^{2k}\theta\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] d\theta$$

$$= (1/2)\, A_{p+2,q+1}^{m+2,n}\left[x\Big|_{(b_j,\beta_j)_{1,q},(1+\rho+\sigma,h+k)}^{\left(\frac{1}{2}+\rho,h\right),\left(\frac{1}{2}+\sigma,k\right),(a_j,\alpha_j)_{1,p}}\right], \tag{4.3.9}$$

provided that $\rho > 0$, $\sigma > 0$, $|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To establish (4.3.9), use (1.2.35) and after changing the order of integration, we get

$$\frac{1}{2\pi i}\int_L x^s\,\theta(s)\left[\int_0^{\frac{\pi}{2}} \sin^{2(\rho+hs)}\theta \cos^{2(\sigma+ks)}\theta\, d\theta\right] ds.$$

Now evaluate the integral in the braces by using the result (4.2.12) and finally interpreting in view of (1.2.35), the integral (4.3.9) is obtained.

Theorem 4.3.6: Prove that

(i) $$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, A_{p,q}^{m,n}\left[x.\sin^{-2h}\theta\cos^{2k}\theta\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] d\theta$$

$$= (1/2)\, A_{p+2,q+1}^{m+1,n+1}\left[x\Big|_{\left(\frac{1}{2}-\rho,h\right),(b_j,\beta_j)_{1,q}}^{\left(\frac{1}{2}+\sigma,k\right),(a_j,\alpha_j)_{1,p},(-\rho-\sigma,h-k)}\right], \qquad (4.3.10)$$

provided that $\rho > 0$, $\sigma > 0$, $|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(ii) $$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, A_{p,q}^{m,n}\left[x.\sin^{2h}\theta\cos^{-2k}\theta\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] d\theta$$

$$= (1/2)\, A_{p+1,q+2}^{m+1,n+1}\left[x\Big|_{\left(\frac{1}{2}-\sigma,k\right),(b_j,\beta_j)_{1,q},(1+\rho+\sigma,h-k)}^{\left(\frac{1}{2}+\rho,h\right),(a_j,\alpha_j)_{1,p}}\right], \qquad (4.3.11)$$

provided that $\rho > 0$, $\sigma > 0$, $|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(iii) $$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, A_{p,q}^{m,n}\left[x.\sin^{-2h}\theta\cos^{-2k}\theta\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] d\theta$$

$$= (1/2)\, A_{p+1,q+2}^{m,n+2}\left[x\Big|_{\left(\frac{1}{2}-\rho,h\right),\left(\frac{1}{2}-\sigma,k\right),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(-\rho-\sigma,h+k)}\right], \qquad (4.3.12)$$

provided that $\rho > 0$, $\sigma > 0$, $|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

The proof of the integrals (4.3.10) to (4.3.12) would run parallel to what we have obtained in theorem 4.3.5.

Theorem 4.3.7: Prove that

$$\int_0^{\pi} \cos(ux)\,(\sin x/2)^{-2\omega_1}\times \quad A_{p,q}^{m,n}\left[z.(\sin x/2)^{2h}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] dx$$

$$= \sqrt{(\pi)}A_{p+2,q+2}^{m+1,n+1}\left[z\Big|_{(1-\omega_1-u,h),(b_j,\beta_j)_{1,q},(1-\omega_1+u,h)}^{\left(\frac{1}{2}-\omega_1,h\right),(a_j,\alpha_j)_{1,p},(1-\omega_1,h)}\right], \qquad (4.3.13)$$

provided that h > 0, $\omega_1 > 0$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

We can express the integrand which contained A-function in term of Mellin-Barnes type integral (1.2.35). Due to absolute convergence of the integrals involved in the process we can interchange the order of integrations, which is justifiable, we get

$$\frac{1}{2\pi i}\int_L x^s\,\theta(s)\left[\int_0^{\pi}\cos(ux)\left(\sin\frac{x}{2}\right)^{-2(\omega_1-hs)}dx\right]ds.$$

Now evaluate the integral in the braces using the formula given in Bajpai [6]:

$$\int_0^{\pi}\cos(ux)\left(\sin\frac{x}{2}\right)^{-2\omega_1}dx = \sqrt{\pi}\,\frac{\Gamma(\omega_1+u)\Gamma(\frac{1}{2}-\omega_1)}{\Gamma(1-\omega_1+u)\Gamma(\omega_1)}$$

and applying (1.2.35), the definition of the A-function, we get the result (4.3.13).

Theorem 4.3.8: Prove that

$$\int_{-1}^{1}(1-y)^{\omega_2}(1+y)^b P_\nu^{(a,b)}(y)\times\quad A_{p,q}^{m,n}\left[z.(1-y)^{-k}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dy$$

$$=\frac{2^{b+\omega_2+1}\Gamma[1+\upsilon+b]}{\upsilon!}$$

$$\times\quad A_{p+2,q+2}^{m+1,n+1}\left[z.2^{-k}\Big|_{(-\omega_2,k),(b_j,\beta_j)_{1,q},(a-\omega_2,k)}^{(a-\omega_2+\upsilon,k),(a_j,\alpha_j)_{1,p},(-1-b-\omega_2-\upsilon,k)}\right], \qquad (4.3.14)$$

provided that k > 0, $\omega_2 > 0$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

We can express the integrand which contained A-function in term of Mellin-Barnes type integral (1.2.35). Due to absolute convergence of the integrals involved in the process we can interchange the order of integrations, which is justifiable, we get

$$\frac{1}{2\pi i}\int_L x^s\,\theta(s)\left[\int_{-1}^{1}(1-y)^{\omega_2-ks}(1+y)^b P_\nu^{(a,b)}(y)\,dy\right]ds.$$

Now evaluate the integral in the braces using the formula given in Bajpai [4]:

$$\int_{-1}^{1}(1-y)^{\omega_2}(1+y)^{b}P_{\nu}^{(a,b)}(y)\,dy=\frac{2^{b+\omega_2+1}\Gamma(1+\nu+b)}{\nu!}\frac{\Gamma(1+\omega_2)\Gamma(a-\omega_2+\nu)}{\Gamma(a-\omega_2)\Gamma(2+b+\omega_2+\nu)},$$

and applying (1.2.35), to get (4.3.14).

Theorem 4.3.9: Prove that

$$\int_0^{\pi}(\sin\theta)^{\rho}\cos u\theta \times A_{p,q}^{m,n}\left[z.(\sin\theta)^{-2\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]d\theta$$

$$=\sqrt{\pi}\cos\left(\frac{\pi u}{2}\right)A_{p+2,q+2}^{m,n+2}\left[z\Big|_{\left(\frac{1-\rho}{2},\delta\right),\left(-\frac{\rho}{2},\delta\right),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},\left(-\frac{\rho+u}{2},\delta\right),\left(-\frac{\rho-u}{2},\delta\right)}\right], \quad (4.3.15)$$

provided that $\rho > -1$, $\delta > 0$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

We can express the integrand which contained A-function in term of Mellin-Barnes type integral (1.2.35). Due to absolute convergence of the integrals involved in the process we can interchange the order of integrations, which is justifiable, we get

$$\frac{1}{2\pi i}\int_L x^s\,\theta(s)\left[\int_0^{\pi}(\sin\theta)^{\rho-2\delta s}\cos u\theta\,d\theta\right]ds.$$

Now evaluate the integral in the braces using the formula (4.2.13), and applying (1.2.35), to get (4.3.15).

Theorem 4.3.10: Prove that

$$\int_0^{\pi}(\sin\theta)^{\rho}\sin u\theta \times A_{p,q}^{m,n}\left[z.(\sin\theta)^{-2\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]d\theta$$

$$=\sqrt{\pi}\sin\left(\frac{\pi u}{2}\right)A_{p+2,q+2}^{m,n+2}\left[z\Big|_{\left(\frac{1-\rho}{2},\delta\right),\left(-\frac{\rho}{2},\delta\right),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},\left(-\frac{\rho+u}{2},\delta\right),\left(-\frac{\rho-u}{2},\delta\right)}\right], \quad (4.3.16)$$

provided that $\rho > -1$, $\delta > 0$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

On applying (4.2.14) instead of the (4.2.13) in theorem (4.3.15) theorem (4.3.16) is established.

Theorem 4.3.11: Prove that

$$\int_0^{\pi} (\sin x)^{\omega-1}\, e^{imx}\ {}_PF_Q\left[{}^{\alpha_P:c(\sin x)^{2h}}_{\beta_Q}\right]\ {}_UF_V\left[{}^{\gamma_U:d(\sin x)^{2k}}_{\delta_V}\right]$$

$$\times \quad A^{m,n}_{p,q}\left[z.(\sin x)^{2\lambda}\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right] dx$$

$$= \sqrt{\pi} e^{im\pi/2} \sum_{r,t=0}^{\infty} \frac{(\alpha_P)_r c^r (\gamma_U)_t d^t}{(\beta_Q)_r r!\,(\delta_V)_t\, t!}$$

$$\times \quad A^{m+2,n}_{p+2,q+2}\left[z\Big|{}^{\left(\frac{\omega+2hr+2kt}{2},\lambda\right),\left(\frac{\omega+2hr+2kt+1}{2},\lambda\right),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},\left(\frac{\omega+2hr+2kt\pm m+1}{2},\lambda\right)}\right] \qquad (4.3.17)$$

where h and k are positive integers, Q > P (or Q + 1 = P, |c| < 1), V < U (or V + 1 = U, |d| < 1), none of the β_Q and δ_V = 0 or < 0 and Re (ω) > 0 and |arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

We can express the integrand which contained A-function in term of Mellin-Barnes type integral (1.2.35). Due to absolute convergence of the integrals involved in the process we can interchange the order of integrations, which is justifiable, we get

$$\frac{1}{2\pi i}\int_L x^s\,\theta(s)\left[\int_0^{\pi}(\sin x)^{\omega+2\lambda s-1}\, e^{imx}\ {}_PF_Q\left[{}^{\alpha_P:c(\sin x)^{2h}}_{\beta_Q}\right]\ {}_UF_V\left[{}^{\gamma_U:d(\sin x)^{2k}}_{\delta_V}\right]dx\right] ds.$$

Evaluate the integral in the braces using the formula (4.2.15) and using Gamma-function's multiplication formula Erdelyi [36, p.4, (11)], we get

$$\sqrt{\pi} e^{im\pi/2} \sum_{r,t=0}^{\infty} \frac{(\alpha_P)_r c^r (\gamma_U)_t d^t}{(\beta_Q)_r r!\,(\delta_V)_t\, t!}$$

$$\frac{1}{2\pi i}\int_L x^s\,\theta(s)\left[\frac{\Gamma\left(\frac{\omega+2hr+2kt}{2}+\lambda s\right)\Gamma\left(\frac{\omega+2hr+2kt+1}{2}+\lambda s\right)}{\Gamma\left(\frac{\omega+2hr+2kt\pm m+1}{2}+\lambda s\right)}\right] ds$$

Now applying (1.2.35), the value of the integral (4.3.17) is obtained.

Theorem 4.3.12: Prove that

$$\int_0^{\pi} \sin(2n+1)\theta\,(\sin\theta)^{1-2u} \times A^{m,n}_{p,q}\left[z.\sin^{2h}\theta\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right] d\theta$$

$$= \sqrt{\pi} A^{m+1,n+1}_{p+2,q+2}\left[z\middle|^{\left(\frac{3}{2}-u,h\right),(a_j,\alpha_j)_{1,p},(1-u,h)}_{(1-u-n,h),(b_j,\beta_j)_{1,q},(2-u+n,h)}\right], \tag{4.3.18}$$

provided that Re(3 – 2u) > 0, n = 0, 1, 2, …, h > 0 and |arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

We can express the integrand which contained A-function in term of Mellin-Barnes type integral (1.2.35). Due to absolute convergence of the integrals involved in the process we can interchange the order of integrations, which is justifiable, we get

$$\frac{1}{2\pi i}\int_L x^s\,\theta(s)\left[\int_0^{\pi}\sin(2n+1)\,\theta\,(\sin\theta)^{1-2(u-hs)}\,d\theta\right]ds.$$

Now evaluate the integral in the braces using the formula (4.2.16), we have

$$\sqrt{\pi}\,\frac{1}{2\pi i}\int_L x^s\,\theta(s)\left[\frac{\Gamma\left(\frac{3}{2}-u+hs\right)\Gamma(u+n-hs)}{\Gamma(u-hs)\Gamma(2-u+n+hs)}\right]ds$$

On applying (1.2.35), the integral (4.3.18) is obtained.

Theorem 4.3.13: Prove that

$$\int_0^{\pi}\cos n\theta\,(\sin\theta/2)^{-2u}\times \quad A^{m,n}_{p,q}\left[z.\sin^{2h}(\theta/2)\middle|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]d\theta$$

$$= \sqrt{\pi} A^{m,n}_{p,q}\left[z\middle|^{\left(\frac{1}{2}-u,h\right),(a_j,\alpha_j)_{1,p},(1-u,h)}_{(1-u-n,h),(b_j,\beta_j)_{1,q},(1-u+n,h)}\right], \tag{4.3.19}$$

provided that Re(1 – 2u) > 0, n = 0, 1, 2, …, h > 0 and |arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

Proceed as in theorem 3.4.2 and using the result (4.2.17)

4.4 DOUBLE INTEGRALS

In this section, we have evaluated nine double integrals involving A-function of one variable by taking the help of some results given in section (4.2). We have proved theorem 4.4.1 and other results can be easily proved by adopting the same lines.

Theorem 4.4.1: Prove that

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{2^{\alpha+\beta+1}}{\pi}e^{i(\alpha-\beta)x}(\cos x)^{\alpha+\beta}e^{i(\sigma+\rho)y}(\sin y)^{\sigma-1}(\cos y)^{\rho-1}$$

$$.A_{p,q}^{m,n}[z\left(2e^{i(x+y)}\cos x\sin y\right)^{\lambda}\left(2e^{i(y-x)}\cos x\cos y\right)^{\mu}]dxdy$$

$$=\frac{\pi e^{\frac{i\pi\sigma}{2}}}{2^{\alpha+\beta+1}}A_{p+3,q+3}^{m+3,n}\left[\frac{ze^{\frac{i\pi\lambda}{2}}}{2^{\lambda+\mu}}\Big|_{(b_j,\beta_j)_{1,q},(\sigma+\rho,\lambda+\mu),(1+\alpha,\lambda),(1+\beta,\mu),}^{(1+\alpha+\beta,\lambda+\mu),(\sigma,\lambda),(\rho,\mu),(a_j,\alpha_j)_{1,p}}\right], \qquad (4.4.1)$$

provided that Re $(\alpha+\beta) > -1$, Re$(\sigma) > 0$, Re$(\rho) > 0$, $\lambda \geq 0$ and $\mu \geq 0$, $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

We can express the integrand which contained A-function in term of Mellin-Barnes type integral (1.2.35). Due to absolute convergence of the integrals involved in the process we can interchange the order of integrations, which is justifiable, we get

$$I=\frac{1}{2\pi i}\int_L \theta(s)\,z^s.$$

$$\left[\int_0^{\frac{\pi}{2}}\frac{2^{(\alpha+\lambda s)+(\beta+\mu s)+1}}{\pi}e^{ix[(\alpha+\lambda s)-(\beta+\mu s)]}(\cos x)^{[(\alpha+\lambda s)+(\beta+\mu s)]}dx\right]$$

$$.\left[\int_0^{\frac{\pi}{2}}e^{iy[(\sigma+\lambda s)+(\rho+\mu s)]}(\sin y)^{(\sigma+\lambda s)-1}(\cos y)^{(\rho+\mu s)-1}dy\right]ds$$

Now using the results (4.2.8), (4.2.9) and interpreting it with the help of (1.2.35), to get R.H.S. of (4.4.1).

Theorem 4.4.2: Prove that

(i) $$\int_0^{t}\int_0^{\pi/2}\frac{2^{\alpha+\beta+1}}{\pi}x^{\rho-1}(t-x)^{\sigma-1}e^{i(\alpha-\beta)y}(\cos y)^{\alpha+\beta}$$

$$.A_{p,q}^{m,n}[z\left(2xe^{iy}\cos y\right)^{\lambda}\left(2(t-x)e^{-iy}\cos y\right)^{\mu}]dxdy$$

$$=\frac{\pi t^{\rho+\sigma-1}}{2^{\alpha+\beta+1}}A_{p+3,q+3}^{m+3,n}\left[z(t/2)^{\lambda+\mu}\Big|_{(b_j,\beta_j)_{1,q},(\sigma+\rho,\lambda+\mu),(1+\alpha,\lambda),(1+\beta,\mu),}^{(1+\alpha+\beta,\lambda+\mu),(\sigma,\lambda),(\rho,\mu),(a_j,\alpha_j)_{1,p}}\right], \qquad (4.4.2)$$

where Re $(\alpha+\beta) > -1$, $\text{Re}(\sigma) > 0$, $\text{Re}(\rho) > 0$, $\lambda \geq 0$ and $\mu \geq 0$, $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(ii) $$\int_0^\infty \int_0^{\pi/2} \frac{2^{\alpha+\beta+1}}{\pi} x^{\rho-1} e^{-x} e^{i(\alpha-\beta)y} (\cos y)^{\alpha+\beta}$$

$$.A_{p,q}^{m,n}[z(2xe^{iy}\cos y)^{\lambda}(2e^{-iy}\cos y)^{\mu}]dxdy$$

$$= \frac{\pi}{2^{\alpha+\beta+1}} A_{p+2,q+2}^{m+2,n}\left[\frac{z}{2^{\lambda+\mu}}\Big|_{(b_j,\beta_j)_{1,q},(1+\alpha,\lambda),(1+\beta,\mu)}^{(1+\alpha+\beta,\lambda+\mu),(\sigma,\lambda),(a_j,\alpha_j)_{1,p}}\right], \quad (4.4.3)$$

provided that Re $(\alpha+\beta) > -1$, $\text{Re}(\sigma) > 0$, $\lambda \geq 0$ and $\mu \geq 0$, $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(iii) $$\int_0^t \int_0^{\pi/2} x^{\rho-1}(t-x)^{\sigma-1} e^{i(\alpha+\beta)y} (\sin y)^{\alpha-1} (\cos y)^{\beta-1}$$

$$.A_{p,q}^{m,n}[z(xe^{iy}\sin y)^{\lambda}((t-x)e^{iy}\cos y)^{\mu}]dxdy$$

$$= e^{\frac{i\pi\alpha}{2}} t^{\rho+\sigma-1} A_{p+4,q+2}^{m+4,n}\left[zt^{\lambda+\mu} e^{\frac{i\pi\lambda}{2}}\Big|_{(b_j,\beta_j)_{1,q},(\sigma+\rho,\lambda+\mu),(\alpha+\beta,\lambda+\mu)}^{(\alpha,\lambda),(\beta,\mu),(\rho,\lambda),(\sigma,\mu),(a_j,\alpha_j)_{1,p}}\right], \quad (4.4.4)$$

where Re $(\alpha) > 0$, Re $(\beta) > 0$, $\text{Re}(\sigma) > 0$, $\text{Re}(\rho) > 0$, $\lambda \geq 0$ and $\mu \geq 0$, $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(iv) $$\int_0^\infty \int_0^{\pi/2} x^{\rho-1} e^{-x} e^{i(\alpha+\beta)y} (\sin y)^{\alpha-1} (\cos y)^{\beta-1}$$

$$.A_{p,q}^{m,n}[z(xe^{iy}\sin y)^{\lambda}(e^{iy}\cos y)^{\mu}]dxdy$$

$$= e^{\frac{i\pi\alpha}{2}} A_{p+3,q+1}^{m+3,n}\left[ze^{\frac{i\pi\lambda}{2}}\Big|_{(b_j,\beta_j)_{1,q},(\alpha+\beta,\lambda+\mu)}^{(\sigma,\lambda),(\alpha,\lambda),(\beta,\mu),(a_j,\alpha_j)_{1,p}}\right], \quad (4.4.5)$$

provided that Re $(\alpha) > 0$, Re $(\beta) > 0$, $\text{Re}(\sigma) > 0$, $\lambda \geq 0$ and $\mu \geq 0$, $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(v) $$\int_0^\infty \int_0^\infty x^{\rho-1} e^{-1/x} y_m(x;a,1) y^{\sigma-1} e^{-1/y} y_n(y;b,1) A_{p,q}^{k,l}[zx^{\lambda}y^{\mu}]\, dxdy$$

$$= (-1)^{m+n} A_{p+2,q+4}^{k+2,l+2}\left[z\Big|_{(1+\rho+m,\lambda),(1+\sigma+n,\mu),(b_j,\beta_j)_{1,q},(2-a-m+\rho,\lambda),(2-b-n+\sigma,\mu)}^{(2-a+\rho,\lambda),(2-b+\sigma,\mu),(a_j,\alpha_j)_{1,p}}\right],$$

(4.4.6)

where $Re(\sigma) < 0$, $Re(\rho) > 0$, $Re(\mu) > 0$, $Re(\lambda) > 0$, $Re(a-\rho) < 2$, $\rho \neq -1, -2, -3, -4, -5 \ldots$, $Re(b-\sigma) < 2$, $\sigma \neq -1, -2, -3, -4, \ldots$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(vi) $$\int_0^\infty \int_0^\infty x^{\rho-1} e^{-1/x} y_m(x; a, 1) y^{\sigma-1} e^{-1/y} y_n(y; b, 1)\, A_{p,q}^{k,l}[zx^{-\lambda} y^{-\mu}]\, dxdy$$

$$= (-1)^{m+n} A_{p+4,q+2}^{k+2,l+2}\left[z\Big|_{(-1-\rho+a,\lambda),(-1-\sigma+b,\mu),(b_j,\beta_j)_{1,q}}^{(-\rho-m,\lambda),(-\sigma-n,\mu),(a_j,\alpha_j)_{1,p},(-1-\rho+a+m,\lambda),(-1-\sigma+b+n,\mu)}\right], \tag{4.4.7}$$

where $Re(\sigma) < 0$, $Re(\rho) > 0$, $Re(\mu) > 0$, $Re(\lambda) > 0$, $Re(a-\rho) < 2$, $\rho \neq -1, -2, -3, -4 \ldots$, $Re(b-\sigma) < 2$, $\sigma \neq -1, -2, -3, -4 \ldots$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(vii) $$\int_0^\infty \int_0^\infty x^{\rho-1} e^{-1/x} y_m(x; a, 1) y^{\sigma-1} e^{-1/y} y_n(y; b, 1)\, A_{p,q}^{k,l}[zx^{\lambda} y^{-\mu}]\, dxdy$$

$$= (-1)^{m+n} A_{p+3,q+3}^{k+2,l+2}\left[z\Big|_{(1+\rho+m,\lambda),(-1-\sigma+b,\mu),(b_j,\beta_j)_{1,q},(2-a-m+\rho,\lambda)}^{(2-a+\rho,\lambda),(-\sigma-n,\mu),(a_j,\alpha_j)_{1,p},(-1-\sigma+b+n,\mu)}\right], \tag{4.4.8}$$

where $Re(\sigma) < 0$, $Re(\rho) > 0$, $Re(\mu) > 0$, $Re(\lambda) > 0$, $Re(a-\rho) < 2$, $\rho \neq -1, -2, -3, -4 \ldots$, $Re(b-\sigma) < 2$, $\sigma \neq -1, -2, -3, -4 \ldots$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(viii) $$\int_0^\infty \int_0^\infty x^{\rho-1} e^{-1/x} y_m(x; a, 1) y^{\sigma-1} e^{-1/y} y_n(y; b, 1)\, A_{p,q}^{k,l}[zx^{-\lambda} y^{\mu}]\, dxdy$$

$$= (-1)^{m+n} A_{p+3,q+3}^{k+2,l+2}\left[z\Big|_{(-1-\rho+a,\lambda),(1+\sigma+n,\mu),(b_j,\beta_j)_{1,q},(2-b-n+\sigma,\mu)}^{(-\rho-m,\lambda),(2-b+\sigma,\mu),(a_j,\alpha_j)_{1,p},(-1-\rho+a+m,\lambda)}\right], \tag{4.4.9}$$

where $Re(\sigma) < 0$, $Re(\rho) > 0$, $Re(\mu) > 0$, $Re(\lambda) > 0$, $Re(a-\rho) < 2$, $\rho \neq -1, -2, -3, -4 \ldots$, $Re(b-\sigma) < 2$, $\sigma \neq -1, -2, -3, -4 \ldots$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

Proceeding on the same lines as in the theorem 4.4.1, the results (4.4.2) to (4.4.9) can be established with the help of (4.2.7), (4.2.8), (4.2.9), (4.2.10) and (4.2.11).

CHAPTER–5

INTEGRATION INVOLVING CERTAIN PRODUCTS AND A-FUNCTION

5.1 INTRODUCTION

Some integrals containing the product of other commonly used hypergeometric functions have been evaluated by Shrivastava [75, 76], Tiwari [81, 82] and several other authors.

In this chapter, we shall establish some integrals containing the products of other hypergeometric functions and A-Function using E-operator on the lines of Shrivastava [75, 76], Tiwari [81, 82] and several other authors.

In section (5.4), some integrals containing the product of A-Function and generalized hypergeometric function have been derived by using E (finite difference operator).

Most of the results in this chapter have been published in Applied Science Periodical [39] in form of a research paper.

5.2 FORMULA USED

From Shrivastava [62, p.426, (1.3); (1.4)] (with z replaced by iz are required in the present work:

$$z^{\lambda} F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} - x^2 z^2, -y^2 z^2\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} J_{\lambda+2n}(2z)\, F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right] \qquad (5.2.1)$$

and

$$z^{\lambda} F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} - x^2 z^2, -y^2 z^2\right]$$

$$= \Gamma(1+\lambda) \sum_{n=0}^{\infty} \frac{z^n}{2^n n!} J_{\lambda+n}(2z)\, F\left[{}^{-n,1+\lambda,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right], \qquad (5.2.2)$$

where $C + A + A' \leq D + B + B'$, $A' + C' + A \leq B' + D' + B$, and for all values of λ with possible exception of zero and negative integers. (a) represents the sequence of A parameters $a_1, a_2, \ldots, a_A$ and this convention will be retained throughout this chapter. Burchnall and Chaundy [13] gives the notation for double hypergeometric function, which was also introduced by Kampe de Feriet [3].

The finite difference operator E is given in [12], with w = 1 has the following operations

$$E_a f(a) = f(a+1),\ E_a^n f(a) = E_a[E_a^{n-1} f(a)]. \tag{5.2.3}$$

5.3 MAIN INTEGRALS

In this section, we have established two integrals containing the products of other hypergeometric functions and A-Function. We have represented these two integrals in another forms also. At the end of this section we have also discussed particular cases.

Theorem 5.3.1: Prove that

$$\int_0^\infty z^{\rho+\lambda-1} \sin 2z\, F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} -x^2z^2, -y^2z^2\right]$$

$$\times\ A_{p,q}^{k,l}\left[\beta z^{-2m}\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right] dz$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\rho-\lambda-2n} n!} F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right]$$

$$A_{p+1,q+3}^{k+1,l+1}\left[\beta z^{2m}\Big|{}^{(\frac{1}{2}-\rho,2m),(a_j,\alpha_j)_{1,p}}_{(\frac{1}{2}-n-\frac{\lambda}{2}-\frac{\rho}{2},m),(b_j,\beta_j)_{1,q},(1+2n+\lambda-\rho,2m),(1-n-\frac{\lambda}{2}-\rho/2,m)}\right], \tag{5.3.1}$$

which is valid under the conditions C + A' + A ≤ D + B' + B, A´ + C´ + A ≤ B´ + D´ + B, $R(\rho+\lambda+\frac{2mb_j}{\beta_j}) > -1$ (for j = 1, 2, 3,…., k), $R(\rho+\lambda+\frac{2m(a_j-1)}{\alpha_j}) < 1$ (for j = 1, 2, 3, …., *l*) and $|\arg(u\,\beta)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (5.3.1), take the expansion (5.2.1), multiply both side by f(z), integrate w.r.t. z from 0 to ∞ and on interchanging the order of summation and integration, we get

$$\int_0^\infty z^\lambda\, F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} - x^2z^2, -y^2z^2\right] f(z)dz$$

$$= \sum_{n=0}^\infty \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right]$$

$$. \int_0^\infty J_{\lambda+2n}(2z)\, f(z)dz, \tag{5.3.2}$$

for $A' + C + A \leq B' + D + B$, $A' + C' + A \leq B' + D' + B$, $R(\lambda + \eta + 1) > 0$ and $R(\lambda + \xi + 1) > 0$, where for large z, $f(z) = O(|z|^\xi)$; and for small z, $f(z) = O(|z|^\eta)$.

The change of integration and summation is justified [12, p.500] because

(i) The series

$$\sum_{k=0}^\infty \frac{(\lambda+2k)\Gamma(\lambda+k)}{k!} J_{\lambda+2k}(2z)\, F\left[{}^{-k,\lambda+k,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right]$$

is uniformly convergent in $0 \leq z \leq N$, N being arbitrary;

(ii) f(z) is a continuous function of z $\forall\ z \geq z_0 > 0$;

(iii) The integral on the left of (5.3.2) converges absolutely under the stated conditions.

Now on taking

$$f(z) = z^{\rho-1}\sin 2z\, A_{p,q}^{k,l}\left[\beta z^{-2m}\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

in (5.3.2), we can express A-function in term of Mellin-Barnes type integral (1.2.35). Due to absolute convergence of the integrals involved in the process we can interchange the order of integrations, which is justifiable, evaluate integral in the braces using [44, p.328(10)] and interpreting it with (1.2.35), we get (5.3.1).

Theorem 5.3.2: Prove that

$$\int_0^\infty z^{\rho+\lambda-1} \cos 2z\, F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} - x^2z^2, -y^2z^2\right]$$

$$.A_{p,q}^{k,l}\left[\beta z^{-2m}\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right] dz$$

$$= \sum_{n=0}^\infty \frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\rho-\lambda-2n}n!} F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right]$$

$$A^{k+1,l+1}_{p+1,q+3}\left[\beta z^{2m}\Big|^{(\frac{1}{2}-\rho,2m),(a_j,\alpha_j)_{1,p}}_{(1-n-\frac{\lambda}{2}-\frac{\rho}{2},m),(b_j,\beta_j)_{1,q},(1+2n+\lambda-\rho,2m),(1-n-\frac{\lambda}{2}-\rho/2,m)}\right], \quad (5.3.3)$$

which is valid under the conditions A´ + C + A ≤ B´ + D + B, A´ + C´ + A ≤ B´ + D´ + B, $R(\rho+\lambda+\frac{2mb_j}{\beta_j}) > 0$ (for j = 1, 2, 3, …., k), $R(\rho+\lambda+\frac{2m(a_j-1)}{\alpha_j}) < 1$ (for j = 1, 2, 3, …., *l*) and $|\arg(u\,\beta)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

If we take

$$f(z) = z^{\rho-1}\cos 2z\, A^{k,l}_{p,q}\left[\beta z^{-2m}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

proceed on the parallel lines as mentioned above and then in the light of the result [45, p.328(11)], we obtain (5.3.3).

On considering the result (5.2.2), proceeding on the parallel lines as mentioned above and making use of the result [45, p.328(10); p.328(11)], we get the following different forms of the integral (5.3.1) and (5.3.3) as

Integral 5.3.1(a)

$$\int_0^\infty z^{\rho+\lambda-1}\sin 2z\, F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} - x^2z^2, -y^2z^2\right]$$

$$.A^{k,l}_{p,q}\left[\beta z^{-2m}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dz$$

$$= \frac{\Gamma(\lambda+1)}{2^{1+\rho-\lambda}}\sum_{n=0}^{\infty}\frac{1}{2^n n!}F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');}x^2, y^2\right]$$

$$A^{k+1,l+1}_{p+1,q+3}\left[\beta z^{2m}\Big|^{(\frac{1}{2}-\rho-n,2m),(a_j,\alpha_j)_{1,p}}_{(\frac{1}{2}-n-\frac{\lambda}{2}-\frac{\rho}{2},m),(b_j,\beta_j)_{1,q},(1+\lambda-\rho,2m),(1-n-\frac{\lambda}{2}-\rho/2,m)}\right], \quad (5.3.4)$$

which is valid under the same conditions as (5.3.1) and

Integral 5.3.1(b)

$$\int_0^\infty z^{\rho+\lambda-1}\cos 2z\, F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} - x^2z^2, -y^2z^2\right]$$

$$.A_{p,q}^{k,l}\left[\beta z^{-2m}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dz$$

$$=\frac{\Gamma(\lambda+1)}{2^{1+\rho-\lambda}}\sum_{n=0}^{\infty}\frac{1}{2^n n!}F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');}x^2,y^2\right]$$

$$\times A_{p+1,q+3}^{k+1,l+1}\left[\beta z^{2m}\Big|_{\left(1-n-\frac{\lambda}{2}-\frac{\rho}{2},m\right),(b_j,\beta_j)_{1,q},(1+\lambda-\rho,2m),(1-n-\frac{\lambda}{2}-\rho/2,m)}^{(\frac{1}{2}-\rho-n,2m),(a_j,\alpha_j)_{1,p}}\right], \qquad (5.3.5)$$

The conditions of validity for (5.3.5) are the same as for (5.3.3).

PARTICULAR CASES

1. For a = b and a´ = b´, the double hypergeometric function in the left breaks up into the product of two generalized hypergeometric functions and from (5.3.1), we thus get

$$\int_0^{\infty} z^{\rho+\lambda-1}\sin 2z\ {}_CF_D\left[{}^{(c);}_{(d);}-x^2z^2\right]{}_{C'}F_{D'}\left[{}^{(c');}_{(d');}-y^2z^2\right]$$

$$.\ A_{p,q}^{k,l}\left[\beta z^{-2m}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dz$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\rho-\lambda-2n}n!}F\left[{}^{-n,\lambda+n,(c);(c');}_{(d);(d');}x^2,y^2\right]$$

$$A_{p+1,q+3}^{k+1,l+1}\left[\beta z^{2m}\Big|_{\left(\frac{1}{2}-n-\frac{\lambda}{2}-\frac{\rho}{2},m\right),(b_j,\beta_j)_{1,q},(1+2n+\lambda-\rho,2m),(1-n-\frac{\lambda}{2}-\rho/2,m)}^{(\frac{1}{2}-\rho,2m),(a_j,\alpha_j)_{1,p}}\right], \qquad (5.3.8)$$

The conditions of validity for (5.3.8) are the same (with A = B, A´ = B´) as given in (5.3.1).

2. On the other hand, since

$$F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');}x,y\right]={}_{A+A'+C}F_{B+B'+D}\left[{}^{(a),(a'),(c);}_{(b),(b'),(d);}x\right],$$

when y = 0.

The special case A = A´ = B = B´ = 0 of (5.3.1) provides us

$$\int_0^{\infty} z^{\rho+\lambda-1}\sin 2z\ {}_CF_D\left[{}^{(c);}_{(d);}-x^2z^2\right].\ A_{p,q}^{k,l}\left[\beta z^{-2m}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dz$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\rho-\lambda-2n}n!}\ {}_{C+2}F_D\left[{}^{-n,\lambda+n,(c);}_{(d);}x^2\right]$$

$$A^{k+1,l+1}_{p+1,q+3}\left[\beta z^{2m}\Big|^{(\frac{1}{2}-\rho,2m),(a_j,\alpha_j)_{1,p}}_{(\frac{1}{2}-n-\frac{\lambda}{2}-\frac{\rho}{2},m),(b_j,\beta_j)_{1,q},(1+2n+\lambda-\rho,2m),(1-n-\frac{\lambda}{2}-\rho/2,m)}\right], \qquad (5.3.9)$$

which is valid under the same conditions as for (5.3.1) with A = A´ = B = B´ = C´ = D´ = 0.

Further, with C = 0, D = 1, $d_1 = 1 + \lambda_1$, x = 1, express ${}_0F_1$ as a Bessel function, evaluate ${}_2F_1$ using Gauss's theorem [56] and after that on a closer examination we find

$$\int_0^\infty z^{\rho+\lambda-1} J_\lambda(2z)\sin 2z\,.\,A^{k,l}_{p,q}\left[\beta z^{-2m}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dz$$

$$=\frac{1}{2^{1+\rho-\lambda}}A^{k+1,l+1}_{p+1,q+3}\left[\beta z^{2m}\Big|^{(\frac{1}{2}-\rho,2m),(a_j,\alpha_j)_{1,p}}_{(\frac{1}{2}-n-\frac{\lambda}{2}-\frac{\rho}{2},m),(b_j,\beta_j)_{1,q},(1+2n+\lambda-\rho,2m),(1-n-\frac{\lambda}{2}-\rho/2,m)}\right],$$

(5.3.10)

provided that $R(\rho+\lambda+\frac{2mb_j}{\beta_j}) > -1$ (j = 1, …., k), $R(\rho+\frac{2m(a_j-1)}{\alpha_j}) < 1/2$ (j = 1, …., *l*) and |arg (u β)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

Similar consequences of the integral (5.3.3)

$$\int_0^\infty z^{\rho+\lambda-1} J_\lambda(2z)\cos 2z\,.\,A^{k,l}_{p,q}\left[\beta z^{-2m}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dz$$

$$=\frac{1}{2^{1+\rho-\lambda}}A^{k+1,l+1}_{p+1,q+3}\left[\beta z^{2m}\Big|^{(\frac{1}{2}-\rho,2m),(a_j,\alpha_j)_{1,p}}_{(1-n-\frac{\lambda}{2}-\frac{\rho}{2},m),(b_j,\beta_j)_{1,q},(1+2n+\lambda-\rho,2m),(1-n-\frac{\lambda}{2}-\rho/2,m)}\right],$$

(5.3.11)

provided that $R(\rho+\lambda+\frac{2mb_j}{\beta_j}) > -1$ (j = 1, …., k), $R(\rho+\frac{2m(a_j-1)}{\alpha_j}) < 1/2$ (j = 1, …., *l*) and |arg (u β)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

3. It may be of interest to conclude with the remark that the integrals (5.3.1) and (5.3.3) provide few fascinating outcomes on reducing few or all the functions that occurred in the integrand and it does not seem out of place to mention that specially in the light of the results [56, p.105,106]

$${}_2F_3\left[{}^{\frac{a}{2}+\frac{b}{2},\ \frac{a}{2}+\frac{b}{2}-\frac{1}{2};}_{a,\ b,\ a+b-1;}4x\right] = {}_0F_1\left[{}^{-;}_{a;}x\right]{}_0F_1\left[{}^{-;}_{b;}x\right]$$

and

$${}_2F_3\left[\begin{matrix}a, & b-a; & \frac{x^2}{4}\\ b, & \frac{b}{2}, & \frac{b}{2}+\frac{1}{2};\end{matrix}\right] = {}_1F_1\left[\begin{matrix}a;\\ b;\end{matrix} - x\right] {}_1F_1\left[\begin{matrix}a;\\ b;\end{matrix} x\right]$$

since ${}_0F_1$ is reduced to Bessel function and by Kummer's second theorem [56, p.126] it can be also transformed to ${}_1F_1$. Then further ${}_1F_1$ can be reduced to a generalized Laguerre polynomial $L_n^{\alpha}(x)$, Whittaker function $M_{k,m}(x)$, Bessel function of first kind $I_n(x)$, Hermite polynomial $H_n(x)$ and Weber's parabolic cylinder function $D_n(x)$.

5.4 INTEGRALS USING FINITE DIFFERENCE OPERATOR E

In this section we evaluate four integrals by using finite difference operator E:

Theorem 5.4.1: Prove that

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta \, A_{p,q}^{m,n}\left[x.\sin^{2h}\theta\cos^{2k}\theta\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$. {}_uF_v\,[e_u; f_v; c \sin^{2\mu}\theta \cos^{2\nu}\theta]\, d\theta$$

$$= \left(\frac{1}{2}\right) \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(e_j,r)c^r}{\prod_{j=1}^{v}(f_j,r)r!} A_{p+2,q+1}^{m+2,n}\left[x\Big|_{(b_j,\beta_j)_{1,q},(1+\rho+\sigma+(\mu+\nu)r,h+k)}^{\left(\frac{1}{2}+\rho+\mu r,h\right),\left(\frac{1}{2}+\sigma+\nu r,k\right),(a_j,\alpha_j)_{1,p}}\right], \quad (5.4.1)$$

provided that $\rho > 0$, $\sigma > 0$, $|\arg (ux)| < \frac{1}{2} \pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

Taking product of (4.3.9) and $\frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda)c^{\lambda}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda)}$ and using the operator $e^{E_\rho^\mu E_\sigma^\nu E_\lambda}$, we get

$$e^{E_\rho^\mu E_\sigma^\nu E_\lambda}\left\{\int_0^{\frac{\pi}{2}} \sin^{2\rho}\theta \cos^{2\sigma}\theta \, A_{p,q}^{m,n}\left[x.\sin^{2h}\theta\cos^{2k}\theta\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] \frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda)c^{\lambda}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda)}\, d\theta\right\}$$

$$= e^{E_\rho^\mu E_\sigma^\nu E_\lambda}\left\{\left(\frac{1}{2}\right) A_{p+2,q+1}^{m+2,n}\left[x\Big|_{(b_j,\beta_j)_{1,q},(1+\rho+\sigma,h+k)}^{\left(\frac{1}{2}+\rho,h\right),\left(\frac{1}{2}+\sigma,k\right),(a_j,\alpha_j)_{1,p}}\right] \frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda)c^{\lambda}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda)}\right\}, \quad (5.4.5)$$

Expanding both sides of (5.4.5) and applying (5.2.3), we have

$$\sum_{r=0}^{\infty}\left\{ \int_0^{\frac{\pi}{2}} \sin^{2(\rho+\mu r)}\theta \cos^{2(\sigma+\nu r)}\theta \, A_{p,q}^{m,n}\left[x.\sin^{2h}\theta\cos^{2k}\theta\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]\right.$$

$$\cdot \ \frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda+r)c^{\lambda+r}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda+r)r!}\ d\theta\}$$

$$=\sum_{r=0}^{\infty}\{\frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda+r)c^{\lambda+r}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda+r)r!}\left(\frac{1}{2}\right) A_{p+2,q+1}^{m+2,n}\left[x\Big|{}_{(b_j,\beta_j)_{1,q},(1+\rho+\sigma+(\mu+\nu)r,h+k)}^{\left(\frac{1}{2}+\rho+\mu r,h\right),\left(\frac{1}{2}+\sigma+\nu r,k\right),(a_j,\alpha_j)_{1,p}}\right]\}.$$

Further, using $(\alpha, n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, on left hand side change the order of summation and integration, then replace$(f_j + \lambda)$ by f_j and $(e_j + \lambda)$ by e_j, to obtain (5.4.1).

Theorem 5.4.2: Prove that

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, A_{p,q}^{m,n}\left[x.\sin^{-2h}\theta\cos^{2k}\theta\Big|{}_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$\times\ {}_uF_v\ [e_u;\ f_v;\ c \sin^{2\mu}\theta \cos^{2\nu}\theta]\ d\theta$$

$$= \left(\frac{1}{2}\right)\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{u}(e_j,r)c^r}{\prod_{j=1}^{v}(f_j,r)r!}A_{p+2,q+1}^{m+1,n+1}\left[x\Big|{}_{\left(\frac{1}{2}-\rho+\mu r,h\right),(b_j,\beta_j)_{1,q}}^{\left(\frac{1}{2}+\sigma+\nu r,k\right),(a_j,\alpha_j)_{1,p},(-\rho-\sigma-(\mu+\nu)r,h-k)}\right], \quad (5.4.2)$$

provided that $\rho > 0$, $\sigma > 0$, $|\arg (ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

Proceed as in theorem 5.4.1 and using the results (4.3.10)

Theorem 5.4.3: Prove that

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, A_{p,q}^{m,n}\left[x.\sin^{2h}\theta\cos^{-2k}\theta\Big|{}_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$\times\ {}_uF_v\ [e_u;\ f_v;\ c \sin^{2\mu}\theta \cos^{2\nu}\theta]\ d\theta$$

$$= \left(\frac{1}{2}\right)\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{u}(e_j,r)c^r}{\prod_{j=1}^{v}(f_j,r)r!}A_{p+1,q+2}^{m+1,n+1}\left[x\Big|{}_{\left(\frac{1}{2}-\sigma-\mu r,k\right),(b_j,\beta_j)_{1,q},(1+\rho+\sigma(\mu+\nu)r,h-k)}^{\left(\frac{1}{2}+\rho+\mu r,h\right),(a_j,\alpha_j)_{1,p}}\right], \quad (5.4.3)$$

provided that $\rho > 0$, $\sigma > 0$, $|\arg (ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

Proceed as in theorem 5.4.1 and using the results (4.3.11)

Integral 5.4.4

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, A_{p,q}^{m,n}\left[x.\sin^{-2h}\theta\cos^{-2k}\theta \middle| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] d\theta$$

$$= \left(\frac{1}{2}\right)\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(e_j,r)c^r}{\prod_{j=1}^{v}(f_j,r)r!} A_{p+1,q+2}^{m,n+2}\left[x\middle| \begin{matrix} (a_j,\alpha_j)_{1,p},(-\rho-\sigma-(\mu+\nu)r,h+k) \\ \left(\frac{1}{2}-\rho-\mu r,h\right),\left(\frac{1}{2}-\sigma-\nu r,k\right),(b_j,\beta_j)_{1,q} \end{matrix}\right], \tag{5.4.4}$$

provided that $\rho > 0$, $\sigma > 0$, $|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

Proceed as in theorem 5.4.1 and using the results (4.3.12)

CHAPTER–6

EXPANSION AND IDENTITIES INVOLVING A-FUNCTION

6.1 INTRODUCTION

Samtani [61], Saxena [63, 64], Srivastava [79], Rathi [57], Agrawal [1], Goyal [23], and several other authors have evaluated some Expansion and Identities for generalized hyper geometric functions.

Looking into the requirement and importance of various properties of expansion and identities in several field, in this chapter we established some new Expansion and Identities involving 'A-Function' of one variable.

We have established some new Expansions for 'A-Function' of one variable in section (6.2).

We have discussed some new Identities involving 'A-Function' of one variable in section (6.3).

Some of the results in this chapter have been published in International Research Journal of Mathematics, Engineering and IT [38] respectively in form of research paper.

6.2 EXPANSION FORMULAE

Expansion Formulae plays an important role in study of special functions in particular A-Function. In this section, we established six Expansion Formula involving A-function of one variable with the help of integrals obtained in chapter 4. In the present investigation, despite of integrals in chapter 4 we also require the following Formulae:

From Rainvile [56]:

$$z\Gamma(z) = \Gamma(z+1), \tag{6.2.1}$$

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}[P_n^{(\alpha,\beta)}(x)]^2 dx = \frac{2^{\alpha+\beta+1}\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}. \tag{6.2.2}$$

Theorem 6.2.1: Prove that

$$(1-x)^{\rho}(1+x)^{\sigma}A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$=\sum_{n=0,k=0}^{\infty}\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\alpha+n+k)}{k!\,\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$A_{p+4,q+4}^{m+2,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\sigma+\beta,\delta),(1-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},}^{(1+\rho+\alpha,\mu),(1+\sigma+\beta+n,\delta),(a_j,\alpha_j)_{1,p},}\right.$$

$$\left.{}_{(2+\rho+\sigma+n+\alpha+\beta,\mu+\delta),(1+n+\rho+\alpha,\mu)}^{(-\alpha-n-k+\sigma,\delta),(1-k+\rho+\sigma+\beta,\mu+\delta)}\right], \qquad (6.2.3)$$

provided that $Re(\beta+1) > 0$, $Re(\alpha+1) > 0$, $Re(\rho+\alpha+\mu+1) > 0$, $Re(\sigma+\beta+n+\delta+1) > 0$, $Re(-\sigma-\delta-n) > 0$, $Re(k-\rho-\sigma-(\mu+\delta)+n) > 0$, $|arg(uz)| < ½\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (6.2.3), consider

$$(1-x)^{\rho}(1+x)^{\sigma}A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$=\sum_{R=0}^{\infty}C_R\,P_R^{(\alpha,\beta)}(x). \qquad (6.2.4)$$

Due to the continuity and bounded variation of expression on the L.H.S. in (−1, 1), equation (6.2.3) is valid. On taking product of (6.2.3) and $(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)$ and integrating between − 1 to 1 with respect to x, using relation (4.3.1) in left hand side, interchanging the order of integration and summation, which is valid under the condition [14, p.176)], using orthogonality property of Jacobi Polynomials, we get

$$C_n=\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}.\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)}{k!}$$

$$A_{p+4,q+4}^{m+2,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\sigma+\beta,\delta),(1-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},}^{(1+\rho+\alpha,\mu),(1+\sigma+\beta+n,\delta),(a_j,\alpha_j)_{1,p},}\right.$$

$$\left.{}_{(2+\rho+\sigma+n+\alpha+\beta,\mu+\delta),(1+n+\rho+\alpha,\mu)}^{(-\alpha-n-k+\sigma,\delta),(1-k+\rho+\sigma+\beta,\mu+\delta)}\right], \qquad (6.2.5)$$

Further using (6.2.5) in (6.2.4), we get the relation (6.2.3).

Theorem 6.2.2: Prove That

(i) $$(1-x)^{\rho}(1+x)^{\sigma}\, A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$=\sum_{n=0,k=0}^{\infty}\frac{2^{\rho+\sigma}(-1)^{n}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\beta+n+k)}{k!\,\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$.\,A_{p+5,q+4}^{m+3,l+2}\left[z2^{\mu+\delta}\Big|_{(1+n+\rho+\alpha,\mu),(1-k-n+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},}^{(1+\sigma+\beta,\delta),(1+\rho+n+\alpha,\mu),(1+n+k+\sigma+\beta,\delta),(a_j,\alpha_j)_{1,p},}\right.,$$

$$\left.{}_{(2+\rho+\sigma+n+\alpha+\beta,\mu+\delta),(1+n+\sigma+\beta,\delta)}^{(-\beta-n-k+\rho,\mu),(1+\alpha-k+\rho+\sigma,\mu+\delta)}\right], \tag{6.2.6}$$

provided that $Re(\beta+1)>0$, $Re(\alpha+1)>0$, $Re(\rho+\alpha+n+\mu+1)>0$, $Re(\sigma+\beta+\delta+1)>0$, $Re(n+k+\sigma+\beta+\mu+1)>0$, $Re(-\rho-\alpha-\mu-n)>0$, $Re(k-\rho-\sigma-(\mu+\delta)+n)>0$, $|arg\ (uz)|<\frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(ii) $$(1-x)^{\rho}(1+x)^{\sigma}\, A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$=\sum_{n=0,k=0}^{\infty}\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\alpha+n+k)}{k!\,\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$A_{p+4,q+3}^{m+4,l}\left[z2^{\mu+\delta}\Big|_{(b_j,\beta_j)_{1,q},(1+n+\rho+\alpha,\mu),(2+k+\rho+\sigma+\alpha+\beta,\mu+\delta),(2+\alpha+\beta+n+k+\sigma,\delta)}^{(1+\rho+\alpha,\mu),(1+\sigma+\beta,\delta),(1+n+k+\rho+\alpha,\mu),(1+k+\sigma,\delta),(a_j,\alpha_j)_{1,p}}\right], \tag{6.2.7}$$

provided that $Re(\beta+1)>0$, $Re(\alpha+1)>0$, $Re(k+\sigma+\delta+1)>0$, $Re(\rho+\alpha+\mu+1)>0$, $Re(\beta+\sigma+\delta+1)>0$, $|arg\ (uz)|<\frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

(iii) $$(1-x)^{\rho}(1+x)^{\sigma}\, A_{p,q}^{m,l}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$=\sum_{n=0,k=0}^{\infty}\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\beta+n+k)}{k!\,\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$A^{m+4,l}_{p+4,q+3}\left[z2^{\mu+\delta}\Big|^{(1+\rho+\alpha,\mu),(1+\sigma+\beta,\delta),(1+n+k+\sigma+\beta,\delta),(1+k+\rho,\mu),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1+n+\sigma+\beta,\delta),(2+n+k+\rho+\alpha+\beta,\mu+\delta),(2+\alpha+\beta+k+\rho+\sigma,\mu)}\right],$$

(6.2.8)

provided that Re(β + 1) > 0, Re(α + 1) > 0, Re(ρ + α + μ + 1) > 0, Re(σ + β + δ + 1) > 0, |arg (uz)| < ½ πh, where h and u are given in (1.2.38) and (1.2.39) respectively.

(iv) $$(1-x)^{\rho}(1+x)^{\sigma} A^{m,l}_{p,q}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$=\sum_{n=0,k=0}^{\infty}\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(-\alpha-\beta-n+k)}{k!\,\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$.\,A^{m+3,l+2}_{p+5,q+4}\left[z2^{\mu+\delta}(1+x)^{\delta}\Big|^{(1+\sigma+\beta,\delta),(1+\rho+n+\alpha,\mu),(1-\beta+\sigma+k+\beta,\delta),(a_j,\alpha_j)_{1,p},}_{(1+n+\rho+\alpha,\mu),(2+k+\rho+\sigma+\alpha+\beta,\mu+\delta),(b_j,\beta_j)_{1,q},}\right.,$$

$$\left.{}^{(2+\rho+\sigma+\alpha+\beta,\mu+\delta),(1+\alpha+\beta+n-k+\rho,\mu)}_{(2+n+\rho+\sigma+\alpha+\beta,\mu+\delta),(-\alpha-n+k+\sigma,\delta)}\right], \qquad (6.2.7)$$

provided that Re(1 + n + ρ + α + μ) >0, Re(− 1 + k − α − β − ρ − σ − (μ + δ)) > 0, Re(1 + σ + β + δ) >0, Re (1 + σ + k +δ) > 0, Re (− ρ − α − n − μ) > 0, Re(1 + n + α + β) > 0, Re(− α − β − n + k) > 0,|arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

(v) $$(1-x)^{\rho}(1+x)^{\sigma} A^{m,l}_{p,q}\left[z(1-x)^{\mu}(1+x)^{\delta}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$=\sum_{n=0,k=0}^{\infty}\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\alpha+\beta-n+k)}{k!\,\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$.\,A^{m+3,l+2}_{p+5,q+4}\left[z2^{\mu+\delta}\Big|^{(1+\rho+\alpha,\mu),(1+\sigma+n+\beta,\delta),(1-\beta+\alpha+\rho+k,\mu),(a_j,\alpha_j)_{1,p},}_{(1+n+\sigma+\alpha,\delta),(2+\alpha+\beta-k+\rho+\sigma,\mu+\delta),(b_j,\beta_j)_{1,q},}\right.,$$

$$\left.{}^{(2+\rho+\sigma+\alpha+\beta,\mu+\delta),(1+2\beta+n-k+\sigma,\delta)}_{(2+n+\rho+\sigma+\alpha+\beta,\mu+\delta),(-\beta-n+k+\rho,\mu)}\right], \qquad (6.2.9)$$

provided that Re(1 + n + α + β) > 0, Re(− α − β − n + k) > 0, Re(1 + ρ + α + μ) > 0, Re(− α − σ − n − δ) > 0, Re(1 + n + σ + β + δ) > 0, Re(1 − β + ρ + α + k + μ) > 0, Re(− 1 − k − ρ − σ α − β − (μ + δ)) > 0, |arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

(i) to (v) Proceed as in theorem 6.2.1 and using the results (4.3.2) to (4.3.6), respectively.

6.3 IDENTITIES

In this section, we have discussed certain properties of A-Function. Going in lines with Kishore and Srivastva [33] we have established nine Identities involving A-function of one variable in form of propositions. We have applied definition of A-Function and properties of Gamma function to obtain these identities.

Theorem 6.3.1: Prove that

$$A_{p+1,q+2}^{m,n+2}\left[x\Big|_{(0,h),(-1+k,\nu),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(k,\nu)}\right]$$

$$= (1-k)A_{p,q+1}^{m,n+1}\left[x\Big|_{(0,h),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$- A_{p+1,q+2}^{m+1,n+1}\left[x\Big|_{(0,h),(b_j,\beta_j)_{1,q},(0,\nu)}^{(1,\nu),(a_j,\alpha_j)_{1,p}}\right], \qquad (6.3.1)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (6.3.1), consider left hand side of (6.3.1), after using (1.2.35), We have

$$\text{L.H.S.} = \frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(1-hs)\Gamma(2-k-\nu s)}{\Gamma(1-k-\nu s)}x^s ds$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\,\Gamma(1-hs)(1-k-\nu s)x^s ds \qquad \text{[On using (6.2.1)]}$$

$$= \frac{(1-k)}{2\pi i}\int_L \theta(s)\,\Gamma(1-hs)x^s ds$$

$$-\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(1-hs)\Gamma(1+\nu s)}{\Gamma(\nu s)}x^s ds,$$

which in the light of (1.2.35) provides right hand side of (6.3.1).

Theorem 6.3.2: Prove that

$$A_{p+1,q+2}^{m+1,n+1}\left[x\Big|_{(0,h),(b_j,\beta_j)_{1,q},(1-k,\nu)}^{(2-k,\nu),(a_j,\alpha_j)_{1,p}}\right]$$

$$=(1-k)A_{p,q+1}^{m,n+1}\left[x\Big|_{(0,h),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$+A_{p+1,q+2}^{m+1,n+1}\left[x\Big|_{(0,h),(b_j,\beta_j)_{1,q},(0,\nu)}^{(1,\nu),(a_j,\alpha_j)_{1,p}}\right], \qquad (6.3.2)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (6.3.2), consider left hand side of (6.3.2), after using (1.2.35), to obtain

$$\text{L.H.S.} = \frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(0-hs)\Gamma(2-k+\nu s)}{\Gamma(1-k+\nu s)}x^s ds$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\,\Gamma(0-hs)(1-k+\nu s)x^s ds \qquad \text{[On using (6.2.1)]}$$

$$=\frac{(1-k)}{2\pi i}\int_L \theta(s)\,\Gamma(1-hs)x^s ds$$

$$-\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(0-hs)\Gamma(1+\nu s)}{\Gamma(\nu s)}x^s ds,$$

which in the light of (1.2.35) provides right hand side of (6.3.2).

Theorem 6.3.3: Prove that

$$k\,A_{p+2,q+2}^{m+2,n+1}\left[x\Big|_{(0,h),(b_j,\beta_j)_{1,q},(1+k,\alpha)}^{(0,\alpha),(k,\alpha),(a_j,\alpha_j)_{1,p}}\right]$$

$$=-A_{p+2,q+2}^{m+2,n+1}\left[x\Big|_{(0,h),(b_j,\beta_j)_{1,q},(1+k,\alpha)}^{(1,\alpha),(k,\alpha),(a_j,\alpha_j)_{1,p}}\right]$$

$$+A_{p+1,q+1}^{m+1,n+1}\left[x\Big|_{(0,h),(b_j,\beta_j)_{1,q}}^{(0,\alpha),(a_j,\alpha_j)_{1,p}}\right], \qquad (6.3.3)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (6.3.3), consider

$$A_{p+2,q+2}^{m+2,n+1}\left[x\middle|\begin{matrix}(1,\alpha),(k,\alpha),(a_j,\alpha_j)_{1,p}\\(0,h),(b_j,\beta_j)_{1,q},(1+k,\alpha)\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(1+\alpha s)\Gamma(k+\alpha s)\Gamma(1-hs)}{\Gamma(1+k+\alpha s)}x^s ds \qquad \text{(on using (1.2.35))}$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\frac{\alpha s\Gamma(\alpha s)\Gamma(-hs)}{k+\alpha s}x^s ds$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\frac{(k+\alpha s-k)}{k+\alpha s}\frac{\Gamma(\alpha s)\Gamma(1-hs)}{}x^s ds \qquad \text{[On using (6.2.1)]}$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\,\Gamma(1-hs)\Gamma(\alpha s)x^s ds$$

$$-\frac{k}{2\pi i}\int_L \theta(s)\frac{\Gamma(1-hs)\Gamma(k+\alpha s)\Gamma(\alpha s)}{\Gamma(1+k+\alpha s)}x^s ds,$$

which in the light of (1.2.35) provides right hand side of (6.3.3).

Theorem 6.3.4: Prove that

$$k\,A_{p+2,q+2}^{m+1,n+2}\left[x\middle|\begin{matrix}(0,\alpha)(a_j,\alpha_j)_{1,p},(-k,\alpha)\\(0,h),(1-k,\alpha),(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

$$=A_{p+1,q+1}^{m+1,n+1}\left[x\middle|\begin{matrix}(0,\alpha)(a_j,\alpha_j)_{1,p}\\(0,h),(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

$$+A_{p+2,q+2}^{m+1,n+2}\left[x\middle|\begin{matrix}(1,\alpha),(a_j,\alpha_j)_{1,p},(-k,\alpha)\\(1-k,\alpha),(0,h),(b_j,\beta_j)_{1,q}\end{matrix}\right],$$

(6.3.4)

$|\arg(ux)| < ½\,\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (6.3.4), consider

$$A_{p+2,q+2}^{m+1,n+2}\left[x\middle|\begin{matrix}(1,\alpha),(a_j,\alpha_j)_{1,p},(-k,\alpha)\\(1-k,\alpha),(0,h),(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(1+\alpha s)\Gamma(k-\alpha s)\Gamma(1-hs)}{\Gamma(1+k-\alpha s)}x^s ds \qquad \text{(on using (1.2.35))}$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\frac{\alpha s\Gamma(\alpha s)\Gamma(1-hs)}{k-\alpha s}x^s ds$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\frac{(k-(k-\alpha s))\Gamma(\alpha s)\Gamma(1-hs)}{k-\alpha s}x^s ds \qquad \text{[On using (6.2.1)]}$$

$$= -\frac{1}{2\pi i}\int_L \theta(s)\,\Gamma(1-hs)\Gamma(\alpha s)x^s ds$$

$$+\frac{k}{2\pi i}\int_L \theta(s)\frac{\Gamma(1-hs)\Gamma(k-\alpha s)\Gamma(\alpha s)}{\Gamma(1+k-\alpha s)}x^s ds,$$

which in the light of (1.2.35) provides right hand side of (6.3.4).

Theorem 6.3.5: Prove that

$$A^{m,n+1}_{p+1,q+1}\left[x\middle|{}^{(a_j,\alpha_j)_{1,p},(-\alpha+2,\sigma)}_{(-\alpha,\sigma),(b_j,\beta_j)_{1,q}}\right]$$

$$-A^{m,n+2}_{p+2,q+2}\left[x\middle|{}^{(a_j,\alpha_j)_{1,p},(-\alpha+k+2,\sigma),(1-\alpha-k,\sigma)}_{(1-\alpha+k,\sigma),(-\alpha-k,\sigma),(b_j,\beta_j)_{1,q}}\right]$$

$$= k(k+1)A^{m,n}_{p,q}\left[x\middle|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right], \qquad (6.3.5)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (6.3.5), consider

$$A^{m,n+1}_{p+1,q+1}\left[x\middle|{}^{(a_j,\alpha_j)_{1,p},(-\alpha+2,\sigma)}_{(-\alpha,\sigma),(b_j,\beta_j)_{1,q}}\right]$$

$$-A^{m,n+2}_{p+2,q+2}\left[x\middle|{}^{(a_j,\alpha_j)_{1,p},(-\alpha+k+2,\sigma),(1-\alpha-k,\sigma)}_{(1-\alpha+k,\sigma),(-\alpha-k,\sigma),(b_j,\beta_j)_{1,q}}\right]$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(1+\alpha-\sigma s)}{\Gamma(1+\alpha-2-\sigma s)}x^s ds$$

$$-\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(\alpha-k-\sigma s)\Gamma(1+\alpha+k-\sigma s)}{\Gamma(1+\alpha-k-2-\sigma s)\Gamma(\alpha+k-\sigma s)}x^s ds \qquad \text{(on using (1.2.35))}$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\,x^s\left\{\frac{\Gamma(1+\alpha-\sigma s)}{\Gamma(1+\alpha-2-\sigma s)} - \frac{\Gamma(\alpha-k-\sigma s)\Gamma(1+\alpha+k-\sigma s)}{\Gamma(1+\alpha-k-2-\sigma s)\Gamma(\alpha+k-\sigma s)}\right\}ds$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\,x^s\{(\alpha-\sigma s)(\alpha-\sigma s-1)-(\alpha-k-1-\sigma s)(\alpha+k-\sigma s)\}ds$$

[On using (6.2.1)]

$$= k(k+1)\frac{1}{2\pi i}\int_L \theta(s)\, x^s ds$$

which in the light of (1.2.35) provides right hand side of (6.3.5).

Theorem 6.3.6: Prove that

$$A_{p+2,q+2}^{m+2,n}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,p},(-\alpha+3/2,\sigma),(1-\alpha,\sigma)\\(-\alpha+\frac{1}{2},\sigma),(-\alpha+2,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]$$

$$-A_{p+2,q+2}^{m+2,n}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,p},(1-\alpha-\beta,\sigma),(-\alpha+\beta+3/2,\sigma)\\(-\alpha-\beta,\sigma),(-\alpha+\beta+1/2,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]$$

$$=\beta(\beta+2)A_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right], \qquad (6.3.6)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

Proceed as in theorem 6.3.5.

Theorem 6.3.7: Prove that

$$A_{p+1,q+1}^{m,n+1}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,p},(-\alpha,\sigma)\\(-\alpha-1,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]$$

$$-A_{p+1,q+1}^{m,n+1}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,p},(-\alpha+\beta+2,\sigma)\\(1-\alpha+\beta,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]$$

$$=(\beta+2)A_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right], \qquad (6.3.7)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (6.3.7), consider left hand side of (6.3.7), after using (1.2.35), to obtain

$$L.H.S. = \frac{1}{2\pi i}\int_L \theta(s)\, x^s\left\{\frac{\Gamma(\alpha+2-\sigma s)}{\Gamma(\alpha+1-\sigma s)}-\frac{\Gamma(\alpha-\beta-\sigma s)}{\Gamma(\alpha-\beta-1-\sigma s)}\right\}ds$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\, x^s\{(\alpha+1-\sigma s)-(\alpha-\beta-1-\sigma s)\}ds \qquad \text{[On using (6.2.1)]}$$

$$= (\beta + 2) \frac{1}{2\pi i} \int_L \theta(s)\, x^s ds$$

which in the light of (1.2.35) provides right hand side of (6.3.7).

Theorem 6.3.8: Prove that

$$A^{m+1,n+1}_{p+1,q+2}[x|^{(2-k,\nu),(a_j,\alpha_j)_{1,p}}_{(1,h),(b_j,\beta_j)_{1,q},(1-k,\nu)}]$$

$$= (1-k) A^{m,n+1}_{p,q+1}[x|^{(a_j,\alpha_j)_{1,p}}_{(1,h),(b_j,\beta_j)_{1,q}}]$$

$$+ A^{m+1,n+1}_{p+1,q+2}[x|^{(1,\nu),(a_j,\alpha_j)_{1,p}}_{(1,h),(b_j,\beta_j)_{1,q},(0,\nu)}], \qquad (6.3.8)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (6.3.8), let us consider left hand side of (6.3.8).

After using (1.2.35), we obtain

$$L.H.S. = \frac{1}{2\pi i} \int_L \theta(s) \frac{\Gamma(2-k+\nu s)\Gamma(-hs)}{\Gamma(1-k+\nu s)} x^s ds$$

$$= \frac{1}{2\pi i} \int_L \theta(s)\, \Gamma(-hs)(1-k+\nu s) x^s ds \qquad \text{[On using (6.2.1)]}$$

$$= \frac{(1-k)}{2\pi i} \int_L \theta(s)\, \Gamma(-hs) x^s ds$$

$$+ \frac{1}{2\pi i} \int_L \theta(s) \frac{\Gamma(-hs)\Gamma(1+\nu s)}{\Gamma(\nu s)} x^s ds,$$

which in the light of (1.2.35) provides right hand side of (6.3.8).

Theorem 6.3.9: Prove that

$$A^{m,n+1}_{p+1,q+2}[x|^{(a_j,\alpha_j)_{1,p},(1-k,\nu)}_{(1,h),\,(-k,\nu),(b_j,\beta_j)_{1,q}}]$$

$$= k\, A^{m,n+1}_{p,q+1}[x|^{(a_j,\alpha_j)_{1,p},}_{(1,h),(b_j,\beta_j)_{1,q}}]$$

$$- A^{m+1,n+1}_{p+1,q+2}[x|^{(1,\nu),(a_j,\alpha_j)_{1,p}}_{(1,h),(b_j,\beta_j)_{1,q},(0,\nu)}], \qquad (6.3.9)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (6.3.9), let us consider left hand side of (6.3.9), after using (1.2.35), we obtain

$$L.H.S. = \frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(-hs)\Gamma(1+k-\nu s)}{\Gamma(k-\nu s)}x^s ds$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\,\Gamma(-hs)(k-\nu s)x^s ds \qquad \text{[On using (6.2.1)]}$$

$$= \frac{k}{2\pi i}\int_L \theta(s)\,\Gamma(-hs)x^s ds$$

$$-\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(-hs)\Gamma(1+\nu s)}{\Gamma(\nu s)}x^s ds$$

which in the light of (1.2.35) provides right hand side of (6.3.9).

CHAPTER–7

APPLICATION OF A-FUNCTION OF ONE VARIABLE IN OBTAINING A SOLUTION OF SOME BOUNDARY VALUE PROBLEMS

7.1 INTRODUCTION

Various problems in science and technology, when formulated mathematically, lead naturally to certain classes of partial differential equations involving one or more unknown functions together with the prescribed conditions (known as boundary conditions) which arise from the physical situation. Several workers have obtained solutions to the equations related to certain problems, which satisfy the given boundary conditions. The classical method in obtaining solutions of the boundary value problems of mathematical physics can be derived from Fourier series.

Another technique using integral transforms, which had its origin in Heaviside's work, has been developed in the past and has certain advantages over the classical method.

The theory developed by Heaviside and Doetsch and others have unified the latter investigations by Bromwhich and Carson in the recent work on the Laplace transformation. Although the Laplace transform has been extensively (and intensively) employed, it is particularly useful for problem associated with ordinary differential equations as well as for problems involving heat conduction. Also, other integral transforms can be utilized while solving the most of the BVP of mathematical physics. This method of solution is really convenient, direct and straightforward than the classical method, which generally requires great ingenuity in assuming at the outset the correct form for the solution.

Several authors such as Arora (1998), Chandel (2002), Chaurasia (1997), Srivastava (1998, 1999, 2000), Tiwari (1993) have used various classes of orthogonal polynomials and generalized hypergeometric functions of one or more variables in finding the solutions of the boundary value problems concerning

(a) heat conduction in

    (i) a non-homogenous finite bar

    (ii) a circular cylinder

(b) free oscillations of water in a circular lake

(c) transverse vibrations in a circular membranes

(d) free symmetrical vibrations in a very large plate

(e) angular displacement in a shaft of circular cross-section

(f) potential theory, etc.

Vishwakarma [83], Tiwari [81, 82], Ronghe [60], Agrawal [1], Srivastava [71], Jain [30], Srivastava [73], Srivastava [74] and several other authors have obtained solutions of boundary value problems involving generalized hypergeometric functions by expressing u(x, t) in terms of known orthogonal polynomials and certain special functions of one and more variables, where $u(x, t) = (k'/k)f(x)g(x)$.

Following Vishwakarma [83], Tiwari [81, 82], Ronghe [60], Agrawal [1], Srivastava [71], Jain [30], Srivastava [73], Srivastava [74] and several other authors, in this chapter we will employ the A-function of one variable in obtaining a solution of some boundary value problems and find new solutions which will be useful for further research.

In section (7.3) first we evaluate an integral involving A-function of one variable and then we make its application to solve two boundary value problems on (i) heat conduction in a bar (ii) deflection of vibrating string under certain conditions. Again in section (7.4) we employ the A-function of one variable in obtaining a solution of a partial differential equation related to heat conduction along with Hermite polynomials. The aim of section (7.5) is to derive the solution of special one-dimensional time dependent Schrodinger equation involving 'A-Function' of one variable and Hermite polynomials, while in (7.6) we employ the 'A-Function' of one variable in obtaining a solution of a Bounded Electrostatic Potential in the Semi-Infinite Space.

Most of the results in section 7.5 and 7.6 of this chapter have been published in The Mathematics Education [36] and in Journal of Indian Academy of Mathematics [34] respectively in form of couple of research papers.

7.2 RESULTS REQUIRED

In the present investigation we require the following results:

From Gradshteyn [25], we have following modified form:

$$\int_0^L (\sin \pi x/L)^{\omega - 1} \sin n\pi x/L \, dx = \frac{L \sin \tfrac{1}{2} n\pi \, \Gamma(\omega)}{2^{\omega - 1} \Gamma\{\tfrac{1}{2}(1 + \omega + n)\} \Gamma\{\tfrac{1}{2}(1 + \omega - n)\}} \tag{7.2.1}$$

where $n \in \mathbb{Z}$.

$$E_a f(a) = f(a + 1); \; E_a^n f(a) = E\,[E_a^{n-1} f(a)], \tag{7.2.2}$$

where E (finite difference operator) is given in Milne-Thamson [47].

Modified form of the integral given by Ronghe [58]:

$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_n(x) \, dx = \frac{\sqrt{\pi}\, 2^{n - 2\rho} \Gamma(2\rho + 1)}{\Gamma(\rho - n/2 + 1)}, \tag{7.2.3}$$

In this chapter, we shall also make application of following modified form of the integral [25, p.372]:

$$\int_0^\pi (\sin y)^{\omega - 1} \sin ny \, dy = \frac{\pi \sin \tfrac{1}{2} n\pi \, \Gamma(\omega)}{2^{\omega - 1} \Gamma\{\tfrac{1}{2}(\omega + n + 1)\} \Gamma\{\tfrac{1}{2}(\omega - n + 1)\}},$$

$$\text{Re}(\omega) > 0. \tag{7.2.4}$$

We will also use the following notation:

$$F[\lambda x^{2\mu}] \equiv {}_UF_V \left[\begin{matrix} A_1, .., A_U; \\ B_1, .., B_V; \end{matrix} \; \lambda x^{2\mu} \right] dx$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{U} (A_j; k)\, \lambda^k\, x^{2\mu k}}{\prod_{j=1}^{V} (B_j; k)\, k!}$$

7.3 APPLICATION OF A-FUNCTION IN BOUNDARY VALUE PROBLEMS

In this section first we evaluate an integral involving A-function of one variable and then we make its application to solve two boundary value problems on (i) heat conduction in a bar (ii) deflection of vibrating string under certain conditions.

First of all we state and prove the following two lemmas which will be used in subsequent sections.

Lemma 7.3.1

$$\int_{-\infty}^{\infty} x^{2\rho}\, e^{-x^2}\, H_n(x)\, A_{p,q}^{m,n}\left[zx^{2\lambda}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] dx$$

$$= \sqrt{\pi}\, 2^{n-2\rho} A_{p+1,q+1}^{m+1,n}\left[z/4^{\lambda}\Big|_{(b_j,\beta_j)_{1,q},(1-\frac{n}{2}+\rho,\lambda)}^{(1+2\rho,2\lambda),(a_j,\alpha_j)_{1,p}}\right] \tag{7.3.1}$$

$|\arg (ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

The result (7.3.1) can be established by replacing the 'A-Function' given in (1.2.35) on the L.H.S., interchanging the order of integral involved in the process, evaluating the integral in the braces using (7.2.3) and applying (1.2.35) the definition of 'A-Function', the value of the integral is obtained.

Now we shall establish the following integral involving the A-function of one variable.

Lemma 7.3.2

$$\int_{0}^{L} \left(\sin\frac{\pi x}{L}\right)^{\omega-1} \sin\frac{n\pi x}{L}\, A_{p,q}^{m,l}\left[z\left(\sin\frac{\pi x}{L}\right)^{\lambda}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= 2^{1-\omega} sin\frac{n\pi}{2} A_{p+1,q+2}^{m+1,l}\left[z2^{-\lambda}\Big|_{(b_j,\beta_j)_{1,q},(\frac{1}{2}+\frac{\omega}{2}\mp\frac{n}{2},\frac{\lambda}{2})}^{(\omega,\lambda),(a_j,\alpha_j)_{1,p}}\right], \tag{7.3.2}$$

provided that Re (ω) > 0, |arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (7.3.2), using 'A-Function' given in (1.2.35), change the order of integration which valid under the given condition, evaluate the inner integral with the help of (7.2.1) and finally interpret it with (1.2.35), to get (7.3.2).

PROBLEM - I

7.3.1 APPLICATION TO HEAT CONDUCTION IN A BAR

Under certain boundary conditions, a problem on heat conduction in a bar is considered in this section. If sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible, then in a uniform bar $0 \le x \le L$, the temperature u(x, t) satisfies the heat equation given below:

$$(\partial^2 u/\partial x^2) = (1/c)(\partial u/\partial t),\ t \ge 0. \qquad (7.3.3)$$

If we take

$$u(0, t) = 0,\ u(L, t) = 0, \qquad (7.3.4)$$

as boundary conditions and

$$u(x, 0) = f(x), \qquad (7.3.5)$$

as initial condition, then partial differential equation (7.3.3) has the solution

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin n\pi x/L \exp[-t(nc\pi)^2/L], \qquad (7.3.6)$$

is given by Prasad [54], where n is any integer and

$$B_n = (2/L) \int_0^L f(x) \sin n\pi x/L\, dx. \qquad (7.3.7)$$

Now we shall consider the problem of determine u(x, t), where

$$u(x, 0) = f(x) = \left(\sin\frac{\pi x}{L}\right)^{\omega-1} A_{p,q}^{m,l}\left[z\left(\sin\frac{\pi x}{L}\right)^{\lambda}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]. \qquad (7.3.8)$$

Solution of the Problem

Combining (7.3.7) and (7.3.8) and making the use of lemma 7.3.2, we derive,

$$B_n = 2^{2-\omega} \sin\frac{n\pi}{2} A^{m+1,l}_{p+1,q+2}\left[z2^{-\lambda}\Big|^{(\omega,\lambda),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(\frac{1}{2}+\frac{\omega}{2}\mp\frac{n}{2},\frac{\lambda}{2})}\right]. \tag{7.3.9}$$

Putting the value of B_n from (7.3.8) in (7.3.6), we get following required solution of the problem

$$u(x,t) = 2^{2-\omega}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\exp\left[-t\frac{(n\pi c)^2}{L}\right]\sin\frac{n\pi}{2}$$

$$.\, A^{m+1,l}_{p+1,q+2}\left[z2^{-\lambda}\Big|^{(\omega,\lambda),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(\frac{1}{2}+\frac{\omega}{2}\mp\frac{n}{2},\frac{\lambda}{2})}\right]. \tag{7.3.10}$$

PROBLEM - II

7.3.2 HOMOGENEOUS WAVE PROBLEM

We shall determine the deflection u(x, t) of vibrating string in this section. If the weight of string due to tension is negligible then the partial differential equation given below is satisfied by deflection u(x, t)

$$(1/c^2)(\partial^2 u/\partial t^2) = (\partial^2 u/\partial x^2),\ 0 < x < L. \tag{7.3.11}$$

Now we choose

$$u(0, t) = 0,\ u(L, t) = 0, \tag{7.3.12}$$

as the boundary conditions and

$$\partial u(x, 0)/\partial t = g(x),\ \text{(initial velocity)} \tag{7.3.13}$$

and

$$u(x, 0) = f(x), \tag{7.3.14}$$

as initial conditions, then partial differential equation (7.3.11) gives the solution

$$u(x, t) = \sum_{n=1}^{\infty} [B_n \cos n\pi ct/L + C_n \sin n\pi ct/L] \sin n\pi x/L, \tag{7.3.15}$$

where B_n is given by (7.3.7) and

$$C_n = (2/n\pi c)\int_0^L g(x)\sin n\pi x/L\,dx. \qquad (7.3.16)$$

The solution (7.3.15) is given by Prasad [54].

Now consider the problem of determining u(x, t), where u(x, 0) [=f(x)] is given by (7.3.8), while

$$g(x) = (\sin\frac{\pi x}{L})^{\omega'-1} A_{P,Q}^{M,N}\left[z(\sin\frac{\pi x}{L})^{\mu}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]. \qquad (7.3.17)$$

After combining (7.3.16) and (7.3.17) and making the use of lemma (7.3.2), we arrive at

$$C_n = 2^{2-\omega'}\frac{L}{n\pi c}\sin\frac{n\pi}{2} A_{P+1,Q+2}^{M+1,N}\left[z2^{-\mu}\Big|_{(b_j,\beta_j)_{1,q},(\frac{1}{2}+\frac{\omega'}{2}\mp\frac{n}{2},\frac{\mu}{2})}^{(\omega',\mu),(a_j,\alpha_j)_{1,p}}\right]. \qquad (7.3.18)$$

Putting the value of B_n and C_n in (7.3.15) to get required solution of the problem in the following form:

$$u(x,t) = 2^{2-\omega}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}\sin\frac{n\pi}{2}$$

$$.\,A_{p+1,q+2}^{m+1,l}\left[z2^{-\lambda}\Big|_{(b_j,\beta_j)_{1,q},(\frac{1}{2}+\frac{\omega}{2}\mp\frac{n}{2},\frac{\lambda}{2})}^{(\omega,\lambda),(a_j,\alpha_j)_{1,p}}\right]$$

$$+\,2^{2-\omega'}\sum_{n=1}^{\infty}\frac{L}{n\pi c}\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\sin\frac{n\pi}{2}$$

$$A_{P+1,Q+2}^{M+1,N}\left[z2^{-\mu}\Big|_{(b_j,\beta_j)_{1,q},(\frac{1}{2}+\frac{\omega'}{2}\mp\frac{n}{2},\frac{\mu}{2})}^{(\omega',\mu),(a_j,\alpha_j)_{1,p}}\right]. \qquad (7.3.19)$$

7.4 HEAT CONDUCTION INVOLVING A-FUNCTION AND HERMITE POLYNOMIALS

Here first of all we shall evaluate an integral containing A-function of one variable and Hermite Polynomials with the help of finite difference operator E and discuss their application in solving a problem on heat conduction considered by Bajpai [1993]. An expansion formula involving A-function of one variable and Hermite Polynomials has also been obtained at the end of this section.

Theorem 7.4.1: Prove that

$$\int_{-\infty}^{\infty} x^{2\rho}\, e^{-x^2} H_n(x) A_{p,q}^{m,n}\left[zx^{2\lambda}\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right] \quad {}_U F_V\left[{}^{A_1,\ldots\ldots,A_U}_{B_1,\ldots\ldots,B_V} \lambda x^{4\mu}\right] dx$$

$$= \sqrt{\pi} 2^{n-2\rho} \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{U}(A_j;l)\lambda^l 2^{-4\mu l}}{\prod_{i=1}^{U}(B_j;l)l!}$$

$$. A_{p+1,q+2}^{m+1,l}\left[z2^{-\lambda}\Big|{}^{(1+2\rho+4\mu l,2\lambda),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1-\frac{n}{2}+\rho+2\mu l,\lambda)}\right], \tag{7.4.1}$$

provided that $|\arg (uz)| < ½ \pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

On multiplying both sides of (7.3.1) by

$$\frac{\prod_{j=1}^{U} (A_j + \delta)\, \lambda^{\delta}}{\prod_{j=1}^{V} (B_j + \delta)}$$

apply the operator $e^{E_\rho{}^{2\mu}E_\delta}$, we get

$$e^{E_\rho{}^{2\mu}E_\delta}\left\{\left(\int_{-\infty}^{\infty} x^{2\rho}\, e^{-x^2}\, H_n(x)\, A_{p,q}^{m,n}\, [zx^{2\lambda}]\, dx\right) \frac{\prod_{j=1}^{U} (A_j + \delta)\, \lambda^{\delta}}{\prod_{j=1}^{V} (B_j + \delta)}\right\}$$

$$= e^{E_\rho{}^{2\mu}E_\delta}\left\{\left(\sqrt{\pi}\, 2^{n-2\rho} A_{p+1,q+1}^{m+1,n}\ \left[|z/4^{\lambda}|{}^{(1+2\rho,\, 2\lambda),(a_i,\, \alpha_i)_{1,p}}_{(b_i,\, \beta_i)_{1,q},\, (1-n/2+\rho,\, \lambda)}\right]\right) \frac{\prod_{j=1}^{U} (A_j + \delta)\, \lambda^{\delta}}{\prod_{j=1}^{V} (B_j + \delta)}\right\} \tag{7.4.2}$$

Expanding both side of (7.4.2) and using $E_a f(a) = f(a + 1)$, we have

$$\sum_{l=0}^{\infty} \left\{\left(\int_{-\infty}^{\infty} x^{2\rho}\, e^{-x^2} H_n(x)\, A_{p,q}^{m,n}\ \left[|zx^{2\lambda}|{}^{(a_j,\, \alpha_j)_{1,p}}_{(b_j,\, \beta_j)_{1,q}}\right] dx\right) \frac{\prod_{j=1}^{U} (A_j + \delta + l) x^{4\mu l}\, \lambda^{\delta+l}}{\prod_{j=1}^{V} (B_j + \delta + l)\, l\,!}\right.$$

$$= (\sqrt{\pi}2^{\,n-2\rho} A^{m+1,\,n}_{p+1,\,q+1} [z/4^{\lambda}|^{(1+2\rho+4\mu\, l,\, 2\lambda),\,(a_j,\,\alpha_j)_{1,\,p}}_{(b_j,\,\beta_j)_{1,\,q},\,(1-n/2+\rho+2\mu\, l,\,\lambda)}]) \left. \frac{\prod_{j=1}^{U} (A_j+\delta+l)\,\lambda^{\delta+l}\, 2^{-4\mu l}}{\prod_{j=1}^{V} (B_j+\delta+l)\, l\,!} \right\} \quad (7.4.3)$$

Now using $(a;n) = \Gamma(a + n)/\Gamma(a)$, altering the summation and integration order in the L.H.S. and replace $(B_j + \delta)$ by B_j and $(A_j + \delta)$ by A_j, to get (7.4.1).

Application to Heat Conduction:

Consider following partial differential equation

$$\partial u/\partial t = k\, [\partial^2 u/\partial x^2 + 2x\, (\partial u/\partial x) + 2u],\ x \in (-\infty, \infty), \quad (7.4.4)$$

where boundary condition is

$$\lim_{|x| \to \infty} u(x) = 0,$$

Equation (4.1) is related to the following equation Carslaw [15]

$$\partial^2 \nu/\partial x^2 - (\partial \nu/\partial x)(U/k) - (1/k)(\nu - \nu_0)\nu - (\partial \nu/\partial t)(1/k) = 0, \quad (7.4.5)$$

where $U = 2kx,\ \nu_0 = 0,\ \nu = -2k,\ (-\infty < x < \infty)$.

The solution of equation (7.4.4) is given by Bajpai [9] as follows:

$$u\,(x, t) = \sum_{n=0}^{\infty} C_n\, e^{-2knt - x^2} H_n(x), \quad (7.4.6)$$

where $H_n(x)$ is the Hermite polynomial and

$$C_n = \frac{1}{2^n\, n!\sqrt{\pi}} \int_{-\infty}^{\infty} u(x)\, Hn(x)\, dx, \quad (7.4.7)$$

Now we shall consider the problem of determining u (x, t), where

$$u(x) = x^{2\rho}\, e^{-x^2} H_n(x)\, A^{m,\,n}_{p,\,q}\, [zx^{2\lambda}|^{\ldots\ldots,\,\ldots\ldots}_{\ldots\ldots,\,\ldots\ldots}]\ {}_UF_V\, [{}^{A_1,\,..,\,A_U;}_{B_1,\,..,\,B_V;}\ \lambda x^{4\mu}]. \quad (7.4.8)$$

Combining (7.4.7) and (7.4.8) and making the use of integral (7.4.1), we derive

$$C_n = [1/(2^{2\rho} n!)] \sum_{l=0}^{\infty} A^{m+1,\, n}_{p+1,\, q+1} [z/4^{\lambda} \left|^{(1+2\rho+4\mu l,\, 2\lambda),\ (a_j, \alpha_j)_{1,p}}_{(b_j, \beta_j)_{1,q},\ (1-n/2+\rho+2\mu l,\, \lambda)}\right.]) \frac{\prod_{j=1}^{U} (A_j; l)\, \lambda^{l}\, 2^{-4\mu l}}{\prod_{j=1}^{V} (B_j; l)\, l\,!} \tag{7.4.9}$$

Putting the value of C_n from (7.4.9) in (7.4.6), we get

$$u(x, t) = (1/2^{2\rho}) \sum_{n=0}^{\infty} (1/n!)\, e^{-2knt - x^2} H_n(x).$$

$$\cdot \left\{ \sum_{l=0}^{\infty} A^{m+1,\, n}_{p+1,\, q+1} [z/4^{\lambda} \left|^{(1+2\rho+4\mu l,\, 2\lambda),\ (a_j, \alpha_j)_{1,p}}_{(b_j, \beta_j)_{1,q},\ (1-n/2+\rho+2\mu l,\, \lambda)}\right.]) \frac{\prod_{j=1}^{U} (A_j; l)\, \lambda^{l}\, 2^{-4\mu l}}{\prod_{j=1}^{V} (B_j; l)\, l\,!} \right\} \tag{7.4.10}$$

Expansion Formula

Making a use of (7.4.8) and (7.4.9) in (7.4.6), we derive the following expansion formula:

$$x^{2\rho} A^{m,\, n}_{p,\, q} [zx^{2\lambda} \left|^{(a_j, \alpha_j)_{1,p}}_{(b_j, \beta_j)_{1,q}}\right.]\ {}_U F_V \left[^{A_1, .., A_U;}_{B_1, .., B_V;}\ \lambda x^{4\mu}\right]$$

$$= (1/2^{2\rho}) \sum_{n=0}^{\infty} (1/n!)\, H_n(x).$$

$$\cdot \left\{ \sum_{l=0}^{\infty} A^{m+1,\, n}_{p+1,\, q+1} [z/4^{\lambda} \left|^{(1+2\rho+4\mu l,\, 2\lambda),\ (a_j, \alpha_j)_{1,p}}_{(b_j, \beta_j)_{1,q},\ (1-n/2+\rho+2\mu l,\, \lambda)}\right.]) \frac{\prod_{j=1}^{U} (A_j; l)\, \lambda^{l}\, 2^{-4\mu l}}{\prod_{j=1}^{V} (B_j; l)\, l\,!} \right\} \tag{7.4.11}$$

7.5 TIME–DEPENDENT SCHRODINGER EQUATION INVOLVING A-FUNCTION

One of the fundamental problems in quantum mechanics is to find solution of Schrodinger equation for different forms of potentials. As a result of the failure of classical physics of predict correctly the result of experiments on microscopic systems, the Schrodinger equation and more general formulation of quantum mechanics have been set up. By testing their predictions of the properties of systems, where in case of failure and success of classical mechanics, they must be verified. In fact whole atomic physics, solid state physics, chemistry and some other branches of applied sciences obey the principals of quantum mechanics or satisfy differential equations similar to the Schrodinger equations, and same is true for nuclear and particle physics.

Making an appeal of Bajpai [7], we obtain the following integrals:

$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2\nu}(x)\, A_{p,q}^{u,v}\left[zx^{2h} \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] dx$$

$$= 2^{2\nu}\, A_{p+2,q+1}^{u+2,v}\left[z \,\middle|\, \begin{matrix}(1/2+\rho,h),(1+\rho,h),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1-\nu+\rho,h)\end{matrix}\right], \tag{7.5.1}$$

and

$$\int_{-\infty}^{\infty} x^{2\rho+1} e^{-x^2} H_{2\nu+1}(x)\, A_{p,q}^{u,v}\left[zx^{2h} \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] dx$$

$$= 2^{2\rho+1}\, A_{p+2,q+1}^{u+2,v}\left[z \,\middle|\, \begin{matrix}(3/2+\rho,h),(1+\rho,h),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1-\nu+\rho,h)\end{matrix}\right], \tag{7.5.2}$$

provided that $\rho>\nu$, $\rho = 0,1,2 \ldots\ldots$, $\nu = 0,1,2,\ldots\ldots\ldots.$, $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

The Special Schrodinger Equation

Let us take the problem of a particle having the potential V(x), where V(x) is given by

$$V(x) = [h^2 / (2m)]\, x^2. \tag{7.5.3}$$

For this system the time dependent Schrodinger equation Rae [55] can be written as:

$$\frac{\partial u}{\partial t} = \frac{-h}{2im}\frac{\partial^2 u}{\partial x^2} + \frac{h}{2im}(x^2 u)\,. \tag{7.5.4}$$

Setting $K = -h/(2im)$ into (7.5.4), we have

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} - Kx^2 u, \tag{7.5.5}$$

provided $u(x,t) \to 0$ for large values of t and $|x| \to \infty$. we also assume that

$u(x, 0) \equiv u(x).$ (7.5.6)

The solution of (7.5.4) is given by Bajpai [10], as under:

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-k(2n+1)t - x^2/2} He_n(\sqrt{2}\, x),$$

where $He_n(x)$ are Chebyshev Hermite polynomials [10]:

$$u(x,t) = \sum_{n=0}^{\infty} B_n e^{-k(2n+1)t - x^2/2} H_n(x), \tag{7.5.7}$$

where $H_n(x)$ are Hermite polynomials.

Also

$$A_n = 1/(n!\sqrt{\pi}) \int_{-\infty}^{\infty} u(x)\, e^{-x^2/2} He_n(\sqrt{2}x)\, dx, \tag{7.5.8}$$

$$B_n = 1/(2^n n!\sqrt{2}) \int_{-\infty}^{\infty} u(x)\, e^{-x^2/2} H_n(x)\, dx. \tag{7.5.9}$$

Solutions in terms of A-Function:

The solution of (7.5.7) leads to the following solutions:

$$u_1(x, t) = \sum_{n=0}^{\infty} B_{2n} e^{-k(4n+1)t - x^2/2} H_{2n}(x), \tag{7.5.10}$$

where

$$B_{2n} = 1/[2^{2n}(2n)!\sqrt{\pi}) \int_{-\infty}^{\infty} u_1(x)\; e^{-x^2/2} H_{2n}(x)\, dx \tag{7.5.11}$$

$$u_2(x, t) = \sum_{n=0}^{\infty} B_{2n+1} e^{-k(4n+3)t - x^2/2} H_{2n+1}(x)\, dx \tag{7.5.12}$$

where

$$B_{2n+1} = 1/[2^{2n+1}(2n+1)!\sqrt{\pi}] \int_{-\infty}^{\infty} u_2(x)\; e^{-x^2/2} H_{2n+1}(x)\, dx \tag{7.5.13}$$

If we substitute

$$u_1(x) = x^{2\rho} e^{-x^2/2} A_{p,q}^{u,v}\left[z x^{2h} \middle| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \tag{7.5.14}$$

and

$$u_2(x) = x^{2\rho+1} e^{-x^2/2} A_{p,q}^{u,v}\left[z x^{2h} \middle| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \tag{7.5.15}$$

in (7.5.11) and (7.5.13) respectively and use the integrals (7.5.1) and (7.5.2), then the solutions corresponding to (7.5.10) and (7.5.12) are given by:

$$u_1(x, t) = 1/(\sqrt{\pi}) \sum_{n=1}^{\rho} [1/(2n)!]\, e^{-k(4n+1)t - x^2/2}$$

$$. A^{u+2, v}_{p+2, q+1}\left[z \,\middle|\, \begin{matrix} (1/2+\rho, h), (1+\rho, h), (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q}, (1-\nu+\rho, h) \end{matrix} \right] H_{2n}(x), \qquad (7.5.16)$$

valid under the conditions of (7.5.1).

$$u_2(x, t) = 1/(\sqrt{\pi}) \sum_{n=1}^{\rho} [1/(2n+1)!]\, e^{-k(4n+3)t - x^2/2}$$

$$. A^{u+2, v}_{p+2, q+1}\left[z \,\middle|\, \begin{matrix} (3/2+\rho, h), (1+\rho, h), (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q}, (1-n+\rho, h) \end{matrix} \right] H_{2n+1}(x), \qquad (7.5.17)$$

valid under the conditions of (7.5.2).

7.6 BOUNDED ELECTROSTATIC POTENTIAL

In this section, with the help of A–function of one variable, in the Semi-Infinite Space we shall obtain a bounded Electrostatic Potential. First of all we shall establish the following integral in form of lemma.

Lemma 7.6.1: Prove that

$$\int_0^{\pi} (\sin y)^{\omega-1} \sin ny\, A^{m, l}_{p, q}\left[z (\sin y)^{\lambda} \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] dy$$

$$= 2^{1-\omega} \pi \sin \tfrac{1}{2} n\pi\, A^{m+1, l}_{p+1, q+2}\left[z\, 2^{-\lambda} \,\middle|\, \begin{matrix} (\omega, \lambda), (aj, \alpha j)_{1,p} \\ (bj, \beta j)_{1,q}, (1/2 + \omega/2 \pm n/2, \lambda/2) \end{matrix} \right], \qquad (7.6.1)$$

provided that $|\arg uz| < \tfrac{1}{2} h\pi$, $\lambda \geq 0$ and $\mathrm{Re}(\omega) > 0$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

Using 'A-Function' given in (1.2.35), alter the order of integration, evaluate the integral (inner) using (7.2.4) and finally interpret it with (1.2.35), to get (7.6.1).

Bounded Electrostatic Potential in the Semi-Infinite Space

Under certain boundary conditions, in the Semi-Infinite Space, we consider a problem on Bounded Electrostatic Potential. When the space is free of charges, in the semi-infinite space $x > 0$, $0 < y < \pi$, let bounded electrostatic potential, which is denoted by $V(x, y)$, so that

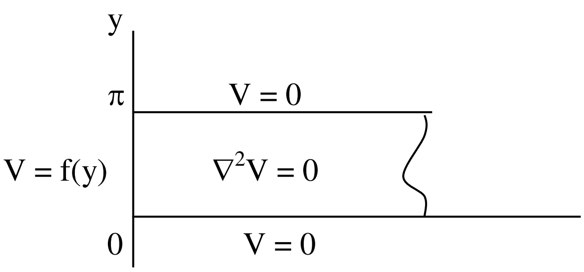
$$V_{xx}(x, y) + V_{yy}(x, y) = 0, \text{ where } x > 0,\ 0 < y < \pi \tag{7.6.2}$$

and suppose that

$$V(x, \pi) = 0,\ V(x, 0) = 0;\ x > 0$$

$$V(0, y) = f(y);\ 0 < y < \pi$$

See the following figure, where boundedness condition serves as a condition at the missing right-hand end of the strip shown there.



Assuming that f is piecewise smooth, then solution of (7.6.2) is given by [16]:

$$V(x, y) = \sum_{n=1}^{\infty} b_n \exp(-nx) \sin ny \tag{7.6.3}$$

where

$$b_n = (2/\pi) \int_0^{\pi} f(y) \sin ny\, dy,\ n = 1, 2, \ldots.. \tag{7.6.4}$$

Now choose

$$f(y) = (\sin y)^{\omega - 1} A^{m, l}_{p, q}\left[z (\sin y)^{\lambda} \Big|^{(a_j, \alpha_j)_{1, p}}_{(b_j, \beta_j)_{1, q}}\right] \tag{7.6.5}$$

Solution of the Problem

Combining (7.6.5) and (7.6.4) and making the use of the lemma 7.6.1, we derive

$$b_n = 2^{2-\omega} \sin \tfrac{1}{2} n\pi\, A^{m+1, l}_{p+1, q+2}\left[z\, 2^{-\lambda} \Big|^{(\omega, \lambda), (aj, \alpha j)_{1, p}}_{(bj, \beta j)_{1, q}, (1/2 + \omega/2 \pm n/2, \lambda/2)}\right], \tag{7.6.6}$$

Putting the value of b_n from (7.6.6) in (7.6.3), we get the following required solution of the problem:

$$V(x, y) = 2^{2-\omega} \sum_{n=1}^{\infty} \Big\{ \sin \tfrac{1}{2} n\pi \; \exp(-nx) \sin ny \times$$

$$\times \; A^{m+1, l}_{p+1, q+2} \left[z\, 2^{-\lambda} \,\middle|\, \begin{matrix} (\omega, \lambda), (aj, \alpha j)_{1, p} \\ (bj, \beta j)_{1, q}, (1/2 + \omega/2 \pm n/2, \lambda/2) \end{matrix} \right], \Big\} \qquad (7.6.7)$$

provided the condition stated with (7.6.1) are satisfied.

CHAPTER–8

FOURIER SERIES INVOLVING A-FUNCTION

8.1 INTRODUCTION

In the study of boundary value problems and special functions, Fourier series for generalized hypergeometric functions plays a vital role. Certain double Fourier series of generalized hypergeometric functions play a vital role in the improvement of the theories of boundary value problems of dimension two and special functions.

Using generalized hypergeometric functions, certain number of Fourier series have been evaluated by Bajpai [5, 11], Taxak [80], Sharma [66], Mishra [49] and others recently.

Looking vital role of Fourier series in the study of boundary value problems and special functions, in this chapter, we shall establish some new Fourier series involving A-function of one variable on the lines of Bajpai [5, 11], Taxak [80], Sharma [66], Mishra [49] and several other authors.

8.2 RESULTS REQUIRED

While deriving Fourier series involving A-Function of one variable following results are required

From Rainville [56]:

$$\int_{-1}^{1} (1-x)^{a}\,(1+x)^{b} P_n^{(a,b)}(x) P_m^{(a,b)}(x)\,dx$$

$= 0$, if $m \neq n$,

$$= \frac{2^{a+b+1}\Gamma(a+n+1)\Gamma(b+n+1)}{n!(a+b+2n+1)\Gamma(a+b+n+1)}, \text{ if } m = n; \tag{8.2.1}$$

where $Re(a) > -1$, $Re(b) > -1$.

The following orthogonality properties given in [43]:

$$\int_0^{\pi} e^{i(m-n)x}\,dx = \begin{cases} \pi, & m = n; \\ \pi, & m = n = 0; \\ 0, & m \neq n; \end{cases} \tag{8.2.2}$$

$$\int_0^{\pi} e^{imx} \cos nx\,dx = \begin{cases} \pi/2, & m = n; \\ \pi, & m = n = 0; \\ 0, & m \neq n; \end{cases} \tag{8.2.3}$$

$$\int_0^{\pi} e^{imx} \sin nx\,dx = \begin{cases} \frac{\pi i}{2}, & m=n; \\ 0, & m\neq n; \end{cases} \tag{8.2.4}$$

provided either both m and n are odd or both m and n are even integers.

From Macrobert [43], [45]:

$$\frac{\sqrt{\pi}\Gamma(2-s)}{2\Gamma(\frac{3}{2}-s)} (sin\theta)^{1-2s} = \sum_{r=0}^{\infty} \frac{(s)_r}{(2-s)_r} \sin(2r+1)\,\theta, \tag{8.2.5}$$

where $0 < \theta \leq \pi$, Re $s \leq \frac{1}{2}$.

$$\frac{\sqrt{\pi}\Gamma(1-s)}{\Gamma(\frac{1}{2}-s)} \left(sin\frac{\theta}{2}\right)^{-2s} = 1 + 2\sum_{r=0}^{\infty} \frac{(s)_r}{(1-s)_r} \cos r\theta, \tag{8.2.6}$$

where $0 < \theta \leq \pi$.

8.3 FOURIER SERIES

In this section, we have established some new Fourier series involving A-function of one variable.

Most of the results have been published in International Journal of Scientific Research and Reviews [42] in form of a research papers.

Fourier series 8.3.1

$$(\sin\frac{x}{2})^{-2w_1}(1-y)^{w_2}$$

$$\times\ A_{p,q}^{m,n}\left[z.(\sin\frac{x}{2})^{2h}(1-y)^{-k}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= \frac{2^{w_2+1}}{\sqrt{\pi}} \sum_{r,t=0}^{\infty} \frac{(a+b+2t+1)\Gamma(a+b+t+1)}{\Gamma(a+t+1)} \cos(rx)\, P_t^{(a,b)}(y)\ \times$$

$$A_{p+4,q+4}^{m+2,n+2}\left[z.2^{-k}\Big|_{(1-w_1-r,h),(-w_2-a,k),(b_j,\beta_j)_{1,q},(1-w_1+r,h),(-w_2,k)}^{(\frac{1}{2}-w_1,h),(-w_2+t,k),(a_j,\alpha_j)_{1,p},(1-w_1,h),(-1-a-b-w_2-t,k)}\right] \tag{8.3.1}$$

provided that h > 0, k > 0, Re(a) > – 1, Re(b) > – 1 and |arg (uz)| < ½ πh, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To establish (8.3.1), let

$$f(x, y) = (\sin\tfrac{x}{2})^{-2w_1}(1-y)^{w_2}$$

$$\times A^{m,n}_{p,q}\left[z.(\sin\tfrac{x}{2})^{2h}(1-y)^{-k}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= \sum_{r,t=0}^{\infty} A_{r,t}\cos(rx)\, P^{(a,b)}_t(y). \qquad (8.3.2)$$

Equation (8.3.2) is valid, since f(x, y) is defined in the region

$0 < x < \pi, -1 < y < 1$.

There are many awkward problems related to writing an expression for a function f(x, y) in terms double Fourier series expansion. With two-variables analogues of well-known Dirichlet's conditions and the Jordan's theorem, convergence of almost all double Fourier series expansions is covered. In this respect, a brief discussion given by Carslaw and Jaeger [15] provide a good coverage of the subject.

Taking the product of (8.3.2) and $(1-y)^a(1+y)^b P^{(a,b)}_v(y)$, integrate w.r.t. y from – 1 to 1, and applying (4.3.14) and (8.2.1), we obtain

$$2^{w_2}(\sin\tfrac{x}{2})^{-2w_1}$$

$$\times\ A^{m+1,n+1}_{p+2,q+2}\left[z.2^{-k}(\sin\tfrac{x}{2})^{2h}\Big|^{(-w_2+v,k),(a_j,\alpha_j)_{1,p},(-1-a-b-w_2-v,k)}_{(-w_2-a,k),(b_j,\beta_j)_{1,q},(-w_2,k)}\right]$$

$$= \sum_{r=0}^{\infty} A_{r,v}\frac{\Gamma(a+v+1)}{(a+b+2v+1)\Gamma(a+b+v+1)}\cos(rx). \qquad (8.3.3)$$

Multiply (8.3.3) by cos(ux), integrate w.r.t. x from 0 to π, and using (4.3.13) and cosine function's orthogonal property, to get

$$A_{u,v} = \frac{2^{w_2+1}}{\sqrt{\pi}}\frac{(a+b+2v+1)\Gamma(a+b+v+1)}{\Gamma(a+v+1)}$$

$$\times A_{p+4,q+4}^{m+2,n+2}\left[z.\,2^{-k}\Big|{}^{\left(\frac{1}{2}-w_1,h\right),(-w_2+v,k),(a_j,\alpha_j)_{1,p},(1-w_1,h),(-1-a-b-w_2-v,k)}_{(1-w_1-u,h),(-w_2-a,k),(b_j,\beta_j)_{1,q},(1-w_1+u,h),(-w_2,k)}\right] \quad (8.3.4)$$

except that $A_{0,v}$ is one-half of the above value. From (8.3.2) and (8.3.4), the Fourier series (8.3.1) is obtained

Fourier series 8.3.2

$$(\sin\theta)^{\rho} A_{p,q}^{m,n}\left[z.\,(\sin\theta)^{-2\delta}\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= \frac{1}{\sqrt{\pi}} A_{p+1,q+1}^{m,n+1}\left[z\Big|{}^{(a_j,\alpha_j)_{1,p},(-\rho/2,\delta)}_{\left(\frac{1}{2}-\frac{\rho}{2},\delta\right),(b_j,\beta_j)_{1,q}}\right]$$

$$+\frac{2}{\sqrt{\pi}}\sum_{r=1}^{\infty} A_{p+2,q+2}^{m,n+2}\left[z\Big|{}^{(a_j,\alpha_j)_{1,p},\left(-\frac{\rho+r}{2},\delta\right),\left(-\frac{\rho-r}{2},\delta\right)}_{\left(1-\frac{1+\rho}{2},\delta\right),\left(-\frac{\rho}{2},\delta\right),(b_j,\beta_j)_{1,q}}\right].\cos(\pi r/2)\cos r\theta, \quad (8.3.5)$$

provided that δ is a positive number and $|\arg(uz)| < ½\,\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To establish (8.3.5), let

$$f(\theta) = (\sin\theta)^{\rho} A_{p,q}^{m,n}\left[z.\,(\sin\theta)^{-2\delta}\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= \frac{C_0}{2} + \sum_{r=1}^{\infty} C_r \cos r\theta, \quad (8.3.6)$$

As $f(\theta)$ is of bounded variation and continuous in $(0, \pi)$, when $\rho > 0$, equation (8.3.6) is valid.

Multiply (8.3.6) by $\cos(u\theta)$, integrate w.r.t. θ from 0 to π, to get

$$\int_0^{\pi}(\sin\theta)^{\rho}\cos(u\theta)\; A_{p,q}^{m,n}\left[z.\,(\sin\theta)^{-2\delta}\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right] d\theta$$

$$= \frac{C_0}{2}\int_0^{\pi}\cos(u\theta)\,d\theta + \sum_{r=1}^{\infty} C_r\int_0^{\pi}\cos r\theta\cos u\theta\,d\theta.$$

Now using (4.3.15) and cosine function's orthogonal property, we get

$$C_u = \frac{2}{\sqrt{\pi}}\cos\frac{\pi u}{2} A_{p+2,q+2}^{m,n+2}\left[z\Big|{}^{(a_j,\alpha_j)_{1,p},\left(-\frac{\rho+u}{2},\delta\right),\left(-\frac{\rho-u}{2},\delta\right)}_{\left(1-\frac{1+\rho}{2},\delta\right),\left(-\frac{\rho}{2},\delta\right),(b_j,\beta_j)_{1,q}}\right] \quad (8.3.7)$$

From (8.3.6) and (8.3.7), the result (8.3.5) is obtained.

Fourier series 8.3.3

$$(\sin\theta)^{\rho} A_{p,q}^{m,n}\left[z.(\sin\theta)^{-2\delta}\Big|{}_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$+\frac{2}{\sqrt{\pi}}\sum_{r=1}^{\infty} A_{p+2,q+2}^{m,n+2}\left[z\Big|{}_{\left(1-\frac{1+\rho}{2},\delta\right),\left(-\frac{\rho}{2},\delta\right),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},\left(-\frac{\rho+r}{2},\delta\right),\left(-\frac{\rho-r}{2},\delta\right)}\right].\sin(\pi r/2)\sin r\theta, \tag{8.3.8}$$

provided that δ is a positive number and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (8.3.8), let

$$f(\theta) = (\sin\theta)^{\rho} A_{p,q}^{m,n}\left[z.(\sin\theta)^{-2\delta}\Big|{}_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= \sum_{r=1}^{\infty} C_r \sin r\theta, \tag{8.3.9}$$

Multiply (8.3.9) by cos(uθ), integrate w.r.t. θ from 0 to π, and using (4.3.16) and sine function's orthogonal property, to get

$$C_u = \frac{2}{\sqrt{\pi}}\sin\frac{\pi u}{2} A_{p+2,q+2}^{m,n+2}\left[z\Big|{}_{\left(1-\frac{1+\rho}{2},\delta\right),\left(-\frac{\rho}{2},\delta\right),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},\left(-\frac{\rho+u}{2},\delta\right),\left(-\frac{\rho-u}{2},\delta\right)}\right], \tag{8.3.10}$$

From (8.3.9) and (8.3.10), the formula (8.3.8) follows immediately.

Fourier series 8.3.4

$$(\sin x)^{w-1}\ {}_PF_Q\left[{}^{\alpha_P:c(\sin x)^{2h}}_{\beta_Q}\right]\ {}_UF_V\left[{}^{\gamma_U:d(\sin x)^{2k}}_{\delta_V}\right]$$

$$\times\ A_{p,q}^{m,n}\left[z.(\sin x)^{2\lambda}\Big|{}_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= \frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\sum_{r,t=0}^{\infty}\frac{(\alpha_P)_r c^r (\gamma_U)_t d^t}{(\beta_Q)_r r!\,(\delta_V)_t\, t!}\ e^{in(\pi/2-x)}$$

$$A_{p+2,q+2}^{m+2,n+2}\left[z\Big|{}_{\left(\frac{\omega+2hr+2kt}{2},\lambda\right),\left(\frac{\omega+2hr+2kt+1}{2},\lambda\right),(b_j,\beta_j)_{1,q}}^{\left(\frac{\omega+2hr+2kt}{2},\lambda\right),\left(\frac{\omega+2hr+2kt+1}{2},\lambda\right),(a_j,\alpha_j)_{1,p}}\right] \tag{8.3.11}$$

where n's are either even or odd in addition to the conditions of validity followed by (4.3.17).

Proof

To prove (8.3.11), let

$$f(x) = (\sin x)^{\omega-1}\ {}_PF_Q\left[{}^{\alpha_P:c(\sin x)^{2h}}_{\beta_Q}\right]\ {}_UF_V\left[{}^{\gamma_U:d(\sin x)^{2k}}_{\delta_V}\right]$$

$$\times A^{m,n}_{p,q}\left[z.(\sin x)^{2\lambda}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= \sum_{n=-\infty}^{\infty} A_n\, e^{-inx}. \quad (8.3.12)$$

As f(x) is of bounded variation and continuous in (0, π), equation (8.3.12) is valid.

Multiply (8.3.12) with e^{imx}, integrate w.r.t. x from 0 to π, to get

$$\int_0^{\pi}(\sin x)^{\omega-1}\, e^{imx}\ {}_PF_Q\left[{}^{\alpha_P:c(\sin x)^{2h}}_{\beta_Q}\right]\ {}_UF_V\left[{}^{\gamma_U:d(\sin x)^{2k}}_{\delta_V}\right]$$

$$\times A^{m,n}_{p,q}\left[z.(\sin x)^{2\lambda}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dx$$

$$= \sum_{n=-\infty}^{\infty} A_n \int_0^{\pi} e^{i(m-n)x}\,dx.$$

Now using (4.3.17) and (8.2.2), we get

$$A_m = \frac{1}{\sqrt{\pi}}\, e^{im\pi/2} \sum_{r,t=0}^{\infty} \frac{(\alpha_P)_r c^r (\gamma_U)_t d^t}{(\beta_Q)_r r!\,(\delta_V)_t\, t!}$$

$$\times\quad A^{m+2,n}_{p+2,q+2}\left[z\Big|^{\left(\frac{\omega+2hr+2kt}{2},\lambda\right),\left(\frac{\omega+2hr+2kt+1}{2},\lambda\right),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},\left(\frac{\omega+2hr+2kt\pm m+1}{2},\lambda\right)}\right] \quad (8.3.13)$$

From (8.3.12) and (8.3.13), the Fourier exponential series (8.3.11) is obtained.

Fourier series 8.3.5

$$(\sin x)^{w-1}\ {}_PF_Q\left[{}^{\alpha_P:c(\sin x)^{2h}}_{\beta_Q}\right]\ {}_UF_V\left[{}^{\gamma_U:d(\sin x)^{2k}}_{\delta_V}\right]$$

$$\times A^{m,n}_{p,q}\left[z.(\sin x)^{2\lambda}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= \frac{1}{\sqrt{\pi}} \sum_{r,t=0}^{\infty} \frac{(\alpha_P)_r c^r (\gamma_U)_t d^t}{(\beta_Q)_r r!\, (\delta_V)_t\, t!} \; A_{p+1,q}^{m+1,n} \left[z \Big|_{(b_j,\beta_j)_{1,q}}^{\left(\frac{\omega+2hr+2kt}{2},\lambda\right),(a_j,\alpha_j)_{1,p}} \right]$$

$$+ \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \sum_{r,t=0}^{\infty} \frac{(\alpha_P)_r c^r (\gamma_U)_t d^t}{(\beta_Q)_r r!\, (\delta_V)_t\, t!} \; e^{in\pi/2} \cos nx$$

$$A_{p+2,q+2}^{m+2,n} \left[z \Big|_{(b_j,\beta_j)_{1,q},\left(\frac{\omega+2hr+2kt\pm n+1}{2},\lambda\right)}^{\left(\frac{\omega+2hr+2kt}{2},\lambda\right),\left(\frac{\omega+2hr+2kt+1}{2},\lambda\right),(a_j,\alpha_j)_{1,p}} \right] \tag{8.3.14}$$

where n's are either even or odd in addition to the conditions of validity followed by (4.3.17).

Proof

To establish (8.3.14), let

$$(\sin x)^{\omega-1} \; {}_PF_Q\left[{}^{\alpha_P : c(\sin x)^{2h}}_{\beta_Q}\right] {}_UF_V\left[{}^{\gamma_U : d(\sin x)^{2k}}_{\delta_V}\right]$$

$$\times \; A_{p,q}^{m,n} \left[z.(\sin x)^{2\lambda} \Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}} \right]$$

$$= \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos nx. \tag{8.3.15}$$

Multiply (8.3.15) with e^{imx}, integrate w.r.t. x from 0 to π, and using (4.3.17) and (8.2.3), we get

$$B_m = \frac{2}{\sqrt{\pi}} e^{im\pi/2} \sum_{r,t=0}^{\infty} \frac{(\alpha_P)_r c^r (\gamma_U)_t d^t}{(\beta_Q)_r r!\, (\delta_V)_t\, t!}$$

$$\times \; A_{p+2,q+2}^{m+2,n} \left[z \Big|_{(b_j,\beta_j)_{1,q},\left(\frac{\omega+2hr+2kt\pm m+1}{2},\lambda\right)}^{\left(\frac{\omega+2hr+2kt}{2},\lambda\right),\left(\frac{\omega+2hr+2kt+1}{2},\lambda\right),(a_j,\alpha_j)_{1,p}} \right] \tag{8.3.16}$$

From (8.3.15) and (8.3.16), the Fourier cosine series (8.3.14) is obtained.

Fourier series 8.3.6

$$(\sin x)^{w-1} \; {}_PF_Q\left[{}^{\alpha_P : c(\sin x)^{2h}}_{\beta_Q}\right] {}_UF_V\left[{}^{\gamma_U : d(\sin x)^{2k}}_{\delta_V}\right]$$

$$\times \; A_{p,q}^{m,n} \left[z.(\sin x)^{2\lambda} \Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}} \right]$$

$$= \frac{2}{i\sqrt{\pi}} \sum_{n=1}^{\infty} \sum_{r,t=0}^{\infty} \frac{(\alpha_P)_r c^r (\gamma_U)_t d^t}{(\beta_Q)_r r!\, (\delta_V)_t\, t!} \; e^{in\pi/2} \sin nx$$

$$\times \quad A^{m+2,n}_{p+2,q+2}\left[z\Big|^{\left(\frac{\omega+2hr+2kt}{2},\lambda\right),\left(\frac{\omega+2hr+2kt+1}{2},\lambda\right),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},\left(\frac{\omega+2hr+2kt\pm n+1}{2},\lambda\right)}\right], \qquad (8.3.17)$$

where n's are either even or odd in addition to the conditions of validity followed by (4.3.17).

Proof

To prove (8.3.17), let

$$(\sin x)^{\omega-1}\ {}_PF_Q\left[{}^{\alpha_P:c(\sin x)^{2h}}_{\beta_Q}\right]\ {}_UF_V\left[{}^{\gamma_U:d(\sin x)^{2k}}_{\delta_V}\right]$$

$$\times \ A^{m,n}_{p,q}\left[z.(\sin x)^{2\lambda}\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= \sum_{n=1}^{\infty} C_n \sin nx. \qquad (8.3.18)$$

Multiply (8.3.18) with e^{imx}, integrate w.r.t. x from 0 to π, and using (4.3.17) and (8.2.4), we get

$$C_m = \frac{2}{i\sqrt{\pi}} e^{im\pi/2} \sum_{r,t=0}^{\infty} \frac{(\alpha_P)_r c^r (\gamma_U)_t d^t}{(\beta_Q)_r r!\,(\delta_V)_t\, t!}$$

$$\times \ A^{m+2,n}_{p+2,q+2}\left[z\Big|^{\left(\frac{\omega+2hr+2kt}{2},\lambda\right),\left(\frac{\omega+2hr+2kt+1}{2},\lambda\right),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},\left(\frac{\omega+2hr+2kt\pm m+1}{2},\lambda\right)}\right] \qquad (8.3.19)$$

From (8.3.18) and (8.3.19), the Fourier sine series (8.3.17) is obtained.

Fourier series 8.3.7

$$(\sin\theta)^{1-2u} A^{m,n}_{p,q}\left[z.\sin^{2h}\theta\Big|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= \frac{2}{\sqrt{\pi}} \sum_{r=0}^{\infty} A^{m+1,n+1}_{p+2,q+2}\left[z\Big|^{\left(\frac{3}{2}-u,h\right),(a_j,\alpha_j)_{1,p},(1-u,h)}_{(1-u-r,h),(b_j,\beta_j)_{1,q},(2-u+r,h)}\right] \sin(2r+1)\theta, \qquad (8.3.20)$$

provided that h is a positive number, $0 \le \theta \le \pi$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (8.3.20), let

$$f(\theta) = (\sin\theta)^{1-2u}\, A_{p,q}^{m,n}\left[z.\sin^{2h}\theta\,\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= \sum_{r=0}^{\infty} \sin(2r+1)\theta\, C_r, \qquad (8.3.21)$$

$R(1 - 2u) > 0,\ 0 \le \theta \le \pi.$

As $f(\theta)$ is of bounded variation and continuous in $(0, \pi)$ when $R(1 - 2u) \ge 0$, equation (8.3.24) is valid.

Multiply (8.3.21) with $\sin(2v+1)\theta$, integrate w.r.t. θ from 0 to π, to get

$$\int_0^{\pi} (\sin\theta)^{1-2u} \sin(2v+1)\theta\, A_{p,q}^{m,n}\left[z.\sin^{2h}\theta\,\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] d\theta$$

$$=\sum_{r=0}^{\infty} C_r \int_0^{\pi} \sin(2v+1)\theta \sin(2r+1)\theta\, d\theta.$$

Now using (4.3.18) and sine function's orthogonal property, we have

$$C_v = \frac{2}{\sqrt{\pi}} A_{p+2,q+2}^{m+1,n+1}\left[z\,\Big|_{(u+v,h),(b_j,\beta_j)_{1,q},(-1+u-v,h)}^{\left(-\frac{1}{2}+u,h\right),(a_j,\alpha_j)_{1,p},(u,h)}\right] \qquad (8.3.22)$$

The result (8.3.20) is obtained with the help of (8.3.21) and (8.3.22).

Fourier series 8.3.8

$$(\sin\theta/2)^{-2u} A_{p,q}^{m,n}\left[z.\sin^{2h}(\theta/2)\,\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= \frac{1}{\sqrt{\pi}} A_{p+1,q+1}^{m+1,n}\left[z\,\Big|_{(b_j,\beta_j)_{1,q},(1-u,h)}^{\left(\frac{1}{2}-u,h\right),(a_j,\alpha_j)_{1,p}}\right]$$

$$+\frac{2}{\sqrt{\pi}}\sum_{r=0}^{\infty} A_{p+2,q+2}^{m+1,n+1}\left[z\,\Big|_{(1-u-r,h),(b_j,\beta_j)_{1,q},(1-u+r,h)}^{\left(\frac{1}{2}-u,h\right),(a_j,\alpha_j)_{1,p},(1-u,h)}\right] \cos r\theta, \qquad (8.3.23)$$

provided that h is a positive number, $0 \le \theta \le \pi$ and $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

To prove (8.3.23), let us consider

$$f(\theta) = (\sin\theta/2)^{-2u}\, A_{p,q}^{m,n}\left[z.\sin^{2h}(\theta/2)\,\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$= \frac{C_0}{2} + \sum_{r=1}^{\infty} C_r \cos r\theta, \tag{8.3.24}$$

$R(2u) > 0$, $0 \le \theta \le \pi$.

Multiply (8.3.24) by $\cos(v\theta)$, integrate w.r.t. θ from 0 to π, and using (4.3.19) and cosine function's orthogonal property, to get

$$C_v = \frac{2}{\sqrt{\pi}} A_{p,q}^{m,n}\left[z\Big|_{(u+v,h),(b_j,\beta_j)_{1,q},(u-v,h)}^{\left(\frac{1}{2}+u,h\right),(a_j,\alpha_j)_{1,p},(u,h)}\right] \tag{8.3.25}$$

From (8.3.24) and (8.3.25), the formula (8.3.23) follows.

Fourier series 8.3.9

$$\sum_{r=0}^{\infty} A_{p+2,q+2}^{m+1,n+1}\left[z\Big|_{\left(-\frac{1}{2},1\right),(b_j,\beta_j)_{1,q},(0,1)}^{(r,1),(a_j,\alpha_j)_{1,p},(-1-r,1)}\right]\sin(2r+1)\theta$$

$$= \frac{\sqrt{\pi}}{2} \sin\theta\, A_{p,q}^{m,n}\left[\frac{z}{\sin^2\theta}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] \tag{8.3.26}$$

provided that $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

Proof

Using (1.2.35), the expression on the left side of (8.3.26) can be written as

$$\sum_{r=0}^{\infty} \frac{1}{2\pi i}\int_L \theta(s)\left[\frac{\Gamma(\frac{3}{2}-s)\Gamma(r+s)}{\Gamma(s)\Gamma(2+r-s)}\sin(2r+1)\,\theta\right] z^s ds$$

On changing the order of integration and summation which is easily seen to be justified, the above expression becomes

$$\frac{1}{2\pi i}\int_L \theta(s)\frac{\Gamma(\frac{3}{2}-s)}{\Gamma(2-s)}\left[\sum_{r=0}^{\infty}\frac{(s)_r}{(2-s)_r}\sin(2r+1)\,\theta\right] z^s ds.$$

and on using the relation (8.2.5), it takes the form

$$\frac{\sqrt{\pi}}{2}\sin\theta . \frac{1}{2\pi i}\int_L \theta(s)\,(z/\sin^2\theta)^s ds.$$

which is just the expression on the right side of (8.3.26). (8.3.26) is the Fourier sine series for the A-function of one variable.

Fourier series 8.3.10

$$A_{p,q+2}^{m,n+1}\left[z\Big|_{\left(\frac{1}{2},1\right),(b_j,\beta_j)_{1,q},(0,1)}^{(a_j,\alpha_j)_{1,p}}\right]$$

$$+2A_{p+2,q+2}^{m+1,n+1}\left[z\Big|_{\left(\frac{1}{2},1\right),(b_j,\beta_j)_{1,q},(0,1)}^{(r,1),(a_j,\alpha_j)_{1,p},(-r,1)}\right] cosr\theta$$

$$= \sqrt{\pi}A_{p,q}^{m,n}\left[\frac{z}{sin^2\frac{\theta}{2}}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]. \qquad (8.3.27)$$

provided that $|\arg(uz)| < \frac{1}{2}\pi h$, where h and u are given in (1.2.37) and (1.2.38) respectively.

The Fourier cosine series (8.3.27) is proved in an analogous manner by using (1.2.35) and (8.2.6).

CHAPTER–9

SUMMARY AND CONCLUSION

9.1 INTRODUCTION

The special functions in mathematics arise in the solution of differential equation governing the behavior of certain physical quantities. Therefore a function ‘special’ when the function has a place in the toolkit of the applied scientist, engineer and the applied mathematician. These are denoted by particular notation and have number of properties. Mathematically, special functions are functions defined on R, the set of rear number or C, the set of complex number and these are not only represented by series representation, but also by integral representations. This thesis is mainly concerned with the A-function and its properties. So the concept of Pochhammer symbols, calculus of residue, Mellin-Barnes integrals and convergence are necessary for the detailed study. Recently the attention of mathematicians towards these functions has increased from both the analytical and numerical point of view due to their wide use.

The present study had been undertaken with the following specific objectives:

- To develop some new generating relations involving A- function of one variable.
- To find some new definite and indefinite integrals involving A-function of one variable.
- To find innovative Fourier Series involving A-function.
- To find some new expansions involving A-function.
- To find some new identities involving A-function.
- To obtain new solutions of some boundary value problems in term of A-function.

9.2 SUMMARY

The present thesis has been divided into nine chapters. In first chapter, the historical background, development and definitions of the A-functions of one variable and polynomials in the context of the research work accomplished in the subsequent chapters of this thesis are given in this chapter. It also provide brief literature of several aspects of special functions.

Generating relations plays an important role in the investigation of various useful properties of the sequences, which they generate. In second chapter, 'Linear and Bilinear Generating Relations involving A-Function' looking into the requirement and importance of various properties of generating relations in the analysis of many problems of mathematics and mathematical physics, we have established eight new linear and four bilinear generating relations involving A-function of one variable.

Several authors have discussed a number of bilateral and trilateral generating relations involving generalized hypergeometric functions time to time. The usefulness of A-Function has inspired us to find some new generating relations. In third chapter, 'Bilateral and Trilateral Generating Relations involving A-Function' some new bilateral and trilateral generating relations have been established involving A-function of one variable and other hypergeometric functions.

Integrals are useful in connection with the study of certain boundary value problems. It is also helpful for obtaining the expansion formula. In fourth chapter 'Definite and Indefinite Integrals involving A-function' we have evaluated some definite, indefinite and double integrals involving the A-function of one variable and other generalized hypergeometric functions.

In Fifth Chapter, 'Integration Involving Certain Products and A-Function' we have established two integrals containing the products of other hypergeometric functions and A-Function. We have represented these two integrals in another forms and also discussed particular cases. We have evaluated new integrals involving A-functions with the help of finite difference operator $[E_a f(a) = f(a + 1)]$.

Looking into the requirement and importance of various properties of expansion and identity in various field, in sixth chapter ‘Expansion and Identities Involving A-Function’ We have established six new expansions and nine new identities involving A-function of one variable.

Various problems in science and technology, when formulated mathematically, lead naturally to certain classes of partial differential equations involving one or more unknown functions together with the prescribed conditions (known as boundary conditions) which arise from the physical situation. Several researchers have obtained solutions to the equations related to certain problems, which satisfy the given boundary conditions. In the seventh chapter ‘Application of A-Function of one variable in obtaining a Solution of some Boundary Value Problems” first we evaluated an integral involving A-function of one variable and then we applied it to get solution of two boundary value problems on (i) heat conduction in a bar (ii) deflection of vibrating string under certain conditions. We have engaged the A-function of one variable in obtaining a solution of a partial differential equation related to heat conduction along with Hermite polynomials. We have derived a solution of special one-dimensional time dependent Schrodinger equation involving Hermite polynomials and A-function of one variable and also obtained a solution of a bounded electrostatic potential in the semi-infinite space.

The subject of Fourier series for generalized hypergeometric functions occupies outstanding place in the literature of special functions and boundary value problems. Certain double Fourier series of generalized hypergeometric functions play vital role in the improvement of the theories of special functions and two-dimensional boundary value problems.

Looking vital role of Fourier series in the literature of special functions and boundary value problems, in eighth chapter, ‘Fourier Series Involving A-Function’ we have established some new Fourier series involving A-Function of one variables on the lines of Bajpai and others.

9.3 CONCLUSION

The conclusions of this thesis are as follows:

- We have evaluated new linear and bilinear generating relations involving A-function of one variable.
- We have established new bilateral and trilateral generating relations involving A-function of one variable.
- New definite and indefinite integrals involving A-function of one variable has been established.
- Innovative Fourier series involving A-function has been derived.
- New expansions and identities involving A-function has been founded.
- New solutions of some boundary value problems involving A-function has been obtained viz. Heat conduction, wave equation, and bounded electrostatic potential in semi-infinite space.

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APPENDIX-I

LIST OF RESEARCH PAPERS

[1] Kishore, K., Srivastava, S. S. (2010). Some new Identities involving A-Function. *Vijnana Parishad Anusandhan Patrika.* 53(2): 121-124.

[2] Kishore, K., Srivastava, S.S. (2011). Bounded Electrostatic potential in the Semi-infinite Space and A-Function. *Journal of Indian Academy of Mathematics.* 33(2): 479-482.

[3] Kishore, K., Srivastava, S.S. (2012). Some Double Integrals involving A-Function. *The Mathematics Education.* 46(4): 191-194.

[4] Kishore, K., Srivastava, S.S. (2013). A-Function, Hermite Polynomials and Time Dependent Schrodinger Equations. *The Mathematics Education*. 47(2):135-147.

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APPENDIX-II

TITLE PAGE OF RESEARCH PAPERS

Vijnana Parishad Anusandhan Patrika, Vol. 53, No. 2, April 2010

A-फलन वाली कुछ नवीन तत्समिकाएं

कमल किशोर,

गणित विभाग, एस.सी.डी. शासकीय कालेज लुधियाना (पंजाब)

तथा

एस.एस.श्रीवास्तव

गणित विभाग, शासकीय स्नातकोत्तर कालेज, शहडोल (म.प्र.)

[प्राप्त – अक्टूबर 15, 2009]

सारांश

प्रस्तुत प्रपत्र का उद्देश्य एक चर वाले A-फलन की कुछ नवीन तत्समिकाओं की स्थापना करना है।

Abstract

Some new identities involving A-function. *By* Kamal Kishore, Department of Mathematics, S.C.D. Government College, Ludhiana (Pb.) and S.S. Srivastava, Department of Mathematics, Government P.G. College, Shahdol (M.P.)

The aim of this paper is to establish some new identities involving A-function of one variable.

1. प्रस्तावना

एक चर वाले A-फलन को गौतमा ने परिभाषित किया है और हम यहाँ पर इसे निम्नवत् प्रदर्शित करते हैं –

Journal of Indian Acad. Math.
Vol. 33, No. 2 (2011) pp. 479-482

Kamal Kishore[1] *and* *S. S. Srivastava*[2]

BOUNDED ELECTROSTATIC POTENTIAL IN THE SEMI-INFINITE SPACE AND A-FUNCTION

Abstract: The aim of this paper is to obtain bounded Electrostatic Potential in the Semi-Infinite Space with the help of A-function of one variable.

Key words: A-function, contour integral, Bounded Electrostatic Potential.

Mathematics Subject Classification: 33D90

1. Introduction

The A-function of one variable is defined by Gautam [3] and we will represent here in the following manner:

$$A^{m,n}_{p,q}\left[x\Big|^{(a_j,\alpha_j)}_{(b_j,\beta_j)}\right] = \frac{1}{2\pi i}\int_L \theta(s)\, x^s\, ds \tag{1}$$

where $i = \sqrt{(-1)}$ and

(i)

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(a_j + s\alpha_j)\prod_{j=1}^{n}\Gamma(1 - b_j - s\beta_j)}{\prod_{j=m+1}^{p}\Gamma(1 - a_j - s\alpha_j)\prod_{j=n+1}^{q}\Gamma(b_j + s\beta_j)} \tag{2}$$

(ii) m, n, p and q are non-negative numbers in which $m \le p$, $n \le q$.

The Mathematics Education
Volume XLVI, No. 4, December 2012

ISSN 0047-6269

Some Double Integrals Involving *A*-Function of One Variable

by **Kamal Kishore,**
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&
S.S. Srivastava,
Department of Mathematics,
Govt. P.G. College, Shahdol - 484001

(Received January 25, 2010)

Abstract :

The aim of this paper is to establish some new double integrals involving A function of one variable.

1. Introduction :

The *A*-function of one variable is defined by Gautam [1] and we will represent here in the following manner :

$$A^{m,n}_{p,q}\left[x\left|\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\theta(s)x^s\,ds \tag{1}$$

where $i=\sqrt{(-1)}$ and

(i)

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(a_j+s\alpha_j)\prod_{j=1}^{n}\Gamma(1-b_j-s\beta_j)}{\prod_{j=m+1}^{p}\Gamma(1-a_j-s\alpha_j)\prod_{j=n+1}^{q}\Gamma(b_j+s\beta_j)} \tag{2}$$

The Mathematics Education
Volume XLVII, No. 3, September 2013 ISSN 0047-6269

A-Function, Hermite Polynomials and Time-dependent Schrodinger Equation

by **Kamal Kishore,**
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(Received January 18, 2011)

Abstract :

The aim of this paper is to derive the solution of special one dimensional time dependent Schrodinger equation involving Hermite polynomials and A-function of one variable.

1. Introduction :

One of the fundamental problems in quantum mechanics is to find solution of Schrodinger equation for different forms of potentials. The Schrodinger equation and more general formulation of quantum mechanics have been set up as a result of the failure of classical physics to predict correctly the result of experiments on microscopic systems. They must be verified by testing their predictions of the properties of systems, where classical mechanics has failed and also where it has succeeded. In fact the whole of atomic physics, solid state physics, chemistry and some other branches of applied sciences obey the principals of quantum mechanics or satisfy differential equations similar to the Schrodinger equations, and the same is almost certainly true for nuclear and particle physics, although the, understanding of very high energy phenomena, where relativistic effects are important, requires a further generalization of theory.

A-function of one variable is defined by Gautam and Goyal [3] as follows :

$$A_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{matrix}\right] = \frac{1}{2\pi i}\int_L \theta(s)\, x^s\, ds \tag{1.1}$$

Applied Science Periodical
Volume XV, No. 1. February 2013

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Some New Bilinear Generating Relations Involving *A*-Function of One Variable

by **Kamal Kishore,**
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&
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Govt. P.G. College,
Shahdol - 484001
(Received February 05, 2011)

Abstract :

The aim of this paper is to establish some new bilinear generating relations involving A-function of one variable.

1. Introduction :

The A-function of one variable is defined by Gautam [1] and we will represent here in the following manner :

$$A^{m,n}_{p,q}\left[x\left|\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \theta(s)x^s ds \quad (1)$$

where $i=\sqrt{(-1)}$ and

(i) $$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(a_j-s\alpha_j)\prod_{j=1}^{n}\Gamma(1-b_j-s\beta_j)}{\prod_{j=m+1}^{p}\Gamma(1-a_j-s\alpha_j)\prod_{j=n+1}^{q}\Gamma(b_j+s\beta_j)} \quad (2)$$

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EXPANSION FORMULAE INVOLVING A-FUNCTION

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ABSTRACT

In this paper, we establish some new some new expansion formulae involving A-function of two variables.

1. INTRODUCTION:

The subject of expansion formulae of generalized hypergeometric functions occupies a vital position in the literature of special functions. Certain two-dimensional expansion formulae of generalized hypergeometric functions participate major role in the growth of the theories of special functions and two-dimensional boundary value problems.

The A-function of one variable is defined by Gautam [2] and we will represent here in the following manner:

$$A_{p,q}^{m,n}\left[x\middle|_{((b_q,\beta_q))}^{((a_p,\alpha_p))}\right] = \frac{1}{2\pi i}\int_L \theta(s)\, x^s\, ds \tag{1}$$

where $i = \sqrt{(-1)}$ and

(i)

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(a_j + s\alpha_j)\prod_{j=1}^{n}\Gamma(1 - b_j - s\beta_j)}{\prod_{j=m+1}^{p}\Gamma(1 - a_j - s\alpha_j)\prod_{j=n+1}^{q}\Gamma(b_j + s\beta_j)} \tag{2}$$

(ii) m, n, p and q are non-negative numbers in which $m \le p$, $n \le q$.

(iii) $x \neq 0$ and parameters a_j, α_j, b_k and β_k (j = 1 to p and k = 1 to q) are all complex.

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Some Integrals Involving Product of Hypergeometric Function and *A*-Function

by **Kamal Kishore,**
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Institute for Excellence in Higher Education,
Bhopal

Abstract :

In this paper we evaluate some integrals involving the products of A function and other hypergeometric functions, while in last section some integrals involving the product of genernlized hypergeometric function and A-function of one variable will be derived by means of finite differance operator E.

1. Introduction :

The A-function of one variable is defined by Gautam [1]

$$A^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \theta(s)\,x^s ds$$

where $i=\sqrt{(-1)}$ and

(i)

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(a_j-s\alpha_j)\prod_{j=1}^{n}\Gamma(1-b_j-s\beta_j)}{\prod_{j=m+1}^{p}\Gamma(a_j-s\alpha_j)\prod_{j=n+1}^{q}\Gamma(b_j-s\beta_j)}$$

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Some New Linear Generating Relations Involving A-Function of One Variable

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Department of Mathematics Institute for Excellence in Higher Education, Bhopal (M.P)

Abstract: *The aim of this paper is to establish some new linear generating relations involving A-function of one variable.*

I. Introduction

The A-function of one variable is defined by Gautam [1] and we will represent here in the following manner:

$$A_{p,q}^{m,n}\left[x\left|\begin{matrix}((a_p,\alpha_p))\\((b_q,\beta_q))\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \theta(s)\,x^s\,ds \qquad (1)$$

where $i=\sqrt{(-1)}$ and

(i)

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(a_j+s\alpha_j)\prod_{j=1}^{n}\Gamma(1-b_j-s\beta_j)}{\prod_{j=m+1}^{p}\Gamma(1-a_j-s\alpha_j)\prod_{j=n+1}^{q}\Gamma(b_j+s\beta_j)} \qquad (2)$$

(ii) m, n, p and q are non-negative numbers in which $m\le p$, $n\le q$.

(iii) $x\neq 0$ and parameters a_j, α_j, b_k and β_k (j = 1 to p and k = 1 to q) are all complex.

The integral in the right hand side of is convergent if

(i) $x\neq 0,\ k=0,\ h>0,\ |\arg(ux)|<\pi h/2$

(ii) $x>0,\ k=0=h,\ (v-\sigma\omega)<-1$

where

$$k=\operatorname{Im}\left(\sum_1^p\alpha_j-\sum_1^q\beta_j\right)$$

$$h=\operatorname{Re}\left(\sum_{j=1}^{m}\alpha_j-\sum_{j=m+1}^{p}\alpha_j+\sum_{j=1}^{n}\beta_j-\sum_{j=m+1}^{q}\beta_j\right)$$

$$u=\prod_1^p\alpha_j^{\alpha_j}\prod_1^q\beta_j^{\beta_j}$$

$$v=\operatorname{Re}\left(\sum_1^p a_j-\sum_1^q b_j\right)-(p-q)/2\,,$$

$$w=\operatorname{Re}\left(\sum_1^q\beta_j-\sum_1^p\alpha_j\right)$$

and $s=\sigma+it$ is on path L when $|t|\to\infty$.

II. Formula Required

From Rainville [2]:

$$(\alpha)_n=(\alpha,n)=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \qquad (3)$$

$$\Gamma(1-\alpha-n)=\frac{(-1)^n\Gamma(1-\alpha)}{(\alpha)_n}, \qquad (4)$$

$$(1-z)^{-a}=\sum_{n=0}^{\infty}(a)_n\frac{z^n}{n!}, \qquad (5)$$

$$(1+z)^{-a}=\sum_{n=0}^{\infty}(a)_n\frac{(-z)^n}{n!}, \qquad (6)$$

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SOME BILATERAL AND TRILATERALGENERATING RELATIONS INVOLVING A-FUNCTION

By
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ABSTRACT

The A-function of one variable plays an important role in the development and study of special functions. The usefulness of this function has inspired us to find some new generating relations. In this paper some new bilateral and trilateral generating relations have been established involving A-function of one variable and other hypergeometric functions.

1. INTRODUCTION:

The A-function of one variable is defined by Gautam [1] and we will represent here in the following manner:

$$A^{m,n}_{p,q}\left[x\,\Big|^{((a_p,\alpha_p))}_{((b_q,\beta_q))}\right] = \frac{1}{\ }\int \theta(s)\, x^s\, ds \qquad (1.1)$$

where $i = \sqrt{(-1)}$ and

(i)

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(a_j + s\alpha_j)\prod_{j=1}^{n}\Gamma(1 - b_j - s\beta_j)}{\prod_{j=m+1}^{p}\Gamma(1 - a_j - s\alpha_j)\prod_{j=n+1}^{q}\Gamma(b_j + s\beta_j)} \qquad (1.2)$$

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Fourier Series Involving A-Function

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ABSTRACT

The object of the present paper is to establish two Fourier series expansion formulae involving A-function of one variable.

KEYWORDS - Hyper-geometric functions, integration and summation, Fourier sine series, Variable

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APPENDIX-III

ATTENDED CONFERENCES

1. International conference on Emerging areas of Mathematics for Science and Technology, January 30 – February 01, 2015, Deptt. of Mathematics, Punjabi University, Patiala.
2. National conference on emerging challenges in Physics and Nano Science, March 4, 2015, PG Deptt. of Physics, JCDAV College, Dasuya.
3. 17^{th} APG meet and national conference, November 4-5, 2016, PG Deptt. of Geography, SCD Govt. College, Ludhiana.
4. International Conference on Recent trends in Mathematical Sciences and Cosmology, December 17-18, 2016, Deptt. of Mathematics and Computer Science, Govt. Model Science College, Rewa (M.P.).
5. International conference of skill in management and applied sciences, April 24-25, 2017, SCD Govt. College, Ludhiana.
6. International conference on Recent advancement in Science and Technology, May 5-7, 2017, Technocrats Institute of Technology, Bhopal.