

**A Mathematical Study of Role of Delay in Stability and  
Bifurcation Analysis of Plant Growth Dynamics under the Effect  
of Toxicants**

A  
Thesis

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## DECLARATION BY THE CANDIDATE

I declare that the thesis entitled "*A Mathematical Study of Role of Delay in Stability and Bifurcation Analysis of Plant Growth Dynamics under the Effect of Toxicants*" submitted for award of degree of Doctor of Philosophy in Mathematics, Lovely Professional University, Phagwara, is my own work conducted under the supervision of Dr. Preety Kalra (Supervisor), Assistant Professor in Department of Mathematics at School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab. I further declare that to the best of my knowledge the thesis does not contain any part of any work which has been submitted for the award of any degree in this University or in any other University/Deemed University without proper citation.

Signature of the Supervisor  
(Dr. Preety Kalra)

Signature of the Candidate  
(Pankaj Kumar)

## CERTIFICATE

This is to certify that the work entitled “*A Mathematical Study of Role of Delay in Stability and Bifurcation Analysis of Plant Growth Dynamics under the Effect of Toxicants*” is a piece of research work done by Mr. Pankaj Kumar under my guidance and supervision for the degree of Doctor of Philosophy of Mathematics , Lovely Professional University, Phagwara (Punjab) India. To the best of my knowledge and belief the thesis:

1. Embodies the work of the candidate himself.
2. Has Duly been completed.
3. Fulfils the requirements of the ordinance related to the Ph.D. degree of the university
4. Is upto the standard both in respect of contents and language for being referred to the examiner.

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**Dedicated**

**To God**

**and**

**My Family**

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Pankaj Kumar

## ABSTARCT

In the proposed work, plant growth dynamics are studied considering delay in nutrient uptake rate, utilization rate and nutrient use efficiency rate under the effect of toxicant present in soil or plant. Plant growth depends largely on the availability of nutrients and favourable resources. The presence of toxicant can hamper the supply of nutrients and their utilization and hence affects the plant growth. This impedance in nutrient supply and utilization are taken as the delay parameter, which plays a key role in this entire study. In the proposed work, the models for single plant growth, tree growth and plant population are formulated considering the delay in growth dynamics under the effect of toxicants. The proposed models have been analysed mathematically and the results have been verified numerically and further validated with already existing statistical data of plant growth under the effect of various toxicants.

Mathematically, the positivity and boundedness of all analytical solutions is established using comparison theorem. All possible and feasible interior and exterior equilibrium are calculated. Local stability analysis of interior equilibrium is established. Stability analysis of interior equilibrium considering delay parameter resulted into Hopf-bifurcation showing the complex dynamical behaviour. Nature of the roots has been studied in detail using Rouché's theorem. Sensitivity analysis of state variables with respect to model parameters is done for almost all the models using 'Direct Method'. Numerical simulation is done using MATLAB where all the model parameters have been assigned different numerical values. This helped to find the critical value of the delay parameter below which the system exhibited stability and above this critical value, the system lost stability and Hopf-bifurcation occurred.

In chapter-1, The general introduction about plant growth dynamics under the effect of toxicants is given. The remarkable work done by the researchers is cited and the gaps have been identified through extensive literature review. All the important concepts of plant physiology and the necessary mathematical concepts required for their study have also been described. It also includes the proposed objectives of the study and summery of all the chapters.

In chapter-2, The effect of exogenic activities on nutrient concentration is studied by considering delay in nutrient utilization. A two-compartment mathematical model is proposed for the study of this phenomenon. It is shown that exogenic activities cause delay in nutrient utilization efficiency due to which the equilibrium losses stability. Sensitivity analysis with respect to model parameters is done using 'Direct Method'. It is found that the state variables of the model are more sensitive to parameters- nutrient transfer rate and consumption rate. It is observed that with increase in nutrient transfer rate and decrease in consumption rate, the system tends to stability for the same value of other parameters. The critical value of the delay parameter comes out to be  $\tau = 0.9$  below which the system is stable and above this critical value, the complex behaviour is shown by the system and Hopf-bifurcation occurred.

In chapter-3, A two-compartment mathematical model is proposed for the study of an individual plant growth dynamics with time lag due to the presence of toxic metals in the soil. It is assumed in the model that nutrient uptake by root is hindered due to presence of toxic metal. This effect is studied by considering a time lag in utilization of nutrients in root in presence of toxic metal. Stability analysis of interior equilibrium is carried out. Sensitivity analysis of state variables with respect to model parameters is done. The nutrient utilization coefficient and consumption coefficient turn out to be the most sensitive model parameter. The value of delay parameter  $\tau = 0.89$  turn out to be the critical value which changes the system behaviour from stability to complex one, leading to Hopf-bifurcation.

In chapter-4, A mathematical model is proposed for analytical study and analysis of plant growth under the effect of toxic metal where the delay is considered in utilization of nutrients. It is assumed that the heavy metals present in the soil have toxic effects on the concentration of nutrients available in the soil for plant growth. This toxic effect in turn affects the plant growth adversely. This effect is studied by introducing the time-lag (delay) in consumption or utilization of nutrients. Stability analysis of feasible interior equilibrium and Hopf bifurcation has been studied. The sensitivity analysis reveals that nutrient concentration is least sensitive to all model parameters as compared to biomass. The system is stable below the critical value of delay parameter  $\tau = 1.373$  and loses stability and show Hopf-bifurcation above this critical value. Simulation is done using MATLAB.

In chapter-5, The reduction in tree biomass is modelled under the effect of toxic metal by considering delay in terms containing intrinsic growth rate. It is assumed that the soil

is contaminated with toxic metal and some amount of toxic metal is already present in the tree itself. It is further assumed that both these concentrations affect the tree biomass. Stability of feasible interior equilibrium is studied and Hopf bifurcation occurs at a critical value of time parameter  $\tau = 3.17$  and the periodic solution results. The sensitivity analysis concludes that tree biomass increases with increase in intrinsic growth rate and decrease in input rate of toxic metal. MATLAB is used to support the analytical results.

In chapter-6, A mathematical model is framed that depicts the ecological phenomenon that and nutrient pool and plant population density are adversely affected by the presence of excessive toxic metal. The system of non-linear delay differential equations is used where the state variables considered are nutrient pool, plant population density and toxic metal. The delay is considered in nutrient utilization. It is shown that as the rate of damage of plant population density due to toxic metal and the input rate of toxic metal increase, the nutrient pool concentration and plant population density decreases. Two feasible equilibriums: uniform equilibrium and interior equilibrium are calculated. The system changes its behaviour from stable to unstable while passing through the critical value of delay parameter  $\tau = 0.49$  and Hopf bifurcation occurred. Sensitivity analysis of state variables in relation to model parameters is done. It reveals that decrease in nutrient use efficiency leads to stability of the system. Model is verified by taking those values of damage rate due to toxic metal (assumed to be Arsenic) which are available in already existing data for growth of winter wheat (*Triticum aestivum* L) and rape (*Brassica napus*) under the effect of high concentrations of Arsenic in soil (Liu et al 2012). Simulation is done using MATLAB.

In chapter-7, A mathematical model is proposed to study the combined effect of acid and toxic metal on plant population considering delay in utilization of favourable resources available in soil and surrounding plant environment. It is assumed that due to presence of toxic metal and acidity in soil, nutrient pool and plant population density gets adversely affected. The stability of the interior equilibrium of the system gets disturbed by the introduction of delay parameter. For the critical value of delay parameter  $\tau = 3.38$ , Hopf bifurcation is observed where by system fluctuates its behaviour from being stable to unstable. The sensitivity of model solutions for different values of model parameters is established using sensitivity analysis. Rate of utilization and rate of interaction between nutrient and resources turn out be crucial model parameters. The model is also verified by taking the uptake rate values of copper using



the existing experimental data of a study conducted on the effect of heavy metals like copper, lead and zinc on growth of Lettuce in Vietnam (Nguyen Xuan Cu). MATLAB code is used for simulation to support analytical results.

## **List of Papers Published and Communicated from the Thesis**

1. Kalra, P. & Kumar, P. *Role of Delay in Plant Growth Dynamics: A Two Compartment Mathematical Model*, AIP Conference Proceedings, 1860, 020045; doi: 10.1063/1.4990344 (2017).

2. Kalra, P. & Kumar, P. *Modelling on Plant Biomass with Time Lag Under the Effect of Toxic Metal*, Ecology, Environment. & Conservation, 24(1): 284-290 (2018).

3. Kalra, P. & Kumar, P. *The Study of Time Lag on Plant Growth Under the Effect of Toxic Metal: A Mathematical Model*, Pertanika- Journal of Science and Technology, 26(3):1131-1154 (2018).

4. Kalra, P. & Kumar, P. *The Study of Effect of Toxic Metal on Plant Growth Dynamics with Time Lag: A Two Compartment Model*, Journal of Mathematical and Fundamental Sciences, 50(30):233-256 (2018).

5. Kalra, P. & Kumar, P. *Modeling and Analysis of Plant Population Growth under the Effect of Toxic Metal with Delay*. [Communicated].

6. Kalra, P. & Kumar, P. *Modelling with Delay on the Plant Population under the Effect of Acid and Toxic Metal* [Communicated].

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# Chapter 1

## General Introduction

### 1.1 Introduction

Plant ecology is a branch of science that encompasses all kinds of possible interactions among organisms and their environment including spread and affluence of plants. It studies the growth of individual plant and plant population under varying environmental conditions. It takes care of the production of quality food, feed and vegetation in genetic, biotic and abiotic factors. Plants are the major producers in an ecosystem. Plant growth depends on local physical environment that includes light, water, temperature and nutrient availability in soil. These factors in turn depend on variables like leaf-area ratio, net assimilation rate, leaching and runoff, accumulation of organic matter, microbial mineralization and immobilization of nutrients in the soil.

Soil consists of the organic and minerals on the surface of earth that acts as a medium for plant growth. Organic matter is formed by decayed microorganisms and plants and minerals are formed by weathering of rocks. Soil holds roots of the plants and provide them the required moisture and nutrients to produce flowers and fruits.

The soil-plant interaction via nutrient demonstrates a relationship that holds good not only at an individual level, but at population and at the ecosystem level too. Nutrient influence an individual plant growth, thereby affecting the plant population dynamics which in turn effects the crop production.

Soil is an important sink for pollutants. For survival of the ever-increasing world population, there is an increase in demand for food. It led to a massive increase in agricultural and industrial activities throughout the world. It resulted into decline of quality and quantity of soil. This soil degradation is brought by process like erosion, salinization, contamination, drainage, acidification, laterization and loss of soil structure or a combination of these.

The toxic chemicals in soil can adversely affect forest, agricultural crops and vegetation by destroying their characteristics and productivity. The direct influence of heavy metal ion on tree metabolism, especially on root physiology, has potential to reduce nutrient uptake. Copper (Cu) and zinc (Zn) are the essential macronutrients which are required for the optimum growth of the plants. Normally, these are found in very low concentration

in soil. Whereas, metals like cadmium (Cd), arsenic (As), chromium (Cr), lead (Pb), nickel (Ni), mercury (Hg) and selenium (Se) are toxic to plants. The various factors including the disposal of municipal and industrial wastes, application of chemical fertilizers, pesticides, insecticides, herbicides, atmospheric deposition and discharge of wastewater on land, resulted in the soil acidification and also elevated the concentration of above-mentioned heavy metals in the soil which in turn has become the main reason for the reduced plant growth.

Cadmium (Cd) is one of the most widely spread heavy metals found in the soil. It is easily accumulated by plants and hinders the growth and ingestion of nutrients by agriculture crops such as Barley. Acid precipitation causes changes in the soil properties that influence root growth or function. The acidification of the soil raises the inorganic aluminium concentration at root zone which, at the sublethal concentration, reduce the cation uptake rate and at lethal concentration, increases the root mortality rate. Increased level of acidity entering the soil system could be the reason of leaching that causes root damages.

It is also observed that the Xenobiotics in soil whose major source is agricultural pesticides can be toxic to soil biota which inhibits seed germination and suppresses plant growth. Acidification of soil is the adverse output of acid rain. It leads to increased ex-change between hydrogen ion and nutrient cations in the soil and can be rapidly leached out in soil solution. The metal concentration in the plants generally increased with decreasing pH and accumulation of metal concentration in plant is inversely proportional to pH of soil. The accumulations of heavy metals in the plants increased with lower pH due to acid rain.

India is basically an agricultural country and about seventy percent of the population is engaged in agriculture practices. To feed such a large population farmer are using extensive amount of pesticides, insecticides, herbicides and chemical fertilizers to increase the yield. Due to excessive use of all these chemicals the fertility of the soil is decreasing, consequently diminishing the yield of agricultural crops which are also containing heavy metals and the other chemical toxicants that are even harmful to human health.

Now it is evident that individual plants and plant population are under different types of stresses, such as low nutrient availability, acidification causing lowering of pH, temperature fluctuations, drought, and presence of many salts, aluminium, heavy metals

and other toxic chemicals in the soil. All these stresses affect plant growth and yield through adverse physiological effects such as reduced leaf area development, reduced root growth or advanced canopy senescence.

In research environments, modelling commonly serves purpose such as integrating knowledge or the quantitative testing hypotheses and for modelling, the system of interest needs to be described. In horticulture and agricultural sciences, the system of interest is commonly a plant and very often a collection of interacting plants, i.e. a row of plants or homogeneous crop canopy.

Time delays of one type or another have been incorporated into biological models by many researchers. In real world, the growth rates of an individual plant or plant population will often not respond immediately to change in its own population, but rather will do so after a time lag. The factors that introduce time delay may include toxic metals in soil, utilization coefficient and nutrient use efficiency in plant growth dynamics. In general, the dynamics exhibited by delay differential equations is much more complicated as compared to the one presented by ordinary differential equations. The time delay forces a stable equilibrium to lose stability and become unstable and cause fluctuations. Role of delay (time lag) in plant growth under the effect of toxic metal is relatively a new field of studies.

Mathematical models are a true replica of the amalgamation of environmental and ecological information. In mathematical model of natural phenomenon, it is not possible to consider all uncountable variables and factors, but only the pertinent information is considered to have a logical understanding of the nature of the dynamics. The proposed research work related to the study of role of delay on plant growth dynamics under the effect of toxicants will be carried out by using mathematical models and the models will be used to determine quantitatively the damage mechanism of plants and reduction in plant yield due to reducing factors such as the presence of toxic chemicals in the soil.



## 1.2 Literature Review

Plant growth studies have their roots right back in ancient times. Early humans, nomads used to observe that crops and plants sown in certain seasons give fruits and feed as compared to other seasons. In the middle ages, Leonardo da Vinci was the first to observe systematically, the seasonal periodicity of growth and some features of plant forms. Theories of the arrangement of leaves on axis or stem, appear already in the 17<sup>th</sup> century. The modelling of the plant-soil interaction started with a single-root scale model proposed by Hiltner [1] for nutrient uptake by roots, known as rhizosphere models. The early modelling work between 1960 and 1970 related to water uptake by single root is credited to Dalton and Gardner [2]. Temperature, Humidity, radiation input, respiration, transpiration, photosynthesis, carbon dioxide etc. are the major environmental factors that affect plant growth. Thornley [3] was the first to propose mathematical models including these factors individually and in combination. Watkinson [4] proposed a novel theoretical model in which the assumptions were that there was a dynamic growth process whose rate was dependent on the size of the plant, diminishing as a plant matured towards an upper size limit, that there was variability between the individual plants in a population, and that competition (particularly for the smaller plants in a dense population), reduced the growth rates, possibly resulting in mortality. Gifford and Evans [5] concluded that soil properties and availability of water are main factors responsible for increasing plant and crop yield through breeding program. It has been shown by Miler [6] that if concentrations of nutrients in long-lived tissues remain relatively constant (constant nutrient-use efficiency), then the contribution of plants to nutrient availability is proportional to accumulation of biomass of long lived tissues. Reynolds and Acock [7], analysed how the agricultural productivity and natural ecosystem have long-term adverse effects of increasing levels of global carbon dioxide concentration. Valentine [8], assumed that a tree can be considered as a sum total of pipes and each pipe undergoes steady-state growth. Makela [9], criticized that the partitioning coefficients defined by Reynolds and Thornley are applicable only to the models that have equal turnover rates of root and shoot. He gave a more general derivation which considers models proposed by Reynolds and Thornley as a special case. and Pugliese [10] has studied age dependent plant growth model, and proposed a continuous-time model of growth and reproduction. Kickert and Krupa [11] proposed and analysed models which deal with general group of

vegetables, instead of individual plant species. Czaran and Bartha [12] studied spatiotemporal dynamic models of plant populations and communities. Thornley [13] described two mathematical models showing root: shoot allocation. Two modes of growth: steady state and exponential growth were considered for the comparison of transport-resistance approach and teleonomic method. Thornley [14] gave a two compartment mathematical model which is based on two processes-transport and chemical conversion which fits well to both theoretical observations and practical application. Thornley [15] proposed a transport-resistance mathematical model for allocation of carbon and nitrogen in root and shoot compartments of plants. Thornley [16] proposed a mathematical model for plants regarding their stem height and growth of diameter. Deleuze and Houllier [17], proposed a model addressing two modelling problems: total wood production which is a function of carbon translocation, described by Munch theory and stem form and wood quality. Diekmann et al. [18] have formulated and analysed general deterministic structured population models. Somma et al.[19]developed a three dimensional model that clubbed the simultaneous growth of root, water and nutrient uptake with soil, water and solute transport.Khush [20]showed how the technological advance has led to the dramatic achievements in world food production over the last 30 years and to the development of high-yielding varieties of wheat and rice. Lacoite[21]concluded that the models proposed by Thornley are specific and cannot be applied over a wide range of plants under different conditions. Tinker and Nye [22] started to consider root hairs in place of single root in models. Luis Garcia-Barrios et al. [23]showed how mathematical models of growth of mixed crop in combination with empirical data can reduce the time and investment required for the task. It illustrates that the spatial disposition of plants in intercrops, and difference in sowing time between species, can strongly affect their ecological inter-actions and, in consequence, the systems viability and performance.Bolker et al. [24] studied a spatial dynamic model for plant communities.Hedden [25] identified the genes responsible for dwarfing of traits which led to the remarkable increase in wheat and rice yields. Ioslovich and Gutman[26] have studied a plant growth model with gradual transition from vegetative to reproductive growth considering the difference of photosynthesis use efficiency when energy is accumulated in the vegetative and in the reproductive organs of a plant. Valentine and Makela [27]formulated a process-based model of tree growth that can be fitted and applied in an empirical mode. Verkroost and Wassan [28] developed a simple plant growth model that verified the linear relationship between relative growth rate and plant nitrogen concentration. Vance and Nevai[29] proposed a

mathematical model for plant population growth considering competition in a light gradient for canopy partitioning. The growth and development of flowering plants depend on phytohormone gibberellin (GA). Overman [30] coupled planted plant biomass and mineral elements by two mathematical models using analytical functions. Harberd et al.[31] studied the mechanism that enable the plants to respond to GA and give them freedom from growth arrest and survival flexibility. Fowler et al.[32] proposed a mathematical model of carbon allocation and storage on the basis of which growth dynamics of trees can be predicted. Liu et al. [33] presented a spatial plant growth model spotted to pattern dynamics by using both mathematical analysis and numerical simulations. Asano et al.[34] used semi dwarf phenotypes as an agronomically important trait for during modern crop breeding of rice ( *Oryza sativa* L.).Pingali [35] provided a detailed account of green revolution, its success and failure in terms of productivity, environment and economic levels. Production of photosynthetic canopy and crop yield both depend on nitrogen. Quilliam et al.[36] showed that the biochar amendment can improve the soil quality, reduce nutrient leaching and increase the crop yield. Clough et al.[37] gave a review of emerging trends and gaps in biochar-nitrogen research which is very important for agronomical productivity and carbon sequestration. Clark et al.[38] synthesized discrete observations of growing season with continuous responses to temperature variation, to predict how increase in the temperature due to global warming accelerates onset of growth. Hawkesford [39] showed that not only the crop production depends on nitrogen, but the focused capture and use of nitrogen also optimises the consumption of this efficient macronutrient. Vanderwel and Purves [40] used a simple data model to predict substantial shift in forest dynamics in next 500 years in united states. King et al.[41], instead of considering the loss of soil phosphorus like many researchers, has put more emphasis on the magnitude and quality of loss of sub surface phosphorus and its adverse effect on plant growth. Serrano-Mislata et al.[42] showed that stem growth and the size of the inflorescence meristem is controlled by DELLA, where flowers initiate. Paxson and Simon [43] showed how computers can control and monitor the growth of plants directly. Sanderman et al.[44], analysed the archived soil samples of crop rotation. The findings suggested the direct feedback between accelerated biological activity, carbon cycling rates and rates of carbon stabilization. Dahiru [45], attempted to review the influence of some important growth regulating substances such as abscisic acid, auxin, cytokinin, ethylene, and gibberellins on plant growth. Ciereszko [46], showed how sugar plays an important role in defense reactions of plants.

The problem of estimating the effects of a toxicant on population by mathematical models began only in the early 1970's. Bazzaz et al. [47] studied the effect of Cadmium on photosynthesis and transpiration of excised leaves of corn and sunflower. Bazzaz et al. [48] studied the Inhibition of photosynthesis in corn and sunflower by lead. In last two chapters of his book, Hewitt [49], covered one essential (Cl) and three non-essential (Si, Co, V) elements and elements with more toxic effects (I, Br, F, Al, Ni, Cr, Se, Pb, Cd) and other heavy metals. The reduction of plant growth due to toxic effect of cadmium on different processes in plants has been studied by Rodecap and Tigey [50]. Hallam et al. [51] presented a three-dimensional model that showed the effect of chronic or acute dose of toxicant on plant population at the system level. Hallam et al. [52] gave a toxicant-population interaction model that depicted the adverse effects of pollutants on population. Hallam and de Luna [53] discussed the effects of toxicants on the population exposed to both, environmental and food chain pathways. De Luna and Hallam [54] proposed three generic toxicant-population models consisting of three state variables: population, toxicant concentration in organism and toxicant concentration in environment. Gatto and Rinaldi [55], using some mathematical models, showed dramatic changes in the forest biomass due to small variations of the human exploitation of a natural forest. Shukla et al. [56] presented a model to show that increasing pollution due to increasing industrialization can lead to extinction of forest biomass and reforestation is the only way out to this issue. Wolf et al. [57] studied the modelling of long-term response of crop to fertilizer and nitrogen. Freedman and Shukla [58] studied the effect of single influx, constant dose and periodic dose of toxicants on single species growth and predator-prey system. De Leo et al. [59] framed a simple mathematical model that clubbed the chemical properties of soil with tree biomass to study the effect of proton concentration on equilibrium. Brune and Deitz [60] conducted an experiment in which Barley seedlings were grown in hydroponic culture in the presence of toxic concentrations of different heavy metals. It revealed some distinct effects on specific leaf and root element content. Shukla et al. [61], presented a mathematical model that analysed how the increasing rate of toxicity due to industrialization leads to decrease in biomass density which in turn results into extinction of resource based species. Curtis and Wang [62] used meta-analytic methods to summarize and interpret more than 500 reports of effects of elevated CO<sub>2</sub> on woody plant biomass accumulation and partitioning, gas exchange, and leaf nitrogen and starch content. Bonnet et al. [63] studied the effect of zinc on growth parameters and chlorophyll concentration of ryegrass. Mossor-Pietraszewska [64] shown that aluminium toxicity is one of the major

factors for reduced plant growth in acidic soils. Pishchik et al.[65] conducted an experiment in which plant growth promoting rhizobacteria were selected to obtain ecologically safe barley crop production on cadmium polluted soils. Dubey et al.[66] proposed a mathematical model to analyse and study the reduction of resource biomass of plants and trees due to industrialization and pollution. Van Ittersum et al.[67] presented an overview of the Wageningen crop and crop-soil modelling approaches, instead of focusing on few models only. Shenker et al.[68], studied Manganese nutrition effects on tomato growth and chlorophyll concentration. Dercole et al.[69] formulated a mathematical model with the help of partial differential-integral equations incorporating the effect of self-shading by leaves. Sheldon and Menzies [70], studied the effect of copper toxicity on the growth and root morphology of Rhodes grass (*Chloris Gayana* Knuth.) in resin buffered solution culture. Thomas et al.[71] proposed a mathematical evolution model for phytoremediation of metals from the soil through the roots. Naresh et al.[72] proposed a nonlinear mathematical model to analyse and study the effect of an intermediate toxic product on the growth of plant biomass. Lauchli and Grattan [73] in one of the chapters, provided an overview of the effect of salinity on growth and development of crop plants. Verma et al. [74] gave a model for accumulation of cadmium in radish, carrot. Spinach and cabbage. A model related to coupling of water and heat transport in a soil-mulch-plant-atmosphere continuum (SMPAC) system has been studied by Wu et al.[75] which is being applied to winter wheat crop. Liu and Zhang [76] have studied the N-species food chain model with feedback control system in a polluted environment and have carried out the persistence and stability analysis. Shukla et al.[77] framed a nonlinear mathematical model with assumption that population and resources are both simultaneously affected by toxicant. Hayat et al.[78] conclude that salicylic acid generates a variety of metabolic and physiological responses in plants and hence affects their growth and development. Nagajyoti et al.[79] studied the occurrence and toxic effects of a number of heavy metals on plants which are an integral part of entire ecosystem. Singh and Agrawal[80] studied the suitability of sewage sludge use for mung bean plants by evaluating the growth and yield responses and heavy metal accumulation at different sewage sludge amendment rates. Sinha et al.[81] proposed a two species competition model which are under the simultaneous effect of toxicant and infectious disease. Tsonev and Lidon [82] measured the extent of plant injury due to increased concentration of zinc. Mishra and Kalra[83] gave a very sound two compartment mathematical model for modelling effect of toxic metal on individual plant growth. Ahmad et al.[84] showed that the uptake of

non-essential element chromium along with other nutrients is one of the main causes of reduced growth and biomass of crops. Guo et al.[85]investigated heavy metals in soils and agricultural productsnear an industrial district in Dongguan City. Pavel et al.[86] studied the phytotoxic effect of chromium and cadmium on germination and growth of *Lepidium Sativum*. Misra and Kalra[87] gave another worth full two compartments mathematical to study the effect of toxic metal on the structural dry weight of a plant. Shukla et al.[88] proposed a nonlinear mathematical model to study the formation of acid rain in the atmosphereand its effect on plant species. Gupta et al.[89] emphasized the requirement of proper management of phosphorus along with its fertilizers which may help the maximum utilization by plants and minimum runoff and wastage. Bedbabis et al.[90] conducted an experiment in Tunisia for a period of ten years in an olive orchard , subjected to two different water treatments: well water (WW) and treated waste water( TWW). Data obtained indicated that standard quality indicesand oil content were not affected significantly by water quality. Boros and Micle [91]determined the *Helianthus Annuus*'s tolerance to copper and its effect on germination of seeds and growth of plant. Sundar and Naresh [92] proposed a nonlinear dynamical model to study the survival of biological population under the environmental pollution. Cu[93] conducted an experiment in Vietnam to study the effect of heavy metals on soil, water and plant biomass.Chi Peng et al. [94], gave a pollution accumulation model (PAM) to simulate the long-term changes of heavy metals concentration in soil. Mustafa and Komatsu [95]showed thatplants have evolved different strategies to cope with the accumulation of heavy metals when they take up heavy metals along with essential elements from the soil. Kumar et al.[96] proposed a non-linear mathematical model to study the effect of toxicant on biological species, some of whose members are already severely affected by the toxicant. Sundar et al.[97]proposed a mathematical model to analyse the effect of population and population density dependent industrialization on forestry resources. Yan et al.[98] evaluated the potential ecological risk of heavy metals accumulation from anaerobic co-digestion of plants by comparing different initial substrate concentrations, digestion temperatures, and mixture ratios.

History of delay differential equations is more than 200 years old. Very first application of delay differential equations is found in the field of geometry and number theory. However, the subject came into lime light after 1940 because of its use in engineering systems and control. Chapter II of the book “Delay-Differential Equations”[99],

presented a general theory of delay-differential equations. The problem of existence of solution and uniqueness properties of solutions of delay-differential equations have been studied. Mackey and Glass [100] used 1st-order nonlinear differential-delay equations to describe dynamical respiratory systems. The equations displayed limit cycle oscillations and chaotic solutions. Glass and Mackey [101] presented a mathematical model for phase locking of a biological oscillator to a sinusoidal stimulus. Model is verified by using experimental data by making comparisons between theory. Cooke and Grossman [102] discussed the importance of taking into account the time delays inherent in the biological, physical and social phenomenon. Donald et al. [103] focused on the theoretical and applied research into populations and ecological systems based on characteristics of individuals. Gopalsamy [104] in his book, presented an overview of recent advances in the stability and oscillation of autonomous delay differential equations. Kuang [105] treated both autonomous and non-autonomous systems with delays in his book. The main topics dealt with stability, coexistence of populations and oscillatory behaviour of the dynamics. Belair et al. [106] developed an age-structure model for erythropoiesis which then got reduced to a system of delay differential equations with two delays. Roussel [107] used Delay Differential Equations in Chemical Kinetics. He [108] concluded that for analytical approximate solutions of delay differential equations, the variational iteration method can be very useful. Li et al. [109] studied the stability and bifurcation of delay differential equations involving two delays. Engelborghs et al. [110] established numerical methods and software package for bifurcation analysis of delay differential equations. Bocharov and Rihan [111] considered those models of biological phenomena whose dynamics is explained better by delay differential equation and numerical approaches are the tools for their solutions. Ruan and Wei [112] discussed the distribution and nature of roots of 3<sup>rd</sup> degree transcendental polynomial. Shampine and Thompson [113] have written a MATLAB code: DDE23 to solve delay differential equations with constant delays. Engelborghs et al. [114] described a MATLAB package: DDE-BIFTOOL, for numerical bifurcation analysis of systems of delay differential equations with several fixed, discrete delays. Kubiacyk and Saker [115] studied oscillation and stability in nonlinear delay differential equations of population dynamics. Kuznetsov [116] studied the nonlinear dynamics system and their bifurcation by varying the values of parameters. Lenbury and Giang [117] studied nonlinear delay differential equations involving population growth. Li and Wei [118] discussed the distribution and nature of roots of 4th degree exponential polynomial. Ruan [119] incorporated time delays of one or the kind in

biological models of several researchers dealing with dynamics of single species populations. Erneux [120] has written this book for researchers working in areas such as biology and engineering, where mathematical and statistical modelling are of main importance. Roose and Szalai [121] studied Continuation and bifurcation analysis of delay differential equations. Balachandran et al. [122] developed a new code for numerical solution of delay differential equations. Zhang et al. [123] discussed the distribution and nature of roots of 5<sup>th</sup> degree transcendental polynomial. Smith [124] focused on the main tools in his book, which are necessary to comprehend the history of delay differential equations and their application in various models. Mallet-Paret and Nussbaum [125] considered a class of autonomous delay-differential equations with state-dependent delays whose solution was assumed to be linearly asymptotically stable. Sieber and Szalai [126] gave and then modified their general construction of characteristic matrix corresponding to system of linear delay differential equation, by pushing the poles in complex plane near to small neighbourhood of origin. Wolfrum et al. [127] investigated the dynamical properties like stability of stationary points with delay differential equations with very large delays. Kuang [128] showed how the popular MATLAB-based dde23 solver developed by Shampine and Thompson for delay differential equations can be used for numerically solving most delay differential equations and stability analysis. Huang et al. [129] studied Global Stability Analysis of Some Nonlinear Delay Differential Equations in Population Dynamics. Berezansky and Braverman [130] studied Boundedness and Persistence of Delay Differential Equations with Mixed Nonlinearity.

Hopf's basic paper appeared in 1942. Hopf [131] described the term Hopf bifurcation as the local birth or death of a periodic solution, which may be a self-excited oscillation, from an equilibrium as a parameter crosses a critical value. Hopf bifurcation occurs when a complex conjugate pair of eigenvalues becomes purely imaginary. This means Hopf bifurcation can only occur in systems of dimension two or higher. Marsden et al. [132] give a reasonably complete, although not exhaustive, discussion of what is commonly referred to as the Hopf bifurcation with applications to specific problems, including stability calculations. Hsu and Hwang [133] took a well-known Holling-Tanner, a prey-predator model and studied the Hopf-bifurcation for this ecological model. Reddy et al. [134] presented a detailed study of the effect of time delay on the collective dynamics of coupled limit cycle oscillators at Hopf bifurcation. Manfredi and Fanti [135] clarified the relations between stability theorems and the notions of simple



and general Hopf bifurcations Wei and Li[136] showed the occurrence of Hopf-bifurcation at the equilibrium point when the delay increased from critical point, in Nicholson's blowflies equation. Gupta and Chandra [137]discussedHopf-bifurcation analysis of a modified Leslie-Gower prey-predator model in the presence of nonlinear harvesting in prey. Xiao et al.[138] studied the Hopf-bifurcation generated by varying the interaction parameter in neural network model. Zhang and Guo [139] studied the direction and stability of Hopf-bifurcation using centre manifold theorem for the classic Van der Pol equation. Wang et al.[140] studied the direction and stability of Hopf bifurcation in a phytoplankton-zooplankton model, using the normal-form theory and centre manifold theorem.

Dickinson and Galinas [141] introduced 'Direct method' for sensitivity analysis of ordinary differential equations involving parameters which are not accurately known. The partial derivatives with respect to model parameters measure the sensitivity of the solution. Baker and Rihan [142] produced a new method to estimate the sensitivity of variables to model parameters and non-linearity effects for delay differential equations. Frey and Patil [143] identified and compared all the sensitivity methods used in various disciplines and prepared a consideration merit for application. Rihan [144]developed a general theory for sensitivity analysis of mathematical models containing delays, using adjoint equations and direct methods. In this paper, the model parameters wereassumed to be constants. Caswell [145] gave a method from matric calculus for sensitivity analysis of transient population. Kepler [146] used both, the adjoint method and direct for sensitivity analysis of mathematical models. Perumal and Gunawan method [147] emphasized that the careless use of parametric sensitivity analysis may not provide the true picture of dynamics, it can lead to incorrect results as well. Rihan [148] discussed the role of delay differential equations in dynamical systems, their computational tools, parameter estimation and sensitivity analysis. Wu [149], in his thesis, developed a methodology sensitivity analysis of functional structural plant models to understand the underlying biological processes. Ingalls et al.[150] developed a new method to analyse the sensitivity of delay differential equations with respect to parameters to investigate direction of steady evolutionary change.

The use of delay differential equations in plant physiology is not very prominent and very frequent. There are only handful of examples where delay differential equations have been used to present plant growth dynamics under the effect of toxicant. Dubey

and Hussain [151] developed a mathematical model to analyse and study the adverse effect of environmental pollution on forestry resource biomass with time delays in diffusive system. Pastor and Walker [152] presented a model depicting the dynamics of delays in nutrient cycling and population oscillations. They suggested that delays in release of nitrogen from decomposing litter, could produce oscillations in populations. Naresh et al. [153] proposed a mathematical model using delay differential equations to study and analyse the effect of toxicant on plant biomass with delay.

### **1.3 Proposed objectives of the study**

On the basis of literature review and research gaps, the following objectives have been proposed in this present study:

1. Mathematical study of plant growth dynamics models with delay.
2. Modelling effects of toxic chemicals on plant growth dynamics with delay.
3. Stability and Bifurcation Analysis of complex behaviour of plant growth dynamics due to delay.
4. Sensitivity analysis of state variables with respect to model parameters.
5. Verification of the model by comparison with existing experimental results.

### **1.4 Basic concepts of general plant physiology used in the thesis**

#### **1.4.1 Structure and Storage**

Plant models are invariably based on a great over-simplification of the real system. A simplification that seems to be some physiological significance is that in which plant material is divided into two categories: structure and storage [3]

Let  $W$  denotes the dry weight of the plant being considered, and  $W_G$  and  $W_S$  be the dry weights of structural and storage components of the total dry weight respectively, so that

$$W = W_G + W_S \quad (1.1)$$

Defining the word “growth” simply in term of dry matter increment, and differentiating Eq. (1.1) with respect to time,  $t$ , it follows that there are two contributions to the total growth rate:

$$\frac{dW}{dt} = \frac{W_G}{dt} + \frac{W_S}{dt} \quad (1.2)$$

The structural growth rate,  $\frac{W_G}{dt}$ , and the storage growth rate,  $\frac{W_S}{dt}$ , together make up the total growth rate,  $\frac{dW}{dt}$ . Thus, it is possible for a plant to have a negative total growth rate, made up of a positive structural growth rate and a negative storage growth rate. An example of this is overnight growth where in many plants the storage material is greatly depleted, there is considerable structural growth, and the overall decrease in dry weight due to respiration. Physiologists traditionally use the symbol  $R_W$  to denote a quantity called the relative growth rate (RGR) or specific growth rate, which is defined by

$$R_G = \frac{1}{W} \frac{dW}{dt} \quad (1.3)$$

Specific growth rate is defined in term of the total dry weight, which includes both structure and storage. It may be more useful to calculate a specific structural growth rate  $\mu_G$ , defined by

$$\mu_G = \frac{1}{W_G} \frac{dW_G}{dt} \quad (1.4)$$

### 1.4.2 Source and Sink strength

The source strength of a region (as a whole) can simply be regarded as the net flux out of the region and can be determined solely by reference to transport processes taking place at the boundaries of the region. Source strength is defined by

$F_X$  = Net rate at which  $X$  is transported out of that region of the plant.

Sink strength is the negative of source strength, so that if  $F_X$  is negative, then a sink of size  $-F_X$  is operating.

### 1.4.3 Utilization of Substrate

The growth and development of a plant organ depends upon the availability in the required amounts of any substrates necessary for these processes. In this section, a phenomenological substrate utilization equation is given that is frequently useful components of plant models.

**Rectangular Hyperbola (Single Substrate)** This is the most useful single equation available to the plant modeler. It is generally known to biologists as the Michaelis-Menten relation. It is often applicable when the process under consideration is dependent upon the level of a single substrate  $X$ . The rate of utilization of  $X$ ,  $U$  is given by

$$U = \frac{kX}{K+X} \quad (1.5)$$

Where  $k$  and  $K$  are constants and the symbol  $X$  denotes the density of the substrate  $X$ .

### 1.4.4 Translocation

Plants would not be able to grow as they do if they were not able to transport materials to sites where they can be used. Transport is essentially a polar phenomenon, although many mechanisms are intrinsically non-polar with polarity being imposed on the system by the arrangement of sources and sinks. In such cases the direction of transport can be changed by changing the sign of the sources and sinks.

### 1.4.5 Light Interception by Plants and Crops

An important part of many models of plant and crop growth is concerned with the interaction of the system with the light climate. The way in which this occurs determines the light flux density falling on each part of the plant or crop surface and this in turn through photosynthesis can greatly affect the extent and pattern of growth.

Let  $I_0$  be the light flux density (assumed uniform) that is incident on a field of area  $A_f$ . If  $J_0$  is the maximum light flux available for absorption, then

$$J_0 = A_f I_0 \quad (1.6)$$

Let  $J_c$  be the actual light flux absorbed by a crop covering a field. The overall efficiency of light absorption by the crop,  $f_c$ , is given by

$$f_c = \frac{J_c}{J_0} \quad (1.7)$$

Let the crop consists of  $n_c$  plants and let  $n_c J_P$  be the light flux absorbed by the crop, so that  $J_P$  is the light flux absorbed by an average plant. If  $f_P$  is the efficiency of light absorption of the plants in the crop,  $f_P$  is given by

$$f_P = \frac{J_c}{n_c J_P} \quad (1.8)$$

### 1.4.6 Photosynthesis

An equation which is frequently used to describe steady-state photosynthetic response to light and carbon dioxide is the rectangular hyperbola

$$P_s = \frac{\alpha I \beta C}{\alpha I + \beta C} \quad (1.9)$$

Where  $P_s$  is the steady-state photosynthetic rate (no distinction is made here between net and gross photosynthesis),  $I$  is the light flux density,  $C$  is the carbon dioxide density,  $\alpha$  and  $\beta$  are constants. This equation is popular because it often gives an acceptable description of actual response and is manageable.

### 1.4.7 Transport of Substrate

The flux of substrate,  $B$  (referred to as the mass transfer rate) from compartment  $i$  to compartment  $(i - 1)$  is given by

$$B = \frac{s_i - s_{i-1}}{r_i} \quad (1.10)$$

Where  $r_i$  is the transport resistance between  $i$ th and  $(i - 1)$ th compartment and  $s_i$  is the substrate concentration in the  $i$ th compartment.

### 1.4.8 Plant Growth Curve

In the beginning (lag phase) the plant growth is slow and sluggish. Then, there comes a swift and expeditious increase (exponential phase) in the plant growth. Henceforth, the

growth rate steadily decreases (stationary phase) due to impediment of nutrients. The typical sigmoid or S- shaped curve obtained by plotting growth and time is known as plant growth curve (Figure 1.1)

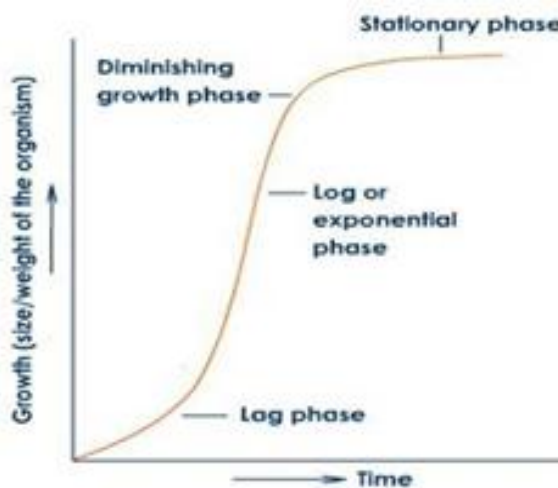


Figure 1.1 Plant Growth Curve

### 1.4.9 Plant Growth Rate

The plant-resource interaction is the effect of a limiting resource on the plants at individual, population, and ecosystem levels of organization. At the individual level, a growth-rate dependency on nutrient availability can be written as [154]

$$r(R) = \eta\mu_m W_r \frac{R}{k_R + R} \quad (1.11)$$

where,  $R$  is the availability of nutrient.  $\eta$  is the nutrient use efficiency.  $W_r$  is the proportion of total biomass allocated to root mass,  $\mu_m$  is the resource-saturated rate of resource uptake per unit of root mass and  $k_R$  is a half-saturation constant for nutrient uptake.

## 1.5 Mathematical Preliminaries

### 1.5.1 Existence of Unique, Bounded and Positive Solution of Delay differential equation

A delay differential equation is a differential equation in which the present time derivative depends on the solution and derivatives of earlier times. Here an initial history function, rather than an initial condition, needs to be defined. A delayed state variable can be used to demonstrate the past dependence of a differential equation. The derivative of the state variable is not required in this case. The corresponding delay differential equation with a single delay  $\tau > 0$  is given by [155]

$$\dot{x}(t) = f(t, x, x(t), x(t - \tau)) \quad (1.12)$$

Assume that  $f(t, x, y)$  and  $f_x(t, x, y)$  are continuous on  $R^3$ . Let  $s \in R$  and  $\phi: [s - \tau, s] \rightarrow R$  be continuous. We seek a solution  $x(t)$  of equation (1.12) satisfying

$$x(t) = \phi(t), t \in [s - \tau, s], x(0) = x_0 \quad (1.13)$$

And satisfying equation (1.12) on  $t \in [s, s + \sigma]$  for some  $\sigma > 0$ .

**Theorem 1.5.1 (Existence of unique solution).** Let  $f(t, x, y)$  and  $f_x(t, x, y)$  are continuous on  $R^3$ . Let  $s \in R$  and  $\phi: [s - \tau, s] \rightarrow R$  be continuous. Then there exists  $\sigma > s$  and a unique solution of the initial-value problem (1.12)- (1.13) on  $[s - \tau, \sigma]$ .

**Theorem 1.5.2 (Boundedness of solution).** Let  $f$  satisfy the hypothesis of theorem 1.5.1 and let  $x: [s - \tau, \sigma] \rightarrow R$  be the noncontinuable solution of the initial value problem (1.12)- (1.13). If  $\sigma < \infty$  then  $\lim_{t \rightarrow \sigma^-} |x(t)| = \infty$

**Remark 1.5.3** Theorems 1.5.1 and 1.5.2 extend immediately to the case that  $x \in R^n$  and  $f: R \times R^n \times R^n \rightarrow R^n$ , it also extends to multiple discrete delays  $\tau_0 < \tau_1 < \dots < \tau_m$  where  $f = f(t, y(t), y(t - \tau_0), y(t - \tau_1), \dots, y(t - \tau_m))$ .

**Theorem 1.5.4 (Positivity of solution).** Suppose that  $f: R \times R_+^n \times R_+^n \rightarrow R^n$  satisfies the hypothesis of theorem 1.5.1 and remark 1.5.3 and for all  $i, t$  and for all  $x, y \in R_+^n$ :

$$x_i = 0 \Rightarrow f_i(t, x, y) \geq 0$$

If the initial data  $\phi$  in equation (1.13) satisfy  $\phi \geq 0$ , then the corresponding solution  $x(t)$  of equation (1.12) satisfy  $x(t) \geq 0$  for all  $t \geq s$  where it is defined.

### 1.5.2 Stability by Variational matrix method

Let an autonomous system of equations be

$$\frac{dy}{dt} = f(y) \quad (1.14)$$

Where  $y$  is an  $n$ -tuple vector i.e.  $y = (y_1, y_2, \dots, y_n)$ . Let  $\phi(t)$  be the solution of system (1.14). The linear part of the expansion of the system (1.14) about  $\phi(t)$  is given by the variational equation of the system (1.14) with respect to  $\phi(t)$ , written as

$$\frac{dx}{dt} = f_y(\phi(t))x \quad (1.15)$$

Where  $f_y(\phi(t)) = \frac{df_i}{(dy_j)_{n \times n}}$  at  $\phi(t)$ . Since the stability of the variational system depicts the stability of any solution of a non-linear system governed by it, so stability of  $x = 0$  of (1.15) determines the stability of  $y = \phi(t)$  of (1.14). Particularly, when  $\phi(t) = \phi_0$ , a constant, the system (1.14) becomes

$$\frac{dx}{dt} = Ax \quad (1.16)$$

Where  $A = f_y(\phi_0)$ . Since a small perturbation of the system (1.14) is represented by system (1.15), so the stability of  $y = \phi_0$  of (1.14) actually gives the stability of the solution of  $x = 0$  of (1.15). The description of stability of every solution of  $x' = Ax$  is given by following theorems. [156]

**Theorem 1.5.5** If all the characteristic roots of  $A$  have negative real parts, then every solution of the system  $x' = Ax$ , where  $A = (a_{ij})$  is a constant matrix, is asymptotically stable.

**Theorem 1.5.6** If all the characteristic roots of  $A$  with multiplicity greater than one has negative real parts and all its roots with multiplicity one has non-positive real parts, then all the solutions of the system  $x' = Ax$  are bounded and hence stable.

Following theorem [156] to determine the sign of real parts of the roots of characteristic equation.



**Theorem 1.5.7 Hurwitz's Theorem.** A necessary and sufficient condition for the negativity of the real parts of all the roots of the polynomial

$L(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n$  with real coefficients is the positivity of all the principal diagonals of the minors of the Hurwitz matrix

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

**Theorem 1.5.8.** Let  $\zeta_1, \zeta_2, \dots, \zeta_m$  are all non-negative and  $\zeta_i^j$  ( $j = 0, 1, 2, \dots, m; i = 1, 2, \dots, n$ ) are constants. As  $(\zeta_1, \zeta_2, \dots, \zeta_m)$  vary, the sum of the orders of the zeros of exponential polynomial  $P(\chi, e^{-\chi\zeta_1}, \dots, e^{-\chi\zeta_m})$  on the open right half plane can change only if a zero appears on or crosses the imaginary axis, where

$$\begin{aligned} P(\chi, e^{-\chi\zeta_1}, \dots, e^{-\chi\zeta_m}) &= \chi^n + \zeta_1^0 \chi^{n-1} + \dots + \zeta_{n-1}^0 \chi^n + \zeta_n^0 \\ &+ [\zeta_1^1 \chi^{n-1} + \dots + \zeta_{n-1}^1 \chi^n + \zeta_n^1] e^{-\chi\zeta_1} + \dots \\ &+ [\zeta_1^m \chi^{n-1} + \dots + \zeta_{n-1}^m \chi^n + \zeta_n^m] e^{-\chi\zeta_m} \end{aligned}$$

Ruan and Wei [112], [157] proved this theorem using Rouches theorem [158].

### 1.5.3 Hopf-Bifurcation

Hopf's crucial contribution was the extension from two dimensions to higher dimensions. Sometimes Hopf bifurcation is also called as "Poincaré-Andronov-Hopf bifurcation". [132] Hopf-bifurcation theorem describes the way that a topological feature of a flow vary as one or more parameters are varied. The fundamental observation of flows is that if the stationary point is hyperbolic, i.e. eigenvalues of the linearized flow at the stationary point all have non-zero real parts, then the local behaviour of the flow is completely determined by the linearized flow. Hence, bifurcations of stationary points can only occur at parameter values for which a stationary point is non-hyperbolic. More, precisely, a bifurcation value of a parameter is a value at which the qualitative nature of the flow changes.

The Hopf bifurcation is several orders of magnitude harder to analyse since it involves a non-hyperbolic stationary point with linearized eigenvalues  $\mp i\omega$ , and thus a two-

dimensional centre manifold, and bifurcating solutions are periodic rather than stationary. **Theorem 1.5.9. Hopf-Bifurcation Theorem.**

Let us consider one parameter family of delay equations

$$x'(t) = F(x_t, \mu) \quad (1.17)$$

Where  $F: C \times R \rightarrow R^n$  is a twice continuously differentiable in its arguments and  $x = 0$  is a steady state for all values of  $\mu: F(0, \mu) \equiv 0$ .

We may linearize  $F$  about  $\emptyset = 0$  as follows

$$F(\emptyset, \mu) = L(\mu)\emptyset + f(\emptyset, \mu)$$

Where  $L(\mu): C \rightarrow R^n$  is a bounded linear operator and  $f$  is higher order:

$$\lim_{\emptyset \rightarrow 0} \frac{|f(\emptyset, \mu)|}{\|\emptyset\|} = 0$$

Following is the characteristic equation about  $L$ :

$$|\lambda I - A(\mu, \lambda)| = 0, A_{ij}(\mu) = L(\mu)_i(e_\lambda e_j)$$

The roots of this equation constitute the main assumption.

**(H)**The characteristic equation will be having a pair of simple roots  $\mp i\omega$  with  $\omega_0 \neq 0$  and no other root that is an integer multiple of  $i\omega_0$  for  $\mu = 0$

Here a root of order one means [159] a simple root. If the characteristic equation is written as  $h(\mu, \lambda) = 0$ , then **(H)** implies  $h_\lambda(0, i\omega_0) \neq 0$ . So, by the implicit function theorem, there exists a continuously differentiable family of roots  $\lambda = \lambda(\mu) = \alpha(\mu) + i\omega(\mu)$  for small  $\mu$  satisfying  $\lambda(0) = i\omega_0$ . In particular,  $\alpha(0) = 0$  and  $\omega(0) = \omega_0$ . Next assumption is that as  $\mu$  increases through zero, the line of imaginary axis is crossed transversally by these roots. Actually, the assumption is:

$$\alpha'(0) > 0 \quad (1.18)$$

In case  $\alpha'(0) < 0$ , we always ensure that equation (1.18) holds by changing the sign of the parameter i.e. we take parameter  $v = -\mu$ . Thus, the positive sign is basically a normalization which ensures that if  $\mu < 0$ , then the pair of roots has a negative real part and if  $\mu > 0$ , then it has positive real part.

**Theorem 1.5.10.** Let **(H)** and equation (1.18) hold. Then there exist  $\varepsilon_0 > 0$ , real valued even function  $\mu(\varepsilon)$  and  $T(\varepsilon) > 0$  satisfying  $\mu(0) = 0$  and  $T(\varepsilon) = 2\pi/\omega_0$ , and a non-constant  $T(\varepsilon)$ - periodic function  $p(t, \varepsilon)$  with all functions being continuously differentiable in  $\varepsilon$  for  $|\varepsilon| < \varepsilon_0$ , such that  $p(t, \varepsilon)$  is a solution of equation (1.17) and  $p(t, \varepsilon) = \varepsilon q(t, \varepsilon)$  where  $q(t, 0)$  is a  $2\pi/\omega_0$ -periodic solution of  $q' = L(0)q$ .

Moreover, there exist  $\mu_0, \beta_0, \delta > 0$  such that if equation (1.17) has a non-constant periodic solution  $x(t)$  of period  $P$  for some  $\mu$  satisfying  $|\mu| < \mu_0$  with  $\max_t |x_t| < \beta_0$  and  $|P - 2\pi/\omega_0| < \delta$ , then  $\mu = \mu(\varepsilon)$  and  $x(t) = p(t + \theta, \varepsilon)$  for some  $|\varepsilon| < \varepsilon_0$  and some  $\theta$ .

If  $F$  is five times continuously differentiable then:

$$\mu(\varepsilon) = \mu_1 \varepsilon^2 + O(\varepsilon^4) \quad (1.19)$$

$$T(\varepsilon) = \frac{2\pi}{\omega_0} [1 + \tau_1 \varepsilon^2 + O(\varepsilon^4)] \quad (1.20)$$

If all other characteristic roots for  $\mu = 0$  have strictly negative real parts except for  $\mp i\omega$  then  $p(t, \varepsilon)$  is asymptotically stable if  $\mu_1 > 0$  and unstable if  $\mu_1 < 0$ .

### 1.5.4 Sensitivity Analysis of State Variables with respect to Model Parameters

Systematic evaluation of the effects of model parameters on system solutions is called sensitivity analysis. There are number of methods to do sensitivity analysis of systems without delay, but there are only a few methods for sensitivity analysis of systems involving delays. The knowledge of how a small change in model parameter can bring change in the state variable, can be a great help in modelling process. It helps in elimination of ineffective and irrelevant parameters. It gives a complete insight into the overall behaviour of the proposed model.

If all the parameters in the given system (1.12)- (1.13) are considered to be constants, then sensitivity analysis includes just the calculation of partial derivatives of solution with respect to each parameter[144]. The matrix of sensitivity functions is of the form:

$$S(t) \equiv S(t, \alpha) = \left[ \frac{\partial}{\partial \alpha} \right]^T x(t, \alpha) \quad (1.21)$$

Its  $j$ th column is: 
$$S_j(t, \alpha) = \left[ \frac{\partial x_j(t, \alpha)}{\partial \alpha_1}, \frac{\partial x_j(t, \alpha)}{\partial \alpha_2}, \dots, \frac{\partial x_j(t, \alpha)}{\partial \alpha_n} \right]^T$$

This column vector gives sensitivity of the solution  $x_j(t, \alpha)$  for small change in parameter  $\alpha_i, i = 1, 2, 3, \dots, n$ .

**Theorem 1.5.11.**  $S(t)$  satisfies the delay differential equation:

$$S'(t) = J(t)S(t) + J_\tau(t)S(t - \tau) + B(t), t \geq 0 \quad (1.22)$$

Where  $J(t) = \frac{\partial}{\partial x} f(t, x, x_\tau)$ ,  $J_\tau(t) = \frac{\partial}{\partial x_\tau} f(t, x, x_\tau)$ ,  $B(t) = \frac{\partial}{\partial \alpha} f(t, x, x_\tau)$

## 1.6 Summary

This thesis consists of seven chapters whose detail is as follows:

In chapter-1, the general introduction of the subject matter is given. Some important concepts of plant physiology are mentioned. All the necessary mathematical concepts have also been described, on the basis which the analytical and numerical analysis of all the proposed mathematical models have become possible. It also includes the detailed study of the earlier work done in the field of plant growth dynamics in the form of literature review.

In chapter-2, a mathematical model is framed for the study and analysis of growth of an individual plant. The plant growth phenomenon is divided into two well-known compartments: root and shoot. Structural dry weight and concentration of nutrients are the two state variables. It is assumed that the exogenic activities interfere with the uptake of nutrients from root compartment. It affects the nutrient use efficiency (utilization coefficients). As a result, there occurs root structural damage. Delay (time-lag) is introduced in utilization parameter to analyse this effect analytically. It is observed that the interior equilibrium is stable in the absence of delay. The equilibrium is still asymptotically stable when the delay is less than a critical value. Once the critical value of delay parameter is crossed, the equilibrium loses stability and Hopf bifurcation occurs. MATLAB code dde23 is used to support the analytical results numerically.

In chapter-3, a two-compartment mathematical model is proposed for the study of an individual plant growth dynamics with time lag due to the presence of toxic metals in the soil. It is assumed in the model that nutrient uptake by root is hindered due to presence of toxic metal. This effect is studied by considering a time lag in utilization

coefficient of nutrient concentration in root in presence of toxic metal. Stability analysis of interior equilibrium is carried out. Sensitivity analysis of state variables with respect to model parameters is done. Hopf bifurcation is studied using various values of delay parameter.

In chapter-4, a mathematical model is proposed for analytical study and analysis of plant growth under the effect of toxic metal. It is assumed that the heavy metals present in the soil have toxic effects on the concentration of nutrients available in the soil for plant growth. This toxic effect in turn affects the plant growth adversely. This effect is studied by introducing the time-lag (delay) in consumption and utilization coefficient. Stability analysis of feasible interior equilibrium and Hopf bifurcation has been studied. The sensitivity of the model solutions is established by taking different values of the parameters appearing in the system. Simulation is done using MATLAB.

In chapter-5, a mathematical model is developed to study and analyse the reduction in plant (tree) biomass under the effect of toxic metals with delays. It is assumed that the soil is contaminated with toxic metal and some amount of toxic metal is already present in the plant itself. It is further assumed that both these concentrations affect the plant (tree) biomass. This effect is studied by introducing the delay in tree biomass. Stability of feasible interior equilibrium is studied and Hopf bifurcation occurs at a critical value of time parameter and the periodic solution results. MATLAB code dde23 is used to support the analytical results.

In chapter-6, a mathematical model is framed that depicts the ecological phenomenon that nutrient pool and plant population density are adversely affected by the presence of excessive toxic metal. The system of non-linear delay differential equations is used where the state variables considered are nutrient pool, plant population density and toxic metal. It is shown that as the rate of damage of plant population density due to toxic metal and the input rate of toxic metal increase, the nutrient pool concentration and plant population density decreases. Two feasible equilibria: uniform equilibrium and interior equilibrium are calculated. Hopf bifurcation is studied by introducing delay parameter in the term involving nutrient utilization. Sensitivity analysis of state variables in relation to model parameters is done. Model is verified by taking those values of damage rate due to toxic metal (assumed to be Arsenic) which are available in already existing data for growth of winter wheat (*Triticum aestivum* L) and rape (*Brassica napus*) under the effect of high concentrations of Arsenic in soil (Liu et al 2012). Simulation is done using MATLAB.

In chapter-7, a mathematical model is proposed to study the combined effect of acid and toxic metal on plant population considering delay in utilization of favourable resources available in soil and surrounding plant environment. It is assumed that due to presence of toxic metal and acidity in soil, nutrient pool and plant population density gets adversely affected. The stability of the interior equilibrium of the system gets disturbed by the introduction of delay parameter. For the critical value of delay parameter, Hopf bifurcation is observed. The sensitivity of model solutions for different values of model parameters is established using sensitivity analysis. The model is also verified by taking the uptake rate values of copper using the existing experimental data of a study conducted on the effect of heavy metals like copper, lead and zinc on growth of Lettuce in Vietnam (Nguyen Xuan Cu). MATLAB code is used for simulation to support analytical results.

In the end, bibliography is given to justify the problems undertaken for the study in this thesis.

## **Chapter 2**

# **The Effect of Exogenic Activities Considering Delay in Utilization of Nutrient Concentration: A Two Compartment Mathematical Model**

### **2.1 Introduction**

Plants need several elements for normal growth. Carbon, hydrogen, and oxygen come from the air and water and other nutrients come from soil. Soil provides nutrients and water and air provides hydrogen, carbon and oxygen. Nutrients are components in food that an organism uses to survive and grow. Plant -soil interaction means the mechanism in which the plants take essential nutrients from the soil through their roots which leads to growth of plants. Plants face problems in getting sufficient number of nutrients due to lack of their mobility. The insufficiency of any of these nutrients may cause decrease in fertility and productivity of plants. Seasonal cycling of growth and certain features of plant patterns were observed by Leonardo da Vinci [160] in middle ages. Hiltner [161] was the first to start soil-plant interaction modelling. The early modelling work in the field of uptake of water by single root was carried out by Dalton and Gardner [2]. Tinker and Nye [22] were the first group to consider roots with hairs in their model. Thornley [162] was the first to apply mathematical modelling to variety of topics in plant physiology to predict the effect of factors such as temperature, humidity, radiation input and concentration of on process rates of photosynthesis, fluid transport, respiration, transpiration and stomatal responses. Lacoite [21] concluded that models suggested by Thornley are designed for a particular plant species and under particular conditions and so they cannot be applicable over a broad range of conditions. The lack of explicit representation of topology and geometry (Plant morphology) was pointed out by Godin et al.[163]. Misra and Kalra [83] studied the effect of toxic metal on growth of individual plant using a very sound 2-compartment mathematical model. Mishra and Kalra [87] further studied the effect of toxic metal on plant's structural dry weight using

two compartment mathematical model. Rouches theorem [158] plays a very important part for the discussion of distribution of roots of exponential polynomials. Roussel [107] used delay differential equations in chemical kinetics. Ruan and Wei [112] used Rouches theorem for the discussion of distribution of roots of exponential polynomials. Kubiacyk and Saker [164] studied stability and oscillations in system of non-linear delay differential equations of population dynamics. Ruan and Wei [157] used Rouches theorem for the discussion of distribution of roots of exponential polynomials for study of stability involving delays. Lenbury and Giang [117] studied system of non-linear delay differential equations for population dynamics . Huang et al.[129]studied and analyzedthe global stability of system of non-linear delay differential equations involving population growth . Berezansky and Braverman [165] analyzed persistence boundedness of delay differential equations having mixed kind of non-linearity . So far, a lot of work has been done in the field of population dynamics where in time lag has been monitored in the various components of mechanisms, may it be a prey-predator models, epidemiology modelling or spread of diseases. The use of Delay differential equations in all these models which involve time lag, has been extremely worth full. Till date, delay differential equations have not been prominently used in the field of soil-plant dynamics and agriculture. It is the need of time and situation, to apply this concept of delay differential equations to have a better analysis of the mechanism of soil-plant dynamics.

In view of the above, mathematical model is framed for the study and analysis of growth of an individual plant. The plant growth phenomenon is divided into two well-known compartments: root and shoot. Structural dry weight and concentration of nutrients are the two state variables. It is assumed that the exogenic activities interfere with the uptake of nutrients from root compartment. It affects the nutrient use efficiency (utilization coefficients). As a result, there occurs root structural damage. Delay (time-lag) is introduced in utilization parameter to analyse this effect analytically. The stability of interior equilibrium is discussed and Hopf bifurcation is observed. Sensitivity analysis is also done. MATLAB code dde23 is used to support the analytical results numerically.

## 2.2 Mathematical Model

Let  $N_2$  and  $N_1$  denote the concentration of nutrients in shoot and root, respectively. Let  $W_2$  and  $W_1$  denote the shoot and root structural dry weights, respectively. The



assumption is that the nutrient is taken up from the root compartment and the exogenic activities hinder the up taking of nutrients that are essential for the plants and adversely affect the nutrient use efficiency (utilization coefficients) resulting into root structural damage. This effect has been studied by introducing the delay (time-lag) in utilization parameter.

These notations lead to the following mathematical model of the plant growth dynamics of non-linear differential equations:

$$\frac{dN_2}{dt} = \frac{T}{R_n} N_1 - \mu W_2 N_2 - d_2 N_2 \quad (2.1)$$

$$\frac{dN_1}{dt} = U_n - \frac{T}{R_n} N_1 - \mu W_1 N_1(t - \tau) - d_1 N_1 \quad (2.2)$$

$$\frac{dW_2}{dt} = r_2(N_2)W_2 - \Delta_2 W_2^2 \quad (2.3)$$

$$\frac{dW_1}{dt} = r_1(N_1)W_1 - \Delta_1 W_1^2 \quad (2.4)$$

With initial conditions, as:

$N_1(0) > 0, N_2(0) > 0, W_1(0) > 0, W_2(0) > 0$  for all  $t > 0$  and  $N_1(t - \tau) = \varepsilon$ ,  
Constant for all  $t \in [0, \tau]$

Here  $r_1(N_1)$  and  $r_2(N_2)$  has the following forms:

$$r_1(N_1) = \rho N_1 - \beta_{10}, r_1'(N_1) > 0 \text{ for } N_1 > 0, \quad r_1(0) = -\beta_{10},$$

$$r_2(N_2) = \rho N_2 - \beta_{20}, r_2'(N_2) > 0 \text{ for } N_2 > 0, \quad r_2(0) = -\beta_{20}.$$

The definitions of system parameters are as follows:

$r_2(N_2)$  and  $r_1(N_1)$  are rates of intrinsic growth of shoot and root, respectively which are dependent on the availability of nutrient.  $T$  is rate of nutrient transfer from root to shoot compartment.  $R_n$  is the resistance to transportation of nutrient,  $U_n$  is the rate of uptake of nutrient by root,  $\mu$  is consumption coefficient or utilization coefficient,  $\rho$  is nutrient-use efficiency,  $\beta_{10}$  is natural decay of  $W_1$ .  $\beta_{20}$  is natural decay of  $W_2$ .  $d_1$  is natural decay of  $N_1$ .  $d_2$  is natural decay of  $N_2$ .  $\Delta_2$  and  $\Delta_1$  are self-limiting growth rates of  $W_2$  and  $W_1$ , respectively.

## 2.3 Boundedness of Solutions

The boundedness of solutions of the model given by (2.1) -(2.4) is given by the lemma stated below:

**Lemma 2.3.1** The model has all its solution lying in the region  $D_1 = \left[ (N_1, N_2, W_1, W_2) \in R_+^4 : 0 \leq N_1 + N_2 + \frac{\mu}{\rho} W_1 + \frac{\mu}{\rho} W_2 \leq \frac{U_n}{\varphi} \right]$ , as  $t \rightarrow \infty$ , for all positive initial values  $\{N_1(0), N_2(0), W_1(0), W_2(0), N_1(t - \tau) = \text{Constant for all } t \in [0, \tau]\} \in D_1 \subset R_+^4$ , where  $\varphi = \min(d_1, d_2, \beta_{10}, \beta_{20})$ .

**Proof.** Consider the following function:

$$F(t) = N_1(t) + N_2(t) + \frac{\mu}{\rho} W_1(t) + \frac{\mu}{\rho} W_2(t)$$

$$\frac{dF(t)}{dt} = \frac{d}{dt} \left[ N_1(t) + N_2(t) + \frac{\mu}{\rho} W_1(t) + \frac{\mu}{\rho} W_2(t) \right]$$

Using Equations (2.1) -(2.4) and  $\varphi = \min(d_1, d_2, \beta_{10}, \beta_{20})$  and assuming that  $N_1(t) \approx N_1(t - \tau)$  as  $t \rightarrow \infty$ , we get

$$\frac{dF(t)}{dt} \leq U_n - \varphi F(t).$$

Applying the comparison theorem, we get as  $t \rightarrow \infty$ : we get

$$F(t) \leq \frac{U_n}{\varphi}$$

$$N_1(t) + N_2(t) + \frac{\mu}{\rho} W_1(t) + \frac{\mu}{\rho} W_2(t) \leq \frac{U_n}{\varphi}.$$

$$\text{So, } 0 \leq N_1(t) + N_2(t) + \frac{\mu}{\rho} W_1(t) + \frac{\mu}{\rho} W_2(t) \leq \frac{U_n}{\varphi}$$

This proves lemma.

## 2.4 Positivity of Solutions

It is essential to prove that all variables are positive for all times as the model explains plant growth dynamics. Positivity means that the system sustains. For positive solutions, we need to show that all solution of system given by Equations. (2.1)– (2.4), where initial condition are  $N_1(0) > 0, N_2(0) > 0, W_1(0) > 0, W_2(0) > 0$  for all  $t >$

0 and  $N_1(t - \tau) = \text{Constant}$  for all  $t \in [0, \tau]$ , the solution  $(N_1(t), N_2(t), W_1(t), W_2(t))$  of the model stays positive for all time  $t > 0$ .

From equation (2.1), we get  $\frac{dN_2}{dt} = \frac{T}{R_n} N_1 - \mu W_2 N_2 - d_2 N_2$

$$\frac{dN_2}{dt} \geq -(\mu W_2 + d_2) N_2$$

$$\frac{dN_2}{dt} \geq -\left(\mu \frac{U_n}{\varphi} + d_2\right) N_2$$

$$N_2 \geq c_1 e^{-(\mu \frac{U_n}{\varphi} + d_2)t}$$

Hence,  $N_2 > 0$  as  $t \rightarrow \infty$

From equation (2.3), we get,  $\frac{dW_2}{dt} = r_2(N_2)W_2 - \Delta_2 W_2^2$

$$\frac{dW_2}{dt} = (\rho N_2 - \beta_{20})W_2 - \Delta_2 W_2^2$$

$$\frac{dW_2}{dt} \geq -(\beta_{20} + \Delta_2 W_2)W_2$$

$$\frac{dW_2}{dt} \geq -\left(\beta_{20} + \Delta_2 \frac{U_n}{\varphi}\right) W_2$$

$$W_2 \geq c_2 e^{-(\beta_{20} + \Delta_2 \frac{U_n}{\varphi})t}$$

Hence,  $W_2 > 0$  as  $t \rightarrow \infty$

From equation (2.4), we get  $\frac{dW_1}{dt} = r_1(N_1)W_1 - \Delta_1 W_1^2$

$$\frac{dW_1}{dt} = (\rho N_1 - \beta_{10})W_1 - \Delta_1 W_1^2$$

$$\frac{dW_1}{dt} \geq -(\beta_{10} + \Delta_1 W_1)W_1$$

$$\frac{dW_1}{dt} \geq -\left(\beta_{10} + \Delta_1 \frac{U_n}{\varphi}\right) W_1$$

$$W_1 \geq c_3 e^{-(\beta_{10} + \Delta_1 \frac{U_n}{\varphi})t}$$

Hence,  $W_1 > 0$  as  $t \rightarrow \infty$

From equation (2.2), we get  $\frac{dN_1}{dt} = U_n - \frac{T}{R_n}N_1 - \mu W_1 N_1(t - \tau) - d_1 N_1$

$$\frac{dN_1}{dt} \geq -\left(\frac{T}{R_n}N_1 + d_1\right)N_1 - \mu W_1 N_1(t - \tau)$$

$$\frac{dN_1}{dt} + \left(\frac{T}{R_n}N_1 + d_1\right)N_1 \geq -\mu W_1 N_1(t - \tau)$$

Let  $\left(\frac{T}{R_n}N_1 + d_1\right) = \delta$

$$\frac{dN_1}{dt} + \delta N_1 \geq -\mu W_1 N_1(t - \tau)$$

$$\frac{d(e^{\delta t} N_1)}{dt} \geq -\mu W_1 N_1(t - \tau)e^{\delta t}$$

$$N_1(t) \geq -\mu \int_0^t N_1(x - \tau)W_1 e^{-\delta(t-x)} dx$$

$$N_1(t) \geq -\mu \left[ \int_0^\tau N_1(u)W_1 e^{-\delta(t-x)} dx + \int_\tau^t N_1(u)W_1 e^{-\delta(t-x)} dx \right]$$

$$N_1(t) \geq -\mu \frac{U_n}{\varphi} \left[ \int_0^\tau N_1(u)e^{-\delta(t-x)} dx + \int_\tau^t N_1(u)e^{-\delta(t-x)} dx \right]$$

Every solution of  $N_1(t)$  of above inequality will converge to 0 as  $t \rightarrow \infty$  if and only if  $N_1(u) < \delta u$  for all  $u > 0$

That is if and only if  $N_1(t - \tau) < \left(\frac{T}{R_n}N_1 + d_1\right)(t - \tau)$  for all  $t > \tau$ .

## 2.5 Interior Equilibrium of Model

We calculate an interior equilibrium  $E_1$  of model. The system of equations (2.1) -(2.4) has one feasible equilibrium  $E_1(N^*_1, N^*_2, W^*_1, W^*_2)$  where

$$W^*_1 = \frac{1}{\Delta_1} [\rho N^*_1 - \beta_{10}] > 0 \text{ provided } \rho N^*_1 > \beta_{10}$$

$$W^*_2 = \frac{1}{\Delta_2} [\rho N^*_2 - \beta_{20}] > 0 \text{ provided } \rho N^*_2 > \beta_{20}$$

$$N^*_1 = \frac{(U_n + \frac{\mu}{\Delta_2} \beta_{20})\epsilon}{\frac{T}{R_n} + \frac{\mu\rho}{\Delta_2} \epsilon + d_2}$$

$$N^*_2 = \frac{-g_2 + \sqrt{g_2^2 - 4g_1g_3}}{2g_1} \text{ where } g_1 = \frac{\mu\rho}{\Delta_1}, g_2 = \left(d_1 - \frac{\mu\beta_{10}}{\Delta_1}\right), g_3 = -\frac{T}{R_n}N^*_1$$

## 2.6 Stability Analysis and Local Hopf Bifurcation

Here, we analyse the dynamical behaviour of the interior equilibrium point  $E_1$  of the model given by (2.1) -(2.4). The exponential characteristic equation about equilibrium  $E_1$  is given by:

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 + de^{-\lambda\tau} = 0 \quad (2.5)$$

Here,

$$A_1 = (P_1 + P_7 + P_4 + P_9),$$

$$A_2 = (P_4P_9 + P_5P_8 + P_1P_7 + P_3P_6 + P_1P_4 + P_1P_9 + P_4P_7 + P_7P_9)$$

$$A_3 = ((P_1 + P_7)(P_4P_9 + P_5P_8) + (P_4 + P_9)(P_1P_7 + P_3P_6))$$

$$A_4 = ((P_4P_9 + P_5P_8)(P_1P_7 + P_3P_6)), d = \mu W^*_1$$

where,  $P_1 = \mu W^*_2 + d_2$ ,  $P_2 = \frac{T}{R_n}$ ,  $P_3 = \mu N^*_2$ ,  $P_4 = \frac{T}{R_n} + \mu W^*_1 + d_1$ ,

$$P_5 = \mu N^*_1, P_6 = \rho W^*_2, P_7 = \Delta_2 W^*_2, P_8 = \rho W^*_1, P_9 = \Delta_1 W^*_1$$

Clearly  $\lambda = i\omega$  is a root of equation (5), So

$$(i\omega)^4 + A_1(i\omega)^3 + A_2(i\omega)^2 + A_3(i\omega) + A_4 + de^{-(i\omega)\tau} = 0$$

$$\omega^4 - iA_1\omega^3 - A_2\omega^2 + iA_3\omega + A_4 + d(\cos \omega\tau - i\sin \omega\tau) = 0$$

Separating real and imaginary parts:

$$\omega^4 - A_2\omega^2 + A_4 = -d \cos \omega\tau \quad (2.6)$$

$$A_1\omega^3 - A_3\omega = -d \sin \omega\tau \quad (2.7)$$

Squaring and adding equation (2.6) and (2.7), we get:

$$\begin{aligned} \omega^8 + (A_1^2 - 2A_2)\omega^6 \\ + (A_2^2 + 2A_4 - 2A_1A_3)\omega^4 \\ + (A_3^2 - 2A_2A_4)\omega^2 \\ + (A_4^2 - d^2) = 0 \end{aligned} \quad (2.8)$$

Let  $\omega^2 = y$  and  $(A_1^2 - 2A_2) = a$ ,  $(A_2^2 + 2A_4 - 2A_1A_3) = b$ ,

$$(A_3^2 - 2A_2A_4) = c, (A_4^2 - d^2) = r$$

Equation (2.8) becomes:

$$y^4 + ay^3 + by^2 + cy + r = 0 \quad (2.9)$$

**Lemma 2.6.1** If  $r < 0$ , Equation (9) contains at least one positive real root.

**Proof.**

$$\text{Let } h(y) = y^4 + ay^3 + by^2 + cy + r$$

$$\text{Here } h(0) = r < 0, \lim_{y \rightarrow \infty} h(y) = \infty$$

$$\text{So, } \exists y_0 \in (0, \infty) \text{ such that } h(y_0) = 0$$

Proof completed.

$$\text{Also } h'(y) = 4y^3 + 3ay^2 + 2by + c$$

$$\text{Let } h'(y) = 0$$

$$\Rightarrow 4y^3 + 3ay^2 + 2by + c = 0 \quad (2.10)$$

$$\Rightarrow x^3 + px + q = 0 \quad (2.11)$$

$$\text{Where } x = y + \frac{3a}{4}, \quad p = \frac{b}{2} - \frac{3a^2}{16}, \quad q = \frac{c}{4} - \frac{ab}{8} + \frac{a^3}{32}$$

Assuming  $x = (m + n)$  be the solution of equation (2.11), we get:

$$m^3 - \frac{p^3}{27m^3} + q = 0$$

$$\text{Assuming } m^3 = z, \text{ we get } z^2 + qz - \frac{p^3}{27} = 0.$$

Three roots of equation (2.11) come out to be:

$$x_1 = \left(-\frac{q}{2} + \sqrt{D}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{D}\right)^{1/3}$$

$$x_2 = \left(-\frac{q}{2} + \sqrt{D}\sigma\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{D}\sigma^2\right)^{1/3}$$

$$x_3 = \left(-\frac{q}{2} + \sqrt{D}\sigma^2\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{D}\sigma\right)^{1/3}$$

$$y_i = x_i - \frac{3a}{4}, \quad i = 1, 2, 3.$$

Where  $D = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$  and  $\sigma = \frac{-1+\sqrt{3}i}{2}$

**Lemma 2.6.2** Suppose  $r \geq 0$

(I) If  $D \geq 0$ , then equation (2.9) has positive roots iff  $y_1 > 0, h(y_1) < 0$ .

(II) If  $D < 0$ , then equation (2.9) has positive roots iff  $\exists$  at least one  $y^* \in (y_1, y_2, y_3)$  such that  $y^* > 0$  and  $h(y^*) \leq 0$ .

(III) If  $D < 0$ , we know that equation (2.11) possesses only three zeros  $x_1, x_2, x_3$  that is equation (2.10) has three roots  $y_1, y_2, y_3$  and at least one of them is real.

**Proof.** (I) If  $D \geq 0$ , then equation (2.11) possesses a unique real root  $x_1$ , which means equation (2.10) possesses unique real root  $y_1$ .

As  $h(y)$  is a differentiable function and  $\lim_{y \rightarrow \infty} h(y) = \infty$ , we have  $y_1$  as the unique critical point of  $h(y)$  which happens to be the minimum point of  $h(y)$ .

Suppose equation (2.9) possesses positive roots. In general, we suppose that it has 4 positive roots denoted by  $y^*_i, i = 1, 2, 3, 4$ . Then equation (2.8) has 4 positive roots  $\omega_i = \sqrt{y^*_i}, i = 1, 2, 3, 4$ .

From equation (2.7) 
$$\sin \omega \tau = \frac{A_3 \omega - A_1 \omega^3}{d}$$

Which gives  $\tau = \frac{1}{\omega} \left[ \sin^{-1} \left( \frac{A_3 \omega - A_1 \omega^3}{d} \right) + 2(j-1)\pi \right]; j = 1, 2, 3, \dots$

Let  $\tau_k^{(j)} = \frac{1}{\omega_k} \left[ \sin^{-1} \left( \frac{A_3 \omega - A_1 \omega^3}{d} \right) + 2(j-1)\pi \right]; k = 1, 2, 3, 4; j = 1, 2, 3, \dots$

Then  $\bar{\tau} i \omega_k$  is a pair of purely imaginary roots of equation (2.5)

Where  $\tau = \tau_k^{(j)}, k = 1, 2, 3, 4; j = 1, 2, 3, \dots$ , We have  $\lim_{j \rightarrow \infty} \tau_k^{(j)} = \infty, k = 1, 2, 3, 4$ .

Thus, we can define  $\tau_0 = \tau_{k_0}^{(j_0)} = \min_{1 \leq k \leq 4, j \geq 1} [\tau_k^{(j)}], \omega_0 = \omega_{k_0}, y_0 = y_{k_0}^*$ .

**Lemma 2.6.3** Suppose that  $A_1 > 0, A_2 > 0, A_3 > 0, (A_4 + d) > 0, A_1 A_2 - A_3 > 0, A_1 A_2 A_3 - A_3^2 - A_1^2 (A_4 + d) > 0$ .

(I) If any one of the following condition holds: (i)  $r < 0$  (ii)  $r \geq 0, D \geq 0, y_1 > 0, h(y_1) \leq 0$  (iii)  $r \geq 0, D < 0$  and there exists a  $y^* \in (y_1, y_2, y_3)$  such that  $y^* > 0$  and

$h(y^*) \leq 0$ , then negative real part will be there in all roots of equation(5) when  $\tau \in [0, \tau_0)$ .

(II) If any one of the conditions (i)-(iii) of (I) are not satisfied, then negative real parts will be there in all roots of equation (5) for all  $\tau \geq 0$ .

**Proof.** When  $\tau = 0$ , equation (5) becomes:

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 + d = 0 \quad (2.12)$$

All roots of equation (12) have negative real parts iff  $A_1 > 0, A_2 > 0, A_3 > 0, (A_4 + d) > 0, A_1A_2 - A_3 > 0, A_1A_2A_3 - A_3^2 - A_1^2(A_4 + d) > 0$ . (Routh-Hurwitz's criteria)

From Lemma 2.3.1 and Lemma 2.6.1, we know that if conditions (i)-(iii) are not satisfied, then none of the roots of equation (2.5) will have zero real part for all  $\tau \geq 0$ .

If one of the conditions (i), (ii) and (iii) holds, when  $\tau \neq \tau_k^{(j)}, k = 1, 2, 3, 4; j \geq 1$ , then none of the roots of equation (2.5) will have zero real part and  $\tau_0$  is the minimum value of  $\tau$  for which the roots of equation (2.5) are purely imaginary.

Let  $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$  be the roots of equation (2.5) satisfying:

$$\alpha(\tau_0) = 0, \quad \omega(\tau_0) = \omega_0$$

**Lemma 2.6.4** Suppose  $h'(y_0) \neq 0$ . If  $\tau = \tau_0$ , then  $\mp i\omega_0$  is a pair of simple purely imaginary roots of equation (2.5). Moreover, If the condition of Lemma 2.6.2 are satisfied, then  $\frac{d}{d\tau}(Re\lambda(\tau_0)) > 0$ .

**Proof.** If  $i\omega_0$  is not a simple root, then  $i\omega_0$  must satisfy:

$$\begin{aligned} \frac{d}{d\lambda}[\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 + de^{-\lambda\tau}]_{\lambda=i\omega_0} &= 0 \\ -4i\omega_0^3 - 3A_1\omega_0^2 + 2iA_2\omega_0 + A_3 - \tau d(\cos \omega_0\tau - i\sin \omega_0\tau) &= 0 \end{aligned}$$

Separating real and imaginary parts:

$$A_3 - 3A_1\omega_0^2 = \tau d \cos \omega_0\tau$$

$$4\omega_0^3 - 2A_2\omega_0 = \tau d \sin \omega_0\tau$$

On dividing we get:



$$\tan \omega_0 \tau = \frac{4\omega_0^3 - 2A_2\omega_0}{A_3 - 3A_1\omega_0^2} \quad (2.13)$$

Also,  $\omega_0$  must satisfy equation (2.6) and (2.7), from where we get:

$$\tan \omega_0 \tau = \frac{A_1\omega_0^3 - A_3\omega_0}{\omega_0^4 - A_2\omega_0^2 + A_4} \quad (2.14)$$

Comparing (2.13) and (2.14) we get

$$4\omega_0^6 + 3(A_1^2 - 2A_2)\omega_0^4 + 2(2A_4 + A_2^2 - 2A_1A_3)\omega_0^2 + (A_3^2 - 2A_2A_4) = 0$$

As we know  $\omega_0^2 = y_0$ ,

$$4y_0^3 + 3(A_1^2 - 2A_2)y_0^2 + 2(2A_4 + A_2^2 - 2A_1A_3)y_0 + (A_3^2 - 2A_2A_4) = 0$$

$$4y_0^3 + 3ly_0^2 + 2my_0 + n = 0$$

Where  $l = (A_1^2 - 2A_2)$ ,  $m = (2A_4 + A_2^2 - 2A_1A_3)$ ,  $n = (A_3^2 - 2A_2A_4)$

Which gives  $h'(y) = 4y^3 + 3ly^2 + 2my + n$

Which is a contradiction as  $h'(y_0) \neq 0$ .

First part of the result is proved.

Differentiating equation (2.5) with respect to  $\tau$ , we get

$$\frac{d\lambda(\tau)}{d\tau} = \frac{d\lambda e^{-\lambda\tau}}{4\lambda^3 + 3A_1\lambda^2 + 2A_2\lambda + A_3 - d\tau e^{-\lambda\tau}}$$

Putting  $\lambda = i\omega$ ,  $\frac{d\lambda(\tau)}{d\tau} = \frac{di\omega(\cos \omega - i\sin \omega\tau)}{(A_3 - 3A_1\omega^2 - d\tau \cos \omega\tau) + i(2A_2 - 4\omega^3 + d\tau \sin \omega\tau)}$

When  $\tau = \tau_0$ ,  $\omega = \omega_0$ ,  $y = y_0$ , we get

$$\frac{d \operatorname{Re}\lambda(\tau_0)}{d\tau} = \frac{\omega_0^2}{\gamma} h'(y_0) \neq 0.$$

Where  $\gamma = (A_3 - 3A_1\omega^2 - d\tau \cos \omega\tau)^2 + (2A_2 - 4\omega^3 + d\tau \sin \omega\tau)^2$

If  $\frac{d \operatorname{Re}\lambda(\tau_0)}{d\tau} < 0$ , then equation (2.5) has a root with positive real part for  $\tau < \tau_0$  and in close proximity of  $\tau_0$  which contradicts Lemma 2.6.2

This completes the proof.

## 2.7 Sensitivity Analysis

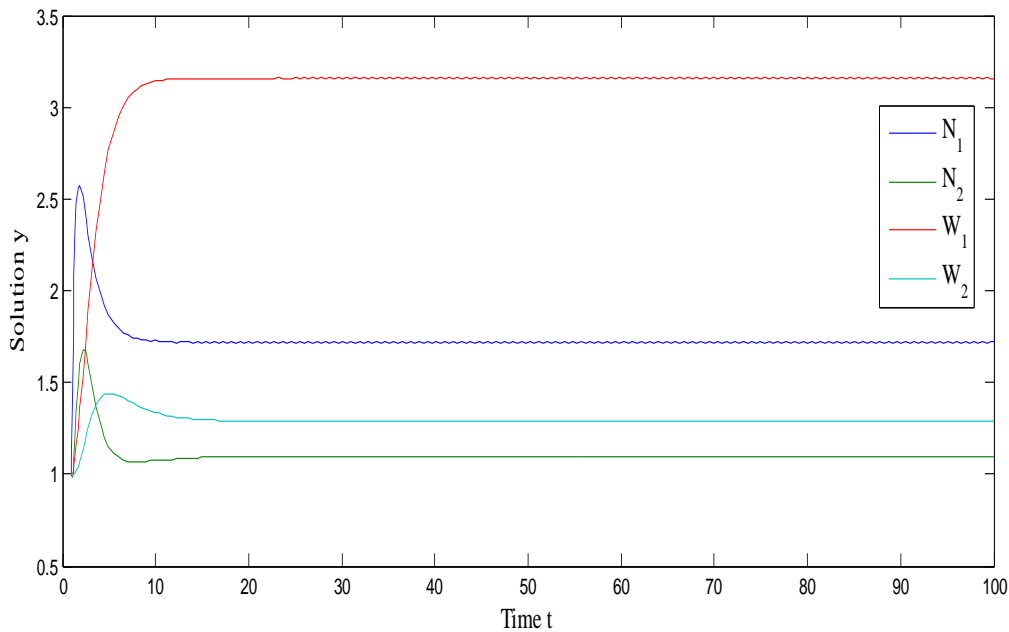
In this section, sensitivity analysis of state variables, namely concentration of  $N_1, N_2$  and structural dry weight  $W_1, W_2$  is done with respect to model parameters, transfer rate  $T$  and consumption coefficient  $\mu$  of delayed nutrients. The values of parameters are perturbed and corresponding changes in state variables are observed. The graphical representation of the same is shown in the section-2.8 with the help of numerical example and detailed discussion is included in section 2.9-Conclusion.

## 2.8 Numerical Example

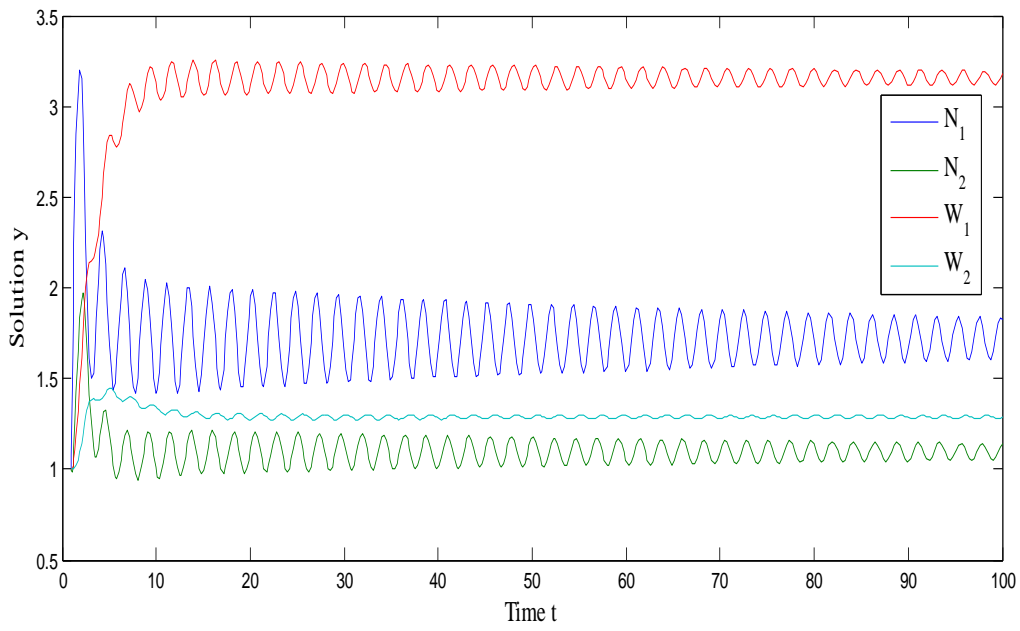
For the following set of values, the behaviour shown by the system is as follows:

$$U_n = 10, T = 1.5, R_n = 1, \mu = 1.05, d_1 = 1, d_2 = 1,$$

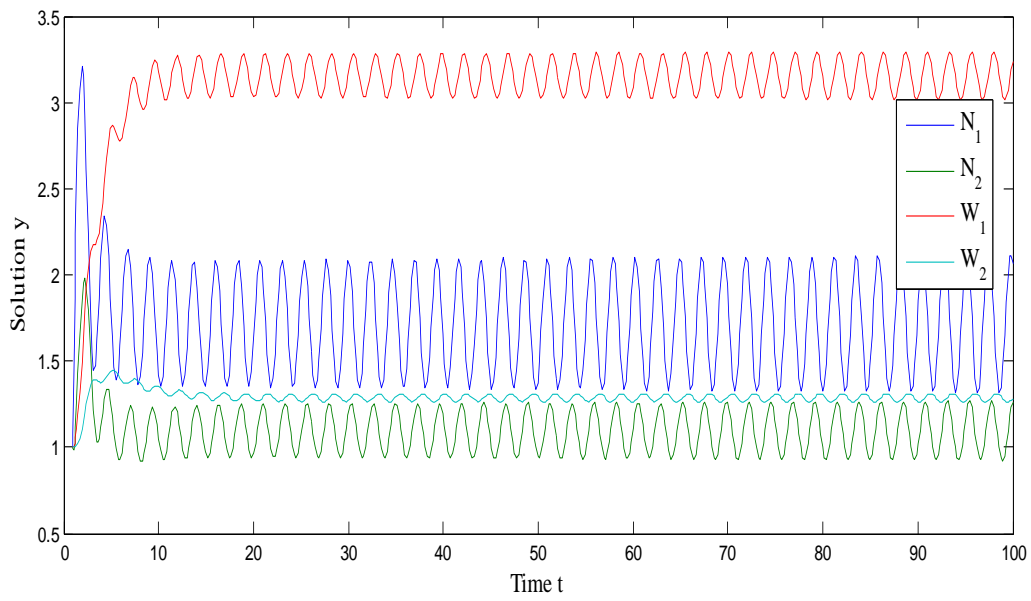
$$\rho = 0.3, \beta_{10} = 0.2, \beta_{20} = 0.2, \Delta_1 = 0.1, \Delta_2 = 0.1$$



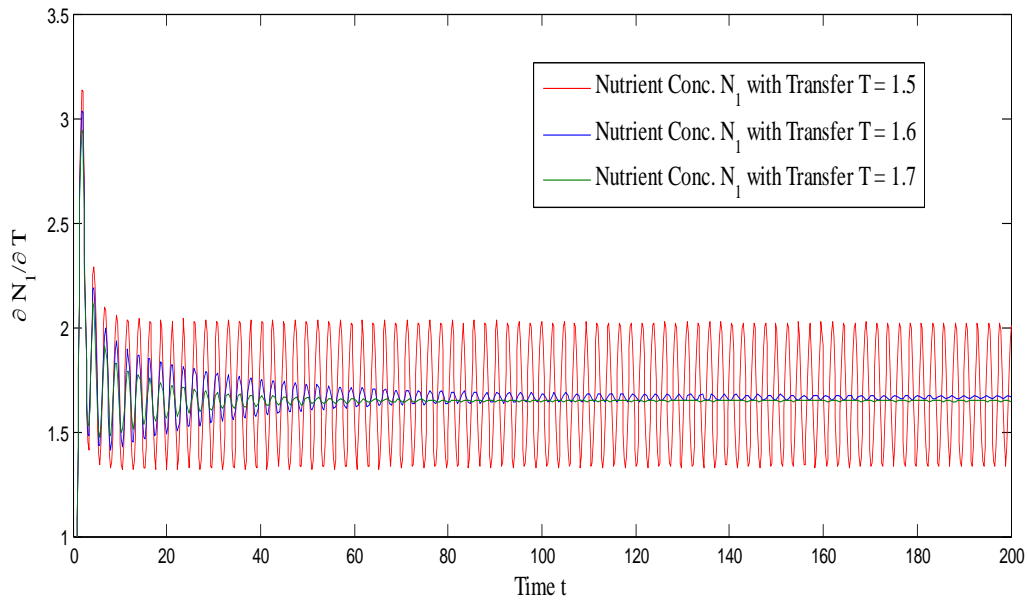
**Figure 2. 1**The system is absolutely stable when there is no delay  $\tau = 0$ .



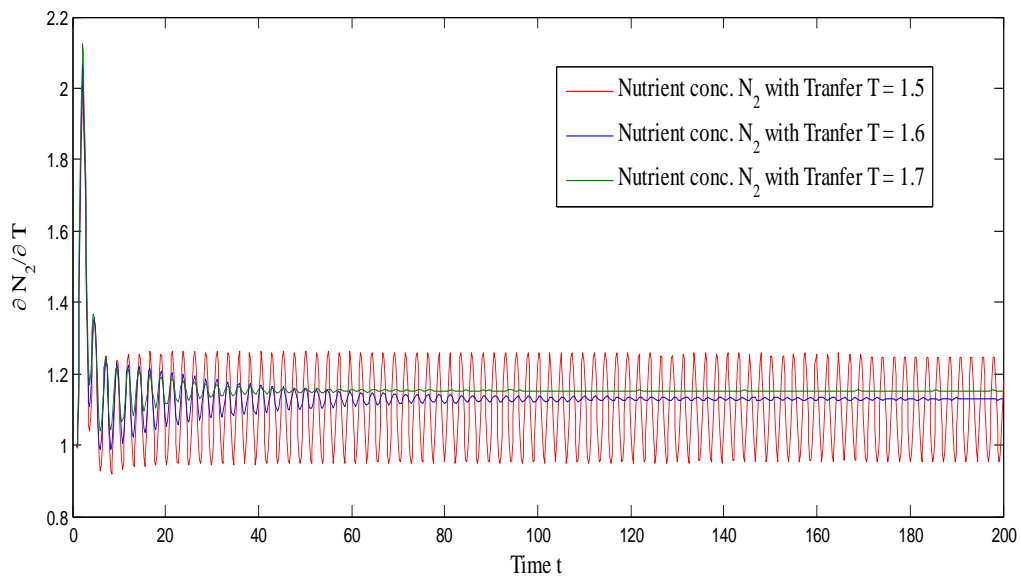
**Figure 2. 2**The system is asymptotically stability when delay  $\tau < 0.9$ .



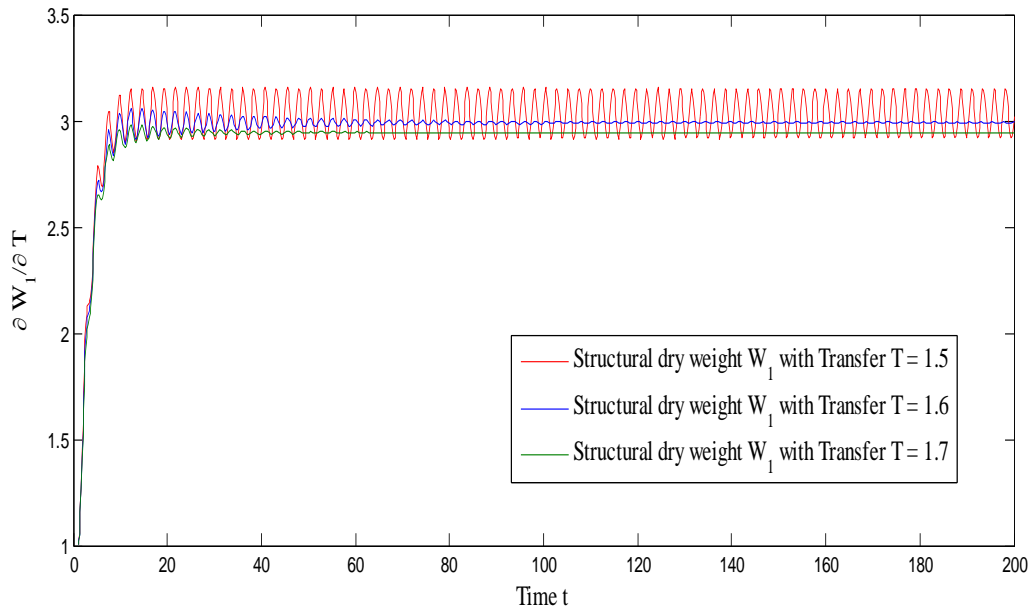
**Figure 2. 3**The system loses stability with delay  $\tau \geq 0.9$  and Hopf bifurcation occurs.



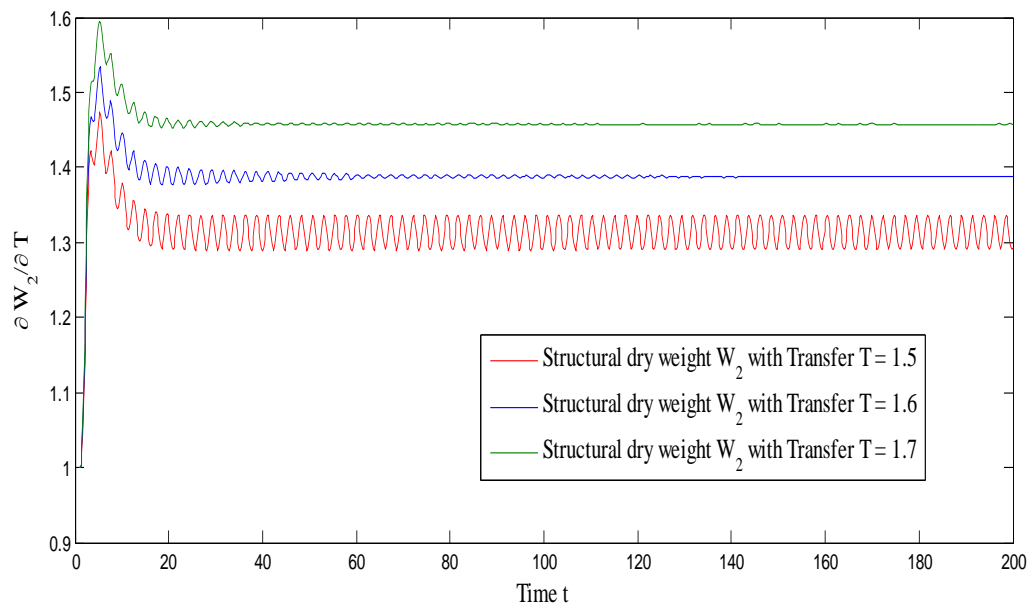
**Figure 2. 4** Time series graph between partial changes in concentration of nutrient  $N_1$  in root compartment and different values of rate of nutrient transfer  $T$ .



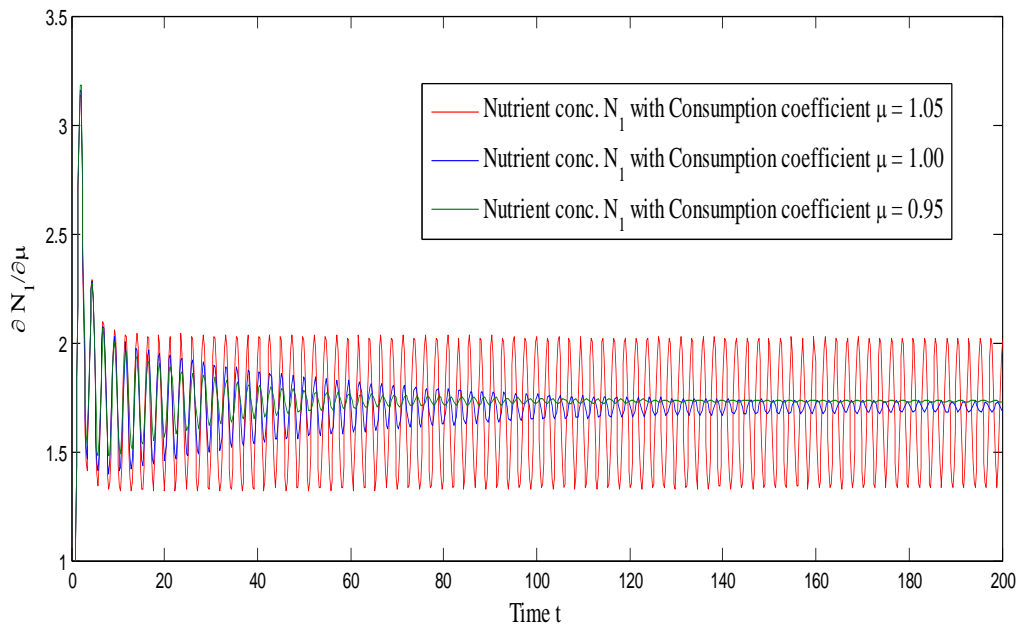
**Figure 2. 5** Time series graph between partial changes in concentration of nutrient  $N_2$  in shoot compartment and different values of rate of nutrient transfer  $T$ .



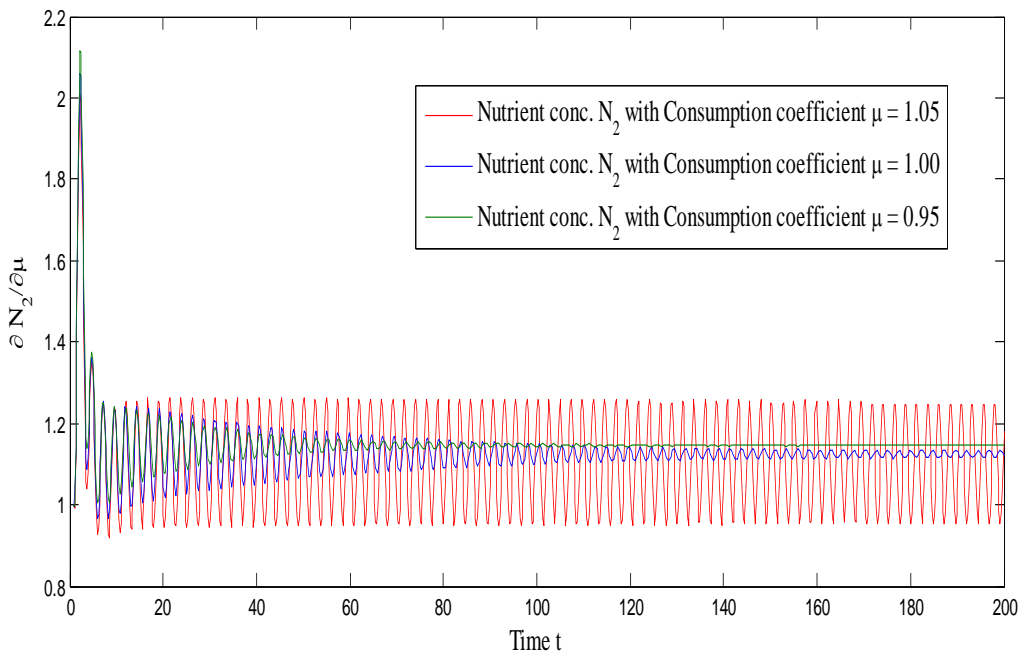
**Figure 2. 6** Time series graph between partial changes in structural dry weight  $W_1$  of root compartment and different values of rate of nutrient transfer  $T$ .



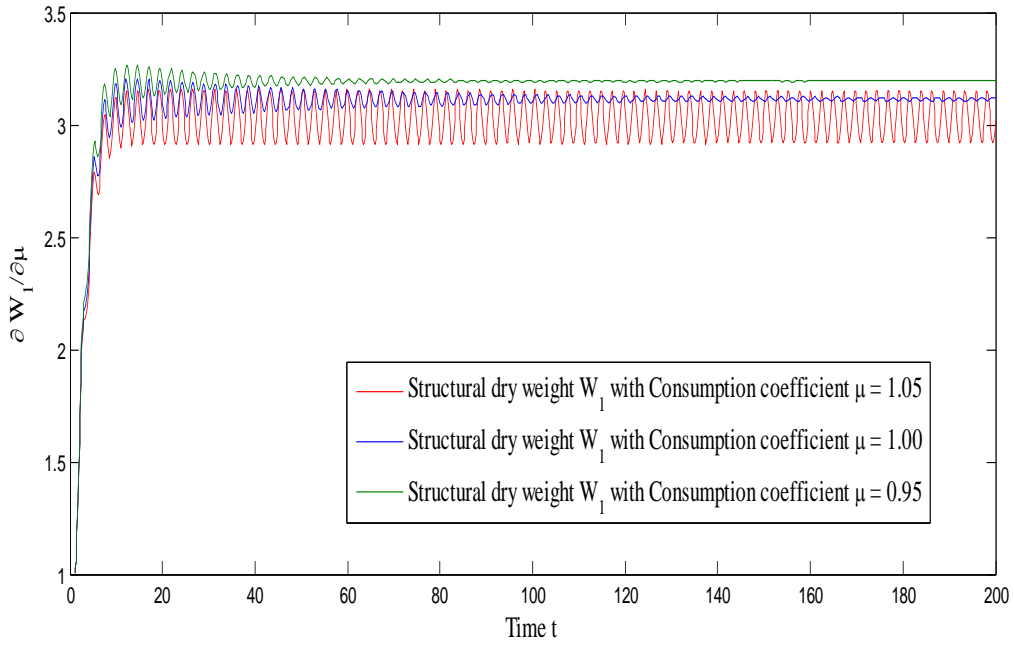
**Figure 2. 7** Time series graph between partial changes in structural dry weight  $W_2$  of shoot compartment and different values of rate of nutrient transfer  $T$ .



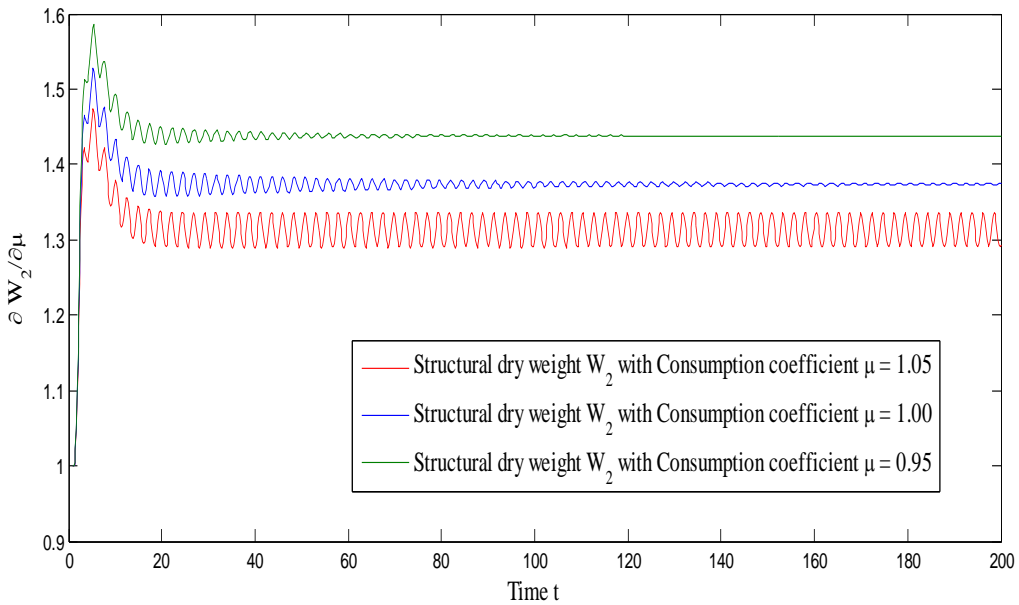
**Figure 2. 8** Time series graph between partial changes in concentration of nutrient  $N_1$  in root compartment and different values of consumption coefficient  $\mu$  of delayed nutrient.



**Figure 2. 9** Time series graph between partial changes in concentration of nutrient  $N_2$  in shoot compartment and different partial values of consumption coefficient  $\mu$ .



**Figure 2.10** Time series graph between partial changes in structural dry weight  $W_1$  of root compartment and different values of consumption coefficient  $\mu$  .



**Figure 2.11** Time series graph between partial changes in structural dry weight  $W_2$  of shoot compartment and different values of consumption coefficient  $\mu$  .

## 2.9 Conclusion

The interior equilibrium  $E_1$  of the model has been studied for stability of the model. It is concluded that the equilibrium  $E_1$  of the model is absolutely stable when there is no

delay ( $\tau = 0$ ) in the system as shown in the Figure2.1. Even after the introduction of delay, there have been two different kinds of behaviour shown by the model. When the value of delay is less than 0.9 ( $\tau < 0.9$ ), the equilibrium  $E_1$  is asymptotically stable as shown in the Figure2.2. This value of delay parameter  $\tau = 0.9$  is the critical value. When delay crosses this critical value ( $\tau \geq 0.9$ ) equilibrium  $E_1$  of the model loses stability and shows oscillations that is Hopf bifurcation occurs as shown in the Figure2.3.

The sensitivity of model solutions is established by taking different values of the parameters appearing in system. It improves the understanding of the role played by specific model parameters.

Sensitivity analysis reveals that with increase in the transfer rate of nutrient  $T$  from root to shoot, the state variables-concentration of nutrients in root and shoot compartments and structural dry weights of root and shoot compartments ( $N_1, N_2, W_1, W_2$ ) tends towards stability for the same set of remaining parameters including time delay  $\tau = 0.9$ . At  $T = 1.5$ , all the above-mentioned state variables show unstable behaviour via Hopf bifurcation. But as the value of  $T$  is increased to  $T = 1.6$ , all the state variables start showing the asymptotic stability and finally for  $T = 1.7$ , all the state variables start converging to a stable equilibrium point as shown by Figures 2.4 to 2.7. Apart from converging to stability, the structural dry weight  $W_2$  of shoot compartment also show increase in its value as we increase the value of  $T$  from 1.5 to 1.7 as shown by Figure 2.7. Similarly, as we decrease the value of consumption coefficient  $\mu$  from  $\mu = 1.05$  to  $\mu = 0.95$ , all the above-mentioned state variables start converging to a stable equilibrium as shown by Figures 2.8 to 2.11. In addition to convergence to stability, the structural dry weights  $W_1$  and  $W_2$  of root and shoot compartment show increase in their values as we decrease the value of consumption coefficient  $\mu$  of delayed nutrient as shown by Figure 2.10 and Figure2.11.



# **Chapter 3**

## **The Study of Effect of Toxicant on Plant Growth Dynamics Considering Time Lag in Nutrient Utilization: A Two Compartment Model**

### **3.1 Introduction**

The survival of the plant population is under great threat as excessive quantity of the toxic metals and contaminants released from industries, acid rain and agricultural fields is entering our ecosystems regularly. Industries are producing heavy metals and radioactive substances. The fertilizers, pesticides and insecticides used in agriculture fields for more production toxic metals which cause harm to plant population. Thornley[3] authored a book in which plant physiology was studied entirely using mathematical modelling. But narrow scope of models given by Thornley was pointed out by Lapointe [21]. The failure to represent the topological and geometrical differences in Lapointe's models was brought into light by Godin et al.[163]. A mathematical model consisting of combined effect of toxic metal and soil chemistry for study of the adverse effect of toxic metal on biomass of trees was given by De Leo et al.[59]. The model proposed by De Leo was further modified and applied to all plants by Guala et al. [166], [167]. A two-compartment mathematical model given by Misra and Kalra [168], [87] studied the adverse effect of toxicity on individual plant growth by showing the overall decrease in uptake and concentration of nutrients and plant biomass in root and shoot compartments. The nature of roots of transcendental and exponential polynomial can be studied using Rouches theorem[158]. Ruan and Wei [112], [157] studied the nature and distribution of roots of exponential polynomials for study of stability with time lags using Rouches theorem. The phenomenon of population dynamics was represented using non-linear delay differential equations and its stability was also studied by Kubiacyk and Saker[115]. The reduction of plant biomass under the effect of toxicant with time lag was studied by Naresh et al.[169]. Shukla et al.[170] studied how the crop yield gets adversely affected by the environmentally degraded soil.

The dynamics of a multi team prey predator system under the effect of time lag was studied by Sikarwar and Misra[171]. Naresh et al.[72] studied how the excessive industrial waste results into toxic uptake by plants and the resulting intermediate toxic product formed affects the intrinsic growth rate plant biomass and carrying capacity. Global stability of population growth with the help of non-linear delay differential equations was studied by Huang et al.[172]. Zhang et al.[123] gave a neural network model where the nature of the roots of a 5<sup>th</sup> degree exponential polynomial was discussed.

Although a lot of work has been done on plant growth under the effect of toxicants, but the use of delay differential equations is rare in this field. In presence of toxic metal in the soil, the nutrient uptake by the plant and the nutrient transfer from root to shoot compartment gets delayed. The nutrient use efficiency is adversely affected too. It leads to decrease in structural dry weight. So, this time delay due to toxic metal in soil is directly responsible for decrease in structural dry weight of plant which is the measure of delayed and reduced plant growth. Considering the above fact, a two-compartment mathematical model is proposed for the study of an individual plant growth. Here delay parameter is introduced in the term containing utilization coefficient and the complex behaviour giving rise to Hopf bifurcation is studied.

### **3.2 Mathematical Model**

It is assumed in the model that nutrient uptake by root is hindered due to presence of toxic metal. It is further assumed that there is less transfer of nutrient from root to shoot compartment due to toxic metal. Nutrient concentration decreases in root as well as in shoot compartment resulting in decrease of structural dry weight of root and shoot respectively. It is further assumed that nutrient use efficiency is also affected with toxic metal resulting into decrease of structural dry weight of shoot.

Let  $N_1$  and  $W_1$  represent the concentration of nutrients and structural dry weight in root compartment respectively. Let  $N_2$  and  $W_2$  represent the concentration of nutrients and structural dry weight in shoot compartment respectively. Let  $H_s$  be the concentration of heavy metal in soil. These notations lead to description of following model (3.1) -(3.5) consisting of system of non-linear delay differential equations:

$$\frac{dN_1}{dt} = (U_n - \alpha H_s) - \frac{T}{R_n} N_1 - \mu W_1 N_1(t) \quad (3.1)$$

$$- \tau) - d_1 N_1$$

$$\frac{dN_2}{dt} = \frac{T(H_s)}{R_n} N_1 - \mu W_2 N_2 - d_2 N_2 \quad (3.2)$$

$$\frac{dW_1}{dt} = r_1(N_1, H_s) W_1 - \Delta_1 W_1^2 \quad (3.3)$$

$$\frac{dW_2}{dt} = r_2(N_2, H_s) W_2 - \Delta_2 W_2^2 \quad (3.4)$$

$$\frac{dH_s}{dt} = I - \alpha_1 H_s N_1 - \Delta H_s \quad (3.5)$$

With initial conditions as:

$N_1(0) > 0, N_2(0) > 0, W_1(0) > 0, W_2(0) > 0, H_s(0) > 0$  for all  $t \geq 0$  and  $N_1(t - \tau = \varepsilon)$ , Constant for all  $t \in [0, \tau]$

Here  $r_1(N_1, H_s)$  and  $r_2(N_2, H_s)$  have the following forms:

$$r_1(N_1, H_s) = \frac{\rho N_1}{1 + \gamma_1 H_s} - \beta_1(H_s), \quad \frac{\partial r_1(N_1, H_s)}{\partial H_s} < 0, \quad \frac{\partial r_1(N_1, H_s)}{\partial N_1} > 0 \text{ for } N_1 > 0, H_s > 0$$

$$r_2(N_2, H_s) = \frac{\rho N_2}{1 + \gamma_2 H_s} - \beta_2(H_s), \quad \frac{\partial r_2(N_2, H_s)}{\partial H_s} < 0, \quad \frac{\partial r_2(N_2, H_s)}{\partial N_2} > 0 \text{ for } N_2 > 0, H_s > 0$$

$$T(H_s) = \frac{T}{1 + T_0 H_s}, \quad \beta_1(H_s) = \beta_{10} + \beta_{11} H_s, \quad \beta_2(H_s) = \beta_{20} + \beta_{21} H_s$$

The definitions of system parameters are as follows:

$r_2(N_2, H_s)$  and  $r_1(N_1, H_s)$  are rates of growth of shoot and root under the effect of heavy metal  $H_s$ , respectively that are dependent on the availability of nutrient.  $T$  is rate of nutrient transfer from root to shoot compartment.  $R_n$  is the resistance to transportation of nutrient.  $T(H_s)$  is rate of nutrient transfer from root to shoot compartment which is hampered by the presence of heavy metal  $H_s$ .  $R_n$  is the aversion to transportation of nutrient,  $(U_n - \alpha H_s)$  is the rate of uptake by plant inhibited due to the presence of heavy metal in soil.  $\mu$  is consumption coefficient or utilization coefficient,  $\rho$  is efficiency of nutrient utilization.  $\beta_{10}$  is natural decay of  $W_1$ .  $\beta_{20}$  is natural decay of  $W_2$ .  $d_1$  is natural decay of  $N_1$ .  $d_2$  is natural decay of  $N_2$ .  $\Delta_2$  and  $\Delta_1$  are self-limiting rates of growth  $W_2$  and  $W_1$ , respectively.  $\beta_{11}$  and  $\beta_{21}$  are damage rates of  $W_1$  and  $W_2$  respectively due to  $H_s$ .  $I$  is the input rate of toxic metals.  $\Delta$  is the first order decay rate of  $H_s$ .  $\alpha_1$  is the depletion rate of  $H_s$  due to reaction between  $H_s$  and  $N_1$ .  $T_0$  is the stress parameter that measures the increase in the resistance to nutrient transport from root to shoot compartment due to

presence of toxic metal in soil.  $\gamma_1$  and  $\gamma_2$  are the parameters measuring the decrease in nutrient use efficiency due to presence of toxic metal in the plants. Here all the parameters  $\alpha, I, \rho, \Delta, \mu, U_n, \alpha_1, T_0, \gamma_1, \gamma_2, \Delta_1, \Delta_2$  are taken to be positive constants.

### 3.3 Boundedness of Solutions

The boundedness of solutions of the model given by (3.1) -(3.5) is given by the following Lemma:

**Lemma 3.3.1** The model has all its solution lying in the region  $D_1 = \left\{ (N_1, N_2, W_1, W_2, H_s) \in R_+^5 : 0 \leq N_1 + N_2 + \frac{\mu}{\rho} W_1 + \frac{\mu}{\rho} W_2 \leq \frac{U_n}{\varphi}, H_{sl} \leq H_s \leq H_{su} \right\}$ , as  $t \rightarrow \infty$ , for all positive initial values  $\{N_1(0), N_2(0), W_1(0), W_2(0), H_s(0), N_1(t - \tau) = \varepsilon \forall t \in [0, \tau]\} \in D_1 \subset R_+^5$ , where  $\varphi = \min(d_1, d_2, \beta_{10}, \beta_{20})$ ,

**Proof:** Consider the following function:

$$F(t) = N_1(t) + N_2(t) + \frac{\mu}{\rho} W_1(t) + \frac{\mu}{\rho} W_2(t)$$

$$\Rightarrow \frac{dF(t)}{dt} = \frac{d}{dt} \left[ N_1(t) + N_2(t) + \frac{\mu}{\rho} W_1(t) + \frac{\mu}{\rho} W_2(t) \right]$$

Using Equations (3.1) -(3.4) and  $\varphi = \min(d_1, d_2, \beta_{10}, \beta_{20})$  and assumption  $N_1(t) \approx N_1(t - \tau)$  as  $t \rightarrow \infty$ ,

$$\Rightarrow \frac{dF(t)}{dt} \leq U_n - \varphi F(t).$$

By usual comparison theorem, when  $t \rightarrow \infty : F(t) \leq \frac{U_n}{\varphi}$

$$\Rightarrow N_1(t) + N_2(t) + \frac{\mu}{\rho} W_1(t) + \frac{\mu}{\rho} W_2(t) \leq \frac{U_n}{\varphi}.$$

Also  $F(t) \geq 0$ . So,

$$0 \leq N_1(t) + N_2(t) + \frac{\mu}{\rho} W_1(t) + \frac{\mu}{\rho} W_2(t) \leq \frac{U_n}{\varphi}$$

From equation (3.5):  $\frac{dH_s}{dt} = I - \alpha_1 H_s N_1 - \Delta H_s$

$$\Rightarrow \frac{dH_s}{dt} \leq I - \Delta H_s$$

Then by usual comparison theorem, we get when  $t \rightarrow \infty$ :

$$H_s \leq \frac{I}{\Delta} = H_{su}$$

Again, from equation (5.5), we get  $\frac{dH_s}{dt} = I - \alpha_1 H_s N_1 - \Delta H_s$

$$\Rightarrow \frac{dH_s}{dt} \geq I - \alpha_1 H_s \frac{U_n}{\varphi} - \Delta H_s$$

$$\Rightarrow \frac{dH_s}{dt} \geq I - \vartheta_3 H_s \text{ where } \vartheta_3 = \left( \frac{\alpha_1 U_n}{\varphi} + \Delta \right)$$

By usual comparison theorem, when  $t \rightarrow \infty$ :

$$H_s \geq \frac{I}{\vartheta_3} = H_{sl}$$

So

$$H_{sl} \leq H_s \leq H_{su}.$$

This completes the proof.

The boundedness lemma proves the fact that since all the quantities-nutrient concentrations in root and shoot, toxic metal in soil and structural dry weights of root and shoot are real quantities, so their individual values as well as their interactional combinations can never be negative and will always be finite for all the time.

### 3.4 Positivity of Solutions

For positive solutions, we need to show that all solution of system given by Equations. (3.1)– (3.5), where initial condition are  $N_1(0) > 0, N_2(0) > 0, W_1(0) > 0, W_2(0) > 0, H_s(0) > 0$  for all  $t > 0$  and  $N_1(t - \tau) = \varepsilon \forall t \in [0, \tau]$ , the solution  $(N_1(t), N_2(t), W_1(t), W_2(t), H_s(t))$  of the model stays positive  $\forall t > 0$ .

From equation (3.2):  $\frac{dN_2}{dt} = \frac{T(H_s)}{R_n} N_1 - \mu W_2 N_2 - d_2 N_2$

$$\Rightarrow \frac{dN_2}{dt} \geq - \left( \mu \frac{U_n}{\varphi} + d_2 \right) N_2$$

$$\Rightarrow N_2 \geq c_1 e^{-\left( \mu \frac{U_n}{\varphi} + d_2 \right) t}$$

Hence  $N_2 > 0$  as  $t \rightarrow \infty$

Similar argument holds for  $N_1, W_1, W_2, H_s$ .

Thus, all the variables remain positive which shows that the system persists.

### 3.5 Interior Equilibrium of the Model

The system of equations (3.1) -(3.5) has one feasible positive interior equilibrium  $E_1(N^*_1, N^*_2, W^*_1, W^*_2, H^*_s)$  where:

$$W^*_1 = \frac{1}{\Delta_1} \left[ \frac{\rho}{1+\gamma_1 H^*_s} N^*_1 - (\beta_{10} + \beta_{11} H^*_s) \right] > 0 ,$$

provided  $\rho N^*_1 > (1 + \gamma_1 H^*_s)(\beta_{10} + \beta_{11} H^*_s)$

$$W^*_2 = \frac{1}{\Delta_2} \left[ \frac{\rho}{1+\gamma_2 H^*_s} N^*_2 - (\beta_{20} + \beta_{21} H^*_s) \right] > 0 ,$$

provided  $\rho N^*_2 > (1 + \gamma_2 H^*_s)(\beta_{20} + \beta_{21} H^*_s)$

$$N^*_1 = \frac{I - \Delta H^*_s}{\alpha_1 H^*_s} \text{ provided } I > \Delta H^*_s,$$

$$N^*_2 = \frac{-g_2 + \sqrt{g_2^2 - 4g_1g_3}}{2g_1} > 0$$

where  $g_1 = \frac{\mu\rho}{\Delta_1(1+\gamma_1 H^*_s)}$ ,  $g_2 = \left( d_1 - \frac{\mu(\beta_{10} + \beta_{11} H^*_s)}{\Delta_1} \right)$ ,  $g_3 = -\frac{T(I - \Delta H^*_s)}{R_n \alpha_1 H^*_s (1 + T_0 H^*_s)}$

The value of  $H^*_s$  is given by positive root of the equation

$$\gamma_2(\alpha + \delta_3 \Delta) H^*_s{}^4 - [\gamma_2(U_n - \delta_1 \Delta + \delta_3 I) - (\alpha + \delta_3 \Delta)] H^*_s{}^3 - [U_n + \delta_3 I + \delta_1(\Delta - I\gamma_2) - \delta_2 \Delta^2] H^*_s{}^2 - I(2\Delta\delta_2 - \delta_1) H^*_s + \delta_2 I^2 = 0$$

Where  $\delta_1 = \frac{1}{\alpha} \left( \frac{T}{R_n} + d_1 - \frac{\mu\beta_{10}}{\Delta_1} \right)$ ,  $\delta_2 = \frac{\mu\rho}{\Delta_1 \alpha_1^2}$ ,  $\delta_3 = \frac{\pi\beta_{11}}{\Delta_1 \alpha_1}$

The 4<sup>th</sup> degree polynomial in  $H^*_s$  will be having at most two positive roots provided:

$$\gamma_2(U_n - \delta_1 \Delta + \delta_3 I) > (\alpha + \delta_3 \Delta), (U_n + \delta_3 I + \delta_1 \Delta) > \delta_1 I \gamma_2 + \delta_2 \Delta^2, k_1 > 2\Delta\delta_2$$

But due to positivity of  $N^*_1$  only one feasible positive root will exist provided  $I > \Delta H^*_s$

### 3.6 Stability Analysis and Local Hopf-Bifurcation

The exponential characteristic equation about equilibrium  $E_1$  is given by:

$$\begin{aligned} & (\lambda^5 + A_1 \lambda^4 + A_2 \lambda^3 + A_3 \lambda^2 + A_4 \lambda + A_5) \\ & + (B_1 \lambda^4 + B_2 \lambda^3 + B_3 \lambda^2 + B_4 \lambda \\ & + B_5) e^{-\lambda\tau} = 0 \end{aligned} \quad (3.6)$$

Here  $A_1 = -(P_1 + P_7 + P_{13} + P_{19} + P_{25})$ ,  $A_2 = (P_7P_{13} + P_{13}P_{25} + P_{25}P_7 + P_{25}P_{10} + P_{11}P_{13} + P_{11}P_7 + P_{11}P_{13} + P_{11}P_{25} + P_7P_{19} + P_{13}P_{19} + P_{19}P_{25})$ ,  $A_3 = -((P_1 + P_{19})(P_7P_{13} + P_{13}P_{25} + P_{25}P_7 + P_{10}P_{25}) + P_1P_{19}(P_7 + P_{13} + P_{25}) + P_1P_{13}P_7 + P_1P_{19} + P_{25} + P_7P_{13}P_{25} + P_{10}P_{13}P_{22})$ ,  $A_4 = P_1P_{19}P_7P_{13} + P_{13}P_{25} + P_{25}P_7 + P_{25}P_{10} + P_1 + P_{19}P_7P_{13}P_{25} + P_{10}P_{13}P_{22} + P_1P_{13}P_{19}P_7 + P_{25} + P_{25}P_7 - P_1P_{19}P_7P_{13}P_{25} + P_{10}P_{13}P_{22} + P_{11}P_{13}P_{19}P_{25}P_7 - P_{10}P_{22}$ ,  $B_1 = \mu W^*_1$ ,  $B_2 = -\mu W^*_1(P_1 + P_{13} + P_{19} + P_{25})$ ,  $B_3 = \mu W^*_1((P_1 + P_{19})(P_{13} + P_{25}) + P_1P_{19} + P_1P_{13} + P_{13}P_{25})$ ,  $B_4 = -\mu W^*_1(P_1P_{19}(P_{13} + P_{25}) + P_{13}(P_1 + P_{19}) + P_1P_{13}(P_{19} + P_{25}))$ ,  $B_5 = \mu W^*_1(P_1P_{19}P_{13}P_{25} + P_{11}P_{19}P_{13}P_{25})$

Where  $P_1 = -(\mu W^*_2 + d_2)$ ,  $P_2 = \frac{T}{R_n(1+T_0H^*_s)}$ ,  $P_3 = -\mu N^*_2$ ,  $P_4 = 0$ ,  $P_5 = \frac{-TT_0N^*_1}{R_n(1+T_0H^*_s)^2}$ ,  $P_6 = 0$ ,  $P_7 = -\left(\frac{T}{R_n} + d_1\right)$ ,  $P_8 = 0$ ,  $P_9 = 0$ ,  $P_{10} = -\alpha$ ,  $P_{11} = \frac{\rho W^*_2}{1+\gamma_2H^*_s}$ ,  $P_{12} = 0$ ,  $P_{13} = \frac{\rho N^*_2}{(1+\gamma_2H^*_s)^2} - (\beta_{20} + \beta_{21}H^*_s) - 2\Delta_2W^*_2$ ,  $P_{14} = 0$ ,  $P_{15} = -\left(\frac{\rho N^*_2W^*_2\gamma_2}{(1+\gamma_2H^*_s)^2} + \beta_{21}W^*_2\right)$ ,  $P_{16} = \frac{\rho W^*_1}{1+\gamma_1H^*_s}$ ,  $P_{17} = 0$ ,  $P_{18} = 0$ ,  $P_{19} = \frac{\rho N^*_1}{(1+\gamma_1H^*_s)^2} - (\beta_{10} + \beta_{11}H^*_s) - 2\Delta_1W^*_1$ ,  $P_{20} = -\left(\frac{\rho N^*_1W^*_1\gamma_1}{(1+\gamma_1H^*_s)^2} + \beta_{11}W^*_1\right)$ ,  $P_{21} = 0$ ,  $P_{22} = -\alpha_1H^*_s$ ,  $P_{23} = 0$ ,  $P_{24} = 0$ ,  $P_{25} = -(\alpha_1N^*_1 + \Delta)$

Let  $\lambda = i\omega$  is a root of equation (3.6), So

$$\begin{aligned} & ((i\omega)^5 + A_1(i\omega)^4 + A_2(i\omega)^3 + A_3(i\omega)^2 + A_4(i\omega) + A_5) \\ & + (B_1(i\omega)^4 + B_2(i\omega)^3 + B_3(i\omega)^2 + B_4(i\omega) + B_5)e^{-(i\omega)\tau} = 0 \\ \Rightarrow & (i\omega^5 + A_1\omega^4 - iA_2\omega^3 - A_3\omega^2 + iA_4\omega + A_5) \\ & + (B_1\omega^4 - iB_2\omega^3 - B_3\omega^2 + iB_4\omega + B_5)(\cos \omega\tau - i\sin \omega\tau) = 0 \end{aligned}$$

Separating real and imaginary parts:

$$\begin{aligned} & (\omega^5 - A_2\omega^3 + A_4\omega) \\ & + (B_4\omega - B_2\omega^3) \cos \omega\tau \\ & - (B_1\omega^4 - B_3\omega^2 \\ & + B_5) \sin \omega\tau = 0 \end{aligned} \tag{3.7}$$

$$\begin{aligned}
& (A_1\omega^4 - A_3\omega^2 + A_5) \\
& + (B_1\omega^4 - B_3\omega^2 \\
& + B_5) \cos \omega\tau \\
& + (B_4\omega - B_2\omega^3) \sin \omega\tau = 0
\end{aligned} \tag{3.8}$$

Squaring and adding equation (5.7) and (5.8), we get:

$$\omega^{10} + a\omega^8 + b\omega^6 + c\omega^4 + d\omega^2 + r = 0 \tag{3.9}$$

Where  $a = (A_1^2 - B_1^2 - 2A_2)$ ,  $b = (A_2^2 - B_2^2 - 2A_4 - 2A_1A_3 + 2B_1B_3)$ ,  $c = (A_3^2 - B_3^2 - 2A_2A_4 + 2B_2B_4 - 2B_1B_5)$ ,  $d = (A_4^2 - B_4^2 - 2A_3A_5 + 2B_3B_5)$ ,  $r = (A_5^2 - B_5^2)$

Let  $\omega^2 = y$ , then equation (5.9) becomes:

$$y^5 + ay^4 + by^3 + cy^2 + dy + r = 0 \tag{3.10}$$

**Lemma 3.6.1** If  $r < 0$ , then equation (3.10) has at least one positive real root.

**Proof.** Let  $h(y) = y^5 + ay^4 + by^3 + cy^2 + dy + r$

Here  $h(0) = r < 0$ ,  $\lim_{y \rightarrow \infty} h(y) = \infty$

So,  $\exists y_0 \in (0, \infty)$  such that  $h(y_0) = 0$

Proof completed.

Also  $h'(y) = 5y^4 + 4ay^3 + 3by^2 + 2cy + d$

$$\begin{aligned}
h'(y) &= 0 \\
&\Rightarrow 5y^4 + 4ay^3 + 3by^2 + 2cy \\
&+ d = 0
\end{aligned} \tag{3.11}$$

Which becomes:

$$x^4 + px^2 + qx + s = 0 \tag{3.12}$$

Where  $x = y + \frac{a}{5}$ ,  $p = \frac{3b}{5} - \frac{6a^2}{25}$ ,  $q = \frac{2c}{5} + \frac{6ab}{25} + \frac{8a^3}{125}$ ,  $s = \frac{d}{5} - \frac{2ac}{25} + \frac{3a^2b}{125} - \frac{3a^4}{625}$

If  $q = 0$ , then, four roots of equation (5.12) come out to be:

$$x_1 = \sqrt{\frac{-p + \sqrt{D}}{2}}, x_2 = -\sqrt{\frac{-p + \sqrt{D}}{2}}, x_3 = \sqrt{\frac{-p - \sqrt{D}}{2}}, x_4 = -\sqrt{\frac{-p - \sqrt{D}}{2}}$$

Thus  $y_i = x_i - \frac{a}{5}$ ,  $i = 1, 2, 3, 4$  are the roots of equation (3.12) where:



$$D = p^2 - 4s$$

**Lemma 3.6.2** Suppose  $r \geq 0$  and  $q = 0$ .

(I) If  $D < 0$ , then equation (3.10) has no positive real roots.

(II) If  $D \geq 0, p \geq 0, s \geq 0$ , then equation (3.10) has no positive real roots.

(III) If (I) and (II) are not satisfied, then equation (3.10) has positive real roots iff  $\exists$  at least one  $y^* \in (y_1, y_2, y_3, y_4)$  such that  $y^* > 0$  and  $h(y^*) \leq 0$ .

**Proof.** (I) If  $D < 0$ , then equation (3.11) has no positive real roots. Since  $\lim_{y \rightarrow \infty} h(y) = \infty$ , we have  $h'(y) > 0$  for  $y \in R$ . Hence  $h(0) = r \geq 0$  implies  $h(y)$  has no zero in  $(0, \infty)$ .

(II) Condition  $D \geq 0, p \geq 0, s \geq 0$  imply that  $h'(y)$  has no zero in  $(-\infty, \infty)$ . It is similar to (I) that  $h(y)$  has no zero in  $(0, \infty)$ .

(III) The sufficiency is obvious. We need only to prove the necessity. If  $D \geq 0$ , we know that equation (3.12) has only four roots  $x_1, x_2, x_3$  and  $x_4$ , that is equation (3.11) has only four roots  $y_1, y_2, y_3$  and  $y_4$  at least  $y_1$  is a real root. Without loss of generality, we assume that  $y_1, y_2, y_3$  and  $y_4$  are all real. This implies that  $h(y)$  has at most four stationary points  $y_1, y_2, y_3$  and  $y_4$ . If it is not true, then we have that either  $y_1 \leq 0$  or  $y_1 > 0$  and  $\min[h(y_i): y_i > 0, i = 1, 2, 3, 4] > 0$ . If  $y_1 \leq 0$ , then  $h'(y)$  has no zero in  $(0, \infty)$ . Since  $h(0) = r \geq 0$  is the strict minimum of  $h(y)$  for  $y \geq 0$  which implies  $h(y) > 0$  in  $(0, \infty)$ . If  $y_1 > 0$  and  $\min[h(y_i): y_i > 0, i = 1, 2, 3, 4] > 0$ , since  $h(y)$  is a derivable function and  $\lim_{y \rightarrow \infty} h(y) = \infty$ , then we have  $\min_{y > 0} h(y) = \min[h(y_i): y_i > 0, i = 1, 2, 3, 4] > 0$ . The necessity is proved.

This completes the proof.

Next, we assume that  $q \neq 0$ . Consider the resolvent of equation (3.12):

$$q^2 - 4(v - p) \left( \frac{v^2}{4} - s \right) = 0$$

$$\Rightarrow v^3 - pv^2 - 4sr + 4ps - q^2 = 0 \quad (3.13)$$

By Cardan formula, equation (3.13) has the following three roots:

$$v_1 = \left( -\frac{q_1}{2} + \sqrt{D_1} \right)^{1/3} + \left( -\frac{q_1}{2} - \sqrt{D_1} \right)^{1/3} + \frac{p}{3},$$

$$v_2 = \sigma \left( -\frac{q_1}{2} + \sqrt{D_1} \right)^{1/3} + \sigma^2 \left( -\frac{q_1}{2} - \sqrt{D_1} \right)^{1/3} + \frac{p}{3},$$

$$v_3 = \sigma^2 \left( -\frac{q_1}{2} + \sqrt{D_1} \right)^{1/3} + \sigma \left( -\frac{q_1}{2} - \sqrt{D_1} \right)^{1/3} + \frac{p}{3}$$

Where  $p_1 = -\frac{p^2}{3} - 4s$ ,  $q_1 = -\frac{2p^3}{27} + \frac{8ps}{3} - q^2$ ,  $D_1 = \frac{p_1^3}{27} + \frac{q_1^2}{4}$ ,  $\sigma = \frac{1+\sqrt{3}i}{2}$

Let  $v_* = v_1 \neq p$ , then equation (3.12) becomes:

$$\begin{aligned} x^4 + v_*x^2 + \frac{v_*^2}{4} & \\ - \left[ (v_* - p)x^2 - qx + \frac{v_*^2}{4} \right. & \\ \left. - s \right] = 0 & \end{aligned} \quad (3.14)$$

For the above equation (3.14), (3.12) implies that the formula in the square brackets is a perfect square. If  $v_* > p$ , then equation (3.14) becomes

$$\left( x^2 + \frac{v_*}{2} \right)^2 - \left( \sqrt{v_* - p}x - \frac{q}{2\sqrt{v_* - p}} \right)^2 = 0$$

After factorization, we get

$$x^2 + \sqrt{v_* - p}x - \frac{q}{2\sqrt{v_* - p}} + \frac{v_*}{2} \text{ and } x^2 - \sqrt{v_* - p}x - \frac{q}{2\sqrt{v_* - p}} + \frac{v_*}{2}$$

So, four roots of the equation (3.12) are

$$x_1 = \frac{-\sqrt{v_* - p} + \sqrt{D_2}}{2},$$

$$x_2 = \frac{-\sqrt{v_* - p} - \sqrt{D_2}}{2}$$

$$x_3 = \frac{-\sqrt{v_* - p} - \sqrt{D_3}}{2}$$

$$x_4 = \frac{-\sqrt{v_* - p} - \sqrt{D_3}}{2}$$

Where  $D_2 = -v_* - p + \frac{q}{2\sqrt{v_* - p}}$  and  $D_3 = -v_* - p - \frac{q}{2\sqrt{v_* - p}}$

Then  $y_i = x_i - \frac{a}{5}$ ,  $i = 1, 2, 3, 4$  are the roots of equation (3.10).

Thus, we have the following result:

**Lemma 3.6.3** Suppose that  $r \geq 0, q_1 \neq 0$  and  $v_* > p$ .

(I) If  $D_2 < 0$  and  $D_3 < 0$ , then equation (3.5) has no positive real roots.

(II) If (I) is not satisfied, then equation (3.5) has positive real roots iff  $\exists$  at least one  $y^* \in (y_1, y_2, y_3, y_4)$  such that  $y^* > 0$  and  $h(y^*) \leq 0$ .

**Proof.** The proof is similar to Lemma 3.6.2. We omit it. Finally, if  $v_* < p$ , then equation (3.14) becomes:

$$\left(x^2 + \frac{v_*}{2}\right)^2 - \left(\sqrt{p - v_*}x - \frac{q}{2\sqrt{p - v_*}}\right)^2 = 0 \quad (3.15)$$

Let 
$$\bar{y} = \frac{q}{2(p - v_*)} - \frac{a}{5}.$$

Hence, we have the following result.

**Lemma 3.6.4** Suppose that  $r \geq 0, q_1 \neq 0$  and  $v_* < p$ , then equation (3.10) has positive real roots iff  $\frac{q^2}{4(p - v_*)^2} + \frac{v_*}{2} = 0$  and  $\bar{y} > 0$  and  $h(\bar{y}) \leq 0$ .

**Proof.** Assume equation (5.14) has a real root  $x_0$  satisfying:

$$\begin{aligned} x_0 &= \frac{q}{2(p - v_*)}, x_0^2 = -\frac{v_*}{2} \\ &\Rightarrow \frac{q^2}{4(p - v_*)^2} + \frac{v_*}{2} = 0. \end{aligned}$$

Therefore, equation (3.14) has a real root  $x_0$  iff  $\frac{q^2}{4(p - v_*)^2} + \frac{v_*}{2} = 0$ .

The rest of the proof is similar to Lemma 3.6.2. We omit it.

Suppose equation (3.10) possesses positive roots. In general, we suppose that it has 5 positive roots denoted by  $y^*_i, i = 1, 2, 3, 4, 5$ . Then equation (3.9) has 5 positive roots  $\omega_i = \sqrt{y^*_i}, i = 1, 2, 3, 4, 5$ .

We have 
$$\cos \omega \tau = \frac{A_6}{(B_4 \omega - B_2 \omega^3)^2 + (B_1 \omega^4 - B_3 \omega^2 + B_5)^2}$$

Which gives 
$$\tau = \frac{1}{\omega} \left[ \cos^{-1} \left( \frac{A_6}{(B_4 \omega - B_2 \omega^3)^2 + (B_1 \omega^4 - B_3 \omega^2 + B_5)^2} \right) + 2j\pi \right]; j = 0, 1, 2, 3, \dots$$

Where  $A_6 = -((B_1\omega^4 - B_3\omega^2 + B_5)(A_1\omega^4 - A_3\omega^2 + A_5) + (B_4\omega - B_2\omega^3)(\omega^5 - A_2\omega^3 + A_4\omega)$

Let  $\tau_k^{(j)} = \frac{1}{\omega_k} \left[ \cos^{-1} \left( \frac{A_6}{(B_4\omega - B_2\omega^3)^2 + (B_1\omega^4 - B_3\omega^2 + B_5)^2} \right) + 2j\pi \right]; k = 1, 2, 3, 4, 5.; j = 0, 1, 2, 3, \dots$

Then  $\bar{\tau}i\omega_k$  is a pair of purely imaginary roots of equation (3.6)

Where  $\tau = \tau_k^{(j)}, k = 1, 2, 3, 4, 5.; j = 1, 2, 3, \dots$

We have  $\lim_{j \rightarrow \infty} \tau_k^{(j)} = \infty, k = 1, 2, 3, 4, 5.$

Thus, we can define:

$$\begin{aligned} \tau_0 = \tau_{k_0}^{(j_0)} &= \min_{1 \leq k \leq 4, j \geq 1} [\tau_k^{(j)}], \\ \omega_0 = \omega_{k_0}, \quad y_0 &= y_{k_0}^* \end{aligned} \quad (3.16)$$

**Lemma 3.6.5** Suppose that  $u_1 > 0, (u_1u_2 - u_3) > 0, u_3(u_1u_2 - u_3) + u_1(u_5 - u_1u_4) > 0, u_2u_5 + u_3u_3u_1u_2 - u_3 + u_1u_4u_5 - u_1u_4 > 0, u_5 > 0.$

Where  $u_1 = (A_1 + B_1), u_2 = (A_2 + B_2), u_3 = (A_3 + B_3), u_4 = (A_4 + B_4), u_5 = (A_5 + B_5).$

(I) If any one of the following condition holds: (i)  $r < 0$  (ii)  $r \geq 0, q = 0, D \geq 0$  and  $p < 0$  or  $s \leq 0$  and there exists a  $y^* \in (y_1, y_2, y_3, y_4)$  such that  $y^* > 0$  and  $h(y^*) \leq 0$  (iii)  $r \geq 0, q \neq 0, v_* > p, D_2 \geq 0$  or  $D_3 \geq 0$  and there exists a  $y^* \in (y_1, y_2, y_3, y_4)$  such that  $y^* > 0$  and  $h(y^*) \leq 0$  (iv)  $r \geq 0, q \neq 0, v_* < p, \frac{q^2}{4(p-v_*)^2} + \frac{v_*}{2} = 0, \bar{y} > 0$  and  $h(\bar{y}) \leq 0$ , then negative real part will be there in all roots of equation(3.1) when  $\tau \in [0, \tau_0)$

(II) If any one of the conditions (i)-(iv) of (I) are not satisfied, then negative real parts will be there in all roots of equation (3.1) for all  $\tau \geq 0.$

**Proof.** When  $\tau = 0$ , equation (3.6) becomes;

$$\begin{aligned} \lambda^5 + (A_1 + B_1)\lambda^4 + (A_2 + B_2)\lambda^3 + (A_3 + B_3)\lambda^2 + (A_4 + B_4)\lambda + (A_5 + B_5) &= 0 \\ \Rightarrow \lambda^5 + u_1\lambda^4 + u_2\lambda^3 + u_3\lambda^2 + u_4\lambda + u_5 & \quad (3.17) \\ &= 0 \end{aligned}$$

All roots of equation (3.17) have negative real parts iff supposition of Lemma 3.6.5 holds (Routh-Hurwitz's criteria).

From Lemmas 3.6.1- 3.6.4, we know that if conditions (i)-(iv) of (I) are not satisfied, then none of the roots of equation (3.6) will have zero real part for all  $\tau \geq 0$ .

If one of the conditions (i)-(iv) holds, when  $\tau \neq \tau_k^{(j)}, k = 1, 2, 3, 4, 5; j \geq 1$ , then none of the roots of equation (3.6) will have zero real part and  $\tau_0$  is the minimum value of  $\tau$  for which the roots of equation (5.6) are purely imaginary. This Lemma is concluded by using Theorem 1.3.8.

$$\text{Let } \lambda(\tau) = \psi(\tau) + i\omega(\tau) \quad (3.18)$$

be the roots of equation (3.6) satisfying:  $\psi(\tau_0) = 0, \omega(\tau_0) = \omega_0$ .

Then we have the following Lemma.

**Lemma 3.6.6** Suppose  $h'(y_0) \neq 0$ . If  $\tau = \tau_0$ , then  $\mp i\omega_0$  is a pair of simple purely imaginary roots of equation (3.6). Moreover, If the condition of Lemma 3.6.5 (I) are satisfied, then  $\frac{d}{d\tau}(Re\lambda(\tau_0)) > 0$ .

**Proof.** Substituting  $\lambda(\tau)$  into equation (3.6) and differentiating both sides with respect to  $\tau$

$$\begin{aligned} & \left(\frac{d\lambda}{d\tau}\right)^{-1} \\ &= \frac{(5\lambda^4 + 4A_1\lambda^3 + 3A_2\lambda^2 + 2A_3\lambda + A_4)e^{\lambda\tau} + (4B_1\lambda^3 + 3B_2\lambda^2 + 2B_3\lambda + B_4)}{(B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5)} - \frac{\tau}{\lambda} \end{aligned}$$

By calculation, we have:

$$\begin{aligned} & [(5\lambda^4 + 4A_1\lambda^3 + 3A_2\lambda^2 + 2A_3\lambda + A_4)e^{\lambda\tau}]_{\tau=\tau_0} \\ &= A_7 \cos \omega_0\tau + A_8 \sin \omega_0\tau + i(-A_8 \cos \omega_0\tau + A_7 \sin \omega_0\tau) \\ & (4B_1\lambda^3 + 3B_2\lambda^2 + 2B_3\lambda + B_4)_{\tau=\tau_0} = B_4 - 3B_2\omega_0^2 + i\omega_0(2B_3 - 4B_1\omega_0^2) \\ & (B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5)_{\tau=\tau_0} \\ &= \omega_0^2(B_2\omega_0^2 - B_5) + i\omega_0(B_5 - B_3\omega_0^2 + B_1\omega_0^4) \end{aligned}$$

Where  $A_7 = (5\omega_0^4 - 3A_3\omega_0^2 + A_4), A_8 = (4A_1\omega_0^3 - 2A_3\omega_0)$

Then, we have:

$$\left(\frac{d \operatorname{Re}\lambda(\tau_0)}{d\tau}\right)^{-1} = \frac{y_0 h'(y_0)}{A_9} \quad (3.19)$$

Where  $A_9 = \omega_0^2 [(B_2 \omega_0^3 - B_5 \omega_0)^2 + (B_5 - B_3 \omega_0^2 + B_1 \omega_0^4)^2]$

Thus, we have:

$$\begin{aligned} \operatorname{sign} \left[ \frac{d \operatorname{Re}\lambda(\tau_0)}{d\tau} \right] &= \operatorname{sign} \left[ \left( \frac{d \operatorname{Re}\lambda(\tau_0)}{d\tau} \right)^{-1} \right] \\ &= \operatorname{sign} \left[ \frac{y_0 h'(y_0)}{A_9} \right] \end{aligned} \quad (3.20)$$

Notice that  $A_9, y_0 > 0$ .

Thus, applying the Lemmas 3.6.1-3.6.6, we have the following theorem:

**Theorem 3.1.** *Let  $\omega_0, y_0, \tau_0$  and  $\lambda(\tau)$  be defined by (3.16) to (3.18), respectively. Assume that the supposition of Lemma 3.6.5 holds.*

(I) *If the conditions (i)-(iv) of Lemma 3.6.5 are not satisfied, then all the roots of equation (3.6) have negative real parts for all  $\tau \geq 0$ .*

(II) *If one of the conditions (i)-(iv) of Lemma 3.6.5 is satisfied, then all the roots of equation (3.6) have negative real parts when  $\tau \in [0, \tau_0)$ ; when  $\tau = \tau_0$  and  $h'(y_0) \neq 0$ , then  $\mp i\omega_0$  is a pair of purely imaginary roots of equation (3.6) and all other roots have negative real parts. In addition,  $\frac{d \operatorname{Re}\lambda(\tau_0)}{d\tau} > 0$  and equation (3.6) has at least one root with positive real part when  $\tau \in (\tau_0, \tau_1)$ , where  $\tau_1$  is the first value of  $\tau > \tau_0$  such that equation (3.6) has purely imaginary roots.*

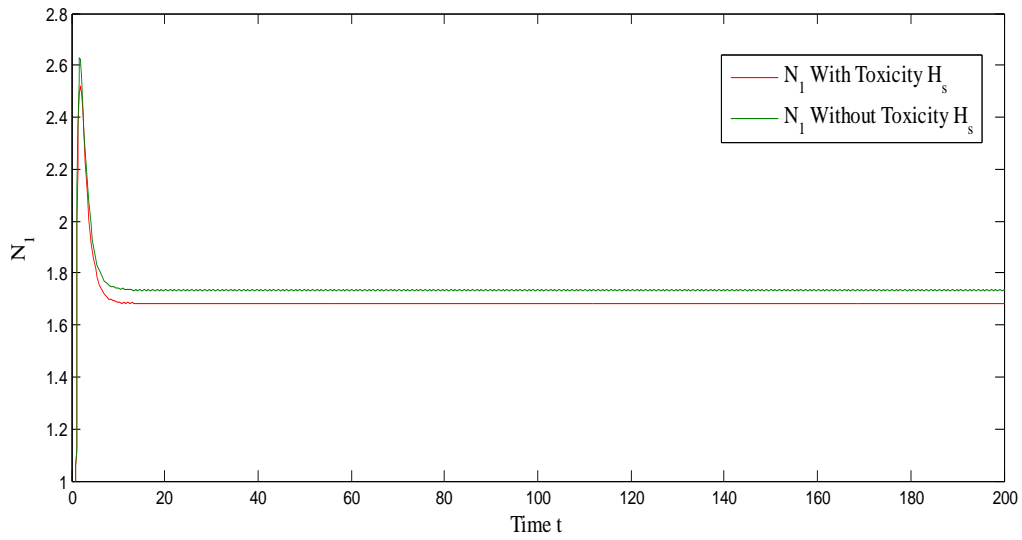
## 3.7 Numerical Example

Numerical method has been used to find the solution of the system of delay differential equations given by equations (3.1) – (3.5), we consider the following values of parameters:

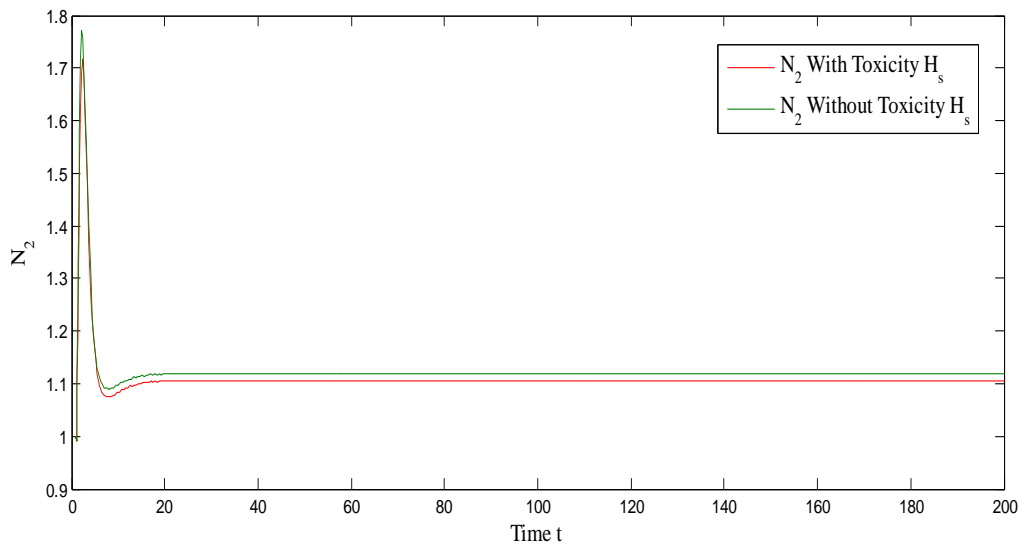
$$U_n = 10, T = 1.5, R_n = 1, \mu = 1.05, d_1 = 0.9, d_2 = 0.9, \rho = 0.3, \beta_{10} = 0.2, \beta_{20} = 0.2, \Delta_1 = 0.1, \Delta_2 = 0.1, I = 2, \alpha_1 = 1.6, \Delta = 0.1$$

For the given parametric values, we have  $E_1: N^*_1 = 1.6830, N^*_2 = 1.1057, W^*_1 = 3.0490, W^*_2 = 1.3172, H^*_s = 0.7161$ .

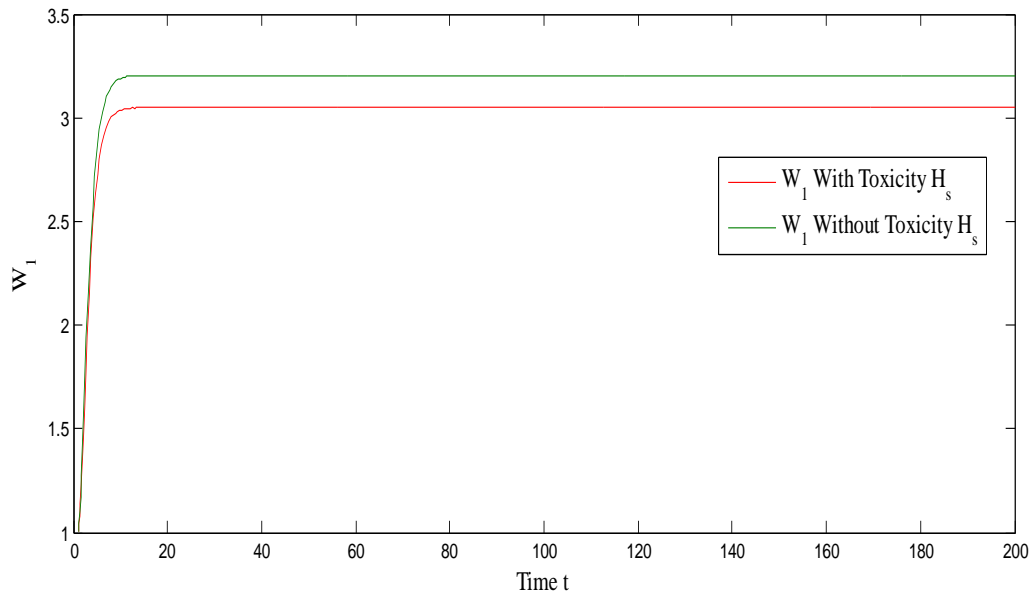
In fact, without toxic effect,  $N^*_1 = 1.7357, N^*_2 = 1.1186, W^*_1 = 3.2041, W^*_2 = 1.3574$ .



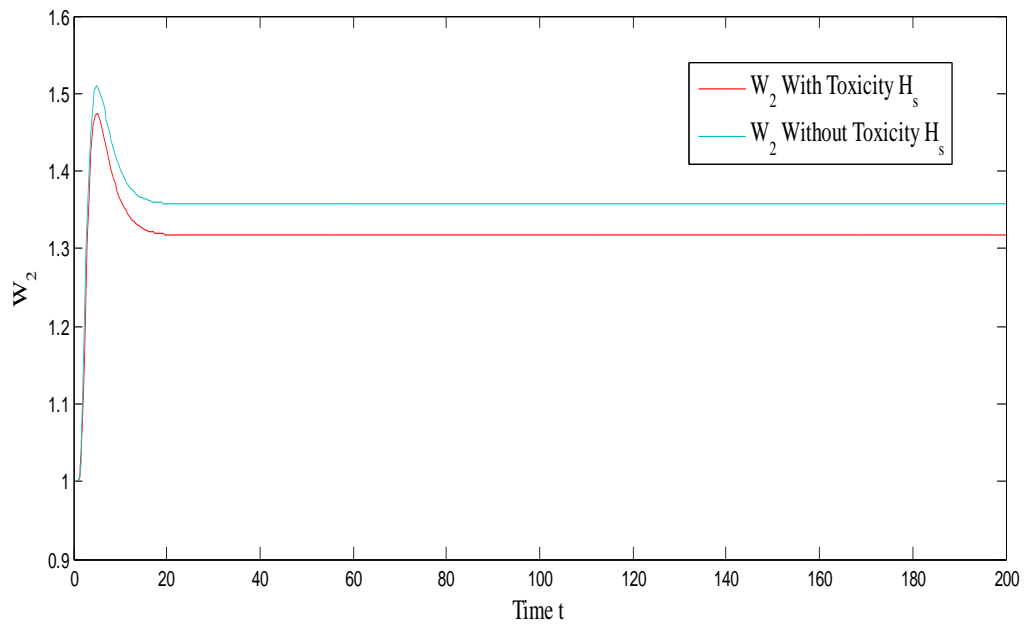
**Figure 3. 1** Graph between nutrient concentration of root  $N_1$  and time  $t$  with toxicity and without toxicity



**Figure 3. 2** Graph between nutrient concentration of shoot  $N_2$  and time  $t$  with toxicity and without toxicity.

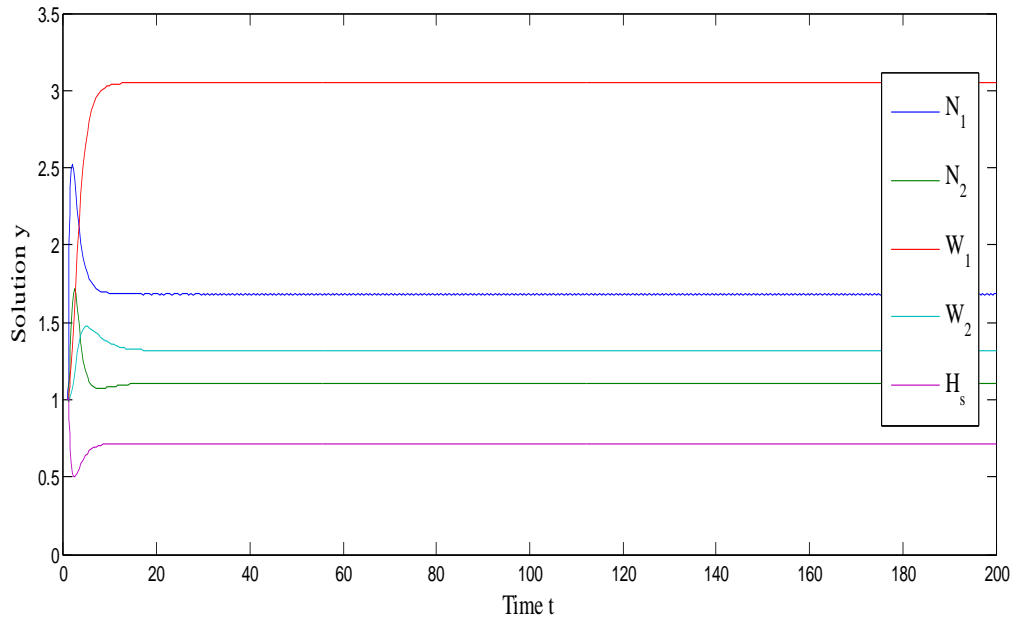


**Figure 3. 3**Graph between structural dry weight of root  $W_1$  and time  $t$  with toxicity and without toxicity.

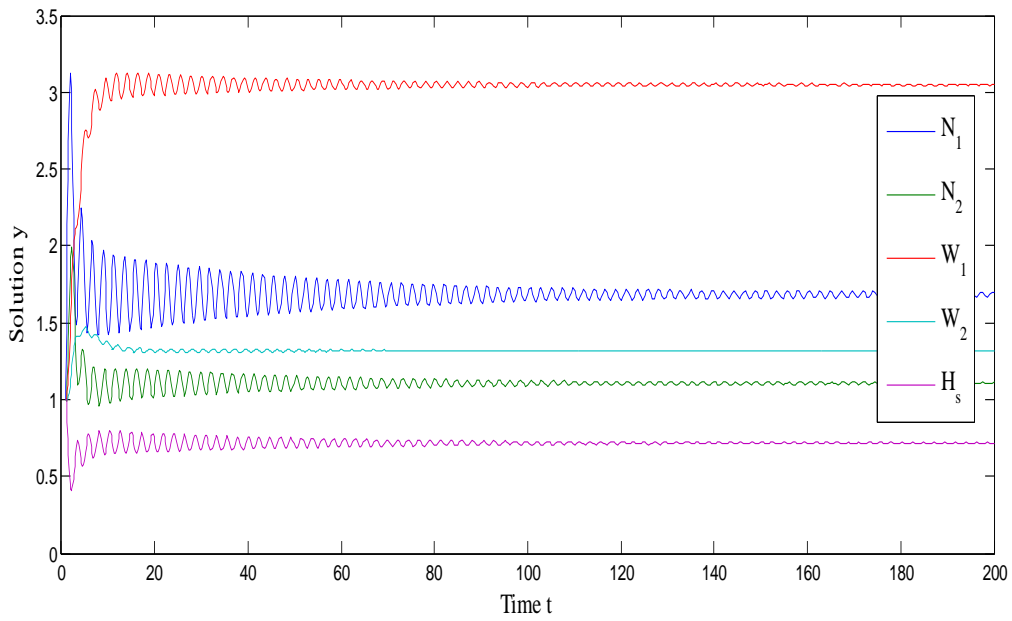


**Figure 3. 4**Graph between structural dry weight of shoot  $W_2$  and time  $t$  with toxicity and without toxicity.

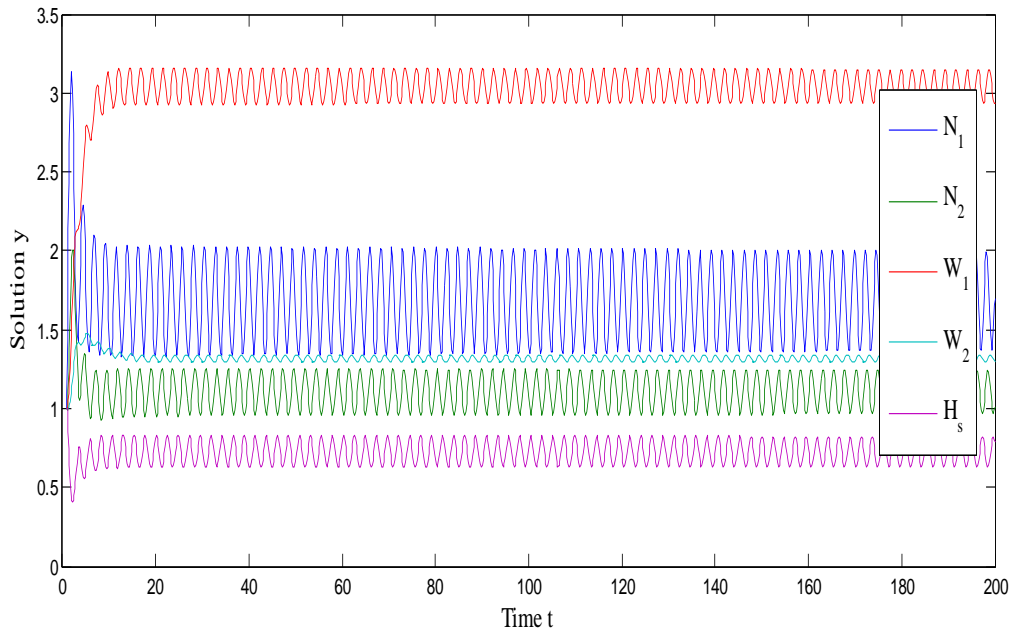




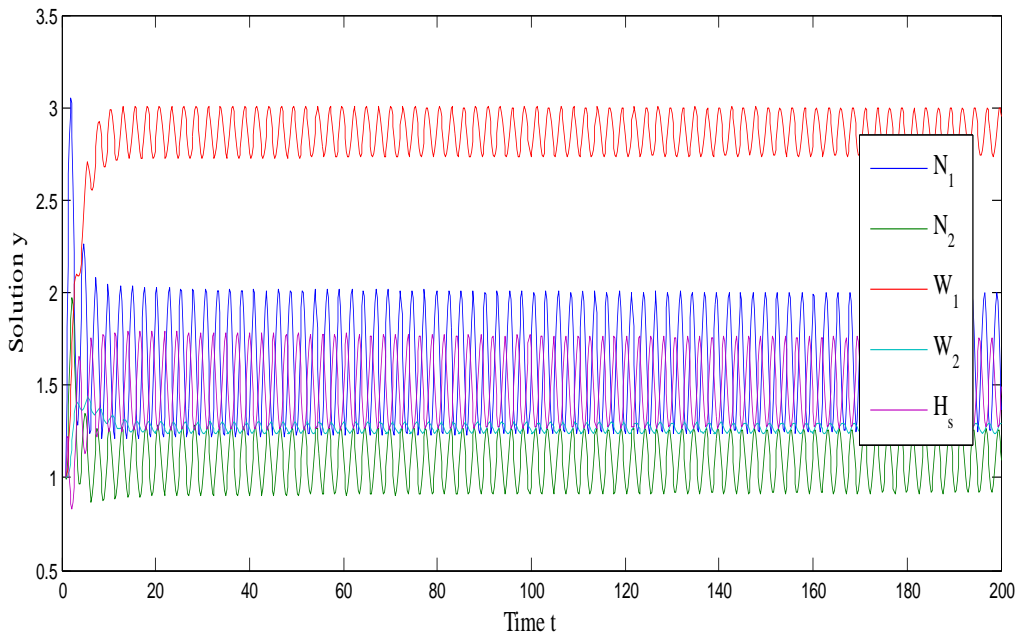
**Figure 3. 5**The interior equilibrium point  $E_1 (1.6830, 1.1057, 3.0490, 1.3172, 0.7161)$  of the system is stable when there is no delay that is  $\tau = 0$ .



**Figure 3. 6**The interior equilibrium point Interior  $E_1 (1.6830, 1.1057, 3.0490, 1.3172, 0.7161)$  is asymptotically stable with delay  $\tau < 0.89$ .



**Figure 3. 7**The interior equilibrium point  $E_1$  (1.6830,1.1057,3.0490,1.3172,0.7161) losses its stability and Hopf- bifurcation occurred with delay  $\tau \geq 0.89$ .

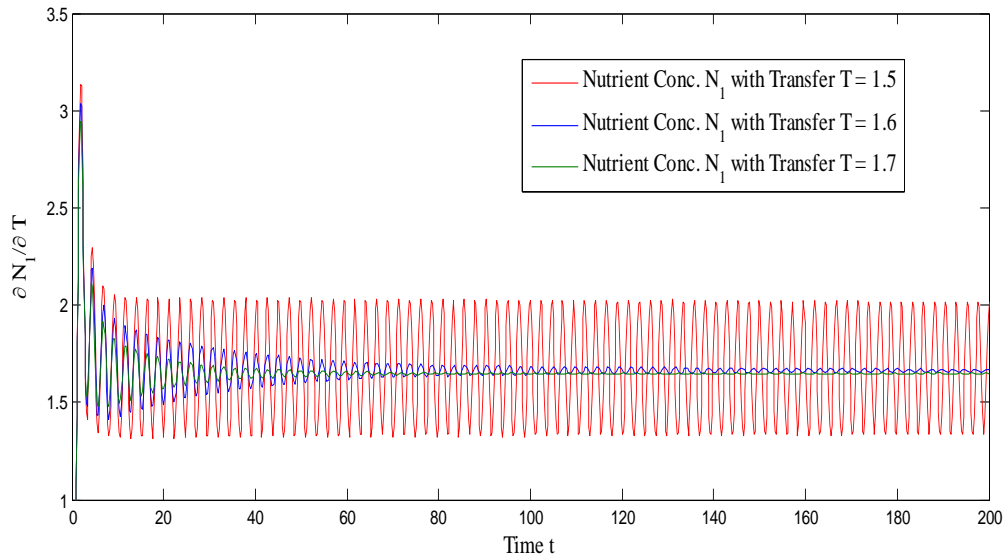


**Figure 3. 8**Increase in intake rate of toxic metal from  $I = 2$  to  $I = 4$ , increases the critical value of delay parameter from  $\tau = 0.89$  to  $\tau = 0.96$ .

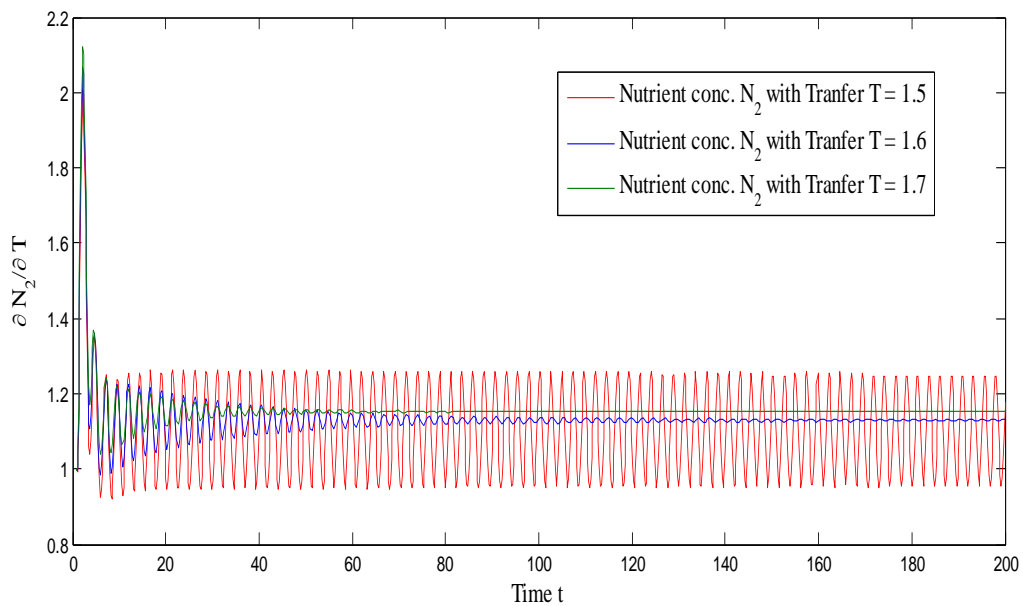
### 3.8 Sensitivity Analysis

Sensitivity analysis helps to know the dependence of system solution on perturbation in model parameters. It tells how the changes in values of other parameters, apart from the key parameter time delay, affects the stability behaviour of state variables. In this

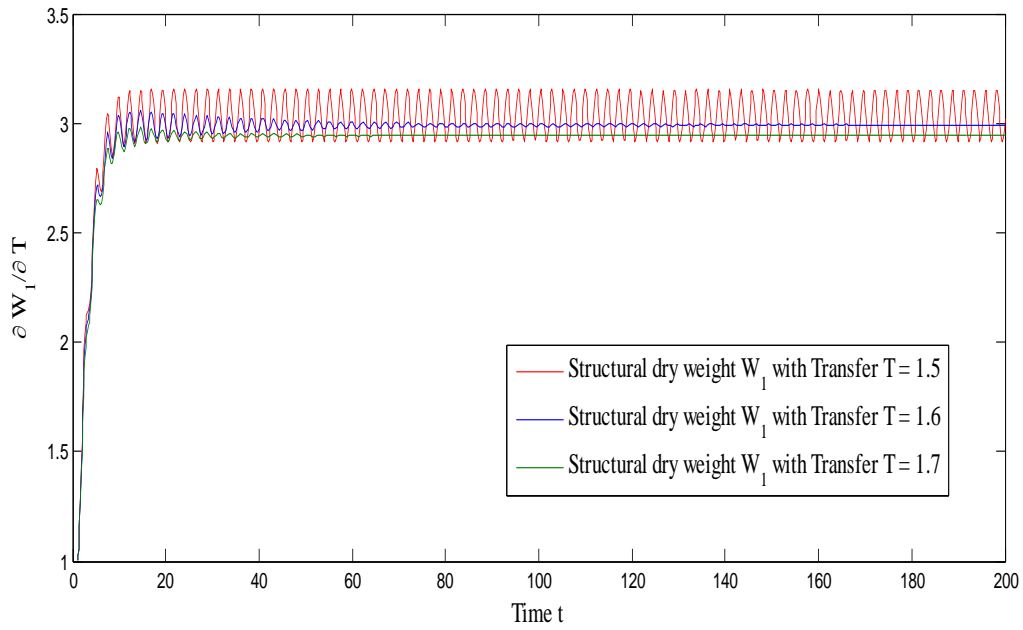
section, two model parameters, namely rate of nutrient transfer from root to shoot  $T$  and consumption coefficient  $\mu$  of delayed nutrient has been perturbed and the corresponding numerical solution of the state variables-concentration of nutrients, structural dry weights and concentration of toxic metal ( $N_1, N_2, W_1, W_2, H_s$ ) has been shown graphically. The main observations have been discussed in detail in Section 3.9-Conclusion.



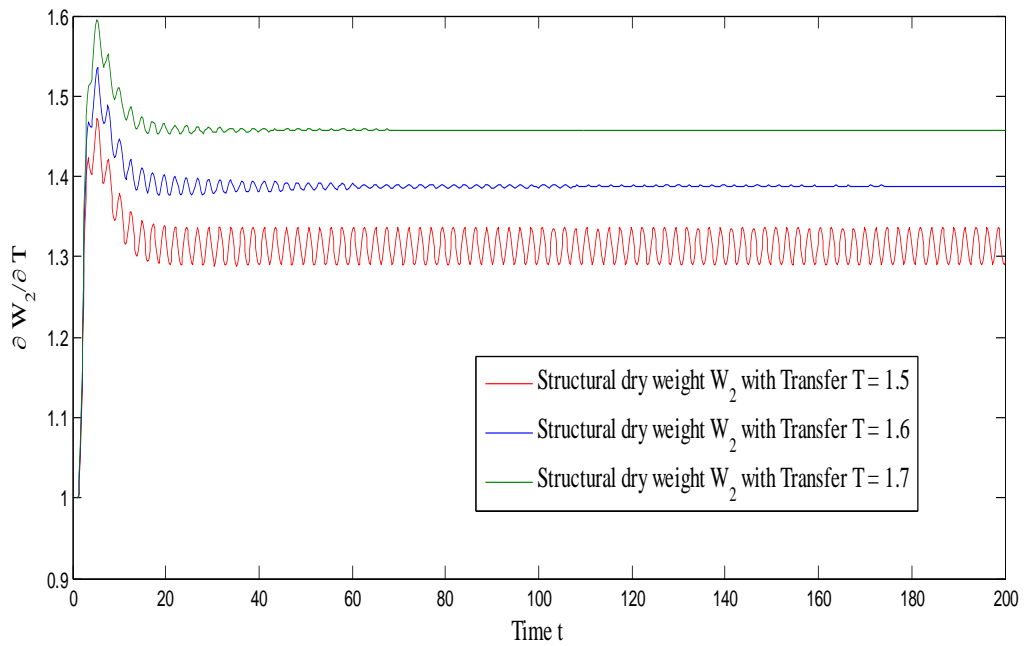
**Figure 3. 9**Time series graph between partial changes in concentration of nutrient  $N_1$  in root compartment and different values of rate of nutrient transfer  $T$ .



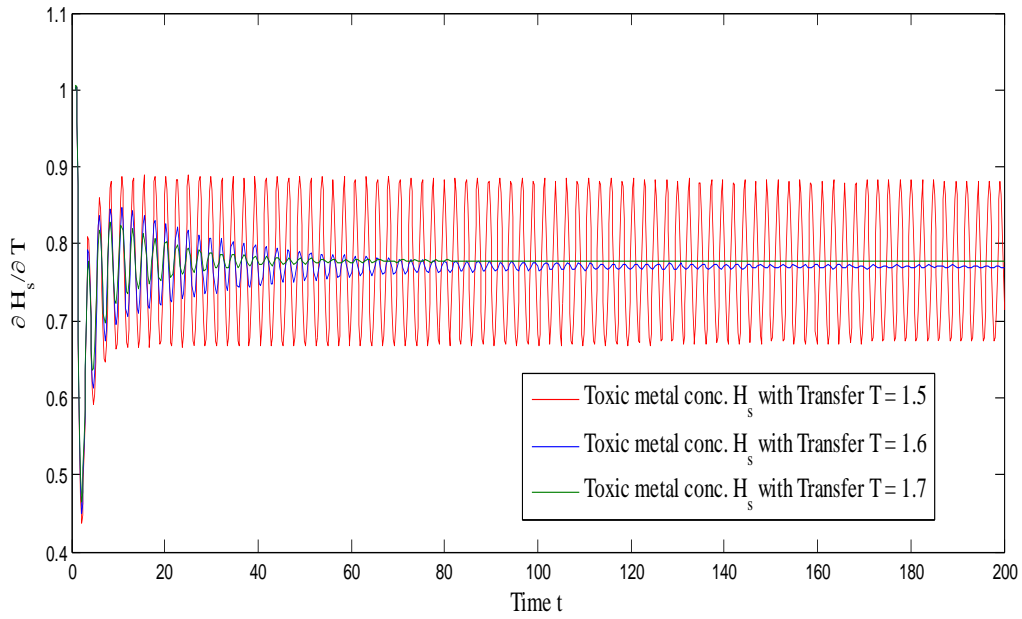
**Figure 3. 10**Time series graph between partial changes in concentration of nutrient  $N_2$  in shoot compartment and different values of rate of nutrient transfer  $T$ .



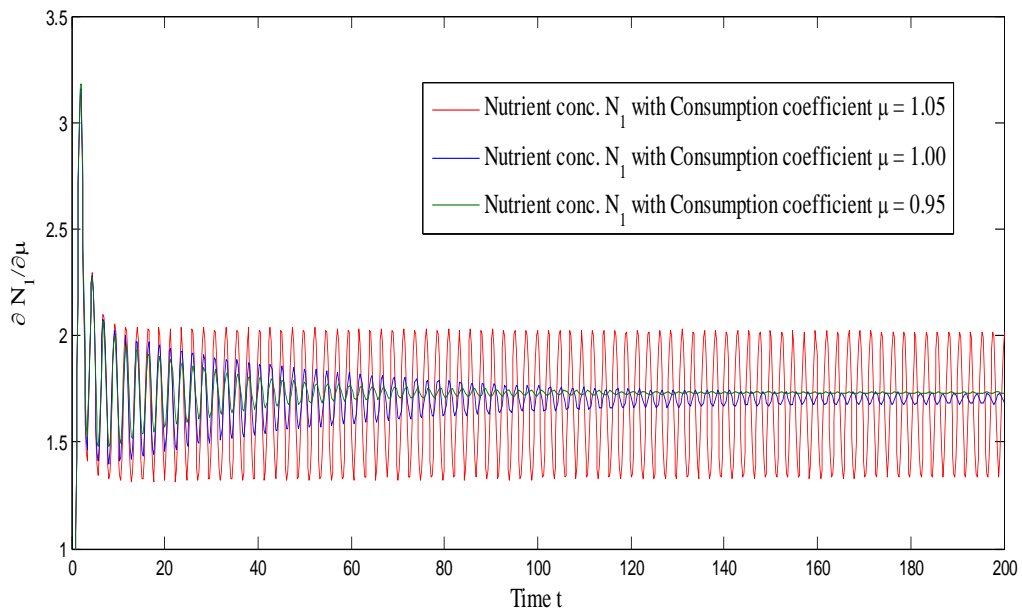
**Figure 3.11** Time series graph between partial changes in structural dry weight  $W_1$  of root compartment and different values of rate of nutrient transfer  $T$ .



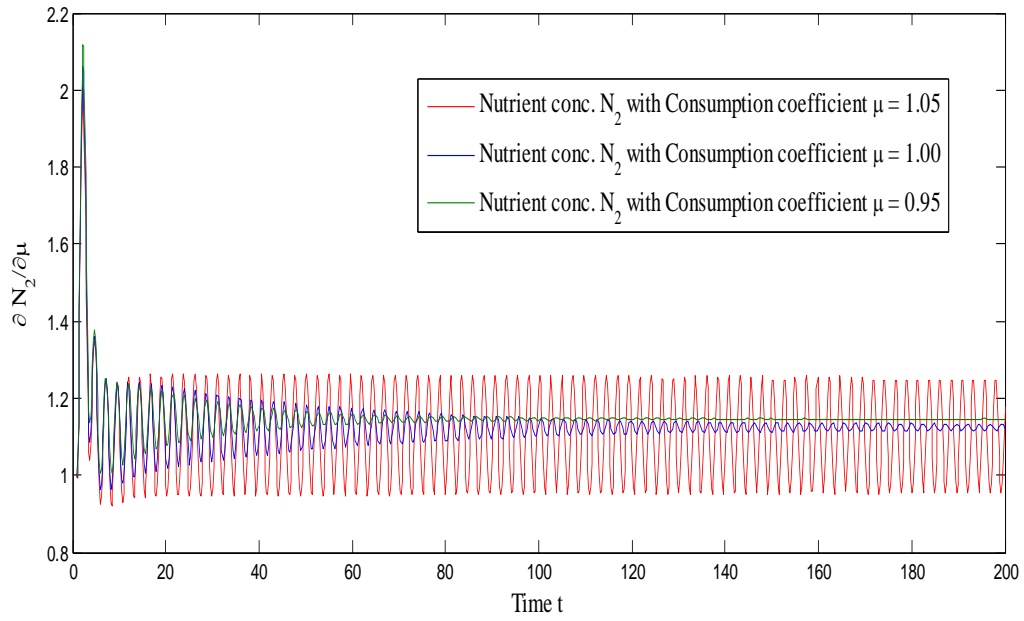
**Figure 3.12** Time series graph between partial changes in structural dry weight  $W_2$  of shoot compartment and different values of rate of nutrient transfer  $T$ .



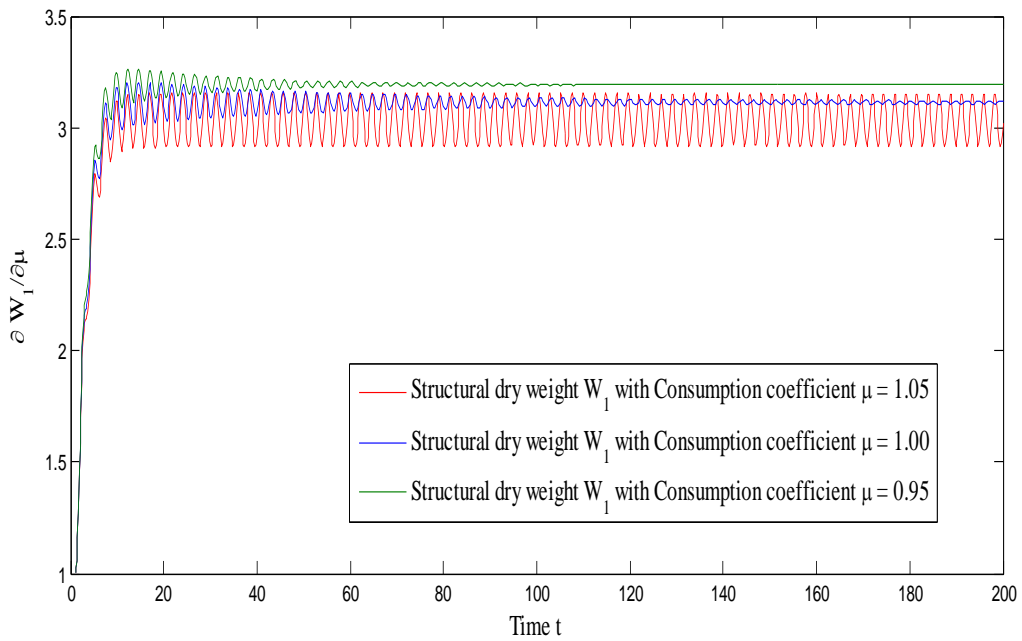
**Figure 3. 13** Time series graph between partial changes in concentration of toxic metal  $H_s$  in soil and different values of rate of nutrient transfer  $T$ .



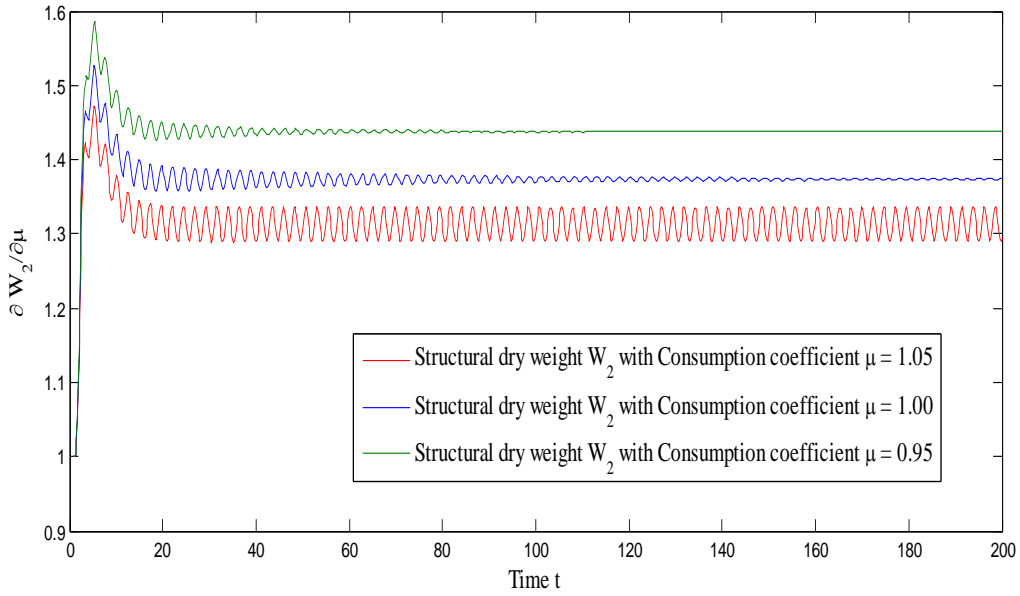
**Figure 3. 14** Time series graph between partial changes in concentration of nutrient  $N_1$  in root compartment and different values of consumption coefficient  $\mu$  of delayed nutrient.



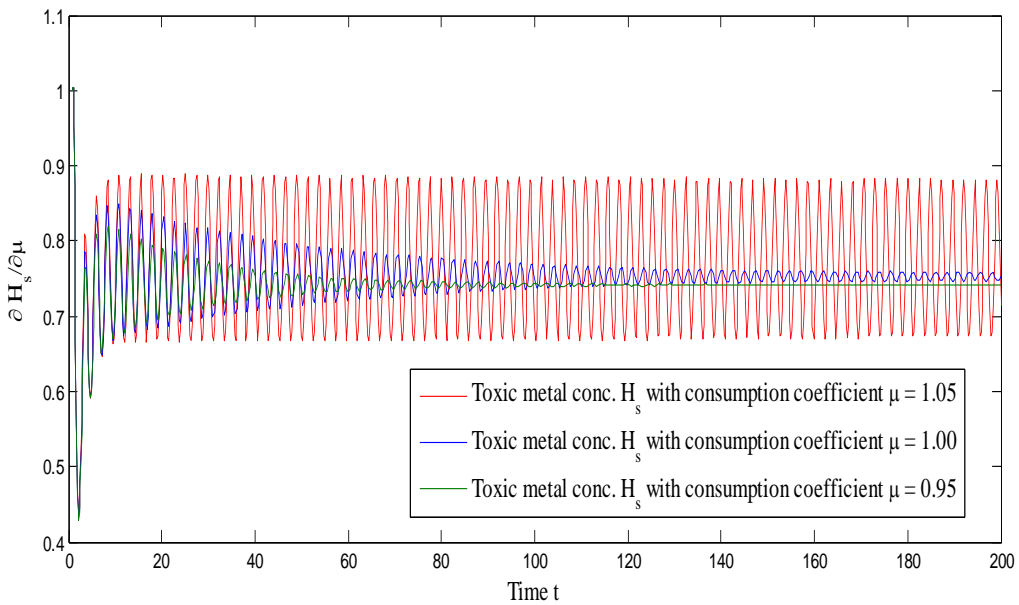
**Figure 3.15** Time series graph between partial changes in concentration of nutrient  $N_2$  in shoot compartment and different values of consumption coefficient  $\mu$ .



**Figure 3.16** Time series graph between partial changes in structural dry weight  $W_1$  of root compartment and different values of consumption coefficient  $\mu$ .



**Figure 3.17** Time series graph between partial changes in structural dry weight  $W_2$  of shoot compartment and different values of consumption coefficient  $\mu$ .



**Figure 3.18** Time series graph between partial changes in concentration of toxic metal  $H_s$  in soil and different values of consumption coefficient  $\mu$ .

### 3.9 Conclusion

In this paper, we concluded that the equilibrium levels of nutrients concentration in root ( $N_1^* = 1.7357$ ) and shoot ( $N_2^* = 1.1186$ ) are more with no toxic effect than equilibrium level of nutrient concentration in root ( $N_1^* = 1.6830$ ) and shoot ( $N_2^* = 1.1057$ ) with toxic effect as shown by Figure 3.1 and Figure 3.2. we also concluded

that the equilibrium levels of structural dry weight in root ( $W^*_1 = 3.2041$ ) and shoot ( $W^*_2 = 1.3574$ ) are more with no toxic effect than equilibrium level of structural dry weight in root ( $W^*_1 = 3.0490$ ) and shoot ( $W^*_2 = 1.3172$ ) with toxic effect as shown by Figure 3.3 and Figure 3.4. It shows that involvement of toxic metal decreases the levels of nutrient concentration and structural dry weights of the plant.

We also studied the stability and Hopf- bifurcation about the interior equilibrium of the system governed by equations (3.1)- (3.5). It has been concluded that when there is no time delay, interior equilibrium  $E_1(1.6830, 1.1057, 3.0490, 1.3172, 0.7161)$  is stable as shown by Figure 3.5 as proved by lemma 3.6.5 using Routh-Hurwitz's criteria. But under the same set of values of parameters, we could find a critical value of the parameter delay below which that is  $\tau < 0.89$ , the system is asymptotically stable as shown by Figure 3.6 and unstable above that critical value of parameter  $\tau \geq 0.89$ , as shown by Figure 3.7 as proved by lemmas 3.6.1-lemma 3.6.4. While passing through the critical value  $\tau = 0.89$  the system showed oscillations, hence Hopf bifurcation occurred. Further with increase in the intake rate of toxic metal, there is more decrease in nutrient concentration and structural dry weight of root and shoot compartment. The same combined adverse effect of increased intake rate of toxic metal and increased time delay is shown in Figure 3.8.

The sensitivity of model solutions is established by taking different values of the parameters appearing in system. It improves the understanding of the role played by specific model parameters.

Sensitivity analysis reveals that with increase in the transfer rate of nutrient  $T$  from root to shoot, the state variables-concentration of nutrients in root and shoot compartments, structural dry weights of root and shoot compartments and concentration of toxic metal in soil ( $N_1, N_2, W_1, W_2, H_s$ ) tends towards stability for the same set of remaining parameters including time delay  $\tau = 0.89$ . At  $T = 1.5$ , all the above-mentioned state variables show unstable behaviour via Hopf bifurcation. But as the value of  $T$  is increased to  $T = 1.6$ , all the state variables start showing the asymptotic stability and finally for  $T = 1.7$ , all the state variables start converging to a stable equilibrium point as shown by Figures 3.9 to 3.13. Apart from converging to stability, the structural dry weight  $W_2$  of shoot compartment also show increase in its value as we increase the value of  $T$  from 1.5 to 1.7 as shown by Figure 3.12. Similarly, as we decrease the value of consumption coefficient  $\mu$  from  $\mu = 1.05$  to  $\mu = 0.95$ , all the above-mentioned state



variables start converging to a stable equilibrium as shown by Figures 3.14 to 3.18. In addition to convergence to stability, the structural dry weights  $W_1$  and  $W_2$  of root and shoot compartment show increase in their values as we decrease the value of consumption coefficient  $\mu$  of delayed nutrient as shown by Figure 3.16 and Figure 3.17.

## Chapter 4

# Modelling on Single Plant Growth Dynamics Considering Delay in the Presence of Toxic Metal

### 4.1 Introduction

Plants need several elements for normal growth like Carbon, hydrogen, and oxygen which come from the air and water and other nutrients which come from soil. Nutrients are components in food that an organism uses to survive and grow. Plant -soil interaction means the mechanism in which the plants take essential nutrients from the soil through their roots which leads to growth of plants. A low concentration of heavy metals is necessary for growth of plants, but excess of these metals adversely affect the soil quality and hence the plant growth gets slowed down considerably. Thornley [162] was the first to apply mathematical modelling to wide range of subjects in plant physiology to predict effect of factors such as temperature, humidity, radiation input and concentration of  $\text{CO}_2$  on process rates of respiration, photosynthesis, transpiration, fluid transport and stomatal responses. Lacoite [21] concluded that models suggested by Thornley are designed for a particular plant species and under particular conditions, so they cannot be applicable to a wider range of conditions. The lack of explicit representation of topology and geometry (Plant morphology) was pointed out by Godin et al. [163]. Deleo et al. [173] gave a simple model that coupled the effect of toxic metal and soil chemistry to study the adverse effect of toxic metal on biomass of trees. Guala et al. [166], [167] further modified the parameters of the model given by Deleo and showed that the model is applicable to not only trees, but all plants in general. Misra and Kalra [83] studied how the toxicity of heavy metals can adversely affect the growth of an individual plant using a two-compartment mathematical model. Rouches theorem [158] plays a very important part for the discussion of distribution of roots of exponential polynomials. Ruan and Wei [112] used Rouches theorem for the discussion of distribution of roots of exponential polynomials. Kubiacyk and Saker [164] studied stability and oscillations in system of non-linear delay differential equations of population dynamics. Ruan and Wei [157] used Rouches theorem for the discussion of distribution of roots of exponential polynomials for study of stability involving delays. Naresh et al. [169] studied the effect of toxicant on plant biomass with time delays. Shukla et al. [170] studied the effect of environmentally degraded soil by rain water and

wind on crop yield. Sikarwar[174] studied the effect of time delay on the dynamics of a multi team prey predator system. Naresh et al. [72] studied the effect of an intermediate toxic product formed by uptake of a toxicant on plant biomass. Huang et al. [129] studied analysis for global stability of system of non-linear delay differential equations involving population growth. Zhang et al. [123] studied the distribution of the roots of a fifth-degree exponential polynomial with applications to a delayed neural network model. The explicit formulae have been derived for determining the properties of the Hopf-bifurcation at the critical value using the normal form theory and manifold reduction by Hassard et al.[175]. Bocharov and Rihan[111] gave adjoint and direct methods for sensitivity analysis in numerical modelling in biosciences using delay differential equations. Rihan[144] did the Sensitivity analysis for dynamic systems with time-lags using adjoint equations and direct methods when the parameters appearing in the model are not only constants but also variables of time. Banks et al.[176] presented theoretical foundations for traditional sensitivity and generalized sensitivity functions for a general class of nonlinear delay differential equations. They Included theoretical results for sensitivity with respect to the delays. Ingalls et al.[150] developed a parametric sensitivity analysis for periodic solutions of delay differential equations.

In the last decade, a lot of work has been done in the field of plant soil interaction under the effect of toxic metals. Till date, delay differential equations have not been prominently used in the field of soil-plant dynamics and agriculture. In the view of above, a mathematical model is proposed for the study of plant growth by introducing delay parameter in the terms containing consumption and utilization coefficient and the complex behaviour giving rise to Hopf bifurcation is studied.

## 4.2 Mathematical Model

The plant growth dynamics is governed by the following system of non-linear delay differential equations involving three state variables: Concentration of nutrients  $N$  in the soil, amount of plant biomass  $W$  and concentration of toxic metal  $M$  in the soil

$$\frac{dN}{dt} = (K_N - K_{NM}M) - \alpha N(t - \tau)W - \delta_1 N \quad (4.1)$$

$$\frac{dW}{dt} = \beta N(t - \tau)W - \delta_2 W \quad (4.2)$$

$$\frac{dM}{dt} = I - \gamma NM - \delta_3 M \quad (4.3)$$

With initial conditions  $N(0) > 0, W(0) > 0, M(0) > 0$  for all  $t$  and  $N(t - \tau) =$  constant for  $t \in [-\tau, 0]$ .

The parameters defined are as:  $K_N$  is the availability of total nutrients and  $(K_N - K_{NM}M)$  is the supply of nutrients hindered due to presence of toxic metal.  $\alpha$  is the consumption coefficient.  $\beta$  is the utilization coefficient for nutrients.  $\gamma$  is the depletion of  $M$  due to interaction between  $M$  and  $N$ .  $I$  is the initial input of toxic metal  $M$  in the soil.  $\delta_1, \delta_2, \delta_3$  are natural decay rates of  $N, W$  and  $M$  respectively. Here all the parameters  $\alpha, \beta, \gamma, K_N, K_{NM}, I, \delta_1, \delta_2, \delta_3$  are taken as positive.

### 4.3 Boundedness of Solution

The boundedness of solutions of the model given by (4.1) -(4.3) is given by the lemma stated below:

**Lemma 4.3.1** The model has all its solution lying in the region  $C = \left[ (N, W, M) \in R_+^3 : 0 \leq N + W \leq \frac{(\delta_3 K_N - I K_{NM})}{\delta_3 \varphi}, 0 \leq M \leq \frac{I}{\delta_3} \right]$ , as  $t \rightarrow \infty$ , for all positive initial values  $\{N(0), W(0), M(0), N(t - \tau) = \text{Constant for all } t \in [-\tau, 0]\} \in C \subset R_+^3$ , where  $\varphi = \min(\alpha, \beta, \delta_1, \delta_2)$ .

**Proof:** Consider the following function:

$$B(t) = N(t) + W(t)$$

$$\frac{dB(t)}{dt} = \frac{d}{dt}[N(t) + W(t)]$$

Using Equations (4.1) -(4.2) and  $\varphi = \min(\alpha, \beta, \delta_1, \delta_2)$  and assuming that  $N_R(t) \approx N_R(t - \tau)$  as  $t \rightarrow \infty$ ,

$$\frac{dW(t)}{dt} \leq \left( K_N - \frac{I K_{NM}}{\delta_3} \right) - \varphi B(t).$$

Applying the comparison theorem, as  $t \rightarrow \infty$

$$B(t) \leq \frac{(\delta_3 K_N - I K_{NM})}{\delta_3 \varphi}$$

So, 
$$0 \leq N(t) + W(t) \leq \frac{(\delta_3 K_N - I K_{NM})}{\delta_3 \varphi}$$

From equation (4.3): 
$$\frac{dM}{dt} = I - \gamma MN - \delta_3 M$$

$$\Rightarrow \frac{dM}{dt} \leq I - \delta_3 M$$

Then by usual comparison theorem, when  $t \rightarrow \infty$ :

$$M \leq \frac{I}{\delta_3}$$

So, 
$$0 \leq M \leq \frac{I}{\delta_3}$$

#### 4.4 Positivity of Solutions

Positivity means that the system sustains. For positive solutions, one needs to show that all solution of system given by Equations. (4.1)– (4.3), where initial condition is  $N(0) > 0, W(0) > 0, M(0) > 0$  for all  $t$  and  $N(t - \tau) = \text{constant}$  for  $t \in [-\tau, 0]$ , the solution  $(N(t), W(t), M(t))$  of the model stays positive for all time  $t > 0$ .

From equation (4.3): 
$$\frac{dM}{dt} \geq -(\gamma N + \delta_3)M$$

$$\Rightarrow \frac{dM}{dt} \geq -\left(\frac{\gamma(\delta_3 K_N - I K_{NM}) + \delta_3^2 \varphi}{\delta_3 \varphi}\right)$$

$$\Rightarrow M \geq c_1 e^{-\left(\frac{\gamma(\delta_3 K_N - I K_{NM}) + \delta_3^2 \varphi}{\delta_3 \varphi}\right)t}$$

This concludes that  $M > 0$  for all  $t$ .

Similar argument holds for  $N$  and  $W$ .

#### 4.5 Interior Equilibrium

We calculate an interior equilibrium  $E^*$  of model. The system of equations (4.1) -(4.3) has one feasible equilibrium  $E^*(N^*, W^*, M^*)$  where:

$$N^* = \frac{\delta_2}{\beta},$$

$$W^* = \left(\frac{K_N(\beta \delta_3 + \gamma \delta_2) - K_{NM} \beta I}{(\beta \delta_3 + \gamma \delta_2)}\right) - \frac{\delta_1}{\alpha},$$

$$M^* = \frac{\beta I}{(\beta \delta_3 + \gamma \delta_2)}.$$

#### 4.6 Stability Analysis and Local Hopf-Bifurcation

Here, the dynamical behaviour of the interior equilibrium points  $E^*(N^*, W^*, M^*)$  of the model given by (4.1) -(4.3) is analysed. The exponential characteristic equation about equilibrium  $E^*$  is given by:

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + (n_1\lambda^2 + n_2\lambda + n_3)e^{-\lambda\tau} = 0 \quad (4.4)$$

Where  $m_1 = (\delta_1 + \delta_2 + \delta_3 + \gamma N^*)$ ,  $m_2 = (\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1 + \delta_1\gamma N^* + KNM\gamma M^*$ ,  $m_3 = \delta_1\delta_2\delta_3 + \delta_2KNM\gamma M^*$ ,  $n_1 = \alpha W^*$ ,  $n_2 = \alpha W^*\delta_1 + \delta_3 + \gamma N^*$ ,  $n_3 = \delta_2\delta_3\alpha W^*$ .

Clearly  $m_1, m_2, m_3, n_1, n_2, n_3$  are all positive.

Clearly  $\lambda = i\omega$  is a solution of equation (4.4) if and only if

$$(i\omega)^3 + m_1(i\omega)^2 + m_2(i\omega) + m_3 + (n_1(i\omega)^2 + n_2(i\omega) + n_3)e^{-i\omega\tau} = 0 \quad (4.5)$$

Separating real and imaginary parts:

$$m_3 - m_1\omega^2 + (n_3 - n_1\omega^2)\cos \omega\tau + n_2\omega \sin \omega\tau = 0 \quad (4.6)$$

$$m_2\omega - \omega^3 + n_2\omega \cos \omega\tau - (n_3 - n_1\omega^2) \sin \omega\tau = 0 \quad (4.7)$$

Which gives:

$$\omega^6 + (m_1^2 - n_1^2 - 2m_2)\omega^4 + (m_2^2 - n_2^2 + 2n_1n_3 - 2m_1m_3)\omega^2 + (m_3^2 - n_3^2) = 0 \quad (4.8)$$

Let  $a = (m_1^2 - n_1^2 - 2m_2)$ ,  $b = (m_2^2 - n_2^2 + 2n_1n_3 - 2m_1m_3)$ ,  $c = (m_3^2 - n_3^2)$

Let  $\omega^2 = y$ , then equation (8) becomes:

$$y^3 + ay^2 + by + c = 0 \quad (4.9)$$

**Lemma 4.6.1** If  $c < 0$ , Equation (4.9) contains at least one positive real root.

**Proof.** Let  $h(y) = y^3 + ay^2 + by + c$

Here  $h(0) = c < 0$ ,  $\lim_{y \rightarrow \infty} h(y) = \infty$

So,  $\exists y_0 \in (0, \infty)$  such that  $h(y_0) = 0$

Proof completed.

**Lemma 4.6.2** If  $c \geq 0$ , then necessary condition for equation (4.9) to have positive real roots is  $D = a^2 - 3b \geq 0$ .

**Proof.** Since  $h(y) = y^3 + ay^2 + by + c$ , therefore  $h'(y) = 3y^2 + 2ay + b$

$$h'(y) = 0 \Rightarrow 3y^2 + 2ay + b = 0 \quad (4.10)$$

The roots of equation (4.10) can be expressed as:

$$y_{1,2} = \frac{-2a \mp \sqrt{4a^2 - 12b}}{6} = \frac{-a \mp \sqrt{D}}{3} \quad (4.11)$$

If  $D < 0$ , then equation (4.10) does not have any real roots. So, the function  $h(y)$  is monotone increasing function in  $y$ . It follows from  $h(0) = c \geq 0$  that equation (4.9) has no positive real roots.

Proof completed.

Clearly if  $D \geq 0$ , then  $y_1 = \frac{-a+\sqrt{D}}{3}$  is local minima of  $h(y)$ . Thus we have the following Lemma.

**Lemma 4.6.3** If  $c \geq 0$ , then equation (4.9) has positive roots if and only if  $y_1 > 0$  and  $h(y_1) \leq 0$ .

**Proof.** The sufficiency is obvious. Only necessity needs to be proved. Otherwise, assume that either  $y_1 \leq 0$  or  $y_1 > 0$  and  $h(y_1) > 0$ . If  $y_1 \leq 0$ , since  $h(y)$  is increasing for  $y \geq y_1$  and  $h(0) = c \geq 0$ , it follows that  $h(y)$  has no positive real zeros. If  $y_1 > 0$  and  $h(y_1) > 0$ , since  $y_2 = \frac{-a-\sqrt{D}}{3}$  is the local maxima value, it follows that  $h(y_1) \leq h(y_2)$ . Hence,  $h(0) = c \geq 0$ ,  $h(y)$  does not have positive real roots.

This completes proof.

**Lemma 4.6.4** Suppose  $y_1$  is defined by equation (4.11).

(I) If  $c < 0$ , Equation (4.9) contains at least one positive real root.

(II) If  $c \geq 0$  and  $D = a^2 - 3b < 0$ , then equation (4.9) has no positive roots.

(III) If  $c \geq 0$ , then equation (4.9) has positive roots if and only if  $y_1 > 0$  and  $h(y_1) \leq 0$ .

**Proof.** Suppose that equation (4.9) has positive roots. Without loss of generality, assume that it has three positive roots, denoted by  $y_1, y_2, y_3$ . Then equation (4.8) has three positive roots, say  $\omega_1 = \sqrt{y_1}, \omega_2 = \sqrt{y_2}, \omega_3 = \sqrt{y_3}$ .

From equation (7):  $\sin \omega \tau = \frac{m_2 \omega - \omega^3}{d}$

Which gives  $\tau = \frac{1}{\omega} \left[ \sin^{-1} \left( \frac{m_2 \omega - \omega^3}{d} \right) + 2(j-1)\pi \right]; j = 1, 2, 3, \dots$

Let  $\tau_k^{(j)} = \frac{1}{\omega_k} \left[ \sin^{-1} \left( \frac{m_2 \omega_k - \omega_k^3}{d} \right) + 2(j-1)\pi \right]; k = 1, 2, 3; j = 0, 1, 2, \dots$

Then  $\bar{\tau}i\omega_k$  is a pair of purely imaginary roots of equation (4.8)

Where  $\tau = \tau_k^{(j)}$ ,  $k = 1, 2, 3$ ;  $j = 0, 1, 2, \dots$ ,  $\lim_{j \rightarrow \infty} \tau_k^{(j)} = \infty$ ,  $k = 1, 2, 3$ .

Thus, we define:

$$\begin{aligned}\tau_0 &= \tau_{k_0}^{(j_0)} = \min_{1 \leq k \leq 3, j \geq 1} [\tau_k^{(j)}], \\ \omega_0 &= \omega_{k_0}, \quad y_0 = y_{k_0}\end{aligned}\tag{4.12}$$

**Lemma 4.6.5** Suppose that  $m_1 > 0$ ,  $(m_3 + d) > 0$ ,  $m_1 m_2 - (m_3 + d) > 0$ .

(I) If  $c \geq 0$  and  $D = a^2 - 3b < 0$ , then all the roots of equation (4.4) have negative real parts for all  $\tau \geq 0$ .

(II) If  $c < 0$  or  $c \geq 0$ ,  $y_1 > 0$  and  $h(y_1) \leq 0$ , then all the roots of equation (4.4) have negative real parts for all  $\tau \in [0, \tau_0)$

**Proof.** When  $\tau = 0$ , equation (4.4) becomes:

$$\lambda^3 + (m_1 + n_1)\lambda^2 + (m_2 + n_2)\lambda + (m_3 + n_3) = 0\tag{4.13}$$

By Routh-Hurwitz's criteria, **(H1)**: All roots of equation (4) have negative real parts if and only if  $(m_3 + n_3) > 0$ ,  $(m_1 + n_1)(m_2 + n_2) - (m_3 + n_3) > 0$ .

If  $c \geq 0$  and  $D = a^2 - 3b < 0$ , Lemma 4.6.4 (II) shows that equation (4.4) has no roots with zero real part for all  $\tau \geq 0$ . When  $c < 0$  or  $c \geq 0$ ,  $y_1 > 0$  and  $h(y_1) \leq 0$ , Lemma 4.6.4 (I) and (III) implies that when  $\tau \neq \tau_k^{(j)}$ ,  $k = 1, 2, 3$ ;  $j \geq 1$ , equation (4.4) has no roots with zero real part and  $\tau_0$  is the minimum value of  $\tau$  so that the equation (4.4) has purely imaginary roots. Applying Theorem 1.5.8, the conclusion of the lemma is obtained.

$$\text{Let } \lambda(\tau) = \psi(\tau) + i\omega(\tau)\tag{4.14}$$

be the roots of equation (4) satisfying:  $\psi(\tau_0) = 0$ ,  $\omega(\tau_0) = \omega_0$

In order to guarantee that  $\bar{\tau}\omega_0$  are simple purely imaginary roots of equation (4.4), with  $\tau = \tau_0$  and  $\lambda(\tau)$  satisfies transversality condition, assume that  $h'(y_0) \neq 0$ .

**Lemma 4.** Suppose  $y_0 = \omega_0^2$ . If  $\tau = \tau_0$ , Then  $\text{Sign} [\psi'(\tau_0)] = \text{Sign} [h'(y_0)]$

**Proof.** Putting  $\lambda(\tau)$  in equation (4.4) and differentiating w.r.t  $\tau$ , it follows that



$$\begin{aligned} \frac{d\lambda}{d\tau} & [3\lambda^2 + 2m_1\lambda + m_2 + ((n_1\lambda^2 + n_2\lambda + n_3)(-\tau) + (2n_1\lambda + n_2))e^{-\lambda\tau}] \\ & = \lambda(n_1\lambda^2 + n_2\lambda + n_3)e^{-\lambda\tau} \\ \Rightarrow \left(\frac{d\lambda}{d\tau}\right)^{-1} & = \frac{(3\lambda^2 + 2m_1\lambda + m_2)e^{\lambda\tau}}{\lambda(n_1\lambda^2 + n_2\lambda + n_3)} + \frac{(2n_1\lambda + n_2)}{\lambda(n_1\lambda^2 + n_2\lambda + n_3)} - \frac{\tau}{\lambda} \end{aligned}$$

From equations (4.6) -(4.8):

$$\begin{aligned} \mu'(\tau_0) & = Re \left[ \frac{(3\lambda^2 + 2m_1\lambda + m_2)e^{\lambda\tau}}{\lambda(n_1\lambda^2 + n_2\lambda + n_3)} \right] + Re \left[ \frac{(2n_1\lambda + n_2)}{\lambda(n_1\lambda^2 + n_2\lambda + n_3)} \right] \\ & = \frac{1}{\Delta} [3\omega_0^6 + 2a\omega_0^4 + b\omega_0^2] \end{aligned}$$

Where  $\Delta = [(n_3 - n_1\omega^2)^2 + (n_2\omega)^2], \Delta > 0$  and  $\omega_0 > 0$ .

It is concluded that:

$$\text{Sign} [\psi'(\tau_0)] = \text{Sign} [h'(y_0)]$$

This proves lemma.

## 4.7 Direction and Stability of Hopf-Bifurcating Solution

In the previous section, a family of periodic solutions is obtained that bifurcate from the positive steady state  $E^*$  at the critical values of  $\tau$ . It is also worthwhile to determine the direction, stability and period of these bifurcating periodic solutions. In this section, an explicit formula is derived to determining the properties of the Hopf-bifurcation at the critical value  $\tau_j$ , using the normal form theory and manifold reduction due to Hassard et al (1981) [175].

Let  $u_1 = N - N^*, u_2 = W - W^*, u_3 = M - M^*$  and normalizing the delay  $\tau$  by time scaling  $t \rightarrow \frac{t}{\tau}$ , equations (4.1) -(4.3) are transformed into:

$$\frac{du_1}{dt} = -\delta_1 u_1 - K_{NM} u_3 - \alpha W^* u_1(t-1) - \alpha u_1(t-1) u_2 \quad (4.15)$$

$$\frac{du_2}{dt} = -\delta_2 u_2 + \beta W^* u_1(t-1) + \beta u_1(t-1) u_2 \quad (4.16)$$

$$\frac{du_3}{dt} = -\gamma M^* u_1 - (\gamma N^* + \delta_3) u_3 - \gamma u_1 u_3 \quad (4.17)$$

Thus, work can be done in the phase  $C = C((-1, 0), R_+^3)$ . Without loss of generality, denote the critical value  $\tau_j$  by  $\tau_0$ .

Let  $\tau = \tau_0 + \mu$ , then  $\mu = 0$  is a Hopf-bifurcation value of the system given by equations (4.15) -(4.17). For the simplicity of notations, rewrite this system as

$$u'(t) = L_\mu(u_t) + F(\mu, u_t) \quad (4.18)$$

Where  $u(t) = (u_1(t), u_2(t), u_3(t))^T \in R^3$ ,  $u_t(\theta) \in C$  is defined by  $u_t(\theta) = u_t(t + \theta)$ , and

$L_\mu: C \rightarrow R$ ,  $F: R \times C \rightarrow R$  are given, respectively by

$$L_\mu \phi = (\tau_0 + \mu) \begin{bmatrix} -\delta_1 & 0 & -K_{NM} \\ 0 & -\delta_2 & 0 \\ -\gamma M^* & 0 & -(\gamma N^* + \delta_3) \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{bmatrix} +$$

$$(\tau_0 + \mu) \begin{bmatrix} -\alpha W^* & 0 & 0 \\ \beta W^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{bmatrix}$$

And  $F(\mu, \phi) = (\tau_0 + \mu) \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$  respectively where  $F_1 = -\alpha \phi_1(-1) \phi_2(0)$ ,

$$F_2 = \beta \phi_1(-1) \phi_2(0), F_3 = -\gamma \phi_1(0) \phi_3(0),$$

$$\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C((-1, 0), R).$$

By the Riesz representation theorem, there exist a function  $\eta(\theta, \mu)$  of bounded variation for  $\theta \in [-1, 0]$ , such that  $L_\mu \phi = \int_{-1}^0 d\eta(\theta, 0) \phi(\theta)$  for  $\phi \in C$ .

In fact, choose

$$\eta(\theta, \mu) = (\tau_0 + \mu) \begin{bmatrix} -\delta_1 & 0 & -K_{NM} \\ 0 & -\delta_2 & 0 \\ -\gamma M^* & 0 & -(\gamma N^* + \delta_3) \end{bmatrix} \delta(\theta) +$$

$$(\tau_0 + \mu) \begin{bmatrix} -\alpha W^* & 0 & 0 \\ \beta W^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta(\theta + 1)$$

Here  $\delta$  is the Dirac delta function.

For  $\phi \in C([-1, 0], R_+^3)$ , define:

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0) \\ \int_{-1}^0 d\eta(\theta, 0)\phi(\theta), & \theta = 0. \end{cases}$$

And 
$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1,0) \\ F(\mu, \phi) & \theta = 0. \end{cases}$$

Then the system (4.18) is equivalent to:

$$u'(t) = A(\mu)\phi + R(\mu)u_t \quad (4.19)$$

For  $\psi \in C^1([-1,0], R_+^3)$ , define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in [-1,0) \\ \int_{-1}^0 d\eta^T(-t, 0)\psi(-t), & s = 0. \end{cases} \text{ and bilinear inner product}$$

$$\begin{aligned} \langle \psi(s), \phi(\theta) \rangle &= \overline{\psi(0)}\phi(0) \\ &\quad - \int_{-1}^0 \int_{\xi=\theta}^{\theta} \overline{\psi(\xi - \theta)}d\eta(\theta)\phi(\xi) d\xi \end{aligned} \quad (4.20)$$

Since  $A^*$  and  $A = A(0)$  are adjoint operators and  $i\omega_0$  are eigen values of  $A(0)$ . Thus they are eigen values of  $A^*$ . Suppose that  $q(\theta) = q(0)e^{i\omega_0\theta}$  is an eigen vector of  $A(0)$  corresponding to the eigen value  $i\omega_0$ . Then  $A(0) = i\omega_0 q(\theta)$ . When  $\theta = 0$ ,

$$\left[ i\omega_0 I - \int_{-1}^0 d\eta(\theta)e^{i\omega_0\theta} \right] q(0) = 0,$$

which yields  $q(0) = (1, \sigma_1, \rho_1)^T$  where:

$$\sigma_1 = \frac{(\alpha W^* + \delta_1 + i\omega_0)}{K_{NM}} \text{ and } \rho_1 = \frac{\beta W^*(\delta_2 - i\omega_0)}{\delta_2^2 + \omega_0^2}$$

Similarly, it can be verified that  $q^*(s) = D(1, \sigma_2, \rho_2)e^{i\omega_0\tau_0 s}$  is the eigen value of  $A^*$  corresponding to  $-i\omega_0$ , where

$$\sigma_1 = \frac{(\alpha W^* + \delta_1 - i\omega_0)}{K_{NM}} \text{ and } \rho_1 = \frac{\beta W^*(\delta_2 + i\omega_0)}{\delta_2^2 + \omega_0^2}$$

In order to assure  $\langle q^*(s), q(\theta) \rangle = 1$ , the value of D needs to be determined.

From equation (4.20),  $\langle q^*(s), q(\theta) \rangle$

$$\begin{aligned} &= \overline{D}(1, \overline{\sigma_2}, \overline{\rho_2})(1, \sigma_1, \rho_1)^T \\ &\quad - \int_{-1}^0 \int_{\xi=\theta}^{\theta} \overline{D}(1, \overline{\sigma_2}, \overline{\rho_2})e^{-i\omega_0\tau_0(\xi-\theta)}d\eta(\theta)(1, \sigma_1, \rho_1)^T e^{i\omega_0\tau_0} d\xi \\ &= \overline{D} \left\{ 1 + \sigma_1 \overline{\sigma_2} + \rho_1 \overline{\rho_2} - \int_{-1}^0 (1, \overline{\sigma_2}, \overline{\rho_2}) \theta e^{i\omega_0\tau_0\theta} (1, \sigma_1, \rho_1)^T \right\} \end{aligned}$$

$$= \overline{D}\{1 + \sigma_1 \overline{\sigma_2} + \rho_1 \overline{\rho_2} + \tau_0 \overline{\sigma_2} W^*(\beta \rho_1 - \alpha \sigma_1) e^{i\omega_0 \tau_0}\}$$

Hence, choose  $\overline{D} = \frac{1}{(1 + \sigma_1 \overline{\sigma_2} + \rho_1 \overline{\rho_2} + \tau_0 \overline{\sigma_2} W^*(\beta \rho_1 - \alpha \sigma_1) e^{i\omega_0 \tau_0})}$

such that  $\langle q^*(s), q(\theta) \rangle = 1, \langle q^*(s), \overline{q(\theta)} \rangle = 0$ .

Following the algorithm given by Hassard et al (1981) [175] and using the same notations as there to compute the coordinates describing the centre manifold  $C_0$  at  $\mu = 0$ .

Let  $u_t$  be the solution of equation (4.18) with  $\mu = 0$ . Define:

$$z(t) = \langle q^*(s), u_t(\theta) \rangle, \quad (4.21)$$

$$W(t, \theta) = u_t(\theta) - 2\text{Re}(z(t)q(\theta))$$

On the centre manifold  $C_0$ ,  $W(t, \theta) = W(z(t), \overline{z(t)}, \theta)$

Where 
$$W(z, \overline{z}, \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z\overline{z} + W_{02}(\theta) \frac{\overline{z}^2}{2} + \dots,$$

$z$  and  $\overline{z}$  are local coordinates for centre manifold  $C_0$  in the direction of  $q^*$  and  $\overline{q^*}$ . Note that  $W$  is real if  $u_t$  is real. Consider only real solution. For solution  $u_t \in C_0$  of equation (4.20), since  $\mu = 0$ ,

$$\begin{aligned} z'(t) &= i\omega_0 \tau_0 z + \langle \overline{q^*}(\theta), F(0, W(z, \overline{z}, \theta) + 2\text{Re}(z(t)q(\theta))) \rangle \\ &= i\omega_0 \tau_0 z + \overline{q^*}(0) F(0, W(z, \overline{z}, 0) + 2\text{Re}(z(t)q(\theta))) \\ &\equiv i\omega_0 \tau_0 z + \overline{q^*}(0) F_0(z, \overline{z}) \end{aligned}$$

Rewrite this equation as:

$$z'(t) = i\omega_0 \tau_0 z(t) + g(z, \overline{z}) \quad (4.22)$$

Where:

$$\begin{aligned} g(z, \overline{z}) &= \overline{q^*}(0) F_0(z, \overline{z}) \\ &= g_{20}(\theta) \frac{z^2}{2} + g_{11}(\theta) z\overline{z} + g_{02}(\theta) \frac{\overline{z}^2}{2} \\ &\quad + g_{21}(\theta) \frac{z^2 \overline{z}}{2} + \dots \end{aligned} \quad (4.23)$$

As  $u_t(\theta) = (u_{1t}, u_{2t}, u_{3t}) = W(t, \theta) + z q(\theta) + \overline{z} \overline{q(\theta)}$

and  $q(0) = (1, \sigma_1, \rho_1)^T e^{i\omega_0 \tau_0 \theta}$ , so

$$u_{1t}(0) = z + \overline{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\overline{z} + W_{02}^{(1)}(0) \frac{\overline{z}^2}{2} + \dots,$$

$$u_{2t}(0) = \sigma_1 z + \overline{\sigma_1} \bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z \bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots,$$

$$u_{3t}(0) = \rho_1 z + \overline{\rho_1} \bar{z} + W_{20}^{(3)}(0) \frac{z^2}{2} + W_{11}^{(3)}(0) z \bar{z} + W_{02}^{(3)}(0) \frac{\bar{z}^2}{2} + \dots,$$

$$u_{1t}(-1) = z e^{-i\omega_0 \tau_0} + \bar{z} e^{i\omega_0 \tau_0} + W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z \bar{z} + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + \dots,$$

$$u_{2t}(-1) =$$

$$\sigma_1 e^{-i\omega_0 \tau_0} z + \overline{\sigma_1} e^{i\omega_0 \tau_0} \bar{z} + W_{20}^{(2)}(-1) \frac{z^2}{2} + W_{11}^{(2)}(-1) z \bar{z} + W_{02}^{(2)}(-1) \frac{\bar{z}^2}{2} + \dots,$$

Thus, comparison of coefficients with equation (4.23) gives:

$$g_{20} = \overline{D}(1, \sigma_1, \rho_1) f_{z^2}, g_{02} = \overline{D}(1, \overline{\sigma_1}, \overline{\rho_2}) f_{\bar{z}^2},$$

$$g_{11} = \overline{D}(1, \overline{\sigma_1}, \overline{\rho_2}) f_{z\bar{z}}, g_{21} = \overline{D}(1, \overline{\sigma_1}, \overline{\rho_2}) f_{z^2\bar{z}}$$

In order to determine  $g_{21}$ , focus needs to be on computation of  $W_{20}(\theta)$  and  $W_{11}(\theta)$ .

From equations (4.19) and (4.21):

$$W' = u_t' - z' q - \bar{z}' q = \begin{cases} AW - 2\text{Re}[\overline{q^*}(0)F_0 q(\theta)], & \theta \in [-1, 0) \\ AW - 2\text{Re}[\overline{q^*}(0)F_0 q(0)] + F_0, & \theta = 0 \end{cases}$$

$$\text{Let } W' = AW + H(z, \bar{z}, \theta) \quad (4.24)$$

Where:

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z \bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + H_{21}(\theta) \frac{z^2 \bar{z}}{2} + \dots, \quad (4.25)$$

On the other hand, on  $C_0$  near the origin  $W' = W_z z' + W_{\bar{z}} \bar{z}'$

Expanding the above series and computing the coefficients, we get

$$[A - 2i\omega_0 I]W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta) \quad (4.26)$$

By equation (4.22), for  $\theta \in [-1, 0)$ ,

$$H(z, \bar{z}, \theta) = -\overline{q^*}(0) \overline{F_0} q(\theta) - \overline{q^*}(0) \overline{F_0} \bar{q}(\theta) = -gq(\theta) - \overline{g} \bar{q}(\theta)$$

Comparing the coefficients with (4.23) for  $\theta \in [-1, 0]$  that

$$H_{20}(\theta) = -g_{20} q(\theta) - \overline{g_{02}} \bar{q}(\theta), \quad H_{11}(\theta) = -g_{11} q(\theta) - \overline{g_{11}} \bar{q}(\theta).$$

From equations (4.22), (4.25) and the definition of  $A$  we obtain

$$W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g_{02}}\overline{q}(\theta)$$

Solving for  $W_{20}(\theta)$ :

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0} q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{02}}}{3\omega_0\tau_0} \overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_1 e^{2i\omega_0\tau_0\theta},$$

And similarly

$$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_0} q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{11}}}{\omega_0\tau_0} \overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_2$$

Where  $E_1$  and  $E_2$  are both three dimensional vectors, and can be determined by setting  $\theta = 0$  in  $H$ .

In fact, since  $H(z, \bar{z}, \theta) = -2Re[\overline{q^*}(0)F_0q(0)] + F_0$ , So

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\overline{q}(\theta) + F_{z^2},$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\overline{q}(\theta) + F_{z\bar{z}}$$

Where 
$$F_0 = F_{z^2} \frac{z^2}{2} + F_{z\bar{z}} z\bar{z} + F_{\bar{z}^2} \frac{\bar{z}^2}{2} + \dots$$

Hence combining the definition of  $A$ ,

$$\int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(0) + g_{20}q(0) + \overline{g_{02}}\overline{q}(0) - F_{z^2} \text{ and}$$

$$\int_{-1}^0 d\eta(\theta)W_{11}(\theta) = g_{11}q(0) - \overline{g_{11}}\overline{q}(0) - F_{z\bar{z}}$$

Notice that

$$\left[ i\omega_0\tau_0 I - \int_{-1}^0 e^{i\omega_0\tau_0\theta} d\eta(\theta) \right] q(0) = 0$$

And 
$$\left[ -i\omega_0\tau_0 I - \int_{-1}^0 e^{-i\omega_0\tau_0\theta} d\eta(\theta) \right] \overline{q}(0) = 0$$

implies

$$\left[ 2i\omega_0\tau_0 I - \int_{-1}^0 e^{2i\omega_0\tau_0\theta} d\eta(\theta) \right] E_1 = F_{z^2}$$

And 
$$-\left[ \int_{-1}^0 d\eta(\theta) \right] E_2 = F_{z\bar{z}}$$

Hence,

$$\begin{aligned} & \begin{bmatrix} (2i\omega_0 + \delta_1 + \alpha W^* e^{-2i\omega_0\tau_0}) & 0 & -K_{NM} \\ \beta W^* e^{-2i\omega_0\tau_0} & (2i\omega_0 + \delta_2) & 0 \\ \gamma M^* & 0 & (2i\omega_0 + \gamma N^* + \delta_3) \end{bmatrix} E_1 \\ & = -2 \begin{bmatrix} \alpha \sigma_1 e^{-i\omega_0\tau_0\theta} \\ -\beta \sigma_1 e^{-i\omega_0\tau_0\theta} \\ -\gamma \rho_1 \end{bmatrix} \end{aligned}$$

and

$$\begin{bmatrix} (\delta_1 + \alpha W^*) & 0 & -K_{NM} \\ \beta W^* & \delta_2 & 0 \\ \gamma M^* & 0 & (\gamma N^* + \delta_3) \end{bmatrix} E_2 = -2 \begin{bmatrix} \alpha \operatorname{Re}\{\sigma_1\} e^{i\omega_0\tau_0\theta} \\ -\beta \operatorname{Re}\{\sigma_1\} e^{i\omega_0\tau_0\theta} \\ -\gamma \operatorname{Re}\{\rho_1\} \end{bmatrix}$$

Thus  $g_{21}$  can be expressed by the parameters.

Based on the above analysis, each  $g_{ij}$  can be determined by the parameters. Thus, following quantities can be computed:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0\tau_0} \left( g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \beta_2 = 2\operatorname{Re}\{C_1(0)\}, \\ T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_0)\}}{\omega_0\tau_0} \end{aligned} \quad (4.27)$$

**Theorem 4.1** The value of  $\mu_2$  determines the direction of the Hopf bifurcation: if  $\mu_2 > 0$  ( $\mu_2 < 0$ ), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for  $\tau > \tau_0$  ( $\tau < \tau_0$ ). The value of  $\beta_2$  determines the stability of bifurcating solutions: the bifurcating periodic solutions are orbitally asymptotically stable (unstable) if  $\beta_2 < 0$  ( $\beta_2 > 0$ ). The value of  $T_2$  determines the period of the bifurcating periodic solutions: the period increases (decreases) if  $T_2 > 0$  ( $T_2 < 0$ )

## 4.8 Sensitivity Analysis

In this paper, the model includes constant parameters. The ‘Direct Method’ to estimate the general sensitivity coefficients is used. The direct method is based on considering all parameters as constants and then the sensitivity coefficients are estimated by solving sensitivity equations simultaneously with the original system. If all the parameters  $(\alpha, \beta, \gamma)$  appearing in the system model (4.1)– (4.3) are taken to be constants, then sensitivity analysis, in this case, may just entail finding the partial derivatives of the solution with respect to each parameter. As an illustration if consider parameter  $\beta$ , then

partial derivatives of the solution  $(N, W, M)$  with respect to  $\beta$  give rise to following set of sensitivity equations:

$$\frac{dS_1}{dt} = -\delta_1 S_1 - \alpha N(t - \tau) S_2 - K_{NM} S_3 - \alpha W S_1(t - \tau) \quad (4.28)$$

$$\frac{dS_2}{dt} = (\beta N(t - \tau) - \delta_2) S_2 + \beta W S_1(t - \tau) + W N(t - \tau) \quad (4.29)$$

$$\frac{dS_3}{dt} = -\gamma M S_1 - (\gamma N + \delta_3) S_3 \quad (4.30)$$

Where

$$S_1 = \frac{\partial N}{\partial \beta}, S_2 = \frac{\partial W}{\partial \beta}, S_3 = \frac{\partial M}{\partial \beta}$$

Then, this system of sensitivity equations (4.28) – (4.30) along with the original system of equations (4.1) – (4.3) is solved to estimate the sensitivity of the state variables  $(N, W, M)$  to the parameter  $\beta$ . The similar procedure and argument holds for estimating the sensitivity of the state variables with respect to the parameters  $\alpha$  and  $\gamma$ .

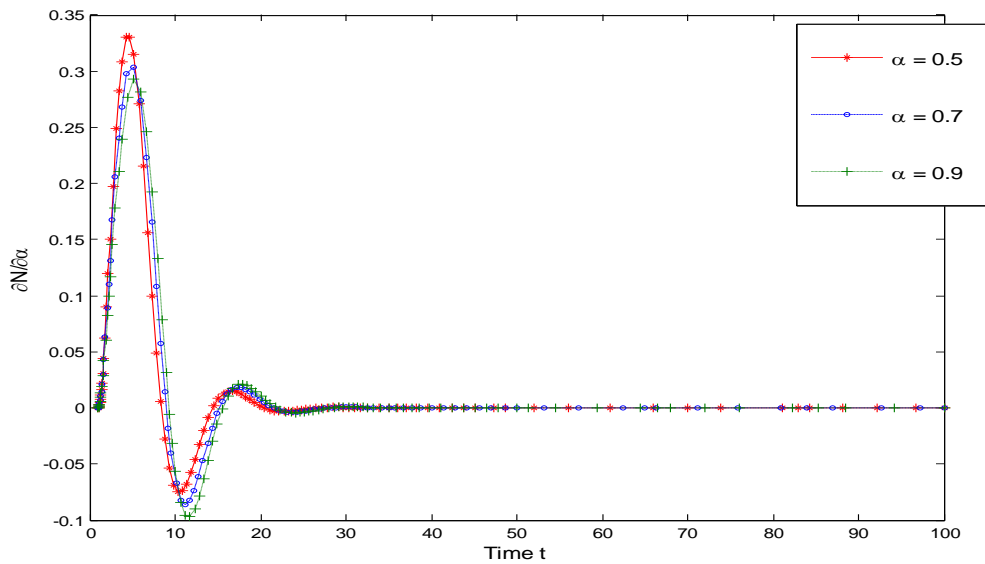
**Sensitivity of Variables to Parameter  $\alpha$ :** As shown by Figures 4.1 and Figure 4.2, the parameter consumption coefficient  $\alpha$  does not lead to much of variation and change in the values of state variables nutrient concentration  $N$  and concentration of toxic metal  $M$  which ultimately remain stable and tends to zero, as we decrease the values of  $\alpha$  from  $\alpha = 0.9$  to  $\alpha = 0.5$ . It predicts the lesser sensitivity of state variables  $N$  and  $M$  to the parameter  $\alpha$ . However, for the same range of values of  $\alpha$ , the state variable amount of plant biomass  $W$  goes under considerable change as shown by Figure 4.3. It shows increase in the rate of plant biomass with decrease in the delayed value of consumption coefficient. It remains stable as well.

**Sensitivity of Variables to Parameter  $\beta$ :** Initially, the rate of nutrient concentration starts losing stability with decrease in value of utilization coefficient, but finally becomes stable and tends to be zero as we decrease the values of parameter utilization coefficient  $\beta$  from  $\beta = 0.7$  to  $\beta = 0.3$  as shown by Figure 4.4. Figure 4.5 shows the increase in rate of concentration of toxic metal  $M$  with decrease in value of utilization coefficient  $\beta$  from  $\beta = 0.7$  to  $\beta = 0.3$ . It starts losing stability as well. Decrease in the rate of plant biomass  $W$  with decreased value of delayed utilization coefficient is shown by the Figure 4.6. It also starts losing stability.

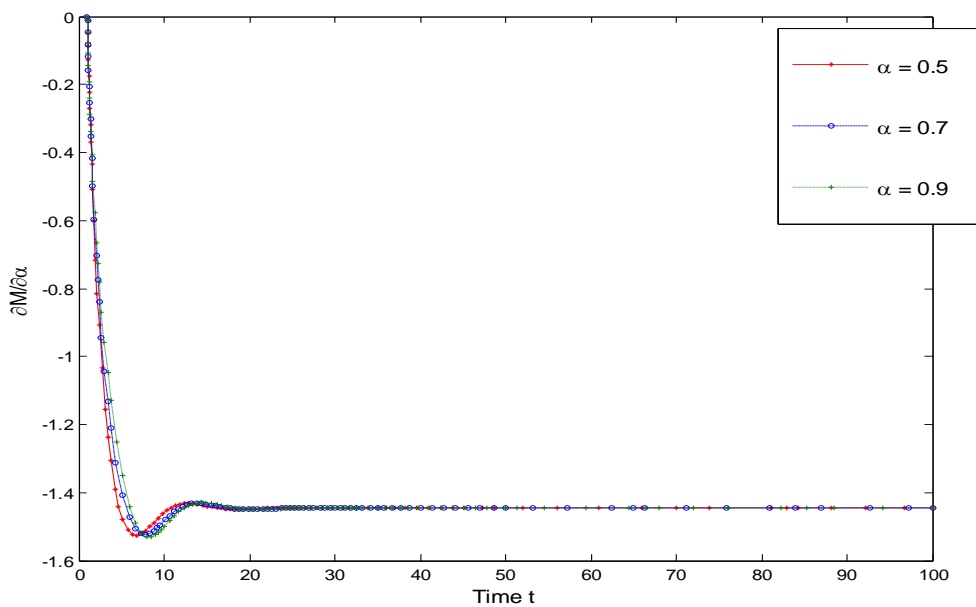
**Sensitivity of Variables to Parameter  $\gamma$ :** Figure 4.7 shows that rate of nutrient concentration is not very much affected by decrease in values of depletion coefficient  $\gamma$  from  $\gamma = 0.2$  to  $\gamma = 0.05$ . It does not lose stability. Figure 4.8 shows decrease in rate of



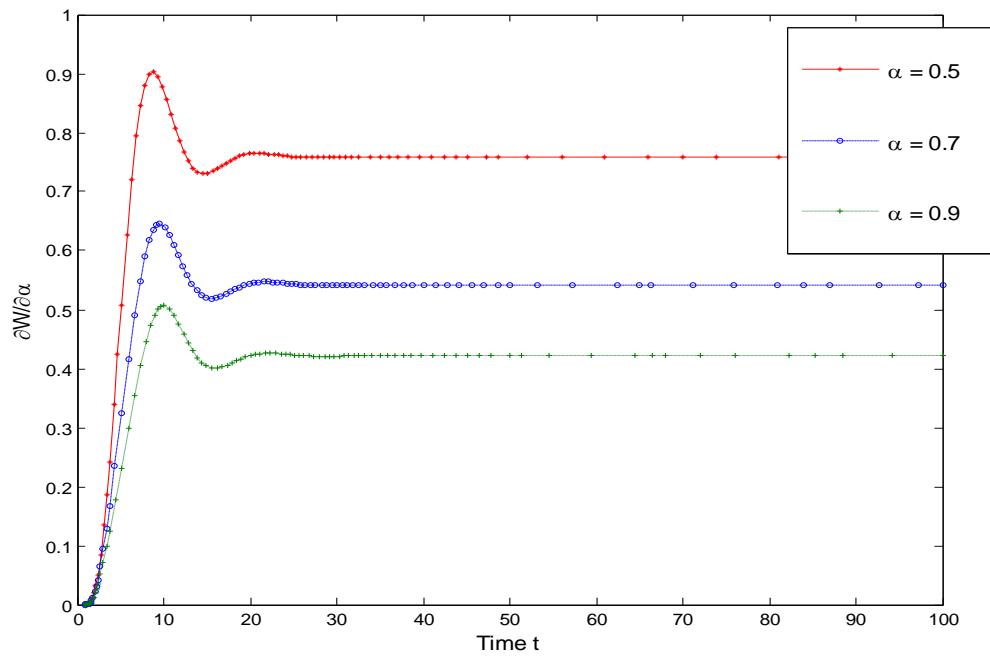
concentration of toxic metal with decrease in values of depletion coefficient of toxic metal due to interaction with nutrients. It stays stable. Figure 4.9 shows Increase in rate of plant biomass with decrease in values of depletion coefficient of toxic metal due to interaction with nutrients. It remains stable as well.



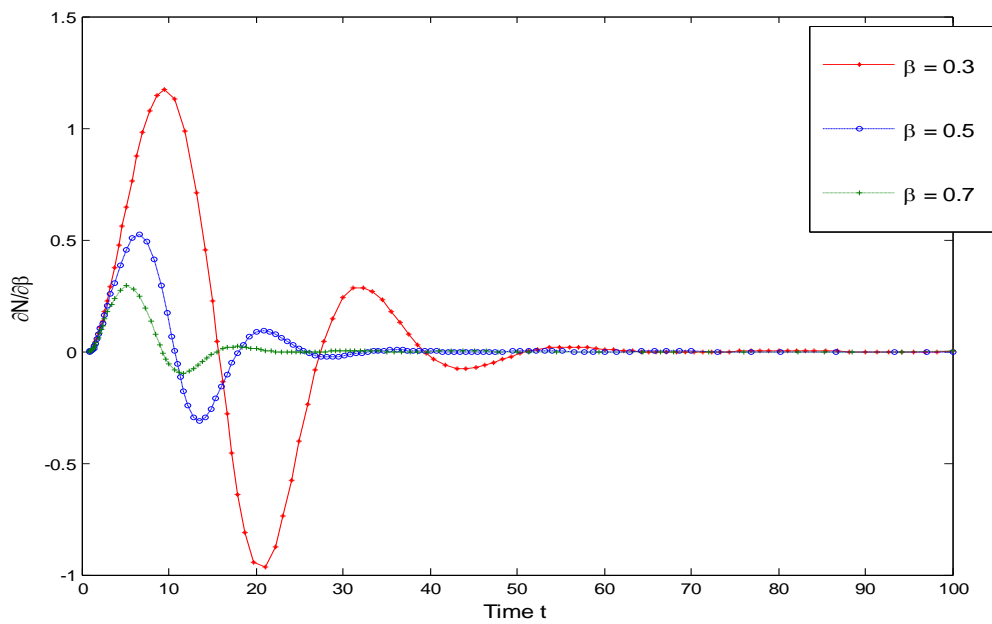
**Figure 4. 1**Time series graph between partial changes in nutrient concentration N for different values of consumption coefficient  $\alpha$



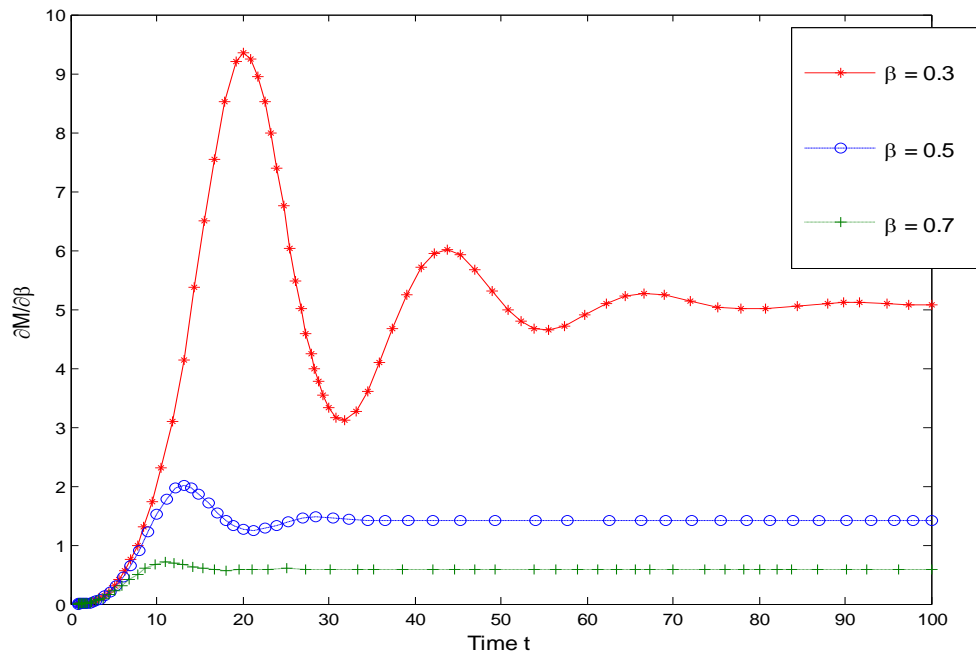
**Figure 4. 2**Time series graph between partial changes in concentration of toxic metal M for different values of consumption coefficient  $\alpha$ .



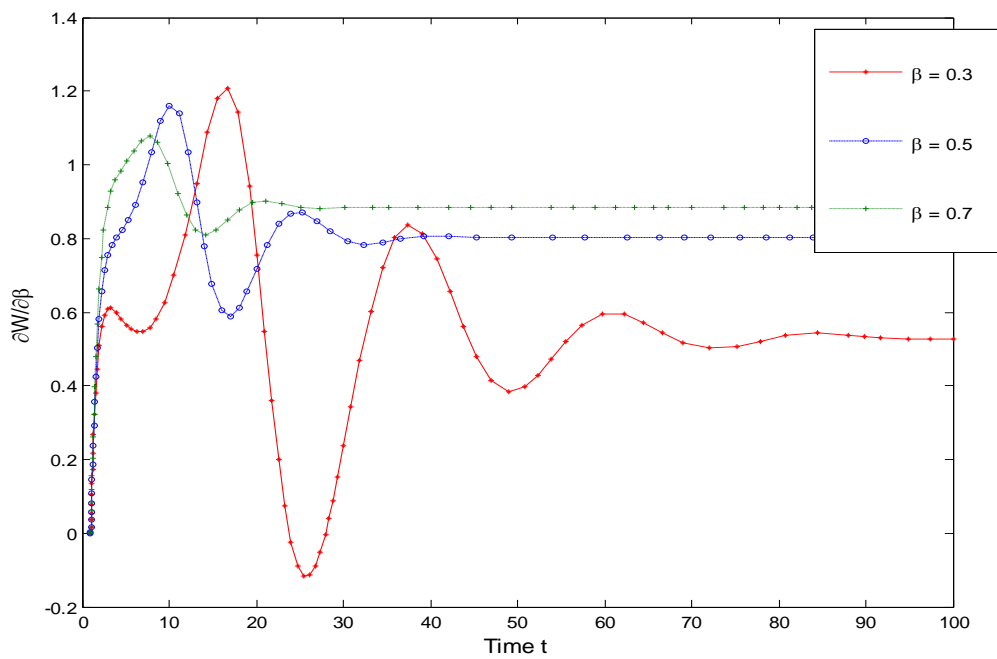
**Figure 4.3** Time series graph between partial changes in plant biomass  $W$  for different values of consumption coefficient  $\alpha$ .



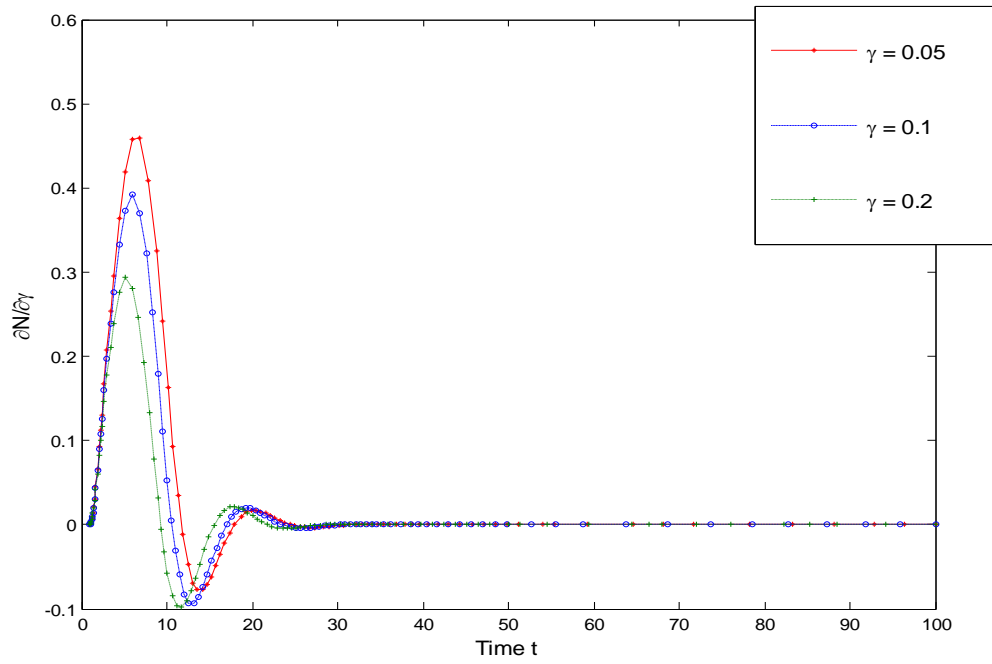
**Figure 4.4** Time series graph between partial changes in nutrient concentration  $N$  for different values of utilization coefficient  $\beta$ .



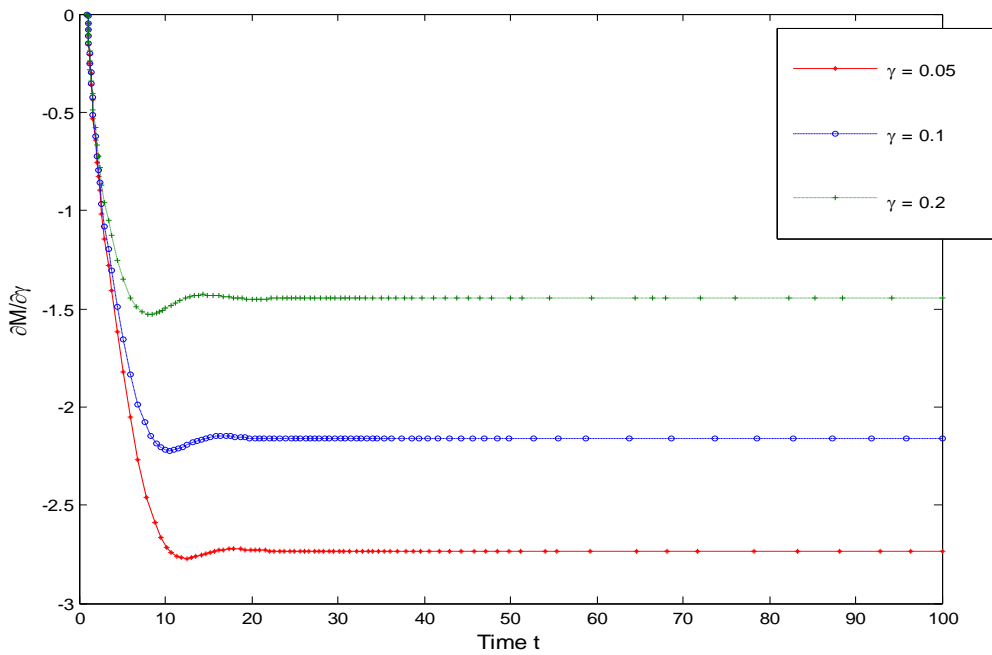
**Figure 4. 5**Time series graph between partial changes in concentration of toxic metal M for different values of utilization coefficient  $\beta$ .



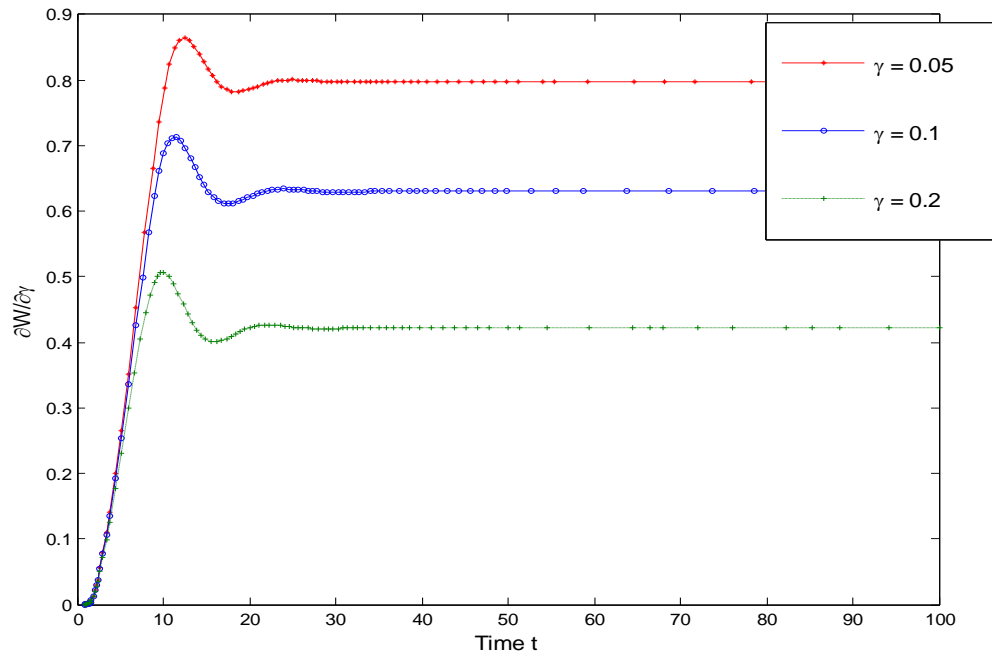
**Figure 4. 6**Time series graph between partial changes in plant biomass W for different values of utilization coefficient  $\beta$ .



**Figure 4. 7**Time series graph between partial changes in nutrient concentration N for different values of depletion coefficient  $\gamma$ .



**Figure 4. 8**Time series graph between partial changes in concentration of toxic metal M for different values of depletion coefficient  $\gamma$ .



**Figure 4.9** Time series graph between partial changes in plant biomass  $W$  for different values of depletion coefficient  $\gamma$ .

## 4.9 Numerical Example

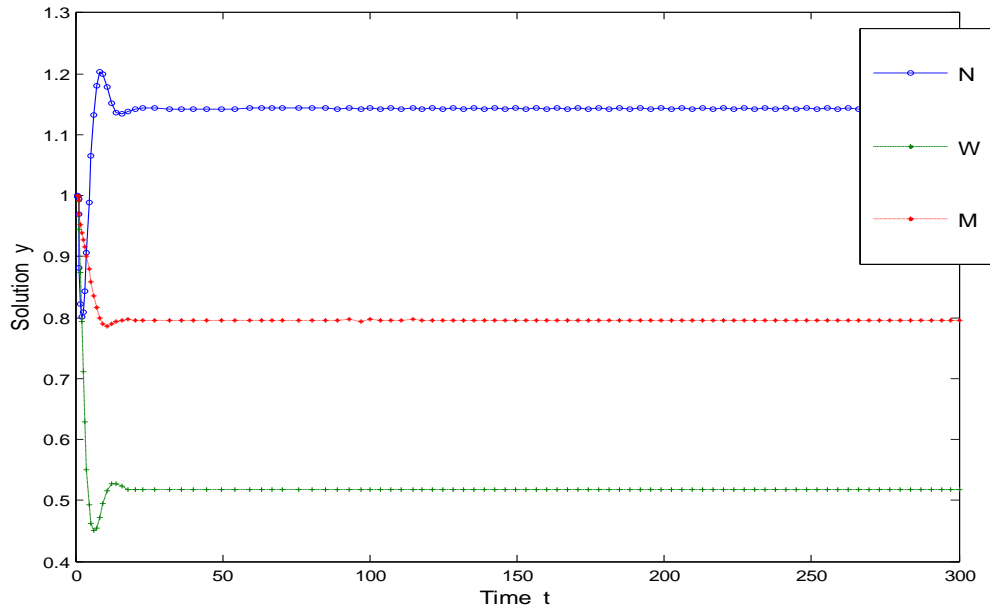
To consolidate the analytical result with the help of a numerical, simulation is done with MATLAB. For the following set of values, the behaviour shown by the system is as follows:

$$K_N = 1, K_{NM} = 0.3,$$

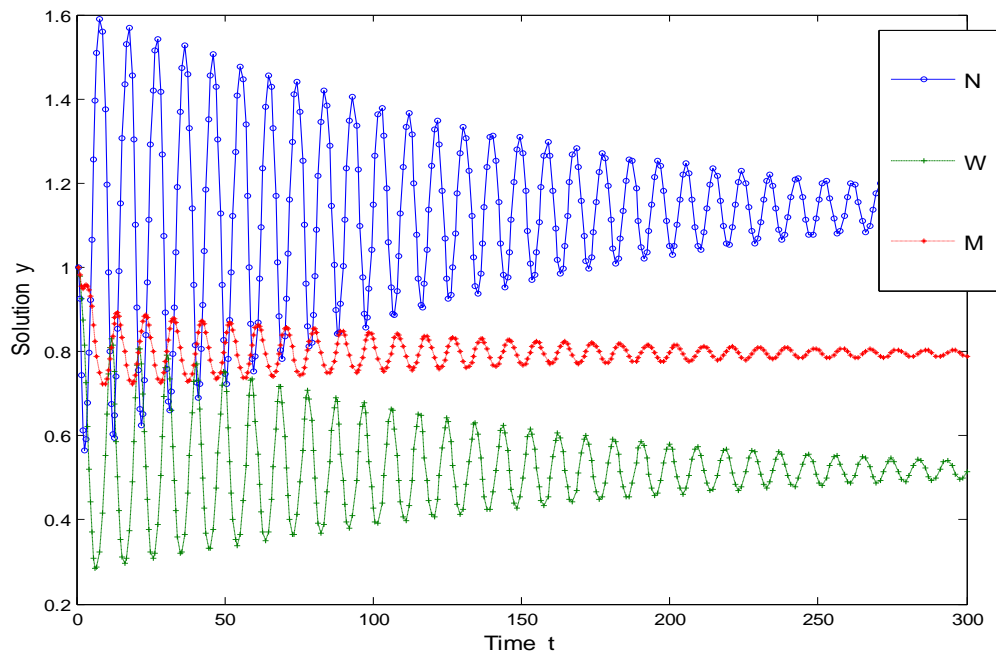
$$\alpha = 0.9, \delta_1 = 0.2, \beta = 0.7, \delta_2 = 0.8, I = 0.5, \gamma = 0.2, \delta_3 = 0.4.$$

The behaviour of the system for different values of delay is shown as under:

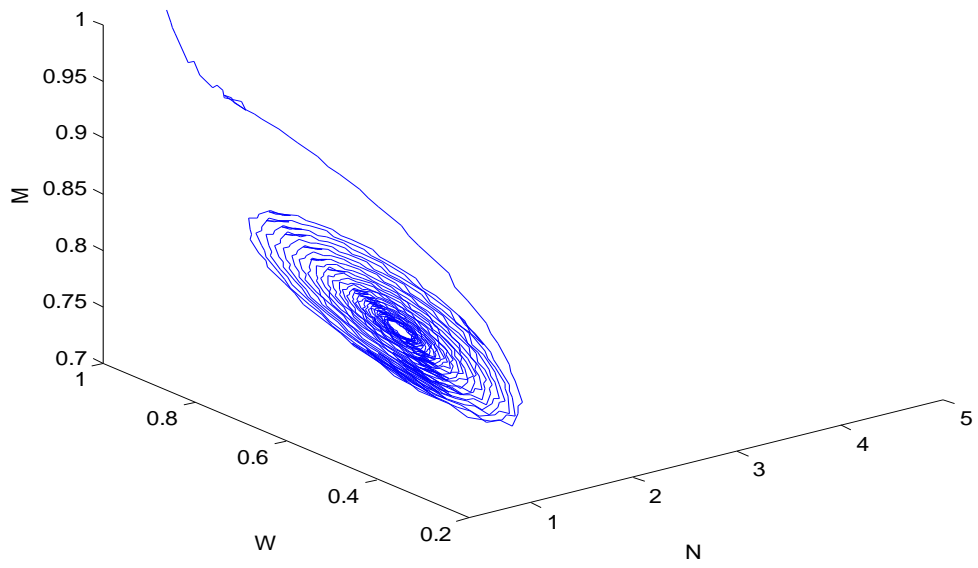
$$E_1(N^* = 1.1426, W^* = 0.5181, M^* = 0.7950)$$



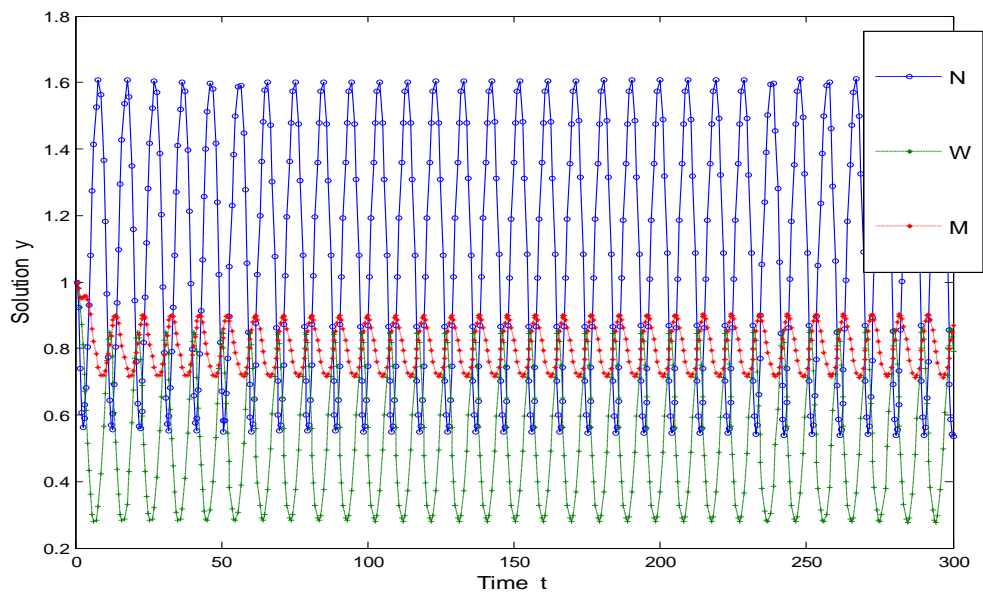
**Figure 4. 10**The interior equilibrium points  $E_1(1.1426, 0.5181, 0.7950)$  of the system is stable when there is no delay that is  $\tau=0$ .



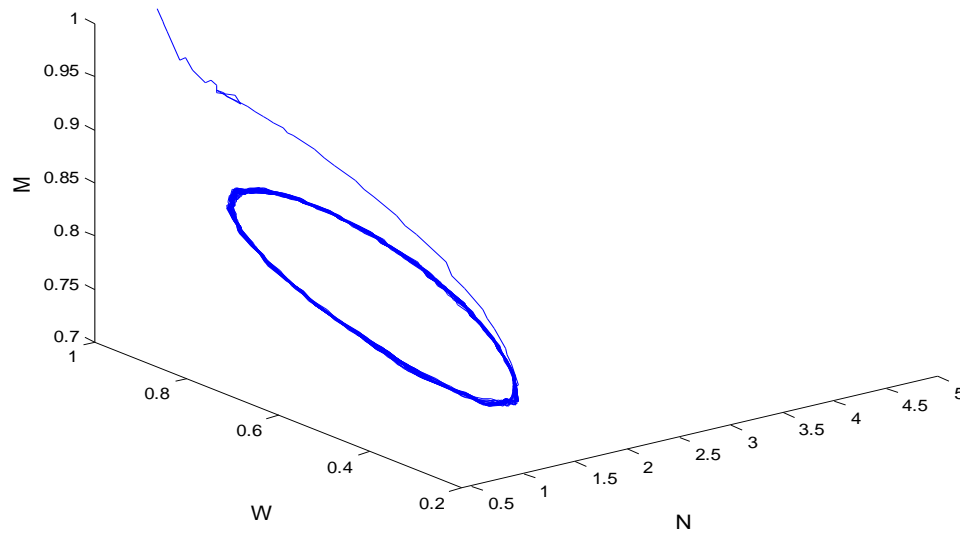
**Figure 4. 11**When delay  $\tau < 1.373$ , the interior equilibrium point  $E_1(1.1426, 0.5181, 0.7950)$  is asymptotically stable.



**Figure 4. 12**Phase space diagram of Nutrient N, Plant Biomass W and Toxic Metal M when delay  $\tau < 1.373$ .



**Figure 4. 13**The interior equilibrium point  $E_1(1.1426, 0.5181, 0.7950)$  loses its stability and Hopf- bifurcation occurred when delay  $\tau \geq 1.373$ .



**Figure 4. 14**Phase space diagram of Nutrient N, Plant Biomass W and Toxic Metal M when delay  $\tau \geq 1.373$ . The bifurcating periodic solution is orbitally, asymptotically stable.

## 4.10 Conclusion

In this paper, a mathematical model is proposed to study the role of delay on the plant growth dynamics under the effect of toxic metal. The stability and Hopf- bifurcation about the interior equilibrium is studied. It has been concluded that when there is no time delay, interior equilibrium  $E_1(1.1426, 0.5181, 0.7950)$  is completely stable (Figure 4.10) as proved by lemma 4.6.5 using Routh-Hurwitz's criteria. But under the same set of parameters, a critical value of the parameter delay is obtained below which the system is asymptotically stable (Figure 4.11 and Figure 4.12) and unstable above that critical value of parameter (Figure 4.13 and Figure 4.14) as proved by lemma 4.6.4 and lemma 4.6.6. While passing through the critical value, the system showed oscillations that is Hopf bifurcation.

In this paper, the sensitivity of model solutions due to perturbing the parameters appearing in delay differential systems is also investigated, using direct method. It is seen how the sensitivity functions enable one in identification of specific parameters and improve the understanding of the role played by specific model parameters. The oscillation and change in values accompanied by the sensitivity of state variables to parameters means that the solution is sensitive to changes in the parameter and that parameter plays an important role in the model. Sensitivity analysis reveals that the state



variable nutrient concentration  $N$  is least sensitive to all parameters  $(\alpha, \beta, \gamma)$  as compared to other two state variables  $W$  and  $M$  who show considerable amount of change in their rates for different sets of values of the parameters. Rate of plant biomass shows increase with decrease in the delayed value of consumption coefficient and stays stable (Figure 4.3) and decrease with decrease in delayed value of utilization coefficient and loses stability (Figure 4.6).

Our theoretical as well as numerical results show that for a certain threshold of parameters, the system possesses asymptotic stability around positive interior equilibrium. Further from stability analysis and numerical simulation, it is concluded that  $\tau$  is a bifurcating parameter for which the interior equilibrium point shows stable oscillatory behaviour when  $\tau \geq \tau_0$ . After considering the effect of time lag in the system, limit cycles appear for interior equilibrium points when time delay crosses some critical value.

In future, the efforts will be made to validate the proposed mathematical model with some existing plant growth data under the effect of toxic metals. The proposed mathematical model dealing with delay in plant soil dynamics under the effect of toxicant will be helpful to farmers, agriculturists, ecologists and scientists to use pesticides, insecticides and chemical fertilizers in an optimal way. The study of the factors due to which the delay is produced, and the components being affected will help the concerned community to plan the remedial measures. Being quantitative in nature, the mathematical model will prove to be economic in terms of time and money being invested on large scale experiments.

# Chapter 5

## Modelling on Tree Biomass with Time Lag due to the Toxic Metal in Soil and Tree

### 5.1 Introduction

The excess of toxic metals adversely affects the soil quality and hence the plant growth gets slowed down considerably. Thornley[162] was the first to apply mathematical modelling to variety of topics in plant physiology. Lacoite [21] concluded that models suggested by Thornley are designed for a particular plant species and under particular conditions and so they cannot be applicable over a broad range of conditions. A simple mathematical model was given by De Leo et al.[59] that coupled the effect of toxic metal and soil chemistry to study the adverse effect of toxic metal on biomass of trees. The parameters of the model given by Leo were further modified and it was shown by Guala et al.[166], [167] that the model is applicable to not only trees, but all plants in general. The effect of toxic metal on growth of individual using a two-compartment mathematical model was studied by Misra and Kalra[168], [177]. Rouches theorem[158] was used for the discussion of distribution of roots of exponential polynomials by Ruan and Wei[112], [157]. The effect of time delay on the dynamics of a multi team prey predator system was studied by Sikarwar and Misra[171].

In view of above, a mathematical model is developed to study the role of delay on plant biomass due to toxic metals and three state variables considered are: concentrations of toxic metal in soil, concentrations of toxic metal in plant and the tree biomass. Hopf bifurcation occurred at a critical value of time parameter. Numerical simulation is done using MATLAB.

### 5.2 Mathematical Model

The model that includes three state variables - concentrations of toxic metal in soil  $M_S$ , concentrations of toxic metal in tree  $T_p$  and tree biomass  $B_M$ , is given by:

$$\frac{dB_M}{dt} = rB_M - \frac{r}{K}B_M^2 - \Delta_1 B_M \quad (5.1)$$

$$\frac{dT_P}{dt} = \alpha T_P - r T_P(t - \tau) \quad (5.2)$$

$$+ r B_M T_P(t - \tau)$$

$$\frac{dM_S}{dt} = I - \alpha M_S T_P - \Delta_2 M_S \quad (5.3)$$

With initial conditions:  $B_M(0) > 0, T_P(0) > 0, M_S(0) > 0$  with  $T_P(t - \tau) = A$  at  $t \in [-\tau, 0]$

The system parameters are taken as:  $r$  is intrinsic growth rate of tree biomass,  $K$  is the carrying capacity for heavy metal.  $\Delta_1$  is natural decay of tree biomass  $B_M$  and  $\Delta_2$  is natural decay of  $M_S$ .  $I$  is the input rate of heavy metals in soil.  $\alpha$  is depletion rate of  $M_S$  due to interaction between  $M_S$  and  $T_P$ . Here all the parameters  $r, K, \Delta_1, \Delta_2, \alpha$ , and  $I$  are taken as positive.

### 5.3 Boundedness of Solutions

The boundedness of the model given by following lemma:

**Lemma 5.3.1** The model has all its solution lying in the region  $C = \left[ (B_M, T_P, M_S) \in R_+^3 : 0 \leq B_M + T_P + M_S \leq \frac{I}{\varphi} \right]$ , as  $t \rightarrow \infty$ , for all positive initial values  $\{B_M(0), T_P(0), M_S(0), T_P(t - \tau) = A, \text{ Constant for all } t \in [-\tau, 0]\} \in C \subset R_+^3$ , where  $\varphi = \min(r, \alpha, \Delta_1, \Delta_2)$ .

**Proof:** Let us assume:  $W(t) = B_M(t) + T_P(t) + M_S(t)$

$$\Rightarrow \frac{dW(t)}{dt} = \frac{d}{dt} [B_M(t) + T_P(t) + M_S(t)]$$

Using Equations (5.1) -(5.2) and  $\varphi = \min(r, \alpha, \Delta_1, \Delta_2)$  and assuming that  $T_P(t) \approx T_P(t - \tau)$  as  $t \rightarrow \infty$ , we get:

$$\frac{dW(t)}{dt} \leq I - \varphi W(t)$$

Applying the comparison theorem, we get:

$$\text{As } t \rightarrow \infty : W(t) \leq \frac{I}{\varphi}$$

Therefore, finally we have  $0 \leq B_M + T_P + M_S \leq \frac{I}{\varphi}$

## 5.4 Positivity of Solutions

Positivity means that the system sustains. For positive solutions, we need to show that all solution of system given by Equations. (5.1)– (5.3), where initial condition are  $B_M(0) > 0, T_P(0) > 0, M_S(0) > 0$  for all  $t > 0$  and  $T_P(t - \tau) = \text{Constant}$  for all  $t \in [-\tau, 0]$ , the solution  $(B_M(t), T_P(t), M_S(t))$  of the model stays positive for all time  $t > 0$ .

$$\begin{aligned} \text{From equation (5.1)} \quad \frac{dB_M}{dt} &\geq -\left(\frac{rI}{K\phi} + \Delta_1\right) B_M \\ &\Rightarrow B_M \geq c_1 e^{-\left(\frac{rI}{K\phi} + \Delta_1\right)t} \end{aligned}$$

$$\begin{aligned} \text{From equation (5.2)} \quad \frac{dT_P}{dt} &\geq -rT_P \\ &\Rightarrow T_P \geq c_2 e^{-rt} \end{aligned}$$

$$\begin{aligned} \text{From equation (5.3)} \quad \frac{dM_S}{dt} &\geq -\left(\frac{\alpha I}{K\phi} + \Delta_2\right) M_S \\ &\Rightarrow M_S \geq c_3 e^{-\left(\frac{\alpha I}{K\phi} + \Delta_2\right)t} \end{aligned}$$

Hence,  $(B_M(t), T_P(t), M_S(t))$  stays positive for all time  $t > 0$ .

## 5.5 Interior Equilibrium

The system of equations (5.1) -(5.3) has one feasible equilibrium  $E^*(B^*_M, T^*_P, M^*_S)$  where:

$$B^*_M = \frac{K(r - \Delta_1)}{r} \text{ provided } r > \Delta_1,$$

$$T^*_P = \frac{A(r - rK + K\Delta_1)}{\alpha}$$

$$M^*_S = \frac{I}{A(r - rK + K\Delta_1) + \Delta_2}$$

## 5.6 Stability Analysis and Local Hopf-Bifurcation

Here, we analyse the dynamical behaviour of interior equilibrium point  $E^*(B^*_M, T^*_P, M^*_S)$  of the model given by (5.1) -(5.3). The exponential characteristic equation about  $E^*$  is given by:

$$\begin{aligned} \lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + (n_1\lambda^2 + n_2\lambda + n_3)e^{-\lambda\tau} \\ = 0 \end{aligned} \quad (5.4)$$

Here  $m_1 = -(P_1 + P_9 + \alpha)$ ,  $m_2 = [P_1P_9 + \alpha(P_1 + P_9)]$ ,  $m_3 = -\alpha P_1P_9$ ,  $n_1 = \left(r - \frac{r}{K}B^*_M\right)$ ,  $n_2 = (P_1 + P_9)\left(\frac{r}{K}B^*_M - r\right)$ ,  $n_3 = P_1P_9\left(r - \frac{r}{K}B^*_M\right)$ .

Here  $m_1, m_2, m_3, n_1, n_2, n_3$  are all positive.

Where  $P_1 = -\left(r - \frac{2r}{K}B^*_M - \Delta_1\right)$ ,  $P_2 = 0$ ,  $P_3 = 0$ ,  $P_4 = \frac{r}{K}e^{-\lambda\tau}$ ,  $P_5 = -\left(\alpha + rKB^*_M - re^{-\lambda\tau}\right)$ ,  $P_6 = 0$ ,  $P_7 = 0$ ,  $P_8 = -\alpha T^*P$ ,  $P_9 = -\alpha T^*P + \Delta_2$

Clearly  $\lambda = i\omega$  is a solution of equation (5.4) if and only if

$$\begin{aligned} (i\omega)^3 + m_1(i\omega)^2 + m_2(i\omega) + m_3 \\ + (n_1(i\omega)^2 + n_2(i\omega) + n_3)e^{-i\omega\tau} \\ = 0 \end{aligned} \quad (5.5)$$

Separating real and imaginary parts we get:

$$\begin{aligned} m_3 - m_1\omega^2 + (n_3 - n_1\omega^2)\cos \omega\tau \\ + n_2\omega \sin \omega\tau = 0 \end{aligned} \quad (5.6)$$

$$\begin{aligned} m_2\omega - \omega^3 + n_2\omega \cos \omega\tau - (n_3 - n_1\omega^2) \sin \omega\tau \\ = 0 \end{aligned} \quad (5.7)$$

Which gives:

$$\begin{aligned} \omega^6 + (m_1^2 - n_1^2 - 2m_2)\omega^4 \\ + (m_2^2 - n_2^2 + 2n_1n_3 \\ - 2m_1m_3)\omega^2 + (m_3^2 - n_3^2) \\ = 0 \end{aligned} \quad (5.8)$$

Let  $\omega^2 = y$  and  $(m_1^2 - n_1^2 - 2m_2) = a$ ,  $(m_2^2 - n_2^2 + 2n_1n_3 - 2m_1m_3) = b$ ,  $(m_3^2 - n_3^2) = c$

Equation (5.8) becomes:

$$y^3 + ay^2 + by + c = 0 \quad (5.9)$$

**Lemma 5.6.1** If  $c < 0$ , Equation (5.9) contains at least one positive real root.

**Proof.** Let  $h(y) = y^3 + ay^2 + by + c$

Here  $h(0) = c < 0$ ,  $\lim_{y \rightarrow \infty} h(y) = \infty$

So,  $\exists y_0 \in (0, \infty)$  such that  $h(y_0) = 0$

Proof completed.

**Lemma 5.6.2** If  $c \geq 0$ , then necessary condition for equation (5.9) to have positive real roots is  $D = a^2 - 3b \geq 0$ .

**Proof.** Since  $h(y) = y^3 + ay^2 + by + c$ ,

We have  $h'(y) = 3y^2 + 2ay + b$

$$h'(y) = 0 \Rightarrow 3y^2 + 2ay + b = 0 \quad (5.10)$$

The roots of equation (5.10) can be expressed as:

$$\begin{aligned} y_{1,2} &= \frac{-2a \mp \sqrt{4a^2 - 12b}}{6} & (5.11) \\ &= \frac{-a \mp \sqrt{D}}{3} \end{aligned}$$

If  $D < 0$ , then equation (5.10) does not have any real roots. So, the function  $h(y)$  is monotone increasing function in  $y$ . It follows from  $h(0) = c \geq 0$  that equation (5.9) has no positive real roots.

Proof completed.

Clearly if  $D \geq 0$ , then  $y_1 = \frac{-a + \sqrt{D}}{3}$  is local minima of  $h(y)$ . Thus we have following Lemma.

**Lemma 5.6.3** If  $c \geq 0$ , then equation (5.9) has positive roots if and only if  $y_1 > 0$  and  $h(y_1) \leq 0$ .

**Proof.** The sufficiency is obvious. We only need to prove necessity. Otherwise, we assume that either  $y_1 \leq 0$  or  $y_1 > 0$  and  $h(y_1) > 0$ . If  $y_1 \leq 0$ , since  $h(y)$  is increasing for  $y \geq y_1$  and  $h(0) = c \geq 0$ , it follows that  $h(y)$  has no positive real zeros. If  $y_1 > 0$  and  $h(y_1) > 0$ , since  $y_2 = \frac{-a - \sqrt{D}}{3}$  is the local maxima value, it follows that  $h(y_1) \leq h(y_2)$ . Hence,  $h(0) = c \geq 0$ , we know that  $h(y)$  does not have positive real roots.

This completes proof.

**Lemma 5.6.4** Suppose  $y_1$  is defined by equation (5.11).

- (I) If  $c < 0$ , Equation (5.9) contains at least one positive real root.
- (I) If  $c \geq 0$  and  $D = a^2 - 3b < 0$ , then equation (5.9) has no positive roots.

(II) If  $c \geq 0$ , then equation (5.9) has positive roots if and only if  $y_1 > 0$  and  $h(y_1) \leq 0$ .

**Proof.** Suppose that equation (5.9) has positive roots. Without loss of generality, we assume that it has three positive roots, denoted by  $y_1, y_2, y_3$ . Then equation (5.8) has three positive roots, say  $\omega_1 = \sqrt{y_1}, \omega_2 = \sqrt{y_2}, \omega_3 = \sqrt{y_3}$ .

From (5.7) 
$$\sin \omega \tau = \frac{m_2 \omega - \omega^3}{d}$$

Which gives: 
$$\tau = \frac{1}{\omega} \left[ \sin^{-1} \left( \frac{m_2 \omega - \omega^3}{d} \right) + 2(j-1)\pi \right]; j = 1, 2, 3, -$$

Let  $\tau_k^{(j)} = \frac{1}{\omega_k} \left[ \sin^{-1} \left( \frac{m_2 \omega_k - \omega_k^3}{d} \right) + 2(j-1)\pi \right]; k = 1, 2, 3.; j = 0, 1, 2, - - -$

Then  $\mp i\omega_k$  is a pair of purely imaginary roots of equation (5.8)

Where  $\tau = \tau_k^{(j)}, k = 1, 2, 3.; j = 0, 1, 2, - - -$

We have  $\lim_{j \rightarrow \infty} \tau_k^{(j)} = \infty, k = 1, 2, 3, 4$ .

Thus, we can define:

$$\begin{aligned} \tau_0 = \tau_{k_0}^{(j_0)} &= \min_{1 \leq k \leq 3, j \geq 1} [\tau_k^{(j)}], \\ \omega_0 = \omega_{k_0}, \quad y_0 = y_{k_0} \end{aligned} \tag{5.12}$$

**Lemma 5.6.5** Suppose that  $m_1 > 0, (m_3 + d) > 0, m_1 m_2 - (m_3 + d) > 0$ .

(I) If  $c \geq 0$  and  $D = a^2 - 3b < 0$ , then all the roots of equation (3.4) have negative real parts for all  $\tau \geq 0$ .

(II) If  $c < 0$  or  $c \geq 0, y_1 > 0$  and  $h(y_1) \leq 0$ , then all the roots of equation (3.4) have negative real parts for all  $\tau \in [0, \tau_0)$ .

**Proof.** When  $\tau = 0$ , equation (5.4) becomes:

$$\begin{aligned} \lambda^3 + (m_1 + n_1)\lambda^2 + (m_2 + n_2)\lambda + (m_3 + n_3) \\ = 0 \end{aligned} \tag{5.13}$$

By Routh-Hurwitz's criteria, **(H1)**: All roots of equation (5.5) have negative real parts if and only if  $(m_3 + n_3) > 0, (m_1 + n_1)(m_2 + n_2) - (m_3 + n_3) > 0$ .

If  $c \geq 0$  and  $D = a^2 - 3b < 0$ , Lemma 5.6.4 (II) shows that equation (5.4) has no roots with zero real part for all  $\tau \geq 0$ . When  $c < 0$  or  $c \geq 0, y_1 > 0$  and  $h(y_1) \leq 0$ , Lemma 5.6.4 (I) and (III) implies that when  $\tau \neq \tau_k^{(j)}, k = 1, 2, 3.; j \geq 1$ , equation (5.4) has no

roots with zero real part and  $\tau_0$  is the minimum value of  $\tau$  so that the equation (5.4) has purely imaginary roots. Applying Theorem 1.5.8., we obtain the conclusion of the lemma.

$$\text{Let } \lambda(\tau) = \beta(\tau) + i\omega(\tau) \quad (5.14)$$

be the roots of equation (4) satisfying:  $\beta(\tau_0) = 0$ ,  $\omega(\tau_0) = \omega_0$

In order to guarantee that  $\mp\omega_0$  are simple purely imaginary roots of equation (5.4), with  $\tau = \tau_0$  and  $\lambda(\tau)$  satisfies transversality condition, we assume that  $h'(y_0) \neq 0$ .

**Lemma 5.6.6** Suppose  $y_0 = \omega_0^2$ . If  $\tau = \tau_0$ , Then  $\text{Sign} [\beta'(\tau_0)] = \text{Sign} [h'(y_0)]$

**Proof.** Let  $\lambda = \lambda(\tau)$  be the root of equation (5.4).

Putting  $\lambda(\tau)$  in equation (5.4) and differentiating w.r.t  $\tau$ , it follows that  $\frac{d\lambda}{d\tau} [3\lambda^2 + 2m_1\lambda + m_2 + n_1\lambda^2 + n_2\lambda + n_3 - \tau + 2n_1\lambda + n_2e^{-\lambda\tau} = \lambda n_1\lambda^2 + n_2\lambda + n_3e^{-\lambda\tau}$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2m_1\lambda + m_2)e^{\lambda\tau}}{\lambda(n_1\lambda^2 + n_2\lambda + n_3)} + \frac{(2n_1\lambda + n_2)}{\lambda(n_1\lambda^2 + n_2\lambda + n_3)} - \frac{\tau}{\lambda}$$

From equations (5.6) -(5.8), we have

$$\beta'(\tau_0) = \text{Re} \left[ \frac{(3\lambda^2 + 2m_1\lambda + m_2)e^{\lambda\tau}}{\lambda(n_1\lambda^2 + n_2\lambda + n_3)} \right] + \text{Re} \left[ \frac{(2n_1\lambda + n_2)}{\lambda(n_1\lambda^2 + n_2\lambda + n_3)} \right]$$

$$= \frac{1}{\gamma} [3\omega_0^6 + 2a\omega_0^4 + b\omega_0^2]$$

Where  $\gamma = [(n_3 - n_1\omega^2)^2 + (n_2\omega)^2]$ , such that  $\gamma > 0$  and  $\omega_0 > 0$ .

We conclude that:  $\text{Sign} [\beta'(\tau_0)] = \text{Sign} [h'(y_0)]$ .

This proves lemma.

## 5.7 Sensitivity Analysis

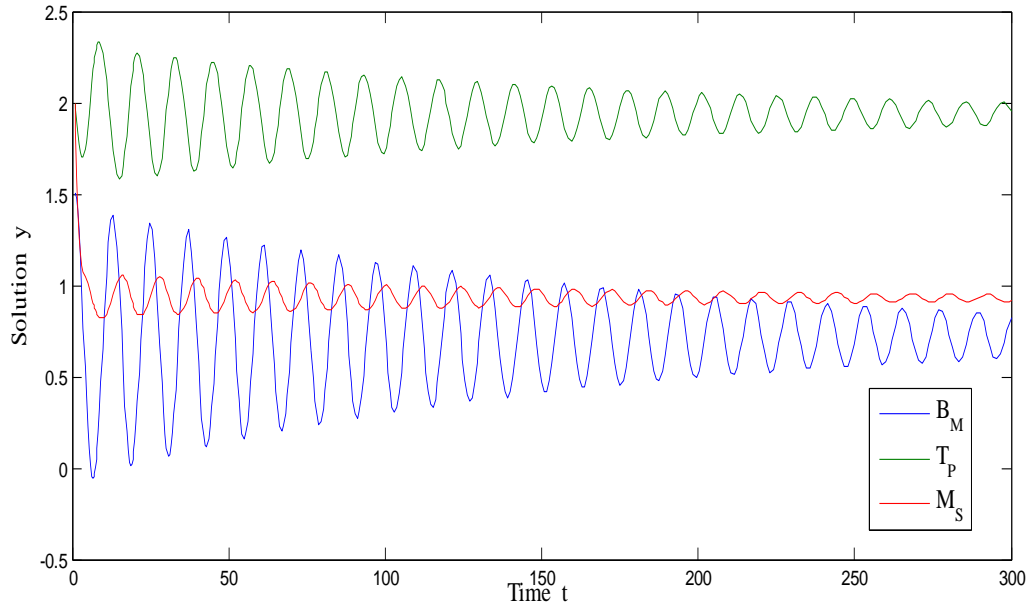
In this section, sensitivity analysis of state variable, namely tree biomass  $B_M$  is done with respect to model parameters, intrinsic growth rate of tree biomass  $r$  and the input rate of heavy metals  $I$  in soil. The values of parameters are perturbed and corresponding changes in state variables are observed. The graphical representation of the same is shown in the section-5.8 with the help of numerical example and detailed discussion is included in section 5.9-Conclusion.



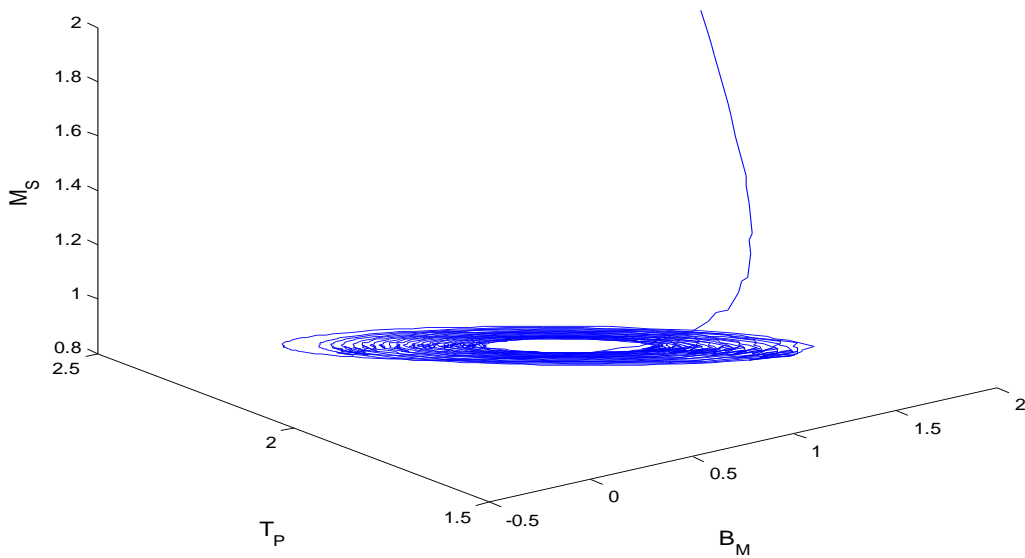
## 5.8 Numerical Example

Numerical simulation is performed by taking set of parametric values as:

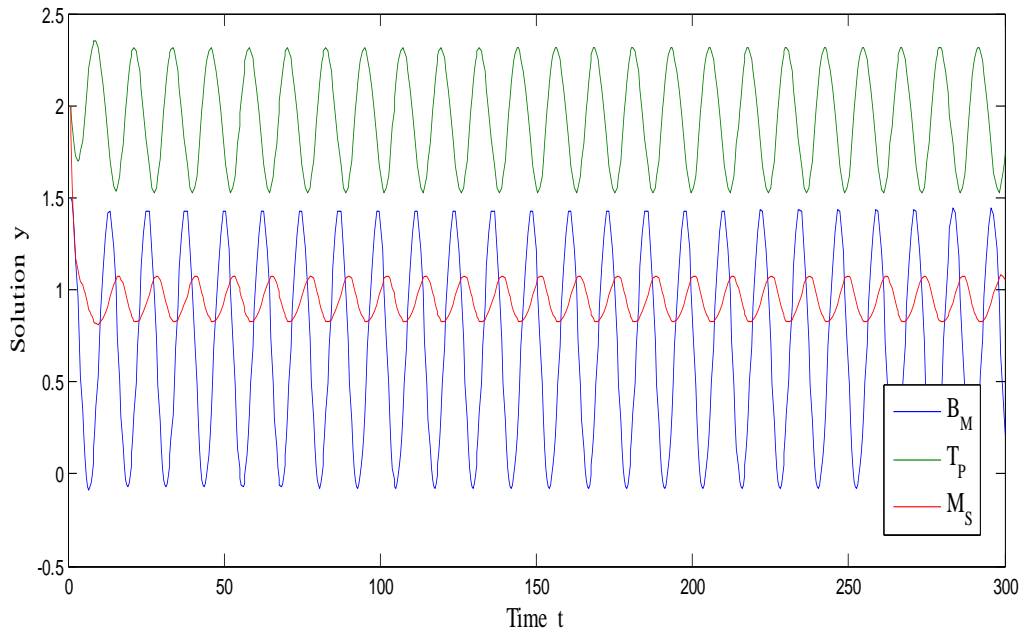
$$\frac{r}{K} = 0.2, \Delta_1 = 0.3, r = 0.9, \Delta_2 = 0.3, I = 1, \alpha = 0.4, A = 3$$



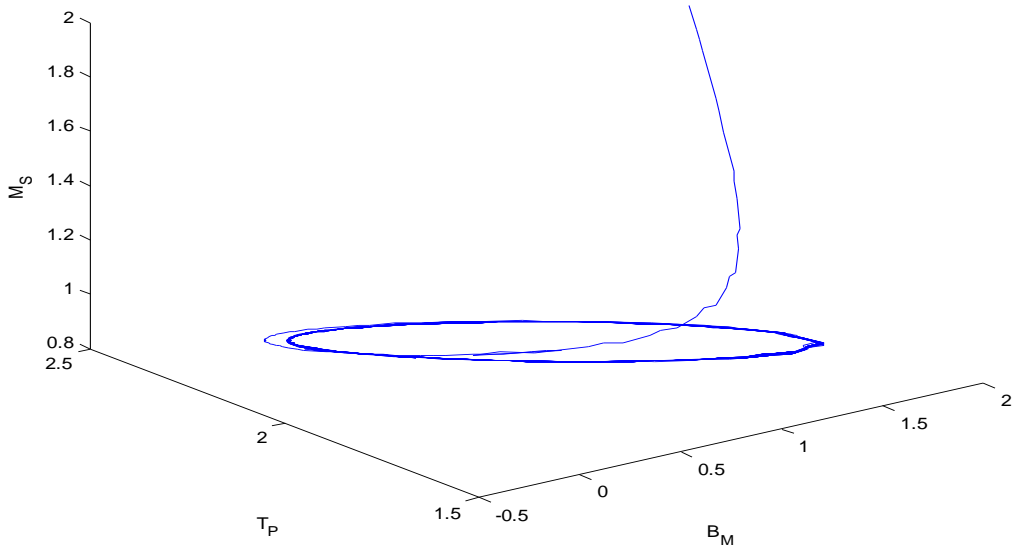
**Figure 5. 1**The interior equilibrium point Interior $E^*(B^*_M, T^*_P, M^*_S)$  is asymptotically stable with delay  $\tau < 3.17$ .



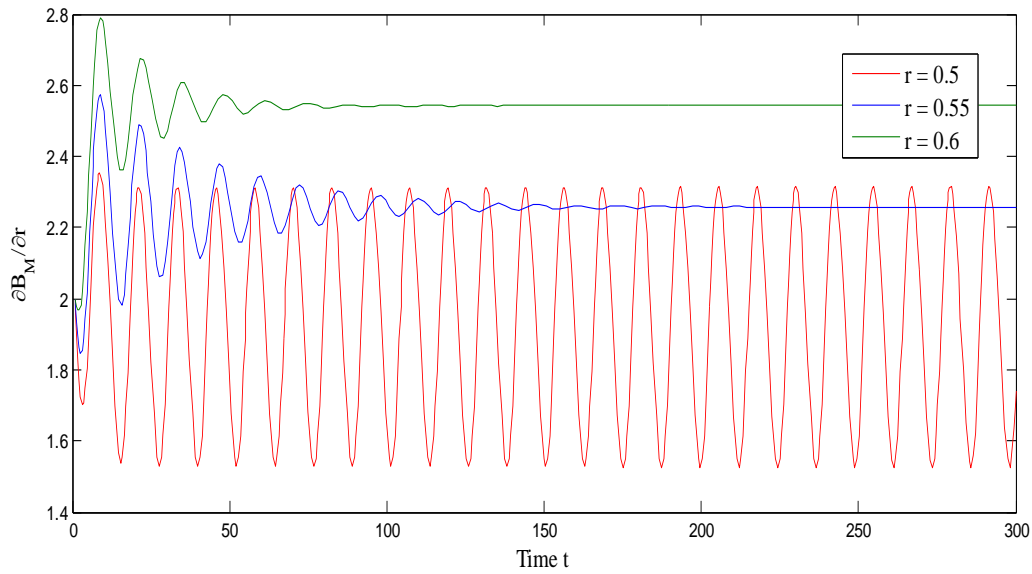
**Figure 5. 2**Phase plane with delay  $\tau < 3.17$



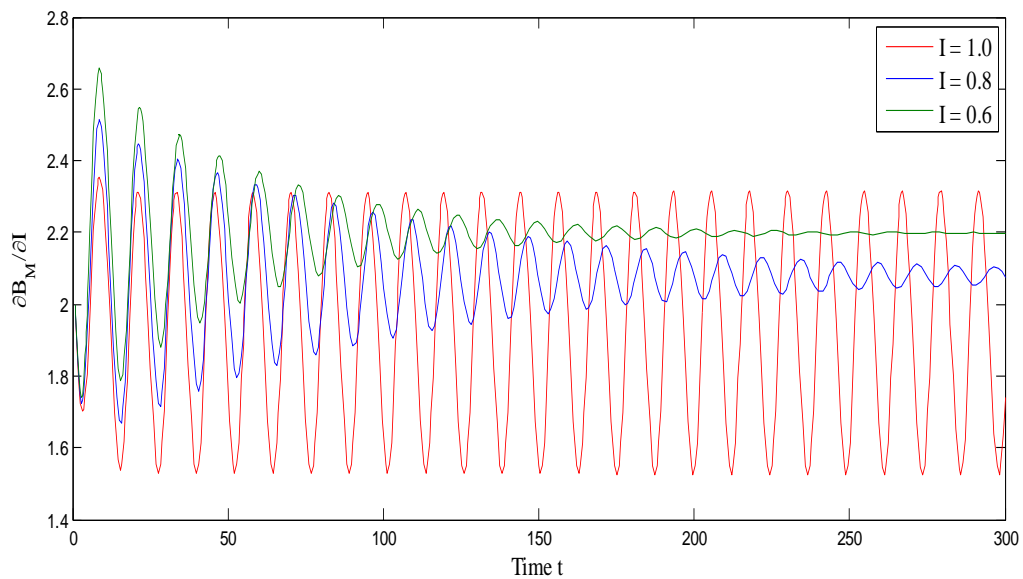
**Figure 5. 3**The interior equilibrium point  $E^*(B_M^*, T_P^*, M_S^*)$  loses its stability and Hopf- bifurcation occurred with delay  $\tau \geq 3.17$



**Figure 5. 4**Phase plane with delay  $\tau \geq 3.17$



**Figure 5.5** Time series graph between partial changes in tree biomass  $B_M$  and different values of internal growth rate  $r$ .



**Figure 5.6** Time series graph between partial changes in tree biomass  $B_M$  and different values of input rate  $I$  of toxic metal.

## 5.9 Conclusion

The interior equilibrium  $E^*$  of the model has been studied for stability of the model. The system represented by equations (5.1) - (5.3) undergoes a Hopfbifurcation at the critical value  $\tau = \tau_0$ . When delay is less than 3.17 i.e.  $\tau < 3.17$  the equilibrium  $E^*(B_M^*, T_P^*, M_S^*)$  is asymptotically stable as shown by Figure 5.1 and Figure 5.2. The moment delay crosses critical value 3.17 i.e.  $\tau \geq 3.17$  the system undergoes bifurcation and periodic solution results as shown by Figure 5.3 and Figure 5.4.

The sensitivity of model solutions is established by taking different values of the parameters appearing in system. It improves the understanding of the role played by specific model parameters.

Sensitivity analysis reveals that with increase in the internal growth rate  $r$  from 0.5 to 0.6, state variable-tree biomass  $B_M$  tends towards stability for the same set of remaining parameters including time delay  $\tau = 3.17$ . At  $r = 0.5$ , tree biomass show unstable behaviour via Hopf bifurcation. But as the value of  $r$  is increased to  $r = 0.55$ , tree biomass start showing the asymptotic stability and finally for  $r = 0.6$ , tree biomass start converging to a stable equilibrium point as shown by Figure 5.5. Apart from converging to stability, the tree biomass  $B_M$  also show increase in its value as we increase the value of  $r$  from 0.5 to 0.6 as shown by Figure 5.5. Similarly, as we decrease the value of input rate  $I$  of toxic metal from  $I = 1.0$  to  $I = 0.6$ , the tree biomass  $B_M$  start converging to a stable equilibrium as shown by Figures 5.6. In addition to convergence to stability, the tree biomass  $B_M$  also show increase in its value as we decrease the value input rate  $I$  of toxic metal from  $I = 1.0$  to  $I = 0.6$  as shown by Figure 5.6.

# Chapter 6

## Dynamics Induced by Delay under the Effect of Toxic Metal on Nutrient Pool and Plant Population Density

### 6.1 Introduction

Production of quality food is one of the topmost priorities for world scientists and agriculturists today, which is under threat. The quality of food is getting compromised and is deteriorating day by day due to persistent presence of toxic metals in the soil. The presence of arsenic in the soil and its toxic effect on the plants is a major concern. Coal fuels and metal smelters are the main sources of excessive arsenic concentration in the soil. Tu and Ma[178] showed that natural formation and anthropogenic activities are the main reasons of entrance of arsenic into terrestrial and aquatic environment. Geng et al.[179] reported that arsenic sensitivity is intimately linked to phosphorus nutrition in plants. Wang and Duan[180] studied the effect of external and internal phosphate status on arsenic toxicity and accumulation in rice seedlings. Pigna et al.[181] reported that applying phosphorus fertilizers could increase arsenic availability in soils and enhance plant uptake of arsenic. Liao et al.[182] studied root distribution and elemental accumulation of Chinese brake from arsenic contaminated soils where forty hectares of arsenic polluted land was irrigated by local farmers. Liu et al.[183] studied the effects of high concentrations of soil arsenic on growth of winter wheat and rape. They concluded that the growth gets reduced by nearly 20 percent. Brune and Dietz[184] did comparative analysis of elemental composition of Barley seedlings in the presence of toxic concentrations of heavy metals like cadmium, molybdenum, nickel and zinc. Pedreno et al.[185] reviewed the main results of heavy metals like cadmium, chromium and nickel on tomato plant. Penget al.[186] developed a pollutant accumulation model based on mass balance theory to simulate the long-term effects of heavy metal concentrations on soil. Cu[93] has shown in his paper that cultivation of crops on contaminated sites may result in both growth inhibition and tissue accumulation that may cause possible risks to human health. It considered the transport and uptake of cadmium by vegetables in Lettuce. Vineeth et al.[187] conducted a pot experiment to study the toxicity nature of heavy metals in *Vigna radiata* (green gram). Seed

germination test was conducted by Boros and Micle[91] to determine plant's tolerance to copper and to investigate its influence on the germination of seeds and growth of sunflower. The extent of plant injury by elevated zinc concentration was assessed by Tsonev and Lidon[82] considering its specific and strong dependence on the environmental conditions and availability of other heavy metals. Pavel et al.[86] used *Lepidium sativum* as a test plant to assess the phytotoxic effects of chromium and cadmium. It was found that root development and dry biomass were adversely affected by toxic stress. Dan et al.[188] studied and established nickel phytotoxicity thresholds for oat (*Avena sativa* L.) in four soil types, each created by blending a low and a high nickel soil, to generate a range of concentrations. A pot study was conducted by Xiong[189] to determine lead bioaccumulation and tolerance on seed germination and growth of *Brassica pekinensis* Rupr. Misra and Kalra[83], [177] studied the growth of an individual plant under the effect of toxic metals using a two-compartment mathematical model consisting of system of ordinary differential equations. Ruan and Wei [112] studied the nature and distribution of roots of third-degree exponential polynomial and applied the result on a delay model for control of testosterone secretion. Rihan[144] did the Sensitivity analysis for dynamic systems with time-lags using adjoint equations and direct methods when the parameters appearing in the model are not only constants but also variables of time. A lot of work has been done on plant growth under the effect of toxicants using ordinary differential equations. The effect of delay on plant growth dynamics under the effect of toxicant has not been widely studied using delay differential equations.

It is an ecological phenomenon that and nutrient pool and plant population density are adversely affected by the presence of excessive toxic metal. This dynamic is mathematically depicted by the model proposed in this study. It is shown that as the rate of damage of plant population density due to toxic metal and the input rate of toxic metal increase, the nutrient pool concentration and plant population density decreases. Hopf bifurcation is studied by introducing delay parameter in the term involving nutrient utilization. Sensitivity analysis of state variables in relation to model parameters is done. Model is verified by taking those values of damage rate due to toxic metal (assumed to be Arsenic) which are available in already existing data for growth of winter wheat (*Triticum aestivum* L) and rape (*Brassica napus*) under the effect of high concentrations of Arsenic in soil (Liu et al 2012). Simulation is done using MATLAB.

## 6.2 Mathematical Model

Let the three state variables be: Nutrient pool  $N$ , Plant population density  $P$  and toxic metal  $T$ . The system governing the dynamics is given by equations (6.1)- (6.3):

$$\frac{dN}{dt} = (K - vNT) - uN(t - \tau)P - \alpha N \quad (6.1)$$

$$\frac{dP}{dt} = r(N, T)P - \beta P^2 \quad (6.2)$$

$$\frac{dT}{dt} = Q - vNT - \gamma T \quad (6.3)$$

Where  $r(N, T) = \rho N - \varepsilon(T)$ ;  $\varepsilon(T) = \varepsilon_0 + \varepsilon_1 T$

With initial conditions:  $N(0) \geq 0, P(0) \geq 0, T(0) \geq 0 \forall t > 0$  and  $N(t - \tau)$  is constant for  $t \in [0, \tau]$ .

The parameters in system of equations (6.1)- (6.3) are:  $K$  is the total available input of nutrient pool,  $(K - vNT)$  is the nutrient availability adversely hampered due to the uptake of toxic metal,  $u$  is the consumption coefficient or the utilization coefficient of nutrient,  $\alpha$  is the natural decay of nutrient,  $r(N, T)$  is the intrinsic growth rate of plant population density which depends on nutrient pool and is adversely affected by toxic metal,  $\rho$  is nutrient use efficiency,  $\varepsilon_0$  is the natural decay of plant population density,  $\varepsilon_1$  is the rate of damage of plant population density due to toxic metal,  $\beta$  is self-limiting rate of plant population density,  $Q$  is the input rate of toxic metal,  $v$  is the deletion rate of toxic metal due to interaction between toxic metal and nutrient pool,  $\gamma$  is natural decay of toxic metal. All the parameters of this system are supposed to be positive.

## 6.3 Boundedness of Solutions

The boundedness of solutions is proved by following Lemma:

**Lemma 6.3.1** All the solutions of the system of equations (6.1)- (6.3) lie in the three-dimensional region  $\mathbb{R} = \left\{ (N, P, T) \in \mathbb{R}^{3+}; 0 \leq N + P \leq \frac{K}{\aleph}, 0 \leq T \leq \frac{Q}{\gamma} \right\}$ ,  $t \rightarrow \infty$ , where  $\aleph = \min.(v, u, \alpha, \rho, \varepsilon_0, \varepsilon_1, \beta)$  with initial conditions  $N \geq 0, P \geq 0, T \geq 0 \forall t > 0$  and  $N(t - \tau) = \text{constant}$  for  $t \in [0, \tau]$ .

**Proof.** Let  $W = N + P$ ;  $N \geq 0, P \geq 0$ .

$$\Rightarrow \frac{dW}{dt} = \frac{dN}{dt} + \frac{dP}{dt}$$

$$\Rightarrow \frac{dW}{dt} \leq K - \aleph W$$

where  $\aleph = \min.(v, u, \alpha, \rho, \varepsilon_0, \varepsilon_1, \beta)$

As  $t \rightarrow \infty, W \leq \frac{K}{\aleph}$  (By usual comparison theorem)

So,  $0 \leq N + P \leq \frac{K}{\aleph}$

From equation (6.3):  $\frac{dT}{dt} \geq Q - \gamma T; T \geq 0.$

As  $t \rightarrow \infty, T \leq \frac{Q}{\gamma}$  (By usual comparison theorem)

So,  $0 \leq T \leq \frac{Q}{\gamma}$

Proof completed.

## 6.4 Positivity of Solutions

From equation (6.3):  $\frac{dT}{dt} \geq -(vN + \gamma)T$

$$\Rightarrow \frac{dT}{dt} \geq -\left(v\frac{K}{\aleph} + \gamma\right)T$$

$$\Rightarrow T \geq e^{-\left(v\frac{K}{\aleph} + \gamma\right)t}$$

Similarly,  $P \geq e^{-\left(\varepsilon_0 + \varepsilon_1\frac{Q}{\gamma} + \beta\frac{K}{\aleph}\right)t}$

and  $N \geq e^{-\left(\alpha + v\frac{Q}{\gamma} + u\frac{K}{\aleph}\right)t}$

So,  $N \geq 0, P \geq 0, T \geq 0 \forall t.$

Hence solution set of the system of equations stay positive for all t. It ensures that system persists.

## 6.5 Equilibrium Points

There are two feasible equilibrium points  $E_1$  and  $E_2$ , out of which  $E_1$  is a uniform equilibrium point in the absence of toxicity and  $E_2$  is the feasible interior equilibrium in the presence of all three state variables. It is assumed that at the points of equilibriums:  $N(t - \tau) \cong N(t).$



### 6.5.1 Equilibrium $E_1(\bar{N} \neq 0, \bar{P} \neq 0, \bar{T} = 0)$

This equilibrium corresponds to natural growth of plant population density through the availability of nutrient pool in the absence of toxic metal where the values of state variables are given by:

$$\bar{P} = \frac{\rho\bar{N} - \varepsilon_0}{\beta} > 0, \text{ provided } \rho\bar{N} > \varepsilon_0,$$

$$\bar{N} = \frac{-f_2 \mp \sqrt{f_2^2 - 4f_1f_3}}{2f_1} \text{ where } f_1 = u\rho, f_2 = (\alpha\beta - u\varepsilon_0), f_3 = -\beta K.$$

### 6.5.2 Equilibrium $E_2(N^* \neq 0, P^* \neq 0, T^* \neq 0)$

This equilibrium corresponds to growth of plant population density in the presence of toxic metal where:

$$N^* = \frac{Q - \gamma T^*}{vT^*} > 0, \text{ provided } Q > \gamma T^*,$$

$$P^* = \frac{1}{\beta} \left[ \frac{\rho(Q - \gamma T^*)}{vT^*} - (\varepsilon_0 + \varepsilon_1 T^*) \right] > 0, \text{ provided } \frac{\rho(Q - \gamma T^*)}{vT^*} > (\varepsilon_0 + \varepsilon_1 T^*),$$

The cubic polynomial in  $T^*$  is:

$$(\beta v^2 \gamma - uv\gamma\varepsilon_1)T^{*3} + (\beta v^2 K - \beta v^2 Q - u\rho\gamma^2 + uv\varepsilon_1 Q - uv\varepsilon_0\gamma + \alpha\beta\gamma v)T^{*2} \\ + (uv\varepsilon_0 Q + 2\rho u\gamma Q - \alpha\beta v Q)T^* - u\rho Q^2 = 0$$

But due to positivity of  $N^*$  only one feasible positive root will exist provided  $Q > \gamma T^*$

## 6.6 Study of Uniform equilibrium $E_1$ and Local Stability Analysis

There is no delay ( $\tau = 0$ ) in the absence of toxic metal ( $\bar{T} = 0$ ). Uniform equilibrium  $E_1(\bar{N} \neq 0, \bar{P} \neq 0, \bar{T} = 0)$  correspond to growth of plant population density through nutrient pool in the absence of toxic metal. The system of equations depicting this dynamic is:

$$\frac{d\bar{N}}{dt} = K - u\bar{N}\bar{P} - \alpha\bar{N} \quad (6.4)$$

$$\frac{d\bar{P}}{dt} = r(\bar{N})\bar{P} - \beta\bar{P}^2 \quad (6.5)$$

The corresponding characteristic equation is given by:

$$\lambda^2 + a\lambda + b = 0 \quad (6.6)$$

Where  $a = (\alpha + u\bar{P} + \varepsilon_0 + 2\beta\bar{P} - \rho\bar{N}) > 0$  provided  $\alpha + u\bar{P} + \varepsilon_0 + 2\beta\bar{P} > \rho\bar{N}$

And  $b = (u\bar{P}\varepsilon_0 + 2\beta u\bar{P}^2 + 2\alpha\beta\bar{P} + \alpha\varepsilon_0 - \alpha\rho\bar{N}) > 0$  provided  $u\bar{P}\varepsilon_0 + 2\beta u\bar{P}^2 + 2\alpha\beta\bar{P} + \alpha\varepsilon_0 > \alpha\rho\bar{N}$

By Routh-Hurwitz criteria all the roots of equation (6.6) will lie in left half plane (having negative real part) and the system (6.4)- (6.5) will be stable if and only if  $a > 0, b > 0$ .

## 6.7 Study of Interior equilibrium $E_2$ and Hopf- Bifurcation Analysis

The interior equilibrium  $E_2(N^* \neq 0, P^* \neq 0, T^* \neq 0)$  represents the growth of plant population density under the effect of toxic metal. The corresponding system of equations is:

$$\frac{dN^*}{dt} = (K - vN^*T^*) - uN^*(t - \tau)P^* - \alpha N^* \quad (6.7)$$

$$\frac{dP^*}{dt} = r(N^*, T^*)P^* - \beta P^{*2} \quad (6.8)$$

$$\frac{dT^*}{dt} = Q - vN^*T^* - \gamma T^* \quad (6.9)$$

The exponential characteristic equation associated with system of equations (6.7)- (6.9) is:

$$(\lambda^3 + g_1\lambda^2 + g_2\lambda + g_3) + e^{-\lambda\tau}(h_1\lambda^2 + h_2\lambda + h_3) = 0 \quad (6.10)$$

Where  $g_1 = -(p_1 + p_5 + p_9)$ ,  $g_2 = (p_1p_5 + p_5p_9 + p_9p_1 - p_3p_7)$ ,  $g_3 = (p_1p_5p_9 + p_3p_5p_7)$ ,  $h_1 = p\tau$ ,  $h_2 = p\tau p_5 + p_9$ ,  $h_3 = -p\tau p_5 p_9$

The values of  $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_\tau$ , are given as:

$$p_1 = -(vT^* + \alpha), p_2 = 0, p_3 = -vN^*, p_4 = \rho P^*, p_5 = \rho N^* - \varepsilon_0 - \varepsilon_1 T^* - 2\beta P^*, p_6 = -\varepsilon_1 P^*, p_7 = -vT^*, p_8 = 0, p_9 = -(vN^* + \gamma), p_\tau = -uP^*$$

As  $\lambda = i\omega$  is a root of equation (6.10), so:

$$\begin{aligned}
& ((i\omega)^3 + g_1(i\omega)^2 + g_2(i\omega) + g_3) \\
& + e^{-(i\omega)\tau}(h_1(i\omega)^2 + h_2(i\omega) \\
& + h_3) = 0
\end{aligned} \tag{6.11}$$

Separation of real and imaginary parts give:

$$\begin{aligned}
& g_3 - g_1\omega^2 \\
& + (h_3 - h_1\omega^2) \cos \omega\tau \\
& + h_2 \omega \sin \omega\tau = 0
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
& g_2\omega - \omega^3 + h_2\omega \cos \omega\tau \\
& - (h_3 - h_1\omega^2) \sin \omega\tau = 0
\end{aligned} \tag{6.13}$$

Elimination of  $\sin \omega\tau$  and  $\cos \omega\tau$  gives:

$$\begin{aligned}
& \omega^6 + (g_1^2 - h_1^2 - 2g_2)\omega^4 \\
& + (g_2^2 - h_2^2 + 2h_1h_3 \\
& - 2g_1g_3)\omega^2 + (g_3^2 - h_3^2) = 0
\end{aligned} \tag{6.14}$$

Let  $y = \omega^2$  and  $l = (g_1^2 - h_1^2 - 2g_2)$ ,  $m = (g_2^2 - h_2^2 + 2h_1h_3 - 2g_1g_3)$ ,  $n = (g_3^2 - h_3^2)$ , then equation (6.14) becomes:

$$y^3 + ly^2 + my + n = 0 \tag{6.15}$$

**Lemma 6.7.1** If  $n < 0$ , Equation (6.15) has at least one positive real root.

**Proof.** Let  $f(y) = y^3 + ly^2 + my + n$

Here  $f(0) = c < 0$ ,  $\lim_{y \rightarrow \infty} f(y) = \infty$

So,  $\exists y_0 \in (0, \infty)$  such that  $h(y_0) = 0$

Hence proved.

**Lemma 6.7.2** If  $n \geq 0$ , then necessary condition for equation (6.15) to have positive real roots is  $D = l^2 - 3m \geq 0$ .

**Proof.** Since  $f(y) = y^3 + ly^2 + my + n$ ,

$$\Rightarrow f'(y) = 3y^2 + 2ly + m$$

$$f'(y) = 0 \Rightarrow 3y^2 + 2ly + m = 0 \tag{6.16}$$

The roots of equation (6.16) can be expressed as:

$$y_{1,2} = \frac{-2l \mp \sqrt{4l^2 - 12m}}{6} = \frac{-l \mp \sqrt{D}}{3} \tag{6.17}$$

If  $D < 0$ , then equation (6.16) does not have any real roots. So, the function  $f(y)$  is monotone increasing function in  $y$ . It follows from  $f(0) = n \geq 0$  that equation (6.15) has no positive real roots.

**Lemma 6.7.3** If  $n \geq 0$ , then equation (6.15) has positive roots if and only if  $y_1 > 0$  and  $f(y_1) \leq 0$ , provided  $D \geq 0$  and  $y_1 = \frac{-l+\sqrt{D}}{3}$  is local minima of  $f(y)$ .

**Proof.** The sufficiency is obvious. Only necessity needs to be proved. Otherwise, assume that either  $y_1 \leq 0$  or  $y_1 > 0$  and  $f(y_1) > 0$ . If  $y_1 \leq 0$ , since  $f(y)$  is increasing for  $y \geq y_1$  and  $f(0) = n \geq 0$ , it follows that  $f(y)$  has no positive real zeros. If  $y_1 > 0$  and  $f(y_1) > 0$ , since  $y_2 = \frac{-l-\sqrt{D}}{3}$  is the local maxima value, it follows that  $f(y_1) \leq f(y_2)$ . Hence,  $f(0) = n \geq 0$ , As  $f(y)$  does not have positive real roots.

**Lemma 6.7.4** Suppose  $y_1$  is defined by equation (6.17).

(I) If  $n < 0$ , Equation (6.10) contains at least one positive real root.

(II) If  $n \geq 0$  and  $D = l^2 - 3m < 0$ , then equation (6.15) has no positive roots.

(III) If  $n \geq 0$ , then equation (6.15) has positive roots if and only if  $y_1 > 0$  and  $h(y_1) \leq 0$ .

**Proof.** Suppose that equation (6.15) has positive roots. Without loss of generality, assume that it has three positive roots, denoted by  $y_1, y_2, y_3$ . Then equation (6.14) has three positive roots, say  $\omega_1 = \sqrt{y_1}, \omega_2 = \sqrt{y_2}, \omega_3 = \sqrt{y_3}$ .

From equation (6.12) and (6.13), we have:

$$\sin \omega \tau = \left[ \frac{(g_1 \omega^2 - g_3) + (g_2 \omega - \omega^3)(h_3 - h_1 \omega^2)}{h_2^2 \omega^2 + (h_3 - h_1 \omega^2)^2} \right]$$

$$\Rightarrow \tau = \frac{1}{\omega} \left[ \sin^{-1} \left( \frac{(g_1 \omega^2 - g_3) + (g_2 \omega - \omega^3)(h_3 - h_1 \omega^2)}{h_2^2 \omega^2 + (h_3 - h_1 \omega^2)^2} \right) + 2(j-1)\pi \right]; j$$

$$= 1, 2, 3, \dots$$

Let 
$$\tau_k^{(j)} = \frac{1}{\omega_k} \left[ \sin^{-1} \left( \frac{(g_1 \omega_k^2 - g_3) + (g_2 \omega_k - \omega_k^3)(h_3 - h_1 \omega_k^2)}{h_2^2 \omega_k^2 + (h_3 - h_1 \omega_k^2)^2} \right) + 2(j-1)\pi \right]; k =$$

$$1, 2, 3; j = 0, 1, 2, \dots$$

Then  $\mp i\omega_k$  is a pair of purely imaginary roots of equation (6.14)

Where  $\tau = \tau_k^{(j)}, k = 1, 2, 3; j = 0, 1, 2, \dots, \lim_{j \rightarrow \infty} \tau_k^{(j)} = \infty, k = 1, 2, 3.$

Thus, we define:

$$\tau_0 = \tau_{k_0}^{(j_0)} = \min_{1 \leq k \leq 3, j \geq 1} [\tau_k^{(j)}], \quad (6.18)$$

$$\omega_0 = \omega_{k_0}, \quad y_0 = y_{k_0}$$

**Lemma 6.7.5** Let  $(g_1 + h_1) > 0, (g_3 + h_3) > 0, (g_1 + h_1)(g_2 + h_2) - (g_3 + h_3) > 0$ .

(I) If  $n \geq 0$  and  $D = l^2 - 3m < 0$ , then all the roots of equation (6.10) have negative real parts for all  $\tau \geq 0$ .

(II) If  $n < 0$  or  $n \geq 0, y_1 > 0$  and  $f(y_1) \leq 0$ , then all the roots of equation (6.10) have negative real parts for all  $\tau \in [0, \tau_0)$ .

**Proof.** When  $\tau = 0$ , equation (6.10) becomes

$$\lambda^3 + (g_1 + h_1)\lambda^2 + (g_2 + h_2)\lambda + (g_3 + h_3) = 0 \quad (6.19)$$

By Routh-Hurwitz's criteria, all roots of equation (6.10) have negative real parts if and only if  $(g_1 + h_1) > 0, (g_3 + h_3) > 0, (g_1 + h_1)(g_2 + h_2) - (g_3 + h_3) > 0$ .

If  $n \geq 0$  and  $D = l^2 - 3m < 0$ , Lemma 6.7.2 (II) shows that equation (6.10) has no roots with zero real part for all  $\tau \geq 0$ . When  $n < 0$  or  $n \geq 0, y_1 > 0$  and  $f(y_1) \leq 0$ , Lemma 6.7.2 (I) and (III) implies that when  $\tau \neq \tau_k^{(j)}, k = 1, 2, 3; j \geq 1$ , equation (6.10) has no roots with zero real part and  $\tau_0$  is the minimum value of  $\tau$  so that the equation (6.10) has purely imaginary roots. Applying Theorem 1.5.8, the conclusion of this Lemma is obtained.

**Lemma 6.7.6** Suppose  $y_0 = \omega_0^2$ . If  $\tau = \tau_0$ , Then  $\text{Sign} [\psi'(\tau_0)] = \text{Sign} [f'(y_0)]$

**Proof.** Let us assume that:

$$\lambda(\tau) = \psi(\tau) + i\omega(\tau) \quad (6.20)$$

be the roots of equation (6.10) satisfying:  $\psi(\tau_0) = 0, \omega(\tau_0) = \omega_0$

To guarantee that  $\mp \omega_0$  are simple purely imaginary roots of equation (6.10), with  $\tau = \tau_0$  and  $\lambda(\tau)$  satisfies transversality condition, assume that  $f'(y_0) \neq 0$ .

Putting  $\lambda(\tau)$  in equation (6.10) and differentiating w.r.t  $\tau$ , it follows that

$$\begin{aligned} \frac{d\lambda}{d\tau} [3\lambda^2 + 2g_1\lambda + g_2 + ((h_1\lambda^2 + h_2\lambda + h_3)(-\tau) + (2h_1\lambda + h_2))e^{-\lambda\tau}] \\ = \lambda(h_1\lambda^2 + h_2\lambda + h_3)e^{-\lambda\tau} \end{aligned}$$

$$\Rightarrow \left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2g_1\lambda + g_2)e^{\lambda\tau}}{\lambda(h_1\lambda^2 + h_2\lambda + h_3)} + \frac{(2h_1\lambda + h_2)}{\lambda(h_1\lambda^2 + h_2\lambda + h_3)} - \frac{\tau}{\lambda}$$

From equations (6.12) -(6.14):

$$\begin{aligned} \psi'(\tau_0) &= Re \left[ \frac{(3\lambda^2 + 2g_1\lambda + g_2)e^{\lambda\tau}}{\lambda(h_1\lambda^2 + h_2\lambda + h_3)} \right] + Re \left[ \frac{(2h_1\lambda + h_2)}{\lambda(h_1\lambda^2 + h_2\lambda + h_3)} \right] \\ &= \frac{[3\omega_0^6 + 2l\omega_0^4 + m\omega_0^2]}{[(h_3 - h_1\omega^2)^2 + (h_2\omega)^2]} \end{aligned}$$

Here  $[(h_3 - h_1\omega^2)^2 + (h_2\omega)^2] > 0$  and  $\omega_0 > 0$ .

It is concluded that  $\text{Sign} [\psi'(\tau_0)] = \text{Sign} [f'(y_0)]$ .

**Theorem 6.1** Let  $\tau_0$  and  $\omega_0$  be defined by equation (6.18) and  $y_0 = \omega_0^2$ . Let  $(g_1 + h_1) > 0, (g_3 + h_3) > 0, (g_1 + h_1)(g_2 + h_2) - (g_3 + h_3) > 0$ .

(I) If  $n \geq 0$  and  $D = l^2 - 3m < 0$ , then the interior equilibrium  $E_2(N^*, P^*, T^*)$  of the system (6.1)- (6.3) is absolutely stable (i.e. asymptotically stable for all  $\tau \geq 0$ ).

(II) If  $n < 0$  or  $n \geq 0, y_1 > 0$  and  $f(y_1) \leq 0$ , then the interior equilibrium  $E_2(N^*, P^*, T^*)$  of the system (6.1)- (6.3) is asymptotically stable for all  $\tau \in [0, \tau_0)$ .

(III) If the conditions of (II) are satisfied for  $\tau = \tau_0$  and  $f'(y_0) \neq 0$ , then the system exhibits Hopf bifurcation at  $E_2(N^*, P^*, T^*)$ .

## 6.8 Sensitivity Analysis

Sensitivity analysis helps to increase the understating of relationship between parameters and variables. It is an essential ingredient of model evaluation and quality assurance. It measures how much of each parameter contributes to the output variable. All the parameters considered in this paper are assumed to be constant. Thus, sensitivity analysis just entails finding the partial derivatives of solution with respect to each variable. For instance, if we take nutrient use efficiency parameter  $\rho$ , then partial derivatives of the solution  $(N, P, T)$  with respect to  $\rho$  give rise to following set of sensitivity equations:

$$\begin{aligned} \frac{dS_1}{dt} &= -(vT + \alpha)S_1 - uN(t - \tau)S_2 - vNS_3 \\ &\quad - uPS_1(t - \tau) \end{aligned} \tag{6.21}$$

$$\frac{dS_2}{dt} = \rho P S_1 + (\rho N - \varepsilon_o - \varepsilon_1 T - 2\beta P) S_2 \quad (6.22)$$

$$\frac{dS_3}{dt} = -v T S_1 - (v N + \gamma) S_3 \quad (6.23)$$

Where  $S_1 = \frac{\partial N}{\partial \rho}$ ,  $S_2 = \frac{\partial P}{\partial \rho}$ ,  $S_3 = \frac{\partial T}{\partial \rho}$

Then, this system of sensitivity equations (6.21) – (6.23) along with the original system of equations (6.1) – (6.3) is solved to estimate the sensitivity of the state variables ( $N, P, T$ ) to the parameter  $\rho$ . The similar procedure and argument holds for estimating the sensitivity of the state variables with respect to the other parameters.

## 6.9 Verification of Model with Existing Experimental Data

In this section, we try to verify the proposed model with already existing experimental data related to the growth of winter wheat (*Triticum aestivum* L) and rape (*Brassica napus*) under the effect of high concentrations of Arsenic in soil[183].

Wheat and rape are two widely grown crops in China. The soil arsenic (As) levels are also very high in these areas. To understand the effect of uptake of As on the growth of these two crops and the soil As bioavailability at different growth stages of wheat and rape, a pot experiment was conducted in the campus of Huazhong agriculture university, China. The results indicated that winter wheat was much more sensitive to As than rape. Crop yields were elevated at low concentrations of As addition (<60mg/kg) but reduced at high rates of As concentrations (80-100mg/Kg). The data related to this dynamic is given by Table 6.1 and Table 6.2.

**Table 6. 1**Effect of Arsenic on biomass and yield of rape:

Arsenic treatments (mg/kg)	Arsenic treatment (% age)	Biomass (g/pot)	% age decrease	Yield (g/pot)	% age decrease
0	0	25.5	0	16.4	0
20	20	23.2	9.0	14.8	9.7
40	40	22.1	13.3	14.1	14.0
60	60	21.0	17.6	13.4	18.2
80	80	20.3	21.4	13.1	20.1
100	100	19.3	24.3	12.1	26.2

**Table 6. 2**Effect of Arsenic on biomass and yield of wheat:

<b>Arsenic treatments (mg/kg)</b>	<b>Arsenic treatment (% age)</b>	<b>Biomass (g/pot)</b>	<b>% age decrease</b>	<b>Yield (g/pot)</b>	<b>% age decrease</b>
0	0	71.3	0	39.9	0
20	20	64.1	10.0	35.7	10.5
40	40	61.4	13.8	33.9	15.0
60	60	58.5	17.9	32.6	18.2
80	80	57.8	18.9	31.2	21.8
100	100	56.4	20.8	29.9	25.0

## 6.10 Numerical Example

The following set of parametric values is taken to represent graphically the dynamics depicted by the system of equations (6.1)- (6.3). The percentage increase in the values of damage rate  $\varepsilon_1$  of plant population density are specifically taken within the range of the values of toxic metal arsenic (As) taken in the above-mentioned experimental data [183].

$$K = 10, u = 1, \alpha = 1, \rho = 0.3, \varepsilon_0 = 0.2, \varepsilon_1 = 0.1, \beta = 0.1, Q = 1, v = 2, \gamma = 1$$

With initial conditions:  $N(0) = 0.5, P(0) = 1, T(0) = 1.5$

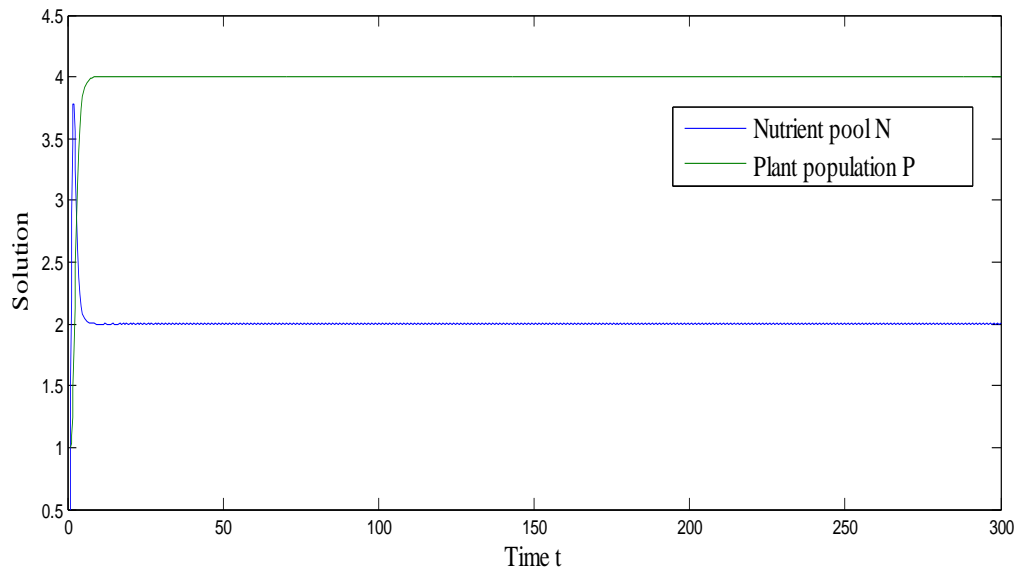
In the absence of toxic metal, the uniform equilibrium  $E_1(\bar{N} \neq 0, \bar{P} \neq 0, \bar{T} = 0)$  has the values:

$$\bar{N} = 2.0288, \bar{P} = 4.0086, \bar{T} = 0.$$

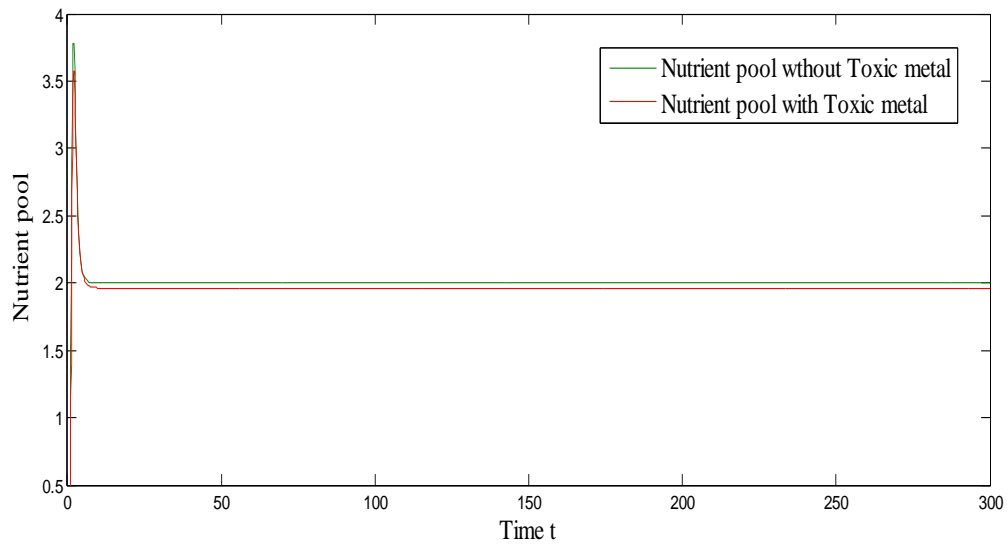
With the introduction of toxic metal, the interior equilibrium  $E_2(N^* \neq 0, P^* \neq 0, T^* \neq 0)$  gets the values:

$$N^* = 1.9634, P^* = 3.6873, T^* = 0.2030$$

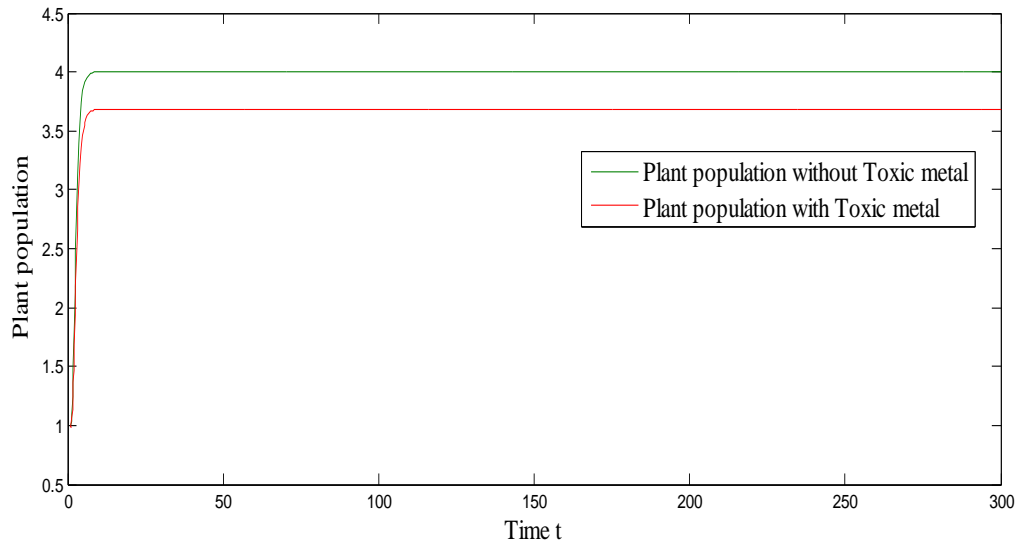




**Figure 6.1** The uniform equilibrium  $E_1(\bar{N}, \bar{P}, 0)$  is absolutely stable in the absence of toxic metal.



**Figure 6.2** Trajectories showing adverse effect of introduction of toxic metal on nutrient pool.

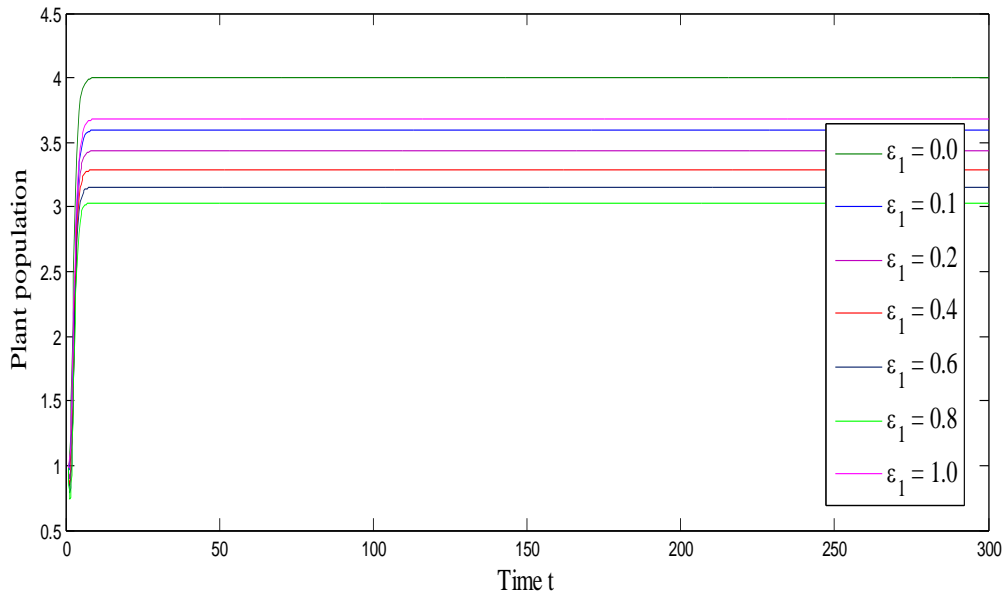


**Figure 6.3** Trajectories showing adverse effect of introduction of toxic metal on plant population.

The following Table 6.3 shows how the growth of plant population gets damaged with increasing rate of  $\varepsilon_1$ . The percentage increase in the values of  $\varepsilon_1$  is taken as per range of soil arsenic metal mentioned in experimental data.

**Table 6. 3** Effect of increasing rate of  $\varepsilon_1$  on plant population:

Toxic metal (uptake rate)	Toxic metal (%age)	Plant population	% age decrease
0.0	0	4.0086	0
0.1	10	3.6873	3.11
0.2	20	3.6001	10.1
0.4	40	3.4378	14.2
0.6	60	3.2901	17.9
0.8	80	3.1552	21.2
1.0	100	3.0313	24.3

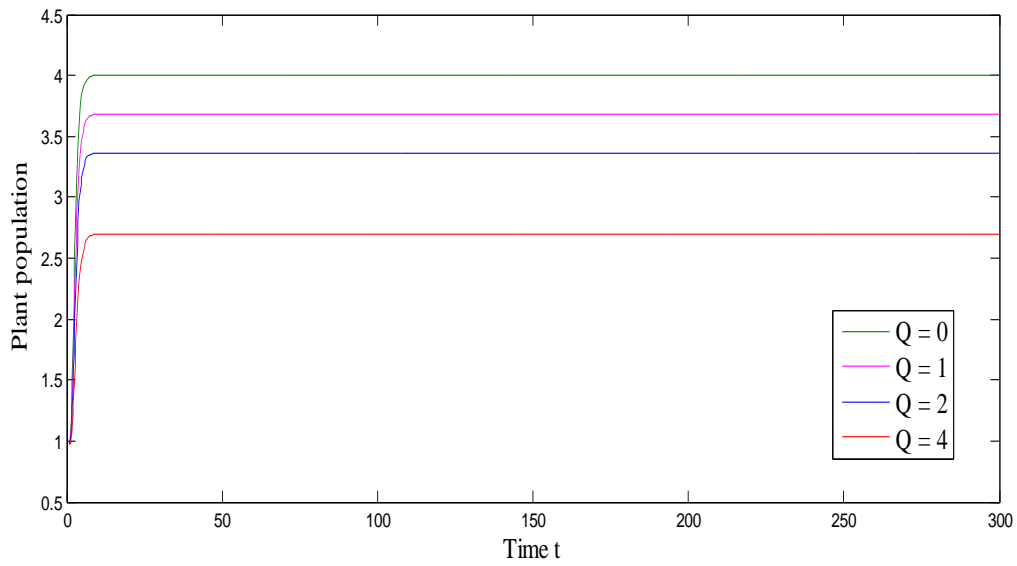


**Figure 6.4** Trajectories of plant population showing decrease in their values with increasing rate of  $\epsilon_1$ .

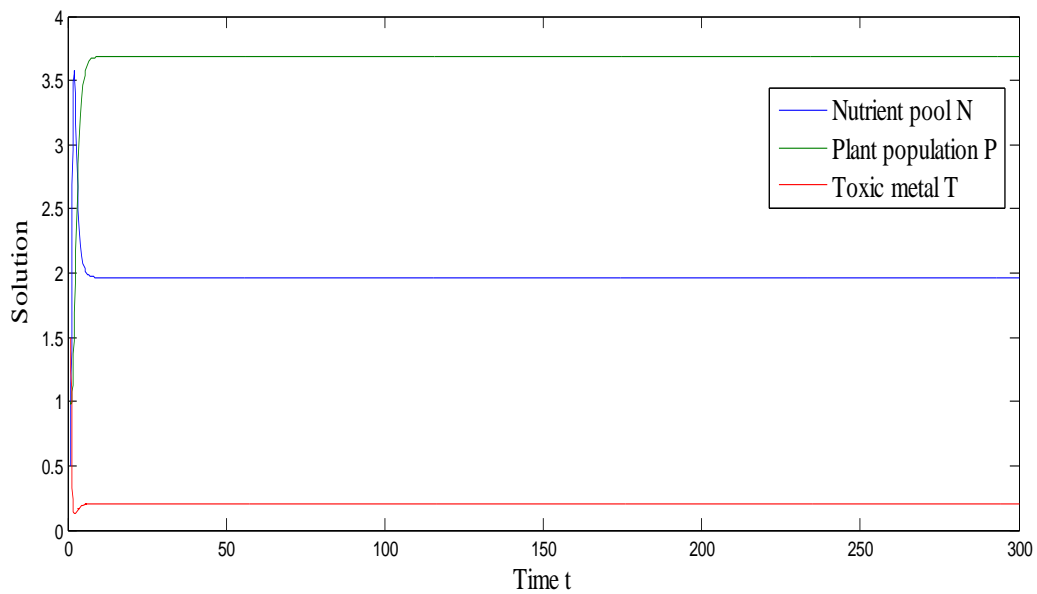
The following Table 6.4 shows how the plant population gets decreased with increase in input rate of toxic metal.

**Table 6. 4** Effect of increase in input rate of toxic metal Q on plant population:

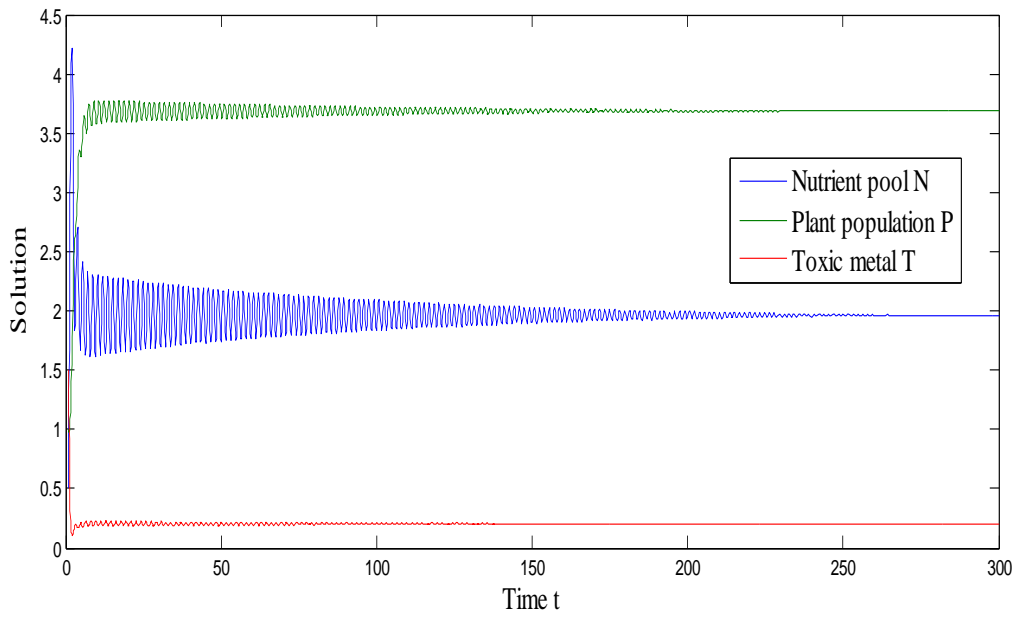
Input rate of toxic metal Q	Toxic metal (%age)	Plant population	% age decrease
0.0	0	4.0086	0
1.0	10	3.6873	3.11
2.0	20	3.3668	16.0
4.0	40	2.7016	32.6



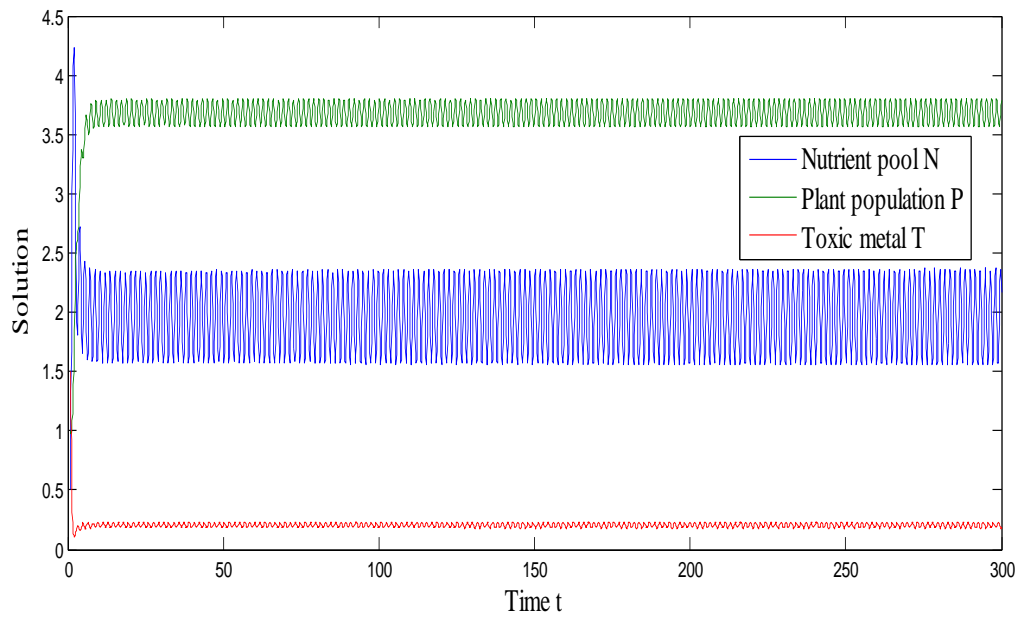
**Figure 6.5** Trajectories of plant population showing decrease in their values with increase in input rate of toxic metal  $Q$ .



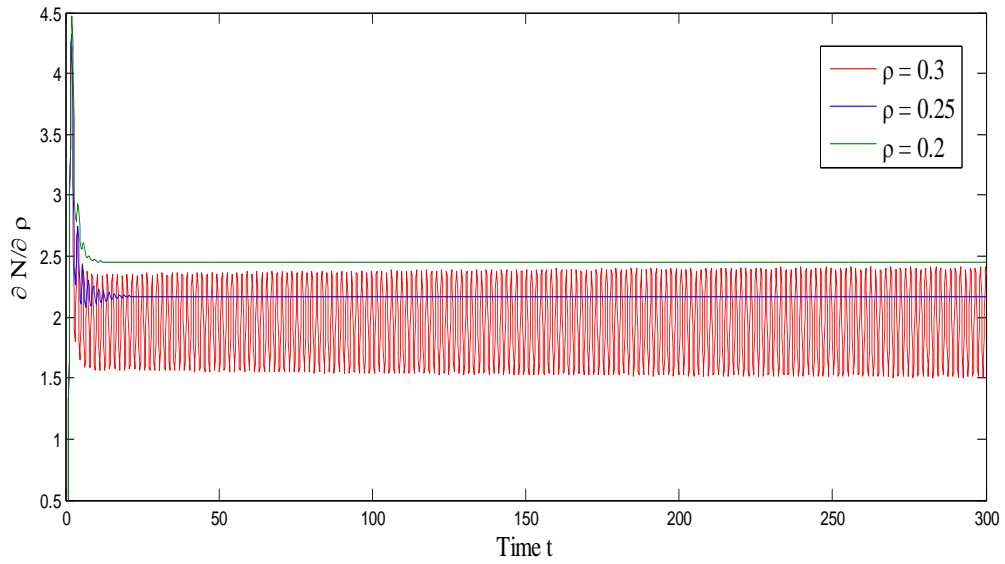
**Figure 6.6** The interior equilibrium  $E_2(N^*, P^*, T^*)$  is absolutely stable when there is no delay i.e.  $\tau = 0$ .



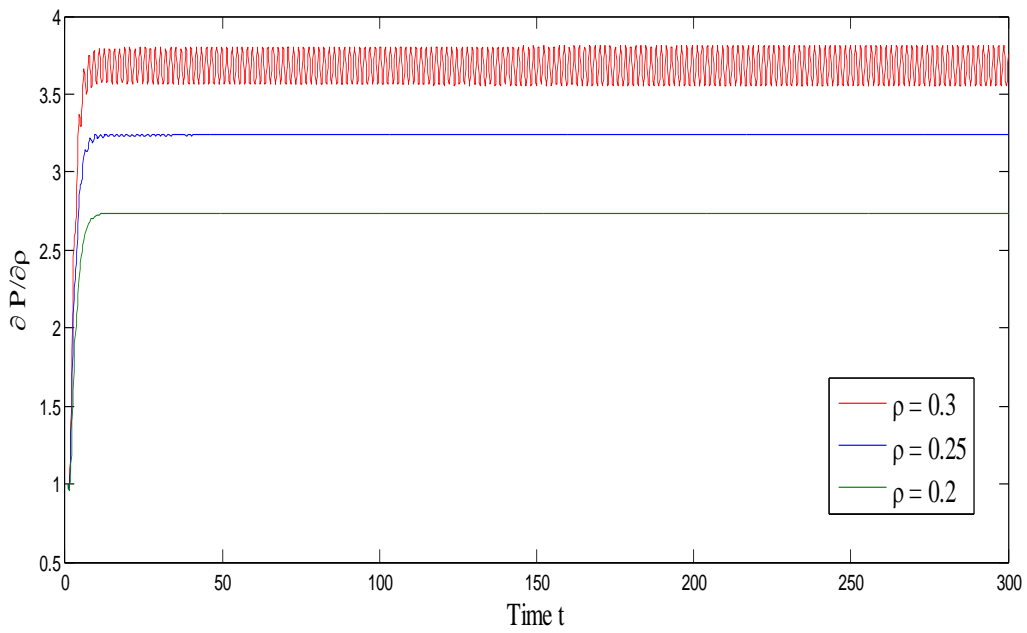
**Figure 6. 7**The interior equilibrium  $E_2(N^*, P^*, T^*)$  is asymptotically stable when delay i.e.  $\tau < 0.49$



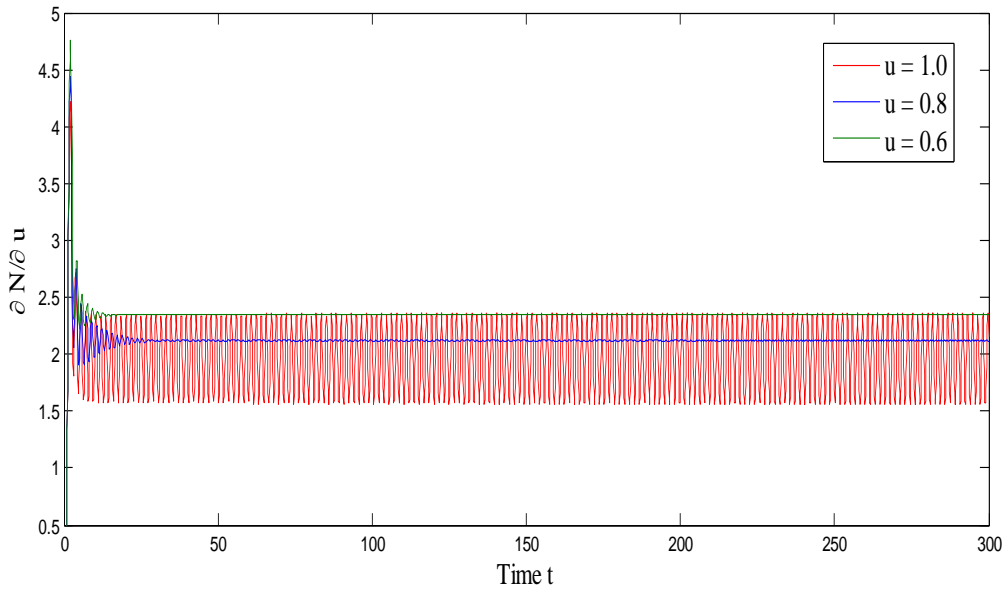
**Figure 6. 8**The interior equilibrium  $E_2(N^*, P^*, T^*)$  exhibits Hopf bifurcation when delay i.e.  $\tau \geq 0.49$



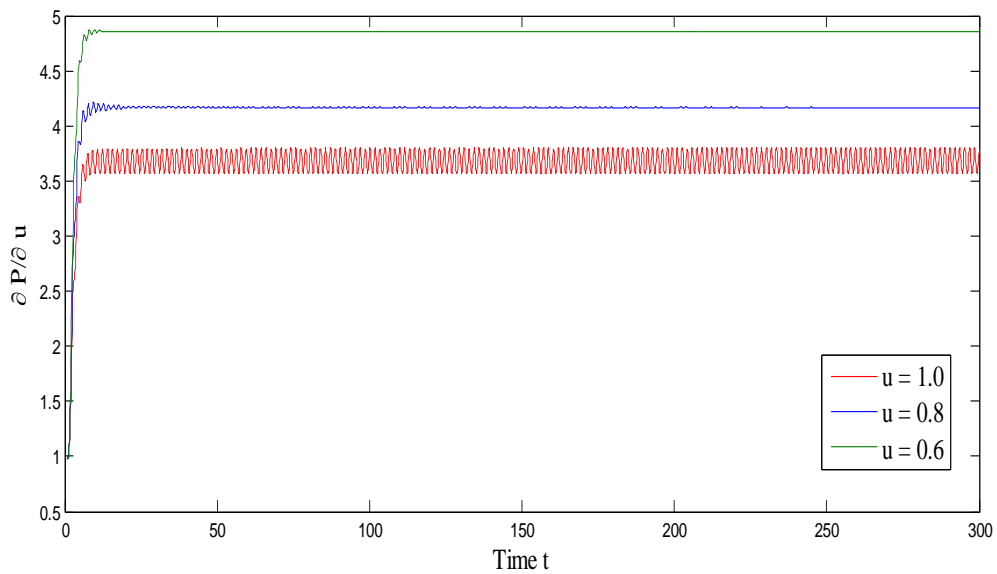
**Figure 6.9** Time series graph between partial changes in N (nutrient pool) for different values of the parameter  $\rho$  (nutrient use efficiency).



**Figure 6.10** Time series graph between partial changes in P (plant population) for different values of the parameter  $\rho$  (nutrient use efficiency).



**Figure 6.11** Time series graph between partial changes in N (nutrient pool) for different values of the parameter u (utilization coefficient).



**Figure 6.12** Time series graph between partial changes in P (plant population) for different values of the parameter u (utilization coefficient).

## 6.11 Conclusion

The role of delay on plant population under the effect of toxic metal is studied with the help of proposed model. The system of non-linear delay differential equations is used where the state variables considered are nutrient pool, plant population density and toxic metal. The boundedness of the system is proved using usual comparison theorem.

Positivity of the solutions shows that all the three variables considered being real in natural phenomenon always remain positive at all the times. Two feasible equilibriums: uniform equilibrium  $E_1$  and interior equilibrium  $E_2$  are calculated. The local stability of uniform equilibrium  $E_1$  is studied. Using Routh Hurwitz criteria, it is shown by Figure 6.1 that equilibrium  $E_1$  is absolutely stable in the absence of toxic metal. The experimental data considered in this paper in Table 6.1 and Table 6.2 showed that how the excessive amount of soil arsenic adversely effects the biomass and yield of wheat and rape. Using the above-mentioned set of parameters, the same ill effect of toxic metal on nutrient pool and plant population is shown by Figure 6.2 and Figure 6.3. The value of nutrient pool decreases from 2.0288 to 1.9634 and plant population decreases from 4.0086 to 3.6873 for the same set of parameters. Rate of the damage to plant population density due to increase in toxicity rate is calculated in Table 6.3. The percentage decrease in plant population from 0% (in absence of toxic metal) to 24.3% (when uptake rate is 1.0) calculated in Table 6.3 in this study, falls well within the range of decrease shown by experimental data for wheat and rape in Table 6.1 and Table 6.2. The same decrease in plant population is shown graphically by Figure 6.4. Table 6.4 and Figure 6.5 shows the adverse effect of increase in input rate of toxic metal on plant population density. The Hopf bifurcation analysis about the interior equilibrium  $E_2$  is done. Figure 6.6 shows that the interior equilibrium is absolutely stable when there is no delay i.e.  $\tau = 0$ . The equilibrium  $E_2$  is asymptotically stable when delay is less than the critical value i.e.  $\tau < 0.91$  as shown by Figure 6.7. The equilibrium  $E_2$  losses stability and shows Hopf bifurcation for values of delay that are more than critical value i.e.  $\tau \geq 0.91$ . This is shown by Figure 6.8. The sensitivity analysis helped to understand the role played by model parameters on the solution space. Figure 6.9 indicates that with decrease in the nutrient use efficiency (from 0.3 to 0.2), not only the nutrient pool tends towards stability, its value increases too. Figure 6.10 indicates how the decrease in nutrient use efficiency (from 0.3 to 0.2) leads to stability of the system and decrease in plant population density. Figure 6.11 shows that with decrease in the utilization of delayed nutrient not only the system tends to stability, the value of nutrient pool increases as well. The increase is more evident in plant population density with decrease in utilization of delayed nutrient as shown by Figure 6.12.



## **Chapter 7**

# **Modelling the Impact of Delay Considered in Utilization of Favourable Resources on Plant Population Density under the Effect of Acid and Toxic Metal**

### **7.1 Introduction**

A minimum level of pH value is essential for survival and growth of plants. But accumulation of acid over a long period of time lowers this pH level and leads to increased level of toxicity of metals. This lower value of pH makes the conditions unsuitable for survival of plants. Weathering of soil, rocks, drainage and mining introduces metal in plant soil dynamics. The solubility of metals is very high in acid solution and thereby increases toxicity of these metals which adversely affects the plant growth and uptake rate of nutrients. Nwachukwu and Agbede[190] did an experiment using a contaminated soil in a pot. It shown significant increase in plant metal uptake and bioaccumulation. In a study by Bolan et al.[191] seven organic amendments were investigated for their effects on the reduction of chromium in a mineral soil, low in organic matter content. Industrial waste and sewage sludge are major source of soil pollution that originate from mining industries, chemical industries, metal processing industries and the like. Brune and Dietz[60] concluded that these industrial waste and sludge include a variety of chemicals like heavy metals, phenolics, acids etc. Excessive concentration of these metals in soils can reduce plant growth and affect soil microorganisms or their activities. Faizan et al.[192] have shown that Cadmium is a highly toxic, metallic soil contaminant which adversely affects the plant growth especially at the early stage reducing the crop production. The soil-nutrient-plant interaction represents a good example of a relationship that operates at individual, population and ecosystem levels. Donald et al.[193] concluded in their book that nutrient influence individual plant growth, which has subsequent effect on population growth dynamics which in turn influence production of standing crop. It is evident from the experimental findings that the toxic chemicals such as heavy metals, pesticides and acids adversely effects the soil nutrient dynamics which in turn reduces the plant population

growth. Agricultural research almost completely depends on experimental and empirical works, combined with statistical analysis and a very few mathematical modelling analyses has been carried out in this direction. Mahmood et al.[194] conducted an experiment to show toxic effects of heavy metals on early growth and tolerance of cereal crops. Nitrogen dynamics in soil, its availability to the crop and the effects of nitrogen deficiency on crop performance were studied by Van Ittersum et al.[195]. Verma et al.[74] gave a mathematical model to show that the crops grown on contaminated soils leads to Phyto-toxicity. Lucia et al.[196] conducted three different experiments to determine the effects of three different heavy metals on plant growth, tissue accumulation and examine their uptake by sunflower in soil. Lu et al.[85] studied heavy metal concentrations in soil and agricultural products near an industrial district in Dongguan city for agriculture products like vegetables and bananas. Cu[197] has shown in his paper that cultivation of crops on contaminated sites may result in both growth inhibition and tissue accumulation that may cause possible risks to human health. It considered the transport and uptake of cadmium by vegetables in Lettuce. Misra and Kalra[168], [177] gave a model which concluded that under the effect of toxicant, both the nutrient concentration and structural dry weight gets adversely affected. Global stability of population growth with the help of non-linear delay differential equations was studied by Huang et al.[172]. Zhang et al.[123] gave a neural network model where the nature of the roots of a 5<sup>th</sup> degree exponential polynomial was discussed. Bocharov and Rihan[198] gave adjoint and direct methods for sensitivity analysis in numerical modelling in biosciences using delay differential equations. Rihan[144] performed the Sensitivity analysis for dynamic systems with time-lags using adjoint equations and direct methods when the parameters appearing in the model are not only constants but also variables of time. Banks et al.[176] presented theoretical foundations for traditional sensitivity and generalized sensitivity functions for a general class of nonlinear delay differential equations. They Included theoretical results for sensitivity with respect to the delays. Ingalls et al.[199] developed a parametric sensitivity analysis for periodic solutions of delay differential equations. Selvarangam et al.[200] complied the oscillation results for second order half liner neutral delay differential equations with maxima. Pramanik and Biswas [201] studied the solutions of some non-linear differential equations in connection to Bruck conjecture.

Although a lot of work has been done on plant growth under the effect of toxicants, but the use of delay differential equations is rare in this field. In the consideration of this

fact, a mathematical model is proposed to study the combined effect of acid and toxic metal on plant population considering delay in utilization of favourable resources available in soil and surrounding plant environment. It is assumed that due to presence of toxic metal and acidity in soil, nutrient pool and plant population density gets adversely affected. The stability of the interior equilibrium of the system gets disturbed by the introduction of delay parameter. For the critical value of delay parameter, Hopf bifurcation is observed. The sensitivity of model solutions for different values of model parameters is established using sensitivity analysis. The model is also verified by taking the uptake rate values of copper using the existing experimental data of a study conducted on the effect of heavy metals like copper, lead and zinc on growth of Lettuce in Vietnam Cu [93]. MATLAB code is used for simulation to support analytical results.

## 7.2 Mathematical Model

The following system of non-linear delay differential equations (8.1) -(8.5) governs the plant population growth having five state variables: Density of favourable resources  $R$  including soil and surrounding environment, Plant population density  $B$ , Concentration of nutrients pool  $N$  in the soil, Concentration of acid  $T$  in the soil and Concentration of toxic metal  $M$  in the soil .

$$\frac{dR}{dt} = \beta_1 NR - \beta_2 R - \alpha_1 BR(t - \tau) \quad (7.1)$$

$$\frac{dB}{dt} = \alpha_1 BR(t - \tau) - \alpha_2 B \quad (7.2)$$

$$\begin{aligned} \frac{dN}{dt} = N_0 - \gamma_1 N - \beta_1 NR - K\delta_2 TN - K\varepsilon_2 MN \\ + K\beta_2 R + K\alpha_2 B \end{aligned} \quad (7.3)$$

$$\frac{dT}{dt} = T_0 - \delta_1 T - \delta_2 TN \quad (7.4)$$

$$\frac{dM}{dt} = M_0 - \varepsilon_1 M - \varepsilon_2 MN \quad (7.5)$$

With the initial conditions:  $R(0) > 0, B(0) > 0, N(0) > 0, T(0) > 0, M(0) > 0$  for all  $t, R(t - \tau) = \text{constant}$  for  $t \in [-\tau, 0]$ .

The parameters are defined as:  $N_0$  = Constant nutrient input in soil,  $T_0$  = Input rate of acid,  $M_0$  = Input rate of metal,  $\delta_1$  = Natural decay rate of acid,  $\varepsilon_1$  = Natural decay rate of metal,  $\delta_2$  = Uptake rate of acid by plant,  $\varepsilon_2$  = Uptake rate of metal by plant,  $\alpha_2$  = Natural decay rate of plant population density,  $\beta_2$  = Natural decay rate of favourable resources,  $\gamma_1$  = Nutrient leaching rate,  $\alpha_1$  = Specific rate of utilization of favourable

resources by plant population density,  $\beta_1 =$  Rate of interaction of nutrient and favourable resources.  $K(0 < K < 1)$  is proportional amount of resource and biomass which is recycled back to nutrient pool after degradation.

### 7.3 Boundedness of Solutions

Boundedness means all the quantities considered in this model being real, their individual values as well as their total will always be finite and non-negative. The following lemma proves the boundedness of solutions of the model given by (7.1) - (7.5):

**Lemma 8.3.1** All the solutions of the model (7.1)- (7.5) lie in the region:  $\Omega = \left[ (R, B, N, T, M) \in R_+^5 : 0 \leq T \leq \frac{T_0}{\delta_1}, 0 \leq M \leq \frac{M_0}{\varepsilon_1}, 0 \leq \frac{N_0 + T_0 + M_0}{\varphi} \leq R + B + N + T + M, 0 \leq R + B + N \leq \frac{N_0}{\varphi_1} \right]$ , as  $t \rightarrow \infty$ , for all positive initial values  $\{R(0), B(0), N(0), T(0), M(0), R(t - \tau) = \text{Constant for all } t \in [-\tau, 0]\} \in \Omega \subset R_+^5$ , where

$$\varphi_1 = \min. \left( (1 - K)\alpha_2, (1 - K)\beta_2, \gamma_1 \right) \text{ and } \varphi = \max \left( \delta_1 + \varepsilon_1 + \frac{(1+K)\delta_2 T_0}{\delta_1} + \frac{(1+K)\varepsilon_2 M_0}{\varepsilon_1}, \alpha_2, \beta_2, \delta_1, \varepsilon_2 \right).$$

**Proof.** From equation (7.4):  $\frac{dT}{dt} \leq T_0 - \delta_1 T$

Applying the comparison theorem, we get:

$$\text{As } t \rightarrow \infty: T \leq \frac{T_0}{\delta_1}$$

From equation (7.5):  $\frac{dM}{dt} \leq M_0 - \varepsilon_1 M$

Applying the comparison theorem, we get:

$$\text{As } t \rightarrow \infty: M \leq \frac{M_0}{\varepsilon_1}$$

$$\text{Let } W(t) = R(t) + B(t) + N(t)$$

$$\frac{dW}{dt} = \frac{d(R + B + N)}{dt}$$

$$= N_0 - \gamma_1 N - K\delta_2 TN - K\varepsilon_2 MN - (1 - K)\alpha_2 B - (1 - K)\beta_2 R$$

$$\text{Let } \varphi_1 = \min. \left( (1 - K)\alpha_2, (1 - K)\beta_2, \gamma_1 \right),$$

Then 
$$\frac{dW}{dt} \leq N_0 - \varphi_1 W$$

Applying the comparison theorem, we get:

$$\text{As } t \rightarrow \infty : W \leq \frac{N_0}{\varphi_1}$$

$$R + B + N \leq \frac{N_0}{\varphi_1}$$

Again, let  $W_1(t) = W(t) + T(t) + M(t)$

$$\frac{dW_1}{dt} = (N_0 + T_0 + M_0) - \delta_1 T - \varepsilon_1 M - (1 + K)\delta_2 TN - (1 + K)\varepsilon_2 MN - \varphi_1 W$$

$$\text{Let } \varphi = \max\left(\delta_1 + \varepsilon_1 + \frac{(1+K)\delta_2 T_0}{\delta_1} + \frac{(1+K)\varepsilon_2 M_0}{\varepsilon_1}, \alpha_2, \beta_2, \delta_1, \varepsilon_2\right),$$

Then 
$$\frac{dW_1}{dt} \geq (N_0 + T_0 + M_0) - \varphi W_1$$

Applying the comparison theorem, we get:

$$\text{As } t \rightarrow \infty : W_1 \geq \frac{(N_0 + T_0 + M_0)}{\varphi}$$

Hence 
$$R + B + N + T + M \geq \frac{(N_0 + T_0 + M_0)}{\varphi}$$

## 7.4 Positivity of Solutions

For system to sustain, solution space must be positive. For positive solutions, we need to show that all solution  $(R(t), B(t), N(t), T(t), M(t))$  of the model governed by Equations. (7.1)– (7.5) stay positive for all time  $t > 0$ , where initial condition is  $R(0) > 0, B(0) > 0, N(0) > 0, T(0) > 0, M(0) > 0$  for all  $t$  and  $R(t - \tau) = \text{constant}$  for  $t \in [-\tau, 0]$ .

From equation (7.4): 
$$\begin{aligned} \frac{dT}{dt} &\geq -(\delta_1 + \delta_2 N)T \\ \Rightarrow \frac{dT}{dt} &\geq -\left(\frac{\delta_1 \varphi_1 + \delta_2 N_0}{\varphi_1}\right)T \\ \Rightarrow T &\geq c_1 e^{-\left(\frac{\delta_1 \varphi_1 + \delta_2 N_0}{\varphi_1}\right)t} \\ \Rightarrow T &> 0 \text{ for all } t. \end{aligned}$$

From equation (7.5): 
$$\frac{dM}{dt} \geq -(\varepsilon_1 + \varepsilon_2 N)M$$

$$\begin{aligned} \Rightarrow \frac{dM}{dt} &\geq -\left(\frac{\varepsilon_1\varphi_1 + \varepsilon_2N_0}{\varphi_1}\right)M \\ \Rightarrow M &\geq c_2 e^{-\left(\frac{\varepsilon_1\varphi_1 + \varepsilon_2N_0}{\varphi_1}\right)t} \\ \Rightarrow M &> 0 \text{ for all } t. \end{aligned}$$

Based on the same logic,  $R > 0, B > 0$  and  $N > 0$ .

## 7.5 Equilibrium Points of the Model

The dynamical system governed by equations (7.1)- (7.5) has two non-negative equilibrium points  $E_2(i = 1,2)$  out of which  $E_1$  is a uniform equilibrium points and  $E_2$  is the interior equilibrium point. At all points of equilibrium:  $R(t - \tau) = R(t)$

(I) The first uniform equilibrium point  $E_1(\bar{R} \neq 0, \bar{B} \neq 0, \bar{N} \neq 0, \bar{T} = 0, \bar{M} = 0)$

$$\bar{R} = \frac{\alpha_2}{\alpha_1}, \bar{B} = \frac{\beta_1\bar{N} - \beta_2}{\alpha_1} \text{ if } \beta_1N^* > \beta_2$$

$$\text{and } \bar{N} = \frac{N_0 + K\beta_2\bar{R} + K\alpha_2\bar{B}}{\gamma_1 + \beta_1\bar{R}}$$

(II) The second interior equilibrium point  $E_2(R^* \neq 0, B^* \neq 0, N^* \neq 0, T^* \neq 0, M^* \neq 0)$

$$T^* = \frac{T_0}{\delta_1 + \delta_2N^*}, M^* = \frac{M_0}{\varepsilon_1 + \varepsilon_2N^*},$$

$$R^* = \frac{\alpha_2}{\alpha_1}, B^* = \frac{\beta_1N^* - \beta_2}{\alpha_1} \text{ if } \beta_1N^* > \beta_2,$$

$$N^* = \frac{N_0 + K\beta_2R^* + K\alpha_2B^*}{\gamma_1 + \beta_1R^* + K\delta_2T^* + K\varepsilon_2M^*}$$

$$\text{such that } N^* = \frac{-h_2 + \sqrt{h_2^2 - 4h_1h_3}}{2h_1} > 0$$

Where  $h_1 = \gamma_1\alpha_1\delta_2 + (1 - K)\alpha_2\beta_1\delta_2$ ,

$$h_2 = \gamma_1\alpha_1\delta_2 + K\delta_2T_0\alpha_1 + K\varepsilon_2M_0 - N_0\delta_2 + (1 - K)\alpha_2\beta_1\delta_1, h_3 = -N_0\delta_1\alpha_1$$

## 7.6 Study of Uniform Equilibrium $E_1$ and Local Stability

In the absence of acid and toxic metal ( $\bar{T} = 0, \bar{M} = 0$ ), there is no delay ( $\tau = 0$ ). The system of equations governing the dynamics about equilibrium  $E_1(\bar{R} \neq 0, \bar{B} \neq 0, \bar{N} \neq 0, \bar{T} = 0, \bar{M} = 0)$  becomes:

$$\frac{d\bar{R}}{dt} = \beta_1 \bar{N} \bar{R} - \beta_2 \bar{R} - \alpha_1 \bar{B} \bar{R}(t) \quad (7.6)$$

$$\frac{d\bar{B}}{dt} = \alpha_1 \bar{B} \bar{R}(t) - \alpha_2 \bar{B} \quad (7.7)$$

$$\frac{d\bar{N}}{dt} = N_0 - \gamma_1 \bar{N} - \beta_1 \bar{N} \bar{R} + K\beta_2 \bar{R} + K\alpha_2 \bar{B} \quad (7.8)$$

The characteristic equation associated with variational matrix about equilibrium  $E_1$  is given by:

$$\lambda^3 + \xi_1 \lambda^2 + \xi_2 \lambda + \xi_3 = 0 \quad (7.9)$$

Where  $\xi_1 = \beta_2 + \alpha_2 + \gamma_1 + \alpha_1 \bar{B} + \beta_1 \bar{R} - \beta_1 \bar{N} - \alpha_1 \bar{R}$

$$\begin{aligned} \xi_2 = & (\beta_1 \bar{N} - \beta_2 - \alpha_1 \bar{B})(\alpha_1 \bar{R} - \alpha_2) - (\alpha_1 \bar{R} - \alpha_2)(\gamma_1 + \beta_1 \bar{R}) \\ & - (\gamma_1 + \beta_1 \bar{R})(\beta_1 \bar{N} - \beta_2 - \alpha_1 \bar{B}) - \beta_1 \bar{R}(\beta_1 \bar{N} - \beta_2 - \alpha_1 \bar{B}) - \alpha_1^2 \bar{B} \bar{R} \end{aligned}$$

$$\begin{aligned} \xi_3 = & (\beta_1 \bar{N} - \beta_2 - \alpha_1 \bar{B})(\alpha_1 \bar{R} - \alpha_2)(\gamma_1 + \beta_1 \bar{R}) + \alpha_1^2 \bar{B} \bar{R}(\gamma_1 + \beta_1 \bar{R}) \\ & + \beta_1 \bar{R}(\alpha_1 \bar{R} - \alpha_2)(K\beta_2 - \beta_1 \bar{N}) \end{aligned}$$

By Routh-Hurwitz's criteria, the real parts of roots of the equation (8.9) will be negative iff  $\xi_1 > 0, \xi_2 > 0, \xi_3 > 0$  and  $\xi_1 \xi_2 - \xi_3 > 0$ .

Hence, the uniform equilibrium  $E_1$  will always be locally stable, provided:

$$\xi_1 > 0, \xi_2 > 0, \xi_3 > 0 \text{ and } \xi_1 \xi_2 - \xi_3 > 0.$$

## 7.7 Study of Interior Equilibrium $E_2$ and Local Hopf-Bifurcation

In the presence of acid and toxic metal ( $T^* \neq 0, M^* \neq 0$ ), it is assumed that there will be delay ( $\tau \neq 0$ ). The system of equations governing the dynamics about equilibrium  $E_2(R^* \neq 0, B^* \neq 0, N^* \neq 0, T^* \neq 0, M^* \neq 0)$  becomes:

$$\frac{dR^*}{dt} = \beta_1 N^* R^* - \beta_2 R^* - \alpha_1 B^* R^*(t - \tau) \quad (7.10)$$

$$\frac{dB^*}{dt} = \alpha_1 B^* R^*(t - \tau) - \alpha_2 B^* \quad (7.11)$$

$$\begin{aligned} \frac{dN^*}{dt} = & N_0 - \gamma_1 N^* - \beta_1 N^* R^* - K\delta_2 T^* N^* \\ & - K\varepsilon_2 M^* N^* + K\beta_2 R^* \\ & + K\alpha_2 B^* \end{aligned} \quad (7.12)$$

$$\frac{dT^*}{dt} = T_0 - \delta_1 T^* - \delta_2 T^* N^* \quad (7.13)$$

$$\frac{dM^*}{dt} = M_0 - \varepsilon_1 M^* - \varepsilon_2 M^* N^* \quad (7.14)$$

The exponential characteristic equation about equilibrium  $E_2$  is given by:

$$\begin{aligned} \mu^5 + \zeta_1 \mu^4 + \zeta_2 \mu^3 + \zeta_3 \mu^2 + \zeta_4 \mu + \zeta_5 \\ + (\eta_1 \mu^4 + \eta_2 \mu^3 + \eta_3 \mu^2 + \eta_4 \mu \\ + \eta_5) e^{-\mu \tau} = 0 \end{aligned} \quad (7.15)$$

Here  $\zeta_1 = -(L_1 + L_7 + L_{13} + L_{19} + L_{25}),$

$$\begin{aligned} \zeta_2 = (L_{15}L_{23} + L_1L_7 + L_{13}L_7 + L_{13}L_{19} + L_{19}L_{25} + L_1L_{25} + L_7L_{19} + L_{25}L_7 + \\ L_{13}L_{25} + L_{11}L_{19} - L_{11}\beta_1R^*, \end{aligned}$$

$$\begin{aligned} \zeta_3 = -(L_1L_{15}L_{23} + L_7L_{15}L_{23} + L_{15}L_{19}L_{23} + L_1L_7L_{13} + L_7L_{13}L_{19} + L_{13}L_{19}L_{25} + \\ L_{11}L_{19}L_{25} + L_{11}L_7L_{25} + L_{11}L_{13}L_{19} + L_7L_{19}L_{25} + L_{11}L_{13}L_{25} + L_7L_{13}L_{25} + L_{14}L_{18} + \beta_1 \\ R^*L_6L_{12} - \beta_1R^*L_7L_{11} - \beta_1R^*L_{11}L_{19} - \beta_1R^*L_{11}L_{25}, \end{aligned}$$

$$\begin{aligned} \zeta_4 = (L_1L_7L_{15}L_{23} + L_7L_{15}L_{19}L_{23} + L_1L_{15}L_{19}L_{23} + L_1L_7L_{13}L_{19} + L_7L_{13}L_{19}L_{25} \\ + L_1L_{13}L_{19}L_{25} + L_1L_7L_{19}L_{25} + L_1L_7L_{13}L_{25} + L_1L_{14}L_{18} + L_7L_{14}L_{18} \\ + \beta_1R^*L_6L_{12}L_{19} + \beta_1R^*L_6L_{12}L_{25} - \beta_1R^*L_7L_{11}L_{19} - \beta_1R^*L_{11}L_{19}L_{25} \\ - \beta_1R^*L_7L_{11}L_{25}) \end{aligned}$$

$$\zeta_5 = -(L_1L_7L_{15}L_{19}L_{23} + L_1L_7L_{13}L_{19}L_{25} + L_1L_7L_{14}L_{18} - \beta_1R^*L_7L_{11}L_{19}L_{25})$$

$$\begin{aligned} \eta_1 = \alpha_1B^*, \eta_2 = \alpha_1B^*(L_7 + L_{13} + L_{19} + L_{25}), \eta_3 \\ = \alpha_1B^*(L_{15}L_{23} + L_7L_{13} + 2L_{19}L_{25} + L_7L_{25} + L_{13}L_{19} + L_{13}L_{25}), \eta_4 \\ = \alpha_1B^*(L_{12}L_{19} + L_{12}L_{25} - L_{14}L_{18} - L_7L_{15}L_{23} - L_{15}L_{19}L_{23} - L_7L_{13}L_{19} \\ - L_{13}L_{19}L_{25} - L_7L_{13}L_{25}), \eta_5 \\ = \alpha_1B^*(\beta_1R^*L_{12}L_{19}L_{25} - L_7L_{15}L_{19}L_{23} - L_7L_{13}L_{19}L_{25} - L_7L_{14}L_{18}) \end{aligned}$$

Where  $L_1 = -(\beta_2 - \beta_1N^*), L_2 = 0, L_3 = \beta_1R^*, L_4 = 0, L_5 = 0, L_6 = \alpha_1B^*, L_7 = -\alpha_2, L_8 = 0, L_9 = 0, L_{10} = 0, L_{11} = (\beta_2K - \beta_1N^*), L_{12} = \alpha_2K, L_{13} = -(\gamma_1 + \beta_1R^* + \delta_2KT^* + \varepsilon_2KM^*), L_{14} = -\delta_2KN^*, L_{15} = -\varepsilon_2KN^*, L_{16} = 0, L_{17} = 0, L_{18} = -\delta_2T^*, L_{19} = -(\delta_1 + \delta_2N^*), L_{20} = 0, L_{21} = 0, L_{22} = 0, L_{23} = -\varepsilon_2KM^*, L_{24} = 0, L_{25} = -(\varepsilon_1 + \varepsilon_2N^*).$

As  $\mu = i\omega$  is a root of equation (7.15), So



$$(i\omega^5 + \zeta_1\omega^4 - i\zeta_2\omega^3 - \zeta_3\omega^2 + i\zeta_4\omega + \zeta_5) \\ + (\eta_1\omega^4 - i\eta_2\omega^3 - \eta_3\omega^2 + i\eta_4\omega + \eta_5)(\cos \omega\tau - i\sin \omega\tau) = 0$$

Separating real and imaginary parts:

$$(\omega^5 - \zeta_2\omega^3 + \zeta_4\omega) + (\eta_4\omega - \eta_2\omega^3) \cos \omega\tau \\ - (\eta_1\omega^4 - \eta_3\omega^2 + \eta_5) \sin \omega\tau \quad (7.16) \\ = 0$$

$$(\zeta_1\omega^4 - \zeta_3\omega^2 + \zeta_5) \\ + (\eta_1\omega^4 - \eta_3\omega^2 + \eta_5) \cos \omega\tau \quad (7.17) \\ + (\eta_4\omega - \eta_2\omega^3) \sin \omega\tau = 0$$

Squaring and adding equation (7.16) and (7.17), we get:

$$\omega^{10} + a\omega^8 + b\omega^6 + c\omega^4 + d\omega^2 + r = 0 \quad (7.18)$$

Where  $a = (\zeta_1^2 - \eta_1^2 - 2\zeta_2)$ ,  $b = (\zeta_2^2 - \eta_2^2 - 2\zeta_4 - 2\zeta_1\zeta_3 + 2\eta_1\eta_3)$ ,

$$c = (\zeta_3^2 - \eta_3^2 - 2\zeta_2\zeta_4 + 2\eta_2\eta_4 - 2\eta_1\eta_5),$$

$$d = (\zeta_4^2 - \eta_4^2 - 2\eta_3\eta_5 + 2\eta_3\eta_5), r = (\zeta_5^2 - \eta_5^2)$$

Let  $\omega^2 = \chi$ , then, equation (7.18) becomes:

$$\chi^5 + \alpha\chi^4 + \beta\chi^3 + \gamma\chi^2 + \delta\chi + \varepsilon = 0 \quad (7.19)$$

**Lemma 7.7.1** If  $\varepsilon < 0$ , Equation (7.19) has contains at least one positive real root.

**Proof.** Let  $f(\chi) = \chi^5 + \alpha\chi^4 + \beta\chi^3 + \gamma\chi^2 + \delta\chi + \varepsilon$

$$\text{Here } f(0) = \varepsilon < 0, \lim_{\chi \rightarrow \infty} f(\chi) = \infty$$

$$\text{So, } \exists \chi_0 \in (0, \infty) \text{ such that } f(\chi_0) = 0.$$

Proof completed.

$$\text{Also } f'(\chi) = 5\chi^4 + 4\alpha\chi^3 + 3\beta\chi^2 + 2\gamma\chi + \delta$$

$$f'(\chi) = 0 \quad (7.20)$$

$$\Rightarrow 5\chi^4 + 4\alpha\chi^3 + 3\beta\chi^2 + 2\gamma\chi \\ + \delta = 0$$

Which becomes:

$$u^4 + au^2 + bu + c = 0 \quad (7.21)$$

Where  $u = \chi + \frac{\alpha}{5}$ ,  $a = \frac{3\beta}{5} - \frac{6\alpha^2}{25}$ ,  $b = \frac{2\gamma}{5} + \frac{6\alpha\beta}{25} + \frac{8\alpha^3}{125}$ ,  $c = \frac{\delta}{5} - \frac{2\alpha\gamma}{25} + \frac{3\alpha^2\beta}{125} - \frac{3\alpha^4}{625}$

If  $b = 0$ , then, four roots of equation (7.21) come out to be:

$$u_1 = \sqrt{\frac{-a + \sqrt{H}}{2}}, u_2 = -\sqrt{\frac{-a + \sqrt{H}}{2}}, u_3 = \sqrt{\frac{-a - \sqrt{H}}{2}}, u_4 = -\sqrt{\frac{-a - \sqrt{H}}{2}}$$

Thus  $\chi_i = u_i - \frac{\alpha}{5}$ ,  $i = 1, 2, 3, 4$  are the roots of equation (8.19) where:

$$H = a^2 - 4c$$

**Lemma 7.7.2** Suppose  $\varepsilon \geq 0$  and  $b = 0$ .

(I) If  $H < 0$ , then equation (7.19) has no positive real roots.

(II) If  $H \geq 0$ ,  $a \geq 0$ ,  $c \geq 0$ , then equation (7.19) has no positive real roots.

(III) If (I) and (II) are not satisfied, then equation (7.19) has positive real roots iff  $\exists$  at least one  $\chi^* \in (\chi_1, \chi_2, \chi_3, \chi_4)$  such that  $\chi^* > 0$  and  $f(\chi^*) \leq 0$ .

**Proof.** (I) If  $H < 0$ , then equation (7.19) has no positive real roots. Since  $\lim_{\chi \rightarrow \infty} f(\chi) = \infty$ , we have  $f'(\chi) > 0$  for  $\chi \in R$ . Hence  $f(0) = \varepsilon \geq 0$  implies  $f(\chi)$  has no zero in  $(0, \infty)$ .

(II) Condition  $H \geq 0$ ,  $a \geq 0$ ,  $c \geq 0$  imply that  $f'(\chi)$  has no zero in  $(-\infty, \infty)$ . It is similar to (I) that  $f(\chi)$  has no zero in  $(0, \infty)$ .

(III) The sufficiency is obvious. We need only to prove the necessity. If  $H \geq 0$ , we know that equation (7.21) has only four roots  $u_1, u_2, u_3$  and  $u_4$ , that is equation (7.20) has only four roots  $\chi_1, \chi_2, \chi_3$  and  $\chi_4$  at least  $\chi_1$  is a real root. Without loss of generality, we assume that  $\chi_1, \chi_2, \chi_3$  and  $\chi_4$  are all real. This implies that  $f(\chi)$  has at most four stationary points  $\chi_1, \chi_2, \chi_3$  and  $\chi_4$ . If it is not true, then we have that either  $\chi_1 \leq 0$  or  $\chi_1 > 0$  and  $\min[f(\chi_i): \chi_i > 0, i = 1, 2, 3, 4] > 0$ . If  $\chi_1 \leq 0$ , then  $f'(\chi)$  has no zero in  $(0, \infty)$ . Since  $f(0) = \varepsilon \geq 0$  is the strict minimum of  $f(\chi)$  for  $\chi \geq 0$  which implies  $f(\chi) > 0$  in  $(0, \infty)$ . If  $\chi_1 > 0$  and  $\min[f(\chi_i): \chi_i > 0, i = 1, 2, 3, 4] > 0$ , since  $f(\chi)$  is a derivable function and  $\lim_{\chi \rightarrow \infty} f(\chi) = \infty$ , then we have  $\min_{\chi > 0} f(\chi) = \min[f(\chi_i): \chi_i > 0, i = 1, 2, 3, 4] > 0$ . The necessity is proved.

This completes the proof.

Next, we assume that  $b \neq 0$ . Consider the resolvent of equation (7.21)

$$b^2 - 4(v - a) \left( \frac{v^2}{4} - c \right) = 0$$

$$\Rightarrow v^3 - av^2 - 4cr + 4ac - b^2 = 0 \quad (7.22)$$

By Cardan formula, equation (7.22) has the following three roots:

$$v_1 = \left( -\frac{b_1}{2} + \sqrt{H_1} \right)^{1/3} + \left( -\frac{b_1}{2} - \sqrt{H_1} \right)^{1/3} + \frac{a}{3},$$

$$v_2 = \sigma \left( -\frac{b_1}{2} + \sqrt{H_1} \right)^{1/3} + \sigma^2 \left( -\frac{b_1}{2} - \sqrt{H_1} \right)^{1/3} + \frac{a}{3},$$

$$v_3 = \sigma^2 \left( -\frac{b_1}{2} + \sqrt{H_1} \right)^{1/3} + \sigma \left( -\frac{b_1}{2} - \sqrt{H_1} \right)^{1/3} + \frac{a}{3}$$

Where  $e_1 = -\frac{a^2}{3} - 4c$ ,  $b_1 = -\frac{2a^3}{27} + \frac{8ac}{3} - b^2$ ,  $H_1 = \frac{e_1^3}{27} + \frac{b_1^2}{4}$ ,  $\sigma = \frac{1+\sqrt{3}i}{2}$

Let  $v_* = v_1 \neq a$ , then equation (8.22) becomes:

$$\begin{aligned} u^4 + v_*u^2 + \frac{v_*^2}{4} & \\ & - \left[ (v_* - a)u^2 - bu + \frac{v_*^2}{4} \right. \\ & \left. - c \right] = 0 \end{aligned} \quad (7.23)$$

For the above equation (7.22), (7.20) implies that the formula in the square brackets is a perfect square.

If  $v_* > a$ , then equation (7.22) becomes:

$$\left( u^2 + \frac{v_*}{2} \right)^2 - \left( \sqrt{v_* - a}u - \frac{b}{2\sqrt{v_* - a}} \right)^2 = 0$$

After factorization, we get:

$$u^2 + \sqrt{v_* - a}u - \frac{b}{2\sqrt{v_* - a}} + \frac{v_*}{2} \text{ and } u^2 - \sqrt{v_* - a}u - \frac{b}{2\sqrt{v_* - a}} + \frac{v_*}{2}$$

So, four roots of the equation (21) are:

$$u_1 = \frac{-\sqrt{v_* - a} + \sqrt{H_2}}{2}, \quad u_2 = \frac{-\sqrt{v_* - a} - \sqrt{H_2}}{2}, \quad u_3 = \frac{-\sqrt{v_* - a} + \sqrt{H_3}}{2}, \quad u_4 = \frac{-\sqrt{v_* - a} - \sqrt{H_3}}{2}$$

Where  $H_2 = -v_* - a + \frac{b}{2\sqrt{v_* - a}}$  and  $H_3 = -v_* - a - \frac{b}{2\sqrt{v_* - a}}$

Then  $\chi_i = u_i - \frac{\alpha}{5}$ ,  $i = 1,2,3,4$  are the roots of equation (8.20).

Thus, we have the following Lemma.

**Lemma 7.7.3** Suppose that  $\varepsilon \geq 0$ ,  $b_1 \neq 0$  and  $v_* > a$ .

(I) If  $H_2 < 0$  and  $H_3 < 0$ , then equation (7.19) has no positive real roots.

(II) If (I) is not satisfied, then equation (7.19) has positive real roots iff  $\exists$  at least one  $\chi^* \in (\chi_1, \chi_2, \chi_3, \chi_4)$  such that  $\chi^* > 0$  and  $f(\chi^*) \leq 0$ .

**Proof.** The proof is similar to lemma 7.7.1. We omit it. Finally, if  $v_* < a$ , then equation (7.23) becomes:

$$\left(u^2 + \frac{v_*}{2}\right)^2 - \left(\sqrt{a - v_*}u - \frac{b}{2\sqrt{a - v_*}}\right)^2 = 0 \quad (7.24)$$

$$\text{Let } \bar{\chi} = \frac{b}{2(a - v_*)} - \frac{\alpha}{5}$$

Hence, we have the following Lemma.

**Lemma 7.7.4** Suppose that  $\varepsilon \geq 0$ ,  $b_1 \neq 0$  and  $v_* < a$ , then equation (7.22) has positive real roots iff  $\frac{b^2}{4(a - v_*)^2} + \frac{v_*}{2} = 0$  and  $\bar{\chi} > 0$  and  $f(\bar{\chi}) \leq 0$ .

**Proof.** Assume equation (7.23) has a real root  $u_0$  satisfying:

$$\begin{aligned} u_0 &= \frac{b}{2(a - v_*)}, u_0^2 = -\frac{v_*}{2} \\ \Rightarrow \frac{b^2}{4(a - v_*)^2} + \frac{v_*}{2} &= 0. \end{aligned}$$

Therefore, equation (7.23) has a real root  $u_0$  iff  $\frac{b^2}{4(a - v_*)^2} + \frac{v_*}{2} = 0$ .

The rest of the proof is similar to Lemma 7.7.1. We omit it.

Suppose equation (7.20) possesses positive roots. In general, we suppose that it has 5 positive roots denoted by  $\chi^*_i, i = 1,2,3,4,5$ . Then equation (7.19) has 5 positive roots  $\omega_i = \sqrt{\chi^*_i}, i = 1,2,3,4,5$ .

$$\text{We have } \cos \omega \tau = \frac{\zeta_6}{(\eta_4 \omega - \eta_2 \omega^3)^2 + (\eta_1 \omega^4 - \eta_3 \omega^2 + \eta_5)^2}$$

$$\Rightarrow \tau = \frac{1}{\omega} \left[ \cos^{-1} \left( \frac{\zeta_6}{(\eta_4 \omega - \eta_2 \omega^3)^2 + (\eta_1 \omega^4 - \eta_3 \omega^2 + \eta_5)^2} \right) + 2j\pi \right]; j = 0,1,2,3, \dots$$

Where  $\zeta_6 = -((\eta_1\omega^4 - \eta_3\omega^2 + \eta_5)(\zeta_1\omega^4 - \zeta_3\omega^2 + \zeta_5) + (\eta_4\omega - \eta_2\omega^3)(\omega^5 - \zeta_2\omega^3 + \zeta_4\omega)$

Let  $\tau_k^{(j)} = \frac{1}{\omega_k} \left[ \cos^{-1} \left( \frac{\zeta_6}{(\eta_4\omega - \eta_2\omega^3)^2 + (\eta_1\omega^4 - \eta_3\omega^2 + \eta_5)^2} \right) + 2j\pi \right]; k = 1, 2, 3, 4, 5.; j = 0, 1, 2, 3, -$

Then  $\bar{\tau}i\omega_k$  is a pair of purely imaginary roots of equation (7.15)

Where  $\tau = \tau_k^{(j)}, k = 1, 2, 3, 4, 5.; j = 1, 2, 3, - - -$ ,

We have  $\lim_{j \rightarrow \infty} \tau_k^{(j)} = \infty, k = 1, 2, 3, 4, 5.$

Thus, we can define:

$$\begin{aligned} \tau_0 = \tau_{k_0}^{(j_0)} &= \min_{1 \leq k \leq 4, j \geq 1} [\tau_k^{(j)}] \quad , \omega_0 & (7.25) \\ &= \omega_{k_0}, \chi_0 = \chi_{k_0}^* \end{aligned}$$

**Lemma 7.7.5** Suppose that  $\varsigma_1 > 0, (\varsigma_1\varsigma_2 - \varsigma_3) > 0, \varsigma_3(\varsigma_1\varsigma_2 - \varsigma_3) + \varsigma_1(\varsigma_5 - \varsigma_1\varsigma_4) > 0, (\varsigma_2\varsigma_5 + \varsigma_3\varsigma_3)(\varsigma_1\varsigma_2 - \varsigma_3) + \varsigma_1\varsigma_4(\varsigma_5 - \varsigma_1\varsigma_4) > 0, \varsigma_5 > 0.$

Where  $\varsigma_1 = (\zeta_1 + \eta_1), \varsigma_2 = (\zeta_2 + \eta_2), \varsigma_3 = (\zeta_3 + \eta_3), \varsigma_4 = (\zeta_4 + \eta_4), \varsigma_5 = (\zeta_5 + \eta_5).$

(I) If any one of the following condition holds: (i)  $\varepsilon < 0$  (ii)  $\varepsilon \geq 0, b = 0, H \geq 0$  and  $a < 0$  or  $c \leq 0$  and there exists a  $\chi^* \in (\chi_1, \chi_2, \chi_3, \chi_4)$  such that  $\chi^* > 0$  and  $f(\chi^*) \leq 0$  (iii)  $\varepsilon \geq 0, b \neq 0, v_* > a, H_2 \geq 0$  or  $H_3 \geq 0$  and there exists a  $\chi^* \in (\chi_1, \chi_2, \chi_3, \chi_4)$  such that  $\chi^* > 0$  and  $f(\chi^*) \leq 0$  (iv)  $\varepsilon \geq 0, b \neq 0, v_* < a, \frac{b^2}{4(a-v_*)^2} + \frac{v_*}{2} = 0, \bar{\chi} > 0$  and  $f(\bar{\chi}) \leq 0$ , then negative real part will be there in all roots of equation (8.15) when  $\tau \in [0, \tau_0).$

(II) If any one of the conditions (i)-(iv) of (I) are not satisfied, then negative real parts will be there in all roots of equation (7.15) for all  $\tau \geq 0.$

**Proof.** When  $\tau = 0$ , equation (7.15) becomes;

$$\begin{aligned} \mu^5 + (\zeta_1 + \eta_1)\mu^4 + (\zeta_2 + \eta_2)\mu^3 + (\zeta_3 + \eta_3)\mu^2 + (\zeta_4 + \eta_4)\mu + (\zeta_5 + \eta_5) &= 0 \\ \mu^5 + \varsigma_1\mu^4 + \varsigma_2\mu^3 + \varsigma_3\mu^2 + \varsigma_4\mu + \varsigma_5 &= 0 \end{aligned} \quad (7.26)$$

All roots of equation (7.26) have negative real parts iff supposition of Lemma 7.7.5 holds (Routh-Hurwitz's criteria).

From Lemmas 7.7.1- 7.7.5, we know that if conditions (i)-(iv) of (I) are not satisfied, then none of the roots of equation (7.26) will have zero-real part for all  $\tau \geq 0$ .

If one of the conditions (i)-(iv) holds, when  $\tau \neq \tau_k^{(j)}, k = 1,2,3,4,5.; j \geq 1$ , then none of the roots of equation (7.26) will have zero real part and  $\tau_0$  is the minimum value of  $\tau$  for which the roots of equation (7.26) are purely imaginary. This lemma is concluded by using Theorem 1.5.8.

$$\text{Let } \mu(\tau) = \psi(\tau) + i\omega(\tau) \quad (7.27)$$

be the roots of equation (8.26) satisfying:  $\psi(\tau_0) = 0, \omega(\tau_0) = \omega_0$ .

Then we have the following lemma.

**Lemma 7.7.6** Suppose  $h'(\chi_0) \neq 0$ . If  $\tau = \tau_0$ , then  $\mp i\omega_0$  is a pair of simple purely imaginary roots of equation (7.26). Moreover, If the condition of Lemma 7.7.5 (I) are satisfied, then  $\frac{d}{d\tau}(Re\mu(\tau_0)) > 0$ .

**Proof.** Substituting  $\lambda(\tau)$  into equation (7.15) and differentiating both sides with respect to  $\tau$

$$\begin{aligned} & \left(\frac{d\mu}{d\tau}\right)^{-1} \\ &= \frac{(5\mu^4 + 4\zeta_1\mu^3 + 3\zeta_2\mu^2 + 2\zeta_3\mu + \zeta_4)e^{\mu\tau} + (4\eta_1\mu^3 + 3\eta_2\mu^2 + 2\eta_3\mu + \eta_4)}{(\eta_1\mu^4 + \eta_2\mu^3 + \eta_3\mu^2 + \eta_4\mu + \eta_5)} - \frac{\tau}{\mu} \end{aligned}$$

By calculation, we have:

$$\begin{aligned} & [(5\mu^4 + 4\zeta_1\mu^3 + 3\zeta_2\mu^2 + 2\zeta_3\mu + \zeta_4)e^{\mu\tau}]_{\tau=\tau_0} \\ &= \zeta_7 \cos \omega_0\tau + \zeta_8 \sin \omega_0\tau + i(-\zeta_8 \cos \omega_0\tau + \zeta_7 \sin \omega_0\tau) \\ & (4\eta_1\mu^3 + 3\eta_2\mu^2 + 2\eta_3\mu + \eta_4)_{\tau=\tau_0} = \eta_4 - 3\eta_2\omega_0^2 + i\omega_0(2\eta_3 - 4\eta_1\omega_0^2) \\ & (\eta_1\mu^4 + \eta_2\mu^3 + \eta_3\mu^2 + \eta_4\mu + \eta_5)_{\tau=\tau_0} \\ &= \omega_0^2(\eta_2\omega_0^2 - \eta_5) + i\omega_0(\eta_5 - \eta_3\omega_0^2 + \eta_1\omega_0^4) \end{aligned}$$

Where  $\zeta_7 = (5\omega_0^4 - 3\zeta_3\omega_0^2 + \zeta_4), \zeta_8 = (4\zeta_1\omega_0^3 - 2\zeta_3\omega_0)$

Then, we have:

$$\left(\frac{d Re\mu(\tau_0)}{d\tau}\right)^{-1} = \frac{\chi_0 h'(\gamma_0)}{\zeta_9} \quad (7.28)$$

Where  $\zeta_9 = \omega_0^2[(\eta_2\omega_0^3 - \eta_5\omega_0)^2 + (\eta_5 - \eta_3\omega_0^2 + \eta_1\omega_0^4)^2]$

Thus, we have:

$$\begin{aligned} \text{sign} \left[ \frac{d \text{Re}\mu(\tau_0)}{d\tau} \right] &= \text{sign} \left[ \left( \frac{d \text{Re}\mu(\tau_0)}{d\tau} \right)^{-1} \right] \\ &= \text{sign} \left[ \frac{\chi_0 f'(\chi_0)}{\zeta_9} \right] \end{aligned} \quad (7.29)$$

Where  $\zeta_9, \chi_0 > 0$ .

Thus, based on the application of lemmas 7.7.1-7.7.6, we have the following theorem:

**Theorem 7.1** *Let the assumptions of lemma 6 hold and  $\omega_0, \chi_0, \tau_0$  and  $\mu(\tau)$  be defined by (7.18) to (7.20), then:*

(I) *All the roots of equation (7.18) will have negative real part for all  $\tau \geq 0$ , if the conditions (i) –(iv) of lemma 7.7.5 are not satisfied.*

(II) *All the roots of equation (7.18) have negative real parts, if one of the conditions (i) –(iv) of lemma 7.7.5 is satisfied and  $\tau \in [0, \tau_0)$ . When  $\tau = \tau_0$  and  $h'(\chi_0) \neq 0$ , then  $\mp i\omega_0$  is a pair of purely imaginary roots of equation (8.18) and all other roots have negative real parts. In addition,  $\frac{d \text{Re}\mu(\tau_0)}{d\tau} > 0$  and equation (7.18) has at least one root with positive real part when  $\tau \in (\tau_0, \tau_1)$ , where  $\tau_1$  is the first value of  $\tau > \tau_0$  such that equation (7.18) has purely imaginary roots.*

## 7.8 Existing Experimental Data for Verification of Model

Lettuce is an annual plant most often grown as a leaf eating vegetable. The consumption of lettuce has become extremely popular in daily diet of Vietnamese people. Lettuce was found to be more responsive than the other vegetables for the accumulation of heavy metals. Waste water and use of excessive fertilizers further enhances the level of heavy metals in soil. Practically, it is also not possible to prevent farmers from cultivation on these soils. Cu (2015) [93] studied the impact of the heavy metals (Cu, Pb, Zn) as stress factors to growth and heavy metal concentration by Lettuce.

The pot experiments were carried out at the green house of Vietnam academy of agricultural sciences in Hanoi, Vietnam. The soil was contaminated by heavy metals in different rates and place in pot experiments (5Kg soil/pot), left it overnight and then sown seeds of Lettuce 30 seeds/pot. After 15 days of sowing pruning conducted to ensure appropriate density for growing plants (10 plants/pot). The soils used in the experiments is red river soil collected at a vegetable growing area of Thanh Tri district,

Hanoi, Vietnam. The treatments were contaminated with heavy metals at levels of 50 ppm (0.05 gm/Kg), 100 ppm (0.1 gm/Kg), 200 ppm (0.2 gm/Kg) for Cu and Pb; and 100 ppm (0.1 gm/Kg), 300 ppm (0.3 gm/Kg), 500 ppm (0.5 gm/Kg) for Zn. The treatments were conducted individually to determine the influence of the levels of added Cu, Pb and Zn. The plants were observed for their growth and yield (above ground) and harvested after 45 days. Heavy metal contents in soil and plant were estimated using atomic absorption spectroscopy, extractions to assess mobilization of  $\text{Cu}^{2+}$ ,  $\text{Pb}^{2+}$ ,  $\text{Zn}^{2+}$  in EDTA solution. The results and discussion regarding this experiment are:

**Table 7. 1**Effect of added Cu on growth of Brassica juncea (fresh weight):

Added Cu: ppm (g/Kg)	Plant Height		Yield	
	cm	%	g/pot	%
0 (0)	19.5	100	70.3	100
50 (0.05)	15.0	77	55.9	80
100 (0.1)	14.7	75	50.0	71
200 (0.2)	12.5	64	33.1	47

**Table 7. 2**Effect of added Pb on growth of Brassica juncea (fresh weight):

Added Pb: ppm (g/Kg)	Plant Height		Yield	
	cm	%	g/pot	%
0 (0)	19.5	100	70.3	100
50 (0.05)	14.7	75	37.3	53
100 (0.1)	13.4	69	34.7	49
200 (0.2)	12.5	56	31.0	44

**Table 7. 3**Effect of added Zn on growth of Brassica juncea (fresh weight):

Added Zn: ppm (g/Kg)	Plant Height		Yield	
	cm	%	g/pot	%
0 (0)	19.5	100	70.3	100
100 (0.1)	19.7	101	75.7	108
300 (0.1)	16.7	86	64.1	91
500 (0.5)	15.5	80	43.8	62

## 7.9 Numerical Example

To support the analytical result, numerical method has been used to solve equations (7.1) -(7.5) and simulation has been done with MATLAB. The following set of parametric values has been considered:



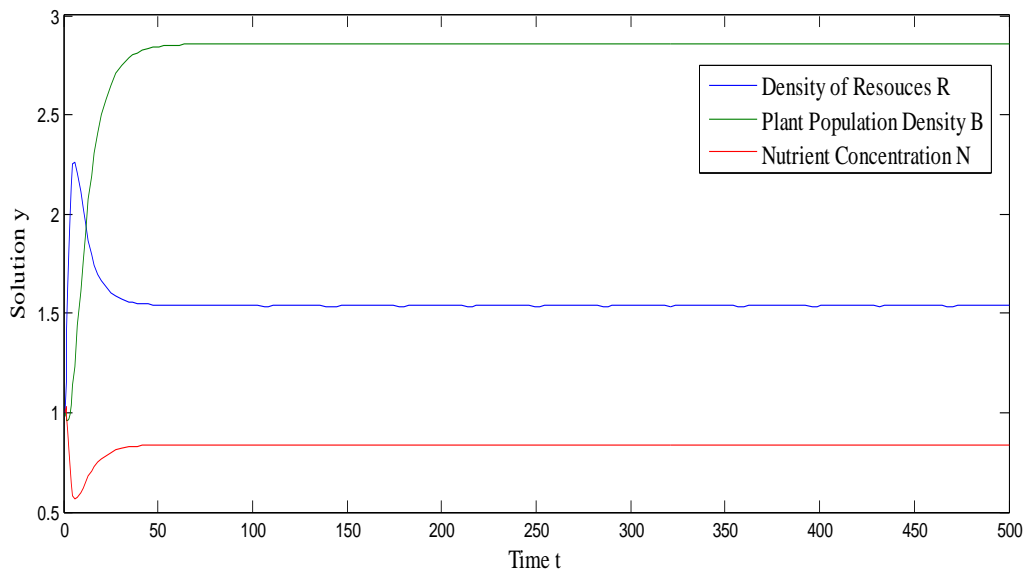
$$N_0 = 3, M_0 = 1.9, T_0 = 1.9, K = 0.1, \alpha_1 = 0.13, \alpha_2 = 0.2, \beta_1 = 0.8, \beta_2 = 0.3, \gamma_1 = 0.1, \delta_1 = 0.4, \delta_2 = 0.1, \varepsilon_1 = 0.45, \varepsilon_2 = 0.1$$

Here the values of uptake rate of acid and metal are taken as per range of values of the heavy metal Copper (Cu) taken in the above-mentioned experimental data on study of growth of Lettuce under the effect of heavy metals [93]

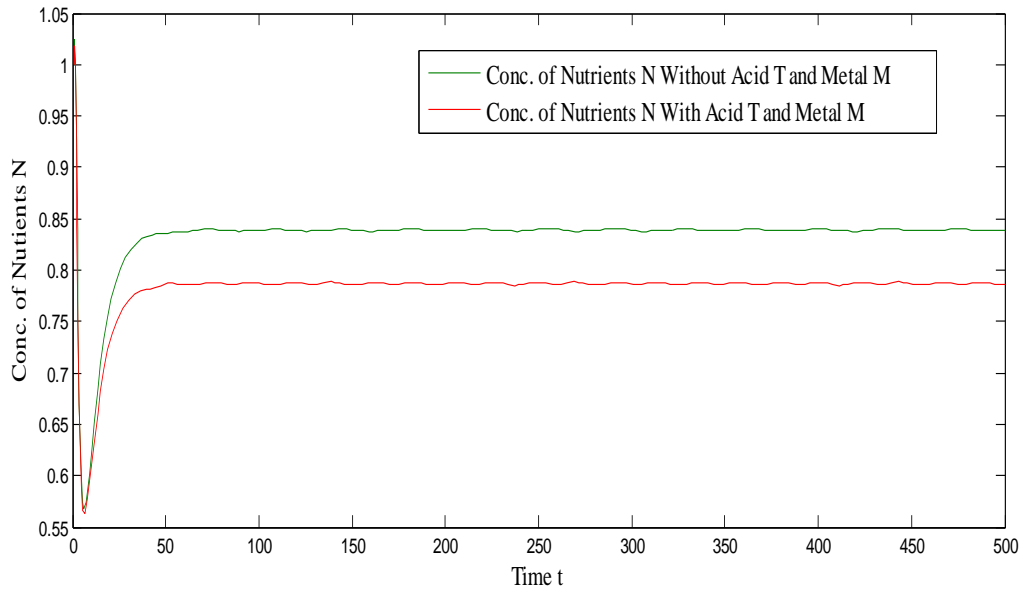
**Behaviour of the system about equilibrium  $E_1(\bar{R} \neq 0, \bar{B} \neq 0, \bar{N} \neq 0, \bar{T} = 0, \bar{M} = 0$ :**

For the above set of parametric values, we obtain the following values of uniform equilibrium point  $E_1: \bar{R} = 1.5891, \bar{B} = 2.8553, \bar{N} = 0.8388$ .

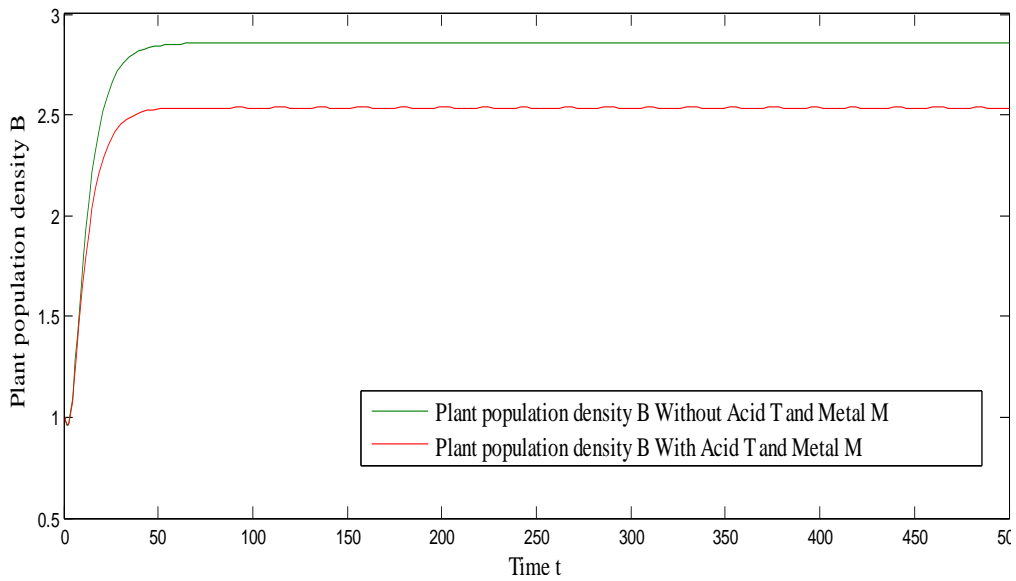
With the initial values:  $R(0) = 1, B(0) = 1, N(0) = 1$



**Figure 7. 1** Trajectories of the model without acid and toxic metal with respect to time shows stable behaviour of the equilibrium  $E_1 (1.5891, 2.8553, 0.8388)$ .



**Figure 7. 2** Trajectories showing adverse effect of acid T and metal M on concentration of nutrient pool N with respect to time.

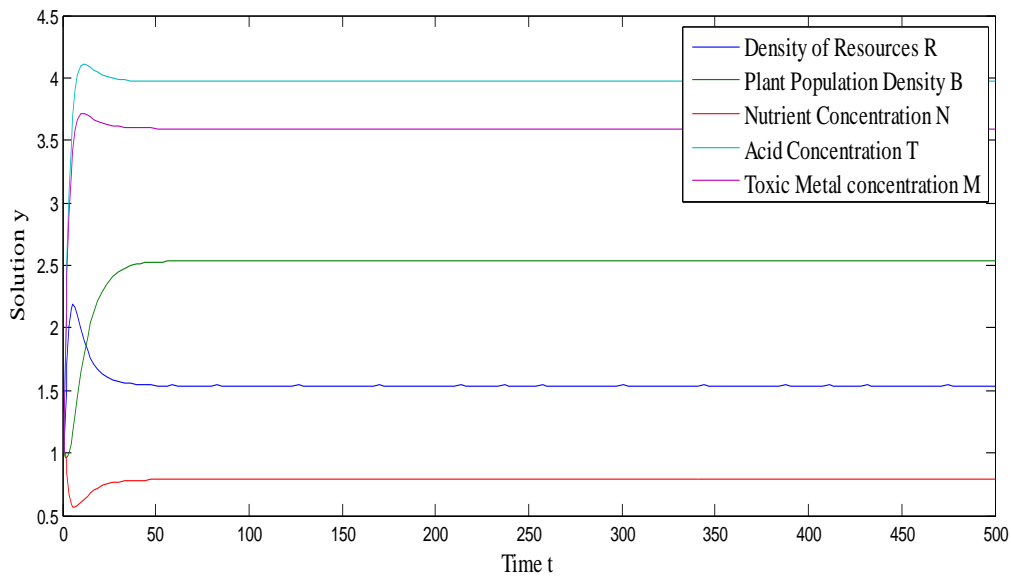


**Figure 7. 3** Trajectories showing adverse effect of acid T and metal M on plant population density B with respect to time t.

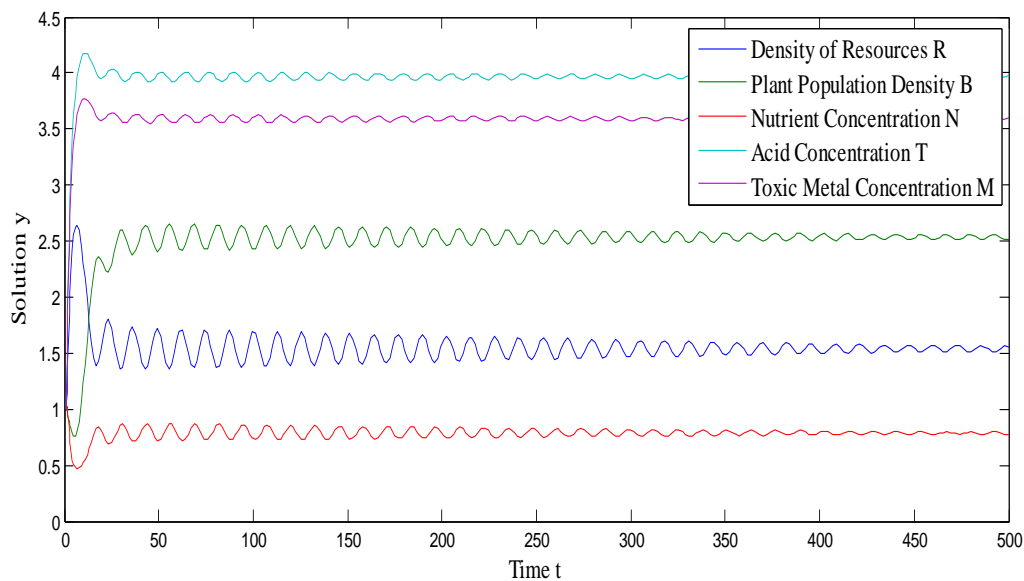
**Behaviour of the system about equilibrium  $E_2(R^* \neq 0, B^* \neq 0, N^* \neq 0, T^* \neq 0, M^* \neq 0$ :**

For the same set of parametric values, we obtain the following values of interior equilibrium point  $E_2: R^* = 1.5891, B^* = 2.5130, N^* = 0.7929, T^* = 3.9177, M^* = 3.5513$ .

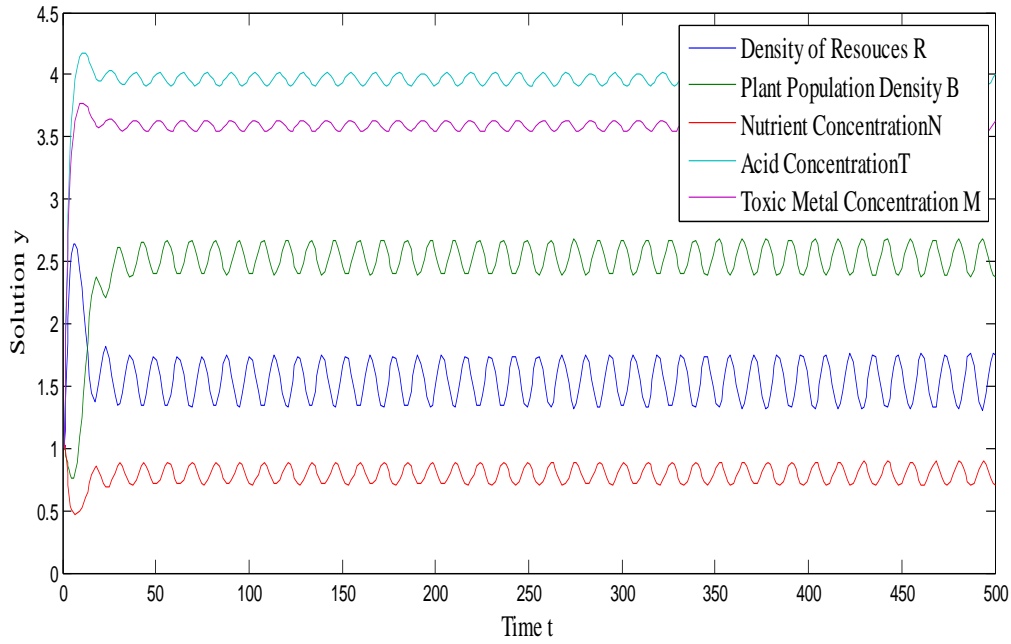
with initial conditions:  $R(0) = 1, B(0) = 1, N(0) = 1, T(0) = 1, M(0) = 1$ .



**Figure 7. 4**The interior equilibrium points  $E_2(1.5891, 2.5130, 0.7929, 3.9177, 3.5513)$  of the system is stable when there is no delay that is  $\tau=0$ .



**Figure 7. 5**The interior equilibrium point  $E_2(1.5891, 2.5130, 0.7929, 3.9177, 3.5513)$  is asymptotically stable with delay  $\tau < 3.38$ .



**Figure 7.6** The interior equilibrium point  $E_2(1.5891, 2.5130, 0.7929, 3.9177, 3.5513)$  loses its stability and Hopf- bifurcation occurred with delay  $\tau \geq 3.38$ .

## 7.10 Sensitivity Analysis

Estimation of the general sensitivity coefficients is done using the ‘Direct Method’ which assumes that all the parameters considered in the model are constants. Here, the sensitivity coefficients can be estimated by solving sensitivity equations simultaneously with the original system. For an instance, the partial derivatives of the solution  $(R, B, N, T, M)$  with respect to  $\beta_1$  (interaction rate between nutrient and resources) give the following set of sensitivity equations:

$$\begin{aligned} \frac{dS_1}{dt} &= (\beta_1 N - \beta_2)S_1 + \beta_1 R S_3 \\ &\quad - \alpha_1 B S_1(t - \tau) + NR \end{aligned} \quad (7.30)$$

$$\frac{dS_2}{dt} = -\alpha_1 S_2 + \alpha_1 B S_1(t - \tau) \quad (7.31)$$

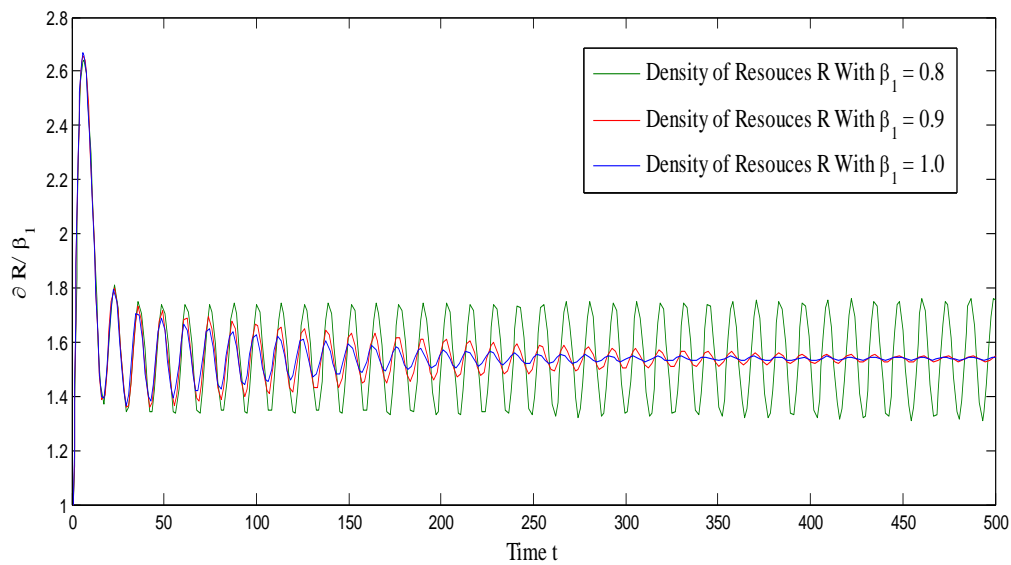
$$\begin{aligned} \frac{dS_4}{dt} &= (K\beta_2 - \beta_1 N)S_1 + K\alpha_2 S_2 \\ &\quad - (\gamma_1 + \beta_1 N + K\delta_2 T \\ &\quad + K\varepsilon_2 M)S_3 - K\delta_2 N S_4 \\ &\quad - K\varepsilon_2 N S_5 \end{aligned} \quad (7.32)$$

$$\frac{dS_4}{dt} = -\delta_2 T S_3 - (\delta_1 + \delta_2 N)S_4 \quad (7.33)$$

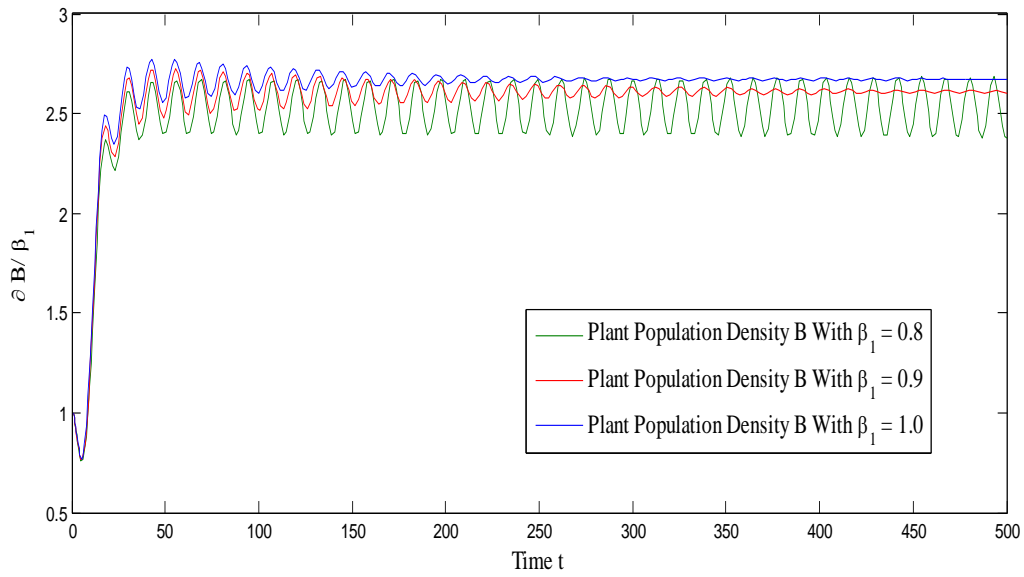
$$\frac{dS_5}{dt} = -\varepsilon_2MS_3 - (\varepsilon_1 + \varepsilon_2N)S_5 \quad (7.34)$$

Where  $S_1 = \frac{\partial R}{\partial a_1}$ ,  $S_2 = \frac{\partial B}{\partial a_1}$ ,  $S_3 = \frac{\partial N}{\partial a_1}$ ,  $S_4 = \frac{\partial T}{\partial a_1}$ ,  $S_5 = \frac{\partial M}{\partial a_1}$

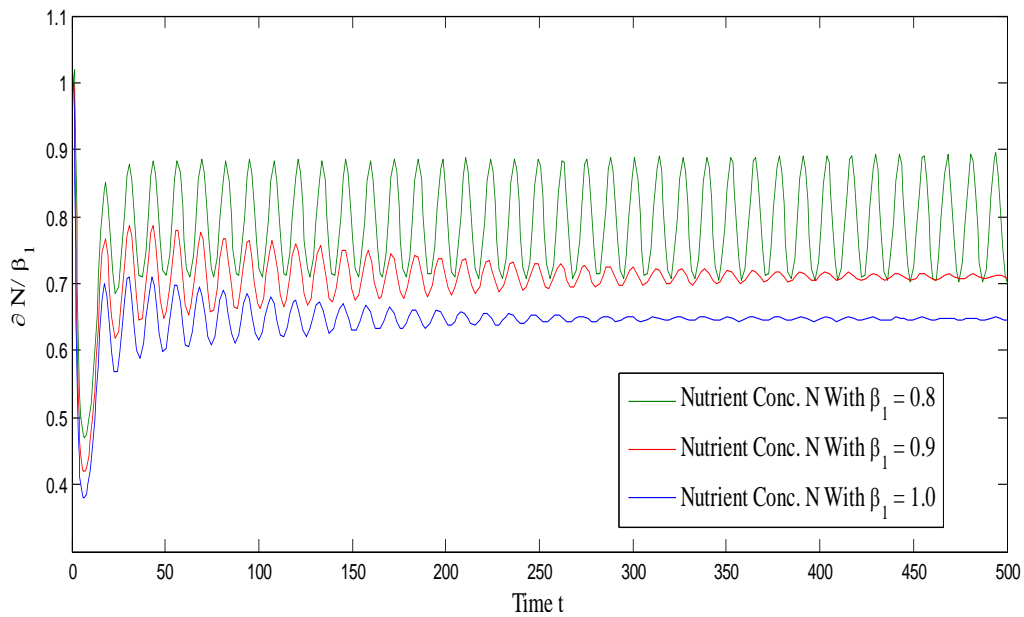
Then, we solve this system of sensitivity equations (7.30) – (7.34) along with the original system of equations (7.1) – (7.5) to estimate the sensitivity of the state variables ( $R, B, N, T, M$ ) to the parameter  $\beta_1$  (interaction rate between nutrient and resources). In the same way, the sensitivity analysis of the state variables with respect to the parameters  $\alpha_1$  (specific rate of utilization of resources by biomass) is also done.



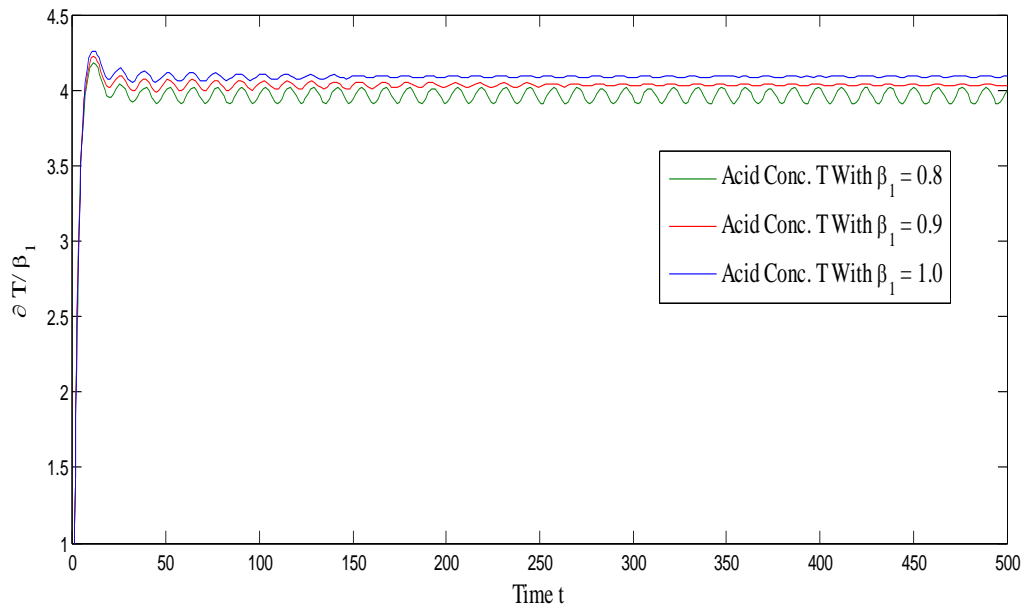
**Figure 7.** Time series graph between partial changes in R (density of favourable resources) and different values of parameter  $\beta_1$  (interaction rate between nutrient and resources).



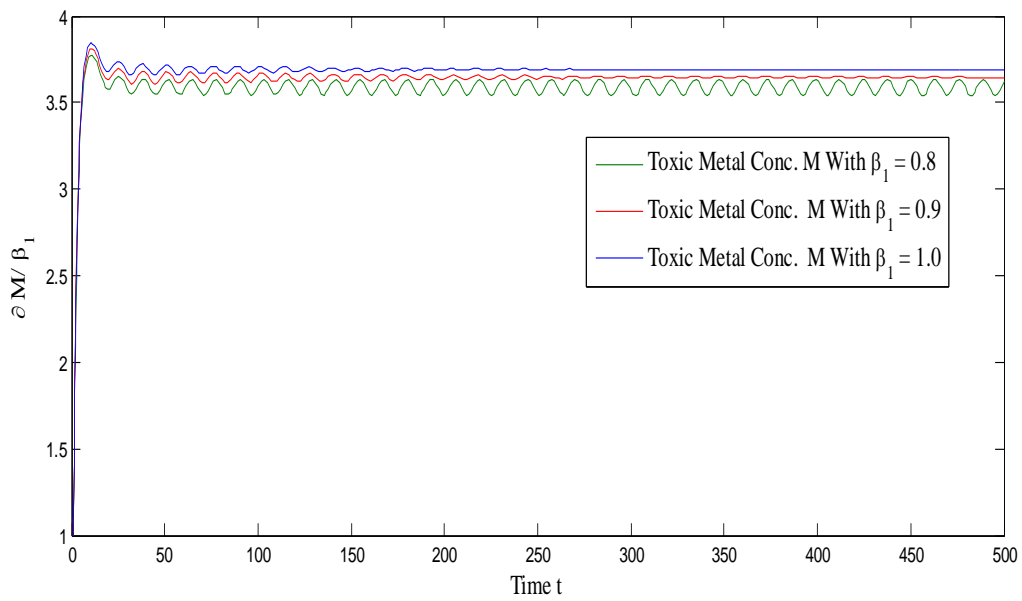
**Figure 7. 8**Time series graph between partial changes in B(plant population density) and different values of parameter  $\beta_1$ (interaction rate between nutrient and resources).



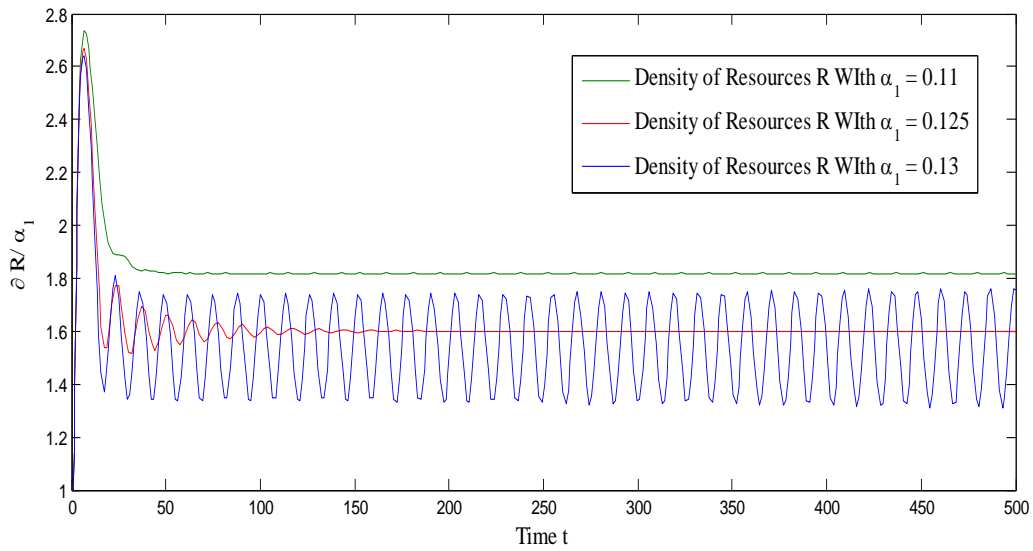
**Figure 7. 9**Time series graph between partial changes in N (concentration of nutrients pool) and different values of parameter  $\beta_1$  (interaction rate between nutrient and resources).



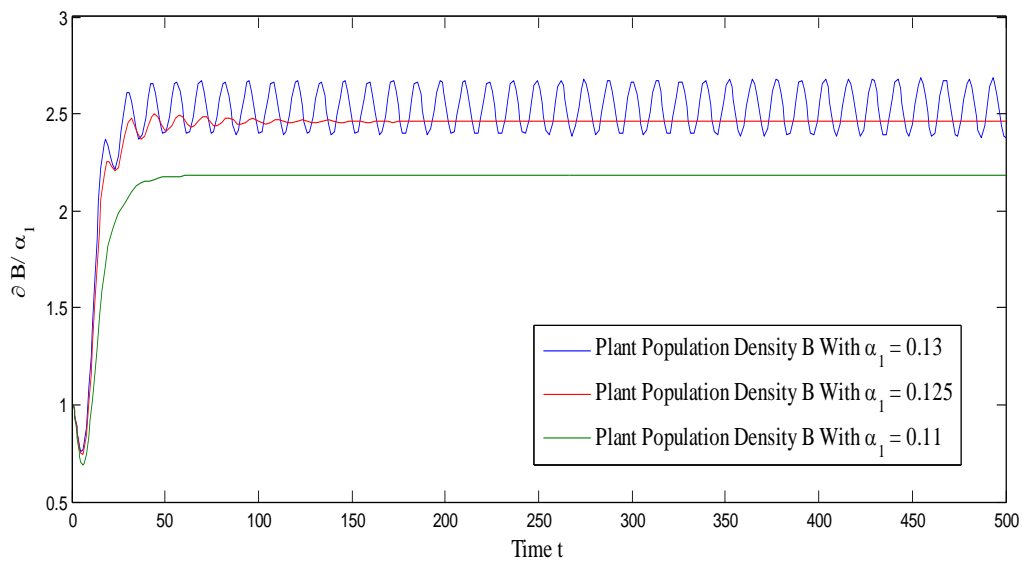
**Figure 7. 10**Time series graph between partial changes in T (concentration of acid in soil) and different values of parameter  $\beta_1$  (interaction rate between nutrient and resources).



**Figure 7. 11**Time series graph between partial changes in M (concentration of toxic metal in soil) and different values of parameter  $\beta_1$  (interaction rate between nutrient and resources).

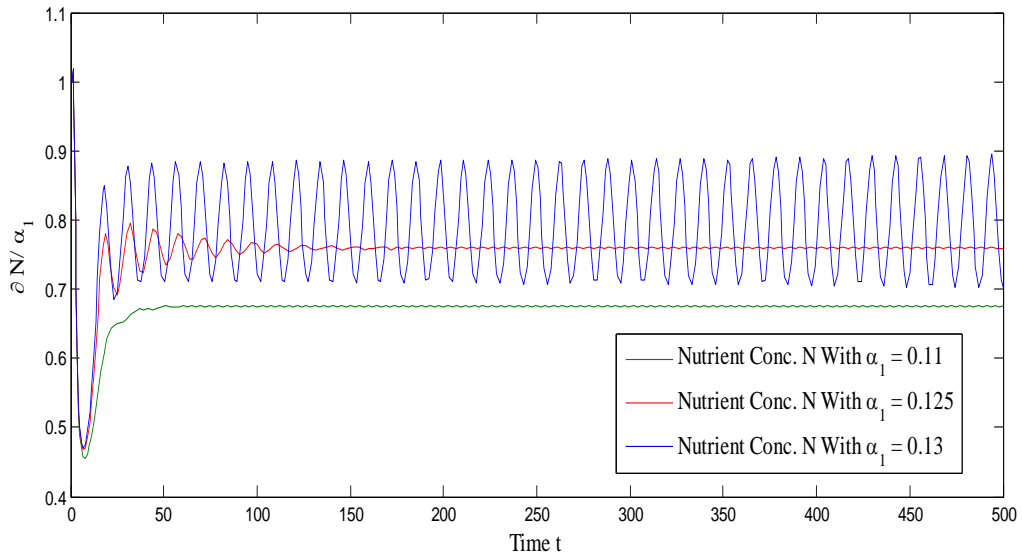


**Figure 7.12** Time series graph between partial changes in R (density of favourable resources) and different values of parameter  $\alpha_1$  (specific rate of utilization of resources by biomass).

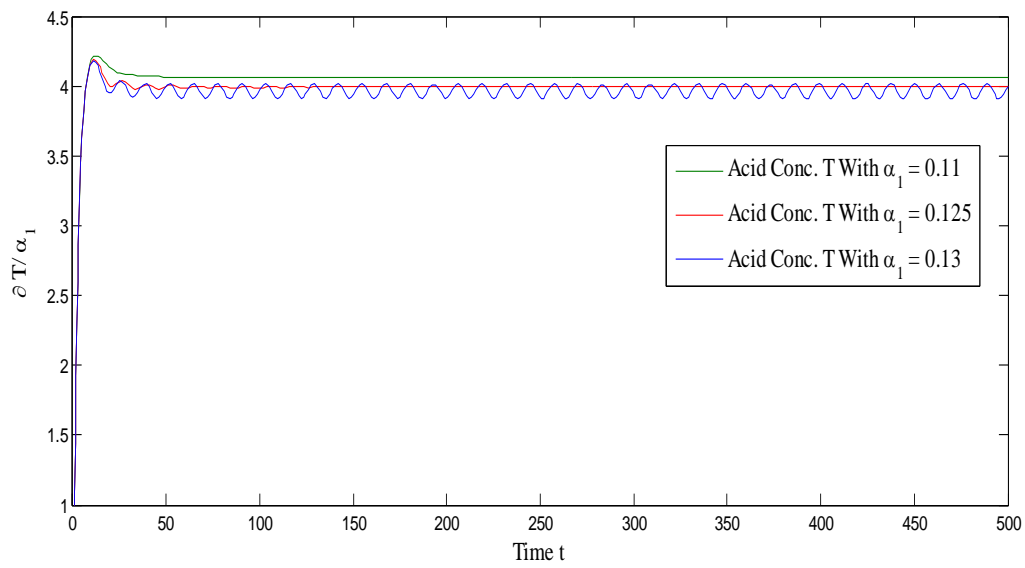


**Figure 7.13** Time series graph between partial changes in B (plant population density) and different values of parameter  $\alpha_1$  (specific rate of utilization of resources by biomass).

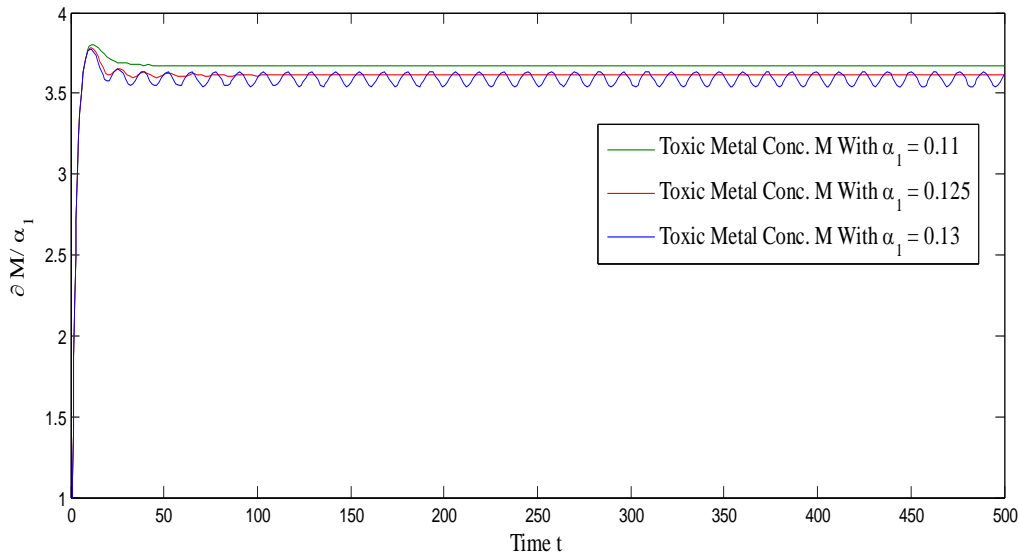




**Figure 7. 14**Time series graph between partial changes in N (concentration of nutrient pool) and different values of parameter  $\alpha_1$ (specific rate of utilization of resources by biomass).



**Figure 7. 15**Time series graph between partial changes in T (concentration of acid in soil) and different values of parameter  $\alpha_1$ (specific rate of utilization of resources by biomass).



**Figure 7. 16**Time series graph between partial changes in M (concentration of toxic metal in soil) and different values of parameter  $\alpha_1$ (specific rate of utilization of resources by biomass).

## 7.11 Conclusion

The role of delay on the plant population growth under the combined effect of acid and toxic metal is studied with the help of proposed mathematical model.

In the absence of acid and toxic metal, the system shows stable behaviour as shown by the Figure 7.1. This result corresponds to the 100% yield in the study of Lettuce considered in this paper. The introduction of the acid and toxic metal has adverse effect on plant growth. It is evident that the value of concentration of nutrient pool decreases (From 0.8388 to 0.7929) under the combined effect of acid and toxic metal as shown by the Figure 7.2. It is also clear from the Figure 7.3 that the plant population density undergoes a decrease in its value (From 2.8553 to 2.5130), which is nearly 14 %, when the rate of uptake of metal and acidity is 0.1. This result also falls well within range of plant yield decrease from 12% to 21%, when Cu input is increased from 50ppm(0.05) to 100ppm (0.1) in the mentioned study.

The local stability of the uniform equilibrium  $E_1$  is studied. It is shown that the equilibrium point  $E_1(\bar{R} = 1.5891, \bar{B} = 2.8553, \bar{N} = 0.8388)$  is stable as shown by Figure 8.1 using Routh-Hurwitz's criteria. The stability and Hopf- bifurcation about the interior equilibrium  $E_2$  is also studied. Using Lemma 7.7.5 (Routh-Hurwitz's criteria), it is shown that interior equilibrium  $E_2(R^* = 1.5891, B^* = 2.5130, N^* = 0.7929, T^* = 3.9177, M^* = 3.5513)$  is stable, in the absence of delay ( $\tau = 0$ ) as shown by Figure

7.4. But the system is asymptotically stable for all values which are below the critical value of delay parameter ( $\tau < 3.38$ ), keeping all the other parameters same as shown by Figure 7.5. Once the critical value of the delay parameter is reached ( $\tau \geq 3.38$ ), the system losses stability and becomes unstable as shown by Figure 7.6. The system shows the periodic oscillation when it passes through that critical value that is Hopf bifurcation occurs.

The sensitivity of model solutions is established by taking different values of the parameters appearing in system. It improves the understanding of the role played by specific model parameters.

As we start increasing the rate of interaction of nutrient and resources, the entire system starts converging to stability. For  $\beta_1 = 0.8$ , the system i.e. the concentration of nutrients, the density of resources and plant population density, concentration of acid and concentration of toxic metal show Hopf bifurcation through periodic oscillations. But as we increase the value of  $\beta_1$  from  $\beta_1 = 0.8$  to  $\beta_1 = 0.9$ , the system starts showing asymptotical stability as the periodic oscillations start dying down and eventually ends up converging to a stable equilibrium point as we further increase the value of  $\beta_1$  from  $\beta_1 = 0.9$  to  $\beta_1 = 1$ . It has also been observed that density of resources remain almost same throughout these increasing values of  $\beta_1$ , but concentration of nutrient pool keep on decreasing with increase in the value of  $\beta_1$ . On the contrary, plant population density, concentration of acid and concentration of toxic metal show similar kind of increase as we increase the values of  $\beta_1$ . This phenomenon is shown by the Figure 7.7- Figure 7.11.

As we start decreasing the specific rate of utilization of delayed resources by plant population density, the entire system starts converging to stability. For  $\alpha_1 = .13$ , the system i.e. the concentration of nutrients pool, the amount of resources and plant population density, concentration of acid and concentration of toxic metal show Hopf bifurcation through periodic oscillations. But as we decrease the value of  $\alpha_1$  from  $\alpha_1 = .13$  to  $\alpha_1 = .125$ , the system starts showing asymptotical stability as the periodic oscillations start dying down and eventually ends up converging to a stable equilibrium point as we further decrease the value of  $\alpha_1$  from  $\alpha_1 = .125$  to  $\alpha_1 = .11$ . It has also been observed that density of resources, concentration of acid and concentration of toxic metal start increasing too with decrease in the value of  $\alpha_1$ , but this increase is more visible in case of resources as compared to concentration of acid and concentration of

toxic metal. On the contrary, plant population density and concentration of nutrient pool show similar kind of decrease in their values with decrease in the value of  $\alpha_1$ . This phenomenon is graphically shown by Figure 7.12-Figure 7.16.

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# Appendices

## Dimensions of Variables and Parameters used in Chapter 2

Variable	Description	Dimensions
$N_1$	Concentration of nutrients in root compartment	$\text{Kg mole } m^{-3}$
$N_2$	Concentration of nutrients in shoot compartment	$\text{Kg mole } m^{-3}$
$W_1$	Concentration of structural dry weight in root compartment	$\text{Kg}$
$W_2$	Concentration of structural dry weight in shoot compartment	$\text{Kg}$
$t$	Time variable	$\text{Day}$

Parameters	Description	Dimensions
$r_1$	Rate of growth of root under the effect of exogenic activities	$mgKg^{-1}day^{-1}$
$\tau$	Time delay due of presence of toxic metal in soil.	$\text{Day}$
$r_2$	Rate of growth of shoot under the effect of exogenic activities	$mgKg^{-1}day^{-1}$
$T$	Rate of nutrient transfer from root to shoot compartment.	$\text{Kg mole } m^{-3}day^{-1}$
$R_n$	The resistance to transportation of nutrient	$\text{Kg mole } m^{-3}$
$\mu$	Consumption coefficient or utilization coefficient	$Kg^{-1}day^{-1}$
$\rho$	Efficiency of nutrient utilization	$mgKg^{-1}day^{-1}$
$\gamma_1$	Measuring the decrease in nutrient use efficiency due to presence of toxic metal in the plants	$mgKg^{-1}day^{-1}$
$\gamma_2$	Measuring the decrease in nutrient use efficiency due to presence of toxic metal in the plants	$mgKg^{-1}day^{-1}$
$\beta_{10}$	Natural decay of $W_1$ .	$mgKg^{-1}day^{-1}$
$\beta_{20}$	Natural decay of $W_2$ .	$mgKg^{-1}day^{-1}$
$d_1$	Natural decay of $N_1$	$Kg \text{ mole}(Kg \text{ mole})^{-1}day^{-1}$
$d_2$	Natural decay of $N_2$	$Kg \text{ mole}(Kg \text{ mole})^{-1}day^{-1}$

$U_n$	Initial availability of nutrient in soil.	$\text{Kg mole } m^{-3} \text{ day}^{-1}$
$\Delta_1$	Self-limiting rate of growth of $W_1$	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\Delta_2$	Self-limiting rate of growth of $W_2$	$\text{mgKg}^{-1} \text{ day}^{-1}$

### Dimensions of Variables and Parameters used in Chapter 3

Variable	Description	Dimensions
$N_1$	Concentration of nutrients in root compartment	$\text{Kg mole } m^{-3}$
$N_2$	Concentration of nutrients in shoot compartment	$\text{Kg mole } m^{-3}$
$W_1$	Concentration of structural dry weight in root compartment	$\text{Kg}$
$W_2$	Concentration of structural dry weight in shoot compartment	$\text{Kg}$
$t$	Time variable	Day
$H_S$	Concentration of heavy metal in soil	$\text{Kg mole } m^{-3}$

Parameters	Description	Dimensions
$r_1$	Rate of growth of root under the effect of exogenic activities	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\tau$	Time delay due of presence of toxic metal in soil.	Day
$r_2$	Rate of growth of shoot under the effect of exogenic activities	$\text{mgKg}^{-1} \text{ day}^{-1}$
$T$	Rate of nutrient transfer from root to shoot compartment.	$\text{mg mole } m^{-3} \text{ day}^{-1}$
$R_n$	The resistance to transportation of nutrient	$\text{mg mole } m^{-3}$
$\mu$	Consumption coefficient or utilization coefficient	$\text{mg}^{-1} \text{ day}^{-1}$
$\rho$	Efficiency of nutrient utilization	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\gamma_1$	Measuring the decrease in nutrient use efficiency due to presence of toxic metal in the plants	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\gamma_2$	Measuring the decrease	$\text{mgKg}^{-1} \text{ day}^{-1}$

	in nutrient use efficiency due to presence of toxic metal in the plants	
$\beta_{10}$	Natural decay of $W_1$ .	$mgKg^{-1}day^{-1}$
$\beta_{20}$	Natural decay of $W_2$ .	$mgKg^{-1}day^{-1}$
$d_1$	Natural decay of $N_1$	$Kg\ mole(Kg\ mole)^{-1}day^{-1}$
$d_2$	Natural decay of $N_2$	$Kg\ mole(Kg\ mole)^{-1}day^{-1}$
$U_n$	Initial availability of nutrient in soil.	$Kg\ mole\ m^{-3}day^{-1}$
$\Delta_1$	Self-limiting rate of growth of $W_1$	$mgKg^{-1}day^{-1}$
$\Delta_2$	Self-limiting rate of growth of $W_2$	$mgKg^{-1}day^{-1}$
$\beta_{11}$	Damage rates of $W_1$ due to $H_s$	$mgKg^{-1}day^{-1}$
$\beta_{21}$	Damage rates of $W_2$ due to $H_s$	$mgKg^{-1}day^{-1}$
$I$	The input rate of toxic metals	$Kg\ mole\ m^{-3}day^{-1}$
$\Delta$	First order decay rate of $H_s$	$mgKg^{-1}day^{-1}$
$\alpha_1$	Depletion rate of $H_s$ due to reaction between $H_s$ and $N_1$	$mgKg^{-1}day^{-1}$
$T_0$	Stress parameter that measures the increase in the resistance to nutrient transport from root to shoot compartment due to presence of toxic metal in soil	$mgKg^{-1}day^{-1}$

#### Dimensions of Variables and Parameters used in Chapter 4

Variable	Description	Dimensions
$N$	Concentration of nutrients in the soil	$Kg\ mole\ m^{-3}$
$W$	Amount of plant biomass	$Kg$
$M$	Concentration of toxic metal in the soil	$Kg\ mole\ m^{-3}$
$t$	Time variable	Day

Parameters	Description	Dimensions
$K_N$	The availability of total nutrients	$Kg\ mole\ m^{-3}day^{-1}$
$\alpha$	The consumption coefficient	$mg^{-1}day^{-1}$
$\beta$	The utilization coefficient	$mg^{-1}day^{-1}$
$\gamma$	The depletion rate of	$Kg\ mole\ m^{-3}day^{-1}$

	toxic metal due to interaction between nutrients and biomass	
$I$	The initial input of toxic metal in the soil	$\text{Kg mole } m^{-3} \text{ day}^{-1}$
$\delta_1$	Natural decay rate of $N$	$mgKg^{-1} \text{ day}^{-1}$
$\delta_2$	Natural decay rate of $W$	$mgKg^{-1} \text{ day}^{-1}$
$\delta_3$	Natural decay rates of $M$	$mgKg^{-1} \text{ day}^{-1}$

### Dimensions of Variables and Parameters used in Chapter 5

Variable	Description	Dimensions
$M_S$	Concentration of toxic metal in the soil	$\text{Kg mole } m^{-3}$
$B_M$	Tree biomass	$\text{Kg}$
$T_P$	Concentration of toxic metal in the tree	$\text{Kg mole } m^{-3}$
$t$	Time variable	Day

Parameters	Description	Dimensions
$r$	Intrinsic growth rate of tree biomass	$KgKg^{-1} \text{ day}^{-1}$
$K$	The carrying capacity	$Kg^{-1} \text{ day}^{-1}$
$\Delta_1$	Natural decay of tree biomass	$KgKg^{-1} \text{ day}^{-1}$
$\Delta_2$	Natural decay of toxic metal in soil	$KgKg^{-1} \text{ day}^{-1}$
$I$	The initial input of heavy metal in the soil	$\text{Kg mole } m^{-3} \text{ day}^{-1}$
$\alpha$	The depletion rate of $M_S$ due to interaction between $M_S$ and $T_P$	$KgKg^{-1} \text{ day}^{-1}$

### Dimensions of Variables and Parameters used in Chapter 6

Variable	Description	Dimensions
$N$	Nutrient pool	$\text{Kg mole } m^{-3}$
$P$	Plant population density	$\text{Kg}$
$T$	Toxic metal	$\text{Kg mole } m^{-3}$
$t$	Time variable	Day



<b>Parameters</b>	<b>Description</b>	<b>Dimensions</b>
$r(N, T)$	The intrinsic growth rate of plant population density which depends on nutrient pool and is adversely affected by toxic metal	$mgKg^{-1}day^{-1}$
$K$	The total available input of nutrient pool	$Kg\ mole\ m^{-3}day^{-1}$
$\alpha$	The natural decay of nutrient pool	$mgKg^{-1}day^{-1}$
$\rho$	Nutrient use efficiency	$mgKg^{-1}day^{-1}$
$\varepsilon_0$	The natural decay of plant population density	$mgKg^{-1}day^{-1}$
$\varepsilon_1$	The rate of damage of plant population density due to toxic metal	$mgKg^{-1}day^{-1}$
$\beta$	Self-limiting rate of plant population density	$mgKg^{-1}day^{-1}$
$Q$	The input rate of toxic metal	$Kg\ mole\ m^{-3}day^{-1}$
$v$	The deletion rate of toxic metal due to interaction between toxic metal and nutrient pool	$mgKg^{-1}day^{-1}$
$\gamma$	Natural decay of toxic metal	$mgKg^{-1}day^{-1}$

### Dimensions of Variables and Parameters used in Chapter 7

<b>Variable</b>	<b>Description</b>	<b>Dimensions</b>
$R$	Density of favourable resources	$Kg\ mole\ m^{-3}$
$B$	Plant population density	$Kg$
$N$	Concentration of nutrient pool	$Kg\ mole\ m^{-3}$
$T$	Concentration of acid	$Kg\ mole\ m^{-3}$
$M$	Concentration of metal	$Kg\ mole\ m^{-3}$
$t$	Time variable	$Day$

<b>Parameters</b>	<b>Description</b>	<b>Dimensions</b>
$N_0$	Constant nutrient	$mgKg^{-1}day^{-1}$

	input in soil	
$T_0$	Input rate of acid	$\text{Kg mole } m^{-3} \text{ day}^{-1}$
$M_0$	Input rate of metal	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\delta_1$	Natural decay rate of acid	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\varepsilon_1$	Natural decay rate of metal	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\delta_2$	Uptake rate of acid by plant	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\varepsilon_2$	Uptake rate of metal by plant	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\alpha_2$	Natural decay rate of plant population density	$\text{Kg mole } m^{-3} \text{ day}^{-1}$
$\beta_2$	Natural decay rate of favourable resources	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\gamma_1$	Nutrient leaching rate	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\alpha_1$	Specific rate of utilization of favourable resources by plant population density	$\text{mgKg}^{-1} \text{ day}^{-1}$
$\beta_1$	Rate of interaction of nutrient and favourable resources	$\text{mgKg}^{-1} \text{ day}^{-1}$