SERIES EXPANSION METHODS FOR SOLVING NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Non-linear partial differential equations (PDEs) play a significant role in portraying most physical phenomenon occurring in the field of engineering and physical sciences. It is difficult to obtain an analytical solution of these mathematical models so many researchers have applied various semi -analytical and numerical techniques to solve these equations. In this thesis, our focus is to obtain the analytical solution as a convergent series solution of non-linear PDEs, non-linear coupled and non-linear fractional PDE. We have used homotopy perturbation method with integral transformation like Laplace transformation, Sumudu transformation and Elzaki transformation for the series solution of the above-said equations. The conditions for the convergence of the series solution have been derived and verified by applying it on some well known physical model. Further, these solutions analyzed using error analysis and are represented in the tabular form and surface graphs.

The first chapter covers the introductory part of partial differential equations (PDEs), perturbation theory and background of homotopy perturbation method. Further, it contains the literature survey of the various semi-analytical techniques like HPM, VIM, DTM,HPTM, HPSTM, HPETM and their modifications which occurred in the last few decades. In the second chapter, HPTM is implemented successfully for the series solution of a higher order non-linear PDE. Finally, the series solution of fifth-order Korteweg-de Vries equation is obtained which describes the model of waves occurring in "shallow water waves". In the next chapter, HPTM

and HPSTM are implemented for the solution of non-linear coupled and fractional PDEs. As an application of HPTM, the solution of Coupled KdV equations of order three, Hirota Satsuma KdV system, 1 and 2-dimensional coupled Burgers equation are obtained. Moreover, we have executed HPSTM to solve fractional K(2, 2) equation, Sawada Kotera equations, KdV equations and 1-D coupled attractor Keller Segel equations. While using these semi-analytical techniques, the solution of the non-linear PDEs is obtained in the form of infinite series. So, for the credibility of the obtained series solution, we have derived the condition of convergence of the series solution obtained by using HPSTM. Then, we have implemented the condition of convergence to find the solution of Newell-Whitehead-Segel equation and Fishers equations. Moreover, we have performed the error analysis and the condition of maximum truncation error is verified. Further, we have achieved the convergence of the HPTM of the series solution of non-linear fractional PDE and then we have actualized the said procedure to solve the Burgers' equation. In the subsequent chapter, HPTM and HPETM are applied to solve the fractional non-linear PDE and comparative analysis has been performed between the two methods. At last, we have proposed a new efficient semi-analytical technique which is hybrid of HPM and "Sumudu transformation" method where we have used a new form of Hes polynomial named as Accelerated Hes polynomial and we conclude that this technique is more efficient semi-analytical technique than other classical techniques. To validate the above argument, we have implemented the proposed technique on a non-linear partial differential equation with proportionate delay. The convergence of the series solution is verified and finally, the error analysis and the statistical analysis has been performed to examine the precision of the proposed technique.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Preliminary

Differential equations are utilized to express many general laws of nature and have numerous applications in physical, social, economic and other dynamical frameworks. Specifically, the origin of the differential equation might be considered as the endeavors of Newton to represent the movement of particles. These equations may give numerous valuable data about the framework if the condition is shaped joining the different vital elements of the framework.

A differential equation depicts a relation between independent, dependent variables and its derivatives. We may characterize the differential equations in two sections:

1. Ordinary differential equation (ODE)

2. Partial differential equation (PDE)

The differential equation which involves only one independent variable, one dependent variable and its derivative with respect to an independent variable is known as an ordinary differential equation. Some well -known examples of ordinary differential equations are exponential decay or growth population model, prey-predator model, Rayleigh's equation (has application in fluid dynamics) and Lane–Emden equation (has application in astrophysics).

A PDE is an equation which involves more than one independent variables like $x_1, x_2, ..., x_n$; a dependent variable u and its partial derivative w.r.t the independent variables such as $F\left(x_1, x_2, ..., x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right) = 0$. PDEs show up as often as possible in every aspect of physics and engineering. Moreover, in recent years we find that partial differential equations have extraordinary significance in numerous areas like biology, chemistry, image processing, graphics and in economics (finance).

These partial differential equations are upgraded by some extra conditions, for example, initial and boundary conditions. As the focus of our research is on partial differential equations only, henceforth, we will give a brief description of partial differential equations.

The analysis of partial differential equations has numerous viewpoints. The established methodology is to make new strategies for finding explicit solutions. Each mathematical advancement that empowers a solution of a new class of partial differential equations prompts a colossal headway in physics because of the tremendous significance of PDEs in physical science. The method of characteristics which was structured by Hamilton incited noteworthy advances in optics and mechanics. The advancement in PDEs has been accomplished with the introduction of numerical techniques. The theoretical analysis of partial differential equations is not only because of educational interest rather it has numerous applications. These PDEs may derive from some physical problems or a model of engineering. Moreover, it is expected in most of the cases that the solution of PDEs ought to be unique and stable under small disturbances of data. So, it is essential to have a complete analysis of the partial differential equations before solving it.

The French mathematician Jacques Hadamard (1865 - 1963) authored the idea

of the well-posedness condition of PDE. The PDE is said to be well-posed if does not depend only on the solution but also depends upon some additional conditions like initial or boundary conditions. The well-posed problem has the following conditions:
Existence: The solution to the problem exists.
Uniqueness: The solution that depicts a specific physical problem must be unique.
Stability: A small change in an equation or auxiliary conditions create only a small change in the solution
The existence and uniqueness of the PDE are given by Cauchy-Kowalevski theorem, which states that the Cauchy problem has a locally unique analytic solution if the PDE coefficients are analytic in the unknown function and its derivatives.

1.1.1 Some basic terminologies

Definition 1.1.1 ***Order of PDE****: It is the order of the highest partial derivative that occurs in the equation. e.g.:* $u_t = c^2 u_{xx}$ *is the partial differential equation (PDE) of order 2, while* $u_t + 2u_{xx} + u_{xxxxx} = 0$ *is PDE of order 5, where* u_t *and* u_x *represent partial derivative with respect to* t *and* x*.*

Definition 1.1.2 ***Linear PDE****:*

A linear PDE is that in which the dependent variable and its partial derivative are linear. Some well-known examples of linear PDEs are given below:

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = 0 \qquad \text{(Transport equation)}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{(Laplace equation)}$$

$$(\nabla^2 + k^2)u = 0 \qquad \text{(Helmholtz equation)}$$

Definition 1.1.3 ***Quasi-linear PDE****:A PDE is said to be quasi-linear if the highest order derivative coefficient does not depend upon the highest order partial* *derivative of dependent variable i.e. if the PDE is of order k then, the coefficient of k^{th} order term contain any function of an independent variable and dependent variable of order less than k. Some of the well-known examples of quasi-linear PDEs are*

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = \nu \frac{\partial^2 \phi}{\partial x^2} \qquad \text{(Burgers' equation)}$$

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^4 \phi}{\partial x^4} - 3\frac{\partial^2 (\phi^2)}{\partial x^2} = 0 \qquad \text{(Boussinesq equation)}$$

Definition 1.1.4 ***Semilinear PDE***: *A PDE is said to be semilinear if the highest order derivative coefficient does not depend upon the dependent variable and its derivative. Some of the well-known examples of semilinear PDEs are*

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u^2 = 0 \qquad \text{(Transport equation)}$$

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} + 6\frac{\partial^3 \phi}{\partial x^3} = 0 \qquad \text{(Korteweg-de Vries equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \text{(Poisson's equation)}$$

Definition 1.1.5 ***Non-linear PDE***: *A PDE is non-linear if the highest order derivative coefficient has non-linearity in the dependent variable. Some examples of non-linear partial differential equations are:*

$$\phi_x^2 + \phi_y^2 = 1, \qquad \text{(Eikonal equation)}$$

$$div\left(\frac{\nabla \phi}{\sqrt{(1 + |\nabla \phi|^2)}}\right) = 0. \qquad \text{(Minimal surface equation)}$$

Definition 1.1.6 ***Solution of PDE***: *The function ϕ is called the solution of the PDE if the function ϕ is continuous and has continuous partial derivative up to the order of the PDE and satisfies the PDE.*

If the function ϕ is discontinuous then ϕ is called weak solution of PDE. For example, Such equation governs the problem of fluid dynamics where discontinuous *solution called shocks waves to develop and propagate across the computational domain. A typical example is the sonic blast created by a supersonic aircraft when it surpasses the speed of sound.*

Definition 1.1.7 ***Initial condition****: If ϕ and all its derivatives are of order $\leq n$ are continuous on domain D contained in space of independent variable of ϕ, then ϕ is said to of space C^n.*

If the independent variable is time and the condition to be satisfied at the initial point, i.e. $t = 0$ then it is called initial condition. A problem which involves the partial differential equation based on initial condition only is called initial value problem.

Definition 1.1.8 ***Boundary condition****: If the conditions are defined on the boundary ∂D of the domain D then, the conditions are called boundary conditions. PDE which includes the boundary conditions is said to be a boundary value problem. There are mainly three types of boundary conditions:*

Definition 1.1.9 ***Dirichlet boundary condition*** *:If the conditions determined the estimation of the dependent variable on the boundary ∂D of the domain, then the conditions are said to be a Dirichlet boundary condition. For example: Consider the following BVP*

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0, x, y \in R, t \geq 0 \tag{1.1}$$

where

$$B(\phi) = 0 \ on \ \partial D \tag{1.2}$$

If $B(\phi) = 0$ stands for the following boundary condition

$$\phi = 0 \ on \ \partial D \tag{Dirichlet condition}$$

Definition 1.1.10 ***Neumann boundary condition*** *:If the conditions specified the derivative of the dependent variable on the boundary ∂D of the domain, then the conditions are said to be a Neumann boundary condition. For example: if in the problem (1.1), the boundary conditions $B(\phi) = 0$ is of the form*

$$\frac{\partial \phi}{\partial x} = 0 \ on \ \partial D \qquad \text{(Neumann condition)}$$

Definition 1.1.11 ***Robin boundary condition*** *:If the conditions involve the dependent variable and its derivative on the boundary ∂D of the domain, then the conditions are said to be a Robin boundary condition or mixed boundary conditions. For example: if in the problem (1.1), the boundary conditions $B(\phi) = 0$ is of the form*

$$\frac{\partial \phi}{\partial x} + \phi = 0 \ on \ \partial D \qquad \text{(Robin condition)}$$

1.2 Fractional calculus

Fractional calculus is the study of the mathematical science that comes out of the customary meaning of the integer-order differentiation and integration. It gives a few tools for fathoming arbitrary order differential and integral equation. The fractional calculus is as old as traditional calculus, however, has gained significant importance amid the previous few decades, because of its immense importance in various assorted fields of science and engineering which include fluid flow, visco-elasticity, solid mechanics, signal processing, probability, statistics, etc. The number of works managing dynamical frameworks portrayed by fractional-order equation that include derivative and integral of arbitrary order as they delineate the memory and innate properties of various substances. In 1695, L'Hopital wrote a letter to Leibnitz in which he used to get some information about a particular notation he published for the n^{th}-order derivative of the linear function. He made an inquiry to

Leibniz, what may the result be if n is half. Leibniz responded by saying that it is an obvious conundrum, which will result in significant outcomes one day. So, this was the first time when fractional derivative came into the picture.

Many researchers utilizing their definitions and notations to present the idea of fractional order derivative and integral. The definitions which have been advanced in the realm of the fractional derivative are the Caputo, Grunwald-Letnikov, and Riemann-Liouville. The Riemann-Liouville definition is for the most part utilized yet this methodology isn't entirely appropriate for physical problems and real-world problems. Caputo introduced the definition at which the initial conditions are defined at the integral order dissimilar to the Riemann-Liouville at which the initial conditions are defined at fractional order. The Grunwald - Letnikov method proceeds towards the problem from the definition of the derivative. This method is used exclusively in numerical algorithms. Grunwald-Letnikov definition is the extension of the definition of derivative for fractional order.

1.2.1 Fractional derivatives and Homotopic function

This section is devoted to the review of the three important definitions of fractional derivatives viz. Riemann-Liouville, Grunwald-Letnikov, and Caputo of fractional derivative and some other basic definitions.

Definition 1.2.1 *A real function $g(t) \in C_\mu, t > 0, \mu \in \mathbb{R}$ if $\exists\ q \in \mathbb{R}; (q > \mu)$, such that $g(t) = t^q k(t)$, where $k(t) \in C[0, \infty)$ and $g(t) \in C_\mu^m$ if $g^{(m)} \in C_\mu, m \in \mathbb{N}$.*

Definition 1.2.2 *Grunwald-Letnikov derivative of fractional order p is given as*

$$f_h^p(t) = \frac{1}{h^p} \sum_{r=0}^{n} (-1)^r\ {}^pC_r f(t - rh)$$

Definition 1.2.3 *Riemann-Liouville (R-L) definition of fractional order derivative* *is*

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\eta)}{(t-\eta)^{\alpha-n+1}} d\eta, (n-1) \le \alpha < n \tag{1.3} $$

Definition 1.2.4 *Caputo definition of fractional order derivative is given as*

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\eta)}{(t-\eta)^{\alpha-n+1}} d\eta, (n-1) \le \alpha < n \tag{1.4} $$

Definition 1.2.5 *The Mittag-Leffler function of two parameter α and β is given by [40]*

$$ E_{\alpha,\beta}(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{\Gamma(\alpha n+\beta)}, \alpha, \beta > 0 \tag{1.5} $$

Definition 1.2.6 ***Homotopy (Homotopic functions)****: Let ϕ and ψ be two continuous functions defined from a (topological) space X into Y, then they are said to be homotopic if*

1. *ϕ and ψ have same initial and final points in X,*

2. *there exist a continuous function,*

$$ H : X \times [0,1] \to Y $$

such that $H(x,0) = \phi(x)$ and $H(x,1) = \psi(x)$

Example: Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be any two continuous real functions. Let us define a function $H : \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$ H(x,t) = (1-t)\phi(x) + t\ \psi(x), 0 \le t \le 1 $$

Clearly, H is continuous as it is composition of two continuous functions. Moreover, $H(x,0) = \phi(x)$ and $H(x,1) = \psi(x)$. Thus, H is a homotopy between ϕ and ψ.

1.3 Perturbation theory

This theory involves techniques to find an estimated solution to the problem by introducing a precise solution to the related simple problem. If the problem cannot be resolved properly, the theory of perturbation is pertinent, be that as it may, can be unraveled by adding a small parameter to the numerical depiction of the critical physical problem. To elucidate the said technique. Consider the following non-linear differential equation.

$$L(x) + \epsilon N(x) = 0, \tag{1.6}$$

where x depends upon t only, i.e. $x = x(t)$, $L(x)$ and $N(x)$ are linear and non-linear operators and ϵ is a small parameter. Here we consider the non-linear term as a perturbation in (1.6). We assume the solution of (1.6) as a power series in small parameter ϵ.

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots \tag{1.7}$$

On substituting (1.7) in (1.6) and comparing the like terms of ϵ, we get various differential equations that can be effectively comprehended to acquire the estimations of $x_0(t), x_1(t), x_2(t), \dots$.

1.4 Series expansion methods for non-linear partial differential equations

In general, obtaining an analytical solution of non-linear PDE is difficult and apart from that, most of the fractional equations don't have the definite solutions, consequently, there is a considerable focus on the numerical and semi-analytical solutions of such equations. Several semi-analytical techniques have been proposed to discover the solution of non-linear, coupled and fractional PDE like

1. Variational Iteration Method

2. Modified Variational Iteration Method
3. Adomian Decomposition Method
4. Laplace Decomposition Method
5. Differential Transform Method
6. Homotopy Analysis Method
7. Homotopy Perturbation Method
8. Homotopy Perturbation Transformation Method
9. Homotopy Perturbation Sumudu Transformation Method
10. Homotopy Perturbation Elzaki Transformation Method

As our focus of research is based on the techniques using Homotopy perturbation method, so in the next section, we will discuss the description of the Homotopy perturbation method, while HPTM, HPSTM, and HPETM will be explained in the subsequent chapters.

1.5 Homotopy perturbation method (HPM)

Consider the following non-linear PDE

$$L(\phi) + N(\phi) = k(\rho), \ \rho \in \Omega \tag{1.8}$$

with boundary condition

$$B\left(\phi, \frac{\partial \phi}{\partial n}\right) = 0, \ \rho \in \Gamma \tag{1.9}$$

where L is a linear and N is non-linear operator and $k(\rho)$ is an analytic function, Γ is the boundary of the domain Ω. In 1999, Dr. He [42], construct a homotopy of eq. (1.8) as $\mathcal{H} : \Omega \times [0, 1] \to \mathbb{R}$ defined as

$$\mathcal{H}(\phi, p) = p(L(\phi) + N(\phi) - k(\rho)) + (1 - p)(L(\phi) - L(\phi_0)) = 0 \quad (1.10)$$

or

$$\mathcal{H}(\phi, p) = pL(\phi_0) + p(N(\phi) - k(\rho)) + (L(\phi) - L(\phi_0)) \quad (1.11)$$

where p is parameter such that $0 \leq p \leq 1$ and ϕ_0 is an initial value of ϕ that satisfies the eq. (1.9). Clearly, from (1.10), we have

$$\mathcal{H}(\phi, 0) = L(\phi) - L(\phi_0) = 0 \quad (1.12)$$

$$\mathcal{H}(\phi, 1) = (L(\phi) + N(\phi) - k(\rho)) = 0 \quad (1.13)$$

As the value of p changes from 0 to 1, ϕ varies from ϕ_0 to $\phi(x, t)$. The basic assumption for this method is that the solution of (1.8) can be expressed as $\phi = \phi_0 + \phi_1 p + \phi_2 p^2 + \phi_3 p^3 + \ldots$. The solution of (1.8) is given by

$$\begin{aligned} \phi(x, t) &= \lim_{p \to 1} (\phi_0 + \phi_1 p + \phi_2 p^2 + \phi_3 p^3 + \ldots) \\ &= \phi_0 + \phi_1 + \phi_2 + \ldots \end{aligned}$$

1.6 Integral transformation

Definition 1.6.1 *The integral transformation (I.T.)of function $f(\tau)$ is defined as a map*

$$\begin{aligned} I &: C_0(f) \to C_0(f) \\ I\{f(\tau)\} &= F(k) = \int_{-\infty}^{\infty} K(\tau, k) f(\tau) d\tau \end{aligned}$$

where $C_0(f)$ is a space of all continuous functions and $K(\tau, k)$ is the kernel of transformation.

1.6.1 Laplace transformation and its properties

Definition 1.6.2 *The Laplace transformation is the I.T. with the kernel of transformation is $K(\tau, k) = e^{-s\tau}$ which is non-zero for the positive value of τ and is defined as*

$$\mathcal{L}[f(\tau)] = \int_0^\infty f(\tau)e^{-s\tau}d\tau = F(s), \tau > 0. \tag{1.14}$$

Properties:

1. $\mathcal{L}\{1\} = \frac{1}{s}$,

2. $\mathcal{L}\{t^m\} = \frac{\Gamma(m+1)}{s^{m+1}}$,

3. $\mathcal{L}[f^{(n)}(\tau)] = s^n\mathcal{L}[f(\tau)] - \sum_{i=0}^{n-1} s^{n-1-i}f^{(i)}(0)$.

4. $\mathcal{L}\left\{\frac{\partial^\alpha}{\partial\tau^\alpha}f(\tau)\right\} = s^\alpha\mathcal{L}\{f(\tau)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(0), n-1 < \alpha \leq n$
   where $\frac{\partial^\alpha}{\partial\tau^\alpha}$ is Caputo fractional derivative [17].

1.6.2 Sumudu transformation and its properties

Definition 1.6.3 *The Sumudu transformation [105] of $f(t)$ is defined as*

$$S[f(t)] = \frac{1}{u}\int_0^\infty f(t)e^{-\frac{t}{u}}dt, t > 0. \tag{1.15}$$

Properties:

1. $S\{1\} = 1$,

2. $S\{\frac{t^m}{\Gamma(m+1)}\} = u^m$,

3. $S\{f^\alpha(t)\} = \frac{1}{u^\alpha}S\{f(t)\} - \sum_{k=0}^{k=n-1}\frac{1}{u^{n-k}}f^k(0), n-1 < \alpha \leq n$,
   where f^α is the α order Caputo fractional derivative [17] of f.

4. $S\{f^n(t)\} = \frac{1}{u^n}S\{f(t)\} - \sum_{k=0}^{k=n-1}\frac{1}{u^{n-k}}f^k(0)$

1.6.3 Elzaki transformation and its properties

Definition 1.6.4 *The Elzaki transformation [23, 24] of $g(\tau)$ is defined as*

$$E[g(\tau)] = v\int_0^\infty e^{\frac{-\tau}{v}} g(\tau)d\tau = F(v), \tau > 0. \tag{1.16}$$

Properties:

1. $E\{1\} = v^2$,

2. $E\{\frac{t^m}{\Gamma(m+1)}\} = v^{m+2}$,

3. $E\left\{\frac{\partial^\alpha}{\partial\tau^\alpha} g(\tau)\right\} = \frac{E\{g(\tau)\}}{v^\alpha} - \sum_{k=0}^{n-1} v^{k-\alpha+2} g^{(k)}(0), n-1 < \alpha \leq n,$
   where $\frac{\partial^\alpha}{\partial\tau^\alpha}$ is Caputo fractional derivative [17]

1.7 Literature review

The homotopy perturbation technique is greatly available to non-mathematicians and engineers. J He, proposed HPM [42], this technique has been considered an incredible scientific tool for different kind of non-linear problems, as it is a promising and advancing technique. In addition to its scientific significance and association with other branches of mathematics, it is broadly utilized in all the ramifications of current science.

Different perturbation methods have been utilized to handle non-linear issues. Sadly, the conventional perturbation methods rely upon the doubt that small parameter should be present, which is over-exacting, making it hard to find more extensive applications because most non-linear equations have no little parameter using any means. Therefore, numerous new techniques have as of late acquainted some ways to wipe out of the small parameter like artificial parameter method which is introduced by Liu [72], Liao [70, 71] proposed HAM, VIM proposed by J He [41], differential transform method (DTM) by Zhou [119],ADM by Adomian

[4] and others. Now we discussed the various authors those who have used these semi-analytical techniques on different classes of physical problems.

G.Adomian [4] proposed a solution for non-linear stochastic equations by considering different types of non-linearity. Further, Adomian [5] has applied the decomposition and asymptotic decomposition method for the different non-linear and the system of non-linear PDEs. Adomian [6] applied decomposition technique on various physical problems like duffing equation, non-linear transport equation, matrix Riccati equation, advection-diffusion and dissipative wave equation in which the non-linear terms are handled with Adomian's polynomial and they conclude that this method gives a viable technique to provide the precise solution of a wide class of dynamical system which represents the real physical problems. Adomian [7] provides the solution of coupled non-linear PDE with uncoupled boundary conditions.

J He [41] proposed VIM for solving non-linear PDE and in [43] applied variation iteration method on various non-linear models like duffing equation, mathematical pendulum, vibrations of the eardrum and then compared the approximation obtained by the proposed method to the Adomian's method and conclude that VIM provides the solution faster than Adomian's method. Further, J He [42] proposed homotopy perturbation method (HPM) by using homotopy concept used in topology and classical perturbation technique. J He [44], [45] and [46] applied HPM successfully on various non-linear differential equations. J He [47], made a comparison between HPM and HAM and conclude that HPM is a better option for non-linear problems than HAM.

Liao [69] compared HAM and HPM and showed that HPM is a special case of HAM. Moreover, he concludes that both the methods give better estimations with just a couple of terms if the conjecture and auxiliary linear operator are adequate.

Ganji and Sadighi [30] applied HPM to fathom coupled systems of non-linear reaction-diffusion equation and compared its result with ADM. Moreover, they conclude that the result obtained from HPM is in good concurrence with those of ADM. Ghorbani and Jafar [33] used HPM for calculating the Adomian polynomial. J He [50] gave a review of the VIM for solving some non-linear problems and he also listed useful iteration formula for some general non-linear problem. Further, he successfully implemented the variational iteration method on the integrodifferential equation, non-linear boundary value problem, oscillator, and wave equations. Wazwaz [107] studied the solution of homogeneous and non-homogenous advection problem using VIM and ADM and also presented the comparative study between these two methods. Babolian et al. [13] proposed some general guidelines to the researcher for choosing the homotopy equation and then applied these guidelines for solving some time-dependent equation like Klein-Gordon (K-G) equation, Emden-Fowler equation, Evolution equation, and Cauchy reaction-diffusion equation.

Ghorbani [32] defines He's polynomial to solve the non-linear problem and conclude that it is an easy and effective technique for the solution of the non-linear problem than Adomian polynomial. Further, Ghorbani presented the comparative study of He's HPM with other methods like ADM, direct method and series solution method on Integro-differential equations and conclude that HPM is more reliable than other traditional methods. Hesameddini and Latifizadeh [53] combined Laplace transformation with VIM to beat the trouble of figuring the Lagrange's multiplier and used for solving non-linear problems. Moreover, they conclude that the proposed technique is more efficient than the variational iteration method. Yildrim [114, 115, 116, 117] pertained HPM for the analytical solution of fractional non-linear Schrödinger equation, time and space fractional advection-dispersion equation and fractional PDEs evolved in liquid mechanics like wave equation, Korteweg-de Vries, Zakharov-Kuznetsov equation, Burgers' equation, and Klein–Gordon (K-G) equation.

Biazar et.al. [16] proposed a modified form of Adomian decomposition method, by this iterative method, the solution of a non-linear problem is obtained without calculating Adomian polynomial separately, this technique is implemented on non-linear partial differential equations and compared with ADM and VIM. Further, they conclude that this technique leads to the outcomes which are equivalent to those acquired by the variational iteration method. Das and Gupta [19] have solved time-fractional diffusion equation having the external force and absorbent term whereas Momani and Yildirim [77] have solved convection-diffusion fractional differential equation using HPM but having non-linear source term.

Chen and Wang [18] have successfully implemented VIM on the neutral differential equation with proportionate delay. Gondal and Khan [36] combine HPM with Laplace transformation to acquire the solution of the non-linear equation and further Pade approximation has been incorporated with HPM and Laplace transform to fastened the convergence of the series solution and named this technique as (HPTPM) homotopy perturbation transform Pade method.

Khan and Mohyud-Din [62] incorporated He's polynomial with Laplace transformation for the solution of MHD viscous fluid and recommended that this technique is more reliable and adequate. Further, Khan and Wu [64] combine HPM with Laplace transformation and named this technique as HPTM. Abazari and Ganji [1] proposed 2-D DTM, 2-D reduced DTM and their properties and implement these techniques for a non-linear partial differential equation with proportionate delay.

Gupta and Gupta [39, 67] employed HPTM on initial boundary value problem where they consider Dirichlet as well as Neumann type boundary condition for solving parabolic and hyperbolic like equations with variable coefficient. Madami et.al. [74]combined Laplace transformation with homotopy perturbation method and named it as Laplace homotopy perturbation method. They used this technique on non-homogeneous NPDE with variable coefficients. Haubold et.al.[40] gave a review on Mittag-Leffler functions, functional relation with Mittag-Leffler functions and its applications in fractional calculus. Cetinkaya et.al applied generalized differential transformation method on non-linear fractional Korteweg -de Vries, modified fractional Korteweg -de Vries equation and K (2,2) equation and conclude that results acquired utilizing the proposed technique exhibited here concur well with the numerical outcomes introduced somewhere else. Gupta and Singh [38] studied the analytical solution of well known Fornberg - Whitham equation with fractional order where fractional derivative was taken in Caputo sense.

Watugala [105] proposed a new form of integral transformation named as Sumudu transformation and its properties. Further, he implemented this transformation on differential equations and control engineering problems to discover the solution of these problems. Kumar et.al. [66] proposed Sumudu homotopy perturbation transformation as a coupling of Sumudu transformation and HPM and applied it on PDEs with variable coefficients. Further, Singh et.al. [97] applied homotopy perturbation Sumudu transformation on homogeneous and nonhomogenous advection problem and conclude that this technique is beneficial for non-linear problems.

Elzaki [23, 24, 25] proposed a transformation called Elzaki transformation and studied its properties and its applicability for solving linear ordinary and PDEs and also discussed its relation with Laplace transformation. Khan et.al.[61] proposed fractional Laplace homotopy perturbation transformation method for the fractional problem where they used modified Reimann–Liouville fractional-order derivative.

Sushila et.al [85] proposed HASTM the blend of HAM and Sumudu transformation to study the analytical solution of Fokker–Planck equations. Mishra and Nagar [76] proposed He-Laplace method as a blend of Laplace transform and HPM and implemented it on some linear and non-linear PDE's. Further, Singh et.al. [95] successfully implemented homotopy perturbation transformation method in time and space fractional reaction– diffusion equation. Dhaigude et.al. [20] applied ADM to study time-fractional, space fractional, time and space fractional Benjamin-Bonamahony-Burger's equations.

Elzaki and Hilal [26] consolidated HPM with Elzaki transformation in understanding non-linear partial differential equations. Grover et.al. [37] implemented HPM for solving linear and non-linear parabolic equations. El-Kalla [57] proposed homotopy perturbation technique in which he used a new form of He's polynomial for calculating the non-linear term which fastened the convergence of the series solution and named this technique as Accelerated homotopy perturbation method. El-Tawil and Huseen proposed a more generalized form of HAM and named it as the q-homotopy analysis method. The series solution obtained through this technique converges rapidly than HAM. They implemented this technique on some non-linear PDE's and they concluded that the region of convergence of series solution increases as the value of the parameter decreases.

Sharma and Kumar [90] used HPM for the solution of the third-order KdV equation. Mishra [75] used He-Laplace method for solving non-linear parabolic–hyperbolic PDE where the non-linear terms are dealt with He's polynomial. Kumar et.al. [68] and Arife et.al. [9] applied homotopy analysis transformation method for studying the solution of the fractional biological population model, fractional diffusion equation and concluded that the solution obtained from HATM is in great concurrence with the exact solution. Further, Singh et.al.[96] applied HPTM on fractional Fornberg-Whitham equation in which the fractional-order derivative was taken in Caputo sense. El-Tawil and Huseen [21] analyzed the convergence of q-homotopy analysis method, they likewise talked about the condition, if the parameter is doled out than the solution merges to the exact solution.

Karbalaie et.al. [58] applied homotopy perturbation Sumudu transformation method for the solution of non-linear fractional Fokker-Plank equation, biological population model, wave equation and linear system of equations. Khan and Usman [63] proposed modified HPTM for the non-linear boundary layer problem. They used a diagonally Pade approximation to deal with the boundary conditions at infinity. Abazari and Kilicman [2] solved first, second and third-order non-linear integrodifferential equations with proportionate delay by applying the differential transformation method (DTM) and conclude that the presented technique lessens the computational challenges of alternate techniques, and every one of the counts can be made straightforward controls.

Patra and Ray [83] applied HPSTM on fractional non-linear energy balance equation of fin temperature and then compared the obtained solution with other semi-analytical techniques like ADM and VIM. Further, they conclude that there is a decent ascension between HPSM results with those of traditional techniques like VIM and ADM.

Rubab et.al. [86] implemented homotopy perturbation Sumudu transformation technique on linear and non-linear inhomogeneous Klien-Gordan equations and calculated the exact solution of these equations. Yousif and Hamed [118] applied HPSTM on time-fractional non-linear Inviscid Burgers' equation, fifth-order KdV equation, etc. and calculated the solution in the closed form using Mittag-Leffler functions.

Adam [3] made a comparative study between He-Laplace method and successive approximation method and conclude that although both the techniques are powerful and efficient for getting a better approximate solution of linear and non-linear PDE's He-Laplace method reduces the volume of computation as compared to successive approximation method. Atangana [10] proposed modified homotopy perturbation method which is a blend of HPM using Abel's integral and decomposition method for solving non-linear Keller-Segel model with different cases of sensitivity functions and analyzed the technique with other semi-analytical techniques.

Ayati and Biazar [12] proposed the condition of convergence of HPM and implemented it for the solution of the Lane-Emden equation. Jassim [55] implemented HPTM for solving linear and non-linear Newell-Whitehead-Segel equation. Filobello-Nino et.al. [28] proposed a modification in Laplace transform HPM to get the precise solution of differential equations. In the proposed method, they introduced an initial approximation as an arbitrary function of a polynomial with some unknown parameter and they demonstrated the proficiency of the proposed technique by effectively executing this method on non-linear differential equations with mixed boundary conditions.

Johnston et.al. [56] implemented LHPM on space fractional-order and time-fractional order Burger's equation and conclude that the solution acquired from LHPM is in concurrence with the solution acquired from VIM and ADM. Neamaty et.al.[81] used HPETM on some time-fractional equation like time fraction advection equation, hyperbolic equation, and Fisher's equation and then compared the obtained solution of these equations with HPM and VIM.

Filobello-Nino et.al. [27] proposed some modification in LHPM to get the analytical solution of some variational problems. In their case study, the pertinence of their work comprised of two points. One point showed that the proposed modification is a very effective technique for linear and non-linear variational problems and secondly, they suggested some mathematical manipulation in the nonhomogenous differential equation with variable coefficient to transform them to the equation which can be easily handled with the proposed technique. Moutsinga et. al.[80] applied Laplace homotopy perturbation method to explain non-linear frameworks of the stiff Riccati differential equation emerging in finance.

Sakar et.al. [87] applied homotopy perturbation technique on time-fractional non-linear PDEs with proportionate delay. The condition of convergence and maximum truncation error has been discussed. Sedeeg [89] implemented homotopy perturbation Elazaki transformation method (HPETM) on time-fractional 1-D heat like equation, 2-D heat equation and 3-D heat like an equation. Moreover, he expressed the solution in the closed compact form using the Mittag-Leffler function. Tripathi and Mishra [103] effectively used Laplace transform HPM to obtain the solution of singular IVP of LaneEmden type differential equations. Martinez et.al. [112] proposed Feng's first integral method to the analytical solution of non-linear coupled space and time-fractional modified KdV equation, in which they have used Reimann-Liouville fractional derivative.

Wang and Liu [104] applied HPM to solve the non-linear time-fractional Fornberg-Whitham equation in which they used a fractional transformation to change the fractional differential equation into PDE and afterward they execute HPM for the solution of the above-said equation where the non-linear term is taken care of with He's polynomial. Wang et.al. [109] proposed the modification of exp–function method for the fractional PDE. The modification in the method is of the form of generalized Kudryashov method, generalized exponential rational function method which is implemented for the solution of fractional Benjamin-Bona-Mahony equation where they used He's fractional derivative. Further, they gave a conclusion that generalized exponential rational function technique is better than a generalized Kudryashov method for FDE.

Liu et.al. [73] implemented an amalgamation of HPM and Laplace transformation for the solution of non-linear PDE's. Tiwana et.al. [102] implemented HPTM for the solution of homogenous and non-homogenous non-linear fractional reaction-diffusion system of Lotka-Volterra type differential equations. Hendi and Qarni [52] proposed a combination of VIM with accelerated HPM i.e variational accelerated HPM to solve non-linear 2-D Volterra-Fredholm integrodifferential equations and the condition of convergence analysis of the analytic solution was also presented.

Gomez–Aguilar and Atangana [34] proposed a new form of fractional derivative via Liouville-Caputo sense and via Riemann-Liouville sense. They used Mittag-Leffler law, power, and exponential decay to model such fractional operator and implemented this fractional operator for the solution of Genesio-Tesi's model, Lotka-Volterra equations and Newton-Leipnik's model. Further, they concluded that this type of operator would be efficient, accurate and very useful to model complex physical problem. Singh and Kumar [94] implemented alternative VIM for the fractional non-linear PDEs with proportionate delay. They used Caputo fractional derivative and the solution obtained from the said technique quickly meets to the precise solution.

Gomez-Aguilar et.al. [35] implemented homotopy perturbation transformation technique to the analytical solution of some well known fractional non-linear PDE like KdV, Klien Gordan, Burgers' equation in which they used Caputo-Fabrizio operator as a fractional differential operator. Atangana and Gomez-Aguilar [11] proposed the numerical estimation of the R-L derivative. They have discussed the numerical approximation of Riemann-Liouville (R-L), Caputo-Fabrizio and Atangana-Baleanu in (R-L) sense. They discussed the application of R-L in mathematics and to model physical problems. Moreover, they executed these techniques for solving the fractional diffusion-advection equation. Srivastava et.al. ([100]) applied q-HAM and Laplace decomposition technique for solving fractional-order vibration equations.

Morales-Delgado et.al.[78] used the homotopy analysis method with Laplace transformation for the analytical solution of Keller Segel model where they used Caputo-Fabrizio differential operator and Atangana-Baleanu operator in Caputo sense. Yepez-Martnez and Gomez-Aguilar [110] implemented HPTM and Adams-Bashforth-Moulton method for the fractional differential equation where they used R-L, Liouville-Caputo, Caputo-Fabrizio, and generalized Mittag-Leffler law fractional operator in Caputo sense. Yepez-Martnez et.al. [111] used the first integral method for fractional non-linear PDE like non-linear fractional Sharma-Tasso-Olver, modified Benjamin-Bona-Mahony equation and Schrodinger equation where a fractional derivative is taken in a beta sense.

Chapter 2

Series Solution of Higher Order Non-linear PDE

A wide variety of problems like non-linear waves arise in gas dynamics, traffic problem, chromatography, water waves, and the biological system can be modeled as first-order non-linear PDE. Second-order non-linear PDE occurs in the study of fluid mechanics, thermodynamics, electrodynamics, internal waves in the deep water and population dynamics. Burgers' equation (Fluid mechanics), Fisher′s equation (Gene propagation), Benjamin- Ono (internal waves in deep water) and Fitzhugh-Nagumo (Biological neuron model), Navier-Stokes equation (Fluid flow, gas flow) etc. are some well known examples of second-order non-linear PDEs.

Third-order non-linear PDE arise during the study of shallow waves, solitons, Peakons, plasma waves. Some of the examples of third-order non-linear PDEs are Camassa-Holm equation (Peakon), Korteweg-de Vries (Shallow waves), Hirota Satsuma equation (shallow water waves) and so on.

In this chapter, we attempt to discover the solution of higher-order non-linear PDE using series solution methods specifically HPTM. Numerous scientist utilize distinctive strategies to unravel the KdV equation [15, 31, 84, 59, 60, 106, 113].

2.1 Homotopy perturbation transformation method (HPTM)

Consider the following general non-linear partial differential equation

$$\frac{\partial^n}{\partial t^n}w + Lw + Nw = f(x,t), t > 0, x \in \mathbb{R},, \tag{2.1}$$

where L and N are linear and non-linear differential operators respectively which satisfy Lipschitz condition and $f(x,t)$ is the source term. Now applying Laplace transform on (2.1), we get

$$\mathcal{L}\left\{\frac{\partial^n}{\partial t^n}w + Lw + Nw\right\} = \mathcal{L}\{f(x,t)\}.$$

Using (1.6.1), we have

$$\mathcal{L}\{w\} = \frac{1}{s^n}\left(\sum_{k=0}^{n-1} s^{n-k-1}w^{(k)}(x,0)\right) + \frac{1}{s^n}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}.$$

$$\mathcal{L}\{w\} = \sum_{k=0}^{n-1} s^{-k-1}w^{(k)}(x,0) + \frac{1}{s^n}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}.$$

Operating inverse Laplace transform , we get

$$w(x,t) = \sum_{k=0}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0) + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}\right\},$$

By applying HPM, we get

$$\begin{aligned} 0 =& (1-p)\Big(w(x,t) - w(x,0)\Big) + p\Big(w(x,t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0) \\ &- \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}\right\}\Big), \end{aligned}$$

$$w(x,t) = w(x,0) + p\left(\sum_{k=1}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0) + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{f(x,t) - Lw - Nw\right\}\right\}\right) \tag{2.2}$$

Let

$$\begin{aligned} w &= \sum_{n=0}^{\infty} p^n w_n, \\ N\ w &= \sum_{n=0}^{\infty} p^n H_n(x,t) \end{aligned} \tag{2.3}$$

where

$$H_n(w(x,t)) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(\sum_{i=0}^{\infty} p^i w_i\right) \tag{2.4}$$

Substituting (2.3) and (2.4) in (2.2), we get

$$\sum_{n=0}^{\infty} p^n w_n = w(x,0) + p\Bigg(\sum_{k=1}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0) \tag{2.5}$$

$$+\mathcal{L}^{-1}\Bigg\{\frac{1}{s^n}\mathcal{L}\Bigg\{f(x,t) - L\Bigg(\sum_{n=0}^{\infty} p^n w_n\Bigg) - \sum_{n=0}^{\infty} p^n H_n\Bigg\}\Bigg\}\Bigg) \tag{2.6}$$

On looking at the coefficients of like power of p, we have

$$\begin{aligned} p^0 : w_0 =& w(x,0); \\ p^1 : w_1 =& \sum_{k=1}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0) + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\Bigg\{f(x,t) - Lw_0 - H_0\Bigg\}\right\}; \\ p^2 : w_2 =& -\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\Bigg\{Lw_1 + H_1\Bigg\}\right\}; \\ p^3 : w_3 =& -\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\Bigg\{Lw_2 + H_2\Bigg\}\right\}, \\ &\vdots \end{aligned}$$

hence, the approximate solution is obtained as $p \to 1$

$$w(x,t) = w_0 + w_1 + w_2 + \ldots.$$

2.2 Application

To illustrate the working procedure and significance of HPTM, we implement this technique on the following well known higher-order non-linear PDEs. The present study demonstrates that HPTM is very proficient for comprehending such non-linear equations.

2.2.1 Fifth-order Korteweg-de Vries (KdV) equation

This equation was first presented by Korteweg and de Vries in 1895. This equation has numerous applications used to portray countless wonders of astrophysical and physical phenomena like wave phenomena in enharmonic crystals, it is also used to describe the waves occur in shallow water waves, ion-acoustic waves occur in plasma, etc. The general form of the fifth-order KdV equation is given as

$$\frac{\partial u}{\partial t} + Au^2\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + Cu\frac{\partial^3 u}{\partial x^3} + D\frac{\partial^5 u}{\partial x^5} = 0 \tag{2.7}$$

with initial condition

$$u(x,0) = h(x) \tag{2.8}$$

The above equation is known as Lax's fifth order KdV equation for $A = 30$, $B = 30$, $C = 10$, $D = 1$ and is known as Sawada-Kotera equation [31] with $A = 45$, $B = 15$, $C = 15$, $D = 1$. Now we impose HPTM for the solution of these two well known equations.

2.2.2 Solution of Sawada Kotera equation

Consider the Sawada Kotera Equation, given by

$$\frac{\partial u}{\partial t} + 45u^2\frac{\partial u}{\partial x} + 15\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 15u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0 \tag{2.9}$$

with initial condition

$$u(x,0) = 2m^2\operatorname{sech}^2(mx) \tag{2.10}$$

By applying the Laplace transformation on eq.(2.9) and using eq.(2.10), we get

$$u(x,s) = \frac{1}{s}(2m^2\operatorname{sech}^2(mx)) - \frac{1}{s}\mathcal{L}\left[u_{xxxxx} + 15uu_{xxx} + 15u_xu_{xx} + 45u^2u_x\right] \tag{2.11}$$

Operating inverse Laplace transformation on eq.(2.11), we get

$$u(x,t) = 2m^2 \operatorname{sech}^2(mx) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[u_{xxxxx} + 15uu_{xxx} + 15u_x u_{xx} + 45u^2 u_x\right]\right] \tag{2.12}$$

Now, we apply HPM on eq.(2.12)

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 2m^2 \operatorname{sech}^2(mx) - p\ \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[(\sum_{n=0}^{\infty} p^n u_n(x,t))_{xxxxx}\right]\right] - p\ \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{2.13}$$

A couple of terms of He's polynomials $H_n(u)$ are given by

$$H_0(u) = 15u_0 u_{0xxx} + 15u_{0x}u_{0xx} + 45u_0^2 u_{0x}$$

$$H_1(u) = 15(u_1 u_{0xxx} + u_{1xxx}u_0) + 15(u_{0x}u_{1xx} + u_{1x}u_{0xx}) + 45(2u_0 u_1 u_{0x} + u_0^2 u_{1x})$$

$$H_2(u) = 15(u_0 u_{2xxx} + u_1 u_{1xxx} + u_2 u_{0xxx}) + 15(u_{0x}u_{2xx} + u_{1x}u_{1xx} + u_{2x}u_{0xx}) + 45\left[(2u_1u_2 + 2u_0u_3)u_{0x} + (u_1^2 + 2u_0u_2)u_{1x} + 2u_0u_1u_{2x} + u_0^2 u_{3x}\right]$$

On looking at the coefficients of like power of p of eq.(2.13), we have

$$p^0 : u_0(x,t) = 2m^2 \operatorname{sech}^2(mx)$$

$$p^1 : u_1(x,t) = 64m^7 \tanh(mx)\operatorname{sech}^2(mx)t$$

$$p^2 : u_2(x,t) = -512m^{12}\operatorname{sech}^4(mx)\left(3 - 2\cosh^2(mx)\right)t^2$$

$$\vdots$$

Therefore, solution of eq. (2.9) when $p \to 1$ is:

$$\begin{aligned} u(x,t) &= 2m^2 \operatorname{sech}^2(mx) + 64m^7 \tanh(mx)\operatorname{sech}^2(mx)t \\ &- 512m^{12}\operatorname{sech}^4(mx)\left(3 - 2\cosh^2(mx)\right)t^2 + \dots \end{aligned}$$

Using the Taylor series, the above solution can be written as:

$$u(x,t) = 2m^2 \operatorname{sech}^2\left(mx - 16m^5 t\right)$$

2.2.3 Solution of Lax's fifth order equation

Consider the Lax's fifth order Equation

$$u_t + 30u^2u_x + 30u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0 \tag{2.14}$$

with condition

$$u(x,0) = 2k^2\left(3\,\mathrm{sech}^2(kx) - 1\right) \tag{2.15}$$

Applying Laplace transformation on eq.(2.14) using initial condition (2.15),we get

$$u(x,s) = \frac{1}{s}(u(x,0)) - \frac{1}{s}\mathcal{L}\left[u_{xxxxx} + 10uu_{xxx} + 30u_xu_{xx} + 30u^2u_x\right] \tag{2.16}$$

where

$$N(u(x,t)) = 30u^2u_x + 30u_xu_{xx} + 10uu_{xxx}. \tag{2.17}$$

Operating inverse Laplace transformation on eq.(2.16), we have

$$u(x,t) = u(x,0) = 2k^2\left(3\,\mathrm{sech}^2(kx) - 1\right) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[N(u(x,t))] + \frac{1}{s}\mathcal{L}[u_{xxxxx}]\right]. \tag{2.18}$$

Now, we apply HPM on eq.(2.18)

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n H_n(u)\right] + \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n u_n(x,t)\right]_{xxxxx}\right] \tag{2.19}$$

Some terms of $H_n(u)$ are given by

$$H_0(u) = 30u_0^2u_{0x} + 30u_{0x}u_{0xx} + 10u_0u_{0xxx}$$

$$H_1(u) = 30(2u_0u_1u_{0x} + u_0^2u_{1x}) + 30(u_{1x}u_{0xx} + u_{0x}u_{1xx}) + 10(u_1u_{0xxx} + u_0u_{1xxx})$$

$$\begin{aligned} H_2(u) \;&=\; 30[(2u_0u_2 + u_1^2)u_{0x} + 2u_0u_1u_{1x} + u_0^2u_{2x}] + 30[u_{2x}u_{0xx} + u_{1x}u_{1xx} + u_{2x}u_{0xx}] \\ &\quad +10[u_2u_{0xxx} + u_1u_{1xxx} + u_0u_{2xxx}] \end{aligned}$$

$$\vdots$$

On looking at the coefficient of like powers of p in eq.(2.19), we have

$$p^0 : u_0(x,t) = 2k^2\left(3\,\mathrm{sech}^2(kx) - 1\right)$$

$$p^1 : u_1(x,t) = 6k^7 t\,\mathrm{sech}^7(kx)\,[7\sinh(5kx) + 141\sinh(3kx) - 586\sinh(kx)]$$

$$\begin{aligned} p^2 : u_2(x,t) &= \frac{k^{12}t^2}{2}\,\mathrm{sech}^{12}(kx)[1602472 - 19327698\cosh(2kx) + 3754368\cosh(4kx) \\ &\quad -330327\cosh(6kx) + 8568\cosh(8kx) - 63\cosh(10kx)] \end{aligned}$$

$$\vdots$$

Therefore, solution of (2.14) when $p \to 1$ is:

$$\begin{aligned} u(x,t) &= 2k^2\left(3\,\mathrm{sech}^2(kx) - 1\right) + 6k^7 t\,\mathrm{sech}^7(kx)[7\sinh(5kx) + 141\sinh(3kx) \\ &\quad -586\sinh(kx)] + \frac{k^{12}t^2}{2}\,\mathrm{sech}^{12}(kx)[1602472 - 19327698\cosh(2kx) \\ &\quad +3754368\cosh(4kx) - 330327\cosh(6kx) + 8568\cosh(8kx) \\ &\quad -63\cosh(10kx)] + \ldots \end{aligned}$$

Using Taylor series, the above solution can be written as:

$$u(x,t) = 2k^2\left(3\,\mathrm{sech}^2(kx - 56k^5t) - 1\right)$$

2.3 Conclusion

1. HPTM is incredibly basic, simple to use and exceptionally precise for solving non-linear problems.

2. HPTM needs less computational work in comparison to other classical techniques.

3. HPTM is very precise and cost proficient tool for taking care of such non-linear problems.

Chapter 3

Series Solution of Coupled Non-linear PDE

3.1 Coupled partial differential equation

A system of PDEs with n variable is said to be coupled PDEs if the solution of one of the variables depends upon the solution of others. For example: consider the following system of equations:

$$\begin{aligned} \frac{\partial u}{\partial t} &= f(u, x, t), \\ \frac{\partial v}{\partial t} &= g(v, x, t) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \frac{\partial z}{\partial t} &= F(z, w, x, t, \frac{\partial z}{\partial x}, \frac{\partial w}{\partial x}), \\ \frac{\partial w}{\partial t} &= G(z, w, x, t, \frac{\partial z}{\partial x}, \frac{\partial w}{\partial x}) \end{aligned} \tag{3.2}$$

So, from equation (3.1) and (3.2), we conclude that equation (3.1) just represents the system of PDEs with variables u and v, whereas equation (3.2) represents, coupled system of partial differential equation because in equation (3.2) , solution of z depends upon the solution of w.

Numerous applications in material science are displayed by non-linear partial differential conditions. Various analysts are willing to comprehend these models, they emphasize finding a definite or estimated solution using diverse numerical or semi-analytical techniques. Here, we use HPTM for solving some system of non-linear coupled PDEs (third-order KdV Equations and coupled Burgers' equations in 1-D and 2-D). A few researchers have utilized HPM to comprehend such sort of non-linear coupled equations [8, 14, 30, 51, 101].

3.2 Application

Presently, we will endeavor to discover the solution for the most famous coupled PDEs with the assistance of HPTM.

3.2.1 Coupled Korteweg-de Vries equation

At the point when a framework bolsters two particular long-wave modes with almost correspondent stage speeds, the weakly non-linear and linear dispersion unfolding conventionally prompts two coupled KdV equations. The coupled Korteweg-de Vries equation of order three is given by

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \frac{\partial^3 \phi}{\partial x^3} + \phi\frac{\partial \phi}{\partial x} + \psi\frac{\partial \psi}{\partial x}\\ \frac{\partial \psi}{\partial t} &= -2\frac{\partial^3 \psi}{\partial x^3} + \phi\frac{\partial \psi}{\partial x}\end{aligned}$$

3.2.2 Solution of coupled KdV equation

Consider the system of KdV equation of order three

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \frac{\partial^3 \phi}{\partial x^3} + \phi\frac{\partial \phi}{\partial x} + \psi\frac{\partial \psi}{\partial x},\\ \frac{\partial \psi}{\partial t} &= -2\frac{\partial^3 \psi}{\partial x^3} + \phi\frac{\partial v}{\partial x}\end{aligned} \tag{3.3}$$

with initial conditions

$$\phi(x,0) = \left(3 - 6\tanh^2\frac{x}{2}\right), \psi(x,0) = -\left(3\iota\sqrt{2}\tanh^2\frac{x}{2}\right). \tag{3.4}$$

By applying Laplace transformation on eq.(3.3) and using (3.4), we get

$$\phi(x,s) = \frac{1}{s}\left(3 - 6\tanh^2\frac{x}{2}\right) + \frac{1}{s}\mathcal{L}\left[\phi_{xxx} + \phi\phi_x + \psi\psi_x\right], \tag{3.5}$$

$$\psi(x,s) = \frac{1}{s}\left(-3\iota\sqrt{2}\tanh^2\frac{x}{2}\right) - \frac{1}{s}\mathcal{L}\left[2\psi_{xxx} + \phi\psi_x\right]. \tag{3.6}$$

operating inverse Laplace transform on eq.(2.11) and (2.12), we get

$$\phi(x,t) = (3 - 6\tanh^2(\frac{x}{2}) + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\phi_{xxx} + \phi\phi_x + \psi\psi_x\right]\right], \tag{3.7}$$

$$\psi(x,t) = (-3\iota\sqrt{2}\tanh^2\frac{x}{2}) - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[2\psi_{xxx} + \phi\psi_x\right]\right]. \tag{3.8}$$

Now, we apply HPM on eq. (3.7) and (3.8), we have

$$\begin{aligned} \phi(x,t) &= \phi_0 + \phi_1 p + \phi_2 p^2 + \ldots, \\ \psi(x,t) &= \psi_0 + \psi_1 p + \psi_2 p^2 + \ldots \end{aligned} \tag{3.9}$$

$$\sum_{n=0}^{\infty} p^n\phi_n(x,t) = \left(3 - 6\tanh^2\left(\frac{x}{2}\right)\right) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\left(\sum_{n=0}^{\infty} p^n\phi_n(x,t)\right)_{xxx} + \sum_{n=0}^{\infty} p^n H_n^1(x,t)\right]\right], \tag{3.10}$$

$$\sum_{n=0}^{\infty} p^n\psi_n(x,t) = \left(-3\iota\sqrt{2}\tanh^2\left(\frac{x}{2}\right)\right) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left(\sum_{n=0}^{\infty} p^n\psi_n(x,t)\right)_{xxx} + \sum_{n=0}^{\infty} p^n H_n^2(x,t)\right]. \tag{3.11}$$

A couple of terms of He's polynomials i.e. $H_n^i, i = 1, 2$, are given by

$$H_0^1(x,t) = \phi_0\phi_{0x} + \psi_0\psi_{0x}$$

$$H_1^1(x,t) = (\phi_1\phi_{0x} + \phi_0\phi_{1x}) + (\psi_1\psi_{0x} + \psi_0\psi_{1x})$$

$$H_2^1(x,t) = (\phi_2\phi_{0x} + \phi_1\phi_{1x} + \phi_0\phi_{2x}) + (\psi_2\psi_{0x} + \psi_1\psi_{1x} + \psi_0\psi_{2x})$$

$$\vdots$$

Similarly,

$$H_0^2(x,t) = \phi_0\psi_{0x}$$

$$H_1^2(x,t) = (\phi_1\psi_{0x} + \phi_0\psi_{1x})$$

$$H_2^2(x,t) = (\phi_2\psi_{0x} + \phi_1\psi_{1x} + \phi_0\psi_{2x})$$

$$\vdots$$

On looking at the coefficients of like power of p of (3.10) and (3.11), we have

$$\begin{aligned} p^0 : \phi_0(x,t) &= \left(3 - 6\tanh^2\frac{x}{2}\right), \\ p^0 : \psi_0(x,t) &= -\left(3\iota\sqrt{2}\tanh^2\frac{x}{2}\right) \end{aligned}$$

$$p^1 : \phi_1(x,t) = -6t\,\mathrm{sech}^2\frac{x}{2}\tanh\frac{x}{2},$$
$$p^1 : \psi_1(x,t) = 3\iota\sqrt{2}t\,\mathrm{sech}^2\frac{x}{2}\tanh(\frac{x}{2})$$

$$p^2 : \phi_2(x,t) = \frac{3}{2}t^2\left(2\,\mathrm{sech}^2\frac{x}{2} + 7\,\mathrm{sech}^4\frac{x}{2} - 15\,\mathrm{sech}^6\frac{x}{2}\right)$$
$$p^2 : \psi_2(x,t) = \frac{3\iota\sqrt{2}}{4}t^2\left(2\,\mathrm{sech}^2\frac{x}{2} + 21\,\mathrm{sech}^4\frac{x}{2} - 24\,\mathrm{sech}^6\frac{x}{2}\right)$$

$$\vdots$$

Setting $p = 1$ results the approximate solution as:

$$\begin{aligned} \phi(x,t) &= \left(3 - 6\tanh^2\frac{x}{2}\right) + -6t\,\mathrm{sech}^2\frac{x}{2}\tanh\frac{x}{2} \\ &\quad + \frac{3}{2}t^2\left(2\,\mathrm{sech}^2\frac{x}{2} + 7\,\mathrm{sech}^4\frac{x}{2} - 15\,\mathrm{sech}^6\frac{x}{2}\right)\ldots \\ \psi(x,t) &= -\left(3\iota\sqrt{2}\tanh^2\frac{x}{2}\right) + 3\iota\sqrt{2}t\,\mathrm{sech}^2\frac{x}{2}\tanh(\frac{x}{2}) \\ &\quad + \frac{3\iota\sqrt{2}}{4}t^2\left(2\,\mathrm{sech}^2\frac{x}{2} + 21\,\mathrm{sech}^4\frac{x}{2} - 24\,\mathrm{sech}^6\frac{x}{2}\right)\ldots \end{aligned}$$

The results are similar to that obtained with HPM [8].

3.2.3 Coupled Hirota Satsuma equation

In 1981, Hirota and Satsuma [54] proposed the Coupled KdV condition and found that it has a 3-soliton arrangement. Further, in 1982, the author[88] discovered that the soliton of coupled Korteweg de Vries condition can be acquired from the

KP equation. Moreover, they demonstrate that after an appropriate scaling of the variables Coupled Hirota Satsuma equation converted to the coupled KdV equation.

This equation has application in shallow water waves.

3.2.4 Solution of coupled Hirota Satsuma KdV equation

Consider the following system of coupled equation.

$$\begin{aligned}
\frac{\partial\psi}{\partial t} &= \frac{1}{2}\frac{\partial^3\psi}{\partial x^3} - 3\psi\frac{\partial\psi}{\partial x} + 3\phi\frac{\partial\zeta}{\partial x} + 3\zeta\frac{\partial\phi}{\partial x},\\
\frac{\partial\phi}{\partial t} &= 3\psi\frac{\partial\phi}{\partial x} - \frac{\partial^3\phi}{\partial x^3},\\
\frac{\partial\zeta}{\partial t} &= 3\psi\frac{\partial\zeta}{\partial x} - \frac{\partial^3\zeta}{\partial x^3}.
\end{aligned} \tag{3.12}$$

subject to the initial condition

$$\psi(x,0) = -\frac{1}{3} + 2\tanh^3 x, \phi(x,0) = \tanh x, \zeta(x,0) = \frac{8}{3}\tanh x. \tag{3.13}$$

By applying the aforesaid method on eq.(3.12) and using(3.13), we get

$$\psi(x,s) = \frac{1}{s}[\psi(x,0)] + \frac{1}{s}\mathcal{L}\left[\frac{1}{2}\psi_{xxx} - 3\psi\psi_x + 3(\phi\zeta)_x\right], \tag{3.14}$$

$$\phi(x,s) = \frac{1}{s}[\phi(x,0)] + \frac{1}{s}\mathcal{L}\left[3\psi\phi_x - \phi_{xxx}\right], \tag{3.15}$$

$$\zeta(x,s) = \frac{1}{s}[\zeta(x,0)] + \frac{1}{s}\mathcal{L}\left[3\psi\zeta_x - \zeta_{xxx}\right]. \tag{3.16}$$

Now operating the inverse Laplace transform on eq.(3.14),(3.15) and (3.16), we get

$$\psi(x,t) = \left(-\frac{1}{3} + 2\tanh^3 x\right) + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{1}{2}\psi_{xxx} - 3\psi\psi_x + 3(\phi\zeta)_x\right]\right], \tag{3.17}$$

$$\phi(x,t) = \tanh x + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[3\psi\phi_x - \phi_{xxx}\right]\right], \tag{3.18}$$

$$\zeta(x,t) = \frac{8}{3}\tanh x + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[3\psi\zeta_x - \zeta_{xxx}\right]\right]. \tag{3.19}$$

Now, we apply HPM on eq.(3.17,3.18,3.19)

$$\sum_{n=0}^{\infty} p^n \psi_n(x,t) = -\frac{1}{3} + 2\tanh^3 x - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n H_n^1(x,t)\right] + \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n \psi_n(x,t)\right]_{xxx}\right], \tag{3.20}$$

$$\sum_{n=0}^{\infty} p^n \phi_n(x,t) = \tanh x + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n H_n^2(x,t)\right] - \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n \phi_n(x,t)\right]_{xxx}\right], \tag{3.21}$$

$$\sum_{n=0}^{\infty} p^n \zeta_n(x,t) = \frac{8}{3}\tanh x + p\mathcal{L}^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} p^n H_n^3(x,t)\right] - \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n \zeta_n(x,t)\right]_{xxx}\right]. \tag{3.22}$$

The couple of terms of He's polynomials $H_n^1(x,t)$, $H_n^2(x,t)$, $H_n^3(x,t)$ are given by

$$H_0^1 = -3\psi_0\psi_{0x} + 3\phi_0\zeta_{0x} + 3\zeta_0\phi_{0x},$$

$$H_1^1 = -3(\psi_1\psi_{0x} + \psi_{1x}\psi_0) + 3(\phi_0\zeta_{1x} + \phi_1\zeta_{0x}) + 3(\zeta_1\phi_{0x} + \zeta_0\phi_{1x}),$$

$$H_2^1 = -3(\phi_2\psi_{0x} + \psi_1\psi_{1x} + \psi_0\psi_{2x}) + 3(\phi_2\zeta_{0x} + \phi_1\zeta_{1x} + \phi_0\zeta_{2x}) + 3(\zeta_2\phi_{0x} + \zeta_1\phi_{1x} + \zeta_0\phi_{2x}),$$

$$\vdots$$

$$H_0^2 = 3\psi_0\phi_{0x},$$

$$H_1^2 = 3(\psi_1\phi_{0x} + \psi_0\phi_{1x}),$$

$$H_2^2 = 3(\psi_2\phi_{0x} + \psi_1\phi_{1x} + \psi_0\phi_{2x}),$$

$$\vdots$$

$$H_0^3 = \psi_0\zeta_{0x},$$

$$H_1^3 = \psi_1\zeta_{0x} + \psi_0\zeta_{1x},$$

$$H_2^3 = \psi_2\zeta_{0x} + \psi_1\zeta_{1x} + \psi_0\zeta_{2x},$$

$$\vdots$$

On looking at the coefficient of p of eq. (3.20,3.21,3.22), we have

$$\begin{aligned} p^0 : \psi_0(x,t) &= -\frac{1}{3} + 2\tanh^3 x, \\ p^0 : \phi_0(x,t) &= \tanh x, \\ p^0 : \zeta_0(x,t) &= \frac{8}{3}\tanh x, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} p^1 : \psi_1(x,t) &= 4t\,\mathrm{sech}^2\, x \tanh x, \\ p^1 : \phi_1(x,t) &= t\,\mathrm{sech}^2\, x, \\ p^1 : \zeta_1(x,t) &= \frac{8}{3}t\,\mathrm{sech}^2\, x, \end{aligned}$$

$$\vdots$$

$$p^2 : \psi_2(x,t) = 4t\,\mathrm{sech}^2\, x(1 - 3\tanh^2 x),$$

$$p^2 : \phi_2(x,t) = -t^2\,\mathrm{sech}^2\, x \tanh x,$$

$$p^2 : \zeta_2(x,t) = -\frac{8}{3}t^2\,\mathrm{sech}^2\, x \tanh x,$$

$$\vdots$$

The approximate solution of eq.(3.12) is acquired, as $p \to 1$ i.e.

$$\psi(x,t) = -\frac{1}{3} + 4t\,\mathrm{sech}^2\, x \tanh x + 4t^2\,\mathrm{sech}^2\, x(1 - 3\tanh^2 x)\ldots \tag{3.23}$$

$$\phi(x,t) = \tanh x + t\,\mathrm{sech}^2\, x - t^2\,\mathrm{sech}^2\, x \tanh x \ldots \tag{3.24}$$

$$\zeta(x,t) = \frac{8}{3}\tanh x + \frac{8}{3}t\,\mathrm{sech}^2\, x - \frac{8}{3}t^2\,\mathrm{sech}^2\, x \tanh x \ldots \tag{3.25}$$

So, from above solution, we analyse that the results obtained in eq. (3.23,3.24,3.25) are similar with HPM [8].

3.2.5 1-D coupled Burgers' equation

Burgers' equation is a non-linear PDE which has a wide application in fluid mechanics. The 1-D coupled Burgers' equation is to be considered as a mathematical model of sedimentation and development of the scaled volumetric concentration of two sorts of particles in liquid suspensions and colloids under the impact of gravity.

The 1-D coupled Burgers' equation is given by

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \frac{\partial^2 \phi}{\partial x^2} - \phi\frac{\partial \psi}{\partial x} - \psi\frac{\partial \phi}{\partial x} + 2\phi\frac{\partial \phi}{\partial x},\\ \frac{\partial \psi}{\partial t} &= \frac{\partial^2 \psi}{\partial x^2} - \phi\frac{\partial \psi}{\partial x} - \psi\frac{\partial \phi}{\partial x} + 2\psi\frac{\partial \psi}{\partial x}.\end{aligned}$$

3.2.6 Solution of 1-D coupled Burgers' equation

Consider the following system

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \frac{\partial^2 \phi}{\partial x^2} + 2\phi\frac{\partial \phi}{\partial x} - \frac{\partial(\phi\psi)}{\partial x},\\ \frac{\partial \psi}{\partial t} &= \frac{\partial^2 \psi}{\partial x^2} + 2\psi\frac{\partial \psi}{\partial x} - \frac{\partial(\phi\psi)}{\partial x}.\end{aligned} \tag{3.26}$$

subjected to the conditions

$$\phi(x,0) = \cos x, \psi(x,0) = \cos x. \tag{3.27}$$

Operating Laplace transform on both equations (3.26) and using initial conditions (3.27)

$$\phi(x,s) = \frac{1}{s}\cos x + \left[\frac{1}{s}\left(\mathcal{L}(\phi_{xx}) + \mathcal{L}(2\phi\phi_x - (\phi\psi)_x)\right)\right], \tag{3.28}$$

$$\psi(x,s) = \frac{1}{s}\cos x + \left[\frac{1}{s}\left(\mathcal{L}(\phi_{xx}) + \mathcal{L}(2\psi\psi_x - (\phi\psi)_x)\right)\right]. \tag{3.29}$$

Now, applying the inverse Laplace transformation on eq.(3.28,3.29)

$$\phi(x,t) = \cos x + \mathcal{L}^{-1}\left[\frac{1}{s}\left(\mathcal{L}(\phi_{xx}) + \mathcal{L}(2\phi\phi_x - (\phi\psi)_x)\right)\right], \tag{3.30}$$

$$\psi(x,s) = \cos x + \mathcal{L}^{-1}\left[\frac{1}{s}\left(\mathcal{L}(\phi_{xx}) + \mathcal{L}(2\psi\psi_x - (\phi\psi)_x)\right)\right]. \tag{3.31}$$

Now, we apply HPM on eq.(3.30,3.31)

$$\sum_{n=0}^{\infty} p^n \phi_n(x,t) = \cos x + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n H_n^1(x,t)\right] + \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n \phi_n(x,t)\right]_{xx}\right], \tag{3.32}$$

$$\sum_{n=0}^{\infty} p^n \psi_n(x,t) = \cos x - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n H_n^2(x,t)\right] + \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n \psi_n(x,t)\right]_{xx}\right]. \tag{3.33}$$

The couple of terms of $H_n^1(x,t)$, $H_n^2(x,t)$ are given below:

$$H_0^1(x,t) = 2\phi_0\phi_{0x} - (\phi_0\psi_{0x} + \phi_{0x}\psi_0),$$

$$H_1^1(x,t) = 2(\phi_1\psi_{0x} + \phi_0\phi_{1x}) - (\phi_1\psi_{0x} + \phi_0\psi_{1x} + \phi_{0x}\psi_1 + \phi_{1x}\psi_0),$$

$$H_2^1(x,t) = 2(\phi_2\psi_{0x}+\phi_1\phi_{1x}+\phi_0\phi_{2x})-(\phi_2\phi_{0x}+\phi_1\psi_{1x}+\phi_0\psi_{2x}+\psi_2\phi_{0x}+\psi_1\phi_{1x}+\psi_0\phi_{2x}),$$

$$\vdots$$

$$H_0^2(x,t) = 2\psi_0\psi_{0x} - (\phi_0\psi_{0x} + \phi_{0x}\psi_0),$$

$$H_1^2(x,t) = 2(\psi_1\psi_{0x} + \psi_0\psi_{1x}) - (\phi_1\psi_{0x} + \phi_0\psi_{1x} + \phi_{0x}\psi_1 + \phi_{1x}\psi_0),$$

$$H_2^2(x,t) = 2(\psi_2\psi_{0x}+\psi_1\psi_{1x}+\psi_0\psi_{2x})-(\phi_2\phi_{0x}+\phi_1\psi_{1x}+\phi_0\psi_{2x}+\psi_2\phi_{0x}+\psi_1\phi_{1x}+\psi_0\phi_{2x}),$$

$$\vdots$$

On looking at the coefficients of like power of p of eq.(3.32,3.33), we have

$$\begin{aligned} p^0 : \phi_0 &= \cos x, \\ p^1 : \phi_1 &= -t\cos x, \\ p^2 : \phi_2 &= \frac{t^2}{2}\cos x, \\ p^3 : \phi_3 &= -\frac{t^3}{6}\cos x, \\ &\vdots \end{aligned}$$

$$\begin{aligned}
p^0 : \psi_0 &= \cos x, \\
p^1 : \psi_1 &= -t\cos x, \\
p^2 : \psi_2 &= \frac{t^2}{2}\cos x, \\
p^3 : \psi_3 &= -\frac{t^3}{6}\cos x, \\
&\vdots
\end{aligned}$$

Therefore, solution of eq. (3.26) is acquired when $p \to 1$ i.e.:

$$\phi(x,t) = \phi_0 + \phi_1 + \phi_2 + \ldots, \psi(x,t) = \psi_0 + \psi_1 + \psi_2 + \ldots.$$

$$\phi(x,t) = \cos x\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \ldots\right), \tag{3.34}$$

$$\psi(x,t) = \cos x\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \ldots\right). \tag{3.35}$$

The solution obtained in eq.(3.34,3.35) in the closed form as $\phi(x,t) = \cos(x)e^{-t}$ and $\psi(x,t) = \cos(x)e^{-t}$. The results are similar to that obtained by HPM [101] and [51].

3.2.7 Solution of 2- dimensional coupled Burgers' equation

Consider the 2- dimensional equation

$$\phi_t - \nabla^2\phi - 2\phi\nabla\phi + (\phi\psi)_x + (\phi\psi)_y = 0, \phi_t - \nabla^2\psi - 2\psi\nabla\psi + (\phi\psi)_x + (\phi\psi)_y = 0. \tag{3.36}$$

subjected to the conditions

$$\phi(x,y,0) = \cos(x+y), \psi(x,y,0) = \cos(x+y). \tag{3.37}$$

Applying the Laplace transformation on equations (3.36) using initial conditions (3.37)

$$\phi(x,y,s) = \frac{1}{s}\cos(x+y) + \left[\frac{1}{s}\left(\mathcal{L}(\nabla^2\phi + 2\phi\nabla\phi - (\phi\psi)_x - (\phi\psi)_y)\right)\right], \tag{3.38}$$

$$\psi(x,y,s) = \frac{1}{s}\cos(x+y) + \left[\frac{1}{s}\left(\mathcal{L}(\nabla^2\psi + 2\psi\nabla\psi - (\phi\psi)_x - (\phi\psi)_y)\right)\right]. \tag{3.39}$$

Now, operating inverse Laplace transformation on eq. (3.38,3.39), we get

$$\phi(x,y,t) = \cos(x+y) + \mathcal{L}^{-1}\left[\frac{1}{s}\left(\mathcal{L}(\nabla^2\phi + 2\phi\nabla\phi - (\phi\psi)_x - (\phi\psi)_y)\right)\right], \quad (3.40)$$

$$\psi(x,y,t) = \cos(x+y) + \mathcal{L}^{-1}\left[\frac{1}{s}\left(\mathcal{L}(\nabla^2\psi + 2\psi\nabla\psi - (\phi\psi)_x - (\phi\psi)_y)\right)\right]. \quad (3.41)$$

Now, we apply HPM on eq. (3.40,3.41), we have

$$\sum_{n=0}^{\infty} p^n\phi_n = \cos(x+y) + p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n H_n^1\right] + \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n\phi_n\right]_{xx}\right]$$
$$+ p\mathcal{L}^{-1}\left\{\mathcal{L}\left[\sum_{n=0}^{\infty} p^n\phi_n\right]_{yy}\right\}, \quad (3.42)$$

$$\sum_{n=0}^{\infty} p^n\psi_n = \cos(x+y) - p\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n H_n^2(\phi,\psi)\right] + \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} p^n\psi_n\right]_{xx}\right]$$
$$- p\mathcal{L}^{-1}\left\{\mathcal{L}\left[\sum_{n=0}^{\infty} p^n\psi_n\right]_{yy}\right\}. \quad (3.43)$$

The couple of terms of H_n^1, H_n^2 are given by

$$H_0^1 = 2\phi_0\nabla\phi_0 - \phi_0\nabla\psi_0 - \psi_0\nabla\phi_0,$$

$$H_1^1 = 2(\phi_0\nabla\phi_1 + \phi_1\nabla\phi_0) - (\phi_0\nabla\psi_1 + \phi_1\nabla\psi_0) - (\psi_0\nabla\phi_1 + \psi_1\nabla\phi_0),$$

$$H_2^1 = 2(\phi_0\nabla\phi_2+\phi_1\nabla\phi_1+\phi_2\nabla\phi_0)-(\phi_0\nabla\psi_2+\phi_1\nabla\psi_1+\phi_2\nabla\psi_0)-(\psi_0\nabla\phi_2+\psi_1\nabla\phi_1+\psi_2\nabla\phi_0),$$

$$\vdots$$

$$H_0^2 = 2\psi_0\nabla\psi_0 - \phi_0\nabla\psi_0 - \psi_0\nabla\phi_0,$$

$$H_1^2 = 2(\psi_0\nabla\psi_1 + \psi_1\nabla\psi_0) - (\phi_0\nabla\psi_1 + \phi_1\nabla\psi_0) - (\psi_0\nabla\phi_1 + \psi_1\nabla\phi_0),$$

$$H_2^2 = 2(\psi_0\nabla\psi_2+\psi_1\nabla\psi_1+\psi_2\nabla\psi_0)-(\phi_0\nabla\psi_2+\phi_1\nabla\psi_1+\phi_2\nabla\psi_0)-(\psi_0\nabla\phi_2+\psi_1\nabla\phi_1+\psi_2\nabla\phi_0),$$

$$\vdots$$

On looking at the coefficients of like power of p of eq. (3.42,3.43), we have

$$p^0 : \phi_0 = \cos(x+y),$$
$$p^0 : \psi_0 = \cos(x+y),$$

$$p^1 : \phi_1 = -2t\cos(x+y),$$
$$p^1 : \psi_1 = -2t\cos(x+y),$$

$$p^2 : \phi_2 = 2t^2\cos(x+y),$$
$$p^2 : \psi_2 = 2t^2\cos(x+y),$$

$$p^3 : \phi_3 = -\frac{4t^3}{3}\cos(x+y),$$
$$p^3 : \psi_3 = -\frac{4t^3}{3}\cos(x+y),$$

$$\vdots$$

Therefore, solution of eq. (3.36) when $p \to 1$ is:

$$\phi(x,y,t) = \phi_0 + \phi_1 + \phi_2 + \ldots, \psi(x,y,t) = \psi_0 + \psi_1 + \psi_2 + \ldots$$

$$\begin{aligned} \phi(x,y,t) &= cos(x+y)\left(1 - 2t + \frac{4t^2}{2!} - \frac{8t^3}{3!}\ldots\right), \\ \psi(x,y,t) &= cos(x+y)\left(1 - 2t + \frac{4t^2}{2!} - \frac{8t^3}{3!}\ldots\right). \end{aligned} \tag{3.44}$$

The solution obtained in eq.(3.44) can be written in the closed form as $\phi(x,y,t) = \cos(x+y)e^{-2t}$ and $\psi(x,y,t) = \cos(x+y)e^{-2t}$. The obtained results are observed to be in great concurrence with HPM [51].

3.2.8 1-D Keller-Segel equations

In 1970, Keller and Segel presented a mathematical formulation of cellular slime mold aggregation process. The simplified form of the Keller Siegel equation in one dimension is given as

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial \chi(\rho)}{\partial x}\right), \tag{3.45}$$

$$\frac{\partial \rho}{\partial t} = b\frac{\partial^2 \rho}{\partial x^2} + cu(x,t) - d\rho(x,t). \tag{3.46}$$

In the above equations $u(x,t)$ and $\rho(x,t)$ represents the concentration of amoebae and chemical substance respectively. The chemo-tactic term $\frac{\partial}{\partial x}\left(u(x,t)\frac{\partial \chi(\rho)}{\partial x}\right)$ indicates the sensitivity of the cells, $\chi(\rho)$ called the sensitivity function of ρ.

3.2.9 Solution of coupled attractor 1-D Keller Segel equation

Consider the following coupled system:

$$\begin{aligned}\frac{\partial v}{\partial t} &= a\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}\left(v\frac{\partial \chi(\rho)}{\partial x}\right),\\ \frac{\partial \rho}{\partial t} &= b\frac{\partial^2 \rho}{\partial x^2} + c\ v - d\ \rho,\end{aligned} \tag{3.47}$$

subject to conditions

$$v(x,0) = m\ exp(-x^2), \rho(x,0) = n\ exp(-x^2). \tag{3.48}$$

Case-I Consider $\chi(\rho) = 1$, then $\frac{\partial}{\partial x}\left(v\frac{\partial \chi(\rho)}{\partial x}\right) = 0$,

hence Keller- Segel equation (3.47) reduces to

$$\begin{aligned}\frac{\partial v}{\partial t} &= a\frac{\partial^2 v}{\partial x^2},\\ \frac{\partial \rho}{\partial t} &= b\frac{\partial^2 \rho}{\partial x^2} + c\ v - d\ \rho.\end{aligned} \tag{3.49}$$

By applying HPTM on eq.(3.49), we have

$$\sum_{n=0}^{\infty} p^n v_n = v(x,0) + p\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{a\left(\sum_{n=0}^{\infty} p^n v_n\right)_{xx}\right\}\right\}, \tag{3.50}$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} p^n \rho_n &= \rho(x,0) + p\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{b\left(\sum_{n=0}^{\infty} p^n \rho_n\right)_{xx}\right\}\right\} \\
&\quad + p\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{c\left(\sum_{n=0}^{\infty} p^n v_n\right) - d\left(\sum_{n=0}^{\infty} p^n \rho_n\right)\right\}\right\}.
\end{aligned} \tag{3.51}
$$

On looking at the coefficients of like power of p of eq.(3.50) & (3.51) and using (3.48), we have

$$
\begin{aligned}
p^0 : v_0 &= m\ exp(-x^2); \\
p^0 : \rho_0 &= n\ exp(-x^2); \\
p^1 : v_1 &= \frac{am\ t}{1}\left(-2\ exp(-x^2) + 4x^2\ exp(-x^2)\right); \\
p^1 : \rho_1 &= \frac{t}{1}\left((c\ exp(-x^2)m - d\ exp(-x^2)n) - 2nb\ exp(-x^2)(2x^2-1)\right); \\
p^2 : v_2 &= \frac{a^2 m\ t^2}{2}\left(12\ exp(-x^2) - 48x^2\ exp(-x^2) + 16x^4\ exp(-x^2)\right); \\
p^2 : \rho_2 &= \frac{t^2}{2}((d\ exp(-x^2)(-cm+dn) + 2a\ exp(-x^2)cm(-1+2x^2) \\
&\quad + 2b\ exp(-x^2)(-1+2x^2)(cm-2dn) + 4b^2\ exp(-x^2)n(3-12x^2+4x^4)); \\
p^3 : v_3 &= \frac{a^3\ exp(-x^2)m\ t^3}{6}(-120+720x^2-480x^4+64x^6);
\end{aligned}
$$

$$
\begin{aligned}
p^3 : \rho_3 &= \frac{t^3}{6}(d^2(cm-dn)\ exp(-x^2) + b\ exp(-x^2)(6-24x^2+8x^4) \\
&\quad + 2acm\ exp(-x^2)(d-2dx^2) + 4b^2\ exp(-x^2)(cm-3dn)(3-12x^2+4x^4) \\
&\quad + 8b^3 n\ exp(-x^2)(-15+90x^2-60x^4+8x^6) + 4a^2\ exp(-x^2)cm(3-12x^2+4x^4) \\
&\quad + 2bd\ exp(-x^2)(-2cm+3dn)(-1+2x^2))), \\
&\ \vdots
\end{aligned}
$$

The approximate solution of eq.(3.49) is obtained as $p \to 1$ i.e.

$$v(x,t) = v_0 + v_1 + v_2 + \ldots,$$

$$\rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \ldots$$

$$v(x,t) = m\ exp(-x^2)\left(1 + a(-2+4x^2)\frac{t}{1} + a^2(12 - 48x^2 + 16x^4)\frac{t^2}{2}\right) + me^{-x^2}\left(a^3(-120 + 720x^2 - 480x^4 + 64x^6)\frac{t^3}{6}\right) + \ldots$$

$$\begin{aligned}
\rho(x,t) &= n\ exp(-x^2) + \frac{t}{1}\big((c\ exp(-x^2)m - d\ exp(-x^2)n) - 2nb\ exp(-x^2)(2x^2-1)\big)\\
&+ \frac{t^2}{2}((d\ exp(-x^2)(-cm+dn) + 2a\ exp(-x^2)cm(-1+2x^2)\\
&+ \frac{t^3}{6}(d^2(cm-dn)\ exp(-x^2) + b\ exp(-x^2)(6 - 24x^2 + 8x^4)\\
&+ 2acm\ exp(-x^2)(d - 2dx^2) + 4b^2\ exp(-x^2)(cm - 3dn)(3 - 12x^2 + 4x^4)\\
&+ 8b^3 n\ exp(-x^2)(-15 + 90x^2 - 60x^4 + 8x^6) + 4a^2\ exp(-x^2)cm\cdot\\
&\quad (3 - 12x^2 + 4x^4) + 2bd\ exp(-x^2)(-2cm + 3dn)(-1 + 2x^2)) + \ldots
\end{aligned}$$

Case-II Consider $\chi(\rho) = \rho$, then $\frac{\partial}{\partial x}\left(v\frac{\partial \chi(\rho)}{\partial x}\right) = \frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2}$,

hence, Keller- Segel equation (3.47) reduces to

$$\begin{aligned}
\frac{\partial v}{\partial t} &= a\frac{\partial^2 v}{\partial x^2} - \left(\frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2}\right),\\
\frac{\partial \rho}{\partial t} &= b\frac{\partial^2 \rho}{\partial x^2} + c\ v - d\ \rho.
\end{aligned} \tag{3.52}$$

Now, for the solution of eq. (3.52), we apply HPTM on eq. (3.52), we have

$$\sum_{n=0}^{\infty} p^n v_n = v(x,0) + p\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{a\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n v_n\right) - \left(\sum_{n=0}^{\infty} p^n H_n\right)\right\}\right\}, \tag{3.53}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} p^n \rho_n &= \rho(x,0) + p\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{b\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n \rho_n\right)\right\}\right\}\\
&\quad + p\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{c\left(\sum_{n=0}^{\infty} p^n v_n\right) - d\left(\sum_{n=0}^{\infty} p^n \rho_n\right)\right\}\right\}.
\end{aligned} \tag{3.54}$$

where

$$\sum_{n=0}^{\infty} p^n H_n(x,t) = \left(\frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2} \right).$$

An initial couple of terms of He's polynomial i.e. $H_n(x,t)$ are given below:

$$\begin{aligned}
H_0(x,t) &= v_{0x}\rho_{0x} + v_0\rho_{0xx};\\
H_1(x,t) &= v_{0x}\rho_{1x} + v_{1x}\rho_{0x} + v_0\rho_{1xx} + v_1\rho_{0xx};\\
H_2(x,t) &= v_{0x}\rho_{2x} + v_{1x}\rho_{1x} + v_{0x}\rho_{2x}\\
&\quad +v_0\rho_{2xx} + v_1\rho_{1xx} + v_1\rho_{0xx},\\
&\vdots
\end{aligned}$$

On looking at the like terms of p of eq. (3.53)& (3.54) and using eq.(3.48) and He's polynomial, we get

$$\begin{aligned}
p^0 : v_0(x,t) &= m\ exp(-x^2);\\
p^0 : \rho_0(x,t) &= n\ exp(-x^2);\\
p^1 : v_1(x,t) &= 2mt\ exp(-2x^2)\ (n - 4nx^2 + a\ exp(x^2)(-1+2x^2))\\
p^1 : \rho_1(x,t) &= t\left(c\ exp(-x^2)m - n(d\ exp(-x^2) + b\ exp(-x^2)(2-4x^2))\right);\\
p^2 : v_2(x,t) &= mt^2\left(-c\ exp(-2x^2)m(-1+4x^2) + 2a^2\ exp(-x^2)(3-12x^2+4x^4)\right)\\
&- mt^2\left(2a\ exp(-2x^2)n(7-58x^2+40x^4) + n(d\ exp(-2x^2)(-1+4x^2)\right)\\
&- mt^2\left(2b\ exp(-2x^2)(3-18x^2+8x^4) + 2n\ exp(-3x^2)(1-18x^2+24x^4)\right);\\
p^2 : \rho_2(x,t) &= \frac{1}{2}\ exp(-2x^2)t^2\left(-cd\ exp(x^2)m + d^2\ exp(x^2)n\right)\\
&+ \frac{1}{2}\ exp(-2x^2)t^2\left(2cmn - 8cmnx^2 + 2ac\ exp(x^2)m(-1+2x^2)\right)\\
&+ \frac{1}{2}\ exp(-2x^2)t^2\left(2bexp(x^2)(cm-2dn)(-1+2x^2)\right)\\
&+ \frac{1}{2}\ exp(-2x^2)t^2\left(4b^2exp(x^2)n(3-12x^2+4x^4)\right)\\
&\vdots
\end{aligned}$$

The solution of eq.(3.52) is obtained as $p \to 1$

$$v(x,t) = v_0 + v_1 + v_2 + \ldots$$

$$\rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \ldots$$

$$\begin{aligned} v(x,t) &= m\ exp(-x^2) + 2mt\ exp(-2x^2)\ (n - 4nx^2 + a\ exp(x^2)(-1+2x^2)) \\ &+ mt^2 \left(-c\ exp(-2x^2)m(-1+4x^2) + 2a^2\ exp(-x^2)(3 - 12x^2 + 4x^4)\right) \\ &- mt^2 \left(2a\ exp(-2x^2)n(7 - 58x^2 + 40x^4) + n(d\ exp(-2x^2)(-1+4x^2)\right) + \ldots \end{aligned}$$

$$\begin{aligned} \rho(x,t) &= n\ exp(-x^2) +\ t\left(c\ exp(-x^2)m - n(d\ exp(-x^2) + b\ exp(-x^2)(2-4x^2))\right) \\ &+ \frac{1}{2}\ exp(-2x^2)t^2\left(-cd\ exp(x^2)m + d^2\ exp(x^2)n\right) \\ &+ \frac{1}{2}\ exp(-2x^2)t^2\left(2cmn - 8cmnx^2 + 2ac\ exp(x^2)m(-1+2x^2)\right) \\ &+ \frac{1}{2}\ exp(-2x^2)t^2\left(2bexp(x^2)(cm - 2dn)(-1+2x^2)\right) \\ &+ \frac{1}{2}\ exp(-2x^2)t^2\left(4b^2exp(x^2)n(3 - 12x^2 + 4x^4)\right) + \ldots \end{aligned}$$

3.3 Conclusion

1. HPTM is extremely simple to handle the non-linear term present in the coupled equations.

2. This semi-analytical technique needs less computation than HPM, only a few iterations leads to the approximate solution of such complex non-linear problems.

3. We discover that solution obtained from this semi-analytical technique is in great concurrence with an exact solution.

4. HPTM is precise and costs proficient tool for taking care of such non-linear problems.

Chapter 4

Series Solution of Fractional PDE

Fractional calculus is an extension of basic calculus of arbitrary order. The physical problems engineering like thermodynamics, ecology, plasma physics occurring in the field of engineering and science so forth are displayed as a non-linear partial or fractional differential equation. In recent years, many researchers have been attracted towards fractional calculus because of its gigantic appropriateness to demonstrate the non-linear phenomenon. The non-linear complex phenomenon assumes an imperative job in physical sciences, and the generalized KdV equation is broadly utilized in the portrayal of waves in non-linear LC circuits, shallow and stratified inward waves, particle acoustic waves. It is hard to tackle these problems analytically henceforth, a few numerical and semi-analytical techniques are created to take care of such problems.

In this chapter, we have applied HPSTM to obtain the solution of some non-linear fractional PDE (Sawada-Kotera equation, KdV equation of fifth-order and K

(2,2) equations all of the time-fractional type).

4.1 Homotopy perturbation Sumudu transform method (HPSTM)

To understand the working procedure of this method, we consider the following fractional non-linear PDE

$$D_t^{\alpha} w + L\ w + N\ w = g(x,t), \tag{4.1}$$

with initial condition

$$\frac{\partial^s}{\partial t^s} w = w^s(x,0), s = 0,1,2,3,...,m-1 \tag{4.2}$$

where $m-1 < \alpha \leq m$, D_t^{α} is the Caputo fractional derivative, $g(x,t)$ is the source term, L and N are linear and non-linear differential operator respectively.

Now operating Sumudu transform on eq.(4.1), we have

$$S[D_t^{\alpha} w + L\ w + N\ w] = S[g].$$

Using (1.6.2) and eq. (4.2), we get

$$S[w(x,t)] = \sum_{k=0}^{m-1} u^k w^{(k)}(x,0) + u^{\alpha} S[g] - u^{\alpha} S[Lw + Nw]. \tag{4.3}$$

Operating inverse Sumudu transformation on eq. (4.3), we have

$$w(x,t) = \sum_{k=0}^{m-1} \frac{t^k}{\Gamma(k+1)} w^{(k)}(x,0) - S^{-1}\left[u^{\alpha} S[Lw + N\ w - g]\right]. \tag{4.4}$$

Now, we apply HPM on eq. (4.4)

$$w(x,t) = \sum_{n=0}^{\infty} p^n w_n, \tag{4.5}$$

where the non-linear term can be expressed as

$$N\ w = \sum_{n=0}^{\infty} p^n H_n. \tag{4.6}$$

A few terms of H_n are given by

$$H_n(w_0, ..., w_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}(p^i w_i)\right)\right]_{p=0}, n = 0, 1, 2, 3... \tag{4.7}$$

Using eq.(4.5) and (4.6) in eq. (4.4), we get

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = w(x,0)$$
$$-p\left(\sum_{k=1}^{m-1}\frac{t^k}{\Gamma(k+1)}w^{(k)}(x,0) + S^{-1}\left[u^\alpha S\left[L\sum_{n=0}^{\infty}p^n w_n(x,t) + \sum_{n=0}^{\infty}p^n H_n(w) - g(x,t)\right]\right]\right). \tag{4.8}$$

On looking at the like terms of p of eq. (4.8), we have

$$p^0 : w_0(x,t) = w(x,0),$$

$$p^1 : w_1 = \sum_{k=1}^{m-1}\frac{t^k}{\Gamma(k+1)}w^{(k)}(x,0) - S^{-1}\left[u^\alpha S[Lw_0 + H_0 - g(x,t)]\right],$$

$$p^2 : w_2 = -S^{-1}\left[u^\alpha S[Lw_1 + H_1]\right],$$

$$p^3 : w_3 = -S^{-1}\left[u^\alpha S[Lw_2 + H_2]\right], \tag{4.9}$$

$$\vdots$$

Hence, solution of (4.1) is obtained as $p \to 1$

$$w(x,t) = w_0 + w_1 + w_2 + w_3 \ldots \tag{4.10}$$

4.2 Application

Now, we implement HPSTM to solve the following well- known non-linear fractional PDE.

4.2.1 $K(2,2)$ equation

The $K(m,n)$ equation which is a speculation of the KdV equation, depicts the advancement of the weakly non-linear and dispersive wave used in different fields mainly plasma physics, fluid mechanics, etc.

The time-fractional $K(m,n)$ equation is given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} - a\frac{\partial(u^n)}{\partial x} + b\frac{\partial^3(u^m)}{\partial x^3} = 0,$$

$K(2,2)$ equation, when $m = 2, n = 2, a = -1$ and $b = 1$ i.e.

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 2u\frac{\partial u}{\partial x} + 2u\frac{\partial^3 u}{\partial x^3} + 6\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} = 0.$$

4.2.2 Solution of fractional K(2,2) equation

Consider the fractional $K(2,2)$ equation, where $0 < \alpha \leq 1$.

$$D_t^\alpha w + 2w\frac{\partial w}{\partial x} + 2w\frac{\partial^3 w}{\partial x^3} + 6\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} = 0, \tag{4.11}$$

with initial condition

$$w(x,0) = x. \tag{4.12}$$

Operating Sumudu transformation on eq.(4.11) and using eq.(4.12), we get

$$S[D_t^\alpha w] = -S\left[2w\frac{\partial w}{\partial x} + 2w\frac{\partial^3 w}{\partial x^3} + 6\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right]. \tag{4.13}$$

Using (1.6.2) and operating inverse Sumudu transform on eq. (4.13), we have

$$w(x,t) = x - S^{-1}\left[u^\alpha S\left\{2w\frac{\partial w}{\partial x} + 2w\frac{\partial^3 w}{\partial x^3} + 6\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right\}\right]. \tag{4.14}$$

By applying HPM on eq. (4.14), we get

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = x - S^{-1}\left[u^\alpha S\left\{\sum_{n=0}^{\infty} p^n H_n(w)\right\}\right]. \tag{4.15}$$

A few components of He's polynomials i.e.$H_n(w)$ are given by

$$H_0^1(w) = 2w_0w_{0x} + 2w_0u_{0xxxx} + 6w_{0x}w_{0xxx},$$

$$H_1^1(w) = 2(w_0u_{1x} + w_1w_{0x}) + 2(w_0u_{1xxx} + w_1w_{0xxx}) + 6(w_{0x}w_{1xx} + w_{1x}w_{0xx}),$$

$$H_2^1(w) = 2(w_0w_{2x} + w_1w_{1x} + w_2w_{0x}) + 2(w_0w_{2xxx} + w_1w_{1xxx} + w_2w_{0xxx}) + 6(w_{0x}w_{2xx} + w_{1x}w_{1xx} + w_{2x}w_{0xx}),$$

$$\vdots$$

On looking at the like terms of p of eq. (4.15), we have

$$p^0 : w_0 = x,$$

$$p^1 : w_1 = \frac{-2x}{\Gamma(1+\alpha)}t^{\alpha},$$

$$p^2 : w_2 = 2^3\frac{x}{\Gamma(1+2\alpha)}t^{2\alpha},$$

$$p^3 : w_3 = -\left(\frac{2^5}{\Gamma(1+3\alpha)} + \frac{2^3\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\right)xt^{3\alpha},$$

$$\vdots$$

As $p \to 1$, the series solution of eq. (4.11) is

$$w(x,t) = x + \frac{-2x}{\Gamma(1+\alpha)}t^{\alpha} + 2^3\frac{x}{\Gamma(1+2\alpha)}t^{2\alpha} - \left(\frac{2^5}{\Gamma(1+3\alpha)} + \frac{2^3\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\right)xt^{3\alpha} + \ldots \tag{4.16}$$

Also when $\alpha = 1$, eq.(4.16) can be rewritten as:

$$w(x,t) = x - 2xt + 4xt^2 - 8xt^3 + \ldots, \tag{4.17}$$

and the solution acquired in eq. (4.17) in closed form is given as $w(x,t) = \frac{x}{1+2t}$.

4.2.3 Solution of time fractional Sawada Kotera equation

Consider the fractional IVP

$$D_t^{\beta} w + 45w^2 w_x + 15w_x w_{xx} + 15ww_{xxx} + w_{xxxxx} = 0; t > 0, 0 < \beta \leq 1, \tag{4.18}$$

where

$$w(x,0) = 2k^2 \operatorname{sech}^2(kx). \tag{4.19}$$

The exact solution of eq. (4.18) for $\beta = 1$ is

$$w(x,t) = 2k^2 \operatorname{sech}^2(k(x - 16kt)). \tag{4.20}$$

Operating Sumudu transformation on eq.(4.18) and using eq.(4.19), we get

$$S[D_t^{\beta} w] = -S[45w^2 w_x + 15w_x w_{xx} + 15ww_{xxx} + w_{xxxxx}]. \tag{4.21}$$

Operating the inverse Sumudu transformation on eq. (4.21), we get

$$w(x,t) = w(x,0) - S^{-1}\left[u^{\beta} S[45w^2 w_x + 15w_x w_{xx} + 15ww_{xxx} + w_{xxxxx}]\right]. \tag{4.22}$$

By applying HPM on eq. (4.22), we get

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = w(x,0) - pS^{-1}\left[u^{\beta} S\left[\left(\sum_{n=0}^{\infty} p^n w_n(x,t)\right)_{xxxxx} + \left(\sum_{n=0}^{\infty} p^n H_n(w)\right)\right]\right], \tag{4.23}$$

where

$$\sum_{n=0}^{\infty} p^n H_n(w) = 45w^2 w_x + 15w_x w_{xx} + 15ww_{xxx}.$$

A few components of He's polynomials i.e.$H_n(w)$ are given by

$$H_0(w) = 45w_0^2 w_{0x} + 15w_{0x} w_{0xx} + 15w_0 w_{0xxx},$$

$$H_1(w) = 45(2w_0 w_1 w_{0x} + w_0^2 w_{1x}) + 15(w_{1x} w_{0xx} + w_{0x} w_{1xx}) + 15(w_1 w_{0xxx} + w_{1xxx} w_0),$$

$$H_2(w) = 45[(2w_1 w_2 + 2w_0 w_3) w_{0x} + (w_1^2 + 2w_0 w_2) w_{1x} + 2w_0 w_1 w_{2x} + w_0^2 w_{3x}]$$
$$+15[w_0 w_{2xxx} + w_1 w_{1xxx} + w_2 w_{0xxx}] + 15[w_{0x} w_{2xx} + w_{1x} w_{1xx} + w_{2x} w_{0xx}, \tag{4.24}$$

$$\vdots$$

on looking at the like terms of eq. (4.23), we have

$$p^0 : w_0(x,t) = 2k^2 \operatorname{sech}^2(kx),$$

$$p^1 : w_1(x,t) = 64k^7 \operatorname{sech}^2(kx) \tanh(kx) \frac{t^\beta}{\Gamma(1+\beta)},$$

$$p^2 : w_2(x,t) = -512 \operatorname{sech}^2(kx)(3\operatorname{sech}^2(kx) - 2) \frac{t^{2\beta}}{\Gamma(1+2\beta)},$$

$$\vdots$$

Therefore, the series solution of eq. (4.18) is

$$w(x,t) = 2k^2 \operatorname{sech}^2(kx) + 64k^7 \operatorname{sech}^2(kx) \tanh(kx) \frac{t^\beta}{\Gamma(1+\beta)} - 512 \operatorname{sech}^2(kx)(3\operatorname{sech}^2(kx) - 2) \frac{t^{2\beta}}{\Gamma(1+2\beta)} + \dots.$$

4.2.4 Solution of time- fractional KdV equation

Consider the following fractional IVP

$$D_t^\beta w + 2w^2 w_x + 6w_x w_{xx} + 3ww_{xxx} + w_{xxxxx} = 0; t > 0, 0 < \beta \leq 1 \tag{4.25}$$

where

$$w(x,0) = 10k^2(3\operatorname{sech}^2(kx) - 1). \tag{4.26}$$

Operating Sumudu transformation on eq. (4.25) and using (4.26), we get

$$S[D_t^\beta w] = -S[2w^2 w_x + 6w_x w_{xx} + 3ww_{xxx} + w_{xxxxx}]. \tag{4.27}$$

Operating the inverse Sumudu transformation on eq. (4.27), we have

$$w(x,t) = w(x,0) - S^{-1}\left[u^\beta S[2w^2 w_x + 6w_x w_{xx} + 3ww_{xxx} + w_{xxxxx}]\right]. \tag{4.28}$$

By applying HPM on eq. (4.28), we get

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = w(x,0) - pS^{-1}\left[u^\beta S\left[\left(\sum_{n=0}^{\infty} p^n w_n(x,t)\right)_{xxxxx} + \left(\sum_{n=0}^{\infty} p^n H_n(w)\right)\right]\right], \tag{4.29}$$

where

$$\sum_{n=0}^{\infty} p^n H_n(w) = 2w^2 w_x + 6w_x w_{xx} + 3w w_{xxx}.$$

A couple of terms of He's polynomials i.e. $H_n(w)$ are given by

$$H_0(w) = 2w_0^2 w_{0x} + 6w_{0x} w_{0xx} + 3w_0 w_{0xxx},$$

$$H_1(w) = 2(2w_0 w_1 w_{0x} + w_0^2 w_{1x}) + 6(w_{1x} w_{0xx} + w_{0x} w_{1xx}) + 3(w_1 w_{0xxx} + w_{1xxx} w_0),$$

$$H_2(w) = 2[(2w_1 w_2 + 2w_0 w_3)w_{0x} + (w_1^2 + 2w_0 w_2)w_{1x} + 2w_0 w_1 w_{2x} + w_0^2 w_{3x}]$$
$$+6[w_0 w_{2xxx} + w_1 w_{1xxx} + w_2 w_{0xxx}] + 3[w_{0x} w_{2xx} + w_{1x} w_{1xx} + w_{2x} w_{0xx},$$

$$\vdots$$

On looking at the like terms of p of eq. (4.29), we have

$$p^0 : w_0(x,t) = 10k^2(3\operatorname{sech}^2(kx) - 1),$$

$$p^1 : w_1(x,t) = 5760k^7 \operatorname{sech}^2(kx)\tanh(kx)\frac{t^\beta}{\Gamma(1+\beta)},$$

$$p^2 : w_2(x,t) = 552960k^{12} \operatorname{sech}^2(kx)(1 - 3\tanh^2(kx))\frac{t^{2\beta}}{\Gamma(1+2\beta)},$$

$$\vdots$$

Hence, the acquired solution of eq. (4.25) is given as

$$w(x,t) = 10k^2(3\operatorname{sech}^2(kx) - 1) + 5760k^7 \operatorname{sech}^2(kx)\tanh(kx)\frac{t^\beta}{\Gamma(1+\beta)} +$$
$$552960k^{12} \operatorname{sech}^2(kx)(1 - 3\tanh^2(kx))\frac{t^{2\beta}}{\Gamma(1+2\beta)} + \dots$$

4.2.5 Solution of fractional attractor 1-D Keller Segel equation

Consider the following Coupled system:

$$\begin{aligned}
\frac{\partial^\beta v}{\partial t} &= a\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}\left(v\frac{\partial \chi(\rho)}{\partial x}\right), \\
\frac{\partial^\beta \rho}{\partial t} &= b\frac{\partial^2 \rho}{\partial x^2} + c\ v - d\ \rho, 0 < \beta \leq 1,
\end{aligned} \tag{4.30}$$

subject to conditions

$$v(x,0) = m\ exp(-x^2), \rho(x,0) = n\ exp(-x^2). \tag{4.31}$$

Case-I Consider the sensitivity function $\chi(\rho) = 1$, then the Chemo-tactic term i.e.$\frac{\partial}{\partial x}\left(v\frac{\partial \chi(\rho)}{\partial x}\right) = 0$, hence Keller- Segel equation reduces to

$$\begin{aligned}
\frac{\partial^\beta v}{\partial t} &= a\frac{\partial^2 v}{\partial x^2}, \\
\frac{\partial^\beta \rho}{\partial t} &= b\frac{\partial^2 \rho}{\partial x^2} + cv(x,t) - d\rho(x,t), 0 < \beta \leq 1.
\end{aligned} \tag{4.32}$$

By applying HPSTM on eq.(4.32), we have

$$\sum_{n=0}^{\infty} p^n v_n = v(x,0) + pS^{-1}\left\{u^\beta S\left\{a\left(\sum_{n=0}^{\infty} p^n v_n\right)_{xx}\right\}\right\}, \tag{4.33}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} p^n \rho_n &= \rho(x,0) + pS^{-1}\left\{u^\beta S\left\{b\left(\sum_{n=0}^{\infty} p^n \rho_n\right)_{xx}\right\}\right\} \\
&\quad + pS^{-1}\left\{u^\beta S\left\{c\left(\sum_{n=0}^{\infty} p^n v_n\right) - d\left(\sum_{n=0}^{\infty} p^n \rho_n\right)\right\}\right\}.
\end{aligned} \tag{4.34}$$

On looking at the like terms of p of eq.(4.33) & (4.34) and using (4.31), we have

$$
\begin{aligned}
p^0 : v_0 &= m\ exp(-x^2);\\
p^0 : \rho_0 &= n\ exp(-x^2);\\
p^1 : v_1 &= \frac{am\ t^\beta}{\Gamma(1+\beta)}\left(-2\ exp(-x^2)+4x^2\ exp(-x^2)\right);\\
p^1 : \rho_1 &= \frac{t^\beta}{\Gamma(1+\beta)}\left((cm-dn)-2nb\ exp(-x^2)(2x^2-1)\right);\\
p^2 : v_2 &= \frac{a^2m\ t^{2\beta}}{\Gamma(1+2\beta)}\left(12\ exp(-x^2)-48x^2\ exp(-x^2)+16x^4\ exp(-x^2)\right);\\
p^2 : \rho_2 &= \frac{t^{2\beta}}{\Gamma(1+2\beta)}((d\ exp(-x^2)(-cm+dn)+2a\ exp(-x^2)cm(-1+2x^2)\\
&\quad +2b\ exp(-x^2)(-1+2x^2)(cm-2dn)+4b^2\ exp(-x^2)n(3-12x^2+4x^4));\\
p^3 : v_3 &= \frac{a^3\ exp(-x^2)m\ t^{3\beta}}{\Gamma(1+3\beta)}(-120+720x^2-480x^4+64x^6);
\end{aligned}
$$

$$
\begin{aligned}
p^3 : \rho_3 &= \frac{t^{3\beta}}{\Gamma(1+3\beta)}(d^2(cm-dn)\ exp(-x^2)+b\ exp(-x^2)(6-24x^2+8x^4)\\
&\quad +2acm\ exp(-x^2)(d-2dx^2)+4b^2\ exp(-x^2)(cm-3dn)(3-12x^2+4x^4)\\
&\quad +8b^3n\ exp(-x^2)(-15+90x^2-60x^4+8x^6)+4a^2\ exp(-x^2)cm(3-12x^2+4x^4)\\
&\quad +2bd\ exp(-x^2)(-2cm+3dn)(-1+2x^2))),\\
&\vdots
\end{aligned}
$$

The approximate solution of eq.(4.32) is obtained as $p \to 1$ i.e.

$$v(x,t) = v_0 + v_1 + v_2 + \dots,$$

$$\rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \dots$$

$$
\begin{aligned}
v(x,t) = m\,exp(-x^2)\left(1 + a(-2+4x^2)\frac{t^\beta}{\Gamma(1+\beta)} + a^2(12-48x^2+16x^4)\frac{t^{2\beta}}{\Gamma(1+2\beta)}\right)\\
+ me^{-x^2}\left(a^3(-120+720x^2-480x^4+64x^6)\frac{t^{3\beta}}{\Gamma(1+3\beta)} + \dots\right),
\end{aligned}
$$

$$
\begin{aligned}
\rho(x,t) &= n\ exp(-x^2) + \frac{t^{\beta}}{\Gamma(1+\beta)}\big((cm-dn) - 2nb\ exp(-x^2)(2x^2-1)\big)\\
&+ \frac{t^{2\beta}}{\Gamma(1+2\beta)}((d\ exp(-x^2)(-cm+dn) + 2a\ exp(-x^2)cm(-1+2x^2)\\
&+ 2b\ exp(-x^2)(-1+2x^2)(cm-2dn) + 4b^2\ exp(-x^2)n(3-12x^2+4x^4))\\
&+ \frac{t^{3\beta}}{\Gamma(1+3\beta)}(d^2(cm-dn)\ exp(-x^2) + b\ exp(-x^2)(6-24x^2+8x^4)\\
&+ 2acm\ exp(-x^2)(d-2dx^2) + 4b^2\ exp(-x^2)(cm-3dn)(3-12x^2+4x^4)\\
&+ 8b^3n\ exp(-x^2)(-15+90x^2-60x^4+8x^6) + 4a^2\ exp(-x^2)cm(3-12x^2+4x^4)\\
&+ 2bd\ exp(-x^2)(-2cm+3dn)(-1+2x^2))) + \ldots
\end{aligned}
$$

Case-II Consider the senstivity function $\chi(\rho) = \rho$, then the Chemo-tactic term i.e. $\frac{\partial}{\partial x}\left(v\frac{\partial\chi(\rho)}{\partial x}\right) = \frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2\rho}{\partial x^2}$, hence, Keller- Segel equation reduces to

$$
\begin{aligned}
\frac{\partial^{\beta} v}{\partial t} &= a\frac{\partial^2 v}{\partial x^2} - \left(\frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2\rho}{\partial x^2}\right),\\
\frac{\partial^{\beta} \rho}{\partial t} &= b\frac{\partial^2 \rho}{\partial x^2} + c\ v - d\ \rho, 0 < \beta \leq 1,
\end{aligned}
\tag{4.35}
$$

Now, for the solution of eq. (4.35), we apply HPSTM on eq. (4.35), we have

$$
\sum_{n=0}^{\infty} p^n v_n = v(x,0) + pS^{-1}\left\{u^{\beta} S\left\{a\left(\sum_{n=0}^{\infty} p^n v_n\right)_{xx} - \left(\sum_{n=0}^{\infty} p^n H_n(x,t)\right)\right\}\right\},
\tag{4.36}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} p^n \rho_n &= \rho(x,0) + pS^{-1}\left\{u^{\beta} S\left\{b\left(\sum_{n=0}^{\infty} p^n \rho_n\right)_{xx}\right\}\right\}\\
&\quad + pS^{-1}\left\{u^{\beta} S\left\{c\left(\sum_{n=0}^{\infty} p^n v_n\right) - d\left(\sum_{n=0}^{\infty} p^n \rho_n\right)\right\}\right\}.
\end{aligned}
\tag{4.37}
$$

where

$$
\sum_{n=0}^{\infty} p^n H_n(x,t) = \frac{\partial v}{\partial x}\frac{\partial \rho}{\partial x} + v\frac{\partial^2 \rho}{\partial x^2}.
$$

An initial couple of terms of He's polynomial i.e. $H_n(x,t)$ are given below:

$$\begin{aligned}
H_0(x,t) &= v_{0x}\rho_{0x} + v_0\rho_{0xx};\\
H_1(x,t) &= v_{0x}\rho_{1x} + v_{1x}\rho_{0x} + v_0\rho_{1xx} + v_1\rho_{0xx};\\
H_2(x,t) &= v_{0x}\rho_{2x} + v_{1x}\rho_{1x} + v_{0x}\rho_{2x}\\
&\quad + v_0\rho_{2xx} + v_1\rho_{1xx} + v_1\rho_{0xx},\\
&\vdots
\end{aligned}$$

On looking at the like terms of p of eq. (4.36)& (4.37) and using eq.(4.31) and He's polynomial, we get

$$\begin{aligned}
p^0 : v_0(x,t) &= m\ exp(-x^2);\\
p^0 : \rho_0(x,t) &= n\ exp(-x^2);\\
p^1 : v_1(x,t) &= \frac{2m\ t^{\beta}}{\Gamma(1+\beta)}(a(2x^2-1) - n\ exp(-x^2)(4x^2-1));\\
p^1 : \rho_1(x,t) &= \frac{t^{\beta}\ exp(-x^2)}{\Gamma(1+\beta)}((cm-dn) + 2nb(-1+2x^2));
\end{aligned}$$

$$p^2 : v_2(x,t) = \frac{2m\ exp(-3x^2)\ t^{2\beta}}{\Gamma(1+2\beta)}(-c\,exp(x^2)m(-1+4x^2) + 2a^2\ exp(2x^2)(3-12x^2+4x^4)$$

$$-2a\ exp(x^2)n(7-58x^2+40x^4) + nd\ exp(-x^2)(-1+4x^2)$$

$$-2nb\ exp(x^2)(3-18x^2+8x^4) + 2n^2(1-18x^2+24x^4));$$

$$p^2 : \rho_2(x,t) = \frac{t^{2\beta}\ exp(-2x^2)}{\Gamma(1+2\beta)}(exp(x^2)d(-cm+nd) + 2cmn(1-4x^2)$$
$$+\ 2\ exp(x^2)(acm + b(cm-2dn)(-1+2x^2) + 4b^2n\ exp(x^2)(3-12x^2+4x^4));$$

$$p^3 : v_3(x,t) = \frac{2m\ exp(-4x^2)\ t^{3\beta}}{\Gamma(1+3\beta)(\Gamma(1+\beta))^2}(-cd\, exp(2x^2)m + d^2 exp(2x^2)n + 14c\, exp(x^2)mn$$
$$-2d\ exp(x^2)n^2 + 4n^3 + 4cd\ exp(2x^2)mx^2 - 4d^2nx^2\ exp(2x^2) + n^3x^4(1056 - 768x^2)$$
$$-156cmnx^2\ exp(x^2) + 36dn^2x^2 exp(x^2) - 248n^3x^2 + 144cmnx^4\ exp(x^2) - 48dn^2x^4\ exp(x^2)$$
$$+4a^3\ exp(3x^2)(-15 + 90x^2 - 60x^4 + 8x^6) - 4b^2n\ exp(2x^2)(-15 + 120x^2 - 100x^4 + 16x^6)$$
$$-4a^2n\ exp(2x^2)(-75 + 924x^2 - 1252x^4 + 336x^6) - 2bcm\ exp(2x^2)(3 - 18x^2 + 8x^4)$$
$$+4nb\ exp(x^2)(d\ exp(x^2)(3 - 18x^2 + 8x^4) - 2bn\ exp(x^2)(-3 + 72x^2 - 148x^4 + 48x^6))$$
$$-2a\ exp(2x^2)cm(9 - 66x^2 + 40x^4) + 4an\ exp(x^2)(d\ exp(x^2)(3 - 24x^2 + 16x^4) - 4b\ exp(x^2)\cdot$$
$$(-6 + 63x^2 - 72x^4 + 16x^6)) - 8an\ exp(x^2)(-7 + 162x^2 - 380x^4 + 168x^6))(\Gamma(1+\beta))^2$$
$$-2\ exp(x^2)(n(-cm + dn)(1 - 18x^2 + 24x^4) + bn^2(6 - 120x^2 + 248x^4 - 96x^6)$$
$$+a\ exp(x^2)(cm - dn)(1 - 10x^2 + 8x^4) + 2bn(-3 + 36x^2 - 52x^4 + 16x^6))\Gamma(1+2\beta)),$$

$$\begin{aligned} p^3 : \rho_3(x,t) \quad = \quad & \frac{t^{3\beta}\ exp(-3x^2)}{\Gamma(1+3\beta)}(cd^2m\ exp(2x^2) + 2c^2m^2\ exp(x^2) - d^2n\ exp(2x^2) \\ & -4cdmn\ exp(x^2)) + 4cmn^2 - 8c^2m^2x^2\ exp(x^2) - 16cdmnx^2\ exp(x^2) \\ & -72cmn^2x^2 + 96cmn^2x^4 + 6bd^2n\ exp(2x^2)n(-1 + 2x^2)\cdot \\ & (-15 + 90x^2 - 60x^4 + 80x^6) + 4a^2cm\ exp(2x^2)(3 - 12x^2 + 4x^4) \\ & +8b^3n\ exp(2x^2) - 4bcm\ exp(x^2)\cdot(d\ exp(x^2)(-1 + 2x^2) \\ & +n(9 - 66x^2 + 40x^4)) + 2acm\ exp(x^2)\cdot(-d\ exp(x^2)(-1 + 2x^2) \\ & +2b\ exp(x^2)(3 - 12x^2 + 4x^4) - 2n(7 - 58x^2 + 40x^4))), \\ \vdots \end{aligned} \tag{4.38}$$

On using eq.(4.38) and (4.38) and as $p \to 1$, the approximate solution of eq.(4.35) is

$$v(x,t) = v_0 + v_1 + v_2 + \dots$$

$$\rho(x,t) = \rho_0 + \rho_1 + \rho_2 + \dots$$

$$v(x,t) = m\,exp(-x^2) + \frac{2m\ t^{\beta}}{\Gamma(1+\beta)}(a(2x^2-1) - n\,exp(-x^2)(4x^2-1)) + \frac{2m\ exp(-3x^2)\ t^{2\beta}}{\Gamma(1+2\beta)}$$
$$(-c\,exp(x^2)m(-1+4x^2) + 2a^2\,exp(2x^2)(3-12x^2+4x^4) - 2a\,exp(x^2)n(7-58x^2+40x^4)$$
$$+\,nd\ exp(-x^2)(-1+4x^2) - 2nb\ exp(x^2)(3-18x^2+8x^4) + 2n^2(1-18x^2+24x^4))$$
$$\frac{2m\ exp(-4x^2)\ t^{3\beta}}{\Gamma(1+3\beta)(\Gamma(1+\beta))^2}(-cd\ exp(2x^2)m + d^2 exp(2x^2)n + 14c\ exp(x^2)mn$$
$$-\,2d\ exp(x^2)n^2 + 4n^3 + 4cd\ exp(2x^2)mx^2 - 4d^2nx^2\ exp(2x^2) + n^3x^4(1056-768x^2)$$
$$-156cmnx^2\,exp(x^2) + 36dn^2x^2 exp(x^2) - 248n^3x^2 + 144cmnx^4\,exp(x^2) - 48dn^2x^4\,exp(x^2)$$
$$+4a^3\,exp(3x^2)(-15+90x^2-60x^4+8x^6) - 4b^2n\,exp(2x^2)(-15+120x^2-100x^4+16x^6)$$
$$-\,4a^2n\ exp(2x^2)(-75+924x^2-1252x^4+336x^6) - 2bcm\ exp(2x^2)(3-18x^2+8x^4)$$
$$+\,4nb\ exp(x^2)(d\ exp(x^2)(3-18x^2+8x^4) - 2bn\ exp(x^2)(-3+72x^2-148x^4+48x^6))$$
$$-\,2a\ exp(2x^2)cm(9-66x^2+40x^4) + 4an\ exp(x^2)(d\ exp(x^2)(3-24x^2+16x^4)$$
$$-4b\,exp(x^2)(-6+63x^2-72x^4+16x^6)) - 8an\,exp(x^2)(-7+162x^2-380x^4+168x^6))(\Gamma(1+\beta))^2$$
$$-\,2\ exp(x^2)(n(-cm+dn)(1-18x^2+24x^4) + bn^2(6-120x^2+248x^4-96x^6)$$
$$+\,a\ exp(x^2)(cm-dn)(1-10x^2+8x^4) + 2bn(-3+36x^2-52x^4+16x^6))\Gamma(1+2\beta)$$

$$\rho(x,t) = n\,exp(-x^2) + \frac{t^{\beta}\ exp(-x^2)}{\Gamma(1+\beta)}((cm-dn) + 2nb(-1+2x^2)) + \frac{2m\ exp(-3x^2)\ t^{2\beta}}{\Gamma(1+2\beta)}$$
$$(-c\,exp(x^2)m(-1+4x^2) + 2a^2\,exp(2x^2)(3-12x^2+4x^4) - 2a\,exp(x^2)n(7-58x^2+40x^4)$$
$$+\,nd\ exp(-x^2)(-1+4x^2) - 2nb\ exp(x^2)(3-18x^2+8x^4) + 2n^2(1-18x^2+24x^4))$$
$$\frac{t^{3\beta}\ exp(-3x^2)}{\Gamma(1+3\beta)}(cd^2m\ exp(2x^2) + 2c^2m^2\ exp(x^2) - d^2n\ exp(2x^2) - 4cdmn\ exp(x^2))$$
$$+4cmn^2 - 8c^2m^2x^2\ exp(x^2) - 16cdmnx^2\ exp(x^2) - 72cmn^2x^2 + 96cmn^2x^4 + 6bd^2n\ exp(2x^2)$$
$$n(-1+2x^2) + 4a^2cm\,exp(2x^2)(3-12x^2+4x^4) + 8b^3n\,exp(2x^2)\cdot(-15+90x^2-60x^4+80x^6)$$
$$-4bcm\,exp(x^2)\cdot(d\,exp(x^2)(-1+2x^2) + n(9-66x^2+40x^4)) + 2acm\,exp(x^2)(-d\,exp(x^2)\cdot$$
$$(-1+2x^2) + 2b\ exp(x^2)(3-12x^2+4x^4) - 2n(7-58x^2+40x^4)))$$

4.3 Conclusion

1. For fractional non-linear PDE, the HPSTM technique is better, extremely straightforward.

2. It seems quite easy to handle non-linear terms.

3. HPSTM needs less computation than HPM, only a few iterations lead to the approximate solution of such complex non-linear problems.

4. HPSTM has fast convergence for solving fractional non-linear PDE.

5. The solution obtained from HPSTM is in great concurrence with an exact solution as fractional-order derivative converges to integer order.

Chapter 5

Convergence Analysis of Series Solution

A vast majority of the problems happening in the field of science and engineering like thermodynamics, liquid mechanics, material science, plasma physical science, environmental science and so forth are displayed as non-linear PDE or fractional PDE. As it is hard to handle these issues logically or numerically henceforth, a few numerical and semi-analytical techniques are proposed to tackle these issues. However, the outcome acquired from these techniques is more precise and worthy than the numerical one. Many researchers have applied various methods like HPM [42, 46], ADM [20, 33], HAM[69], HPTM [64], HPSTM[97] for the series solution of such equations.

In this chapter, we have applied HPSTM and HPTM for the series solution of non-linear PDE and fractional PDE. However, for the validity of the acquired series solution, the condition of convergence and uniqueness is derived. Accuracy is achieved in the context of convergence and error analysis.

Firstly, we have derived the condition of the convergence of the series solution of PDE using HPSTM and then it is verified by implementing it on well known Newell-Whitehead-Segel equation[82] and Fisher′s equation [65]. Further, in the same manner, the condition of convergence of HPTM is derived and implemented on well known fractional Burgers' equation. Moreover, the maximum truncation error and the error analysis have been done and the results are also interpreted in the form of surface graphs. For the results related to convergence analysis of HPM we refer the reader to [12, 22, 98].

5.1 Convergence analysis of HPSTM

Here we emphasize the condition of convergence of HPSTM for the series solution of non-linear PDE. For this, consider the following general non-linear PDE:

$$\frac{\partial^n U(x,t)}{\partial t^n} + L\ U(x,t) + N\ U(x,t) = f(x,t), t > 0, x \in \mathbb{R}, \tag{5.1}$$

Consider the Banach space $C[0,T]$ of all continuous real-valued functions on $[0,T]$ with supremum norm. Throughout this section, we consider $U(x,t)$, $U_n(x,t) \in C[0,T]$, $\forall\ n \in \mathbb{N}$.

Theorem 5.1.1 *(Uniqueness theorem) The solution obtained by HPSTM of partial differential equation (5.1) has a unique solution, whenever* $0 < \gamma < 1$.

Proof*: The solution of eq.(5.1) is of the form* $U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t)$ *. Here,*

$$U(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} U^{(k)}(x,0) + S^{-1}\Big\{u^n\Big(S\Big\{f(x,t) - LU(x,t) - NU(x,t)\Big\}\Big)\Big\}.$$

Let U and V be the distinct solution of the eq.(5.1) then

$$|U - V| = \Big| - S^{-1}\Big[u^n\Big(S\Big[L(U-V) + N(U-V)\Big]\Big)\Big]\Big|.$$

Using convolution theorem,

$$\begin{aligned} |U - V| &\leq \int_0^t \Big(|L(U-V)| + |N(U) - N(V)|\Big)\Big|\frac{(t-\tau)^n}{n!}\Big| d\tau \\ &\leq \int_0^t (\eta|(U-V)| + \delta|U-V|)\Big|\frac{(t-\tau)^n}{n!}\Big| d\tau \end{aligned}$$

where L is a bounded operator i.e. $|L(U) - L(V)| \leq \eta|U - V|$, N *satisfies Lipschitz condition with* $\delta > 0$ *such that* $|N(U) - N(V)| \leq \delta|U - V|$.

$$|U - V| \;\leq\; \int_0^t (\eta + \delta)(|U - V|)\left|\frac{(t-\tau)^n}{n!}\right| d\tau$$

Using mean value theorem of integral calculus, $|U - V| \leq [(\eta+\delta)|U - V|]MT$, *where* $M = \max(t-\tau)^n$ *and* $t \in [0, T]$. *Hence,* $|U - V| \leq |U - V|\gamma$, *where* $\gamma = (n+\delta)MT$. *So* $(1-\gamma)|U - V| \leq 0$, *implies* $U = V$ *whenever,* $0 < \gamma < 1$.

Theorem 5.1.2 *Let* U *and* $U_n(x,t)$ *be defined in Banach space* $\boldsymbol{B}$, *the condition that the series solution* $\sum_{n=0}^{\infty} U_n$, *converges to the solution* U *is* $||U_{n+1}|| \leq \gamma\ ||U_n||$ *where* $\gamma \in (0, 1)$ *and* $n \in \mathbb{N}$.

Proof*: For the convergence of sequence* $\{s_n\}$ *of the partial sums of the series* $U(x,t) = \sum_{n=0}^{\infty} U_n$, *we prove that* $\{s_n\}$ *is a Cauchy sequence in* $(C[0,T], ||\cdot||)$. *As,*

$$\begin{aligned} ||s_{n+1} - s_n|| &= ||U_{n+1}|| \;\leq\; \gamma||U_n|| \\ &\leq \gamma^2||U_{n-1}|| \;\leq\; \cdots \leq \gamma^{n+1}||U_0|| \end{aligned}$$

Hence,

$$\begin{aligned} ||s_n - s_m|| &= ||\Sigma_{i=m+1}^{n} U_i|| \leq \Sigma_{i=m+1}^{n}||U_i|| \\ &\leq \gamma^{m+1}(\Sigma_{i=0}^{n-m}\gamma^i)||U_0|| = \gamma^{m+1}\frac{(1-\gamma^{n-m})}{1-\gamma}||U_0||, n, m \in \mathbb{N} \end{aligned}$$

Since , $0 < \gamma < 1$, *hence* $||s_n - s_m|| \leq \frac{\gamma^{m+1}}{1-\gamma}||U_0||$. *Also* U_0 *is bounded, therefore* $||s_{n+1} - s_n|| \to 0$ *as* $m, n \to \infty$. *So* $\{s_n\}$ *is a Cauchy sequence in* $C[0,T]$, *hence* $\sum_{n=0}^{\infty} U_n(x,t)$ *is convergent.*

Remark 5.1.3 *The maximum truncated error of* $U(x,t) = \sum_{n=0}^{\infty} U_n$ *is given by* $\frac{\gamma^{m+1}}{1-\gamma}||U_0||$.

5.2 Application

In order to understand the functioning and significance of the HPSTM, we apply the said technique to present the solution of well-known Newell-Whitehead-Segel equation.

5.2.1 Newell-Whitehead-Segel equation

Uniform, oscillatory and pattern states are very common in non-equilibrium systems. Many stripes patterns such as swells in sand, stripes of seashells emerge in an assortment of spatially expanded frameworks which can be displayed by an arrangement of conditions called amplitude condition. In 2-D system, the amplitude equation, i.e. Newell-Whitehead -Segel equations depicts the presence of stripe design. The equation derived by Newell, Whitehead and Segel is of the form.

$$\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial t^2} + b\,U - c\,U^m.$$

where a, b are real numbers, c and m are positive integers.

5.2.2 Solution of Newell-Whitehead-Segel equation

Consider the following equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + 2U - 3U^2 \text{ with initial condition } U(x,0) = \lambda. \tag{5.2}$$

By applying the Sumudu transformation on eq. (5.2) with the initial conditions, we get

$$S\{U(x,t)\} = \frac{1}{1-2u}\lambda + \frac{u}{1-2u}S\{U_{xx} - 3U^2\} \tag{5.3}$$

Operating inverse Sumudu transformation on eq. (5.3), we have

$$U(x,t) = e^{2t}\lambda + S^{-1}\{\frac{u}{1-2u}S\{U_{xx} - 3U^2\}\} \tag{5.4}$$

Now, we apply HPM, eq.(5.4) becomes

$$\sum_{n=0}^{\infty} U_n(x,t) = e^{2t}\lambda + pS^{-1}\left\{\frac{u}{1-2u} S\left\{\left(\sum_{n=0}^{\infty} p^n U_n(x,\ t)\right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(x,t)\right\}\right\} \tag{5.5}$$

A few couple of terms of He's polynomial H_n is given by

$$\begin{aligned}
H_0 &= 3U_0^2,\\
H_1 &= 6U_0U_1,\\
H_2 &= 6U_0U_2 + 3U_1^2,\\
H_3 &= 6U_0U_3 + 6U_1U_2,\\
&\vdots
\end{aligned}$$

On looking at the coefficients of like power of p of eq. (5.5), we have

$$\begin{aligned}
p^0 : U_0 &= e^{2t}\lambda,\\
p^1 : U_1 &= -\frac{3}{2}\lambda^2(e^{2t}(e^{2t}-1)),\\
p^2 : U_2 &= \frac{9}{4}\lambda^3(e^{2t}(e^{2t}-1)^2),\\
p^3 : U_3 &= -\frac{27}{8}\lambda^4(e^{2t}(e^{2t}-1)^3),\\
p^4 : U_4 &= \frac{81}{16}\lambda^5(e^{2t}(e^{2t}-1)^4),\\
&\vdots
\end{aligned}$$

For $t = \frac{1}{2}\ln(1+\frac{2\gamma}{3\lambda})$, where $0 < \gamma < 1$. Consider

$$
\begin{aligned}
||s_1 - s_0|| &= ||U_1|| = \left|\left| -\frac{3}{2}\lambda^2(e^{2t}(e^{2t}-1))\right|\right| = \left|\left|(e^{2t}\lambda)(-\frac{3}{2}\lambda(e^{2t}-1))\right|\right| \\
&\leq ||(e^{2t}\lambda)|||\gamma| = \gamma||U_0||, \\
||s_2 - s_1|| &= ||U_2|| = \left|\left|\frac{9}{4}\lambda^3(e^{2t}(e^{2t}-1)^2)\right|\right| = \left|\left|(e^{2t}\lambda)(\frac{9}{4}\lambda^2(e^{2t}-1)^2)\right|\right| \\
&\leq ||U_0|||\gamma^2| = \gamma^2||U_0||, \\
||s_3 - s_2|| &= ||U_3|| = \left|\left| -\frac{27}{8}\lambda^4(e^{2t}(e^{2t}-1)^3)\right|\right| = \left|\left|(e^{2t}\lambda)(-\frac{27}{8}\lambda^3(e^{2t}-1)^3)\right|\right| \\
&\leq \gamma^3||U_0||, \\
||s_4 - s_3|| &= ||U_4|| = \left|\left|\frac{81}{16}\lambda^5(e^{2t}(e^{2t}-1)^4)\right|\right| = \left|\left|(e^{2t}\lambda)(\frac{81}{16}\lambda^4(e^{2t}-1)^4)\right|\right| \\
&\leq \gamma^4||U_0||, \\
&\vdots
\end{aligned}
$$

Consider

$$
\begin{aligned}
||s_n - s_m|| &\leq ||s_n - s_{n-1}|| + ||s_{n-1} - s_{n-2}|| + \cdots + ||s_{m+1} - s_m|| \\
&= ||U_n|| + ||U_{n-1}|| + ||U_{n-2}|| + \cdots + ||U_{m+1}|| \\
&\leq \gamma^{m+1}(1 + \gamma^1 + \gamma^2 + \cdots + \gamma^{n-m-1})||U_0|| \\
&\leq \frac{\gamma^{m+1}}{1-\gamma}||U_0||
\end{aligned}
$$

Hence, $||s_n - s_m|| \to 0$ as $m, n \to \infty$, which implies that $\{s_n\}$ is a Cauchy sequence. Hence, the approximate solution of eq. (5.2) is

$$
\begin{aligned}
U(x,t) = e^{2t}\lambda - \frac{3}{2}\lambda^2 e^{2t}(e^{2t}-1) + \frac{9}{4}\lambda^3 e^{2t}(e^{2t}-1)^2 - \frac{27}{8}\lambda^4 e^{2t}(e^{2t}-1)^3 \\
+ \frac{81}{16}\lambda^5(e^{2t}(e^{2t}-1)^4) + \cdots \quad (5.6)
\end{aligned}
$$

Hence, the series solution obtained in eq.(5.6) converges to $U(x,t) = \frac{e^{2t}\lambda}{1+\frac{3}{2}\lambda(e^{2t}-1)}$.

5.2.3 Fisher's equation

Fisher(1937) proposed a non-linear equation to portray the spread of a viral mutant in an interminably long habitat. This equation is experienced in different applications, such as gene equation, tissue engineering, and neurophysiology.

Fisher's equation is the partial differential equation of the form

$$\frac{\partial \phi}{\partial t} - a\frac{\partial^2 \phi}{\partial t^2} = b\phi(1-\phi).$$

It belongs to the class of reaction-diffusion equation.

5.2.4 Solution of Fisher's equation

Consider the follwing IVP

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \phi(1-\phi), \tag{5.7}$$

where $\phi(x,0) = \beta$.

Operating Sumudu transformation on eq. (5.7) and using initial conditions, we get

$$S\{\phi(x,t)\} = \frac{1}{1-u}\beta + \frac{u}{1-u}S\{\phi_{xx} - \phi^2\}. \tag{5.8}$$

Now, operating inverse Sumudu transformation on eq. (5.8), we have

$$\phi(x,t) = e^t\beta + S^{-1}\{\frac{u}{1-u}S\{\phi_{xx} - \phi^2\}\}. \tag{5.9}$$

We apply HPM, eq. (5.9) becomes

$$\sum_{n=0}^{\infty} \phi_n(x,t) = e^t\beta + pS^{-1}\left\{\frac{u}{1-u}S\left\{\left(\sum_{n=0}^{\infty} p^n\phi_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(x,t)\right\}\right\}. \tag{5.10}$$

A couple of terms of $H_n(x,t)$ are given by

$$\begin{aligned} H_0 &= \phi_0^2, \\ H_1 &= 2\phi_0\phi_1, \\ H_2 &= 2\phi_0\phi_2 + \phi_1^2, \\ H_3 &= 2\phi_0\phi_3 + 2\phi_1\phi_2, \\ &\vdots \end{aligned}$$

On looking at the like terms of eq. (5.10), we have

$$\begin{aligned} p^0 : \phi_0 &= e^t\beta, \\ p^1 : \phi_1 &= -\beta^2(e^t(e^t-1)), \\ p^2 : \phi_2 &= \beta^3(e^t(e^t-1)^2), \\ p^3 : \phi_3 &= -\beta^4(e^t(e^t-1)^3), \\ p^4 : \phi_4 &= \beta^5(e^t(e^t-1)^4), \\ &\vdots \end{aligned}$$

For $t = \ln(1+\frac{\gamma}{\beta})$, where $0 < \gamma < 1$. Let us consider

$$\begin{aligned} ||s_1 - s_0|| &= ||\phi_1|| = \Big|\Big| - \beta^2(e^t(e^t-1))\Big|\Big| = \Big|\Big|(e^t\beta)(-\beta(e^{2t}-1))\Big|\Big| \\ &\leq ||(e^t\beta)|||\gamma| = \gamma||\phi_0||, \\ ||s_2 - s_1|| &= ||\phi_2|| = \Big|\Big|\beta^3(e^t(e^t-1)^2)\Big|\Big| = \Big|\Big|(e^t\beta)(\beta^2(e^t-1)^2)\Big|\Big| \\ &\leq ||\phi_0|||\gamma^2| = \gamma^2||\phi_0||, \\ ||s_3 - s_2|| &= ||\phi_3|| = \Big|\Big| - \beta^4(e^t(e^t-1)^3)\Big|\Big| = \Big|\Big|(e^t\beta)(\beta^3(e^t-1)^3)\Big|\Big| \\ &\leq \beta^3||\phi_0||, \\ ||s_4 - s_3|| &= ||\phi_4|| = \Big|\Big|\beta^5(e^t(e^t-1)^4)\Big|\Big| = \Big|\Big|(e^t\beta)(\beta^4(e^t-1)^4)\Big|\Big| \\ &\leq \beta^4||\phi_0||, \\ &\vdots \end{aligned}$$

Consider

$$\begin{aligned} ||s_n - s_m|| &\leq ||\phi_n|| + ||\phi_{n-1}|| + ||\phi_{n-2}|| + \ +||\phi_{m+1}|| \\ &\leq \gamma^{m+1}(1 + \gamma^1 + \gamma^2 + \ +\gamma^{n-m-1})||\phi_0|| \\ &\leq \frac{\gamma^{m+1}}{1-\gamma}||\phi_0|| \end{aligned}$$

Hence, $||s_n - s_m|| \to 0$ as $m, n \to \infty$, so $\{s_n\}$ is a Cauchy sequence. The approximate solution of eq. (5.7) is

$$\phi(x,t) = e^t\beta - \beta^2 e^t(e^t-1) + \beta^3 e^t(e^t-1)^2 - \beta^4 e^t(e^t-1)^3 + \beta^5(e^t(e^t-1)^4) + \cdots \quad (5.11)$$

Hence the series solution converges to the exact solution $\phi(x,t) = \frac{e^t\beta}{1+\beta(e^t-1)}$.

Table 5.1: Numerical solution of Newell-Whitehead-Segel equation (5.2) for $\lambda = 2$

t	U_{exact}	$U_{HPSTM} = s_5$	*truncation error*	$U_5 = \|s_5 - s_4\|$	$U_4 = \|s_4 - s_3\|$	$U_3 = \|s_3 - s_2\|$
0	2	2	0	0	0	0
0.01	1.9238	1.9238	0	0	0.0005	0.0075
0.02	1.8546	1.8546	0	0.0005	0.0038	0.0312
0.03	1.7914	1.7918	0.0004	0.0025	0.0136	0.0731
0.04	1.7335	1.7351	0.0016	0.0084	0.0338	0.1353
0.05	1.6802	1.6855	0.0053	0.0219	0.0694	0.22
0.06	1.6311	1.6445	0.0134	0.0483	0.1262	0.3299
0.07	1.5857	1.6152	0.0295	0.095	0.2108	0.4676
0.08	1.5436	1.6025	0.0589	0.1723	0.331	0.6359
0.09	1.5044	1.6134	0.109	0.2934	0.4959	0.8382
0. 1	1.4678	1.6576	0.1898	0.4755	0.7158	1.0777

Table 5.2: Numerical solution of Fisher's equation (5.7) for $\beta = 3$

t	ϕ_{exact}	$\phi_{HPSTM} = s_5$	*truncation error*	$\phi_5 = \|s_5 - s_4\|$	$\phi_4 = \|s_4 - s_3\|$	$\phi_3 = \|s_3 - s_2\|$
0	2	2	0	0	0	0
0.025	1.9518	1.9518	0	0	0.0003	0.0053
0.05	1.907	1.907	0	0.0002	0.0023	0.0221
0.075	1.8652	1.8654	0.0002	0.0013	0.0081	0.0523
0.1	1.8262	1.827	0.0008	0.0043	0.0206	0.0978
0.125	1.7897	1.7921	0.0024	0.0114	0.0428	0.1607
0.15	1.7555	1.7617	0.0062	0.0255	0.0788	0.2434
0.175	1.7233	1.7374	0.0141	0.051	0.1333	0.3486
0.2	1.6931	1.7219	0.0288	0.0939	0.2121	0.479
0.225	1.6646	1.7191	0.0545	0.1624	0.3219	0.6379
0.25	1.6377	1.7346	0.0969	0.2674	0.4707	0.8287

5.3 Convergence of HPTM for the series solution of fractional PDE

Here, we emphasize on the condition of convergence of HPTM for the series solution of non-linear fractional PDE. For that, we consider the following non-linear fractional PDE.

$$\frac{\partial^\alpha}{\partial t^\alpha} w(x,t) + L\ w(x,t) + N\ w(x,t) = f(x,t), t > 0, x \in \mathbb{R}, n-1 < \alpha \leq n, \quad (5.12)$$

here, $\frac{\partial^\alpha}{\partial t^\alpha}$, is the Caputo fractional derivative with respect to t, L and N are linear and non-linear differential operators respectively which satisfy Lipschitz condition, $f(x,t)$ is the source term. Consider the Banach space $C[0,T]$ of all continuous functions on $[0,T]$ with supremum norm. Throughout this section, we consider $w(x,t)$, $w_n(x,t) \in C[0,T]$.

Theorem 5.3.1 *(Uniqueness theorem) The solution obtained by HPTM of fractional partial differential equation (5.12) has a unique solution, whenever* $0 < \gamma < 1$.

Proof*: The solution of eq.(5.12) is of the form* $w(x,t) = \sum_{n=0}^{\infty} p^n w_n(x,t)$ *. Here,*

$$w(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} w^{(k)}(x,0) + \mathcal{L}^{-1}\Big\{ \frac{1}{s^\alpha}\Big(\mathcal{L}\Big\{ f(x,t) - L\ w(x,t) - N\ w(x,t) \Big\}\Big)\Big\}.$$

Let W *and* V *be the distinct solutions of the eq.(5.12) then*

$$|W - V| = | - \mathcal{L}^{-1}[\frac{1}{s^\alpha}(\mathcal{L}[L(W-V) + N(W-V)])]|.$$

Using convolution theorem,

$$\begin{aligned} |W - V| &\leq \int_0^t \Big(|L(W-V)| + |N(W) - N(V)| \Big) \left| \frac{(t-\tau)^{n-1}}{(n-1)!} \right| d\tau \\ &\leq \int_0^t (\eta |(U-V)| + \delta |U - V|) \left| \frac{(t-\tau)^n}{n!} \right| d\tau \end{aligned}$$

$\Big\{ L$ *is a bounded operator i.e.* $|L(W) - L(V)| \leq \eta |W - V|$, N *satisfies Lipschitz condition with* $\delta > 0$ *such that* $|N(W) - N(V)| \leq \delta |W - V| \Big\}$.

$$|W - V| \ \leq \ \int_0^t (\eta + \delta)|W - V|) \left| \frac{(t-\tau)^{n-1}}{(n-1)!} \right| d\tau$$

Using mean value theorem of integral calculus, $|W - V| \leq [(\eta + \delta)|W - V|]\ M\ T$, *where* $M = \max(t-\tau)^n$ *and* $t \in [0, T]$. *Hence,* $|W - V| \leq |W - V|\gamma$, *where* $\gamma = (n + \delta)M\ T$. *So* $(1-\gamma)|W - V| \leq 0$, *implies* $W = V$ *whenever,* $0 < \gamma < 1$.

Theorem 5.3.2 *Let* w *and* $w_n(x,t)$ *be defined in Banach space* $\boldsymbol{B}$, *the condition that the series solution* $\sum_{n=0}^{\infty} w_n$ *, converges to the solution* w *is* $||w_{n+1}|| \leq \gamma\ ||w_n||$ *where* $\gamma \in (0,1)$ *and* $n \in \mathbb{N}$.

Proof*: Let*

$$s_n = w_0 + w_1 + w_2 + \cdots + w_n \tag{5.13}$$

be the partial sum of the series solution $\sum_{n=0}^{\infty} w_n$. *The convergence of sequence* $\{s_n\}$ *of the partial sums will be proved, if we show that* $\{s_n\}$ *is a Cauchy sequence in* $(C[0,T], || \, ||)$. *Consider*

$$\begin{aligned} ||s_{n+1} - s_n|| &= ||w_{n+1}|| \leq \gamma ||w_n|| \\ &\leq \gamma^2 ||w_{n-1}|| \leq \cdots \leq \gamma^{n+1} ||w_0|| \end{aligned}$$

Hence,

$$\begin{aligned} ||s_n - s_m|| &= ||\Sigma_{i=m+1}^{n} w_i|| \leq \Sigma_{i=m+1}^{n} ||w_i|| \\ &\leq \gamma^{m+1} (\Sigma_{i=0}^{n-m} \gamma^i) ||w_0|| = \gamma^{m+1} \frac{(1-\gamma^{n-m})}{1-\gamma} ||w_0||, n, m \in \mathbb{N} \end{aligned}$$

Since , $0 < \gamma < 1$, *hence* $||s_n - s_m|| \leq \frac{\gamma^{m+1}}{1-\gamma} ||w_0||$. *Also* w_0 *is bounded, therefore* $||s_{n+1} - s_n|| \to 0$ *as* $m, n \to \infty$. *which shows that* $\{s_n\}$ *is a Cauchy sequence in* $C[0,T]$, *hence* $\sum_{n=0}^{\infty} w_n(x,t)$ *is convergent.*

5.4 Application

To understand the effectiveness of the HPTM, we impose this technique to the following well-known equation.

5.4.1 Burgers' equation

The Burgers' equation $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = c \frac{\partial^2 u}{\partial x^2}$, which is a balance between time evolution, non -linearity, and diffusion. This is a non-linear mathematical model used for diffusive waves especially occur in fluid dynamics. In 1948, Burgers' proposed this equation to illuminate turbulence depicted by the interaction of two inverse impacts of convection and diffusion. The term uu_x represents the shocking impact that will make waves break while the term cu_{xx} represents the diffusion.

5.4.2 Solution of fractional Burgers' equation

Consider the Burgers' equation of fractional order

$$\frac{\partial^\eta u}{\partial t^\eta} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, 0 < \eta \leq 1 \tag{5.14}$$

with initial condition

$$u(x,0) = 2x \tag{5.15}$$

Operating Laplace transformation on eq.(5.14), we have

$$\mathcal{L}\left\{\frac{\partial^\eta u}{\partial t^\eta} + u\frac{\partial u}{\partial x}\right\} = \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} \tag{5.16}$$

Using (1.6.1) on (5.15), we have

$$s^\eta \mathcal{L}\{u\} - s^{\eta-1}u(x,0) = \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right\}$$

or

$$\mathcal{L}\{u\} = \frac{1}{s}2x + \frac{1}{s^\eta}\mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right\} \tag{5.17}$$

Operating inverse Laplace transformation on eq.(5.17), we get

$$u(x,t) = 2x + \mathcal{L}^{-1}\left\{\frac{1}{s^\eta}\mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right\}\right\} \tag{5.18}$$

Now, we apply HPM , we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 2x + p\mathcal{L}^{-1}\left\{\frac{1}{s^\eta}\mathcal{L}\left\{\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(x,t)\right\}\right\} \tag{5.19}$$

where $H_n(x,t)$ represents He's polynomial used for non-linear term present in eq. (5.18), i.e.

$$\sum_{n=0}^{\infty} p^n H_n(x,t) = u\frac{\partial u}{\partial x} = \left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_x \tag{5.20}$$

A few couples of terms of He's polynomial using eq.(5.20) are given below:

$$\begin{aligned}
H_0(x,t) &= u_0 u_{0x};\\
H_1(x,t) &= u_1 u_{0x} + u_1 u_{0x};\\
H_2(x,t) &= u_2 u_{0x} + u_1 u_{1x} + u_0 u_{2x};\\
&\vdots
\end{aligned}$$

On looking at the coefficient of like power of p of eq.(5.19), we have

$$\begin{aligned}
u_0(x,t) &= 2x;\\
u_1(x,t) &= \frac{-4x\ t^{\eta}}{\Gamma(\eta+1)};\\
u_2(x,t) &= \frac{16x\ t^{2\eta}}{\Gamma(2\eta+1)};\\
u_3(x,t) &= \frac{16x\ t^{3\eta}}{\Gamma(3\eta+1)}\left(4+\frac{\Gamma(2\eta+1)}{(\Gamma(\eta+1))^2}\right);\\
u_4(x,t) &= 64xt^{4\eta}\left[\frac{2\Gamma(3\eta+1)}{\Gamma(\eta+1)\Gamma(2\eta+1)\Gamma(4\eta+1)}+\frac{1}{\Gamma(4\eta+1)}\left(4+\frac{\Gamma(2\eta+1)}{(\Gamma(\eta+1))^2}\right)\right],\\
&\vdots
\end{aligned} \tag{5.21}$$

For $\eta = 1$ and $t = \frac{\gamma}{2}$, where $0 < \gamma < 1$, let us consider

$$\begin{aligned}
||s_1 - s_0|| &= ||u_1|| = \left|\left|\frac{-4x\ t^{\eta}}{\Gamma(\eta+1)}\right|\right| = \left|\left|(2x)(\frac{-2t^{\alpha}}{\Gamma(\alpha+1)})\right|\right|\\
&\leq ||2x|| \left|2(\frac{\gamma}{2})\right| = \gamma||u_0||,\\
||s_2 - s_1|| &= ||u_2|| = \left|\left|\frac{16x\ t^{2\eta}}{\Gamma(2\eta+1)}\right|\right| = \left|\left|(2x)\left(\frac{8t^2}{\Gamma 3}\right)\right|\right|,\\
&\leq ||u_0||\left(\frac{8(\frac{\gamma}{2})^2}{2}\right) = \gamma^2||u_0||\\
||s_3 - s_2|| &= ||u_3|| = \left|\left|\frac{16x\ t^{3\eta}}{\Gamma(3\eta+1)}\left(4+\frac{\Gamma(2\eta+1)}{(\Gamma(\eta+1))^2}\right)\right|\right|,\\
&= \left|\left|(2x)\frac{8\ t^3}{\Gamma(4)}\left(4+\frac{\Gamma(3)}{(\Gamma(2))^2}\right)\right|\right| \leq \gamma^3||u_0||,
\end{aligned}$$

$$\begin{aligned}
||s_4 - s_3|| &= ||u_4|| = \left|\left|64xt^{4\eta}\left[\frac{2\Gamma(3\eta+1)}{\Gamma(\eta+1)\Gamma(2\eta+1)\Gamma(4\eta+1)}\right.\right.\right. \\
&\quad \left.\left.\left.+\frac{1}{\Gamma(4\eta+1)}\left(4+\frac{\Gamma(2\eta+1)}{(\Gamma(\eta+1))^2}\right)\right]\right|\right|, \\
&= \left|\left|(2x)(32t^4)\left[\frac{2\Gamma 4}{\Gamma 2\Gamma 3\Gamma 5}+\frac{1}{\Gamma 5}\left(4+\frac{\Gamma 3}{(\Gamma 2)^2}\right)\right]\right|\right| \\
&\leq ||u_0||\left|32\left(\frac{\gamma}{2}\right)^4\left[\frac{1}{4}+\frac{1}{4}\right]\right| = \gamma^4||u_0||; \\
&\vdots
\end{aligned}$$

Consider

$$\begin{aligned}
||s_n - s_m|| &\leq ||u_n|| + ||u_{n-1}|| + \cdots + ||u_{m+1}|| \\
&\leq ||u_0||\gamma^n + ||u_0||\gamma^{n-1} + \cdots + ||u_0||\gamma^{m+1} \\
&= \gamma^{m+1}(1+\gamma+\gamma^2+\cdots+\gamma^{n-m-1})||u_0|| \\
&\leq \frac{\gamma^{m+1}}{1-\gamma}||u_0||
\end{aligned}$$

Hence $||s_n - s_m|| \to 0$ as $n, m \to \infty$, which shows that $\{s_n\}$ is a Cauchy sequence. So, the approximate solution of (5.14) is given as

$$\begin{aligned}
u(x,t) &= 2x - \frac{4x\ t^{\eta}}{\Gamma(\eta+1)} + \frac{16x\ t^{2\eta}}{\Gamma(2\eta+1)} - \frac{16x\ t^{3\eta}}{\Gamma(3\eta+1)}\left(4+\frac{\Gamma(2\eta+1)}{(\Gamma(\eta+1))^2}\right) \\
&+ 64xt^{4\eta}\left[\frac{2\Gamma(3\eta+1)}{\Gamma(\eta+1)\Gamma(2\eta+1)\Gamma(4\eta+1)}+\frac{1}{\Gamma(4\eta+1)}\left(4+\frac{\Gamma(2\eta+1)}{(\Gamma(\eta+1))^2}\right)\right]+\ldots
\end{aligned} \tag{5.22}$$

when $\eta = 1$, solution (5.22) of Burgers' equation (5.14) reduces to

$$u(x,t) = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 - \cdots = \frac{2x}{(1+2t)}. \tag{5.23}$$

From eq. (5.23),we discover that series solution obtained in eq.(5.22) converges to the exact solution of Burgers' equation when $\eta = 1$.Further fig. (5.1) and (5.2) shows that the surface graphs of the solution of eq.(5.14) up to fifth-order approximation for different values of η. It also shows that as the value of η approaches to one the surface graph approaches to surface graph of the exact solution. Further, the approximate solutions for $\eta = 1$ obtained for different values of x and t is presented in table 5.3 which supports our analytical results regarding the convergence of the method.

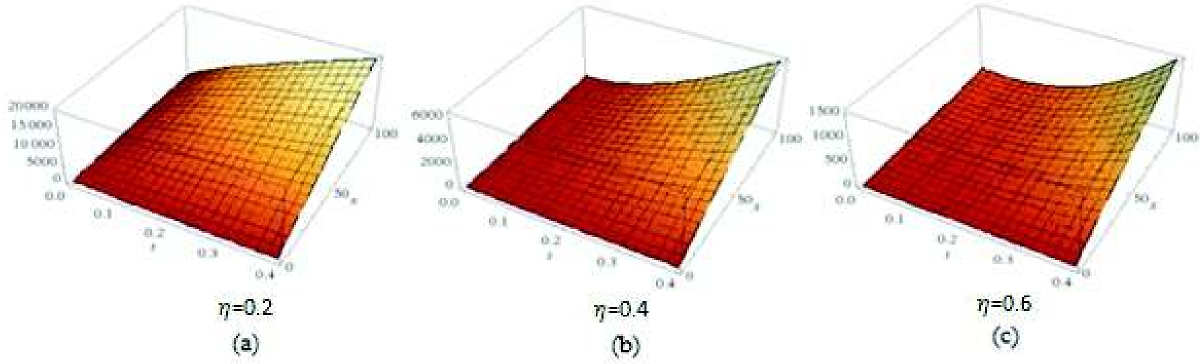


Figure 5.1: Surface graph of fractional Burgers' equation for $\eta = 0.2, 0.4$ and 0.6

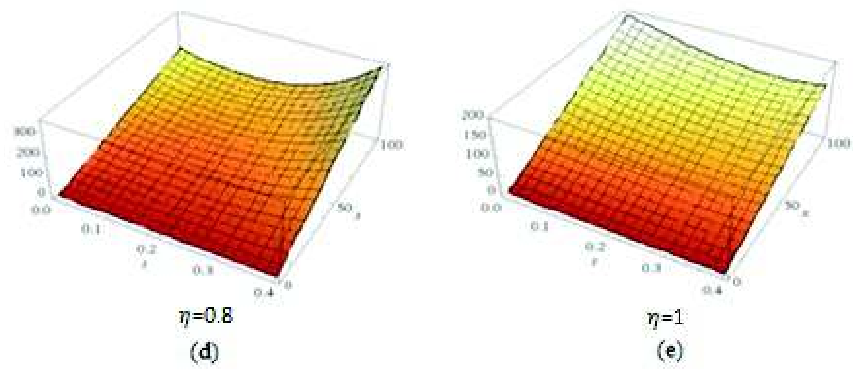


Figure 5.2: Surface graph of fractional Burgers' equation for $\eta = 0.8$ and 1

Table 5.3: Approximate solution of fractional Burgers' equation (5.14) up to fourth order

t	x	s_3	s_4	s_5	u_{exact}	$\lvert s_4 - s_3\rvert$	$\lvert s_5 - s_4\rvert$
	0	0	0	0	0	0	0
	0.1	0.1664	0.1667	0.1667	0.1667	0.0003	0
0.1	0.2	0.3328	0.3334	0.3333	0.3333	0.0006	0.0001
	0.3	0.4992	0.5002	0.5	0.5	0.001	0.0002
	0.4	0.6656	0.6669	0.6666	0.6667	0.0013	0.0003
	0.5	0.832	0.8336	0.8333	0.8333	0.0016	0.0003
	0	0	0	0	0	0	0
	0.1	0.1392	0.1443	0.1423	0.1429	0.0051	0.002
0.2	0.2	0.2784	0.2886	0.2845	0.2857	0.0102	0.0041
	0.3	0.4176	0.433	0.4268	0.4286	0.0154	0.0062
	0.4	0.5568	0.5773	0.5691	0.5714	0.0205	0.0082
	0.5	0.696	0.7216	0.7114	0.7143	0.0256	0.0102
	0	0	0	0	0	0	0
	0.1	0.1088	0.1347	0.1192	0.125	0.0259	0.0155
0.3	0.2	0.2176	0.2694	0.2383	0.25	0.0518	0.0311
	0.3	0.3264	0.4042	0.3575	0.375	0.0778	0.0467
	0.4	0.4352	0.5389	0.4767	0.5	0.1037	0.0622
	0.5	0.544	0.6736	0.5958	0.625	0.1296	0.0778

5.5 Conclusion

1. The convergence and uniqueness of HPSTM and HPTM are expressed analytically.

2. The estimated results of a series solution with maximum truncation error are obtained and reported in table 5.1 and 5.2.

3. Results acquired about the convergence and error analysis of HPTM and HPSTM are numerically illustrated.

4. The obtained results are approaching to exact results and are closed.

Chapter 6

Comparative Study of HPTM with HPETM

The fractional calculus is an important tool to refine the description of most of the natural phenomenon. Fractional PDEs attracted the interest of many researchers because of their successive appearance in diverse fields of science and engineering. Many numerical and semi-analytical methods are utilized to obtain solutions of linear and non-linear PDEs.

In this chapter, we apply HPTM [64, 92, 93] and HPETM [23, 24, 25, 26],[81] to discover the solution of fractional Fisher's equation, time-fractional Fornberg-Whitham equation, and time-fractional Inviscid Burgers' equation and we obtain a power series solution is a rapidly convergent series and just a couple of iterations leads to a more accurate solution. In these techniques, there is no need for the algorithm like discretizing the problem, no linearization is required for the non-linear problems. There are much symbolic computation software like Maple, Mathematica, etc. with which we can easily calculate more terms very easily, hence it reduces the computational cost for solving such a complex problem. Finally, we compare the result obtained by these methods.

6.1 Homotopy perturbation Elzaki transform method (HPETM)

Consider the following general fractional non-linear partial differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha} w(x,t) + Lw(x,t) + Nw(x,t) = f(x,t), t > 0, x \in \mathbb{R}, n-1 < \alpha \le n, \tag{6.1}$$

here, $\frac{\partial^\alpha}{\partial t^\alpha}$, is the Caputo fractional derivative with respect to t. Now applying Elzaki transform, we get

$$E\left\{\frac{\partial^\alpha}{\partial t^\alpha} w + Lw + Nw\right\} = E\big\{f(x,t)\big\}.$$

Using (1.6.3), we have

$$E\big\{w\big\} = \sum_{k=0}^{n-1} v^{k+2} w^{(k)}(x,0) + v^\alpha\left(E\left\{f(x,t) - Lw - Nw\right\}\right).$$

Applying the inverse Elzaki transform, we have

$$w = \sum_{k=0}^{n-1} \frac{t^k}{k!} w^{(k)}(x,0) + E^{-1}\left\{v^\alpha\left(E\left\{f(x,t) - Lw - Nw\right\}\right)\right\}. \tag{6.2}$$

By applying HPM, we get

$$\begin{aligned} 0 =& (1-p)\Big(w(x,t) - w(x,0)\Big) + p\Bigg(w(x,t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} w^{(k)}(x,0) \\ &- E^{-1}\left\{v^\alpha E\left\{f(x,t) - Lw - Nw\right\}\right\}\Bigg), \end{aligned}$$

Let

$$\begin{aligned} w(x,t) &= \sum_{n=0}^{\infty} p^n w_n(x,t), \qquad (6.3) \\ Nw(x,t) &= \sum_{n=0}^{\infty} p^n H_n(w(x,t)) \end{aligned}$$

where

$$H_n(w(x,t)) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(\sum_{i=0}^{\infty} p^i w_i\right)_{(p=0)}, n = 0, 1, 2, 3, \ldots \tag{6.4}$$

So, (6.2) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} p^n w_n &= w(x,0) + p\sum_{k=1}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0)\\ &+E^{-1}\left\{v^{\alpha}\left(E\left\{f(x,t) - L\sum_{n=0}^{\infty}p^n w_n - \sum_{n=0}^{\infty}p^n H_n(w)\right\}\right)\right\}\end{aligned}$$

On equating the coefficients of like powers of p, we have

$$\begin{aligned} p^0 &: w_0 = w(x,0);\\ p^1 &: w_1 = \sum_{k=1}^{n-1}\frac{t^k}{k!}w^{(k)}(x,0) + E^{-1}\left\{v^{\alpha}\left(E\left\{f(x,t) - Lw_0(x,t) - H_0\right\}\right)\right\};\\ p^2 &: w_2 = -E^{-1}\left\{v^{\alpha}\left(E\left\{Lw_1(x,t) + H_1\right\}\right)\right\};\\ p^3 &: w_3 = -E^{-1}\left\{v^{\alpha}\left(E\left\{Lw_2(x,t) + H_2\right\}\right)\right\},\\ &\ \ \vdots \end{aligned}$$

therefore, the HPETM series solution is obtained as $p \to 1$

$$w(x,t) = w_0 + w_1 + w_2 + w_3 + \dots.$$

6.2 Convergence analysis

In this section, we emphasis on the condition of convergence of the proposed method for the series solution of eq. (6.1).

Theorem 6.2.1 *Let w and $w_n(x,t)$ be defined in Banach space $\boldsymbol{B}$, the condition that series solution given by eq. (6.3) converges to the solution is such that $||w_{n+1}|| \leq \eta\ ||w_n||$ where $\eta \in (0,1)$. The condition of convergence is proved in [98, 99].*

Remark 6.2.2 *The maximum truncated error of $w(x,t) = \sum_{n=0}^{\infty} w_n$ is given by $\frac{\eta^{m+1}}{1-\eta}||w_0||$.*

6.3 Application

To understand the effectiveness of the said technique, we apply this technique to the following famous equations.

6.3.1 Fisher's equation

In (1937), Fisher proposed a non-linear equation to portray the spread of a viral mutant in an interminably long habitat. This equation is experienced in different applications, such as gene equation, tissue engineering, and neurophysiology. Fisher's equation is the partial differential equation of the form

$$\frac{\partial U}{\partial t} - a\frac{\partial^2 U}{\partial t^2} = bU(1-U)$$

6.3.2 Solution of time fractional Fisher's equation using HPETM

Consider the time fractional non-linear Fisher's equation [81]

$$\frac{\partial^\beta w}{\partial t^\beta} = \frac{\partial^2 w}{\partial x^2} + 6w(1-w), t > 0, x \in \mathbb{R}, 0 < \beta \leq 1, \tag{6.5}$$

with $w(x,0) = \frac{1}{(1+e^x)^2}$.

By applying HPETM on (6.5), we have

$$\sum_{n=0}^{\infty} p^n w_n = \frac{1}{(1+e^x)^2} + p\Bigg(E^{-1}\Bigg\{v^\beta\Bigg(E\Bigg\{\sum_{n=0}^{\infty}(p^n w_n)_{xx} + 6\Bigg\{\sum_{n=0}^{\infty} p^n w_n - 6\sum_{n=0}^{\infty} p^n H_n(x,t)\Bigg\}\Bigg\}\Bigg)\Bigg\}\Bigg), \tag{6.6}$$

The initial couple of terms of components of $H_n(w)$ are given by

$$\begin{aligned} H_0 &= w_0^2; \\ H_1 &= 2w_0 w_1; \\ H_2 &= w_1^2 + 2w_0 w_2, \\ &\vdots \end{aligned}$$

Comparing the like terms of (6.6), we have

$$
\begin{aligned}
p^0 : w_0 =& \frac{1}{(1+e^x)^2},\\
p^1 : w_1 =& 10\frac{e^x t^\beta}{(1+e^x)^3\Gamma(\beta+1)},\\
p^2 : w_2 =& 50\frac{e^x(2e^x-1)t^{2\beta}}{(1+e^x)^4\Gamma(2\beta+1)},\\
p^3 : w_3 =& \left(50\frac{e^x(-16e^{3x}-15e^{2x}+30e^x+5)}{(1+e^x)^6}+600e^{2x}\frac{\Gamma(2\beta+1)}{(1+e^x)^6\Gamma(\beta+1)^2}\right)\frac{t^{3\beta}}{\Gamma(3\beta+1)}.\\
&\vdots
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
w(x,t) =& \frac{1}{(1+e^x)^2}+10\frac{e^x t^\beta}{(1+e^x)^3\Gamma(\beta+1)}+50\frac{e^x(2e^x-1)t^{2\beta}}{(1+e^x)^4\Gamma(2\beta+1)}\\
&+\left(50\frac{e^x(-16e^{3x}-15e^{2x}+30e^x+5)}{(1+e^x)^6}+600e^{2x}\frac{\Gamma(2\beta+1)}{(1+e^x)^6\Gamma(\beta+1)^2}\right)\frac{t^{3\beta}}{\Gamma(3\beta+1)}+\dots.
\end{aligned}
\tag{6.7}
$$

The above solution at $\beta = 1$ converges to $w(x,t) = \frac{1}{(1+e^{x-5t})^2}$.

6.3.3 Fornberg-Whitham equation

Notably, the KdV equation and some Camassa Holm type equations concede soliton solutions which keep up a consistent shape and move at steady speed; and numerous Camassa Holm type conditions have peakon (peaked soliton) solution. It is fascinating that the Fornberg-Whitham equation does not just permit traveling wave solutions, yet besides, has peakon solution.

6.3.4 Solution of time fractional Fornberg-Whitham equation using HPETM

Consider the time-fractional Fornberg-Whitham equation [96]

$$
\frac{\partial^\beta}{\partial t^\beta}w(x,t) = \frac{\partial^3 w}{\partial x^2\partial t}-\frac{\partial w}{\partial x}+w\frac{\partial^3 w}{\partial x^3}-w\frac{\partial w}{\partial x}+3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}, t>0, x\in\mathbb{R}, 0<\beta\le 1, \tag{6.8}
$$

with initial condition $w(x,0) = e^{\frac{x}{2}}$.

By applying HPETM on (6.8), we have

$$\sum_{n=0}^{\infty} p^n w_n = e^{\frac{x}{2}} + p\Bigg(E^{-1}\Bigg\{v^{\beta}\Bigg(E\Bigg\{\sum_{n=0}^{\infty} p^n\Bigg((w_n)_{xxt} + (-w_n)_x + H_n(w)\Bigg)\Bigg\}\Bigg)\Bigg\}\Bigg), \quad (6.9)$$

The initial a couple of terms of components of H_n are given by

$$\begin{aligned} H_0 &= w_0 w_{0xxx} - w_0 w_{0x} + 3w_{0x} w_{0xx}; \\ H_1 &= w_0 w_{1xxx} + w_1 w_{0xxx} - w_0 w_{1x} - w_1 w_{0x} + 3w_{0x} w_{1xx} + 3w_{1x} w_{0xx}; \end{aligned}$$

$$\begin{aligned} H_2 = {} & w_0 w_{2xxx} + w_1 w_{1xxx} + w_2 w_{0xxx} - w_0 w_{2x} - w_1 w_{1x} - w_2 w_{0x} \\ & + 3w_{2x} w_{0xx} + 3w_{1x} w_{1xx} + 3w_{0x} w_{2xx}. \end{aligned}$$

On looking at the like terms of (6.9), we have

$$\begin{aligned} p^0 : w_0 &= e^{\frac{x}{2}}; \\ p^1 : w_1 &= \frac{-e^{\frac{x}{2}}}{2} \frac{t^{\beta}}{\Gamma(\beta+1)}; \\ p^2 : w_2 &= \frac{-e^{\frac{x}{2}}}{8} \frac{t^{2\beta-1}}{\Gamma 2\beta} + \frac{e^{\frac{x}{2}}}{4} \frac{t^{2\beta}}{\Gamma(2\beta+1)}; \\ p^3 : w_3 &= e^{\frac{x}{2}} \left(\frac{-1}{32} \frac{t^{3\beta-2}}{\Gamma(3\beta-1)} + \frac{1}{8} \frac{t^{3\beta-1}}{\Gamma(3\beta)} - \frac{1}{8} \frac{t^{3\beta}}{\Gamma(3\beta+1)} \right), \\ &\vdots \end{aligned}$$

Hence, the solution is

$$\begin{aligned} w(x,t) = e^{\frac{x}{2}} & - \frac{e^{\frac{x}{2}}}{2} \frac{t^{\beta}}{\Gamma(\beta+1)} - \frac{e^{\frac{x}{2}}}{8} \frac{t^{2\beta-1}}{\Gamma 2\beta} + \frac{e^{\frac{x}{2}}}{4} \frac{t^{2\beta}}{\Gamma(2\beta+1)} \\ & + e^{\frac{x}{2}} \left(\frac{-1}{32} \frac{t^{3\beta-2}}{\Gamma(3\beta-1)} + \frac{1}{8} \frac{t^{3\beta-1}}{\Gamma(3\beta)} - \frac{1}{8} \frac{t^{3\beta}}{\Gamma(3\beta+1)} \right) + \dots \end{aligned} \quad (6.10)$$

From the above solution, it is clear that the approximate solution obtained from the abovesaid technique is converging to exact solution i.e. $w(x,t) = e^{\frac{1}{2}(x - \frac{4t}{3})}$ for $\beta = 1$.

6.3.5 Inviscid Burgers' equation

It is worth mentioning that Inviscid Burgers' equation is a prototype of the conservation law i.e. $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\int f(u)du\right) = 0$.The Inviscid Burgers' equation is given as

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0$$

It is a non-linear hyperbolic equation. It has application in gas dynamics and traffic flow.

6.3.6 Solution of time fractional Inviscid Burgers' equation using HPETM

Consider the following non-homogeneous time fractional Inviscid Burgers' equation [118]

$$D_t^{\beta} w + ww_x = 1 + x + t, w(x,0) = x, 0 < \beta \leq 1 \tag{6.11}$$

By applying HPETM on eq.(6.11), we have

$$\sum_{n=0}^{\infty} p^n w_n = x + p\Bigg(E^{-1}\Bigg\{v^{\beta}E\Bigg\{1 + x + t\Bigg\} - v^{\beta}E\Bigg\{\sum_{n=0}^{\infty} p^n H_n(x,t)\Bigg\}\Bigg\}\Bigg) \tag{6.12}$$

where

$$ww_x = \sum_{n=0}^{\infty} p^n H_n(x,t)$$

The initial couple of terms of $H_n(w)$ are given as

$$\begin{aligned}
H_0 &= w_0 w_{0x} \\
H_1 &= w_0 w_{1x} + w_1 w_{0x} \\
H_2 &= w_0 w_{2x} + w_1 w_{1x} + w_2 w_{0x} \\
&\vdots
\end{aligned}$$

On comparing the like terms of (6.12), we have

$$
\begin{aligned}
p^0 &: w_o = x \\
p^1 &: w_1 = (1+x)\frac{t^\beta}{\Gamma(1+\beta)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} - E^{-1}\Big\{v^\beta E\{H_0\}\Big\} \\
&= \frac{t^\beta}{\Gamma(1+\beta)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} \\
p^2 &: w_2 = -E^{-1}\Big\{v^\beta E\{H_1\}\Big\} \\
&= -\left(\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\right) \\
p^3 &: w_3 = -E^{-1}\Big\{v^\beta E\{H_2\}\Big\} \\
&= \left(\frac{t^{3\beta}}{\Gamma(3\beta+1)} + \frac{t^{3\beta+1}}{\Gamma(3\beta+2)}\right) \\
&\vdots
\end{aligned}
$$

Hence the solution of (6.11) is

$$
w(x,t) = x + \frac{t^\beta}{\Gamma(1+\beta)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} - \left(\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\right) + \left(\frac{t^{3\beta}}{\Gamma(3\beta+1)} + \frac{t^{3\beta+1}}{\Gamma(3\beta+2)}\right) + \dots
$$

or

$$
\begin{aligned}
w(x,t) = x + \left(\frac{t^\beta}{\Gamma(1+\beta)} - \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \dots\right) \\
+ \left(\frac{t^{\beta+1}}{\Gamma(\beta+2)} - \frac{t^{2\beta+1}}{\Gamma(2\beta+2)} + \frac{t^{3\beta+1}}{\Gamma(3\beta+2)} + \dots\right)
\end{aligned}
$$

or

$$
w(x,t) = x - \sum_{n=1}^{\infty} \frac{(-1)^n t^{n\beta}}{\Gamma(n\beta+1)} - t\sum_{n=1}^{\infty} \frac{(-1)^n t^{n\beta}}{\Gamma(n\beta+2)}
$$

$$
w(x,t) = x + 1 + t - E_{\beta,1}(-t^\beta) - tE_{\beta,2}(-t^\beta) \tag{6.13}
$$

where $E_{\beta,2}(-t^\beta)$ in eq. (6.13) is Mittag-Leffler function defined in (1.5). When $\beta = 1$, the exact solution of (6.11) is $w(x,t) = x + t$.

6.3.7 Solution of time fractional Fisher's equation using HPTM

Consider the time fractional non-linear Fisher's equation[81]

$$\frac{\partial^\beta w}{\partial t^\beta} = \frac{\partial^2 w}{\partial x^2} + 6w(1-w), t > 0, x \in \mathbb{R}, 0 < \beta \le 1, \tag{6.14}$$

with initial condition $w(x,0) = \frac{1}{(1+e^x)^2}$.

By applying HPTM on (6.14), we have

$$\sum_{n=0}^{\infty} p^n w_n = \frac{1}{(1+e^x)^2} + p\Bigg(\mathcal{L}^{-1}\Bigg\{\frac{1}{s^\beta}\Bigg(\mathcal{L}\Bigg\{\sum_{n=0}^{\infty}(p^n w_n)_{xx} + 6\Bigg\{\sum_{n=0}^{\infty} p^n w_n - \sum_{n=0}^{\infty} p^n H_n(w)\Bigg\}\Bigg\}\Bigg)\Bigg\}\Bigg) \tag{6.15}$$

The first few components of $H_n(w)$ are given by

$$\begin{aligned} H_0(w) &= w_0^2; \\ H_1(w) &= 2w_0w_1; \\ H_2(w) &= w_1^2 + 2w_0w_2, \\ &\vdots \end{aligned}$$

Comparing the like powers of p on both sides of (6.15), we have

$$\begin{aligned} p^0 &: w_0 = \frac{1}{(1+e^x)^2}; \\ p^1 &: w_1 = 10\frac{e^x t^\beta}{(1+e^x)^3\Gamma(\beta+1)}; \\ p^2 &: w_2 = 50\frac{e^x(2e^x-1)t^{2\beta}}{(1+e^x)^4\Gamma(2\beta+1)}; \\ p^3 &: w_3 = \left(50\frac{e^x(8e^{2x}-11e^x+5)}{(1+e^x)^5} + 600e^{2x}\frac{\Gamma(2\beta+1)}{(1+e^x)^6\Gamma(\beta+1)^2}\right)\frac{t^{3\beta}}{\Gamma(3\beta+1)}, \\ &\vdots \end{aligned}$$

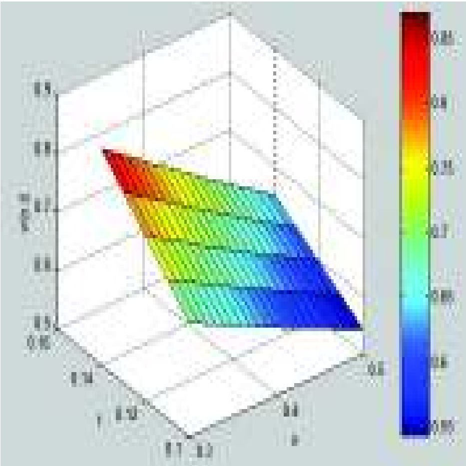


Figure 6.1: Surface graph of $w(x,t)$ of eq. (6.14), when $\beta = 0.6$

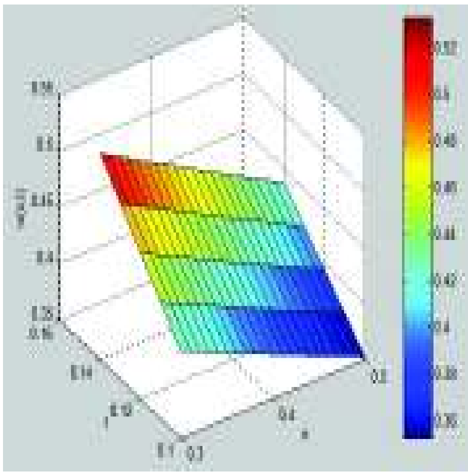


Figure 6.2: Surface graph of $w(x,t)$ of eq. (6.14), when $\beta = 0.8$

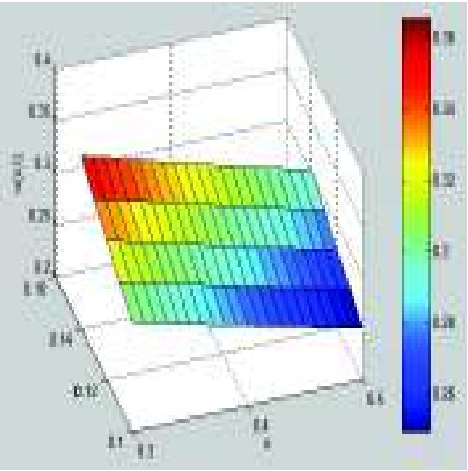


Figure 6.3: Surface graph of $w(x,t)$ of eq. (6.14), when $\beta = 1$

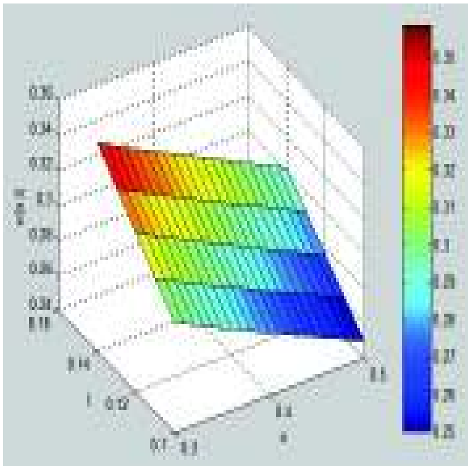


Figure 6.4: Surface graph of $w(x,t)$ of eq. (6.14), when $\beta = 1$(exact solution)

Hence, the solution is given as

$$w(x,t) = \frac{1}{(1+e^x)^2} + 10\frac{e^x t^\beta}{(1+e^x)^3\Gamma(\beta+1)} + 50\frac{e^x(2e^x-1)t^{2\beta}}{(1+e^x)^4\Gamma(2\beta+1)}$$
$$+\left(50\frac{e^x(-16e^{3x}-15e^{2x}+30e^x+5)}{(1+e^x)^6} + 600e^{2x}\frac{\Gamma(2\beta+1)}{(1+e^x)^6\Gamma(\beta+1)^2}\right)\frac{t^{3\beta}}{\Gamma(3\beta+1)}+\dots. \tag{6.16}$$

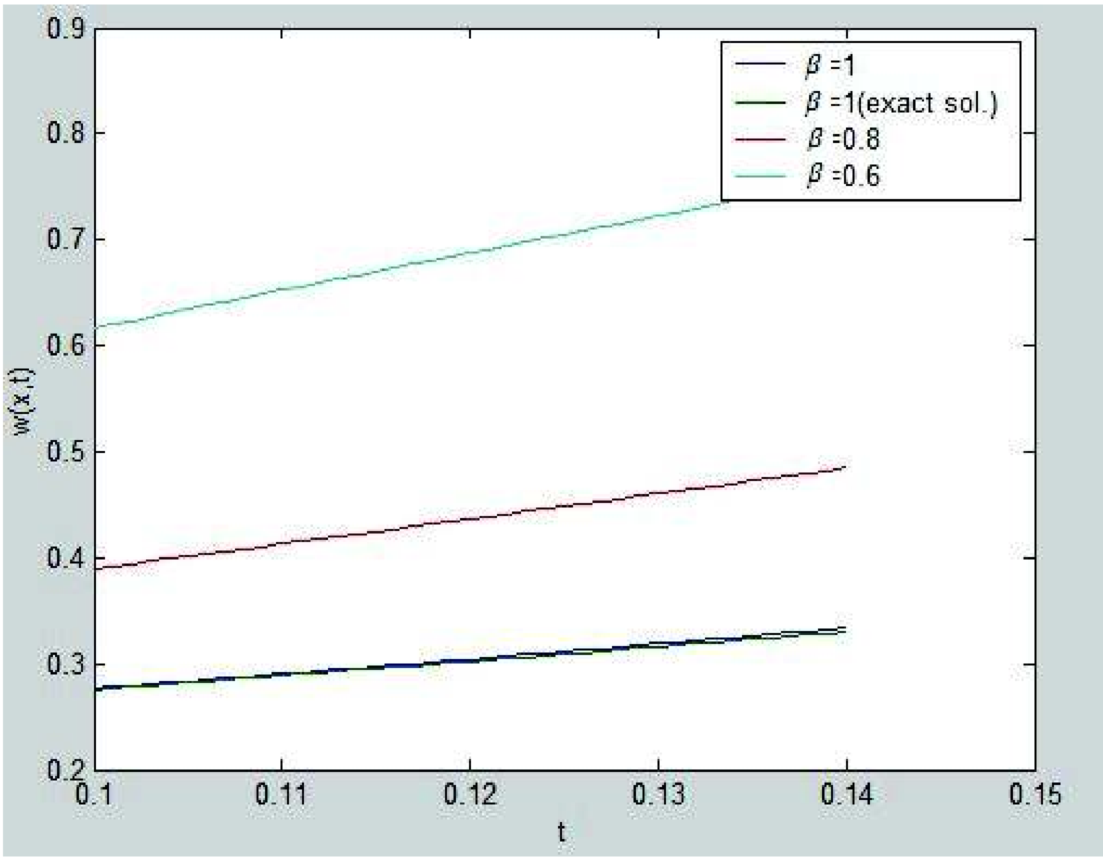


Figure 6.5: Plot of of $w(x,t)$ of eq. (6.14) when $x = 0.4$ and $\beta = 0.6, 0.8$ and 1

6.3.8 Solution of time fractional Fornberg-Whitham equation using HPTM

Consider the time-fractional Fornberg-Whitham equation [96]

$$\frac{\partial^\beta}{\partial t^\beta}w(x,t) = \frac{\partial^3 w}{\partial x^2 \partial t} - \frac{\partial w}{\partial x} + w\frac{\partial^3 w}{\partial x^3} - w\frac{\partial w}{\partial x} + 3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}, t > 0, x \in \mathbb{R}, 0 < \beta \leq 1, \tag{6.17}$$

with initial condition $w(x,0) = e^{\frac{x}{2}}$.

By applying HPTM on (6.17), we have

$$\sum_{n=0}^{\infty} p^n w_n = e^{\frac{x}{2}} + p\Bigg(\mathcal{L}^{-1}\Bigg\{\frac{1}{s^\beta}\Bigg(\mathcal{L}\Bigg\{\sum_{n=0}^{\infty} p^n\Bigg((w_n)_{xxt} + (-w_n)_x + H_n(w)\Bigg)\Bigg\}\Bigg)\Bigg\}\Bigg) \tag{6.18}$$

Table 6.1: Approximate solution of Fisher's equation (6.5) and (6.14) up to fourth order(when $\beta = 1$)

x	t	w_{HPETM} (approx.)	w_{HPTM} (approx.)	w (exact sol.)	abs.error	$\|\|w_1\|\|$	$\|\|w_2\|\|$	$\|\|w_3\|\|$
	0.1	0.304691131	0.304691131	0.302317425	0.002373706	0.104031064	1.88E-02	7.49E-04
	0.11	0.319292625	0.319292625	0.316042418	0.003250207	0.11443417	2.28E-02	9.97E-04
0.3	0.12	0.334319781	0.334319781	0.329984205	0.004335576	0.124837277	2.71E-02	1.29E-03
	0.13	0.34977709	0.34977709	0.344120184	0.005656906	0.135240383	3.18E-02	1.65E-03
	0.14	0.365669045	0.365669045	0.358426914	0.007242131	0.145643489	3.69E-02	2.05E-03
	0.1	0.276611064	0.276611064	0.275603147	0.001007917	9.64E-02	1.92E-02	4.91507E-05
	0.11	0.29026645	0.29026645	0.288830839	0.001435611	0.106061562	2.32E-02	6.54196E-05
0.4	0.12	0.304302372	0.304302372	0.302317425	0.001984947	0.115703523	2.76E-02	8.49324E-05
	0.13	0.318718535	0.318718535	0.316042418	0.002676117	0.125345483	3.24E-02	0.000107984
	0.14	0.333514645	0.333514645	0.329984205	0.00353044	0.134987443	3.76E-02	0.00013487
	0.1	0.249765515	0.249765515	0.25	0.000234485	8.87E-02	1.92E-02	0.000734094
	0.11	0.262435106	0.262435106	0.262653581	0.000218475	9.76E-02	2.33E-02	0.00097708
0.5	0.12	0.275441031	0.275441031	0.275603147	0.000162116	0.10646815	2.77E-02	0.001268515
	0.13	0.288778885	0.288778885	0.288830839	5.19537E-05	0.115340496	3.25E-02	0.001612806
	0.14	0.302444264	0.302444264	0.302317425	0.000126839	0.124212842	3.77E-02	0.002014355

The inital couple of terms of $H_n(w)$ are given by

$$
\begin{aligned}
H_0(w) &= w_0 w_{0xxx} - w_0 w_{0x} + 3w_{0x} w_{0xx};\\
H_1(w) &= w_0 w_{1xxx} + w_1 w_{0xxx} - w_0 w_{1x} - w_1 w_{0x} + 3w_{0x} w_{1xx} + 3w_{1x} w_{0xx};\\
H_2(w) &= w_0 w_{2xxx} + w_1 w_{1xxx} + w_2 w_{0xxx} - w_0 w_{2x} - w_1 w_{1x} - w_2 w_{0x}\\
&\quad +3w_{2x} w_{0xx} + 3w_{1x} w_{1xx} + 3w_{0x} w_{2xx}.
\end{aligned}
$$

On looking at the like terms of (6.18), we have

$$p^0 : w_0 = e^{\frac{x}{2}};$$

$$
\begin{aligned}
p^1 : w_1 =& \frac{-e^{\frac{x}{2}}}{2}\frac{t^\beta}{\Gamma(\beta+1)};\\
p^2 : w_2 =& \frac{-e^{\frac{x}{2}}}{8}\frac{t^{2\beta-1}}{\Gamma 2\beta}+\frac{e^{\frac{x}{2}}}{4}\frac{t^{2\beta}}{\Gamma(2\beta+1)};\\
p^3 : w_3 =& e^{\frac{x}{2}}\left(\frac{-1}{32}\frac{t^{3\beta-2}}{\Gamma(3\beta-1)}+\frac{1}{8}\frac{t^{3\beta-1}}{\Gamma(3\beta)}-\frac{1}{8}\frac{t^{3\beta}}{\Gamma(3\beta+1)}\right),\\
&\vdots
\end{aligned}
$$

Hence, the solution is given as

$$
\begin{aligned}
w(x,t) = e^{\frac{x}{2}} &- \frac{e^{\frac{x}{2}}}{2}\frac{t^\beta}{\Gamma(\beta+1)} - \frac{e^{\frac{x}{2}}}{8}\frac{t^{2\beta-1}}{\Gamma 2\beta}+\frac{e^{\frac{x}{2}}}{4}\frac{t^{2\beta}}{\Gamma(2\beta+1)}\\
&+ e^{\frac{x}{2}}\left(\frac{-1}{32}\frac{t^{3\beta-2}}{\Gamma(3\beta-1)}+\frac{1}{8}\frac{t^{3\beta-1}}{\Gamma(3\beta)}-\frac{1}{8}\frac{t^{3\beta}}{\Gamma(3\beta+1)}\right)+\ldots.
\end{aligned} \tag{6.19}
$$

6.3.9 Solution of time fractional Inviscid Burgers' equation using HPTM

Consider the non-linear non-homogeneous time fractional Inviscid Burgers' equation[118]

$$
D_t^\beta w + w w_x = 1 + x + t, w(x,0) = x, 0 < \beta \leq 1 \tag{6.20}
$$

By applying HPTM on eq.(6.20), we have

$$
\sum_{n=0}^{\infty} p^n w_n = x + p\Bigg(\mathcal{L}^{-1}\bigg\{\frac{1}{s^\beta}\mathcal{L}\Big\{1+x+t\Big\} - \frac{1}{s^\beta}\mathcal{L}\Big\{\sum_{n=0}^{\infty} p^n H_n(w)\Big\}\bigg\}\Bigg) \tag{6.21}
$$

where

$$
w w_x = \sum_{n=0}^{\infty} p^n H_n(w)
$$

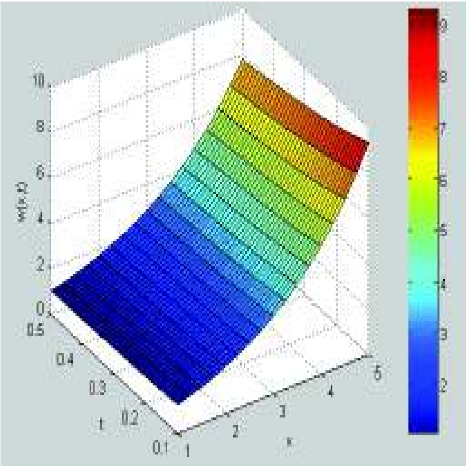


Figure 6.6: Surface graph of $w(x,t)$ of eq. (6.17), when $\beta = 0.6$

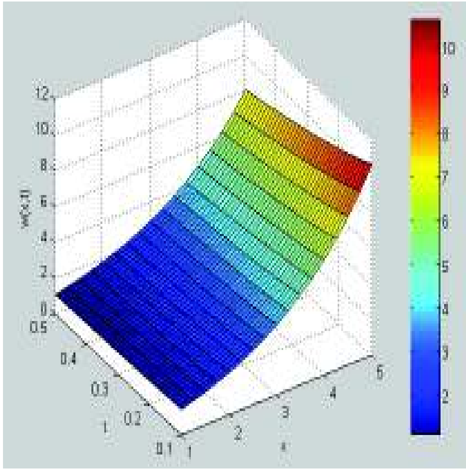


Figure 6.7: Surface graph of $w(x,t)$ of eq. (6.17), when $\beta = 0.8$

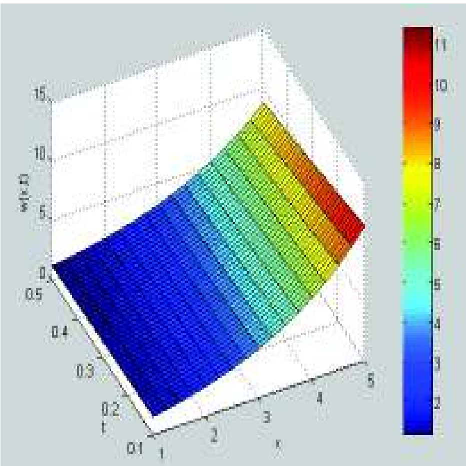


Figure 6.8: Surface graph of $w(x,t)$ of eq. (6.17), when $\beta = 1$

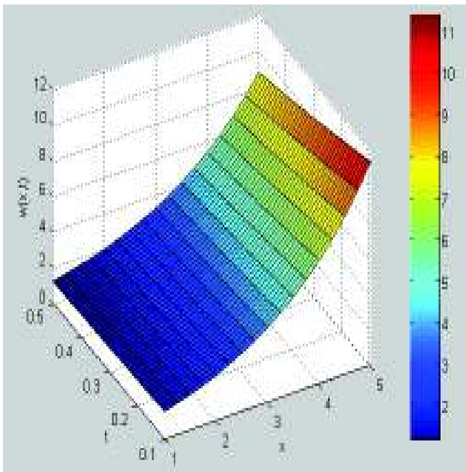


Figure 6.9: Surface graph of $w(x,t)$ of eq. (6.17), when $\beta = 1$(exact solution)

The first few components of He's polynomial i.e. $H_n(w)$ are given as

$$
\begin{aligned}
H_0 &= w_0 w_{0x} \\
H_1 &= w_0 w_{1x} + w_1 w_{0x} \\
H_2 &= w_0 w_{2x} + w_1 w_{1x} + w_2 w_{0x} \\
&\vdots
\end{aligned}
$$

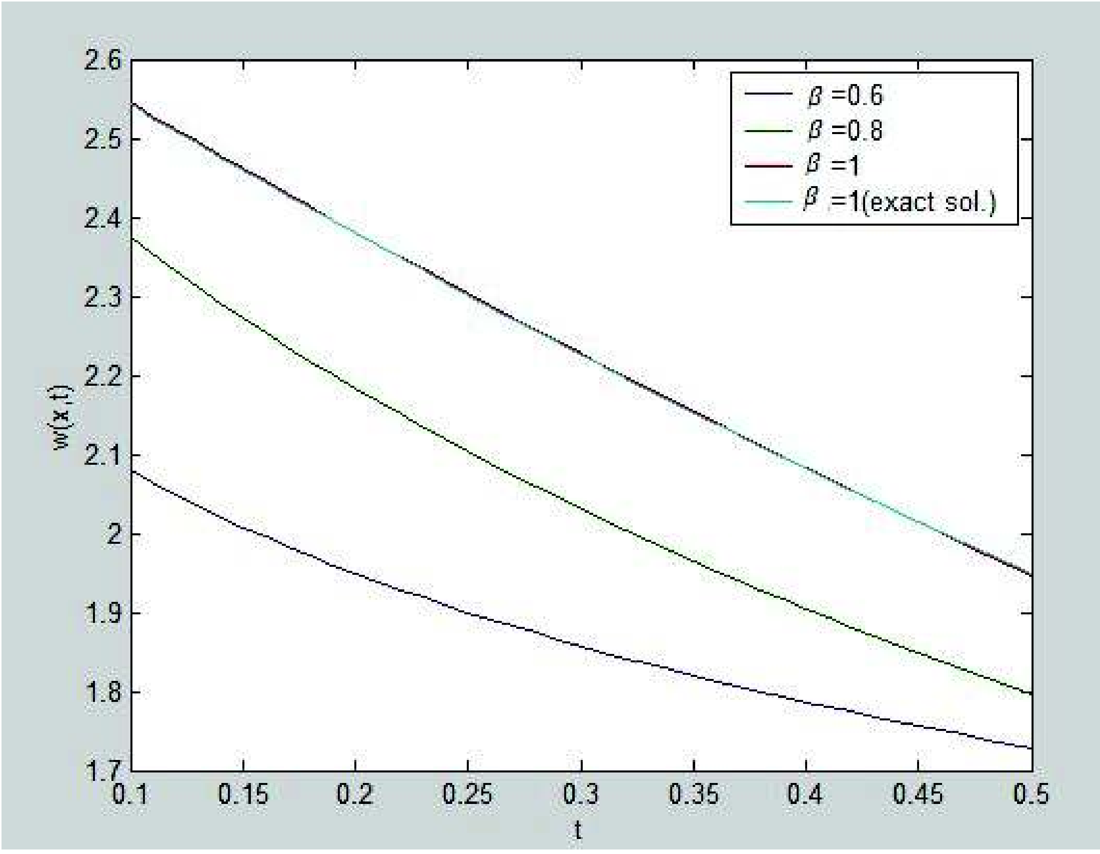


Figure 6.10: Plot of of $w(x,t)$ of eq. (6.17) when $x = 2$ and $\beta = 0.6, 0.8$ and 1

On comparing the like powers of p on both sides of (6.21), we have

$$\begin{aligned}
&p^0 : w_o = x\\
&p^1 : w_1 = (1+x)\frac{t^\beta}{\Gamma(1+\beta)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} - \mathcal{L}^{-1}\Big\{\frac{1}{s^\beta}\mathcal{L}\{H_0\}\Big\} = \frac{t^\beta}{\Gamma(1+\beta)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)}\\
&p^2 : w_2 = -\mathcal{L}^{-1}\Big\{\frac{1}{s^\beta}\mathcal{L}\{H_1\}\Big\} = -\Big(\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\Big)\\
&p^3 : w_3 = \mathcal{L}^{-1}\Big\{\frac{1}{s^\beta}\mathcal{L}\{H_2\}\Big\} = \Big(\frac{t^{3\beta}}{\Gamma(3\beta+1)} + \frac{t^{3\beta+1}}{\Gamma(3\beta+2)}\Big)\\
&\qquad\vdots
\end{aligned}$$

Hence the solution of (6.20) is

$$w(x,t) = x + \frac{t^\beta}{\Gamma(1+\beta)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} - \Big(\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\Big) + \Big(\frac{t^{3\beta}}{\Gamma(3\beta+1)} + \frac{t^{3\beta+1}}{\Gamma(3\beta+2)}\Big) + \dots$$

Table 6.2: Approximate solution of Fornberg-Whitham equation (6.8) and (6.17) up to fourth order (when $\beta = 1$)

t	x	w_{HPETM} (approx.)	w_{HPTM} (approx.)	w (exact sol.)	abs.error	$\|\|w_1\|\|$	$\|\|w_2\|\|$	$\|\|w_3\|\|$
	1	1.543580941	1.543580941	1.542390265	1.19E-03	0.082436064	0.018548114	0.004156152
	2	2.544934731	2.544934731	2.542971638	1.96E-03	0.135914091	0.030580671	0.006852335
0.1	3	4.195888024	4.195888024	4.19265143	3.24E-03	0.224084454	0.050419002	0.011297591
	4	6.917849834	6.917849834	6.912513593	5.34E-03	0.369452805	0.083126881	0.018626579
	5	11.40560617	11.40560617	11.3968082	8.80E-03	0.609124698	0.137053057	0.030710037
	1	1.351024036	1.351024036	1.349858808	1.17E-03	0.247308191	0.043278933	0.00711011
	2	2.227462066	2.227462066	2.225540928	1.92E-03	0.407742274	0.071354898	0.01172259
0.3	3	3.672464088	3.672464088	3.669296668	3.17E-03	0.672253361	0.117644338	0.019327284
	4	6.054869657	6.054869657	6.049647464	5.22E-03	1.108358415	0.193962723	0.031865304
	5	9.982792395	9.982792395	9.974182455	8.61E-03	1.827374094	0.319790467	0.052537005
	1	1.180724868	1.180724868	1.181360413	6.36E-04	0.412180318	0.05152254	0.004293545
	2	1.946686205	1.946686205	1.947734041	1.05E-03	0.679570457	0.084946307	0.007078859
0.5	3	3.209542954	3.209542954	3.211270543	1.73E-03	1.120422268	0.140052783	0.011671065
	4	5.291641738	5.291641738	5.29449005	2.85E-03	1.847264025	0.230908003	0.019242334
	5	8.72444229	8.72444229	8.729138364	4.70E-03	3.04562349	0.380702936	0.031725245

or

$$w(x,t) = x + \left(\frac{t^{\beta}}{\Gamma(1+\beta)} - \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \dots\right) + \left(\frac{t^{\beta+1}}{\Gamma(\beta+2)} - \frac{t^{2\beta+1}}{\Gamma(2\beta+2)} + \frac{t^{3\beta+1}}{\Gamma(3\beta+2)} + \dots\right)$$

or

$$w(x,t) = x - \sum_{n=1}^{\infty} \frac{(-1)^n t^{n\beta}}{\Gamma(n\beta+1)} - t\sum_{n=1}^{\infty} \frac{(-1)^n t^{n\beta}}{\Gamma(n\beta+2)}$$

or

$$w(x,t) = x + 1 + t - E_{\beta,1}(-t^{\beta}) - tE_{\beta,2}(-t^{\beta})$$

When $\beta = 1$,solution of (6.20) reduces to $w(x,t) = x + t$.

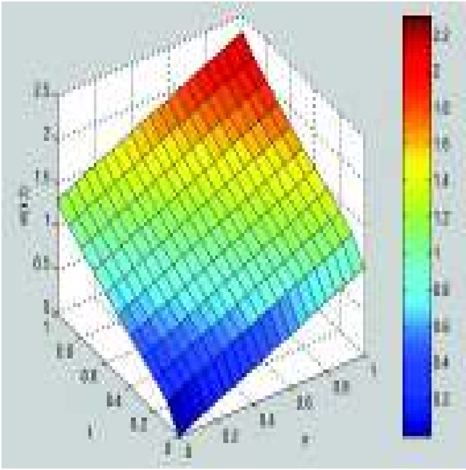


Figure 6.11: Surface graph of $w(x,t)$ of eq. (6.20), when $\beta = 0.6$

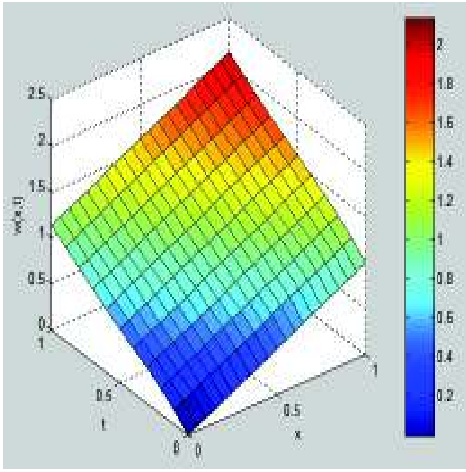


Figure 6.12: Surface graph of $w(x,t)$ of eq. (6.20), when $\beta = 0.8$

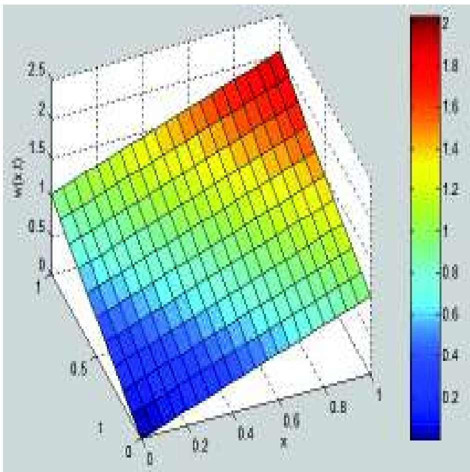


Figure 6.13: Surface graph of $w(x,t)$ of eq. (6.20), when $\beta = 1$

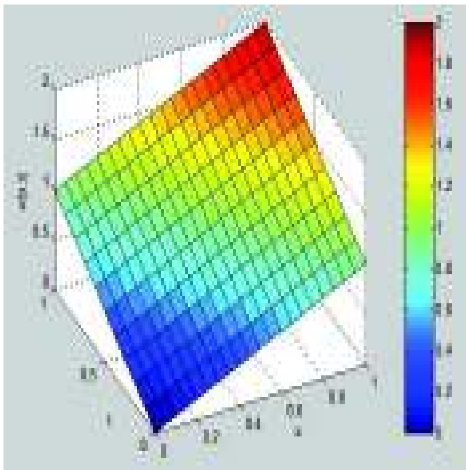


Figure 6.14: Surface graph of $w(x,t)$ of eq. (6.20), when $\beta = 1$(exact solution)

6.4 Analysis

We Know that

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) = \bar{f}(s) \tag{6.22}$$

Also from (1.16) and (6.22), we have

$$E\{f(t)\} \;=\; F(v) = v\bar{f}\left(\frac{1}{v}\right) \tag{6.23}$$

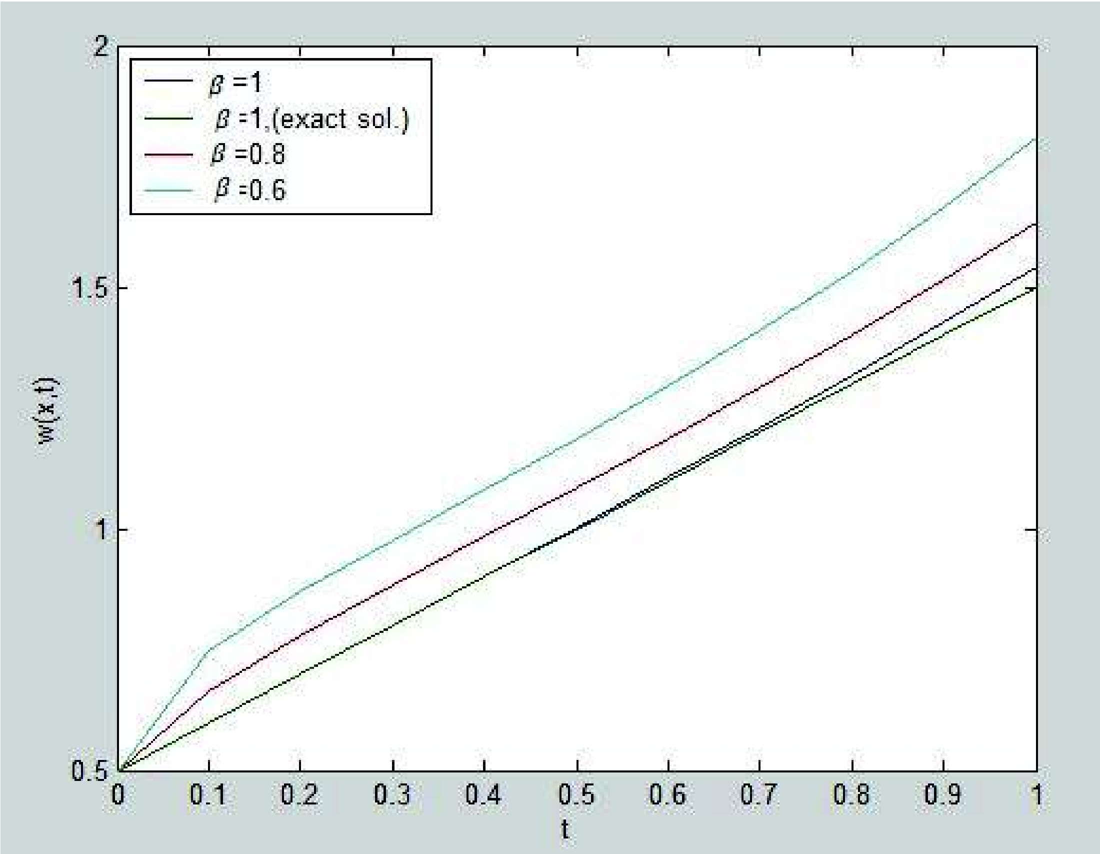


Figure 6.15: Plot of of $w(x,t)$ of eq. (6.20) when $x = 0.5$ and $\beta = 0.6, 0.8$ and 1

or

$$\bar{f}\left(\frac{1}{v}\right) = \frac{1}{v}F(v)$$

$$\Rightarrow \bar{f}(s) = \frac{1}{v}F(v), \text{ where } v = \frac{1}{s}$$

Hence

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}. \tag{6.24}$$

using (6.23)

$$\Rightarrow E\{t^n\} = v^{n+2}\Gamma(n+1),$$

$$\mathcal{L}\left\{\frac{\partial^\beta}{\partial t^\beta}f(t)\right\} = s^\beta \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{\beta-k-1}f^{(k)}(0), n-1 < \beta \leq n,$$

Table 6.3: Approximate solution of Inviscid Burgers' equation (6.11) and (6.20) up to fourth order($\beta = 1$)

x	t	w_{HPETM} (approx.)	w_{HPTM} (approx.)	w (exact sol.)	abs.error	$\|\|w_1\|\|$	$\|\|w_2\|\|$	$\|\|w_3\|\|$
	0.25	0.50016276	0.50016276	0.5	0.00016276	0.28125	0.033854167	0.002766927
0.25	0.5	0.752604167	0.752604167	0.75	0.002604167	0.625	0.145833333	0.0234375
	0.75	1.013183594	1.013183594	1	0.013183594	1.03125	0.3515625	0.083496094
	1	1.291666667	1.291666667	1.25	0.041666667	1.5	0.666666667	0.208333333
	0.25	0.75016276	0.75016276	0.75	0.00016276	0.28125	0.033854167	0.002766927
0.5	0.5	1.002604167	1.002604167	1	0.002604167	0.625	0.145833333	0.0234375
	0.75	1.263183594	1.263183594	1.25	0.013183594	1.03125	0.3515625	0.083496094
	1	1.541666667	1.541666667	1.5	0.041666667	1.5	0.666666667	0.208333333
	0.25	1.00016276	1.00016276	1	0.00016276	0.28125	0.033854167	0.002766927
0.75	0.5	1.252604167	1.252604167	1.25	0.002604167	0.625	0.145833333	0.0234375
	0.75	1.513183594	1.513183594	1.5	0.013183594	1.03125	0.3515625	0.083496094
	1	1.791666667	1.791666667	1.75	0.041666667	1.5	0.666666667	0.208333333

using (6.23), we have

$$\begin{aligned}
\Rightarrow \frac{1}{v}E\{f^{\beta}(t)\} &= \frac{1}{v^{\beta}}\frac{F(v)}{v} - \sum_{k=0}^{n-1}\left(\frac{1}{v}\right)^{\beta-k-1} f^{(k)}(0), n-1 < \beta \leq n, \\
\Rightarrow E\{f^{\beta}(t)\} &= \frac{F(v)}{v^{\beta}} - \sum_{k=0}^{n-1} v^{k-\beta+2} f^{(k)}(0), n-1 < \beta \leq n.
\end{aligned}$$

Fig. 6.1-6.4, represent the surface graphs of approximate solution of (6.14) for various estimations of β and the exact solution for $\beta = 1$ and we find that approximate solution up to order four converges to exact solution for $\beta = 1$, in table 6.1, the condition of convergence is verified i.e. we analyse that $||w_1|| < ||w_2|| < ||w_3||$. Moreover, from Fig.6.5, we conclude that with the decrease in the estimation of β, $w(x,t)$ increases. On the other hand, Fig. 6.6-6.9 and Fig. 6.11-6.14 represent the surface graph of (6.17) and (6.20) for various estimations of β and the exact solution for $\beta = 1$, the approximate solution of $w(x,t)$ approaches to exact solution when

$\beta = 1$, but by slightly decreasing the value of β, the value of $w(x,t)$ also decreases which is shown in the Fig. 6.10 and Fig.6.15.

6.5 Conclusion

1. We made a comparative study of two powerful semi-analytical techniques i.e. HPTM and HPETM for the solution of non-linear fractional PDE.

2. Elzaki transformation and its properties could be derived from Laplace transformation. Hence both the semi-analytical techniques give the same series solution.

3. The series solution obtained from HPTM and HPETM satisfied conditions of convergence that are reported in obtained results mentioned in tables 6.1, 6.2 and 6.3.

4. HPTM and HPETM both the techniques are equally competent for homogeneous and non-homogeneous non-linear fractional PDE.

Chapter 7

Accelerated HPSTM for the Series Solution of Non-linear PDE

The model including delay differential equations may display physical frameworks for which the advancement does rely upon the present and past circumstances. This type of the model is found in the area of population dynamics and epidemiology, where the delay is due to the gesture or maturity period, or is in numerical control, where there is a delay in taking care of in the controller input circle. The partial differential equation with proportionate delay is a particular case of delay differential equation emerge uniquely in the field of medicine, populace ecology, control frameworks, biology, and climate models[108]. Some of the authors have adopted numerical techniques like HPM, VIM, and DTM for solving delay partial differential equations.

Here we apply a new form of a semi-analytic technique named as Accelerated HPSTM to study the following type of PDE with proportional delay.

$$\begin{aligned} w_t(x,t) &= F(w(\alpha_1 x, \beta_1 t), w_x(\alpha_2 x, \beta_2 t), w_{xx}(\alpha_3 x, \beta_3 t), \dots), \qquad (7.1) \\ w(x,0) &= g(x), \end{aligned}$$

$\alpha_i, \beta_j \in (0,1), i, j \in \mathbb{N}$, and F is the partial differential operator.

7.1 Accelerated homotopy perturbation Sumudu transform method (AHPSTM)

To elucidate the proposed technique, let us consider the following non-linear equation

$$\frac{\partial^n \psi(x,t)}{\partial t^n} + L\psi(x,t) + N\psi(x,t) = f(x,t), t > 0, x \in \mathbb{R}, \tag{7.2}$$

Now applying Sumudu transform, we get

$$S\left\{\frac{\partial^n \psi(x,t)}{\partial t^n} + L\psi + N\psi\right\} = S\{f(x,t)\}.$$

Using (1.6.2), we have

$$S\{\psi(x,t)\} = u^n \sum_{k=0}^{n-1} u^{k-n} \psi^k(x,0) + u^n S\{f(x,t) - L\psi(x,t) - N\psi(x,t)\} \tag{7.3}$$

Operating inverse Sumudu transform on (7.3), we have

$$\psi(x,t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} \psi^{(k)}(x,0) + S^{-1}\left\{u^n \left(S\left\{f(x,t) - L\psi(x,t) - N\psi(x,t)\right\}\right)\right\}. \tag{7.4}$$

By applying HPM, we get

$$\begin{aligned} 0 =&(1-p)\Big(\psi(x,t) - \psi(x,0)\Big) + p\left(\psi(x,t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \psi^{(k)}(x,0)\right. \\ &\left. - S^{-1}\left\{u^n S\left\{f(x,t) - L\psi(x,t) - N\psi(x,t)\right\}\right\}\right), \end{aligned}$$

where $p \in [0,1]$ is parameter and $\psi(x,0), \psi^k(x,0), k = 1,2,\ldots,n-1$ are the initial approximations of the given differential equation (7.2). Let

$$\psi(x,t) = \sum_{n=0}^{\infty} p^n \psi_n(x,t), \tag{7.5}$$

$$N\psi(x,t) = \sum_{n=0}^{\infty} p^n \tilde{H}_n(\psi(x,t)) \tag{7.6}$$

Here, we use $\tilde{H}_n$ given in [57] also known as accelerated He's polynomial.

$$\sum_{n=0}^{\infty} \tilde{H}_n(\psi) = \sum_{n=0}^{\infty} H_n(\psi),$$

with

$$\tilde{H}_n(\psi_0, \psi_1, \psi_2, \ldots, \psi_n) = N(S_n) - \sum_{k=0}^{n-1} \tilde{H}_k, n \geq 1 \tag{7.7}$$

here $H_n(\psi)$ is He's polynomial

Using (7.5) and (7.7), eq.(7.4) becomes

$$\sum_{n=0}^{\infty} p^n \psi_n(x,t) = \psi(x,0) + p\{\sum_{k=1}^{n-1} \frac{t^k}{k!} \psi^{(k)}(x,0) + \\ S^{-1}(u^n(S\{f(x,t) - L\sum_{n=0}^{\infty} p^n \psi_n(x,t) - \sum_{n=0}^{\infty} p^n \tilde{H}_n(\psi)\}))\}.$$

On looking at the coefficient of like power of p, we have

$$\begin{aligned}
p^0 : \psi_0 =& \psi(x,0); \\
p^1 : \psi_1 =& \sum_{k=1}^{n-1} \frac{t^k}{k!} \psi^{(k)}(x,0) + S^{-1}\left\{u^n\left[S\left\{f(x,t) - L\psi_0(ax,bt) - \tilde{H}_0\right\}\right]\right\}; \\
p^2 : \psi_2 =& -S^{-1}\left\{u^n\left[S\left\{L\psi_1(x,t) + \tilde{H}_1\right\}\right]\right\}; \\
p^3 : \psi_3 =& -S^{-1}\left\{u^n\left[S\left\{L\psi_2(x,t) + \tilde{H}_2\right\}\right]\right\}. \\
& \vdots
\end{aligned}$$

Hence the solution is obtained by taking the limit $p \to 1$ and we have

$$\psi(x,t) = \sum_{i=0}^{\infty} \psi_i(x,t).$$

To elucidate the efficiency of the proposed method and importance of Accelerated He's polynomial over He's polynomial. Let us consider the non-linear term as

$$N\psi(x,t) = \psi^2 \psi_x$$

If we write an initial couple of terms of He's polynomial, we have

$$\begin{aligned}
H_0(\psi) &= \psi_0^2\psi_0 x;\\
H_1(\psi) &= 2\psi_0\psi_1\psi_{0x} + \psi_0^2\psi_{1x};\\
H_2(\psi) &= \psi_0^2\psi_{2x} + 2\psi_0\psi_1\psi_{1x} + 2\psi_0\psi_2\psi_{0x} + \psi_1^2\psi_{0x};\\
H_3(\psi) &= 2\psi_0\psi_3\psi_{0x} + 2\psi_1\psi_2\psi_{0x} + 2\psi_0\psi_2\psi_{1x} + 2\psi_0\psi_1\psi_{2x} + \psi_0^2\psi_{3x} + \psi_1^2\psi_{1x},\\
&\vdots
\end{aligned} \tag{7.8}$$

A couple of terms of Accelerated He's polynomial are

$$\begin{aligned}
\tilde{H}_0(\psi) &= \psi_0^2\psi_0 x;\\
\tilde{H}_1(\psi) &= 2\psi_0\psi_1\psi_{0x} + \psi_0^2\psi_{1x} + \psi_1^2\psi_{0x} + \psi_1^2\psi_{1x} + 2\psi_0\psi_1\psi_{1x};\\
\tilde{H}_2(\psi) &= 2\psi_0\psi_2\psi_{0x} + 2\psi_1\psi_2\psi_{0x} + 2\psi_0\psi_2\psi_{1x} + 2\psi_1\psi_2\psi_{1x}\\
&\quad +2\psi_0\psi_1\psi_{2x} + \psi_2^2\psi_{1x} + \psi_2^2\psi_{2x} + \psi_0^2\psi_{2x} + \psi_1^2\psi_{2x}\\
\tilde{H}_3(\psi) &= 2\psi_1\psi_3\psi_{1x} + 2\psi_2\psi_3\psi_{1x} + 2\psi_1\psi_3\psi_{2x} + 2\psi_2\psi_3\psi_{2x}\\
&\quad +2\psi_1\psi_2\psi_{3x} + 2\psi_1\psi_3\psi_{3x} + 2\psi_2\psi_3\psi_{3x} + (\psi_{1x} + \psi_{2x})\psi_3^2\\
&\quad +(\psi_0^2 + \psi_1^2)\psi_{3x} + (\psi_2^2 + \psi_3^2)\psi_{3x} + \psi_{0x}\psi_3(2\psi_1 + 2\psi_2 + \psi_3)\\
&\quad +2\psi_0(\psi_{0x}\psi_3 + \psi_{1x}\psi_3 + \psi_{2x}\psi_3 + \psi_{3x}\psi_1 + \psi_{3x}\psi_2 + \psi_{3x}\psi_3)\\
&\vdots
\end{aligned} \tag{7.9}$$

on comparing (7.8) and (7.9), it is clear that $\tilde{H}_0, \tilde{H}_1, \tilde{H}_2$ and $\tilde{H}_3$ of accelerated He's polynomial contains not only the terms present in H_0, H_1, H_2, H_3 but also the terms which might be present in $H_4, H_5, H_6, \ldots$ which shows that the solution obtained using accelerated He's polynomial would converge faster than using He's polynomial.

7.2 Convergence analysis

In this section, we insist on the position of the convergence of the proposed method for chain solution. Now, we emphasize on the position of convergence of the proposed method for the series solution of eq. (7.2).

Theorem 7.2.1 *Let ψ and $\psi_n(x,t)$ be defined in Banach space $\boldsymbol{B}$, the condition that the series solution given by eq. (7.5) converges to the solution is $||\psi_{n+1}|| \leq \eta\ ||\psi_n||$ where $\eta \in (0,1)$.The condition of convergence has been proved in [98, 99].*

Remark 7.2.2 *The condition of absolute truncation error is given below:*

$$||\psi - \sum_{k=0}^{n} \psi_k|| \leq \frac{\eta^{n+1}}{1-\eta} ||\psi_0||$$

.

7.3 Application

7.3.1 Solution of generalized Burgers' equation with proportional delay

Consider the following initial value problem [1]

$$\frac{\partial \psi(x,t)}{\partial t} = \psi_{xx}(x,t) + \psi_x\big(x, \tfrac{t}{2}\big)\psi\big(\tfrac{x}{2}, \tfrac{t}{2}\big) + \frac{1}{2}\psi(x,t), t > 0, x \in \mathbb{R}, \tag{7.10}$$

where $\psi(x,0) = x$. By applying Sumudu transformation on eq. (7.10), we have

$$S\big\{\frac{\partial \psi(x,t)}{\partial t} - \frac{1}{2}\psi(x,t)\big\} = S\big\{\psi_{xx}(x,t) + \psi_x\big(x, \tfrac{t}{2}\big)\psi\big(\tfrac{x}{2}, \tfrac{t}{2}\big)\big\}, \tag{7.11}$$

$$S\big\{\psi(x,t)\big\} = x\frac{2}{2-u} + \frac{2u}{2-u}\bigg(S\big\{\psi_{xx}(x,t) + \psi_x\big(x, \tfrac{t}{2}\big)\psi\big(\tfrac{x}{2}, \tfrac{t}{2}\big)\big\}\bigg). \tag{7.12}$$

By applying inverse Sumudu transformation on eq.(7.12), we have

$$\psi(x,t) = xe^{\frac{t}{2}} + S^{-1}\bigg\{\frac{2u}{2-u}\bigg(S\big\{\psi_{xx}(x,t) + \psi_x\big(x, \tfrac{t}{2}\big)\psi\big(\tfrac{x}{2}, \tfrac{t}{2}\big)\big\}\bigg)\bigg\}. \tag{7.13}$$

Now, we apply AHPSTM on (7.10), we have

$$\sum_{n=0}^{\infty} p^n \psi_n(x,t) = xe^{\frac{t}{2}} + p\ S^{-1}\bigg\{\frac{2u}{2-u}\bigg(S\bigg\{\sum_{n=0}^{\infty}(p^n\psi_n)_{xx}(x,t) + \sum_{n=0}^{\infty} p^n \tilde{H}_n(\psi)\bigg\}\bigg)\bigg\}, \tag{7.14}$$

where the initial couple of terms of $\tilde{H}_n$ are given as

$$
\begin{aligned}
\tilde{H}_0(\psi) &= \psi_{0x}\big(x,\tfrac{t}{2}\big)\psi_0\big(\tfrac{x}{2},\tfrac{t}{2}\big),\\
\tilde{H}_1(\psi) &= \psi_{0x}\big(x,\tfrac{t}{2}\big)\psi_1\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{1x}\big(x,\tfrac{t}{2}\big)\psi_0\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{1x}\big(x,\tfrac{t}{2}\big)\psi_1\big(\tfrac{x}{2},\tfrac{t}{2}\big),\\
\tilde{H}_2(\psi) &= \psi_{0x}\big(x,\tfrac{t}{2}\big)\psi_2\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{1x}\big(x,\tfrac{t}{2}\big)\psi_2\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{2x}\big(x,\tfrac{t}{2}\big)\psi_2\big(\tfrac{x}{2},\tfrac{t}{2}\big)\\
&+ \psi_{2x}\big(x,\tfrac{t}{2}\big)\psi_0\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{2x}\big(x,\tfrac{t}{2}\big)\psi_1\big(\tfrac{x}{2},\tfrac{t}{2}\big),\\
\tilde{H}_3(\psi) &= \psi_{0x}\big(x,\tfrac{t}{2}\big)\psi_3\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{1x}\big(x,\tfrac{t}{2}\big)\psi_3\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{2x}\big(x,\tfrac{t}{2}\big)\psi_3\big(\tfrac{x}{2},\tfrac{t}{2}\big)\\
&+ \psi_{3x}\big(x,\tfrac{t}{2}\big)\psi_3\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{3x}\big(x,\tfrac{t}{2}\big)\psi_2\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{3x}\big(x,\tfrac{t}{2}\big)\psi_1\big(\tfrac{x}{2},\tfrac{t}{2}\big)\\
&+ \psi_{3x}\big(x,\tfrac{t}{2}\big)\psi_0\big(\tfrac{x}{2},\tfrac{t}{2}\big)\\
\tilde{H}_4(\psi) &= \psi_{0x}\big(x,\tfrac{t}{2}\big)\psi_4\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{1x}\big(x,\tfrac{t}{2}\big)\psi_4\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{2x}\big(x,\tfrac{t}{2}\big)\psi_4\big(\tfrac{x}{2},\tfrac{t}{2}\big)\\
&+ \psi_{3x}\big(x,\tfrac{t}{2}\big)\psi_4\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{4x}\big(x,\tfrac{t}{2}\big)\psi_4\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{4x}\big(x,\tfrac{t}{2}\big)\psi_3\big(\tfrac{x}{2},\tfrac{t}{2}\big)\\
&+ \psi_{4x}\big(x,\tfrac{t}{2}\big)\psi_2\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{4x}\big(x,\tfrac{t}{2}\big)\psi_1\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{4x}\big(x,\tfrac{t}{2}\big)\psi_0\big(\tfrac{x}{2},\tfrac{t}{2}\big)\\
&\ \vdots
\end{aligned}
$$

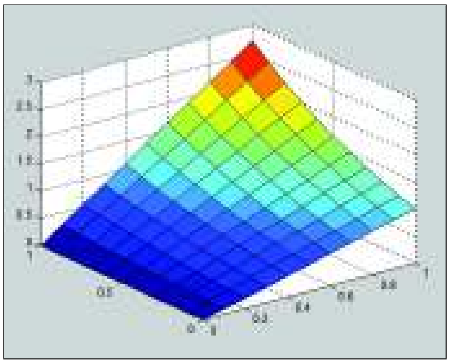
On looking at the like powers of p of eq. (7.14), we have

$$
\begin{aligned}
&p^0 : \psi_0 = xe^{\frac{t}{2}};\\
&p^1 : \psi_1 = \left(\frac{t}{2}\right)xe^{\frac{t}{2}};\\
&p^2 : \psi_2 = xe^{\frac{t}{2}}\left(\left(\frac{t^2}{2^2 2!}\right)+\frac{1}{2}\left(\frac{t^3}{2^3 3!}\right)\right);\\
&p^3 : \psi_3 = xe^{\frac{t}{2}}\left(\frac{1}{2}\left(\frac{t^3}{2^3 3!}\right)+\frac{7}{8}\left(\frac{t^4}{2^4 4!}\right)+\frac{5}{8}\left(\frac{t^5}{2^5 5!}\right)+\frac{5}{16}\left(\frac{t^6}{2^6 6!}\right)+\frac{5}{64}\left(\frac{t^7}{2^7 7!}\right)\right);\\
&p^4 : \psi_4 = xe^{\frac{t}{2}}\left(\frac{1}{8}\left(\frac{t^4}{2^4 4!}\right)+\frac{23}{64}\left(\frac{t^5}{2^5 5!}\right)+\frac{5}{8}\left(\frac{t^6}{2^6 6!}\right)+\frac{395}{512}\left(\frac{t^7}{2^7 7!}\right)+\frac{2455}{4096}\left(\frac{t^8}{2^8 8!}\right)+\dots\right);\\
&\qquad\vdots
\end{aligned}
\tag{7.15}
$$

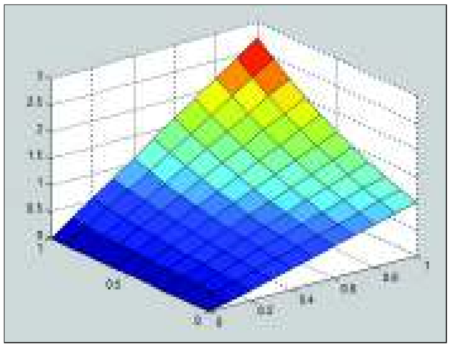
As $p \to 1$, we get the series solution of (7.14) as

$$\psi(x,t) = \sum_{i=0}^{\infty}\psi_i(x,t),$$

Figure 7.1: Solution of generalized Burgers' equation (7.10) with proportionate delay



(a) Approximate solution using AHPSTM up to fourth order



(b) (Exact sol.)

using (7.15), we get

$$\begin{aligned}\psi(x,t) &= xe^{\frac{t}{2}} + xe^{\frac{t}{2}}\left(\frac{t}{2}\right) + xe^{\frac{t}{2}}\left(\left(\frac{t^2}{2^2 2!}\right) + \frac{1}{2}\left(\frac{t^3}{2^3 3!}\right)\right) \\ &+ xe^{\frac{t}{2}}\left(\frac{1}{2}\left(\frac{t^3}{2^3 3!}\right) + \frac{7}{8}\left(\frac{t^4}{2^4 4!}\right) + \frac{5}{8}\left(\frac{t^5}{2^5 5!}\right) + \frac{5}{16}\left(\frac{t^6}{2^6 6!}\right) + \frac{5}{64}\left(\frac{t^7}{2^7 7!}\right)\right) \\ &+ xe^{\frac{t}{2}}\left(\frac{1}{8}\left(\frac{t^4}{2^4 4!}\right) + \frac{23}{64}\left(\frac{t^5}{2^5 5!}\right) + \frac{5}{8}\left(\frac{t^6}{2^6 6!}\right) + \frac{395}{512}\left(\frac{t^7}{2^7 7!}\right) + \frac{2455}{4096}\left(\frac{t^8}{2^8 8!}\right) + \dots\right)\end{aligned}$$

$$\psi(x,t) = xe^{\frac{t}{2}}\left(1+\frac{t}{2}+\left(\frac{t^3}{2^3 3!}\right)+\left(\frac{t^4}{2^4 4!}\right)+\frac{63}{64}\left(\frac{t^5}{2^5 5!}\right)+\frac{15}{16}\left(\frac{t^6}{2^6 6!}\right)+\frac{435}{512}\left(\frac{t^7}{2^7 7!}\right)+\dots\right) \tag{7.16}$$

The exact solution of the eq(7.10) in closed form is

$$\psi(x,t) = xe^{t} \tag{7.17}$$

On comparing (7.17) and (7.16), we find that the series solution rapidly converges to the actual solution. So, we discover that the accelerated homotopy perturbation Sumudu transformation method provides us the faster rate of convergence which can be seen in table 7.1 that the value of ψ_n (APSTM) decreases rapidly.

Table 7.1: Approximate solution of (7.10) using AHPSTM

x	t	ψ_2 (AHPSTM)	ψ_3 (AHPSTM)	ψ_4 (AHPSTM)
	0.25	0.002259289	4.8675E-05	3.87068E-07
0.25	0.5	0.010449426	0.00046536	7.54081E-06
	0.75	0.027174522	0.00187525	4.64816E-05
	1	0.055816085	0.00530269	0.000178857
	0.25	0.004518577	9.73499E-05	7.74136E-07
0.5	0.5	0.020898851	0.000930721	1.50816E-05
	0.75	0.054349045	0.003750499	9.29631E-05
	1	0.111632169	0.01060538	0.000357713
	0.25	0.006777866	0.000146025	1.1612E-06
0.75	0.5	0.031348277	0.001396081	2.26224E-05
	0.75	0.081523567	0.005625749	0.000139445
	1	0.167448254	0.015908069	0.00053657

7.3.2 Solution of non-linear PDE with proportional delay

Consider the following initial value problem[1]

$$\frac{\partial\psi(x,t)}{\partial t} = \psi_{xx}\Big(x, \frac{t}{2}\Big)\psi\Big(x, \tfrac{t}{2}\Big) - \psi(x,t), t > 0, x \in \mathbb{R}, \tag{7.18}$$

with initial condition $\psi(x,0) = x^2$.

By applying Sumudu transformation on both sides of (7.18), we get

$$S\bigg\{\frac{\partial\psi(x,t)}{\partial t} + \psi(x,t)\bigg\} = S\bigg\{\psi_{xx}\Big(x, \frac{t}{2}\Big)\psi\Big(x, \tfrac{t}{2}\Big)\bigg\}, \tag{7.19}$$

$$\Big(\frac{1}{u} + 1\Big)S\Big\{\psi(x,t)\Big\} - \Big(\frac{1}{u}\Big)\psi(x,0) = S\bigg\{\psi_{xx}\Big(x, \frac{t}{2}\Big)\psi\Big(x, \tfrac{t}{2}\Big)\bigg\}, \tag{7.20}$$

$$S\Big\{\psi(x,t)\Big\} = x^2\frac{1}{1+u} + \frac{u}{1+u}\bigg(S\bigg\{\psi_{xx}\Big(x, \frac{t}{2}\Big)\psi\big(x, \tfrac{t}{2}\big)\bigg\}\bigg). \tag{7.21}$$

By applying inverse Sumudu transformation, we have

$$\psi(x,t) = x^2e^{-t} + S^{-1}\bigg\{\frac{u}{1+u}\bigg(S\bigg\{\psi_{xx}\Big(x, \frac{t}{2}\Big)\psi\big(x, \tfrac{t}{2}\big)\bigg\}\bigg)\bigg\}. \tag{7.22}$$

Now , we apply AHPSTM on (7.18), we have

$$\sum_{n=0}^{\infty} p^n\psi_n(x,t) = x^2e^{-t} + p\ S^{-1}\bigg\{\frac{u}{1+u}\bigg(S\bigg\{\sum_{n=0}^{\infty} p^n\tilde{H}_n(\psi)\bigg\}\bigg)\bigg\}, \tag{7.23}$$

where the initial couple of terms of $\tilde{H}_n$ are given as

$$\begin{aligned}
\tilde{H}_0(\psi) &= \psi_{0xx}\big(x, \tfrac{t}{2}\big)\psi_0\big(x, \tfrac{t}{2}\big),\\
\tilde{H}_1(\psi) &= \psi_{0xx}\big(x, \tfrac{t}{2}\big)\psi_1\big(x, \tfrac{t}{2}\big) + \psi_{1xx}\big(x, \tfrac{t}{2}\big)\psi_0\big(x, \tfrac{t}{2}\big) + \psi_{1xx}\big(x, \tfrac{t}{2}\big)\psi_1\big(x, \tfrac{t}{2}\big),\\
\tilde{H}_2(\psi) &= \psi_{0xx}\big(x, \tfrac{t}{2}\big)\psi_2\big(x, \tfrac{t}{2}\big) + \psi_{1xx}\big(x, \tfrac{t}{2}\big)\psi_2\big(x, \tfrac{t}{2}\big) + \psi_{2xx}\big(x, \tfrac{t}{2}\big)\psi_2\big(x, \tfrac{t}{2}\big)\\
&\quad + \psi_{2xx}\big(x, \tfrac{t}{2}\big)\psi_0\big(x, \tfrac{t}{2}\big) + \psi_{2xx}\big(x, \tfrac{t}{2}\big)\psi_1\big(x, \tfrac{t}{2}\big),\\
&\vdots
\end{aligned}$$

On looking at the like powers of p of (7.23), we have

$$
\begin{aligned}
p^0 : \psi_0 =& x^2e^{-t};\\
p^1 : \psi_1 =& x^2e^{-t}(2t);\\
p^2 : \psi_2 =& x^2e^{-t}\left(\frac{2^2t^2}{2!}+\frac{1}{2}\frac{2^3t^3}{3!}\right);\\
p^3 : \psi_3 =& x^2e^{-t}\left(\frac{1}{2}\frac{2^3t^3}{3!}+\frac{7}{8}\frac{2^4t^4}{4!}+\frac{5}{8}\frac{2^5t^5}{5!}+\frac{5}{16}\frac{2^6t^6}{6!}+\frac{5}{64}\frac{2^7t^7}{7!}\right);\\
p^4 : \psi_4 =& x^2e^{-t}\left(\frac{1}{8}\frac{2^4t^4}{4!}+\frac{23}{64}\frac{2^5t^5}{5!}+\frac{5}{8}\frac{2^6t^6}{6!}+\frac{395}{512}\frac{2^7t^7}{7!}+\frac{2455}{4096}\frac{2^8t^8}{8!}+\dots\right);\\
&\vdots
\end{aligned}
\tag{7.24}
$$

Hence the series solution of (7.18)is obtained by

$$
\begin{aligned}
\psi(x,t) = & x^2e^{-t}+x^2e^{-t}(2t)+x^2e^{-t}\left(\frac{2^2t^2}{2!}+\frac{1}{2}\frac{2^3t^3}{3!}\right)\\
&+x^2e^{-t}\left(\frac{1}{2}\frac{2^3t^3}{3!}+\frac{7}{8}\frac{2^4t^4}{4!}+\frac{5}{8}\frac{2^5t^5}{5!}+\frac{5}{16}\frac{2^6t^6}{6!}+\frac{5}{64}\frac{2^7t^7}{7!}\right)\\
&+x^2e^{-t}\left(\frac{1}{8}\frac{2^4t^4}{4!}+\frac{23}{64}\frac{2^5t^5}{5!}+\frac{5}{8}\frac{2^6t^6}{6!}+\frac{395}{512}\frac{2^7t^7}{7!}+\frac{2455}{4096}\frac{2^8t^8}{8!}+\dots\right)
\end{aligned}
$$

$$
\psi(x,t) = x^2e^{-t}\left(1+2t+\frac{2^2t^2}{2!}+\frac{2^3t^3}{3!}+\frac{2^4t^4}{4!}+\frac{63}{64}\frac{2^5t^5}{5!}+\frac{15}{16}\frac{(2t)^6}{6!}+\frac{435}{512}\frac{(2t)^7}{7!}+\dots\right)
\tag{7.25}
$$

The exact solution of the eq(7.10) in closed form is

$$
\psi(x,t) = x^2e^{t}
\tag{7.26}
$$

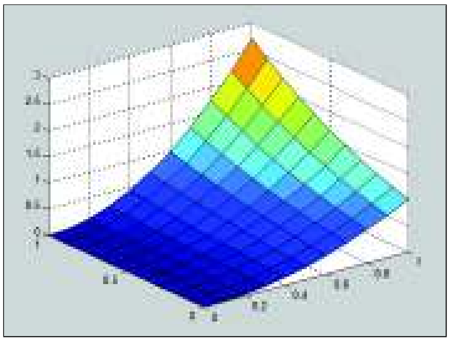
On comparing (7.26) and (7.25), it is clear that the series approaches to the exact solution. Also from the table 7.2, it is clear that $||\psi_4|| < ||\psi_3|| < ||\psi_2||$ i.e. the series solution satisfy the condition of convergence.

7.3.3 Solution of non-linear PDE with proportional delay

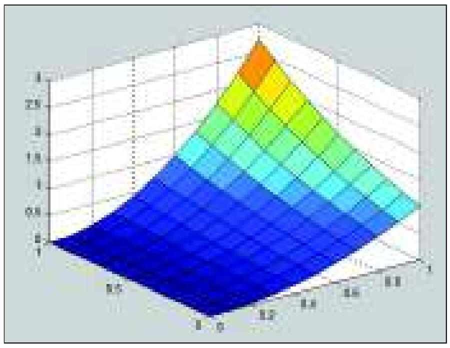
Consider the following equation [1]

$$
\frac{\partial\psi(x,t)}{\partial t} = \psi_{xx}\left(\frac{x}{2},\frac{t}{2}\right)\psi_x\left(\frac{x}{2},\frac{t}{2}\right)-\psi_x(x,t)-\psi(x,t), t>0, x\in\mathbb{R},
\tag{7.27}
$$

Figure 7.2: Solution of Non-linear PDE (7.18) with proportionate delay



(a) Approximate solution using AHPSTM up to fourth order



(b) (Exact sol.)

Table 7.2: Approximate solution of (7.18) up to fourth order

x	t	ψ_2 (AHPSTM)	ψ_3 (AHPSTM)	ψ_4 (AHPSTM)
	0.25	0.006591413	0.000626203	2.11215E-05
0.25	0.5	0.022113097	0.004755562	0.000350289
	0.75	0.041516592	0.015073801	0.001831803
	1	0.06131324	0.033256958	0.005952034
	0.25	0.026365652	0.002504813	8.44858E-05
0.5	0.5	0.088452388	0.019022246	0.001401156
	0.75	0.166066366	0.060295204	0.007327211
	1	0.245252961	0.133027834	0.023808135
	0.25	0.059322716	0.00563583	0.000190093
0.75	0.5	0.199017873	0.042800054	0.003152601
	0.75	0.373649324	0.135664209	0.016486225
	1	0.551819162	0.299312626	0.053568305

with initial condition $\psi(x,0)=x^2$.

On operating Sumudu transformation on both sides of (7.27), we have

$$S\bigg\{\frac{\partial\psi(x,t)}{\partial t}+\psi(x,t)\bigg\}=S\bigg\{\psi_{xx}\bigg(\frac{x}{2},\frac{t}{2}\bigg)\psi_x\bigg(\tfrac{x}{2},\tfrac{t}{2}\bigg)-\psi_x(x,t)\bigg\}, \tag{7.28}$$

$$\bigg(\frac{1}{u}+1\bigg)S\bigg\{\psi(x,t)\bigg\}-\bigg(\frac{1}{u}\bigg)\psi(x,0)=S\bigg\{\psi_{xx}\bigg(\frac{x}{2},\frac{t}{2}\bigg)\psi_x\bigg(\tfrac{x}{2},\tfrac{t}{2}\bigg)-\psi_x(x,t)\bigg\}, \tag{7.29}$$

$$S\bigg\{\psi(x,t)\bigg\}=x^2\frac{1}{1+u}+\frac{u}{1+u}\bigg(S\bigg\{\psi_{xx}\bigg(\frac{x}{2},\frac{t}{2}\bigg)\psi_x\bigg(\tfrac{x}{2},\tfrac{t}{2}\bigg)-\psi_x(x,t)\bigg\}\bigg), \tag{7.30}$$

By applying inverse Sumudu transformation, we have

$$\psi(x,t)=x^2e^{-t}+S^{-1}\bigg\{\frac{u}{1+u}\bigg(S\bigg\{\psi_{xx}\bigg(\frac{x}{2},\frac{t}{2}\bigg)\psi_x\bigg(\tfrac{x}{2},\tfrac{t}{2}\bigg)-\psi_x(x,t)\bigg\}\bigg)\bigg\}. \tag{7.31}$$

Now , we apply here AHPSTM on (7.27), we have

$$\sum_{n=0}^{\infty}p^n\psi_n(x,t)=x^2e^{-t}+p\ S^{-1}\bigg\{\frac{u}{1+u}\bigg(S\bigg\{\sum_{n=0}^{\infty}p^n\tilde{H}_n(\psi)-\psi_x(x,t)\bigg\}\bigg)\bigg\}, \tag{7.32}$$

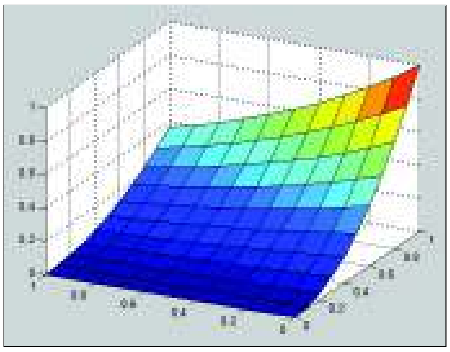
The initial couple of terms of $\tilde{H}_n$ are given by

$$\begin{aligned}
\tilde{H}_0(\psi) &= \psi_{0x}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\psi_{0xx}\big(\tfrac{x}{2},\tfrac{t}{2}\big),\\
\tilde{H}_1(\psi) &= \psi_{0x}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\psi_{1xx}\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{1x}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\psi_{0xx}\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{1x}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\psi_{1xx}\big(\tfrac{x}{2},\tfrac{t}{2}\big),\\
\tilde{H}_2(\psi) &= \psi_{0x}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\psi_{2xx}\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{1x}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\psi_{2xx}\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{2x}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\psi_{2xx}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\\
&\quad+\psi_{2x}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\psi_{0xx}\big(\tfrac{x}{2},\tfrac{t}{2}\big)+\psi_{2x}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\psi_{1xx}\big(\tfrac{x}{2},\tfrac{t}{2}\big)\\
&\ \ \vdots
\end{aligned}$$

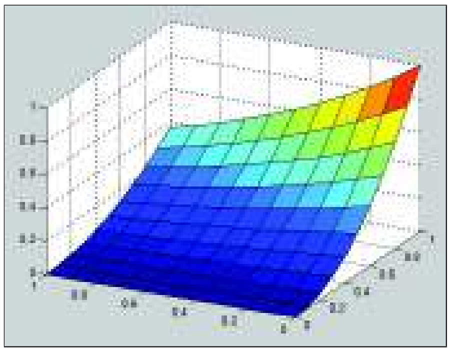
On looking at like powers of p of (7.32), we have

$$p^0:\psi_0=x^2e^{-t};$$

Figure 7.3: Solution of Non-linear PDE (7.27) with proportionate delay



(a) Approximate solution using AHPSTM up to fourth order



(b) (Exact sol.)

$$p^1 : \psi_1 = 0;$$
$$p^2 : \psi_2 = 0;$$
$$p^3 : \psi_3 = 0;$$
$$\vdots$$

Therefore, the series solution of (7.27) is

$$\psi(x,t) = x^2 e^{-t}. \tag{7.33}$$

Also the exact solution of the eq.(7.27) in closed form is

$$\psi(x,t) = x^2 e^{-t} \tag{7.34}$$

So from eq.(7.33) and eq.(7.34),we have found this exact solution in only one iteration.

7.4 Statistical Analysis

In order to validate the solution obtained from the semi-analytic technique AHPSTM and to investigate the techniques(AHPSTM, HPM and DTM) for giving better outcomes in regard of solution of non-linear problem considered in eq.(7.10),(7.18)and

Table 7.3: Approximate solution of (7.10) up to fourth order

x	t	Exact sol	AHPSTM	DTM[1]	HPM[87]	Abs.Err. (AHPSTM)	Abs.Err. (DTM)	Abs.Err. (HPM)
	0.25	0.3210063542	0.3210063530	0.3210042318	0.3210042318	1.22e-9	2.12e-6	2.12e-6
0.25	0.5	0.4121803177	0.4121802694	0.4121093750	0.4121093750	4.83e-8	7.09e-5	7.09e-5
	0.75	0.52925000412	0.5292495523	0.5286865234	0.5286865234	4.52e-7	5.63e-4	5.63e-4
	1	0.6795704571	0.6795681075	0.6770833333	0.6770833333	2.35e-6	2.49e-3	2.49e-3
	0.25	0.6420127083	0.6420127059	0.6420084635	0.6420084635	2.4e-9	4.24e-6	4.24e-6
0.5	0.5	0.8243606354	0.8243605388	0.8242187500	0.8242187500	9.66e-8	1.42e-4	1.42e-4
	0.75	1.058500008	1.058499105	1.057373047	1.057373047	9.03e-7	1.13e-3	1.13e-3
	1	1.359140914	1.359136215	1.354166667	1.354166667	4.70e-6	4.97e-3	4.97e-3
	0.25	0.9630190625	0.9630190588	0.9630126953	0.9630126953	3.7e-9	6.36e-6	6.36e-6
0.75	0.5	1.236540953	1.236540808	1.236328125	1.236328125	1.45e-7	2.13e-4	2.13e-4
	0.75	1.587750012	1.587748657	1.586059570	1.586059570	1.36e-6	1.69e-3	1.69e-3
	1	2.038711371	2.038704323	2.031250000	2.031250000	7.05e-6	7.46e-3	7.46e-3

(7.27) we have employed a statistical technique i.e. paired student's t-test at 5% level of significance to the data of tables 7.3, 7.4 and 7.5. The null hypothesis has been defined as under

Null Hypothesis:

$$H_0^A : \mu_1^A = \mu_{2j}^A, H_0^B : \mu_1^B = \mu_{2j}^B, H_0^C : \mu_1^C = \mu_{2j}^C,$$

where $\mu_1^k, k = A, B, C$ denotes the exact solution of (7.10),(7.18) and (7.27) respectively while $\mu_{2j}^k, K = A, B, C, j = 1, 2, 3$ denotes the approximate solution of eq. (7.10),(7.18) and (7.27) via AHPSTM, DTM and HPM respectively. The considered degree of freedom is $n_k - 1 = 12 - 1 = 11$ and the tabulated value of t at $\alpha = 5\%$ is $|t_{tab.}| = 2.201$. The calculated values of test statistic of (7.10),(7.18) and (7.27) for pair AHPSTM with exact solution A_i, DTM with exact solution B_i

Table 7.4: Approximate solution of (7.18) up to fourth order

x	t	Exact sol	AHPSTM	DTM[1]	HPM[87]	Abs.Err. (AHPSTM)	Abs.Err. (DTM)	Abs.Err. (HPM)
	0.25	0.0802515885	0.0802513111	0.0802510579	0.0802510579	2.77e-7	5.30e-7	5.30e-7
0.25	0.5	0.1030450794	0.1030352801	0.1030273438	0.1030273438	9.80e-6	1.77e-5	1.77e-5
	0.75	0.1323125010	0.1322294700	0.1321716308	0.1321716308	8.30e-5	1.41e-4	1.41e-4
	1	0.1698926143	0.1694996494	0.1692708333	0.1692708333	3.93e-4	6.22e-4	6.22e-4
	0.25	0.3210063542	0.3210052443	0.3210042318	0.3210042318	1.11e-6	2.12e-6	2.12e-6
0.5	0.5	0.4121803177	0.4121411203	0.4121093750	0.4121093750	3.92e-5	7.09e-5	7.09e-5
	0.75	0.5292500042	0.5289178802	0.5286865234	0.5286865234	3.32e-4	5.63e-4	5.63e-4
	1	0.6795704571	0.6779985974	0.6770833333	0.6770833333	1.57e-3	2.49e-3	2.49e-3
	0.25	0.7222642969	0.7222617997	0.7222595215	0.7222595215	2.50e-6	4.78e-6	4.78e-6
0.75	0.5	0.9274057148	0.9273175206	0.9272460938	0.9272460938	8.82e-5	1.60e-4	1.60e-4
	0.75	1.190812509	1.190065230	1.189544678	1.189544678	7.47e-4	1.27e-3	1.27e-3
	1	1.529033528	1.525496844	1.523437500	1.523437500	3.54e-3	5.60e-3	5.60e-3

and HPM with exact solution $C_i, i = 1, 2, 3$ are given below:

$$|t_{cal.}(A_1)| = 2.192, |t_{cal.}(B_1)| = 2.282, |t_{cal.}(C_1)| = 2.282,$$

$$|t_{cal.}(A_2)| = 1.884, |t_{cal.}(B_2)| = 1.914, |t_{cal.}(C_2)| = 1.914,$$

$$|t_{cal.}(B_3)| = 1.954, |t_{cal.}(C_3)| = 1.954$$

From the above analysis, it is clear that null hypothesis H_0 is accepted for eq. (7.10) only for pair AHPSTM solution and exact solution but rejected for DTM with exact solution and HPM with exact solution while for eq. (7.18) and (7.27), null Hypothesis is accepted in all the three cases (Note:For eq. (7.27), as we get exact solution with AHPSTM , so we do not test statistically). Hence, with this statistical analysis we conclude that AHPSTM gives better solution than other semianalytical technique like DTM and HPM.

Table 7.5: Approximate solution of (7.27) up to fourth order

x	t	Exact sol	AHPSTM	DTM[1]	HPM[87]	Abs.Err. (AHPSTM)	Abs.Err. (DTM)	Abs.Err. (HPM)
	0.25	0.0486750489	0.0486750489	0.0486755371	0.0486755371	0	4.88e-7	4.88e-7
0.25	0.5	0.0379081662	0.0379081662	0.0379231771	0.0379231771	0	1.50e-5	1.50e-5
	0.75	0.0295229096	0.0295229096	0.0296325684	0.0296325684	0	1.10e-4	1.10e-4
	1	0.0229924651	0.0229924651	0.0234375000	0.0234375000	0	4.45e-4	4.45e-4
	0.25	0.1947001958	0.1947001958	0.1947021484	0.1947021484	0	1.95e-6	1.95e-6
0.5	0.5	0.1516326649	0.1516326649	0.1516927083	0.1516927083	0	6.00e-5	6.00e-5
	0.75	0.1180916382	0.1180916382	0.1185302734	0.1185302734	0	4.39e-4	4.39e-4
	1	0.09196986029	0.09196986029	0.0937500000	0.0937500000	0	1.78e-3	1.78e-3
	0.25	0.4380754405	0.4380754405	0.4380798340	0.4380798340	0	4.39e-6	4.39e-6
0.75	0.5	0.3411734961	0.3411734961	0.3413085938	0.3413085938	0	1.35e-4	1.35e-4
	0.75	0.2657061859	0.2657061859	0.2666931152	0.2666931152	0	9.87e-4	9.87e-4
	1	0.2069321857	0.2069321857	0.2109375000	0.2109375000	0	4.01e-3	4.01e-3

7.5 Conclusion

1. We obtained a power series solution is a rapidly convergent series and analyzed that only a few iterations yield a high precision solution.

2. The proposed technique converges faster than other semi-analytical techniques like HPM, VIM, and DTM.

3. To approve and elucidate the effectiveness of the method, we have implemented the proposed technique on non-linear PDEs.

4. The series solution satisfies the condition of convergence which is reported in the results mentioned in tables 7.1 and 7.2.

5. The approximate results obtained by the semi-analytical technique are approaching to the exact solutions. These results are nearly equal.

6. The AHPSTM is faster than HPM, VIM, and DTM as it needs less number of iterations to obtain the convergent results.
7. The proposed method gives a better result to the solution of nonlinear PDEs as no discretizing algorithm and no linearization is required for non-linear problems.
8. Only a few iterations lead to the solution and it can be easily calculated and hence it reduces the computational cost.
9. AHPSTM is equally competent for linear and non-linear PDEs.

Concluding Remarks

All the physical problems can be modeled in the form of non-linear PDEs. It is exceptionally hard to acquire the analytical solution of these non-linear problems. To overcome this difficulty, various semi-analytical techniques and numerical techniques have been implemented by many authors. Here we have used Homotopy perturbation method as the main tool with different integral transformations like Laplace, Sumudu and Elzaki transformation to obtain the solution of various non-linear higher ordered, coupled and fractional PDEs in which we have a high degree of non-linearity. We have analyzed that with these methodologies we can undoubtedly discover the solution in the form of series expansion which rapidly converges to a precise solution. The condition of convergence of these semi-analytical techniques is derived and verified by applying these techniques on the problems of non-linear partial and fractional PDEs.

Finally, we propose a new semi-analytical technique which is more efficient than the classical semi-analytical techniques like HPM, VIM, and DTM, as only a few numbers of iterations are required to get convergent results. AHPSTM gives a better outcome for non-linear PDE solution as no discretizing algorithm and no linearization is required for non-linear problems. Only a few iterations will lead to the solution, and these can be easily calculated and hence reduces the computational cost.

Future Scope

We aim to make progressively utilization of the proposed technique for the solution of some new form of fractional PDEs using Caputo-Fabrizio, Atangana-Baleanu fractional operator and fractional integro-differential equations. Further, the status of the proposed technique, the present structure will likewise be explored. The proposed technique will be explored with integral transform like Fourier transform, Mellin transform and will be compared with numerical techniques or semi-analytical techniques like Finite element method, Haar wavelet, etc. Moreover, statistical analysis (like paired t-test, ANOVA) can also be performed to validate the results obtained from the method.

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List of Publications and Paper Presentations

Publication

1. Homotopy perturbation transform Method with He's polynomial for solution of coupled nonlinear partial differential equations. Nonlinear Engineering (2016), 5(1), 17-23.

2. Solution of fifth-order Korteweg and de Vries equation by homotopy perturbation transform method using He's polynomial. Nonlinear Engineering (2017), 6(2), 89-93.

3. Homotopy perturbation Sumudu transform method with He's polynomial for solutions of some fractional nonlinear partial differential equations. International Journal of Nonlinear Science (2016), 21(2), 91-97.

4. On the problem of convergence of series solution of non-linear fractional partial differential equation. In AIP Conference Proceedings (2017) (Vol. 1860, No. 1, p. 020027). AIP Publishing.

5. Convergence and Error Analysis of Series Solution of Nonlinear Partial Differential Equation. Nonlinear Engineering (2018), 7(4), 303-308.

6. Comparative study of homotopy perturbation transformation with homotopy perturbation Elzaki transform method for solving nonlinear fractional PDE (Accepted)

7. An efficient semi-anlytical technique for the solution of nonlinear partial differential equation. (Communicated)

Oral Presentation

1. International Conference on **Recent Trends in Engineering and Material Sciences** held at Jaipur National University, Jaipur on March 17-19, 2016 was attended and paper entitled **Solution of Fractional Attractor One-Dimensional Keller-Segel Equations Using Homotopy Perturbation Sumudu Transforms Method** was presented.

2. International Conference on **Recent Advances in Fundamental and Applied Sciences** held at Lovely Professional University, Phagwara on November 25-26,2016 was attended and paper entitled **On the Problem of Convergence of Series Solution of Non-Linear Fractional Partial Differential Equation** was presented.

3. International conference on **Recent Advances in Theoretical & Computational Partial Differential equations with Applications** held at Panjab University, Chandigarh on December 4-9,2016. Paper entitled **Comparative Study of Homotopy Perturbation Transformation with Elzaki Transforms Method for the Solution of Nonlinear Fractional PDE** was presented.

4. A national conference **Recent Trends in Numerical Analysis and Computational Techniques** held at DAV Institute of Engineering and Technology, Jalandhar on March 28-29, 2017 was attended and paper entitled **Convergence and Error Analysis of Series Solution of Certain Partial Differential Equation** was presented.

5. **National Conference on Innovation in Applied Science and Engineering** held at Dr. B.R. Ambedkar NIT, Jalandhar on April 27-28, 2019 was attended and paper entitled **Accelerated HPSTM: An Efficient Semi-Analytical Technique for Nonlinear Partial Differential Equation** was presented.