INVESTIGATION OF SECOND AND THIRD HARMONIC GENERATION OF SHORT PULSE LASER IN PLASMA

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DECLARATION

I hereby declared that the thesis entitled "INVESTIGATION OF SECOND AND THIRD HARMONIC GENERATION OF SHORT PULSE LASER IN PLASMA", has been prepared by me under the guidance of Dr. Niti Kant, Professor of Physics, Lovely Professional University. No part of the thesis has formed the basis for the award of any degree or fellowship previously.

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CERTIFICATE

This is to certify that Mr. Vinay Sharma has completed his Ph.D. thesis entitled, “INVESTIGATION OF SECOND AND THIRD HARMONIC GENERATION OF SHORT PULSE LASER IN PLASMA”, for the award of Ph.D. degree of the Lovely Professional University under my guidance and supervision. To the best of my knowledge, the present work is the result of his original investigation and study. No part of the thesis has ever been submitted for any other degree or fellowship previously at any university.

The thesis is fit for the submission and partial fulfilment of the condition for the award of Ph.D. degree in Physics.

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Abstract

Self-focusing, second harmonic generation (SHG) and third harmonic generation (THG) using different laser profiles like Gaussian laser beam, Hermite-Gaussian (HG) laser beam, cosh-Gaussian (chG) laser beam and Hermite-cosh-Gaussian (HchG) laser beam for optimum values of laser and plasma parameters have been analysed. Using paraxial approximation we have derived the coupled equations for beam width parameter and normalized amplitude for SHG and THG. We solved these equations numerically and results are analysed graphically. When we allow an intense laser beam to propagate through plasma, it exerts ponderomotive force on electrons and expel them away from the axial region, results density perturbations. Due to the presence of electrostatic force by positive ions electrons acquire the quiver velocity which coupled with density perturbation results SHG and THG at frequencies $2\omega_1$ and $3\omega_1$ respectively, where ω_1 the frequency of incident laser is. Due to variation in plasma density the dielectric properties of plasma changes and variation in refractive index takes place. This results the self-focusing of incident laser along axial region, under the effect of ponderomotive force and relativistic mass variation, known as ponderomotive and relativistic self-focusing respectively.

We have analysed the effect of density ripple on pulse slippage of third harmonic pulse for Gaussian laser and cosh-Gaussian laser beam, at optimum value of different laser and plasma parameters. It is observed that the efficiency gain is more at higher value of density ripple factor for d5fferent values of laser and plasma parameters. Density ripple provide the additional momentum to the photons of third harmonic generation and satisfy the resonance condition. It has also been observed that peak value of normalized amplitude for third harmonic pulse shifts toward higher value of normalized distance for higher values of density ripple factor. When we studied the cosh-Gaussian laser beam, we observed that the gain in efficiency for cosh-Gaussian laser is more at higher values of density ripple factor as compared to Gaussian laser beam. It is also observed that cosh-Gaussian beam shows oscillatory behaviuor at shorter Rayleigh length, whereas as Gaussian laser beam maintain its self-focusing upto longer distance. Self focusing of

Gaussian laser under relativistic self-focusing and exponential density ramp has been analysed. Our observations show that the exponential density ramp plays a dominant role in increasing the Rayleigh length and in the reduction of spot size. Density ramp vary the plasma density and variation of dielectric properties of plasma takes place, due to which the refractive index increases and self-focusing becomes stronger. We analysed the SHG for Hermite-Gaussian laser beam and cosh-Gaussian laser beam for different values of laser and plasma parameters. The outcome of our study shows that efficiency gain for cosh-Gaussian laser beam is very high as compared to Hermite-Gaussian laser beam. Self-focusing of incident laser is more stronger for cosh-Gaussian laser beam and also it maintain its self-focusing upto longer Rayleigh length as compared to Hermite-Gaussian laser beam. The effect of linear absorption has been studied for the self-focusing of Hermite-Gaussian laser beam. We observed that self-focusing becomes more stronger and maintained up to longer Rayleigh length, with increasing value of linear absorption coefficient. In the presence of linear absorption the reduction in spot size is significant at higher values of intensity of incident laser and normalized plasma frequency, whereas the self-focusing is also gets more stronger. It is also observed that oscillatory behavior of Hermite-Gaussain laser beam decreases at higher values of these laser parameters. The normalized radius of incident laser has been analysd under critical condition with intensity of incident laser, at different values of normalized plasma density. Results shows that at higher values of intensity of incident laser the radius decreases to appreciably small value and reduction is further increases when we increase the normalized plasma frequency. We analysed the beam width parameter of Hermite-cosh-Gaussian laser beam, at different mode indices and observed that spot size shows a sharp reduction at higher mode indices and maintain self-focusing for longer Rayleigh length as compare to cosh-Gaussian laser beam, whereas cosh-Gaussian laser beam shows the oscillatory behaviour at shorter Rayleigh length as compared to Hermite-cosh-Gaussian laser beam.

Higher efficiency gain is always desirable for different applications of SHG and THG in various fields. In our study we analysed the effeiciency gain of different laser profiles. It is observed that the gain in normalized amplitude of SHG and THG is significant at higher values of intensity of incident laser. Intense laser beam exert stronger ponderomotive force on electrons, where the phase matching condition is satisfied by the density ripple and wiggler field due to which increase in amplitude of second and third harmonic pulse is significant. SHG for Hermite-Gaussian and cosh-Gaussian laser beam has been analysed for optimum values of laser and plasma parameters. With increase in intensity of incident laser at higher values of normalized plasma density, it is observed that efficiency of SHG is much higher in cosh-Gaussian laser beam whereas Herimte-Gaussian laser beam shows oscillatory behavior upto longer Rayleigh length, as compared to cosh-Gaussian laser beam. For Gaussian and Hermite-cosh-Gaussian laser, the study of THG for relativistic self-focusing, at the different values of wiggler field and exponential density ramp has been done. It is observed that gain is more prominent when we introduce the exponential density ramp. The outcome of our study reveals that the gain is appreciably higher for Hermite-cosh-Gaussian laser beam, as compared to Gaussian laser beam, at higher values of Intensity of incident laser and wiggler field. Higher intensity of incident laser provides stronger ponderomotive force, results higher quiver velocity. Stronger quiver velocity of electrons couples with density perturbation results third harmonic generation of high intensity. Whereas wiggler magnetic field provides the additional momentum to the photons of third harmonic pulse and satisfies the phase matching condition and also confined the electrons within plasma region to maintain the cyclotron frequency. Due to phase matching condition satisfy by the wiggler magnetic field, The higher gain in efficiency of SHG and THG for different laser profiles has been achieved. The effect of decentred parameter has also been studied for cosh-Gaussian and Hermite-cosh-Gaussian beam laser. The small variation of decentred parameter result significant variation in intensity for SHG and THG and this shows the sensitivity of decentred parameter. Our study shows that efficiency of SHG and THG is more prominent at higher value of intensity of incident laser pulse, on increasing wiggler magnetic field and increasing the plasma density. Our study shows that amongst different laser profiles the intensity gain is very high for cosh-Gaussian and Hermite-cosh-Gaussian laser beam. Our study can be useful for different applications of SHG and THG in medical science, biomedical field, microscopic imaging and various other applications.

PREFACE

Effect of different parameters on second and third harmonic generations has been studied in present thesis. Interaction of short laser pulse propagating through plasma results nonlinearity due to ponderomotive force and relativistic self-focusing. When density perturbation couple with incident laser frequency result second and third harmonic generation which have a wide range of applications in medical field, microscopic imaging, biological sensing, optical imaging, eye safe and technologically relevant telecommunication bands, laser industry, next generation electronics and opto-electronic. Our research work has been aimed to study the affect of various laser parameters on self-focusing and the efficiency of second and third harmonic generations.

I am thankful to Dr. Niti Kant for his valuable guidance. I am also thankful to Prof. V.K. Tripathi for valuable discussions and his useful suggestions, to Dr. Vishal Thakur for his valuable support, to Mr. Shivam Sharma for his motivation and to Manoj Rattan for his extensive support. I am thankful to all my family members for their immense cooperation through the period of my research work.

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2. V. Thakur, S. Vij, V. Sharma, N. Kant, Influence of exponential density ramp on second harmonic generation by a short pulse laser in magnetized plasma, *Optik* **171** (2018) 523.

3.V. Sharma, V. Thakur and N. Kant Third harmonic generation of a relativistic self-focusing laser in plasma in the presence of wiggler magnetic field, *High Energ. Dens. Phys*. 32 (2019) 51.

4.V. Sharma, V. Thakur and N. Kant, Effect of linear absorption on self-focusing of Hermite Gaussian laser beam in plasma in relativistic and ponderomotive regime, *Optik* **194** (2019). 163076

Conference Papers

5. V. Sharma, V. Thakur and N. Kant, *Self-focusing of Hermite-Gaussian Laser Beam in Plasma in Relativistic and Ponderomotive, International Conference on Photonics, Meta-materials and Plasmonics* 2019.

Chapter-1

Introduction and Review

1.1 Introduction

With the invention of short pulse laser, the laser plasma interaction became a very important and wide area of research in last few decades. Plasma is the fourth state of matter and it consists of charged particles which get influenced by electric and magnetic field. Irving Langmuir first used the term plasma in 1928. Plasma behaves as an electrical fluid due to electrostatic interaction of every charged particle with its neighbour. Plasma frequency is the most important parameter of plasma, and these rapid oscillations of the electrons are also known as Langmuir waves.

The phase velcoity of an electromagnetic wave in a plasma, is given by $v_\varphi = \omega_1 / k_1$. where ω_1 is the electromagnetic wave frerquency and $k_1 = c\omega_1^{-1}\left(1 - \omega_p^2/\omega_1^2\right)^{-1/2}$ is the wave number and phase velocity is $v_\varphi = c\left(1 - \omega_p^2/\omega_1^2\right)^{-1/2}$. ω_p is plasma frequency given by $\omega_p = \sqrt{n_0 e^2 / m_e \varepsilon_0}$ where c is the velocity of light, m_e is the mass of electron, e is charge of electron, ε_0 is permittivity of the plasma region and n_0 is the equilibrium density of the plasma. By using the definition of refractive index given by $\eta = c/v_\phi$ this can be represented as $\eta = \left(1 - \omega_p^2/\omega_1^2\right)^{1/2}$. The group velocity is given by the formula $v_g = \partial\omega_1/\partial k_1 = \eta c$.

Plasma density oscillations are electromagnetic in nature and plasma frequency is critical to laser plasma interactions. If $\omega_1 > \omega_p$ then electrons will not be able to shield the electromagnetic field and plasma is known as under dense plasma. If $\omega_1 < \omega_p$, then plasma electrons respond fast and reflect the electromagnetic wave and plasma is said to be over dense. At $\omega_1 = \omega_p$ the density of the plasma is said to be critical and given as $n_c = \omega_1^2 m_e \varepsilon_0 / e^2$. Interactions with plasma densities below this critical density i.e. $n_e < n_c$ are called under dense plasma interactions and refractive index of the plasma is said to have its real value. For $n_e > n_c$ plasma is said to be over dense and the refractive index becomes complex. Therefore, it damps the propagating field, resulting in reflection of the laser pulse.

By measuring the effect of laser plasma interactions on driving laser pulse, it is possible to diagnose some of the important properties of plasma. When intense short laser pulse propagates through plasma, it exerts ponderomotive force on electrons resulting in flushing out of electrons from axial region and due to electrostatic force electrons undergo oscillations resulting in density perturbation. With this variation in plasma density the nonlinearity arises resulting in various nonlinear phenomena like relativistic self-focusing [1-2], laser driven plasma-based accelerators [3-5], harmonic generations [6-7], x-ray generations [8-9], advanced laser fusion [10] etc. having their usage in wide range of applications. Amongst harmonic generations the Second Harmonic Generation (SHG) and Third harmonic generation (THG) have a specific importance in the field of research. Some of these important applications are in the field of medical sciences [11], resonance imaging [12], probing of different surfaces [13], high speed optical communication and signal processing [14] and developing high quality 2-D materials [15]. Very high amplitudes of the SHG and THG can help us in deeper analysis of the various field of research that have important significance in life. That is why we are motivated to undertake the study of different laser profiles namely Gaussian, Hermite-Gaussian (HG), cosh-Gaussian (chG) and Hermite-cosh-Gaussian (HchG) beam to study the various laser parameters affecting the self-focusing and efficiency of SHG and THG.

1.2 Nonlinear Phenomena

Whenever there is change in the optical properties of any material medium it results nonlinearity and give rise to different nonlinear phenomena. Frankin *et al.* [16] were the first who discovered the SHG in 1961. The discovery of short pulse laser opens a broad area of research of nonlinear phenomena during short pulse laser plasma interaction. The SHG takes places due to atomic vibrations which vary quadratically with the intensity of optical field. Electric field provided by the intense laser is strong enough for Kerr effect to occur and is known as Optical Kerr Effect (OKE). As intensity of SHG varies as square of the intensity of electric field. Therefore, intensity dependent refractive index is given as

$$\eta = \eta_0 + \eta_2 I \tag{1.2.1}$$

Where η_0 is the linear refractive index, I the intensity of laser pulse and η_2 the material dependent Kerr coefficient. In high intensity laser the electromagnetic field is strong enough to induce nonlinearity's in material and allow phenomena like wave mixing and self-focusing. Induced dipole moment per unit volume P (t) that is the polarization density is given as

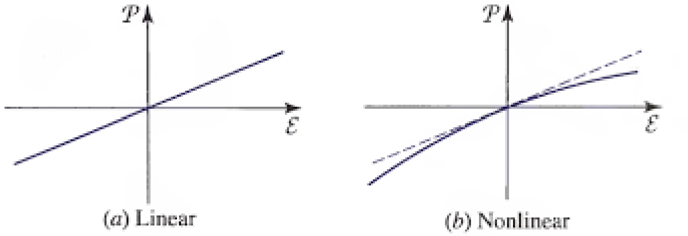

Fig. 1.1 Nonlinear dielectric medium is characterised by a nonlinear relation between P and E.

(https://encryptedtbn0.gstatic.com/images?q=tbn:ANd9GcToTmPc_eVfALksEead11l9CH5tcYrzF0cy_mo7EVZ-lzGwrBlPYA)

$$P(t) = \varepsilon_0 \chi^{(1)} E \tag{1.2.2}$$

Where E is the electric field, $\chi^{(1)}$ is the linear susceptibility and term ε_0 is said to be the permittivity of free space. When nonlinearity arises then polarization $P(t)$ is generalized as a power series of electric field is given as

$$P(t) = \varepsilon_0 \chi^{(1)} E(t) + \chi^{(2)} E_2(t) + \chi^{(3)} E_3(t) + --$$

$$P(t) = P_{(2)} + P_{(2)}(t) + P_{(3)}(t) + -- \tag{1.2.3}$$

Where $\chi^{(2)}$ and $\chi^{(3)}$ are second- and third-order nonlinear optical susceptibilities, respectively. Second-order nonlinearity takes place in non-Centro-symmetric crystals as these crystals do not posses inversion symmetry. Therefore, for non-Centro-symmetric crystals we have $\chi^{(2)} = 0$. Whereas odd harmonic generations like THG can take place for both Centro-symmetric and non-C=Centro-symmetric crystals.

1.2.1 Self-focusing of laser

When intense laser beam propagates through plasma then plasma electrons experience ponderomotive force and density perturbation takes place due to which non uniform intensity distribution occur. This result variation of refractive index results very important nonlinear phenomena known as self-focusing of incident laser as shown in Fig.1.3 Any medium for which nonlinear refractive index is positive the convergence takes place and beam is said to be self focused. Due to high intensity of focal spot, the optical material medium may get damaged. At critical power P_{cr} the diffraction spreading of the optical beam is balanced by nonlinear refraction maintaining its original diameter. If incident laser have the power lesser then the critical power there no self-focusing. However if power of the incident laser $P > P_{cr}$ it results in self-focusing. Intensity dependent refractive index and filament production give rise to phenomena such as ponderomotive self-focusing, Relativistic self-focusing and Thermal Relativistic self-focusing.

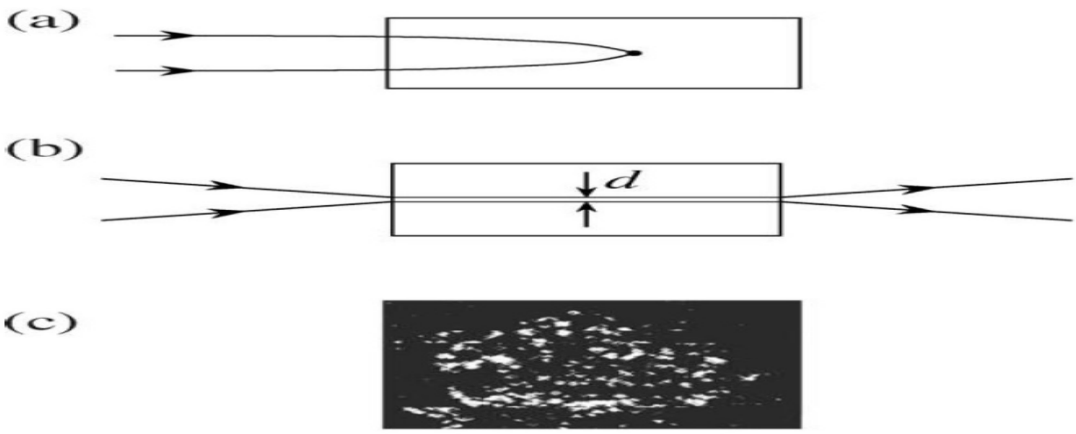

Fig. 1.2 Phenomenon of self-focusing
(http://www.physics.ttk.pte.hu/files/TAMOP/FJ_Nonlinear_Optics/Fig 5p2.jpg)

1.2.2 Ponderomotive Self-Focusing

The ponderomotive self-focusing can be understood by considering an initially incident laser which is little more intense in one region say by an amount δI, as shown in Fig.1.4. Intense laser exerts ponderomotive force and expel electrons away from the region of higher intensity and creating a lower electron density space. Since the index of refraction η of a plasma depends on the local electron density n_e, $\eta = (1 - n_e/n_c)^{1/2}$, the depletion of plasma electrons at the place where δI is large raises the index of refraction there, slows down the phase velocity of the light wave due to which wave-front form a curvature and get focused. This results increase in intensity of laser along the axis.

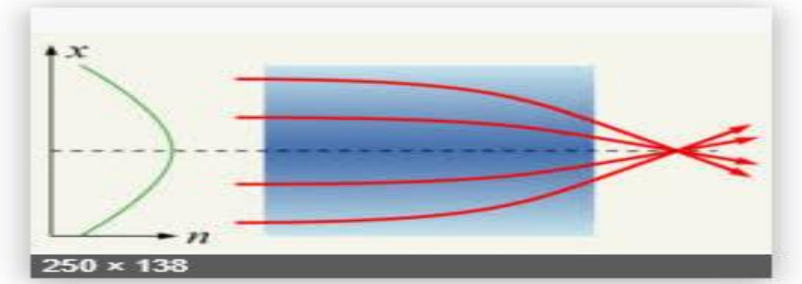

Fig. 1.3 Ponderomotive self-focusing (https://upload.wikimedia.org/wikipedia/commons/thumb/2/2c/Grin-lens.png/250px-Grin-lens.png)

1.2. 3 Thermal self-focusing

When short pulse laser propagates through plasma, it results non-uniform heating of plasma electrons and creating the temperature gradients. This results in the density perturbations and variation of dielectric properties of plasma takes place. The self-focusing of the laser beam that arises due to temperature gradient is said to be thermal self-focusing. Refraction of laser light acts to enhance the density perturbation

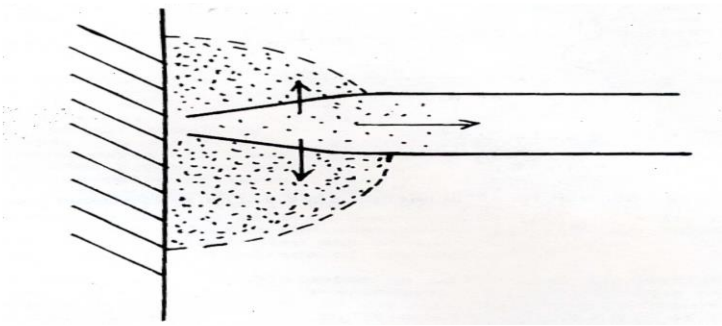

Fig.1.4 Thermal self-focusing (www.laserphy.uni-duesseldorf.de)

The characteristic minimum scale-size for TF is the mean free pathλ, of conduction electrons. Simultaneous occurring of Thermal and Ponderomotive self-focusing mechanisms takes place and variation in their relative strengths depends upon on the density and temperature of the plasma.

1.2.4 Relativistic Self-Focusing

When incident laser is very intense then the motion of the plasma electrons in the field of an ultra-intense laser-pulse becomes relativistic and this results in the change in mass of electrons causing change in plasma density. Change in density of plasma results in nonlinear optical behaviour of plasma and variation in dielectric properties occur. Due to change in refractive index the incident laser undergoes self-focusing known as relativistic self-focusing. Since in general the intensity will be higher at the centre of the laser spot than in the outer part of the spot, the value of the refractive index will exhibit a maximum on the laser axis. As seen before, this condition causes the pulse to self-focus. In the week relativistic regime and in a very under dense plasma, where $\eta = \sqrt{1-\omega_p^2/\gamma\omega_1^2}$ where $\gamma = \sqrt{1+(eE/m\omega_1 c)^2/2}$ and $(eE/m\omega_1 c)$ is normalized intensity of the laser pulse. If we allow the incident laser having power greater then critical power $P_c = 17(\omega_1^2/\omega_p^2)$GW, then the result is relativistic self-focusing (RSF). The RSF does not occur when the frequency of incident laser is more compared to ω_p even if its power is higher than the critical power. For $\omega_1 < \omega_p$ the self-focusing occur even for powers lower than P_c.

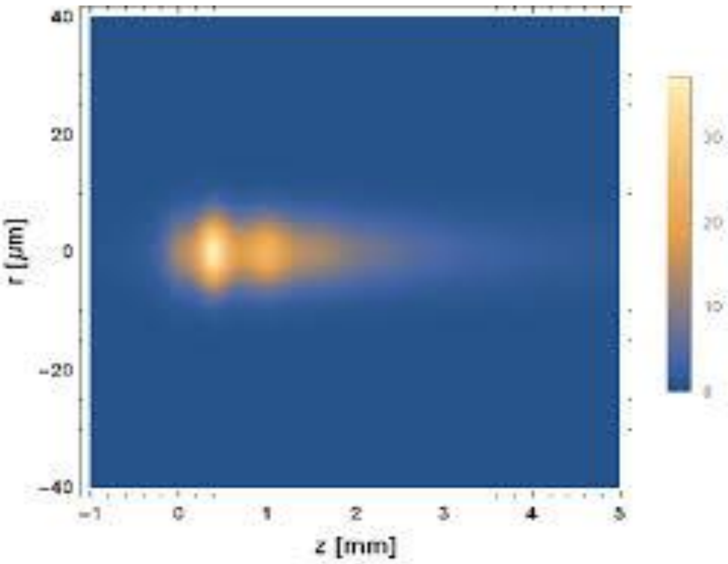

Fig. 1.5 Relativistic self-focusing (Epj-conferences.org) (https://encryptedtbn0.gstatic.com/images?q=tbn:ANd9GcTPTi536Toi03E6uEeaKgtqLhVxNCEluMtXaheAUx-2WQ45kRws)

1.2.5 Second Harmonic Generation

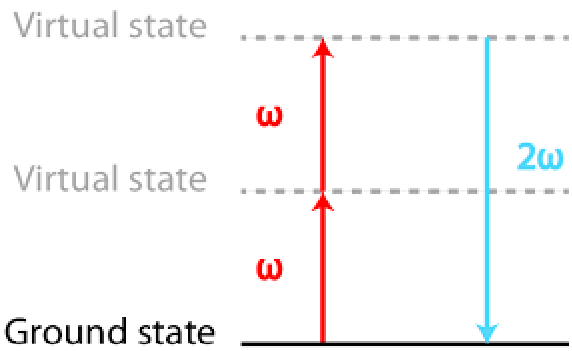

Fig. 1.6 Second harmonic generations

(*https://www.researchgate.net/profile/Stefaan_Vandendriessche/publication/262308876/figure/fig3/AS:613898407387157@1523376194837/In-second-harmonic-generation-two-photons-at-frequency-o-combine-and-are-emitted-as.png*)

Another important nonlinear phenomenon that takes place during propagation of intense laser beam through plasma is known as second harmonic generation (SHG) as illustrated in Fig.1.7 The nonlinear polarization is given as $P_2(t) = \varepsilon_0 \chi^{(2)} E_2(t)$, where $\chi^{(2)}$ is second order susceptibility and $E_2(t)$ is the electric field of the incident laser. In SHG total power of the fundamental laser of frequency ω_1 is converted to a radiation with nearly twice the incident frequency i.e. $2\omega_1$. SHG can modify the output of a laser of invisible spectrum to a visible spectral region a radiation of wavelength 10600A^{o} gets converted to a wavelength of 5200A$^{o.}$ This wavelength lies almost in the visible range. SHG can be better understood as the exchange of photons between the different frequency components of the laser field. The destruction of two photons of frequency ω_1 results a simultaneous formation of a single photon of frequency $2\omega_1$. The thick line in the Fig. 1.7 represents the atomic ground energy level where as the dotted lines represent virtual energy levels. In Fig. 1.7 If the medium is lossless then both at the fundamental frequency ω_1 and at the second harmonic frequency $\omega_2 = 2\omega_1$, then the nonlinear susceptibility obeys the condition of full permutation symmetry.

1.2.6 Third harmonic Generation

For the formation of THG, the third-order nonlinear polarization is given as $P_{(3)}(t)=\varepsilon_0\chi^{(3)}E_{(3)}(t)$, $E_{(3)}(t)$ consist of several different frequency components, the expression for is polarization density $P_{(3)}(t)$ becomes complicated. For this reason, we first consider the simple case in which the applied field is

$$E_3(t)=E_0\cos\omega_1 t$$

Using identity $\cos^3\omega_1=\cos 3\omega_1 t/4+3\cos\omega_1 t/4$, nonlinear polarization will have two terms. The first term will give rise to THG is illustrated in Fig.1.8. From this figure, it is clear that three incident photons each having frequency ω_1 combine to form one photon having the frequency $3\omega_1$.

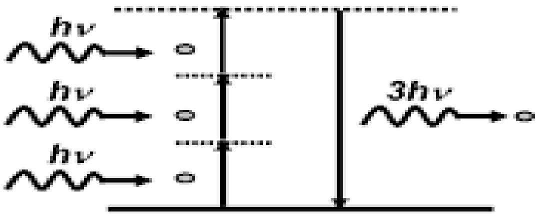

Fig.1.7 Formation of THG and Energy level of different energy states (https://www.researchgate.net/profile/ChiKuang_Sun/publication/44589034/figure/fig1/AS:394183799001088@1470992149155/Energy-level-diagrams-describing-a-second-harmonic-generation-and-b-third-harmonic.png)

1.3 Wiggler Magnetic Field

Wigglers consist of arrangements of magnets placed in series with poles of adjoining magnets alternate in their direction. Charged particle, when passing is deflected laterally to accelerate and emit intense radiations of high frequency possesed very high energy. When electrons are subjected to alternate magnets, they undergo periodic oscillations and emit strong radiations that are useful in many experiments. Effect of wiggler is widely used in harmonic generation. It satisfies the phase matching conditions by providing additional momentum to electrons to get in resonance with incident laser beam. It also confined electrons within plasma region and maintain cyclotron frequency.

Various researchers have investigated SHG and THG and studied the effect phase matching conditions that enhances the efficiency.

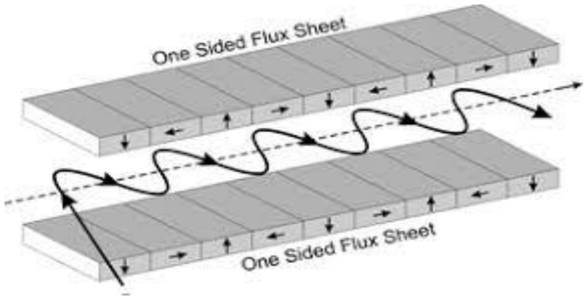

Fig. 1.8 Wiggler magnetic field
(https://upload.wikimedia.org/wikipedia/commons/thumb/e/e1/HalbachArrayFEL2.png/300px-HalbachArrayFEL2.png)

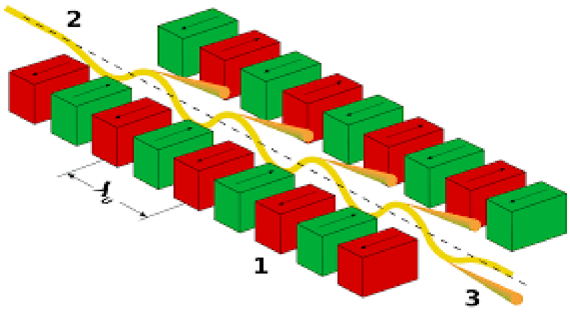

Fig.1.9 Undulator
(https://upload.wikimedia.org/wikipedia/commons/thumb/9/9f/Undulator.png/300px-Undulator.png)

1.4 Plasma Density Ramp

Various researchers analysed the self-focusing under different types of density ramp profiles like linear, parabolic and exponential. Researchers observed that plasma density ramp affect the nonlinearity in short pulse laser plasma interaction and the density ramp helps in overcoming the defocusing of incident laser beam [17]. Intense laser beam undergoes self-focusing and defocusing while propagating through plasma. With the introduction of density ramp self-focusing becomes stronger. [18-19]. because the dielectric properties of plasma changes. This results in an increase in self-focusing of incident laser and also decreases the defocusing [20-21]. Hashemzadeha *et al.* [22] studied the electric field, magnetic field and absorption rate in plasma under the effect of linear, parabolic and exponential density ramp

1.5 Literature Review

Erokhin *et al.* (1969) [23] obtained the second harmonic power, with nonlinear dispersion results second harmonic generation. Their result showed that the coefficient of energy transformation into second harmonic can be of the order of unity. They considered the fixed position of ions and with short duration's process, density perturbation results nonlinear effects and SHG takes place.

Sodha *et al.* (1978) [24] investigated the generation of second harmonic due to the propagation of Gaussian beam in a hot collision less plasma. They showed the self-focusing of beam if the incident pulse has power greater than the critical power. They considered that if the power of the incident laser is greater than critical power, than it results self-focusing. Their study shows that efficiency of SHG also depends upon the power of incident laser.

Basov *et al.* (1979) [25] investigated the SHG, theoretically as well as experimentally. They used solid targets and Kalmar laser and were pioneer to study the various phenomena taking place in laser plasma interaction, under critical density conditions. They experimentally irradiated the solid targets, using Kalmar laser and became very first to identify the laser plasma interaction in critical plasma density region.

Sodha *et al.* (1979) [26] had studied SHG during intense laser plasma interaction, where nonlinearity is due to ponderomotive self-focusing. They studied the dependence of beam width parameter on inhomogeneity of plasma and intensity of incident laser pulse. They observed that the electron density decreases as beam propagates deep inside the plasma, result increase in beam width parameter. Whereas with increase in the intensity of incident laser the beam width parameter reduces significantly.

Thakur and Sharma (1981) [27] investigated the SHG electrostatic pulse in hot plasma by a Gaussian EM pulse. On account of $\left(\vec{v}\times\vec{B}\right)$ force where oscillatory velocity of electrons is v in the EM pulse and B is the magnetic field of the pulse, an electrostatic pulse is generated at twice the frequency of the incident pulse. They observed that for an EM pulse, having uniform intensity distribution, along its wave front, the pulse width of the generated Gaussian electrostatic pulse is $\left(1/\sqrt{2}\right)$ times the initial phase width. The generated pulse again interact with incident EM pulse generates a third harmonic pulse.

Heinz and Loy (1988) [28] studied the nonlinear process of SHG which is very useful to investigate the surfaces and interfaces when the medium is Centro-symmetric. They collected the data useful for the development and applications of the technique for semiconductors. In their study they clearly observed sensitivity of SHG for different electronic properties and the dependence of nonlinearity on the chemical nature of the semiconductor surfaces, studied under controlled conditions.

Mohebi *et al.* (1988) [29] investigated self-focusing under linear absorption they experimentally showed that power loss due to linear absorption is responsible for power gain during self-focusing. They also observed that critical power is independent of linear absorption. They reported that the gain in power during self-focusing by linear absorption, due to power loss of the incident laser, before fundamental laser gets self-focused.

Liu *et al*. (1993) [30] using picosecond and microsecond laser pulse interacts in Neutral and Ionized Gases had investigated the harmonic generation. They experimentally studied the variation of the efficiency of THG. They observed that gain is higher only if phase matching conditions are taken along with different plasma effects. They studied that higher intensity of incident laser s necessary for the relativistic harmonic generations.

Krylov *et al.* (1995) [31] studied theoretically and experimentally the SHG and reported how the efficiency of second harmonic pulse is affected by phase modulation of incident laser pulse. They also observed that at high intensity, due to continuum generation, the decrease in efficiency of SHG takes place. Experimentally they observed that gain is lesser than 50%, which may be limited by the phase modulation by incident pulse.

Birulin, *et al.* (1996) [32**]** presented a theoretical model of high-harmonic generation when two waves interfere in a homogeneous or inhomogeneous dispersive medium and higher efficiency, of generated harmonics, was reported. They had shown experimentally that how the interference of two waves results harmonic generations, by the complete or partial cancellations of dispersion medium. They observed higher efficiency of harmonic generation, with the increase of interaction length.

Malka *et al.* (1996) [33] studied the interaction of second harmonic generation with relativistic plasma. They observed the second harmonic due to interaction of second harmonic light with relativistic plasma wave. They had taken the images of second harmonic pulse at some angle with axis, and observed the propagation of pulse over few

Rayleigh length. In their study, due to ionization and ponderomotive force, the radial electron density gets created, which is responsible for the power of SHG. The efficiency of second harmonic pulse and the effect of intensity of incident laser on the efficiency of SHG are in agreement with theoretical study.

Ramachandran (1996) [34**]** studied the second harmonic Raman scattering through laser plasma interaction and reported the maximum growth for side scattering that occurs above the one fourth critical density. When plasma is underdense, the Raman process is more pronounced than SHG. For collisional plasma both the Raman as well as SHG becomes unimportant.

Brandi and Neto (1997) [35] had studied THG through laser-plasma interaction: beyond the low-density approximation. By considering the different values of the electron plasma density, the intense laser pulse propagates through plasma without any change in its shape; they studied the oscillatory behaviour of THG. They reported that, due to fine spectral linewidth, the oscillatory behaviour of THG reduced. Their results for THG were restricted to moderate values of incident laser intensity.

Parashar and Sharma (1998) [36] had given the SHG by an obliquely incident laser on a vacuum-plasma interface. An intense laser incident on plasma at some angle results SHG in reflected component. They observed that second harmonic power gets vanishes at normal and grazing angle. Their study showed that second harmonic power is sensitive to the angle of incidence. The power efficiency increases with electron density is maximum at critical density and again decreases with increase in density.

Osman *et al.* (1998) [37] studied the self-focusing of intense laser in relativistic and ponderomotive regime. They derived the expressions for nonlinear refractive index, considering relativistic conditions and comparison was done with pondromotive nonlinearity. Their study revealed that on increasing the intensity of fundamental laser the ponderomotive self-focusing is stronger. But when oscillatory velocity of electrons increases, then relativistic self-focusing is also dominant. At higher order approximations, the alternating self-focusing and de-focusing was observed.

Kim and Lee (1999) [38] studied the Hermite-Gaussian and Laguerre–Gaussian laser beam. In their results beam power of incident laser is constant along axis of propagation. Using Lommel's lemma they derived the power series which is in close agreement with Lax's equations. Using transverse electric field, they obtained the solutions of Helmholtz equations for z > 0 and proved that total power is conserved along the direction of propagation of beam. The terms obtained by paraxial approximations are dominant. They obtained the electric field vector of the wave propagating along z-axis and it satisfies the Lax's equations.

Gupta and Abrol, (2000) [39] studied interaction of ultra-intense short-pulse laser beam in a tunnel- ionized plasma. Using paraxial approximation they derived the equations to analyse intensity along the axis with linear propagation distance. They obtained the Laplace term for beam width parameter, and their analysis showed that the self-focusing became stronger when intensity of incident laser is increased along axis.

Asthana *et al.* (2000) [40] studied the interaction of intense short pulses propagating in under dense plasmas. They analytically investigated the self-focusing of intense laser interacting with under dense plasmas. Using W.K.B and paraxial approximation they derived the expression for beam width parameter, under relativistic self-focusing. They analysed the beam width parameter with linear distance of propagation, at different values of different laser parameters. They observed the oscillatory behaviour of beam width parameter and also observed the self-channelling of the incident beam, due to the mutual cancellation of diffraction and self-focusing.

Singh *et al.* (2002) [41] studied the Phase-matched relativistic SHG in a plasma. The propagation of high intensity laser beam through plasma results density perturbation which combine with oscillatory velocity at fundamental frequency results second harmonic generation. Density ripples are generated by acoustic waves, which are introduced internally or externally. Their study showed that efficiency of second harmonic pulse is effected by inclination that incident laser makes with the density ripple. They analysed the propagation of intense laser under the different nonlinear effects like self-focusing, charged particle acceleration etc.

Banerjee *et al*. (2002) [42] studied the interaction of highly intense laser with plasma electrons under the relativistic condition. They showed experimentally that at high intensities the electrons density played important role for harmonic generations. Their results show that VUV radiations are produced by the relativistic Thomson scattering, affected by high energy electrons. They had given the estimated efficiency of high harmonic generations of the order of 10^{-7} and reported the beam size, nearly of 3mm.

Kant and Sharma (2004) [43] analysed the resonant SHG when intense laser beam propagates through plasma in the presence of a wiggler field. Density gradient along the axis is responsible for SHG. They studied the effect of self-focusing on second harmonic under wiggler magnetic field where wiggler field was responsible for the Gain in intensity of second harmonic pulse. In their study they observed the oscillatory behaviour of self focussed beam. The variation of normalized amplitude is also periodic and they observed the maximum increase in the value of normalized amplitude, as beam propagates in plasma.

Upadhyay and Tripathi (2005) [44] studied SHG in a laser channel. They analysed the interaction of short pulse laser beam through the plasma of low density results SHG that generated in the region of high density and propagate at low-angle with respect to channel axis. Under ponderomotive self-focusing, they observed the oscillatory behaviour of second harmonic with the density of the channel. The gain in efficiency is appreciable at higher values of densities and they also observed the linear dependence of peak efficiency on channel radius. Their analysis is limited to self guided laser, but not for a converging beam.

Sandhu *et al.* (2005) [45] presented the SHG and hard x rays generated during the interaction of an intense laser pulse with a pre plasma created on a solid target. They presented the investigation of hard x-rays and SHG and developed a relation between them depending on scale length. They chosen specific value of length scale at which second harmonic becomes minimum and hard x-ray maximizes. Their experimental analysis was in close agreement with theoretical results. The efficiency of SHG is quite high at the higher values of frequency of incident laser pulse.

Carrasco *et al.* (2006) [46] studied the SHG and THG using vector-Gaussian beams profile and considered harmonic generations by illuminated optical crystal by a vector Gaussian beam. Using first born approximation they obtain equations for terms that gave the geometrical parameters for a nonlinear optical crystal and explained nonlinear susceptibility tensor. They studied the importance of Guoy phase for THG for centrosymmetric crystals. In their study the efficiency of SHG and THG depends upon geometrical factors that become small with decrease in crystal size.

Dhaiyaa *et al.* (2007) [47] studied SHG and THG in the presence of density ripple which provided the phase matching. They had given the theoretical study for the efficiency of

SHG and THG and presented the particle in cell simulations that leads to even and odd harmonic generations. Under relativistic conditions they reported maximum efficiency when phase matching is satisfied by density ripple. The order of efficiency is even greater as compared when phase matching condition is not satisfied.

Gupta *et al.* (2007) [48] investigated the THG when high intensity Gaussian beam interacted with hot colliosionless plasma. With paraxial approximation they obtained the expressions for incident laser under relativistic and ponderomotive nonlinearities. They analysed the THG and filamentation. Due to Lorentz force, second harmonic wave is generated in plasma which gets coupled with oscillatory velocity at fundamental frequency and produce THG. In their study the maximum power obtained for THG was of the order of 10^{-6}.

Pallavi *et al.* (2007) [49] studied the SHG during the interaction of high intensity linearly polarized laser in magnetized plasma. Second order nonlinear current density arises in transverse direction results SHG in magnetized plasma. They derived the expression for normalized amplitude of SHG and observed its oscillatory behaviour. They also observed the increase in efficiency with the increasing values of magnetic field and intensity of incident laser.

Kaur and Sharma (2008) [50] had given the analytical study of THG when intense laser propagate through plasma generated from thin metallic film. When intense laser gets reflected form critical layer in plasma, the generation of density ripples takes place. The presence of density ripple satisfies the phase matching condition and is responsible for the gain in efficiency. Superposition of reflected laser from critical layer results THG. Higher efficiency of THG is reported when the density scale is increased. Under non-relativistic condition, the efficiency of THG is obtained as the fifth power of the intensity of incident laser.

Sodha *et al.* (2009) [51] investigated the THG and had given the effect of various laser parameters on efficiency of third harmonic pulse. They studied how the power of third harmonic pulse varies with different parameters of fundamental laser. They reported the enhancement in efficiency of THG due to the self-focusing and observed that THG is located near the axis of the beam.

Verma & Sharma (2009) [52] studied SHG in a rippled density magnetized plasma. Strong magnetic field is responsible for SHG. Density ripples provided the phase matching condition that results the enhancement in the efficiency of SHG. When intense laser beam propagates then under the combined effect of ponderomotive force, magnetic field and density ripple results SHG. They reported the enhancement in SHG, due to coupling between the laser and the magnetic field. They also observed the increase in SHG with the rise in electron density.

Rajput *et al.* (2009) [53] studied resonant THG when high intensity laser propagates through plasma. Phase matching condition were satisfied by wiggler filed that enhances the efficiency and due to mismatch between velocity of incident and third harmonic pulse, the pulse slippage takes place. They observed that third harmonic pulse moves for a longer propagation distance. Their study showed that the enhancement in the efficiency is significant with increasing wiggler field.

Prashar (2009) [54] studied the second harmonic generation when plasma is filled with parallel plane wave guide in the presence of density ripple that provided the phase matching. They had given the decrease in efficiency is inversely to sixth power of frequency of incident pulse, and for higher efficiency the plasma frequency should be greater than 0.5 times the frequency of incident pulse.

Patil *et al.* (2009) [55] studied the Hermite-Gaussian laser beam and analysed the oscillatory behaviour of laser beam propagating though collision-less plasma. They used the W.K.B. and paraxial approximation to derive the expressions for beam width parameter and study the behaviour of beam width parameter with linear propagation distance, at different values of decentred parameter and linear absorption coefficient. In their study under ponderomotive force the nonlinearity arises and they observed the defocusing of odd as well as even p-values giving oscillatory behaviour of beam width parameter.

Kaur *et al.* (2010) [56] used the short pulse laser propagating through rippled density plasma to study the THG. Where density ripple provides additional momentum to electrons and this satisfied the phase matching condition results increase in self-focusing that enhanced the efficiency of THG. They observed that, at high intensity of incident laser, the plasma density decreases along axis and, resonance condition for THG reduces, result decrease in efficiency. Nonlinearity depends upon the intensity of incident laser, and is independent of the frequency of incident laser. In their study, at very high intensity of incident laser the relativistic nonlinearity is dominant.

Askari and Norzooi (2010) [57] studied the SHG where ponderomotive force is responsible for nonlinearity which led to harmonic generation. The phase matching condition is provided by wiggler magnetic field due to which the efficiency of SHG rises sufficiently. They showed that efficiency of second harmonic pulse depends upon laser and plasma frequency.

Kant *et al.* (2011) [58] studied the Gaussian laser beam for second harmonic generation with density transition. They reported the higher intensity of second harmonic pulse and on introducing the plasma density ramp the self-focusing increases and also the beam extends to several Rayleigh lengths. They observed the appreciable increment in the efficiency of second harmonic pulse, in the presence of exponential density ramp.

Kant *et al.* (2012) [59] investigated the interaction of Hermit-Gaussian laser beams and analysed the affect of plasma density ramp on self-focusing of laser. They observed that on introducing the plasma density ramp the beam exhibit stronger self-focusing. They studied the variation of beam width parameter at optimum vales of different laser parameters, and observed that laser pulse propagates through several Rayleigh lengths without showing diffraction.

Kant *et al.* (2012) [60] studied the resonant THG, when intense laser propagate through plasma, generated in semiconductor called electron-hole plasma. Phase matching condition was satisfied by wiggler field results high efficiency of third harmonic pulse. They studied for the maximum efficiency of third harmonic pulse at different optimum values of different laser parameters.

Sharma and Sharma (2012) [61] investigated the laser plasma interaction, when intense short pulse laser propagating through magnetized plasma and analysed the SHG under the ponderomotive and relativistic regime. They obtained the coupled equations of beam width parameter and efficiency of second harmonic pulse, by using paraxial approximation. In their study the power of plasma wave show variation when different values of magnetic field is chosen. Their study revealed that the power of the SHG is considerably affected by the variation of strength of wiggler magnetic field.

Sharma *et al.* (2013) [62] analysed the THG at different values of magnetic field when intense Gaussian laser pulse propagates through plasma. Where wiggler magnetic field provided additional momentum to plasma electrons to satisfy the phase matching condition due to which efficiency gain is appreciable. Their study showed that Rayleigh length is more when wiggler magnetic field is applied.

Mihailescu *et al.* (2013) [63] studied the higher harmonic generations through intense laser plasma interaction and studied the impact of different laser parameters on power output of harmonic generation. In their work harmonic generations takes place with the reflection of femtosecond Ti-Sapphire terawatt laser and they had given the effect of different laser parameters on efficiency of harmonic generations

Jha *et al.* (2014) [64] investigated chirped laser pulses in a magnetized plasma channel. They used variational technique and obtain the equations that describe the spot size of incident laser, length of the pulse and chirp parameter. They also studied the spot size, pulse length and chirping for chirped as well as unchirped laser, under the effect of external magnetic field. Their significant observation is that, in external magnetic field, the unchirped laser pulse attains a positive and negative chirp.

Chen *et al.* (2014) [65] studied SHG and THG using chirped nonlinear photonic crystals. They had designed the fabricated chirped poled lithium niobate (CPPLN) nonlinear photonic crystal, that was used for phase matching with finite bandwidth, results second and third harmonic generations. They showed how the chirped rate affects the efficiency and bandwidth. They showed how CPPLN leads to the generation of different primary coloured lasers, to be used in large screen laser displays.

Thakur and Kant (2015) [66] studied THG during the high intensity laser plasma interaction and analysed self-focusing of laser beam by using density transition. In their study the phase matching condition is satisfied by using the wiggler magnetic field due to which efficiency gain is achieved. They had shown the pulse slippage due to velocity mismatch of fundamental and third harmonic pulse.

Kant and Wani (2015) [67] used the Hermite-Gaussian beam and studied under relativistic self-focusing. In their study the relativistic mass variation is responsible for non-linearity. Using paraxial approximation they obtained the expression for beam width parameter and studied the self-focusing for different laser and plasma parameters. They observed the effect of density ramp, linear absorption coefficient aand decnetred parameter. Their study shows the earlier and strongers elf-focusing when density ramp was introduced. They also showed that self focusing is stronger at higher values of linear absorption coefficient.

Aggarwal *et al.* (2015) [68] studied wiggler assisted SHG, when highly intense laser beam propagate through plasma in the presence of atomic clusters. Wiggler field is responsible for phase matching condition due to which efficiency of SHG increases. They studied the effect of intensity of fundamental laser and size of the clusters on the efficiency of second harmonic pulse. Enhancement in efficiency of SHG is observed and due to velocity mismatch of fundamental laser and second harmonic pulse the pulse slippage is observed.

Singh *et al.* (2015) [69] investigated the SHG and THG under relativistic self-focusing, when an intense laser beam propagates through a magnetized plasma. Relativistic mass variation and presence of magnetic field are responsible for nonlinearity, result a plasma channel due to self-focusing. The density perturbation arises, which coupled with quiver velocity of electrons, at fundamental frequency give rise to THG. Their study showed that the magnetic field of plasma has a significant effect on the efficiency of THG.

Verma *et al.* (2015) [70] studied the intense laser plasma interaction, in the presence of transversely distributed electric field. This results the density perturbation, which couple with the quiver velocity of electrons at fundamental frequency and SHG takes place. The time period of applied electric field is responsible for phase matching condition, results significantly higher efficiency of second harmonic pulse.

Gupta *et al.* (2015) [71] studied intense q-Gaussian laser beam for the SHG, by considering parabolic plasma channel and collisions are responsible for nonlinear absorption. Non uniform irradiance is responsible for non uniform heating of plasma electrons, results self-focusing and density gradients that excite plasma wave at fundamental frequency to get interact with incident laser to results SHG.

Vasari *et al.* (2015) [6] studied the laser plasma interaction by using the self twisted laser beam. The incident laser pulse results density perturbation due to which nonlinearity arises and second harmonic generation takes place. They used paraxial approximation to derive the couple equations for beam width parameter and efficiency for second harmonic pulse. They studied the effect of intensity of fundamental beam, vortex charge and plasma density on the efficiency of second harmonic pulse.

Patil *et al.* [2015] [72] studied the Gaussian laser beam propagating through plasma under the effect of light absorption and analyzed the relativistic self-focusing. Using W. K.B. and paraxial approximation, they derived beam width parameter equation and investigated the effect of normalized intensity of incident laser and normalized plasma dielectric function on the beam width parameter. In their study they observed that light absorption weakens the relativistic self-focusing.

Singh and Gupta (2016) [73] analysed SHG when high intensity cosh-Gaussian laser beam penetrating deep inside the underdense plasma, give rise to relativistic self-focusing due to which SHG takes place. They used W.K.B. approximation and moment theory and they obtained the differential equations for the beam width parameter. Using Runge-Kutta method, they numerically solved the equations and studied the effect of decentred parameter, intensity of incident laser pulse and normalized frequency on the efficiency of SHG. Shifting of peak intensity in transverse direction is observed by the variation of decentred parameter and stronger self-focusing with higher efficiency gain is observed.

Rathore and Kumar (2016) [74] had given the study of THG using quantum hydrodynamic model. Third harmonic current is produced when Gaussian beam propagates through plasma in normal wiggler field that produce transverse current and provided the phase matching. They observed the maximum efficiency when phase matching condition is satisfied. Quantum effects in harmonic generation were also studied in their work.

Purohit *et al.* (2016) [75] studied SHG by self-focusing of intense hollow Gaussian laser beam, with zero intensity at centre, in collision-less plasma. Using paraxial approximation, they obtained the coupled differential equations of beam width parameter and investigated the effect of self focused hollow Gaussian laser beam, exciting electron plasma wave, and studied the second harmonic generation under ponderomotive and relativistic nonlinearity. They reported that electric field linked with electron plasma wave and the power of SHG is very sensitive.

Vij *et al.* (2016) [76] studied THG when short pulse laser propagates through plasma in the presence of clusters with density ripple. Where ripples in plasma provided the Phase matching and higher electron density outside the cluster enhance the amplitude of THG. With the increase in collisional frequency of electrons, the absorption decreases and efficiency of THG decreases. The efficiency of THG is maximum at low collisional frequency. They reported the group velocity mismatch of THG and fundamental laser, due to which pulse slippage is observed.

Hassan *et al.* (2016) [77] studied the influence on ponderomotive self-focusing of laser beam through Plasma, in the presence of static magnetic field. They obtained the expression for beam width parameter of magnetized plasma and studied the increase in self-focusing of incident laser at different values of different laser parameters. In their study stronger self-focusing was observed with increasing transverse as well as longitudinal magnetic field. Their study outcome showed that the increase of longitudinal magnetic field is more effective than the transverse magnetic field on the self-focusing of incident laser pulse.

Song *et al.* (2015) [78] used non uniform partially coherent, Hermite-Gaussian laser beam to study its propagation characteristics. They had done analysis in both free as well as turbulent atmosphere to investigate the use of spectral intensity distribution. In their study, the different order Hermite function controls the intensity maxima. They observed that the beam is weakly affected by the turbulence, when linear propagation distance is lesser than 100m and beam evolution can be predicted by structure parameters.

Kaur *et al.* (2016) [79] considered the Hermite-cosh-Gaussian laser beam and investigated the focusing and defocusing in rippled density plasma. They analysed the propagation characteristics of Hermite-cosh-Gaussian laser beam. They studied the beam width parameter under the effect of different laser parameters and outcome of their study shows the oscillatory behaviour of beam width parameter. They showed that the ripple density increases the Rayleigh length and self-focusing becomes stronger under relativistic effect.

Kant and Thakur (2016) [80] analysed SHG considering chirped laser pulse plasma interaction, under relativistic self-focusing. They studied the effect of chirped parameter and normalized plasma frequency on the efficiency of second harmonic pulse. They had shown the significance of cyclotron frequency, in the enhancement of efficiency of second harmonic pulse, due to phase matching condition satisfied by wiggler field.

Kumar *et al.* (2016) [81] studied Harmonic generation in magnetized quantum plasma using quantum hydrodynamic model. They studied SHG by using the linearly polarized beam through dense plasma, under normal magnetic field, Dispersion was reported to be reduced when Fermi pressure and electron spin was considered. They also reported the velocity mismatch of second harmonic pulse with the fundamental laser; due to which pulse slippage occur. In their study amplitude of SHG is maximum at resonance when ω_1 and $2\omega_1$ equal to cyclotron frequency.

Shen *et al.* (2017) [82] studied the Gain-phase modulation in chirped-pulse amplification. They demonstrated both experimentally and theoretically the cross modulation between the gain and chirped phase in chirped pulse amplification. They used the nonlinear Schrodinger equation coupled with gain phase modulation. They observed the distortion of incident pulse compensated by amplification that can be explained by gain phase modulation. Moreover their experimental results are in close agreement with theoretical results.

Patil *et al.* (2018) [83] studied the turning point temperature of self-focusing under effect of light absorption, considering the Gaussian laser plasma interaction. Using the WKB and paraxial approximation they obtained the expressions for beam width parameter and presented the variation of beam width parameter with linear propagation distance at different values of normalized intensity of incident laser, dielectric function of plasma, under the effect of light absorption. Their study outcomes shows the oscillatory behavior of beam width parameter

Varaki and Jafari (2018) [84] studied SHG when the linearly polarized-Gaussian laser pulse interacted with magnetized plasma in a planar magneto static wiggler field. They used the theory of perturbation. Their results showed appreciable rise in efficiency of second harmonic pulse when planar magneto-static wiggler field was considered responsible for resonance condition. Their study also revealed that efficiency of second harmonic pulse is higher at higher plasma frequency.

Hashemzadeh *et al.* (2018) [22] studied interaction of high intensity laser pulse in inhomogeneous plasma. Using Maxwell's equations, they derived the electric, magnetic field and absorption rate equations. They studied the effect of linear, parabolic and density profile on spatial scale length. Outcome of their study showed that resonance absorption is more important than bremsstrahlung effect at intensity above 10^{16}W/cm^2.

1.6 Research Gap

There was an extensive research in self-focusing, second harmonic generation and Third harmonic generation for Gaussian laser beam but very little attention was given to investigate SHG and THG using other laser profiles. Self-focusing of cosh-Gaussian and Hermite-cosh-Gaussian beam has been studied by no of researchers. Appreciable work on improving the efficiency of second and third harmonic generation using Gaussian laser profile is done by various researchers but little attention was given to study the SHG and THG using other laser profiles like Hermite-Gaussian, cosh-Gaussian and Hermite-cosh-Gaussian laser beams. It is noticed that the self-focusing is stronger for other profiles of laser beams, when analysed under the effect of different laser parameters. Higher efficiency gain is always desired in the study outcome for different harmonic generations. This motivated us to investigate different laser profiles such as Hermite-Gaussian, cosh-Gaussian, Hermite-cosh-Gaussian by using paraxial approximation.

1.7 Objectives

The main objective of the present thesis is to enhance the efficiency of second and third order harmonic generation of laser beam in plasma by optimizing various laser and plasma parameters. We prefer our study using different laser profiles for enhanced self-focusing of the incident laser beam which leads to produce efficient second and third harmonic generation.

Following 7 problems are analyzed in order to achieve the objectives.

1. Third harmonic generation of a relativistic self-focusing laser in plasma.
2. Effect of Density Ripple on Pulse Slippage of Third Harmonic Generation in Plasma.
3. Second harmonic generation of self focused Hermite-Gaussian laser beam in plasma.
4. Effect of linear absorption and temperature on self-focusing of Hermite-Gaussian beam in plasma with relativistic and ponderomotive regime.
5. Third harmonic generation of a relativistic self-focusing laser in plasma under exponential density ramp.
6. Second harmonic generation of cosh-Gaussian laser beam in rippled density magnetized plasma.
7. Third harmonic generation of self-focused Hermite-cosh-Gaussian laser beam in plasma.

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Chapter 2

2.1 Method and materials

In our work, we use the Maxwell's wave equations, governing electromagnetic fields in a dielectric medium are given as

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \tag{2.1}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{2.2}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{2.3}$$

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \tag{2.4}$$

$$\vec{D} = \epsilon_0\vec{E} + 4\pi\vec{P} \text{ and } \vec{B} = \mu_0\vec{H} + 4\pi\vec{M}\ .$$

where, c is the velocity of light in vacuum, $\vec{E}$ and $\vec{H}$ are electric and magnetic fields of em waves, $\vec{D}$ and $\vec{B}$ are electric displacement and magnetic induction vectors, $\vec{P}$ and $\vec{M}$ are electric and magnetic dipole moments per unit volume that are caused by the displacement and orientation of bound electrons of neutral atoms and molecules, and ρ and $\vec{J}$ are charge and current densities due to free electrons and ions, related through the equation of continuity

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{D} = 0 \tag{2.5}$$

In a fully ionized plasma, $\vec{P} = \vec{M} = 0$.

Using equation of continuity the expressions for nonlinear current density can be obtained for SHG and THG

Now, the wave equation governing the propagation of electromagnetic waves in plasmas can be obtained by taking the curl of the third Maxwell's equation and using the fourth, we can write,

$$\nabla^2 \vec{E}_2 = \frac{4\pi}{c^2}\frac{\partial \vec{J}_2}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \vec{E}_2}{\partial t^2} \tag{2.6}$$

In plasma the equation (5) reduces to,

$$\nabla^2 \vec{E} - \frac{\varepsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{\omega_P^2}{c^2}\vec{E} \tag{2.7}$$

2.2 Gaussian Beams

Radiation confined to the fundamental (TEM_{00}) Gaussian mode and of monochromatic wavelength are said to be Gaussian beam. Gaussian beam posses' electric and magnetic field transverse to its direction of propagation. The output of the most of the laser is described by transverse mode of field due to which Gaussian beam can be focused coherently. Change in transverse phase takes place when the beam is converge by a lens and the formation of new Gaussian profile takes place. Beam waist parameter w_0 gives the amplitude of electric and magnetic field.

We consider the Gaussian beam for small divergence so that we can apply the paraxial approximation. This allows the omission of the second order derivative and first order differential equation of propagation is obtained. Gaussian beams are important due to its various significant properties. There is no alternation in Gaussian profile after it passing through simple optical lenses. During its propagation only beam radius changes whereas its intensity remains same along the axis.

2.2.1 Hermite-Gaussian beam (HG)

Hermite-Gaussian beam has different mode indices and electric field variation is along both x and y- axis. Hermite-Gaussian modes, with their rectangular symmetry, are especially suited for the modal analysis of radiation from lasers whose cavity design is asymmetric in a rectangular fashion. Electric field for Hermite-Gaussian beam is given as

$$\vec{E} = A(x,y,z)\exp\left[i\left(\omega_1 t - k_1 z\right)\right] \tag{2.8}$$

$$A = A_0 \exp(-iks),$$

Where $\vec{k}_1$ is the wave vector of fundamental laser pulse $A = A_0 \exp(-iks)$ is the amplitude of the fundamental laser pulse. Where A_0 and s are the functions of x, y and z, A_0 is given as

$$A_0^2 = \frac{E_0^2}{f_1(z) f_2(z)} H_m\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right) H_m\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right) \exp\left[-\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]. \tag{2.9}$$

f_1 & f_2 are the beam width parameters and r_0 represents the radius of the incident laser pulse.

2.2.2 cosh-Gaussian beam (chG)

chG beam profile resembles very closely to the flat-top field distributions under suitable parameters. When two coherent decentred Gaussian beams of equal waist width get superimposed we obtain chG beam, having centres lie at positions (b/2, 0) and (-b/2, 0). Electric field for chG beam is given as

$$\vec{E}_1 = \hat{x} A(z) \exp[-i(\omega_1 t - k_1 z)]$$

$$A(z) = A_0(z, r) \exp[-ikS(r, z)]$$

$$\therefore\ A_{0p}^2 = \frac{E_0^2}{f^2(z)} Exp\left[\frac{b^2}{2}\right]$$

$$\left\{Exp\left[-2\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] + Exp\left[-2\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] + 2Exp\left[-\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right]\right\}$$

$$\vec{E}_1 = \hat{x} \frac{E_0}{f(z)} Exp\left[\frac{b^2}{4}\right]$$

$$\left\{\begin{array}{l} Exp\left[-2\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] + Exp\left[-2\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] \\ + 2Exp\left[-\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2} \exp[-ikS(r, z)](z) \exp[-i(\omega_1 t - k_1 z)]. \tag{2.10}$$

2.2.3 Hermite-cosh-Gaussian beam (HchG)

Hermite-Gaussian beams are obtained from the solution of Helmholtz equations with paraxial approximations. For these beams, the field is expresses in the rectangular coordinate system. This is the generalized form of the Hermite-Gaussian and sinusoidal functions of complex argument. The Gaussian field can be written as the multiple of Hermite polynomial, sinusoidal and Gaussian functions of complex argument. Hermite-sinusoidal Gaussian beams is known as Hermite-cosh-Gaussian (HchG) laser beam when distributes at z = 0 and given as

$$E_3 = A_3 e^{i(\omega t - kz)}$$

Where $A_3 = A_{30}(r,z) e^{-ikS(r,z)}$

$$A_{30}^2 = \frac{E_{30}^2}{f^2(z)} \left[H_m \left(\frac{\sqrt{2} r}{r_0 f_{(z)}} \right) \right]^2 e^{\left(\frac{b^2}{2}\right)} \left\{ \exp\left[-2\left(\frac{r}{r_0 f(z)} + \frac{b}{2} \right)^2 \right] + \exp\left[-2\left(\frac{r}{r_0 f(z)} - \frac{b}{2} \right)^2 \right] \right\}$$

$$+ 2\exp\left[-\left(\frac{2r^2}{r_0^2 f(z)^2} + \frac{b^2}{2} \right) \right]$$

$$E_3 = A_{30}^2 = \frac{E_{30}^2}{f^2(z)} \left[H_m \left(\frac{\sqrt{2} r}{r_0 f_{(z)}} \right) \right] e^{\left(\frac{b^2}{4}\right)} \left\{ \begin{array}{l} \exp\left[-2\left(\frac{r}{r_0 f(z)} + \frac{b}{2} \right)^2 \right] \\ + \exp\left[-2\left(\frac{r}{r_0 f(z)} - \frac{b}{2} \right)^2 \right] \end{array} \right\}^{1/2} . \qquad (2.11)$$

$$+ 2\exp\left[-\left(\frac{2r^2}{r_0^2 f(z)^2} + \frac{b^2}{2} \right) \right] e^{-ikS(r,z)} e^{i(\omega t - kz)}$$

m is defined to be the mode index related to Hermite polynomial function.

Using above laser profiles, by using the paraxial approximation, we have derived the expressions for beam width parameter and normalized amplitude of SHG and THG for different laser profiles. These expressions have been solved numerically using Mathematica software. Graphs have been plotted using ORIGIN software to analyse the result.

Chapter -3

Effect of density ripple on pulse slippage of third harmonic generation in plasma

3.1. Introduction

Harmonic generation during laser plasma interaction has been a fascinating field of research for last few decades. When intense short-range laser pulse interacts with plasma, interesting nonlinear phenomena arises i.e. X-ray generation [1], neutron production [2], Laser plasma accelerators [3] and harmonic generation [4-7] etc. arise. Harmonic generation is one of the most important nonlinear phenomena as it has wide range applications. It converts infrared lasers to shorter wavelengths in the visible and ultraviolet region and widely use in microscopy imaging. Saytashev *et al.* [8] report on multimodal imaging of blood using sub-50 fs pulses centred at 1060 nm wavelength. They find that red blood cells appear dark on Second harmonic images while on third harmonic images of blood provide bright signal and good contrast. Graham *et al.* [9] studied the three- dimensional imaging of direct- written photonic structures where third harmonic generation microscopy is used to analyze the morphology of photonic structures created using the femto-second laser direct- write technique. Due to homogenous nature of the medium odd harmonics can only be generated and third harmonic is dominant. When large amplitude electromagnetic wave of frequency ω_1 and wave vector $\vec{k}_1$ propagates through the nonlinear medium it produces electromagnetic waves at harmonic frequencies say third harmonic, $3\omega_1$. The efficiency is not very high due to a phase mismatch $(\vec{k}_3 > 3\vec{k}_1)$ between the third harmonic and fundamental pulses. For high efficiency phase matching needs to be satisfied. Various researchers proposed different schemes to satisfy the phase matching condition and to overcome the phase mismatch problem [10-13]. Milchberg *et al.* [10] have proposed the idea of using extra degree of freedom available in plasma fiber for phase matching of nonlinear conversion processes. Chen *et al.* [14] have first time experimentally observed the phase-matched relativistic third-harmonic generation in forward direction by varying the temporal delay and the energy of the laser pulse. Prashar and Pandey [15] suggested density ripple present in plasma and applied Wiggler magnetic field to satisfy the phase matching condition for second harmonic. They studied the process of second harmonic generation in plasma having a density ripple that could provide additional momentum required for generating resonant harmonic photons. Verma and Sharma [17] studied the second harmonic generation which is increased by applying magnetic field and density ripple provide the phase matching between the fundamental and second harmonic pulses. Shim *et al.* [18] Reported controlled enhancement of optical third harmonic generation (THG) from hydro-dynamically expanding clusters of ∼6×105 noble-gas atoms several hundred femto-seconds following ionization and heating by ultra-short pump pulses. Simulations show that the nonlinear susceptibility $\chi^{(3)}$ of the individual clusters and the coherence length of the clustered plasma medium are optimized nearly simultaneously as the clusters expand, and both contribute to the observed THG enhancement. Panwar *et al.* [19] in their study. The self-focusing and compression of the fundamental pulse periodically enhances the intensity of the third-harmonic pulse at lower powers of main laser. In a deeper plasma channel, the third harmonic power is less effective by self-focusing and the compression of main laser and increase with main laser pulse power. Liu and Tripathi [20] reported third harmonic generation in a plasma with density ripple created by a machining beam. The presence of a density ripple of suitable wave number turns the third harmonic generation into a resonant process, raising the harmonic conversion efficiency by an order of magnitude even at mildly relativistic intensities. Dhaiya *et al.* [21] studied the generation of second and third harmonic laser radiation in pre-ionized under-dense plasma having density ripple in the direction of laser propagation and provide the additional momentum between a harmonic photon and combining fundamental photon result resonance condition and they observed that the efficiency of phase matched harmonic is higher than the one without phase matching. Singh *et al.* [22] have obtained the high efficiency of harmonic generations when the pump laser is incident at critical angle and the angle between the ripple wave factor and the surface has a specific value of $\phi = \phi_m$, at which phase matching condition is satisfied. Shibu and Tripathi [23] studied the third-harmonic generation of the laser radiation in a plasma channel in the presence of a density ripple where the ripples provide the uncompensated momentum between the third harmonic photon and fundamental photons and leads to resonant enhancement of harmonic power. Vij *et al.* [24] studied the resonant third harmonic generation of short pulse in cluster plasma, due to density ripple and plasma electron density outside the cluster, the phase matching condition for third harmonic process is satisfied which results high efficiency. Thakur and Kant [25] studied the effect of pulse slippage on density transition based resonant third harmonic generation of short pulse laser in plasma and wiggler magnetic field is used to satisfy the phase matching condition in order to enhance the amplitude of third harmonic pulse.

In the present work, we investigate the effect of density ripple on pulse slippage of third harmonic generation in plasma. Short range intense laser pulse incident on plasma produce density ripple and laser exerts ponderomotive force $-e\vec{F}_{2\omega,2\vec{k}_1} \times \vec{B}_{\omega,\vec{k}_1}$ on electrons and they attain oscillatory velocity $\vec{v}_{2\omega,2\vec{k}_1.}$ This velocity along with n_0 produces a density perturbation $n_{2\omega,2\vec{k}_1+\vec{k}_0}$, the density perturbation beats with $v_{\omega,\vec{k}}$ to produce third harmonic current density $J_{3\omega,3\vec{k}_1+\vec{k}_0} = -en_{2\omega,2\vec{k}_1+\vec{k}_0}\vec{v}_{\omega,\vec{k}_1}$, resulting resonant third harmonic generation having electric field $\vec{E}_{3\omega,\vec{k}_3}$. Effect of density ripple on pulse slippage has been discussed for different laser-plasma parameters.

In the Presentwork we are presenting a mathematical model for normalized amplitude of third harmonic pulse using nonlinear current density for third harmonic wave and effect of density ripple is included. The results have been obtained graphically and the variation of normalized amplitude of third harmonic pulse with normalized distance for different values of density ripple factor and constant α and discussed theoretically.

3.2 Theoretical considerations

Consider an intense laser beam propagating in plasma with electric and magnetic fields are as

$$\vec{E}_1 = \hat{x}A_1(z,t)\exp[-i(\omega_1 t - \vec{k}_1 z)], \tag{3.1}$$

$$\vec{B}_1 = \frac{\vec{k}_1 \times \vec{E}_1}{\omega_1},$$

where ω_1 is the frequency and $\vec{k}_1$ is the wave vector of incident laser pulse. Third harmonic obey the linear dispersive relation $\left(\omega_1^2/c^2\right)\left(1-\omega_p^2/\omega_1^2\right)$ and wave vector increases non linearly with fundamental frequency ω_1 hence, $\vec{k}_3 > \vec{k}_1$ where $\vec{k}_3$ is the wave vector of third harmonic pulse. Density ripple factor n_0^0 provides angular momentum to the third harmonic photon to satisfy phase matching condition, so $\vec{k}_3 = 3\vec{k}_1 + \vec{k}_0$. Here

$\omega_p = \left(4\pi n_0 e^2 / m\right)^{1/2}$ is the plasma frequency, e and m are the charge and rest mass of the electron. Under ponderomotive force electrons attain oscillatory velocity

$$\vec{v}_1 = e\,\frac{\vec{E}_1}{mi(\omega_1 + i\nu)} \quad . \tag{3.2}$$

Laser exerts ponderomotive force on electrons and electrons acquire oscillatory velocity at 2ω and this velocity beats with density ripple produces density perturbation at 2ω. This density perturbation beats with $\vec{v}_{\omega,k}$ to produce third harmonic current density $\vec{J}_3$ driving resonant condition. Where $\vec{J}_3^{L}$ and $\vec{J}_3^{NL}$ [28] are obtained as

$$\vec{J}_3^{L} = \frac{n_0 e^2 \vec{E}_3}{m3i\omega_1} \ , \tag{3.3}$$

$$\vec{J}_3^{NL} = \frac{-n_0^0 e^4 \vec{E}_1^3 k_1 (3k_1 + k_0)}{16 i m^3 \omega_1^5}, \tag{3.4}$$

$$\nabla^2 \vec{E}_3 = \frac{4\pi}{c^2}\frac{\partial \vec{J}_3}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \vec{E}_3}{\partial t^2}, \tag{3.5}$$

where $\vec{J}_3 = \vec{J}_3^{L} + \vec{J}_3^{NL}$, and $\vec{J}_3^{L}$ is the linear current density due to self-consistent field $\vec{E}_3$;

$$\nabla^2 \vec{E}_3 = \frac{4\pi}{c^2}\frac{\partial \vec{J}_3^{\,L}}{\partial t} + \frac{4\pi}{c^2}\frac{\partial \vec{J}_3^{\,NL}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \vec{E}_3}{\partial t^2},$$

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} + \frac{\partial^2 \vec{E}_3}{\partial r^2} + \frac{\partial \vec{E}_3}{r\partial r} = \frac{\partial^2 \vec{E}_3}{c^2 \partial t^2} + \frac{4\pi\partial \vec{J}_3^{L}}{c^2 \partial t^2}\frac{4\pi\partial \vec{J}_3^{NL}}{c^2 \partial t^2},$$

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} + \frac{\partial^2 \vec{E}_3}{\partial r^2} + \frac{\partial \vec{E}_3}{r\partial r} - \frac{\partial^2 \vec{E}_3}{c^2 \partial t^2} - \frac{4\pi\partial \vec{J}_3^{L}}{c^2 \partial t^2} = \frac{4\pi\partial \vec{J}_3^{NL}}{c^2 \partial t^2}, \tag{3.6}$$

where $\vec{E}_3 = \hat{x}A_3(z,t)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]$,

$$\frac{\partial \vec{E}_3}{\partial z} = \hat{x}A_3\left[(i(3k_1 + k_0)\right]z,t)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}] + \hat{x}(z,t)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3}{\partial z},$$

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} = \hat{x} A_3(z,t)\left(\left[(i(3k_1 + k_0)\right]\right)^2 \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]$$
$$+ \hat{x}(z,t)\left[(i(3k_1 + k_0)\right]\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3}{\partial z}$$
$$+ \hat{x}(z,t)\left[(i(3k_1 + k_0)\right]\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3}{\partial z}$$
$$+ \hat{x}(z,t)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial^2 A_3}{\partial z^2},$$

Neglecting small term containing $\partial^2 A_3 / \partial z^2$, we obtain

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} = \hat{x} A_3(z,t)\left(\left[(i(3k_1 + k_0)\right]\right)^2 \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]$$
$$+ 2\hat{x}(z,t)\left(i(3k_1 + k_0)\right)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3}{\partial z}, \tag{3.7}$$

$$\frac{\partial \vec{E}_3}{r\partial r} = \frac{1}{r}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3}{\partial r}, \tag{3.8}$$

$$\frac{\partial^2 \vec{E}_3}{\partial r^2} = \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial^2 A_3}{\partial r^2}, \tag{3.9}$$

$$\frac{\partial \vec{E}_3}{\partial t} = A_3 \hat{x}(-i3\omega_1)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}] + \hat{x}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3}{\partial t},$$

$$\frac{\partial^2 \vec{E}_3}{c^2 \partial t^2} = \frac{A_3}{c^2}(-i3\omega_1)^2 \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}] + \frac{(-i3\omega_1)}{c^2}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3}{\partial t} +$$
$$\frac{(-i3\omega_1)}{c^2}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3}{\partial t} + \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial^2 A_3}{\partial t^2},$$

$$\frac{\partial^2 \vec{E}_3}{c^2 \partial t^2} = \frac{A_3}{c^2}(-i3\omega_1)^2 \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]$$
$$+ \frac{(-i6\omega_1)}{c^2}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3}{\partial t} +$$
$$+ \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial^2 A_3}{\partial t^2},$$

$\partial^2 A_3 / \partial t^2$ being small is neglected

$$\frac{\partial^2 \vec{E}_3}{c^2 \partial t^2} = -\frac{9A_3}{c^2}\omega_1{}^2 \exp[-i\{3\omega_1 t-(3k_1+k_0)z\}] - \frac{i6\omega_1}{c^2}\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_3}{\partial t}, \qquad (3.10)$$

$$\vec{J}_3^L = \frac{n_0 e^2 \vec{E}_3}{m3i\omega_1} = \frac{n_0 e^2 A_3 \exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]}{m3i\omega_1},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} =$$

$$\frac{4\pi n_0 e^2 A_3(-3i\omega_1)\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]}{c^2 m3i\omega_1} + \frac{4\pi n_0 e^2 \exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]}{c^2 m3i\omega_1}\frac{\partial A_3}{\partial t},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \begin{array}{l} -\dfrac{\omega^2{}_p A_3 \exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]}{c^2} \\ +\dfrac{\omega^2{}_p \exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]}{c^2 3i\omega_1}\dfrac{\partial A_3}{\partial t} \end{array}. \qquad (3.11)$$

$$\vec{J}_3^{NL} = \frac{-n_0^0 e^2 \vec{E}_1^3 k_1[(i(3k_1+k_0)]}{16im^3\omega_1^5},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \frac{-4\pi n_0^0 e^2 k_1[(i(3k_1+k_0)]}{16c^2 im^3\omega_1^5}\frac{\partial \vec{E}_1^3}{\partial t},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \frac{-4\pi n_0^0 e^2 k_1[(i(3k_1+k_0)]}{16c^2 im^3\omega_1^5}\frac{\partial A_1^3(z,t)\exp[-3i(\omega_1 t-k_1 z)]}{\partial t},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \frac{-4\pi n_0 n_0^0 e^2 k_1[(i(3k_1+k_0)]}{16n_0 c^2 im^3\omega_1^5}\frac{\partial A_1^3(z,t)\exp[-3i(\omega_1 t-k_1 z)]}{\partial t},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \frac{-\omega_p^2 n_0^0 e^2 k_1[(i(3k_1+k_0)]}{16n_0 c^2 im^2\omega_1^5}\frac{\partial A_1^3(z,t)\exp[-3i(\omega_1 t-k_1 z)]}{\partial t},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \frac{-\omega_p^2 n_0^0 e^2 k_1[(i(3k_1+k_0)]}{16n_0 c^2 im^2\omega_1^5}\frac{(-3i\omega_1)A_1^3(z,t)\exp[-3i(\omega_1 t-k_1 z)]}{}$$

$$\frac{-\omega_p^2 n_0^0 e^2 k_1[(i(3k_1+k_0)]}{16n_0 c^2 im^2\omega_1^5}\frac{(z,t)\exp[-3i(\omega_1 t-k_1 z)]3A_1^2}{\partial t}\frac{\partial A_1}{\partial t},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \frac{-3\omega_p^2 n_0^0 e^2 k_1\left[(i(3k_1+k_0)\right]}{16n_0 c^2 i m^2\omega_1^4}\frac{A_1^3(z,t)\exp[-3i(\omega_1 t - k_1 z)]]}{}$$

$$\frac{-\omega_p^2 n_0^0 e^2 k_1(3k_1+k_0)(z,t)\exp[-3i(\omega_1 t - k_1 z)]3A_1^2}{16n_0 c^2 i m^2 \omega_1^5}\frac{\partial A_1}{\partial t},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \frac{-3\omega_p^2 n_0^0 e^2 k_1(i(3k_1+k_0)}{16c^2 n_0 i m^2\omega_1^4}\frac{\exp[-3i(\omega_1 t - k_1 z)]A_1^2}{}\left(A_1 - \frac{\partial A_1}{i\omega_1 \partial t}\right). \tag{3.12}$$

Using Eqs. (3.6), (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12) into Eq. (3.5), we obtain

$$A_3\left[(i(3k_1+k_0)\right]^2\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]$$
$$+2\left[(i(3k_1+k_0)\right]\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_3}{\partial z}+\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial^2 A_3}{\partial r^2}$$
$$+\frac{1}{r}\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_3}{\partial r}+\frac{A_3}{c^2}(9\omega_1)^2\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]$$
$$+\frac{\omega_p^2}{c^2\omega_1}\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]-\frac{6i\omega_1)}{c^2}\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_3}{\partial t}$$

$$=\frac{-3\omega_p^2 n_0^0 e^2 k_1(3k_1+k_0)}{16c^2 n_0 i m^2\omega_1^4}\frac{\exp[-3i(\omega_1 t-k_1 z)]A_1^2}{}[A_1-\frac{\partial A_1}{i\omega_1\partial t}],$$

$$A_3(i(3k_1+k_0))^2\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]+2(i(3k_1+k_0))\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_3}{\partial z}$$
$$+\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial^2 A_3}{\partial r^2}$$
$$\frac{1}{r}\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_3}{\partial r}+\frac{A_3}{c^2}(9\omega_1)^2\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]$$
$$+\frac{\omega_p^2}{c^2\omega_1}\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]+\frac{1}{r}\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_3}{\partial r}$$
$$+\frac{A_3}{c^2}(9\omega_1)^2\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]+\frac{\omega_p^2}{c^2\omega_1}\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]$$
$$-\frac{6i\omega_1}{2c^2}\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_3}{\partial t}$$

$$=\frac{-3\omega_p^2 n_0^0 e^2 k_1(3k_1+k_0)}{16c^2 n_0 i m^2\omega_1^4}\frac{\exp[-3i(\omega_1 t-k_1 z)]A_1^2}{}\left(A_1-\frac{\partial A_1}{i\omega_1\partial t}\right),$$

$$\left[\begin{array}{l} -A_3{k_3}^2 + 2(i(3k_1+k_0))\dfrac{\partial A_3}{\partial z} + \exp\dfrac{\partial^2 A_3}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial A_3}{\partial r} + \dfrac{A_3}{c^2}9{\omega_1}^2 \\ + \dfrac{\omega_p^2 A_3}{c^2\omega_1} - \dfrac{6i\omega_1}{c^2}\dfrac{\partial A_3}{\partial t} \end{array}\right]\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]$$

$$= \frac{-3\omega_p^2 n_0^0 e^2 k_1(3k_1+k_0)}{16c^2 n_0 i m^2\omega_1^4}\exp[-3i(\omega_1 t-k_1 z)]A_1^2\left(A_1-\frac{\partial A_1}{i\omega_1\partial t}\right),$$

$$\left[\begin{array}{l} -\dfrac{A_3(3k_1+k_0)^2}{2ik_3} + \dfrac{2(i(3k_1+k_0))}{2i(3k_1+k_0)}\dfrac{\partial A_3}{\partial z} + \dfrac{1}{2i(3k_1+k_0)}\exp\dfrac{\partial^2 A_3}{\partial r^2} \\ + \dfrac{1}{2i(3k_1+k_0)r}\dfrac{\partial A_3}{\partial r} \\ + \dfrac{A_3}{2i(3k_1+k_0)c^2}9{\omega_1}^2 + \dfrac{\omega_p^2 A_3}{2i(3k_1+k_0)c^2\omega_1} \\ - \dfrac{6i\omega_1}{2i(3k_1+k_0)c^2}\dfrac{\partial A_3}{\partial t} \end{array}\right]\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}] \quad (3.13)$$

$$= \frac{-3\omega_p^2 n_0^0 e^2 k_1(3k_1+k_0)}{(2ik_3)16c^2 n_0 i m^2\omega_1^4}\exp[-i(3\omega_1 t-k_1 z)]A_1^2\left(A_1-\frac{\partial A_1}{i\omega_1\partial t}\right),$$

where $-A_3{k_3}^2/2ik_3 + A_3 9{\omega_1}^2/2ik_3c^2$ gets cancelled out, therefore taking following terms on left hand side of Eq. (3.13) becomes

$$\Rightarrow\left[\frac{\partial A_3}{\partial z} - \frac{3\omega_1}{k_3c^2}\frac{\partial A_3}{\partial t}\right]$$

where $k_3 = \sqrt{3\omega_1^2/c^2\left[1-\omega_p^2/9\omega_1^2\right]}$

$$\Rightarrow \frac{\partial A_3}{\partial z} - \frac{3\omega_1}{c^2\sqrt{\dfrac{9\omega_1^2}{c^2}\left[1-\dfrac{\omega_p^2}{9\omega_1^2}\right]}}\frac{\partial A_3}{\partial t}$$

$$\Rightarrow \frac{\partial A_3}{\partial z} - \frac{1}{v_{g3}}\frac{\partial A_3}{\partial t}$$

where $v_{g3} = \dfrac{1}{c\sqrt{\left[1-\omega_p^2/9\omega_1^2\right]}}$

R.H.S. of Eq.(3.13) is

$$= \frac{-3\omega_p^2 n_0^0 e^2 k_1(3k_1+k_0)}{(2ik_3)16c^2 n_0 i m^2 \omega_1^4} \frac{\exp[-3i(\omega_1 t - k_1 z)]A_1^2}{} [A_1 - \frac{\partial A_1}{i\omega_1 \partial t}] = \alpha n_0^0 A_1^2\, [A_1 - \frac{\partial A_1}{i\omega_1 \partial t}]$$

$$\alpha = \frac{-3\omega_p^2 n_0^0 e^2 k_1(3k_1+k_0)}{(2ik_3)16c^2 n_0 i m^2 \omega_1^4} \text{ and } A_1^3 = \left[F(z - v_{g1}t)\right]^3$$

The ratio of the third harmonic amplitude

$$\frac{|A_3|}{|A_0|} = \frac{\alpha_3 A_0 \xi \sqrt{\pi} n_0^0}{2\beta}\left[1 - \frac{2(z'-t')}{\omega'}\right]\underline{\left[erf\,(z'-t) - erf\,\{(1-\beta)z'-t\}\right]} \tag{3.14}$$

3.3 Result and discussion

Eq.(3.14) has been studied graphically for normalized third harmonic amplitude A_3/A_0 with z' at different values of n_0^0. The normalized amplitude for third harmonic pulse increases by increasing n_0^0 and due to phase matching condition satisfied by the density ripple. Due to group velocity mismatch between the fundamental and third harmonic pulse the third harmonic pulse slipped out of the domain of fundamental pulse where the group velocity mismatch is significant at higher values of n_0^0. Density ripple plays an important role in phase matching for third harmonic generation. Fig. 1 shows the variation of A_3/A_0 with z' at $t = 2$ for different values of α and n_0^0. For α = 0.05 at n_0^0 = 0.01, 0.02 and α = 0.15 at n_0^0 = 0.01, 0.02 the value of A_3/A_0 increases with increase in n_0^0 as shown in table 1. Fig. 2 and Fig. 3 show the significant rise in A_3/A_0 at $t = 6$ and 10 for given values of α and n_0^0 and shows that group velocity mismatch between the fundamental pulse and third harmonic wave increases at t =10 due to which third harmonic pulse slips out of the domain of the fundamental pulse. Similar results were reported by Aggarwal *et al.* [26] for second harmonic generation with the effect of wiggler magnetic field. It is clear from the results presented in this manner that the power of third harmonic pulse increases with the higher values of n_0^0 andα. Thakur and Kant [25] show the similar results where wiggler magnetic field is used to provide the phase matching condition and reported the phenomenon of pulse slippage. Vij. *et al.* [27] also observed that A_3/A_0 decreases on decreasing the value of n_0^0.

Table 3.1. Peak value of A_3/A_0 at different values of t forgiven values of α and n_0^0

α	t =2		t =6		t =10	
	A_3/A_0 at n_0^0=0.01	A_3/A_0 at n_0^0=0.02	A_3/A_0 at n_0^0=0.01	A_3/A_0 at n_0^0=0.02	A_3/A_0 at n_0^0=0.01	A_3/A_0 at n_0^0=0.02
0.05	0.0011	0.0018	0.0048	0.0088	0.0087	0.0158
0.15	0.0023	0.0052	0.0142	0.029	0.027	0.058

Fig. 3.4 shows the variation of A_3/A_0 for different values of n_0^0 =0.01, 0.02, 0.03. Significant rise in A_3/A_0 is observed at n_0^0= 0.03.Third harmonic further increases with t at given values of n_0^0 also the peak value of third harmonic amplitude is shifted towards higher value of z'. Thakur and Kant [25] reported the similar result and they observed the rise in amplitude with t where wiggler magnetic field is used to provide the phase matching condition. Rajput *et al.* [28] also reported the increase in A_3/A_0 with z' using wiggler magnetic field for the phase matching conditions. In the present work we have optimized the different parameters in order to increase the efficiency of third harmonic generations and results are also shown in table 2.

Table 3.2. A_3/A_0 With n_0^0 and laser pulse length.

t =4			t =8			t =10		
n_0^0	z'	A_3/A_0	n_0^0	z'	A_3/A_0	n_0^0	z'	A_3/A_0
0.01	9	0.0062	0.01	17	0.019	0.01	22	0.25
0.02	9	0.015	0.02	17	0.043	0.02	22	0.059
0.03	9	0.023	0.03	17	0.064	0.03	22	0.083

Fig. 3.5. Shows the variation of A_3/A_0 with z' at $\alpha = 0.05$ and $n_0^0 = 0.02$. The third harmonic pulse slips out of the domain of fundamental pulse due to the group velocity mismatch between fundamental and the third harmonic pulses. In table 3, comparative

study of A_3/A_0 shows that amplitude increase when we increase α for given values of n_0^0. This shows that peak values of A_3/A_0 also increases with increased value of α.

Table 3.3. Variation of A_3/A_0 with τ' at given values $n_0^0 = 0.02$

τ'	A_3/A_0 at α=0.05	A_3/A_0 at α=0.15
0	0.00026	0.005
4	0.0059	0.0158
6	0.029	0.029
10	0.073	0.0157

3.4 Conclusion:

In this study we observed that amplitude of third harmonic pulse increases due to phase matching condition provided by density ripple. Increase in A_3/A_0 is observed with increasing n_0^0. We have shown graphically the significant rise in A_3/A_0 when n_0^0 increases. Also the peak value of A_3/A_0 is shifted towards the higher value of z' with increasing n_0^0. The group velocity mismatch between the fundamental and third harmonic pulse is significant at higher values of n_0^0 and third harmonic pulse slip out of the domain of the fundamental pulse.

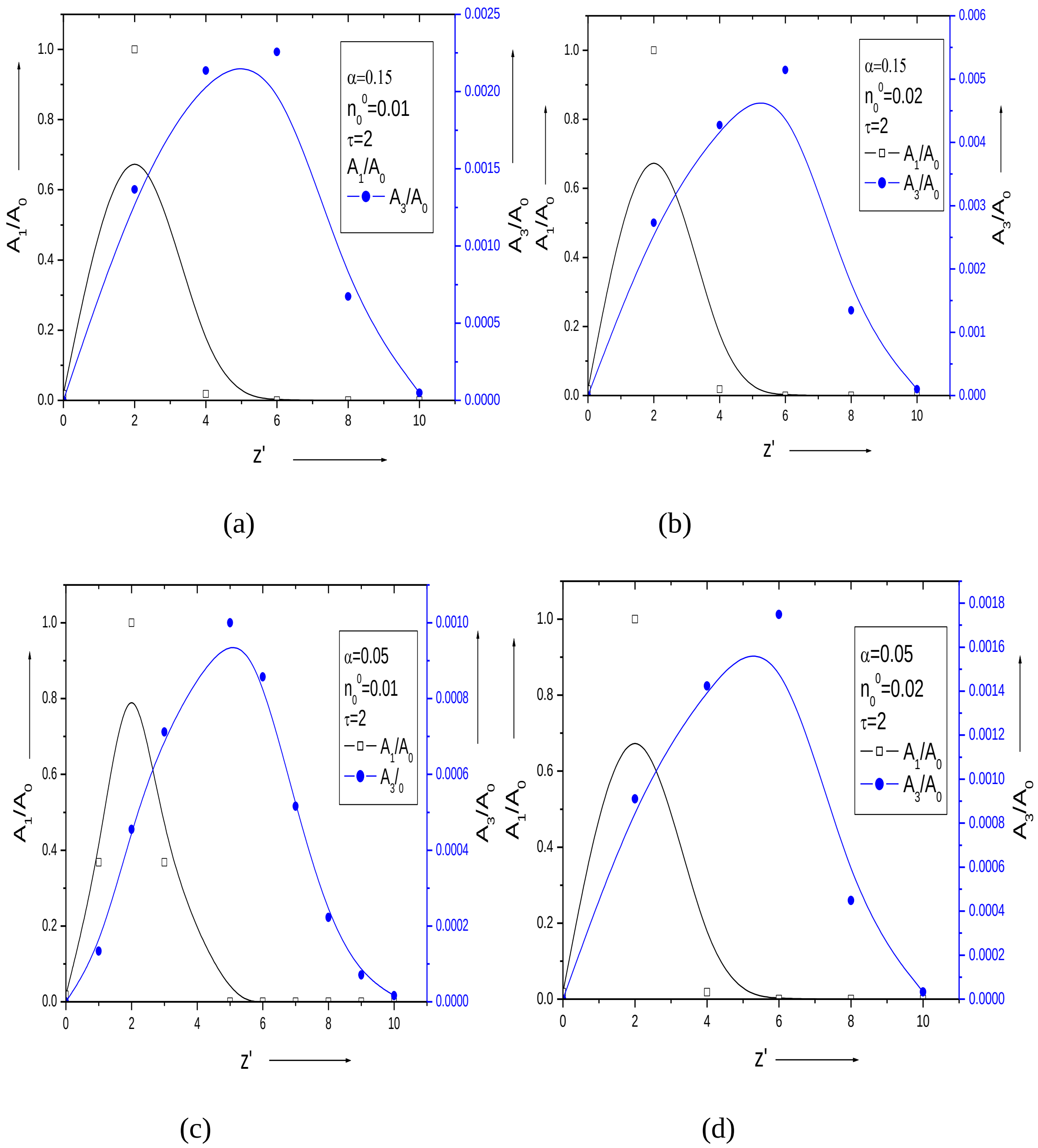


Fig. 3.1. Variation of normalized amplitude of fundamental and third harmonic pulses with normalized propagation distance at α =0.15 for n_0^0 =0.01 & 0.02 and at α =0.15 for n_0^0 =0.01 & 0.02 t =2.

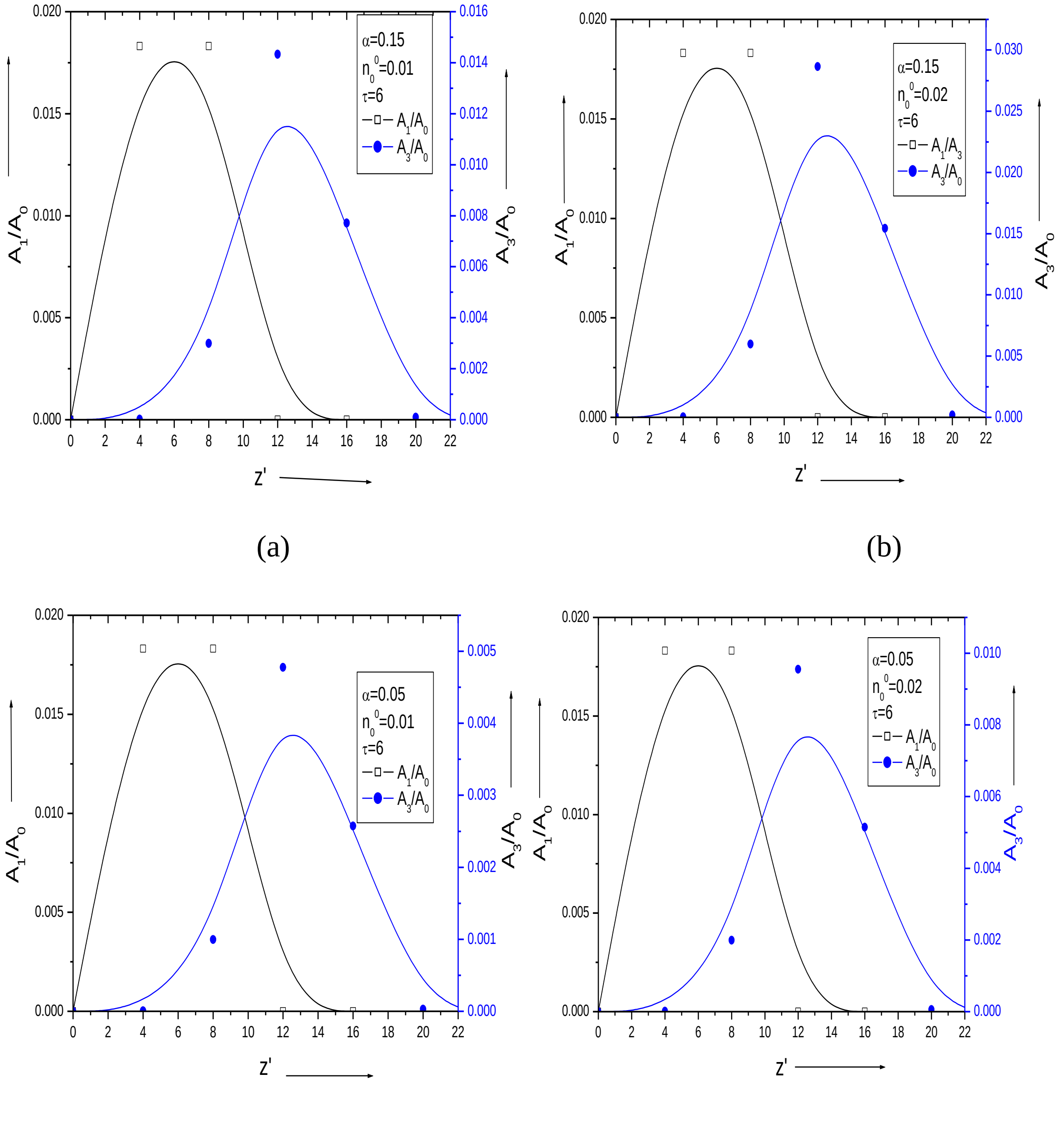


Fig 3.2. Variation of normalized amplitude of fundamental and third harmonic pulses with normalized propagation distance at α =0.15 for n_0^0 =0.01 & 0.02 and at α =0.05 for n_0^0 =0.01 & 0.02 at t = 6

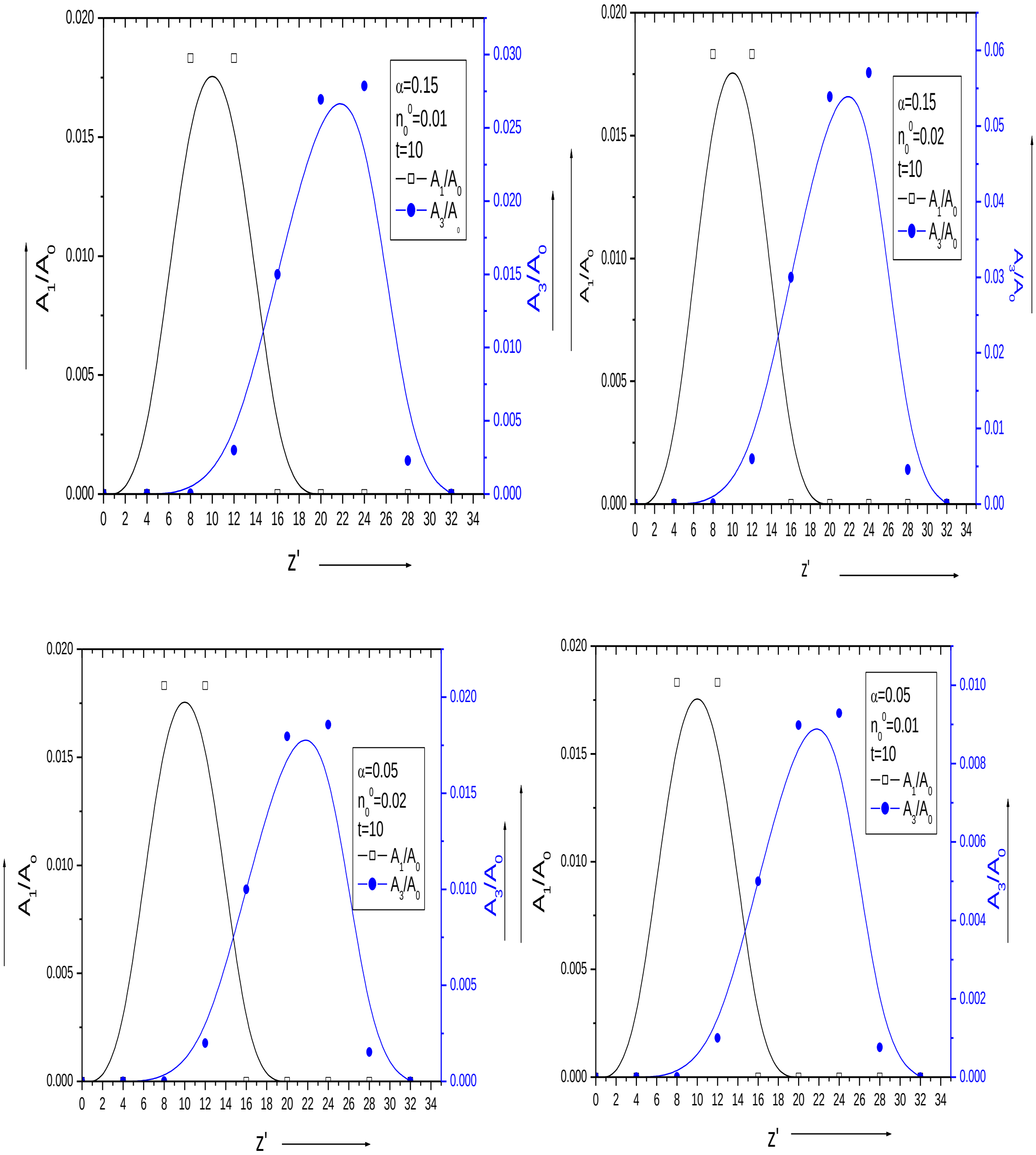
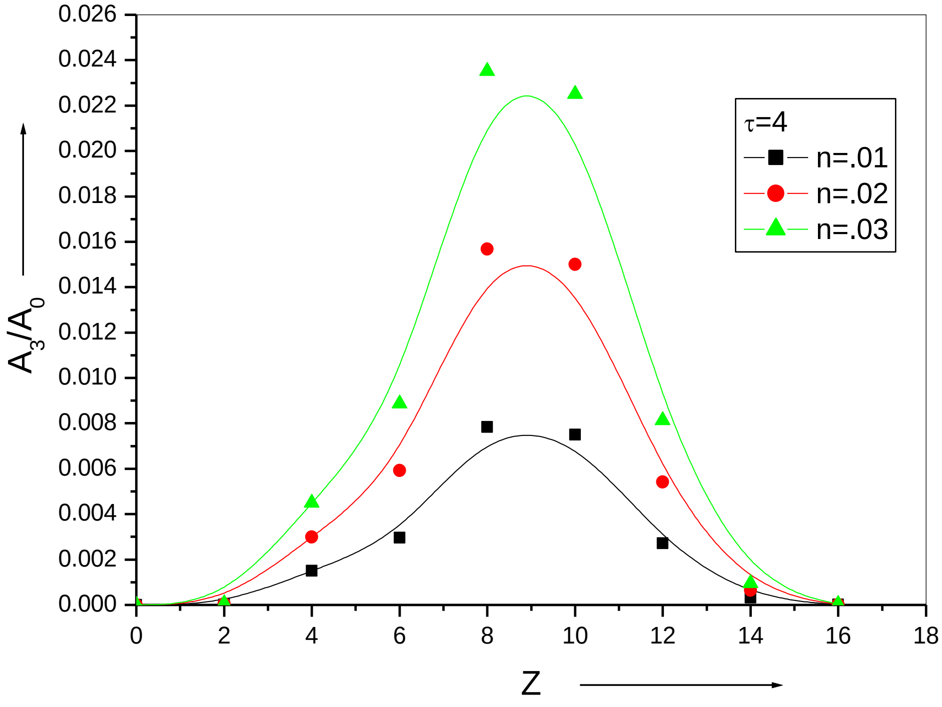
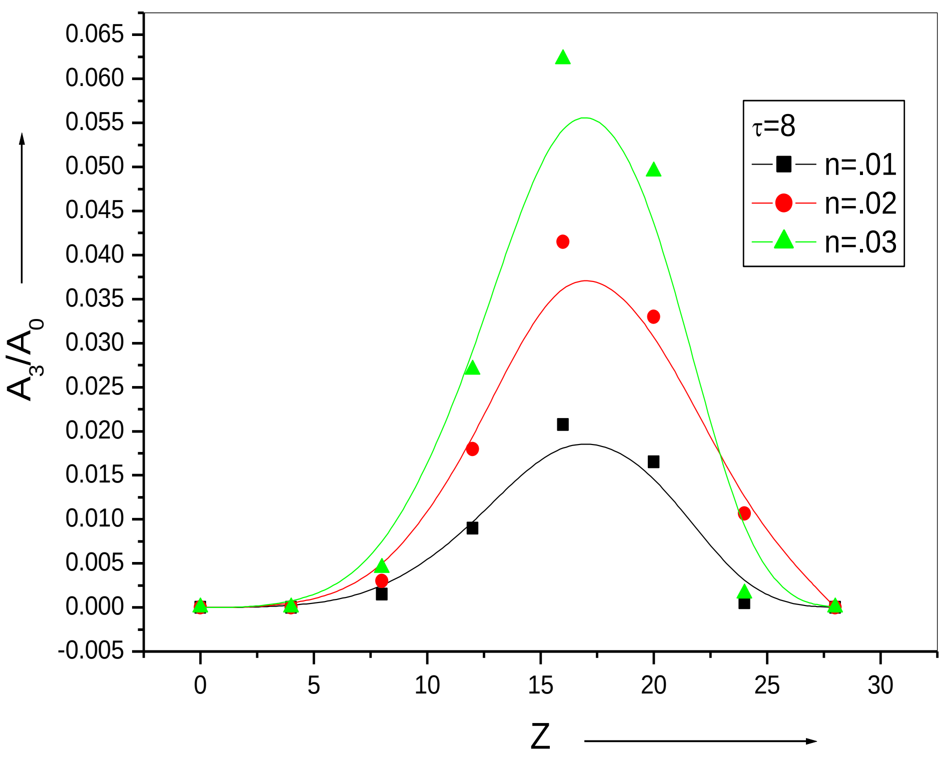


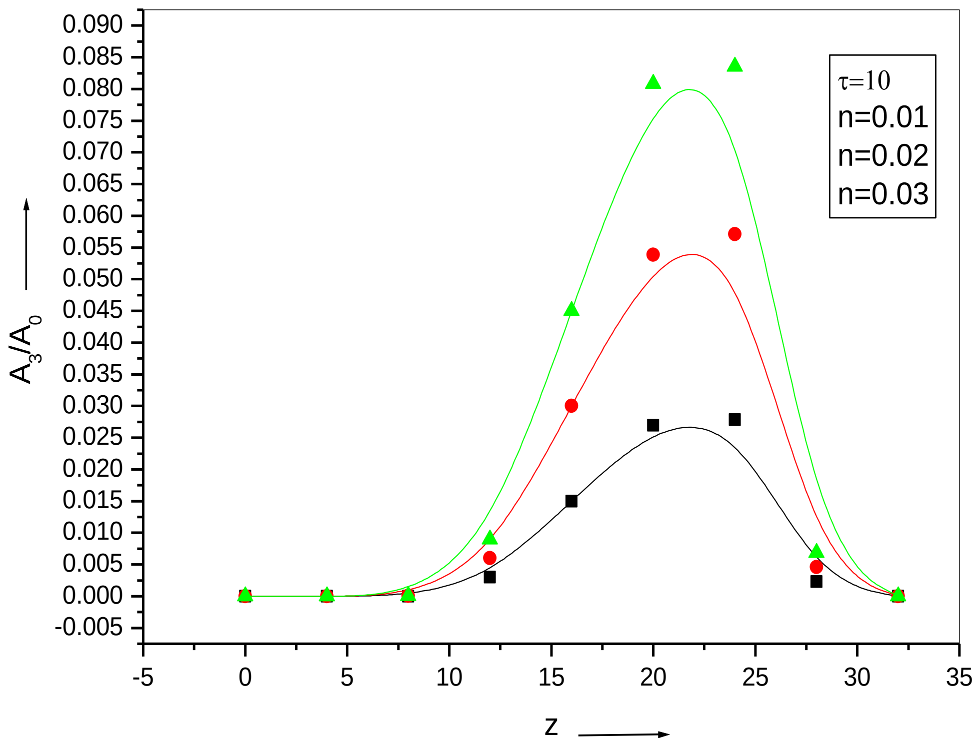
Fig. 3.3. Variation of normalized amplitude of fundamental and third harmonic with normalized propagation distance at α =0.15 for n_0^0 =0.01 and 0.02 at $t = 10$



4.(a)



4.(b)



4. (c)

Fig.3.4 Comparison of variation of normalized amplitude of third harmonic pulse with normalized propagation distance at different values of n_0^0 =.01, .02, .03

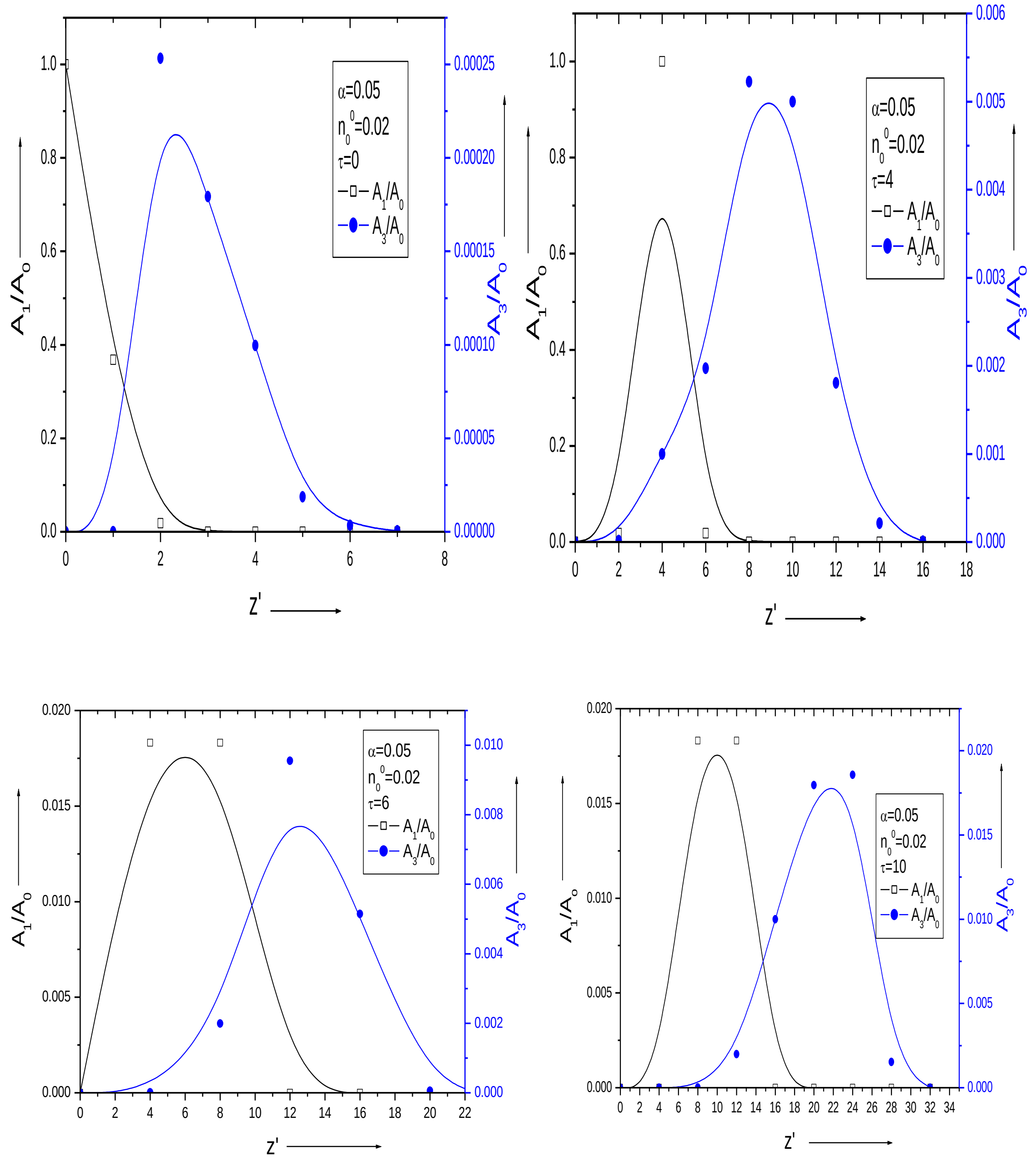


Fig. 3.5 Variation of normalized amplitude of fundamental and third harmonic pulses with normalized propagation distance at $\alpha = 0.05$ and $n_0^0 = 0.02$

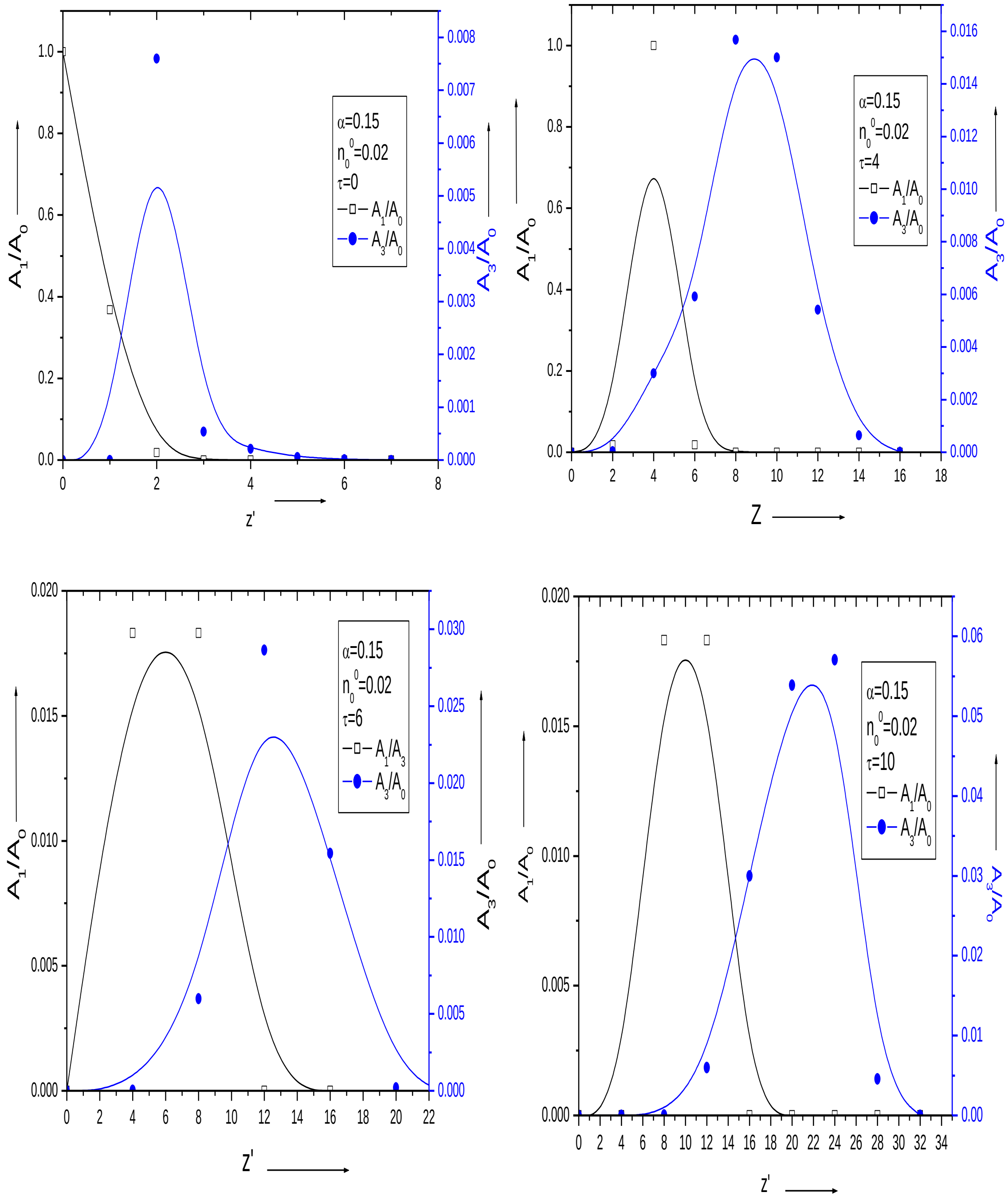


Fig. 3.6 Variation of ratio of normalized amplitude fundamental and third harmonic pulses with normalized propagation distance at $\alpha = 0.15$ & $n_0^0 = 0.02$ at $t = 0, 4,6,10$.

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Chapter-4

Third harmonic generation of a relativistic self-focusing laser in plasma

4.1 Introduction

In present study we are investigating the effect of relativistic self-focusing on third harmonic generation in plasma. An intense Gaussian laser beam propagating through a plasma channel gets self focused due to relativistic nonlinearity and the third harmonic pulse is generated through the beating of ponderomotive force-induced second harmonic density oscillations with the quiver velocity of electron at the fundamental frequency. The relativistic mass effect and the magnetic field contribute to the nonlinear dielectric response of plasma and create a plasma channel due to the relativistic self-focusing. Enhancement in the intensity of the third harmonic generation is observed on account of ponderomotive and relativistic self-focusing of the fundamental wave in the presence of wiggler magnetic field. Higher efficiency of third harmonic is reported in the region where self-focusing dominates and wiggler magnetic field provides the phase matching condition.

With the invent of short laser pulse, the interaction of laser with plasma becomes an important area of research in last few decades. The nonlinear interaction of highly intense laser beams with plasma is gaining importance due to its wide range of novel applications. These include laser driven plasma based particle accelerators [1-3], laser plasma channelling [4-7], super continuum generation [8-9], inertial confinement fusion [10-12], higher harmonic generation [13-17], nonlinear optics in extreme ultra violet region [18], photoelectron spectroscopy [19], and opacity measurements of high density matter [20-21].

Strong electric field of intense laser propagating through plasma results ponderomotive force on electrons in plasma forced them to oscillate with relativistic velocity results number of high order harmonic generations and due to its wide range of application the third harmonic has its unique place in the research related to laser plasma interaction. Esarey *et al.* [22] analyzed the relativistic harmonic generation by intense lasers in plasma. A nonlinear cold fluid model, valid for ultrahigh intensities, is formulated and used to analyze the current driven harmonic generation. Kant *et al.[*23] had studied the resonant second harmonic generation of a short pulse laser in a plasma channel and analyzed the generation of second harmonic in plasma in the presence of wiggler magnetic field, where wiggler magnetic field provide the phase matching condition result enhancement of the intensity of second harmonic wave. If the laser intensity is quite high enough then the plasma electron motion can be at relativistic level. They considered the case where the relativistic effects are dominant in laser-plasma interactions. Prashar [24] studied the effect of self-focusing in laser third harmonic generation in a clustered gas. In their work the efficiency of the harmonic generation is sensitive to the focusing of the fundamental laser beam where self-focusing enhances the efficiency by ten times. Singh *et al.* [25] studied the relativistic third harmonic generation of a laser in a self sustained magnetized plasma. They reported the significant effect of the self-sustained magnetized plasma channel on the efficiency of third-harmonic generation of the laser beam. Singh *et al.* [26] studied the effect of laser- plasma channelling on third harmonic radiation generation where an intense Gaussian laser beam, propagating through a magnetized plasma, becomes self-focused due to the ponderomotive force on the electrons. Their results show that the efficiency of third-harmonic generation of the laser beam is affected significantly due to the self-sustained plasma channel. The strength of magnetic field plays a crucial role in efficiency enhancement of third-harmonic generation. Kant *et al.* [27] observed the resonant third-harmonic generation of a short pulse laser from electron hole plasmas in the presence of wiggler magnetic field. They observed that for a specific wiggler wave number value, the phase matching conditions for the process are satisfied, leading to resonant enhancement in energy conversion efficiency.

In present manuscript effect of ponderomotive and relativistic self-focusing on the harmonic generation of third order of Gaussian laser beam in a plasma channel is analyzed. The phase matching condition is provided by wiggler magnetic field and the relativistic self-focusing of the pump beam helps in enhancing the intensity of the third harmonic pulse. Using paraxial approximarion we have obtained the expressions for beam width parameter of the third harmonic pulse and for normalized amplitude of third harmonic pulse. Results are given in graphical form.

4.2 Theoretical Considerations

In plasma Maxwell fourth equation is written as

$$\nabla^2 \vec{E}_3 = \frac{4\pi}{c^2}\frac{\partial \vec{J}_3}{\partial t} + \frac{\varepsilon}{c^2}\frac{\partial^2 \vec{E}_3}{\partial t^2},$$

where $\vec{J}_3 = \vec{J}_3^{L} + \vec{J}_3^{NL}$, and $\vec{J}_3^{L}$ [28] is the linear current density due to self-consistent field $\vec{E}_3$ where

$\varepsilon = \varepsilon_0 + \phi\left(E_1 E_1^*\right)$

$$\nabla^2 \vec{E}_3 = \left[\frac{\varepsilon_0 + \phi\left(E_1 E_1^*\right)}{c^2}\right]\frac{\partial^2 E}{\partial t^2} + \frac{4\pi \partial \vec{J}^{L}}{c^2 \partial t} + \frac{4\pi \partial \vec{J}_3^{NL}}{c^2 \partial t} \quad , \tag{4.1}$$

where $\varepsilon_0 = \left(1 - \omega_P^2 / \omega_1^2\right)$, $\omega_P^2 = 4\pi n_o e^2 / m$ and

$$\phi\left(\vec{E}_1 \vec{E}_1^{\ *}\right) = \frac{A_{10}^2 r^2}{2 r_0^2 f_1^4}\phi\frac{A_{10}^2}{2 f_1^2} \qquad , \tag{4.2}$$

$$\vec{J}_3^{L} = -\frac{n_0 e^2 \vec{E}_3}{m 3 i \omega_1} = \frac{-n_0 e^2 A_3 \hat{x} \exp(-i k_3 S_3) \exp[-i(3\omega_1 t - k_3 z)]}{m 3 i \omega_1}, \tag{4.3}$$

$$\vec{J}_3^{NL} = \frac{-n_0 e^5 B_w k_1 \vec{E}_1^3}{16 c i m^4 \omega_1^4 \left(\omega_1 + i v\right)}\left[\frac{5 k_1}{18 \omega_1} + \frac{k_1 + k_0}{\omega_1 + i v}\right]\hat{x} \ . \tag{4.4}$$

The beam width parameter [29] for fundamental laser is given as

$$\frac{d^2 f}{dz^2} - \frac{1}{R_d^2 f^3} + \frac{A_{10}^2}{\in_0 r_0^2 f^3}\phi'\left(\frac{A_{10}^2}{2 f^2}\right) = 0 . \tag{4.4a}$$

Complimentary solution of (4.1) is

$$\vec{E}_3 = x A\left(z,t\right)\exp\left[-i\left(3\omega_1 t - kz\right)\right]\exp\left(-i k_3 s\right), \tag{4.5}$$

$$\frac{\partial \vec{E}_3}{\partial z} = \hat{x}A_3(ik_3\hat{)}\exp(-ik_3S_3)\exp[-i(3\omega_1 t - k_3 z)] + \hat{x}A_3(-ik_3)\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial z} +$$
$$\hat{x}\exp(-ik_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial z}$$

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} = \hat{x}A_3(ik_3)^2\exp(-ik_3S_3)\exp[-i(3\omega_1 t - k_3 z)] + \hat{x}A_3(-ik_3)(ik_3)\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial z} +$$
$$\hat{x}(ik_3)\exp(-ik_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial z} + \hat{x}A_3(-ik_3)(ik_3)\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial z} +$$
$$\hat{x}A_3(-ik_3)(-ik_3)\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\left(\frac{\partial S_3}{\partial z}\right)^2 + (-ik_3)\hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z} +$$
$$\hat{x}A_3(-ik_3)\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial^2 S_3}{\partial z^2} + \hat{x}(ik_3)\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial z} +$$
$$\hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial^2 A_3}{\partial z^2} + \hat{x}(-ik_3)\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z},$$

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} = (ik_3)^2\vec{E}_3 + (-ik_3)(ik_3)\vec{E}_3\frac{\partial S_3}{\partial z} + (ik_3)\hat{x}\exp(-ik_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial z}$$
$$+ k_3^2\vec{E}_3\frac{\partial S_3}{\partial z} - k_3^2\vec{E}_3\left(\frac{\partial S_3}{\partial z}\right)^2 + 2(-ik_3)\hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z} +$$
$$(-ik_3)\frac{\vec{E}_3\partial^2 S_3}{\partial z^2} + (ik_3)\hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial z} +$$
$$\hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial^2 A_3}{\partial z^2},$$

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} = -k_3^{\ 2}\vec{E}_3 + 2k_3^{\ 2}\vec{E}_3\frac{\partial S_3}{\partial z} + (ik_3)\hat{x}\exp(-ik_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial z} -$$
$$k_3^2\vec{E}_3\left(\frac{\partial S_3}{\partial z}\right)^2 + 2(-ik_3)\hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z} +$$
$$ik_3\vec{E}_3\frac{\partial^2 S_3}{\partial z^2} + (ik_3)\hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial z} +$$
$$\hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial^2 A_3}{\partial z^2},$$

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} = -k_3^{\ 2}\vec{E}_3 + 2k_3^{\ 2}\vec{E}_3\frac{\partial S_3}{\partial z} + 2(ik_3)\hat{x}\exp(-ik_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial z} -$$
$$k_3^2\vec{E}_3\left(\frac{\partial S_3}{\partial z}\right)^2 + 2(-ik_3)\hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z} + \tag{4.6}$$
$$-ik_3\vec{E}_3\frac{\partial^2 S_3}{\partial z^2} + \hat{x}\exp(-ik_3S)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial^2 A_3}{\partial z^2},$$

$$\frac{\partial \vec{E}_3}{\partial r} = (-ik_3)\hat{x}A_3 \exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial r}$$
$$+ \hat{x}\exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial r}, \tag{4.7}$$

$$\frac{\partial^2 \vec{E}_3}{\partial r^2} = (-ik_3)^2 \hat{x}A_3 \exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\left(\frac{\partial S_3}{\partial r}\right)^2 +$$
$$(-ik_3)\hat{x}\exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial r}\frac{\partial A_3}{\partial r}$$

$$(-ik_3)\hat{x}A_3 \exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial^2 S_3}{\partial r^2} + (-ik_3)\hat{x}\exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial r}\frac{\partial A_3}{\partial r}$$
$$\hat{x}\exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial^2 A_3}{\partial r^2},$$
$$\frac{\partial^2 \vec{E}_3}{\partial r^2} = -k_3{}^2 \vec{E}_3\left(\frac{\partial S_3}{\partial r}\right)^2 + -2ik_3\hat{x}\exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial S_3}{\partial r}\frac{\partial A_3}{\partial r}$$
$$(-ik_3)\vec{E}_3\frac{\partial^2 S_3}{\partial r^2} - \hat{x}\exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial^2 A_3}{\partial r^2}. \tag{4.8}$$

Using Eq. (4.3) we obtain

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = -\frac{4\pi}{c^2}\frac{n_0 e^2 \hat{x}A_3(-3i\omega_1)\exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]}{m3i\omega_1},$$
$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \frac{4\pi}{mc^2}\frac{n_0 e^2 \vec{E}_3}{},$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^L}{\partial t} = \frac{\omega_p^2}{c^2}\vec{E}_3, \tag{4.9}$$

now

$$\frac{\partial \vec{E}_3}{\partial t} = \frac{\partial}{\partial t}\left[\{\hat{x}A_3 \exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\}\right],$$
$$\frac{\partial \vec{E}_3}{\partial t} = (-3i\omega_1)\{\hat{x}A_3 \exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\},$$
$$\frac{\partial^2 \vec{E}_3}{\partial t^2} = \frac{(-3i\omega_1)}{}\frac{\partial}{\partial t}\{\hat{x}A_3 \exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\},$$
$$\frac{\partial^2 \vec{E}_3}{\partial t^2} = 9\omega_1^2\{\hat{x}A_3 \exp(-ik_3 S_3)\exp[-i(3\omega_1 t - k_3 z)]\},$$

$$\partial^2 \vec{E}_3 \big/ \partial t^2 = -9\omega_1{}^2 \vec{E}_3 \ . \tag{4.10}$$

Taking first term on R.H.S. of Eq. (4.1), putting $\varepsilon_0 = \left(1 - \omega_P^2 / \omega_1^2\right)$and using Eq. (4.10) we obtain

$$\left[\frac{\varepsilon_0 + \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)}{c^2}\right]\frac{\partial^2 \vec{E}_3}{\partial t^2} = \frac{1}{c^2}\left(1 - \frac{\omega_p^2}{\omega_1^2}\right)\frac{\partial^2 \vec{E}_3}{\partial t^2} + \frac{\phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)}{c^2}\frac{\partial^2 \vec{E}_3}{\partial t^2} \ ,$$

$$\left[\frac{\varepsilon_0 + \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)}{c^2}\right]\frac{\partial^2 \vec{E}_3}{\partial t^2} = \frac{1}{c^2}\left(1 - \frac{\omega_p^2}{\omega_1^2}\right)\left(-9\omega_1{}^2 \vec{E}_3\right) + \frac{\phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)}{c^2}\left(-9\omega_1{}^2 \vec{E}_3\right),$$

$$\left[\frac{\varepsilon_0 + \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)}{c^2}\right]\frac{\partial^2 \vec{E}_3}{\partial t^2} = \frac{-9\omega_1{}^2 \vec{E}_3}{c^2} + \frac{9\omega_p^2 \vec{E}_3}{c^2} - \frac{9\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)\vec{E}_3}{c^2} \ ,$$

$$\left[\frac{\varepsilon_0 + \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)}{c^2}\right]\frac{\partial^2 \vec{E}_3}{\partial t^2} = \frac{-9\omega_1{}^2 \vec{E}_3}{c^2} + \frac{9\omega_p^2 \vec{E}_3}{c^2} - \frac{9\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)\vec{E}_3}{c^2} \ . \tag{4.11}$$

Using Eqs. (4.9) and (4.11)

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^{\,L}}{\partial t} + \left[\frac{\varepsilon_0 + \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)}{c^2}\right]\frac{\partial^2 \vec{E}_3}{\partial t^2} = \frac{\omega_p^2}{c^2}\,\vec{E}_3 \ \frac{-9\omega_1{}^2 \vec{E}_3}{c^2} + \frac{9\omega_p^2 \vec{E}_3}{c^2} - \frac{9\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}_1^{*}\right)\vec{E}_3}{c^2} \ ,$$

$$\frac{4\pi}{c^2}\frac{\partial \vec{J}_3^{\,L}}{\partial t} + \left[\frac{\varepsilon_0 + \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)}{c^2}\right]\frac{\partial^2 \vec{E}_3}{\partial t^2} = \frac{10\omega_p^2}{c^2}\,\vec{E}_3 \ \frac{-9\omega_1{}^2 \vec{E}_3}{c^2} - \frac{9\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)\vec{E}_3}{c^2} \ , \tag{4.12}$$

putting Eq. (4.12) in Eq. (4.1) we obtain

$$\nabla^2 \vec{E}_3 = \frac{10\omega_p^2}{c^2}\,\vec{E}_3 \ \frac{-9\omega_1{}^2 \vec{E}_3}{c^2} - \frac{9\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)\vec{E}_3}{c^2} + \frac{4\pi \partial \vec{J}_3^{\,NL}}{c^2 \partial t} \ ,$$

$$\nabla^2 \vec{E}_3 + \left[\left(\frac{9\omega_1{}^2 - 10\omega_p^2}{c^2}\right) + \frac{9\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}^{\,*}_{\,1}\right)}{c^2}\right]\vec{E}_3 = \frac{4\pi \partial \vec{J}_3^{\,NL}}{c^2 \partial t} \ . \tag{4.13}$$

From Eq. (4.4)

$$\vec{J}_3^{NL} = \frac{-n_0 e^5 B_w k_1 \vec{E}_1^3}{16 c i m^4 \omega_1^4 (\omega_1 + i\nu)} \left[\frac{5k_1}{18\omega_1} + \frac{k_1 + k_0}{\omega_1 + i\nu} \right] \hat{x} \,, \tag{4.14}$$

$$\vec{E}_1^3 = A_1^3 \hat{x} \exp(-3ik_1 S_1) \exp[-i3(\omega_1 t - k_1 z)]$$

$$\frac{\partial \vec{E}_1^3}{\partial t} = (-3i\omega_1) \hat{x} A_1^3 \exp(-3ik_1 S_1) \exp[-i3(\omega_1 t - k_1 z)]$$

$$\frac{4\pi}{c^2} \frac{\partial \vec{J}_3^{NL}}{\partial t} = \frac{-12\pi i \omega_1}{c^2} \left(\frac{-n_0 e^5 B_w k_1}{16 c i m^4 \omega_1^4 (\omega_1 + i\nu)} \right) \left[\frac{5k_1}{18\omega_1} + \frac{k_1 + k_0}{\omega_1 + i\nu} \right] \vec{E}_1^3 . \tag{4.15}$$

Putting Eq. (4.15) in Eq. (4.13)

$$\nabla^2 \vec{E}_3 + \left[\left(\frac{9\omega_1^2 - 10\omega_p^2}{c^2} \right) + \frac{9\omega_1^2 \phi(\vec{E}_1 \vec{E}_1^*)}{c^2} \right] \vec{E}_3 = \frac{-12\pi i \omega_1}{c^2} \left(\frac{-n_0 e^5 B_w k_1}{16 c i m^4 \omega_1^4 (\omega_1 + i\nu)} \right) \left[\frac{5k_1}{18\omega_1} + \frac{k_1 + k_0}{\omega_1 + i\nu} \right] \vec{E}_1^3 ,$$

therefore,

$$\left[\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} \right) \vec{E}_3 + \left(\frac{9\omega_1^2 - 10\omega_p^2}{c^2} \right) + \frac{9\omega_1^2 \phi(\vec{E}_1 \vec{E}_1^*)}{c^2} \right] \vec{E}_3$$

$$= \frac{-12\pi i \omega_1}{c^2} \left(\frac{-n_0 e^5 B_w k_1}{16 c i m^4 \omega_1^4 (\omega_1 + i\nu)} \right) \left[\frac{5k_1}{18\omega_1} + \frac{k_1 + k_0}{\omega_1 + i\nu} \right] \vec{E}_1^3 . \tag{4.16}$$

Using Eqs. (4.6), (4.7), (4.8), Eq. (4.16) becomes

$$-k_3{}^2\vec{E}_3+2k_3{}^2\vec{E}_3\frac{\partial S_3}{\partial z}+2(ik_3)\hat{x}\exp(-ik_3)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial A_3}{\partial z}+$$

$$k_3^2\vec{E}_3\left(\frac{\partial S_3}{\partial z}\right)^2+2(-ik_3)\hat{x}\exp(-ik_3 S)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z}+$$

$$-ik_3\vec{E}_3\frac{\partial^2 S_3}{\partial z^2}+\hat{x}\exp(-ik_3 S)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial^2 A_3}{\partial z^2}-k_3{}^2\vec{E}_3\left(\frac{\partial S_3}{\partial r}\right)^2+$$

$$-2ik_3\hat{x}\exp(-ik_3 S_3)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial S_3}{\partial r}\frac{\partial A_3}{\partial r}+(-ik_3)\vec{E}_3\frac{\partial^2 S_3}{\partial r^2}$$

$$-\hat{x}\exp(-ik_3 S_3)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial^2 A_3}{\partial r^2}+\frac{(-ik_3)\vec{E}_3}{r}\frac{\partial S_3}{\partial r}$$

$$+\hat{x}\exp(-ik_3 S_3)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial A_3}{r\partial r}+\left[\left(\frac{9\omega_1{}^2-10\omega_p^2}{c^2}\right)+\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]\vec{E}_3 \tag{4.17}$$

$$=\frac{-12\pi i\omega_1}{c^2}\left(\frac{-n_{,0}e' B_, k_1}{16cim^4\omega_1^4(\omega_1+i\nu)}\right)\left[\frac{5k_1}{18\omega_1}+\frac{k_1+k_0}{\omega_1+i\nu}\right]\vec{E}_1^3.$$

Taking real Part

$$-k_3{}^2\vec{E}_3+2k_3{}^2\vec{E}_3\frac{\partial S_3}{\partial z}+k_3^2\vec{E}_3\left(\frac{\partial S_3}{\partial z}\right)^2+\frac{\vec{E}_3}{A_3}\frac{\partial^2 A_3}{\partial z^2}-k_3{}^2\vec{E}_3\left(\frac{\partial S_3}{\partial r}\right)^2-\frac{\vec{E}_3}{A_3}\frac{\partial^2 A_3}{\partial r^2}$$

$$\frac{\vec{E}_3}{rA_3}\frac{\partial A_3}{\partial r}+\left[\left(\frac{9\omega_1{}^2-10\omega_p^2}{c^2}\right)+\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]\vec{E}_3=0.$$

Divide by $k_3^2\vec{E}_3$, we obtain

$$-1+2\frac{\partial S_3}{\partial z}+\left(\frac{\partial S_3}{\partial z}\right)^2+\frac{}{k_3{}^2A_3}\frac{\partial^2 A_3}{\partial z^2}-\left(\frac{\partial S_3}{\partial r}\right)^2-\frac{}{k_3^2A_3}\frac{\partial^2 A_3}{\partial r^2}$$

$$\frac{1}{rk_3{}^2A_3}\frac{\partial A_3}{\partial r}+\frac{1}{k_3{}^2}\left[\left(\frac{9\omega_1{}^2-10\omega_p^2}{c^2}\right)+\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]=0$$

$$2\frac{\partial S_3}{\partial z}+\left(\frac{\partial S_3}{\partial z}\right)^2+\frac{}{k_3{}^2A_3}\frac{\partial^2 A_3}{\partial z^2}-\left(\frac{\partial S_3}{\partial r}\right)^2$$

$$-\frac{}{k_3^2A_3}\left[\frac{\partial^2 A_3}{\partial r^2}-\frac{1}{r}\frac{\partial A_3}{\partial r}\right]+\frac{1}{k_3{}^2}\left[\left(\frac{9\omega_1{}^2-10\omega_p^2}{c^2}\right)+\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_{\,1}\right)}{c^2}-1\right]=0\,\cdot \qquad (4.18a)$$

Taking imaginary part of Eq. (4.17)

$$2(ik_3)\hat{x}\exp(-ik_3)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial A_3}{\partial z}+2(-ik_3)\hat{x}\exp(-ik_3 S)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z}+$$

$$-ik_3\vec{E}_3\frac{\partial^2 S_3}{\partial z^2}+-2ik_3\hat{x}\exp(-ik_3S_3)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial S_3}{\partial r}\frac{\partial A_3}{\partial r}$$

$$(-ik_3)\vec{E}_3\frac{\partial^2 S_3}{\partial r^2}-\hat{x}\exp(-ik_3S_3)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial^2 A_3}{\partial r^2}+(-ik_3)\vec{E}_3\frac{\partial S_3}{r\partial r}=0,$$

neglecting the small term containing $\partial^2 A_3/\partial r^2$

$\hat{x}\exp(-ik_3S_3)\exp[-i(3\omega_1 t-k_3 z)]\partial^2 A_3/\partial r^2$, we obtain

$$2(ik_3)\hat{x}\exp(-ik_3)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial A_3}{\partial z}+2(-ik_3)\hat{x}\exp(-ik_3 S)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z}+$$

$$-ik_3\vec{E}_3\frac{\partial^2 S_3}{\partial z^2}-2ik_3\hat{x}\exp(-ik_3S_3)\exp[-i(3\omega_1 t-k_3 z)]\frac{\partial S_3}{\partial r}\frac{\partial A_3}{\partial r}$$

$$(-ik_3)\vec{E}_3\frac{\partial^2 S_3}{\partial r^2}+(-ik_3)\vec{E}_3\frac{\partial S_3}{r\partial r}=0,$$

$$2(ik_3)\frac{\vec{E}_3}{A_3}\frac{\partial A_3}{\partial z}-2\frac{\vec{E}_3}{A_3}\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z}-ik_3\vec{E}_3\frac{\partial^2 S_3}{\partial z^2}-2ik_3\frac{\vec{E}_3}{A_3}\frac{\partial S_3}{\partial r}\frac{\partial A_3}{\partial r}$$

$$(-ik_3)\vec{E}_3\frac{\partial^2 S_3}{\partial r^2}+(-ik_3)\vec{E}_3\frac{\partial S_3}{r\partial r}=0,$$

$$2A_3\frac{\partial A_3}{\partial z}-2A_3\frac{\partial S_3}{\partial z}\frac{\partial A_3}{\partial z}-A_3^2\frac{\partial^2 S_3}{\partial z^2}-2A_3\frac{\partial S_3}{\partial r}\frac{\partial A_3}{\partial r}-A_3^2\left[\frac{\partial^2 S_3}{\partial r^2}+\frac{\partial S_3}{r\partial r}\right]=0\,,$$

$$\frac{\partial A_3^2}{\partial z}-\frac{\partial S_3}{\partial z}\frac{\partial A_3^2}{\partial z}-A_3^2\frac{\partial^2 S_3}{\partial z^2}-\frac{\partial S_3}{\partial r}\frac{\partial A_3^2}{\partial r}-A_3^2\left[\frac{\partial^2 S_3}{\partial r^2}+\frac{\partial S_3}{r\partial r}\right]=0, \qquad (4.18b)$$

$$A_3^2=\frac{A_{30}^2}{f_3^2}\exp\left[\frac{-3r^2}{2r_0^2 f_3^2}\right], \qquad (4.19)$$

$$S_3 = \frac{\beta(z) r^2}{2} + \phi(z),$$

$$\frac{\partial A_3^2}{\partial z} = \frac{-2A_{30}^2}{f_3^3}\frac{\partial f_3}{\partial z}\exp\left[\frac{-3r^2}{2r_0^2 f_3^2}\right] + \frac{2A_{30}}{f_3^2}\frac{\partial A_{30}}{\partial z}\exp\left[\frac{-3r^2}{2r_0^2 f_3^2}\right]$$
$$+\frac{A_{30}^2}{f_3^2}\left(\frac{-2}{f_3^2}\right)\left(\frac{-3r^2}{2r_0^2}\right)\exp\left[\frac{-3r^2}{2r_0^2 f_3^2}\right]\frac{\partial f_3}{\partial z},$$

$$\frac{\partial A_3^2}{\partial z} = \frac{-2A_3^2}{f_3^2}\frac{\partial f_3}{\partial z} + \frac{2A_3^2}{A_{30}}\frac{\partial A_{30}}{\partial z} + \frac{3r^2 A_3^2}{r_0^2 f_3^4}\frac{\partial f_3}{\partial z}, \qquad (4.20a)$$

$$S_3 = \frac{\beta(z) r^2}{2} + \phi(z) \quad \text{and} \quad \frac{\partial S_3}{\partial z} = \frac{r^2}{2}\frac{\partial \beta(z)}{\partial z} + \frac{\partial \phi_3(z)}{\partial z}, \qquad (4.20b)$$

$$\frac{\partial^2 S_3}{\partial z^2} = \frac{r^2}{2}\frac{\partial^2 \beta(z)}{\partial z^2} + \frac{\partial^2 \phi_3(z)}{\partial z^2}, \qquad (4.20c)$$

$$\frac{\partial A_3^2}{\partial r} = \frac{A_{30}^2}{f_3^2}\left(\frac{-3r}{f_3^2 r_0^2}\right)\exp\left[\frac{-3r^2}{2r_0^2 f_3^2}\right],$$
$$\frac{\partial A_3^2}{\partial r} = \left(\frac{-3r}{r_0^2}\right)\frac{A_{30}^2}{f_3^4}, \qquad (4.20d)$$

$$\frac{\partial S_3}{\partial r} = r\beta(z), \qquad (4.20e)$$

$$\frac{\partial^2 S_3}{\partial r^2} = \beta(z), \qquad (4.20f)$$

$$A_3 = \frac{A_{30}}{f_3}\exp\left[\frac{-3r^2}{4r_0^2 f_3^2}\right],$$
$$\frac{\partial A_3}{\partial r} = \frac{A_{30}}{f_3}\left[\frac{-3r}{2r_0^2 f_3^2}\right]\exp\left[\frac{-3r^2}{4r_0^2 f_3^2}\right],$$

$$\frac{\partial A_3}{\partial r} = \left[\frac{-3rA_3}{2r_0^2 f_3^3}\right], \qquad (4.20g)$$

$$\frac{\partial^2 A_3}{\partial r^2} = \frac{-3A_{30}}{2r_0^2 f_3^3}\exp\left[\frac{-3r^2}{4r_0^2 f_3^2}\right] + \frac{A_{30}}{f_3^3}\left[\frac{-3r}{r_0^2}\right]\left[\frac{-3r}{2r_0^2 f_3^2}\right]\exp\left[\frac{-3r^2}{4r_0^2 f_3^2}\right],$$

$$\frac{\partial^2 A_3}{\partial r^2} = \frac{-3A_{30}}{2r_0^2 f_3^3}\exp\left[\frac{-3r^2}{4r_0^2 f_3^2}\right] + \frac{A_{30}}{f_3^5}\left[\frac{9r^2}{2r_0^4}\right]\exp\left[\frac{-3r^2}{4r_0^2 f_3^2}\right],$$

$$\frac{\partial^2 A_3}{\partial r^2} = \frac{-3A_3}{2r_0^2 f_3^3} + \frac{9A_3}{2f_3^5}\left[\frac{r^2}{r_0^4}\right] \quad ,$$

$$\frac{\partial^2 A_3}{\partial r^2} = \frac{9A_3}{2f_3^5}\left[\frac{r^2}{r_0^4}\right] - \frac{3A_3}{2r_0^2 f_3^3}, \tag{4.20h}$$

$$\frac{\partial A_3^2}{\partial z} - \frac{\partial S_3}{\partial z}\frac{\partial A_3^2}{\partial z} - A_3^2\frac{\partial^2 S_3}{\partial z^2} - \frac{\partial S_3}{\partial r}\frac{\partial A_3^2}{\partial r} - A_3^2\left[\frac{\partial^2 S_3}{\partial r^2} + \frac{\partial S_3}{r\partial r}\right] = 0.$$

Using Eqs. 4.20(a), 4.20(d), 4.20(e), 4.20(f) in Eq. (4.18b), we obtain

$$\frac{-2A_3^2}{f_3^2}\frac{\partial f_3}{\partial z} + \frac{2A_3^2}{A_{30}}\frac{\partial A_{30}}{\partial z} + \frac{6r^2 A_3^2}{r_0^2 f_3^4}\frac{\partial f_3}{\partial z} - \frac{\partial S_3}{\partial z}\frac{\partial A_3^2}{\partial z} - A_3^2\frac{\partial^2 S_3}{\partial z^2} \; - \left(\frac{6r}{r_0^2}\right)\frac{A_3^2}{f_3^4}r\beta(z).$$

$$A_3^2\left[\beta(z) + \beta(z)\right], \tag{4.21}$$

equating the coefficient of ‘ r^2 ‘ from both sides

$$\frac{6A_3^2}{r_0^2 f_3^4}\left[\frac{\partial f_3}{\partial z} - f_3\beta(z)\right] = 0 \quad ,$$

$$\beta(z) = \frac{\partial f_3}{f_3 \partial Z} \quad , \tag{4.22}$$

$$\frac{\partial \beta(z)}{\partial z} = \frac{\partial^2 f_3}{f_3 \partial z^2} - \frac{1}{f_3^2}\left(\frac{\partial f_3}{\partial z}\right)^2, \tag{4.23}$$

putting Eqs. (4.2) (4.20b), (4.20e), (4.20g), (4.20h) into Eq. (4.18a)

$$2\left[\frac{r^2}{2}\frac{\partial\beta(z)}{\partial z} + \frac{\partial\phi_3(z)}{\partial z}\right] + \left(\frac{\partial S_3}{\partial z}\right)^2 + \frac{1}{k_3^2 A_3}\frac{\partial^2 A_3}{\partial z^2} - \left(r\beta(z)\right)^2$$

$$-\frac{1}{k_3^2 A_3}\left[\frac{9A_3}{2f_3^5}\left[\frac{r^2}{r_0^4}\right] - \frac{3A_3}{2r_0^2 f_3^3} + \frac{1}{r}\left[\frac{-3rA_3}{2r_0^2 f_3^3}\right]\right] + \frac{1}{k_3^2}\left[\begin{array}{c}\left(\dfrac{9\omega_1^2 - 10\omega_p^2}{c^2}\right) \\ + \dfrac{9\omega_1^2}{c^2}\dfrac{A_{10}^2 r^2}{2r_0^2 f_1^4}\phi\dfrac{A_{10}^2}{2f_1^2}\end{array}\right] - 1 = 0,$$

$$r^2\frac{\partial\beta(z)}{\partial z}+\frac{2\partial\phi_3(z)}{\partial z}+\left(\frac{\partial S_3}{\partial z}\right)^2+\frac{}{k_3{}^2A_3}\frac{\partial^2A_3}{\partial z^2}-r^2\beta^2(z)$$

$$-\frac{9A_3}{2k_3^2A_3f_3^5}\left[\frac{r^2}{r_0^4}\right]+\frac{3A_3}{2k_3^2A_3r_0^2f_3^3}+\frac{1}{2k_3^2A_3r}\left[\frac{3rA_3}{r_0^2f_3^3}\right]+\frac{1}{k_3{}^2}\left[\left(\frac{9\omega_1{}^2-10\omega_p^2}{c^2}\right)+\frac{9\omega_1{}^2}{c^2}\frac{A_{10}^2r^2}{2r_0^2f_1{}^4}\phi\frac{A_{10}^2}{2f_1^2}\right]-1=0,$$

$$r^2\frac{\partial\beta(z)}{\partial z}+\frac{2\partial\phi_3(z)}{\partial z}+\left(\frac{\partial S_3}{\partial z}\right)^2+\frac{}{k_3{}^2A_3}\frac{\partial^2A_3}{\partial z^2}-r^2\beta^2(z)-\frac{9}{2k_3^2f_3^5}\left[\frac{r^2}{r_0^4}\right]+$$

$$\frac{3}{2k_3^2r_0^2f_3^3}+\frac{1}{2k_3^2r}\left[\frac{3r}{r_0^2f_3^3}\right]+\frac{1}{k_3{}^2}\left[\left(\frac{9\omega_1{}^2-10\omega_p^2}{c^2}\right)+\frac{9\omega_1{}^2}{c^2}\frac{A_{10}^2r^2}{2r_0^2f_1{}^4}\phi\frac{A_{10}^2}{2f_1{}^2}\right]-1=0\ ,$$

equating coefficient of r^2 and putting Eqs. 4.22a) and (4.23), we obtain

$$\frac{\partial\beta(z)}{\partial z}-\beta^2(z)-\frac{9}{2k_3^2r_0^4f_3^5}+\frac{1}{k_3{}^2}\left[\frac{9\omega_1{}^2}{c^2}\frac{A_{10}^2r^2}{2r_0^2f_1{}^4}\phi\frac{A_{10}^2}{2f_1{}^2}\right]=0\,,$$

$$-\frac{\partial^2f_3}{f_3\partial z^2}+\frac{1}{f_3^2}\left(\frac{\partial f_3}{\partial z}\right)^2-\frac{1}{f_3^2}\left(\frac{\partial f_3}{\partial z}\right)^2-\frac{9}{2k_3^2r_0^4f_3^5}+\frac{1}{k_3{}^2}\left[\frac{9\omega_1{}^2}{c^2}\frac{A_{10}^2r^2}{2r_0^2f_1{}^4}\phi\frac{A_{10}^2}{2f_1^2}\right]=0\,,$$

$$\frac{\partial^2f_3}{f_3\partial z^2}=-\frac{9}{2k_3^2r_0^4f_3^5}+\frac{1}{k_3{}^2}\left[\frac{9\omega_1{}^2}{c^2}\frac{A_{10}^2}{2r_0^2f_1{}^4}\phi\left(\frac{A_{10}^2}{2f_1{}^2}\right)\right]=0\,. \qquad (4.24)$$

where

$$\phi\left[\frac{A_{10}^2}{2f_1^2}\right]=\frac{(\varepsilon_s\varepsilon_2)}{\varepsilon_0}\exp\left[\frac{-\varepsilon_2A_{10}^2}{2\varepsilon_0f_1^2}\right].$$

Particular integral of Eq.(4.16) is

$$\vec{E}_3=\hat{x}A_3^{'}\exp[-i\{3\omega_1t-(3k_1+k_0)z\}], \qquad (4.25)$$

$A_3^{'} = A_{30}^{'}(z)\psi_3$ Where $\psi_3 = \exp\left[-3r^2/2r_0^2 f_1^2\right]\exp\left(-3ik_1 S_1\right)$ (4.26)

$$A_3^{'} = A_{30}^{'}(z)\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp\left(-3ik_1 S_1\right), \tag{4.26a}$$

$$\frac{\partial \vec{E}_3}{\partial z} = \hat{x}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3^{'}}{\partial z} + A_3^{'}\hat{x}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\{i(3k_1 + k_0)\},$$

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} = \hat{x}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial^2 A_3^{'}}{\partial z^2} + \hat{x}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\{i(3k_1 + k_0)\}\frac{\partial A_3^{'}}{\partial z},$$
$$+ \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\{i(3k_1 + k_0)\}\frac{\partial A_3^{'}}{\partial z} + A_3^{'}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\{i(3k_1 + k_0)\}^2$$

$\partial^2 A_3^{'}/\partial z^2$ being small gets neglected

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} = 2\hat{x}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\{i(3k_1 + k_0)\}\frac{\partial A_3^{'}}{\partial z} + A_3^{'}\hat{x}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\{i(3k_1 + k_0)\}^2, \tag{4.27}$$

using Eq. (4.26)

$$\frac{\partial A_3^{'}}{\partial z} = \exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp\left(-3ik_1 S_1\right)\frac{\partial A_{30}^{'}}{\partial z} + A_{30}^{'}(z)\left(-3ik_1\right)\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp\left(-3ik_1\right)\frac{\partial S_1}{\partial z}$$
$$+ A_{30}^{'}\left[\frac{3r^2}{r_0^2 f_1^3}\right]\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp\left(-3ik_1 S_1\right)\frac{\partial f_1}{\partial z},$$

$$\frac{\partial A_3^{'}}{\partial z} = \frac{A_3^{'}}{A_{30}^{'}}\frac{\partial A_{30}^{'}}{\partial z} + A_3^{'}(z)\left(-3ik_1\right)\frac{\partial S_1}{\partial z} + A_3^{'}\left[\frac{3r^2}{r_0^2 f_1^3}\right]\frac{\partial f_1}{\partial z}, \tag{4.28}$$

using Eq. (4.28) in Eq. (4.27)

$$\frac{\partial^2 \vec{E}_3}{\partial z^2} = 2\hat{x}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\{i(3k_1 + k_0)\}\left[\begin{array}{l}\frac{A_3^{'}}{A_{30}^{'}}\frac{\partial A_{30}^{'}}{\partial z} + A_3^{'}(z)\left(-2ik_1\right)\frac{\partial S_1}{\partial z} \\ + A_3^{'}\left[\frac{3r^2}{r_0^2 f_1^3}\right]\frac{\partial f_1}{\partial z}\end{array}\right] \tag{4.29}$$
$$+ A_3^{'}\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\{i(3k_1 + k_0)\}^2,$$

using Eqs. (4.25) and (4.26)

$$\frac{\partial \vec{E}_3}{\partial r} = \hat{x}\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial A_3}{\partial r},$$

$$\frac{\partial \vec{E}_3}{\partial r} = \exp[-i(3\omega_1 t - k_3 z)] A_{30}^{'} \frac{\partial \psi_3}{\partial r}, \tag{4.30}$$

$$\frac{\partial^2 \vec{E}_3}{\partial r^2} = \hat{x} \exp[-i(3\omega_1 t - k_3 z)] \frac{A_{30}^{'} \partial^2 \psi_3}{\partial r^2}, \tag{4.31}$$

using Eqs. (4.29), (4.30) and (4.31) into Eq. (4.13), we obtain

$$2\psi_3 \{i(3k_1 + k_0)\} \frac{\partial A_{30}^{'}}{\partial z} + \left[\left(\frac{9\omega_1{}^2 - 10\omega_p^2}{c^2} \right) + \frac{9\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}_1^{\,*}\right)}{c^2} + \{i(3k_1 + k_0)\}^2 \right] A_{30}^{'} \psi_3$$

$$+ A_{30}^{'} \frac{\partial^2 \psi_3}{\partial r^2} + A_{30}^{'} \frac{\partial \psi_3}{r \partial r} = \frac{12\pi i n_0^0 e^5 B_w k_1 [5k + (k + k_0)]}{c^2 16 i m^4 \omega_1^4} \frac{\partial \vec{E}_1^3}{\partial t},$$

$$2\{i(3k_1 + k_0)\} \frac{\partial A_{30}^{'}}{\partial z} \int \psi_3 r \psi_3^* dr + \left[\left(\frac{9\omega_1{}^2 - 10\omega_p^2}{c^2} \right) + \frac{9\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}_1^{\,*}\right)}{c^2} + \{i(3k_1 + k_0)\}^2 \right] A_{30}^{'} \int \psi_3 r \psi_3^* dr \tag{4.32}$$

$$+ A_{30}^{'} \frac{\partial^2 \psi_3}{\partial r^2} \int r \psi_3^* dr + A_{30}^{'} \frac{\partial \psi_3}{r \partial r} \int r \psi_3^* dr = \frac{12\pi i n_0^0 e^5 B_w k_1 [5k + (k + k_0)]}{c^2 16 i m^4 \omega_1^4} \frac{\partial \vec{E}_1^3}{\partial t} \int r \psi_3^* dr.$$

Multiply ψ_3 by $r\psi_3^*$ and integrate with respect to ‘r’ we obtain

$$2\int \psi_3 r \psi_3^* dr = 2\int r \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] dr = 2\int \frac{r(r_0^2 f_1^2)}{-6r} \exp[x] dx = 2\int \frac{(r_0^2 f_1^2)}{-3} \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] dx$$

$$2\int \psi_3 r \psi_3^* dr = \frac{-2r_0^2 f_1^2}{3} \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right]. \tag{4.33}$$

Using Eq. (4.26) we obtain

$$\frac{\partial \psi_3}{\partial r} = \left(\frac{-3r}{r_0^2 f_1^2} \right) \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] \exp(-3ik_1 S_1) + (-3ik_1) \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] \exp(-3ik_1 S_1) \left(\frac{r\partial f}{f \partial z} \right), \tag{4.34}$$

$$\frac{\partial \psi_3}{r\partial r} = \left(\frac{1}{r} \right) \left(\frac{-3r}{r_0^2 f_1^2} \right) \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] \exp(-3ik_1 S_1) + (-3ik_1) \left(\frac{1}{r} \right) \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] \exp(-3ik_1 S_1) \left(\frac{r\partial f}{f \partial z} \right),$$

multiply above equation by $r\psi_3^*$ and integrate with respect to ‘r’

$$\int \frac{r\psi_3^* \partial \psi_3 \partial r}{r\partial r} = \int \left(\frac{-3r}{r_0^2 f_1^2} \right) \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] \partial r - \int \ (3ik_1) r \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] \left(\frac{r\partial f}{f\partial z} \right) \partial r \,,$$

$$\int \frac{r\psi_3^* \partial \psi_3 \partial r}{r\partial r} = \left(\frac{-3}{r_0^2 f_1^2} \right) \int \frac{r\left(r_0^2 f_1^2\right)}{(-3r)} \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] \partial x - (3ik_1) \left(\frac{\partial f}{f\partial z} \right) \int \left(r \frac{\left(r_0^2 f_1^2\right)}{(-3r)} \right) \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] \partial r,$$

$$\int \frac{r\psi_3^* \partial \psi_3 \partial r}{r\partial r} = \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] + ik_1 \left(\frac{\partial f}{f\partial z} \right) \left(\frac{\left(r_0^2 f_1^2\right)}{} \right) \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right],$$

$$\int \frac{r\psi_3^* \partial \psi_3 \partial r}{r\partial r} = \exp\left[\frac{-3r^2}{2r_0^2 f_1^2} \right] \left[1 + ik_1 \left(\frac{\partial f}{f\partial z} \right) \left(\frac{\left(r_0^2 f_1^2\right)}{} \right) \right]. \tag{4.35}$$

Differentiate Eq. (4.34) with respect to ‘r’, we obtain

$$\frac{\partial^2 \psi_3}{\partial r^2} = -\frac{3}{r_0^2 f_1^2} \exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1) + 9\left(\frac{r}{r_0^2 f_1^2} \right)^2 \exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1)$$

$$-\frac{3r}{r_0^2 f_1^2}(-3ik_1)\exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1)\frac{\partial S_1}{\partial r} + (-3ik_1)\left(\frac{-3r}{r_0^2 f_1^2} \right)\exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1)\frac{\partial S_1}{\partial r}$$

$$+(-3ik_1)^2 \exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1)\left(\frac{\partial S_1}{\partial r} \right)^2 + (-3ik_1)\exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1)\frac{\partial^2 S_1}{\partial r^2},$$

$$\frac{\partial^2 \psi_3}{\partial r^2} = -\frac{3}{r_0^2 f_1^2} \exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1) + \frac{9r^2}{\left(r_0^2 f_1^2\right)^2} \exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1)$$

$$+\frac{12ik_1 r}{r_0^2 f_1^2} \exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-2ik_1 S_1)\frac{\partial S_1}{\partial r} \tag{4.36}$$

$$-9k_1^{\ 2} \exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1)\left(\frac{\partial S_1}{\partial r} \right)^2 + (-3ik_1)\exp(\frac{-3r^2}{2r_0^2 f_1^2}) \exp(-3ik_1 S_1)\frac{\partial^2 S_1}{\partial r^2},$$

neglecting $\partial^2 S_1 / \partial r^2$ term and multiply Eq. (4.36) by $r\psi_3^*$ and integrate with respect to ‘r’

$$\int \frac{r\psi_3^* \partial^2 \psi_3}{\partial r^2} = -\int \frac{3r}{r_0^2 f_1^2} \exp\left(\frac{-3r^2}{r_0^2 f_1^2} \right) + \int \frac{9r^3}{\left(r_0^2 f_1^2\right)^2} \exp\left(\frac{-3r^2}{r_0^2 f_1^2} \right)$$

$$+\int \frac{12ik_1 r^3}{r_0^2 f_1^2} \exp\left(\frac{-3r^2}{r_0^2 f_1^2} \right) \left(\left(\frac{\partial f_3}{f_3 \partial z} \right) \right) - \int 9k_1^{\ 2} r^3 \exp\left(\frac{-3r^2}{r_0^2 f_1^2} \right) \left(\left(\frac{\partial f_3}{f_3 \partial z} \right) \right)^2,$$

$$\int \frac{r\psi_3^* \partial^2 \psi_3}{\partial r^2} = \exp\left(\frac{-3r^2}{r_0^2 f_1^2}\right) + 3\left[\left(\frac{-3r^2}{r_0^2 f_1^2}\right) + 1\right]\exp\left(\frac{-3r^2}{r_0^2 f_1^2}\right)$$

$$+ \frac{4(ik_1)\left(r_0^2 f_1^2\right)}{3}\left(\frac{\partial f_1}{f_3 \partial z}\right)\left[\frac{3r^2}{r_0^2 f_1^2} + 1\right]\exp\left(\frac{-3r^2}{r_0^2 f_1^2}\right) - 6k_1^2\left(r_0^2 f_1^2\right)^2\left[\frac{3r^2}{r_0^2 f_1^2} + 1\right]\exp\left(\frac{-3r^2}{r_0^2 f_1^2}\right)\left(\frac{\partial f_1}{f_3 \partial z}\right)^2,$$

$$\int \frac{r\psi_3^* \partial \psi_3 \partial r}{r \partial r} + \int \frac{r\psi_3^* \partial^2 \psi_3}{\partial r^2} = \begin{bmatrix} \left[1 + \frac{ik_1}{}\left(\frac{\partial f}{f \partial z}\right)\left(\frac{\left(r_0^2 f_1^2\right)}{}\right)\right] \\ + 1 + \frac{3}{r_0^2 f_1^2}\left[\left(\frac{3r^2}{r_0^2 f_1^2}\right) + 1\right] \\ + \frac{4(ik_1)\left(r_0^2 f_1^2\right)}{3}\left(\frac{\partial f_1}{f_3 \partial z}\right)\left[\frac{3r^2}{r_0^2 f_1^2} + 1\right] \\ - 6k_1^2\left(r_0^2 f_1^2\right)^2\left[\frac{3r^2}{r_0^2 f_1^2} + 1\right]\left(\frac{\partial f_1}{f_3 \partial z}\right)^2 \end{bmatrix} \exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right], \qquad (4.37)$$

using Eqs. (4.33), (4.35) and (4.37) into Eq. (4.32), we obtain

$$\frac{-2r_0^2 f_1^2}{3}\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\left\{i\left(3k_1 + k_0\right)\right\}\frac{\partial A_{30}'}{\partial z} + \begin{bmatrix} \left(\frac{9\omega_1^2 - 10\omega_p^2}{c^2}\right) + \frac{9\omega_1^2 \phi\left(\vec{E}_1 \vec{E}_1^{\,*}\right)}{c^2} \\ + \left\{i\left(3k_1 + k_0\right)\right\}^2 \end{bmatrix} A_{30}' \frac{-2r_0^2 f_1^2}{3}\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]$$

$$+ A_{30}'\left[\begin{bmatrix} \left[1 + \frac{ik_1}{}\left(\frac{\partial f}{f \partial z}\right)\left(\frac{\left(r_0^2 f_1^2\right)}{}\right)\right] + 1 + 3\left[\left(\frac{3r^2}{r_0^2 f_1^2}\right) + 1\right] \\ + \frac{4(ik_1)\left(r_0^2 f_1^2\right)}{3}\left(\frac{\partial f_1}{f_3 \partial z}\right)\left[\frac{3r^2}{r_0^2 f_1^2} + 1\right] - 6k_1^2\left(r_0^2 f_1^2\right)^2\left[\frac{3r^2}{r_0^2 f_1^2} + 1\right]\left(\frac{\partial f_1}{f_3 \partial z}\right)^2 \end{bmatrix}\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\right] =$$

$$\frac{12\pi i}{c^2}\frac{n_0^0 e^5 B_w k_1}{16cim^4\omega_1^3\left(\omega_1 + iv\right)}\left[\frac{5k_1}{18\omega_1} + \frac{k_1 + k_0}{\omega_1 + iv}\right]\frac{\vec{E}_1^3}{}\int r\psi_3^* dr,$$

$$\frac{-2r_0^2 f_1^2}{3}\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\{i(3k_1+k_0)\}\frac{\partial A'_{30}}{\partial z}$$

$$+\left[\left(\frac{9\omega_1{}^2-10\omega_p^2}{c^2}\right)+\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_{\,1}\right)}{c^2}+\{i(3k_1+k_0)\}^2\right]A'_{30}\frac{-2r_0^2 f_1^2}{3}\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]$$

$$+A'_{30}\left[\left[\left[1+\frac{ik_1}{}\left(\frac{\partial f}{f\partial z}\right)\left(\frac{\left(r_0^2 f_1^2\right)}{}\right)\right]+1+3\left[\left(\frac{3r^2}{r_0^2 f_1^2}\right)+1\right] \atop +\frac{4(ik_1)\left(r_0^2 f_1^2\right)}{3}\left(\frac{\partial f_1}{f_3\partial z}\right)\left[\frac{3r^2}{r_0^2 f_1^2}+1\right]-6k_1{}^2\left(r_0^2 f_1^2\right)^2\left[\frac{3r^2}{r_0^2 f_1^2}+1\right]\left(\frac{\partial f_1}{f_3\partial z}\right)^2\right]\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\right]=$$

$$\frac{12\pi i\omega_1}{c^2}\frac{n_0^0 e^5 B_w k_1}{16cim^4\omega_1^3(\omega_1+iv)}\left[\frac{5k_1}{18\omega_1}+\frac{k_1+k_0}{\omega_1+iv}\right]\frac{A_{10}^3\left[r_0^2 f_1^2\right]}{(-3)}\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right],$$

$$\frac{-2}{3}\{i(3k_1+k_0)\}\frac{\partial A'_{30}}{\partial \xi}$$

$$\frac{-2}{3}\left[\left(\frac{9\omega_1{}^2-10\omega_p^2}{c^2}\right)+\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_{\,1}\right)}{c^2}+\{i(3k_1+k_0)\}^2\right]A'_{30}$$

$$+A'_{30}\left[\left[\left[\frac{1}{r_0^2 f_1^2}+\frac{ik_1}{}\left(\frac{\partial f}{f\partial\xi}\right)\right]+\frac{1}{r_0^2 f_1^2}+\frac{3}{\left(r_0^2 f_1^2\right)^2}\left[\left(\frac{3r^2}{r_0^2 f_1^2}\right)+1\right] \atop +\frac{4(ik_1)}{3}\left(\frac{\partial f_1}{f_3\partial\xi}\right)\left[\frac{3r^2}{r_0^2 f_1^2}+1\right] \atop -6k_1{}^2\left(r_0^2 f_1^2\right)\left[\frac{3r^2}{r_0^2 f_1^2}+1\right]\left(\frac{\partial f_1}{f_3\partial\xi}\right)^2\right]\right]=$$

$$-\frac{4\pi i\omega_1}{c^2}\frac{n_0^0 e^5 B_w k_1 A_{10}^3}{16cim^4\gamma^3 c\omega_1^3(\omega_1+iv)}\left[\frac{5k_1}{18\omega_1}+\frac{k_1+k_0}{\omega_1+iv}\right],$$

$$
\frac{-2i}{3}\frac{\partial A''_{30}}{\partial \xi}
$$

$$
+\left(\frac{-2}{3}\right)\left[\left(\frac{r_0^2\omega_1^2}{c^2}\right)\left(9-\frac{10\omega_p^2}{\omega_1^2}\right)+\left(\frac{r_0^2\omega_1^2}{c^2}\right)\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)-\left(\frac{9\omega_1^2 r_0^2}{c^2}\right)\left(1-\frac{\omega_p^2}{9\omega_1^2}\right)^2\right]A''_{30}
$$

$$
+A''_{30}\left[\left[5+i\left(\frac{\partial f}{f\partial \xi}\right)+\frac{4(i)}{3}\left(\frac{\partial f_1}{f_3\partial \xi}\right)-6f_1^2\left(\frac{\partial f_1}{f_3\partial \xi}\right)^2\right]\right]= \qquad (4.37)
$$

$$
-\frac{1}{16}\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{r_0^2\omega_1^2}{c^2}\right)\left(\left(1-\omega_p^2/\omega_1^2\right)^{1/2}\left(\frac{eB_w}{cm\omega_1}\right)\left(\frac{e^2A_{10}^2}{m^2\omega_1^2c^2}\right)\left[\left[\begin{matrix}3\left(1-\dfrac{\omega_p^2}{9\omega_1^2}\right)^{1/2}\\ -31\left(1-\dfrac{\omega_p^2}{\omega_1^2}\right)^{1/2}\end{matrix}\right]\right].\right.
$$

4.3 Results and discussion

Eqs (4.24) & (4.37) are the coupled equations of beam width parameter and normalized amplitude of third harmonic pulse. we solved these equations numerically and variation of beam width parameter is studied graphically with normalized propagation distance. Plasma is irradiated by Nd: YAG laser with a pulse duration of 2.5 ps and laser spot size of 45µm, under a wiggler magnetic field B_w = 10T, the Wiggler period was ~0.2cm. Fig. 4.1 shows the variation of beam width parameter with linear propagation distance. Due to the relativistic effect of the self-focusing, beam width shows its minimum value at $\xi = 0.8$ that is the spot size is minimum. With further increase in value of ξ the beam width parameter increases due to defocusing of beam. Kaur and Kaur [30] reported the similar results for Q-Gaussian beam in a density transition under relativistic and ponderomotive self-focusing for different values of q using paraxial ray approximation. Fig. 4.2 shows the variation of normalized amplitude with wiggler magnetic field and peak is maximum at $\xi = 0.8$ when spot size is minimum and increase in amplitude is also due to the phase matching condition provided by the wiggler magnetic field. Rajput *et.al* [28] studied the resonant third harmonic generation of a short pulse laser in plasma by applying wiggler magnetic field where wiggler magnetic provide the phase matching condition results enhancement in third harmonic pulse amplitude. Patil *et al.* [31] studied the combined effect of ponderomotive and relativistic self-focusing as laser beam propagates in plasma where ponderomotive self-focusing contributes in the relativistic self-focusing of the laser beam. Fig. 4.3 shows the variation of normalized amplitude with propagation distance at different values of intensity of incident beam. Increase in intensity results increase in refractive index result self-focusing and minimum spot size. The amplitude peak is maximum at $\xi = 0.8$ where beam width is minimum. Thakur and Kant [27] shows the similar results that is with the increase in intensity of the fundamental laser beam the efficiency of third harmonic increases. S. Vij [32] studied the Wiggler magnetic field assisted third harmonic generation in expanding clusters. The efficiency of third harmonic generation is enhanced due to cluster Plasmon resonance and by phase matching due to wiggler magnetic field. In Fig. 4.4 the variation of normalized amplitude with ω_p/ω_1 is shown where normalized amplitude of third harmonic pulse shows significant rise with increase in ω_p/ω_1 as plasma density increases. Thakur and Kant [27] reported the significant increase of normalized amplitude of third harmonic pulse with normalized propagation distance at increasing values of ω_p/ω_1. Aggarwal *et al.* [33] reported the similar results for second harmonic generation through a gas embedded with atomic clusters, converts it into plasma balls where wiggler magnetic field makes the process resonant and enhancement in the amplitude of second harmonic takes place. Singh *et al.* [34] reported the significant rise in amplitude of second and third harmonic pulse with ω_p/ω_1 under the effect of relativistic self-focusing where density ripple provide the phase matching condition. Rajput *et al.* [28] reported the similar results where wiggler magnetic field provided the phase matching condition and normalized amplitude of third harmonic shows rise with ω_p/ω_1.

4.4 Conclusion

Due to the ponderomotive force and relativistic self-focusing beam width parameter attain minimum value at $\xi = 0.8$ where the spot size minimum. At similar value of $\xi = 0.8$ the normalized amplitude of third harmonic is maximum when variation is studied with wiggler magnetic field and intensity of incident pulse. Wiggler magnetic field provide the phase matching condition and enhances the relativistic effect whereas intensity of incident laser pulse results increase of non linear refractive index results self-focusing. Enhancement in the normalized amplitude results with increase in intensity of incident laser pulse results as ultra short laser increases the relativistic self-focusing.

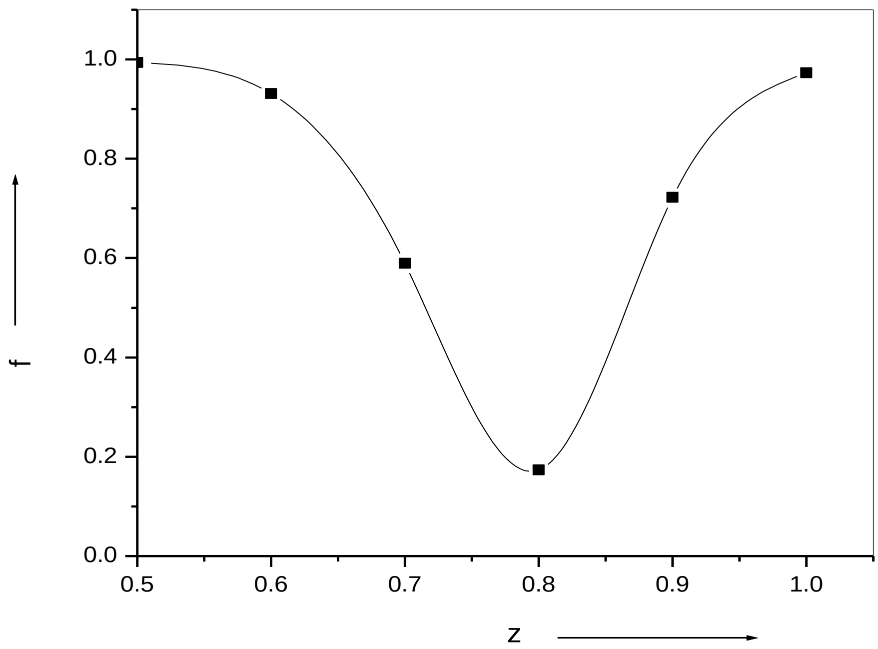

Fig. 4.1 Variation of beam width parameter with nromalized propagation distance ξ

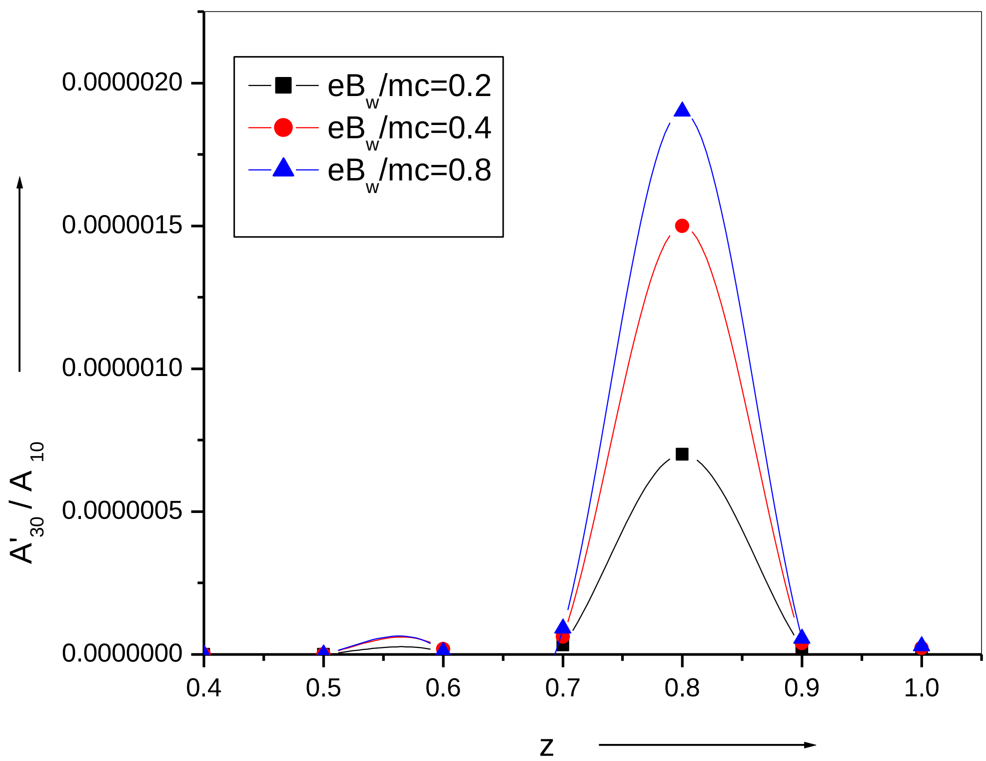

Fig. 4.2 Variation of normalized amplitude of third harmonic pulse with ξ at given value of B_w

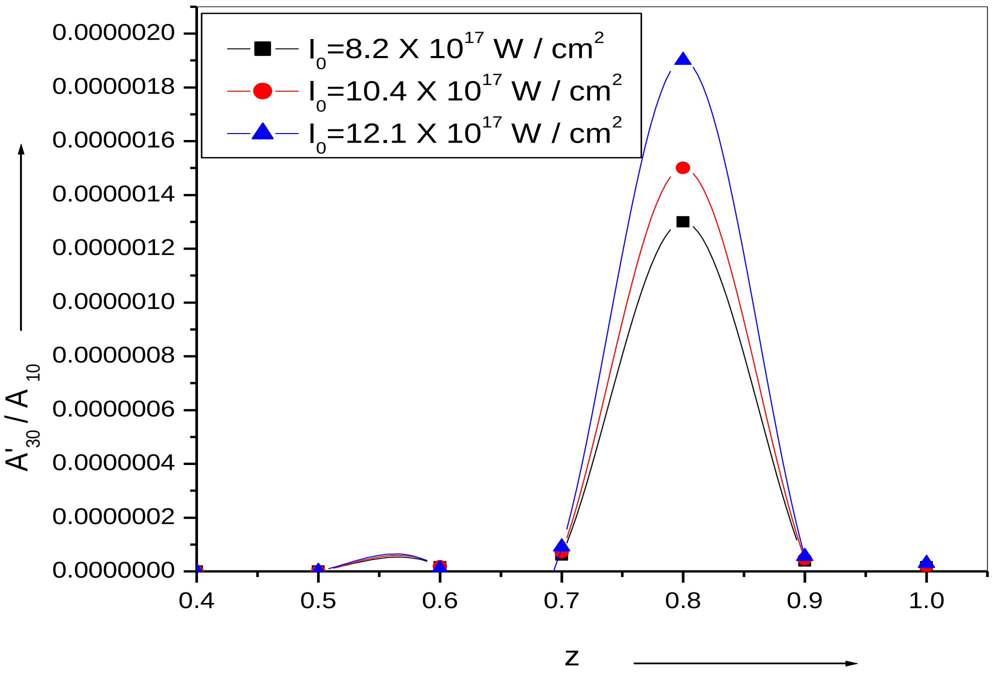

Fig. 4.3 Variation of normalized amplitude of third harmonic pulse with ξ at given value of intensity.

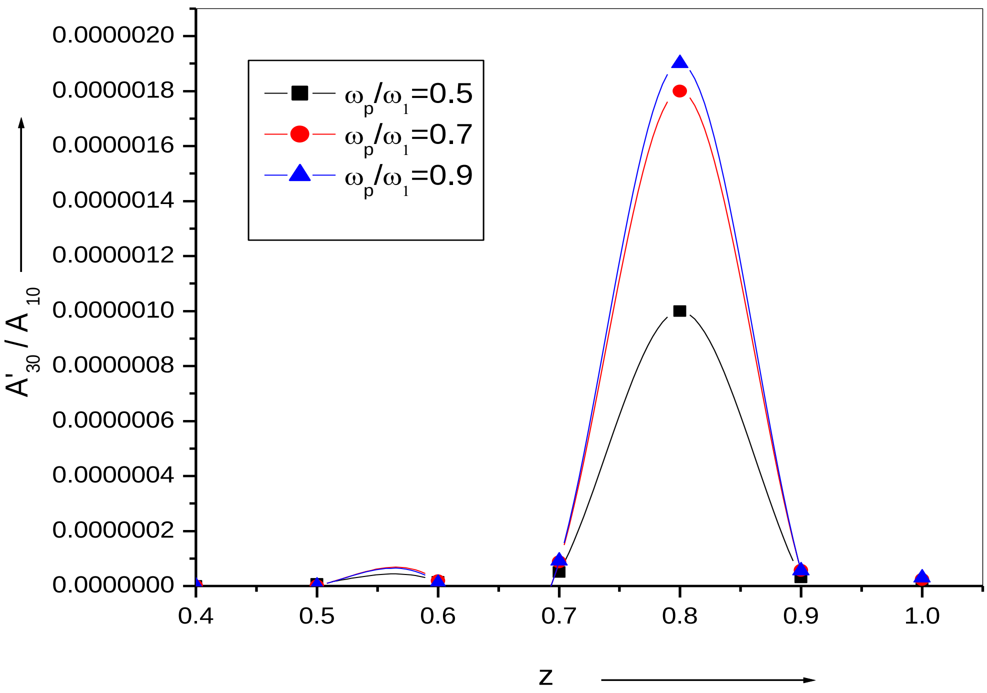

Fig. 4.4 Variation of normalized amplitude of third harmonic pulse with ξ at given value of ω_p/ω_1

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Chapter-5

Second harmonic generation of a self focused Hermite-Gaussian laser beam in plasma

5.1 Introduction

Propagation of intense laser pulses through plasma results different important nonlinear phenomena arises which have important applications e.g. x-ray laser [1-3], inertial confinement fusion [4-5], laser plasma channelling [6], photoelectron spectroscopy [7-9], nonlinear optics in extreme ultra violet region [10], optical measurement of high density matter [11], higher harmonic generations [12-13] etc. Generation of harmonic radiations is of vital importance and second harmonic generation (SHG) has its specific significance due to various applications in medical sciences [14]. SHG is important in biomedical science due to its ability to detect low concentrations of analytes such as proteins, peptides and small molecules [15] and also helps us to understand tumour microenvironment [16]. In last few decades various researchers investigated the SHG for different profiles and for different laser parameters. Using Gaussian profile short pulse laser plasma interaction in magnetized plasma was investigated by Jha *et al*. [17] and they reported the SHG with significant conversion efficiency when an intense laser pulse interacts with homogeneous plasma embedded in a transverse magnetic field. The study of second harmonic radiations under density transition for a self-focused laser in plasma was given by Kant *et al*. [18]. They studied the affect of density ramp on the efficiency of second harmonic pulse. Kant *et al*. [19] reported that the amplitude of second harmonic acquire maximum values as wiggler field satisfy reoinance condition by providing additional momentum to electrons. It was observed that different laser profiles may have different impact on SHG. Steinbach *et al.* [20] used the elliptical-Gaussian laser beams to investigate the SHG where elliptical focusing results in the higher conversion efficiency and crystal damage risk is reduced. Nguyen *et al.* [21] studied the SHG with the zeroth order Bessel- profile beams in uniaxial crystal and showed that the conversion efficiency is always less than that of the SHG with the Gaussian beam and they studied that the zeroth-order Bessel beam is not the optimum beam for the SHG in uniaxial crystals. Gupta *et al.* [22] studied SHG in collisional plasma due to nonlinear absorption by using the q-Gaussian laser beam. They studied the effect nonlinear absorption on the amplitude of second harmonics and self-focusing using the moment theory. The relativistic self-focusing effect on SHG for ChG laser beam was analysed considering under dense plasma was studied by Singh *et al.* [23]. Their results showed that the maximum intensity of chG laser beam shift in the mormal direction by varying the decentred parameter and significant changes were observed in efficiency of second harmonics. Beam width parameter for chG laser profile in collision less plasma was also investigated by Singh *et al.* [24]. Using moment theory they derived beam width parameter equations and presented intensity variation of cosh-Gaussian laser beam by varying decentred parameter.

In recent past, HG laser beam attracted researchers and studied exclusively. It has motivated us to analyze SHG by using the profile of HG laser beam propagating in plasma and to analyze effect of self-focusing. Expressions for the beam width parameter and efficiency of normalize amplitude of SHG is investigated. Graphical analysis of Self-focusing and efficiency of SHG is analysed with linear propagation distance at optimum value of different laser and plasma perameters.

5.2 Theoretical considerations

When HG laser beam travel through the collisionless plasma, the ponderomotive force is responsible for nonlinearity where linearity and nonlinearity is due to ε_0 and Φ respectively and dielectric constant is given as $\varepsilon = \varepsilon_0 + \Phi(EE^*)$ where $\varepsilon_0 = 1 - \omega_p^2 / \omega_1^2$, ω_p is the plasma frequency given as $\omega_p^2 = 4\pi n_0 e^2 / m$, $\Phi(EE^*) = \omega_p^2 / \omega_1^2 \left(1 - \exp\left(3m/4M\, \alpha EE^*\right)\right)$ and $\alpha = (e^2 M / 6m^2 \omega_1{}^2 K_b T_0)$ ω_1 is the frequency of fundamental laser, e is the charge on the electron, equilibrium plasma density is n_0, K_b is the Boltzmann constant, T_0 is the equilibrium a temperature of plasma, m is the rest mass of the electron, and M is the mass of the ions. Electric field for HG laser beam moving in plasma is directed z- is given as,

$$\vec{E} = A(x, y, z)\exp\left[i\left(\omega_1 t - k_1 z\right)\right], \tag{5.1}$$

$$\vec{B}_w = \hat{y} B_0 \exp(ik_0 z), \tag{5.2}$$

where $\vec{k}_1$ is the wave number of fundamental laser pulse, $\vec{B}_w$ is the wiggler magnetic field, k_0 is the wiggler wave number and $A = A_0 \exp(-iks)$ where A_0 is the constant amplitude of the incident laser pulse. s is the functions of x , y and z . A_0 is given as

$$A_0^2 = \frac{E_0^2}{f_1(z) f_2(z)} H_m\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right) H_m\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right) \exp\left[-\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]. \quad (5.3)$$

5.3 Equation of beam width parameter for self-focusing

From Maxwell's equations we have

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \quad (5.4)$$

taking curl of eqn. (5.4) we get

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right) \text{ where we have taken } B = H \quad (5.5)$$

Or $\vec{\nabla}\left(\vec{\nabla}.\vec{E}\right) - \overrightarrow{\nabla^2 E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right),$

But $\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t},$

taking J=0 we get

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t},$$

therefore, from Eqn.(5.5) we get

$$\vec{\nabla}\left(\vec{\nabla}..\vec{E}\right) - \nabla^2 \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \vec{D}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \vec{D}}{\partial t^2}, \quad (5.6)$$

also, $\vec{\nabla}.\vec{D} = 4\pi\rho = 0$ (Because ρ =0)

But $\vec{D} = \varepsilon \vec{E}$, (5.7)

therefore, $\vec{\nabla}.\left(\varepsilon \vec{E}\right) = 0$,

Or $\varepsilon\left(\vec{\nabla}.\vec{E}\right) - \vec{E}.\vec{\nabla}\varepsilon = 0,$

Or $\vec{\nabla}.\vec{E} = -\frac{1}{\varepsilon}\vec{E}.\vec{\nabla}\varepsilon$, (5.8)

using Eqs. (5.7) and (5.8) in Eq. (5.6) we get

$$\nabla^2\vec{E} - \frac{\varepsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \nabla\left(\frac{\vec{E}.\vec{\nabla}\varepsilon}{\varepsilon}\right) = 0,$$

from Eq. (5.1), $\partial^2 E/\partial t^2 = \omega_1^2 E$,

using above two equations

$$\nabla^2\vec{E} + \left(\frac{\omega_1^2}{c^2}\right)\varepsilon\vec{E} + \nabla\left(\frac{\vec{E}\varepsilon}{\varepsilon}\right) = 0 \ . \tag{5.9}$$

Also, $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$,

taking only y-component of eqn. (5.9) we get

$$\nabla^2\vec{E}_y + \left[\frac{\varepsilon\omega_1^2}{c^2} + \nabla\frac{\partial}{\partial y^2}(\ln\varepsilon)\right]\vec{E} = 0, \tag{5.10}$$

$k_1 = (\omega_1/c)\varepsilon^{1/2}$,

but, $k_1 = (\omega_1/c)\varepsilon^{1/2}$, therefore, from eqn. (5.10) we get

$$\nabla^2\vec{E}_y + k_1^2\left(1 + k_1^{-2}\nabla^2\ln\varepsilon\right)\vec{E}_y = 0, \tag{5.11}$$

$k_1^{-2}\nabla^2(\ln\varepsilon) << 1$, thus Eq. (5.11) reduce to

$$\nabla^2\vec{E}_y + k_1^2\vec{E}_y = 0 \text{ Or } \nabla^2\vec{E}_y + \left(\frac{\omega_1^2}{c^2}\right)\varepsilon\vec{E}_y = 0 \ , \tag{5.12}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, (5.13)

therefore, from eqn. (5.12) we get

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} + \left(\frac{\omega_1^2}{c^2}\right)\varepsilon \vec{E}, \tag{5.14}$$

using Eq. (5.1)

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \exp[i(\omega_1 t - k_1 z)]\frac{\partial^2 A}{\partial x^2}, \tag{5.15}$$

$$\frac{\partial^2 \vec{E}}{\partial y^2} = \exp[i(\omega_1 t - k_1 z)]\frac{\partial^2 A}{\partial y^2}, \tag{5.16}$$

$$\frac{\partial \vec{E}}{\partial z} = \exp[i(\omega_1 t - k_1 z)]\left\{\frac{\partial A}{\partial z} + A(-ik_1)\right\},$$

$$\frac{\partial^2 \vec{E}}{\partial z^2} = \exp[i(\omega_1 t - k_1 z)]\left\{\frac{\partial^2 A}{\partial z^2} + (-ik_1)\frac{\partial A}{\partial z} + (-ik_1)\frac{\partial A}{\partial z} + A(-ik_1)^2\right\},$$

$$\frac{\partial^2 \vec{E}}{\partial z^2}\exp[i(\omega_1 t - k_1 z)] = \left\{\frac{\partial^2 A}{\partial z^2} + (-ik_1)\frac{\partial A}{\partial z} + (-ik_1)\frac{\partial A}{\partial z} + A(-ik_1)^2\right\}, \tag{5.17}$$

neglecting $\partial^2 A/\partial z^2$

$$\frac{\partial^2 \vec{E}}{\partial z^2} = 2(-ik_1)\exp[i(\omega_1 t - k_1 z)]\frac{\partial A}{\partial z} + A(-ik_1)^2 \exp[i(\omega_1 t - k_1 z)], \tag{5.18}$$

using Eqs. (5.15), (5.16) & (5.18) into Eq. (5.14), we obtain

$$\exp[i(\omega_1 t - k_1 z)]\frac{\partial^2 A}{\partial x^2} + \exp[i(\omega_1 t - k_1 z)]\frac{\partial^2 A}{\partial y^2} + \left(\frac{\omega^2}{c^2}\right)\Phi(EE^*)\vec{E}$$
$$+\left[2(-ik_1)\exp[i(\omega_1 t - k_1 z)]\frac{\partial A}{\partial z} + A(-ik_1)^2 \exp[i(\omega_1 t - k_1 z)]\right] = 0,$$

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \left(\frac{\omega_1^2}{c^2}\right)\Phi(AA^*)A = -\left[2(-ik_1)\frac{\partial A}{\partial z} + A(-ik_1)^2\right]. \tag{5.19}$$

Using

$A(x, y, z) = A_0(x, y, z)\exp(-ik_1 s)$, we obtain (5.20)

$$\frac{\partial A}{\partial x}=\exp\left(-ik_1 s\right)\frac{\partial A_0}{\partial x}+A_0\left(-ik_1\right)\exp\left(-ik_1 s\right)\frac{\partial s}{\partial x},$$

$$\frac{\partial^2 A}{\partial x^2}=\exp\left(-ik_1 s\right)\frac{\partial^2 A_0}{\partial x^2}+\left(-ik_1\right)\exp\left(-ik_1 s\right)\frac{\partial A_0}{\partial x}\frac{\partial s}{\partial x}+\left(-ik_1\right)\exp\left(-ik_1 s\right)\frac{\partial s}{\partial x}\frac{\partial A_0}{\partial x}$$
$$+A_0\left(-ik_1\right)^2\exp\exp\left(-ik_1 s\right)\left(\frac{\partial s}{\partial x}\right)^2+$$
$$A_0\left(-ik_1\right)\exp\left(-iks\right)\frac{\partial^2 s}{\partial x^2},$$

$$\frac{\partial^2 A}{\partial x^2}=\exp\left(-ik_1 s\right)\frac{\partial^2 A_0}{\partial x^2}+2\left(-ik_1\right)\exp\left(-ik_1 s\right)\frac{\partial s}{\partial x}\frac{\partial A_0}{\partial x}+A_0\left(-ik_1\right)^2\exp\left(-ik_1 s\right)\left(\frac{\partial s}{\partial x}\right)^2+$$
$$A_0\left(-ik_1\right)\exp\left(-ik_1 s\right)\frac{\partial^2 s}{\partial x^2}. \tag{5.21}$$

Similarly

$$\frac{\partial^2 A}{\partial y^2}\exp\left(-ik_1 s\right)\frac{\partial^2 A_0}{\partial y^2}+2\left(-ik\right)\exp\left(-ik_1 s\right)\frac{\partial s}{\partial y}\frac{\partial A_0}{\partial y}+A_0\left(-ik_1\right)^2\exp\left(-ik_1 s\right)\left(\frac{\partial s}{\partial y}\right)^2+$$
$$A_0\left(-ik_1\right)\exp\left(-ik_1 s\right)\frac{\partial^2 s}{\partial y^2}. \tag{5.22}$$

$$\frac{\partial A}{\partial z}=\exp\left(-ik_1 s\right)\frac{\partial A_0}{\partial z}+A_0\left(-ik_1\right)\exp\left(-ik_1 s\right)\frac{\partial s}{\partial z}. \tag{5.23}$$

Using Eqs. (5.20), (5.21), (5.22 & (5.23) into Eq. (5.19), we obtain

$$\exp\left(-ik_1 s\right)\left[\frac{\partial^2 A_0}{\partial x^2}+\frac{\partial^2 A_0}{\partial y^2}\right]+2\left(-ik_1\right)\exp\left(-ik_1 s\right)\left[\frac{\partial A_0}{\partial x}\frac{\partial s}{\partial x}+\frac{\partial A_0}{\partial y}\frac{\partial s}{\partial y}\right]$$
$$+A_0\left(-ik_1\right)^2\exp\left(-ik_1 s\right)\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right]+$$
$$A_0\left(-ik_1\right)\exp\left(-ik_1 s\right)\left[\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial x^2}\right]+\frac{\omega^2}{c^2}\Phi\left(AA^*\right)A_0\exp\left(-ik_1 s\right)=$$
$$-\left[2\left(-ik_1\right)\exp\left(-ik_1 s\right)\frac{\partial A_0}{\partial z}+2\left(-ik_1\right)^2\exp\left(-iks\right)\frac{\partial s}{\partial z}+A_0\left(-ik_1\right)^2\exp\left(-ik_1 s\right)\right],$$

$$\left[\frac{\partial^2 A_0}{\partial x^2}+\frac{\partial^2 A_0}{\partial y^2}\right]+2\left(-ik_1\right)\left[\frac{\partial A_0}{\partial x}\frac{\partial s}{\partial x}+\frac{\partial A_0}{\partial y}\frac{\partial s}{\partial y}\right]+A_0\left(-ik_1\right)^2\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right]+$$
$$A_0\left(-ik_1\right)\left[\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial x^2}\right]+\frac{\omega^2}{c^2}\Phi\left(AA^*\right)A_0=-\left[2\left(-ik\right)\frac{\partial A_0}{\partial z}+2\left(-ik_1\right)^2\frac{\partial s}{\partial z}+A_0\left(-ik_1\right)^2\right]. \tag{5.24}$$

Real part

$$\frac{1}{A_0 k^2}\left[\frac{\partial^2 A_0}{\partial x^2}+\frac{\partial^2 A_0}{\partial y^2}\right]-\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right]+\Phi\left(A_0{}^2\right)-2\frac{\partial s}{\partial z}-1=0, \tag{5.25}$$

imaginary part

$$\frac{\partial A_0^2}{\partial x}\frac{\partial s}{\partial x}+\frac{\partial A_0^2}{\partial y}\frac{\partial s}{\partial y}+A_0^2\frac{\partial^2 s}{\partial x^2}+A_0^2\frac{\partial^2 s}{\partial x^2}+\frac{\partial A_0^2}{\partial z}=0, \tag{5.26}$$

$$A_0^2=\frac{E_0^2}{f_1(z)f_2(z)}H_m\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right)H_m\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)\exp\left[-\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right]. \tag{5.27}$$

From Eq. (4.20e)

$$\frac{\partial s}{\partial x}=x\beta_1;\frac{\partial s}{\partial y}=y\beta_2;\frac{\partial s}{\partial z}=x^2\frac{\partial \beta_1}{\partial z}+y^2\frac{\partial \beta_2}{\partial z}+\frac{\partial \Phi}{\partial z}, \tag{5.28}$$

from Eq. (4.22) we have

$$\beta_1(z)=\frac{1}{f_1}\frac{\partial f_1}{\partial z};\ \beta_2(z)=\frac{1}{f_2}\frac{\partial f_2}{\partial z}\ \text{and}\ \frac{\partial \beta(z)}{\partial z}=\frac{\partial^2 f_3}{f_3 \partial z^2}-\frac{1}{f_3^2}\left(\frac{\partial f_3}{\partial z}\right)^2, \tag{5.29}$$

using Eqs. (5.28) into Eq. (5.25), we get

$$\frac{1}{k_1{}^2 A_0}\left[\frac{\partial^2 A_0}{\partial x^2}+\frac{\partial^2 A_0}{\partial y^2}\right]+\Phi\left(A_0^2\right)-\left[x^2\beta_1^2+y^2\beta_2^2\right]+\frac{x^2}{2}\frac{\partial \beta_1}{\partial z}+\frac{y^2}{2}\frac{\partial \beta_2}{\partial z}+\frac{\partial \Phi}{\partial z}+1=0, \tag{5.30}$$

for m = 0 we have $H_0\left(\sqrt{2}x/r_0 f_1(z)\right)=H_0\left(\sqrt{2}y/r_0 f_2(z)\right)=1$ and putting in Eq. (5.27), we get

$$A_0^2=\frac{E_0^2}{f_1(z)f_2(z)}\exp\left[-\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right],$$

$$A_0=\frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}}\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right],$$

$$\frac{\partial A_0}{\partial x}=\frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}}\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{-x}{r_0^2 f_1^2}\right),$$

$$\frac{\partial^2 A_0}{\partial x^2} = -\frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}}\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{1}{r_0^2 f_1^2}\right)$$

$$+\frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}}\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left[\frac{x^2}{\left(r_0^2 f_1^2\right)^2}\right],$$

$$\frac{\partial^2 A_0}{\partial x^2} = -\frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}}\left(\frac{1}{r_0^2 f_1^2}\right)\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left[1-\frac{x^2}{\left(r_0^2 f_1^2\right)}\right],$$

$$\frac{\partial^2 A_0}{\partial x^2} = -\left(\frac{A_0}{r_0^2 f_1^2}\right)\left[1-\frac{x^2}{\left(r_0^2 f_1^2\right)}\right] \text{ and } \frac{\partial^2 A_0}{\partial y^2} = -\left(\frac{A_0}{r_0^2 f_2^2}\right)\left[1-\frac{y^2}{\left(r_0^2 f_2^2\right)}\right], \qquad (5.31)$$

for m= 1 Eq. (5.27) reduces to

$$H_1\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right) = \frac{\sqrt{2}x}{r_0 f_1(z)} \textit{ and } H_1\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right) = \frac{\sqrt{2}y}{r_0 f_2(z)},$$

$$A_0 = E_2 \frac{2\sqrt{2xy}}{r_0 f_1(z) f_2(z)}\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right],$$

$$\frac{\partial A_0}{\partial x} = E_2 \frac{2\sqrt{2}y}{r_0 f_1(z) f_2(z)}\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\frac{x^{-1/2}}{2}+$$

$$E_2 \frac{2\sqrt{2}\sqrt{y}\sqrt{x}}{r_0 f_1(z) f_2(z)}\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{-x}{r_0^2 f_1^2(z)}\right),$$

$$\frac{\partial A_0}{\partial x} = E_2 \frac{2\sqrt{2}y}{r_0 f_1(z) f_2(z)}\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\frac{x^{-1/2}}{2}-$$

$$E_2 \frac{2\sqrt{2}\sqrt{y}(x)^{3/2}}{r_0^3 f_1^3(z) f_2(z)}\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)}+\frac{y^2}{r_0^2 f_2^2(z)}\right)\right],$$

$$\frac{\partial^2 A_0}{\partial x^2} = E_2 \frac{2\sqrt{2}y}{r_0 f_1(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{x^{-3/2}}{-4}\right) +$$

$$E_2 \frac{2\sqrt{2}y}{r_0 f_1(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{x^{-1/2}}{2}\right)\left(\frac{-x}{r_0^2 f_1^2(z)}\right)$$

$$-E_2 \frac{3\times 2\sqrt{2}\sqrt{y}(x)^{1/2}}{2r_0^3 f_1^3(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]$$

$$-E_2 \frac{2\sqrt{2}\sqrt{y}(x)^{3/2}}{r_0^3 f_1^3(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{-x}{r_0^2 f_1^2(z)}\right),$$

$$\frac{\partial^2 A_0}{\partial x^2} = E_2 \frac{\sqrt{2}y}{r_0 f_1(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{x^{-3/2}}{-2}\right) +$$

$$E_2 \frac{\sqrt{2}yx^{-1/2}}{r_0 f_1(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{-x}{r_0^2 f_1^2(z)}\right)$$

$$-E_2 \frac{3\sqrt{2}\sqrt{y}(x)^{1/2}}{r_0^3 f_1^3(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]$$

$$-E_2 \frac{2\sqrt{2}\sqrt{y}(x)^{3/2}}{r_0^3 f_1^3(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{-x}{r_0^2 f_1^2(z)}\right),$$

$$\frac{\partial^2 A_0}{\partial x^2} = E_2 \frac{\sqrt{2}y}{r_0 f_1(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{x^{-3/2}}{-2}\right)$$

$$-4E_2 \frac{3\sqrt{2}\sqrt{y}(x)^{1/2}}{2r_0^3 f_1^3(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]$$

$$+E_2 \frac{2\sqrt{2}\sqrt{y}(x)^{1/2}(x)^2}{r_0^5 f_1^5(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right],$$

$$\frac{\partial^2 A_0}{\partial x^2} = E_2 \frac{\sqrt{2}y}{r_0 f_1(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{x^{-3/2}}{-2}\right)$$

$$-4E_2 \frac{\sqrt{2}\sqrt{y}(x)^{1/2}}{2r_0^3 f_1^3(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]$$

$$+E_2 \frac{2\sqrt{2}\sqrt{y}(x)^{1/2}(x)^2}{r_0^5 f_1^5(z) f_2(z)} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right],$$

$$\frac{\partial^2 A_0}{\partial x^2} = A_0\left[\frac{-1}{x^2} - \frac{2}{r_0^2 f_1^2(z)} + \frac{x^2}{r_0^4 f_1^4(z)}\right] \tag{5.32a}$$

similarly we can obtain

$$\frac{\partial^2 A_0}{\partial y^2} = A_0\left[\frac{-1}{y^2} - \frac{2}{r_0^2 f_2^2(z)} + \frac{y^2}{r_0^4 f_2^4(z)}\right], \tag{5.32b}$$

$$\Phi\left(A_0^2\right) = \Phi\left(\frac{E_0^2}{f_1 f_2}\right) - \frac{E_0 x^2 \Phi'\left(A_0^2\right)}{r_0^2 f_1^3 f_2} - \frac{E_0 y^2 \Phi'\left(A_0^2\right)}{r_0^2 f_2^3 f_1}, \tag{5.33}$$

using Eqs. (5.32), (5.32b) and (5.33) into Eq. (5.30), we obtain

$$-\frac{1}{k_1{}^2 A_0}\left[\left(\frac{A_0}{r_0^2 f_1^2}\right)\left[1 - \frac{x^2}{\left(r_0^2 f_1^2\right)}\right] + \left(\frac{A_0}{r_0^2 f_2^2}\right)\left[1 - \frac{y^2}{\left(r_0^2 f_2^2\right)}\right]\right]$$
$$+\Phi\left(\frac{E_0^2}{f_1 f_2}\right) - \frac{E_0 x^2 \Phi'\left(A_0^2\right)}{r_0^2 f_1^3 f_2} - \frac{E_0 y^2 \Phi'\left(A_0^2\right)}{r_0^2 f_2^3 f_1} - \left[x^2\beta_1^2 + y^2\beta_2^2\right] + \frac{x^2}{2}\frac{\partial \beta_1}{\partial z} + \frac{y^2}{2}\frac{\partial \beta_2}{\partial z} + \frac{x^2}{2}\frac{\partial \beta_1}{\partial z} + \frac{\partial \Phi}{\partial z} + 1 = 0,$$

and

$$-\frac{1}{k_1{}^2 A_0}\left[\left(\frac{A_0}{r_0^2 f_1^2}\right)\left[\frac{-1}{x^2} - \frac{2}{r_0^2 f_1^2(z)} + \frac{x^2}{r_0^4 f_1^4(z)}\right] + \left(\frac{A_0}{r_0^2 f_2^2}\right)\left[1 - \frac{y^2}{\left(r_0^2 f_2^2\right)}\right]\right]$$
$$+\Phi\left(\frac{E_0^2}{f_1 f_2}\right) - \frac{E_0 x^2 \Phi'\left(A_0^2\right)}{r_0^2 f_1^3 f_2} - \frac{E_0 y^2 \Phi'\left(A_0^2\right)}{r_0^2 f_2^3 f_1} - \left[x^2\beta_1^2 + y^2\beta_2^2\right] + \frac{x^2}{2}\frac{\partial \beta_1}{\partial z} + \frac{y^2}{2}\frac{\partial \beta_2}{\partial z} + \frac{\partial \Phi}{\partial z} + 1 = 0,$$

taking coefficient of ‘ x^2 ’

$$-\frac{1}{k^2}\left[\frac{1}{\left(r_0^2 f_1^2\right)^2}\right] - \frac{E_0 \Phi'\left(A_0^2\right)}{r_0^2 f_1^3 f_2} - \beta_1^2 + \frac{\partial \beta_1}{\partial z} = 0, \tag{5.34}$$

using Eq. (5.29) into Eq. (5.34)

$$-\frac{1}{k_1^2}\left[\frac{1}{\left(r_0^2 f_1^2\right)^2}\right] - \frac{E_0 \Phi'\left(A_0^2\right)}{r_0^2 f_1^3 f_2} - 2\left(\frac{1}{f_1}\frac{\partial f_1}{\partial z}\right)^2 + \frac{1}{f_1}\frac{\partial^2 f_1}{\partial z^2}. \tag{5.35}$$

Similarly, we can obtain

$$-\frac{1}{k_1^2}\left[\frac{1}{\left(r_0^2 f_2^2\right)^2}\right] - \frac{E_0 \Phi'\left(A_0^2\right)}{r_0^2 f_2^3 f_1} - 2\left(\frac{1}{f_2}\frac{\partial f_2}{\partial z}\right)^2 + \frac{1}{f_2}\frac{\partial^2 f_2}{\partial z^2}. \tag{5.36}$$

Also $z = \xi R_d$ we obtain

$$-\frac{1}{k_1^2}\left[\frac{1}{\left(r_0^2 f_1^2\right)^2}\right]-\frac{E_0\Phi'\left(A_0^2\right)}{r_0^2 f_1^3 f_2}-2\left(\frac{1}{f_1}\frac{\partial f_1}{R_d\partial\xi}\right)^2+\frac{1}{f_1}\frac{\partial^2 f_1}{R_d^2\partial\xi^2}\ , \tag{5.37}$$

$$-\frac{R_d^2}{k_1^2}\left[\frac{1}{\left(r_0^2 f_1^2\right)^2}\right]-\frac{R_d^2 E_0\Phi'\left(A_0^2\right)}{r_0^2 f_1^3 f_2}-2\left(\frac{1}{f_1}\frac{\partial f_1}{\partial\xi}\right)^2+\frac{1}{f_1}\frac{\partial^2 f_1}{\partial\xi^2},$$

$$-\frac{R_d^2}{k_1^2}\frac{1}{r_0^4 f_1^3}-\frac{R_d^2 E_0\Phi'\left(A_0^2\right)}{r_0^2 f_1^2 f_2}-\frac{2}{f_1}\left(\frac{\partial f_1}{\partial\xi}\right)^2+\frac{\partial^2 f_1}{\partial\xi^2}=0\ , \tag{5.38}$$

divide the equation by ‘ ω_p^2 ’ and multiply by 2 we get

$$-\frac{2c^2R_d^2}{\omega_p^2}\frac{1}{r_0^4 f_1^3}-\frac{2R_d^2\omega_1^2E_0\Phi'\left(A_0^2\right)}{\omega_p^2 r_0^2 f_1^2 f_2}-\frac{4\omega_1^2}{\omega_p^2 f_1}\left(\frac{\partial f_1}{\partial\xi}\right)^2+\frac{2\omega_1^2}{\omega_p^2}\frac{\partial^2 f_1}{\partial\xi^2}=0, \tag{5.39}$$

similarly

$$-\frac{2c^2R_d^2}{\omega_p^2}\frac{1}{r_0^4 f_2^3}-\frac{2R_d^2\omega_1^2E_0\Phi'\left(A_0^2\right)}{\omega_p^2 r_0^2 f_2^2 f_1}-\frac{4\omega_1^2}{\omega_p^2 f_2}\left(\frac{\partial f_2}{\partial\xi}\right)^2+\frac{2\omega_1^2}{\omega_p^2}\frac{\partial^2 f_2}{\partial\xi^2}=0. \tag{5.40}$$

Now

$$\Phi\left(EE*\right)=\Phi\left(AA*\right)=\Phi\left(A_0^2\right)=\frac{\omega_p^2}{\omega_1^2}\left[1-\exp\left(\frac{-3m}{4M}\alpha EE*\right)\right],$$
$$\Phi\left(EE*\right)=\Phi\left(AA*\right)=\Phi\left(A_0^2\right)=\frac{\omega_p^2}{\omega_1^2}\left[1-\exp\left(\frac{-3m}{4M}\alpha A_0^2\right)\right],$$

$$\Phi'\left(A_0^2\right)=\frac{\partial\Phi\left(A_0^2\right)}{\partial A_0^2}=\left(\frac{3m}{4M}\alpha\right)\frac{\omega_p^2}{\omega_1^2}\left[\exp\left(\frac{-3m\alpha E_0^2}{4M_1 f_1 f_2}\right)\right]. \tag{5.41}$$

Putting Eq. (5.41) into Eqs. (5.39) and (5.40)

$$-\frac{1}{\left(\frac{r_o\omega_p}{c}\right)^2}\left(\frac{R_d}{r_0}\right)^2\left(\frac{2}{f_1^3}\right)-\left(\frac{R_d}{r_0}\right)^2\left(\frac{3mE_0^2}{2M}\alpha\right)\frac{1}{f_1^2 f_2}\left[\exp\left(\frac{-3m\alpha E_0^2}{4M_1 f_1 f_2}\right)\right]-\frac{4\omega_1^2}{\omega_p^2 f_1}\left(\frac{\partial f_1}{\partial\xi}\right)^2+\frac{2\omega_1^2}{\omega_p^2}\frac{\partial^2 f_1}{\partial\xi^2}=0, \tag{5.42}$$

$$-\frac{1}{\left(\frac{r_o\omega_p}{c}\right)^2}\left(\frac{R_d}{r_0}\right)^2\left(\frac{2}{f_2^3}\right)-\left(\frac{R_d}{r_0}\right)^2\left(\frac{3mE_0^2}{2M}\alpha\right)\frac{1}{f_2^2 f_1}\left[\exp\left(\frac{-3m\alpha E_0^2}{4M_1 f_1 f_2}\right)\right]-\frac{4\omega_1^2}{\omega_p^2 f_1}\left(\frac{\partial f_2}{\partial\xi}\right)^2+\frac{2\omega_1^2}{\omega_p^2}\frac{\partial^2 f_2}{\partial\xi^2}=0, \quad (5.43)$$

also $(R_d/r_0)^2=(r_0\omega/c)^2(1-\omega_p^2/\omega^2)$, putting in Eqs. (5.42) and (5.43), we get

$$-\frac{1}{\left(\frac{r_o\omega_p}{c}\right)^2}\left(\frac{r_0\omega_1}{c}\right)^2\left(1-\frac{\omega_p^2}{\omega^2}\right)\left(\frac{2}{f_1^3}\right)-\left(\frac{r_0\omega_1}{c}\right)^2\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{3mE_0^2}{2M}\alpha\right)\frac{1}{f_1^2 f_2}\left[\exp\left(\frac{-3m\alpha E_0^2}{4M_1 f_1 f_2}\right)\right]$$

$$-\frac{4\omega_1^2}{\omega_p^2 f_1}\left(\frac{\partial f_1}{\partial\xi}\right)^2+\frac{2\omega_1^2}{\omega_p^2}\frac{\partial^2 f_1}{\partial\xi^2}=0, \quad (5.44)$$

$$-\frac{1}{\left(\frac{r_o\omega_p}{c}\right)^2}\left(\frac{r_0\omega}{c}\right)^2\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{2}{f_2^3}\right)-\left(\frac{r_0\omega}{c}\right)^2\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{3mE_0^2}{2M}\alpha\right)\frac{1}{f_2^2 f_1}\left[\exp\left(\frac{-3m\alpha E_0^2}{4M_1 f_1 f_2}\right)\right]$$

$$-\frac{4\omega_1^2}{\omega_p^2 f_1}\left(\frac{\partial f_2}{\partial\xi}\right)^2+\frac{2\omega_1^2}{\omega_p^2}\frac{\partial^2 f_2}{\partial\xi^2}=0, \quad (5.45)$$

multiply Eq (5.44) and (5.45) by $\frac{\omega_p^2}{\omega^2}$ we obtain

$$-\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{2}{f_1^3}\right)-\left(\frac{r_0\omega_1}{c}\right)^2\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{3mE_0^2}{2M}\alpha\right)\frac{1}{f_1^2 f_2}\left[\exp\left(\frac{-3m\alpha E_0^2}{4M_1 f_1 f_2}\right)\right]-\frac{4}{f_1}\left(\frac{\partial f_1}{\partial\xi}\right)^2+2\frac{\partial^2 f_1}{\partial\xi^2}=0. \quad (5.46)$$

$$-\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{2}{f_2^3}\right)-\left(\frac{r_0\omega_1}{c}\right)^2\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{3mE_0^2}{2M}\alpha\right)\frac{1}{f_2^2 f_1}\left[\exp\left(\frac{-3m\alpha E_0^2}{4M_1 f_1 f_2}\right)\right]-\frac{4}{f_1}\left(\frac{\partial f_2}{\partial\xi}\right)^2+\frac{2}{}\frac{\partial^2 f_2}{\partial\xi^2}=0. \quad (5.47)$$

Wave equation is given as

$$\nabla^2\vec{E}_2=\frac{4\pi}{c^2}\frac{\partial\vec{J}_2}{\partial t}+\frac{\varepsilon}{c^2}\frac{\partial^2\vec{E}_2}{\partial t^2},$$

$$\nabla^2\vec{E}_2=\frac{4\pi}{c^2}\frac{\partial\vec{J}_2}{\partial t}+\varepsilon_0+\phi\left(E_1E_1^*\right)\frac{\partial^2\vec{E}_2}{c^2\partial t^2} \quad (5.48)$$

where $\varepsilon=\varepsilon_0+\phi\left(E_1E_1^*\right)$, (5.49)

$$\nabla^2 \vec{E}_{\backslash 2} = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} \right) \vec{E}_2 , \quad (5.50)$$

$$E_2 = \exp[-i(2\omega_1 t - k_2 z)], \quad (5.51)$$

$\vec{J}_2 = \vec{J}_2^{\,L} + \vec{J}_2^{\,NL}$, and $\vec{J}_3^{\,L}$ and $\vec{J}_3^{\,L}$ [25] the linear and nonlinear current densities .

$$\vec{J}_2^{\,L} = \frac{-n_0 e^2 \vec{E}_2}{m 2 i \omega_1} = \frac{-n_0 e^2 \hat{x} A_2 \exp(-ik_2 S_2) \exp[-i(2\omega_1 t - k_2 z)]}{m 2 i \omega_1}, \quad (5.52)$$

$$\vec{J}_2^{\,NL} = \frac{n_0 e^4 B_W E_1^2}{4ic\omega_1^2 m^3 (\omega_1 + i\upsilon)} \left(\frac{3k_1}{4\omega_1} + \frac{k_1 + k_0}{\omega_1 + i\upsilon} \right) . \hat{x} , \quad (5.53)$$

using Eq. (5.52)

$$\frac{4\pi}{c^2} \frac{\partial \vec{J}_2^{\,L}}{\partial t} = \frac{4\pi}{mc^2} n_0 e^2 \vec{E}_2 ,$$

$$\frac{4\pi}{c^2} \frac{\partial \vec{J}_2^{\,L}}{\partial t} = \frac{\omega_p^2 \vec{E}_2}{c^2} , \quad (5.54)$$

differentiating Eq. (5.51)

$$\frac{\partial \vec{E}_2}{\partial t} = \frac{\partial \{ \hat{x} A_2 \exp(-ik_2 S_2) \exp[-i(2\omega_1 t - k_2 z)] \}}{\partial t} ,$$

$$\frac{\partial \vec{E}_2}{\partial t} = (-2i\omega_1) \{ \hat{x} A_2 \exp(-ik_2 S_2) \exp[-i(2\omega_1 t - k_2 z)] \} ,$$

$$\frac{\partial^2 \vec{E}_2}{\partial t^2} = (-2i\omega_1) \frac{\partial}{\partial t} \{ \hat{x} A_3 \exp(-ik_3 S_3) \exp[-i(2\omega_1 t - k_3 z)] \} ,$$

$$\frac{\partial^2 \vec{E}_2}{\partial t^2} = (-2i\omega_1)^2 \{ \hat{x} A_2 \exp(-ik_2 S_2) \exp[-i(2\omega_1 t - k_2 z)] \} ,$$

$$\frac{\partial^2 \vec{E}_2}{\partial t^2} = -4\omega_1{}^2 \{ \hat{x} A_2 \exp(-ik_2 S_2) \exp[-i(\omega_1 t - k_2 z)] \} ,$$

$$\frac{\partial^2 \vec{E}_2}{\partial t^2} = -2\omega_1{}^2 \vec{E}_2 . \quad (5.55)$$

we also have

$$\varepsilon_0 = \frac{1}{c^2} \left(1 - \frac{\omega_p^2}{\omega_1^2} \right) \quad (5.56)$$

taking second term on R.H.S. of Eq. (5.48) and using Eq. (5.56) we obtain

$$\left[\frac{\varepsilon_0+\phi\left(\vec{E}_1\vec{E}^*_{\ 1}\right)}{c^2}\right]\frac{\partial^2\vec{E}_2}{\partial t^2}=\frac{1}{c^2}\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\frac{\partial^2\vec{E}_2}{\partial t^2}+\frac{\phi\left(\vec{E}_1\vec{E}^*_{\ 1}\right)}{c^2}\frac{\partial^2\vec{E}_2}{\partial t^2},$$

$$\left[\frac{\varepsilon_0+\phi\left(\vec{E}_1\vec{E}^*_{\ 1}\right)}{c^2}\right]\frac{\partial^2\vec{E}_2}{\partial t^2}=\frac{1}{c^2}\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(-4\omega_1^{\ 2}\vec{E}_2\right)+\frac{\phi\left(\vec{E}_1\vec{E}^*_{\ 1}\right)}{c^2}\left(-4\omega_1^{\ 2}\vec{E}_2\right),$$

$$\left[\frac{\varepsilon_0+\phi\left(\vec{E}_1\vec{E}^*_{\ 1}\right)}{c^2}\right]\frac{\partial^2\vec{E}_2}{\partial t^2}=\frac{-4\omega_1^{\ 2}\vec{E}_2}{c^2}+\frac{4\omega_p^2\vec{E}_2}{c^2}-\frac{4\omega_1^{\ 2}\phi\left(\vec{E}_1\vec{E}^*_{\ 1}\right)\vec{E}_2}{c^2}, \qquad (5.57)$$

add Eq. (5.54) and Eq. (5.57)

$$\frac{4\pi}{c^2}\frac{\partial\vec{J}^{\,L}_2}{\partial t}+\left[\frac{\varepsilon_0+\phi\left(\vec{E}_1\vec{E}^*_{\ 1}\right)}{c^2}\right]\frac{\partial^2\vec{E}_2}{\partial t^2}=\frac{5\omega_p^2\vec{E}_2}{c^2}\frac{-4\omega_1^{\ 2}\vec{E}_2}{c^2}-\frac{4\omega_1^{\ 2}\phi\left(\vec{E}_1\vec{E}^*_{\ 1}\right)\vec{E}_2}{c^2}. \qquad (5.58)$$

Now putting (5.58) in (5.48) we get

$$\nabla^2\vec{E}_2+\left[\left(\frac{4\omega_1^{\ 2}-5\omega_p^2}{c^2}\right)+\frac{4\omega_1^{\ 2}\phi\left(\vec{E}_1\vec{E}^*_{\ 1}\right)}{c^2}\right]\vec{E}_2=\frac{4\pi\partial\vec{J}^{\,NL}_2}{c^2\partial t}, \qquad (5.59)$$

$$\vec{E}_1^2=A_1^2\hat{x}\exp(-2ik_1S_1)\exp\left[-i2\left(\omega_1 t-k_1z\right)\right]$$

$$\frac{\partial\vec{E}_1^2}{\partial t}=\left(-2i\omega_1\right)\hat{x}A_1^2\exp(-2ik_1S_1)\exp\left[-i\left(2\omega_1 t-k_1z\right)\right],$$

$$\frac{\partial\vec{E}_1^2}{\partial t}=\left(-2i\omega_1\right)E_1^3,$$

using Eq. (5.53)

$$\frac{4\pi}{c^2}\frac{\partial\vec{J}^{\,NL}_3}{\partial t}=-\frac{8\pi\omega_1 n_0e^4B_W}{4c^3\omega_1^2m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\vec{E}_1^3, \qquad (5.60)$$

putting (5.60) into Eq. (5.59)

$$\nabla^2 \vec{E}_2 + \left[\left(\frac{4\omega_1{}^2 - 5\omega_p^2}{c^2} \right) + \frac{4\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}_1^{\,*}\right)}{c^2} \right] \vec{E}_2 = -\frac{8\pi\omega_1 n_0 e^4 B_W}{4c^3 \omega_1^2 m^3 (\omega_1 + i\upsilon)} \left(\frac{3k_1}{4\omega_1} + \frac{k_1 + k_0}{\omega_1 + i\upsilon} \right) \vec{E}_1^3,$$

using Eq. (5.50) we obtain

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} \right) \vec{E}_2 + \left[\left(\frac{4\omega_1{}^2 - 5\omega_p^2}{c^2} \right) + \frac{4\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}_1^{\,*}\right)}{c^2} \right] \vec{E}_3$$

$$= -\frac{8\pi\omega_1 n_0 e^4 B_W}{4c^3 \omega_1^2 m^3 (\omega_1 + i\upsilon)} \left(\frac{3k_1}{4\omega_1} + \frac{k_1 + k_0}{\omega_1 + i\upsilon} \right) \vec{E}_1^3. \quad (5.61)$$

Eq. (5.59) have the particular integral solution

$$\vec{E}_2 = A_2^{'} \exp[-i\{2\omega_1 t - (2k_1 + k_0)\}], \quad (5.62)$$

$$\psi_2 = \exp\left[\frac{-x^2}{2r_0^2 f_1^2} \frac{-y^2}{2r_0^2 f_2^2} \right] \exp(-2iks), \quad (5.63)$$

$$A_2^{'} = A_{20}^{'} \psi_2 = A_{20}^{'} \exp\left[\frac{-x^2}{2r_0^2 f_1^2} \frac{-y^2}{2r_0^2 f_2^2} \right] \exp(-2iks), \quad (5.64)$$

$$\vec{E}_2 = A_{20}^{'} \exp\left[\frac{-x^2}{2r_0^2 f_1^2} \frac{-y^2}{2r_0^2 f_2^2} \right] \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}] \exp(-2iks),$$

from Eq. (5.62)

$$\frac{\partial \vec{E}_2}{\partial z} = \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}] \exp(-2iks) \frac{\partial A_2^{'}}{\partial z}$$
$$+ A_2^{'} \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}]\{i(2k_1 + k_0)\} \exp(-2iks),$$

$$\frac{\partial^2 \vec{E}_2}{\partial z^2} = \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}] \left\{ \begin{array}{l} \frac{\partial^2 A_2^{'}}{\partial z^2} \exp(-2iks) + A_2^{'} \{i(2k_1 + k_0)\}^2 \exp(-2iks) \\ + \{i(2k_1 + k_0)\} \exp(-2iks) \frac{\partial A_2^{'}}{\partial z} + \{i(2k_1 + k_0)\} \frac{\partial A_2^{'}}{\partial z} \exp(-2iks) \end{array} \right\},$$

$$\frac{\partial^2 \vec{E}_2}{\partial z^2} = A_2' \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}]\{i(2k_1 + k_0)\}^2 \exp(-2iks)$$

$$+ 2\exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}]\{i(2k_1 + k_0)\}\exp(-2iks)\frac{\partial A_2'}{\partial z},$$

using Eq. (5.64)

$$\frac{A_2'}{\partial z} = \exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-2iks)\frac{\partial A_{20}'}{\partial z} + A_{20}'\frac{\partial f_1}{\partial z}\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\left[\frac{-2x^2}{2r_0^2 f_1^2}\right]\exp(-2iks),$$

$$\frac{A_2'}{\partial z} = \left\{\frac{\partial A_{20}'}{\partial z}\exp(-2iks) + A_{20}'\frac{\partial f_1}{\partial z}\left[\frac{-x^2}{f_1^2}\right]\right\}\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-2iks),$$

$$\frac{\partial^2 \vec{E}_2}{\partial z^2} = \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}]\{i(2k_1 + k_0)\}^2 A_{20}' \exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-2iks)$$

$$+ 2\exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}]\{i(2k_1 + k_0)\}\left\{\frac{\partial A_{20}'}{\partial z} + A_{20}'\frac{\partial f_1}{\partial z}\left[\frac{-x^2}{r_0^2 f_1^2}\right]\right\}\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-2iks),$$

$$\frac{\partial^2 \vec{E}_2}{\partial z^2} = \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}]\{i(2k_1 + k_0)\}^2 A_{20}' \exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-2iks)$$

$$+ 2\exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}]\{i(2k_1 + k_0)\}\left\{\begin{array}{l}\dfrac{\partial A_{20}'}{\partial z} \\ + A_{20}'\dfrac{\partial f_1}{\partial z}\left[\dfrac{-x^2}{r_0^2 f_1^2}\right]\end{array}\right\}\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-2iks). \quad (5.65)$$

Using Eq. (5.63)

$$\frac{\partial \psi_2}{\partial x} = -A_{20}'\frac{x}{r_0^2 f_1^2}\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-ikS), \quad (5.66)$$

$$\frac{\partial^2 \psi_2}{\partial x^2} = -A_{20}'\frac{1}{r_0^2 f_1^2}\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right] + A_{20}'\left[\frac{x^2}{r_0^4 f_1^4}\right]\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-ikS),$$

$$\frac{x\partial^2 \psi_2}{\partial x^2} = -A_{20}'\frac{x}{r_0^2 f_1^2}\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right] + A_{20}'\left[\frac{x^2}{r_0^4 f_1^4}\right]\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-ikS), \quad (5.67)$$

from Eqs. (5.66) and (5.67)

$$\frac{\partial\psi_2}{x\partial x}+\frac{\partial^2\psi_2}{\partial x^2}=-A'_{20}\frac{1}{r_0^2 f_1^2}\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-ikS)$$

$$-A'_{20}\frac{1}{r_0^2 f_1^2}\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]+A'_{20}\left[\frac{x^2}{r_0^4 f_1^4}\right]\exp\left[\frac{-x^2}{2r_0^2 f_1^2}\frac{-y^2}{2r_0^2 f_2^2}\right]\exp(-ikS),$$

$$\frac{\partial\psi_2}{x\partial x}+\frac{\partial^2\psi_2}{\partial x^2}=-2A'_{20}\frac{1}{r_0^2 f_1^2}\psi_2+A'_{20}\left[\frac{x^2}{r_0^4 f_1^4}\right]\psi_2, \tag{5.68}$$

$$\frac{\partial E_2}{\partial x}=A'_{20}\exp\left[-i\{2\omega_1 t-(2k_1+k_0)z\}\right]\frac{\partial\psi_2}{\partial x}, \tag{5.69}$$

$$\frac{\partial^2 E_2}{\partial x^2}=A'_{20}\exp\left[-i\{2\omega_1 t-(2k_1+k_0)z\}\right]\frac{\partial^2\psi_2}{\partial x^2}, \tag{5.70}$$

using Eqs (5.65), (5.68), (5.69) & (5.70) into Eq. (5.61), we obtain

$$\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]\{i(2k_1+k_0)\}^2 A'_{20}\psi_2$$
$$+2\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]\{i(2k_1+k_0)\}\left\{\frac{\partial A'_{20}}{\partial z}+A'_{20}\frac{\partial f_1}{\partial z}\left[\frac{-2x^2}{2r_0^2 f_1^2}\right]\right\}\psi_2$$
$$\left\{-2A'_{20}\frac{1}{r_0^2 f_1^2}+A'_{20}\left[\frac{x^2}{r_0^4 f_1^4}\right]\right\}\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]\psi_2$$
$$-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1*)\right]\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]A'_{20}\psi_2$$
$$=-\frac{8i\pi n_0 e^4 B_W E_1^2}{4ic^3\omega_1 m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\hat{x}, \tag{5.71}$$

multiply by $x\psi_2^*$ and integrate Eq. (5.65)with respect to 'x'

$$\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]\{i(2k_1+k_0)\}^2 A'_{20}\int x\psi_2\psi_2^* dx$$
$$+2\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]\{i(2k_1+k_0)\}\left\{\frac{\partial A'_{20}}{\partial z}+A'_{20}\frac{\partial f_1}{\partial z}\left[\frac{-2x^2}{2r_0^2 f_1^2}\right]\right\}\int x\psi_2\psi_2^* dx$$
$$\left\{-2A'_{20}\frac{1}{r_0^2 f_1^2}+A'_{20}\left[\frac{x^2}{r_0^4 f_1^4}\right]\right\}\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]\int x\psi_2\psi_2^* dx$$
$$-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1*)\right]\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]A'_{20}\int x\psi_2\psi_2^* dx$$
$$=-\frac{8i\pi n_0 e^4 B_W E_1^2}{4ic^3\omega_1 m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\int x\psi_2^* dx,$$

$$\{i(2k_1+k_0)\}^2 A'_{20}\int x\psi_2\psi_2^* dx + 2\{i(2k_1+k_0)\}\left\{\frac{\partial A'_{20}}{\partial z}\right\}\int x\psi_2\psi_2^* dx +$$

$$2\{i(2k_1+k_0)\}\left\{A'_{20}\frac{\partial f_1}{\partial z}\left[\frac{-2x^2}{2r_0^2 f_1^2}\right]\right\}\int x\psi_2\psi_2^* dx - 2A'_{20}\frac{1}{r_0^2 f_1^2}\int x\psi_2\psi_2^* dx$$

$$+A'_{20}\left[\frac{x^2}{r_0^4 f_1^4}\right]\int x\psi_2\psi_2^* dx$$

$$-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1{}^*)\right]A'_{20}\int x\psi_2\psi_2^* dx$$

$$=-\frac{8i\pi n_0 e^4 B_W E_1^2}{\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]4ic^3\omega_1 m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\int x\psi_2^* dx,$$

$$\{i(2k_1+k_0)\}^2 A'_{20}\int x\psi_2\psi_2^* dx + 2\{i(2k_1+k_0)\}\left\{\frac{\partial A'_{20}}{\partial z}\right\}\int x\psi_2\psi_2^* dx -$$

$$2\{i(2k_1+k_0)\}\left\{A'_{20}\frac{\partial f_1}{\partial z}\left[\frac{1}{r_0^2 f_1^2}\right]\right\}\int x^3\psi_2\psi_2^* dx - 2A'_{20}\frac{1}{r_0^2 f_1^2}\int x\psi_2\psi_2^* dx$$

$$+A'_{20}\left[\frac{1}{r_0^4 f_1^4}\right]\int x^3\psi_2\psi_2^* dx$$

$$-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1{}^*)\right]A'_{20}\int x\psi_2\psi_2^* dx$$

$$=-\frac{8i\pi n_0 e^4 B_W E_1^2}{\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]4ic^3\omega_1 m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\int x\psi_2^* dx,$$

$$\{i(2k_1+k_0)\}^2 A'_{20}\int x\psi_2\psi_2^* dx + 2\{i(2k_1+k_0)\}\left\{\frac{\partial A'_{20}}{\partial z}\right\}\int x\psi_2\psi_2^* dx -$$

$$2\{i(2k_1+k_0)\}\left\{A'_{20}\frac{\partial f_1}{\partial z}\left[\frac{1}{r_0^2 f_1^2}\right]\right\}\int x^3\psi_2\psi_2^* dx - 2A'_{20}\frac{1}{r_0^2 f_1^2}\int x\psi_2\psi_2^* dx$$

$$+A'_{20}\left[\frac{1}{r_0^4 f_1^4}\right]\int x^3\psi_2\psi_2^* dx-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1{}^*)\right]A'_{20}\int x\psi_2\psi_2^* dx$$

$$=-\frac{8i\pi n_0 e^4 B_W A_{10}^2}{4ic^3\omega_1 m^3(\omega_1+i\upsilon)}\left[\begin{matrix} H_0\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right) \\ H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)\exp\left[\begin{matrix}\frac{-x^2}{2r_0^2 f_1^2}+ \\ \frac{-y^2}{2r_0^2 f_2^2}\end{matrix}\right]\left(\begin{matrix}\frac{3k_1}{4\omega_1} \\ +\frac{k_1+k_0}{\omega_1+i\upsilon}\end{matrix}\right)\int x\psi_2^* dx\end{matrix}\right],$$

$$\{i(2k_1+k_0)\}^2 A_{20}^{'} \frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$+2\{i(2k_1+k_0)\}\left\{\frac{\partial A_{20}^{'}}{\partial z}\right\}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$-2\{i(2k_1+k_0)\}\left\{A_{20}^{'}\frac{\partial f_1}{\partial z}\left[\frac{1}{r_0^2 f_1^2}\right]\right\}\int x^3\psi_2\psi_2^* dx - 2A_{20}^{'}\frac{1}{r_0^2 f_1^2}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$+A_{20}^{'}\left[\frac{1}{r_0^4 f_1^4}\right]\int x^3\psi_2\psi_2^* dx$$

$$-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1{}^*)\right]A_{20}^{'}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$=-\frac{8i\pi n_0 e^4 B_W A_{10}^2}{4ic^3\omega_1 m^3(\omega_1+i\upsilon)}\left[\begin{array}{l} H_0\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right) \\ H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)\exp\left[\begin{array}{l}\frac{-x^2}{2r_0^2 f_1^2}+ \\ \frac{-y^2}{2r_0^2 f_2^2}\end{array}\right]\left(\begin{array}{l}\frac{3k_1}{4\omega_1} \\ +\frac{k_1+k_0}{\omega_1+i\upsilon}\end{array}\right)\int x\psi_2^* dx.\end{array}\right],$$

$$\{i(2k_1+k_0)\}^2 A_{20}^{'} \frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$+2\{i(2k_1+k_0)\}\left\{\frac{\partial A_{20}^{'}}{\partial z}\right\}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2} - 2A_{20}^{'}\frac{1}{r_0^2 f_1^2}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1*)\right]A_{20}^{'}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$=-\frac{8i\pi n_0 e^4 B_W A_{10}^2}{4ic^3\omega_1 m^3(\omega_1+i\upsilon)} H_0\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right)$$

$$H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)\exp\begin{pmatrix}\frac{3k_1}{4\omega_1}\\ +\frac{k_1+k_0}{\omega_1+i\upsilon}\end{pmatrix}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2},$$

$$\{i(2k_1+k_0)\}^2 A_{20}^{'} \frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$+2\{i(2k_1+k_0)\}\left\{\frac{\partial A_{20}^{'}}{\partial z}\right\}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$-2A_{20}^{'}\frac{1}{r_0^2 f_1^2}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1*)\right]A_{20}^{'}\frac{\left(-r_0^2 f_1^2 \exp\left[\frac{-x^2}{r_0^2 f_1^2}+\frac{-y^2}{r_0^2 f_2^2}\right]\right)}{2}$$

$$.\quad =-\frac{8i\pi n_0 e^4 B_W A_{10}^2}{4ic^3\omega_1 m^3(\omega_1+i\upsilon)} H_0\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right)$$

$$H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)\exp\begin{bmatrix}\frac{-x^2}{2r_0^2 f_1^2}+\\ \frac{-y^2}{2r_0^2 f_2^2}\end{bmatrix}\begin{pmatrix}\frac{3k_1}{4\omega_1}\\ +\frac{k_1+k_0}{\omega_1+i\upsilon}\end{pmatrix}\int x\begin{bmatrix}\frac{-x^2}{2r_0^2 f_1^2}+\\ \frac{-y^2}{2r_0^2 f_2^2}\end{bmatrix}dx,$$

$$2\{i(2k_1+k_0)\}\left\{\frac{\partial A_{20}^{'}}{\partial z}\right\}-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1*)+\frac{2}{r_0^2 f_1^2}-\{i(2k_1+k_0)\}^2\right]A_{20}^{'}$$

$$=-\frac{\omega_p^2 e^2 B_W A_{10}^2}{2c^3\omega_1 m^2(\omega_1+i\upsilon)}\frac{8xy}{r_0^2 f_1\ f_2}\exp\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right),$$

$$2\{i(2k_1+k_0)\}\left\{\frac{\partial A_{20}^{''}}{\partial z}\right\}-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1*)+\frac{2}{r_0^2 f_1^2}-\{i(2k_1+k_0)\}^2\right]A_{20}^{''}$$

$$=-\frac{\omega_p^2}{2c}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{\omega_1 mc}\right)\begin{pmatrix}H_0\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right)\\ H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)\exp\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\end{pmatrix},$$

$$2\{i(2k_1+k_0)\}\left\{\frac{\partial A_{20}^{''}}{R_d\partial\xi}\right\}-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1*)+\frac{2}{r_0^2 f_1^2}-\{i(2k_1+k_0)\}^2\right]A_{20}^{''}$$

$$=-\frac{\omega_p^2}{2c}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{\omega_1 mc}\right)\begin{bmatrix}H_0\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right)\\ H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)\exp\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\end{bmatrix},$$

$$2\frac{\{i(2k_1+k_0)\}}{kr_0^2}\left\{\frac{\partial A_{20}^{''}}{\partial\xi}\right\}-\left[\frac{4\omega_1^2-5\omega_p^2}{c^2}+\frac{4\omega_1^2}{c^2}\phi(E_1E_1*)+\frac{2}{r_0^2 f_1^2}-\{i(2k_1+k_0)\}^2\right]A_{20}^{''}$$

$$=-\frac{\omega_p^2}{2c}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{\omega_1 mc}\right)\begin{bmatrix}H_0\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right)\\ H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)\exp\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\end{bmatrix},$$

$$2i\left\{\frac{\partial A_{20}^{''}}{\partial\xi}\right\}-\left[\frac{4r_0^2\omega_1^2-5r_0^2\omega_p^2}{c^2}+\frac{4r_0^2\omega_1^2}{c^2}\phi(E_1E_1*)+\frac{2}{f_1^2}-r_0^2\{i(2k_1+k_0)\}^2\right]A_{20}^{''}$$

$$=-\frac{\omega_p^2}{2c}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{\omega_1 mc}\right)\frac{8xy}{f_1\ f_2}\exp\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right),$$

$$2i\left\{\frac{\partial A''_{20}}{\partial \xi}\right\}-\left[\frac{4r_0^2\omega_1^2}{c^2}-\frac{5r_0^2\omega_p^2}{c^2}+\frac{4r_0^2\omega_1^2}{c^2}\phi(E_1E_1*)+\frac{2}{f_1^2}-r_0^2\{i(2k_1+k_0)\}^2\right]A''_{20}$$

$$=-\frac{\omega_p^2}{2c}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{\omega_1 mc}\right)\left[\begin{array}{l} H_0\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right) \\ H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)\exp\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right) \\ \end{array}\right]. \quad (5.72)$$

For m = 0

$$H_0\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right)=H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)=1,$$

$$2i\left\{\frac{\partial A''_{20}}{\partial \xi}\right\}-\left[\begin{array}{l}\frac{4r_0^2\omega_1^2}{c^2}-\frac{5r_0^2\omega_p^2}{c^2}+\frac{4r_0^2\omega_1^2}{c^2}\phi(E_1E_1*)+\frac{2}{f_1^2} \\ -r_0^2\{i(2k_1+k_0)\}^2\end{array}\right]A''_{20}$$

$$=-\frac{\omega_p^2}{2c}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{\omega_1 mc}\right)\left[\exp\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\right]. \quad (5.73)$$

For m = 1

$$H_1\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right)=\frac{2\sqrt{2}x}{r_0 f_1(z)} \text{ and } H_0\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right)=\frac{2\sqrt{2}y}{r_0 f_2(z)},$$

we get

$$2i\left\{\frac{\partial A''_{20}}{\partial \xi}\right\}-\left[\begin{array}{l}\frac{4r_0^2\omega_1^2-5r_0^2\omega_p^2}{c^2}+\frac{4r_0^2\omega_1^2}{c^2}\phi(E_1E_1*)+\frac{2}{f_1^2} \\ -r_0^2\{i(2k_1+k_0)\}^2\end{array}\right]A''_{20}$$

$$=-\frac{\omega_p^2}{2c}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{\omega_1 mc}\right)\frac{8xy}{f_1\ f_2}\exp\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right). \quad (5.74)$$

5.4 Results and discussion

Eqs. (5.46), (5.47) and (5.74) are coupled equations of beam width parameter f and normalized amplitude $A_{20}^{'}/A_{10}$ of second harmonic pulse. Eqs. (5.46), (5.47) and (5.74) have been solved numerically whereas the results are presented in graphical form. Variation of f for fundamental laser pulse with normalized propagation distance ξ for $\omega_1 r_0 / c = 25$, $eA_{10} / m\omega_1 c = 5$, $\varepsilon_2 A_{10}^2 / \varepsilon_0 = 1$, $eB_w / m\omega_1 c = 0.005$ and $\omega_p / \omega_1 = 0.8$ is shown in Fig. 5.1. It is observed that f shows reduction in spot size at $\xi = 0.25$ due to stronger self-focusing. Kant *et al.* [26] had also presented the similar study for the variation of f for HChG laser beam and study its variation by introducing density ramp. Variation of normalized second harmonic amplitude with ξ at different values of $A_{20}^{'}/A_{10}$ of fundamental pulse is presented in Fig. 5.2. The results give the $A_{20}^{'}/A_{10}$ for SHG with normalized amplitude of fundamental laser and it is clear that it attains its maximum value at $\xi = 0.25$. This is due to rise in ponderomotive force with increase in amplitude of fundamental laser pulse and self-focusing becomes more stronger. Aggarwal *et al.* [27] had given the similar results where self-focusing increases on increasing value of $A_{20}^{'}/A_{10}$ of incident laser pulse. The study of $A_{20}^{'}/A_{10}$ for second harmonic pulse varies with ξ for increasing values of wiggler magnetic field is presented in Fig. 5.3. It is observed that the $A_{20}^{'}/A_{10}$ for second harmonic pulse significantly rise as wiggler field satisfy the phase matching conditions. Aggarwal *et al.* [27] presented the similar results where $A_{20}^{'}/A_{10}$ with ξ rises different values of wiggler field. They studied the positive effect of wiggler magnetic field on normalized second harmonic amplitude. Fig. 5.4 shows how $A_{20}^{'}/A_{10}$ is affected by increasing plasma density with ξ. It is observed that the SHG is more prominent in high density plasma and shows sharp rise at $\xi = 0.25$. Askari *et al.* [28] presented the variations of second harmonic pulse normalized amplitude in terms of ξ, at variable values of ω_p / ω_1.

5.5 Conclusion

In our results, the f decreases significantly with ξ due to stronger self-focusing. Ponderomotive nonlinearity results stronger self-focusing of incident laser pulse and increases the efficiency of SHG. Wiggler field is highly significant as it provides the phase matching condition that results increasing $A_{20}^{'}/A_{10}$. In addition to this the role of intensity of incident laser pulse is dominant in increasing the $A_{20}^{'}/A_{10}$ as observed in the present study.

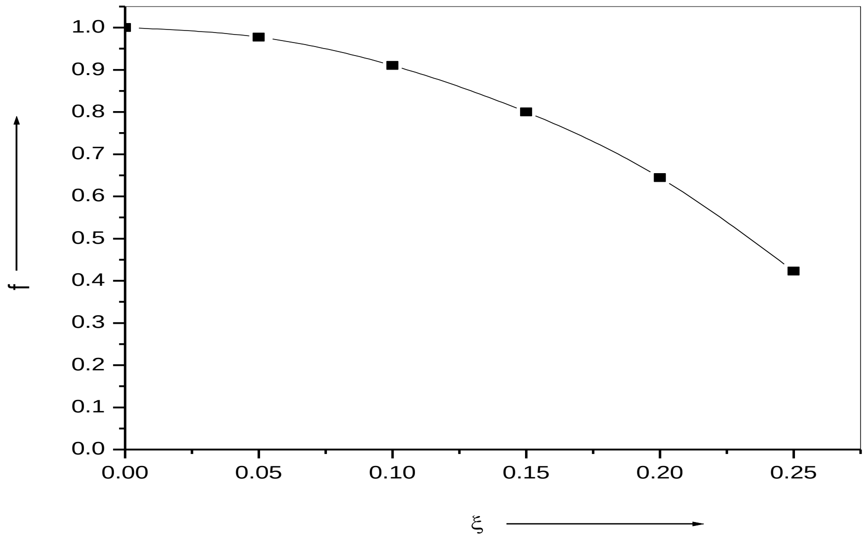

Fig. 5.1 Variation of f with ξ. The other parameters are $\omega_1 r_0/c = 25$, $eA_{10}/m\omega_1 c = 5$, $eB_w/m\omega_1 c = 0.005$, $\varepsilon_2 A_{10}^2/\varepsilon_0 = 1$ and $\omega_p/\omega_1 = 0.8$.

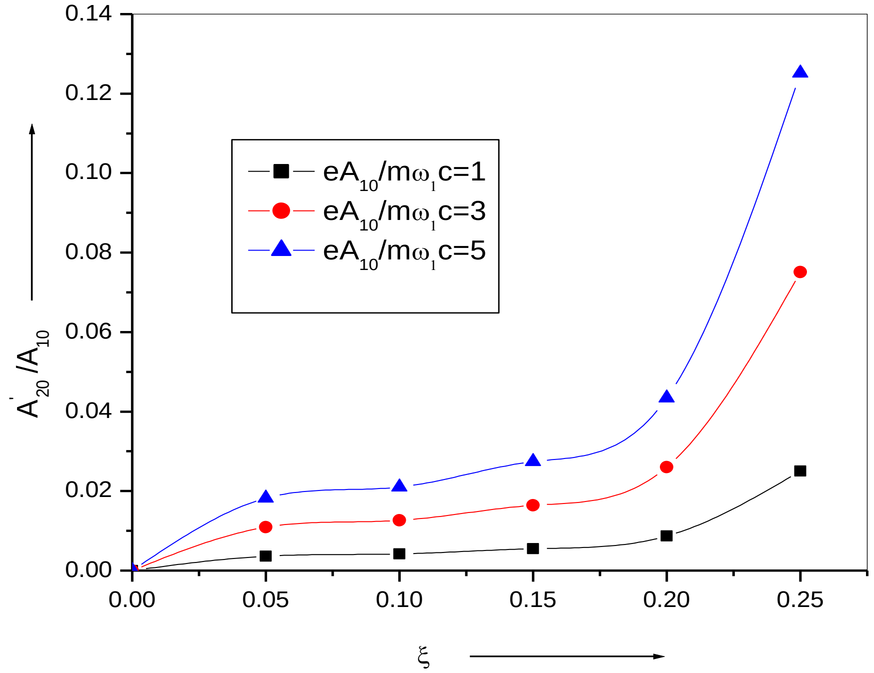

Fig. 5.2 Variation of $A_{20}^{'}/A_{10}$ with ξ for different values at $eA_{10}/m\omega_1 c = 1, 3\ \&\ 5$. The other parameters remain same as in Fig. 5.1.

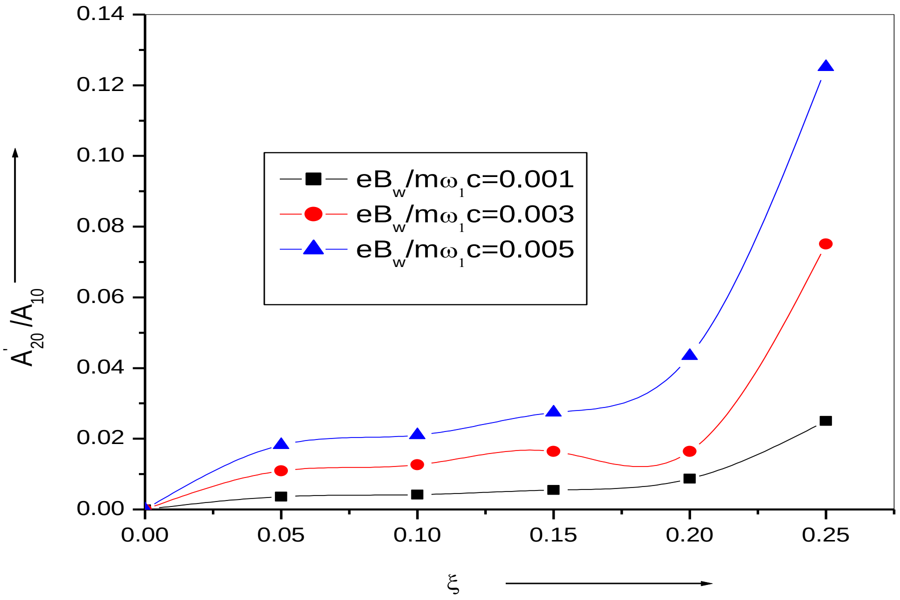

Fig. 5.3 Variation of $A_{20}^{'}/A_{10}$ with ξ at different values of $eB_w/m\omega_1 c = 0.001,$ $0.003 \,\&\, 0.005$. The other parameters remain same as in Fig. 5.1.

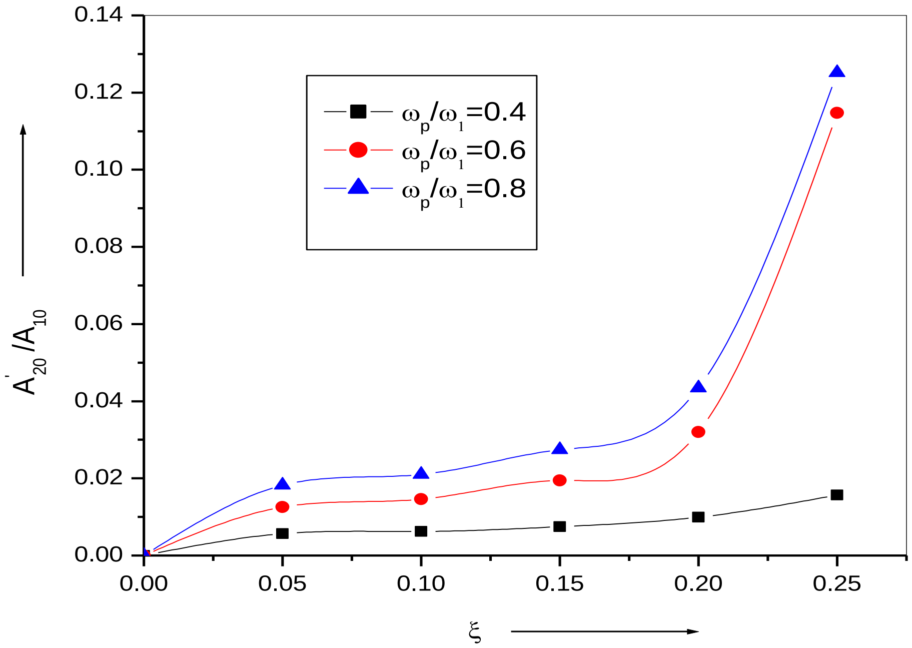

Fig. 5.4 variation of $A_{20}^{'}/A_{10}$ with ξ at different values of $\omega_p/\omega_1 = 0.4, 0.6 \,\&\, 0.8$. The other parameters remain same as in Fig. 1.

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Chapter-6

Self-focusing of Hermite-Gaussian laser beam under the effect of linear absorption in relativistic and ponderomotive regime.

6.1 Introduction

Recent advancements in the laser technology have been drawing much attentions as it evolves new horizon due to optical nonlinearity during the propagation of short pulse laser through plasma. When intense laser beam propagates through plasma results ponderomotive force on electrons due to which electrons acquire quiver velocity and results relativistic mass variation of electrons. This result variation in plasma dielectric properties giving rise to important nonlinear phenomena like, laser driven inertial confinement fusion [1], laser-electron acceleration [2], x-rays lasers [3], harmonic generation [4] etc. where the ponderomotive force [5] and relativistic self-focusing [6-7]. Various researchers investigated self-focusing and beam width parameter has with different laser parameters. Wani *et al.* [8] investigated the Hermite-Gaussian (HG) laser beam in plasma under relativistic effect and density transition where density ramp played a significant role to beam gets self-focused with suitable laser and plasma parameters. Song *et al.* [9] Chosen the (HG) beam to studied itst propagation characteristics and presented the spectral intensity distribution. The mode transformation of HG beam was studied by Wang *et al.* [10] and they found that infinite number of normalized orthogonal modes can be obtained by modulation of characteristics function. Takale *et al.* [11] while studying HG beam had shown that defocusing of beam width parameter occur for modes with odd p-values whereas oscillatory as well as defocusing of beam width parameters was observed with even p values exhibit character. Self-focusing under relativistic nonlinearity of HG laser beam was studied by Sharma [12] by using paraxial approximation and they analysed the second order beam width parameter. The study of relativistic nonlinearity using short pulse laser in non isothermal, under dense collisional plasma was undertaken by Abedi *et al.* [13]. Kant *et al.* [14] presented the importance of density ramp on beam width parameter of HG laser beam.

In recent past the effect of linear absorption was also studied by the number of researchers. The study of self-focusing in CS_2 under the effect of linear absorption was given by Mohebi [15] and his experimental study revealed that linear absorption results power loss due to which threshold power for self-focusing was increased. Patil *et al.* [16]

undertaken the study of beam width parameter of Gaussian laser beam under light absorption for weaker relativistic and ponderomotive effect. Kant *et al.* [17] using ChG laser beam analysed beam width parameter and study the effect of linear absorption. They had also given the variation of beam width parameter by varying plasma density, decentered parameter. Aggarwal *et al.* [18] studied the Q-Gaussian laser beam interaction with magnetized plasma and study the variation of beamwidth parameter under light absorption. Quahid *et al.* [19] studied the Airy-Gaussian beam propagating through plasma and analysed the relativistic and ponderomotive self-focusing. Patil et al. [20] study the effect of absorption on beam width parameter of Gaussian beam in weak relativistic ponderomotive nonlinearity. Absorption of intense short laser pulse in near critical inhomogeneous plasma at resonance was given by Hashmzadeh *et al.* [21]. They studied the effect of different density on power absorption and their study revealed that absorption rate is better when density profile is parabolic. Madhavi *et al.* [22] studied the linear absorption mechanisms in laser plasma interactions that shows some restrictions on linear absorption and in their study collisional or resonance absorption is dominant.

Presently, using HG laser, we are investigating the variation of beam width parameter and normalized amplitude under the linear absorption of light in relativistic and ponderomotive regime. Deriving the expression for second derivative of f_1 & f_2 and by using the critical condition $\partial^2 f_1/\partial\xi^2 = \partial^2 f_2/\partial\xi^2 = 0$ we obtained the expression for normalized radius ρ . Variation of beam width parameter with linear propagation distance at different values of linear absorption coefficient has been analysed. Study of beam width parameter and normalized radius, with linear propagation distance at optimum values of different laser parameters has been done.

6.2Theoretical Considerations

Hermite-Gaussian laser beam is considered to be propagating plasma and its electric field is given as

$$\vec{E} = A(x,y,z)\exp\left[i\left(\omega_1 t - k_1 z\right)\right], \qquad (6.1)$$

where A is the amplitude of the incident pulse given as $A = A_0 \exp(-iks)$

$$A_0^2 = \frac{E_0^2}{f_1(z) f_2(z)} H_m\left(\frac{\sqrt{2}x}{r_0 f_1(z)}\right) H_m\left(\frac{\sqrt{2}y}{r_0 f_2(z)}\right) \exp\left[-\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right] \exp(-2ik_i z). \quad (6.2)$$

Where A_0 is the constant amplitude of incident laser. r_0 is defined as radius of fundamental laser pulse, f_1 & f_2 are the beam width parameter along axis normal to z-axis, k_i is the linear absorption coefficient and $k_1 = \omega_1/c \sqrt{1-\omega_p^2/\omega_1^2}$, is the wave vector of incident laser pulse given as HG laser beam propagating in collision less plasma is characterized by dielectric constant of the form $\varepsilon = \varepsilon_0 + \Phi(EE^*)$ and $\varepsilon_0 = 1-\omega_p^2/\omega_1^2$ where $\omega_p^2 = 4\pi n_0 e^2/\gamma m$, γ is the relativistic factor given as $\gamma = (1+a^2/2)^{1/2}$ and $a = e^2 |A|^2 / m^2 \omega_1{}^2 c^2$ is the laser field strength parameter, m is the rest mass of the electron, e is the charge on electron, n_0 is the equilibrium plasma density, ε_0 and Φ gives the linear and nonlinear parts of the dielectric constant respectively. Where $\Phi(EE^*) = \omega_p^2/\omega_1^2 \left[1-\exp\left(3\gamma m/4M\,\alpha EE^*\right)\right]$, K_b denotes the Boltzmann constant, M denote the mass of the ion in the plasma, $\alpha = e^2 M / 6\gamma m^2 \omega_1{}^2 K_b T_0$, frequency of incident laser is ω_1, and T_0 plasma temperature in equilibrium condition.

Using Eq. (5.13)

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} + \left(\frac{\omega_1^2}{c^2}\right)\varepsilon\vec{E}$$

$$A_0^2 = \frac{E_0^2}{f_1(z) f_2(z)} \exp\left[-\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right] \exp(-2ik_i z),$$

$$A_0 = \frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right] \exp(-ik_i z),$$

$$\frac{\partial A_0}{\partial x} = \frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{-x}{r_0^2 f_1^2}\right) \exp(-ik_i z),$$

$$\frac{\partial^2 A_0}{\partial x^2} = -\frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}}\exp(-ik_i z)\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left(\frac{1}{r_0^2 f_1^2}\right)$$

$$+ \frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}}\exp(-ik_i z)\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left[\frac{x^2}{\left(r_0^2 f_1^2\right)^2}\right],$$

$$\frac{\partial^2 A_0}{\partial x^2} = -\frac{E_2}{\sqrt{f_1(z)}\sqrt{f_2(z)}}\exp(-ik_i z)\left(\frac{1}{r_0^2 f_1^2}\right)\exp\left[-\frac{1}{2}\left(\frac{x^2}{r_0^2 f_1^2(z)} + \frac{y^2}{r_0^2 f_2^2(z)}\right)\right]\left[1-\frac{x^2}{\left(r_0^2 f_1^2\right)}\right],$$

$$\frac{\partial^2 A_0}{\partial x^2} = -\left(\frac{A_0}{r_0^2 f_1^2}\right)\left[1-\frac{x^2}{\left(r_0^2 f_1^2\right)}\right] \text{ and } \frac{\partial^2 A_0}{\partial y^2} = -\left(\frac{A_0}{r_0^2 f_2^2}\right)\left[1-\frac{y^2}{\left(r_0^2 f_2^2\right)}\right], \tag{6.3}$$

also

$$\Phi'\left(A_0^2\right) = \left(\frac{3m\gamma}{4M}\alpha\right)\frac{\omega_p^2}{\omega_1^2}\left[\exp\left(\frac{-3m\gamma\alpha\frac{E_0^2}{f_1 f_2}\exp\left[-\left(\frac{x^2}{r_0^2 f_1^2} + \frac{y^2}{r_0^2 f_2^2}\right)\right]\exp\left(-2ik_i'\xi\right)}{4M_1 f_1 f_2}\right)\right], \tag{6.4}$$

substituting into Eqs. (5.47) and (5.48) we obtain

$$\frac{\partial^2 f_1}{\partial \xi^2} = \left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{1}{f_1^3}\right)$$

$$+\left(\frac{r_0\omega_1}{c}\right)^2\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{3\gamma m E_0^2}{2M}\alpha\right)\frac{1}{2f_1^2 f_2}\left[\exp\left(\frac{-3\gamma m\alpha\frac{E_0^2}{f_1 f_2}\exp\left[-\left(\frac{x^2}{r_0^2 f_1^2} + \frac{y^2}{r_0^2 f_2^2}\right)\right]\exp\left(-2ik_i'\xi\right)}{4M_1 f_1 f_2}\right)\right] \tag{6.5}$$

$$+\frac{2}{f_1}\left(\frac{\partial f_1}{\partial \xi}\right)^2.$$

and

$$\frac{\partial^2 f_2}{\partial \xi^2} = \left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{1}{f_2^3}\right)$$

$$+\left(\frac{r_0\omega_1}{c}\right)^2\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{3\gamma m E_0^2}{2M}\alpha\right)\frac{1}{2f_2^2 f_1}\left[\exp\left(\frac{-3\gamma m\alpha\frac{E_0^2}{f_1 f_2}\exp\left[-\left(\frac{x^2}{r_0^2 f_1^2}+\frac{y^2}{r_0^2 f_2^2}\right)\right]\exp(-2ik_i z)}{4Mf_1 f_2}\right)\right] \quad (6.6)$$

$$+\frac{2}{f_1}\left(\frac{\partial f_2}{\partial \xi}\right)^2.$$

where $k_i^{'} = k_i(r_0\omega/c)$ is the normalized absorption coefficient. Using critical condition $\partial^2 f_1/\partial\xi^2 = \partial^2 f_2/\partial\xi^2 = 0$ we obtain the normalized radius ρ as

$$(\rho)^2 = \frac{-2\left(1-\frac{\omega_p^2}{\omega_1^2}\right)}{\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{3\gamma m A_0^2}{2M}\alpha\right)\left[\exp\left(\frac{-3\gamma m\alpha A_0^2}{4M}\right)\right]}. \quad (6.7)$$

6.3 Results and discussion

Eqs. (6.5) and (6.6) are the beam width parameter (f_1 & f_2) equations and Eq. (6.7) gives the normalized radius (ρ). We solved these equations numerically and analysed them graphically under linear absorption at different values of different laser parameter. Fig. 6.1, Fig. 6.2 and Fig. 6.3 gives the variation of f_1 & [1] f_2 with ξ at k_i ,= 3,4 & 5, normalized intensity $eA_0/m\omega_1 c$ and normalized plasma density ω_p/ω_1. The parameters chosen are $\omega_p/\omega_1 = 0.8, \omega_1 r_0/c = 20$ and $eA_0/m\omega_1 c = 3.$ Variation of f_1 &

f_2 with ξ at $k_i^{'}$ = 3, 4 & 5is analysed is shown in Fig.6.1 and we observe that f_1 & f_2 shows the oscillatory behaviour and its values decreases with increasing values of $k_i^{'}$. At $k_i^{'}$ = 3, 4 & 5 the values of f_1 & f_2 are observed to be 0.5, 0.4 and 0.15 respectively. It is quite clear that for higher values $k_i^{'}$. the self-focusing becomes stronger as the incident beam penetrate deep inside the plasma. Patil *et al.* [23] had presented similar behaviour of beam width parameter at $k_i^{'} =$ 0, 0.5, 1.0 and 1.5. In their study beam width parameter exhibit oscillatory behaviour which is lost due to absorption effect and with increase in the absorption level. In their results at higher temperature the self-focusing shows reverse behaviour and at higher values of $k_i^{'}$ the beam width parameter increases. Fig.6.2 shows how f_1 & f_2 vary with ξ at different values of $eA_0 / m\omega_1 c$ is shown where other parameters remain same. Oscillatory behaviour of f_1, f_2 is observed and self-focusing becomes stronger with increasing values of $eA_0 / m\omega_1 c$. Due to stronger self-focusing we observe f_1 & f_2 = 0.32, 0.28 and 0.15 at $eA_0 / m\omega_1 c$ = 1, 2, and 3 respectively. Habibi *et al.* [24] given the similar behaviour of the f with ξ at different values of power of incident laser pulse and observed that intensity of incident laser play dominant role to decrease f of chG laser beam. Strong self-focusing was observed when density of plasma is higher. high density plasma as shown in ig. 6.3. For the higher value of plasma frequency ω_p / ω_1 = 0.8, the f_1 & f_2 attains its minimum value i.e. 0.18 which supports the result obtained by Habibi *et al.* [24]. Fig. 6.4 represents the variation of $\rho = (r_0 \omega_1 / c)$ with normalized intensity αE_0^2 of incident laser pulse at different values of ω_p / ω_1. We observe that ρ decreases with increasing value of αE_0^2 and at higher value of αE_0^2 the value of ρ becomes constant. Value of ρ decreases significantly at higher values of ω_p / ω_1. Ouahid *et al.* [20] studied the normalized beam width radius with αE_0^2 by varying the electron temperature and their results show the similar behaviour where ρ is inigdependent of αE_0^2 at its higher values.

6.4 Conclusion

In our present study we use the paraxial ray approximation to obtain the expressions for f_1 & f_2 for HG laser beam to study the effect of linear absorption, plasma frequency and intensity of incident laser pulse on self-focusing. These parameter results stronger self-focusing under ponderomotive regime and our results are useful to study harmonic generations when intense laser pulse propagates through plasma.

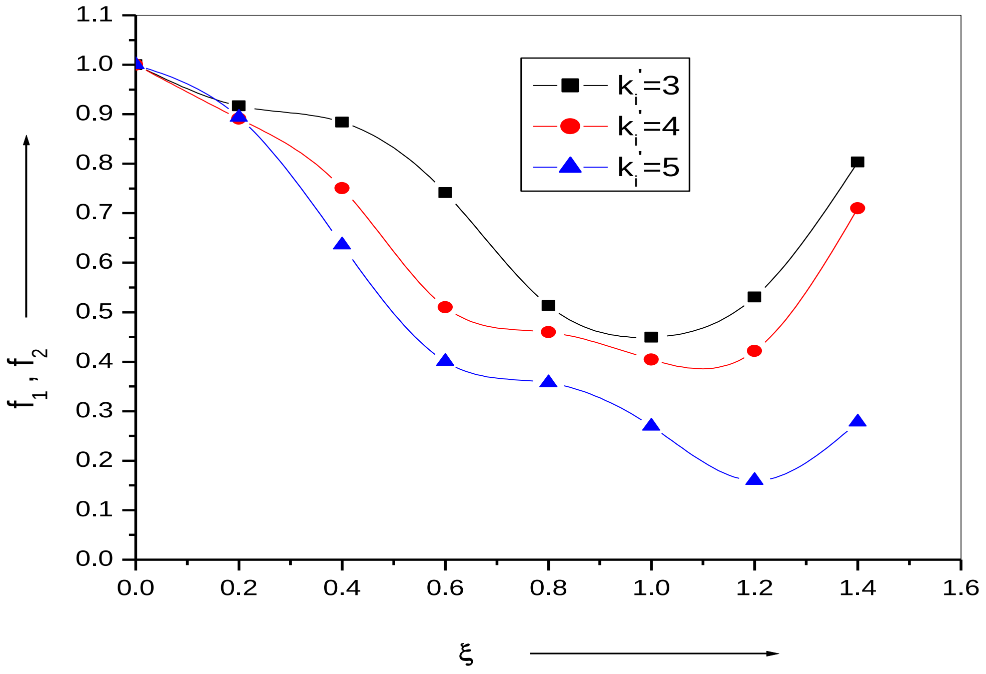

Fig. 6.1 Dependence of f_1 and f_2 on ξ for various values of $k_i^{'} = 3, 4$ & 5. Other parameters are $\omega_p / \omega_1 = 0.8$, $\omega_1 r_0 / c = 20$ and $eA_0 / m\omega_1 c = 3$.

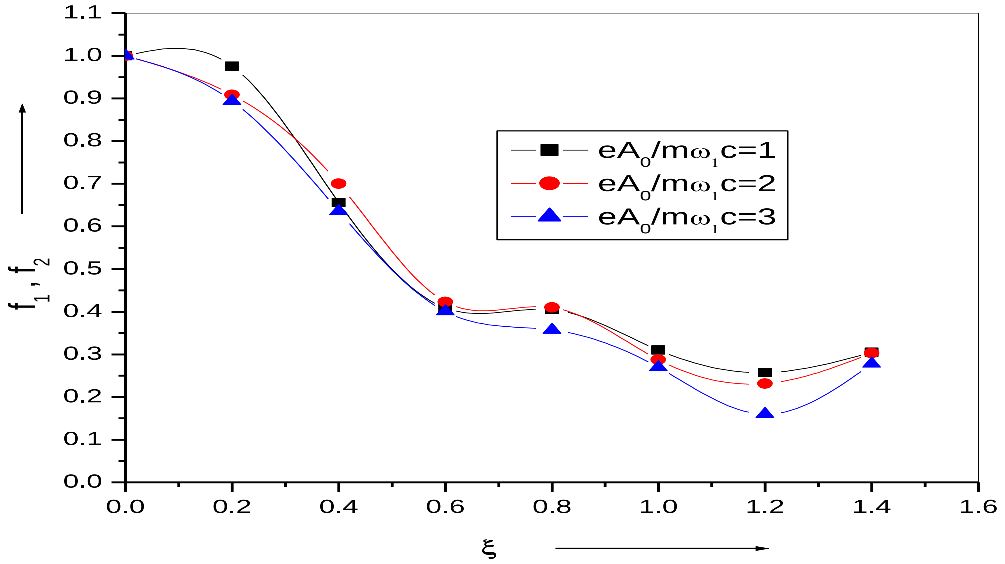

Fig. 6.2 Behaviour of f_1 and f_2 on ξ for different values of $eA_0 / m\omega_1 c = 1, 2$ & 3. Rest of the parameters are same as considered in Fig. 1.

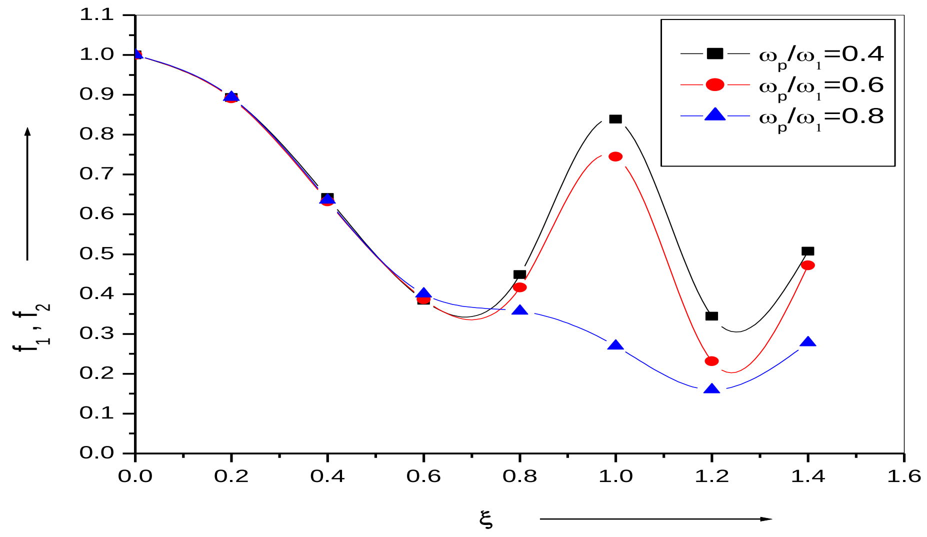

Fig. 6.3 Dependence of f_1 and f_2 on ξ for various values of ω_p / ω_1. Rest of the parameters are same as considered in Fig. 1.

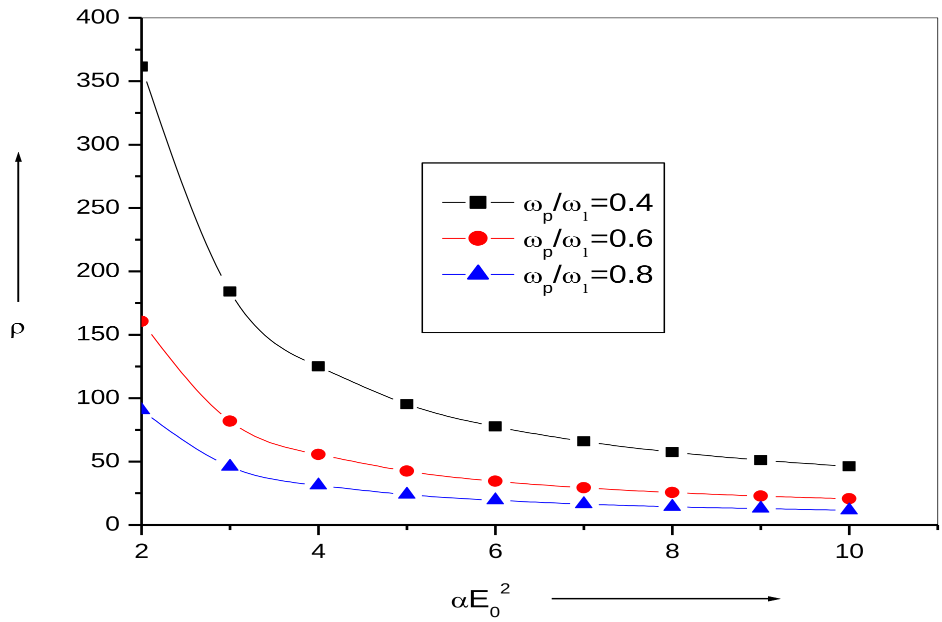

Fig. 6.4 Dependence of ρ $(r_0\omega_1 / c)$ on αE_0^2 for various values of ω_p / ω_1. Rest of the parameters are same as considered in Fig. 1

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Chapter -7

Third harmonic generation of a relativistic self-focusing laser in plasma in plasma under exponential density ramp.

7.1 Introduction

The interaction of short pulse laser and plasma is an prominent area of research in last few decades. Propagation of short pulse laser through plasma results nonlinearity that give rise to various nonlinear phenomena like inertial confinement fusion, x-rays, laser plasma accelerator, harmonic generation etc. [1-5] and due to their wide range of application is highly attracted by research workers in last few decades. Due to their significant applications third harmonic generation (THG) have specific importance amongst the different harmonic generations. Yelin *et al.* [6] used the laser scanning microscope used THG to produced high resolution images of transparent biological specimens. Also, the detailed and clear images of live neurons, internal orgalles of yeast cells were also imaged. Clay *et al.* [7] studied the spectroscopy of THG and presented existence of resonances in compounds in which images are obtained. They confirmed two-photon resonance in the ratio of third- order susceptibilities of rhodamine B, Fura-2 and haemoglobin. Canek *et al.* [8] studied the THG and its applications in optical imaging and they use the time-gated imaging through scattering media. Konopsky et al [9] studied the THG under phase matching condition through resonant optical surface modes in 1D photonic crystals. They had given 1-D photonic crystal through which fundamental and THG can move in same phase. Dhaiya et al [10] studied the SHG and THG in plasmas with density ripple which provides the phase matching conditions and presented the SHG and THG when intense laser pulse interact with underdense plasma and density ripple provide the phase matching conditions for enhanced apmitude. Verma *et al.* [11] studied the impact of laser self-focusing on THG in a tunnel ionizing gas where strong radial non uniformity in electron density was developed by the ionization that result laser defocusing and efficiency of harmonic generation gets limited.

Feit *et al.* [12] studied the relativistic self-focusing in underdense plasma. They emphasized the electron cavitation as a result of ponderomotive forces and due to cavitation filamentation gets suppressed and increases the Rayleigh length. Effect of wiggler magnetic field on THG during intense laser plasma interaction was given by Rajput *et al.* [13]. In their study due to strong ponderomotive force at second harmonic oscillations that coupled with electron velocity at fundamenatl frequency produce a nonlinear current results THG. Rodel *et al.* [14] studied the harmonic generation from relativistic plasma surfaces in ultra steep plasma density gradients. Their observations demonstrate that while the efficient generation of high order harmonics from relativistic surfaces requires steep plasma density scale-lengths ($L_p/\lambda < 1$) the absolute efficiency of the harmonics declines for the steepest plasma density scale-length $L_p \rightarrow 0$, thus demonstrating that near-step like density gradients can be achieved for interactions using high-contrast high-intensity laser pulses. Kant *et al.* [15] investigated the affect of density ramp on resonant SHG in plasma depending on density transition. Kant *et al.* [16] studied the resonant THG of a short-pulse laser from electron hole plasmas and THG takes place due to interaction of high-power laser radiation through a semi-conductor plasma. phase matching condition was satisfied by wiggler field that enhances the amplitude of third harmonic pulse. Kant [17] analysed the THG when intense pulse laser propagates in a ionizing plasma. Ionization is caused by the electric field along the axis and less ionization off the axis. Due to ionization density gradient set up which is maximum on the axis of propagation results focusing and defocusing of the third harmonic pulse. Thakur *et al.* [18] investigated the self-focusing under relativistic conditions and SHG in plasma under exponential density profile exponential density ramp. When density ramp is introduced it results stronger self-focusing and enhanced amplitude of SHG due to relativistic self-focusing was reported. Thakur at al. [19] studied the SHG under the influence of exponential density ramp when a intense pulse laser propagates in magnetized plasma and their results shows that increase efficiency of SHG results in the existence of plasma density ramp.

In present study we are investigating the efficiency THG by intense laser in plasma under relativistic conditions and effect of exponential density ramp is also analysed. We have derived the expressions for beam width parameter and normalized amplitude of third harmonic taking density ramp in consideration. Comparative analysis of beam width parameter is done with and without density ramp. Wiggler magnetic field provides additional momentum to phototns of third harmonic pulse to satisfy the Phase matching condition, result gain in efficiency of third harmonic pulse.

7.2 Theoretical considerations

Consider an intense Gaussian pulse laser beam propagating through plasma having electric and magnetic field as $\vec{E}_1$ and $\vec{B}_1$ given as

$$\vec{E}_1 = \hat{x}A_1(z,t)\exp[-i(\omega_1 t - k_1 z)], \tag{7.1}$$

$$A_1 = A_0 \exp\left[-ik_1 s\right] \tag{7.1a}$$

$$A_0^2 = \frac{A_{10}^2}{f^2(z)}\, e^{-r^2/r_0^2 f^2} \tag{7.1b}$$

$$A_0 = \frac{A_{10}}{f(z)}\, e^{-r^2/2r_0^2 f^2} \tag{7.1c}$$

$$\vec{B}_1 = \frac{c\vec{k}_1 \times \vec{E}_1}{\omega_1}, \tag{7.2}$$

$$\vec{B}_w = \hat{y}B_0 \exp(ik_0 z), \tag{7.3}$$

Where ω_1 is the frequency of pump laser, $\vec{k}_1$ is the wave vector of pump laser pulse, $\vec{B}_w$ is the wiggler field and $\vec{k}_0$ is the wiggler number that provides the phase matching condition, A_{10} is the maximum amplitude of the incident laser pulse, f_1 is the beam width parameter of the main laser pulse and r_0 is the spot size of the main laser pulse.

Due to electromagnetic field of short pulse laser, electrons experience ponderomotive force and acquire quiver velocity $v = e\hat{E}/mi\omega_1\gamma$ at $2\omega_1$ and on beating with density oscillations results perturbed density at $2\omega_1$. When perturbed density beats with quiver velocity at fundamental frequency incident produce third harmonic current density $\vec{J}_3$ driving resonant condition. $\vec{J}_3$ contain linear current density $\vec{J}_3^L$ [20] and nonlinear current density $\vec{J}_3^{NL}$ [20] given as $\vec{J}_3 = \vec{J}_3^L + \vec{J}_3^{NL}$

$$\vec{J}_3^L = -\frac{n_0 e^2 \vec{E}_3}{m\gamma 3i\omega_1}. \tag{7.4}$$

$$\vec{J}_3^{NL} = \frac{-n_0^0 e^5 B_w k_1 \vec{E}_1^3}{16cim^4\gamma^4\omega_1^4(\omega_1 + iv)}\left[\frac{5k_1}{18\omega_1} + \frac{k_1 + k_0}{\omega_1 + iv}\right]\hat{x}. \tag{7.5}$$

Where γ is the relativistic constant given as $\gamma = (1 + \left(e^2\ |A|^2 / m^2\omega^2 c^2\right)^2 / 2)^{1/2}$ is the laser field strength parameter.

From Eq. (4.13)

$$\nabla^2\vec{E}_3 + \left[\left(\frac{9\omega_1{}^2 - 10\omega_p^2}{c^2}\right) + \frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^*{}_1\right)}{c^2}\right]\vec{E}_3 = \frac{4\pi\partial\vec{J}_3^{NL}}{c^2\partial t},$$

where

$$\nabla^2\vec{E}_3 = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r}\right)\vec{E}_3,$$

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r}\right)\vec{E}_3 = \left[\frac{\varepsilon_0 + \phi\left(E_1E_1^*\right)}{c^2}\right]\frac{\partial^2 E}{\partial t^2} + \frac{4\pi\partial\vec{J}^L}{c^2\partial t} + \frac{4\pi\partial\vec{J}_3^{NL}}{c^2\partial t}. \tag{7.6}$$

and $\phi\left(\vec{E}_1\vec{E}^*{}_1\right) = \phi\left(\frac{A_{10}^2}{2f^2}\right) - \frac{A_{10}^2 r^2}{2r_0^2 f^4}\phi'\left(\frac{A_{10}^2}{2f^2}\right)$. (7.6a)

Where

$k_1 = \omega_1/c\left(1 - \omega_p^2/\omega_1^2\right)^{1/2}$, $\omega_p = \left(4\pi n_0 e^2/m_0\gamma\right)\exp\left(z/dR_d\right)$, $\omega_p^2 = \left(4\pi n_0 e^2/m_0\gamma\right)\exp\left(z/dR_d\right)$
and $\omega_{p0}^2 = \left(4\pi n_0 e^2/m_0\right)$

Therefore, $k_1 = \omega_1/c\left(1 - \omega_{p0}^2/\gamma\exp\left(z/dR_d\right)\right)^{1/2}$. (7.7)

Using Eq. (7. 1)

$$\frac{\partial \vec{E}}{\partial z} = \hat{x}\exp[-i(\omega_1 t - k_1 z)]\frac{\partial A_1}{\partial z} + \hat{x}A(i)\exp[-i(\omega_1 t - k_1 z)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right], \qquad (7.\ 8)$$

$$\frac{\partial^2 \vec{E}}{\partial z^2} = \hat{x}\exp[-i(\omega_1 t - k_1 z)]\frac{\partial^2 A_1}{\partial z^2} + \hat{x}(i)\exp[-i(\omega_1 t - k_1 z)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right]\frac{\partial A_1}{\partial z}$$

$$+\hat{x}(i)\exp[-i(\omega_1 t - k_1 z)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right]\frac{\partial A_1}{\partial z} + \hat{x}A_1(i)^2\exp[-i(\omega_1 t - k_3 z)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right]^2$$

$$+\hat{x}(i)A_1\exp[-i(\omega_1 t - k_1 z)]\left[2\frac{\partial k_1}{\partial z} + z\frac{\partial^2 k_1}{\partial z^2}\right],$$

$$\frac{\partial^2 \vec{E}}{\partial z^2} = \hat{x}\exp[-i(\omega_1 t - k_1 z)]\frac{\partial^2 A_1}{\partial z^2} + 2\hat{x}(i)\exp[-i(\omega_1 t - k_1 z)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right]\frac{\partial A_1}{\partial z}$$

$$+\hat{x}A_1(i)^2\exp[-i(\omega_1 t - k_1 z)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right]^2 + \hat{x}(i)A_1\exp[-i(\omega_1 t - k_1 z)]\left[2\frac{\partial k_1}{\partial z} + z\frac{\partial^2 k_1}{\partial z^2}\right],$$

$\partial^2 A_1/\partial z^2$ being small is neglected

$$\frac{\partial^2 \vec{E}}{\partial z^2} = 2\hat{x}(i)\exp[-i(\omega_1 t - k_1 z)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right]\frac{\partial A_1}{\partial z}$$

$$+\hat{x}A_1(i)^2\exp[-i(\omega_1 t - k_1 z)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right]^2 + \hat{x}(i)A_1\exp[-i(\omega_1 t - k_1 z)]\left[2\frac{\partial k_1}{\partial z} + z\frac{\partial^2 k_1}{\partial z^2}\right], \qquad (7.\ 9)$$

using Eq. (7.7)

$$\frac{\partial k_1}{\partial z} == -\frac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}}, \qquad (7.10)$$

$$\frac{\partial^2 k_1}{\partial z^2} == -\frac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma(dR_d)^2\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}} + \frac{\left(\omega_{p0}^2\exp(z/dR_d)\right)^2}{4c(\gamma dR_d)^2\omega_1^3\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{3/2}}, \qquad (7.11\ )$$

using Eq. (7.1a)

$$\frac{\partial A_1}{\partial z} = \exp[-ik_1 s)]\left[A_0(-i)(z,t)\left[s\frac{\partial k_1}{\partial z} + k_1\frac{\partial s}{\partial z}\right] + \frac{\partial A_0}{\partial z}\right], \qquad (7.\ 12)$$

putting Eq. (7.12) and (7.1a) in Eq. (7.9), we get

$$\frac{\partial^2 \vec{E}}{\partial z^2} = 2\hat{x}(i)\exp[-i(\omega_1 t - k_1 z)]\exp[-ik_1 s)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right]\left[A_0(-i)(z,t)\left[s\frac{\partial k_1}{\partial z} + k_1\frac{\partial s}{\partial z}\right] + \frac{\partial A_0}{\partial z}\right]$$

$$+ \hat{x}A_0(i)^2 \exp[-i(\omega_1 t - k_1 z)]\exp[-ik_1 s)]\left[k_1 + z\frac{\partial k_1}{\partial z}\right]^2 \tag{7.13}$$

$$+ \hat{x}(i)A_0 \exp[-i(\omega_1 t - k_1 z)]\exp[-ik_1 s)]\left[2\frac{\partial k_1}{\partial z} + z\frac{\partial^2 k_1}{\partial z^2}\right],$$

Putting Eqs. (7.7), (7.10) and (7.12) into Eq. (7.13), we get

$$\frac{\partial^2 \vec{E}}{\partial z^2} = R\left\{\begin{array}{l}
A_0\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2} - z\frac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\right] \\
\left\{A_0\left[\begin{array}{l} -s\frac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}} \\ +\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\frac{\partial s}{\partial z}\end{array}\right] + \frac{\partial A_0}{\partial z}\right\} \\
-\hat{x}A_0\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2} - z\frac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\right]^2 \\
+\hat{x}(i)A_0\left[\begin{array}{l} 2-\frac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}} - z\frac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma(dR_d)^2\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}} \\ +z\frac{\left(\omega_{p0}^2\exp(z/dR_d)\right)^2}{4c(\gamma dR_d)^2\omega_1^3\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{3/2}}\end{array}\right]
\end{array}\right\}, \tag{7.14}$$

R = $\hat{x}(z,t)\exp[-ik_3 s_3)]\exp[-i(3\omega_1 t - k_3 z)]$,

using Eq. (7.1)

$$\frac{\partial \vec{E}_1}{\partial r} = \hat{x}(z,t)\exp[-i(\omega_1 t - k_1 z)]\frac{\partial A_1}{\partial r}, \tag{7.15}$$

$$\frac{\partial^2 \vec{E}_1}{\partial r^2} = \hat{x}(z,t)\exp[-i(\omega_1 t - k_1 z)]\frac{\partial^2 A_1}{\partial r^2}, \tag{7.16}$$

from Eq. (7.1a) we obtain

$$\frac{\partial A}{\partial r} = \frac{\partial A_0}{\partial r}(z,t)\exp[-ik_1 s)] + A_0(-ik_1)\exp[-(ik_1 s)]\frac{\partial s}{\partial r}, \tag{7.17}$$

$$\frac{\partial^2 A_1}{\partial r^2} = \frac{\partial^2 A_0}{\partial r^2}(z,t)\exp[-ik_1 s)] + (-ik_1)\exp[-(ik_1 s)]\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r} + A_0(-ik_1)^2\exp[-(ik_1 s)]\left(\frac{\partial s}{\partial r}\right)^2$$
$$+ A_0(-ik_1)\exp[-(ik_1 s)]\frac{\partial^2 s}{\partial r^2} + (-ik_1)\exp[-(ik_1 s)]\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r},$$

$$\frac{\partial^2 A_1}{\partial r^2} = \frac{\partial^2 A_0}{\partial r^2}(z,t)\exp[-ik_1 s)] + 2(-ik_1)\exp[-(ik_1 s)]\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r} + A_0(-ik_1)^2\exp[-(ik_1 s)]\left(\frac{\partial s}{\partial r}\right)^2$$
$$+ A_0(-ik_1)\exp[-(ik_1 s)]\frac{\partial^2 s}{\partial r^2}, \tag{7.18}$$

putting Eq. (7.17) in Eq. (7.15) we get

$$\frac{\partial \vec{E}_3}{\partial r} = \hat{x}(z,t)\exp[-i(\omega_1 t - k_1 z)]\exp[-(ik_1 s)]\left[\frac{\partial A_0}{\partial r}(z,t) + A_0(-ik_1)\frac{\partial s}{\partial r}\right], \tag{7.19}$$

Now putting Eq. (7.18) into Eq. (7.16) we get

$$\frac{\partial^2 \vec{E}_1}{\partial r^2} = \hat{x}(z,t)\exp[-i(\omega_1 t - k_1 z)]\exp[-ik_1 s)]\left[\begin{array}{l}\frac{\partial^2 A_0}{\partial r^2}(z,t) + 2(-ik_1)\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r} + A_0(-ik_1)^2\left(\frac{\partial s}{\partial r}\right)^2 \\ + A_0(-ik_1)\frac{\partial^2 s}{\partial r^2},\end{array}\right] \tag{7.20}$$

Putting Eqs. (7.15), (7.19) and (7.20) into Eq. (7.6)

$$
R\left\{
\begin{array}{l}
\left[\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2} - z\,\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right] \\
\left\{A_0\left[\begin{array}{l} -s\,\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} \\ +\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\dfrac{\partial s}{\partial z}\end{array}\right] + \dfrac{\partial A_0}{\partial z}\right\} \\
-\hat{x}A_0\left[\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2} - z\,\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right]^2 \\
+\hat{x}(i)A_0\left[\begin{array}{l} -2\,\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} \\ -z\,\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma\left(dR_d\right)^2\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} \\ +z\,\dfrac{\left(\omega_{p0}^2\exp\left(z/dR_d\right)\right)^2}{4c\left(\gamma dR_d\right)^2\omega_1^3\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{3/2}}\end{array}\right]
\end{array}
\right\}
$$

$$
+\hat{x}(z,t)R\left[\frac{\partial^2 A_0}{\partial r^2}(z,t)+2\left(-ik_1\right)\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r}+A_0\left(-ik_1\right)^2\left(\frac{\partial s}{\partial r}\right)^2+A_0\left(-ik_1\right)\frac{\partial^2 s}{\partial r^2}\right] \tag{7.21}
$$

$$
+\hat{x}(z,t)R\frac{1}{r}\left[\frac{\partial A_0}{\partial r}(z,t)+A_0\left(-ik_1\right)\frac{\partial s}{\partial r}\right]-\left[\frac{\varepsilon_0+\phi\left(E_1E_1^*\right)}{c^2}\right]\frac{\partial^2 E}{\partial t^2}-\frac{4\pi\partial\vec{J}^{\,L}}{c^2\partial t}=\frac{4\pi\partial\vec{J}_3^{\,NL}}{c^2\partial t},
$$

from Eq. (4.12)

$$
\frac{4\pi}{c^2}\frac{\partial\vec{J}_3^{\,L}}{\partial t}+\left[\frac{\varepsilon_0+\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]\frac{\partial^2\vec{E}_3}{\partial t^2}=\frac{10\omega_p^2}{c^2}\vec{E}_3\,\frac{-9\omega_1^{\,2}\vec{E}_3}{c^2}-\frac{9\omega_1^{\,2}\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)\vec{E}_3}{c^2}, \tag{7.22}
$$

putting Eq. (7.22) into Eq. (7.21) we obtain

$$
R\left\{
\begin{aligned}
&\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}-z\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right] \\
&\left\{A_0\left[\begin{aligned}&-s\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\\ &+\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\frac{\partial s}{\partial z}\end{aligned}\right]+\frac{\partial A_0}{\partial z}\right\} \\
&-\hat{x}A_0\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}-z\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right]^2 \\
&+\hat{x}(i)A_0\left[\begin{aligned}&-2\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}-z\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma\left(dR_d\right)^2\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\\ &+z\frac{\left(\omega_{p0}^2\exp\left(z/dR_d\right)\right)^2}{4c\left(\gamma dR_d\right)^2\omega_1^3\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{3/2}}\end{aligned}\right]
\end{aligned}
\right\}
$$

$$
+\hat{x}(z,t)R\left[\frac{\partial^2 A_0}{\partial r^2}(z,t)+2\left(-ik_1\right)\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r}+A_0\left(-ik_1\right)^2\left(\frac{\partial s}{\partial r}\right)^2+A_0\left(-ik_1\right)\frac{\partial^2 s}{\partial r^2}\right]
$$

$$
+\hat{x}(z,t)R\frac{1}{r}\left[\frac{\partial A_0}{\partial r}(z,t)+A_0\left(-ik_1\right)\frac{\partial s}{\partial r}\right]-\left[\frac{10\omega_p^2-9\omega_1{}^2}{c^2}-\frac{9\omega_1{}^2\,\phi\left(\vec{E}_1\vec{E}^{\,*}_{\,1}\right)}{c^2}\right]\vec{E}_1=\frac{4\pi\partial\vec{J}_3^{\,NL}}{c^2\partial t},
$$

equating real and imaginary parts the real part is

$$
R\left\{\begin{array}{l}
\left[\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2} - z\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right] \\
\left\{A_0\left[-s\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} + \dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\dfrac{\partial s}{\partial z}\right]\right\} \\
-\hat{x}A_0\left[\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2} - z\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right]^2
\end{array}\right\}
$$

$$
+\hat{x}(z,t)R\left[\frac{\partial^2 A_0}{\partial r^2}(z,t) + A_0\left(-ik_1\right)^2\left(\frac{\partial s}{\partial r}\right)^2\right] \tag{7.23}
$$

$$
+\hat{x}(z,t)R\frac{1}{r}\left[\frac{\partial A_0}{\partial r}(z,t)\right] - \left[\frac{10\omega_p^2-9\omega_1{}^2}{c^2} - \frac{9\omega_1{}^2\,\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]\vec{E}_1 = \frac{4\pi\partial\vec{J}_3^{NL}}{c^2\partial t},
$$

using Eqs. (4.20b) and (4.20e) into Eq. (7.23) we obtain

$$
R\left\{\begin{array}{l}
\left[\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2} - z\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right] \\
\left\{A_0\left[\begin{array}{l} -\left(\beta(z)\dfrac{r^2}{2}+\phi(z)\right)\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} \\ +\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\left(\dfrac{r^2}{2}\dfrac{\partial\beta}{\partial z}+\dfrac{\partial\phi}{\partial z}\right)\end{array}\right]\right\} \\
-\hat{x}A_0\left[\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2} - z\dfrac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right]^2
\end{array}\right\}
$$

$$
+\hat{x}(z,t)R\left[\frac{\partial^2 A_0}{\partial r^2}(z,t) - A_0 {k_1}^2 (r\beta)^2\right] \tag{7.24}
$$

$$
+\hat{x}(z,t)R\frac{1}{r}\left[\frac{\partial A_0}{\partial r}(z,t)\right] - \left[\frac{10\omega_p^2 - 9{\omega_1}^2}{c^2} - \frac{9{\omega_1}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_{\,1}\right)}{c^2}\right]\vec{E}_1 = 0,
$$

using Eq. (7.1c) we obtain

$$
\frac{\partial A_0}{\partial r} = \frac{A_{10}}{f(z)}e^{-r^2/2r_0^2f^2}\left(\frac{-r}{r_0^2 f^2}\right),
$$

$$
\frac{\partial A_0}{\partial r} = \frac{-A_{10}r}{r_0^2 f^3}e^{-r^2/2r_0^2f^2}, \tag{7.25}
$$

$$
\frac{\partial^2 A_0}{\partial r^2} = \frac{-A_{10}}{r_0^2 f^3}e^{-r^2/2r_0^2f^2} + \frac{A_{10}r^2}{r_0^4 f^5}e^{-r^2/2r_0^2f^2},
$$

$$\frac{\partial^2 A_0}{\partial r^2} = \left(\frac{A_{10} r^2}{r_0^4 f^5} - \frac{A_{10}}{r_0^2 f^3}\right) e^{-r^2/2r_0^2 f^2}, \tag{7.26}$$

putting Eqs. (7.25), (7.26) and (7.6a) into Eq. (7.24) we obtain

$$R\left\{\begin{array}{l}\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2} - z\frac{\omega_{p0}^2 \exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\right] \\ \left\{A_{10}e^{-r^2/2r_0^2 f^2}\left[\begin{array}{l}-\left(\frac{1}{f}\frac{\partial f}{\partial z}\frac{r^2}{2}+\phi(z)\right)\frac{\omega_{p0}^2 \exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}} \\ +\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\left(\frac{r^2}{2}\left(\frac{1}{f}\frac{\partial^2 f}{\partial z^2}\right)+\frac{\partial\phi}{\partial z}\right)\end{array}\right]\right\} \\ -\hat{x}A_{10}e^{-r^2/2r_0^2 f^2}\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2} - z\frac{\omega_{p0}^2 \exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\right]^2\end{array}\right\}$$

$$+\hat{x}(z,t)R\left[\left(\frac{A_{10} r^2}{r_0^4 f^5} - \frac{A_{10}}{r_0^2 f^3}\right) e^{-r^2/2r_0^2 f^2} - A_{10}e^{-r^2/2r_0^2 f^2} k_1^{\,2}\left(r\frac{1}{f}\frac{\partial f}{\partial z}\right)^2\right]$$

$$+\hat{x}(z,t)R\frac{1}{r}\left[\frac{-A_{10} r}{r_0^2 f^3}e^{-r^2/2r_0^2 f^2}\right] - \left[\frac{10\omega_p^2 - 9\omega_1{}^2}{c^2} - \frac{9\omega_1{}^2}{c^2}\phi\left(\frac{A_{10}^2}{2f^2}\right) - \frac{9\omega_1^2}{c^2}\frac{A_{10}^2 r^2}{2r_0^2 f^4}\phi'\left(\frac{A_{10}^2}{2f^2}\right)\right] RA_{10}e^{-r^2/2r_0^2 f^2} = 0,$$

$$
\left\{
\begin{aligned}
&\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2} - z\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right] \\
&\left\{A_{10}\left[\begin{aligned}&-\left(\frac{1}{f}\frac{\partial f}{\partial z}\frac{r^2}{2}+\phi(z)\right)\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\\ &+\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\left(\frac{r^2}{2}\left(\frac{1}{f}\frac{\partial^2 f}{\partial z^2}\right)+\frac{\partial\phi}{\partial z}\right)\end{aligned}\right]\right\} \\
&-\hat{x}A_{10}\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2} - z\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\,\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right]^2
\end{aligned}
\right\}
$$

$$
+\hat{x}(z,t)\left[\left(\frac{A_{10}r^2}{r_0^4 f^5}-\frac{A_{10}}{r_0^2 f^3}\right)-A_{10}{k_1}^2\left(r\frac{1}{f}\frac{\partial f}{\partial z}\right)^2\right]
$$

$$
+\hat{x}(z,t)\frac{1}{r}\left[\frac{-A_{10}r}{r_0^2 f^3}\right]-\left[\frac{10\omega_p^2-9{\omega_1}^2}{c^2}-\frac{9{\omega_1}^2}{c^2}\phi\left(\frac{A_{10}^2}{2f^2}\right)-\frac{9\omega_1^2}{c^2}\frac{A_{10}^2 r^2}{2r_0^2 f^4}\phi'\left(\frac{A_{10}^2}{2f^2}\right)\right]A_{10}=0,
$$

$$
\left\{
\begin{aligned}
&\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2} - z\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right] \\
&\left\{A_{10}\left[\begin{aligned}&-\left(\frac{1}{f}\frac{\partial f}{\partial z}\frac{r^2}{2}+\phi(z)\right)\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} \\ &+\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\left(\frac{r^2}{2}\left(\frac{1}{f}\frac{\partial^2 f}{\partial z^2}\right)+\frac{\partial\phi}{\partial z}\right)\end{aligned}\right]\right\} \\
&-\hat{x}A_{10}\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2} - z\frac{\omega_{p0}^2\exp\left(z/dR_d\right)}{2c\gamma dR_d\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}}\right]^2
\end{aligned}
\right\}
$$

$$
+\hat{x}(z,t)\left[\left(\frac{A_{10}r^2}{r_0^4 f^5}-\frac{A_{10}}{r_0^2 f^3}\right)-A_{10}k_1^{\,2}\left(r\frac{1}{f}\frac{\partial f}{\partial z}\right)^2\right]
$$

$$
+\hat{x}(z,t)\frac{1}{r}\left[\frac{-A_{10}r}{r_0^2 f^3}\right]-\left[\frac{10\omega_p^2-9\omega_1^{\,2}}{c^2}-\frac{9\omega_1^{\,2}}{c^2}\phi\left(\frac{A_{10}^2}{2f^2}\right)-\frac{9\omega_1^2}{c^2}\frac{A_{10}^2 r^2}{2r_0^2 f^4}\phi'\left(\frac{A_{10}^2}{2f^2}\right)\right]A_{10}=0,
$$

equating coefficient of $r^2 = 0$

$$\left\{\begin{array}{l}\left[\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2} - z\dfrac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\right] \\ \left\{A_{10}\left[\begin{array}{l}-\left(\dfrac{1}{f}\dfrac{\partial f}{\partial z}\dfrac{r^2}{2}\right)\dfrac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}} \\ +\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\left(\dfrac{r^2}{2}\left(\dfrac{1}{f}\dfrac{\partial^2 f}{\partial z^2}\right)\right)\end{array}\right]\right\}\end{array}\right\}$$

$$+\hat{x}(z,t)\left[\left(\frac{A_{10}r^2}{r_0^4 f^5}\right)-A_{10}k_1{}^2\left(r\frac{1}{f}\frac{\partial f}{\partial z}\right)^2\right]$$

$$+\left[-\frac{9\omega_1^2}{c^2}\frac{A_{10}^2 r^2}{2r_0^2 f^4}\phi'\left(\frac{A_{10}^2}{2f^2}\right)\right]A_{10}=0,$$

$$\phi'\left(\frac{A_{10}^2}{2f^2}\right)=\frac{\in_2\in_S}{\in_0}\exp\left(\frac{-\in_2}{2\in_0}\frac{A_{10}^2}{f^2}\right),$$

$$\phi'\left(\frac{A_{10}^2}{2f^2}\right)=\frac{\in_2\in_S}{\in_0}\exp\left(\frac{-\in_2}{2\in_0}\frac{A_{10}^2}{f^2}\right),$$

$$\left[\begin{array}{l}\dfrac{\omega_1}{c}\left(1-\dfrac{1}{\omega_1^2}\left(\dfrac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2} \\ -zW\end{array}\right]\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\left(\frac{1}{2}\left(\frac{1}{f}\frac{\partial^2 f}{\partial z^2}\right)\right)$$

$$=A_{10}\left\{\left[\frac{\omega_1}{c}\left(1-\frac{1}{\omega_1^2}\left(\frac{4\pi n_0 e^2}{m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}-zW\right]\left\{\left[-\left(\frac{1}{f}\frac{\partial f}{\partial z}\frac{1}{2}\right)W\right]\right\}\right\}$$

$$+\hat{x}(z,t)\left[\left(\frac{A_{10}}{r_0^4 f^5}\right)-A_{10}k_1{}^2\left(\frac{1}{f}\frac{\partial f}{\partial z}\right)^2\right]+\left[-\frac{9\omega_1^2}{c^2}\frac{A_{10}^2}{2r_0^2 f^4}\frac{\in_2\in_S}{\in_0}\exp\left(\frac{-\in_2}{2\in_0}\frac{A_{10}^2}{f^2}\right)\right]A_{10}=0,$$

where $W=\dfrac{\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp(z/dR_d)\right)^{1/2}}$

$$\begin{bmatrix} \dfrac{\omega_1}{c}\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\exp\left(\dfrac{\xi}{d}\right)\right)^{1/2} \\ -zW \end{bmatrix}\dfrac{\omega_1}{c}\left(1-\dfrac{\omega_{p0}^2}{\omega_1^2\gamma}\exp\left(\dfrac{\xi}{d}\right)\right)^{1/2}\left(\dfrac{1}{2}\left(\dfrac{1}{f}\dfrac{\partial^2 f}{\partial z^2}\right)\right)$$

$$= A_{10}\left\{\left[\dfrac{\omega_1}{c}\left(1-\dfrac{\omega_{p0}^2}{\omega_1^2\gamma}\exp\left(\dfrac{\xi}{d}\right)\right)^{1/2} - zW\right]\left\{\left[-\left(\dfrac{1}{f}\dfrac{\partial f}{\partial z}\dfrac{1}{2}\right)W\right]\right\}\right\}$$

$$+\hat{x}(z,t)\left[\left(\dfrac{A_{10}}{r_0^4 f^5}\right)-A_{10}k_1{}^2\left(\dfrac{1}{f}\dfrac{\partial f}{\partial z}\right)^2\right]+\left[-\dfrac{9\omega_1^2}{c^2}\dfrac{A_{10}^2}{2r_0^2 f^4}\dfrac{\in_2\in_S}{\in_0}\exp\left(\dfrac{-\in_2}{2\in_0}\dfrac{A_{10}^2}{f^2}\right)\right]A_{10}=0,$$

$$\left(\dfrac{\omega_1}{c}\right)^2\begin{bmatrix} \left(\dfrac{\omega_1}{c}\right)^2\left(1-\dfrac{\omega_{p0}^2}{\omega_1^2\gamma}\exp\left(\dfrac{\xi}{d}\right)\right)^{1/2} \\ -\xi\dfrac{\omega_{p0}^2}{\omega_1^2}\dfrac{\exp\left(\dfrac{\xi}{d}\right)}{2\gamma d\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\left(\dfrac{\xi}{d}\right)\right)^{1/2}} \end{bmatrix}\left(1-\dfrac{\omega_{p0}^2}{\omega_1^2\gamma}\exp\left(\dfrac{\xi}{d}\right)\right)^{1/2}\left(\dfrac{1}{2}\left(\dfrac{1}{R_d^2 f}\dfrac{\partial^2 f}{\partial\xi^2}\right)\right)$$

$$= A_{10}\left(\dfrac{\omega_1}{c}\right)^2\left\{\left[\left(1-\dfrac{\omega_{p0}^2}{\omega_1^2\gamma}\exp\left(\dfrac{\xi}{d}\right)\right)^{1/2}-\dfrac{\omega_{p0}^2}{\omega_1^2}\dfrac{\xi\exp\left(\dfrac{\xi}{d}\right)}{2\gamma d\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\left(\dfrac{\xi}{d}\right)\right)^{1/2}}\right]\left\{\left[-\left(\dfrac{1}{f}\dfrac{\partial f}{\partial\xi}\dfrac{1}{2}\right)\dfrac{\omega_{p0}^2}{\omega_1^2}\dfrac{\exp\left(\dfrac{\xi}{d}\right)}{2\gamma dR_d^2\left(1-\dfrac{\omega_{p0}^2}{\gamma\omega_1^2}\left(\dfrac{\xi}{d}\right)\right)^{1/2}}\right]\right\}\right\}$$

$$+\hat{x}(z,t)\left[\left(\dfrac{A_{10}}{r_0^4 f^5}\right)-A_{10}k_1{}^2\left(\dfrac{1}{f}\dfrac{\partial f}{\partial z}\right)^2\right]+\left[-\dfrac{9\omega_1^2}{c^2}\dfrac{A_{10}^2}{2r_0^2 f^4}\dfrac{\in_2\in_S}{\in_0}\exp\left(\dfrac{-\in_2}{2\in_0}\dfrac{A_{10}^2}{f^2}\right)\right]A_{10}=0,$$

$$\left(\frac{\omega_1}{c}\right)^2\left[\begin{matrix}\left(\frac{\omega_1}{c}\right)^2\left(1-\frac{\omega_{p0}^2}{\omega_1^2\gamma}\exp\left(\frac{\xi}{d}\right)\right)^{1/2}\\ -\xi\frac{\omega_{p0}^2}{\omega_1^2}\frac{\exp\left(\frac{\xi}{d}\right)}{2\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\left(\frac{\xi}{d}\right)\right)^{1/2}}\end{matrix}\right]\left(1-\frac{\omega_{p0}^2}{\omega_1^2\gamma}\exp\left(\frac{\xi}{d}\right)\right)^{1/2}\left(\frac{1}{2}\left(\frac{1}{R_d^2 f}\frac{\partial^2 f}{\partial\xi^2}\right)\right)$$

$$=A_{10}\left(\frac{\omega_1}{c}\right)^2\left\{\left[\left(1-\frac{\omega_{p0}^2}{\omega_1^2\gamma}\exp\left(\frac{\xi}{d}\right)\right)^{1/2}-\frac{\omega_{p0}^2}{\omega_1^2}\frac{\xi\exp\left(\frac{\xi}{d}\right)}{2\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\left(\frac{\xi}{d}\right)\right)^{1/2}}\right]\left\{\left[-\left(\frac{1}{f}\frac{\partial f}{\partial\xi}\frac{1}{2}\right)\frac{\omega_{p0}^2}{\omega_1^2}\frac{\exp\left(\frac{\xi}{d}\right)}{2\gamma dR_d^2\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\left(\frac{\xi}{d}\right)\right)^{1/2}}\right]\right\}\right\}$$

$$+\hat{x}(z,t)\left[\left(\frac{A_{10}}{r_0^4 f^5}\right)-\frac{A_{10}k_1{}^2}{R_d^2}\left(\frac{1}{f}\frac{\partial f}{\partial\xi}\right)^2\right]+\left[-\frac{9\omega_1^2}{c^2}\frac{A_{10}^2}{2r_0^2 f^4}\frac{\in_2\in_S}{\in_0}\exp\left(\frac{-\in_2}{2\in_0}\frac{A_{10}^2}{f^2}\right)\right]A_{10}=0,$$

$$\frac{1}{\left(1-\frac{\omega_p^2}{\omega_1^2}\right)}\left[\begin{matrix}\left(\frac{\omega_1}{c}\right)^2\left(1-\left(\frac{\omega_{p0}^2}{\omega_1^2\gamma}\right)\exp\left(\frac{\xi}{d}\right)\right)^{1/2}\\ -\xi\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\frac{\exp\left(\frac{\xi}{d}\right)}{2\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\left(\frac{\xi}{d}\right)\right)^{1/2}}\end{matrix}\right]\left(1-\left(\frac{\omega_{p0}^2}{\omega_1^2\gamma}\right)\exp\left(\frac{\xi}{d}\right)\right)^{1/2}\left(\frac{1}{2}\left(\frac{1}{f}\frac{\partial^2 f}{\partial\xi^2}\right)\right)$$

$$=A_{10}\frac{1}{\left(1-\frac{\omega_p^2}{\omega_1^2}\right)}\left\{\left[\begin{matrix}\left(1-\left(\frac{\omega_{p0}^2}{\omega_1^2\gamma}\right)\exp\left(\frac{\xi}{d}\right)\right)^{1/2}\\ -\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\frac{\xi\exp\left(\frac{\xi}{d}\right)}{2\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\left(\frac{\xi}{d}\right)\right)^{1/2}}\end{matrix}\right]\left\{\left[-\left(\frac{1}{f}\frac{\partial f}{\partial\xi}\frac{1}{2}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\frac{\exp\left(\frac{\xi}{d}\right)}{2\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma\omega_1^2}\left(\frac{\xi}{d}\right)\right)^{1/2}}\right]\right\}\right\}$$

$$+\hat{x}(z,t)\left[\begin{matrix}\left(\frac{A_{10}}{f^5}\right)\\ -A_{10}\left(\frac{1}{f}\frac{\partial f}{\partial\xi}\right)^2\end{matrix}\right]+\left[-\frac{9\omega_1^2 r_0^2}{c^2}\frac{A_{10}^2}{2f^4}\frac{\in_2\in_S}{\in_0}\exp\left(\frac{-\in_2}{2\in_0}\frac{A_{10}^2}{f^2}\right)\right]A_{10}=0. \tag{7.27}$$

Particular integral of Eq (7.6) is

$$E_{33} = A_3^{'} \exp\left[-i\left\{3\omega_1 t - \left(3k_1 + k_0\right)z\right\}\right], \tag{7.28}$$

$$A_3^{'} = A_{30}^{'}(z)\psi_3\,,\ \psi_3 = \exp\left[-3r^2/2r_0^2 f_1^2\right]\exp\left(-3ik_1S_1\right), \tag{7.29}$$

using Eq. (7.28)

$$\frac{\partial \vec{E}_{33}}{\partial z} = \exp\left[-i\left\{3\omega_1 t - \left(3k_1 + k_0\right)z\right\}\right]\left[\frac{\partial A_3^{'}}{\partial z} + iA_3^{'}\left(z\frac{\partial k_3}{\partial z} + k_3\right)\right],$$

$$\frac{\partial^2 \vec{E}_{33}}{\partial z^2} = \exp\left[-i\left\{3\omega_1 t - \left(3k_1 + k_0\right)z\right\}\right]\left[\begin{aligned}&\frac{\partial^2 A_3^{'}}{\partial z^2} + i\exp\left[-i\left\{3\omega_1 t - \left(3k_1 + k_0\right)z\right\}\right]\left(z\frac{\partial k_3}{\partial z} + k_3\right)\frac{\partial A_3^{'}}{\partial z}\\ &+ iA_3^{'}\left(z\frac{\partial^2 k_3}{\partial z^2} + \frac{\partial k_3}{\partial z} + \frac{\partial k_3}{\partial z}\right) + i\left(z\frac{\partial k_3}{\partial z} + k_3\right)\frac{\partial A_3^{'}}{\partial z}\\ &+ i^2 A_3^{'}\left(z\frac{\partial k_3}{\partial z} + k_3\right)^2,\end{aligned}\right]$$

neglecting the small term containing $\partial^2 A_3^{'}/\partial z^2$

$$\frac{\partial^2 \vec{E}_{33}}{\partial z^2} = \left\{2i\left(z\frac{\partial k_3}{\partial z} + k_3\right)\frac{\partial A_3^{'}}{\partial z} + iA_3^{'}\left(z\frac{\partial^2 k_3}{\partial z^2} + 2\frac{\partial k_3}{\partial z}\right) + i^2 A_3^{'}\left(z\frac{\partial k_3}{\partial z} + k_3\right)^2\right\}\left[-i\left\{3\omega_1 t - \left(3k_1 + k_0\right)z\right\}\right], \tag{7.30}$$

using eq. (7.29)

$$\frac{\partial A_3^{'}}{\partial z} = \frac{\partial A_{30}^{'}}{\partial z}\psi_3 + A_{30}^{'}\frac{\partial \psi_3}{\partial z}\,,$$

$$\begin{aligned}\frac{\partial A_3^{'}}{\partial z} &= \frac{\partial A_{30}^{'}}{\partial z}\psi_3 + A_{30}^{'}\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp\left(-3ik_1S_1\right)\left(\frac{3r^2}{2r_0^2 f_1^3}\right)\frac{\partial f_1}{\partial z}\\ &- A_{30}^{'}\left(-3i\right)\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp\left(-3ik_3S_1\right)\left[\frac{\partial k_3}{\partial z} + \frac{\partial S_1}{\partial z}\right],\end{aligned} \tag{7.31}$$

using Eqs. (7.31), (4.20b), (4.22), (7.12) & (7.13) into Eq. (7.30) we obtain

$$\frac{\partial^2 \vec{E}_{33}}{\partial z^2} = 2iR\left(-z\frac{3\omega_{p0}^2 \exp\left(z/dR_d\right)}{2c\gamma dR_d 9\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} + \left(\frac{3\omega_1}{c}\right)\left[\left(1-\left(\frac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\right]\right)\left\{ \begin{array}{l} \frac{\partial A_{30}^{'}}{\partial z}\psi_3 \\ + A_{30}^{'}\psi_3\left(\frac{3r^2}{r_0^2 f_1^3}\right)\frac{\partial f_1}{\partial z} \\ - A_{30}^{'}\psi_3 3i\left(-\frac{3\omega_{p0}^2 \exp\left(z/dR_d\right)}{2c\gamma dR_d 9\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} + \frac{r^2}{2}\frac{\partial \beta}{\partial z} + \frac{\partial \phi}{\partial z}. \right) \end{array} \right\}$$

$$+ iA_{30}^{'}\psi_3 R\left(z\left(-\frac{3\omega_{p0}^2 \exp\left(z/dR_d\right)}{2c\gamma\left(dR_d\right)^2 9\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} - \frac{3\left(\omega_{p0}^2 \exp\left(z/dR_d\right)\right)^2}{4c\omega_1^3\left(\gamma dR_d 9\right)^2\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp\left(z/dR_d\right)\right)^{3/2}} \right) - 2\frac{3\omega_{p0}^2 \exp\left(z/dR_d\right)}{2c\gamma dR_d 9\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} \right)$$

$$+ i^2 A_{30}^{'}\psi_3 R\left(-z\frac{3\omega_{p0}^2 \exp\left(z/dR_d\right)}{2c\gamma dR_d 9\omega_1\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp\left(z/dR_d\right)\right)^{1/2}} + \left(\frac{3\omega_1}{c}\right)\left[\left(1-\left(\frac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\right]\right)^2 , \tag{7.32}$$

$$\frac{\partial E_{33}}{\partial r} = \exp\left[-i\left\{3\omega_1 t - \left(3k_1 + k_0\right)z\right\}\right]A_{30}^{'}(z)\left(\frac{-3r}{r_0^2 f_1^2}\right)\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp\left(-3ik_1 S_1\right)$$
$$+ A_{30}^{'}(z)\left(-3ik_1\right)\exp\left[-i\left\{3\omega_1 t - \left(3k_1 + k_0\right)z\right\}\right]\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp\left(-3ik_1 S_1\right)\frac{\partial S_1}{\partial r}, \tag{7.33}$$

using Eqs. (4.20e) and (4.22) into Eq. (7.33)

$$\frac{\partial E_{33}}{\partial r} = \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]A'_{30}(z)\left(\frac{-3r}{r_0^2 f_1^2}\right)\psi_3$$

$$+ A'_{30}(z)(-3ik_1)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\psi_3\left(r\frac{\partial f_1}{f\partial z}\right), \tag{7.34}$$

using Eq. (7.33)

$$\frac{\partial^2 E_{33}}{\partial r^2} = \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]A'_{30}(z)\left(\frac{-3}{r_0^2 f_1^2}\right)\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp(-3ik_1 S_1)$$

$$+ A'_{30}(z)\left(\frac{-3r}{r_0^2 f_1^2}\right)^2 \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp(-3ik_1 S_1)$$

$$+ A'_{30}(z)(-3ik_1)\left(\frac{-3r}{r_0^2 f_1^2}\right)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp(-3ik_1 S_1)\frac{\partial S_1}{\partial r}$$

$$+ A'_{30}(z)(-3ik_1)\left(\frac{-3r}{r_0^2 f_1^2}\right)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp(-3ik_1 S_1)\frac{\partial S_1}{\partial r}$$

$$+ A'_{30}(z)(-3ik_1)^2 \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp(-3ik_1 S_1)\left(\frac{\partial S_1}{\partial r}\right)^2$$

$$+ A'_{30}(z)(-3ik_1)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\exp\left[\frac{-3r^2}{2r_0^2 f_1^2}\right]\exp(-3ik_1 S_1)\frac{\partial^2 S_1}{\partial r^2},$$

using Eqs. (4.20e), (4.20f) and (4.22) we get

$$\frac{\partial^2 E_{33}}{\partial r^2} = \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]A'_{30}(z)\left(\frac{-3}{r_0^2 f_1^2}\right)\psi_3$$

$$+ A'_{30}(z)\left(\frac{-3r}{r_0^2 f_1^2}\right)^2 \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\psi_3$$

$$+ 2A'_{30}(z)(-3ik_1)\left(\frac{-3r}{r_0^2 f_1^2}\right)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\psi_3\left(r\frac{\partial f_1}{f_1\partial z}\right)$$

$$+ A'_{30}(z)(-3ik_1)^2 \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\psi_3\left(r\frac{\partial f_1}{f_1\partial z}\right)^2$$

$$+ A'_{30}(z)(-3ik_1)\exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\psi_3\left(\frac{\partial f_1}{f_1\partial z}\right),$$

$$\frac{\partial^2 E_{33}}{\partial r^2}+\frac{\partial E_{33}}{r\partial r}=\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]A_{30}^{'}(z)\left(\frac{-3}{r_0^2 f_1^2}\right)\psi_3$$
$$+A_{30}^{'}(z)\left(\frac{-3r}{r_0^2 f_1^2}\right)^2\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]\psi_3$$
$$+2A_{30}^{'}(z)(-3ik_1)\left(\frac{-3r}{r_0^2 f_1^2}\right)\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]\psi_3\left(r\frac{\partial f_1}{f_1\partial z}\right)$$
$$+A_{30}^{'}(z)(-3ik_1)^2\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]\psi_3\left(r\frac{\partial f_1}{f_1\partial z}\right)^2$$
$$+A_{30}^{'}(z)(-3ik_1)\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]\psi_3\left(\frac{\partial f_1}{f_1\partial z}\right)+\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]A_{30}^{'}(z)\left(\frac{-3}{r_0^2 f_1^2}\right)\psi_3$$
$$+A_{30}^{'}(z)(-3ik_1)\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]\psi_3\left(\frac{\partial f_1}{f\partial z}\right),$$

$$\frac{\partial^2 E_{33}}{\partial r^2}+\frac{\partial E_{33}}{r\partial r}=\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]A_{30}^{'}(z)\left(\frac{-3}{r_0^2 f_1^2}\right)\psi_3$$
$$+A_{30}^{'}(z)\left(\frac{-3r}{r_0^2 f_1^2}\right)^2\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]\psi_3$$
$$+2A_{30}^{'}(z)(-3ik_1)\left(\frac{-3r}{r_0^2 f_1^2}\right)\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]\psi_3\left(r\frac{\partial f_1}{f_1\partial z}\right) \quad (7.35)$$
$$+A_{30}^{'}(z)(-3ik_1)^2\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]\psi_3\left(r\frac{\partial f_1}{f_1\partial z}\right)^2$$
$$+2A_{30}^{'}(z)(-3ik_1)\exp\left[-i\{3\omega_1 t-(3k_1+k_0)z\}\right]\psi_3\left(\frac{\partial f_1}{f_1\partial z}\right),$$

putting Eqs. (7.32), (7.34) and (7.35) into Eq. (7.6) we obtain

$$2iR\left(\begin{array}{l}-z\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\,9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\\+\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0e^2}{9\omega_1^2m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\end{array}\right)\left\{\begin{array}{l}\dfrac{\partial A_{30}^{'}}{\partial z}\psi_3\\+A_{30}^{'}\psi_3\left(\dfrac{3r^2}{r_0^2f_1^3}\right)\dfrac{\partial f_1}{\partial z}\\-A_{30}^{'}\psi_3\,3i\left(\begin{array}{l}-\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\,9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\\+\dfrac{r^2}{2}\dfrac{\partial\beta}{\partial z}+\dfrac{\partial\phi}{\partial z}.\end{array}\right)\end{array}\right\}$$

$$+iA_{30}^{'}\psi_3R\left(\begin{array}{l}z\left(-\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma(dR_d)^2 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}-\dfrac{3\left(\omega_{p0}^2\exp(z/dR_d)\right)^2}{4c\omega_1^3(\gamma dR_d 9)^2\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{3/2}}\right)\\-2\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\,9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\end{array}\right)$$

$$+i^2A_{30}^{'}\psi_3R\left(-z\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d\,9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}+\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0e^2}{9\omega_1^2m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\right)^2$$

$$+RA_{30}^{'}(z)\left(\dfrac{-3}{r_0^2f_1^2}\right)\psi_3+A_{30}^{'}(z)\left(\dfrac{-3r}{r_0^2f_1^2}\right)^2R\psi_3+2A_{30}^{'}(z)(-3ik_1)\left(\dfrac{-3r}{r_0^2f_1^2}\right)R\psi_3\left(r\dfrac{\partial f_1}{f_1\partial z}\right)$$

$$+A_{30}^{'}(z)(-3ik_1)^2R\psi_3\left(r\dfrac{\partial f_1}{f_1\partial z}\right)^2+2A_{30}^{'}(z)(-3ik_1)R\psi_3\left(\dfrac{\partial f_1}{f_1\partial z}\right)+-\left[\dfrac{10\omega_p^2}{c^2}\dfrac{-9\omega_1^2}{c^2}-\dfrac{9\omega_1^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]RA_{30}^{'}\psi_3=$$

$$\dfrac{4\pi\partial\vec{J}_3^{NL}}{c^2\partial t},$$

above equation is multiplied by $r\psi_3^*$ and integration is done with respect to 'r'

$$2iR\left(\begin{array}{l} -z\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}} \\ +\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\end{array}\right)\left\{\begin{array}{l}\dfrac{\partial A'_{30}}{\partial z}\int r\psi_3\psi_3^* + A'_{30}\left(\dfrac{3r^2}{r_0^2 f_1^3}\right)\dfrac{\partial f_1}{\partial z}\int r\psi_3\psi_3^* \\ -A'_{30}3i\left(\begin{array}{l} -\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}} \\ +\dfrac{r^2}{2}\dfrac{\partial\beta}{\partial z}+\dfrac{\partial\phi}{\partial z}.\end{array}\right)\int r\psi_3\psi_3^*\end{array}\right\}$$

$$+iA'_{30}R\left(\begin{array}{l} z\left(-\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma(dR_d)^2 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}-\dfrac{3\left(\omega_{p0}^2\exp(z/dR_d)\right)^2}{4c\omega_1^3(\gamma dR_d 9)^2\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{3/2}}\right) \\ -2\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\end{array}\right)\int r\psi_3\psi_3^*$$

$$+i^2A'_{30}R\left(-z\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}+\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\right)^2\int r\psi_3\psi_3^*$$

$$+RA'_{30}(z)\left(\frac{-3}{r_0^2 f_1^2}\right)\int r\psi_3\psi_3^* + A'_{30}(z)\left(\frac{-3r}{r_0^2 f_1^2}\right)^2 R\int r\psi_3\psi_3^* + 2A'_{30}(z)(-3ik_1)\left(\frac{-3r}{r_0^2 f_1^2}\right)R\left(r\frac{\partial f_1}{f_1\partial z}\right)\int r\psi_3\psi_3^*$$

$$+A'_{30}(z)(-3ik_1)^2R\left(r\frac{\partial f_1}{f_1\partial z}\right)^2\int r\psi_3\psi_3^* + 2A'_{30}(z)(-3ik_1)R\left(\frac{\partial f_1}{f_1\partial z}\right)\int r\psi_3\psi_3^*$$

$$+\left[\frac{10\omega_p^2}{c^2}\frac{-9\omega_1^{\,2}}{c^2}-\frac{9\omega_1^{\,2}\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]RA'_{30}\int r\psi_3\psi_3^* = \frac{4\pi\partial\vec{J}_3^{NL}}{c^2\partial t}\int r\psi_3^*,$$

$$\int r^3\psi_3\psi_3^* dr = \int r^3\exp\left[\frac{-3r^2}{r_0^2 f_1^2}\right] = \left(\frac{r_0^4 f_1^4}{18}\right)\int\frac{-3r^2}{\left(r_0^2 f_1^2\right)}\exp\left[\frac{-3r^2}{r_0^2 f_1^2}\right] = \left(\frac{r_0^4 f_1^4}{18}\right)\exp\left[\frac{-3r^2}{r_0^2 f_1^2}\right]\left(\frac{-3r^2}{r_0^2 f_1^2}-1\right)$$

,

$$2iR\left(\begin{array}{l} -z\dfrac{3\omega_{p0}^{2}\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^{2}}{\gamma 9\omega_1^{2}}\exp(z/dR_d)\right)^{1/2}} \\ +\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\end{array}\right)\left\{\begin{array}{l} \dfrac{\partial A_{30}^{'}}{\partial z}\left(\dfrac{r_0^2 f_1^2}{-6}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right] \\ +A_{30}^{'}\left(\dfrac{3r^2}{r_0^2 f_1^3}\right)\dfrac{\partial f_1}{\partial z}\left(\dfrac{r_0^4 f_1^4}{18}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]\left(\dfrac{-3r^2}{r_0^2 f_1^2}-1\right) \\ -A_{30}^{'}3i\left(\begin{array}{l} -\dfrac{3\omega_{p0}^{2}\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^{2}}{\gamma 9\omega_1^{2}}\exp(z/dR_d)\right)^{1/2}} \\ +\dfrac{r^2}{2}\dfrac{\partial\beta}{\partial z}+\dfrac{\partial\phi}{\partial z}. \end{array}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]\left(\dfrac{r_0^2 f_1^2}{-6}\right) \end{array}\right\}$$

$$+iA_{30}^{'}R\left(\begin{array}{l} z\left(-\dfrac{3\omega_{p0}^{2}\exp(z/dR_d)}{2c\gamma (dR_d)^2 9\omega_1\left(1-\dfrac{\omega_{p0}^{2}}{\gamma 9\omega_1^{2}}\exp(z/dR_d)\right)^{1/2}}-\dfrac{3\left(\omega_{p0}^{2}\exp(z/dR_d)\right)^2}{4c\omega_1^3(\gamma dR_d 9)^2\left(1-\dfrac{\omega_{p0}^{2}}{\gamma 9\omega_1^{2}}\exp(z/dR_d)\right)^{3/2}}\right) \\ -2\dfrac{3\omega_{p0}^{2}\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^{2}}{\gamma 9\omega_1^{2}}\exp(z/dR_d)\right)^{1/2}} \end{array}\right)\left(\dfrac{r_0^2 f_1^2}{-6}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]$$

$$+i^2A_{30}^{'}R\left(-z\dfrac{3\omega_{p0}^{2}\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^{2}}{\gamma 9\omega_1^{2}}\exp(z/dR_d)\right)^{1/2}}+\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\right)^2\left(\dfrac{r_0^2 f_1^2}{-6}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]$$

$$+RA_{30}^{'}(z)\left(\dfrac{-3}{r_0^2 f_1^2}\right)\left(\dfrac{r_0^2 f_1^2}{-6}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]+A_{30}^{'}(z)\left(\dfrac{-3}{r_0^2 f_1^2}\right)^2 R\left(\dfrac{r_0^4 f_1^4}{18}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]\left(\dfrac{-3r^2}{r_0^2 f_1^2}-1\right)$$

$$+2A_{30}^{'}(z)(-3ik_1)\left(\dfrac{-3}{r_0^2 f_1^2}\right)R\left(\dfrac{\partial f_1}{f_1\partial z}\right)\left(\dfrac{r_0^4 f_1^4}{18}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]\left(\dfrac{-3r^2}{r_0^2 f_1^2}-1\right)$$

$$+A_{30}^{'}(z)(-3ik_1)^2R\left(\dfrac{\partial f_1}{f_1\partial z}\right)^2\left(\dfrac{r_0^4 f_1^4}{18}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]\left(\dfrac{-3r^2}{r_0^2 f_1^2}-1\right)+2A_{30}^{'}(z)(-3ik_1)R\left(\dfrac{\partial f_1}{f_1\partial z}\right)\left(\dfrac{r_0^2 f_1^2}{-6}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]$$

$$+-\left[\dfrac{10\omega_p^2}{c^2}\dfrac{-9\omega_1{}^2}{c^2}-\dfrac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_1\right)}{c^2}\right]RA_{30}^{'}\left(\dfrac{r_0^2 f_1^2}{-6}\right)\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]=\dfrac{4\pi\partial\vec{J}_3^{NL}}{c^2\partial t}\int r\psi_3^{*}$$

$$2iR\left(\begin{array}{l}-z\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\\+\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\end{array}\right)\left\{\begin{array}{l}\dfrac{\partial A_{30}^{'}}{\partial z}\left(\dfrac{r_0^2 f_1^2}{-6}\right)\\+A_{30}^{'}\left(\dfrac{3r^2}{r_0^2 f_1^3}\right)\dfrac{\partial f_1}{\partial z}\left(\dfrac{r_0^4 f_1^4}{18}\right)\left(\dfrac{-3r^2}{r_0^2 f_1^2}-1\right)\\-A_{30}^{'}3i\left(\begin{array}{l}-\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\\+\dfrac{r^2}{2}\dfrac{\partial\beta}{\partial z}+\dfrac{\partial\phi}{\partial z}.\end{array}\right)\left(\dfrac{r_0^2 f_1^2}{-6}\right)\end{array}\right\}$$

$$+iA_{30}^{'}R\left(\begin{array}{l}z\left(-\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma(dR_d)^2 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}-\dfrac{3\left(\omega_{p0}^2\exp(z/dR_d)\right)^2}{4c\omega_1^3(\gamma dR_d 9)^2\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{3/2}}\right)\\-2\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\end{array}\right)\left(\dfrac{r_0^2 f_1^2}{-6}\right)$$

$$+i^2A_{30}^{'}R\left(-z\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}+\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\right)^2\left(\dfrac{r_0^2 f_1^2}{-6}\right)$$

$$+\frac{1}{2}RA_{30}^{'}(z)+\frac{1}{2}A_{30}^{'}(z)\left(\frac{-3r^2}{r_0^2 f_1^2}-1\right)$$

$$+A_{30}^{'}(z)(ik_1)R\left(\frac{\partial f_1}{f_1\partial z}\right)\left(r_0^2 f_1^2\right)\left(\frac{-3r^2}{r_0^2 f_1^2}-1\right)$$

$$-A_{30}^{'}(z)k_1^2R\left(\frac{\partial f_1}{f_1\partial z}\right)^2\left(\frac{r_0^4 f_1^4}{2}\right)\left(\frac{-3r^2}{r_0^2 f_1^2}-1\right)+A_{30}^{'}(z)(ik_1)R\left(\frac{\partial f_1}{f_1\partial z}\right)\left(\frac{r_0^2 f_1^2}{-6}\right)$$

$$+-\left[\frac{10\omega_p^2}{c^2}\frac{-9\omega_1^2}{c^2}-\frac{9\omega_1^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]RA_{30}^{'}\left(\frac{r_0^2 f_1^2}{-6}\right)=\frac{4\pi\partial\vec{J}_3^{NL}}{\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]c^2\partial t}\int r\psi_3^*,$$

neglecting coefficient of r^2

$$2iR\left(\begin{array}{l}-z\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\\+\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\end{array}\right)\left\{\begin{array}{l}\dfrac{\partial A_{30}^{'}}{\partial z}\left(\dfrac{r_0^2 f_1^2}{-6}\right)-A_{30}^{'}\left(\dfrac{3r^2}{r_0^2 f_1^3}\right)\dfrac{\partial f_1}{\partial z}\left(\dfrac{r_0^4 f_1^4}{18}\right)\\-A_{30}^{'}3i\left(\begin{array}{l}-\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\\+\dfrac{\partial\phi}{\partial z}.\end{array}\right)\left(\dfrac{r_0^2 f_1^2}{-6}\right)\end{array}\right\}$$

$$+iA_{30}^{'}R\left(\begin{array}{l}z\left(-\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma(dR_d)^2 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}-\dfrac{3\left(\omega_{p0}^2\exp(z/dR_d)\right)^2}{4c\omega_1^3(\gamma dR_d 9)^2\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{3/2}}\right)\\-2\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\end{array}\right)\left(\dfrac{r_0^2 f_1^2}{-6}\right)$$

$$+i^2A_{30}^{'}R\left(-z\dfrac{3\omega_{p0}^2\exp(z/dR_d)}{2c\gamma dR_d 9\omega_1\left(1-\dfrac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}+\left(\dfrac{3\omega_1}{c}\right)\left[\left(1-\left(\dfrac{4\pi n_0 e^2}{9\omega_1^2 m_0\gamma}\right)\exp\left(\dfrac{z}{dR_d}\right)\right)^{1/2}\right]\right)^2\left(\dfrac{r_0^2 f_1^2}{-6}\right)$$

$$+\frac{1}{2}RA_{30}^{'}(z)-\frac{1}{2}A_{30}^{'}R-A_{30}^{'}(z)(ik_1)R\left(\frac{\partial f_1}{f_1\partial z}\right)\left(r_0^2 f_1^2\right)$$

$$+A_{30}^{'}(z)k_1^2R\left(\frac{\partial f_1}{f_1\partial z}\right)^2\left(\frac{r_0^4 f_1^4}{2}\right)+A_{30}^{'}(z)(ik_1)R\left(\frac{\partial f_1}{f_1\partial z}\right)\left(\frac{r_0^2 f_1^2}{-6}\right)+-\left[\frac{10\omega_p^2}{c^2}\frac{-9\omega_1^2}{c^2}-\frac{9\omega_1^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]RA_{30}^{'}\left(\frac{r_0^2 f_1^2}{-6}\right)=$$

$$\frac{4\pi\partial\vec{J}_3^{\,NL}}{\exp\left[\dfrac{-3r^2}{r_0^2 f_1^2}\right]c^2\partial t}\int r\psi_3^*,$$

$$\frac{\partial^2 \vec{E}_{33}}{\partial z^2} = 2i\left(-\frac{\xi \exp(z/dR_d)}{6\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)+\left[\left(1-\left(\frac{1}{9\gamma}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\right]\right)\left\{\frac{\partial A_{30}^{'}}{\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\partial\xi}\left(\frac{f_1^2}{3}\right)\right\}$$

$$-2i\left(\begin{array}{l}-\frac{\xi \exp(z/dR_d)}{6\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\\ +\left[\left(1-\left(\frac{1}{9\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\right]\end{array}\right)\left\{\frac{A_{30}^{'} f_1^4}{\left(1-\frac{\omega_p^2}{\omega_1^2}\right)}\left(i\frac{\partial f_1}{f_1\partial\xi}\frac{1}{4}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)^2\left(\frac{\exp\left(\frac{z}{dR_d}\right)}{2\gamma d\left(1-\left(\frac{1}{\gamma}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}}\right)\right\}$$

$$++iA_{30}^{'}\left(\begin{array}{l}-\frac{3\xi\omega_{p0}^2 f_1^2\exp(z/dR_d)}{2\gamma d^2\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}9\omega_1^2\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)-\\ \frac{3f_1^2\xi(\exp(z/dR_d))^2}{4(\gamma d9)^2\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{3/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)^2\\ -\frac{f_1^2\exp(z/dR_d)}{3\gamma d\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\end{array}\right)\left(-\frac{1}{6}\right)$$

$$i^2 A_{30}^{'} R\left(-\frac{\xi\exp(z/dR_d)}{6\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)^2+9\left[\left(1-\left(\frac{\omega_{p0}^2}{9\omega_1^2\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\right]\right)^2\left(-\frac{r_0^2\omega_1 f_1^2}{c^2 6}\right)$$

$$+A_{30}^{'}(z)+A_{30}^{'}(z)\left(\frac{3}{2}\right)-\left[\frac{10\omega_p^2}{\omega_1^2}-9-9\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)\right]A_{30}^{'}(z)\left(-\frac{f_1^2}{6}\right)\left(\frac{r_0^2\omega_1^2}{c^2}\right)$$

$$=-\frac{1}{16\gamma^4}\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{r_0^2\omega_1^2}{c^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\left(\frac{e^2A_{10}^2}{m^2c^2\omega_1^2}\right)\left(\frac{eB_w}{mc\omega_1}\right)\left[3\left(1-\frac{\omega_p^2}{9\omega_1^2}\right)^{1/2}-31\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\right]A_{10},$$

$$2i\left(-\frac{\xi\exp(z/dR_d)}{6\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)+\left[\left(1-\left(\frac{1}{9\gamma}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\right]\right)\left\{\frac{\partial A_{30}^{''}}{\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\partial\xi}\left(\frac{f_1^2}{3}\right)\right\}$$

$$-2i\left(\begin{array}{l}-\dfrac{\xi\exp(z/dR_d)}{6\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\\ +\left[\left(1-\left(\frac{1}{9\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\right]\end{array}\right)\left\{\frac{A_{30}^{''}f_1^4}{\left(1-\frac{\omega_p^2}{\omega_1^2}\right)}\left(i\frac{\partial f_1}{f_1\partial\xi}\frac{1}{4}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)^2\left(\frac{\exp\left(\frac{z}{dR_d}\right)}{2\gamma d\left(1-\left(\frac{1}{\gamma}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}}\right)\right\}$$

$$++iA_{30}^{''}\left(\begin{array}{l}-\dfrac{3\xi\omega_{p0}^2 f_1^2\exp(z/dR_d)}{2\gamma d^2\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}9\omega_1^2\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\\ -\dfrac{3f_1^2\xi(\exp(z/dR_d))^2}{4(\gamma d9)^2\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{3/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)^2\\ -\dfrac{f_1^2\exp(z/dR_d)}{3\gamma d\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\end{array}\right)\left(-\frac{1}{6}\right)$$

$$i^2A_{30}^{''}R\left(-\frac{\xi\exp(z/dR_d)}{6\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(z/dR_d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)^2+9\left[\left(1-\left(\frac{\omega_{p0}^2}{9\omega_1^2\gamma}\right)\exp\left(\frac{z}{dR_d}\right)\right)^{1/2}\right]\right)^2\left(-\frac{r_0^2\omega_1 f_1^2}{c^2 6}\right)$$

$$+A_{30}^{''}(z)+A_{30}^{'}(z)\left(\frac{3}{2}\right)-\left[\frac{10\omega_p^2}{\omega_1^2}-9-9\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)\right]A_{30}^{''}(z)\left(-\frac{f_1^2}{6}\right)\left(\frac{r_0^2\omega_1^2}{c^2}\right)$$

$$=-\frac{1}{16\gamma^4}\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{r_0^2\omega_1^2}{c^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\left(\frac{e^2A_{10}^2}{m^2c^2\omega_1^2}\right)\left(\frac{eB_w}{mc\omega_1}\right)\left[3\left(1-\frac{\omega_p^2}{9\omega_1^2}\right)^{1/2}-31\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\right],$$

$$2i\left(-\frac{\xi\exp(\xi/d)}{6\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(\xi/d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)+\left[\left(1-\left(\frac{1}{9\gamma}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\exp(\xi/d)\right)^{1/2}\right]\right)\left\{\frac{\partial A_{30}''}{\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\partial\xi}\left(\frac{f_1^2}{3}\right)\right\}$$

$$-\frac{i\xi}{\gamma d}\left(\begin{array}{l}-\frac{1}{6\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(\xi/d)\right)^{1/2}}\\+\left[\left(1-\left(\frac{1}{9\gamma}\right)\exp(\xi/d)\right)^{1/2}\right]\end{array}\right)\left\{\frac{A_{30}'' f_1^4}{\left(1-\frac{\omega_p^2}{\omega_1^2}\right)}\left(i\frac{\partial f_1}{f_1\partial\xi}\frac{1}{4}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)^2\left(\frac{(\exp(\xi/d))^2}{\left(1-\left(\frac{1}{\gamma}\right)\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\exp(\xi/d)\right)^{1/2}}\right)\right\}$$

$$+i\frac{A_{30}''}{\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\left(\begin{array}{l}-\frac{3\xi\omega_{p0}^2 f_1^2\exp(\xi/d)}{2\gamma d^2 9\omega_1^2\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(\xi/d)\right)^{1/2}}\\-\frac{3f_1^2\xi(\exp(\xi/d))^2}{4(\gamma d9)^2\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(\xi/d)\right)^{3/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)\\-\frac{f_1^2\exp(\xi/d)}{3\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(\xi/d)\right)^{1/2}}\end{array}\right)\left(-\frac{1}{6}\right)$$

$$i^2A_{30}''R\left(-\frac{\xi\exp(\xi/d)}{6\gamma d\left(1-\frac{\omega_{p0}^2}{\gamma 9\omega_1^2}\exp(\xi/d)\right)^{1/2}}\left(\frac{\omega_{p0}^2}{\omega_1^2}\right)^2+9\left[\left(1-\left(\frac{\omega_{p0}^2}{9\omega_1^2\gamma}\right)\exp(\xi/d)\right)^{1/2}\right]\right)^2\left(-\frac{r_0^2\omega_1 f_1^2}{c^2 6}\right)$$

$$+A_{30}''(z)+A_{30}'(z)\left(\frac{3}{2}\right)-\left[\frac{10\omega_p^2}{\omega_1^2}-9-9\phi\left(\vec{E}_1\vec{E}_1^*\right)\right]A_{30}''(z)\left(-\frac{f_1^2}{6}\right)\left(\frac{r_0^2\omega_1^2}{c^2}\right)$$

$$=-\frac{1}{16\gamma^4}\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{r_0^2\omega_1^2}{c^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\left(\frac{e^2A_{10}^2}{m^2c^2\omega_1^2}\right)\left(\frac{eB_w}{mc\omega_1}\right)\left[3\left(1-\frac{\omega_p^2}{9\omega_1^2}\right)^{1/2}-31\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\right].$$

(7.36)

7.3 Results and discussion

Eqs. (7.27) and (7.36) are the derived coupled expressions of f and $A_{30}^{'}/A_{10}$. We solved this equation numerically and results have been expressed graphically. Fig. 7.1 gives the variation of with f with normalized propagation distance ξ whereas $\in_2 A_{10}^2/\in_0=1$, $\in_S/\in_0=0.2$, $\omega_1 r_0/c=18$, $eA_{10}/m\omega_1 c=5$, $eB_w/m\omega_1 c=0.005$ and $\omega_p/\omega_1=0.8$. It is observed that f attain its minimum value 0.42 at ξ = 1.2 and in the presence of density ramp minimum value of f at ξ = 1.2 is 0.1 This shows that with increase in plasma density, the nonlinear refractive index increases result stronger self-focusing. Due to ponderomotive force electrons expelled from low plasma density region and due to highly intense beam the mass variation takes place. Due to this relativistic nonlinearity arises results stronger self-focusing. On introducing the density tamp plasma density further increases results decrease in refractive index and self-focusing becomes more stronger and f decreases significantly. Valkunde *et al.* [23] presented the similar results for q-Gaussian laser pulse under normal density and exponential density profile and their study showed that under exponential density ramp self-focusing is stronger. Gupta *et al.* [24] showed that f shows oscillatory behaviour whereas on introducing density ramp the f maintains its minimum value for a longer distance. Variation of $A_{30}^{'}/A_{10}$ with ξ at different values of $eA_{10}/mc\omega_1$ = 1, 3, 5 is shown in Fig. 7.2 whereas other parameters remain the same as in Fig. 7.1. It is observed that at $eA_{10}/mc\omega_1$ =1, 3 & 5 the corresponding values of $A_{30}^{'}/A_{10}$ = 0.005, 0.065 & 0.1 respectively at ξ = 1.2 and after attaining peak value the defocusing starts taking place. This shows that ponderomotive force becomes stronger with increasing intensity of incident pulse that results the plasma density and results stronger self-focusing. [25] Thakur and Kant studied the increase in normalized amplitude for second harmonic pulse, under plasma density ramp, at different values of $eA_{10}/mc\omega_1$. Fig. 7.3 represents variation of $A_{30}^{'}/A_{10}$ with ξ at different values of $eB_w/mc\omega_1$ whereas other parameters remain the same as in Fig. 7.1. It is observed that $A_{30}^{'}/A_{10}$ = 0.022, 0.079 & 01 at ξ = 1.2 for $eB_w/mc\omega_1$ = 0.001, 0.003 & 0.005 respectively. The gain is apprehensive when wiggler field provides the phase matching condition by providing additional momentum to the THG. Vaziri *et al*. [26] studied the SHG at different values of wiggler field and outcome of their study showed that the normalized amplitude of SHG increase with rise magnetic field. In Fig. 7.4 we are analyzing the graphical variation of A'_{30}/A_{10} with ξ at different values of ω_p/ω_1, whereas other parameters remain the same as in Fig. 7.1. A'_{30}/A_{10} =0.018, 0.07 & 0.97 at ω_p/ω_1 = 0.4, 0.6 & 0.8 respectively. This shows that at higher value of ω_p/ω_1 the gain is significant at higher plasma frequency. Due to strong ponderomotive force, expelling electrons from plasma region results density variation of plasma that introduce nonlinearity results harmonic generation. [27] Kant *et al.* presented the similar study for SHG. Their results showed that normalized amplitude increases significantly at higher values of wiggler magnetic field when density ramp is introduced.

7.4 Conclusion

From our study we observed that incident beam gets strongly focused in the presence of exponential density ramp due o variation in plasma density and nonlinearity becomes more stronger. A'_{30}/A_{10} gains significantly when incident beam gets more intense due to stronger self-focusing and gains further increases at higher values of wiggler field. Finally it is also evident that at higher values of ω_p/ω_1 the gain in A'_{30}/A_{10} is further increased.

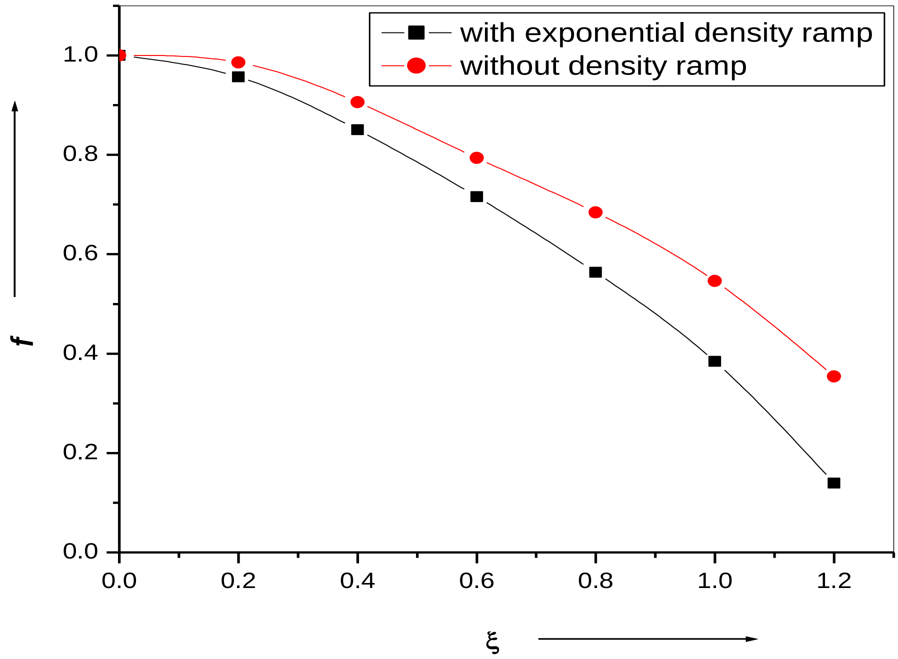

Fig. 7.1 Variation of f with normalized propagation distance. The other parameters are $\omega_1 r_0 / c = 18,\ \varepsilon_2 A_{10}^2 / \varepsilon_0 = 1,\ eA_{10} / m\omega_1 c = 5,\ eB_w / m\omega_1 c = 0.005$ and $\omega_p / \omega_1 = 0.8$.

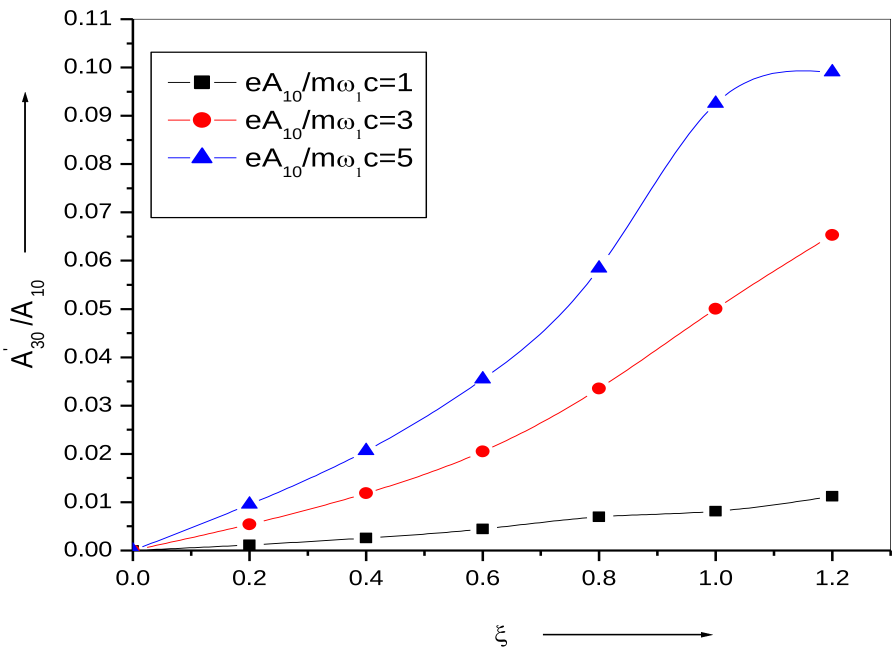

Fig. 7.2 Variation of $A_{30}^{'} / A_{10}$ with ξ at different values of $eA_{10} / m\omega_1 c = 1, 3 \,\&\, 5$. The other parameters are same as taken in Fig. 1.

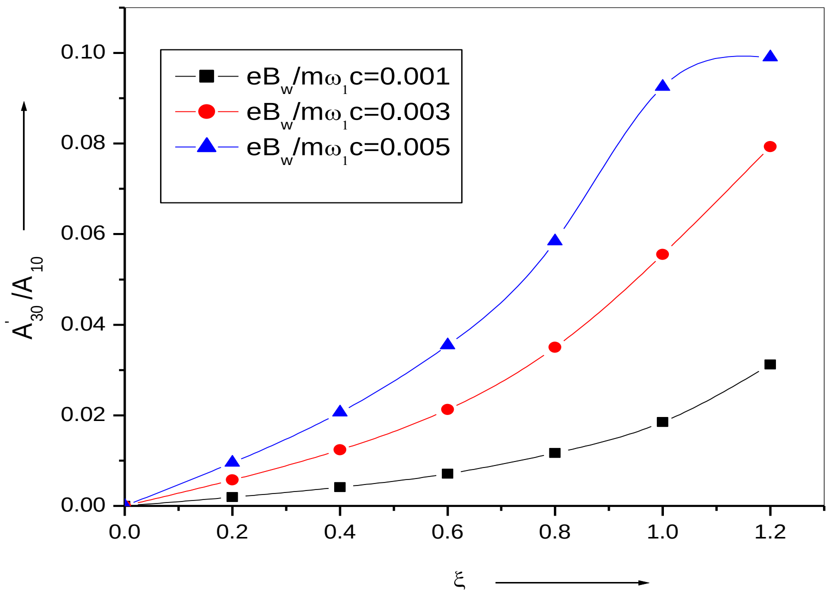

Fig. 7.3 Variation of A'_{30}/A_{10} with ξ at different values of $eB_w/m\omega_1 c = 0.001, 0.003 \,\&\, 0.005$. The other parameters are same as taken in Fig. 1.

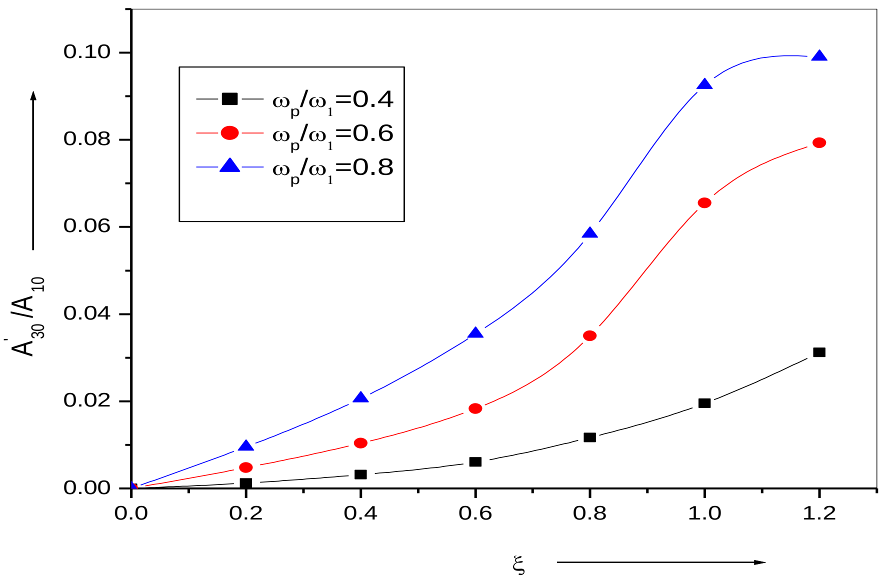

Fig. 7. 4 Variation of A'_{30}/A_{10} with ξ atr different values of $\omega_p/\omega_1 = 0.4, 0.6 \,\&\, 0.8$. The other parameters are same as taken in Fig. 1.

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Chapter-8

Second harmonic generation of cosh-Gaussian laser beam in ripple density magnetized plasma

8.1 Introduction

Short pulse laser propagating through plasma results harmonic generation and due to their wide range of applications, has created great interest amongst the different workers. Amongst different harmonic generations the second harmonic generation (SHG) have specific importance due to its application, such as microscopic resonance imaging [1,2,3], in medical science [4], to probe different surface by [5], in optoelectronics [6], to probe molecular structure[7] etc. various workers had investigated SHG for different applications and under different conditions. Nahata and Heinz [8] applied the SHG to measure ultrafast electrical signals due to sensitivity of second harmonic pulse for electric field.. Balnc *et al.* [9] studied the material structure using SHG, and applied the study for aluminum nitride thin films and their theoretical results are very close to the experimental results. Marrucci *et al.* [10] optically analysed the surfaces by SHG, having their wide applications in lubricant industry. They investigated the absorption by solid surface from liquid medium. Paul Campagnola [11] used the SHG for microscopic imaging for disease diagnostics. Microscopic imaging completely describe the chemical and physical properties, and is a very useful tool for the diagnosis of brain diseases. Simon *et al.* [12] investigated the application of SHG for phase diagrams. Which are very useful to microstructures. Their study shows that efficiency of SHG of several binary organile power mixtures shows different behaviour of melting and freezing compares to their individual behaviour. Lee *et al.* [13] probed the insoluble fibrous protein of vertebrates and substance of bones with the help of SHG nonlinear microscopy. Their study assessed the adult and ambryonic chick corneas through SHG. Tran *et al.* [14] applied the SHG for biological sensing useful for developments in bio sensors and bioassays as SHG are surface sensitive.

Due to their significance in different fields, the SHG is investigated by various researchers using different profiles. Askari *et al*. [15] studied the SHG in laser plasma interaction and analysed the impact of different laser parameters on efficiency of second harmonic pulse. Varaki *et al.* [16] studied the SHG when linearly polarized wave interacted with magnetized plasma. Their results show that the conversion efficiency of second harmonic pulse shows significant rise with increase in wiggler magnetic field and plasma frequency. Rathore and Kumar [17] studied the phase matched THG using Gaussian profile considering quantum plasma of high density. They showed that the harmonic radiations show maximum value under the condition of resonance. Singh *et al.* [18] studied the harmonic generation from laser plasma interaction under relativistic conditions and also analysed the effect of density ripple. They observed the increase in efficiency of second harmonic pulse, when pump laser incident at specific value of angle, between q and normal surface. Jha *et al.* [19] studied the SHG, which takes place when intense laser pulse interacts with magnetized plasma. Kaur *et al.* [20] studied the effect of self-focusing on resonant THG of laser in a tippled density plasma. In their study the self-focusing of laser enhances the third harmonic power and at higher intensity plasma density reduced on axis and weakens the THG. Salih *et al.* [21] studied the SHG when intense laser interact with magnetized plasma and second harmonic is generated. Vij. *et al.* [22] studied the resonant THG in clusters with density ripple. Group velocity of the third harmonic wave differs with the fundamental pulse results pulse slippage.

In the present work we have undertaken the study of cosh-Gaussian laser beam to study the self-focusing and efficiency of SHG propagating through rippled density magnetized plasma. We have obtained the expressions for normalized amplitude and using the expressions of beam width parameter, as derived by other coworkers, we have solved these equations numerically and results have been discussed graphically.

8.2 Theoretical considerations

Electric field cosh-Gaussian laser beam incident through plasma region is given as

$$\vec{E}_1 = \hat{x}A_1(z)\exp[-i(\omega_1 t - k_1 z)] , \tag{8.1}$$

where k_1 represents the wave number of incident laser, ω_1 is the fundamental frequency and A_1 is the amplitude of Gaussian wave given as

$$A(z) = A_0(z,r)Exp[-ikS(r,z)], \tag{8.2}$$

where A_0 and S are the real functions of $'r'$ and 'z'. A_0 is maximum amplitude of incident laser and given as

$$A_{0p}^2 = \frac{E_0^2}{f^2(z)} Exp\left[\frac{b^2}{2}\right] \left\{ \begin{array}{l} Exp\left[-2\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] + Exp\left[-2\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] \\ + 2Exp\left[-\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right] \end{array} \right\}. \tag{8.3}$$

Where beam width parameter is [22]

$$\frac{\partial^2 f(z)}{\partial \xi^2} = \left[4 - 4b^2 - \frac{6\alpha E_0^2 m_0}{M}\left(\frac{\omega_1^2 r_0^2}{c^2}\right)\left(\frac{\omega_p^2}{\omega_1^2}\right) Exp\left[\frac{b^2}{2}\right]\right] \frac{1}{f^3(z)}. \tag{8.4}$$

From Eq (5.59)

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r}\right)\vec{E}_2 + \left[\left(\frac{4\omega_1{}^2 - 5\omega_p^2}{c^2}\right) + \frac{4\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}_1^{\,*}\right)}{c^2}\right]\vec{E}_2 = \frac{4\pi\partial \vec{J}_2^{NL}}{c^2 \partial t}. \tag{8.5}$$

where $\vec{J}_2 = \vec{J}_2^L + \vec{J}_2^{NL}$, and $\vec{J}_2^L$ and $\vec{J}_2^{NL}$ [23] are the linear and nonlinear current density .

The particular integral of Eq (8.5) is

$$\vec{E}_2 = \hat{x} A_2^{'}(z,t) \exp[-i(2\omega_1 t - k_2 z)], \quad A_2^{'} = A_{20}^{'}(z)\psi_2, \tag{8.6}$$

where

$$\psi_2 = \left\{ \begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] \\ + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right] \end{array} \right\}^{1/2} \exp(-ikS),$$

$$\frac{\partial \psi_2}{\partial r} = \frac{\partial}{\partial r} \left\{ \begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] \\ + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right] \end{array} \right\}^{1/2} \exp(-ikS),$$

$$\frac{\partial\psi_2}{\partial r}=\frac{1}{2}\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{-1/2} \exp(-ikS)$$

$$\left[\begin{array}{l} \frac{\partial Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]}{\partial r}+\frac{\partial Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]}{\partial r} \\ +\frac{\partial 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]}{\partial r} \end{array}\right]\exp(-ikS)$$

$$+(-ik_3)\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2} \exp(-ikS)\frac{\partial S}{\partial r},$$

$$\frac{\partial\psi_2}{\partial r}=\frac{1}{2}\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]+2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{-1/2} \exp(-ikS)$$

$$\left[\begin{array}{l} Exp\left[-2\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\left(-\frac{8r}{r_0^2 f^2(z)}-\frac{4b}{r_0 f(z)}\right) \\ +Exp\left[-2\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\left(-\frac{8r}{r_0^2 f^2(z)}+\frac{4b}{r_0 f(z)}\right) \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\left(-\frac{8r}{r_0^2 f^2(z)}\right) \end{array}\right]\exp(-ikS)$$

$$+(-ik_3)\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2} \exp(-ikS)\frac{\partial S}{\partial r},$$

$$\frac{\partial \psi_2}{\partial r} = \frac{1}{2}\gamma^{-1/2}\exp(-ik_3 S) + \frac{1}{2}\left(-\frac{8r}{r_0^2 f^2(z)}\right)\gamma^{1/2}\exp(-ik_3 S)$$

$$+\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\exp(-ikS)$$

$$+(-ik_3)\gamma^{1/2}\exp(-ik_3 S)\frac{\partial S}{\partial r},$$

$$\frac{\partial \psi_2}{\partial r} = \frac{1}{2}\gamma^{-1/2}\exp(-ik_3 S) + \frac{1}{2}\left(-\frac{8r}{r_0^2 f^2(z)}\right)\psi_3$$

$$+\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\exp(-ikS))$$

$$+(-ik_3)\psi_3\frac{\partial S}{\partial r}. \tag{8.7}$$

$$Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] = 1 - -4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2 = 1 - \frac{4r^2}{r_0^2 f^2(z)} - b^2 - \frac{4br}{r_0 f(z)}, \tag{8.8}$$

$$Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] = 1 - -4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2 = 1 - \frac{4r^2}{r_0^2 f^2(z)} - b^2 + \frac{4br}{r_0 f(z)}, \tag{8.9}$$

$$2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] = 2\left(1-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right) = 2 - \frac{8r^2}{r_0^2 f^2(z)} - 2b^2, \tag{8.10}$$

$$\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{matrix}\right\} = 4 - \frac{16r^2}{r_0^2 f^2(z)} - 4b^2, \tag{8.11}$$

$$Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] = -\frac{8br}{r_0 f(z)}, \tag{8.12}$$

$$Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] = 2 - \frac{8r^2}{r_0^2 f^2(z)} - 2b^2. \tag{8.13}$$

Multiply Eq. (8.7) by $r\psi_2^*$ to integrate w.r.t 'r'

$$\int \frac{r\psi_2^*\partial\psi_2}{r\partial r} == \frac{1}{2}\int\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]+2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{array}\right\}$$

$$+\int\frac{1}{2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}+(-ik_3)\frac{\partial S}{\partial r}\int\psi_3\psi_3^* dr$$

$$\int \frac{r\psi_2^*\partial\psi_2}{r\partial r} == \frac{1}{2}\int\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]+2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{array}\right\}$$

$$+\int\frac{1}{2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}$$

$$+(-ik_3)\frac{\partial S}{\partial r}\int\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]+2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{array}\right\}dr,$$

using Eqs. (8.11) and (8.12) we obtain

$$\int \frac{r\psi_2^*\partial\psi_2}{r\partial r} == \frac{1}{2}\int\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}+\int\frac{1}{2}\left\{-\frac{8br}{r_0 f(z)}\right\}$$

$$+(-ik_3)\frac{\partial S}{\partial r}\int\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr,$$

$$\int \frac{r\psi_2^*\partial\psi_2}{r\partial r} == \frac{1}{2}\int\left\{-\frac{32r}{r_0^2 f^2(z)}+\frac{128r^3}{r_0^2 f^2(z)}+\frac{32b^2 r}{r_0^2 f^2(z)}\right\}+\int\frac{1}{2}\left\{-\frac{8br}{r_0 f(z)}\right\}$$

$$+(-ik_3)\frac{\partial S}{\partial r}\int\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr,$$

$$\int \frac{r\psi_2^{*}\partial\psi_2}{r\partial r} == \left\{-\frac{32r^2}{4r_0^2 f^2(z)}+\frac{128r^4}{8r_0^2 f^2(z)}+\frac{32b^2r^2}{4r_0^2 f^2(z)}\right\}+\frac{1}{2}\left\{-\frac{8br^2}{2r_0 f(z)}\right\}$$
$$+(-ik_3)\frac{\partial S}{\partial r}\left\{4r-\frac{16r^3}{3r_0^2 f^2(z)}-4b^2r\right\}. \tag{8.14}$$

Differentiate Eq. (8.7) with respect to 'r'

$$\frac{\partial^2\psi_2}{\partial r^2}=\frac{1}{2}\gamma^{-1/2}\exp(-ik_3S)+\frac{1}{2}\frac{\partial}{\partial r}\left\{\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{1/2}\right\}\exp(-ikS))$$

$$+\frac{1}{2}\frac{\partial}{\partial r}\left[\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{-1/2}\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\end{matrix}\right\}\right]\exp(-ikS)$$

$$+(-ik_3)\frac{d}{dr}\left[\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]+Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{1/2}\exp(-ikS)\frac{\partial S}{\partial r}\right],$$

$$\frac{\partial^2\psi_2}{\partial r^2}=\frac{\partial D}{\partial r}+\frac{\partial H}{\partial r}+\frac{\partial B}{\partial r}+\frac{dC}{dr}, \tag{8.15}$$

$$\frac{\partial D}{\partial r}=\frac{\partial}{\partial r}\left[\frac{1}{2}\gamma^{-1/2}\exp(-ikS)\right],$$

$$\frac{\partial D}{\partial r}=\left[\frac{1}{2}\gamma^{-1/2}\exp(-ikS)\right](-ikS)\frac{\partial S}{\partial r}+\left[\frac{1}{4}\gamma^{-3/2}\exp(-ik_3 S)\right]\left(-\frac{8r}{r_0^2 f^2(z)}\right)\gamma$$

$$-\frac{4b}{r_0 f(z)}\left[\frac{1}{4}\gamma^{-3/2}\exp(-ikS)\right]\left[\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\end{array}\right], \tag{8.16}$$

$$H=\frac{1}{2}\frac{\partial}{\partial r}\left\{\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{array}\right\}^{1/2}\right\}\exp(-ikS),$$

$$\frac{\partial H}{\partial r}=\frac{1}{2}\left\{\left(-\frac{8}{r_0^2 f^2(z)}\right)\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]+2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{1/2}\right\}\exp(-ikS)$$

$$+\frac{1}{4}\left\{\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{-1/2}\right\}\left[\left(-\frac{8r}{r_0^2 f^2(z)}-\frac{4b}{r_0 f(z)}\right)\right]Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\exp(-ikS)$$

$$+\frac{1}{4}\left\{\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{-1/2}\right\}\left[\left(-\frac{8r}{r_0^2 f^2(z)}+\frac{4b}{r_0 f(z)}\right)\right]Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\exp(-ikS)$$

$$+\frac{1}{4}\left\{\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{-1/2}\right\}\left(-\frac{8r}{r_0^2 f^2(z)}\right)2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\exp(-ikS)$$

$$+\frac{1}{2}\left\{\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]+2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{1/2}\right\}(-iks)\frac{\partial S}{\partial r}\exp(-ikS),$$

$$\frac{\partial H}{\partial r}=\frac{1}{2}\left(-\frac{8}{r_0^2 f^2(z)}\right)\gamma^{1/2}\exp(-ikS)+\frac{1}{4}\left(-\frac{8r}{r_0^2 f^2(z)}\right)^2\gamma^{1/2}\exp(-ikS)$$
$$+\left(\frac{32br}{r_0^3 f^3(z)}\right)\frac{E_0}{4f^2(z)}Exp\left[\frac{b^2}{4}\right]\gamma^{-1/2}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]$$
$$-\left(\frac{32br}{r_0^3 f^3(z)}\right)\frac{1}{4}\gamma^{-1/2}Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\exp(-ikS)$$
$$+\frac{1}{2}\left\{\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left\{\begin{array}{l}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\+Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\\+2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{array}\right\}^{1/2}\right\}(-iks)\frac{\partial S}{\partial r}\exp(-ikS),$$

$$\frac{\partial H}{\partial r}=\left(-\frac{4}{r_0^2 f^2(z)}\right)\gamma^{1/2}+\left(\frac{16r^2}{r_0^4 f^4(z)}\right)\gamma^{1/2}$$
$$+\left(\frac{8br}{r_0^3 f^3(z)}\right)\gamma^{-1/2}\left[Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right]$$
$$+\frac{1}{2}\left\{\left(-\frac{8r}{r_0^2 f^2(z)}\right)\gamma^{1/2}\right\}(-iks)\frac{\partial S}{\partial r}\exp(-ikS).$$

(8.17)

$$\frac{\partial B}{\partial r}=\frac{1}{2}\frac{\partial}{\partial r}\left[\left\{\begin{array}{l}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\+Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\\+2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{array}\right\}^{-1/2}\left\{\begin{array}{l}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\end{array}\right\}\right]\exp-ikS)$$

$$\frac{\partial B}{\partial r}=\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\left(-\frac{8r}{r_0^2 f^2(z)}-\frac{4b}{r_0 f(z)}\right)\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\right\}\gamma^{-1/2}$$

$$-\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\left(-\frac{8r}{r_0^2 f^2(z)}+\frac{4b}{r_0 f(z)}\right)\left(Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right)\left[\gamma^{-1/2}\right]$$

$$+\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-4b}{r_0 f(z)}\right)\left(\begin{matrix}-\dfrac{8r}{r_0^2 f^2(z)}\\ -\dfrac{4b}{r_0 f(z)}\end{matrix}\right)\left[\gamma^{-3/2}\left\{\begin{matrix}Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right]\\ -Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\end{matrix}\right\}\right]Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]$$

$$+\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-4b}{r_0 f(z)}\right)\left(\begin{matrix}-\dfrac{8r}{r_0^2 f^2(z)}\\ +\dfrac{4b}{r_0 f(z)}\end{matrix}\right)\left[\gamma^{-3/2}\left\{\begin{matrix}Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right]\\ -Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\end{matrix}\right\}\right]\left(Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right)$$

$$+\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-4b}{r_0 f(z)}\right)\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left[\gamma^{-3/2}\left\{\begin{matrix}Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right]\\ -Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\end{matrix}\right\}\right]2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]$$

$$+\frac{1}{2}\left[\left\{\begin{matrix}Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right]\\ +Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\\ +2Exp\left[-2\left(\dfrac{2r^2}{r_0^2 f^2(z)}+\dfrac{b^2}{2}\right)\right]\end{matrix}\right\}^{-1/2}\left\{\begin{matrix}Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right]\\ -Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\end{matrix}\right\}\right](-ikS)\exp-ikS)\frac{\partial S}{\partial r},$$

$$\frac{\partial B}{\partial r}=\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\left(-\frac{8r}{r_0^2 f^2(z)}-\frac{4b}{r_0 f(z)}\right)\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\right\}\gamma^{-1/2}$$

$$-\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\left(-\frac{8r}{r_0^2 f^2(z)}+\frac{4b}{r_0 f(z)}\right)\left(Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right)\left[\gamma^{-1/2}\right]$$

$$+\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-4b}{r_0 f(z)}\right)\left(-\frac{8r}{r_0^2 f^2(z)}\right)\left[\gamma^{-3/2}\left\{\begin{matrix}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\ -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\end{matrix}\right\}\right]\gamma$$

$$+\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-4b}{r_0 f(z)}\right)\left(-\frac{4b}{r_0 f(z)}\right)\gamma^{-3/2}\left[\left\{\begin{matrix}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\ -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\end{matrix}\right\}\right]^2$$

$$+\frac{1}{2}\left[\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right](-ikS)\exp-ikS)\frac{\partial S}{\partial r},$$

$$\frac{\partial B}{\partial r}=\frac{1}{2}\left(\frac{32br}{r_0^3 f^3(z)}\right)\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\right\}\gamma^{-1/2}+\frac{1}{2}\left(\frac{16b^2}{r_0^2 f^2(z)}\right)\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\right\}\gamma^{-1/2}$$

$$+\frac{1}{2}\left(\frac{32br}{r_0^3 f^3(z)}\right)\left(-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right)\left[\gamma^{-1/2}\right]-\frac{1}{2}\left(\frac{16b^2}{r_0^2 f^2(z)}\right)\left(-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right)\left[\gamma^{-1/2}\right]$$

$$-\left(\frac{8br}{r_0^3 f^3(z)}\right)\left[\gamma^{-3/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right]\gamma$$

$$-\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\left[\gamma^{-3/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right]^2$$

$$+\frac{1}{2}\left[\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right](-ikS)\exp-ikS)\frac{\partial S}{\partial r}$$

,

$$\frac{\partial B}{\partial r}=\left\{\left(\frac{16br}{r_0^3 f^3(z)}\right)+\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\right\}\gamma^{-1/2}$$

$$+\left\{\left(\frac{16br}{r_0^3 f^3(z)}\right)-\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\left(-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right)\left[\gamma^{-1/2}\right]$$

$$-\left(\frac{8br}{r_0^3 f^3(z)}\right)\left[\gamma^{-3/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\gamma\right]$$

$$-\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\left[\gamma^{-3/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right]^2$$

$$+\frac{1}{2}\left[\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right](-ikS)\exp-ikS)\frac{\partial S}{\partial r},$$

$$\frac{\partial B}{\partial r}=\left\{\left(\frac{16br}{r_0^3 f^3(z)}\right)\right\}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\gamma^{-1/2}$$

$$+\frac{E_0}{2f^2(z)}Exp\left[\frac{b^2}{4}\right]\left\{\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]+Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\left[\gamma^{-1/2}\right]$$

$$-\frac{E_0}{2f^2(z)}Exp\left[\frac{b^2}{4}\right]\left(\frac{8br}{r_0^3 f^3(z)}\right)\left[\gamma^{-3/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\gamma\right]$$

$$-\frac{E_0}{2f^2(z)}Exp\left[\frac{b^2}{4}\right]\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\gamma^{-3/2}\gamma\exp(-2)$$

$$+\frac{1}{2}\left[\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right](-ikS)\exp-ikS)\frac{\partial S}{\partial r},$$

$$\frac{\partial B}{\partial r}=\left\{\left(\frac{16br}{r_0^3 f^3(z)}\right)\right\}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\gamma^{-1/2}$$

$$+\left\{\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\left[\gamma^{-1/2}\right]$$

$$-\left(\frac{8br}{r_0^3 f^3(z)}\right)\left[\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right] \qquad (8.18)$$

$$-\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\gamma^{-1/2}\exp(-2)+\frac{1}{2}\left[\gamma^{-1/2}\left\{\begin{matrix}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\ -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\end{matrix}\right\}\right](-ikS)\exp-ikS)\frac{\partial S}{\partial r}.$$

$$C=(-ik_3)\left[\left\{\begin{matrix}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{1/2}\exp(-ik_3 S)\frac{\partial S}{\partial r}\right],$$

$$\frac{dC}{dr}=(-ik_3)\frac{d}{dr}\left[\left\{\begin{matrix}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{matrix}\right\}^{1/2}\exp(-ik_3 S)\frac{\partial S}{\partial r}\right],$$

$$\frac{dC}{dr}=(-ik_3)\exp(-ik_3S)\frac{\partial S}{\partial r}\frac{d}{dr}\left[\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2}\right]$$

$$+(-ik_3)^2\exp(-ik_3S)\left[\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2}\right]\left(\frac{\partial S}{\partial r}\right)^2$$

$$+(-ik_3)\exp(-ik_3S)\left[\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2}\right]\frac{\partial^2 S}{\partial r^2},$$

$$\frac{dC}{dr}=(-ik_3)\exp(-ik_3S)\frac{\partial S}{\partial r}\left[\left\{\begin{array}{l} \left(-\frac{8r}{r_0^2 f^2(z)}-\frac{4}{r_0 f(z)}\right)Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +\left(-\frac{8r}{r_0^2 f^2(z)}+\frac{4}{r_0 f(z)}\right)Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +\left(-\frac{8r}{r_0^2 f^2(z)}\right)2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2}\right]$$

$$+(-ik_3)\exp(-ik_3S)\frac{\partial S}{2\partial r}\left[\left\{\begin{array}{l} \left(-\frac{8r}{r_0^2 f^2(z)}-\frac{4}{r_0 f(z)}\right)Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ +\left(-\frac{8r}{r_0^2 f^2(z)}+\frac{4}{r_0 f(z)}\right)Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +\left(-\frac{8r}{r_0^2 f^2(z)}\right)2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{-1/2}\right]$$

$$+(-ik_3)^2\exp(-ik_3S)\left[\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2}\right]\left(\frac{\partial S}{\partial r}\right)^2$$

$$+(-ik_3)\exp(-ik_3S)\left[\ ,\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2}\right]\frac{\partial^2 S}{\partial r^2},$$

$$
\frac{dC}{dr} = (-ik_3)\exp(-ik_3 S)\left(-\frac{8r}{r_0^2 f^2(z)}\right)\frac{\partial S}{\partial r}\left[\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{matrix}\right\}^{1/2}\right]
$$

$$
-(-ik_3)\exp(-ik_3 S)\left(\frac{2b}{r_0 f(z)}\right)\frac{\partial S}{\partial r}\left[\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}^{1/2}\right]
$$

$$
+(-ik_3)\exp(-ik_3 S)\frac{\partial S}{2\partial r}\left[\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{matrix}\right\}^{-1/2}\right]
$$

$$
+(-ik_3)^2\exp(-ik_3 S)\left[\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{matrix}\right\}^{1/2}\right]\left(\frac{\partial S}{\partial r}\right)^2
$$

$$
+(-ik_3)\exp(-ik_3 S)\left[\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{matrix}\right\}^{1/2}\right]\frac{\partial^2 S}{\partial r^2},
$$

$$
\frac{dC}{dr} = (-ik_3)\exp(-ik_3 S)\left(-\frac{8r}{r_0^2 f^2(z)}\right)\frac{\partial S}{\partial r}\left[\gamma^{1/2}\right]
$$

$$
-(-ik_3)\exp(-ik_3 S)\left(\frac{2b}{r_0 f(z)}\right)\frac{\partial S}{\partial r}\left[\left\{\begin{matrix} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \end{matrix}\right\}^{1/2}\right] \qquad (8.19)
$$

$$
+(-ik_3)\exp(-ik_3 S)\frac{\partial S}{2\partial r}\left[\gamma^{-1/2}\right] + (-ik_3)^2\exp(-ik_3 S)\left[\gamma^{1/2}\right]\left(\frac{\partial S}{\partial r}\right)^2
$$

$$
+(-ik_3)\exp(-ik_3 S)\left[\lambda^{1/2}\right]\frac{\partial^2 S}{\partial r^2}.
$$

Using Eqs. (8.16), (8.17), (8.18) and (8.19) in Eq. (8.15) and neglecting derivative of 's' we obtain

$$\frac{\partial^2\psi_2}{\partial r^2} = \left(-\frac{4}{r_0^2 f^2(z)}\right)\gamma^{1/2}\exp(-ikS) + \left(\frac{16r^2}{r_0^4 f^4(z)}\right)\gamma^{1/2}\exp(-ikS)$$

$$+\left(\frac{8br}{r_0^3 f^3(z)}\right)\gamma^{-1/2}\left[Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right]\exp(-ikS)$$

$$+\left\{\left(\frac{16br}{r_0^3 f^3(z)}\right)\right\}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\gamma^{-1/2}\exp(-ikS)$$

$$+\left\{\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\left[\gamma^{-1/2}\right]\exp(-ikS)$$

$$-\left(\frac{8br}{r_0^3 f^3(z)}\right)\left[\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right]\exp(-ikS)$$

$$-\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\gamma^{-1/2}\exp(-2)\exp(-ikS),$$

multiply above Eq. by $r\psi_2^*$ and integrating with respect to 'r'

$$\frac{\partial^2\psi_2}{\partial r^2} = \left(-\frac{4}{r_0^2 f^2(z)}\right)\int r\psi_2^*\gamma^{1/2}\exp(-ikS) + \left(\frac{16r^2}{r_0^4 f^4(z)}\right)\int r\psi_2^*\gamma^{1/2}\exp(-ikS)$$

$$+\left(\frac{8br}{r_0^3 f^3(z)}\right)\int r\psi_2^*\gamma^{-1/2}\left[Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right]\exp(-ikS)$$

$$+\left\{\left(\frac{16br}{r_0^3 f^3(z)}\right)\right\}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\int r\psi_2^*\gamma^{-1/2}\exp(-ikS)$$

$$+\left\{\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\int r\psi_2^*\left[\gamma^{-1/2}\right]\exp(-ikS)$$

$$-\left(\frac{8br}{r_0^3 f^3(z)}\right)\int r\psi_2^*\left[\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right]\exp(-ikS)$$

$$-\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\int r\psi_2^*\gamma^{-1/2}\exp(-2)\exp(-ikS),$$

$$\int r\psi_2^* \frac{\partial^2\psi_2}{\partial r^2} = \left(-\frac{4}{r_0^2 f^2(z)}\right)\int r\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}dr$$

$$+\left(\frac{16}{r_0^4 f^4(z)}\right)\int r^3\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] \\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}dr$$

$$+\left(\frac{8b}{r_0^3 f^3(z)}\right)\int r^2\left[Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right]dr$$

$$+\left\{\left(\frac{16b}{r_0^3 f^3(z)}\right)\right\}\int r^2\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}dr$$

$$+\left\{\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\int r\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}dr$$

$$-\left(\frac{8b}{r_0^3 f^3(z)}\right)\int r^2\left[\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] - Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right]dr \tag{8.20}$$

$$-\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\exp(-2)\int rdr,$$

using Eqs. (8.11), (8.12) & (8.13) into Eq (8.20), we get

$$\int r\psi_2^* \frac{\partial^2\psi_2}{\partial r^2} = \left(-\frac{4}{r_0^2 f^2(z)}\right)\int r\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr$$

$$+\left(\frac{16}{r_0^4 f^4(z)}\right)\int r^3\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr + \left(\frac{8b}{r_0^3 f^3(z)}\right)\int r^2\left[-\frac{8br}{r_0 f(z)}\right]dr$$

$$+\left\{\left(\frac{16b}{r_0^3 f^3(z)}\right)\right\}\int r^2\left\{-\frac{8br}{r_0 f(z)}\right\}dr \tag{8.21}$$

$$+\left\{\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\int r\left\{2-\frac{8r^2}{r_0^2 f^2(z)}-2b^2\right\}dr$$

$$-\left(\frac{8b}{r_0^3 f^3(z)}\right)\int r^2\left[\left\{-\frac{8br}{r_0 f(z)}\right\}\right]dr$$

$$-\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\exp(-2)\int rdr.$$

Using Eq. (8.6)

$$\frac{\partial E_{22}}{\partial z} = \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}]\left[\frac{\partial A_2'}{\partial z} + A_2'\left(i - (2k_1 + k_0)\right)\right],$$

$$\frac{\partial E_{22}}{\partial z} = \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}]\begin{bmatrix}\frac{\partial^2 A_2'}{\partial z^2} + \left(-i(2k_1 + k_0)\right)\frac{\partial A_2'}{\partial z} + \\ A_2'\left(-i(2k_1 + k_0)\right)^2 + \left(-i(2k_1 + k_0)\right)\frac{\partial A_2'}{\partial z}\end{bmatrix},$$

Neglecting $\partial^2 A_2 / \partial z^2$

$$\begin{aligned}\frac{\partial E_{22}}{\partial z} &= -2i(2k_1 + k_0)\exp[-i\{2\omega_1 t - (2k_1 + k_0)z\}\frac{\partial A_2'}{\partial z} \\ &+ A_2'\left(-i(2k_1 + k_0)\right)^2 \exp[-i\{2\omega_1 t - (2k_1 + k_0)z\},\end{aligned} \tag{8.22}$$

where $A_2' = A_{20}'\psi_2$

$$A_2 = A_{20}'\left\{\begin{matrix}Exp\left[-4\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right]\end{matrix}\right\}^{1/2} \exp(-ik_1 S_1),$$

$$\frac{\partial A_2}{\partial z} =$$

$$+\left\{\begin{matrix}Exp\left[-4\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right]\end{matrix}\right\}^{1/2} \exp(-ik_1 S_1)\frac{\partial A_{30}'}{\partial z}$$

$$+ A_{30}'\frac{1}{2}\gamma^{-1/2}\frac{\partial}{\partial z}\left\{\begin{matrix}Exp\left[-4\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right]\end{matrix}\right\}\exp(-ik_1 S_1)$$

$$+ A_{30}'(-ik_1 S_1)\left\{\begin{matrix}Exp\left[-4\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right]\end{matrix}\right\}^{1/2} \exp(-ik_1 S_1)\frac{\partial S_1}{\partial z},$$

$$\frac{\partial A_2}{\partial z} =$$

$$+\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2} \frac{\partial A_{30}^{'}}{\partial z} +$$

$$+ A_{30}^{'}\frac{1}{2}\gamma^{-1/2}\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\left[\left(\frac{-4r^2}{r_0^2}\right)\left(\frac{-2}{f^3(z)}\right)\left(\frac{\partial f}{\partial z}\right)-\left(\frac{2br}{r_0}\right)\left(\frac{-1}{f^2(z)}\right)\left(\frac{\partial f}{\partial z}\right)\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\left[\left(\frac{-4r^2}{r_0^2}\right)\left(\frac{-2}{f^3(z)}\right)\left(\frac{\partial f}{\partial z}\right)+\left(\frac{2br}{r_0}\right)\left(\frac{-1}{f^2(z)}\right)\left(\frac{\partial f}{\partial z}\right)\right] \\ + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\left(\frac{-4r^2}{r_0^2}\right)\left(\frac{-2}{f^3(z)}\right)\left(\frac{\partial f}{\partial z}\right) \end{array}\right\}$$

$$+ A_{30}^{'}(-ik_1 S_1)\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2} \exp(-ik_1 S_1)\frac{\partial S_1}{\partial z},$$

$$\frac{\partial A_2}{\partial z} =$$

$$+\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2} \frac{\partial A_{30}^{'}}{\partial z} +$$

$$+ A_{30}^{'}\frac{1}{2}\gamma^{-1/2}\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\left[\left(\frac{8r^2}{r_0^2 f^3(z)}\right)\left(\frac{\partial f}{\partial z}\right)+\left(\frac{2br}{r_0 f^2(z)}\right)\left(\frac{\partial f}{\partial z}\right)\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\left[\left(\frac{8r^2}{r_0^2 f^3(z)}\right)\left(\frac{\partial f}{\partial z}\right)-\left(\frac{2br}{r_0 f^2(z)}\right)\left(\frac{\partial f}{\partial z}\right)\right] \\ + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\left(\frac{8r^2}{r_0^2 f^3(z)}\right)\left(\frac{\partial f}{\partial z}\right) \end{array}\right\}$$

$$+ A_{30}^{'}(-ik_1 S_1)\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ + Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right] + 2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right] \end{array}\right\}^{1/2} \exp(-ik_1 S_1)\frac{\partial S_1}{\partial z},$$

$$\frac{\partial A_2}{\partial z}=$$

$$+\psi_2\frac{\partial A_{30}^{'}}{\partial z}++A_{30}^{'}\frac{1}{2}\psi_2\left(\frac{8r^2}{r_0^2 f^3(z)}\right)\left(\frac{\partial f}{\partial z}\right)$$

$$+A_{30}^{'}\frac{1}{2}\gamma^{-1/2}\left(\frac{2br}{r_0 f^2(z)}\right)\left(\frac{\partial f}{\partial z}\right)\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}$$

$$+A_{30}^{'}(-ik_1S_1)\left\{\begin{array}{l}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\ +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\\ +2Exp\left[-2\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{array}\right\}^{1/2}\exp(-ik_1S_1)\frac{\partial S_1}{\partial z},\qquad(8.23)$$

putting Eq. (8.23) in Eq. (8.22) we get

$$\frac{\partial E_{22}}{\partial z}=-2i(2k_1+k_0)\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}\left[\begin{array}{l}+\psi_2\frac{\partial A_{30}^{'}}{\partial z}+\\ A_{30}^{'}\frac{1}{2}\psi_2\left(\frac{8r^2}{r_0^2 f^3(z)}\right)\left(\frac{\partial f}{\partial z}\right)\\ +A_{30}^{'}\frac{1}{2}\gamma^{-1/2}\left(\frac{2br}{r_0 f^2(z)}\right)\left(\frac{\partial f}{\partial z}\right)\left\{\begin{array}{l}Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\ -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\end{array}\right\}\end{array}\right]\qquad(8.24)$$

$$+A_2{}'\left(-i(2k_1+k_0)\right)^2\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}.$$

Eq. (8.5) can be written as

$$\frac{\partial^2\vec{E}_2}{\partial z^2}+\frac{A_{20}^{'}\partial^2\psi_2}{\partial z^2}+\frac{A_{20}^{'}\partial\psi_2}{r\partial r}+\left[\left(\frac{4\omega_1{}^2-5\omega_p^2}{c^2}\right)+\frac{4\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_1\right)}{c^2}\right]\vec{E}_2=\frac{4\pi\partial\vec{J}_2^{NL}}{c^2\partial t},\qquad(8.24a)$$

multiply Eq. (8.24) by $r\psi_2^*$ and integrate with respect to 'r' and using Eqs. (8.14), (8.21) and (8.24) in above Eq. (8.24a), we obtain

$$
-2i(2k_1+k_0)\left[\begin{array}{l}
+\int r\psi_2^*\psi_2\dfrac{\partial A_{30}'}{\partial z}++A_{30}'\dfrac{1}{2}\int r\psi_2^*\psi_2\left(\dfrac{8r^2}{r_0^2f^3(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\\
+A_{30}'\dfrac{1}{2}\gamma^{-1/2}\left(\dfrac{2br}{r_0f^2(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\int r\psi_2^*\left\{\begin{array}{l}Exp\left[-4\left(\dfrac{r}{r_0f(z)}+\dfrac{b}{2}\right)^2\right]\\-Exp\left[-4\left(\dfrac{r}{r_0f(z)}-\dfrac{b}{2}\right)^2\right]\end{array}\right\}
\end{array}\right]
$$

$$
+A_2'\left(-i(2k_1+k_0)\right)^2
$$

$$
+A_{20}'\left\{\begin{array}{l}
\left(-\dfrac{4}{r_0^2f^2(z)}\right)\int r\psi_2^*\psi_2+\left(\dfrac{16r^2}{r_0^4f^4(z)}\right)\int r\psi_2^*\psi_2\\
+\left(\dfrac{8br}{r_0^3f^3(z)}\right)\int r\psi_2^*\gamma^{-1/2}\left[Exp\left[-4\left(\dfrac{r}{r_0f(z)}+\dfrac{b}{2}\right)^2\right]-Exp\left[-4\left(\dfrac{r}{r_0f(z)}-\dfrac{b}{2}\right)^2\right]\right]\\
+\left\{\left(\dfrac{16br}{r_0^3f^3(z)}\right)\right\}\int r\psi_2^*\left\{Exp\left[-4\left(\dfrac{r}{r_0f(z)}+\dfrac{b}{2}\right)^2\right]-Exp\left[-4\left(\dfrac{r}{r_0f(z)}-\dfrac{b}{2}\right)^2\right]\right\}\gamma^{-1/2}\\
+\left\{\left(\dfrac{8b^2}{r_0^2f^2(z)}\right)\right\}\int r\psi_2^*\left\{Exp\left[-4\left(\dfrac{r}{r_0f(z)}+\dfrac{b}{2}\right)^2\right]+Exp\left[-4\left(\dfrac{r}{r_0f(z)}-\dfrac{b}{2}\right)^2\right]\right\}\left[\gamma^{-1/2}\right]\\
-\left(\dfrac{8br}{r_0^3f^3(z)}\right)\int r\psi_2^*\left[\gamma^{-1/2}\left\{Exp\left[-4\left(\dfrac{r}{r_0f(z)}+\dfrac{b}{2}\right)^2\right]-Exp\left[-4\left(\dfrac{r}{r_0f(z)}-\dfrac{b}{2}\right)^2\right]\right\}\right]\\
-\left(\dfrac{4b^2}{r_0^2f^2(z)}\right)\int r\psi_2^*\gamma^{-1/2}\exp(-2)
\end{array}\right\}
$$

$$
+A_{20}'\left\{\begin{array}{l}
\dfrac{1}{2}\left(-\dfrac{8r}{r_0^2f^2(z)}\right)\int r\psi_2^*\psi_2\\
+\dfrac{1}{2}\left(\dfrac{-4b}{r_0f(z)}\right)\int r\psi_2^*\gamma^{-1/2}\left\{Exp\left[-4\left(\dfrac{r}{r_0f(z)}+\dfrac{b}{2}\right)^2\right]-Exp\left[-4\left(\dfrac{r}{r_0f(z)}-\dfrac{b}{2}\right)^2\right]\right\}
\end{array}\right\}
$$

$$
+\left[\left(\dfrac{4\omega_1^2-5\omega_p^2}{c^2}\right)+\dfrac{4\omega_1^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]A_{20}'\int r\psi_2^*\psi_2=\dfrac{4\pi}{\exp[-i\{2\omega_1t-(2k_1+k_0)z\}]c^2}\dfrac{\partial\vec{J}_2^{NL}}{\partial t}\int r\psi_2^*,
$$

$$-2i(2k_1+k_0)\left[\begin{array}{l}+\frac{\partial A_{30}^{'}}{\partial z}\int r\psi_2^*\psi_2 ++A_{30}^{'}\frac{1}{2}\left(\frac{8}{r_0^2 f^3(z)}\right)\left(\frac{\partial f}{\partial z}\right)\int r^3\psi_2^*\psi_2 \\ +A_{30}^{'}\frac{1}{2}\gamma^{-1/2}\left(\frac{2b}{r_0 f^2(z)}\right)\left(\frac{\partial f}{\partial z}\right)\int r^2\psi_2^*\left\{\begin{array}{l} Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] \\ -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\end{array}\right\}\end{array}\right]$$

$$+A_{20}'\left(-i(2k_1+k_0)\right)^2\int r\psi_2^*\psi_2$$

$$+A_{20}^{'}\left\{\begin{array}{l}\left(-\frac{4}{r_0^2 f^2(z)}\right)\int r\psi_2^*\psi_2+\left(\frac{16}{r_0^4 f^4(z)}\right)\int r^3\psi_2^*\psi_2 \\ +\left(\frac{8b}{r_0^3 f^3(z)}\right)\int r^2\psi_2^*\gamma^{-1/2}\left[Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]-Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right] \\ +\left\{\left(\frac{16b}{r_0^3 f^3(z)}\right)\right\}\int r^2\psi_2^*\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\gamma^{-1/2} \\ +\left\{\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\int r\psi_2^*\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] +Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\left[\gamma^{-1/2}\right] \\ -\left(\frac{8b}{r_0^3 f^3(z)}\right)\int r^2\psi_2^*\left[\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\right] \\ -\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\int r\psi_2^*\gamma^{-1/2}\exp(-2)\end{array}\right\}$$

$$+A_{20}^{'}\left\{\begin{array}{l}\frac{1}{2}\left(-\frac{8}{r_0^2 f^2(z)}\right)\int r^2\psi_2^*\psi_2 \\ +\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\int \psi_2^*\gamma^{-1/2}\left\{Exp\left[-4\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right] -Exp\left[-4\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\}\end{array}\right\}$$

$$+\left[\left(\frac{4\omega_1{}^2-5\omega_p^2}{c^2}\right)+\frac{4\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^*{}_1\right)}{c^2}\right]A_{20}^{'}\int r\psi_2^*\psi_2=\frac{4\pi}{\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]c^2}\frac{\partial \vec{J}_2^{NL}}{\partial t}\int r\psi_2^*,$$

$$
-2i(2k_1+k_0)\left[\begin{array}{l}
+\dfrac{\partial A_{30}'}{\partial z}\int r\psi_2^*\psi_2 dr + +A_{30}'\dfrac{1}{2}\left(\dfrac{8}{r_0^2 f^3(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\int r^3\psi_2^*\psi_2 dr \\
+A_{30}'\dfrac{1}{2}\left(\dfrac{2b}{r_0 f^2(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\int r^2\left\{\begin{array}{l} Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right] \\ -Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\end{array}\right\}dr
\end{array}\right]
$$

$$
+A_{20}'\left(-i(2k_1+k_0)\right)^2\int r\psi_2^*\psi_2 dr
$$

$$
+A_{20}'\left\{\begin{array}{l}
\left(-\dfrac{4}{r_0^2 f^2(z)}\right)\int r\psi_2^*\psi_2 dr + \left(\dfrac{16}{r_0^4 f^4(z)}\right)\int r^3\psi_2^*\psi_2 dr \\
+\left(\dfrac{8b}{r_0^3 f^3(z)}\right)\int r^2\left[Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right]-Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\right]dr \\
+\left\{\left(\dfrac{16b}{r_0^3 f^3(z)}\right)\right\}\int r^2\left\{Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right]-Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\right\}dr \\
+\left\{\left(\dfrac{8b^2}{r_0^2 f^2(z)}\right)\right\}\int r\left\{Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right]+Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\right\}dr \\
-\left(\dfrac{8b}{r_0^3 f^3(z)}\right)\int r^2\left\{Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right]-Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\right\}dr \\
-\left(\dfrac{4b^2}{r_0^2 f^2(z)}\right)\exp(-2)\int r dr
\end{array}\right\}
$$

$$
+A_{20}'\left\{\dfrac{1}{2}\left(-\dfrac{8}{r_0^2 f^2(z)}\right)\int r^2\psi_2^*\psi_2 dr + \dfrac{1}{2}\left(\dfrac{-4b}{r_0 f(z)}\right)\int\left\{\begin{array}{l} Exp\left[-4\left(\dfrac{r}{r_0 f(z)}+\dfrac{b}{2}\right)^2\right] \\ -Exp\left[-4\left(\dfrac{r}{r_0 f(z)}-\dfrac{b}{2}\right)^2\right]\end{array}\right\}dr\right\}
$$

$$
+\left[\left(\dfrac{4\omega_1^2-5\omega_p^2}{c^2}\right)+\dfrac{4\omega_1^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]A_{20}'\int r\psi_2^*\psi_2 dr = \dfrac{4\pi}{\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]c^2}\dfrac{\partial \vec{J}_2^{NL}}{\partial t}\int r\psi_2^* dr \,,
$$

$$-2i(2k_1+k_0)\left[\begin{array}{l}+\dfrac{\partial A'_{30}}{\partial z}\displaystyle\int r\left\{4-\dfrac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr+ \\ A'_{30}\dfrac{1}{2}\left(\dfrac{8}{r_0^2 f^3(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\displaystyle\int r^3\left\{4-\dfrac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr \\ +A'_{30}\dfrac{1}{2}\left(\dfrac{2b}{r_0 f^2(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\displaystyle\int r^2\left\{-\dfrac{8br}{r_0 f(z)}\right\}dr\end{array}\right]+A_{20}'\left(-i(2k_1+k_0)\right)^2\int r\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr$$

$$+A'_{20}\left\{\begin{array}{l}\left(-\dfrac{4}{r_0^2 f^2(z)}\right)\displaystyle\int r\left\{4-\dfrac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr+\left(\dfrac{16}{r_0^4 f^4(z)}\right)\displaystyle\int r^3\left\{4-\dfrac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr \\ +\left(\dfrac{8b}{r_0^3 f^3(z)}\right)\displaystyle\int r^2\left[-\dfrac{8br}{r_0 f(z)}\right]dr+\left\{\left(\dfrac{16b}{r_0^3 f^3(z)}\right)\right\}\displaystyle\int r^2\left\{-\dfrac{8br}{r_0 f(z)}\right\}dr \\ +\left\{\left(\dfrac{8b^2}{r_0^2 f^2(z)}\right)\right\}\displaystyle\int r\left\{2-\dfrac{8r^2}{r_0^2 f^2(z)}-2b^2\right\}dr-\left(\dfrac{8b}{r_0^3 f^3(z)}\right)\displaystyle\int r^2\left\{-\dfrac{8br}{r_0 f(z)}\right\}dr \\ -\left(\dfrac{4b^2}{r_0^2 f^2(z)}\right)\exp(-2)\displaystyle\int r\,dr\end{array}\right\}$$

$$+A'_{20}\left\{\frac{1}{2}\left(-\frac{8}{r_0^2 f^2(z)}\right)\int r^2\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr+\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\int\left\{-\frac{8br}{r_0 f(z)}\right\}dr\right\}$$

$$+\left[\left(\frac{4\omega_1{}^2-5\omega_p^2}{c^2}\right)+\frac{4\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_{\ 1}\right)}{c^2}\right]A'_{20}\int r\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr$$

$$=\frac{4\pi}{\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]c^2}\frac{n_0 e^4 B_W E_1^2}{4ic\omega_1^2 m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\frac{d\exp[-i\{2\omega_1 t-(2k_1+k_0)z\}]}{dt}\int r\psi_2^* dr,$$

$$
-2i(2k_1+k_0)\left[\begin{array}{l}
+\dfrac{\partial A_{30}^{'}}{\partial z}\displaystyle\int r\left\{4-\dfrac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr+\\
A_{30}^{'}\dfrac{1}{2}\left(\dfrac{8}{r_0^2 f^3(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\displaystyle\int r^3\left\{4-\dfrac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr\\
+A_{30}^{'}\dfrac{1}{2}\left(\dfrac{2b}{r_0 f^2(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\displaystyle\int r^2\left\{-\dfrac{8br}{r_0 f(z)}\right\}dr
\end{array}\right]
$$

$$
+A_{20}'\left(-i(2k_1+k_0)\right)^2\int r\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr
$$

$$
+A_{20}^{'}\left\{\begin{array}{l}
\left(-\dfrac{4}{r_0^2 f^2(z)}\right)\displaystyle\int r\left\{4-\dfrac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr+\left(\dfrac{16}{r_0^4 f^4(z)}\right)\displaystyle\int r^3\left\{4-\dfrac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr\\
+\left(\dfrac{8b}{r_0^3 f^3(z)}\right)\displaystyle\int r^2\left[-\dfrac{8br}{r_0 f(z)}\right]dr+\left\{\left(\dfrac{16b}{r_0^3 f^3(z)}\right)\right\}\displaystyle\int r^2\left\{-\dfrac{8br}{r_0 f(z)}\right\}dr\\
+\left\{\left(\dfrac{8b^2}{r_0^2 f^2(z)}\right)\right\}\displaystyle\int r\left\{2-\dfrac{8r^2}{r_0^2 f^2(z)}-2b^2\right\}dr-\left(\dfrac{8b}{r_0^3 f^3(z)}\right)\displaystyle\int r^2\left\{-\dfrac{8br}{r_0 f(z)}\right\}dr\\
-\left(\dfrac{4b^2}{r_0^2 f^2(z)}\right)\exp(-2)\displaystyle\int r\,dr
\end{array}\right\}
$$

$$
+A_{20}^{'}\left\{\frac{1}{2}\left(-\frac{8}{r_0^2 f^2(z)}\right)\int r^2\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr+\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\int\left\{-\frac{8br}{r_0 f(z)}\right\}dr\right\}
$$

$$
+\left[\left(\frac{4\omega_1{}^2-5\omega_p^2}{c^2}\right)+\frac{4\omega_1{}^2\,\phi\left(\vec{E}_1\vec{E}^{\,*}_{\,1}\right)}{c^2}\right]A_{20}^{'}\int r\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr
$$

$$
=\frac{4\pi n_0 e^4 B_W A_{10}^2}{4ic^3\omega_1^2 m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)(-2i\omega_1)\int r\left\{4-\frac{16r^2}{r_0^2 f^2(z)}-4b^2\right\}dr,
$$

$$-2i(2k_1+k_0)\left[\begin{array}{l} +\dfrac{\partial A_{30}^{'}}{\partial z}\left\{2r^2-\dfrac{16r^3}{3r_0^2 f^2(z)}-2b^2r^2\right\}+ \\ A_{30}^{'}\dfrac{1}{2}\left(\dfrac{8}{r_0^2 f^3(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\left\{r^4-\dfrac{16r^6}{6r_0^2 f^2(z)}-r^4b^2\right\} \\ +A_{30}^{'}\dfrac{1}{2}\left(\dfrac{2b}{r_0 f^2(z)}\right)\left(\dfrac{\partial f}{\partial z}\right)\left\{-\dfrac{8br^4}{4r_0 f(z)}\right\} \end{array}\right]$$

$$+A_{20}'\left(-i(2k_1+k_0)\right)^2\left\{2r^2-\frac{16r^3}{3r_0^2 f^2(z)}-2r^2b^2\right\}dr$$

$$+A_{20}^{'}\left\{\begin{array}{l} \left(-\dfrac{4}{r_0^2 f^2(z)}\right)\left\{2r^2-\dfrac{16r^3}{3r_0^2 f^2(z)}-2r^2b^2\right\}+\left(\dfrac{16}{r_0^4 f^4(z)}\right)\left\{r^4-\dfrac{16r^6}{4r_0^2 f^2(z)}-r^4b^2\right\} \\ +\left(\dfrac{8b}{r_0^3 f^3(z)}\right)\left[-\dfrac{8br^4}{4r_0 f(z)}\right]+\left\{\left(\dfrac{16b}{r_0^3 f^3(z)}\right)\right\}\left[-\dfrac{8br^4}{4r_0 f(z)}\right] \\ +\left\{\left(\dfrac{8b^2}{r_0^2 f^2(z)}\right)\right\}\left\{r^2-\dfrac{8r^3}{2r_0^2 f^2(z)}-r^2b^2\right\}-\left(\dfrac{8b}{r_0^3 f^3(z)}\right)\left\{-\dfrac{8br^4}{4r_0 f(z)}\right\} \\ -\left(\dfrac{4b^2}{r_0^2 f^2(z)}\right)\exp(-2)\dfrac{r^2}{2} \end{array}\right\}$$

$$+A_{20}^{'}\left\{\frac{1}{2}\left(-\frac{8}{r_0^2 f^2(z)}\right)\left\{2r^2-\frac{16r^3}{3r_0^2 f^2(z)}-2r^2b^2\right\}+\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\left\{-\frac{4br^2}{r_0 f(z)}\right\}\right\}$$

$$+\left[\left(\frac{4\omega_1{}^2-5\omega_p^2}{c^2}\right)+\frac{4\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_{\;1}\right)}{c^2}\right]A_{20}^{'}\left\{2r^2-\frac{16r^3}{3r_0^2 f^2(z)}-2r^2b^2\right\}$$

$$=\frac{4\pi n_0 e^4 B_W A_{10}^2}{4ic^3\omega_1^2 m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)(-2i\omega_1)\left\{2r^2-\frac{16r^3}{3r_0^2 f^2(z)}-2r^2b^2\right\},$$

equating coefficient of r^2

$$-2i(2k_1+k_0)\left[\frac{\partial A_{30}^{'}}{\partial z}\left\{2r^2-2b^2r^2\right\}\right]+A_{20}{}'\left(-i(2k_1+k_0)\right)^2\left\{2r^2-2r^2b^2\right\}dr$$

$$+A_{20}^{'}\left\{\begin{array}{l}\left(-\frac{4}{r_0^2 f^2(z)}\right)\left\{2r^2-2r^2b^2\right\}\\ +\left\{\left(\frac{8b^2}{r_0^2 f^2(z)}\right)\right\}\left\{r^2-r^2b^2\right\}-\left(\frac{4b^2}{r_0^2 f^2(z)}\right)\exp(-2)\frac{r^2}{2}\end{array}\right\}$$

$$+A_{20}^{'}\left\{\frac{1}{2}\left(-\frac{8}{r_0^2 f^2(z)}\right)\left\{2r^2-2r^2b^2\right\}+\frac{1}{2}\left(\frac{-4b}{r_0 f(z)}\right)\left\{-\frac{4br^2}{r_0 f(z)}\right\}\right\}$$

$$+\left[\left(\frac{4\omega_1{}^2-5\omega_p^2}{c^2}\right)+\frac{4\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]A_{20}^{'}\left\{2r^2-2r^2b^2\right\}$$

$$=\frac{4\pi n_0 e^4 B_W A_{10}^2}{4ic^3\omega_1^2 m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)(-2i\omega_1)\left\{2r^2-2r^2b^2\right\},$$

$$-4i(2k_1+k_0)\left[\frac{\partial A_{30}^{'}}{\partial z}\left\{1-b^2\right\}\right]+2A_{20}{}'\left(-i(2k_1+k_0)\right)^2\left\{1-b^2\right\}$$

$$+A_{20}^{'}\left\{\left(-\frac{8}{r_0^2 f^2(z)}\right)\left\{1-b^2\right\}+\left\{\left(\frac{16b^2}{r_0^2 f^2(z)}\right)\right\}\left\{1-b^2\right\}-\left(\frac{2b^2}{r_0^2 f^2(z)}\right)\exp(-2)\right\}$$

$$+A_{20}^{'}\left\{\left(-\frac{8}{r_0^2 f^2(z)}\right)\left\{1-b^2\right\}+\left\{\frac{8b^2}{r_0^2 f^2(z)}\right\}\right\}+\left[\left(\frac{4\omega_1{}^2-5\omega_p^2}{c^2}\right)+\frac{4\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]2A_{20}^{'}\left\{1-b^2\right\}$$

$$=\frac{4\pi n_0 e^4 B_W A_{10}^2}{c^3\omega_1 m^3(\omega_1+i\upsilon)}\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1+i\upsilon}\right)\left\{1-b^2\right\},$$

$$
-\frac{4i(2k_1+k_0)}{k\,r_0^2}\left[\frac{\partial A_{20}^{''}}{\partial\xi}\{1-b^2\}\right]+\left\{\begin{array}{l}
-\frac{8\omega_1^2}{c^2}\left(1-\frac{\omega_p^2}{4\omega_1^2}\right)\{1-b^2\}\\
+\left\{\begin{array}{l}\left(-\frac{8}{r_0^2 f^2(z)}\right)\{1-b^2\}+\left\{\left(\frac{16b^2}{r_0^2 f^2(z)}\right)\right\}\{1-b^2\}\\
-\left(\frac{2b^2}{r_0^2 f^2(z)}\right)\exp(-2)\end{array}\right\}\\
+\left\{\left(-\frac{8}{r_0^2 f^2(z)}\right)\{1-b^2\}+\left\{\frac{8b^2}{r_0^2 f^2(z)}\right\}\right\}\\
+\left[\left(\frac{4\omega_1{}^2-5\omega_p^2}{c^2}\right)+\frac{4\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]2\{1-b^2\}
\end{array}\right\}A_{20}^{''}
$$

$$
=\frac{\omega_p^2}{c}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{m\omega_1 c}\right)\left(\frac{3k_1}{4\omega_1}+\frac{k_1+k_0}{\omega_1}\right)\{1-b^2\},
$$

$$
-4i\left[\frac{\partial A_{20}^{''}}{\partial\xi}\{1-b^2\}\right]+\left\{\begin{array}{l}
-\frac{8r_0^2\omega_1^2}{c^2}\left(1-\frac{\omega_p^2}{4\omega_1^2}\right)\{1-b^2\}\\
+\left\{\begin{array}{l}\left(-\frac{8}{f^2(z)}\right)\{1-b^2\}+\left\{\left(\frac{16b^2}{f^2(z)}\right)\right\}\{1-b^2\}\\
-\left(\frac{2b^2}{f^2(z)}\right)\exp(-2)\end{array}\right\}\\
+\left\{\left(-\frac{8}{f^2(z)}\right)\{1-b^2\}+\left\{\frac{8b^2}{f^2(z)}\right\}\right\}\\
+\frac{\omega_1{}^2 r_0^2}{c^2}\left[\left(4-\frac{5\omega_p^2}{\omega_1{}^2}\right)+4\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)\right]2\{1-b^2\}
\end{array}\right\}A_{20}^{''}
$$

$$
=\frac{r_0^2\omega_p^2}{c^2}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{m\omega_1 c}\right)\left(+2\left(1-\frac{\omega_p^2}{4\omega_1^2}\right)^{1/2}-\frac{1}{4}\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\right)\{1-b^2\},
$$

$$
-4i\left[\frac{\partial A_{20}^{''}}{\partial \xi}\left\{1-b^2\right\}\right]+\left\{\begin{array}{l}
-\frac{8r_0^2\omega_1^2}{c^2}\left(1-\frac{\omega_p^2}{4\omega_1^2}\right)\left\{1-b^2\right\} \\
+\left\{\begin{array}{l}\left(-\frac{16}{f^2(z)}\right)\left\{1-b^2\right\}+\left\{\left(\frac{24b^2}{f^2(z)}\right)\right\}\left\{1-b^2\right\} \\ -\left(\frac{2b^2}{f^2(z)}\right)\exp(-2)\end{array}\right\} \\
+\frac{\omega_1^2 r_0^2}{c^2}\left[\left(4-\frac{5\omega_p^2}{\omega_1^2}\right)+4\phi\left(\vec{E}_1\vec{E}_1^*\right)\right]2\left\{1-b^2\right\}
\end{array}\right\}A_{20}^{''}
$$

$$
=\frac{r_0^2\omega_p^2 n_0}{c^2}\left(\frac{eB_W}{m\omega_1 c}\right)\left(\frac{eA_{10}}{m\omega_1 c}\right)\left(+2\left(1-\frac{\omega_p^2}{4\omega_1^2}\right)^{1/2}-\frac{1}{4}\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\right)\left\{1-b^2\right\}. \tag{8.26}
$$

8.3 Results and discussion

Eqs (8.4) and (8.26) are coupled differential equations for beam width parameter f and normalized amplitude of second harmonic pulse, $A_{20}^{''}/A_{10}$ of second harmonic pulse. We have solved these equations numerically at optimum values of different laser parameters $\omega_1 r_0 / c = 18$, $\varepsilon_2 A_{10}^2 / \varepsilon_0 = 1$, $eA_{10} / m\omega_1 c = 5, eB_w / m\omega_1 c = 3$ and $\omega_p / \omega_1 = 0.8$ and results have been interpreted graphically. Fig. 8.1 shows the variation of f with ξ and it is observed that f attain its minimum value = 0.13 at ξ = 0.02 and continue to show oscillatory behavior due to self-focusing and defocusing, as electrons oscillates transverse to axis results oscillatory variation in plasma density and refractive index. Rawat & Purohit [24] studied the self-focusing of cosh Gaussian beam in magnetized plasma and analyzed the behaviour of f for different laser parameter such as laser intensity, decentered parameter and magnetic field. Their presented that self focusing become stronger with increasing values of these parameters. In Fig. 8.2. the variation of normalized amplitude of second harmonic pulse with ξ distance at different values of normalized intensity of incident laser = 1, 3 and 5 is shown. Results shows that normalized amplitude of second harmonic pulse rises significantly at higher values of normalized intensity of incident laser, the ponderomotive force becomes stronger, which pushes the electrons away from the axial region, therefore increasing the refractive index and stronger self-focusing in induced. Due to stronger self-focusing the gain in efficiency is appreciable. Thakur & Kant [25] presented the similar study for Gaussian beam and observed that increase in intensity of incident laser results appreciable rise in normalized amplitude for second harmonic. Fig. 8.3 presents the graphical variation of normalized amplitude of second harmonic pulse at different values of normalized wiggler field, $eB_W/m\omega_1 c$ = 1, 3 & 5 where other parameters remain the same as in Fig. 8.1. Wiggler magnetic field maintains the cyclotron frequency to confine the electrons in plasma region results strong self-focusing and gain in efficiency is significant. Abedi et al. [15] in the outcome of their study observed that the efficiency of second-harmonic increases significantly with increasing of the wiggler field strength. Fig. 8.4 gives the variation of normalized amplitude of second harmonic pulse with ξ at different values of normalized plasma density, ω_p/ω_1 and our results shows significant rise in normalized amplitude of second harmonic pulse, at higher values of ω_p/ω_1 due to increase in plasma density. Thakur & Kant [26] studied the variation of normalized second harmonic amplitude with the propagation distance for different values of ω_p/ω_1 = 0.2, 0.4, 0.6. The other parameters are same as taken in Fig. 8.1. A sharp increase in the A''_{20}/A_{10} =0.33 is observed for ω_p/ω_1 = 0.6 at ξ = 0.05. Therefore, efficiency of second harmonic generation increases greatly with the increase in plasma frequency in the focal region of fundamental laser beam. Fig. 8.5 shows the variation of normalized amplitude of second harmonic pulse with ξ at different values of decentered parameter. It is observed that, with a small change in the value of decentred parameter the normalized amplitude of second harmonic pulse increases. This shows the sensitiveness of decentred parameter. Singh and Gupta [27] presented the similar study and they showed that normalized amplitude of second harmonic increases for 0 < b < 1 where as the decrease in normalized amplitude is observed for b > 1. Fig. 8.6 shows the variation of normalized amplitude of second harmonic pulse with ξ at different values of density ripple factor. For n = 0.01, 0.02 and 0.03 A''_{20}/A_{10} attain the values 0.09, 0.21 and 0.33. Density ripples provides the phase matching condition by providing the additional momentum to the electrons, results rise in normalized amplitude of second harmonic pulse of second harmonic pulse. Kaur and Kaur [28] presented the similar study for Hermite Gaussian beam where increase in density ripple strengthens the self-focusing.

8.4 Conclusion

From given study we observed the oscillatory behaviour of beam width parameter with normalized propagation distance due to regular self focusing and defocusing of incident laser pulse. Also it is observed the oscillatory behavior of normalized amplitude of second harmonic pulse with intensity of incident pulse, wiggler field, normalized plasma density, decentred parameter and density ripple. This is due to periodic variation of plasma density as electrons are under oscillatory motion, transverse to the direction of propagation of fundamental laser. This result increase and decrease in refractive index due to which SHG shows the oscillatory behaviour.

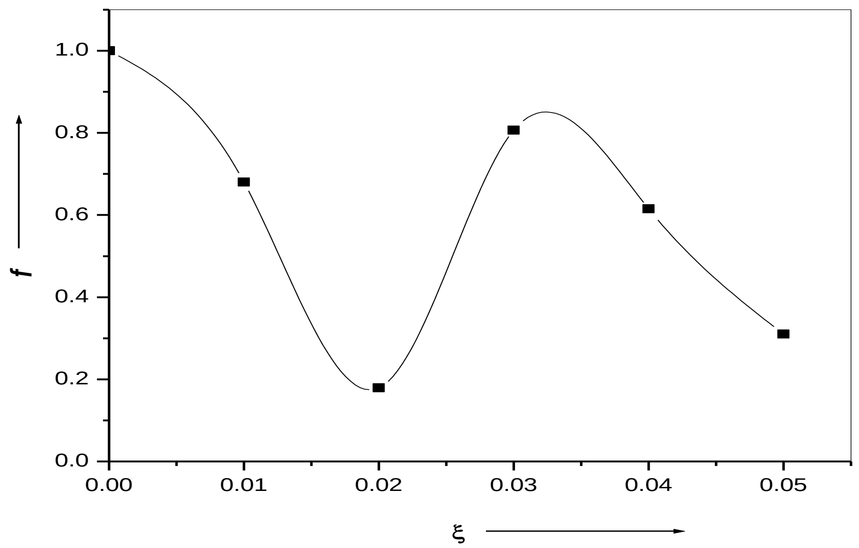

Fig. 8.1 Variation of beam width parameter of the pump laser beam with normalized propagation distance. The other parameters are $\omega_1 r_0 / c = 18,$ $\varepsilon_2 A_{10}^2 / \varepsilon_0 = 1,$ $eA_{10} / m\omega_1 c = 5,$ $eB_w / m\omega_1 c = 3$ and $\omega_p / \omega_1 = 0.8.$

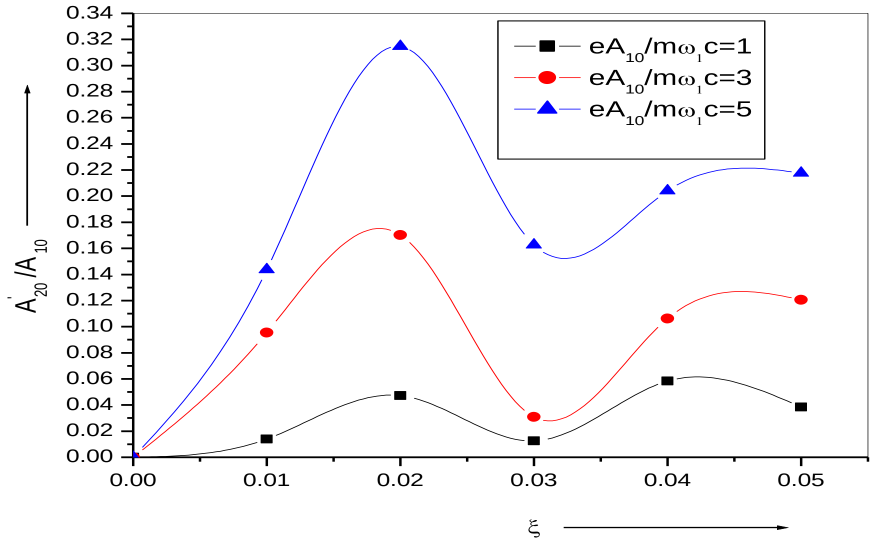

Fig. 8.2 Variation of normalized second harmonic amplitude with normalized propagation distance for different values of $eA_{10} / m\omega_1 c = 1, 3 \,\&\, 5$. The other parameters are same as taken in Fig. 1.

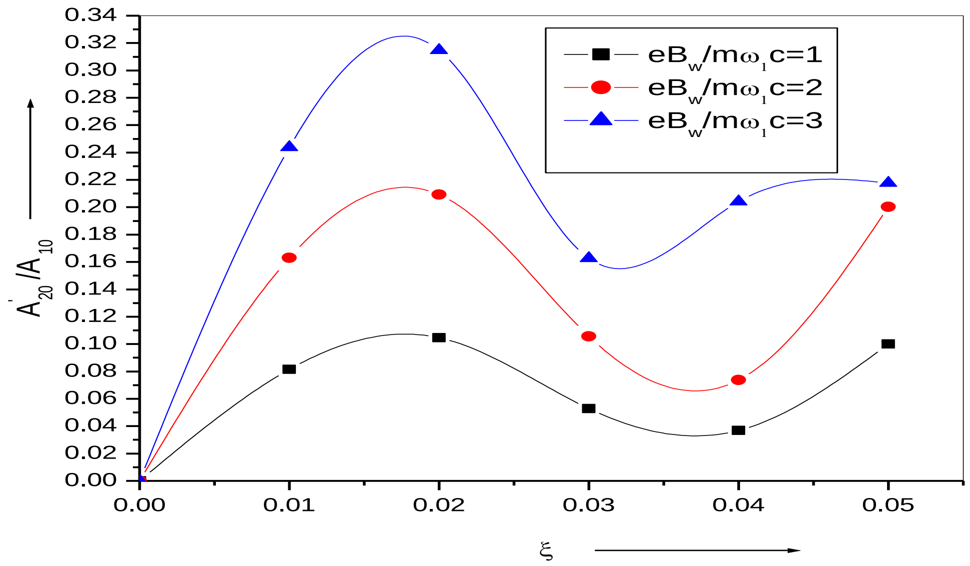

Fig. 8.3 Variation of normalized second harmonic amplitude with normalized propagation distance for different values of $eB_w / m\omega_1 c = 1, 2 \& 3$. The other parameters are same as taken in Fig. 1.

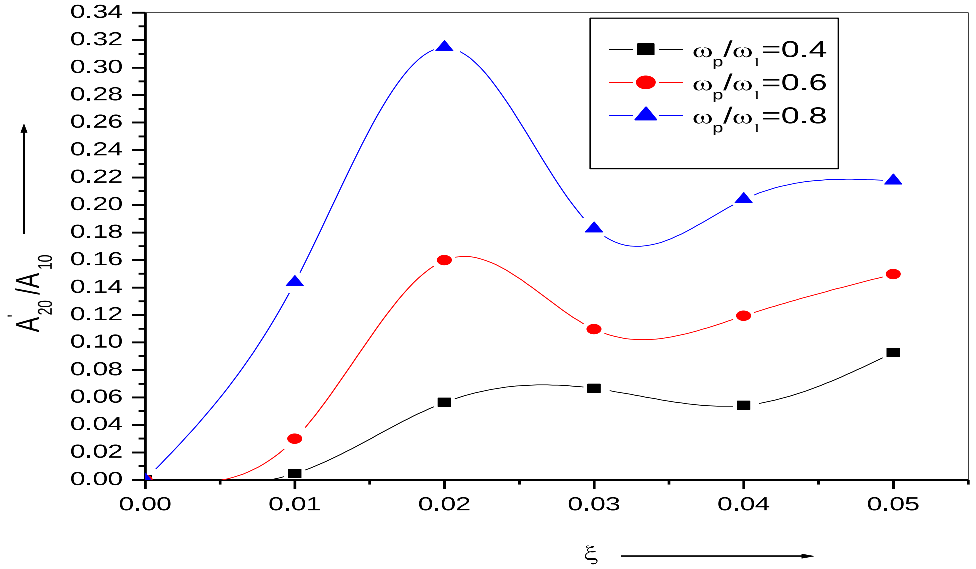

Fig. 8.4 Variation of normalized second harmonic amplitude with normalized propagation distance for different values of $\omega_p / \omega_1 = 0.4, 0.6 \& 0.8$. The other parameters are same as taken in Fig. 1.

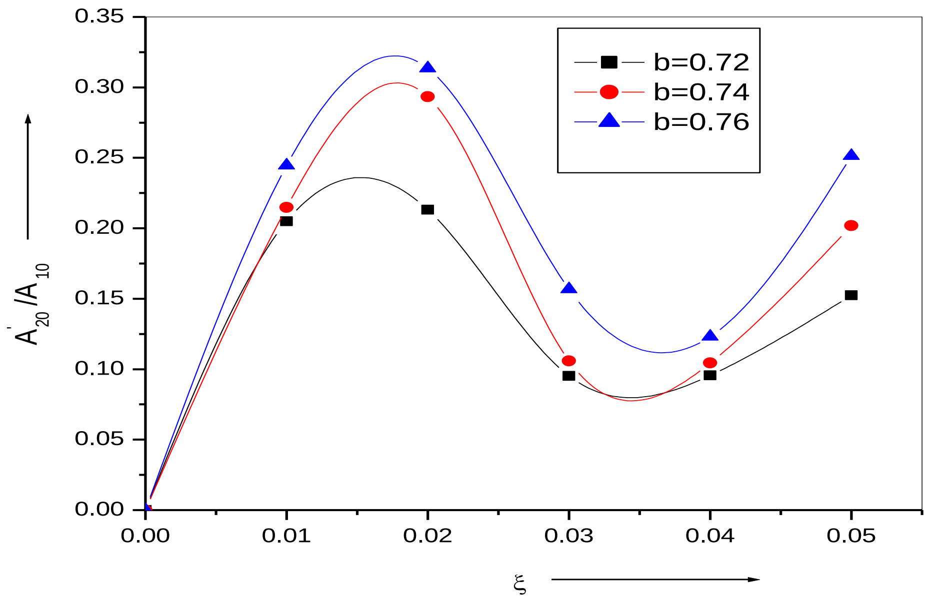

Fig. 8.5 Variation of normalized second harmonic amplitude with normalized propagation distance for different values of decentered parameter $b = 0.7, 0.8 \,\&\, 0.9$. The other parameters are same as taken in Fig. 1.

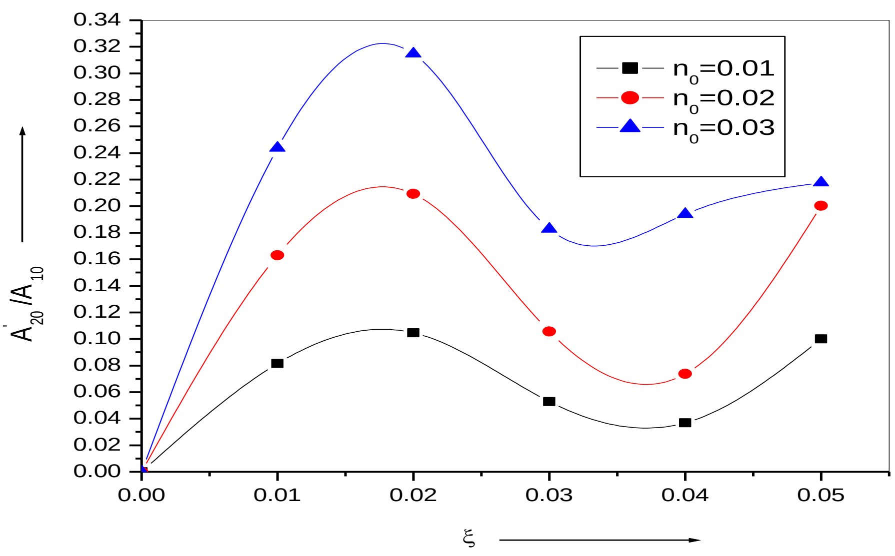

Fig. 8.6 Variation of normalized second harmonic amplitude with normalized propagation distance for different values of rippled density factor $n_0 = 0.01, 0.02 \,\&\, 0.03$. The other parameters are same as taken in Fig. 1.

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Chapter-9

Relativistic self-focused Hermite-cosh-Gaussian laser beam and third harmonic generation

9.1 Introduction

Propagation of high intensity laser through plasma, has been widely studied in last few decades by number of researchers. When short pulse laser propagates through plasma, it results density perturbations and nonlinearity arises. Variation in dielectric properties of plasma results important nonlinear phenomena, which have their wide range of applications like inertial confinement fusion [1], laser plasma accelerator [2], microscopic resonance imaging [3] by using second harmonic generation (SHG) and THG, laser driven accelerators [4], relativistic self-focusing [5], ponderomotive self-focusing [6], harmonic generations [7] etc. Amongst different harmonic generations the THG is a specific topic of research due to its important applications in medical science [8], telecommunication [9], optoelectronics [10] etc. Kant *et al*. [11] studied the THG when short-pulse laser propagates through plasma created by electron-hole pair in semiconductor medium. Wiggler field satisfied the resonant condition, results enhancement in the energy conversion efficiency. Dhaiya *et al*. [12] studied phase matched SHG and THG in plasma in the presence of density ripple. Density ripple provide the phase matching condition, result increase in the efficiency of harmonic generations. Vij *et al.* [13] studied the resonant THG in cluster density. Presence of density ripples in cluster density, satisfy the phase matching condition, which is responsible for the enhancement in harmonic generation. They observed that the velocity of fundamental laser differ from group velocity results the third harmonic pulse to slips out of the domain of the fundamental laser pulse. Thakur *et al.* [14] studied THG of short pulse laser in plasma. In the presence of density ramp in plasma the enhancement of normalized amplitude of third harmonic pulse takes place. In their study wiggler field provided the necessary phase matching condition and pulse slippage of third harmonic pulse was reported due to the difference in velocity of third harmonic pulse with fundamental laser pulse. Shibhu *et al.* [15] investigated the Phase matched THG of laser radiation in a plasma channel, where background density perturbation satisfy the phase matching condition. Rajput *et al.* [16] studied the resonant THG of short pulse laser in plasma. Phase matching condition was satisfied by the wiggler magnetic field result enhancement in the efficiency of third harmonic pulse. Sharma *et al.* [17] studied THG under relativistic self-focusing, when short pulse laser interacts with plasma. In their study, the wiggler field satisfied the phase matching conditions, results enhancement in the normalized amplitude of third harmonic pulse. Stronger self-focusing was reported at higher intensity of incident laser pulse.

Number of research workers investigated the self-focusing of HchG laser beam using different laser parameters, under different conditions. Nanda *et al.* [18] studied the relativistic self-focusing of HchG laser beam in plasma under density transition and studied the effect of different laser parameters on beam width parameter, where plasma density transition played vital role in self-focusing. Nanda *et al.* [19] studied the self-focusing of HchG laser beam in magneto-plasma with density ramp profile. They showed that for m = 0, 1 and 2, the diffraction overcomes nonlinear term for b = 0, where b is the decentred parameter. Wani *et al.* [20] studied the self-focusing of HchG laser beam in plasma under density transition, where plasma density ramp reduces the defocusing and smaller spot size is obtained upto several Rayleigh length. Kaur *et al.* [21] studied the self-focusing of HchG laser beam under relativistic nonlinearity and observed the self-focusing for m = 0, using a particular set of de-cantered parameter. Nanda *et al.* [22] studied the effect of decentred parameter on relativistic self-focusing of HchG laser beam in plasma and stronger self-focusing was reported for m=2. Patil *et al.* [23] studied the self-focusing of HchG laser beam in collision less magneto plasma and their study comprises the combined effect of nonlinearity and spatial diffraction. Kant *et al.* [24] presented the stronger self-focusing of HchG laser beam in plasma and analyzed the behaviour of beam width parameter at different values of decentred parameter and frequency of incident laser beam.

This has motivated us to study the effect of different laser parameters on normalized amplitude of THG, for intense HchG laser beam propagating through plasma. Using paraxial approximation we have obtained the expression of normalized amplitude for THG using HchG laser beam, results have been analyzed graphically. Behaviour of beam width parameter with linear propagation distance has been done at different mode indices. Analysis of variation of efficiency of third harmonic pulse, with linear propagation distance, at optimum values of different laser and plasma parameters is also done in present work.

9.2 Theoretical considerations

Hermite-Gaussian beam is considered to be propagating in plasma with the field distribution as

$$\vec{E}_1 = \hat{x}A_1(z)\exp[-i(\omega_1 t - k_1 z)]\,, \qquad (9.1)$$

$$\vec{B}_w = \hat{y}B_0\exp(ik_0 z)\,, \qquad (9.2)$$

where $A = A_0(r,z)\exp(-ikS)$,

$$A_0^2 = \frac{E_{30}^2}{f^2(z)}\left[H_m\left(\frac{\sqrt{2}r}{r_0 f_{(z)}}\right)\right]^2 e^{\left(\frac{b^2}{2}\right)}\left\{\exp\left[-2\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]+\exp\left[-2\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\right\} . \qquad (9.3)$$
$$+2\exp\left[-\left(\frac{2r^2}{r_0^2 f(z)^2}+\frac{b^2}{2}\right)\right]$$

where $\vec{k}_1$ is the wave vector of fundamental laser pulse, $\vec{B}_w$ is the wiggler magnetic field and $\vec{k}_0$ is the wiggler wave number A is the amplitude of the incident pulse. Due to electromagnetic field of incident laser electrons attain oscillatory velocity given as

$$\vec{v}_1 = e\,\frac{\vec{E}_1}{mi(\omega_1 + iv)}\,. \qquad (9.4)$$

Density perturbation beats with $\vec{v}$ to produce third harmonic current density $\vec{J}_3$ driving and beam width parameter [24] for for different mode is given as

For m =0

$$\frac{\partial^2 f(z)}{\partial \xi^2} = \left[4 - 4b^2 - \frac{6\alpha E_0^2\omega_1^2 r_0^2 m_0\omega_p^2}{\omega_1^2\gamma c^2 M}Exp\left[\frac{b^2}{2}\right]\right]\frac{1}{f^3(z)}. \qquad (9.5)$$

For m =1

$$\frac{\partial^2 f(z)}{\partial \xi^2} = \left[4 - 4b^2 - \frac{12\alpha E_0^2\omega_1^2 r_0^2 m_0\omega_p^2}{\omega_1^2\gamma c^2 M}Exp\left[\frac{b^2}{2}\right]\left(b^2-2\right)\right]\frac{1}{f^3(z)}. \qquad (9.6)$$

For m =2

$$\frac{\partial^2 f(z)}{\partial \xi^2} = \left[-8b^2 - \frac{24\alpha E_0^2 \omega_1^2 r_0^2 m_0 \omega_p^2}{\omega_1^2 \gamma c^2 M} Exp\left[\frac{b^2}{2}\right]\left(5-2b^2\right)\right] \frac{1}{f^3(z)}. \tag{9.7}$$

Using Eq. (5.13)

$$\nabla^2 \vec{E}_3 + \left[\left(\frac{9\omega_1{}^2 - 10\omega_p^2}{c^2} \right) + \frac{9\omega_1{}^2 \phi\left(\vec{E}_1 \vec{E}^{\,*}_1\right)}{c^2} \right] \vec{E}_3 = \frac{4\pi \partial \vec{J}_3^{NL}}{c^2 \partial t} \ , \tag{9.8}$$

where $\vec{J}_3^L$ and $\vec{J}_3^{NL}$ [16] are obtained as

$$\vec{J}_3^L = \frac{n_0 e^2 \vec{E}_3}{m 3 i \omega_1} \ , \tag{9.9}$$

$$\vec{J}_3^{NL} = \frac{-n_0^0 e^5 B_w k_1 \vec{E}_1^3}{16 c i m^4 \gamma^4 \omega_1^4 (\omega_1 + i v)} \left[\frac{5k_1}{18\omega_1} + \frac{k_1 + k_0}{\omega_1 + i v} \right] \hat{x}. \tag{9.10}$$

Particular integral of Eq. (9.8) is

$$\vec{E}_3 = \hat{x} A_3^{'}(z,t) \exp[-i(3\omega_1 t - k_3 z)], \tag{9.11}$$

$$A_3^{'} = A_{30}^{'}(z)\psi_3, \tag{9.12}$$

$$\psi_3 = \frac{E_0}{f(z)} Exp\left[\frac{b^2}{4}\right] \left\{ \begin{array}{l} Exp\left[-6\left(\frac{r}{r_0 f(z)} + \frac{b}{2} \right)^2 \right] \\ + Exp\left[-6\left(\frac{r}{r_0 f(z)} - \frac{b}{2} \right)^2 \right] + 2Exp\left[-3\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2} \right) \right] \end{array} \right\}^{1/2} , \tag{9.13}$$

$$\frac{\partial E_3}{\partial z} = \exp\left[-i\left\{ 3\omega_1 t - \left(3k_1 + k_0\right) z \right\} \right] \left[\frac{\partial A_3^{'}}{\partial z} + i\left(3k_1 + k_0\right) A_3^{'} \right],$$

$$\frac{\partial^2 E_3}{\partial z^2} = \exp\left[-i\left\{ 3\omega_1 t - \left(3k_1 + k_0\right) z \right\} \right] \left[\frac{\partial^2 A_3^{'}}{\partial z^2} + 2i\left(3k_1 + k_0\right) \frac{\partial A_3^{'}}{\partial z} + \left[i\left(3k_1 + k_0\right) \right]^2 A_3^{'} \right],$$

neglecting term containing $\partial^2 A_3^{'} / \partial z^2$, being small we get

$$\frac{\partial^2 E_3}{\partial z^2} = 2i(3k_1 + k_0)A_3' \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}]\frac{\partial A_3'}{\partial z} + [i(3k_1 + k_0)]^2 A_3' \exp[-i\{3\omega_1 t - (3k_1 + k_0)z\}], \quad (9.14)$$

$$\frac{\partial A_3'}{\partial z} = \frac{\partial A_{30}'}{\partial z}\left\{\begin{array}{l} Exp\left[-6\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] \\ + Exp\left[-6\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] + 2Exp\left[-3\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right]\end{array}\right\}^{1/2} \exp(-iks)$$

$$+ A_{30}' \frac{\partial f}{\partial z}\left(\frac{1}{2}\right)\gamma^{-1/2}\left\{\begin{array}{l} Exp\left[-6\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right]\left[\begin{array}{l}\left(\frac{-6r^2}{r_0^2 f^2(z)}\right)\left(\frac{-2}{f}\right)\left(\frac{\partial f}{\partial z}\right) \\ -\left(\frac{6br}{r_0 f(z)}\right)\left(\frac{-1}{f}\right)\left(\frac{\partial f}{\partial z}\right)\end{array}\right] \\ + Exp\left[-6\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right]\left(\frac{-6r^2}{r_0^2 f^2(z)}\right)\left(\frac{-2}{f}\right)\left(\frac{\partial f}{\partial z}\right) \\ + \left(\frac{6br}{r_0 f(z)}\right)\left(\frac{-1}{f}\right)\left(\frac{\partial f}{\partial z}\right) \\ + 2Exp\left[-3\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right]\left(\frac{-6r^2}{r_0^2 f^2(z)}\right)\left(\frac{-2}{f}\right)\left(\frac{\partial f}{\partial z}\right)\end{array}\right\}\exp(-iks),$$

where

$$\gamma = \left\{\begin{array}{l} Exp\left[-6\left(\frac{r}{r_0 f(z)} + \frac{b}{2}\right)^2\right] \\ + Exp\left[-6\left(\frac{r}{r_0 f(z)} - \frac{b}{2}\right)^2\right] + 2Exp\left[-3\left(\frac{2r^2}{r_0^2 f^2(z)} + \frac{b^2}{2}\right)\right]\end{array}\right\},$$

$$\frac{\partial A_3'}{\partial z} = \frac{\partial A_{30}'}{\partial z}\left\{\begin{aligned}&Exp\left[-6\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\&+Exp\left[-6\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]+2Exp\left[-3\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{aligned}\right\}^{1/2}\exp(-iks)$$

$$+A_{30}'\frac{\partial f}{\partial z}\left(\frac{1}{2}\right)\gamma^{-1/2}\gamma\left(\frac{-6r^2}{r_0^2 f^2(z)}\right)\left(\frac{-2}{f}\right)\left(\frac{\partial f}{\partial z}\right)\exp(-iks)$$

$$+A_{30}'\frac{\partial f}{\partial z}\left(\frac{1}{2}\right)\gamma^{-1/2}\left[-\left(\frac{6br}{r_0 f(z)}\right)\left(\frac{-1}{f}\right)\left(\frac{\partial f}{\partial z}\right)\right]\left\{\begin{aligned}&Exp\left[-6\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\&-Exp\left[-6\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]\end{aligned}\right\}\exp(-iks), \quad (9.15)$$

putting Eq. (9.15) into Eq. (9.14) we obtain

$$\frac{\partial^2 E_3}{\partial z^2} = 2i(3k_1+k_0)\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{E_0}{f(z)}Exp\left[\frac{b^2}{4}\right]\frac{\partial A_{30}'}{\partial z}\gamma^{1/2}\exp(-iks)$$

$$-A_{30}'\frac{E_0}{f^2(z)}Exp\left[\frac{b^2}{4}\right]\frac{\partial f}{\partial z}\gamma^{1/2}\exp(-iks)$$

$$+[i(3k_1+k_0)]^2 A_3'\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\exp(-iks),$$

$$\frac{\partial^2 E_3}{\partial z^2} = 2i(3k_1+k_0)\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_{30}'}{\partial z}\gamma^{1/2}\exp(-iks)$$

$$+[i(3k_1+k_0)]^2 A_3'\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\exp(-iks),$$

$$\frac{\partial^2 E_3}{\partial z^2} = 2i(3k_1+k_0)A_3'\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\psi_3\frac{\partial A_{30}'}{\partial z}$$

$$+[i(3k_1+k_0)]^2 A_{30}'\psi_3\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}],$$

multiply by $r\psi_3^*$ and integrate with respect to ‘r’

$$\int r\psi_3^*\frac{\partial^2 E_3}{\partial z^2} = 2i(3k_1+k_0)A_3'\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\frac{\partial A_{30}'}{\partial z}\int r\psi_3^*\psi_3$$

$$+[i(3k_1+k_0)]^2 A_{30}'\exp[-i\{3\omega_1 t-(3k_1+k_0)z\}]\int r\psi_3^*\psi_3, \quad (9.16)$$

$$\int r\psi_3^*\psi_3 dr = \int r\left\{\begin{aligned}&Exp\left[-6\left(\frac{r}{r_0 f(z)}+\frac{b}{2}\right)^2\right]\\ &+Exp\left[-6\left(\frac{r}{r_0 f(z)}-\frac{b}{2}\right)^2\right]+2Exp\left[-3\left(\frac{2r^2}{r_0^2 f^2(z)}+\frac{b^2}{2}\right)\right]\end{aligned}\right\}dr,$$

$$\int r\psi_3^*\psi_3 dr = \int r\left\{4-24\frac{r^2}{r_o^2 f^2(z)}-6b^2\right\}dr,$$

$$\int r\psi_3^*\psi_3 dr = \int\left\{4r-24\frac{r^3}{r_o^2 f^2(z)}-6rb^2\right\}dr,$$

$$\int r\psi_3^*\psi_3 dr = \left\{2r^2-6\frac{r^4}{r_o^2 f^2(z)}-r^2b^2\right\}, \tag{9.17}$$

differentiating Eq. (9.11) with respect to r

$$\frac{\partial E_3}{\partial r} = A_{30}^{'}\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial\psi_3}{\partial r} \quad , \tag{9.18}$$

$$\frac{\partial^2 E_3}{\partial r^2} = A_{30}^{'}\exp[-i(3\omega_1 t - k_3 z)]\frac{\partial^2\psi_3}{\partial r^2}, \tag{9.19}$$

putting Eqs. (9.16), (9.18) and (9.19) into Eq. (9.8) We obtain

$$\frac{\partial^2 E_3}{\partial z^2}\int r\psi_3^* + A_{30}^{'}\exp[-i(3\omega_1 t - k_3 z)]\int\frac{r\psi_3^*\partial\psi_3}{r\partial r} + A_{30}^{'}\exp[-i(3\omega_1 t - k_3 z)]\int r\psi_3^*\frac{\partial^2\psi_3}{\partial r^2}$$
$$-\left[\frac{10\omega_p^2}{c^2}\frac{-9\omega_1{}^2}{c^2}-\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]\vec{E}_3\int r\psi_3^* = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\left[\frac{-n_0^0 e^5 B_w k_1 \vec{E}_1^3}{16cim^4\gamma^4\omega_1^4(\omega_1+iv)}\left[\frac{5k_1}{18\omega_1}+\frac{k_1+k_0}{\omega_1+iv}\right]\right]\int r\psi_3^*,$$

$$2i(3k_1+k_0)A_3^{'}\int r\psi_3^*\psi_3\,\frac{\partial A_{30}^{'}}{\partial z}$$

$$+[i(3k_1+k_0)]^2A_{30}^{'}\int r\psi_3^*\psi_3+A_{30}^{'}\int\frac{r\psi_3^*\partial\psi_3}{r\partial r}+A_{30}^{'}\int r\psi_3^*\frac{\partial^2\psi_3}{\partial r^2}$$

$$-\left[\left(\frac{10\omega_p^2}{c^2}\frac{-9\omega_1{}^2}{c^2}-\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\;*}\right)}{c^2}\right)\right]A_{30}^{'}\int r\psi_3^*\psi_3=$$

$$\frac{4\pi}{c^2}\frac{\partial}{\partial t}\left\{\frac{-n_0^0e^5B_wk_1\vec{E}_1^3}{\exp[-i\{3\omega_1t-(3k_1+k_0)z\}]16cim^4\gamma^4\omega_1^4(\omega_1+iv)}\left[\frac{5k_1}{18\omega_1}+\frac{k_1+k_0}{\omega_1+iv}\right]\right\}\int r\psi_3^*,$$

$$\int r\psi_3^*\frac{\partial^2\psi_3}{\partial r^2}=\left\{\frac{54b^2}{2r_0^2f^2(z)}-\frac{27b^4}{r_0^2f^2(z)}-\frac{24}{2r_0^2f^2(z)}-\frac{9b^2}{2r_0^2f^2(z)}\exp(-2)+\{2-b^2\}(-ik)\frac{r^2\partial f_3}{f_3\partial z}\right\},$$

$$\int\frac{r\psi_3^*\partial\psi_3}{r\partial r}==\left\{4-24\frac{r^2}{r_o^2f^2(z)}-6b^2\right\},$$

using Eq. (9.17), we get

$$2i(3k_1+k_0)A_3^{'}\frac{\partial A_{30}^{'}}{\partial z}\left\{2r^2-6\frac{r^4}{r_o^2f^2(z)}-r^2b^2\right\}$$

$$+[i(3k_1+k_0)]^2A_{30}^{'}\left\{2r^2-6\frac{r^4}{r_o^2f^2(z)}-r^2b^2\right\}+A_{30}^{'}\left\{-\frac{24}{r_o^2f^2(z)}\right\}$$

$$+A_{30}^{'}\left\{\begin{matrix}\frac{36b^2}{2r_0^2f^2(z)}+\frac{18b^2}{r_0^2f^2(z)}-\frac{24}{2r_0^2f^2(z)}-\frac{27b^4}{r_0^2f^2(z)}\\-\frac{9b^2}{2r_0^2f^2(z)}\exp(-2)\end{matrix}\right\}$$

$$-\left[\left(\frac{10\omega_p^2}{c^2}\frac{-9\omega_1{}^2}{c^2}-\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\;*}\right)}{c^2}\right)\right]A_{30}^{'}\{2-b^2\}=$$

$$\frac{4\pi}{c^2}\frac{\partial}{\partial t}\left\{\frac{-n_0^0e^5B_wk_1\vec{E}_1^3}{\exp[-i\{3\omega_1t-(3k_1+k_0)z\}]16cim^4\gamma^4\omega_1^4(\omega_1+iv)}\left[\frac{5k_1}{18\omega_1}+\frac{k_1+k_0}{\omega_1+iv}\right]\right\}\int r\psi_3^*. \tag{9.20}$$

$$\int r\psi_3^* \frac{\partial^2\psi_3}{\partial r^2} = \left\{ \begin{array}{l} \frac{36b^2}{2r_0^2 f^2(z)} + \frac{18b^2}{r_0^2 f^2(z)} - \frac{24}{2r_0^2 f^2(z)} - \frac{27b^4}{r_0^2 f^2(z)} \\ -\frac{9b^2}{2r_0^2 f^2(z)}\exp(-2) + \{2-b^2\}(-ik)\frac{r^2\partial f_3}{f_3\partial z} \end{array} \right\},$$

where

$$\int r\psi_3^* = \int\left\{4r - 24\frac{r^3}{r_o^2 f^2(z)} - 6b^2 r\right\}dr = \int\left\{2r^2 - 6\frac{r^4}{r_o^2 f^2(z)} - 3b^2 r^2\right\}dr, \qquad (9.21)$$

from Eq. (9.21) and Eq. (9.20), we obtain

$$2i\frac{(3k_1+k_0)}{R_d}\frac{\partial A'_{30}}{\partial \xi}\{2-b^2\} + \left[\begin{array}{l} [i(3k_1+k_0)]^2\{2-b^2\} + A'_{30}\left\{-\frac{24}{r_o^2 f^2(z)}\right\} \\ +\left\{ \begin{array}{l} \frac{36b^2}{2r_0^2 f^2(z)} + \frac{18b^2}{r_0^2 f^2(z)} - \frac{24}{2r_0^2 f^2(z)} - \frac{27b^4}{r_0^2 f^2(z)} \\ -\frac{9b^2}{2r_0^2 f^2(z)}\exp(-2) \end{array} \right\} \\ -\left[\frac{10\omega_p^2}{c^2}\frac{-9\omega_1{}^2}{c^2} - \frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_{\,1}\right)}{c^2} \right]\{2-b^2\} \end{array} \right] A'_{30} =$$

$$\left(\frac{4\pi}{c^2}\right)\frac{-n_0^0 e^5 B_w k_1 A_1^3(-3i\omega_1)}{16cim^4\gamma^4\omega_1^4(\omega_1+iv)}\left[\frac{5k_1}{18\omega_1} + \frac{k_1+k_0}{\omega_1+iv}\right]\{2-3b^2\},$$

$$2i\frac{\partial A'_{30}}{\partial \xi}\{2-b^2\} + \left[\begin{array}{l} [i(3k_1+k_0)]^2 r_0^2\{2-b^2\} + A'_{30}\left\{-\frac{24}{r_o^2 f^2(z)}\right\} \\ +\left\{\frac{36b^2}{2f^2(z)} + \frac{18b^2}{f^2(z)} - \frac{27b^4}{f^2(z)} - \frac{24}{2f^2(z)} - \frac{9b^2}{2f^2(z)}\exp(-2)\right\} \\ -\left[\frac{10\omega_p^2}{c^2}\frac{-9\omega_1{}^2}{c^2} - \frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}^{\,*}_{\,1}\right)}{c^2} \right] r_0^2\{2-b^2\} \end{array} \right] A'_{30} =$$

$$\left(\frac{4\pi}{c^2}\right)\frac{-n_0^0 e^5 B_w k_1 A_1^3(-3i\omega_1)r_0^2}{16cim^4\gamma^4\omega_1^4(\omega_1+iv)}\left[\frac{5k_1}{18\omega_1} + \frac{k_1+k_0}{\omega_1+iv}\right]\{2-3b^2\},$$

$$2i\frac{\partial A_{30}^{'}}{\partial\xi}\left\{2-b^2\right\}+\left[\begin{array}{l}\left[i(3k_1+k_0)\right]^2 r_0^2\left\{2-b^2\right\}+A_{30}^{'}\left\{-\frac{24}{r_o^2 f^2(z)}\right\}\\ +\left\{\begin{array}{l}\frac{36b^2}{2f^2(z)}+\frac{18b^2}{f^2(z)}-\frac{27b^4}{f^2(z)}-\frac{24}{2f^2(z)}\\ -\frac{9b^2}{2f^2(z)}\exp(-2)\end{array}\right\}\\ -\left[\frac{10\omega_p^2}{c^2}\frac{-9\omega_1{}^2}{c^2}-\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]r_0^2\left\{2-b^2\right\}\end{array}\right]A_{30}^{'}=$$

$$\frac{3A_{10}}{16\gamma^4}\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{\omega_1^2 r_0^2}{c^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\left(\frac{e^2A_{10}^2}{m^2\omega_1^2c^2}\right)\left(\frac{eB_w}{m\omega_1 c}\right)\left[+3\left(1-\frac{\omega_p^2}{9\omega_1^2}\right)^{1/2}-\frac{31}{18}\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\right]\left\{2-3b^2\right\},$$

$$2i\frac{\partial A_{30}^{''}}{\partial\xi}\left\{2-b^2\right\}+\left[\begin{array}{l}-\left(\frac{9\omega_1^2 r_0^2}{c^2}\right)\left(1-\frac{\omega_p^2}{9\omega_1^2}\right)^2\left\{2-b^2\right\}+A_{30}^{'}\left\{-\frac{24}{r_o^2 f^2(z)}\right\}\\ +\left\{\begin{array}{l}\frac{36b^2}{2f^2(z)}+\frac{18b^2}{f^2(z)}-\frac{27b^4}{f^2(z)}-\frac{24}{2f^2(z)}\\ -\frac{9b^2}{2f^2(z)}\exp(-2)\end{array}\right\}\\ -\left[\frac{10\omega_p^2}{c^2}\frac{-9\omega_1{}^2}{c^2}-\frac{9\omega_1{}^2\phi\left(\vec{E}_1\vec{E}_1^{\,*}\right)}{c^2}\right]r_0^2\left\{2-b^2\right\}\end{array}\right]A_{30}^{''}=$$

$$\frac{3}{16\gamma^4}\left(\frac{\omega_p^2}{\omega_1^2}\right)\left(\frac{\omega_1^2 r_0^2}{c^2}\right)\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\left(\frac{e^2A_{10}^2}{m^2\omega_1^2c^2}\right)\left(\frac{eB_w}{m\omega_1 c}\right)\left[\begin{array}{l}+3\left(1-\frac{\omega_p^2}{9\omega_1^2}\right)^{1/2}\\ -\frac{31}{18}\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{1/2}\end{array}\right]\left\{2-3b^2\right\}, \qquad (9.22)$$

9.3 Results and discussion

Eqs (9.5) (9.6) and (9.7) are the coupled differential equations for beam width parameter for mode index m = 0, 1, 2 and Eq (9.22) is the derived differential equation for normalized amplitude of third harmonic pulse. We have solved these equations numerically at different values of different laser parameters as $\omega_1 r_0 / c = 27,\ \ \varepsilon_2 A_{10}^2 / \varepsilon_0 = 1,\ \ eA_{10} / m\omega_1 c = 5,\ eB_w / m\omega_1 c = 3$ and $\omega_p / \omega_1 = 0.8$ and interpreted the results graphically. Fig. 1 Shows the variation of beam width parameter, f with normalized propagation distance ξ at m = 0, 1 and 2 respectively. For m = 0, 1 and 2 the beam width parameter = 0.7, 0.38 and 0.1 respectively. This shows that self-focusing is stronger for m = 2. Patil *et al.* [23] studied the self focusing for Hermite cosh Gaussian laser beam in magneto-plasma for m = 0, m = 1 and m = 2. Their result shows similar behavior for self focusing, which is more stronger for m = 2. The variation of normalized amplitude of third harmonic pulse with ξ at different values of normalized intensity of incident pulse $eA_{10} / m\omega_1 c$ = 1, 3, 5 is plotted in Fig. 2, where other parameters are same as in Fig.1. It is observed that normalized amplitude of third harmonic pulse attain its values 0.049, 0.26 0.55 at normalized intensity of incident pulse = 1, 3, 5 respectively. As incident laser becomes more intense, the electrons experience stronger ponderomotive and higher quiver velocity, results stronger self-focusing. Singh et al. [25] presented the similar results for SHG of Hermite cosh Gaussian laser beam, where the normalized amplitude of second harmonic pulse increases significantly with increase in intensity of incident laser pulse due to stronger self focusing. Fig. 3 Shows the variation of normalized amplitude of third harmonic pulse with ξ at different values of normalized wiggler magnetic field given as, $eB_w / m\omega_1 c =\ 1, 2\ \&\ 3$, where other parameters are same as in Fig. 1. For $eB_w / m\omega_1 c = 1, 2\ \&\ 3$ the values of normalized amplitude for third harmonic pulse = 0.15, 0.34 & 0.54 respectively. This shows that normalized amplitude of third harmonic pulse rises significantly at higher values of normalized wiggler magnetic field, due to resonance condition satisfied by wiggler field. Sharma et al. [26] studied the THG for Gaussian laser pulse under the effect of wiggler magnetic field and observed that gain is significant at higher values of wiggler magnetic field. The peak value of normalized amplitude for third harmonic pulse is 0.132, where as in present work, for Hermite cosh Gaussian laser beam the peak value of normalized amplitude is nearly 0.54. This shows that gain is more prominent when we use the Hermite cosh Gaussian laser profile. Fig. 4 shows the variation of normalized amplitude of third harmonic pulse, with linear propagation distance, at different values of normalized plasma density ω_p / ω_1. Normalized amplitude for third harmonic pulse attain its peak values = 0.14, 0.37 and 0.55 at ω_p / ω_1 = 0.4, 0.6 & 0.8 respectively. With increase in plasma density the phase velocity of pulse decreases, results increase in refractive index due to which gain in efficiency increases. Singh et al. [25] studied the normalized amplitude of SHG for Hermite cosh Gaussian laser beam with normalized plasma density. They obtained the peak value nearly 0.38, where as our results shows maximum value nearly 0.55, which is better than the previous work. Fig. 5 shows the variation of normalized amplitude of third harmonic pulse, with linear propagation distance, at different values of decentred parameter taken as $b = 0.52, 0.54 \,\&\, 0.56$. It is observed that the small variation of beam width parameter results change in normalized amplitude, thus showing the sensitivity of decentred parameter. Nanda et al. [27] presented the similar study for Hermite cosh Gaussian beam in magneto-plasma and had given the sensitivity of decentered parameter by studying the beam width parameter with the variation of decentred parameter.

Conclusion

With the graphical analysis of beam width parameter, it is observed that the self-focusing is weaker for m =1 and is stronger at higher mode indices. It is observed that beam width parameter of HchG laser beam attain a very low value, nearly 0.1, is stronger at m = 2. It is due to stronger ponderomotive force due to result increase in refractive index. Gain in normalized amplitude of THG is significantly high for m =2 as compare to m =1, at different values of normalized intensity of incident laser, normalized wiggler field and normalized plasma density. On comparison we have seen that gain in efficiency of HchG laser beam is more as compared to Gaussian or cosh Gaussian laser beam. The variation of normalized amplitude of third harmonic pulse with small variation of decentred parameter shows the sensitivity of b.

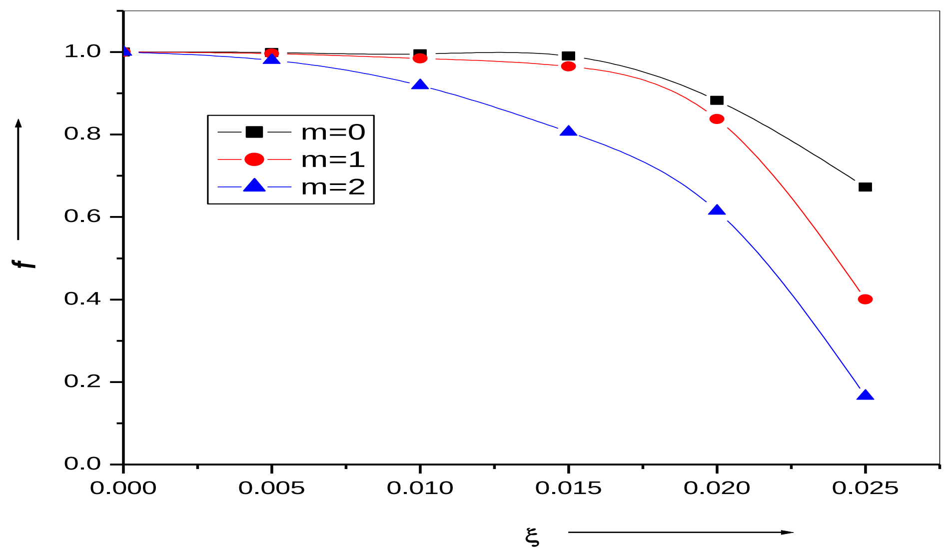

Fig. 9.1 Variation of beam width parameter f of the pump laser beam with normalized propagation distance ξ for different values of $m = 0, 1 \& 2$. The other parameters are $\omega_1 r_0 / c = 27$, $\varepsilon_2 A_{10}^2 / \varepsilon_0 = 1$, $eA_{10} / m\omega_1 c = 5$, $eB_w / m\omega_1 c = 3$ and $\omega_p / \omega_1 = 0.8$.

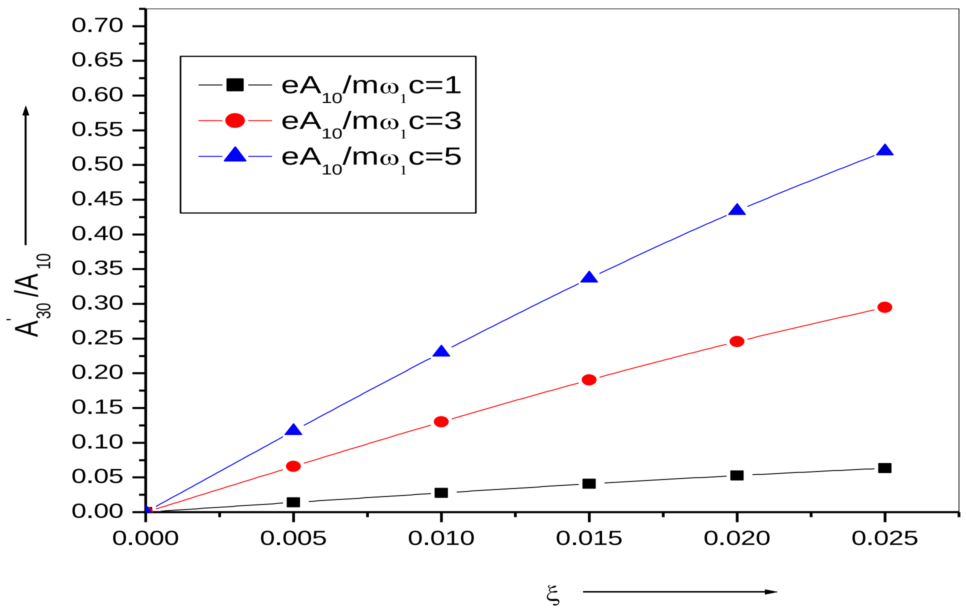

Fig. 9.2 Variation of normalized third harmonic amplitude $A_{30}^{'} / A_{10}$ with normalized propagation ξ distance for different values of normalized intensity of incident laser, $eA_{10} / m\omega_1 c = 1, 3 \& 5$. The other parameters are same as taken in Fig. 1.

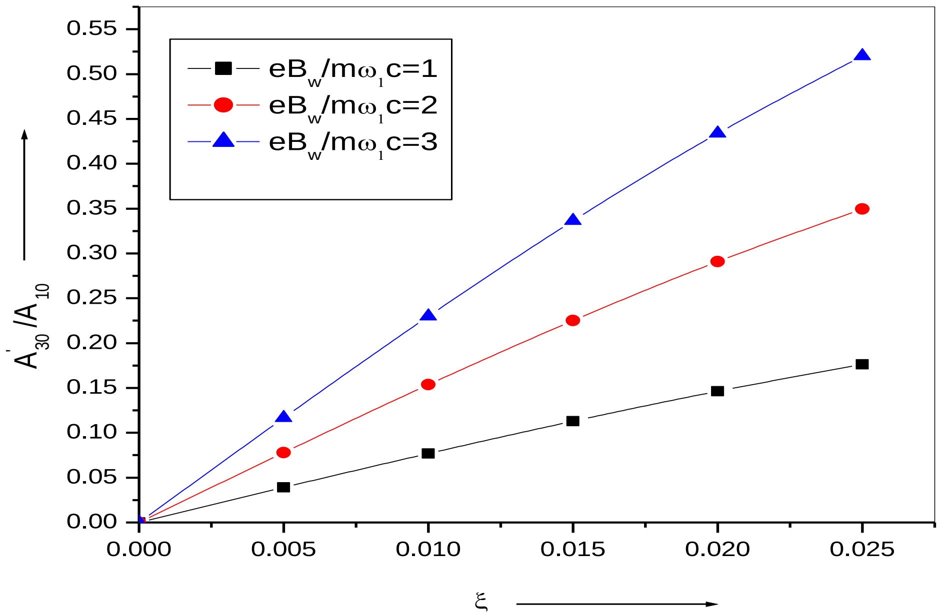

Fig. 9.3 Variation of normalized third harmonic amplitude with normalized A'_{30}/A_{10} propagation distance for different values of normalized wiggler magnetic field, $eB_w/m\omega_1 c = 1,\ 2\ \&\ 3$. The other parameters are same as taken in Fig. 1.

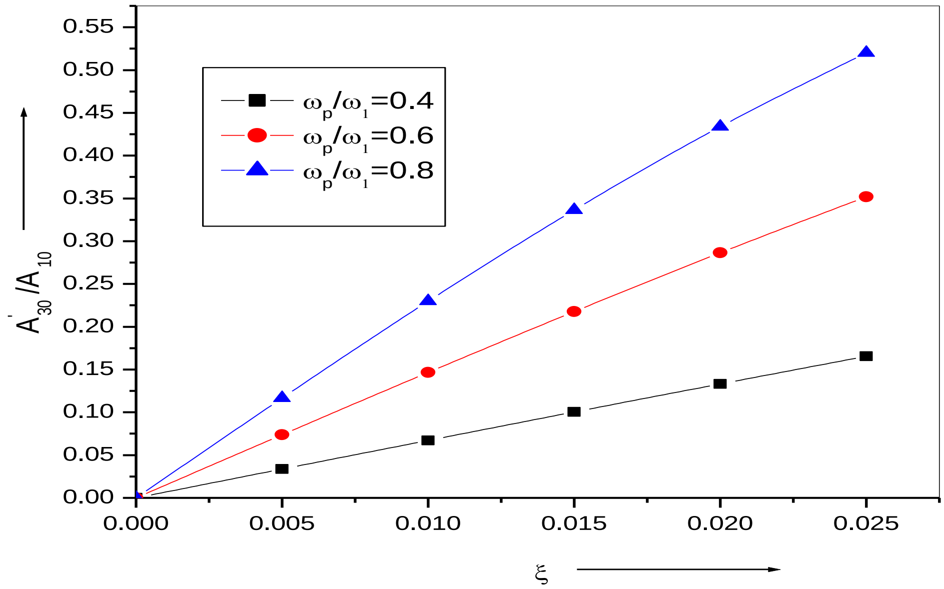

Fig. 9.4 Variation of normalized third harmonic amplitude A'_{30}/A_{10} with normalized propagation distance for different values of normalized plasma density, $\omega_p/\omega_1 = 0.4, 0.6\ \&\ 0.8$. the other parameters are same as taken in Fig. 1.

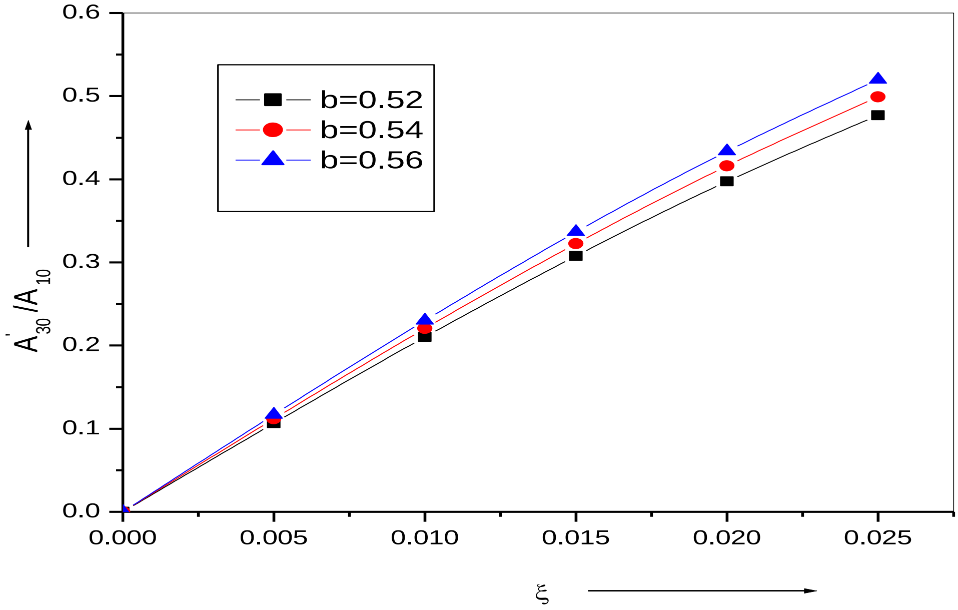

Fig. 9.5 Variation of normalized third harmonic amplitude $A_{30}^{'}/A_{10}$ with normalized propagation distance for different values of decentred parameter $b = 0.52, 0.54 \,\&\, 0.56.$ the other parameters are same as taken in Fig. 1.

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Chapter-10

Suummary and Conclusion

This discourse comprise the theoretical work on self-focusing, SHG and THG for different laser profiles. When short pulse laser propagates through plasma, the nonlinearity produces important phenomenon called self-focusing and it affects the efficiency of second and third harmonic generations. Therefore, it becomes a very important to study the different laser parameters that affect the self-focusing. When intense laser beam propagates through plasma it results in density perturbation that in turn produces transverse nonlinear current density, coupled with fundamental frequency giving rise to SHG and THG of low amplitude. While studying the effect of density ripple on harmonic generations we have observed that there is a rise in amplitude because of the additional momentum is provided by density ripples. The increase in intensity of incident laser compel electrons to get expelled from high density region. The variation in dielectric constant of plasma gives rise to stronger self-focusing and incident laser is able to travel greater Rayleigh length through plasma.

In the present work, we have analyzed the self-focusing and normalized amplitude of SHG and THG of different laser profiles like Gaussian beam, HG beam, chG beam and HchG beam under the effect of laser and plasma parameters such as wiggler field, density ripple, linear absorption, decentred parameter and exponential density ramp. Uni-axial Hermite-Gaussian beam spread in x and y direction exhibits self-focusing and defocusing oscillatory behaviour. It is observed that on introducing exponential plasma density ramp the Gaussian laser beam propagated through plasma for longer distance with reduced spot size due to rise in ramp slope of plasma density. The field of intense laser pulse is electromagnetic in nature and by increasing the intensity of incident laser the electrons experience stronger ponderomotive force that increases the frequency of oscillations and the oscillatory velocity of electrons is near to the order of velocity of light. This results the relativistic mass variation, increase in plasma frequency and rise in inrefractive index of the medium rises that enhances the intensity of SHG and THG. Comparing the results, we observe that efficiency is maximum for HchG when intensity of incident laser is increased. whereas oscillatory behaviours of chG and HchG beam is greater than the Gaussian beam. Wiggler field confined the electrons within the plasma region and maintains the cyclotron frequency. Wiggler field also provides the additional momentum to electrons resulting in significant rise in normalized amplitude of second and third harmonics for all Gaussian profiles. Analysis shows that Hermite-cosh-Gaussian beam exhibits higher frequency output for similar variation of laser parameters as compared to Gaussian profile, Hermite-Gaussian profile and cosh-Guassian profile.

The present work is of significance in microscopic imaging to study the tissue cell structure, to detect the disease by resonance imaging, to probe the inner material structure, optoelectronic, optical analysis of surfaces by SHG etc. In order to determine the significant applications of harmonic generations the focus-ability must be examined. The higher amplitude pulses can be used to probe plasmas to higher densities. The Doppler measurements of reflecting surfaces could be performed over a wider range of laser and plasma conditions using circularly polarized laser. Combining this with target reflectivity/transmissivity diagnostics would give a wide knowledge of plasma.

For THG bandwidth of tunable gate is from ~ 1300 nm to 1650 nm and it covers the fibre telecommunications range that falls in 1550 nm spectral range. It is desired to have cost effective, compact, better performing optical devices to reduce losses and have high band width. Future research for THG will be very useful to achieve these requirements. In our work we have analyzed how different parameters of laser affect the efficiency of SHG and THG which will be useful in the development of nano-photonics and nano-physics. Optical nonlinearity is useful for enhancement in high quality for 2-D materials useful in nanodevice construction which can be use in advances in metrology, sensing, imaging, quantum technology and telecommunications.