MODELING WITH IMPULSIVE DIFFERENTIAL EQUATIONS ON POPULATION DYNAMICS AND ITS STABILITY ANALYSIS

A Thesis

Submitted

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Mathematics

By

Maninderjit Kaur

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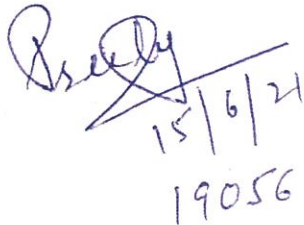### Supervised By

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LOVELY PROFESSIONAL UNIVERSITY
PUNJAB
2021

DECLARATION BY THE CANDIDATE

I declare that the thesis entitled "Modeling with Impulsive Differential Equations on Population Dynamics and its Stability Analysis" submitted for award of degree of Doctor of Philosophy in Mathematics, Lovely Professional University, Phagwara, is my own work conducted under the supervision of Dr. Preety Kalra (Supervisor), Assistant Professor in Department of Mathematics at School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab. I further declare that to the best of my knowledge the thesis does not contain any part of any work which has been submitted for the award of any degree in this University or in any other University/Deemed University without proper citation.

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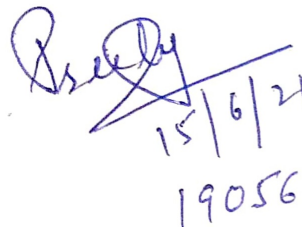
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CERTIFICATE

This is to certify that the work entitled "Modeling with Impulsive Differential Equations on Population Dynamics and its Stability Analysis" is a piece of research work done by Mrs. Maninderjit kaur under my guidance and supervision for the degree of Doctor of Philosophy of Mathematics, Lovely Professional University, Phagwara (Punjab) India. To the best of my knowledge and belief the thesis.

1. Embodies the work of the candidate herself.
2. Has duly been completed.
3. Fulfills the requirements of the ordinance relating to the Ph.D. degree of the University, and
4. Is up to the standard both in respect of contents and language for being referred to the examiner.

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ABSTRACT

In the present study, we analyzed the role of impulsive differential equations in population dynamics. Single-species and multi-species ecological models under the affect of impulsive perturbations at fixed moments of time are proposed and analyzed mathematically. The study of the underlying system consisting of three major inter-related components, viz. population dynamics, impulse and integrated pest management is carried out by using the concept of system analysis in which the system is studied by defining its borders, by distinguishing its major components, characterizing the change in them by mathematical equations and then interconnecting the representative equations in order to obtain a model of the original system. Once the model is governed by impulsive differential equations, these equations are solved assuming the initial positivity of all the state variables. The boundednes of the solutions obtained are checked using integral inequalities of IDE. The stability and permanence of the systems was studied with the help of stroboscopic map, comparison analysis technique and Floquet theory of impulsive differential equations. Also, numerical solutions of the proposed models were obtained using numerical techniques and MATLAB.

The proposed research work dealing with modeling of impulsive differential equations in population dynamics will be helpful to prevent the extinction of a particular species. It can also be helpful to agriculturists, ecologists and scientists to use pesticides and insecticides in an optimal way to control pests. The study of the factors under which impulse is applied and the components being effected will help the concerned community to plan the remedial measures. Being quantitative in nature, the mathematical model will prove to be economic in terms of time and money being invested on large scale experiments. In the view of above, first mathematical model is formulated on population dynamics considering single population and interacting populations, that is, prey-predator and competition. These formulated models will be analyzed with different stability analysis criterion. On the basis of literature review and research gaps, the following objectives have been proposed in this present study:

1. Modeling on single species population using impulsive differential equations and their stability analysis by using Lyapnuov function.

2. Stability Analysis of single Species population model with impulse by using Comparison theorems.

3. Modeling on interacting population (prey-predator) with impulse and stability criterion by using Lyapnuov Direct Method and comparison theorems.

4. Modeling on interacting population (Competition) with impulse and stability criterion by using Lyapnuov Direct Method and comparison theorems.

In chapter-1, basic introduction about population dynamics, mathematical modeling, impulsive differential equations and pest management has been described. The basic terms used in this study are also discussed. Under the section of literature review, the work done by many renowned researchers till date at national and international level has been extensively examined. After analyzing the research gaps, objectives of the present study have been proposed. To check the boundedness, stability and permanence of the formulated models, mathematical preliminaries of impulsive differential equations have been given. The last section includes the summary of the study.

In chapter-2, Dynamics of single-species population with predation considering Holling type-III functional response using impulsive differential equations is studied. Existence of positive and periodic solution is proved using fixed point with the help of Brower's fixed point theorem. Then sufficient conditions are established for the global stability of solution using Lyapunov function and comparison analysis technique of impulsive differential equations. To validate theoretical results, numerical simulation is done using MATLAB.

In chapter-3, a single-species population model under the influence of constant and linear impulsive perturbations at fixed moments of time is analyzed in which Holling type-IV or Monod Haldane functional response is taken as predation term. The conditions required for the permanence of species are established by using comparison analysis technique. Numerical example is given to substantiate the

theoretical findings.

In chapter-4, microbial and biological pest control techniques are applied simultaneously by impulsively releasing natural enemies and infected pests to prevent the outbreak of pest population. Therefore a SIN (prey-predator) model considering infection in prey with two classes (susceptible-infected) and stage structure in predator is analyzed for the cause of integrated pest management. Prey acts as pest and predator plays the role of a natural enemy. Firstly, local and global stability of pest extinction periodic solution is carried out, then condition for the permanence of system is derived using Stroboscopic map, comparison analysis technique and Floquet theory of impulsive differential equations. Further, it is observed that there exists a threshold value of the impulsive period which has significant effect on the dynamics of the system. Finally, for validating the established results, numerical simulation is done using MATLAB.

In chapter-5, a two-prey one-predator ecological model is proposed with mixed functional responses (Holling IV and Beddington-DeAngelis) in context to pest management. Firstly, it is proved that prey free periodic solution is locally stable by using Floquet theory of impulsive differential equations. Secondly, two threshold values of the impulsive release amount of natural enemies are calculated to establish the global stability of prey free boundary equilibrium. This is achieved by implementing the concept of the stroboscopic map and comparison analysis technique. Finally, the permanence of the system is established. Further, validation of theoretically proved results is done using MATLAB.

In chapter-6, a three tropic level food chain model is proposed considering plant, pest and natural enemies. Two different type of functional responses are taken for middle and top predator. Threshold value of the impulsive period is calculated for the annihilation of middle predator using Floquet theory, Lyapnuov functions and stroboscopic map. Mid level predator plays the role of pest. Permanence of system is also established. Some complex dynamics is also observed at higher value of impulsive period greater than threshold value. Further, validation of theoretical findings is done using MATLAB.

In chapter-7, two species periodic competitive model with Beddington-DeAngelis inter-inhibition term in the presence of toxic substances is analyzed. It was found

in various studies that some phytoplankton species release toxic substances in the presence of other phytoplankton species to hinder their growth. This phenomenon ultimately leads to the extinction of concerned species. The proposed model is considered under the effect of impulsive perturbations. Sufficient conditions for the eradication of one species and permanence of other are derived using suitable Lyapunov functions and comparison analysis technique. Also, it is observed that impulses effects the dynamics of the system. To substantiate theoretical findings, numerical simulation is done using MATLAB.

To God
and
My Family

ACKNOWLEDGMENT

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Chapter 1

General Introduction

1.1 Introduction

Ecology is a branch of science comprising interactions of organisms with their environment and other organisms in the community. Due to these interactions, individuals of a species exert selective pressures on each other. There are numerous biotic and abiotic components of the environment which affect the evolution of these organisms. Biotic factors consists of other living creatures in the environment and abiotic elements consists of physical aspects such as temperature, soil, water, topography and others. The ecology, in general, can be categorised into two parts: auto-ecology that deals with single species population; and syn-ecology that studies the interactions of two or more species. The researchers of this discipline inquisitively explore natural phenomena in which different types of species coexist together in a common habitat.

Population refers to all the individuals of particular species or group who live together under same environmental conditions and are capable to reproduce. Population dynamics is the quantitative sub-discipline of ecology that studies variations in age and size of the populations in time and space. Here, the population

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may be of different age groups as age is a variable which change over time. It is also apparent that the size of population may increase or decrease due to various biological and ecological factors like birth rates, death rates, immigration, evacuation and others. Thus, the investigation on the variations of age and size of the population is defined as population dynamics. The study of population dynamics relates with the interactions of two or more species in the same environment. The different types of population interactions are illustrated below:

•Predation: It is an interaction between individuals of same or different species in which population of one species serves nutrition to other species. Thus, the population of one species grows and the other shrinks and it is known as prey-predator relationship. The organism that is eradicated or reduced either for the time being or everlastingly due to the fact to provide food to another species (predator) is known as the prey. Sometimes, the prey adopt defence mechanism to protect themselves from the attacks of the predator like refuge effect, indestructible class of prey, heterogeneity in the environment and others. On the other side, the predators are living organisms that hunt and feed diverse prey species. They contribute in controlling the harmful prey populations of agricultural crops and act as good bio-control agents. Normally, they are bigger in size than their prey.

•Competition: When individuals of closely related species fight for the same living resources, for example, food and space, this type of population interaction is known as competition. There is decline in population density of both the species.

•Mutualism: If members of two different species interact to boost the growth of each other, then this type of interaction is called mutualism. To exemplify, interaction between plants and animals where some animals depend on plants for their food and in turn they help in the pollination of the plants.

•Parasitism: It is defined as an interaction where one individual, known as parasite, harms the other which is called host, for its growth. The parasite interferes with the biological process of the host by feeding on its tissues or cells, resulting in degradation of the host's health. For example, interaction between pests and plants.

•Commensalism: It refers to the interaction of two different species in which one species gains while the other one is at no loss no gain. For example, birds oftenly

take shelter on tress but it does not effect the later in any way.

•Amensalism : Amensalism is defined as an interaction of two different populations where one population bears loses whereas the other remains unaffected.

Apart from these different types of two-species interactions, more complex networks of multi-species interactions exist namely, food-chain and food-web.

•Food-chain : It is a series of organisms having different trophic levels where the species at lower trophic level acts as a food for the one at higher trophic level. Only the first member of the series is producer whereas all others are consumers.

•Food-web : It refers to the interconnecting network of different food chains in an ecosystem.

The survival of the most of the species depends on the natural food-chains and food-webs existing in the environment. The removal of an element of these species' networks can lead to ecological imbalance. Therefore, food-chain system is utmost important for maintaining ecological balance, for example, about one lakh species of decomposers help in splitting the organic matter into valuable nutrients which further contribute to increase the fertility of the land.

Changes are the inheritance property of species within due period of time. It is evident that there is always limit in the growth of population of species due to many environmental and biological attributes. There are number of stress factors which affect the evolution of living organisms. Depending on the population density of species, these can be particularly categorized into two types, density dependent and density independent. In the former, we include food availability, predation and competition. These factors could have favorable or detrimental impacts to population. It does influence the fertility rate, mortality rate and migration of the species. To exemplify, it is observed that due to the increase in population density, some female red squirrels, who got less nutritious food, in the forests of Europe and Asia, responded negatively to reproduction rate [155]. Further it is also found that there is positive co-relation between the population density and mortality rate, and migration of species. In the latter, there are environmental stressors such as pollution and catastrophic stressors like fires, floods and hurricanes. For example, the soil and water pollution caused by chemical pes-
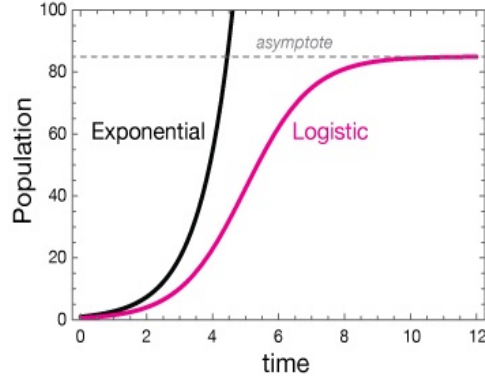
Figure 1.1: Graphical representation of exponential and logistic growth model

ticides hamper the growth of amphibians and aquatic bodies, resulting in increase in mortality rate. On the other hand, environmental catastrophes also contribute in death and destruction of habitat of species.

The large number of mathematical models using ordinary differential equations are proposed to solve ecological problems. Ahmad and Rao [3] thoroughly described the applications of ordinary differential equations in biology. In this way, mathematical models are considered as an important tool to investigate and forecast the nature of different species and their impact in the ecological system. In past few decades, exhaustive studies on different types of species interactions have been done which examined the behavior of these species in diverse ways. The first single-species exponential growth model was proposed by Malthus [4]. It is obvious that species experience several stress factors such as nutrition and space available for their survival in a particular region. Thus, the model developed by Malthus was then modified by Verhulst [5] who introduced the concept of carrying capacity of the environment. The comparison between these two models is shown in figure 1.1. Interactions among species for their survival is very important aspect in the universe. Hence, the study of interdependence of species has acquired a central stage for the researchers in which they eagerly want to know the symptomatic results of these interactions on the evolution of species. Emphasizing on the study of two-species interactions, the first classical prey-predator and competition interaction models were proposed by Lotka and Volterra using ordinary differential

equations. Apart from this, number of food-chain and food-web models consisting of multi-species dependence are proposed by various scholars by assuming same or different functional responses at different trophic levels of the species interaction. It is found that dynamics of these systems strongly depends on the death rate of species [156]. Also, it is observed that in two-prey one-predator food web model, chaotic behavior of the system can be controlled with harvesting of predator [157].

Futher, it is impossible to study continually, the evolutionary processes of species which always remain dynamic due to various interventions imposed by humans and existing environment. Such sudden interventions are referred as harvesting and stocking of species by mankind, and natural calamities in the environment. Hence the study of differential equations with impulses assist to produce remarkable results about evolution of species with instantaneous disturbances. Impulsive differential equations (IDE) are more appropriate mathematical tool than ordinary differential equations as it covers wider and comprehensive patterns of growth of species. These differential equations with impulses were introduced in the middle of twentieth century and after that, there is rigorous development in the study of their applications in different fields of science. IDE contribute a lot to the dynamics of single-species and multi-species ecological models. These are helpful to maintain ecological balance, for example, to prevent the eradication of particular species and to encourage coexistence of species. Impulsive species interaction models (single -species, prey-predator, competition, food-chain and food-web) are proposed and analyzed by various researchers.

Furthermore, impulsive prey-prdator, food-chain and food-web ecological models are also applied to study pest management. The concept of pest management largely relates with the agriculture and its allied sectors. Agriculture is the backbone of Indian economy where about 70 per cent of the population depends on it for their subsistence which accounts 14 per cent in gross national product of the nation. As the population of country is increasing robustly, the area under cultivation is decreased and still decreasing which stipulated towards the improvement and intensification of agriculture sector. Therefore, in order to meet the needs of vast growing Indian population, it was necessary to take some

corrective measures in various directions like increase the productivity of the crops, technological advancement in agriculture, increase cropping intensity and others. Thus pest management was one of the most significant mechanisms to increase the efficiency of agriculture and allied activities along with to maintain the pest population in the crop. Under it, different kinds of agro-chemicals likes insecticides, fungicides and herbicides are used in order to increase the yield productivity and maximize profits. On the contrary, the excessive use of these chemicals contributed in environmental degradation along with hazards to human health. Thus the concept of integrated pest management (IPM) came into existence which combines two or more pest control measures.

Mathematics is an unambiguous and universal language. Mathematical modeling is the technique to represent the real world phenomenon in terms of mathematical equations. With the advancement in technology, these well framed models are used to perform numerical simulations on computers, which ultimately simplifies the complex and tedious process of lengthy calculations. Although some elements of compromise exist in modeling because while obtaining mathematical representation of any real world interacting system, it is too complicate to model in entirety. But only the pertinent information is considered to have a logical understanding of the nature of the dynamics of the system. This is the best way to understand many arduous processes in this ever changing universe. Basically, two categories of mathematical models are well known namely, deterministic models and stochastic models. Mathematical models are being implemented in different fields of science and engineering, that is, ecology, population dynamics, optimal control, physics and chemistry.

The proposed research work related to the modeling on population dynamics using IDE will be carried out by using mathematical models and these models will be used to determine quantitatively the impact of impulses on single-species and multi-species interactions. In the present study, prey-predator interactions under the influence of impulses are studied in the context of pest management.

1.2 Basic terminology of population dynamics and pest management used in the thesis

1. **Impulse**: It is defined as a force or disturbance which acts for short period of time.

2. **Pest**: These pest populations refers to the insects that are responsible to inflict the damage to the agricultural yield which ultimately effect on the quantity as well as quality of the crops. This is the reason that many techniques such as integrated pest management, pest control, bio-control, physical or mechanical control are used to regulate their number.

3. **Pesticides**: The word pesticide includes several nomenclatures like insecticides, fungicides, herbicides and others. They can be divided into two categories: synthetic or chemical pesticides and bio or organic pesticides. The former has many adverse affects on earth, air, water, flora and fauna and human population while the latter is an alternate to the chemical pesticides and are more beneficial, safer and environment-friendly. There are many sectors in which pesticides are used such as agriculture, forestry, industry, public health and others.

4. **Cultural Control**: Cultural control method makes crop fields unattractive to pest by various means such as crop rotation, soil solarisation, selecting best suitable sowing and harvesting times, using built-in pest resistant varieties of seeds and plants, eradication of old crops residues etc. All these measure are used to hamper pest population's reproduction, nourishment and shelter in the respective fields.

5. **Chemical Control**: Chemical control is a method of IPM to control insect pest problems below economically injury level. Under this technique, synthetic pesticides are used which provide quick results to protect crops from pests and diseases along with the improvement of yield productivity. Some of pesticides which are used for spraying the crops in India are Fenthion, Monocrotophos, Diazinon and Endosulphan. However, the rampant use of

chemicals adversely affect to human health and environment. Thus, it is essential to note that this technique should be used in limited way without associating ecological degradation and risk to human beings.

6. **Biological Control**: Biological control means the use of natural enemies in order to reduce or eradicate harmful prey population which inflict massive damage to the yield. These bio- control agents can be categorized as predators, parasitoids, pathogens, fungi, bacteria etc. These are able to inhibit the growth of the prey resulting to enhancing the yield of the crop. These insects play an important role not only in controlling pests/ diseases but also to minimizing the use of synthetic pesticides. Thus, this strategy is considered to be economical and environmentally sound.

7. **Integrated Pest Management (IPM)**: Integrated pest management is an eco-friendly technique used to control pests. These pests can be generally divided into different categories such as, bugs, rodents, weeds, nematodes and virus. The term 'integrated' means combined control action. Thus integrated pest management is defined as a significant ecological approach to pest control which brings together different methods, that is, biological, cultural, physical and chemical into a whole. This approach has many advantages like reduces danger of pesticide resistance, minimizes the pesticide expenses, growing of organic food products, increases agricultural yield, minimizes environmental deterioration and many others.

8. **Economic Threshold (ET)**: Economic Threshold value is used to mark a fix point (level) without adverse effect to the specific yield. It is defined as the pest density at which protective mechanisms are taken into account to check the growth of prey population from embracing the Economic Injury Level. In fact, it points out that the time has come to take actions and this is the reason that some researchers [59] referred it as action threshold

9. **Economic Injury Level (EIL)**: It is a point where number of pests that cause economic damage more than the cost to control the pest.

1.3　Literature Review

Mathematical biology has become an interesting area of interdisciplinary research during the last three decades. Several mathematicians, biologists and engineers have made great achievements on mathematical modeling in population dynamics and ecology for the benefit of the evolving world. It was started with the single-species exponential growth model in which there was no restriction on the living resources [4]. But there are several population limiting factors in the environment as Dilao [2] stated that the population densities of species (human or animals) undergo noticeable variations with the passage of time. These changes may happen due to the different attributes of the environment such as, climate change, diseases, the available environmental resources for their survival and interactions of individuals within or outside their community. Then the concept of carrying capacity of the environment was introduced and more realistic single-species logistic growth model came into existence [5]. The approach given by Ludwig et al. [6] of single-species model was more practical. The authors propounded an outbreak model of Spruce budworm which is the major problem in some parts of United States and Canada. These insects desolate conifer flora of North America and Western Canada resulting privation of growth and sometimes tree fatality. They represented the budworm growth with single-species logistic model, assuming predation by birds. In this model, it was presumed that, firstly, predation increase with budworm population density and then become constant for larger prey population. This provided the idea for insect control either by spraying pesticide on foliage or by increasing the threshold number of predators, which ultimately lower the reproduction rate of pest population. Further, researchers observed that the study of dynamics of population interactions was more natural as compared to single-species. Out of all types of population interactions, prey-predator and competition are extensively studied by various researchers. Classical prey-predator and competition models were proposed by Lotka and Volterra. The qualitative analysis of this classical prey-predator model was carried out with the assumption that in the absence of predators, prey population grow infinitely and predator species decline exponentially in the absence of food (prey) [1]. In two-species competition

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model, species were in competition for the same restricted living resources. By considering logistic growth of both of the species, it is concluded that the superior species would ever dominate which eventually lead to the extinction of inferior species. This is called principle of competition exclusion. Now we move towards the study of origin of impulsive differential equations and their importance in modeling different ecological problems.

Many naturally occurring evolutionary processes in the real world does experience a sudden variation of state at some specific time events. Although, the time period of occurrence of these instantaneous disturbances is small, but they effect the dynamics of the whole process. These short term external disturbances are known as impulses. Thus, it is very natural to implement differential equations comprising impulsive effects–that is, impulsive differential equations to model these real-world phenomena. The traits of the study of impulsive differential equations (IDE) were pioneered by Mil'man and Myshkis [7] in the middle of 20th century. Lakshmikantham et al. presented a general theory of these equations in their book "Theory of Impulsive Differential Equations" [8]. The authors gave the example of Kruger-Thiemer model of drug distribution to introduce the concept of IDE. They categorized IDE into two parts, one with impulses at fixed time events and second with impulses depending on the state of system. It was concluded that the study of impulsive differential equations is more interesting than the corresponding ordinary differential equations because it includes some new characteristics such as noncontinuability of solutions. Along with this, for the existence and stability of solutions of impulsive differential equations, the authors outlined different methods such as upper and lower solutions, impulsive integral inequalities and discontinuous Lyapunov functions. Milev and Bainov [9] identified more precised criteria for the staility of solutions of IDE with impulses at fixed moments of time. Bainov and Simenov [10] emphasized on periodic impulsive differential equations and proposed the notion of Stroboscopic map and Floquet theory to study the stability of these systems. Also, they suggested some numerical-analytical methods to find periodic solutions of these systems. Later on, Samoilenko and Perestyuk [11] gave insight into almost periodic solutions and optimum control problems of

impulsive differential equations. Further, researchers analyzed that solving impulsive differential equations analytically is very lengthy and tedious process and even sometimes, it is not possible to find the analytical solution. Therefore, they paid attention towards numerical solution of IDE. Randelovic et al. [12] provided general algorithm to solve impulsive differential systems numerically. Continuing with this approach, for better convergence, number of scholars developed different numerical techniques to solve systems with impulses at fixed time events [13, 14, 19]. They also compared the results with analytic solutions and found good degree of accuracy. The parameterized form of IDE is explored by several researchers [15, 16, 17]. Singh and Srivastava, in their paper [18] reiterated on the decay rate of solutions of IDE. They used comparison analysis technique and Lyapunov functions to establish strict stability criteria for the solutions of IDE. Li t al. [20], by enforcing Lyapunov's second method, achieved existence and global attractiveness of periodic solution of IDE. Impulsive differential equations have plethora of applications in the field of epidemiology, ecology, physics, chemistry, pharmacokinetics, population dynamics and optimal control. So, these equations are continuously attracting the attention of several researchers from different areas of science and engineering. Zeng et al. [21] put forward his approach to apply impulsive vaccination control on susceptible-infected-recovered (SIR) epidemic model to regulate the disease outbreak. The scholars concluded that pulse vaccination at fixed moments of time was helpful to eliminate the disease and derived sufficient conditions for global asymptotic stability of epidemic-eradication periodic solution. They also observed chaotic behavior of the system depending on the amount of pulse vaccination. Shi et al. [22], in their research paper, proposed and investigated two SIR epidemic models, one without impulsive vaccination and the other under the influence of impulsive vaccination. They calculated the threshold value of impulsive vaccination rate and observed that if the rate of impulsive vaccination is greater than threshold value, the epidemic would be eliminated otherwise it will prevail and become permanent. Li et al. [23] explained that how impulsive control is beneficial for chemostat model. Dishlieva [24] gave the detailed description of various applications of impulsive differential equations in different fields along with brief summary of the difficulties arising in the study of solutions of these

equations. Continuing in this direction, the importance of impulsive vaccination in prevention of infectious diseases, particularly Cholera, is recently discussed by Sisodiya et al. [25]. They categorized the total human population into four classes namely, susceptible, exposed, infected and recovered. In this research paper, two preventive measures are taken against Cholera. One is inoculation of vaccine at fixed impulsive moments and second is the use of disinfectants to kill the harmful bacteria.

This study explores the importance of impulsive differential equations in modeling on population dynamics. The research also focuses on the stability analysis and applications of impulsive ecological models of single-species and species interactions. It is the natural process that population interactions undergo sudden changes in the population densities of the concerned species. These changes may happen due to fire, flood, seasonal birth patterns of species or human interventions such as harvesting and stocking of population. To maintain ecological balance, harvesting and stocking is required, accordingly. Therefore, one of the appropriate mathematical tool to study the effect of these instantaneous disturbances are impulsive differential equations. Several researchers have done recognized work in modeling on single-species models under the effect of impulsive perturbations. Yan and Zhao [26] constituted periodic impulsive Lotka-Volterra type single-species model and derived sufficient conditions for the global stability of its periodic solution by using suitable Lyapunov function. Wang et al. [27] explained the importance of spatial factors in population dynamics. It is quite often that the species migrate to different places in search of better environment for their survival. Mostly, this happens in regular pulses. Thus, the authors, in this paper considered the impulsive diffusion of species from one patch to another and proved the persistence of species in both the patches. Liu et al. [28] showed how species living in a weak environment can be prevented from extinction with the help of impulsive stocking at fixed moments of time. Tan et al. [29] proposed a single-species model under the effect of external perturbations where species are living in periodically varying environment. They have taken predation of species according to Holling II functional response. By applying Brower's fixed point theo-

rem and Lyapunov method, it was proved that system has globally stable periodic solution and species were permanent in the presence of impulses. Extending the above work, Tan et al. [30] studied the effect of constant impulsive perturbations at fixed time events on single-species population model and compared the results with those obtained in [29]. Liu et al. [31] presented a single-species population model with impulsive and random perturbations. They used stochastic impulsive systems to frame the model and established conditions for the global stability of the periodic solution. Now we make a shift towards the study of interactions of same or different species under the influence of impulsive perturbations.

Our planet comprises of infinite number of species of various kinds. To maintain ecological balance, it is requisite to have their inter-dependency on each other. Thus, researchers are paying more attention towards species interactions. Depending upon the number of species interacting, these are divided into two parts, one is two species interaction (for eg. prey-predator and competition) and other is multi-species (food-chain and food-web). The important factor to be considered, while studying species interactions, is functional response. This refers to the intake rate of the predator as a function of prey density. Dawes and Souza [32], in their paper, described three different types of functional responses given by C.S. Holling. These are known as–Holling Type I, Holling Type II and Holling Type III. In type I, the consumption rate of predator is linearly dependent on prey density while in type II and III, time required to search and handle prey is considered. Zhijun and Ronghua [33] investigated a prey-predator model by taking into account the phenomenon of group defense by the prey population using Monod-Haldane functional response that is also known as Holling Type IV. It involves a decrease in predation rate because the ability of prey species to defend themselves get enhanced in a group. Apart from that, Beddington [34] and DeAngelis [35] also put forward a new kind of functional response called Beddington-DeAngelis functional response which includes interference in the predator population during predation. Ballinger and Liu [36] were the first to explain permanence in impulsive prey-predator models where impulse occurs at fixed time events. They provided sufficient conditions for permanence by using Lyapunov functions and also discussed how an unbounded

growth of population of species can be controlled with the help of impulses. Xinzhi and Rohlf [37] showed that the only way to control two-species or multi-species dynamics is taking impulses on the population of first species. These impulses can be in the form of addition or removal of some members of the species. They have also given an example of fishery model to explain that the population of particular types of fish species can be controlled with impulses (by adding prey species of these fish species) to attract more anglers in a resort. Shuwen et al. [38] analyzed a prey-predator model by considering impulsive stocking of the predator and group defense by prey population using Monod-Haldane functional response. They derived critical value of impulsive period for the permanence of the system. Moreover, when the value of impulsive period is increased, the system exhibited chaotic behavior. Dong et al. [39] have studied the more generalized situation, where both prey and predator population are under impulsive perturbations. Prey is impulsively stocked and predator is harvested impulsively at fixed moments of time. After performing numerical simulation, the scholars identified that predator extinction solution become globally stable under specified conditions on impulsive period. A multi-species interaction model with impulses consisting of two prey and one predator population was proposed by Song and Li [40]. They consider Holling Type II function for the predation of both the preys by predator. By implementing Floquet theory and comparison analysis technique of IDE, they have shown that impulsive perturbations encourage co-existence of all the populations. Negi and Gakkhar [41] examined a prey-predator model with Beddington-DeAngelis functional response. Firstly, they established global asymptotic stability of predator-free solution and then identified that with the increase in impulsive period, the system lead towards bifurcation. Zheng et al. [42] studied the impact of delay on impulsive prey-predator model and identified that combined effect of delay and impulses pushed the system towards more chaos. Beak and Lee [43] investigated a food chain system consisting of three-species with linear functional response. The system is considered with impulsive stocking of mid-level predator. They derived sufficient conditions for the permanence of the system. Hunki and Do [44] extended the above work by considering Holling type IV functional response. Along with this, they incorporated impulsive harvesting of all the species and impulsive

stocking of top-predator respectively at two different fixed time events. With the help of Floquet theory and comparison analysis technique, they presented appropriate conditions for the stability of prey and predator-free boundary equilibrium points. Xiaong et al. [45] examined the food chain model for its permanence with simultaneous impulsive harvesting of prey and mid-level predator, and stocking of top predator. Li et al. [47] constituted a predator-prey system by taking into account the mutual interference between preys and predators under the influence of impulsive perturbations. Sufficient conditions for global attractivity of the system are identified using Lyapunov functions and comparison theorems. It is well known fact that most of the existing species go through different stages such as juvenile and adult in their life cycle. So, it was obvious to include age-structure in modeling on population dynamics. Jiao et al. [46] proposed prey-predator model considering two life stages of predator–that is, immature and mature. The prey is impulsively stocked and predator is impulsively harvested. They also incorporated the time required for the maturity of predator in the form of delay. Further, it was assumed that immature predators depend on mature ones for their survival. They obtained the threshold value of the amount of impulsive stocking of prey for the permanence of the system. Zhao [48] developed a prey-predator model considering stage-structure and time delay in predator. He assumed impulsive harvesting for the predator and performing numerical simulation, concluded that under prescribed conditions, predator extinction boundary equilibrium is globally attractive. Du and Feng [49] confirmed the existence of periodic solution in a delayed predator-prey model with Beddington-DeAngelis functional response. It is already confirmed that certain prey species prefer to travel in groups to avoid predation. Cosner et al. [50], in their paper, mentioned that some species of predators also travel in groups. Zhou et al. [51] inspected a three-species food chain model incorporating Hassell-Varley function response. They applied impulsive harvesting to all the thre species and impulsive stocking to middle predator, respectively at two different moments of time. Using small perturbation technique and Floquet theory, they calculated threshold value of impulsive stocking amount of mid-level predator for the permanrnce of the prey and top-predator free equilibrium state. They further observed that system exhibit bifurcation and chaotic behavior which

depend on the period of impulsive release. Now we make a shift towards pest management as an application of prey-predator interactions.

Pest management is a major cause of concern in these days because pests adversely effect plants, crops and natural vegetation. They are responsible to inflict the damage to human beings, animals and yields either directly or indirectly. Therefore researchers at the global level are engaged to solve this problem of pests through several techniques and tactics such as chemical control method, biological control method, integrated pest management and others. Pesticides play an important role to contain the effect of pests on crop yield. Aktar et al. [52], mentioned in their paper that India first time started to produce pesticides at a plant established in Calcutta in 1952. They described that the immediate effects of pesticides are high yield and better quality food over healthy crop. With the overall increase in production, the revenue of the farmers also increases which ultimately ensure them to meet their expenses. In contrast to the benefits of pesticides, these have deleterious impacts on human health as well as environment. As pesticides ingressed in the food chain, so the number of deaths caused by chronic diseases arising from pesticide poisoning have increased gradually. Aquatic bodies also endure the threat due to water contamination by the excessive use of pesticides. Thus, it is to mention here that to save our environment and human beings, the use of these pesticides must be limited. Kalmakoff and Longworth [53] explained the use of microbial agents to regularize insect pests. The authors categorized the insect pests into two types, that is, Lepidoptera such as butterflies and moths and Coleoptera such as beetles. They specified that several pathogens can be used to generate infectious diseases among pest population. These are protozoa, bacteria, fungi and nematodes. Theses insect pathogens resides on the host pest and hamper their growth by intervening in their biological process, which ultimately leads to the death of host pest. Cherry et al. [54] carried out a study on the potential of certain pathogens for integrated pest management (IPM) of stem borers in West Africa. They found that most of the small scale farmers in Africa could not afford to buy pesticides due to very high cost. Therefore, IPM is its best alternative where unrestrained use of pesticides can be gradually reduced by efficiently combining

with biological control. They concluded that the use of insect pathogen *Beauveria*, which is specific fungi, is very effective for IPM strategy to stop the damage of maize plants from pests. Burges and Hussey [55] discussed two ways to insert these insect pathogens in the target pest population. These can either be added in the marginal amount to the pest population to create an epidemic or can be used as bio-pesticides when the targeted pests reach at an economically significant level. Freedman [56] explained that the other way to control pests biologically is through some specific natural enemies which act as predators for the targeted pests. To exemplify, Caterpillars and Aphids are among the common pests of Tomato and Cabbage, respectively. It is found that their natural enemies, Parasitic Wasp and Hover Fly Larva are being used by farmers to control these pests. In this paper, the authors proposed basic prey-predator model for pest control. Prey acts as pest and predator plays the role of natural enemy. Further, studies suggested that if the natural enemies are not available in enough numbers, they can be bred in suitable environment and instantaneously released in the field from outside. This process is called augmentation of natural enemies. Lenteren [58] discussed the concept of augmentation of natural enemies for biological control in two ways. He specified that approximately 125 species are identified as natural enemies of specific pests. First is inundative release where the natural enemies of the targeted pests are reared in large numbers and released periodically in the field for immediate pest control. This method is generally applied to the crops where only one particular type of pest species are prevalent. Second is seasonal inoculative release which includes periodical release of these natural pest control agents. Apart from providing the immediate control, this method helped to regularize the pest population throughout the season. He concluded that the scope of bio-pest control is very wide because market demand of vegetables grown with minimized or no use of pesticides is continuously rising. Thus, the dynamics of pest control models can be effectively studied with the help of impulsive prey-predator interactions. The concept of IPM came into limelight in the middle of twentieth century. After that, there is continuous development in this field of IPM. Barclay [57] studied a stage-structured prey-predator model with release of predator, pesticides and habitat management for pest control strategy. He has taken two life stages of

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pest (prey) population and identified the various factors to be considered while applying prey-predator systems to regulate pests. The first successful attempt in the field of IPM was made by Tang et al [59]. They constructed two impulsive models of IPM with fixed and unfixed time events, respectively. In the first model, the authors adopted the strategies of biological as well as chemical methods in order to reduce pest population in the crop with fixed impulsive effects. They took threshold values of impulsive period for the global stability of prey neutral periodic solution. The authors stressed to use these measures integrally as it not only cut down pest population to tolerable level but is environmentally safe and economically feasible. In the second model, they examined the case of state-dependent impulsive control events just after the preys attain an economic threshold value. In this model, the strategies of releasing natural enemies, regular field monitoring and spraying pesticides are taken into account simultaneously. Hong et al. [60] showed how certain insect pathogens are beneficial to obstruct the growth of targeted pests by applying continuous and impulsive release of infected pests. These can be germinated in laboratories under suitable conditions and are not capable to cause damage to crops. To exemplify, the viruses of dengue fever are transmitted by Tiger mosquito and spraying with bacterium Bacillus thuringiensis help to control flocks of these mosquitoes. The authors achieved that by controlling the impulsive release amount of infected pests, pest population could be maintained at acceptably low levels. Georgescu and Morosanu [61], in their research article, studied an integrated pest management (IPM) model with biological and chemical control methods which are used in an impulsive and periodic manner. They included pesticide spraying and impulsive release of infected pests at two different time moments and applied the Floquet theory of impulsive differential equations to check the stability of the system. Bhattacharya and Bhattacharya [62] conducted a study to examine the problems in an agro-ecosystem. They examined that sterile insect techniques (SIT) is very viable biological control method in controlling pest population. They also favoured in the use of pesticides provided it should be in low volume. The study stressed that if these control measures are adopted in plausible manner, it could be helpful in increasing the production of yield also. The study further indicated towards the vulnerability of the system in which climatic

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factors, particularly effects of temperature has discussed in an interesting way. Depending on the richness of available survival resources, it is evident to consider the patchy distribution of species in an environment. Thus, Georgescu and Zhang [63] extended the work done in [61] by dividing the environment into two different patches. They proposed a susceptible-infected (SI) model where movement of susceptible pests is allowed between patches but infected pests are forced to stay to their respective patches. Spraying of pesticides and impulsive release of infected pests is incorporated at two different time events. They discussed the effect of dispersal rate from one patch to other on the stability of pest free equilibrium state and highly recommended the diffusion of susceptible pests from unstable to stable patch for effective pest control strategy. Another interesting SI model for IPM was proposed by Jiao et al. [64] by considering Holling type II function as incidence rate and identified sufficient conditions to maintain the pest population at tolerable level. Georgescu and Zhang [65] presented susceptible pest-infected pest-natural enemy (SIN) model with two life stages of natural enemies–that is, immature and mature for IPM. The researchers assumed that immature predator could not haunt on prey and completely depend on mature predator for their survival. They summed up that when density of pest population increases, high voracity of predators would encourage timely eradication of pests. Wang et al. [66] examined impulsive SI model for efficient pest control with more generalized nonlinear transmission rate from susceptible to infected pests. Gao and Tang [67] compared two different prey-predator models with instantaneous release of predators to curb the growth of pest population. First is the periodic Lotka-Volterra type prey-predator model where fixed amount of natural enemies are released at fixed impulsive moments and in second case, the release amount is proportional to the predator density. It is found that proportional impulsive release is more adequate for annihilation of pests. Zhao et al. [68] explored a three-species food chain model with mixed functional response and infection in the prey species for pest management. Infected prey (pests) are impulsively released at fixed time events. An extensive numerical simulation is performed to observe the chaotic behavior of system which depends on the impulsive release amount of infected prey population. Further, Georgescua and Zhang [69] featured the possibility of availability of more

than one species of natural enemies for prey species and developed an impulsive pest control model with 'n' number of predator species which haunt on common prey. Tang et al. [70] embodied the duration of residual effects of pesticides on the growth of pests and natural enemies. They mentioned that repeated application of pesticides could be diminished if these have strong and long residual effects. They also considered the decay rate of pest population with pesticide spraying and established permanence of the system. Yang and Yang [71] constructed an impulsive pest management model. The authors studied periodically impulsive releasing of predators and harvesting of pests at two different fixed time periods. Jatav and Dhar [72] in a comprehensive study explored a three level plant-pest-natural enemy model with stage structure and impulsive perturbation. They investigated impulsive releasing of predators and impulsive harvesting of pests in numerical terms. The authors also examined the simultaneous effects of biological and chemical control measures on annihilation and permanence of pests. The study further highlighted that hybrid pest control strategies, in which two or more methods used, are very productive and cost-effective techniques. It is more rational that immature individuals of any species take some time to get mature, thus, Dhar et al. [73] assimilated delay time for maturation of immature to mature pests and analyzed a prey-predator model with stage structure in pest population with impulsive release of predator for pest control. They calculated the threshold value of impulsive period for the extinction of pests depending on significant factors such as, maturation delay time of pests and impulsive release amount of predators. Because, complete extermination of pests is not appreciated economically and biologically, thus, they recapitulated that short time period of maturation or release of natural enemies at small scale would encourage the permanence of the system. Li et al. [74] studied an eco-epidemic model with double impulsive control method. They examined global stability of the susceptible prey extinction periodic solution. In the study, the scholars adopted a strategy in which infected preys and predators are acquitted periodically at diverse points. The study confirmed that integrated pest management is a cost-effective and environment-friendly tactic. Akman et al. [75], in their review article on pest management models, generalised a number of impulsive differential equation pest control models given by many renowned

scholars. In this paper, they discussed random fluctuations in the birth rate of the species. At the end, the authors suggested some valuable points to select the best suitable models to study a problem. They highly stressed on the use of an accurate and precise modelling parameters and simulated numerical techniques. Mathur [76] made an inquisitive approach in which two-prey one –predator model is proposed. He used organic and synthetic pest control techniques at two different time periods. The author determined the threshold values for the eradication of the pests and permanence of the system. Jiao et al. [77] explored a predator-prey model by using impulsive diffusion which is linked with two patches. They found that the diffusive rate of the natural enemy contributes a lot in controlling pest population. Chaves et al. [78] conducted a study on the developments occurred in the field of IPM as it contributed a lot in the formation of plethora of mathematical models using in pest control. They discussed differential equations used in agricultural ecosystems right from classical to modern times. The study is quite useful to understand a number of aspects relating to the parametric values of pest control, its impact on environment and economy etc. Apart from these studies, the perception of delay in gestation time of natural enemies is also acknowledged by some researchers. Because, if an infectious disease exists in pest (prey) population, it is evident that the natural enemies would also get infected after the consumption of diseased pests which effects their breeding. Kumar et al. [79] proposed and analyzed a food chain model consisting plant, pest and natural enemies with disease in pest population for pest control. They assimilated the gestation delay time for natural enemies and studied the stability of different equilibrium states. Jose and Usha [80] studied the nature of plant-pest virus, bio pesticides and predators food chain model. The major focus of the study was to examine plant pest extinction and pest eradication periodic solution. They applied Floquet theory of impulsive differential equations. The study concluded that the use of organic pesticides is very effective as it is not just to minimize the effect of pest but also to increase the period of impulsive release of infected pests and natural enemies. Akman et al. [81] in their study made an attempt to propose an integrated pest management (IPM) model with prey refuge effect which describes the capability of pests to camouflage themselves from the attack of pesticides and predators. The authors constructed a

stage structured impulsive differential equation model which assumes that in order to control pest population, it is essential to use pesticides and discharge predators in the field at specified time period. The authors claimed that the present study has the potential to extend in medical immunotherapy which helps in reducing the risk of cancer and other harmful cells in human body. Chavez et al. [82] examined the similarities and differences of two ecological models which are based on chemical and biological control, implemented in periodic and impulsive manner. They applied Path-Following technique which provides numerical explanation to system parameters in pest control. It is essential to know that the authors focused on the effect of the impulse period on the ecosystems with the help of a branching point (BP). Kumari et al. [83, 87] studied integrated pest management approach which significantly suppresses pest population and prevents pest resistance to yield. They opined that these control measures are proved to be more effective in reducing pest population if they applied in combination. This approach also led to positive economic and environment outcomes. Kumar et al. [84] extended his work done in [79] assuming gestation delay time in natural enemies as well as pest population. They proved boundedness of the system and carried out the bifurcation analysis. Tariq et al. [85] conducted a study to find out the harmful effects induced by rodents in the fields of sugarcane. They conducted an experiment on three varieties of sugarcane namely, Thatta-10, BL-4 and BF-129. It was observed that BL-4 variety was heavily destroyed by the rodents because of its softness as compared to other two. The study further explored that egg mixed brodifacoum bait was proved the most successful IPM strategy in reducing the rodents. Tiana et al. [86] studied nonlinear impulsive control actions which are based on the density of the prey and predator. They opined that these measures could develop complex switching pattern which warns about the possibility of an epidemic of the prey population because of ecological challenge. Gupta et al. [88] studied a stage structured pest management model with mixed type of functional responses. The authors examined that impulsive control method has acquired special significance in the extinction and permanence of pests. The results in general of the study brought out that the annihilation of susceptible pests (immature or mature) and exposed preys completely leans on the pulse releasing amount and impulsive pe-

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riod. The study further pointed out that biological control methods which include releasing of predators or infected pests are very effective to suppress the pest population. Khan and Tang [89] proposed a prey- predator model with ratio-dependent Action Threshold (AT) which is based on the pest density and its variation rate. The authors formed Poincare map which proved to be very beneficial in exploring the impulsive dynamics of the system. The study highlighted that it is possible to control the whole pest population with the help of control action strategies. In this study, the authors made an attempt to overcome the drawbacks of Economic Threshold (ET) model which had particularly two major flaws. The first was to if there was substantial number of population, the change rate was small. In the second, the number of population was insignificant whereas the change rate was high. With the advancement science and technology, researchers are still working to find more efficient pest control methods which cause minimal environmental pollution. Working in this direction, Anandhi et al. [90] made an attempt towards the use of nano-pesticides in pest control. It is observed that many of the chemical pesticides are sensitive towards various factors such as light, humidity and others, so very low percentage of applied pesticide reaches the target. Thus, nano-particles can be used to overcame these difficulties. They also provided some examples in their paper where nano-pesticides are successfully used for pest control. Now we move towards the study of impulsive perturbations on ecological models where species compete for the same survival resources, that is competition models.

The first classical competition model was proposed by Lotka and Volterra. Murray [1] analyzed this model with the assumption that both the species are growing logistically in the absence of each other and are competing for the same survival resources. He described that the species having greater competitive effect on other, would be the superior and the second would be the inferior. He epitomized that the superior species dominate the inferior, which leads to its extinction. This is called principle of competition exclusion. Ahmad [91] analyzed the basic Lotka-Volterra competition model under more pragmatic situation when changes in the species environment effect their growth. Thus, he explored the periodic two-species competition model and derived required conditions for the system to

follow the principle of competition exclusion. Further, it is found that many a times, species experience instantaneous disturbances which bring about positive or negative effects on population density of these living organisms. So, it is better to study these systems with impulsive differential equations. Liu and Chen [92] examined an impulsive periodic Lotka-Volterra competition model. They concluded that under impulsive perturbations, both the species could coexist. Jin et al. [93] identified the influence of impulsive perturbations on the persistence and extinction of species in non-autonomous competition model. The authors have taken into account the average growth rate of the species. Liu et al. [94] explored the linear stability of trivial and non-trivial solutions of impulsive two-species competition model using Floquet theory of IDE. Liu et al. [95] introduced the concept of delay in competition system and presented an impulsive periodic two-species competition model with delay in both the species. Liu et al. [96] also inspected an impulsive delayed competition model and concluded that the influence of time delays on the system is almost negligible while impulses contributed towards permanence of the system. As earlier discussed, role of functional response of a species is very momentous in species interactions. Inspired by prey-predator interactions with different functional responses, researchers also examined competition models with non-linear inter-inhibition terms. Yu and Chen [97] developed two-species competition model with Beddington-DeAngelis inter-specific competition term. Furthermore, when species in a particular region compete for the available living resources, they exhibit specific behaviors to hamper the growth of their competitors. Thus, some species discharge toxic substances in the presence of their competitors to harm them. The study of competition interactions in the presence of toxic secretions by species was firstly initiated by Chattopadhyay [98]. Liu et al. [99] investigated a delayed two-species competition model with inhibitory toxic effects of species under the influence of impulses. Various mathematical studies on competition models considering the effect of toxic substances released by species in the presence of their competitors proved the partial extinction of the system [100, 101, 102, 103].

After doing this extensive literature review, we realized that there is still need to explore in the area of mathematical modeling on ecological models comprising

single species and species interactions with impulsive effects.

1.4   Proposed objectives of the study

On the basis of literature review and research gaps, the following objectives have been proposed in this present study:

1. Modeling on single species population using impulsive differential equations and their stability analysis by using Lyapnuov function.

2. Stability Analysis of single species population model with impulse by using Comparison theorems.

3. Modeling on interacting population (prey-predator) with impulse and stability criterion by using Lyapnuov Direct Method and comparison theorems.

4. Modeling on interacting population (Competition) with impulse and stability criterion by using Lyapnuov Direct Method and comparison theorems.

1.5   Mathematical Preliminaries

1.5.1   Impulsive Differential Equation

"An impulsive differential equation consists of two parts, first is an ordinary differential equation which describes the state of evolutionary process within impulsive moments and second is the function defining the system at impulsive moments. Consider that the state of a metamorphic system is described by an ordinary differential equation

$$x'(t) = f(t, \ x),$$

where $t \in R$, $x \in \Omega \subset R^n$, $f \colon R_+ \times \Omega \to R^n$ and impulsive moments for the solution $x(t)$ occurs at $t = t_k$ $(k \in N)$.

Define $I(t,\ x) \colon R \times \Omega \to \Omega$, where

$$(t,\ x) \to (t,\ x + I(t,\ x))$$

is the transformation of the solution before the impulsive moment, $x(t_k^-)$, to after the impulse effect, $x(\tau_k^+)$. Then

$$\Delta x(t_k) = I(t_k,\ x(t_k))$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$.

1.5.2 Types of Impulsive Differential Equations [8]

Impulsive differential equations are of three types:

*Type*1: *Equation with impulses at fixed times*

$$\begin{cases} x'(t) = f(t,\ x), & t \neq t_k, \\ \Delta x = I_k(x), & t = t_k, \quad k \in N. \end{cases} \tag{1.1}$$

The moments of impulsive effects $t_1 < t_2 < t_3 < \cdots < t_k$ are priory fixed and $t_k \to \infty$ as $k \to \infty$. The solution $x(t)$ of the system (1.1) satisfies the equation $x'(t) = f(t,\ x)$ for $t \in (t_k,\ t_{k+1}]$ and $x(t)$ satisfies the relation $x(t_k^+) - x(t_k^-) = I(t_k,\ x(t_k))$ for $t = t_k$.

*Type*2: *Equation with impulses at variable times*

$$\begin{cases} x'(t) = f(t,\ x), & t \neq t_k(x), \\ \Delta x = I_k(x), & t = t_k(x), \quad k \in N \end{cases} \tag{1.2}$$

where $t_k \colon \Omega \to R$ and $t_1 < t_2 < t_3 < \cdots < t_k$ are the moments of impulsive effects. The impulses occur when the mapping point (t, x) meets some hyper surface σ_k of the equation$t = t_k(x)$.

$$Type3 \colon \ Autonomous \ \ impulsive \ differential \ equations$$

$$\begin{cases} x'(t) = f(x), & x \notin \sigma, \\ \Delta x = I_k(x), & x \in \sigma, \end{cases} \qquad (1.3)$$

where σ is an $(n-1)$ dimensional manifold contained in the phase space $\Omega \subset R^n$. The impulses occur when the solution $x(t)$ meets the manifold σ.

1.5.3 Solution of Impulsive Differential Equation [8]

In the present study, we will deal with impulsive differential equations with fixed moments. A function $x \colon (t_0, \ t_0 + a) \to R^n$, $t_0 \geq 0$, $a > 0$, is said to be solution of system (1.1) if

(i) $x(t_0^+) = x_0$ and $(t, \ x(t)) \in R_+ \times \Omega$ for $t \in [t_0, \ t_0 + a)$.

(ii) $x(t)$ is continuously differentiable and satisfies $x'(t) = f(t, \ x(t))$ for $t \in [t_0, \ t_0 + a)$ and $t \neq t_k$.

(iii) If $t \in [t_0, \ t_0 + a)$ and $t = t_k$, then $x(\tau_k^+) = x(t_k) + I_k(x(t))$, and at such $t's$ we always assume that $x(t)$ is left continuous.

1.5.4 Stability and Permanence of the Solution of Impulsive Differential Equation [8]

Definition 1.5.4.1. *Let $x_0(t) = x(t, \ t_0, \ y_0)$ be a given solution of the system (1.1) with $x(t_0^+) = x_0$ existing for $t \geq t_0$. Then $x_0(t)$ is said to be*

(S_1) *stable, if for each $\epsilon > 0$ and $t_0 \in R_+$, there exist a $\delta = \delta(t_0, \ \epsilon) > 0$ such that $\mid x_0 - y_0 \mid < \delta$ implies $\mid x(t) - x_0(t) \mid < \epsilon$ for $t \geq t_0$, where $x(t) = x(t, \ t_0, \ x_0)$ is any solution of system (1.1) existing for $t \geq t_0$;*

(S_2) *uniformly stable, if δ in S_1 is independent of t_0;*

(S_3) *attractive, if for each $\epsilon > 0$ and $t_0 \in R_+$ there exist $\delta_0 = \delta_0(t_0) > 0$ and a $T = T(t_0, \ \epsilon) > 0$ such that for $\mid x_0 - y_0 \mid < \delta_0$ implies $\mid x(t) - x_0(t) \mid < \epsilon$ for $t \geq t_0 + T$;*

(S_4) *uniformly attractive, if δ_0 and T in S_3 are independent of t_0;*

(S_5) *asymptotically stable, if S_1 and S_3 holds;*

(S_6) *uniformly asymptotically stable, if S_2 and S_4 holds.*

Definition 1.5.4.2. *The system (1.1) is said to be permanent if there exists constants $Q \geq q > 0$ such that $q \leq x(t) \leq Q$ for sufficiently large t, where $x(t)$ is any solution of system (1.1) with $x(0^+) > 0$.*

Definition 1.5.4.3. *Let $V_0 = \{V : R_+ \times R_+^n, \ continuous \ on \ (t_k, \ t_{k+1}] \times R_+^n, \ and \ \lim_{(t, \ y) \to (t_k, \ x), \ t > t_k} V(t, \ x) = V(t_k^+, \ x) exists\}$. If $V \in V_0$, then for $(t, \ x) \in (t_k, \ t_{k+1}] \times R_+^n$, the upper right derivative of V(t, x) with respect to the impulsive differential system (1.1) is defined as*

$$D^+V(t, \ x) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, \ x + hf(t, \ x)) - V(t, \ x)]. \qquad (1.4)$$

1.5.5   Impulsive Integral Inequality [8]

Theorem 1.5.1. *Assume that*

(A_0) *the sequence t_k satisfies $0 \leq t_1 < t_2 < t_3 < \cdots$, with $\lim_{k \to \infty} t_k = \infty$;*

(A_1) *$m \in PC^1[R_+, \ R]$ and $m(t)$ is left continuous at $t = t_k$;*

(A_2) *for $k = 1, 2..., \ t \geq t_0$,*

$$\begin{cases} m'(t) \leq p(t)m(t) + q(t), & t \neq t_k \\ m(t_k^+) \leq d_k m(t_k) + b_k, & t = t_k. \end{cases} \quad (1.5)$$

where $q, p \in PC[R_+, \ R]$, $d_k \geq 0$ and b_k are constants

Then,

$$\begin{aligned} m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_0^t p(s)ds\right) \\ + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^t p(s)ds\right)\right) b_k \\ + \int_{t_0}^t \prod_{s < t_k < t} d_k \exp\left(\int_s^t p(\sigma)d\sigma\right) q(s)ds, \ t \geq t_0. \end{aligned} \quad (1.6)$$

1.5.6 Maximal and Minimal Solution of Impulsive Differential Equations [8]

Definition 1.5.6.1. *Let* $r(t) = r(t, \ t_0, x_0)$ *be the solution of system (1.1) on* $[t_0, \ t_0+a)$. *Then* $r(t)$ *is said to be the maximal solution of (1.1), if for any solution* $x(t) = x(t, \ t_0, \ x_0)$ *of (1.1) existing on* $[t_0, \ t_0 + a)]$, *the inequality* $x(t) \leq r(t)$ *holds. A minimal solution of system (1.1) can be similarly defined by reversing the inequality.*

1.5.7 Comparison Principle [8]

Theorem 1.5.2. *Assume that* (A_0) *and* (A_1) *holds. Suppose that* $g \colon R_+ \times R \to R$ *and* $\psi_k \colon R \to R$, $\psi_k(u)$ *is non decreasing in* u *and for each* $k = 1, 2, 3, \ldots$

$$\begin{cases} D^+ m(t) \leq g(t, \ m(t)), & t \neq t_k, \ m(t_0) \leq u_0, \\ m(t_k^+) \leq \psi_k(m(t_k)), & t = t_k. \end{cases} \quad (1.7)$$

Let $r(t)$ be the maximal solution of the impulsive differential system

$$\begin{cases} u' = g(t, \ u), & t \neq t_k, \ u(t_0) = u_0, \\ u(t_k^+) \leq \psi_k(u(t_k)), & t = t_k. \end{cases} \tag{1.8}$$

Then $m(t) \leq r(t), \ t_0 \leq t < \infty$.

1.5.8 Floquet Theory of Impulsive Differential Equations [10]

Consider the linear T-periodic impulsive system with fixed moments of impulse

$$\begin{cases} \frac{dx}{dt} = A(t)x, & t \neq \tau_k, \ t \in R, \\ \Delta x = B_k(x), & t = \tau_k, \ k \in Z, \end{cases} \tag{1.9}$$

subject to the following conditions

H1.1 The matrix $A(.) \in PC(R, \ C^{n \times n})$ and $A(t + T) = A(t)$ for $t \in R$.

H1.2 $B_k \in C^{n \times n}$, det $(E + B_k) \neq 0, \ \tau_k < \tau_{k+1}$.

H1.3 There exists an integer $q \in N$ such that $B_{k+q} = B_k$ and $\tau_{k+q} = \tau_k + T$ for $k \in Z$.

Definition 1.5.8.1. *Let $x_1(t), \ x_2(t), \ \cdots, \ x_n(t)$ be solutions of system (1.9) on the interval $(0, \ \infty)$. Let $X(t) = x_1(t), \ x_2(t), \cdots, \ x_n(t)$ are linearly independent iff det $X(t_0^+) \neq 0$. Then $X(t)$ is said to be fundamental matrix of (1.9).*

Theorem 1.5.3. *Let conditions H1.1-H1.3 hold. Then each fundamental matrix of system (1.9) can be represented in the form*

$$X(t) = \phi(t)e^{\Lambda t} \quad (t \in R), \tag{1.10}$$

where the matrix $\Lambda \in C^{n \times n}$ is constant and the matrix $\phi(.) \in PC^1(R, \ C^{n \times n})$ is non-singular and T-periodic.

To the fundamental matrix $X(t)$ there corresponds a unique matrix M such that $X(t + T) = MX(t)$ for all $t \in R$. Here, M is called the monodromy matrix of system of equations (1.9). The eigen values $\mu_1, \ \mu_2, \cdots, \mu_n$ of the monodromy matrix M are called Floquet multipliers of (1.9). The eigen values $\lambda_1, \ \lambda_2, \cdots, \lambda_n$ of matrix Λ are called Floquet exponents of (1.9).

Remark 1.5.8.1. *In order to calculate the multipliers $\mu_1, \ \mu_2, \cdots, \mu_n$ of (1.9), we have to choose an arbitrary fundamental matrix $X(t)$ of (1.9) and calculate the eigenvalues of the matrix*

$$M = W(t_0 + T, \ t_0) = X(t_0 + T)X^{-1}(t_0), \tag{1.11}$$

where $t_0 \in R$ is fixed. If $X(0) = E$, then we can choose $M = X(T)$ as the monodromy matrix of (1.9).

Theorem 1.5.4. *Let conditions H1.1-H1.3 hold. Then $\mu \in C$ is a Floquet multiplier of (1.9) if and only if there exists a non-trivial solution $\varphi(t)$ of (1.9) such that $\varphi(t + T) = \mu\varphi(t)$ for all $t \in R$.*

Theorem 1.5.5. *Let conditions H1.1-H1.3 hold. Then the system (1.9) has a non- trivial kT-periodic solution if and only if the kth power of at least one of its multipliers equals 1.*

The multipliers of (1.9) completely characterize its stability. This is seen from the following theorem and relation

$$\frac{1}{T} \ln \mid \mu_j \mid = Re(\lambda_j), \ \ (j = 1, 2, \cdots, n). \tag{1.12}$$

Theorem 1.5.6. *Let conditions H1.1-H1.3 hold. Then the linear T-periodic impulsive system (1.9) is*

1. *stable if and only if all multipliers μ_j ($j = 1, 2, \cdots, n$) of the system ([1.9](#)) satisfy the inequality $\mid \mu_j \mid \leq 1$;*

2. *asymptotically stable if and only if all multipliers μ_j ($j = 1, 2, \cdots, n$) of the system ([1.9](#)) satisfy the inequality $\mid \mu_j \mid < 1$;*

3. *unstable if $\mid \mu_j \mid > 1$ for some $j = 1, 2, \cdots, n$.*

Remark 1.5.8.2. *For impulsive differential equations with fixed impulsive moments, the definition of various types of stability of the solutions coincide with the known notion of stability of the solutions of ordinary differential equations without impulses. So, below we will define Lyapunov's method for the stability of differential systems.*

1.5.9 Lyapunov's Direct Method of Stability [3, 104]

Let Ω be an open set in R^n containing the origin. Suppose $V(x)$ is a scalar continuous function defined on Ω. Then

Definition 1.5.9.1. *A scalar function $V(x)$ is said to be positive definite on the set Ω if and only if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ and $x \in \Omega$.*

Definition 1.5.9.2. *A scalar function $V(x)$ is said to be positive semidefinite on the set Ω when V has the positive sign throughout Ω, except at points where it is zero.*

Definition 1.5.9.3. *A scalar function $V(x)$ is said to be negative definite (negative semidefinite) on the set Ω if and only if $-V(x)$ is positive definite (positive semidefinite) on Ω.*

Theorem 1.5.7. *If on a neighborhood Ω of the origin, there exist a function $V : \Omega \to R$ and a constant a such that*

(i) $V(x)$ and its partial derivatives are continuous for all $x \in \Omega$.

(ii) $V(x) \geq 0$ *for all* $x \in \Omega$ *and* $V(0) = 0$.

(iii) $\dot{V}(t, x) \leq -a(V(x))$ *for all* $x \in \Omega$ *and for all* $t \geq 0$.

Then the zero solution of system (1.1) is uniformly asymptotically stable.

Definition 1.5.9.4. *The function* $V \in C^1(\Omega, R)$ *satisfying the above three conditions is called strong Lyapunov function.*
If in the third condition of Theorem 1.5.7, $V(x)$ is negative semidefinite. Then $V(x)$ is called weak Lyapunov function.

1.5.10 Important Lemmas [10, 60]

Lemma 1.5.1. *If $V(t)$ be any solution of system (1.1) with $V(0) \geq 0$ then $V(t) \geq 0$ for all $t \geq 0$. Also $V(t) > 0$ for all $t \geq 0$ if $V(0) > 0$.*

Lemma 1.5.2. *([10]) Consider the following impulsive system*

$$\begin{cases} \frac{dx(t)}{dt} = x(t)r(t)\left(1 - \frac{x(t)}{k(t)}\right), & t \neq \tau_k, \\ v(\tau_k^+) = (1 + c_k)v(\tau_k), & t = \tau_k, \ k \in Z_+. \end{cases} \tag{1.13}$$

with initial condition $x(0) > 0$, $r(t)$ and $k(t)$ are continuous functions that are periodic with period $\overline{T}$, $k(t) \geq 0 \ \forall \ t \geq 0$, $c_k > -1$ and $\tau_{k+r} = \tau_k + \overline{T}$, $c_{k+r} = c_k$ for some $r \in N$. If $\prod_{k=1}^{r}(1 + c_k)\exp\left(\int_0^{\overline{T}} r(t)dt\right) > 1$, then the system (1.13) has a solution which is positive, globally asymptotically stable and periodic with period $\overline{T}$.

Lemma 1.5.3. *([60]) Consider the following impulsive system*

$$\begin{cases} I'(t) = -wI(t), & t \neq n\tau, \\ \Delta I(t) = \mu, & t = n\tau, \\ n \in Z_+. \end{cases} \tag{1.14}$$

with initial condition $I(0^+) = I_0$ and τ is the period of impulsive moments. It has periodic solution $\overline{I}(t)$ and for any solution $I(t)$ of (1.14), $\mid I(t) - \overline{I}(t) \mid \to 0$ as $t \to \infty$ where

$$\overline{I}(t) = \frac{\mu \exp(-w(t - n\tau))}{1 - \exp-(w\tau)} \ and \ \overline{I}(0^+) = \frac{\mu}{1 - \exp(-w\tau)}$$

Thus $\overline{I}(t)$ is globally stable."

1.6 Summary

The study of the underlying system consisting of three major inter-related components, viz. population dynamics, impulse and integrated pest management was carried out by using the concept of system analysis in which the system is studied by defining its borders, by distinguishing its major components, characterizing the change in them by mathematical equations and then interconnecting the representative equations in order to obtain a model of the original system. For the proposed study, it is planned to construct deterministic and dynamic mathematical models using systems of linear and nonlinear impulsive differential equations in order to predict the changes in the attributes of the inter-related objects of the system. Once the model is governed by impulsive differential equations, these equations were solved assuming the initial positivity of all the state variables. The boundedness of the solutions obtained were checked using integral inequalities of IDE. The stability and permanence of the systems was studied with the help of stroboscopic map, comparison analysis technique and Floquet theory of impulsive differential equations. Also, numerical solutions of the proposed models were obtained using numerical techniques and MATLAB. This thesis consists of seven chapters whose detail is as follows:

In chaper-1, basic introduction about population dynamics, mathematical modeling, impulsive differential equations and pest management has been described. The basic terms used in this study are also discussed. Under the section of literature review, the work done by many renowned researchers till date at national

and international level has been extensively examined. After analyzing the research gaps, objectives of the present study have been proposed. To check the boundedness, stability and permanence of the formulated models, mathematical preliminaries of impulsive differential equations have been given. The last section includes the summary of the study.

In chapter-2, a single-species model under the effect of periodically changing environment and impulsive perturbations is proposed and investigated in this paper considering Holling type III functional response as predation term. It is assumed that self-inhibition rate must be positive. Existence of positive periodic solution is proved with Brower's fixed point theorem. Sufficient conditions for the global asymptotic stability of system are derived using suitable Lyapunov function and comparison principle.

In chapter-3, a single-species population model under the influence of constant and linear impulsive perturbations at fixed moments of time is explored taking Monod Haldane functional response as predation term. It is assumed that species is living in weak environmental conditions. The species become extinct in the absence of impulsive perturbations. Sufficient conditions for the permanence of the species surviving under unfavorable conditions are obtained. It is shown that in case of linear impulses at fixed time events, higher value of impulsive perturbations expedite the attainment of permanence of the system. Thus, we proved that extinction of species can be prevented with impulsive stocking.

In chapter-4, a susceptible pest-infected pest-natural enemy (prey-predator) model considering infection in prey with two classes (susceptible-infected) and stage structure in predator is examined for the cause of integrated pest management. Prey acts as pest and predator plays the role of a natural enemy. Firstly, local and global stability of pest extinction periodic solution is carried out, then condition for the permanence of system is derived using Stroboscopic map, comparison analysis technique and Floquet theory. It is found that instead of using pesticides, microbial control agents along with natural enemies are more efficient to suppress the growth of pests. The threshold value of impulsive period is obtained and it is established that susceptible pests can coexist with infected pests and natural enemies. Also, the effect of releasing the number of infected pests and natural

enemies is discussed and found that greater releasing amount of infected pests and natural enemies supports pest eradication. Numerical simulations are performed using MATLAB.

In chapter-5, a two-prey one-predator model is proposed with mixed functional responses (Holling IV and Beddington-DeAngelis). Firstly, it is proved that prey neutral periodic solution is locally stable by using Floquet theory of impulsive differential equations. Secondly, two threshold values of the impulsive release amount of predator are calculated to establish the global stability of prey free boundary equilibrium. Also, conditions for the permanence of the system are derived. Numerical solutions are also obtained using MATLAB.

In chapter-6, a three tropic level food chain model is analyzed considering plant, pest and natural enemies for integrated pest management. Two different type of functional responses are taken for mid level and top level predator. Threshold value of the impulsive period is calculated for extinction of mid level predator using Floquet theory of impulsive differential equations, Lyapnuov functions and stroboscopic map. Mid level predator plays the role of pest. Permanence of system is also established. Effect of spraying amount of pesticides and natural enemies is discussed and found that greater releasing amount or small impulsive period support pest eradication. Some complex dynamics is also observed at higher value of impulsive period greater than threshold value. Further, validation of theoretical findings is done using MATLAB.

In chapter-7, a two species periodic competitive model with Beddington-DeAngelis inter-inhibition term in the presence of toxic substances is examined. The system is considered under the effect of impulsive perturbations. Sufficient conditions for the extinction of one species and permanence of other are derived using suitable Lyapunov functions and comparison theorem of impulsive differential equations. Therefore, extension of principle of competition exclusion is done for two species competition system with Beddington-DeAngelis functional response under the effect impulsive perturbations and toxic substances released by both the species. Also, it is observed that impulses effects the dynamics of the system. To substantiate theoretical findings, numerical simulation is done using MATLAB.

Chapter 2

Periodicity and Stability of Single-species Model with Holling Type III Predation Term using Impulse

2.1 Introduction

Mathematical modeling is an important term to study population dynamics. For the past years, modeling with ordinary differential equations is playing an important part in the study of both single species and multi-species population dynamics. Two main interactions among species are competition and prey-predator. Dynamics of single-species exponential growth model and that of two-species interactions, that is, Lotka-Volterra prey predator and competition model is studied. But, many evolutionary processes taking place in nature undergo a change of state abruptly at certain moments of time. These are sometimes influenced by short term perturbations which act instantaneously in the form of impulses. The time span of these perturbations is small in relation to that of the studied process, but they

effect the dynamics of the process. So these processes can better be analyzed using impulsive differential equations. Hence, these equations has wider scope and greater practical importance than ordinary differential equations without impulses because these are more closer to the practical problems. Initially, Mil'man and Myshkis [7] worked in the field of quantitative study of impulsive differential equations. Further, work is done on solutions of impulsive differential equations and their comparison theorems, see [8]. After this there is continuous development in the study of impulsive differential equations. These are being used in modeling in population dynamics, ecology and other applied sciences [24]. The Lotka-Volterra periodic competing population model with impulsive perturbations in the form of human activities is investigated by Liu and Chen [92]. They removed the continuous interference of humans in biological processes with impulsive effect as species are being harvested or stocked mostly seasonally. It occurs in regular pulses. In the recent years, work has also been done on single species population.

Moving further, constant and linear impulses at fixed moments of time in a periodic single-species model has analyzed by Liu et al.[153]. The authors established conditions for the permanence of system using comparison theorems of impulsive differential equations. They concluded that species can be protected from extinction by linear and constant impulsive perturbations. In a single species model, the periodicity and stability with impulsive perturbations by considering the predation term as Holling Type-II functional response using comparison theorems and Lyapnuov function is examined by Tan et al [29]. In continuation of this, Tan et al. [30] compared the effect of linear and constant impulsive perturbations on single species population density considering Holling Type-II functional response.

In the view of above, in this chapter, impulsive perturbations in a single-species model are taken and its stability is discussed with the help of comparison systems and Lyapnuov function by considering Holling Type –III functional response as the predation term.

2.2   Mathematical Model

Simple and useful framework to investigate the behavior of single-species model is represented by the first order differential equation

$$n'(t) = n(t)f(n(t)) - h(n(t)), \tag{2.1}$$

where $n(t)$ denotes the population density of species at any time t. In the absence of external factors $f(n(t))$ represents per capita rate at which the species grows and $h(n(t))$ represents the cumulative effect of the outside, in the form of predation or harvesting. The most suitable choice of $f(n(t))$ is given by $f(n(t)) = p(t) - q(t)n(t)$, $q(t) > 0$ giving rise to logistic growth model. Consider that predation is represented by the term $h(n(t))$. The predation term generally is an increasing function in which saturation comes as the population density of prey species increases. At low prey density, the predation term $h(n(t))$ decreases more rapidly. As introduced by Ludwig et al. [6], Murray [1] took the form of $h(n(t))$ as $\frac{\alpha n(t)^2}{\beta + n(t)^2}$, so the dynamics of the species n is governed by the equation:

$$n'(t) = n(t)(p(t) - q(t)n(t)) - \frac{\alpha n(t)^2}{\beta + n(t)^2}, \tag{2.2}$$

where the predation term $\frac{\alpha n(t)^2}{\beta + n(t)^2}$ is Holling type III functional response, $p(t)$ is intrinsic growth rate, $q(t)$ is self-inhibition rate and the positive constants α, β measures the saturation value. In equation (2.2), the coefficients are assumed to be constants. But in real, due to periodic variations in the environment, these cannot be taken as fixed constants. So, we assume that the coefficients $\alpha(t)$, $\beta(t)$, $q(t)$ and $p(t)$ are all periodic in nature. The species experience abrupt changes in such a fluctuating environment. Thus, considering all these factors, the single species model we obtained is governed by impulsive differential system is as follows:

$$\begin{cases} n'(t) = n(t)(p(t) - q(t)n(t)) - \frac{\alpha(t)n(t)^2}{\beta(t)+n(t)^2}, & t \neq \tau_k, \\ n(\tau_k^+) = (1+\gamma_k)n(\tau_k), & t = \tau_k, \ k \in Z_+. \end{cases} \tag{2.3}$$

where $n(0) > 0$, τ_k represents the moment of time when impulse occurs for every k and $0 < \tau_1 < \tau_2 < \tau_3 < \ldots < \tau_k$. $\beta(t)$, $\alpha(t)$, $q(t)$, & $p(t)$ are positive and periodic coefficients with period $\overline{T}$ for $t \geq 0$. $\gamma_k > -1$ for biological relevance. Here $\gamma_k n(\tau_k) < 0$ represents the harvesting of species and $\gamma_k n(\tau_k) > 0$ represents stocking of species at time τ_k. Sufficient conditions required for the existence of positive periodic solution and global asymptotic stability of system (2.3) are derived in this chapter. Thus, firstly, it is proved that the system (2.3) has solution which is periodic with period $\overline{T}$ by using Brower's fixed point theorem. Then, global asymptotic stability of positive $\overline{T}$ periodic solution has been established by using Lyapunov function.

2.3 Existence of positive periodic solution

In this section, the existence of positive periodic solution of impulsive system (2.3) is established. Consider the corresponding non-impulsive system of (2.3)

$$\phi'(t) = \phi(t)(p(t) - Q(t)\phi(t)) - \frac{\alpha(t)\gamma(t)(\phi(t))^2}{\beta(t) + (\gamma(t)\phi(t))^2}, \tag{2.4}$$

where $\gamma(t) = \prod_{0 < \tau_k < t}(1 + \gamma_k)$, $Q(t) = q(t)\gamma(t)$. It is very easy to prove that $\phi(0) > 0$ implies that $\phi(t) > 0 \; \forall \; t \geq 0$.

Lemma 2.3.1. *If the system (2.4) has solution $\phi(t)$, then $n(t) = \prod_{0 < \tau_k < t}(1 + \gamma_k)\phi(t)$ is the solution of system (2.3). Reverse is also true.*

Proof. Suppose that the above non impulsive system (2.4) has solution $\phi(t)$. Let $n(t) = \prod_{0 < \tau_k < t}(1 + \gamma_k)\phi(t)$, then if $t \neq \tau_k$, we have

$$n'(t) - n(t)\left(p(t) - q(t)n(t) - \frac{\alpha(t)n(t)}{\beta(t) + n(t)^2}\right)$$

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$$= \prod_{0 < \tau_k < t} (1 + \gamma_k) \left(\phi'(t) - \phi(t) \left(p(t) - q(t) \prod_{0 < \tau_k < t} (1 + \gamma_k) \phi(t) \right. \right. \tag{2.5}$$

$$\left. \left. - \frac{\alpha(t) \prod_{0 < \tau_k < t} (1 + \gamma_k) \phi(t)}{\beta(t) + \left(\prod_{0 < \tau_k < t} (1 + \gamma_k) \phi(t) \right)^2} \right) \right)$$

$$= \prod_{0 < \tau_k < t} (1 + \gamma_k) \left(\phi'(t) - \phi(t) \left(p(t) - Q(t) \phi(t) - \frac{\alpha(t) \gamma(t) \phi(t)}{\beta(t) + (\gamma(t) \phi(t))^2} \right) \right)$$

$$= \prod_{0 < \tau_k < t} (1 + \gamma_k)(0) = 0 \tag{2.6}$$

On the other hand for every $t = \tau_k$, we obtain

$$n(\tau_k^+) = \lim_{t \to \tau_k^+} \prod_{0 < \tau_j < t} (1 + \gamma_j) \phi(t) = \prod_{0 < \tau_j \leq \tau_k} (1 + \gamma_j) \phi(\tau_k) = (1 + \gamma_k) \prod_{0 < \tau_j < \tau_k} (1 + \gamma_j) \phi(\tau_k)$$

Thus for $k = 1, 2, 3 \ldots$, we obtain

$$n(\tau_k^+) = (1 + \gamma_k) n(\tau_k) \tag{2.7}$$

It follows from equations (2.6) and (2.7) that $n(t)$ is solution of system (2.3). This proves lemma. $\qquad \square$

Lemma 2.3.2. *Consider that $\phi(t)$ be the solution of the system (2.4). Then both $\phi_1(t) = \prod_{t \leq \tau_k < t + \overline{T}} (1 + \gamma_k) \phi(t + \overline{T})$ and $\phi_2(t) = \prod_{t - \overline{T} \leq \tau_k < t} (1 + \gamma_k) \phi(t - \overline{T})$ are also the solutions of system (2.4). Also $\phi_1(0) = \phi_2(\overline{T}) = \prod_{0 \leq \tau_k < \overline{T}} (1 + \gamma_k) \phi(0)$.*

Proof. It is very easy to verify that $\phi_1(t)$ is continuous. Now

$$\phi_1'(t) = \prod_{t \leq \tau_k < t + \overline{T}} (1 + \gamma_k) \phi'(t + \overline{T})$$

$$= \prod_{t \leq \tau_k < t + \overline{T}} (1 + \gamma_k) \phi(t + \overline{T}) \left(p(t + \overline{T}) - q(t + \overline{T}) \prod_{0 < \tau_k < t + \overline{T}} (1 + \gamma_k) \phi(t + \overline{T}) \right.$$

$$\left. - \frac{\alpha(t + \overline{T}) \phi(t + \overline{T}) \gamma(t + \overline{T})}{\beta(t + \overline{T}) + (\gamma(t + \overline{T}) \phi(t + \overline{T}))^2} \right)$$

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$$= \prod_{t \le \tau_k < t+\overline{T}} (1+\gamma_k)\phi(t+\overline{T}) \left(p(t) - q(t) \prod_{0<\tau_k<t} (1+\gamma_k) \prod_{t \le \tau_k < t+\overline{T}} (1+\gamma_k)\phi(t+\overline{T}) \right.$$

$$\left. - \frac{\alpha(t)\phi(t+\overline{T})\gamma(t+\overline{T})}{\beta(t+\overline{T}) + \left(\gamma(t+\overline{T})\phi(t+\overline{T})\right)^2} \right)$$

$$= \prod_{t \le \tau_k < t+\overline{T}} (1+\gamma_k)\phi(t+\overline{T}) \left(p(t) - q(t) \prod_{0<\tau_k<t} (1+\gamma_k)\phi_1(t) \right.$$

$$\left. - \frac{\alpha(t)\prod_{0<\tau_k<t}(1+\gamma_k)\prod_{t\le\tau_k<t+\overline{T}}(1+\gamma_k)\phi(t+\overline{T})}{\beta(t) + \left(\prod_{0<\tau_k<t}(1+\gamma_k)\prod_{t\le\tau_k<t+\overline{T}}(1+\gamma_k)\phi(t+\overline{T})\right)^2} \right)$$

$$= \prod_{t \le \tau_k < t+\overline{T}} (1+\gamma_k)\phi(t+\overline{T}) \left(p(t) - Q(t)\phi_1(t) - \frac{\alpha(t)\gamma(t)\phi_1(t)}{\beta(t) + (\gamma(t)\phi_1(t))^2} \right)$$

$$= \phi_1(t) \left(p(t) - Q(t)\phi_1(t) - \frac{\alpha(t)\gamma(t)\phi_1(t)}{\beta(t) + (\gamma(t)\phi_1(t))^2} \right)$$

Thus $\phi_1(t)$ is the solution of system (2.4). Clearly, $\phi_1(0) = \prod_{0 \le \tau_k < \overline{T}} \phi(0)$. Similarly, it can be proved for $\phi_2(t)$. This completes the proof of lemma 2.3.2. $\qquad\square$

Lemma 2.3.3. *If* $\ln B > 0$, *then* $\exists\ \phi_1(0) > 0$ *such that* $\prod_{0<\tau_k<\overline{T}}(1+\gamma_k)\phi_1(\overline{T}) \ge \phi_1(0)$, *where* $\phi_1(t)$ *is the solution of system (2.4) and*

$$B = \prod_{0<\tau_k<\overline{T}} (1+\gamma_k) \exp\left(\int_0^{\overline{T}} p(t)dt \right).$$

Proof. Let $G = \max_{t\in[0,\ \overline{T}]} \prod_{0<\tau_k<\overline{T}}(1+\gamma_k)$, $g = \min_{t\in[0,\ \overline{T}]}\prod_{0<\tau_k<\overline{T}}(1+\gamma_k)$. If $\ln B > 0$, then we have $B > 1$. So, it is possible to find a positive constant δ such that

$$B_\delta = \prod_{0<\tau_k<\overline{T}} (1+\gamma_k) \exp\left(\int_0^{\overline{T}} (p(s) - \delta)ds \right) > 1. \qquad (2.8)$$

Consider a constant $d_1 > 0$ such that

$$q^U G\phi(t) + \frac{\alpha^U}{\beta^L} \le \delta,\ \text{if } \phi(t) \le d_1, \qquad (2.9)$$

where G is defined above. Suppose $0 < \phi_1(0) \leq d_1 \exp\left(-\int_0^{\overline{T}} p(s)ds\right)$. Because $\phi_1(t)$ be the solution of system (2.4). This implies that

$$\phi_1'(t) \leq p(t)\phi_1(t)$$

Solving this, we obtain

$$\phi_1(t) \leq \phi_1(0) \exp\left(\int_0^{\overline{T}} p(s)ds\right) \leq \phi_1(0) \exp\left(\int_0^{\overline{T}} p(s)ds\right) \leq d_1, \ t \in [0, \ \overline{T}].$$
(2.10)

Thus from equation (2.9) and system (2.4), we obtain

$$\phi_1'(t) = \phi_1(t)\left(p(t) - q(t)\prod_{0<\tau_k<t}(1+\gamma_k)\phi_1(t) - \frac{\alpha(t)\gamma(t)\phi_1(t)}{\beta(t) + \left(\prod_{0<\tau_k<t}(1+\gamma_k)\phi_1(t)\right)^2}\right)$$
$$\geq \phi_1(t)\left(p(t) - \left\{q^U G\phi_1(t) + \frac{\alpha^U}{\beta^L}\right\}\right)$$
$$\geq \phi_1(t)(p(t) - \delta), \ t \in [0, \ \overline{T}],$$

which implies that

$$\phi_1(\overline{T}) \geq \phi_1(0) \exp\left(\int_0^{\overline{T}}(p(s) - \delta)ds\right)$$
(2.11)

From equations (2.8) and (2.11), we obtain

$$\prod_{0<\tau_k<\overline{T}}(1+\gamma_k)\phi_1(\overline{T}) \geq \phi_1(0)\prod_{0<\tau_k<\overline{T}}(1+\gamma_k)\exp\left(\int_0^{\overline{T}}(p(s) - \delta)ds\right)$$
$$\geq \phi_1(0)B_\delta \geq \phi_1(0)$$

Thus $\prod_{0<\tau_k<\overline{T}}(1+\gamma_k)\phi_1(\overline{T}) \geq \phi_1(0)$. This completes the proof of lemma 2.3.3. $\square$

Lemma 2.3.4. *If the assumptions $0 < \ln B < \frac{q^L g}{2q^U G}$ and $\frac{\alpha(t)}{\beta(t)} < p(t)$ holds true, then there exists $\phi_2(0) > 0$ such that $\prod_{0<\tau_k<\overline{T}}(1+\gamma_k)\phi_2(\overline{T}) \leq \phi_2(0)$.*

Proof. It is possible to choose a positive constant δ_1 such that

$$\ln B < \delta_1 \overline{T} < \frac{q^L g}{2q^U G}. \tag{2.12}$$

As a result

$$B_{\delta_1} = \prod_{0 < \tau_k < \overline{T}} (1 + \gamma_k) \exp\left(\int_0^{\overline{T}} (p(s) - \delta_1)ds\right) < 1. \tag{2.13}$$

Assume that $d_2 = \frac{1}{2q^U G \overline{T}} > 0$ such that

$$q^L g\phi(t) \geq \delta_1 \text{ if } \phi(t) \geq d_2. \tag{2.14}$$

Let $\phi_2(0) \geq 2d_2$. Because $\phi_2(t)$ is also the solution of equation (2.4), therefore

$$
\begin{aligned}
\phi_2'(t) &= \phi_2(t)\left(p(t) - q(t)\prod_{0 < \tau_k < t}(1 + \gamma_k)\phi_2(t) - \frac{\alpha(t)\gamma(t)\phi_2(t)}{\beta(t) + (\gamma(t)\phi_2(t))^2}\right) \\
&\geq \phi_2(t)\left(p(t) - \frac{\alpha(t)}{\beta(t)}\right) - q(t)\prod_{0 < \tau_k < t}(1 + \gamma_k)(\phi_2(t))^2 \\
&\geq -q(t)\prod_{0 < \tau_k < t}(1 + \gamma_k)(\phi_2(t))^2 \\
&\geq -q^U G(\phi_2(t))^2
\end{aligned}
$$

Thus,

$$\phi_2'(t) \geq -q^U G(\phi_2(t))^2. \tag{2.15}$$

Integrating Eq. (2.15)

$$\frac{1}{\phi_2(t)} \leq \frac{1}{\phi_2(0)} + q^U G\overline{T} \leq \frac{1}{d_2}, \ t \in [0, \ \overline{T}].$$

Therefore,

$$\phi_2(t) \geq d_2, \ t \in [0, \ \overline{T}]. \tag{2.16}$$

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From system (2.4), equations (2.14) and (2.16), we obtain

$$\phi_2'(t) \leq \phi_2(t)\left(p(t) - Q(t)\phi_2(t)\right) = \phi_2(t)\left(p(t) - q(t)\prod_{0<\tau_k<t}(1+\gamma_k)\phi_2(t)\right)$$

$$\leq \phi_2(t)(p(t) - \delta_1), \ t \in [0, \overline{T}].$$

Thus, we have

$$\phi_2(\overline{T}) \leq \phi_2(0)\exp\left(\int_0^{\overline{T}}(p(s) - \delta_1)ds\right),$$

which combining with equation (2.13), leads to

$$\prod_{0<\tau_k<\overline{T}}(1+\gamma_k)\phi_2(\overline{T}) \leq \phi_2(0)\prod_{0<\tau_k<\overline{T}}(1+\gamma_k)\exp\left(\int_0^{\overline{T}}(p(s) - \delta_1)ds\right)$$

$$= \phi_2(0)B_{\delta_1}$$

$$\leq \phi_2(0)$$

This completes the proof. $\qquad\square$

Theorem 2.3.1. *If the assumptions $0 < \ln B < \frac{q^L g}{2q^U G}$ and $\frac{\alpha(t)}{\beta(t)} < p(t)$ holds true, then there exists solution of the system (2.3) that is periodic with period $\overline{T}$.*

Proof. Let the solution of the system (2.4) be $\phi(t)$ with $\phi_1(0) < \phi(0) < \phi_2(0)$. Using Kamke's Theorem, $\phi_1(t) < \phi(t) < \phi_2(t)$.
Define a map $L\colon [\phi_1(0), \ \phi_2(0)] \to [\phi_1(0), \ \phi_2(0)]$ such that

$$L(\phi_0) = \prod_{0<\tau_k<\overline{T}}(1+\gamma_k)\phi(\overline{T}).$$

As the map L is continuous, so, by applying Brower's Fixed Point Theorem, we get $\phi_0^* \to [\phi_1(0), \ \phi_2(0)]$ such that

$$L(\phi_0^*) = \phi_0^*,$$

that is

$$\phi_0^* = \prod_{0 < \tau_k < \overline{T}} (1 + \gamma_k) \phi^*(\overline{T}), \qquad (2.17)$$

where $\phi^*(t)$ is the solution of system (2.4) with $\phi^*(0) = \phi_0^*$. Also $\overline{\phi(t)} = \prod_{t \leq \tau_k < t + \overline{T}} (1 + \gamma_k) \phi^*(t + \overline{T})$ is the solution of system (2.4) with

$$\overline{\phi(0)} = \prod_{0 \leq \tau_k < t + \overline{T}} (1 + \gamma_k) \phi_{\overline{T}}^* = \phi^*(0). \qquad (2.18)$$

By the uniqueness theorem, we have

$$\overline{\phi(t)} = \phi^*(t) = \prod_{t \leq \tau_k < t + \overline{T}} (1 + \gamma_k) \phi^*(t + \overline{T}). \qquad (2.19)$$

By Lemma (2.3.1), $n^*(t) = \prod_{0 \leq \tau_k < \overline{T}} (1 + \gamma_k) \phi^*(t)$ is the solution of system (2.3). Further, from equation (2.19)

$$n^*(t + \overline{T}) = \prod_{0 < \tau_k < t + \overline{T}} (1 + \gamma_k) \phi^*(t + \overline{T}) = \prod_{0 < \tau_k < t} (1 + \gamma_k) \prod_{t < \tau_k < t + \overline{T}} (1 + \gamma_k) \phi^*(t + \overline{T})$$

$$= \prod_{0 < \tau_k < t} \phi^*(t) = n^*(t).$$

This means $n^*(t)$ is periodic with period $\overline{T}$. This completes the proof of Theorem (2.3.1). $\qquad \square$

2.4 Global asymptotic stability of positive periodic solution

In this section, the discussion on the global asymptotic stability of positive and periodic solution of system (2.3) is held. Firstly, lemma is proved which is useful for the proof of main results.

Lemma 2.4.1. *Assume that*

$$\prod_{k=1}^{r}(1+\gamma_k)\exp\left(\int_0^{\overline{T}}\left(p(t)-\frac{\alpha(t)}{\beta(t)}\right)dt\right) > 1 \tag{2.20}$$

holds, then there exists two positive constants κ, K and finite time $\breve{t}$ such that any positive solution $n(t)$ of the system (2.3) satisfies $\kappa \le n(t) \le K \; \forall \; t \ge \breve{t}$.

Proof. Assume that $n(t)$ is any positive solution of the system (2.3) with $n(0) > 0$. From system (2.3), we obtain

$$\begin{cases} n'(t) \le n(t)(p(t) - q(t)n(t)), & t \ne \tau_k, \\ n(\tau_k^+) = (1+\gamma_k)n(\tau_k), & t = \tau_k, \; k \in Z_+. \end{cases} \tag{2.21}$$

Consider the comparison impulsive system (2.22) of system (2.21) with $n(0) = v(0)$.

$$\begin{cases} v'(t) = v(t)(p(t) - q(t)v(t)), & t \ne \tau_k, \\ v(\tau_k^+) = (1+\gamma_k)v(\tau_k), & t = \tau_k, \; k \in Z_+. \end{cases} \tag{2.22}$$

Because of condition (2.20), using Lemma (1.5.2), the system (2.22) has a unique positive solution $v^*(t)$ that is periodic with period $\overline{T}$. Thus for any positive solution $v(t)$ of system (2.22), $\lim_{t\to\infty} \mid v(t) - v^*(t) \mid \to 0$. Therefore, there exists sufficiently small $\grave{\epsilon} > 0$ and $\breve{t}_1$ such that

$$v(t) \le v^*(t) + \grave{\epsilon} \; \forall \; t \ge \breve{t}_1. \tag{2.23}$$

By the comparison theorem of impulsive differential equations

$$n(t) \le v(t) \le v^*(t) + \grave{\epsilon} \; \forall \; t \ge \breve{t}_1 \tag{2.24}$$

Because of condition (2.20) and Lemma 1.5.2, the comparison system of (2.3) given below in equation (2.25) has unique globally asymptotically stable periodic solution

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$\psi^*(t)$. Thus for any positive solution $\psi(t)$ of system (2.25),

$$\lim_{t\to\infty} \mid \psi(t) - \psi^*(t) \mid \to 0.$$

$$\begin{cases} \psi'(t) = \psi(t)\left(p(t) - q(t)\psi(t) - \frac{\alpha(t)}{\beta(t)}\right), & t \neq \tau_k, \\ \psi(\tau_k^+) = (1 + \gamma_k)\psi(\tau_k), & t = \tau_k, \ k \in Z_+. \end{cases} \qquad (2.25)$$

Therefore, there exists sufficiently small $\grave{\epsilon}_1 > 0$ and $\breve{t}_2$ such that

$$\psi^*(t) - \grave{\epsilon}_1 \leq u(t) \ \forall \ t \geq \breve{t}_2. \qquad (2.26)$$

By comparison theorem of impulsive differential equations, if $\psi(\breve{t}_2) = n(\breve{t}_2)$, then we have

$$\psi^*(t) - \grave{\epsilon}_1 \leq u(t) \leq n(t) \ \forall \ t \geq \breve{t}_2. \qquad (2.27)$$

Let $\breve{t} = \max\left\{\breve{t}_1, \ \breve{t}_2\right\}$, and

$$\kappa = \min_{t\in[0, \ \overline{T}]} \left\{\psi^*(t) - \grave{\epsilon}_1\right\}, \ \ K = \max_{t\in[0, \ \overline{T}]} \left\{v^*(t) + \grave{\epsilon}\right\}. \qquad (2.28)$$

From equation (2.28),

$$\kappa \leq n(t) \leq K \ \forall \ T \geq \breve{t}. \qquad (2.29)$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Theorem 2.4.1. *In addition to conditions in Lemma 2.3.4 and Lemma 2.4.1, assume that*

$$q(t) > \frac{(K^2 - \beta(t))\alpha(t)}{(\beta(t) + \kappa^2)^2}, \qquad (2.30)$$

then the system (2.3) has positive periodic solution which is globally asymptotically stable.

Proof. Let $n(t)$ be any positive solution of system (2.3). By lemma 2.4.1, there

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exists sufficiently large $\mathring{t}$ such that for all $t > \mathring{t} \geq \check{t}$,

$$\kappa \leq n(t) \leq K, \ \kappa \leq n^*(t) \leq K. \tag{2.31}$$

By using mean value theorem and Eq. (2.31), we obtain

$$\frac{1}{K} \mid n(t) - n^*(t) \mid \leq \mid \ln n(t) - \ln n^*(t) \mid \leq \frac{1}{\kappa} \mid n(t) - n^*(t) \mid . \tag{2.32}$$

Consider the Lyapunov function $\varrho(t)$ defined as $\varrho(t) = \mid \ln n(t) - \ln n^*(t) \mid$. Calculating the Dini derivative $D^+\varrho(t)$ of $\varrho(t)$ for $t > \mathring{t}$ and $t \neq \tau_k$,

$$\begin{aligned}
D^+\varrho(t) &= \mathrm{sgn}\,(n(t) - n^*(t)) \left(\frac{n'(t)}{n(t)} - \frac{n^{*'}(t)}{n^*(t)} \right) \\
&= \mathrm{sgn}\,(n(t) - n^*(t)) \left[-q(t)(n(t) - n^*(t)) \right. \\
&\quad + \left. \frac{\alpha(t)n(t)n^*(t)(n(t) - n^*(t)) - \alpha(t)\beta(t)(n(t) - n^*(t))}{(\beta(t) + (n^*(t))^2)(\beta(t) + (n(t))^2)} \right] \\
&\leq - \left[q(t) - \frac{\alpha(t)(K^2 - \beta(t))}{(\beta(t) + \kappa^2)^2} \right] \mid n(t) - n^*(t) \mid . \tag{2.33}
\end{aligned}$$

From equation (2.30), it is possible to find a constant $\omega > 0$, such that

$$q(t) > \frac{\alpha(t)(K^2 - \beta(t))}{(\beta(t) + \kappa^2)^2} + \omega \tag{2.34}$$

Combining equation (2.33) and equation (2.34), we obtain

$$D^+\varrho(t) \leq -\omega \mid n(t) - n^*(t) \mid \leq -\omega \mid \ln n(t) - \ln n^*(t) \mid \leq -\omega\kappa\varrho(t) \tag{2.35}$$

For $t = \tau_k$, we have

$$\begin{aligned}
\varrho(\tau_k^+) &= \mid \ln n(\tau_k^+) - n^*(\tau_k^+) \mid = \mid \ln(1 + \gamma_k)n^*(\tau_k) - \ln(1 + \gamma_k)n(\tau_k) \mid \\
&= \mid \ln n(\tau_k) - n^*(\tau_k) \mid = \varrho(\tau_k), \tag{2.36}
\end{aligned}$$

which means $\varrho(t)$ is continuous. Thus, using differential inequality theorem, it

follows from equations (2.35) and (2.36) that for $t > \mathring{t}$

$$\varrho(t) \leq \varrho(\tau_j) \exp(-\omega\kappa(t - \tau_j)), \tag{2.37}$$

where τ_j is the impulsive point and $\varrho(t)$ is continuous.

From equation (2.31), we have

$$\varrho(\tau_j) = \mid \ln n(\tau_k^+) - n^*(\tau_k^+) \mid \leq \mid \ln n(\tau_j) \mid + \mid \ln n^*(\tau_j) \mid \leq \max\{2 \mid \ln \kappa \mid, \ 2 \mid \ln K \mid\}, \tag{2.38}$$

which implies that $\varrho(\tau_j)$ is bounded. Thus, we get

$$\varrho(\tau_j) \exp(-\omega\kappa(t - \tau_j)) \to 0, \ \text{as } t \to \infty. \tag{2.39}$$

From equations (2.37) and (2.39), we obtain

$$\mid \ln n(t) - \ln n^*(t) \mid \to 0, \ \text{as } t \to \infty \tag{2.40}$$

which combining with equation 2.32, gives

$$\frac{1}{K} \mid n(t) - n^*(t) \mid \leq \mid \ln n(t) - \ln n^*(t) \mid \to 0, \ \text{as } t \to \infty. \tag{2.41}$$

As a result,

$$\lim_{t \to \infty} \mid n(t) - n^*(t) \mid \to 0 \tag{2.42}$$

Therefore, $n(t)$ is globally asymptotically stable. This completes the proof of Theorem 2.4.1. $\qquad\square$

2.5 Numerical Example

To substantiate the theoretical findings, consider the following system

$$
\begin{cases}
n'(t) = n(t)(0.2 + 0.1\cos\pi t - (1.4 + 0.1\sin\pi t)n(t)) - \frac{(0.11+0.01\cos\pi t)n(t)^2}{1.5+0.1(\cos\pi t)+n(t)^2}, & t \neq \tau_k, \\
n(\tau_k^+) = (1 + \gamma_k)n(\tau_k), & t = \tau_k, \\
k \in Z_+.
\end{cases}
$$

(2.43)

with $n(0) = 0.3$. Clearly $\overline{T} = 2$. Let $r = 2$, $\tau_k = k \in N$. Therefore, $\tau_{k+2} = \tau_k + 2$. If $k = 2s-1$, choose $\gamma_{2s-1} = -0.2$ and if $k = 2s$, choose $\gamma_{2s} = 0.2$, $s \in N$. It is verified that assumptions in Lemma 2.3.4 are true. By applying Theorem 2.3.1, the system 2.3 has a positive solution $n^*(t)$ that is periodic with period 2 as shown in Figure 2.1. Graphical representation of $n^*(t)$ for $t \in [40, 50]$ is shown in Figure (2.2). Also, assumptions in Lemma 2.4.1 and Theorem 2.4.1 are true for the system 2.43. By Theorem 2.4.1, the positive periodic solution $n^*(t)$ is globally asymptotically stable. As shown in Figure 2.3, there exists a positive solution $n(t)$ of the system 2.3 with initial value $n(0) = 0.2$ which approaches to the positive periodic solution $n^*(t)$ with initial value $n^*(0) = 0.3$. For $t \in [40, 50]$, $n(t)$ completely merges in $n^*(t)$ as shown in Figure 2.4.
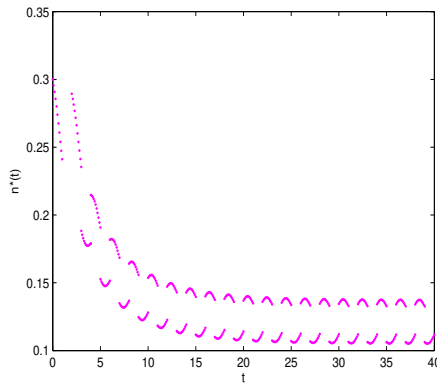
Figure 2.1: Graphical representation of solution of the system (2.43) with $n^*(0) = 0.3$, $t \in [0, 40]$.
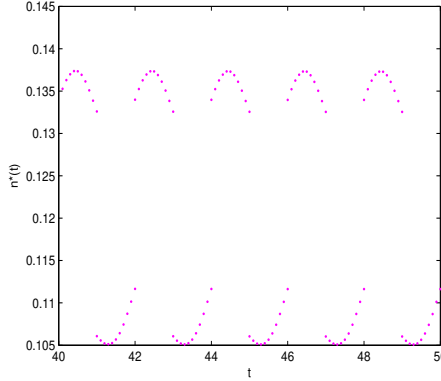
Figure 2.2: Graphical representation of solution of the system (2.43) with $n^*(0) =$ 0.3, $t \in [40, \ 50]$.

2.6   Conclusion

A single-species model under the effect of impulsive perturbations is proposed and analyzed in this paper considering Holling type III functional response as predation term. The system (2.3) represents the variation in the population density of species in a periodically changing environment and the system (2.4) reflects that how the population density varies under the effect of impulsive perturbations. Figures 2.1 and 2.2 show that a positive 2-periodic solution $n^*(t)$ exists under suitable impulsive perturbations. Figures 2.3 and 2.4 reflect the global asymptotic stability of positive solution $n^*(t)$, periodic with period 2. It is assumed that self-inhibition rate $q(t)$ must be positive. The growth rate $p(t)$ can also be negative in the weak environmental conditions. Thus, it is concluded that extinction of species can be prevented with linear impulsive perturbations. Therefore, impulsive effect is taken on a single-species living in a periodically changing environment. Mathematical modeling is done using impulsive differential equations where impulse is applied at fixed time events. By making use of comparison analysis technique, Lyapunov function and Brower's fixed point theorem, existence and global asymptotic of solution of the system (3.2) that is positive periodic is achieved.

Further, in future, results can also be established by considering impulsive
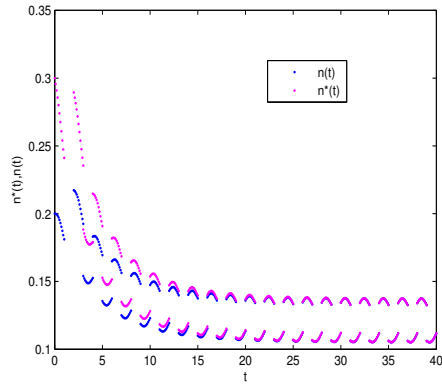
Figure 2.3: Graphical representation of the solutions $n^*(t)$ and $n(t)$ of the system (2.43) with $n^*(0) = 0.3$, $n(0) = 0.2$, $t \in [0, \ 40]$.

forms that are more general and non linear.
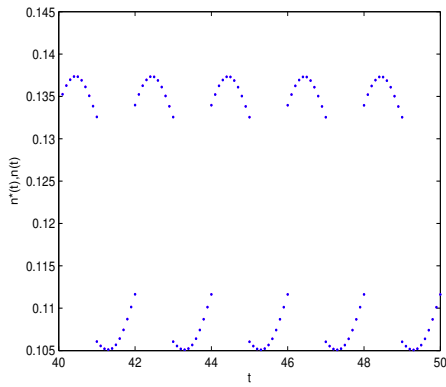
Figure 2.4: Graphical representation of the solutions $n^*(t)$ and $n(t)$ of the system (2.43) with $n^*(0) = 0.3$, $n(0) = 0.2$, $t \in [40, \ 50]$.

Chapter 3

Impact of Constant and Linear Impulsive Perturbations Considering Holling Type IV Functional Response: A Single Species Model

3.1 Introduction

Impulsive differential equations are more suitable tool than ordinary differential equations to model population ecology because evolution of species encounter different sudden short term changes in population density in this ever changing environment. There are wide range of applications of IDE in different fields of science, see [24, 92, 154, 25]. In a single species model, the periodicity and stability under the influence of linear impulsive perturbations at fixed moments of time by considering the predation term as Holling Type-II functional response using comparison theorems and Lyapnuov function is discussed by Tan et al. [29]. In continuation

of this, the effect of linear and constant impulsive perturbations on single species population density considering Holling Type-II functional response is compared by Tan et al. [30]. Liu et al. [153] analyzed the constant and linear impulses at fixed moments of time by taking single species model in which Holling Type III functional response is taken as predation term and permanence has been established using comparison theorems of impulsive differential equations. They concluded that species can be protected from extinction by linear and constant impulsive perturbations. Liu et al. [31] investigated a single species non-autonomous model with impulsive and random perturbations in which environmental noise is also taken.

In continuation of above, in this paper, impulsive perturbations at fixed moments of time in a single-species model are taken and permanence of the species is discussed with the help of comparison theorem on by considering Holling Type IV or Monod-Haldane functional response as the predation term.

3.2   Mathematical Model

Considering all these facors, the single species model we obtain is governed by the following impulsive system

$$
\begin{cases}
\Psi'(t) = \Psi(t)(\sigma(t) - \omega(t)\Psi(t)) - \frac{\mu(t)\Psi(t)}{\beta(t)+\Psi(t)\gamma(t)+\delta(t)\Psi(t)^2}, & t \neq t_n, \\
\Psi(t_n^+) = \Psi(t_n^-) + Q(\Psi(t_n)), & t = t_n,\ n \in Z_+,
\end{cases}
\tag{3.1}
$$

where the predation term $\frac{\mu(t)\Psi(t)}{\beta(t)+\Psi(t)\gamma(t)+\delta(t)\Psi(t)^2}$ is Holling type IV functional response, $\Psi(0) > 0$, and t_n represents the moments of time of impulse for every $0 < t_1 < t_2 < t_3 < \ldots < t_n$. Consider the set of positive integers Z_+. The variables $\beta(t)$, $\gamma(t)$, $\delta(t)$, and $\omega(t)$ are positive and periodic functions with period $\overline{T}$ for $t \geq 0$ as some seasonal changes can occur in environment. The species are considered to be living in weak environment, so the growth rate $\sigma(t)$ is taken as negative. $\omega(t) > 0$ is the self-inhibiting rate. $\beta(t)$, $\mu(t)$, $\gamma(t)$, and $\delta(t)$ are measures of

saturation value. $Q(\Psi(t_n))$ is impulse function that is positive and represents the amount of species increased at impulsive moment t_n. It can be linear as well as constant depending on whether the impulsive increase in the population density of species is directly proportional to the population at present or it is constant. The aim of this chapter is to establish required sufficient conditions for the permanence of system (3.1) with constant and linear impulsive perturbations. Further, it is proved that impulse has significant impact on permanence of the species and their extinction can also be prevented. Through out this paper, it is assumed that $\{r_n\}$ is the sequence of numbers with $r_n > -1$ and e is an integer such that $t_{n+e} = t_n + \overline{T}$.

3.3 Permanence of system (3.1) with $Q(\Psi(t_n)) = r$

Here, in this subsection, sufficient conditions required for the permanence of the system (3.1) with constant impulsive perturbations are established. Consider the following lemma.

Lemma 3.3.1. *(Lemma 2.3 in [28]) Consider the impulsive system*

$$\begin{cases} \Psi'(t) = \Psi(t)\sigma(t), & t \neq t_n, \\ \Psi(t_n^+) = \Psi(t_n) + p, & t = t_n, \ n \in Z_+, \end{cases} \tag{3.2}$$

where $\sigma(t)$ is continuous and periodic function with period $\overline{T}$. If $\int_0^{\overline{T}} \sigma(t)dt < 0$, then the system (3.2) has a unique periodic solution that is positive and globally asymptotically stable.

For constant impulsive perturbations, system (3.1) becomes

$$\begin{cases} \Psi'(t) = \Psi(t)(\sigma(t) - \omega(t)\Psi(t)) - \frac{\mu(t)\Psi(t)}{\beta(t) + \Psi(t)\gamma(t) + \delta(t)\Psi(t)^2}, & t \neq t_n, \\ \Psi(t_n^+) = \Psi(t_n^-) + r, & t = t_n, \ n \in Z_+. \end{cases} \tag{3.3}$$

Theorem 3.3.1. *System (3.3) is permanent.*

Proof. For $t \in (t_{n-1},\ t_n]$, from system (3.3), we obtain

$$
\begin{cases}
\Psi'(t) = \Psi(t_{n-1}) \exp \left\{ \int_{t_{n-1}}^{t_n} \left[\sigma(u) - \omega(u)\Psi(u) - \frac{\mu(u)}{\beta(u)+\Psi(u)\gamma(u)+\delta(u)\Psi(u)^2} \right] du \right\}, & t \neq t_n, \\
\Psi(t_n^+) = \Psi(t_n^-) + Q(\Psi(t_n)), & t = t_n, \\
n \in Z_+.
\end{cases}
$$

$$(3.4)$$

Because exponential function is always positive, thus, $\Psi(t) > 0$ if $\Psi(0) > 0$ for $t > 0$. Now we obtain the upper and lower bounds of any positive solution of the system (3.3). From system (3.3), we have

$$
\begin{cases}
\Psi'(t) \leq \Psi(t)\sigma(t), & t \neq t_n, \\
\Psi(t_n^+) = \Psi(t_n^-) + r, & t = t_n,\ n \in Z_+.
\end{cases}
$$

$$(3.5)$$

Consider its comparison impulsive system

$$
\begin{cases}
\nu'(t) = \nu(t)\sigma(t), & t \neq t_n, \\
\nu(t_n^+) = \nu(t_n^-) + r, & t = t_n,\ n \in Z_+.
\end{cases}
$$

$$(3.6)$$

Because species are living in weak environment, thus, $\int_0^{\overline{T}} \sigma(t)dt < 0$, by Lemma 3.3.1, system (3.6) has a unique positive solution $\hat{\nu}(t)$ which is periodic and globally asymptotically stable. This implies that for suitably small $\tilde{\epsilon} > 0$ there exists $\mathring{t}_1 > 0$ such that

$$\nu(t) < \hat{\nu}(t) + \tilde{\epsilon} \ \forall \ t \geq \mathring{t}_1, \qquad (3.7)$$

where $\nu(t)$ is any positive solution of the system (3.6) with $\nu(0) = \Psi(0)$. Using comparison analysis technique of impulsive differential equations and equation (3.7), it is obtained that

$$\Psi(t) \leq \hat{\nu}(t) + \tilde{\epsilon} \ \forall \ t \geq \mathring{t}_1. \qquad (3.8)$$

Let $H = \max_{t \in [0,\ \overline{T}]}[\hat{\nu}(t) + \tilde{\epsilon}]$. Therefore, from equation (3.8)

$$\Psi(t) \leq H \ \forall \ t \geq \mathring{t}_1. \qquad (3.9)$$

Further, to obtain lower bound of $\Psi(t)$, for $t \geq \mathring{t}_1$, from the system (3.3)

$$\begin{cases} \Psi'(t) \geq \Psi(t)\left(\sigma(t) - \omega(t)H - \frac{\mu(t)}{\beta(t)}\right) \geq \Psi(t)\left(\sigma(t) - \eta - \omega(t)H - \frac{\mu(t)}{\beta(t)}\right), & t \neq t_n, \\ \Psi(t_n^+) = \Psi(t_n^-) + r, & t = t_n, \end{cases}$$

(3.10)

Consider the following comparison impulsive differential system of (3.10)

$$\begin{cases} C'(t) = C(t)\left(\sigma(t) - \eta - \omega(t)H - \frac{\mu(t)}{\beta(t)}\right), & t \neq t_n, \\ C(t_n^+) = C(t_n^-) + r, & t = t_n, \ n \in Z_+, \end{cases}$$

(3.11)

where it is possible to find a constant $\eta > 0$ such that

$$\int_0^{\overline{T}} \left(\sigma(t) - \eta - \omega(t)H - \frac{\mu(t)}{\beta(t)}\right) dt < 0.$$

(3.12)

By Lemma 3.3.1, the system (3.11) has a unique positive periodic solution which is globally asymptotically stable. Thus, there exists $\mathring{t}_2 > \mathring{t}_1$ such that

$$C(t) \geq \hat{C}(t) - \tilde{\epsilon} \ \forall \ t \geq \mathring{t}_2.$$

(3.13)

where $C(t)$ is any positive solution of the system (3.6) with $C(\mathring{t}_2) = \Psi(\mathring{t}_2)$. Again, using comparison principle of IDE and equation (3.13), it is obtained that

$$\Psi(t) \geq \hat{C}(t) - \tilde{\epsilon} \ \forall \ t \geq \mathring{t}_2.$$

(3.14)

Let $\hat{C} = \min_{t \in [0, \ \overline{T}]}[\hat{C}(t) - \tilde{\epsilon}]$. Therefore, from equation (3.14)

$$\Psi(t) \geq \hat{C} \ \forall \ t \geq \mathring{t}_2.$$

(3.15)

From equations (3.9) and (3.15),

$$\hat{C} \leq \Psi(t) \leq H \ \forall \ t \geq \mathring{t}_2.$$

(3.16)

This completes the proof of theorem 3.3.1. $\qquad\square$

3.4 Permanence of the system (3.1) with $Q(\Psi(t_n)) = r_n\Psi(t_n)$

In this subsection, conditions required for the permanence of system (3.1) are established when impulsive increase in the population density of the species is directly proportional to population density at present. Therefore, consider the following impulsive system

$$
\begin{cases}
\Psi'(t) = \Psi(t)(\sigma(t) - \omega(t)\Psi(t)) - \frac{\mu(t)\Psi(t)}{\beta(t)+\Psi(t)\gamma(t)+\delta(t)\Psi(t)^2}, & t \neq t_n, \\
\Psi(t_n^+) = \Psi(t_n^-) + r_n\Psi(\tau_n^-), & t = t_n, \ n \in Z_+.
\end{cases}
\tag{3.17}
$$

Theorem 3.4.1. *If the inequality*

$$
\prod_{n=1}^{e}(1 + r_n)\exp\left\{\int_0^{\overline{T}}\left[\sigma(t) - \frac{\mu(t)}{\beta(t)}\right]dt\right\} > 1
\tag{3.18}
$$

holds true, then the system (3.17) is permanent.

Proof. For $t \in (t_{n-1}, \ t_n]$, from system (3.3), we obtain

$$
\begin{cases}
\Psi'(t) = \Psi(t_{n-1})\exp\left\{\int_{t_{n-1}}^{t_n}\left[\sigma(u) - \omega(u)\Psi(u) - \frac{\mu(u)}{\beta(u)+\Psi(u)\gamma(u)+\delta(u)\Psi(u)^2}\right]du\right\}, & t \neq t_n, \\
\Psi(t_n^+) = (1 + r_n)\Psi(t_n^-), & t = t_n, \\
n \in Z_+,
\end{cases}
\tag{3.19}
$$

Because, exponential function is always positive, thus, $\Psi(t) > 0$ if $\Psi(0) > 0$ for $t > 0$. Now we obtain the upper and lower bounds of any positive solution of the system (3.17). From system (3.17), we have

$$
\begin{cases}
\Psi'(t) \leq \Psi(t)(\sigma(t) - \omega(t)\Psi(t)), & t \neq t_n, \\
\Psi(t_n^+) = (1 + r_n)\Psi(t_n^-), & t = t_n, \ n \in Z_+.
\end{cases}
\tag{3.20}
$$

Consider its comparison impulsive system

$$
\begin{cases}
D'(t) = D(t)(\sigma(t) - \omega(t)D(t)), & t \neq t_n, \\
D(t_n^+) = (1 + r_n)D(t_n^-), & t = t_n, \ n \in Z_+.
\end{cases}
\tag{3.21}
$$

Now inequality (3.18) implies that $\prod_{n=1}^{e}(1 + r_n)\exp\left\{\int_0^{\overline{T}}\sigma(t)dt\right\} > 1$ is true. Therefore, by Lemma 1.5.2, the system (3.21) has a unique positive solution $D^*(t)$ which is periodic and globally asymptotically stable. Thus, $\exists \mathring{t}_3 > 0$ such that for suitably small $\tilde{\epsilon}_1$

$$
D(t) < D^*(t) + \tilde{\epsilon}_1 \ \forall \ t \geq \mathring{t}_3.
\tag{3.22}
$$

Using comparison analysis technique of IDE, we obtain

$$
\Psi(t) \leq D(t) < D^*(t) + \tilde{\epsilon}_1 \ \forall \ t \geq \mathring{t}_3.
\tag{3.23}
$$

From system (3.17)

$$
\begin{cases}
\Psi'(t) \geq \Psi(t)(\sigma(t) - \frac{\mu(t)}{\beta(t)} - \omega(t)\Psi(t)), & t \neq t_n, \\
\Psi(t_n^+) = (1 + r_n)\Psi(t_n^-), & t = t_n, \ n \in Z_+,
\end{cases}
\tag{3.24}
$$

Consider its comparison impulsive system

$$
\begin{cases}
q'(t) = q(t)(\sigma(t) - \frac{\mu(t)}{\beta(t)} - \omega(t)q(t)), & t \neq t_n, \\
q(t_n^+) = (1 + r_n)q(t_n^-), & t = t_n, \ n \in Z_+.
\end{cases}
\tag{3.25}
$$

By inequality (3.18) and Lemma 1.5.2, system (3.25) has unique positive periodic solution $q^*(t)$ which is globally asymptotically stable. Thus, for suitably small $\tilde{\epsilon}_2 > 0$, there exists $\mathring{t}_4$ such that

$$
q^*(t) - \tilde{\epsilon}_2 \leq q(t) \leq \Psi(t) \ \forall \ t \geq \mathring{t}_4.
\tag{3.26}
$$

Let

$$
\lambda^* = \min_{t \in [0, \overline{T}]} \{q^*(t) - \tilde{\epsilon}_2\}, \quad \gamma^* = \max_{t \in [0, \overline{T}]} \{D^*(t) + \tilde{\epsilon}_1\}
\tag{3.27}
$$

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Table 3.1: Expressions of the coefficients in system (3.3)

Coefficient	Expression
$\sigma(t)$	$-0.17+0.16cos(\pi t)$
$\omega(t)$	$1.5+0.1sin(\pi t)$
$\mu(t)$	$0.17+0.1cos(\pi t)$
$\beta(t)$	$1.4+cos(\pi t)$
$\gamma(t)$	$1.7+cos(\pi t)$
$\delta(t)$	$1.5+cos(\pi t)$

Therefore, it is obtained that $\lambda^* \leq \Psi(t) \leq \gamma^*$. This completes the proof of theorem 3.4.1. $\qquad\square$

3.5 Numerical Example and Discussion

A single-species model under the effect of impulse is proposed in which Holling type IV functional response is taken as predation term. It is considered that species are living in weak environment, that is, survival conditions are not favorable to growth of species. Sufficient conditions are found for the permanence of single species model under the effect of constant as well as linear impulsive disturbances using comparison analysis technique. It is observed that in case of linear perturbations, bigger value of r_n favors the permanence of the system (3.17). Bigger the value of r_n, more quickly the permanence is achieved even in weak environment. But, under constant impulsive perturbations, permanence of the species is independent of the value of r_n. To substantiate the theoretical results, the values of different parameters of systems (3.3) and (3.17) are given in Table 3.1. Also the initial condition is $\Psi(0) = 0.3$. Here, $t_n = n \in Z_+$, $\overline{T} = 2$ and $e = 2$ because $t_{n+2} = t_n + 2$.

Further, it is established that the inequality in Lemma 2.4.1 holds true for the give numerical values . Thus, permanence of the system (3.17) is achieved. Figure 3.1 depicts that in the absence of impulsive perturbations, the density of the species $\Psi(t) \to 0$, that is species will driven towards extinction. Figure (3.2)
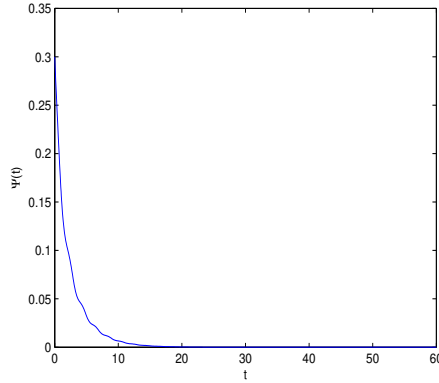
Figure 3.1: Graphical representation of population density $\Psi(t)$ with time when $\Psi(0) = 0.3$ in the absence of impulse.

graphically represents the population density $\Psi(t)$ when $r_n = 0.15$. Thus, it is clear from Figures (3.2) and (3.3) that lower value of impulsive perturbations does not favors the permanence of species. Higher the value of the impulsive perturbation, more quickly the permanence of system (3.17) is achieved (see Figure 3.4). Figure (3.5) represents the permanence of system (3.3).

Hence, it is observed that the role of impulsive perturbations is very important in the dynamics of single-species model and also to prevent the extinction of species.
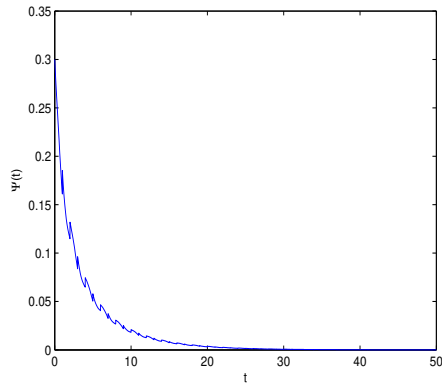
Figure 3.2: Graphical representation of population density $\Psi(t)$ of system (3.17) for $\Psi(0) = 0.3$ and $r_n = 0.15$.
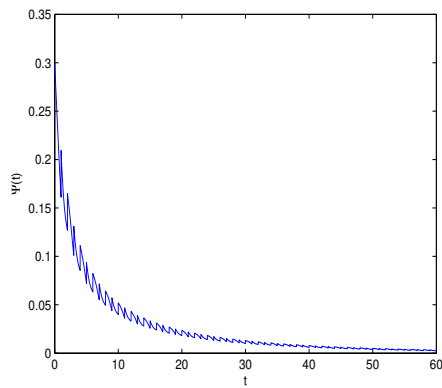
Figure 3.3: Graphical representation of population density $\Psi(t)$ of system (3.17) for $\Psi(0) = 0.3$ and $r_n = 0.3$.
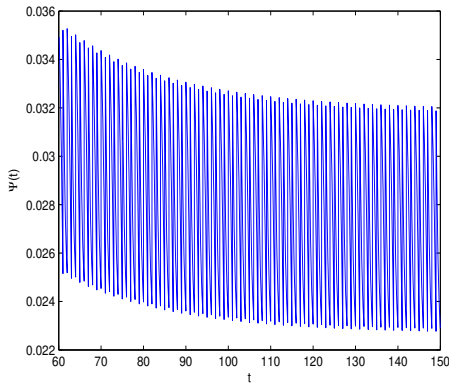
Figure 3.4: Graphical representation of population density $\Psi(t)$ of system (3.17) for $\Psi(0) = 0.3$ and $r_n = 0.4$.
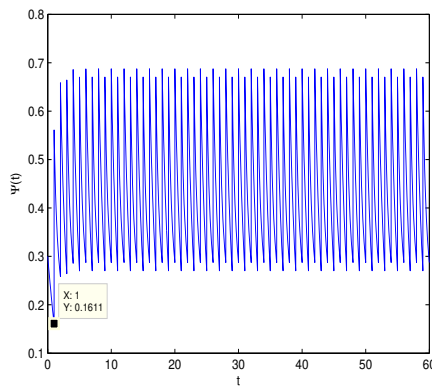
Figure 3.5: Graphical representation of population density $\Psi(t)$ of system (3.3) for $\Psi(0) = 0.3$ and $r_n = 0.4$.

Chapter 4

Stability Analysis of an Eco-Epidemiological SIN Model with Impulsive Control Strategy for Integrated Pest Management Considering Stage-Structure in Predator

4.1 Introduction

Eradication of agricultural pests is a matter of great concern over the past few decades. Many times their outbreak has resulted in production loss because of the destruction of crops and economic impoverishment due to spending on measures to avert these losses. Therefore, with the advancement in agricultural technology, farmers are acquiring the best pest control techniques. One such widely used technique is chemical control consisting of spraying pesticides. Also, biological

control using specific living organisms as natural enemies of pests is implemented on a large scale nowadays. Other techniques include physical control by killing and removing pests with hand using manpower and remote sensing. Big achievements have been made by eminent researchers in this regard. Although, McEwen [106] observed that pesticides are proved to be very effective to eradicate pests. But they are contributing a lot to environmental pollution, identified as a paramount health hazard to mankind and also harmful for certain beneficial pests such as pollinators as described in [53]. Several pest species have become resistant to pesticides due to long term use. Cherry [54] also studied that due to high cost, small scale farmers are finding it hard to use chemical pesticides.

Therefore, biological control is the best alternative. It is executed in two ways. Freedman [56] explained that the first way includes some specific natural enemies and these act as predators for the targeted pests. Second is the microbial control that involve spreading of some infectious diseases in pests using viruses. These are bacteria, fungi, nematodes and protozoa. Again, there are two ways to insert these insect pathogens in the targeted pest community. In the first method, to create an epidemic in the pest population, a marginal number of pathogens are inserted in the pest population. In the second method, pathogens are used as bio-pesticides. In this approach, the pathogen is applied when the targeted pests reach at an economically significant level and the pathogen cannot survive for a long time in the environment as explained in [55] and [105]. Therefore, these organisms are capable of creating an epidemic in the pest population by interfering with their biological process. One such example is given in [119] that entomopathogenic bacteria Bacillus Thuringiensis acts as strong microbial control agent against many species of Lepidopteran pests (Cotton Bollworm, Pink Bollworm). Moreover, Biological pest control is considered as a boon for both open crop fields and greenhouses. Lanteren and Woets [109] stated that more than fifty percent of the world's greenhouse area is covered by the Netherlands and United Kingdom. Biological control had been a great success in these countries as the parasitoid *Encassia Formosa* is widely used to control tomato pest *Trialeurodes Vaporariorum.*

Mary and Robert [108] explained that Integrated Pest Management is to

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suppress the pest population below the acceptable range which is called economic injury level (EIL) in order to avoid major economic and yield loss. Because complete eradication of pests is very expensive, so integrated pest management (IPM) is emerging as a topic of broad interest for the past few years. Many researchers are working in this area and they have provided different strategies to hinder the growth of the targeted pests using a combination of chemical and biological control. Nandi et al. [120] developed an ecological model consisting of predator-prey interaction with two-stage infection in prey for pest management using ordinary differential equations. The authors have analyzed the dynamics of the system at five different equilibria. They found that when natural enemies are absent, there was rapid increase in the pest population below some critical value of the carrying capacity.

Further, the dynamics of pest control models using biological and chemical control techniques is studied effectively with the help of impulsive differential equations as these techniques involve the instantaneous implication of viruses or natural predators of specified pests. Impulsive differential equations have a plethora of applications in modeling in ecology, population dynamics and other applied sciences described in [8, 10, 24]. These act as a good mathematical tool to represent several real-life phenomena that undergo short term perturbations, see [110, 39, 46, 29]. Liu et al. [111] have studied the dynamics of the prey-dependent consumption model with impulsive control strategy. The authors have established that the pest population can be suppressed by taking an impulsive period greater than the specified threshold value to prevent an outbreak. Similarly, valuable results in terms of threshold impulsive period and the release amount of infected pests and natural enemies are obtained in [64, 112, 113, 114, 117, 118] to check the pest population. Recently, the researchers have paid more attention in the implementation of microbial pest control by dividing the pest population into two or three parts as susceptible-infected (SI), susceptible-exposed-infected (SEI) and susceptible-exposed-infected -natural enemy (SEIN) models. Wang and Song [116] have studied an impulsive SEI model for pest management considering nonlinear incidence rate and established threshold impulsive period which was the

key parameter for the permanence of the system. Extending this work, Mathur and Dhar [121] analyzed an eco-epidemiological SEIN model considering impulsive control and have observed that predators have significant role to calculate the threshold value of the impulsive period.

Furthermore, a good biological understanding of different life stages of pests and natural enemies must be there for the effectiveness of biological pest control. Hence, Jatav and Dhar [72] considered a stage-structured plant-pest-natural enemy (food-chain) model to find the conditions for the permanence of the system. Again, Bhanu et al. [88] extended the work done in [121] by analyzing stage- structure in susceptible pest population.

In this chapter, therefore, a stage-structured predator-prey model is taken into consideration by acknowledging infection in prey for IPM. Prey acts as pest and predator plays the role of a natural enemy. Also, the functional response of the prey population to predator plays a significant role in predator-prey interactions. It can be prey dependent or predator dependent. In this paper, the functional response of susceptible pest population to the predator is taken as Holling type IV. It incorporates the situation of group defense by prey species. Thus, there is decrease in predation rate because the ability of prey species to defend themselves get enhanced in a group. In this modeling process, stage structuring in the predator population is considered as it is proposed by Ma et al. [115]. They have taken two stages of predator, immature larvae and mature adults. The death rate of immature and mature predator population is taken as same. Therefore, the following predator-prey model is proposed and examined in this chapter by taking an immature and mature class of predator and infection in prey. Let $x_s(t)$, $x_i(t)$, $y_{em}(t)$, $y_{ea}(t)$ be the population densities of susceptible prey, infected prey, immature and mature natural enemies, respectively at time t with initial conditions $x_s(0) > 0$, $x_i(0) > 0$, $y_{em}(0) > 0$ and $y_{ea}(0) > 0$. The model is formulated

under some assumptions as follows :

$$
\left\{
\begin{array}{l}
\dfrac{dx_s(t)}{dt} = \alpha x_s \left(1 - \dfrac{x_s}{\beta}\right) - \dfrac{\beta_i x_s x_i}{1 + \gamma_4 x_s} - \dfrac{\alpha_n x_s y_{ea}}{1 + \gamma_2 x_s + \gamma_3 x_s^2}, \\[3mm]
\dfrac{dx_i(t)}{dt} = \dfrac{\beta_i x_s x_i}{1 + \gamma_4 x_s} - \delta_1 x_i, \\[3mm]
\dfrac{dy_{em}(t)}{dt} = \dfrac{\gamma_1 \alpha_n x_s y_{ea}}{1 + \gamma_2 x_s + \gamma_3 x_s^2} - \mu_{em} y_{em} - \delta_2 y_{em}, \\[3mm]
\dfrac{dy_{ea}(t)}{dt} = \mu_{em} y_{em} - \delta_2 y_{ea}, \\[3mm]
\left.
\begin{array}{l}
\Delta x_s(t) = 0, \\[1mm]
\Delta x_i(t) = e_1, \\[1mm]
\Delta y_{em}(t) = e_2, \\[1mm]
\Delta y_{ea}(t) = e_3,
\end{array}
\right\} \; t = n\tau, \; n \in \; Z_+.
\end{array}
\right\} \quad t \neq n\tau,
\tag{4.1}
$$

(a) The logistic growth of prey population is taken in the absence of infection.

(b) The infected prey population is neither able to reproduce nor recover. Also they do not contribute towards carrying capacity of the total prey population.

(c) Mature predator only catch susceptible pest and immature predator is not capable of predation. So, their growth mainly depends on mature predator.

(d) There are four different kinds of Holling type functional responses are available depending on the situation. In this paper, the crowding effect of the susceptible pest population is incorporated. Therefore, Holling II type incidence rate is considered for transmission from susceptible to infected pest population.

(e) Functional response of prey population to predator is taken as Holling type IV.

(f) For the integrated pest control, infected pests, immature and mature natural enemies are released periodically at time $t = n\tau$ with intensities e_1, e_2, e_3 respectively. $\Delta x_s(t) = x_s(t^+) - x_s(t)$, $\Delta x_i(t) = x_i(t^+) - x_i(t)$, $\Delta y_{em}(t) = y_{em}(t^+) - y_{em}(t)$, $\Delta y_{ea}(t) = y_{ea}(t^+) - y_{ea}(t)$, where τ is the impulsive period.

The different parameters used in system (4.1) are defined as follows:

(i) $\alpha > 0$ is the internal growth rate of susceptible pests and $\beta > 0$ is the carrying capacity.

(ii) $\alpha_n > 0$ measures the efficiency of the prey to avoid predator's attack.

(i) $\gamma_2 > 0$, $\gamma_3 > 0$, $\gamma_4 > 0$ are the half saturation constants from Holling type IV and II functional responses.

(ii) μ_{em} is the conversion rate from immature to mature predator.

(iii) β_i is the transmission rate from susceptible to infected pest and δ_1 is the death rate of infected prey.

(iv) δ_2 is the death rate of immature and mature natural enemies.

(v) γ_1 represents the fraction of prey available to immature predator.

Let $R_+ = [0, \infty)$, $R_+^4 = \{x \in R^4 \colon x \geq 0\}$, $\Omega = int R_+^4$. The map defined by the right hand of the system (4.1) is given as $g = (g_1, g_2, g_3, g_4)^T$. Let $S_0 = \{V \colon R_+ \times R_+^4 \mapsto R_+$, continuous on $(n\tau, (n+1)\tau] \times R_+^4$ and $\lim_{(t, y) \to (n\tau, x), t > n\tau} S(t, x) = S(n\tau^+, x)$ exits$\}$.

Our main aim here is to suppress the pests in a targeted region beneath a tolerable limit so that it does not cause major production loss.

4.2 Boundedness and Global Stability

4.2.1 Upper bound of all the variables

Here, in this section, firstly, upper bound for all the variables of system (4.1) are obtained in the coming lemma.

Lemma 4.2.1. *For sufficiently large t, $\exists\; Q_0 > 0$ such that $x_s(t) \leq Q_0$, $x_i(t) \leq Q_0$, $y_{em}(t) \leq Q_0$, $y_{ea}(t) \leq Q_0$. That is there is an upper bound for every solution of (4.1).*

Proof. Consider $X(t) = (x_s(t),\; x_i(t),\; y_{em}(t),\; y_{ea}(t))$ as any solution of (4.1). Let $W(t,\; X(t)) = x_s(t) + x_i(t) + y_{em}(t) + y_{ea}(t)$ for $t \neq n\tau$,

$$
\begin{aligned}
D^+ W(t) + \theta W(t) = {}& \alpha x_n(t) - \frac{\alpha x_n(t)^2}{\beta} - \beta_i x_n(t) x_i(t) - \frac{\alpha_n x_n(t) y_{ea}(t)}{1 + \gamma_2 x_n(t)} \\
& + \beta_i x_n(t) x_i(t) - \delta_1 x_i(t) + \frac{\gamma_1 \alpha_n x_n(t) y_{ea(t)}}{1 + \gamma_2 x_n(t)} - \delta_2 y_{em}(t) \\
& + \mu_{em} y_{em}(t) + \theta(x_n(t) + x_i(t) + y_{em}(t) + y_{ea}(t)) \\
& - \mu_{em} y_{em}(t) \\
= {}& \alpha x_n(t) - \frac{\alpha x_n(t)^2}{\beta} - (\delta_2 - \theta)(y_{em}(t) + y_{ea}(t)) \\
& - (1 - \gamma_1)\frac{\alpha_n x_n(t) y_{ea}(t)}{1 + \gamma_2 x_n(t)} - (\delta_1 - \theta) x_i(t) \\
\leq {}& (\alpha + \theta) x_n(t) - \frac{\alpha x_n(t)^2}{\beta} \quad (\gamma \leq 1) \\
\leq {}& \frac{\beta(\alpha + \theta)^2}{4\alpha} = L_0
\end{aligned}
$$

Also, $W(t^+) = W(t) + e_1 + e_2 + e_3$ for $t = n\tau$. Therefore using impulsive integral

inequality from Theorem 1.5.1, we obtain

$$W(t) \leq W(0) \exp\left(\int_0^t (-\theta)\right) ds + (e_1 + e_2 + e_3) \sum_{0 < n\tau < t} \exp\left(\int_{n\tau}^t (-\theta) ds\right)$$

$$+ \int_0^t \left(L_0 \exp \int_s^t (-\theta d\sigma)\right) ds$$

$$\leq W(0) \exp(-\theta t) + (e_1 + e_2 + e_3) \sum_{0 < n\tau < t} \exp(-\theta(t - n\tau)) + \frac{L_0}{\theta}(1 - \exp(-\theta t))$$

$$\leq W(0) \exp(-\theta t) + \frac{L_0}{\theta}(1 - \exp(-\theta t)) + \frac{(e_1 + e_2 + e_3)(\exp(-\theta(t - \tau)))}{1 - \exp(-\theta\tau)}$$

$$+ \frac{(e_1 + e_2 + e_3)(\exp(\theta t))}{\exp(\theta\tau) - 1}$$

$$\to \frac{L_0}{\theta} + \frac{(e_1 + e_2 + e_3)(\exp(\theta\tau))}{\exp(\theta\tau) - 1} = Q_0 \text{ as } t \to \infty$$

Thus, W(t) is uniformly bounded. Hence, $\exists$ the constant Q_0 such that $x_s(t) \leq Q_0$, $x_i(t) \leq Q_0$, $y_{em}(t) \leq Q_0$, $y_{ea}(t) \leq Q_0$. This completes the proof. $\qquad\square$

After using microbial and natural pest control, when susceptible pest population becomes extinct, then $x_s(t) = 0$, the impulsive system (4.1) reduces to

$$\begin{cases}
\dfrac{dx_i(t)}{dt} = -\delta_1 x_i(t), \\
\dfrac{dy_{em}(t)}{dt} = -\mu_{em} y_{em}(t) - \delta_2 y_{em}(t), \\
\dfrac{dy_{ea}(t)}{dt} = \mu_{em} y_{em}(t) - \delta_2 y_{ea}(t),
\end{cases} \quad t \neq n\tau, \\
\left.\begin{aligned}
\Delta x_i(t) &= e_1, \\
\Delta y_{em}(t) &= e_2, \\
\Delta y_{yea}(t) &= e_3,
\end{aligned}\right\} t = n\tau, \ n \in \ Z_+.
\tag{4.2}$$

From first and fourth equations of system (4.2) and using Lemma 1.5.3, we get globally asymptotically stable periodic solution $\bar{x}_i(t)$ as:

$$\bar{x}_i(t) = \frac{e_1 \exp((-\delta_1)(t - n\tau))}{1 - \exp(-\delta_1\tau)}, \quad \bar{x}_i(0^+) = \frac{e_1}{1 - \exp(-\delta_1 t)}. \tag{4.3}$$

Similarly, applying Lemma 1.5.3 on second and fifth equations of system (4.2), we have

$$\bar{y}_{em}(t) = \frac{e_2 \exp(-(\mu_{em} + \delta_2)(t - n\tau))}{1 - \exp((-\mu_{em} + \delta_2)\tau)}, \quad \bar{y}_{em}(0^+) = \frac{e_2}{1 - \exp((-\mu_{em} + \delta_2)\tau)}. \quad (4.4)$$

Now substituting the value of $\bar{y}_{em}(t)$ in third equation of system (4.2), we get the following subsystem

$$\begin{cases} \frac{dy_{ea}(t)}{dt} = \mu_{em}\bar{y}_{em}(t) - \delta_2 y_{ea}(t), & t \neq n\tau, \\ \Delta y_{ea}(t) = e_3, & t = n\tau, \ n \in Z_+. \end{cases} \quad (4.5)$$

Integrating first equation of system (4.5) on $t \in (n\tau, (n+1)\tau]$,

$$\begin{aligned} y_{ea}(t) = &\frac{e_2(\exp(-\delta_2(t - n\tau))) - \exp(-(\mu_{em} + \delta_2)(t - n\tau))}{1 - \exp(-(\mu_{em} + \delta_2)\tau)} \\ &+ y_{ea}(n\tau_+)\exp(-\delta_2(t - n\tau)). \end{aligned} \quad (4.6)$$

After solution is effected by impulse at time $(n+1)\tau$, the stroboscopic map is given as :

$$\begin{aligned} y_{ea}(t) &= \frac{e_2(\exp(-\delta\tau) - \exp(-(\mu_{em} + \delta_2)\tau)}{1 - \exp(-(\mu_{em} + \delta_2)\tau)} + y_{ea}(n\tau_+)\exp(-\delta_2\tau) + e_3 \\ &= h(y_{ea}(n\tau_+)), \ n\tau < t \leq (n+1)\tau. \end{aligned} \quad (4.7)$$

The system (4.7) has a unique fixed point y_{ea}^*. Therefore,

$$h(y_{ea}^*) = y_{ea}^*$$

This implies

$$y_{ea}^* = \frac{e_2(1 - \exp(-\mu_{em}\tau))\exp(-\delta_2\tau)}{(1 - \exp-(\delta_2 + \mu_{em})\tau)(1 - \exp(-\delta_2\tau))} + \frac{e_3}{1 - \exp(-\delta_2\tau)}.$$

As from equation (4.7), $h(y_{ea})$ is an increasing function, therefore $0 < y_{ea} < y_{ea}^*$ implies $y_{ea} < h(y_{ea}) < y_{ea}^*$ and $y_{ea} > y_{ea}^*$ implies $y_{ea}^* < h(y_{ea}) < y_{ea}$. Thus by [107],

y_{ea}^* is globally stable. Hence, the corresponding periodic solution of system (4.5) is

$$\bar{y}_{ea}(t) = \frac{-e_2 \exp((-\delta_2 + \mu_{em})(t - n\tau))}{1 - \exp(-(\delta_2 + \mu_{em})\tau)} + \frac{(e_2 + e_3) \exp(-\delta_2(t - n\tau))}{1 - \exp(-\delta_2\tau)}, \text{ where}$$

(4.8)

$$\bar{y}_{ea}(0^+) = y_{ea}^* = \frac{-e_2}{1 - \exp(-\delta_2 + \mu_{em})\tau} + \frac{e_2 + e_3}{1 - \exp(-\delta_2\tau)}, \quad t \in (n\tau, (n+1)\tau],$$

(4.9)

which is globally asymptotically stable.

Theorem 4.2.1. *There exists a threshold value (τ_{max}) of the impulsive period such that if $\tau \le \tau_{max}$, then the susceptible pest eradication solution $(0, \bar{x}_i(t), \bar{y}_{em}(t), \bar{y}_{ea}(t))$ is locally asymptotically stable and if $\tau > \tau_{max}$, it is unstable where*

$$\tau_{max} = \frac{1}{\alpha} \left[\frac{\beta_i e_1}{\delta_1} - \frac{\alpha_n e_2}{\delta_2 + \mu_{em}} + \frac{\alpha_n(e_2 + e_3)}{\delta_2} \right].$$

Proof. Here, we use small perturbation method to prove the local stability of the required solution. Let $(\zeta_1(t), \zeta_2(t), \zeta_3(t), \zeta_4(t))$ be the small perturbations in the periodic solution $(0, \bar{x}_i(t), \bar{y}_{em}(t), \bar{y}_{ea}(t))$ respectively.

$$\begin{cases} \dfrac{d\zeta_1(t)}{dt} = \alpha\zeta_1(t) - \beta_i\bar{x}_i(t)\zeta_1(t) - \alpha_n\bar{y}_{ea}(t)\zeta_1(t), \\[2mm] \dfrac{d\zeta_2(t)}{dt} = \beta_i\bar{x}_i(t)\zeta_1(t) - \delta_1\zeta_2(t), \\[2mm] \dfrac{d\zeta_3(t)}{dt} = \gamma_1\alpha_n\bar{y}_{ea}(t)\zeta_1(t) - \mu_{em}\zeta_3(t) - \delta_2\zeta_3(t), \\[2mm] \dfrac{d\zeta_4(t)}{dt} = \mu_{em}\zeta_3(t) - \delta_2\zeta_4(t), \\[2mm] \left.\begin{array}{l} \zeta_1(t^+) = \zeta_1(t), \\ \zeta_2(t^+) = \zeta_2(t), \\ \zeta_3(t^+) = \zeta_3(t), \\ \zeta_4(t^+) = \zeta_4(t), \end{array}\right\} t = n\tau, \ n \in \ Z_+. \end{cases} \quad t \ne n\tau,$$

(4.10)

Then, we obtain the system (4.10) by substituting

$$x_s(t) = \zeta_1(t), \quad x_i(t) = \bar{x}_i + \zeta_2(t), \quad y_{em}(t) = \bar{y}_{em}(t) + \zeta_3(t), \quad y_{ea}(t) = \bar{y}_{ea}(t) + \zeta_4(t).$$

The above system represents system of linear differential equations, which can be written in matrix form. Hence, for $t \neq n\tau$, the coefficient matrix is given as

$$B = \begin{bmatrix} \alpha - \beta_i \bar{x}_i(t) - \alpha_n \bar{y}_{ea}(t) & 0 & 0 & 0 \\ \beta_i \bar{x}_i(t) & -\delta_1 & 0 & 0 \\ \gamma_1 \alpha_n \bar{y}_{ea}(t) & 0 & -(\mu_{em} + \delta_2) & 0 \\ 0 & 0 & \mu_{em} & -\delta_2 \end{bmatrix},$$

and for $t = n\tau$,

$$\begin{bmatrix} \zeta_1(t^+) \\ \zeta_2(t^+) \\ \zeta_3(t^+) \\ \zeta_4(t^+) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \zeta_3(t) \\ \zeta_4(t) \end{bmatrix}.$$

Let $\Phi(t)$ be the fundamental solution of system (4.10), then

$$\frac{d\Phi(t)}{dt} = B\Phi(t), \tag{4.11}$$

$$\Phi(\tau) = \Phi(0) \exp\left(\int_0^T B \, dt\right),$$

with $\Phi(0) = I$, the identity matrix. Solving, we have

$$\Phi(\tau) = \begin{bmatrix} e^{(\int_0^\tau \alpha - \beta_i \bar{x}_i(t) - \alpha_n \bar{y}_{ea}(t) dt)} & 0 & 0 & 0 \\ e^{(\int_0^\tau \beta_i \bar{x}_i(t) dt)} & e^{(\int_0^\tau -\delta_1 dt)} & 0 & 0 \\ e^{(\int_0^\tau \gamma_1 \alpha_n \bar{y}_{ea}(t))} & 0 & e^{(\int_0^\tau -(\mu_{em} + \delta_2) dt)} & 0 \\ 0 & 0 & e^{(\int_0^\tau \mu_{em} dt)} & e^{(\int_0^\tau -\delta_2 dt)} \end{bmatrix},$$

which is upper triangular matrix. Now according to Floquet theory of impulsive differential equations (Remark 1.5.8.1 and Theorem 1.5.5), if absolute values of all the eigen values of Monodromy matrix M are less than one, then the required

solution is globally stable where

$$M = [\Phi(0)]^{-1}\Phi(\tau).$$

Since M is an upper triangular matrix, therefore, eigen values of M are

$$\lambda_1 = exp\left(\int_0^\tau \alpha - \beta_i \bar{x}_i(t) - \alpha_n \bar{y}_{ea}(t)dt\right),$$

$$\lambda_2 = \exp\left(\int_0^\tau -\delta_1 dt\right),$$

$$\lambda_3 = \exp\left(\int_0^\tau -(\mu_{em} + \delta_2)dt\right),$$

$$\lambda_4 = \exp\left(\int_0^\tau -\delta_2 dt\right). \tag{4.12}$$

Now, it is obvious from (4.12), that $\mid \lambda_2 \mid < 1$, $\mid \lambda_3 \mid < 1$, $\mid \lambda_4 \mid < 1$ and $\mid \lambda_1 \mid < 1$ if $\tau \leq \tau_{max}$. This completes the proof. $\qquad\square$

4.2.2 Global Stability

Theorem 4.2.2. *There is a threshold value $(\check{\tau})$ of the impulsive period such that if $\tau < \check{\tau}$, then the susceptible pest extinction solution $(0, \bar{x}_i(t), \bar{y}_{em}(t), \bar{y}_{ea}(t))$ is globally asymptotically stable where,*

$$\check{\tau} = \frac{1}{\alpha}\left[\frac{\beta_i e_1}{\delta_1(1 + \gamma_4\beta)} - \left(\frac{1}{1 + \gamma_2\beta + \gamma_3\beta^2}\right)\left(\frac{\alpha_n e_2}{\delta_2 + \mu_{em}} + \frac{\alpha_n(e_2 + e_3)}{\delta_2}\right)\right].$$

Proof. Let $(x_s(t), x_i(t), y_{em}(t), y_{ea}(t))$ be an arbitrary solution of (4.1). Given that $\tau < \check{\tau}$, so, it is possible to find sufficiently small $\check{\varepsilon} > 0$ such that

$$\int_0^\tau \left(\alpha - \frac{\beta_i(\bar{x}_i(t) - \check{\varepsilon})}{1 + \gamma_4\beta} - \frac{\alpha_n(\bar{y}_{ea}(t) - \check{\varepsilon})}{1 + \gamma_2\beta + \gamma_3\beta^2}\right)dt = \varrho_1 < 0. \tag{4.13}$$

From (4.1),

$$\begin{cases} \frac{dx_i(t)}{dt} \geq -\delta_1 x_i(t), & t \neq n\tau \\ \Delta x_i(t) = e_1, & t = n\tau, \ n \in Z_+. \end{cases} \tag{4.14}$$

Consider the corresponding comparison impulsive system of (4.14) as

$$\begin{cases} \frac{dw_i(t)}{dt} = (-\delta_1 w_i), & t \neq n\tau \\ \Delta w_i(t) = e_1, & t = n\tau. \end{cases} \tag{4.15}$$

Applying Lemma 1.5.3, system (4.15) has periodic solution

$$\bar{w}_i(t) = \frac{e_1 \exp((-\delta_1)(t - n\tau))}{1 - \exp(-\delta_1 \tau)}, \ t \in (n\tau, (n+1)\tau],$$

which is globally asymptotically stable. Therefore, using comparison principle (Theorem 1.5.2), $x_i(t) \geq w_i(t) \rightarrow \bar{w}_i(t)$ and $\bar{w}_i(t) = \bar{x}_i(t)$. Hence, $\exists$ a positive integer κ such that $x_i(t) > \bar{x}_i(t) - \dot{\varepsilon} \ \forall \ t \geq \kappa\tau$. Again from (4.1)

$$\begin{cases} \frac{dy_{em}(t)}{dt} \geq -(\mu_{em} + \delta_2)y_{em}(t), & t \neq n\tau, \\ \Delta y_{em}(t) = e_2, & t = n\tau. \end{cases} \tag{4.16}$$

Consider the following comparison system of (4.16)

$$\begin{cases} \frac{dw_{em}(t)}{dt} = -(\mu_{em} + \delta_2)w_{em}(t), & t \neq n\tau, \\ \Delta w_{em}(t) = e_2, & t = n\tau. \end{cases} \tag{4.17}$$

Similarly, by using Lemma 1.5.3 and comparison analysis technique of impulsive differential equations, it is obtained that $y_{em}(t) \geq w_{em}(t) \rightarrow \bar{w}_{em}(t)$ and $\bar{w}_{em}(t) = \bar{y}_{em}(t)$. Hence, $\exists$ a positive integer κ_1 such that $y_{em}(t) > \bar{y}_{em}(t) - \dot{\varepsilon} \ \forall \ t \geq \kappa_1\tau$. From fourth equation of system (4.1), we have,

$$\begin{cases} \frac{dy_{ea}(t)}{dt} \geq \mu_{em}(\bar{y}_{em}(t) - \dot{\varepsilon}) - \delta_2)y_{ea}(t), & t \neq n\tau, \\ \Delta y_{ea}(t) = e_3, & t = n\tau. \end{cases} \tag{4.18}$$

Consider its corresponding comparison impulsive system

$$
\begin{cases}
\frac{dw_{ea}(t)}{dt} = \mu_{em}(\bar{y}_{em}(t) - \grave{\varepsilon}) - \delta_2)w_{ea}(t), & t \neq n\tau, \\
\Delta w_{ea}(t) = e_3, & t = n\tau.
\end{cases}
\tag{4.19}
$$

Applying Lemma 1.5.3, we get periodic solution of system (4.19)

$$
\bar{w}_{ea}(t) = \frac{-e_2 \exp((-\delta_2 + \mu_{em})(t - n\tau))}{1 - \exp(-(\delta_2 + \mu_{em})\tau)} + \frac{(e_2 + e_3)\exp(-\delta_2(t - n\tau))}{1 - \exp(-\delta_2\tau)} - \frac{\mu_{em}\grave{\varepsilon}}{\delta_2}.
$$

Applying comparison theorem of IDE, $y_{ea}(t) \geq w_{ea}(t) \rightarrow \bar{w}_{ea}(t)$. Hence, $\exists$ a positive integer κ_2 such that $y_{ea}(t) > \bar{y}_{ea}(t) - \grave{\varepsilon} \; \forall \; t \geq \kappa_2\tau \; (\kappa_2 > \kappa_1 > \kappa)$. Therefore, for $t \geq \kappa_2\tau$, first equation of (4.1) gives

$$
\frac{dx_s(t)}{dt} \leq \left[\alpha - \frac{\beta_i(x_i - \grave{\varepsilon})}{1 + \gamma_4\beta} - \frac{\alpha_n(y_{ea} - \grave{\varepsilon})}{1 + \gamma_2\beta + \gamma_3\beta^2}\right] x_s.
\tag{4.20}
$$

Integration of equation (4.20) on $(\kappa_2\tau, (\kappa_2 + 1)\tau]$ gives

$$
x_s((\kappa_2 + q)\tau) \leq x_s(\kappa_2\tau)\exp(q\varrho_1) \rightarrow 0 \text{ as } t \rightarrow \infty \; (\because \varrho_1 < 0).
\tag{4.21}
$$

This implies that there exists a positive integer $\kappa_3 > \kappa_2$ and sufficiently small $\grave{\varepsilon}_1 > 0$ such that $x_s(t) < \grave{\varepsilon}_1$ for $t \geq \kappa_3$ and $\grave{\varepsilon}_1 < \frac{\delta_1}{\beta_i}$. Using maximum value of $x_s(t)$ in the second equation of system (4.1), we get

$$
\begin{cases}
\frac{dx_i(t)}{dt} \leq (\beta_i\grave{\varepsilon}_1 - \delta_1)x_i, & t \neq n\tau, \\
\Delta x_i(t) = e_1, & t = n\tau.
\end{cases}
$$

Analyzing again its comparison impulsive system

$$
\begin{cases}
\frac{du_i(t)}{dt} = (\beta_i\grave{\varepsilon}_1 - \delta_1)x_i, & t \neq n\tau, \\
\Delta u_i(t) = e_1, & t = n\tau.
\end{cases}
\tag{4.22}
$$

Applying Lemma 1.5.3, system (4.22) has periodic solution $\bar{u}_i(t)$, which is globally

asymptotically stable.

$$\bar{u}_i(t) = \frac{e_1 \exp((\beta_i \hat{\varepsilon}_1 - \delta_1)(t - n\tau))}{1 - \exp(\beta_i \hat{\varepsilon}_1 - \delta_1 \tau)}, \ t \in (n\tau, \ (n+1)\tau].$$

Therefore, by Theorem 1.5.2, $x_1(t) \leq u_i(t) \to \bar{u}_i(t)$. Hence, $\exists$ a positive integer κ_4 such that

$$x_i(t) < \bar{u}_i(t) + \grave{\varepsilon} \ \forall \ t \geq \kappa_4 \tau. \tag{4.23}$$

From third equation of system (4.1)

$$\begin{cases} \frac{dy_{em}(t)}{dt} \leq (\gamma_1 \alpha_n \grave{\varepsilon}_1 Q_0) - (\mu_{em} + \delta_2)y_{em}(t), & t \neq n\tau, \\ \Delta y_{em}(t) = e_2, & t = n\tau. \end{cases}$$

Applying the same argument, $\exists$ a positive integer κ_5 such that

$$y_{em}(t) \leq \bar{u}_{em}(t) + \grave{\varepsilon} \ \forall \ t \geq \kappa_5 \tau, \tag{4.24}$$

where

$$\bar{u}_{em}(t) = \frac{\gamma_1 \alpha_n \grave{\varepsilon}_1 Q_0}{\mu_{em} + \delta_2} + \frac{e_2 exp(-(\mu_{em} + \delta_2)(t - n\tau))}{1 - exp(-(\mu_{em} + \delta_2)\tau)}.$$

From fourth equation of system (4.1), we have

$$\begin{cases} \frac{dy_{ea}(t)}{dt} \leq \mu_{em}(\bar{u}_{em}(t) + \grave{\varepsilon}) - \delta_2)y_{ea}(t), & t \neq n\tau, \\ \Delta y_{ea}(t) = e_3, & t = n\tau. \end{cases} \tag{4.25}$$

Similarly, as above $\exists$ a positive integer κ_6 such that

$$y_{ea}(t) \leq \bar{u}_{ea}(t) + \grave{\varepsilon}_2 \forall \ t \geq \kappa_6 \tau, \tag{4.26}$$

where

$$\begin{aligned} \bar{u}_{ea}(t) = {} & \frac{-e_2 \exp((-\delta_2 + \mu_{em})(t - n\tau))}{1 - \exp(-(\delta_2 + \mu_{em})\tau)} + \frac{(e_2 + e_3) \exp(-\delta_2(t - n\tau))}{1 - \exp(-\delta_2 \tau)} \\ & + \frac{\mu_{em}}{\delta_2} \left(\frac{\gamma_1 \alpha_n \grave{\varepsilon}_1 Q_0}{\mu_{em} + \delta_2} + \grave{\varepsilon} \right). \end{aligned}$$

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As $\dot{\varepsilon} > 0, \dot{\varepsilon}_1 > 0$ and $\dot{\varepsilon}_2 > 0$ are sufficiently small, therefore, $\bar{u}_{em}(t) \to \bar{y}_{em}(t)$ and $\bar{u}_{ea}(t) \to \bar{y}_{ea}(t)$ as $t \to \infty$ ($\dot{\varepsilon}_1 \to 0$). Hence, it is established that $x_s(t) \to 0$, $x_i(t) \to \bar{x}_i(t)$, $y_{em}(t) \to \bar{y}_{em}(t)$, and $y_{ea} \to \bar{y}_{ea}(t)$ as $t \to \infty$. This completes the proof. $\qquad\square$

4.3 Permanence

The required condition for the system to be permanent is established as follows

Theorem 4.3.1. *The system (4.1) is permanent if $\tau > \tau_{max}$.*

Proof. Upper bound of $x_s(t)$, $x_i(t)$, $y_{em}(t)$, $y_{ea}(t)$ of the system is already been obtained in Lemma 4.2.1. Also in the above section, it is proved that

$$x_i(t) > \bar{x}_i(t) - \dot{\varepsilon} = q_1 \ \forall \ t \geq \kappa_4\tau,$$
$$y_{em}(t) > \bar{y}_{em}(t) - \dot{\varepsilon} = q_2 \ \forall \ t \geq \kappa_5\tau,$$
$$y_{ea}(t) > \bar{y}_{ea}(t) - \dot{\varepsilon} = q_3 \ \forall \ t \geq \kappa_6\tau. \tag{4.27}$$

Thus, for the permanence of the system (4.1), there must exists a constant $q_4 < min\left(\beta, \frac{\delta_1}{\beta_i}\right)$ such that $x_s(t) \geq q_4$ for sufficiently large t. This is done in two steps as follows:

Step I To start with, assume that $x_s(t) \geq q_4$ is not true $\forall \ t$. Thus $\exists$ a positive integer l_1 such that $x_s < q_4 \ \forall \ t \geq l_1\tau$. Considering this assumption, from system (4.1), we have

$$\begin{cases} \frac{dx_i(t)}{dt} \leq -(\delta_1 - \beta_i q_4), & t \neq n\tau, \\ \Delta x_i(t) = e_1, & t = n\tau. \end{cases}$$

Consider the following impulsive system

$$\begin{cases} \frac{d\breve{u}_i(t)}{dt} = -(\delta_1 - \beta_i q_4)\breve{u}_i, & t \neq n\tau, \\ \Delta \breve{u}_i(t) = e_1, & t = n\tau. \end{cases} \tag{4.28}$$

Applying Lemma 1.5.3, system (4.28) has periodic solution

$$\bar{\bar{u}}_i(t) = \frac{e_1 \exp(-(\delta_1 - \beta_i q_4)(t - n\tau))}{1 - \exp(-(\delta_1 - \beta_i q_4)\tau)}, \ t \in (n\tau, (n+1)\tau]$$

which is globally asymptotically stable. Therefore by comparison principle, $x_i(t) \leq u_i(t) \rightarrow \breve{u}_i(t)$. Hence, $\exists$ a positive integer l_2 such that

$$x_i(t) \leq \bar{\bar{u}}_i(t) + \dot{\varepsilon}_3 \ \forall \ t \geq l_2\tau. \tag{4.29}$$

From the third equation of the system (4.1)

$$\begin{cases} \frac{dy_{em}(t)}{dt} \leq (\gamma_1 \alpha_n q_4 Q_0) - (\mu_{em} + \delta_2) y_{em}(t), & t \neq n\tau, \\ \Delta y_{em}(t) = e_2, & t = n\tau. \end{cases}$$

Now, consider the following impulsive system

$$\begin{cases} \frac{dv_{em}(t)}{dt} = (\gamma_1 \alpha_n q_4 Q_0) - (\mu_{em} + \delta_2) y_{em}(t), & t \neq n\tau, \\ \Delta v_{em}(t) = e_2, & t = n\tau. \end{cases} \tag{4.30}$$

Applying the same argument, $\exists$ a positive integer l_2 such that

$$y_{em}(t) \leq \bar{v}_{em}(t) + \dot{\varepsilon}_3 \ \forall \ t \geq \ l_2\tau, \tag{4.31}$$

where

$$\begin{aligned} \bar{v}_{em}(t) &= \frac{\gamma_1 \alpha_n q_4 Q_0}{\mu_{em} + \delta_2} + \frac{e_2 exp(-(\mu_{em} + \delta_2)(t - n\tau))}{1 - exp(-(\mu_{em} + \delta_2)\tau)}, \\ \bar{v}_{em}(0^+) &= \frac{\gamma_1 \alpha_n q_4 Q_0}{\mu_{em} + \delta_2} + \frac{e_2}{1 - exp(-(\mu_{em} + \delta_2)\tau)}. \end{aligned} \tag{4.32}$$

Now, from fourth equation of system (4.1), we have

$$\begin{cases} \frac{dy_{ea}(t)}{dt} \leq \mu_{em}(\bar{v}_{em}(t) + \dot{\varepsilon}_3) - \delta_2) y_{ea}(t), & t \neq n\tau, \\ \Delta y_{ea}(t) = e_3, & t = n\tau. \end{cases}$$

Considering again its comparison system as below

$$\begin{cases} \frac{dv_{ea}(t)}{dt} = \mu_{em}(\bar{v}_{em}(t) + \dot{\varepsilon}_3) - \delta_2)v_{ea}(t), & t \neq n\tau, \\ \Delta v_{ea}(t) = e_3, & t = n\tau. \end{cases} \tag{4.33}$$

Applying Lemma 1.5.3 and comparison theorem, $\exists$ a positive integer l_3 such that

$$y_{ea}(t) \leq \bar{v}_{ea}(t) + \dot{\varepsilon}_3 \forall \, t \geq l_3\tau, \tag{4.34}$$

where

$$\bar{v}_{ea}(t) = \frac{-e_2 \exp(-(\delta_2 + \mu_{em})(t - n\tau))}{1 - \exp(-(\delta_2 + \mu_{em})\tau)} + \frac{(e_2 + e_3)\exp(-\delta_2(t - n\tau))}{1 - \exp(-\delta_2\tau)}$$
$$+ \frac{\mu_{em}}{\delta_2}\left(\frac{\gamma_1 \alpha_n \dot{q}_4 Q_0}{\mu_{em} + \delta_2} + \dot{\varepsilon}_3\right). \tag{4.35}$$

Therefore, for $t \geq l_2\tau$, first equation of system (4.1) gives

$$\frac{dx_s(t)}{dt} \geq \left[\alpha - \frac{\alpha q_4}{\beta}\beta_i(\bar{\bar{u}} - \dot{\varepsilon}_3) - \alpha_n(\bar{v}_{ea} + \dot{\varepsilon}_3)\right]x_s(t)$$

Integration of above equation on $(l_2\tau, (l_2 + 1)\tau]$ gives

$$x_s((l_2 + 1)\tau) \geq x_s(l_2\tau)\exp\left(\int_{l_2\tau}^{(l_2+1)\tau}\left[\alpha - \frac{\alpha q_4}{\beta} - \beta_i(\bar{\bar{u}} + \dot{\varepsilon}_3) - \alpha_n(\bar{v}_{ea} + \dot{\varepsilon}_3)\right]dt\right)$$
$$\geq x_s(l_2\tau)\exp(\varrho_2).$$

where

$$\varrho_2 = \int_{l_2\tau}^{(l_2+1)\tau}\left[\alpha - \frac{\alpha q_4}{\beta} - \beta_i(\bar{\bar{u}} + \dot{\varepsilon}_3) - \alpha_n(\bar{v}_{ea} + \dot{\varepsilon}_3)\right]dt.$$

Because $\tau > \tau_{max}$, so it is possible to find $q_4 > 0$ and $\dot{\varepsilon}_3 > 0$ such that $\varrho_2 > 0$. This implies

$$x_s[(l_2 + l)\tau] \geq x_s(l_2\tau)\exp(l\varrho_2) \to \infty$$

as $l \to \infty$. This is in contradiction to our assumption that $x_s(t) < q_4 \, \forall \, t \geq l_1\tau, \, (l_2 > l_1)$. Hence $\exists \, \mathring{t} > l_1\tau$ such that $x_s(\mathring{t}) \geq q_4$.

Step II There is nothing to prove if $x_s(t) \geq q_4 \ \forall \ t > \mathring{t}$. But if this is not true, let $\mathring{t}_1 = \inf\{t \mid x_s(t) < q_4 \ ; \ t > \mathring{t}\}$. Thus, $x_s(t) \geq q_4 \ \forall \ t \in [\mathring{t}, \ \mathring{t}_1], \ \mathring{t}_1 \in (\check{n}_1\tau, \ (n_1 + 1)\tau]$. $x_s(\mathring{t}_1) = q_4$, because of continuity of $x_s(t)$. Let $\tau^* = (\check{n}_2 + \check{n}_3)\tau$ where $\check{n}_2 = \check{n}_{21} + \check{n}_{22} + \check{n}_{23}$ and $\check{n}_{21}, \ \check{n}_{22}, \ \check{n}_{23}, \ \check{n}_3$ satisfying the following conditions

$$
\begin{aligned}
(\check{n}_{21})\tau &> -\left(\frac{1}{\delta_1 - \beta_i q_4}\right) \ln \frac{\grave{\varepsilon}_3}{Q_0 + e_1}, \\
(\check{n}_{22})\tau &> -\left(\frac{1}{\mu_{em} + \delta_2}\right) \ln \frac{\grave{\varepsilon}_3}{Q_0 + e_2}, \\
(\check{n}_{23})\tau &> -\left(\frac{1}{\mu_{em} + \delta_2}\right) \ln \frac{\grave{\varepsilon}_3}{Q_0 + e_3}, \\
\exp(\check{n}_3 \varrho_2 - \upsilon(\check{n}_2 + 1)\tau) &> 1, \ \upsilon = \left(\frac{\alpha q_4}{\beta} + \beta_i Q_0 + \alpha_n Q_0\right).
\end{aligned}
\tag{4.36}
$$

Now, we will prove that $\exists \ \mathring{t}_2 \in ((\check{n}_1 + 1)\tau, \ (\check{n}_1 + 1)\tau + \tau^*]$ such that $x_s(\mathring{t}_2) \geq q_4$. Suppose this is not true, then $x_s(t) < q_4 \ \forall \ t \in ((\check{n}_1 + 1)\tau, \ (\check{n}_1 + 1)\tau + \tau^*]$. If system (4.28) is considered with $\breve{u}_i((\check{n}_1 + 1)\tau^+) = x_i((\check{n}_1 + 1)\tau^+)$, then using Lemma 1.5.3 for $t \in ((\check{n}_1 + 1)\tau, \ (n_1 + 1)\tau + \tau^*]$, we have

$$
\breve{u}_i(t) = \left[\breve{u}_i(\check{n}_1 + 1)\tau^+) - \frac{e_1}{1 - \exp(-(\delta_1 - \beta_i q_4)\tau)}\right] \exp(-(\delta_1 - \beta_i q_4)(t - (n + 1)\tau))
$$
$$
+ \overline{\overline{u}}_i(t)
$$

This implies

$$
\mid \breve{u}_i(t) - \overline{\overline{u}}_i(t) \mid \leq (Q_0 + e_1)\exp(-(\delta_1 - \beta_i q_4)(t - n\tau))
$$
$$
\leq \grave{\varepsilon}_3.
$$

which depicts that

$$
x_i(t) \leq \breve{u}_i(t) < \overline{\overline{u}}_i(t) + \grave{\varepsilon}_3, \ (\check{n}_1 + \check{n}_{21} + 1)\tau \leq t \leq (\check{n}_1 + 1)\tau + \tau^*.
$$

Now consider the system (4.30) with $v_{em}((\check{n}_1 + \check{n}_{21} + 1)\tau^+) = y_{em}((\check{n}_1 + \check{n}_{21} + 1)\tau^+)$,

then using Lemma 1.5.3, we have,

$$v_{em}(t) = \left[v_{em}((\check{n}_1 + \check{n}_{21} + 1)\tau^+) - \overline{v}_{em}(0^+)\right]\exp(-(\mu_{em}+\delta_2)(t-(\check{n}_1+\check{n}_{21}+1)\tau))\overline{v}_{em}(t).$$

Hence,

$$| v_{em}(t) - \overline{v}_{em}(t) | \le (Q_0 + e_2)\exp(-(\mu_{em} + \delta_2)(t - (\check{n}_1 + \check{n}_{21} + 1)\tau))$$
$$\le \grave{\varepsilon}_3 \forall (\check{n}_1 + \check{n}_{21} + \check{n}_{22} + 1) \le t \le (\check{n}_1 + 1)\tau + \tau^*$$

which concludes that $y_{em}(t) \le \overline{v}_{em}(t) + \grave{\varepsilon}_3$. Finally, consider (4.33) with $v_{ea}((\check{n}_1 + \check{n}_{21} + \check{n}_{22} + 1)\tau^+) = y_{ea}((\check{n}_1 + \check{n}_{21} + \check{n}_{22} + 1)\tau^+) \ge 0$, then using Lemma 1.5.2, we have,

$$v_{ea}(t) = (v_{ea}(\check{n}_1 + \check{n}_{21} + \check{n}_{22} + 1)\tau^+ - \overline{v}_{ea}(0^+))$$
$$\times \exp(-(\mu_{em} + \delta_2)(t - (\check{n}_1 + \check{n}_{21} + \check{n}_{22} + 1)\tau) + \overline{v}_{ea}(t)$$

This implies

$$| v_{ea}(t) - \overline{v}_{ea}(t) | \le (Q_0 + e_3)\exp(-(\mu_{em} + \delta_2)(t - (\check{n}_1 + \check{n}_{21} + \check{n}_{22} + 1)\tau))$$
$$\le \grave{\varepsilon}_3 \forall (\check{n}_1 + \check{n}_2 + 1)\tau \le t \le (\check{n}_1 + 1)\tau + \tau^*$$

Therefore,

$$y_{ea}(t) \le \overline{v}_{ea}(t) + \grave{\varepsilon}_3.$$

Hence, from first equation of system (4.1), we obtain

$$\frac{dx_s(t)}{dt} \ge \left[\alpha - \frac{\alpha q_4}{\beta}\beta_i(\overline{\overline{u}} - \grave{\varepsilon}_3) - \alpha_n(\overline{v}_{ea} + \grave{\varepsilon}_3)\right]x_s(t).$$

Integrating the above equation on $[(\check{n}_1 + \check{n}_2 + 1)\tau, (\check{n}_1 + \check{n}_2 + \check{n}_3 + 1)\tau]$, we get

$$x_s((\check{n}_1 + \check{n}_2 + \check{n}_3 + 1)\tau) \ge x_s((\check{n}_1 + \check{n}_2 + 1)\tau)\exp(\varrho_2 \check{n}_3). \tag{4.37}$$

Further, for $t \in [\mathring{t}_1, (\check{n}_1 + 1)\tau]$, two possibilities are there

Case(i) If $x_s(t) < q_4 \ \forall \ t \in [\mathring{t}_1, (\check{n}_1 + 1)\tau]$, then from above assumption $x_s(t) <$

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$q_4 \ \forall \ t \in [\mathring{t}_1, \ (\tilde{n}_1 + 1)\tau + \tau^*]$. This implies

$$\frac{dx_s(t)}{dt} \geq \left(-\frac{\alpha q_4}{\beta} - \beta_i Q_0 - \alpha_n Q_0\right) x_s(t). \tag{4.38}$$

Integrating equation (4.38) on $[\mathring{t}_1, \ (\tilde{n}_1 + \tilde{n}_2 + 1)\tau]$, we have

$$x_s((\tilde{n}_1 + \tilde{n}_2 + 1)\tau) \geq x_s(\mathring{t}_1) \exp(-\upsilon(\tilde{n}_2 + 1)\tau). \tag{4.39}$$

Using (4.39) in (4.37)

$$x_s((\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 + 1)\tau) \geq x_s(\mathring{t}_1) \exp(\varrho_2 \tilde{n}_3) \exp(-\upsilon(\tilde{n}_2 + 1)\tau) > q_4.$$

But this contradicts our assumption. Therefore, $x_s(t) \geq q_4$ in $[\mathring{t}_1, \ (\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 + 1)\tau]$ for some t. Let $\mathring{t}_3 = \inf\{t \mid x_s(t) \geq q_4; \ t > \mathring{t}_2\}$. Due to continuity of $x_s(t)$, $x_s(\mathring{t}_3) = q_4$. Now integration of equation (4.38) on the interval $[\mathring{t}_2, \ \mathring{t}_3]$ gives

$$\begin{aligned}
x_s(t) &\geq x_s(\mathring{t}_2) \exp((-\upsilon)(t - \mathring{t}_2) \\
&\geq q_4 \exp((-\upsilon)(t - \mathring{t}_2) \\
&\geq q_4 \exp(\upsilon(\tilde{n}_2 + \tilde{n}_3 + 1)\tau) = \overline{q}_4.
\end{aligned}$$

Since $x_s(\mathring{t}_3) \geq \overline{q}_4$, so similar process can be continued for $t > \mathring{t}_3$. Hence $x_s(t) \geq \overline{q}_4 \ \forall \ t > t_1$.

Case(ii) If $\exists \ \mathring{t}_4 \in [\mathring{t}_1, (\tilde{n}_1 + 1)\tau]$ such that $x_s(\mathring{t}_4) \geq q_4$, then let $\mathring{t}_5 = \inf\{t \mid x_s(t) \geq q_4; t > \mathring{t}_2\}$. Therefore, $x_s(t) < q_4$ for $t \in [\mathring{t}_2, \ \mathring{t}_5]$ and $x_s(\mathring{t}_5) = q_4$. Now, integration of equation (4.38) on the interval $[\mathring{t}_2, \ \mathring{t}_5]$ gives

$$x_s(t) \geq x_s(\mathring{t}_2) \exp((-\upsilon)(t - \mathring{t}_2)) \geq q_4 \exp(\upsilon\tau) = \overline{q}_4.$$

Because $x_s(\mathring{t}_5) \geq \overline{q}_4$, so, similar argument can be followed for $t > \mathring{t}_5$. Hence, it is concluded that $x_s(t) \geq \overline{q}_4 \ \forall \ t > \mathring{t}$.

Step III Let $a = \min\{q_1, \ q_2, \ q_3, \ \overline{q}_4\}, \Theta = \{R_+^3 : a \leq x_i(t), \ x_s(t), \ y_{em}(t), \ y_{ea}(t) \leq Q_0\}$. Thus, from above steps and Lemma 4.2.1, it is proved that each solution of system (4.1) will always remain in region Θ. Therefore, by Definition 1.5.4.2,

system (4.1) is permanent. □

4.4 Numerical Analysis and Discussion

In this chapter, a prey-predator model, with stage structure in predator and infection in prey, is constituted and investigated to control the outbreak of pest population. Susceptible prey is considered as a pest and predator acts as the natural enemy. The main aim is to inspect how the period of impulsive perturbations and releasing amounts of infected pests and natural enemies population is beneficial for integrated pest management.

It is analyzed that in the absence of impulsive release of infected pests and natural enemies, stable limit cycles exist for susceptible pest and infected pest population while immature and mature predator is driven towards extinction as shown in Figure 4.1. The global stability of pest-free equilibrium point is established and then it is derived that the system (4.1) is permanent. For ensuring the same, the threshold value of the impulsive period is calculated that depends on number of infected pests and natural enemies.The values of different parameters of the system (4.1) used for numerical simulation are given in Table 4.1. The initial values of population densities of susceptible prey, infected prey, immature and mature predator are $x_s(0) = 0.5$, $x_i(0) = 0.8$, $y_{em}(0) = 0.8$, $y_{ea}(0) = 4$. Therefore, by using all these numerical values, it is observed that if there is no impulsive release, that is $e_1 = e_2 = e_3 = 0$, then there exist stable limit cycles for susceptible and infected pest population. But immature and mature predators become extinct as shown in figure (4.1). Further from Theorems 4.2.2 and 4.3.1 we get $\check{\tau} = 1.836$ and $\tau_{max} = 4.55$. Therefore, it is found that susceptible pest free solution is globally stable if $\tau > \check{\tau}$ as depicted in Figure (4.2). Also, phase portrait of susceptible pest verses infected pest in Figure (4.2e) shows that stable limit cycle moves towards chaotic behavior. But complete extinction of pests is not encouraged biologically. Thus, Theorem 4.3.1 implies that if $\tau > \tau_{max}$ system (4.1) is permanent as shown in Figure (4.3) and exhibit chaotic behavior (see Figures 4.3e and 4.3f).

Table 4.1: Values of different parameters used in system(4.1)

| Parameter	Representation	Its Value (per week)
α	Reproduction rate of susceptible pest	1.7
β	Carrying capacity of susceptible pest	3
β_i	Contact rate of susceptible pest per unit time infected pest	2.6
α_n	Rate of predation by mature natural enemy	0.3
γ_2	half saturation constant by Holling IV	0.1
δ_1	Death rate of infected pest	0.5
γ_1	Conversion rate of pest to immature natural enemy	0.7
μ_{em}	Conversion rate of immature to mature natural enemy	0.4
δ_2	Death rate of mature and immature natural enemy	0.3
e_1	Impulsive releasing amount of infected pests	0.5
e_2	Impulsive releasing amount of immature natural enemy	2
e_1	Impulsive releasing amount of mature natural enemy	4
γ_3	half saturation constant	0.2
γ_4	half saturation constant	0.1

Apart from this, it is also analyzed that if $e_1 = 0$, that is only natural enemies are released then $\tau_{max} = 3.02511$. As shown in Figure (4.4), the system is again permanent and shows chaotic behavior but threshold value of impulsive period decreased which means that natural enemies are to be released at a fast pace. But this is not always feasible specially when natural enemies are not native species and are being reared. Similarly, if $e_2 = 0$ then $\tau_{max} = 3.88$. Permanence and chaotic behavior of the system is shown graphically in Figure (4.5). If $e_3 = 0$, then $\tau_{max} = 2.20168$ and permanence of the system (2.2) is shown in Figure (4.6). In all these situations, permanence of the system is achieved but at relatively low values of τ_{max} than the situation when we released both infected pests and natural enemies (immature and mature) simultaneously. Also numerical simulation is performed to examine the effect of impulsive releasing of natural enemies and infected pest population on the extinction of susceptible pest population. It is observed that susceptible pest population moves towards extinction as impulsive release is increased as shown in Figure (4.2f). The results obtained in this chapter supports those obtained by Mathur and Dhar [121]. But the model (4.1) incorporates the concept of stage-structure in natural enemies that makes it more realistic. Additionally, crowding effect of susceptible pest population and mutual interference between natural enemies is considered with the help of Holling II and Holling IV functional responses. As a result the threshold value of the impulsive period for complete eradication of susceptible pests $\check{\tau}$ is decreased while the threshold value for the coexistence of pest and natural enemies population τ_{max} is increased. Thus infected pests and natural enemies are to be released after longer period.

4.5 Conclusion

The war between pests and humans is going on from several decades. From time to time, different pest control techniques are acquired by mankind. Working on the same path, here, we investigated a stage structure predator-prey model for the purpose of integrated pest management. It is found that instead of using pesticides, microbial control agents along with natural enemies are more efficient
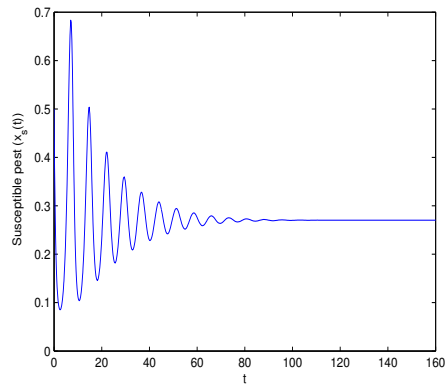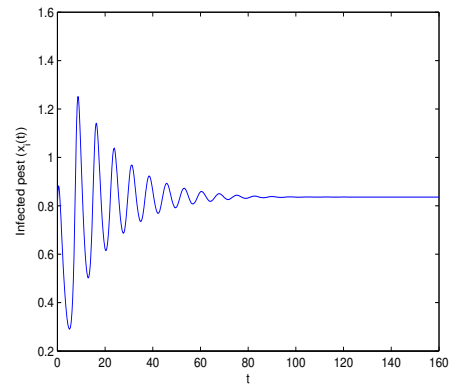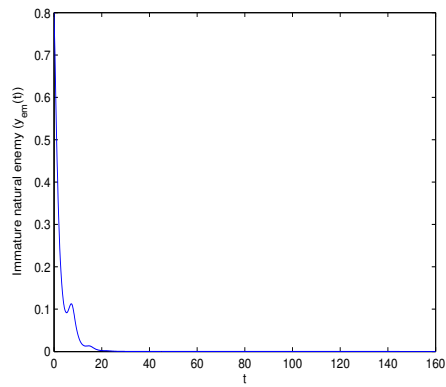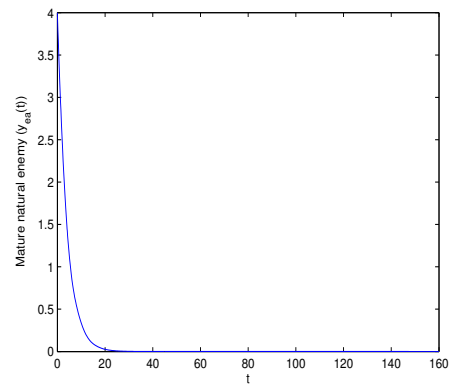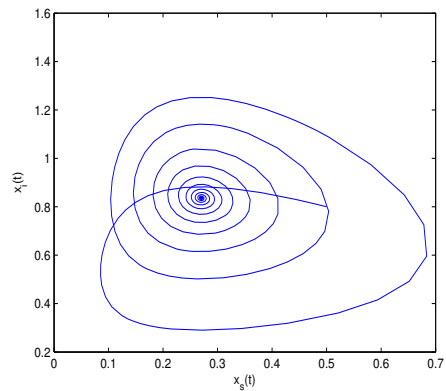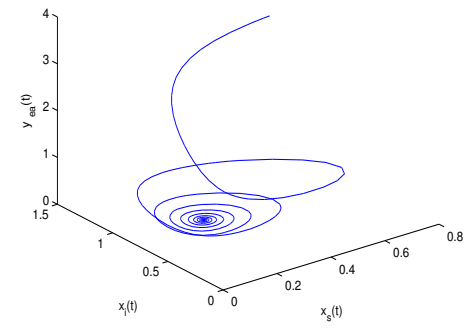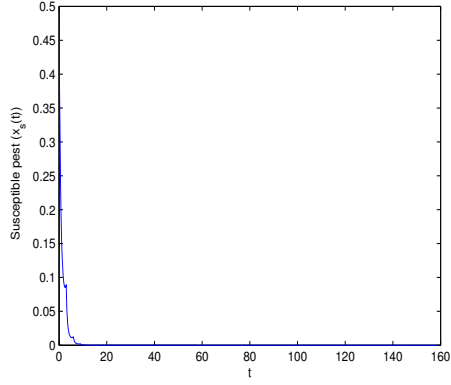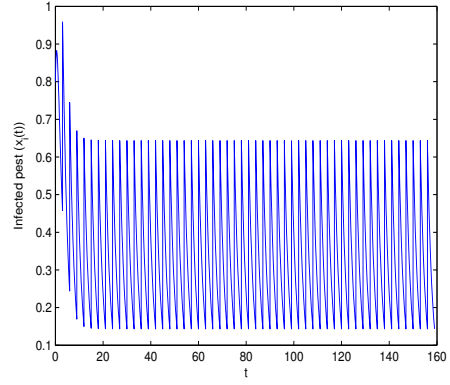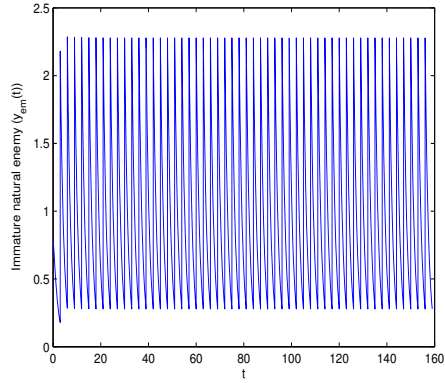
Figure 4.1: Stable limit cycles of susceptible pests and infectious pests when $e_1 = e_2 = e_3 = 0$ and $x_s(0) = 0.5$, $x_i(0) = 0.8$, $y_{em}(0) = 0.8$, $y_{ea}(0) = 4$.
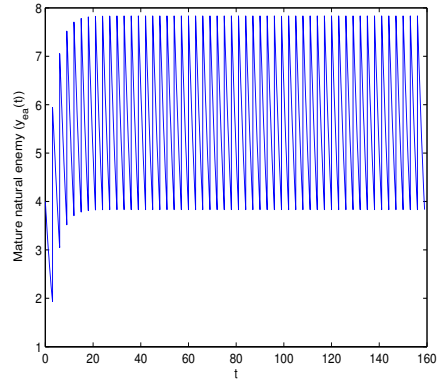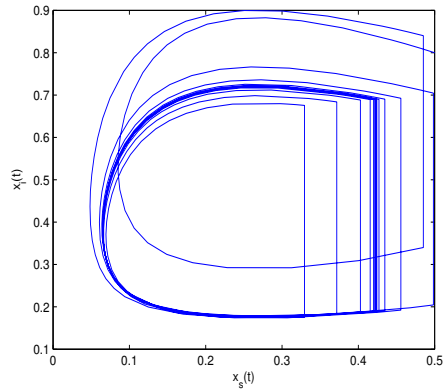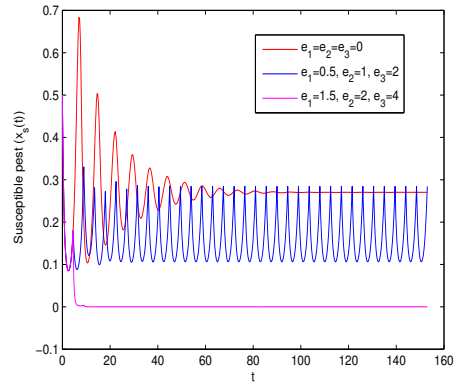
Figure 4.2: Global stability of pest extinction periodic solution $(0, \ x_i(t), \ y_{em}(t), \ y_{ea}(t))$ of system (4.1) at $\tau \ < \ \check{\tau}(= \ 4.7)$ with $e_1 = 0.5, \ e_2 = 2, \ e_3 = 4$.
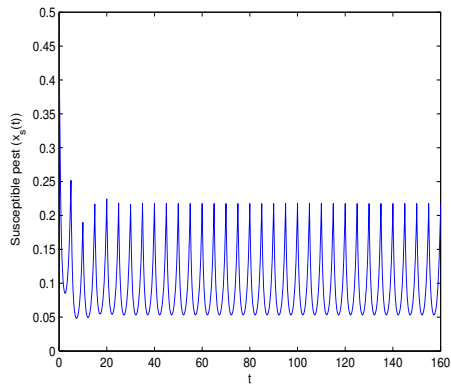
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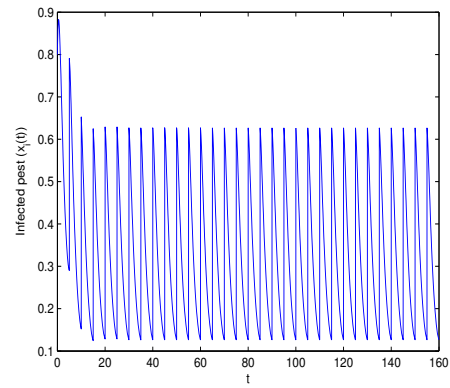
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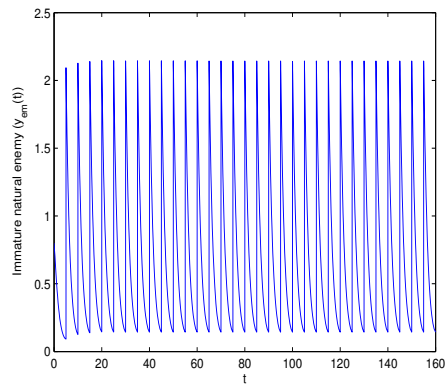
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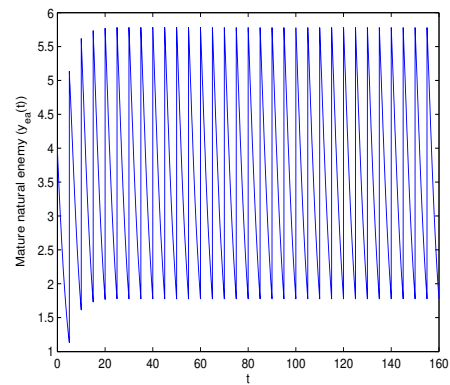
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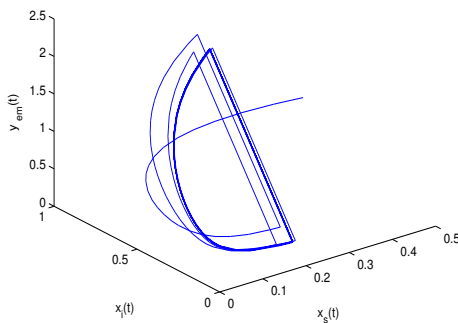
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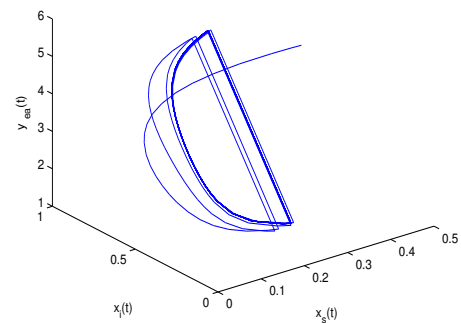
(f)

Figure 4.3: Permanence of the system (4.1) at $\tau > \tau_{max}(= 4.55)$ with $x_s(0^+ = 0.5)$, $x_i(0^+) = 0.8$, $y_{em}(0^+) = 0.8$, $y_{ea}(0^+) = 4$ with $e_1 = 0.5$, $e_2 = 2, e_3 = 4$
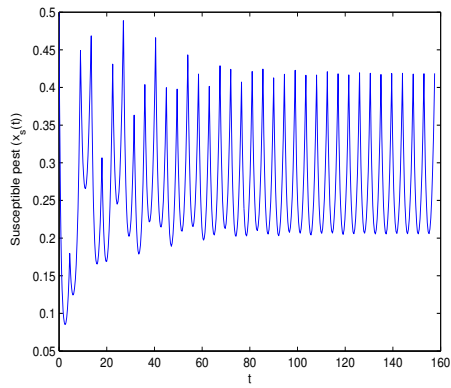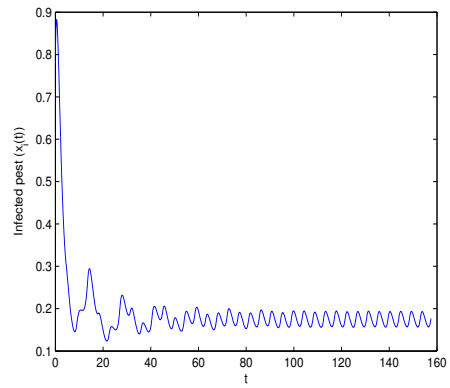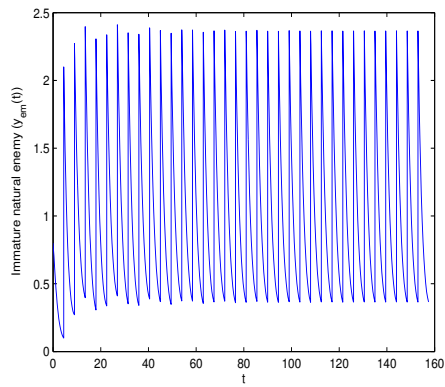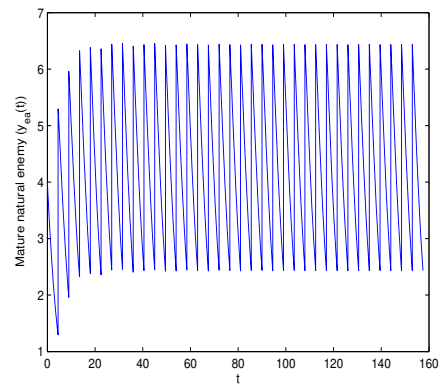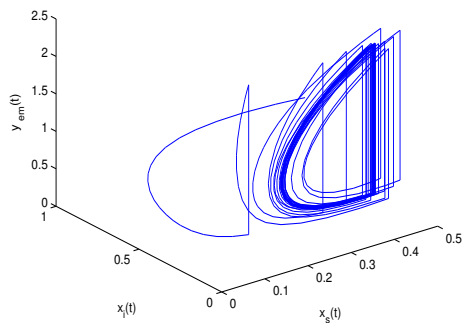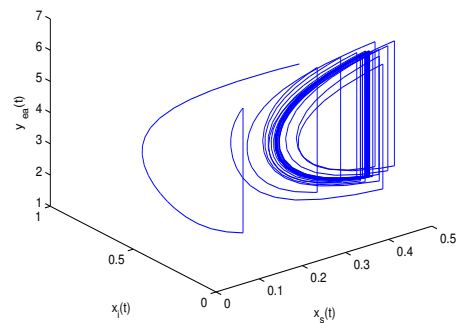
Figure 4.4: Permanence of the system (4.1) at $\tau > \tau_{max}(= 3.02511)$ with $x_s(0^+) = 0.5$, $x_i(0^+) = 0.8$, $y_{em}(0^+) = 0.8$, $y_{ea}(0^+) = 4$ and $e_1 = 0$, $e_2 = 2$, $e_3 = 4$.
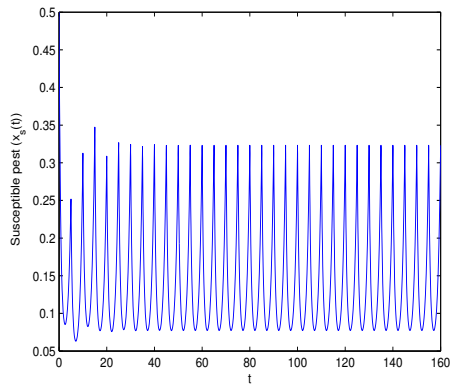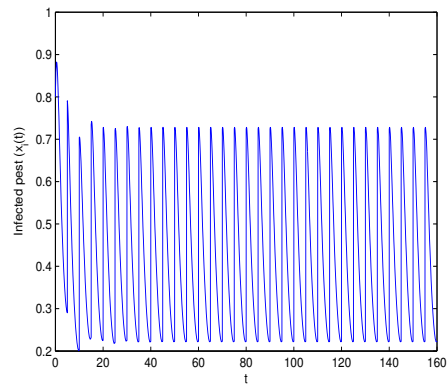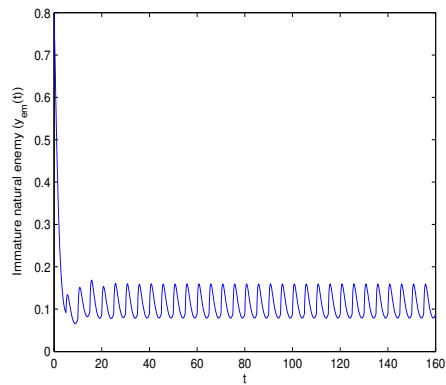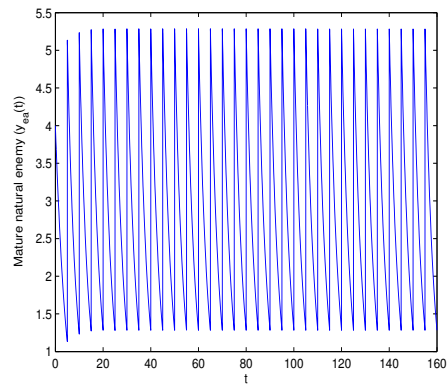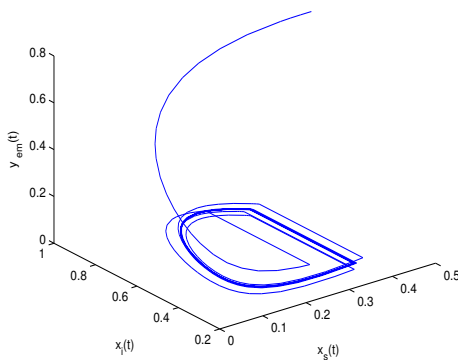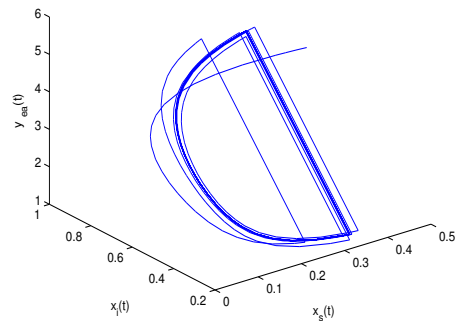
Figure 4.5: Permanence of the system (4.1) at $\tau > \tau_{max}(= 3.88)$ with $x_s(0^+) = 0.5$, $x_i(0^+) = 0.8$, $y_{em}(0^+) = 0.8$, $y_{ea}(0^+) = 4$ and $e_1 = 0.5$, $e_2 = 0$, $e_3 = 4$.
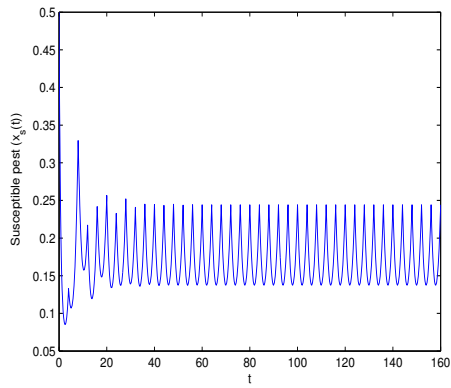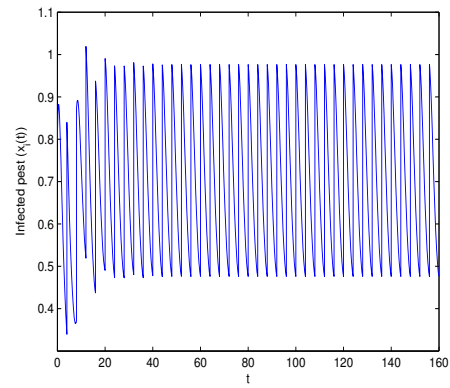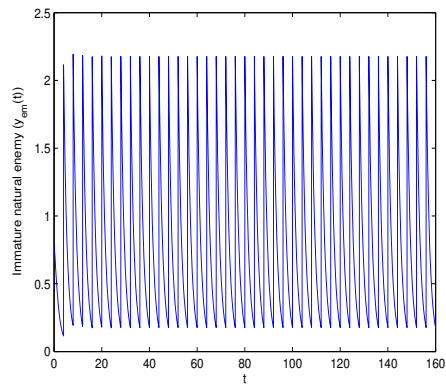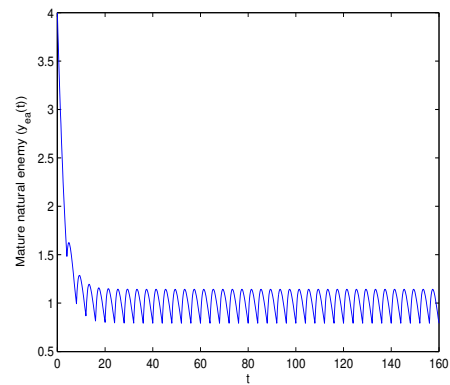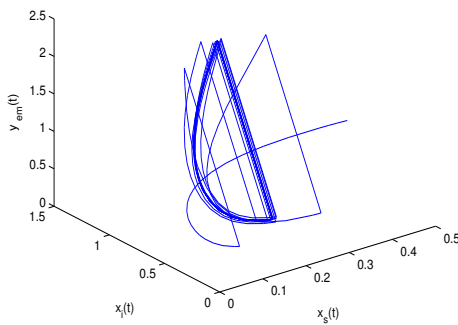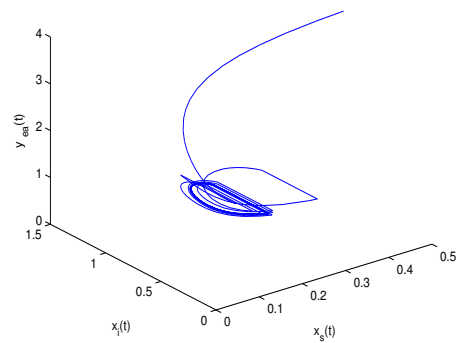
Figure 4.6: Permanence of the system (4.1) at $\tau > \tau_{max}(= 2.20168)$ with $x_s(0^+) = 0.5$, $x_i(0^+) = 0.8$, $y_{em}(0^+) = 0.8$, $y_{ea}(0^+) = 4$ and $e_1 = 0.5$, $e_2 = 2$, $e_3 = 0$.

in pest control. In Theorem 3, the threshold value of impulsive period (τ_{max}) is obtained and it is established that susceptible pests can coexist with infected pests and natural enemies if $\tau > \tau_{max}$. Also, the effect of releasing the number of infected pests and natural enemies is discussed and found that greater releasing amount supports pest eradication.

Chapter 5

A Two-prey One-predator Ecological Model with Holling IV and Beddington-DeAngelis Functional Response for Impulsive Pest Control Strategy

5.1 Introduction

Study of ecological models is becoming an interesting area of research over the past few decades. It all started with the single species population model proposed by Malthus [4] in which there is no restriction on living resources. This was not quite realistic. Then Verhulst [5] analyzed a more realistic logistic growth model considering limitation on living resources. Because, it was more natural to consider population interactions, so the simplest prey-predator model was firstly proposed by Lotka and Volterra with some biologically relevant assumptions, see [127]. After that, many researchers have extensively studied three species ecological models.

It was found that while studying population interactions, the important factor is functional response. Initially, Holling suggested three different types of functional responses as described in [32]. In type I, the consumption rate of predator is linearly dependent on prey density while in type II and III, time required to search and handle prey is considered. Then came the Monod-Haldane functional response that is also known as Holling type IV . It incorporated the situation of group defense by prey species. Liu et al. [33] investigated a prey-predator model by taking into account the phenomenon of group defense by the prey population. It involves a decrease in predation rate because the ability of prey species to defend themselves get enhanced in a group. Also, the recognition of an individual prey by predator becomes difficult in a cluster. One such example is decreased potential of wolves to attack small herds of lone musk ox as explained in [122]. Apart from that, Beddington [34] and DeAngelis [35] also put forward a new kind of functional response called Beddington-DeAngelis functional response which includes interference in the predator population during predation.

Moving further, it is absolutely natural to consider that prey-predator interactions undergo sudden changes in population density of concerned species. These changes can be in the form of fire, flood, seasonal birth or due to human interventions such as harvesting and stocking of population. Many of times, harvesting and stocking are done to maintain ecological balance. Therefore, impulsive differential equations are an appropriate mathematical tool to analyze these instantaneous disturbances. These are thoroughly studied by various researchers [8, 10]. There are plethora of applications of impulsive differential equations in ecology and other applied sciences as explained in [24]. Dong et al. [39] investigated predator-prey model where stocking of prey and harvesting of predator was done impulsively. The authors established required conditions for predator-free boundary solution and then for the permanence of the system. Song and Li [40] studied the dynamic behavior of a food web model under impulsive perturbations. They observed that if the impulsive period was less than the calculated threshold value, both the prey populations became extinct. Also, the authors studied the permanence of the system. Xiaong et al. [45] examined a Holling Type IV food chain model con-

sidering the impulsive effect. The researchers established global stability of prey and middle predator extinction solution. So, there is extensive literature available on three species population interactions under impulsive perturbations, see [43, 44, 134, 135].

Further, the preservation of non-renewable resources for future generations while meeting their present needs for food is the main goal of sustainable agriculture. Pest management is an essential component to accomplish this goal. Pesticides are widely being used to kill pests but studies found that these act as a major component of environmental pollution. Also, pests become resistant to certain pesticides after long term use [52]. So, Integrated Pest Management (IPM) is its best alternative. In IPM, the use of pesticides is combined appropriately with biological control to check the growth of pest population. Biological control includes the use of specified natural enemies to kill targeted pests. Because natural enemies and spraying of pesticides are to be done instantaneously, so prey-predator interactions with impulsive effect were used to constitute pest control models. Prey acted as pest and predator played the role of a specific natural enemy.

Therefore, many researchers have inspected three species food chain and food web models considering different Holling type functional responses for impulsive pest control strategies as described in [123, 124, 125, 126, 144, 130, 131, 133]. They found different threshold values of impulsive period for global stability of the prey (pest) extinction periodic solution and permanence of the respective system. Shulin and Cuihua [132] proposed a two-prey one-predator model with Beddington-DeAngelis functional response for both the prey populations. They established required conditions for the permanence of the system and also provided a threshold value of the impulsive release amount of natural enemy for the extinction of one of the prey populations. Similarly, Gupta et al. [136] inspected Susceptible-Exposed-Infected-Natural enemy (SEIN) model considering Beddington-DeAngelis functional response with impulsive control for IPM. Yu et al. [129] proposed a community food web model with two different prey populations and one predator population for impulsive pest control strategy. The authors applied biological pest control technique by periodically releasing natural enemies of the targeted

pest population. Because both the prey populations have the same natural enemy, so apparent competition between prey species is considered in the above model. The researchers established the condition for the global stability of prey free equilibrium. Further, it was found that the system exhibit rich dynamics such as period-doubling bifurcation and chaos. However, in this paper, biological pest control technique is combined effectively with chemical pest control to suppress the pest population. Also, as both the prey (pest) species shares the same resources for their survival, so direct competition is taken into account. Motivated by above, a two-prey one-predator model is formulated in this paper by taking into account the direct competition between prey populations because they are competing for the same natural resources. Also, different functional responses are considered for both the preys. The first prey population exhibits the phenomenon of group defense, so, Holling type IV functional response is considered as predation term. While for the predation of second prey, mutual interference between the predator population is involved, so Beddington-DeAngelis functional response is taken for predation. Also, spraying of pesticides is effectively combined with the biological pest control for Integrated Pest Management. The model and different parameters used in it are defined as follows:

(i) $x_p(t)$, $x_q(t)$, $y_{ne}(t)$ be the population densities of two preys species and the predator species at time t.

(ii) $\alpha_1 > 0$ and $\alpha_2 > 0$ are the intrinsic reproduction rate of two prey populations and $\beta > 0$ is the carrying capacity.

(iii) $\alpha_p > 0$, $\alpha_q > 0$ be the predation rates of first and second prey respectively.

(iv) $\alpha_3 > 0$ and $\alpha_4 > 0$ are the parameters representing competition between two prey species.

(v) β_1 is the saturation constant by Holling and β_2 measures the rate of group defense.

(vi) $\eta_1 > 0$ is the measure of interference between predators for predation and β_3 is the saturation constant

(vii) $\gamma_1 > 0$ and $\gamma_2 > 0$ denotes the rates of conversing prey into predator

(viii) $b_{ne} > 0$ be the death rate of predator.

$$
\begin{cases}
\begin{rcases}
\begin{aligned}
\frac{dx_p(t)}{dt} &= \alpha_1 x_p(t)\left(1 - \frac{x_p(t)}{\beta}\right) - \frac{\alpha_p x_p(t) y_{ne}(t)}{1 + \beta_1 x_p(t) + \beta_2 x_p^2(t)} \\
&\quad - \alpha_3 x_p(t) x_q(t) \\
\frac{dx_q(t)}{dt} &= \alpha_2 x_q(t)\left(1 - \frac{x_q(t)}{\beta}\right) - \frac{\alpha_q x_q(t) y_{ne}(t)}{\beta_3 + x_q(t) + \eta_1 y_{ne}(t)} \\
&\quad - \alpha_4 x_p(t) x_q(t) \\
\frac{dy_{ne}(t)}{dt} &= \frac{\gamma_1 \alpha_p x_p(t) y_{ne}(t)}{1 + \beta_1 x_p(t) + \beta_2 x_p^2(t)} + \frac{\gamma_2 \alpha_q x_q(t) y_{ne}(t)}{\beta_3 + x_q(t) + \eta_1 y_{ne}(t)} \\
&\quad - b_{ne} y_{ne}(t)
\end{aligned}
\end{rcases} t \neq n\tau, \\
\begin{rcases}
\Delta x_p(t) = -e_1 x_p(t) \\
\Delta x_q(t) = -e_2 x_q(t) \\
\Delta y_{ne}(t) = e_3
\end{rcases} t = n\tau, \quad n \in Z_+.
\end{cases} \tag{5.1}
$$

Let $R_+ = [0, \infty)$, $R_+^3 = \{x \in R^3 : x \geq 0\}$, $\Omega = int R_+^3$. The map defined by the right hand of the system (5.1) is given as $g = (g_1, \ g_2, \ g_3)^T$. Let $S_0 = \{V : R_+ \times R_+^3 \mapsto R_+$, continuous on $(n\tau, (n+1)\tau] \times R_+^3$ and $\lim_{(t, \ y) \to (n\tau, \ x), \ t > n\tau} S(t, \ x) = S(n\tau^+, \ x)$ exits$\}$.

5.2 Boundedness and Global Stability

5.2.1 Upper bound of all the variables

Here, in this section, firstly, upper bound for all the variables of system (5.1) are obtained in the coming lemma.

Lemma 5.2.1. *For sufficiently large t, there exists a constant $N > 0$ such that $x_p(t) \leq N$, $x_q(t) \leq N$, $y_{ne}(t) \leq N$. That is there is an upper bound for every solution of (5.1).*

Proof. Consider $X(t) = (x_p(t), \; x_q(t), \; y_{ne}(t))$ as an arbitrary solution of (5.1). Let $Q(t, \; X(t)) = x_p(t) + x_q(t) + y_{ne}(t)$ then for $t \neq n\tau$,

$$D^+ Q(t) + \theta Q(t) = (\alpha_1 + \theta)x_p + (\alpha_2 + \theta)x_q - \frac{\alpha_1 x_p^2}{\beta} - \frac{\alpha_2 x_q^2}{\beta}$$

$$- (1 - \gamma_1)\frac{\alpha_p x_p(t) y_{ne}(t)}{1 + \beta_1 x_p(t) + \beta_2 x_p^2(t)} - (b_{ne} - p)y_{ne}(t)$$

$$- (1 - \gamma_2)\frac{\alpha_q x_q(t) y_{ne}(t)}{\beta_3 + x_q(t) + \eta_1 y_{ne}(t)} - (\alpha_3 + \alpha_4)x_p x_q$$

$$\leq (\alpha_1 + \theta)x_p + (\alpha_2 + \theta)x_q - \frac{\alpha_1 x_p^2}{\beta} - \frac{\alpha_2 x_q^2}{\beta} \quad (p < b_{ne} \text{ and } \gamma \leq 1)$$

$$\leq \frac{\beta}{4}\left[\frac{(\alpha_1 + \theta)^2}{\alpha_1} + \frac{(\alpha_2 + \theta)^2}{\alpha_2}\right] = L_0$$

Also, $Q(t^+) = Q(t) + e_3$ for $t = n\tau$. Therefore by Theorem 1.5.1

$$Q(t) \leq Q(0) \exp\left(\int_0^t (-\theta)\right) ds + e_3 \sum_{0 < n\tau < t} \exp\left(\int_{n\tau}^t (-\theta)ds\right)$$

$$+ \int_0^t \left(L_0 \exp \int_s^t (-\theta d\sigma)\right) ds$$

$$\leq W(0) \exp(-\theta t) + (e_3) \sum_{0 < n\tau < t} \exp(-\theta(t - n\tau)) + \frac{L_0}{\theta}(1 - \exp(-\theta t))$$

$$\leq W(0) \exp(-\theta t) + \frac{L_0}{\theta}(1 - \exp(-\theta t)) + \frac{(e_3)(\exp(-\theta(t - \tau)))}{1 - \exp(-\theta\tau)} + \frac{(e_3)(\exp(\theta t))}{\exp(\theta\tau) - 1}$$

$$\rightarrow \frac{L_0}{\theta} + \frac{(e_3)(\exp(\theta\tau))}{\exp(\theta\tau) - 1} = N \text{ as } t \rightarrow \infty$$

This implies Q(t) is uniformly bounded. Hence, $\exists$ the constant N such that $x_p(t) \leq N$, $x_q(t) \leq N$, $y_{ne}(t) \leq N$. This completes the proof. $\qquad\square$

After spraying pesticides and releasing natural enemies, when pest population becomes extinct, then $x_p(t) = 0$ and $x_q(t) = 0$, the impulsive system (5.1) reduces

to

$$\begin{cases} \frac{dy_{ne}(t)}{dt} = -b_{ne}y_{ne}(t), & t \neq n\tau, \\ \Delta y_{ne}(t) = e_3, & t = n\tau, \ n \in Z_+. \end{cases} \tag{5.2}$$

Now, using Lemma 1.5.3, system (5.2) has periodic solution $\overline{y}_{ne}(t)$ which is globally asymptotically stable.

$$\overline{y}_{ne}(t) = \frac{e_3 \exp((-b_{ne})(t - n\tau))}{1 - \exp((-b_{ne})\tau)}, \ \overline{y}_{ne}(0^+) = \frac{e_3}{1 - \exp((-b_{ne})\tau)}. \tag{5.3}$$

Theorem 5.2.1. *Let $(x_p(t), \ x_q(t), \ y_{ne}(t))$ be an arbitrary solution of system (5.1). Then the boundary equilibrium $(0, \ 0, \ \overline{y}_{ne}(t))$ is locally asymptotically stable if the following conditions are satisfied.*

$$\ln(1 - e_1) + \alpha_1\tau - \frac{\alpha_p e_3}{b_{ne}} < 0. \tag{5.4}$$

and

$$\ln(1 - e_2) + \alpha_2\tau + \frac{\alpha_q}{b_{ne}\eta_1} \left[\ln\left(1 - \frac{\eta_1 e_3(1 - \exp(-b_{ne}\tau))}{\beta_3(1 - \exp(-b_{ne}\tau)) + \eta_1 e_3}\right)\right] < 0. \tag{5.5}$$

Proof. To prove that the required solution is locally stable, we use small perturbation method.

$$\begin{cases} \frac{d\zeta_1(t)}{dt} = \alpha_1\zeta_1(t) - \alpha_p\overline{y}_{ne}(t)\zeta_1(t), \\ \frac{d\zeta_2(t)}{dt} = \alpha_2\zeta_2 - \frac{\alpha_q\zeta_2\overline{y}_{ne}(t)}{\beta_3 + \eta_1\overline{y}_{ne}(t)}, & t \neq n\tau, \\ \frac{d\zeta_3(t)}{dt} = \alpha_p\gamma_1\overline{y}_{ne}(t)\zeta_1 + \frac{\gamma_2\alpha_q\zeta_2\overline{y}_{ne}(t)}{\beta_3 + \eta_1\overline{y}_{ne}(t)} - b_{ne}\zeta_3, \\ \zeta_1(t^+) = (1 - e_1)\zeta_1(t), \\ \zeta_2(t^+) = (1 - e_2)\zeta_2(t), & t = n\tau, \quad n \in Z_+. \\ \zeta_3(t^+) = \zeta_3(t), \end{cases} \tag{5.6}$$

Let $\zeta_1(t)$, $\zeta_2(t)$, $\zeta_3(t)$ be the small perturbations in 0, 0, $\overline{y}_{ne}(t)$, respectively. Then

$$x_p(t) = \zeta_1(t), \ x_q(t) = \zeta_2(t), \ \text{and} \ y_{ne}(t) = \overline{y}_{ne}(t) + \zeta_3(t).$$

Putting these values in system (5.1), it reduces to system (5.6). The system (5.6) represents system of linear differential equations, which can be written in matrix form. Hence, for $t \neq n\tau$, the coefficient matrix is given as

$$B = \begin{bmatrix} \alpha_1 - \alpha_p\overline{y}_{ne}(t) & 0 & 0 \\ 0 & \alpha_2 - \frac{\alpha_q\overline{y}_{ne}(t)}{\beta_3 + \eta_1\overline{y}_{ne}(t)} & 0 \\ \alpha_p\gamma_1\overline{y}_{ne}(t) & \frac{\gamma_2\alpha_q\overline{y}_{ne}(t)}{\beta_3 + \eta_1\overline{y}_{ne}(t)} & -b_{ne} \end{bmatrix},$$

and for $t = n\tau$,

$$\begin{bmatrix} \zeta_1(t^+) \\ \zeta_2(t^+) \\ \zeta_3(t^+) \\ \zeta_4(t^+) \end{bmatrix} = \begin{bmatrix} 1 - e_1 & 0 & 0 \\ 0 & 1 - e_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \zeta_3(t) \end{bmatrix}.$$

Let $\Phi(t)$ be the fundamental solution of system (5.6), then

$$\frac{d\Phi(t)}{dt} = B\Phi(t), \tag{5.7}$$

$$\Phi(\tau) = \Phi(0)\exp(\int_0^T B \ dt),$$

with $\Phi(0) = I$, the identity matrix. Solving, we have

$$\Phi(\tau) = \begin{bmatrix} \exp(\int_0^\tau \alpha_1 - \alpha_p\overline{y}_{ne}(t)dt) & 0 & 0 \\ 0 & \exp(\int_0^\tau \alpha_2 - \frac{\alpha_q\overline{y}_{ne}(t)}{\beta_3 + \eta_1\overline{y}_{ne}(t)}dt) & 0 \\ \exp(\int_o^\tau \alpha_p\gamma_1\overline{y}_{ne}(t)dt) & \exp(\int_0^\tau \frac{\gamma_2\alpha_q\overline{y}_{ne}(t)}{\beta_3 + \eta_1\overline{y}_{ne}(t)}dt) & \exp(\int_0^\tau -b_{ne}dt) \end{bmatrix},$$

which is upper triangular matrix. Now according to Floquet theory of impulsive differential equations (Remark 1.5.8.1 and Theorem 1.5.5), if absolute values of all the eigen values of Monodromy matrix M are less than one, then the required

solution is globally stable where

$$M = \begin{bmatrix} 1 - e_1 & 0 & 0 \\ 0 & 1 - e_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \phi(\tau)$$

Since M is an upper triangular matrix, therefore, eigen values of M are

$$\lambda_1 = (1 - e_1) \exp\left(\int_0^\tau \alpha_1 - \alpha_p \overline{y}_{ne}(t) dt\right),$$
$$\lambda_2 = (1 - e_2) \exp\left(\int_0^\tau \alpha_2 - \frac{\alpha_q \overline{y}_{ne}(t)}{\beta_3 + \eta_1 \overline{y}_{ne}(t)} dt\right), \tag{5.8}$$
$$\lambda_3 = \exp\left(\int_0^\tau -(b_{ne}) dt\right).$$

It is obvious from (5.8), that $| \lambda_3 | < 1$. Now $| \lambda_1 | < 1$ and $| \lambda_2 | < 1$ if

$$\ln(1 - e_1) + \alpha_1 \tau - \frac{\alpha_p e_3}{b_{ne}} < 0$$

and

$$\ln(1 - e_2) + \alpha_2 \tau + \frac{\alpha_q}{b_{ne} \eta_1} \left[\ln\left(1 - \frac{\eta_1 e_3 (1 - \exp(-b_{ne}\tau))}{\beta_3 (1 - \exp(-b_{ne}\tau)) + \eta_1 e_3}\right) \right] < 0.$$

This completes the proof. □

5.2.2 Global Stability

Theorem 5.2.2. *If conditions (5.4 and 5.5) are satisfied and*

$$e_3 > max\left[\frac{\alpha_1 \tau b_{ne}}{\alpha_p}(1 + \beta_1 L + \beta_2 L^2), \frac{\alpha_2 \tau b_{ne}}{\alpha_q}(\beta_3 + L + \eta_1 L) \right],$$

then the boundary equilibrium $(0, 0, \overline{y}_{ne}(t))$ *is globally asymptotically stable.*

Proof. By analyzing system (5.1), we have

$$
\begin{cases}
\frac{dy_{ne}(t)}{dt} \geq -b_{ne}y_{ne}(t), & t \neq n\tau \\
\Delta y_{ne}(t) = e_3, & t = n\tau, \ n \in Z_+.
\end{cases}
\tag{5.9}
$$

Consider the corresponding comparison impulsive system of (5.9) as

$$
\begin{cases}
\frac{dw_{ne}(t)}{dt} = -b_{ne}y_{ne}(t), & t \neq n\tau \\
\Delta w_{ne}(t) = e_3, & t = n\tau, \ n \in Z_+.
\end{cases}
\tag{5.10}
$$

Applying Lemma 1.5.3, (5.10) has periodic solution

$$
\overline{w}_{ne}(t) = \frac{e_3 \exp((-b_{ne})(t - n\tau))}{1 - \exp(-b_{ne}\tau)}, \ t \in (n\tau, (n+1)\tau],
$$

which is globally asymptotically stable. Therefore by Theorem 1.5.2, $y_{ne}(t) \geq w_{ne}(t) \to \overline{w}_{ne}(t)$ and $\overline{w}_{ne}(t) = \overline{y}_{ne}(t)$. Hence, $\exists$ a positive integer κ_1 such that

$$
y_{ne}(t) > \overline{y}_{ne}(t) - \dot{\varepsilon} \ \forall \ t \geq \kappa_1\tau.
$$

Let $W(t) = x_p(t) + x_q(t)$, then for $t \neq n\tau$, we have

$$
\begin{aligned}
W' \mid_{(5.1)} &= \alpha_1 x_p - \frac{\alpha_1 x_p^2}{\beta} - \frac{\alpha_p x_p y_{ne}}{1 + \beta_1 x_p + \beta_2 x_p^2} - \alpha_3 x_{px_q} + \alpha_2 x_q - \frac{\alpha_2 x_q^2}{\beta} \\
&\quad - \frac{\alpha_q x_q y_{ne}}{\beta_3 + x_q + \eta_1 x_q} - \alpha_4 x_{px_q} \\
&\leq \left(\alpha_1 - \frac{\alpha_p(\overline{y}_{ne} - \dot{\varepsilon})}{1 + \beta_1 x_p + \beta_2 x_p^2} \right) x_p + \left(\alpha_2 - \frac{\alpha_q(\overline{y}_{ne} - \dot{\varepsilon})}{\beta_3 + x_q + \eta_1 x_q} \right) x_q.
\end{aligned}
\tag{5.11}
$$

Let $M_1 = \left(\alpha_1 - \frac{\alpha_p(\overline{y}_{ne} - \dot{\varepsilon})}{1 + \beta_1 x_p + \beta_2 x_p^2} \right)$ and $M_2 = \left(\alpha_2 - \frac{\alpha_q(\overline{y}_{ne} - \dot{\varepsilon})}{\beta_3 + x_q + \eta_1 x_q} \right)$. If

$$
e_3 > \max \left[\frac{\alpha_1 \tau b_{ne}}{\alpha_p}(1 + \beta_1 L + \beta_2 L^2), \ \frac{\alpha_2 \tau b_{ne}}{\alpha_q}(\beta_3 + L + \eta_1 L) \right],
$$

then $M_1 < 0$ and $M_2 < 0$. Assume that $M = \min\{| M_1 |, | M_2 |\}$. Then from

equation (5.11), for $t \neq n\tau$ we obtain

$$W' \mid_{(5.1)} \leq -MW(t)$$

for $t \geq \kappa_2 \tau$. Therefore we have the following system

$$\begin{cases} \frac{dW(t)}{dt} \leq -MW(t), & t \neq n\tau \\ W(t^+) \leq W(t), & t = n\tau, \ n \in Z_+. \end{cases} \tag{5.12}$$

So, by using Theorem 1.5.2, we obtain, $W(t) \leq W_0 \exp(-Mt) \to 0$ as $t \to \infty$. This implies that $x_p(t), \ x_q \to 0$ as $t \to \infty$. So, here exists $\grave{\varepsilon}_1 > 0$ and $\grave{\varepsilon}_2 > 0$ such that $x_p(t) < \grave{\varepsilon}_1$ and $x_q(t) < \grave{\varepsilon}_2$ for $t \geq \kappa_2 \tau$. Again, from system (5.1), we obtain that

$$\begin{cases} \frac{dy_{ne}(t)}{dt} \leq (\gamma_1 \alpha_p \grave{\varepsilon}_1 + \gamma_2 \alpha_q \grave{\varepsilon}_2 - b_{ne}) y_{ne}(t), & t \neq n\tau \\ \Delta y_{ne}(t) = e_3, & t = n\tau, \ n \in Z_+. \end{cases} \tag{5.13}$$

Consider its correspondence comparison system

$$\begin{cases} \frac{dv_{ne}(t)}{dt} \leq (\gamma_1 \alpha_p \grave{\varepsilon}_1 + \gamma_2 \alpha_q \grave{\varepsilon}_2 - b_{ne}) v_{ne}(t), & t \neq n\tau \\ \Delta v_{ne}(t) = e_3, & t = n\tau, \ n \in Z_+. \end{cases} \tag{5.14}$$

Applying Lemma 1.5.3, system (5.14) has periodic solution

$$\overline{v}_{ne}(t) = \frac{e_3 \exp((-b_{ne} - (\gamma_1 \alpha_p \grave{\varepsilon}_1 + \gamma_2 \alpha_q \grave{\varepsilon}_2))(t - n\tau))}{1 - \exp((-b_{ne} - (\gamma_1 \alpha_p \grave{\varepsilon}_1 + \gamma_2 \alpha_q \grave{\varepsilon}_2))\tau)}, \ t \in (n\tau, (n+1)\tau],$$

which is globally asymptotically stable. Therefore by Theorem 1.5.2, $y_{ne}(t) \leq v_{ne}(t) \to \overline{v}_{ne}(t)$. Hence, $\exists$ a positive integer κ_3 such that

$$y_{ne}(t) < \overline{v}_{ne}(t) + \grave{\varepsilon} \ \forall \ t \geq \kappa_3 \tau, \ (\kappa_3 > \kappa_2 > \kappa_1)$$

Therefore, as $t \to \infty$, $x_p \to 0$, $x_q \to 0$ and $y_{ne}(t) \to \overline{y}_{ne}(t)$. Thus $(0, \ 0, \ \overline{y}_{ne}(t))$ is globally asymptotically stable. This completes the proof. $\qquad \square$

5.3 Permanence

Firstly, condition for the system to be permanent is established as follows:

Theorem 5.3.1. *The system (5.1) is permanent if*

$$\ln(1 - e_1) + \alpha_1 \tau - \frac{\alpha_p e_3}{b_{ne}} > 0. \tag{5.15}$$

and

$$\ln(1 - e_2) + \alpha_2 \tau + \frac{\alpha_q}{b_{ne}\eta_1} \left[\ln \left(1 - \frac{\eta_1 e_3 (1 - \exp(-b_{ne}\tau))}{\beta_3 (1 - \exp(-b_{ne}\tau)) + \eta_1 e_3} \right) \right] > 0. \tag{5.16}$$

Proof. Upper bound of $x_p(t)$, $x_q(t)$, $y_{ne}(t)$ of the system is already been obtained in Lemma 5.2.1. Also in the above section, it is proved that

$$y_{ne}(t) > \overline{y}_{ne}(t) - \mathring{\varepsilon} = r_1 \ \forall \ t \geq \kappa_1 \tau, \tag{5.17}$$

Thus, for permanence of the system (5.1), there must exists a constant r_2, r_3 such that $x_p(t) \geq r_2$ and $x_q(t) \geq r_3$ for sufficiently large t. Here, lower bound of x_p is attained. This is achieved in two steps as follows:

Step I Firstly, we will prove that $\exists \ \mathring{t}_1 \in (0, \infty)$ such that $x_p(\mathring{t}_1) \geq r_2$. If this is not true, then $x_p(t) \leq r_2 \forall \ t > 0$. Considering this assumption, from system of equations (5.1), we have

$$\begin{cases} \frac{dy_{ne}(t)}{dt} \leq (\gamma_1 \alpha_p r_2 + \gamma_2 \alpha_q N - b_{ne}) y_{ne}(t), & t \neq n\tau \\ \Delta y_{ne}(t) = e_3, & t = n\tau, \ n \in Z_+. \end{cases}$$

Consider the following impulsive system

$$\begin{cases} \frac{du_{ne}(t)}{dt} = (\gamma_1 \alpha_p r_2 + \gamma_2 \alpha_q N - b_{ne}) u_{ne}(t), & t \neq n\tau \\ \Delta u_{ne}(t) = e_3, & t = n\tau, \ n \in Z_+. \end{cases} \tag{5.18}$$

Here r_2 should be such that $\gamma_1 \alpha_p r_2 + \gamma_2 \alpha_q N - b_{ne} < 0$. Applying Lemma 1.5.3,

([5.18](#)) has periodic solution

$$\overline{u}_{ne}(t) = \frac{e_3 \exp(-(b_{ne} - (\gamma_1\alpha_p r_2 + \gamma_2\alpha_q N)(t - n\tau)))}{1 - \exp(-(b_{ne} - (\gamma_1\alpha_p r_2 + \gamma_2\alpha_q L)\tau)}, \ t \in (n\tau, (n+1)\tau].$$

which is globally asymptotically stable. Therefore by Theorem [1.5.2](#), $y_{ne}(t) \leq u_{ne}(t) \to \overline{u}_{ne}(t)$. Hence, $\exists$ a positive integer κ_4 such that

$$y_{ne}(t) \leq \overline{u}_{ne}(t) + \dot{\varepsilon}_3 \ \forall \ t \geq \kappa_4\tau. \tag{5.19}$$

Therefore, for $t > \kappa_4\tau$, first equation of system ([5.1](#)) gives

$$\begin{cases} \frac{dx_p(t)}{dt} \geq x_p \left[\alpha_1 \left(1 - \frac{N}{\beta} \right) - \alpha_p(\overline{u}_{ne}(t) + \dot{\varepsilon}_3) - \alpha_3 N \right], & t \neq n\tau \\ \Delta x_p(t) = -e_1 x_p(t), & t = n\tau, \ n \in Z_+. \end{cases} \tag{5.20}$$

Integration of first equation of system ([5.20](#)) on $(\kappa_4\tau, (\kappa_4+1)\tau]$ gives the following stroboscopic map

$$x_p[(\kappa_4+1)\tau] \geq x_p(\kappa_4\tau)(1 - e_1)$$
$$\times \exp\left(\int_{\kappa_4\tau}^{(\kappa_4+1)\tau} \left(\alpha_1 \left(1 - \frac{N}{\beta} \right) - \alpha_p(\overline{u}_{ne}(t) + \dot{\varepsilon}_3) - \alpha_3 N \right) dt \right)$$
$$\geq x_p(\kappa_4\tau)\varrho_1 \tag{5.21}$$

where

$$\varrho_1 = (1 - e_1) \exp\left(\int_{\kappa_4\tau}^{(\kappa_4+1)\tau} \left(\alpha_1 \left(1 - \frac{N}{\beta} \right) - \alpha_p(\overline{u}_{ne}(t) + \dot{\varepsilon}_3) - \alpha_3 N \right) dt \right)$$

Because of condition given in equation ([5.15](#)), it is possible to find $\dot{\varepsilon}_3$ and N such that $\varrho_1 > 1$. Therefore, for some positive integer q, from equation ([5.21](#)),

$$x_p[(\kappa_4+q)\tau] \geq x_p(\kappa_4\tau)(\varrho_1)^q \to \infty \text{ as } q \to \infty.$$

This is contradiction to our assumption. Hence, there must exist $\mathring{t}_1 > 0$ such that

$x_p(\mathring{t}_1) \geq r_2$.

Step II There is nothing to prove if $x_p(t) \geq r_2 \; \forall \; t > \mathring{t}_1$. But if this is not true, let $\mathring{t}_2 = \inf\{t \mid x_p(t) < r_2; \; t > \mathring{t}_1\}$. Thus $x_p(t) \geq r_2 \; \forall \; t \in [\mathring{t}_1, \; \mathring{t}_2]$, $\mathring{t}_2 \in (\check{n}_1\tau, \; (\check{n}_1+1)\tau]$. $x_p(\mathring{t}_2) = r_2$, because of continuity of $x_p(t)$. Let $\tau^* = (\check{n}_2 + \check{n}_3)\tau$ where $\check{n}_2$ and $\check{n}_3$ satisfy the following conditions

$$(\check{n}_2)\tau > - \left(\frac{1}{b_{ne} - (\gamma_1 r_1 \alpha_p + \gamma_2 N \alpha_q)} \right) \ln \frac{\grave{\varepsilon}_3}{L + e_3}, \tag{5.22}$$

$$(1 - e_1)^{\check{n}_2+1}(\varrho_1)^{\check{n}_3}(\exp(\upsilon(\check{n}_2 + 1)\tau)) > 1$$

where

$$\upsilon = \left[\alpha_1 \left(1 - \frac{r_2}{\beta} \right) - (\alpha_p + \alpha_3 N) \right] < 0.$$

Now, we will prove that $\exists \; \mathring{t}_3 \in ((\check{n}_1 + 1)\tau, \; (\check{n}_1 + 1)\tau + \tau^*]$ such that $x_p(\mathring{t}_3) \geq r_2$. Suppose this is not true, then $x_p(t) < r_2 \; \forall \; t \in ((\check{n}_1+1)\tau, \; (\check{n}_1+1)\tau+\tau^*]$. If system (5.18) is considered with $u_{ne}((\check{n}_1 + 1)\tau^+) = y_{ne}((\check{n}_1 + 1)\tau^+)$, then using Lemma 1.5.3 for $t \in ((\check{n}_1 + 1)\tau, \; (\check{n}_1 + 1)\tau + \tau^*]$, we have

$$u_{ne}(t) = \left[u_{ne}(\check{n}_1 + 1)\tau^+) - \frac{e_3}{1 - \exp(-b_{ne} - (\gamma_1 r_1 \alpha_p + \gamma_2 N \alpha_q)\tau)} \right] \times$$
$$\exp(-b_{ne} - (\gamma_1 r_1 \alpha_p + \gamma_2 N \alpha_q)(t - n\tau))) + \overline{u}_{ne}(t)$$

This implies

$$| u_{ne}(t) - \overline{u}_{ne}(t) | \leq (N + e_3) \exp(-b_{ne} - (\gamma_1 r_1 \alpha_p + \gamma_2 N \alpha_q)(t - n\tau)))$$
$$\leq \grave{\varepsilon}_3.$$

which depicts that $y_{ne}(t) \leq u_{ne}(t) < \overline{u}_{ne}(t) + \grave{\varepsilon}_3$, $(\check{n}_1 + \check{n}_2 + 1)\tau \leq t \leq (\check{n}_1 + 1)\tau + \tau^*$. Hence system of equations (5.20) holds for $(\check{n}_1 + \check{n}_2 + 1)\tau \leq t \leq (\check{n}_1 + 1)\tau + \tau^*$. Thus, integrating first equation of system (5.20) in the interval $(\check{n}_1 + \check{n}_2 + 1)\tau, \; (\check{n}_1 + 1)\tau + \tau^*)$ and using jump condition, we get the following stroboscopic map

$$x_p(\check{n}_1 + \check{n}_2 + \check{n}_3 + 1)\tau \geq x_p(\check{n}_1 + \check{n}_2 + 1)\varrho_1^{\check{n}_3} \tag{5.23}$$

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Further, for $t \in [\mathring{t}_2, (\check{n}_1 + 1)\tau]$, two possibilities are there

Case(i) If $x_p(t) < r_2 \ \forall \ t \in [\mathring{t}_2, \ (\check{n}_1 + 1)\tau]$, then from the above assumption $x_p(t) < r_2 \ \forall \ t \in [\mathring{t}_2, \ (\check{n}_1 + 1)\tau + \tau^*]$. This implies $x_p(t) < r_2 \ \forall \ t \in [\mathring{t}_2, \ (\check{n}_1 + \check{n}_2 + 1)\tau]$. Thus, from the system of equations (5.1), we have

$$
\begin{cases}
\frac{dx_p(t)}{dt} \geq x_p \left[\alpha_1 \left(1 - \frac{r_2}{\beta} \right) - \alpha_p N - \alpha_3 N \right], & t \neq n\tau \\
\Delta x_p(t) = -e_1 x_p(t), & t = n\tau, \ n \in Z_+.
\end{cases}
\tag{5.24}
$$

Solving the system (5.24) in $[\mathring{t}_2, \ (\check{n}_1 + \check{n}_2 + 1)\tau]$, we have

$$
x_p((\check{n}_1 + \check{n}_2 + 1)\tau) \geq (1 - e_1)^{\check{n}_2 + 1} x_p(\mathring{t}_2) \exp(\upsilon(\check{n}_2 + 1)\tau).
\tag{5.25}
$$

Using (5.25) in (5.23)

$$
x_p((\check{n}_1 + \check{n}_2 + \check{n}_3 + 1)\tau) \geq x_p(\mathring{t}_2)(1 - e_1)^{\check{n}_2 + 1} \varrho_1^{\check{n}_3} \exp(\upsilon(\check{n}_2 + 1)\tau) > r_2.
$$

But this contradicts our assumption. Therefore, $\exists \ \mathring{t}_3 \in (\check{n}_1 \tau, \ (\check{n} + 1)\tau)$ such that $x_p(\mathring{t}_3) \geq r_2$. Let $\mathring{t}_4 = \inf\{t \mid x_p(t) \geq r_2; \ t > \mathring{t}_2\}$. Due to continuity of $x_p(t)$, $x_s(\mathring{t}_4) = r_2$. Now integration of equation (5.24) on the interval $[\mathring{t}_2, \ \mathring{t}_4]$ gives

$$
\begin{aligned}
x_p(t) & \geq \ x_p(\mathring{t}_2) \exp((\upsilon)(t - \mathring{t}_2)) \\
& \geq \ r_2 \exp((\upsilon)(t - \mathring{t}_2)) \\
& \geq \ r_2 \exp(\upsilon(\check{n}_2 + \check{n}_3 + 1)\tau) = \bar{r}_4.
\end{aligned}
$$

Since $x_s(\mathring{t}_4) \geq \bar{r}_2$, so similar process can be continued for $t > \mathring{t}_4$. Hence $x_p(t) \geq \bar{r}_2 \ \forall \ t > \mathring{t}_1$.

Case(ii) if $\exists \ \mathring{t}_5 \in [\mathring{t}_2, \ (\check{n}_1 + 1)\tau]$ such that $x_p(\mathring{t}_5) \geq r_2$, then let $\mathring{t}_6 = \inf\{t \mid x_p(t) \geq r_2; t > \mathring{t}_2\}$. Therefore, $x_p(t) < r_2$ for $t \in [\mathring{t}_2, \ \mathring{t}_6]$ and $x_p(\mathring{t}_6) = r_2$. Again, integration of equation (5.24) on the interval $[\mathring{t}_2, \ \mathring{t}_6]$ gives

$$
x_p(t) \geq x_p(\mathring{t}_2) \exp((\upsilon)(t - \mathring{t}_2)) \geq r_2 \exp(\upsilon\tau) = \bar{r}_2.
$$

Because $x_s(\mathring{t}_6) \geq \bar{r}_2$, so, similar argument can be followed for $t > \mathring{t}_6$. Hence, it is

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concluded that $x_p(t) \geq \bar{r}_2 \; \forall \; t > \mathring{t}_1$. Similarly, we can prove that $x_q(t) \geq \bar{r}_3$ for some t.

Step III Let $r_4 = \min\{r_1, \; \bar{r}_2, \; \bar{r}_3\}, \Theta = \{R_+^3 : r_4 \leq x_p(t), \; x_q(t), \; y_{ne}(t) \leq N\}$. Thus, from above steps and Lemma 5.2.1, it is proved that each solution of system (5.1) will always remain in region Θ. Therefore, by Definition 1.5.4.2, system (5.1) is permanent. □

5.4 Numerical Analysis and Discussion

A food web model consisting of two-prey species and a predator species considering Holling IV and Beddington-DeAngelis type functional responses is examined in this chapter for impulsive pest control strategy. The dynamics of the required ecological model is studied numerically by obtaining the solution of system (5.1) with initial values lying in the first octant. Biologically feasible values of different parameters are given in Table 5.1. From Theorem 5.2.2, it is obtained that the prey (pest) free equilibrium $(0, \; 0, \; \bar{y}_{ne}(t))$ is globally stable. So, it is found that if $e_3 > 13.5$, the equilibrium point $(0, \; 0, \; \bar{y}_{ne}(t))$ is globally asymptotically stable (see Figure 5.1). Again, the results of Theorem 5.3.1 are verified numerically as if $e_3 < 6.7$, the system (5.1) is permanent (see Figure 5.2). Therefore both the prey and predator species coexist. Thus we have obtained two threshold values of impulsive release amount of natural enemy i.e. e_3 as $e_3^* = 6.7$ and $e_3^{**} = 13.5$.

Further, numerically, it is observed that because of the principle of competition exclusion, second prey species goes to extinction if $6.7 \leq e_3 < 8$ and if $8 \leq e_3 \leq 13.5$, the extinction of first prey species takes place (see Figures 5.3 and 5.4). Thus the results obtained will be helpful to control the pest population in order to avoid major production or economic loss.

Table 5.1: Values of different parameters used in system (5.1)

Parameter	Representation	Its Value
α_1	Intrinsic reproduction rate of the first prey population	0.9
α_2	Intrinsic reproduction rate of the second prey populations	1.2
β	Carrying capacity	15
α_p	Predation rate of first prey	0.6
β_1	Saturation constant by Holling	0.3
β_2	Rate of group defense by first prey population	0.1
α_3	Effect of first prey species on second due to competition	0.01
α_q	Predation rate of second prey	0.8
β_3	Saturation constant by Beddington-DeAngelis	1.5
η_1	Measure of interference between predators	0.08
α_4	Effect of second prey species on first due to competition	0.02
γ_1	Rates of conversing first prey species into predator	0.35
γ_2	Rates of conversing second prey species into predator	0.45
b_{ne}	Death rate of predator	0.45
e_1	Impulsive harvesting rate of first prey species	0.01
e_2	Impulsive harvesting rate of second prey species	0.05
e_3	Impulsive release amount of mature natural enemy	13.6
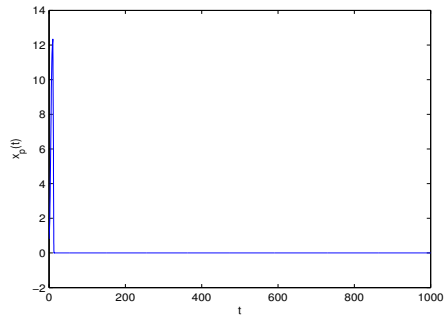
(a) Time series of species x_p
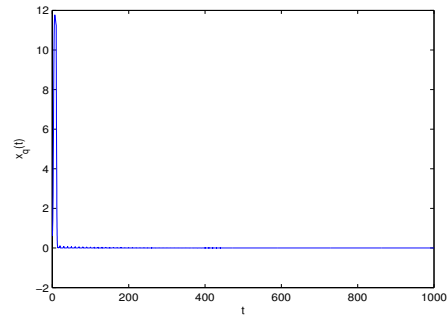
(b) Time series of pecies x_q
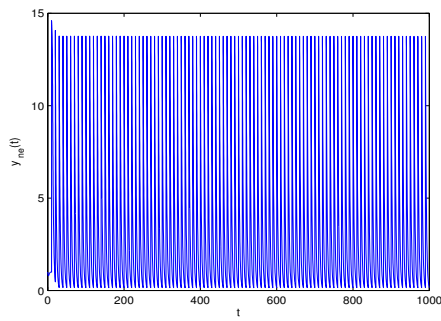
(c) Time series of species y_{ne}

Figure 5.1: Global stability of prey extinction periodic solution $(0,\ 0,\ \overline{y}_{ne}(t))$ of system (5.1) at $e_3(=13.6) > e_3^{**}$ with $x_p(0^+ = 0.8)$, $x_q(0^+) = 0.6$, $y_{ne}(0^+) = 1$

(a) Time series of species x_p

(b) Time series of species x_q

(c) Time series of species y_{ne}

(d) Phase portrait of the system (5.1) for $e_1 = 0.01$, $e_2 = 0.05$, $e_3 = 6.5$
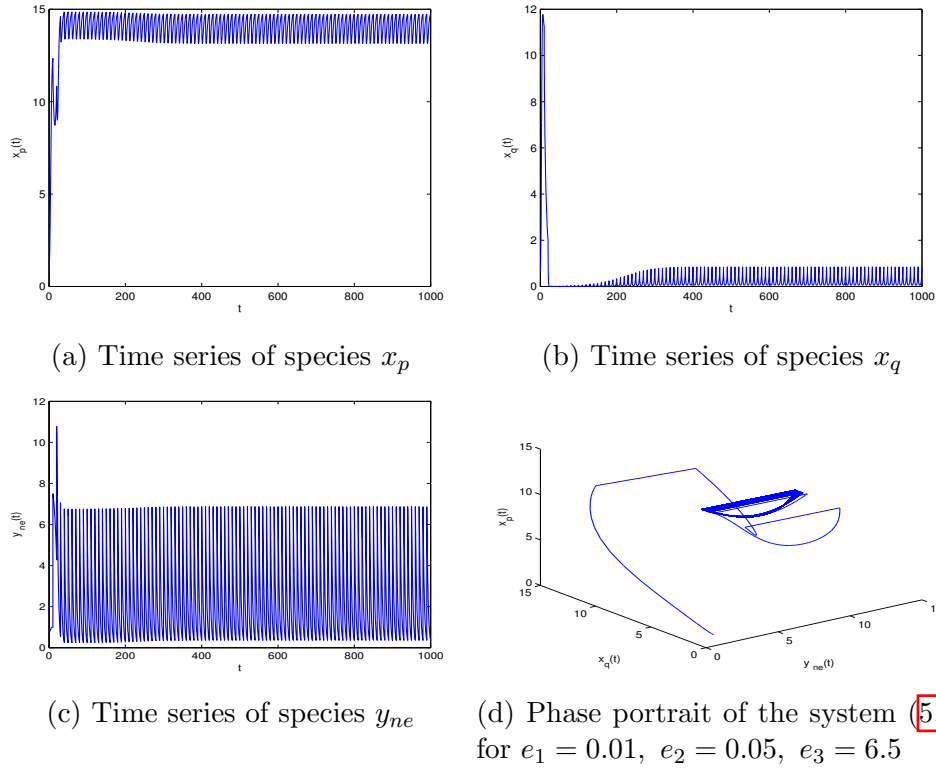
Figure 5.2: Permanence of the system (5.1) at $e_3(= 6.5) < e_3^*$ with $x_p(0^+ = 0.8)$, $x_q(0^+) = 0.6, y_{ne}(0^+) = 1$

(a) Time series of species x_p

(b) Time series of species x_q
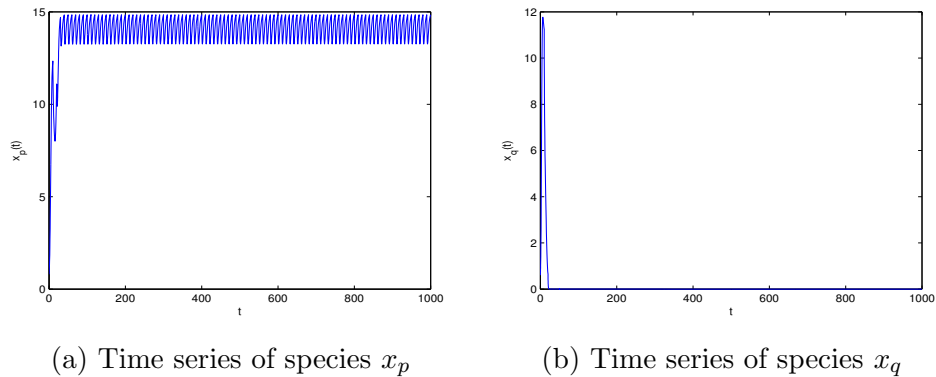
Figure 5.3: Graphical representation of the solution of the system (5.1) at $e_3 = 7$ $i.e. e_3^* \leq e_3 < 8$ with $x_p(0^+) = 0.8, x_q(0^+) = 0.6, y_{ne}(0^+) = 1$ and $e_1 = 0.01, e_2 = 0.05$.
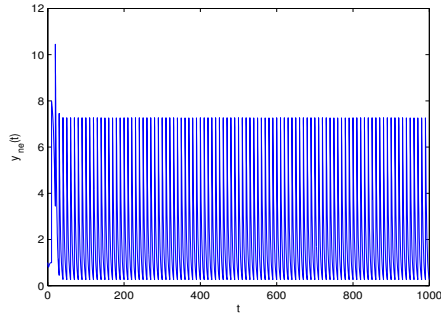
(c) Time series of species y_{ne}

Figure 5.3: Continued
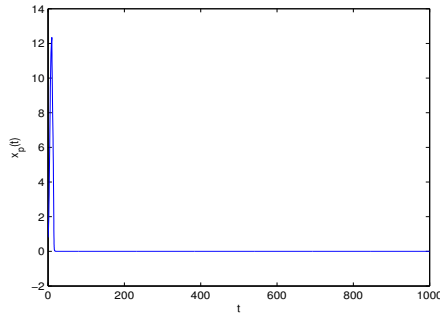
(a) Time series of species x_p
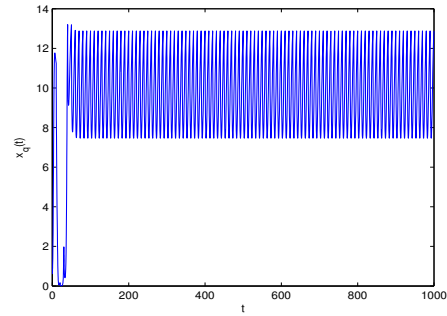
(b) Time series of species x_q
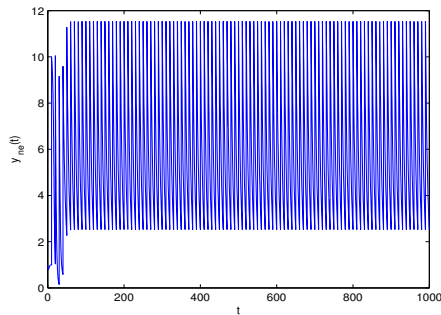
(c) Time series of species $y_{ne}(t)$

Figure 5.4: Graphical representation of the solution of the system (5.1) at $e_3 = 9$ $i.e. 8 \leq e_3 < e_3^{**}$ with $x_p(0^+) = 0.8, x_q(0^+) = 0.6, y_{ne}(0^+) = 1$ and $e_1 = 0.01, e_2 = 0.05$.

5.5 Conclusion

The dynamics of a food web model under the influence of impulsive perturbations is studied analytically and numerically. Here, we have taken mixed functional response as first prey species exhibit the phenomenon of group defense while in the second case, mutual interference in predators is considered. Hence, Holling Type IV functional response is used for the predation of first prey species while Beddington-DeAngelis type in the predation of second prey species. It is proved that the prey free equilibrium $(0,\ 0,\ \overline{y}_{ne}(t))$ is globally stable if the conditions in Theorem 5.2.2 are satisfied. Also, conditions for the permanence of the system (5.1) are established. It is also concluded that the system (5.1) follows the principle of competitive exclusion.

Chapter 6

A Three Tropic Level Food Chain Model Considering Holling Type II and IV Functional Response for Impulsive Pest Control Strategy

6.1 Introduction

Preservation of non-renewable resources and protection of environment for coming generations while satisfying human requirements for fodder is the main aim of sustainable agriculture. It's biggest component is pest management. In order to prevent major economic and production loss, it is the need of hour to control pest population. Pesticides are widely being used to eradicate pests [60, 106]. But there are some big issues with the use of pesticides. Firstly, these are responsible for environmental pollution up to great extent and identified as a health hazard to mankind. Secondly, aquatic bodies suffer due to water pollution caused by pesticides. Pesticides are harmful to beneficial insects such as pollinators. Further, due to high cost, small scale farmers are finding it hard to use chemical pesticides

[54]. Moreover, after long term use, pests even became resistant to pesticides.

Therefore, chemical pesticides must be combined with some other pest control techniques to get maximum benefit and minimum loss. This is called integrated pest management. Biological control is proved to be boon for te same. It includes identifying specific natural enemies of the targeted pest population. These enemies can be predators, parasites or some microbial control agents [137]. All these help to suppress growth of pest population. Natural enemies either kill the pests or hinder their biological process resulting in death of pests. Biological control is used for both open crop field crops and greenhouses. In Netherlands and United Kingdom, the parastoid Encarsia Formosa is used on wider scale to control tomato pest Trialeurodes Vaporariorum [109].

In this chapter, pesticides are applied along impulsive release of natural enemies to manage the pest population. It is observed that many of these insect pests do not cause much damage in their native habitat. But, the problem becomes serious when they migrate into the region where there are no natural enemies. Hence, specified natural enemies can be reared or stocked under favorable environmental conditions and then released periodically in targeted regions to kill pests [138]. Therefore, in our work threshold value of impulsive period for impulsive release of natural enemies is calculated in order to check the pest population. Since pesticides and natural enemies are released periodically, so this can be well analyzed using impulsive differential equations. [8, 10]. There are plethora of applications of impulsive differential equations in Ecology and other applied sciences [24]. Also pest management can be studied effectively with the help of perturbed prey-predator interactions. Great achievements have been made by eminent researchers by considering prey as pest and natural enemies as predators.

Further, functional response of prey population to predator has an important role in predation. This response can be prey dependent (Holling type) or both prey and predator dependent (Beddington-DeAngelis type). Liu and Chen [110] analyzed Lotka-Volterra predator-prey system with impulsive perturbations using Holloing Type II functional response and studied the chaotic behavior of system.

Zhang [139] established two pest-one natural enemy model, and found threshold value of impulsive period for pest free equilibrium. Similarly, valuable results have been obtained in [140, 123, 124, 141] considering food chain and food web models for impulsive pest control strategy. Zhang [142] studied the bifurcation analysis of prey-predator impulsive pest control model with Holling type IV functional response. He found that bifurcation depends on the impulsive release amount of natural enemies. Differnt threshold values of impulsive period have been obtained in [143, 144] for permanence of the system.

Furthermore, good biological understanding of different life stages (immature larva, mature adult) of pests and natural enemies must be there for effectiveness of biological pest control. Hence, Jatav and Dhar [72] considered a stage structured (in natural enemies) plant-pest-natural enemy (food-chain) model to find the conditions for permanence of the system. Again, Bhanu et.al. [136] extended the above work by analyzing stage- structure in pests also.

In view of the above, a three tropic level plant-pest-natural enemy food chain model is developed using Holling type II and IV functional responses for impulsive pest control strategy. Pesticides and natural enemies are released periodically and simultaneously with impulsive period to manage pest population.

6.2   Mathematical Model

The following predator-prey food chain model is proposed in this chapter. Here, prey act as plant crop, mid level prey plays the role of pest and top predator is the specified natural enemy. The model is formulated under some assumptions as follows

(A_1) The prey (plant) grows logistically in the absence of predator.

(A_2) Prey response to mid level predator is Holling type II and mid level prey response to top predator is Holling type IV.

(A_3) Pesticides do not cause any harm to natural predators.

(A_4) For the integrated pest control, pesticides are sprayed and natural enemies are released periodically at time $t = n\tau$ with intensities θ_1 and θ_2, respectively where τ is the impulsive period.

$$
\begin{cases}
\dfrac{dx_c}{dt} = \alpha x_c \left(1 - \dfrac{x_c}{\beta}\right) - \dfrac{\alpha_c x_c y_p}{1 + \gamma_1 x_c}, \\[2mm]
\dfrac{dy_p}{dt} = \dfrac{\alpha_c x_c y_p}{1 + \gamma_1 x_c} - \dfrac{\alpha_p y_p z_{ne}}{1 + \gamma_2 y_p^2} - \delta_1 y_p, \\[2mm]
\dfrac{dz_{ne}}{dt} = \dfrac{\alpha_p y_p z_{ne}}{1 + \gamma_2 y_p^2} - \delta_2 z_{ne}, \\[2mm]
\left.\begin{array}{l}\Delta y_p(t) = -\theta_1 y_p, \\[1mm] \Delta z_{ne}(t) = \theta_2,\end{array}\right\} \quad t = n\tau, \ n \in \ Z_+.
\end{cases}
\Bigg\} \ t \neq n\tau,
\tag{6.1}
$$

The different parameters used in (6.1) are defined as follows

(i) x_c, y_p, z_{ne} be the densities of prey, mid level predator and top predator at time t.

(ii) $\alpha > 0$ is the intrinsic reproduction rate of susceptible pests and $\beta > 0$ is the carrying capacity.

(iii) α_c and α_p be the discovery rates by Holling and γ_1, $\gamma_2 > 0$ are the half saturation constants.

(iv) δ_1, δ_2 be the death rates of mid level predator and top predator, respectively.

Let $R_+ = [0, \infty)$, $R_+^3 = \{x \in R^3 : x \geq 0\}$, $\Omega = int R_+^3$. The map defined by the right hand of the system (2.1) is given as $g = (g_1, g_2, g_3)^T$. Let $S_0 = \{V \colon R_+ \times R_+^3 \mapsto R_+$, continuous on $(n\tau, (n+1)\tau] \times R_+^3$ and $\lim_{(t,y)\to(n\tau,x),t>n\tau} S(t,x) = S(n\tau^+, x)$ exits$\}$.

Our main aim here is to suppress the pests in a targeted region beneath a tolerable limit so that it does not cause major production loss.

6.3 Boundedness and Global Stability

6.3.1 Upper bound of all the variables

Here, in this section, firstly, upper bound for all the variables of system(6.1) are obtained in the following lemma.

Lemma 6.3.1. *For sufficiently large t, there exists a constant $L > 0$ such that $x_c \leq L$, $y_p \leq L$, $z_{ne} \leq L$. That is there is an upper bound for every solution of (6.1).*

Proof. Suppose $(x_c(t), y_p(t), z_{ne}(t))$ be any solution of (6.1). Let $Q(t) = x_c(t) + y_p(t) + z_{ne}(t)$ then for $t \neq n\tau$,

$$D^+Q(t) + pQ(t) = \alpha x_c - \frac{\alpha x_c^2}{\beta} - \frac{\alpha_c x_c y_p}{1 + \gamma_1 x_c} + \frac{\alpha_c x_c y_p}{1 + \gamma_1 x_c} - \frac{\alpha_p y_p z_{ne}}{1 + \gamma_2 y_p^2}$$

$$+ \frac{\alpha_p y_p z_{ne}}{1 + \gamma_2 y_p^2} - \delta_1 y_p - \delta_2 z_{ne} + p(x_c + y_p + z_{ne})$$

$$= (\alpha + p)x_c - \frac{\alpha x_c^2}{\beta} - (\delta_1 - p)y_p - (\delta_2 - p)(z_{ne}$$

$$\leq (\alpha + p)x_c - \frac{\alpha x_c^2}{\beta}$$

$$\leq \frac{\beta(\alpha + p)^2}{4\alpha} = L_0$$

$Q(n\tau^+) = Q(n\tau) + \theta_2, for \ t = n\tau.$

Therefore by Theorem 1.5.1,

$$Q(t) \leq Q(0) \exp\left(\int_0^t (-p)\right) ds + \theta_2 \sum_{0 < n\tau < t} \exp \int_{n\tau}^t (-p) ds + \int_0^t \left(L_0 \exp \int_s^t (-pd\sigma)\right) ds$$

$$\leq Q(0) \exp(pt) + (\theta_2) \sum_{0 < n\tau < t} \exp(-p(t - n\tau)) + \frac{L_0}{p}(1 - \exp(-pt))$$

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$$\leq Q(0)\exp(-pt) + \frac{L_0}{p}(1 - \exp(-pt)) + \frac{\theta_2(\exp(-p(t-\tau)))}{1 - \exp(-p\tau)} + \frac{\theta_2(\exp(pt))}{\exp(p\tau) - 1}$$

$$\rightarrow \frac{L_0}{p} + \frac{\theta_2(\exp(p\tau))}{\exp(p\tau) - 1} \text{ as } t \rightarrow \infty$$

This implies

$$Q(t) \leq L, \ L = \frac{L_0}{p} + \frac{\theta_2(\exp(p\tau))}{\exp(p\tau) - 1}$$

Thus Q(t) is uniformly bounded. Hence, $\exists$ the constant L such that $x_c \leq L$, $y_p \leq L$, $z_{ne} \leq L$. This completes the proof. $\square$

After using chemical pesticides and natural enemies, when pest population becomes extinct, then $y_p = 0$, the impulsive system (6.1) reduces to

$$\begin{cases} \left. \begin{aligned} \frac{dx_c}{dt} &= \alpha_c \left(1 - \frac{x_c}{\beta}\right) \\ \frac{dz_{ne}}{dt} &= -\delta_2 z_{ne}, \end{aligned} \right\} t \neq n\tau, \\ \Delta z_{ne} = \theta_2, \quad \left. \right\} t = n\tau, \ n \in \ Z_+. \end{cases} \tag{6.2}$$

First equation of (6.2) is simply logistic population growth model. It has two equilibrium points 0 and β. $x_c = 0$ is unstable while $x_c = \beta$ is stable equilibrium point. Also applying Lemma 1.5.3 on second and third equations of (6.2), we get globally asymptotically stable periodic solution $\overline{z}_{ne}(t)$ as

$$\overline{z}_{ne}(t) = \frac{\theta_2 \exp((-\delta_2)(t - n\tau))}{1 - \exp((-\delta_2)\tau)}, \ \overline{z}_{ne}(0^+) = \frac{\theta_2}{1 - \exp((-\delta_2)\tau)}. \tag{6.3}$$

Now, system (6.1) has two pest extinction equilibrium points $(0, \ 0, \ \overline{z}_{ne}(t))$ and $(\beta, \ 0, \ \overline{z}_{ne}(t))$.

Theorem 6.3.1. *Let $(x_c(t), \ y_p(t), \ z_{ne}(t))$ be any solution of system (6.1), then*

(i) *$\overline{X}_1 = (0, \ 0, \ \overline{z}_{ne}(t))$ is unstable.*

(ii) *There exists a threshold value(τ_{max}) of the impulsive period such that if $\tau \leq \tau_{max}$, then the pest eradication solution $\overline{X}_2 = (\beta, \ 0, \ \overline{z}_{ne}(t))$ is locally*

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asymptotically stable and if $\tau > \tau_{max}$, it is unstable where

$$\tau_{max} = \left(\frac{\theta_2 \alpha_p}{\delta_2} - \ln(1 - \theta_1) \right) \left(\frac{1 + \gamma_1 \beta}{\alpha_c \beta - \delta_1 - \delta_1 \gamma_1 \beta} \right).$$

Proof. (i) Here, we use small perturbation method to prove the local stability of the required solution. Let $\zeta_1(t)$, $\zeta_2(t)$, $\zeta_3(t)$ be the small perturbations in $0, 0$ and $\bar{z}_{ne}$, respectively. Then

$$x_c(t) = \zeta_1(t), \ y_p(t) = \zeta_2(t), \ z_{ne}(t) = \bar{z}_{ne}(t) + \zeta_3(t).$$

Putting these values in system (6.1) and after linearization it reduces to

$$\begin{cases} \dfrac{d\zeta_1(t)}{dt} = \alpha \zeta_1(t), \\[2mm] \dfrac{d\zeta_2(t)}{dt} = -\alpha_p \bar{z}_{ne}(t)\zeta_2 - \delta_1 \zeta_2, \\[2mm] \dfrac{d\zeta_3(t)}{dt} = \alpha_p \bar{z}_{ne}(t)\zeta_2 - \delta_2 \zeta_3, \\[2mm] \zeta_1(n\tau^+) = \zeta_1(t), \\[2mm] \zeta_2(n\tau^+) = (1 - \theta_1)\zeta_2(t), \\[2mm] \zeta_3(n\tau^+) = \zeta_3(t), \end{cases} \left. \begin{array}{l} \\ \\ t \neq n\tau, \\ \\ \\ \\ t = n\tau, \ n \in \ Z_+. \end{array} \right\} \tag{6.4}$$

Then (6.4) represents system of linear differential equations, which can be written in matrix form. Hence for $t \neq n\tau$, the coefficient matrix is given as

$$B = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & -\alpha_p \bar{z}_{ne}(t) - \delta_1 & 0 \\ 0 & \alpha_p \bar{z}_{ne}(t) & -\delta_2 \end{bmatrix},$$

and for $t = n\tau$

$$\begin{bmatrix} \zeta_1(n\tau^+) \\ \zeta_2(n\tau^+) \\ \zeta_3(n\tau^+) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_1(n\tau) \\ \zeta_2(n\tau) \\ \zeta_3(n\tau) \end{bmatrix},$$

Let $\Phi(t)$ be the fundamental solution matrix of system (6.4), then

$$\frac{d\Phi(t)}{dt} = B\Phi(t), \tag{6.5}$$

$$\Phi(t) = \Phi(0) \exp(\int_0^T A \, dt),$$

with $\Phi(0) = I$, the identity matrix. Solving for $\Phi(t)$, we have

$$\Phi(t) = \begin{bmatrix} exp(\int_0^\tau \alpha dt) & 0 & 0 \\ 0 & exp(\int_0^\tau -\alpha_p \bar{z}_{ne}(t) - \delta_1 dt) & 0 \\ 0 & \exp(\int_0^\tau \alpha_p \bar{z}_{ne}(t) dt) & \exp(\int_0^\tau -\delta_2 dt) \end{bmatrix},$$

Now according to Floquet Theory of impulsive differential equations(Remark 1.5.8.1 and Theorem 1.5.5), if absolute values of all the eigen values of Monodromy matrix M are less than one, then the required solution is globally stable where

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - \theta_1 & 0 \\ 0 & 0 & 1\phi(\tau) \end{bmatrix} \phi(\tau).$$

The eigen values of M are

$$\lambda_1 = exp\left(\int_0^\tau \alpha dt\right),$$

$$\lambda_2 = (1 - \theta_1) \exp\left(\int_0^\tau -\alpha_p \bar{z}_{ne}(t) - \delta_1 dt\right), \tag{6.6}$$

$$\lambda_3 = \exp\left(\int_0^\tau -\delta_2) dt\right).$$

Now it is obvious from (6.6) that $|\lambda_1| > 1(\alpha > 0)$. Hence the pest extinction solution $(0, 0, \bar{z}_{ne}(t))$ is unstable.

(ii) Similarly, we can discuss the local stability of second pest extinction equilibrium point $(\beta, 0, \bar{z}_{ne}t))$. Here,

$$x_c(t) = \zeta_1(t) + \beta, \ \ y_p(t) = \zeta_2(t), \ \ z_{ne}(t) = \bar{z}_{ne}(t) + \zeta_3(t).$$

Proceeding similarly as above, the Monodromy matrix M in this case is

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1-\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\alpha & -\frac{\alpha_c \beta}{1+\gamma_1 \beta} & 0 \\ 0 & \frac{\alpha_c \beta}{1+\gamma_1 \beta} - \alpha_p \overline{z}_{ne}(t) - \delta_1 & 0 \\ 0 & \alpha_p \overline{z}_{ne}(t) & -\delta_2 \end{bmatrix} \qquad (6.7)$$

Therefore, the eigen values of M are

$$\lambda_1 = -\alpha\tau < 0,$$
$$\lambda_2 = (1-\theta_1)\exp\left(\int_0^\tau \frac{\alpha_c \beta}{1+\gamma_1 \beta} - \alpha_p \overline{z}_{ne}(t) - \delta_1 dt\right), \qquad (6.8)$$
$$\lambda_3 = -\delta_2 \tau.$$

Now, it is obvious from (6.8) that $\mid \lambda_1 \mid < 1$, $\mid \lambda_3 \mid < 1$ and $\mid \lambda_2 \mid < 1$ if $\tau \leq \tau_{max}$. Hence the required result.

$\square$

6.3.2 Global Stability

Theorem 6.3.2. *There is a threshold value($\check{\tau}$) of the impulsive period such that if $\tau < \check{\tau}$, then the pest eradication solution $(\beta, 0, \overline{z}_{ne}(t))$ is globally asymptotically stable where*

$$\check{\tau} = \left(\frac{\theta_2 \alpha_p}{\delta_2} - \ln(1-\theta_1)\right)\left(\frac{1}{\alpha_c \beta - \delta_1}\right)$$

.

Proof. Let $(x_c(t), y_p(t), z_{ne}(t))$ be arbitrary solution of (6.1). Given that $\tau < \check{\tau}$, so, it is possible to find sufficiently small $\tilde{\varepsilon}_1 > 0$ such that

$$\int_0^\tau (\alpha_c(\beta + \tilde{\varepsilon}_1) - \alpha_p(\overline{z}_{ne}(t) - \tilde{\varepsilon}_1 - \delta_1)) \, dt = \varrho_1 < 0. \qquad (6.9)$$

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From (6.1),

$$\frac{dx_c}{dt} \leq \alpha x_c \left(1 - \frac{x_c}{\beta}\right) \tag{6.10}$$

Consider its comparison differential equation

$$\frac{du_c}{dt} = \alpha u_c \left(1 - \frac{u_c}{\beta}\right) \tag{6.11}$$

Using comparison analysis technique of ordinary differential equations, $x_c \leq u_c \to \beta$ as $t \to \infty$. Therefore,

$$x_c \leq \beta + \tilde{\varepsilon}_1 \text{ for } t > \kappa_1 \tau.$$

From (6.1),

$$\begin{cases} \frac{dz_{ne}(t)}{dt} \geq -\delta_2 z_{ne}(t), & t \neq n\tau, \\ \Delta z_i(t) = \theta_2, & t = n\tau, \ n \in Z_+. \end{cases} \tag{6.12}$$

Using comparison analysis technique of impulsive differential equations and Lemma 1.5.3, solution of (6.12) must satisfies $z_{ne}(t) \geq \bar{z}_{ne}(t) - \tilde{\varepsilon}_1 \ \forall \ t \geq \kappa_2 t$. Again from system (6.1)

$$\begin{cases} \frac{dy_p(t)}{dt} \leq \alpha_c(\beta + \tilde{\varepsilon}_1) - \alpha_p(\bar{z}_{ne}(t) - \tilde{\varepsilon}_1 - \delta_1), & t \neq n\tau, \\ \Delta y_p(t) = -\theta_1 y_p, & t = n\tau, \ n \in Z_+. \end{cases} \tag{6.13}$$

Integrating first equation of system (6.13) on $(\kappa_2\tau, \ (\kappa_2 + 1)\tau]$ gives

$$y_p((\kappa_2 + 1)\tau) \leq y_p(\kappa_2\tau) \exp\left(\int_{\kappa_2\tau}^{(\kappa_2+1)\tau} \alpha_c(\beta + \tilde{\varepsilon}_1) - \alpha_p(\bar{z}_{ne}(t) + \tilde{\varepsilon}_1 - \delta_1)dt\right) \tag{6.14}$$

After using impulsive jump condition from second equation of system (6.13), we obtain the stroboscopic map

$$y_p((\kappa_2 + 1)\tau) \leq (1 - \theta_1)y_p(\kappa_2\tau) \exp(\varrho_1)$$

This implies, for some positive integer q, we get

$$y_p((\kappa_2 + q)\tau) \leq (1 - \theta_1)^q y_p(\kappa_2 \tau) \exp(q\varrho_1) \to 0 \text{ as } t \to \infty \ (\because \ \varrho_1 < 0) \qquad (6.15)$$

This implies, there exists a positive integer $\kappa_3 > \kappa_2$ and sufficiently small $\tilde{\varepsilon}_2 > 0$ such that $y_p(t) < \tilde{\varepsilon}_2$ for $t \geq \kappa_3$ and $\tilde{\varepsilon}_2 < \frac{\delta_2}{\alpha_p}$. Using maximum value of $y_p(t)$ in first equation of system (6.1), we get

$$\frac{dx_c}{dt} \geq \alpha x_c \left(1 - \frac{x_c}{\beta} - \alpha_c \tilde{\varepsilon}_2 \right).$$

So $\lim_{t \to \infty} x_c(t) = \beta$. Now, consider the following subsystem of (6.1)

$$\begin{cases} \frac{dz_{ne}(t)}{dt} \leq (\alpha_p \tilde{\varepsilon}_2 - \delta_2) z_{ne}(t), & t \neq n\tau, \\ \Delta z_i(t) = \theta_2, & t = n\tau, \ n \in Z_+. \end{cases} \qquad (6.16)$$

Likewise, applying comparison analysis technique and Lemma 1.5.3, system (6.16) has periodic solution

$$\overline{w}_{ne}(t) = \frac{\theta_2 exp(-(\delta_2 - \alpha_p \tilde{\varepsilon}_2)(t - n\tau))}{1 - exp(-(\delta_2 - \alpha_p \tilde{\varepsilon}_2)\tau)}.$$

such that $z_{ne}(t) < \overline{w}_{ne}(t) + \tilde{\varepsilon}_3 \ \forall \ t \geq \kappa_4 \tau$. As $\tilde{\varepsilon}_1 > 0$, $\tilde{\varepsilon}_2 > 0$ and $\tilde{\varepsilon}_3 > 0$ are sufficiently small, therefore, $\overline{w}_{ne}(t) \to \overline{z}_{ne}(t)$ as $t \to \infty$. Hence it is proved that $x_c \to \beta$, $y_p(t) \to 0$ and $z_{ne}(t) \to \overline{z}_{ne}(t)$ as $t \to \infty$. $\qquad \square$

6.4 Permanence and Numerical Simulation

Firstly, condition for the system to be permanent is established as follows

Theorem 6.4.1. *The system (6.1) is permanent if $\tau > \check{\tau}$.*

Proof. Upper bound of $x_c(t)$, $y_p(t)$, $z_{ne}(t)$ of the system is already been obtained

in Lemma 6.3.1. Also in the above section, it is proved that

$$z_{ne}(t) > \bar{z}_{ne}(t) - \dot{\varepsilon} = r_1 \; \forall \; t \geq \kappa_2 \tau. \tag{6.17}$$

Also

$$\frac{dx_c}{dt} \geq \alpha x_c \left(1 - \frac{x_c}{\beta} - \alpha_c L \right).$$

This implies $x_c > (1 - \alpha_c L)\beta = r_2$ for sufficiently large t. Thus, for permanence of the system (6.1), there must exists a constant $r_3 < \frac{\delta_2}{\alpha_p}$ such that $y_p(t) \geq r_3$ for sufficiently large t. This is done in two steps as follows:

Step I To start with, assume that $y_s(t) \geq r_3$ is not true $\forall \; t$. Thus $\exists$ a positive integer $\tilde{t}_1$ such that $y_p < r_3 \; \forall \; t \geq \tilde{t}_1$. Considering this assumption, from (6.1), we have

$$\begin{cases} \frac{dz_{ne}(t)}{dt} \leq (\alpha_p r_3 - \delta_2) z_{ne}), & t \neq n\tau, \\ \Delta z_{ne}(t) = \theta_2, & t = n\tau. \end{cases}$$

Consider the following impulsive system

$$\begin{cases} \frac{du_{ne}(t)}{dt} = (\alpha_p r_3 - \delta_2) u_{ne}, & t \neq n\tau, \\ \Delta u_{ne}(t) = \theta_2, & t = n\tau. \end{cases} \tag{6.18}$$

Applying Lemma 1.5.3, (6.18) has periodic solution

$$\bar{u}_{ne}(t) = \frac{\theta_2 \exp(-(\delta_2 - \alpha_p r_3)(t - n\tau))}{1 - \exp(-(\delta_2 - \alpha_p r_3)\tau)}, \; t \in (n\tau, (n+1)\tau].$$

which is globally asymptotically stable. Therefore by Theorem 1.5.2, $z_{ne}(t) \leq u_{ne}(t) \to \bar{u}_{ne}(t)$. Hence, $\exists$ a positive integer κ_5 such that

$$z_{ne}(t) \leq \bar{u}_{ne}(t) + \tilde{\varepsilon}_4 \forall t \geq \kappa_5 \tau. \tag{6.19}$$

Therefore, $x_c > r_2$ implies that for $t > \kappa_5 \tau$, we have the following subsystem of

$$\begin{cases} \dfrac{dy_p(t)}{dt} \geq \alpha_c r_2 - \alpha_p(\overline{u}_{ne}(t) - \tilde{\varepsilon}_4 - \delta_1)y_p, & t \neq n\tau, \\ \Delta y_p(t) = -\theta_1 y_p, & t = n\tau,\ n \in Z_+. \end{cases} \tag{6.20}$$

Integration of (6.20) on $(\kappa_5\tau,\ (\kappa_5 + 1)\tau]$ gives the stroboscopic map

$$y_p((\kappa_5 + 1)\tau) \geq y_p(\kappa_5\tau)(1 - \theta_1) \exp\left(\int_{\kappa_5\tau}^{(\kappa_5+1)\tau} [\alpha_c r_2 - \alpha_p(\overline{u}_{ne}(t) + \tilde{\varepsilon}_4) - \delta_1]\,dt \right)$$

$$\geq y_p(\kappa_5\tau)\varrho_3.$$

where

$$\varrho_3 = (1 - \theta_1) \exp\left(\int_{\kappa_5\tau}^{(\kappa_5+1)\tau} [\alpha_c r_2 - \alpha_p(\overline{u}_{ne}(t) + \tilde{\varepsilon}_4) - \delta_1]\,dt \right).$$

Because $\tau > \check{\tau}$, so it is possible to find $r_2 > 0$ and $\tilde{\varepsilon}_4 > 0$ such that $\varrho_2 > 1$. This implies

$$y_p[(\kappa_5 + q)\tau] \geq y_p(\kappa_5\tau)\varrho_2^q \to \infty$$

as $q \to \infty$ This is in contradiction to our assumption that $y_p < r_3 \ \forall \ t \geq \tilde{t}_1$. Hence $\exists\ \tilde{t}_2 > \tilde{t}_1$ such that $y_p(\tilde{t}_2) \geq r_3$.

Step II There is nothing to prove if $y_p(t) \geq r_3 \ \forall\ t > \tilde{t}_2$. But if this is not true, let $\tilde{t}_3 = \inf\{t \mid y_p(t) < r_3; t > \tilde{t}_2\}$. Thus $y_p(t) \geq r_3 \ \forall\ t \in [\tilde{t}_2,\ \tilde{t}_3], \tilde{t}_3 \in (\check{n}_1\tau, (n_1 \overset{\vee}{+} 1)\tau]$. $y_p(\tilde{t}_3) = r_3$, because of continuity of $y_p(t)$. Let $\tau^* = (\check{n}_2 + \check{n}_3)\tau$ where $\check{n}_2$ and $\check{n}_3$ satisfies the following conditions

$$(\check{n}_2)\tau > -\left(\frac{1}{\delta_2 - \alpha_p r_3} \right) \ln \frac{\tilde{\varepsilon}_4}{L + \theta_2},$$

$$(1 - \theta_1)^{\check{n}_2 + \check{n}_3 + 1} \exp(\check{n}_3\varrho_3 - \upsilon(\check{n}_2 + 1)\tau) > 1. \tag{6.21}$$

where

$$\upsilon = (\alpha_c r_2 - \alpha_p L - \delta_1) < 0$$

Now, we will prove that $\exists\ \tilde{t}_4 \in ((\check{n}_1 + 1)\tau,\ (\check{n}_1 + 1)\tau + \tau^*]$ such that $y_p(\tilde{t}_4) \geq r_3$. Suppose this is not true, then $y_p(\tilde{t}_4) < r_3 \ \forall\ t \in ((\check{n}_1 + 1)\tau,\ (\check{n}_1 + 1)\tau + \tau^*]$. If system

(6.18) is considered with $u_{ne}((\check{n}_1+1)\tau^+) = x_s((\check{n}_1+1)\tau^+)$, then using Lemma 1.5.3 for $t \in ((\check{n}_1 + 1)\tau, \ (n_1 \overset{\vee}{+} 1)\tau + \tau^*]$, we have

$$u_{ne}(t) = \left[u_{ne}((\check{n}_1 + 1)\tau^+) - \frac{\theta_2}{1 - exp(-(\delta_2 - \alpha_p r_3)\tau)} \right]$$
$$\times \exp(-(\delta_2 - \alpha_p r_3)(t - (\check{n}_1 + 1)\tau)) + \overline{u}_{ne}(t)$$

This implies

$$\mid u_{ne}(t) - \overline{u}_{ne}(t) \mid \leq (L + \theta_2)\exp(-(\delta_2 - \alpha_p r_3)(t - (\check{n}_1 + 1)\tau))$$
$$\leq \tilde{\varepsilon}_4.$$

which depicts that

$$z_{ne}(t) \leq u_{ne}(t) < \overline{u}_{ne}(t) + \tilde{\varepsilon}_4, \ (\check{n}_1 + \check{n}_2 + 1)\tau \leq t \leq (\check{n}_1 + 1)\tau + \tau^*.$$

Integrating (6.20) on $[(\check{n}_1 + \check{n}_2 + 1)\tau, (\check{n}_1 + \check{n}_2 + \check{n}_3 + 1)\tau]$, we get

$$y_p((\check{n}_1 + \check{n}_2 + \check{n}_3 + 1)\tau) \geq y_p((\check{n}_1 + \check{n}_2 + 1)\tau)(1 - \theta_1)^{\check{n}_3} \exp(\varrho_3 \check{n}_3) \qquad (6.22)$$

Now, for $t \in [t_3, \ (\check{n}_1 + 1)\tau]$, there are two possibilities described below :
Case(i) If $y_p(t) < r_3 \ \forall \ t \in [\tilde{t}_3, \ (\check{n}_1 + 1)\tau]$, then from above assumption $y_p(t) < r_3 \ \forall \ t \in [\tilde{t}_3, \ (\check{n}_1 + 1)\tau + \tau^*]$. This implies

$$\begin{cases} \frac{dy_p(t)}{dt} \geq (\alpha_c r_2 - \alpha_p L - \delta_1)y_p, & t \neq n\tau, \\ \Delta y_p(t) = -\theta_1 y_p, & t = n\tau, \ n \in Z_+. \end{cases} \qquad (6.23)$$

Integrating first equation of system (6.23) in $[\tilde{t}_3, (\check{n}_1 + \check{n}_2 + 1)\tau]$ and using impulse condition, we obtain the stroboscopic map

$$y_p((\check{n}_1 + \check{n}_2 + 1)\tau) \geq y_p(\tilde{t}_3)(1 - \theta_1)^{\check{n}_2+1} \exp(-\upsilon(\check{n}_2 + 1)\tau). \qquad (6.24)$$

Using (6.24) in (6.22)

$$y_p((\check{n}_1 + \check{n}_2 + \check{n}_3 + 1)\tau) \geq y_p(\tilde{t}_3)(1 - \theta_1)^{\check{n}_1 + \check{n}_2 + 1} \exp(\varrho_3 \check{n}_3) \exp(\upsilon(\check{n}_2 + 1)\tau) > r_3.$$

But this contradicts our assumption. Therefore, $y_p(t) \geq r_3$ in $[\tilde{t}_3, (\check{n}_1 + \check{n}_2 + \check{n}_3 + 1)\tau]$ for some t. Let $\tilde{t}_5 = \inf\{t \mid y_p(t) \geq r_3, t > \tilde{t}_4\}$. Due to continuity of $y_p(t)$, $y_p(\tilde{t}_5) = r_3$. Now integration of equation (6.23) on the interval $[\tilde{t}_3, \tilde{t}_5]$ gives

$$\begin{aligned}
y_p(t) &\geq (1 - \theta_1) y_p(\tilde{t}_3) \exp((\upsilon)(t - \tilde{t}_3)) \\
&\geq (1 - \theta_1) r_3 \exp((\upsilon)(t - \tilde{t}_3)) \\
&\geq (1 - \theta_1) r_3 \exp(\upsilon(\check{n}_2 + \check{n}_3 + 1)\tau) = \overline{r_3}.
\end{aligned}$$

Since $y_p(\tilde{t}_5) \geq \overline{r}_3$, so similar process can be continued for $t > \tilde{t}_5$. Therefore, $y_p(t) \geq \overline{r}_3 \forall t > \tilde{t}_2$.

Case(ii) If $\exists \tilde{t}_6 \in [\tilde{t}_3, (\check{n}_1 + 1)\tau]$ such that $y_p(\tilde{t}_6) \geq r_3$, then let $\tilde{t}_7 = \inf\{t \mid y_p(t) \geq r_3, t > \tilde{t}_3\}$. Therefore, $y_p(\tilde{t}_7) = r_3$. Again, integrating equation (6.23) on the interval $[\tilde{t}_3, \tilde{t}_7]$, we get

$$y_p(t) \geq y_p(\tilde{t}_3) \exp((\upsilon)(t - \tilde{t}_3)) \geq r_3 \exp(\upsilon\tau) = \overline{r}_3.$$

So, similar argument can be followed for $t > \tilde{t}_7$. Hence, it is concluded that $y_p(t) \geq \overline{r}_3 \ \forall \ t > \tilde{t}_2$.

Step III Let $a = \min\{r_1, r_2, \overline{r}_3\}$, $\Theta = \{R_+^3, : a \leq x_c(t), y_p(t), z_{ne}(t) \leq L\}$. Thus, from above steps and Lemma 6.3.1, it is proved that each solution of system (6.1) will always remain in region Θ. Therefore, by Definition 1.5.4.2, system 6.1 is permanent. $\qquad\square$

6.5 Numerical Analysis and Discussion

A prey-predator food chain model with harvesting of middle prey and stocking of top predator is constituted and investigated in this chapter to control the out-

Table 6.1: Values of different parameters used in system (6.1)

Parameter	Representation	Its Value (per week)
α	Reproduction rate of prey	1.1
β	Carrying capacity	1.1
α_c	Predation rate by mid level predator (pest)	0.9
α_p	Rate of predation by top predator (natural enemy)	0.9
γ_1	Half saturation constant for Holling II predation	0.1
γ_2	Half saturation constant for Holling IV predation	0.2
δ_1	Death rate of mid level predator	0.4
δ_2	Death rate of top predator (natural enemy)	0.6
θ_1	Impulsive spraying amount of pesticides	0.1
θ_2	Impulsive release amount of natural enemies	3

break of pest population. Mid level prey is taken as pest and top predator plays the role of natural enemy. Use of pesticides is combined with impulsive release of natural enemies for Integrated pest management. Firstly, global stability of mid level predator (pest) free solution is established and then condition for the permanence of system is derived. For this, threshold value of impulsive period is found that depends on releasing amounts of pesticides and natural enemies population. The initial values of population densities of prey, mid level and top predator are $x_c(0^+) = 0.5, y_p(0^+) = 0.5, z_{ne}(0^+) = 1$. The values of different parameters that are used in (6.1) are given in Table 6.1. After simple calculation, we get $\check{\tau} = 7.805$ and $\tau_{max} = 9.256653$. Therefore, by Theorem 6.3.1, it is obtained that pest eradication solution is locally stable if $\tau \leq 9.256653$. Also, Theorem 6.3.2 is verified here that is the pest free solution is globally stable if impulsive period $\tau < 7.805$ (see Figure 6.1). Hence impulsive perturbations contribute a lot to the dynamics of the system since some complex dynamics is there at higher values of impulsive period greater than threshold value. Thus, combination of chemical and natural

(a) Time series of species x_c

(b) Time series of pecies x_p

(c) Time series of species z_{ne}
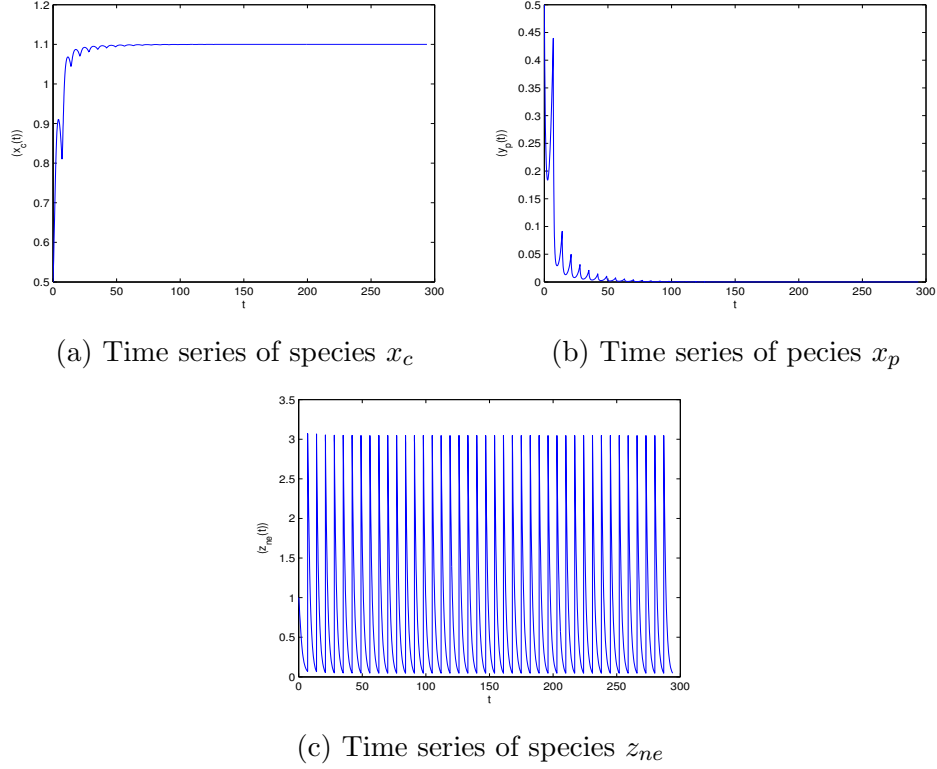
Figure 6.1: Global stability of pest extinction periodic solution $(\beta, 0, z_{ne}(t))$ of system (6.1) at $\tau < \check{\tau}(= 7.805)$

control is very effective for pest control.

6.6   Conclusion

The war between pests and humans is going on from several decades and time to time, different pest control techniques are acquired by mankind. Working on the same path, here we investigated a predator-prey three tropic level model for the purpose of integrated pest management. It is found that instead of using pesticides alone, combination of chemical control along with natural enemies is more efficient in pest control. In Theorem 6.4.1, threshold value of impulsive period $\check{\tau}$ is obtained and it is established that pests can coexist with infected pests

(a) Time series of species x_c

(b) Time series of pecies y_p

(c) Time series of species z_{ne}

(d) Phase plane of species x_c and y_p

(e) Phase plane of species y_p and z_{ne}

(f) Phase portrait of system (6.1)
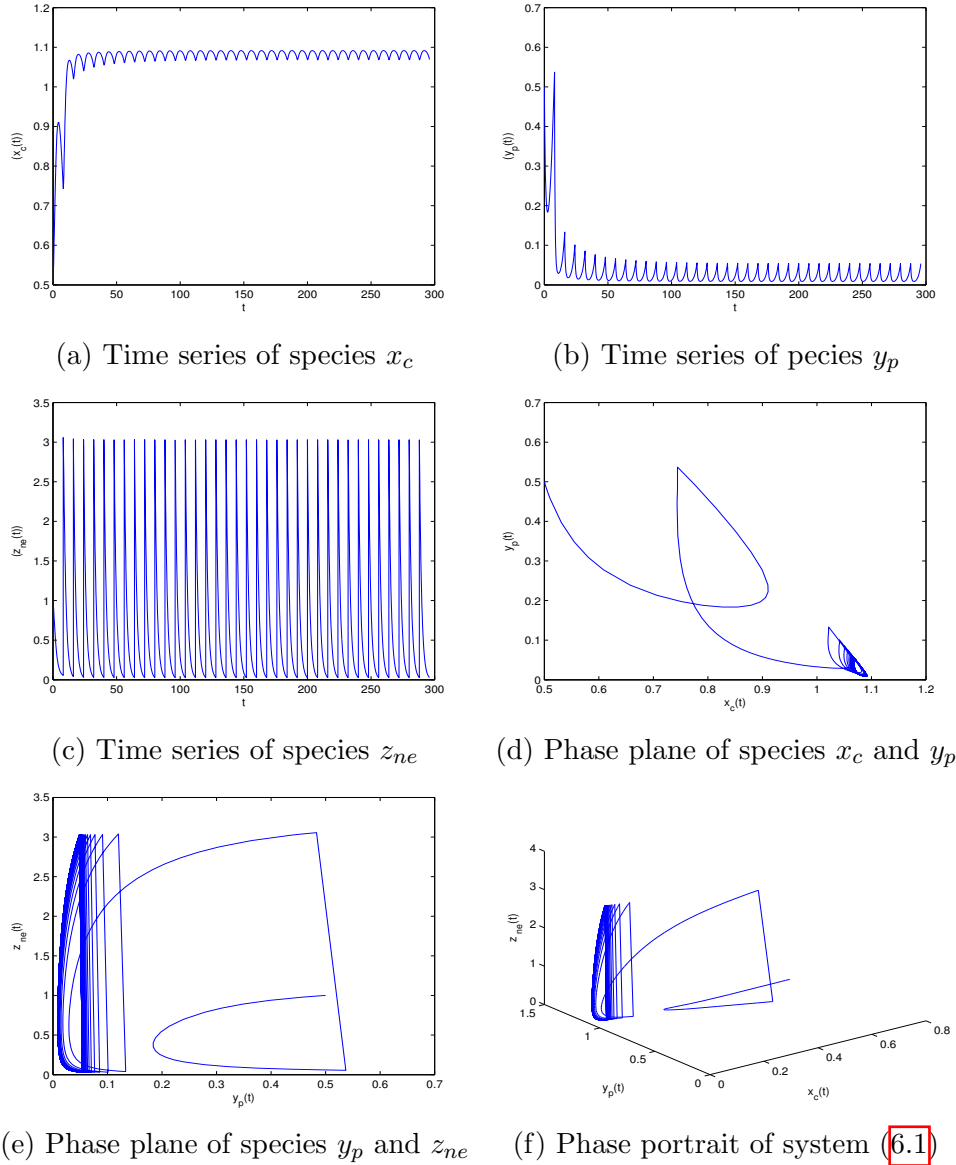
Figure 6.2: Permanence of the system (6.1) at $\tau > \check{\tau}(= 7.805)$ with $x_c(0^+ = 0.5), y_p(0^+) = 0.5, z_{ne}(0^+) = 1$, phase portrait of mid level predator and top predator and phase portrait when system (6.1) is permanent.

and natural enemies if $\tau > \check{\tau}$. Also effect of spraying amount of pesticides and natural enemies is discussed and found that greater releasing amount or small impulsive period support pest eradication.

Chapter 7

Dynamics of Two Species Competitive System Considering Beddington-DeAngelis inter-specific Competition Term under the Effect of Toxic Substances and Impulsive Perturbations

The study of the dynamical behavior of ecological models has remained an interesting area of research over the past few decades. It was started with the single-species model proposed by Malthus [4]. Then Verhulst [5] modified this model to make it more realistic. Further, researchers observed that the study of dynamics of population interactions rather than single-species was more natural. Out of all types of population interactions, prey-predator and competition are extensively studied by

various researchers. The simplest two-species competition model was firstly proposed by Lotka and Volterra. In this model, two species were in competition for the same resources. They concluded that the superior species will ever dominate which eventually lead to the extinction of inferior species. This is called principle of competition exclusion. Sahir Ahmad [91] explored the non-autonomous competitive Lotka-Volterra model. Thus, he established sufficient conditions for the system to follow the principle of competition exclusion.

Furthermore, it was absolutely natural that population interactions undergo sudden changes in the population densities of the concerned species. These changes may happen due to fire, flood, seasonal birth patterns of species or human interventions such as harvesting and stocking of population. To maintain ecological balance, harvesting and stocking is required, accordingly. Therefore, one of the significant mathematical tool to study the effect of these instantaneous disturbances, are impulsive differential equations. These are thoroughly studied by Lakshmikantham and Bainov [8, 10]. There are plethora of applications of impulsive differential equations in ecology and other applied sciences, see [24, 39, 40, 144] and references therein. Liu and Chen [92] studied the dynamics of impulsive periodic Lotka-Volterra competition model. Using Floquet theory of differential equations, they established that impulsive perturbations are helpful to prevent extinction of inferior competitor, so these perturbations encourage avoidance of principle of competition exclusion. Jin et al. [93] studied persistence and extinction in Lotka-Volterra competition model under the influence of impulsive perturbations. In this paper, the authors focused on the average growth rates of the species. Apart from this, Liu et al. [94] confirmed that impulsive perturbations can promote the coexistence of the species during competition.

In addition to this, it was found that while analyzing prey-predator population interactions, the important factor to be considered, is functional response. These interactions are examined with linear functional response, Holling type and ratio dependent also. Therefore, inspired by this, many authors chosen non-linear inter-specific competition term for the two species competition models. Liu et al. [95] derived sufficient conditions for the permanence and extinction of impul-

sive Lotka-Volterra competition model considering Holling type II inter-inhibition term. After that, it was found that some species have to mature to hinder the growth of other competing species. So, delay factor was introduced in impulsive competition systems but it was observed that delays do not have distinguishable effect on the dynamics of system [96]. Apart from the Holling's four different types of functional responses, Beddington and DeAngelis put forward a new kind of functional response that includes the combined effect of Holling II and ratio dependent functional responses. Several scholars studied prey-predator interactions considering Beddington-DeAngelis functional response which includes mutual interference in predator population, see [145, 132, 150]. Kharbanda and Kumar [149] investigated a food web model consisting of one prey and two predators. They considered two different types of functional responses, that is, Beddington-DeAngelis for the first predator and Crowley-Martin for the second predator. They derived sufficient conditions for the local stability of different equilibrium points and global stability of inner equilibrium point. Motivated by this, Yu and Chen [97] examined two species competition model by taking Beddington De-Angelis functional response as inter-specific competition and then they established sufficient conditions for the partial extinction and permanence of the system. He et al. [146] examined the permanence, extinction and global attractivity of nonautonomous competitive system with impulsive effect and infinite delays.

Moving further, it is analyzed that presence of toxic substances in the environment always has prominent effect on the dynamics of the species interactions. Chattopadyhay [98] firstly introduced this concept that some species produce toxic substances in the presence of other to hamper the growth of their competitors. He derived that the effect of toxic substances do contribute in the stability of the competition systems. Kar and Chaudhuri [147] mentioned in their paper that certain aquatic species release toxicants. To exemplify the unicellular green algae *Chlorella vulgaris* releases a toxin that has detrimental effects on its own population as well as on some planktonic algae, *Asterionella formos* and *Nitzschia frusttulum*. Many scholars studied the extinction of species in competition system under the combined influence of toxic substances produced by both the species

in the presence of each other and impulses, see [148, 100, 99, 151, 102, 152]. Li-juan et al. [103] considered the possibility that only one species releases toxin in the presence of other and studied the extinction of species. Fengde et al. [101] proposed two species competition model under the influence of toxic substances. The authors have used Beddington- DeAngelis function as inter-inhibition term and derived sufficient conditions for the extinction of one species and permanence of the other which verified the concept of principle of competition exclusion. But they have ignored the effect of outside disturbances on the population density of the species such as harvesting or stocking of species. These small perturbations have significant effect on the dynamics of population interaction models.

It is very often that ecological systems are enormously disturbed by human exploring activities. Thus, it is more appropriate to consider the model analyzed by Fengde et al. [101] under the influence of impulsive perturbations. Therefore, in this paper, we have analyzed the periodic two species competition model with Beddington-DeAngelis inter-specific competition under the influence of impulsive perturbations.

7.1 Mathematical Model

Let $y_f(t)$ and $y_s(t)$ be the population densities of two species competing for the same living resources. The assumption is that both the species produce toxins in the presence of other. The terms $\gamma_f(t)y_f^2(t)y_s(t)$ and $\gamma_s(t)y_f(t)y_s^2(t)$ represents the effect of toxicans produced by species y_s on y_f and y_f on y_s, respectively. This leads to the following mathematical model with initial conditions $y_f(0) > 0$ & $y_s(0) > 0$. Here, τ_k represents the moments of impulsive effect. The terms $(1 + l_{1k})y_f(\tau_k)$ and $(1 + l_{2k})y_s(\tau_k)$ denotes amount of impulsive harvesting or stocking of both the species at time $t = \tau_k$. If $l_{1k}y_f(\tau_k) > 0$, $l_{2k}y_s(\tau_k) > 0$, then both the species are stocked impulsively and if $l_{1k}y_f(\tau_k) < 0$, $l_{2k}y_s(\tau_k) < 0$, then harvesting of species is done. Also for biological relevance, $(1 + l_{ik}) > 0$, $i = 1, 2$. The biological interpretation of parameters used in the model (7.1) is as follows:

1. $\alpha_1(t)$ and $\alpha_2(t)$ be the intrinsic reproduction rate of species y_f and y_s, respectively.

2. $\beta_1(t)$ and $\beta_2(t)$ be the rates of intra-specific competition within species.

3. $\gamma_f(t)$ and $\gamma_s(t)$ are the measures of the toxic effect that one species exert on another.

4. Because the species are considered to be living in a periodically varying environment, therefore, the coefficients $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$, $a_f(t)$, $a_s(t)$, $\gamma_f(t)$, $\gamma_s(t)$ are all positive periodic continuous functions with period $\check{T}$. There exists a positive integer m such that $\tau_{k+m} = \tau_k + \check{T}$.

$$
\begin{cases}
\dfrac{dy_f(t)}{dt} = y_f(t)\left[\alpha_1(t) - \beta_1(t)y_f(t) - \dfrac{a_f(t)y_s(t)}{b_f(t) + c_f(t)y_f(t) + d_f(t)y_s(t)}\right] \\
\qquad \left(-\gamma_f(t)y_f^2(t)y_s(t)\right), \\
\dfrac{dy_s(t)}{dt} = y_s(t)\left[\alpha_2(t) - \beta_2(t)y_s(t) - \dfrac{a_s(t)y_f(t)}{b_s(t) + c_s(t)y_f(t) + d_s(t)y_s(t)}\right] \\
\qquad - \left(\gamma_s(t)y_f(t)y_s^2(t)\right), \\
y_f(\tau_k^+) = (1 + l_{1k})y_f(\tau_k), \\
y_s(\tau_k^+) = (1 + l_{2k})y_s(\tau_k),
\end{cases}
\begin{array}{l} \\ \\ \left.\rule{0pt}{2.2em}\right\} \, t \neq \tau_k, \\ \\ \\ \left.\rule{0pt}{1.2em}\right\} \, t = \tau_k, \quad k \in N. \end{array}
$$

$$(7.1)$$

Through out this paper, it is assumed that for any function $h(t)$, $h^L = \inf_{-\infty < t < \infty} h(t)$ and $h^M = \sup_{-\infty < t < \infty} h(t)$. Further, average intrinsic growth rate of both the species is considered in this paper.

7.2 Positivity of Solutions

Lemma 7.2.1. *Suppose $Y(t) = (y_f(t),\ y_s(t))$ is any solution of (7.1) with $y_f(0) \geq 0$ and $y_s(0) \geq 0$, then $y_f(t) \geq 0$, $y_s(t) \geq 0 \ \forall\ t > 0$. Also $y_f(t) > 0$, $y_s(t) > 0 \ \forall\ t >$*

0 *if* $y_f(0) > 0$ *and* $y_s(0) > 0$.

Proof. Let

$$P(t) = \left[\alpha_1(t) - \beta_1(t)y_f(t) - \frac{a_f(t)y_{s(t)}}{b_f(t) + c_f(t)y_f(t) + d_f(t)y_s(t)} - \gamma_f(t)y_f(t)y_s(t) \right],$$

$$Q(t) = \left[\alpha_2(t) - \beta_2(t)y_s(t) - \frac{a_s(t)y_{f(t)}}{b_s(t) + c_s(t)y_f(t) + d_s(t)y_s(t)} - \gamma_s(t)y_f(t)y_s(t) \right].$$

Thus if $Y(t) = (y_f(t), y_s(t))^T$ is any solution of (7.1) with $y_f(0) \geq 0$ and $y_s(0) \geq 0$, then integrating first equation of system (7.1) in (τ_n, τ_{n+1}) and using equation third of system (7.1), we obtain

$$y_f(t) = y_f(\tau_n^+) \exp \int_{\tau_n}^t (P(s)ds), \ t \in (\tau_n, \ \tau_{n+1}] \tag{7.2}$$

and

$$y_f(\tau_n^+) = (1 + l_{1k})y_f(\tau_n). \tag{7.3}$$

Therefore, from equation (7.3), we have

$$y_f(t) = (1 + l_{1k})y_f(\tau_n) \exp \int_{\tau_n}^t (P(s)ds), \ t \in (\tau_n, \ \tau_{n+1}] \tag{7.4}$$

This implies

$$y_f(t) = (1 + l_{1k})^2 y_f(\tau_{n-1}) \exp \int_{\tau_n}^t (P(s)ds), \ t \in (\tau_n, \ \tau_{n+1}] \tag{7.5}$$

By repeating this process, we obtain

$$y_f(t) = (1 + l_{1k})^n y_f(0) \exp \int_0^t (P(s)ds), \ t \in (\tau_n, \ \tau_{n+1}] \tag{7.6}$$

Because $y_f(0^+) \geq 0$ and exponential function is always positive. Hence $y_f(t) \geq 0 \ \forall \ t > 0$. Similarly it can be proved for $y_s(t)$. $\qquad \square$

Lemma 7.2.2. *(refer [99]) Let* $Y(t) = (y_f(t), ys(t))$ *is any solution of* (7.1) *with*

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$y_f(t) > 0$ and $y_s(t) > 0$, then we have

$$\lim_{t \to \infty} \sup y_f(t) \leq \phi_{[\alpha_1]}^M \tag{7.7}$$

$$\lim_{t \to \infty} \sup y_s(t) \leq \phi_{[\alpha_2]}^M \tag{7.8}$$

where $\phi_{[\alpha_1]}$ is the unique positive solution of impulsive system

$$\begin{cases} \frac{dy_f(t)}{dt} = y_f(t) \left(\alpha_1(t) - \beta_1(t)y_f(t) \right), & t \neq \tau_k, \\ y_f(\tau_k^+) = (1 + l_{1k})y_f(\tau_k), & t = \tau_k, \ k \in N \end{cases}$$

and $\phi_{[\alpha_2]}$ is the unique positive solution of impulsive system

$$\begin{cases} \frac{dy_s(t)}{dt} = y_s(t) \left(\alpha_2(t) - \beta_2(t)y_s(t) \right), & t \neq \tau_k, \\ y_s(\tau_k^+) = (1 + l_{2k})y_s(\tau_k), & t = \tau_k, \ k \in N \end{cases}$$

7.3   Extinction of one species and Permanence of other

Theorem 7.3.1. *If*

$$\frac{\overline{\alpha_1^L}}{\overline{\alpha_2^M}} \geq max \left\{ \frac{\beta_1^M (b_s^M + c_s^M \phi_{[\alpha_1]}^M + d_s^M \phi_{[\alpha_2]}^M)}{a_s^L}, \ \frac{\gamma_f^M}{\gamma_s^L}, \ \frac{a_f^M}{b_f^L \beta_2^L} \right\}, where \tag{7.9}$$

$$\overline{\alpha_1^L} = \alpha_1^L + \frac{1}{\check{T}} \sum_{k=1}^m \ln(1 + l_{1k}) > 0 \tag{7.10}$$

$$\overline{\alpha_2^M} = \alpha_2^M + \frac{1}{\check{T}} \sum_{k=1}^m \ln(1 + l_{2k}) > 0 \tag{7.11}$$

then for any solution $(y_f(t), y_s(t))$ of the system (7.1), species y_s will be driven to extinction.

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Proof. By using lemma 7.2.2, there exists a positive constant $\acute{\epsilon}_1$ such that

$$y_f(t) < \phi^M_{[\alpha_1]} + \acute{\epsilon}_1 \tag{7.12}$$

$$y_s(t) < \phi^M_{[\alpha_2]} + \acute{\epsilon}_1 \tag{7.13}$$

Now, the given conditions (7.9) can be written as

$$\frac{\beta^M_1}{\frac{a^M_s}{b^M_s + c^M_s(\phi^M_{[\alpha_1]} + \acute{\epsilon}_1) + d^M_s(\phi^M_{[\alpha_2]} + \acute{\epsilon}_1)}} \leq \frac{\overline{\alpha^L_1}}{\underline{\alpha^M_2}}, \quad \frac{\frac{a^M_f}{b^L_f}}{\beta^L_2} \leq \frac{\overline{\alpha^L_1}}{\underline{\alpha^M_2}}, \quad \frac{\gamma^M_f}{\gamma^L_s} \leq \frac{\overline{\alpha^L_2}}{\underline{\alpha^M_2}}.$$

Therefore, it is possible to find two constants ϱ and μ such that

$$\frac{\beta^M_1}{\frac{a^M_s}{b^M_s + c^M_s(\phi^M_{[\alpha_1]} + \acute{\epsilon}_1) + d^M_s(\phi^M_{[\alpha_2]} + \acute{\epsilon}_1)}} \leq \frac{\varrho}{\mu} \leq \frac{\overline{\alpha^L_1}}{\underline{\alpha^M_2}}, \tag{7.14}$$

$$\frac{\frac{a^M_f}{b^L_f}}{\beta^L_2} \leq \frac{\varrho}{\mu} \leq \frac{\overline{\alpha^L_1}}{\underline{\alpha^M_2}}, \qquad \frac{\gamma^M_f}{\gamma^L_s} \leq \frac{\varrho}{\mu} \leq \frac{\overline{\alpha^L_1}}{\underline{\alpha^M_2}}. \tag{7.15}$$

Thus

$$\mu\beta^M_1 - \frac{\varrho a^M_s}{b^M_s + c^M_s(\phi^M_{[\alpha_1]} + \acute{\epsilon}_1) + d^M_s(\phi^M_{[\alpha_2]} + \acute{\epsilon}_1)} \leq 0, \quad \frac{\mu\alpha^M_1}{b^L_f} - \varrho\beta^L_2 \leq 0, \tag{7.16}$$

$$\mu\gamma^M_f - \varrho\gamma^L_s \leq 0 \text{ and } \varrho\overline{\alpha^M_2} - \mu\overline{\alpha^L_1} \leq 0. \tag{7.17}$$

From equation (7.17), it is possible to find a positive constant κ such that

$$\varrho\left(\alpha^M_2 + \frac{1}{\check{T}}\sum_{k=1}^m \ln(1 + l_{2k})\right) - \mu\left(\alpha^L_1 + \frac{1}{\check{T}}\sum_{k=1}^m \ln(1 + l_{1k})\right) < -\frac{\kappa}{\check{T}} < 0. \tag{7.18}$$

From system (7.1) and equation (7.16), we obtain

$$\frac{d}{dt}\left[\ln \frac{(y_s(t))^\varrho}{(y_f(t))^\mu}\right] = [\varrho\alpha_2(t) - \mu\alpha_1(t)] + [\mu\gamma_f(t) - \varrho\gamma_s(t)]y_f(t)y_s(t)$$

$$+ y_f(t)\left[\mu\beta_1(t) - \frac{\varrho a_s(t)}{b_s(t) + c_s(t)y_f(t) + d_s(t)y_s(t)}\right]$$

$$+ x_s(t)\left[\frac{\mu a_f(t)}{b_f(t) + c_f(t)y_f(t) + d_f(t)y_s(t)} - \varrho\beta_2(t)\right] \quad (7.19)$$

$$\leq [\varrho\alpha_2(t) - \mu\alpha_1(t)] + [\mu\gamma_f^M(t) - \varrho\gamma_s^L(t)]y_f(t)y_s(t)$$

$$+ y_f(t)\mu\beta_f^M(t) + y_s(t)\left[\frac{\mu a_f^M(t)}{b_f^L(t)} - \varrho\beta_2^L(t)\right]$$

$$- \left[\frac{\varrho a_s^L(t)y_f(t)}{b_s^M(t) + c_s^M(t)(\phi_{[\alpha_1]}^M + \dot{\epsilon}_1) + d_s^M(t)(\phi_{[\alpha_2]}^M + \dot{\epsilon}_1)}\right]$$

$$\leq \varrho\alpha_2(t) - \mu\alpha_1(t), \ t \neq \tau_k \quad (7.20)$$

and

$$\ln\left(\frac{(y_s(\tau_k^+))^\varrho}{(y_f(\tau_k^+))^\mu}\right) = \ln\left(\frac{(1 + l_{1k})^\varrho}{(1 + l_{2k})^\mu}\right) + \ln\left(\frac{(y_s(\tau_k))^\varrho}{(x_f(\tau_k))^\mu}\right), \ t = \tau_k \quad (7.21)$$

Now, for any $t \in [\tau_\delta, \ \tau_{\delta+1})$ where $\tau_\delta \in [n_1\check{T}, \ (n_1 + 1)\check{T})$, $n_1 \in N$. Integrating inequality (7.20) over the intervals $[0, \ \tau_1)$, $[\tau_1, \ \tau_2)$, $[\tau_{\delta-1}, \ \tau_\delta)$ and $[\tau_\delta, t)$, respectively, we obtain

$$\ln\left(\frac{(y_s(\tau_1))^\varrho}{(y_f(\tau_1))^\mu}\right) - \ln\left(\frac{(y_s(0))^\varrho}{(y_f(0))^\mu}\right) \leq \int_0^{\tau_1}(\varrho\alpha_2(t) - \mu\alpha_1(t))dt$$

$$\ln\left(\frac{(y_s(\tau_2))^\varrho}{(y_f(\tau_2))^\mu}\right) - \ln\left(\frac{(y_s(\tau_1^+))^\varrho}{(y_f(\tau_1^+))^\mu}\right) \leq \int_{\tau_1}^{\tau_2}(\varrho\alpha_2(t) - \mu\alpha_1(t))dt$$

$$\vdots$$

$$\ln\left(\frac{(y_s(t))^\varrho}{(y_f(t))^\mu}\right) - \ln\left(\frac{(y_s(\tau_\delta^+))^\varrho}{(y_f(\tau_\delta^+))^\mu}\right) \leq \int_{\tau_\delta}^{t}(\varrho\alpha_2(t) - \mu\alpha_1(t))dt$$

Using equation (7.21) in all the above inequalities and adding the resulting inequalities, we have

$$\ln\left(\frac{(y_s(t))^\varrho}{(y_f(t))^\mu}\right) - \ln\left(\frac{(y_s(0))^\varrho}{(y_f(0))^\mu}\right) \leq \int_0^t (\varrho\alpha_2(t) - \mu\alpha_1(t))dt + \ln\frac{\prod_{0<\tau_k<t}(1+l_{2k})^\varrho}{\prod_{0<\tau_k<t}(1+l_{1k})^\mu}$$

$$= \int_0^{n_1\check{T}} (\varrho\alpha_2(t) - \mu\alpha_1(t))dt + n_1\varrho\sum_{k=1}^m \ln(1+l_{2k})$$

$$+ \int_{n_1\check{T}}^t (\varrho\alpha_2(t) - \mu\alpha_1(t))dt - n_1\mu\sum_{k=1}^m \ln(1+l_{1k})$$

$$+ \ln\frac{\prod_{n_1\check{T}<\tau_k<t}(1+l_{2k})^\varrho}{\prod_{n_1\check{T}<\tau_k<t}(1+l_{1k})^\mu}$$

$$\leq n_1\varrho\left(\check{T}\alpha_2^M + \sum_{k=1}^m \ln(1+l_{2k})\right)$$

$$- n_1\mu\left(\check{T}\alpha_1^L + \sum_{k=1}^m \ln(1+l_{1k})\right) + C$$

$$\leq -n_1\kappa + C \qquad (7.22)$$

where

$$C = \max_{0\leq V\leq\check{T}}\left(\int_0^V (\varrho\alpha_2(t) - \mu\alpha_1(t))dt + \ln\frac{\prod_{0\leq\tau_k<V}(1+l_{2k})^\varrho}{\prod_{0\leq\tau_k<t}(1+l_{1k})^\mu}\right).$$

This implies

$$\frac{(y_s(t))^\varrho}{(y_f(t))^\mu} \leq \exp\left(-n_1\kappa + C\right)\frac{(y_s(0))^\varrho}{(y_f(0))^\mu}$$

$$(y_s(t)^\varrho) \leq (y_f(t))^\mu \exp(-n_1\kappa + C)\frac{(y_s(0))^\varrho}{(y_f(0))^\mu} \qquad (7.23)$$

Now, from Lemma 7.2.2, $y_f(t)$ is bounded above, therefore from (7.23)

$$\lim_{t\to\infty}(y_s(t)) \to 0.$$

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Thus, species y_s will become extinct. □

Theorem 7.3.2. *If the condition (7.10) is satisfied, then the first species y_f is permanent, that is, there exists a constant $\nu > 0$ such that for any solution $(y_f(t), \, y_s(t))^T$ of system (7.1), $y_f(t)$ satisfies*

$$\nu \leq \liminf_{t \to \infty} y_f(t) \leq \limsup_{t \to \infty} y_f(t) \leq \phi^M_{[\alpha_1]},$$

where $\phi^M_{[\alpha_1]}$ is defined in Lemma 7.2.2.

Proof. From Theorem 7.3.1, $\lim_{t \to \infty} y_s(t) = 0$, therefore, there exists $\grave{\epsilon}_2 > 0$ such that

$$y_s(t) \leq \grave{\epsilon}_2 \; \forall, \; t > \check{T}_1. \tag{7.24}$$

Also from condition (7.10), it is possible that

$$\alpha_1^L - \frac{a_1^M \grave{\epsilon}_2}{b_f^L} + \frac{1}{\check{T}} \sum_{k=1}^{m} \ln(1 + l_{1k}) > 0. \tag{7.25}$$

From system (7.1), we obtain the following impulsive subsystem

$$\begin{cases} \frac{dy_f(t)}{dt} \geq y_f(t) \left(\alpha_1^L - \frac{a_f^M \grave{\epsilon}_2}{b_f^L} - (\beta_1 + \gamma_f \grave{\epsilon}_2) y_f(t) \right), & t \neq \tau_k \\ y_f(\tau_k^+) = (1 + l_{1k}) y_f(\tau_k), & t = \tau_k, \; k \in N \end{cases} \tag{7.26}$$

Now consider the corresponding comparison system of (7.26)

$$\begin{cases} \frac{du_f(t)}{dt} = u_f(t) \left(\alpha_1^L - \frac{a_f^M \grave{\epsilon}_2}{b_f^L} - (\beta_1 + \gamma_f \grave{\epsilon}_2) u_f(t) \right), & t \neq \tau_k \\ u_f(\tau_k^+) = (1 + l_{1k}) u_f(\tau_k), & t = \tau_k, \; k \in N \end{cases} \tag{7.27}$$

By using condition (7.25), Theorem 1.5.2 and Lemma 1.5.2, the impulsive system (7.26) has a unique periodic solution $u_f^*(t)$ which is globally stable. Therefore, $y_f(t) > u_f^*(t) - \grave{\epsilon}_3 \; \forall \; t > \check{T}_2$, $(\check{T}_2 > \check{T}_2)$. Let $\nu = \inf(u_f^*(t) - \grave{\epsilon}_3), t \in [0, \check{T}]$. This implies $\lim_{t \to \infty} y_f(t) \geq \nu$. This completes the proof. □

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Next, we will discuss the global attractivity of the species $y_f(t)$.

Theorem 7.3.3. *The species y_f is globally attractive if condition (7.10)is true. Therefore, for any positive solution $(y_f(t), y_s(t))^T$ of the system (7.1) and any positive solution $v(t)$ of*

$$\begin{cases} \frac{dv(t)}{dt} = v(t)\left(\alpha_1(t) - \beta_1(t)v(t)\right), & t \neq \tau_k, \\ v(\tau_k^+) = (1 + l_{1k})v(\tau_k), & t = \tau_k, \ k \in N \end{cases} \tag{7.28}$$

we have, $\lim_{t\to\infty}(y_f(t) - v(t)) = 0$.

Proof. Consider the Lyapunov function $W(t) = |\ln y_f(t) - \ln v(t)|$. Now, for $t > \check{T}_2$ and $t \neq \tau_k, \ k \in N$, the Dini derivative of $W(t)$ is

$$D^+W(t) = \text{sgn}(\ln(y_f(t)) - \ln(v(t)))\left(\frac{y_f'(t)}{y_f(t)} - \frac{v'(t)}{v(t)}\right)$$

$$= \text{sgn}(y_f(t) - v(t))\left(-\beta_1(t)(y_f(t) - v(t))\right)$$

$$\quad - \text{sgn}(y_f(t) - v(t))\left(\frac{a_f(t)y_s(t)}{b_f(t) + c_f(t)y_f(t) + d_f(t)y_s(t)} - \gamma_f(t)y_f(t)y_s(t)\right)$$

$$\leq -\beta_1(t) \mid y_f(t) - v(t) \mid +m(t) \tag{7.29}$$

where $m(t) = v(t)) + \frac{a_f(t)y_s(t)}{b_f(t)+c_f(t)y_f(t)+d_f(t)y_s(t)} - \gamma_f(t)y_f(t)y_s(t)$. $m(t) \to 0$ because $y_s(t) \to 0$ as $t \to \infty$. For $t > \check{T}_2$, we have $v(t) > y_f(t) > \frac{\nu}{2}$. Therefore, by using mean value theorem

$$\mid \ln y_f(t) - \ln v(t) \mid \leq \frac{2}{\nu} \mid y_f(t) - v(t) \mid .$$

Now, from equation(7.29)

$$D^+W(t) \leq -\frac{\beta_1(t)\nu}{2} \mid \ln y_f(t) - \ln v(t) \mid +m(t)$$
$$\leq -\frac{\beta_1(t)\nu}{2}W(t) + m(t)$$
$$\leq -\theta W(t) + m(t)$$

$$(7.30)$$

Further, for $t = \tau_k$, $W(\tau_k^+) = W(\tau_k)$. Therefore for $t > \check{T}_2$, we obtain

$$D^+W(t) + \theta W(t) \leq m(t) \tag{7.31}$$

Applying differential inequality theorem to equation (7.31) for $t > \check{T}_2$, we get

$$W(t) \leq \exp(-\theta(t - \check{T}_2)) \left(\int_{\check{T}_2}^t m(s) \exp(-\theta(t - \check{T}_2))ds + W(\check{T}_2) \right) \tag{7.32}$$

This implies $W(t) \to 0$ as $t \to \infty$. Thus $y_f(t) \to v(t)$ as $t \to \infty$. Hence, the species y_f is globally stable. $\qquad\square$

Theorem 7.3.4. *Assume that*

$$\frac{\overline{\alpha_1^M}}{\overline{\alpha_2^L}} \leq \frac{a_f^L}{(b_s^M + c_s^M \phi_{[\alpha_1]}^M + d_s^M \phi_{[\alpha_2]}^M)} \quad and \quad \frac{\overline{\alpha_1^M}}{\overline{\alpha_2^L}} \leq \frac{\beta_1^L + b_s^L}{a_s^M} \tag{7.33}$$

then the species y_f will extends towards extinction and species y_s will be permanent. That is for any solution $(y_f(t), y_s(t))^T$ of system (7.1) $y_s(t) \to u(t)$ where $u(t)$ is any positive solution of the system

$$\begin{cases} \frac{du(t)}{dt} = u(t)\left(\alpha_2(t) - \beta_2(t)u(t)\right), & t \neq \tau_k \\ u(\tau_k^+) = (1 + l_{1k})u(\tau_k), & t = \tau_k, \ k \in N \end{cases} \tag{7.34}$$

Proof. Proof is obtained by interchanging the role of $y_f(t)$ and $y_s(t)$ in Theorems 7.3.1, 7.3.2 and 7.3.3. $\qquad\square$

Table 7.1: Values of the coefficients in system (7.1)

Coefficient	Value
α_1	1.6
α_2	0.8
β_1	$1.7+0.4sin(2\pi t)$
β_2	$1.4+0.6sin(2\pi t)$
a_f	$1.2+0.3sin(2\pi t)$
a_s	$1.5+0.7sin(2\pi t)$
b_f	1
b_s	1
c_f	0.2
c_s	0.2
d_f	0.2
d_s	0.2
γ_f	$\sqrt{2}+cos(2\pi t)$
γ_s	1.5

7.4   Numerical Analysis and Discussion

Motivated by the work done in [102, 101], here we have examined the extinction of two species Lotka-Volterra model with Beddington-DeAngelis type inter-specific competition under the effect of toxic substances and impulsive perturbations. For validation of theoretical findings, consider the system (7.1) with the coefficients given in the Table (7.1). Here, all the coefficients are periodic functions with period $\check{T}=1$. Let $\tau_k = k$. So there exists an integer $m=1$ such that $\tau_{k+m} = \tau_k + \check{T}$. There are four different cases.

Case(i) First if $l_{1k} = 0.7$ and $l_{2k} = -0.5$, i.e. species y_f is stocked while y_s is harvested, through simple computation, we obtain

$$\phi^M_{[\alpha_1]} = 1.23076, \ \phi^M_{[\alpha_2]} = 1,$$

$$\frac{\gamma^M_f}{\gamma^L_s} = 1.60947, \ \frac{a^M_f}{b^L_f \beta^L_2} = 1.875,$$

$$\frac{\beta_1^M(b_s^M + c_s^M \phi_{[\alpha_1]}^M + d_s^M \phi_{[\alpha_2]}^M)}{a_s^L} = 3.79615, \ \frac{\overline{\alpha_1^L}}{\alpha_2^M} = 12.6425$$

Therefore, it can be easily concluded from above values that conditions given in Theorem 7.3.1 are satisfied. Thus, species y_s is driven to extinction while species y_f is permanent as shown in Figure 7.1.

Case (ii) Now, if roles of y_f and y_s are interchanged in system (7.1), then by retaining the same values of all the coefficients and taking impulsive factor $l_{1k} = -0.7$, $l_{2k} = 0.5$, that is first species undergo impulsive harvesting and other is stocked impulsively, we obtain

$$\frac{\beta_1^L b_s^L}{a_s^M} = 0.59091, \ \frac{a_f^L}{b_f^M + c_f^M \phi_{[\alpha_1]}^M + d_f^M \phi_{[\alpha_2]}^M} = 0.6223 \ \text{and} \ \frac{\overline{\alpha_1^M}}{\alpha_2^L} = 0.32856.$$

This implies that conditions in Theorem 7.3.4 are satisfied. Hence, species y_f will become extinct while species y_s is permanent as shown in Figure 7.2.

The results obtained in the above two cases supports those obtained by Fengde et al. [101], in which the authors have not analyzed the effect of impulsive perturbations on two species competition model. Additionally, we have established that both the species can become extinct or achieve permanence depending on the value of external perturbations. Thus rich dynamical behavior is observed for different values of impulsive perturbations l_{1k} and l_{2k} as discussed in the following two cases.

Case (iii) It is found that if in system (7.1), both the species undergo impulsive harvesting, that is, $l_{1k} = l_{2k} = -0.9$, then,

$$\overline{\alpha_1} = \alpha_1 + \frac{1}{\tilde{T}} \sum_{k=1}^{m} \ln(1 + l_{1k}) = -0.7026 \ \text{and} \ \overline{\alpha_2} = \alpha_2 + \frac{1}{\tilde{T}} \sum_{k=1}^{m} \ln(1 + l_{2k}) = -1.5026$$

So, the conditions in Theorem 7.3.1 and Theorem 7.3.4 are not satisfied. Thus both the species tends to extinction as shown in Figure 7.3.

Case (iv) On the other hand, if in system (7.1), the values of impulsive perturbations are $l_{1k} = 0.7$ and $l_{2k} = -0.1$, then,

$$\overline{\alpha_1} = \alpha_1 + \frac{1}{\tilde{T}} \sum_{k=1}^{m} \ln(1 + l_{1k}) = 2.1306 \ \text{and} \ \overline{\alpha_2} = \alpha_2 + \frac{1}{\tilde{T}} \sum_{k=1}^{m} \ln(1 + l_{2k}) = 0.6946$$

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So, neither the conditions in Theorem 7.3.1 nor in Theorem 7.3.4 are satisfied. The system (7.1) becomes permanent as shown in Figure 7.4. Thus from Figures 7.1 and 7.2, it is clear that one species will survive and other will go to extinction which supports the results obtained without impulses in [100]. Therefore, in this paper
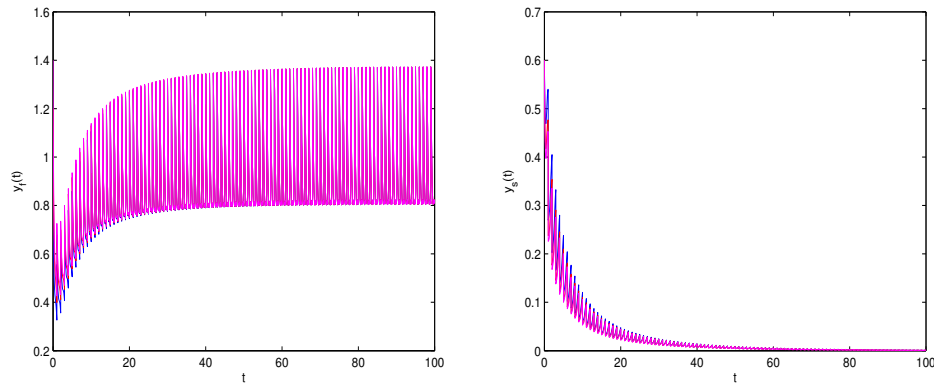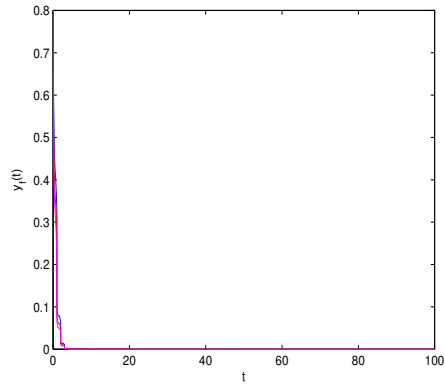

(a) Species y_f is permanent
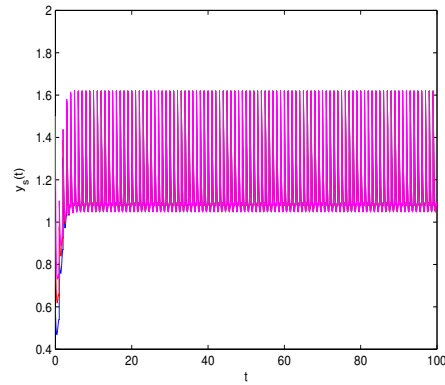
(b) species y_s is extinct
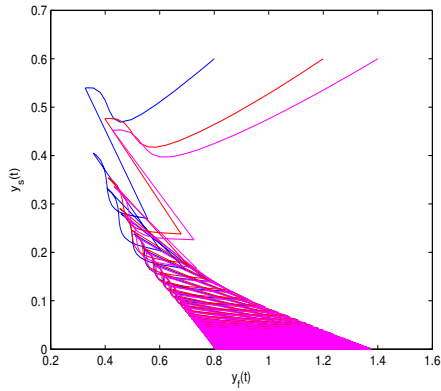
(c) Phase portrait of system (7.1)

Figure 7.1: Global attractivity of species y_f and extinction of species y_s of system (7.1) when $l_{1k} = 0.7$, $l_{2k} = -0.5$ with initial conditions $(y_f(0), \ y_s(0))^T = (0.8, \ 0.6)^T$, $(1.2, \ 0.6)^T$, $(1.4, \ 0.6)^T$

extension of principle of competition exclusion is done for two species competition system with Beddington-DeAngelis functional response under the effect impulsive perturbations and toxic substances released by both the species. Further, from Figures 7.1, 7.2, 7.3, 7.4, it can be easily observed that impulsive perturbations

(a) Species y_f is extinct

(b) species y_s is permanent
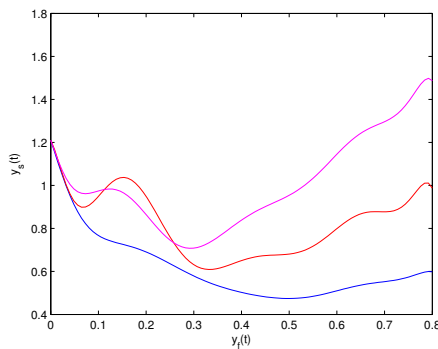
(c) Phase portrait of system (7.1)

Figure 7.2: Global attractivity of species y_s and extinction of species y_f of system (7.1) when $l_{1k} = -0.7$, $l_{2k} = 0.5$ with initial conditions $(y_f(0), \ y_s(0))^T = (0.8, \ 0.6)^T$, $(0.8, \ 1.0)^T$, $(0.8, \ 1.5)^T$

have great impact on the dynamics of the competition models.
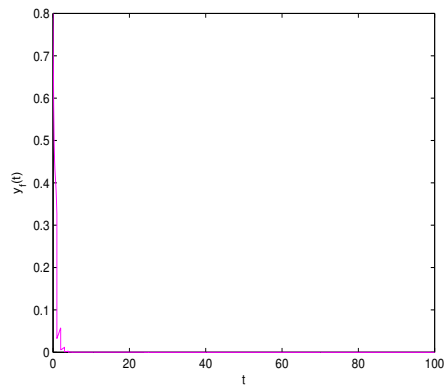
(a) Species y_f is extinct
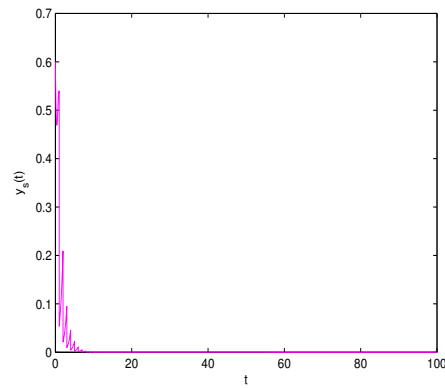
(b) species y_s is extinct
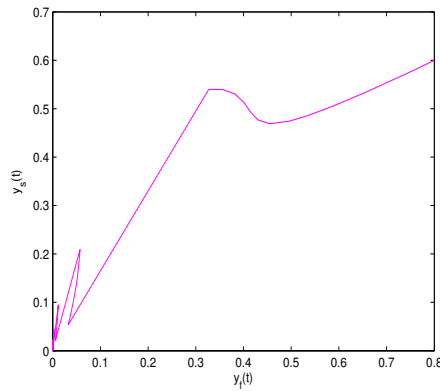
(c) Phase plane of system (7.1)

Figure 7.3: Extinction of species y_f and y_s of system (7.1) when $l_{1k} = -0.9$, $l_{2k} = -0.9$ with initial conditions $(y_f(0),\ y_s(0))^T = (0.8,\ 0.6)^T$
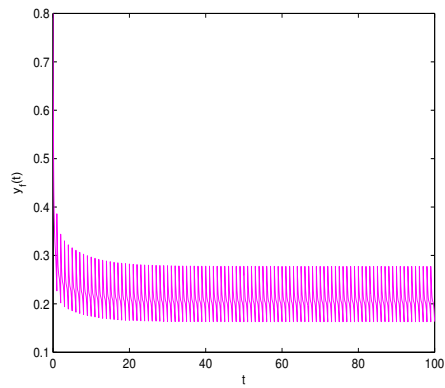
(a) Species y_f is permanent
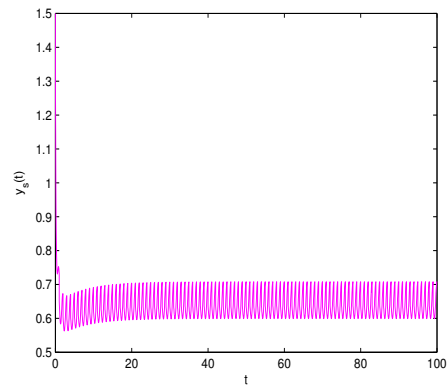
(b) species y_s is permanent
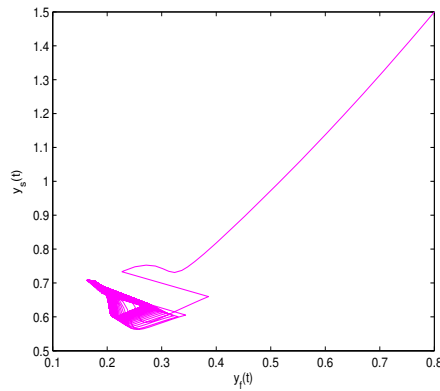
(c) Phase portrait of system (7.1)

Figure 7.4: Permanence of the system (7.1) when $l_{1k} = 0.7$, $l_{2k} = -0.1$ with initial conditions $(y_f(0),\ y_s(0))^T = (0.8,\ 0.6)^T$

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2. Preety Kalra and Maninderjit Kaur, *Impact of Constant and Linear Perturbations considering Holling Type IV Functional Response in Predation Term: A Single Species Model*, Indian Journal of Science and Technology (WoS indexed), [Communicated].

3. Preety Kalra and Maninderjit Kaur, *A three tropic level food chain model considering Holling type II and IV functional response for impulsive pest control strategy* European Journal of Molecular and Clinical Medicine, Vol. 7(7), 3743-3756 (2020).

4. Preety Kalra and Maninderjit Kaur, *Stability analysis of an eco-epidemiological SIN model with impulsive control strategy for integrated pest management considering stage-structure in predator*, Indian Journal of Mathematical Modelling and Numerical Optimization (Scopus Indexed), [Accepted, In production Manuscript No. IJMMNO-302137].

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