(EXISTENCE OF COMMON FIXED POINT THEOREMS IN METRIC SPACES, G-METRIC SPACES, GENERALIZED METRIC SPACES AND PARTIAL METRIC SPACES)

A Thesis

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By

Rashmi Sharma
(11512764)

Supervised By
Dr. Sanjay Mishra

Co-Supervised by
Dr. Manoj Kumar

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Declaration of Authorship

I, Rashmi Sharma, declare that this dissertation titled, " Existence of common fixed point theorems in Metric spaces, G-metric spaces, Generalized metric spaces and Partial metric spaces" and the work presented in it are my own. I confirm that:

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*Author:*Rashmi Sharma
Registration No.:11512764

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This is to certify that Rashmi Sharma,has completed the dissertation titled "Existence of common fixed point theorems in Metric spaces, G-metric spaces, Generalized metric spaces and Partial metric spaces" under my guidance and supervision. To the best of my knowledge, the present work is the result of his original investigation and study. No part of this dissertation has ever been submitted for any other degree at any University. The thesis is fit for the submission and the partial fulfilment of the conditions for the award of Doctor of Philosophy in Mathematics.

Signed:

Supervisor: Dr. Sanjay Mishra

Date: March 2021

Signed:

Co-Supervisor: Dr. Manoj Kumar

Date: March 2021

Abstract

In this thesis, several FPT for WC maps, contractive type mapping, proximal generalized contraction and CFP in different spaces are proved. In the first chapter, a history of MS, g.m.s., G-M.S., PMS and a brief survey of Banach Contraction rule are defined. On the other hand, in the first chapter, a brief introduction to FPT and literature review, we talk about the basic ideas that are very important for the work done in the following chapters. To achieve the objectives of our investigative work, the following five chapters are given. The main purpose of thesis is to establish approximation theorems for continuous and non-continuous self maps in the setting of various spaces by considering several conditions.

Along with the added PC in CMS, we first add new notions of PC of kind-R and kind-M, as well as proving the existence of g_α-best similarity for a pair of maps. We illustrate FPT in CMS with the help of these principles, achieving the first goal. Furthermore, several examples showed the validity of our findings.

Now, we use SF and the PC of first and second kind in CMS to implement a new class of generalized $\beta_a - \phi_a - \mathcal{Z}$-contractive pair of mappings. We also use SF to implement the idea of $\mathcal{Z}$-contraction in G-M.S. We also prove some FPT in these spaces, as well as applications to further demonstrate these findings in fixed point theory, achieving the second research goal.

Some common FPT have been proved with the help of some new notions like modified $\alpha-(\psi_0, g_0)$-PC of type-I and type-II. An application is also given to show the genuineness of our results. In G-M.S., we also introduce the definition of $G - \upsilon - \psi$-proximal cyclic weak contractive mapping. With the help of this new definition, cyclic (ψ_1, ϕ_1, I, J)-rational contraction in the sense of PMS can be achieved. The third goal is accomplished by proving such FPT in these spaces.We develop a general case for four WC self-maps that satisfy a general contractive condition, utilizing the same approach as Altun et al. We use this research to show that WC maps have a common FPT, as well as E.A. and (CLR) properties.

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Contents

Abbreviations

MS	**M**etric **S**pace.
CMS	**Complete** **M**etric **S**pace.
POSET	**P**artial **O**rdered **S**et.
g.m.s.	**g**eneralized **m**etric **s**pace.
G-M.S.	**G** **M**etric **S**pace.
PMS	**P**artial **M**etric **S**pace.
FPT	**F**ixed **P**oint **T**heorems.
CS	**C**auchy **S**equence.
BCP	**B**anach **C**ontraction **P**rinciple.
SF	**S**imulation **F**unction.
CFP	**C**ommon **F**ixed **P**oint.
WLOG	**W**ithout **L**oss **O**f **G**enerality.
WC	**W**eakly **C**ompatible.
PC	**P**roximal **C**ontraction.
BPP	**B**est **P**roximity **P**oint.

Chapter 1

Introduction

First chapter is basically early on in nature. In this category, we fix our documentations, test certain simple concepts and condense a portion of the natural established and ongoing outcomes identified with our research work. Moreover, we define some basic notions.

It comprises of three phases. We're dealing with a summary of the principle of fixed points in first phase. In phase two, we give some notations, preliminaries and fundamental definitions which are utilized all through the text of the dissertation. In third phase, we talk about different sort of mappings which are helpful all through the content of our thesis.

1.1 Introduction

1.1.1 Origin of Fixed Point Theory

Fixed point theory itself is a lovely blend of investigation, geometrics and topography. Five years decade, this theory has been a very influential and significant technique in the analysis of non-linear phenomena. In many fields, such as genetics, chemistry, economics, electronics etc. fixed-point structures have been linked together. The point at which the $y = f(x)$ curve intersects with the $y = x$ line intersects provides the curve solution, and the point of convergence is the curve fixed point. Thanks to the advent of detailed methods for finding set focuses the importance of solid applications has grown tremendously.

Fixed point theory is quickly moving into the standard of arithmetic chiefly as a result of its applications in assorted fields which incorporate numerical techniques like Newton-Raphson strategy, setting up Picard's Presence hypothesis, presence of arrangement of essential conditions and an arrangement of direct conditions.

In many cases, it is impossible to find a particular solution; therefore, it is necessary to develop appropriate algorithms to find out the desired result. This is closely related to the control and optimization problems that arise in different scientific and engineering problems. Many situations in the study of nonlinear equations, variational calculus, partial differential equations, optimal control and inverse problems can be expressed by fixed-point problems or optimization. Fixed point theory is a powerful tool to determine uniqueness of solutions to dynamical systems and is widely used in theoretical and applied analysis. So it must be applicable to mathematical biology as well.

1.1.2 Importance of Fixed/Invariant Points

The points which remains invariant under a transformation. We note that $f(x) = 0$ is equal to problems with fixed points and the root discovery problems.

The investigation is actually disclosing what kind of problems the fixed point has. The issues with the fixed point can be described in the accompanying way:

1. What features/maps have fixed point?
2. Where will we determine the point set?
3. Is the single point fixed?

"First, we claim a conclusion that allows us the certainty of fixed point remaining in life. Assume g is a continuous representation of yourself on $[\bar{a}, \bar{b}]$. We then get the following conclusions:

1. If the mapping set $\hat{y} = g(\hat{x})$ for all $\hat{x} \in [\bar{a}, \bar{b}]$ matches $\hat{y} \in [\bar{a}, \bar{b}]$, then g has a fixed point in $[\bar{a}, \bar{b}]$.
2. Suppose $g(\hat{x})$ is defined above $(\bar{a}, \bar{b})$ and a positive constant $k < 1$ occurs with k for all $\hat{x} \in (\bar{a}, \bar{b})$, then g has a single fixed point p in $[\bar{a}, \bar{b}]$."

"Now, presume (X, d) is a complete MS and $T\colon X \to X$ is a mapping. The T mapping satisfies a requirement of Lipschitz with $\alpha \geq 0$ unchanged, so that $d(Tx, Ty) \leq \alpha d(x, y)$, for all x, y in X. For various α values, we have the cases below :

1. Contraction mapping is called T if $\alpha < 1$;
2. When $\alpha \leq 1$ is not expansive, T is named non expansive;
3. T is alluded to as contractive if $\alpha = 1$."

"Clearly, that contraction the $\Rightarrow$ contractive $\Rightarrow$ non-expansive $\Rightarrow$ Lipschitz contraction is. In this case, though, converse can not be valid as:

1. Let $I : X \to X$ is an identity map, where X is a MS, is non-expansive but not contractional.
2. Suppose $M = [0, \infty)$ be a complete MS equipped with the absolute metric value. Set, $\bar{f} : X \to X$ to $\bar{f}(x) = x + \frac{1}{x}$, $\bar{f}$ is instead a contractive map, while $\bar{f}$ is not a contraction."

There are two important FPT: one is Brouwers, and the other Banachs FPT. Brouwers FPT is existential by its nature. "Brouwer1912: Any continuous self map on the ball of the closed unit $C = \{x : ||x|| \leq 1\}$ in $\mathbb{R}^n$ has a fixed point. Elegant Banach FPT solution is:

1. problems with the nature of a single solution to an equation,
2. provides a practical way of getting approximate solutions and
3. provides an easy way to acquire estimated answers."

The uses of the Banach's fixed theorem and its generalizations are very significant in numerous scientific, methodological, technical and economic disciplines. Banach [8] proved a FPT in 1922 and named it Banach FPT/BCP which is considered the mile stone. This hypothesis states "If T is self mapping of a complete MS (X, d) and there exists a number $h \in [0, 1)$, such that for all $x, y \in X$,

$$d(Tx, Ty) \leq hd(x, y), \tag{1.1}$$

then T has a unique fixed point, i.e., every contraction map on a complete MS has a fixed point."

This theorem offers a methodology in mathematical sciences and engineering for solving a number of practical problems. This hypothesis was expanded by various scholars and developed in different ways. There have been several implementations of this principle but it has a disadvantage-The description requires T to be constant.

Definition 1.1. If $\tilde{T}$ is a self map on a non-void set X, then a point $x \in X$ satisfying $\tilde{T}x = x$ is called a $\tilde{T}$ fixed point.

Example 1.1. *Examples of fixed points are as follows:*

1. *A mapping $I : F \to F$ identified by $Ix = x$, has indefinitely many fixed points, i.e. each domain point is a fixed I point.*
2. *A mapping $I : F \to F$ specified by $Ix = \frac{x}{p} - (p - 1)$, if p is a positive integer, otherwise $x = p$ is a fixed point.*
3. *A mapping $I : F \to F$ equal to $Ix = x^2$ has two 0 and 1 fixed point.*
4. *There is no fixed point in a localization projection, that is, $Tx = x + 3$ for all $x \in R$.*

We may therefore conclude from the above definitions that a mapping may be a particular fixed point, may be more than us, may be an infinite number of points, and may not be a fixed point. Theorems concerned with the life and development of a solution for a $Tx = x$ operator equation shape the part of the fixed point theory. We notice that every mapping of contractions is consistent and universally constant, but need not be valid to converse. Kannan [31] provided the first answer to this problem in 1968, which proved to be a FPT for operators who don't have to be continuous.

Kannan1968 [31]: "If T is self mapping of a CMS X satisfying

$$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)] \tag{1.2}$$

for all x, y in X and $0 \leq k < 12$, then T has unique fixed point in X."

We notice that every mapping of contractions is consistent and universally constant, but need not be valid to converse. Kannan [31] provided the first answer to this problem in 1968, which proved to be a FPT for operators who don't have to be continuous. After Kannan, Chatterjea [17] proved to be an operator's FPT that fulfills the condition: "there exists $c \in [0, 12)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)];" \tag{1.3}$$

Rhoades [51] had suggested these three requirements (1.1), (1.2) and (1.3) are independent. "Zamfirescu [61] combined the conditions (1.1), (1.2) and (1.3) as follows: there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b < \frac{1}{2}$, and $0 \leq c < \frac{1}{2}$; such that for each $x, y \in X$ at least one of the following is true:

$$(z1) d(Tx, Ty) \leq ad(x, y);$$
$$(z2) d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)];$$
$$(z3) d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)].\text{"}$$

Another generalisation of the BCP in 1983 was given by Rus [52], which replaced the condition (1.1) with conditions: "there is a comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) \tag{1.4}$$

for all $x, y \in X$.

$$d(Tx, Ty) \leq a \ max\{d(x, Tx), d(y, Ty)\}. \tag{1.5}$$

In such conditions the operator T has a fixed point which is unique."

1.1.3 Various Types of Spaces

We accentuation our research essentially on the accompanying spaces:

1. MS
2. g.m.s.
3. G-M.S.
4. PMS

1.1.3.1 Metric Space

In 1906, a French mathematician, Maurice Frechet (1878-1973), invented the notion of MS, derived from the term metor (measure). In fact, he led the analysis of these spaces and their applications to different mathematics fields. The description currently in use, though, was provided in 1914 by the German mathematician, Felix Hausdroff (1868-1942).

"Let X be an arbitrary set. Let $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:

1. $d(x, y) \geq 0; d(x, y) = 0$ iff $x = y$,

2. $d(x, y) = d(y, x)$,

3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. The set X together with the metric d, i.e., (X, d) is called a MS."

"Let (X, d) be a MS. A sequence $\{x_n\}$ in X is said to be

1. Convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $\{x_n\} \to x$ as $n \to \infty$ or $lim_{n\to\infty} x_n = x$.

2. Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number $n(\varepsilon)$ such that for all $n > m > n(\varepsilon)$, $d(x_n, x_m) < \varepsilon$.

3. Complete if every Cauchy sequence is convergent in X."

1.1.3.2 Generalized Metric Space

The definition of g.m.s. was proposed by Branciari [15] in 2000 as follows:

"Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and for all distinct point $u, v \in X$, each of them different from x and y, one has

1. $d(x, y) = 0$ if and only if $x = y$,

2. $d(x, y) = d(y, x)$,

3. $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$(the rectangular inequality).

Then (X, d) is called a g.m.s. ."

"Let (X, d) be a g.m.s.. A sequence $\{x_n\}$ in X is said to be

1. g.m.s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $\{x_n\} \to x$ as $n \to \infty$ or $li_{n\to\infty} x_n = x$.

2. g.m.s. Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number $n(\varepsilon)$ such that for all $n > m > n(\varepsilon)$, $d(x_n, x_m) < \varepsilon$.

3. complete g.m.s. if every g.m.s. Cauchy sequence is g.m.s. convergent in X."

1.1.3.3 G-Metric Space

(Mustafa and Sims [43]) showed in 2003 that many D-MS Dhage tests were invalid. Consequently, they implemented an updated variant of the generic MS system, and named it G-M.S..

The G-M.S. description was introduced in 2006 by (Mustafa and Sims [44]) as follows: "Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all x, y in X with $z \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all x, y, z in X with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all x, y, z, a in X (rectangle inequality).

Then the function G is called a G-M.S. or, more specifically, a G-metric on X and the pair (X, G) is called a G-MS.

Assuming (X, G) is a G-M.S.. Then for $\tilde{x}_0 \in X$, $\tilde{r} > 0$, the G-ball with center $\tilde{x}_0$ and $\tilde{r}$ is the radius

$$B_G(\tilde{x}_0, \tilde{r}) = \{\tilde{y} \in M; G(\tilde{x}_0, \tilde{y}, \tilde{y}) < \tilde{r}\}."$$

"Let (X, G) be a G-M.S.. Then a sequence $\{x_n\}$ is

1. G-convergent to x if $lim_{m,n\to\infty} G(x, x_n, x_m) = 0$; i.e., for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ (set of natural numbers) such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$. We call x as the limit of the sequence and write $x_n \to x$ or $limn \to \infty x_n = x$.
2. said to be G-Cauchy if for each $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$ that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.
3. said to be G-complete(or a complete G-M.S.) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G)."

1.1.3.4 Partial Metric Space

"Matthews [39] introduced the notion of PMS as follows:

Let X be a non-empty set and $p : X \times X \to [0, \infty)$ satisfy the following:

(p_1) $x = y$ iff $p(x,x) = p(x,y) = p(y,y)$;

(p_2) $p(x,x) \leq p(x,y)$;

(p_3) $p(x,y) = p(y,x)$;

(p_4) $p(x,y) \leq p(x,z) + p(z,y)p(z,z)$, for all $x, y, z \in X$. Then p is called a partial metric and the pair (X,p) is called a PMS.

We note that the function $d_p(x,y) = 2p(x,y)p(x,x)p(y,y)$ satisfies the conditions of a MS X and hence this is a regular metric for X."

"Let (X,p) be a PMS. Then, the sequence x_n is:

1. A sequence $\{x_n\}$ in the PMS (X,p) converges to x if and only if $p(x,x) = lim_{n\to\infty} p(x,x_n)$.
2. A sequence $\{x_n\}$ in the PMS (X,p) is called a Cauchy sequence if $lim_{n,m\to\infty} p(x_m,x_n)$ exists and finite.
3. A PMS (X,p) is called complete, if every Cauchy sequence $\{x_n\}$ in X converges."

1.2 Review of Literature

1.2.1 Various types of mappings

We discuss now some types of mappings which are basic tools for our further study of different spaces. The different types of mappings used in different chapter are similar to MS and these mappings can also be defined on the same line in other spaces.

Firstly, we discuss various forms of mapping in MS. The basic concept of metric FPT is given by Banach namely BCP, which is the basis of the theory. This principle gives:

1. The nature and singularity of fixed points.
2. Methods to get estimated fixed points .

The theory of contraction has had numerous implications that are spread through nearly all branches of mathematics.

1.2.2 Diverse forms of metric space mapping

Jungck [30] is proved to be a general FPT for map exchange, which summarizes Banach's FPT.

Das and Naik [21] have generalized Jungck's test. Many scholars have since suggested and researched various generalizations of commuting mappings. On the other side, Sessa [55] described, in 1982, the principle of weak commutativity as: "Two self-mappings f and g of a MS (X, d) are said to be weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all x in X."

"A new type of fixed-point issue was realised by [34] during 1984, with the aid of the control function, called the distance altering function.

A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

1. $\psi(0) = 0$,
2. ψ is continuous and monotonically non-decreasing."

The next FPT has been proved by Khan et al. [34] uses the following distance modifier function:

"Let (X, d) be a CMS. Let ψ be an altering distance function and $f : X \to X$ be a self-mapping which satisfies the following inequality:

$$\psi(d(fx, fy)) \leq c\ \psi(d(x, y)) \tag{1.6}$$

for all $x, y \in X$ and for some $0 < c < 1$. Then f has a unique fixed point."

"Chaudhary et al. [18] and [18] add two variables and three variables to the notion of altering distance."

Jungck [28] invented the concept of consistent maps in 1986 and demonstrated to be common FPT connected with such maps. "Two self-mappings f and g of a MS (X, d) are said to be compatible if $lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = t$ for some t in X."

This definition helps to realize the FPT of compatible mapping that satisfies the contractive circumstances, and at least suggests a consistency of mapping. Kannan 's paper is established that certain maps be not continous, but have some points, except for being constant at a fixed level, the maps involved in each case are. In the next two decades, his papers became the catalyst for many fixed-point papers.

"During 1994, the definition of R-weakly commuting mapping in MS was introduced by Pant [46], to start with, to extend the field of analysis of particular FPT from the compatible class to the broader class of R-weakly commuting. Second, the maps are not necessarily fixed-level constant.

A pair of self-mappings (f, g) of a MS (X, d) is said to be R-weakly commuting if there exists some $R \geq 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all x in X."

Jungck [29] launched a definition in 1996 as follows: "Two self maps f and g are said to be WC if they commute at coincidence points."

"In 1997, Alber and Guerre-Delabriere [5] introduced the notion of weakly contraction as follows:

Presume (X, d) be a MS. A mapping $f : X \to X$ is said to be φ-weakly contraction, if there exists a map $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(fx, fy) \leq d(x, y)\varphi(d(x, y)), \ \textit{for all } x, y \textit{ in } X."$$

Suppose (M, d) reflect a MS. If there is particular number $a > 1$, a mapping $C : M \to M$ on M is considered to be expansive, so that $\tilde{d}(Cp, Cq) \geq a\tilde{d}(p, q)$ occurs.

"Let (X, d) be a MS. A mapping $f : X \to X$ is said to be φ- weakly expansive, if there exists a map $\varphi : [0, \infty) \to (-\infty, 0]$ with $\varphi(0) = 0$ and $\varphi(t) < 0$ for all $t > 0$ such that

$$d(fx, fy) \geq d(x, y) - \varphi(d(x, y)), \ \textit{for all } x, y \textit{ in } X."$$

Karapinar et al. [32] and others suggested a definition of triangular map, as depicted below: "Let $\alpha : X \times X \to \mathbb{R}$ be a function. We say that a self - mapping $T : X \to X$ is triangular α - admissible if

1. $x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \alpha(Tx, Ty) \geq 1,$

2. $x, y, z \in X,$

$$\begin{cases} \alpha(x, z) \geq 1, \\ \alpha(z, y) \geq 1 \end{cases} \Longrightarrow \alpha(x, y) \geq 1."$$

Samet et al. [53] presented the following principles:

"Let (X, d) be a MS and $T : X \to X$ and if there exist two functions $\alpha : X \times X \to [0, \infty)$ be a given mapping. We say that

1. T is α-admissible if, for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1) \Rightarrow \alpha(Tx, Ty) \geq 1.$$

2. T is a $\alpha - \psi$-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq (d(x, y)), \ for\ all\ x, y \in X."$$

"Priya Shahi et al. [56] present a definition about α-admissible w.r.t.g and generalizing $\alpha - \psi$-contractive mapping pair as follows:

Let $f, g : X \times X \to [0, \infty)$. We say that f is α-admissible w.r.t. g it for all $x, y \in X$, we have

$$\alpha(gx, gy) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1."$$

Definition 1.2. "Let (X, d) be a MS and $f, g : X \to X$ be given mappings. We say that the pair (f, g) is a generalized $\alpha - \psi-$contractive pair of mappings if there exists two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have

$$\alpha(gx, gy)d(fx, fy) \leq \psi(M(gx, gy)),$$

where $M(gx, gy) = max\{d(gx, gy), \frac{d(gx,fx)+d(gy,fy)}{2}, \frac{d(gx,fy)+d(gy,fx)}{2}\}$."

Popa [48] presented the concept of implicit functions that are beneficial in the deduction of criteria for contraction.

In 2002 E.A. property was found by Aamri and Moutawakil [1]. In addition to relaxing the commutativity criterion at the correlated points, it obtains preservation of ranges beyond the need for continuum. A pair enjoying E.A. property usually doesn't need to follow the trend of integrating the dimension of one map into another's dimension. In addition, the space's criterion for completeness is diminished to a normal state of space range completeness. We note also that the E.A. property does not need to satisfy the compatible property. "Two self-mappings f and g of a MS (X, d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = t$ for some t in X."

The theorem was proven by Dutta et al. [22] as: "Let (X, d) be a complete MS and let $T : X \to X$ be a self-mapping satisfying the inequality

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \tag{1.7}$$

where, $\psi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Then T has a unique fixed point."

Khojasteh et al. [35] have recently implemented a new mapping type, named SF. Later in the description of SF, Argoubi et al. [7], by eliminating a condition, slightly modified the definition of SF.

Let $\mathcal{Z}^*$ be a collection of the Argoubi et al. [7] SF.

Definition 1.3. "A SF is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(t, s) < s - t$ for all $t, s > 0$
(ζ_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that

$$lim_{n\to\infty}\{t_n\} = lim_{n\to\infty}\{s_n\} = l \in (0, \infty),$$

then

$$lim_{n\to\infty} sup\zeta(t_n, s_n) < 0."$$

"The concept of weak contraction presented by Berinde [12], but in [13], the author renames it as an 'almost contraction' which is apposite. Shatanawi [58] presented some FPT for a nonlinear weakly $\mathcal{C}$-contraction type mapping in MS. Ciric et al. [20] introduced the concept of almost generalized contractive condition on mappings and proved some existential theorems on fixed points of such mappings in an ordered complete MS."

Example 1.2. *[24] "Let R be the set of all real numbers. Define $G : R \times R \times R \to R^+$ by*
$G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$.
Then, it is clear that (R, G) is a G-MS."

Definition 1.4. [47] "Let X be a nonempty set. Then $(X, \preceq, d)$ is called an ordered MS if and only if:

1. (X, d) is a MS, and

2. $(X, \preceq)$ is a partially ordered set.

$(X, \preceq, d)$ is called an ordered complete MS if $(X, \preceq, d)$ is an ordered MS, and (X, d) is a complete metric space."

Definition 1.5. "Let A and B be two non empty subsets of a metric space (X, d). A non-self mapping $T : A \to B$ is said to be a contraction if $d(Tx, Ty) \leq kd(x, y), \quad$ for all $x, y \in X$, where $k \in [0, 1]$. "

Definition 1.6. "Let A and B be two non-empty subsets of a metric space (X, d). A non-self-mapping $T : A \to B$ is said to be a contraction of the kind-R if

$$\begin{cases} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \Longrightarrow d(u, v) \leq k[d(x, Ty) + d(y, Tx)],\ for\, all\, u, v, x, y \in A,\ where\, k \in [0, 1]."$$

Definition 1.7. "Let A and B be two non-empty subsets of a metric space (X, d). A non-self-mapping $T : A \to B$ is said to be a contraction of the kind-M if

$$\begin{cases} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \Longrightarrow d(u, v) \leq k[d(x, Tx) + d(y, Ty)],\ for\, all\, u, v, x, y \in A,\ where\, k \in [0, 1]."$$

"In 2012, Amini Harrandi [25] introduced a generalization to the partial metric spaces, which is called metric-like spaces and he proved some fixed point theorems in such spaces.

Definition 1.8. [25] Let X be a nonempty set, a function $\sigma : X \times X \to \mathbb{R}_+$ is said to be a metric-like on X if the following conditions satisfied:
(i) $\sigma(x, y) = 0$ implies that $x = y$
(ii) $\sigma(x, y) = \sigma(y, x)$
(iii) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$,
the space (X, σ) is said to be a metric-like space."

Example 1.3. *[25] "Let $X = \{0, 1\}$, define $\sigma : X \times X \to \mathbb{R}_+$ as follows:*

$$Tx = \begin{cases} 2, & if\ x = y = 0, \\ 1, & otherwise. \end{cases}$$

Then σ is metric-like on X, since $\sigma(0, 0) > \sigma(0, 1)$ then σ is not partial metric."

"In 2011, Sintunavarat et al. [59] presented the definition of the (CLRf) property as follows: Two self-mappings f and g of a MS (X, d) are said to satisfy $(CLRf)$ property if there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = fx$ for some x in X."

In 2008, Di. C. Bari [9] introduced the notion of WC maps.

Definition 1.9. [9] "Two self maps f and g are said to be WC if they commute at coincidence points."

Definition 1.10. [33] "Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that T is non decreasing with respect to $\preceq$ if

$$x, y \in X, x \preceq y \Rightarrow Tx = Ty."$$

Example 1.4. *[44] "Let $X = [0, \infty)$. The function $G : X \times X \times X \to [0, +\infty)$, defined by*

$$G(x, y, z) = |x - y| + |y - z| + |z - x|,$$

for all $x, y, z \in X$, is a G - metric on X."

Definition 1.11. ([10]) "A mapping $T : A \to B$ is said to be a PC of first kind if there exists $\alpha \in [0, 1)$ such that, for all $a, b, x, y \in A$,

$$\begin{cases} d(a, Tx) = d(A, B) \\ d(b, Ty) = d(A, B), \end{cases}$$

implies that,

$$d(a, b) \leq \alpha d(x, y)."$$

Definition 1.12. ([10]) "A mapping $T : A \to B$ is said to be a strong PC of first kind if there exists $\alpha \in [0, 1)$ such that, for all $a, b, x, y \in A$,

$$\begin{cases} d(a, Tx) = d(A, B) \\ d(b, Ty) = d(A, B), \end{cases}$$

implies that,

$$d(a, b) \leq \alpha d(x, y) + (\beta - 1)d(A, B))."$$

Definition 1.13. ([10]) "A mapping $T : A \to B$ is said to be a PC of second kind if there exists $\alpha \in [0, 1)$ such that, for all $a, b, x, y \in A$,

$$\begin{cases} d(a, Tx) = d(A, B) \\ d(b, Ty) = d(A, B), \end{cases}$$

implies that,

$$d(a, b) \leq \alpha d(Tx, Ty).$$

The necessary condition for a self-mapping T to be a PC of the second kind is that

$$d(T^2x, T^2y) \leq \alpha d(Tx, Ty)$$

for all x, y in the domain of T."

Definition 1.14. ([10]) "Given $T : A \to B$ and an isometry $g : A \to A$, the mapping T is said to preserve isometric distance with respect to g if

$$d(Tgx, Tgy) = (Tx, Ty)$$

for all $x, y \in A$."

"Manro et al. [57] gave the cyclic (ψ, ϕ, A, B)-contractions in PMS.

Definition 1.15. [57] Let (X, p) be a PMS and A, and let B be X non-empty closed subsets. A mapping $T : X \to X$ is referred to as a cyclic (ψ, ϕ, A, B)-contraction if

1. ψ and ϕ are altering distance functions.
2. $A \cup B$ has a cyclic representation w.r.t. T; that is, $T(A) \subseteq B$ and $T(B) \subseteq A$; and
3. $$\begin{aligned}\psi(p(Tx, Ty)) \leq \psi(max\{p(x, y), p(x, Tx), P(y, Ty), \frac{1}{2}(p(x, Ty) + p(Tx, y))\}) \\ -\phi(max\{p(x, y), p(y, Ty)\})\end{aligned}$$

for all $x \in A$ and $y \in B$."

Lemma 1.16. *([39],[45]) "Let (X, p) be a PMS.*

1. *$\{x_n\}$ is a (X, p) CS if and only if it is a (X, d_p) CS in MS.*
2. *A partial (X, p) MS is complete if and only if the (X, d_p) MS is complete. Additionally $lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if*

$$p(x, x) = lim_{n\to\infty} p(x_n, x) = lim_{n,m\to\infty} p(x_n, x_m)."$$

Definition 1.17. [11] "Let $(X, d, \preceq)$ be a partially ordered MS. We say that a non - self mapping $T : A \to B$ is proximally ordered - preserving if and only if, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = d(A, B), \Longrightarrow u_1 \preceq u_2. \\ d(u_2, Tx_2) = d(A, B)." \end{cases}$$

Theorem 1.18. *[41] "Let A, B be two non - empty closed subsets of a partially ordered complete MS $(X, d, \preceq)$ such that A_0 is non - empty. Assume that $T : A \to B$ satisfies the following conditions:*

1. *T is continuous and proximally ordered - preserving such that $T(A_0) \subseteq B_0$,*
2. *there exists elements $x_0, x_1 \in A_0$ such that*
   $d(gx_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$,
3. *for all $x, y, u, v \in A$,*

$$\begin{cases} gx \preceq gy, \\ d(u, Tx) = d(A, B), \Longrightarrow d(u, v) \leq \frac{1}{2}(d(gx, v) + d(gy, u)) - \psi(d(gx, v), d(gy, u)). \\ d(gy, Ty) = d(A, B). \end{cases}$$

Then T has a BPP.

Proof. Define $\alpha : A \times A \to [0, +\infty)$ by

$$\alpha(gx, gy) = \begin{cases} 1, & if\ x \preceq y, \\ 0, & otherwise. \end{cases}$$

Firstly we prove that T is a triangular $\alpha - (\psi, g)$ - proximal admissible mapping. To this aim, assume

$$\begin{cases} \alpha(gx, gy) \geq 1, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Therefore, we have

$$\begin{cases} x \preceq y, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Now, since T is proximally ordered - preserving, then $u \preceq v$, that is, $\alpha(u, v) \geq 1$. Consequently, condition (T1) of Definition (3.8) holds. Also, assume

$$\begin{cases} \alpha(gx, z) \geq 1, \\ \alpha(z, gy) \geq 1, \end{cases}$$

so that

$$\begin{cases} gx \preceq z, \\ z \preceq gy, \end{cases}$$

and consequently $x \preceq y$, that is, $\alpha(gx, gy) \geq 1$. Hence, condition (T2) of Definition (3.2) holds. Further, by (ii) we have $d(gx_1, Tx_0) = d(A, B)$ and $\alpha(gx_0, gx_1) \geq 1$.
Moreover, from (3) we get

$$\begin{cases} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A, B), \Longrightarrow d(u, v) \leq \frac{1}{2}(d(gx, v) + d(gy, u)) - \psi(d(gx, v), d(gy, u)). \\ d(v, Ty) = d(A, B) \end{cases}$$

Thus all the conditions of Theorem (3.11) hold and T has a BPP." □

Definition 1.19. "Let (X, G) be a G-MS, $f : X \to X$ a mapping and $\zeta \in \mathbb{Z}$. Then f is called a $\mathbb{Z}$-contraction with respect to ζ if the following condition is satisfied

$$\zeta(G(fx, fy, fz), G(x, y, z)) \geq 0 \text{ for all } x, y, z \in X." \tag{1.8}$$

1.3 Research Gap

Banach introduced a contraction principle and We boost the results by specifying $mathcalZ$-contraction PC of kind-R and kind-M in the sense of CMS (X, d) and also using the g_α-best proximity condition for a pair of maps in MS. "Samet et al. [53] introduced the $(\alpha - \phi)$-contractive form mapping group in 2012. We improve it by introducing new notions $\beta - \phi - \mathcal{Z}$-contractive mappings with SF in MS. With respect to generalized metric spaces, Karapinar [32] given an analog of the principle of $(\alpha - \psi)$-contractive mappings. Then we use these Type-I and Type-II contractive mappings in g.m.s.. Manro et al. [57] gave the cyclic (ψ, ϕ, A, B)-contractions in PMS, but we introduced cyclic (ψ_1, ϕ_1, I, J)-rational contractions in PMS. With the support of the E.A. and (CLR) properties, Altun et al. proved a few fixed point results in the MS in 2007. We utilize this for four WC self maps satisfying a general contractive condition due to the same method introduced by Altun et al.. By introducing the new notions, the contractive conditions present in this literature are weakened and rather common fixed points are obtained."

1.4 Objectives of Study

1. Introduction of new notions of proximal contractions of kind-R and kind-M and to prove fixed point theorems using g-best proximity condition for a pair of maps in Metric spaces.

2. Introduction of new notions using simulation functions in metric spaces, G-metric spaces, generalized metric spaces and partial metric spaces and to prove certain fixed point theorems using these new notions in these spaces.

3. To extend and unify the results of various authors present in metric spaces, G-metric spaces, generalized metric spaces and partial metric spaces.

4. To prove some FPT using E.A. property and (CLR) property.

Chapter 2

Theorems on Fixed Points in Metric Spaces

We prove four WC self-maps for some common fixed theorems, add a new class of generalized $\beta_a - \phi_a - \mathcal{Z}$-contractive pair of SF mappings together with compatible/WC general contracting state and E.A. and (CLR) properties. We also prove theorems of these fixed points in MS equipped with a partial order.

It is composed of five sections. In first section, we use the same method introduced by Altun et al. [6] to derive a general case for four WC self-maps which satisfy a general contractive condition. In section two together with E.A. and (CLR) of WC properties maps of particular common FPT are proved. The third section introduces a new generalized class of $\beta_a - \phi_a - \mathcal{Z}$-contractive mapping pair. We use simulation method in this segment to show some FPT for a number of mappings. In the last section, we illustrate some FPT in partly ordered MS.

2.1 Four Self-Maps Weakly Compatible Satisfactory to a General Condition

Because of the same approach proposed by Altun et al. [6], for four WC self-maps, we obtain an overall case that satisfies a simple contractive condition.

Altun et al. [6] implemented four WC maps which fulfill an integral form of general contractive condition.

We explain our findings in the following general way:

Theorem 2.1. *Let A_0, B_0, S_0 and T_0 be the self-mapping of MS $(M, \hat{d})$ satisfying the following conditions:*

$$S_0 M \subseteq B_0 M, T_0 M \subseteq A_0 M, \tag{2.1}$$

$$\forall\ \breve{x}\ \breve{y}\ in\ M,\ continous\ existence\ functions\ \psi_a, \phi_a : \mathbb{R}^+ \to \mathbb{R}^+,$$
$$\psi_a(0) = 0 = \phi_a(0),\ \psi_a(s) < s,\ \phi_a(s) < s\ for\ s > 0\ so \tag{2.2}$$

$$\psi_a(\hat{d}(S_0\breve{x}, T_0\breve{y}) \leq \psi_a(m_\delta(\breve{y}, \breve{y}))\phi_a(m_\delta(\breve{x}, \breve{y})),$$

$m_\delta(\breve{y}, \breve{y}) = max\{\hat{d}(A_0\breve{x}, B_0\breve{y}), \hat{d}(S_0\breve{x}, A_0\breve{y}), \hat{d}(T_0\breve{y}, B_0\breve{y}), \frac{1}{2}(\hat{d}(S_0\breve{x}, B_0\breve{y}) + \hat{d}(T_0\breve{y}, A_0\breve{x}))\}$. *If one of A_0M, B_0M, S_0M or T_0M is complete subspace of M, there is a coincidence for (A_0, S_0) or (B_0, T_0).*

In addition, if (A_0, S_0) and (B_0, T_0) pairs are WC, then A_0, B_0, S_0 and T_0 have a unique CFP.

Proof. Assume that $\breve{x}_0 \in M$ be any point of M. The series $\breve{y}_{\breve{n}}$ can be constructed into M from (2.1):

$$\breve{y}_{(2\breve{n}+1)} = S_0\breve{x}_{2\breve{n}} = B_0\breve{x}_{(2\breve{n}+1)}, \breve{y}_{(2\breve{n}+2)} = T_0\breve{x}_{(2\breve{n}+1)} = A_0\breve{x}_{(2\breve{n}+2)},\ \forall\ \breve{n} = 0, 1, 2, ... \tag{2.3}$$

Let's start with $\hat{d}_{\breve{n}} = \hat{d}(y\breve{n}, \breve{y}_{(\breve{n}+1)})$.

Presume $\hat{d}_{2\breve{n}} = 0$ for some $\breve{n}$. Then $\breve{y}_{2\breve{n}} = \breve{y}_{(2\breve{n}+1)}$, that is, $T_0\breve{x}_{(2\breve{n}-1)} = A_0\breve{x}_{2\breve{n}} = S_0\breve{x}_{2\breve{n}} = B_0\breve{x}_{(2\breve{n}+1)}$, both A_0, S_0 have a point of coincidence.

Alike, if $\hat{d}_{(2\breve{n}+1)} = 0$, later there is a match point between B_0 and T_0.

For every $\breve{n}$, presume that $\hat{d}_{\breve{n}} \neq 0$.

We've got from (2.2) and (2.9),

$$\psi_a(\hat{d}(S_0\breve{x}_{2\breve{n}}, T_0\breve{x}_{(2\breve{n}+1)}) \leq \psi_a(m_\delta(\breve{x}_{2\breve{n}}, \breve{x}_{(2\breve{n}+1)})) - \phi_a(m_\delta(\breve{x}_{2\breve{n}}, \breve{x}_{2\breve{n}+1)})), \tag{2.4}$$

$$\begin{aligned} m_\delta(\breve{x}_{2\breve{n}}, \breve{x}_{2\breve{n}+1)}) = &\ max\{\hat{d}(A_0\breve{x}_{2\breve{n}}, B_0\breve{x}_{(2\breve{n})}), \hat{d}(S_0\breve{x}_{2\breve{n}}, A_0\breve{x}_{2\breve{n}}), \hat{d}(T_0\breve{x}_{(2\breve{n}+1)}, B_0\breve{x}_{(2\breve{n}+1)}), \\ &\ \frac{\hat{d}(S_0\breve{x}_{2\breve{n}}, B_0\breve{x}_{(2\breve{n}+1)}) + \hat{d}(T_0\breve{x}_{(2\breve{n}+1)}, A_0\breve{x}_{2\breve{n}})}{2}\} \\ = &\ max\{\hat{d}_{2\breve{n}}, \hat{d}_{(2\breve{n}+1)}\}. \end{aligned} \tag{2.5}$$

So, we get from (2.4),

$$\psi_a(\hat{d}(S_0\breve{x}_{2\breve{n}}, T_0\breve{x}_{(2\breve{n}+1)}) \leq \psi_a(max\{\hat{d}_{2\breve{n}}, \hat{d}_{2\breve{n}+1)}\}) - \phi_a(max\{\hat{d}_{2\breve{n}}, \hat{d}_{(2\breve{n}+1)}\}). \tag{2.6}$$

Presently, if $\hat{d}_{(2\breve{n}+1)} \geq \hat{d}_{2\breve{n}}$, for any $\breve{n}$, we've got from (2.6),

$$\psi_a(\hat{d}_{(2\breve{n}+1)}) \leq \psi_a(\hat{d}_{(2\breve{n}+1)}) - \phi_a(\hat{d}_{(2\breve{n}+1)}) < \psi_a(\hat{d}_{(2\breve{n}+1)}), \ a \ contradiction. \tag{2.7}$$

Thus, $\hat{d}_{2\breve{n}} > \hat{d}_{(2\breve{n}+1)}$ $\forall$ $\breve{n}$, and we obtain it from (2.6)

$$\psi_a(\hat{d}_{(2\breve{n}+1)}) \leq \psi_a(\hat{d}_{2\breve{n}}) - \phi_a(\hat{d}_{2\breve{n}}), \ for \ all \ \breve{n} \in \mathbb{N}. \tag{2.8}$$

Similarly,

$$\psi_a(\hat{d}_{2\breve{n}}) \leq \psi_a(\hat{d}_{(2\breve{n}-1)}) - \psi_a(\hat{d}_{(2\breve{n}-1)}),$$

$$\psi_a(\hat{d}_{(2\breve{n}-1)}) \leq \psi_a(\hat{d}_{(2\breve{n}-2)}) - \phi_a(\hat{d}_{(2\breve{n}-2)}).$$

In general, all of us have $\breve{n} = 1, 2, ...,$

$$\psi_a(\hat{d}_{\breve{n}}) \leq \psi_a(\hat{d}_{(\breve{n}-1)}) - \phi_a(\hat{d}_{(\breve{n}-1)}) < \psi_a(\hat{d}_{(\breve{n}-1)}). \tag{2.9}$$

Consequently the sequence $\{\psi_a(\hat{d}_{\breve{n}})\}$ is monotonically decreasing and bounded below. Thus, $\exists$, $r \geq 0$, s.t.

$$lim_{\breve{n}\to\infty}\psi_a(\hat{d}_{\breve{n}}) = r. \tag{2.10}$$

From (2.9), we infer

$$0 \leq \phi_a(\hat{d}_{(\breve{n}-1)}) \leq \psi_a(\hat{d}_{(\breve{n}-1)}) - \psi_a(\hat{d}_{\breve{n}}).$$

On applying limit as $\breve{n} \to \infty$ and using (2.10), we get

$$lim_{\breve{n}\to\infty}\phi_a(\hat{d}_{(\breve{n}-1)}) = 0, \ implies \ that, \ lim_{\breve{n}\to\infty}\phi_a(\hat{d}_{(\breve{n}-1)}) = lim_{\breve{n}\to\infty}(\hat{d}(\breve{y}_{(\breve{n}-1)}, \breve{y}_{\breve{n}})) = 0, or \tag{2.11}$$

$$lim_{\breve{n}\to\infty}\hat{d}_{\breve{n}} = lim_{\breve{n}\to\infty}\hat{d}(\breve{y}_{\breve{n}}, \breve{y}_{(\breve{n}+1)}) = 0. \tag{2.12}$$

Now we're showing that $\{\breve{y}_{\breve{n}}\}$ is the CS. For this reason, it is sufficient to prove $\{\breve{y}_{2\breve{n}}\}$ is the CS. Try to make, $\{\breve{y}_{2\breve{n}}\}$ is not a CS. Then there is $\epsilon > 0$, there is an even integer $2k_A$ and $2m_\delta(k_A) > 2\breve{n}(k_A) > 2k_A$, so that there is an integer $2k_A$

$$\hat{d}((\breve{y}_{(2\breve{n}(k_A)}, \breve{y}_{(2m_\delta(k_A))}) \geq \epsilon. \tag{2.13}$$

For each even number $2k_A$, assume that $2m_\delta(k_A)$ is the smallest positive integer that satisfies $2\breve{n}(k_A)$ fulfilling (2.13) s.t.

$$\hat{d}(\breve{y}_{2\breve{n}(k_A)}, \breve{y}_{(2m_\delta(k_A)-2)}) < \epsilon. \tag{2.14}$$

From (2.13), we've got a

$$\begin{aligned}\epsilon \leq & \hat{d}(\check{y}_{2\check{n}(k_A)}, \check{y}_{2m_\delta(k_A)}) \\ \leq & \hat{d}(\check{y}_{2\check{n}(k_A)}, \check{y}_{(2m_\delta(k_A)-2)}) + \hat{d}(\check{y}_{(2m_\delta(k_A)-2)}, \check{y}_{(2m_\delta(k_A)-1)}) + \hat{d}(\check{y}_{(2m_\delta(k_A)-1)}, \check{y}_{2m_\delta(k_A)}).\end{aligned}$$

Using (2.12), (2.14) in the above inequality, it becomes

$$lim_{\acute{k}\to\infty}\tilde{d}(\check{y}_{2\check{n}(k_A)}, \check{y}_{2m(k_A)}) = \epsilon. \tag{2.15}$$

In addition, because of the triangle inequality,

$$|\hat{d}(\check{y}_{(2\check{n}(k_A))}, \check{y}_{(2m_\delta(k_A)-1)}) + \hat{d}(\check{y}_{(2\check{n}(k_A)}, \check{y}_{(2m_\delta(k_A)})| \leq \hat{d}_{(2m_\delta(k_A)-1)},$$

$$|\hat{d}(\check{y}_{(2\check{n}(k_A)+1)}, \check{y}_{(2m_\delta(k_A)-1)}) + \hat{d}(\check{y}_{(2\check{n}(k_A))}, \check{y}_{(2m_\delta(k_A))})| \leq \hat{d}_{(2m_\delta(k_A)-1)} + \hat{d}_{2m_\delta(k_A)}. \tag{2.16}$$

If (2.12) is used, we're going to get

$$lim_{k_A\to\infty}\hat{d}(\check{y}_{2\check{n}(k_A)}, \check{y}_{(2m_\delta(k_A)-1)}) = lim_{k_A\to\infty}\hat{d}(\check{y}_{(2\check{n}(k_A)+1)}, \check{y}_{(2m_\delta(k_A)-1)}) = \epsilon. \tag{2.17}$$

From (2.2), we have

$$\psi_a(\hat{d}(S_0\check{x}_{2\check{n}(k_A)}, T_0\check{x}_{(2m_\delta(k_A)-1)}) \leq \psi_a(m_\delta(\check{x}_{(2\check{n}(k_A))}, \check{x}_{(2m_\delta(k_A)-1)}) - \phi_a(m_\delta(\check{x}_{(2\check{n}(k_A)}, \check{x}_{(2m_\delta(k_A)-1)}), \tag{2.18}$$

where

$$\begin{aligned}m_\delta(\check{x}_{2\check{n}(k_A)}, \check{x}_{(2m_\delta(k_A)-1)}) =& max\{\hat{d}(\check{x}_{2\check{n}(k_A)}, B_0\check{x}_{(2m_\delta(k_A)-1)}), \hat{d}(S_0\check{x}_{2\check{n}(k_A)}, A_0\check{x}_{2\check{n}(k_A)}), \\ & \hat{d}(T_0\check{x}_{(2m_\delta(k_A)-1)}, B_0\check{x}_{(2m_\delta(k_A)-1)}), \\ & \frac{(\hat{d}(S_0\check{x}_{2\check{n}(k_A)}, B_0\check{x}_{(2m_\delta(k_A)-1)}) + \hat{d}(T_0\check{x}_{2\check{n}(k_A)}, A_0\check{x}_{(m_\delta(k_A)-1)})}{2}\} \\ =& max\{\hat{d}(\check{y}_{2\check{n}(k_A)}, \check{y}_{(2m_\delta(k_A)-1)}), \hat{d}(\check{y}_{2\check{n}(k_A)}, \check{y}_{(2\check{n}(k_A)+1)}), \\ & \hat{d}(\check{y}_{(2m_\delta(k_A)-1)}, \check{y}_{2m_\delta(k_A)}), \\ & \frac{(\hat{d}(\check{y}_{(2\check{n}(k_A)+1)}, \check{y}_{(2m_\delta(k_A)-1)}) + \hat{d}(\check{y}_{2\check{n}(k_A)}, \check{y}_{(2m_\delta(k_A)-1)})}{2}\}.\end{aligned}$$

Set the limit as $k_A \to \infty$ and taking (2.17), we get it

$$\psi_a(\epsilon) \leq \psi_a(\epsilon) - \phi_a(\epsilon),\ an\ inconsistency,\ since\ \epsilon > 0.$$

Consequently, $\{\check{y}_{\check{n}}\}$ is a CS and so $\{\check{y}_{\check{n}}\}$ is a CS.

Currently, $A_0(M)$ is believed to have been completed. Note that $\{\check{y}_{2\check{n}}\}$ is included in $A_0(M)$ and has a restriction in $A_0(M)$, say $\bar{u}$, $lim_{\check{n}\to\infty}\check{y}_{2\check{n}} = \bar{u}$.

Presume $\bar{v} \in A_0^{(-1)}\bar{u}$. Then $A_0\bar{v} = \bar{u}$.

We are going to show this $S_0\bar{v} = \bar{u}$.

Suppose to be feasible, $S_0\bar{v} \neq \bar{u}$, that is, $\hat{d}(S_0\bar{v}, \bar{u}) = \hat{p} > 0$.

If we take $\breve{x} = \bar{v}$, $\breve{y} = \breve{x}_{(2\breve{n}-1)}$ in (2.2), we have

$$\psi_a(\hat{d}(S_0\bar{v}, T_0\breve{x}_{(2\breve{n}-1)}) \leq \psi_0(m_\delta(\bar{v}, \breve{x}_{(2\breve{n}-1)}) - \phi_a(m_\delta(\bar{v}, \breve{x}_{(2\breve{n}-1)}).$$

Allowing a limit like $\breve{n} \to \infty$, we have

$$lim_{\breve{n}\to\infty}\psi_a(\hat{d}(S_0\bar{v}, T_0\breve{x}_{(2\breve{n}-1)}) \leq lim_{\breve{n}\to\infty}\psi_a(m_\delta(\bar{v}, \breve{x}_{(2\breve{n}-1)}) - lim_{\breve{n}\to\infty}\phi_a(m_\delta(\bar{v}, \breve{x}_{(2\breve{n}-1)}), \tag{2.19}$$

$$\begin{aligned} lim_{\breve{n}\to\infty}m_\delta(\bar{v}, \breve{x}_{(2\breve{n}-1)}) =& lim_{\breve{n}\to\infty}[max\{\hat{d}(\bar{u}, \breve{y}_{(2\breve{n}-1)}), \hat{d}(S_0\bar{v}, \bar{u}), \hat{d}(\breve{y}_{2\breve{n}}, \breve{y}_{(2\breve{n}-1)}), \\ & \frac{(\hat{d}(S_0\bar{v}, \breve{y}_{(2\breve{n}-1)}) + \hat{d}(\breve{y}_{2\breve{n}}, \bar{u}))}{2}\}] \\ =& max\{\hat{d}(\bar{u}, \bar{u}), \hat{d}(S_0\bar{v}, \bar{u}), \hat{d}(\bar{u}, \bar{u}), \frac{1}{2}(\hat{d}(S_0\bar{v}, \bar{u}) + \hat{d}(\bar{u}, \bar{u}))\} \\ =& \hat{d}(S_0\bar{v}, \bar{u}) \\ =& \hat{p}. \end{aligned}$$

Thus, from (2.19), we have

$$\psi_a(\hat{d}(S_0\bar{v}, \bar{u}) \leq \psi_a(\hat{p}) - \phi_a(\hat{p}),\ that\ is,$$

$$\psi_a(\hat{p}) \leq \psi_a(\hat{p}) - \phi_a(\hat{p}),\ a\ contradiction,\ since\ \hat{p} > 0.$$

Thus, $S_0\bar{v} = \bar{u} = A_0\bar{v}$.

Therefore, $\bar{u}$ is the matching point of the (A_0, S_0) pair.

Since $S_0M \subseteq B_0M$, $S_0\bar{v} = \bar{u}$, that is, $\bar{u} \in B_0M$.

Assume $\bar{w} \in B_0^{(-1)}\bar{u}$. Next $B_0\bar{w} = \bar{u}$. Before utilizing indistinguishable contentions from over, can be easily verified, $T_0\bar{w} = \bar{u} = B_0\bar{w}$, so $\bar{u}$ is the matching point of (B_0, T_0).

If we conclude that B_0M is total instead of A_0M, similar outcomes hold.

If T_0M is complete, then by (2.1), $\bar{u} \in T_0M \subseteq A_0M$.

In essence, if S_0M is completed, then $\bar{u} \in S_0M \in B_0M$.

Now, since the pairs (A_0, S_0) and (B_0, T_0) are WC, so

$$\bar{u} = S_0\bar{v} = A_0\bar{v} = T_0\bar{w} = B_0\bar{w},$$

then

$$\begin{aligned} A_0\bar{u} &= A_0S_0\bar{v} = S_0A_0\bar{v} = S_0\bar{u}, \\ B_0\bar{u} &= B_0T_0\bar{w} = T_0B_0\bar{w} = T_0\bar{u}. \end{aligned} \tag{2.20}$$

Now, we're going to argue $T_0\bar{u} = \bar{u}$.

If possible, $T_0\bar{u} \neq \bar{u}$.

We got it from (2.2)

$$\begin{aligned} \psi_a(\hat{d}(\bar{u}, T_0\bar{u}) =& \psi_a(\hat{d}(S_0\bar{v}, T_0\bar{u}) \\ &\leq \psi_a(m_\delta(\bar{v}, \bar{u})) - \phi_a(m_\delta(\bar{v}, \bar{u})), \ \textit{where} \end{aligned}$$

$$\begin{aligned} m_\delta(\bar{v}, \bar{u}) =& max\{\hat{d}(A_0\bar{v}, B_0\bar{u}), \hat{d}(S_0\bar{v}, A_0\bar{v}), \hat{d}(T_0\bar{u}, B_0\bar{u}), \frac{1}{2}(\hat{d}(S_0\bar{v}, B_0\bar{u}) + \hat{d}(T_0\bar{u}, A_0\bar{v}))\} \\ =& max\{\hat{d}(\bar{u}, T_0\bar{u}), \hat{d}(\bar{u}, \bar{u}), 0, \frac{1}{2}(\hat{d}(\bar{u}, T_0\bar{u}) + \hat{d}(T_0\bar{u}, \bar{u}))\} \\ =& \hat{d}(\bar{u}, T_0\bar{u}). \end{aligned}$$

So, we have

$$\begin{aligned} \psi_a(\hat{d}(\bar{u}, T_0\bar{u}) \leq& \psi_a(\hat{d}(\bar{u}, T_0\bar{u})) - \phi_a(\hat{d}(\bar{u}, T_0\bar{u})) \\ <& \psi_a(\hat{d}(\bar{u}, T_0\bar{u})), \ \textit{a contradiction.} \end{aligned}$$

So, $T_0\bar{u} = \bar{u}$.

Similarly, $S_0\bar{u} = \bar{u}$.

Thus, we get $A_0\bar{u} = S_0\bar{u} = B_0\bar{u} = T_0u = \bar{u}$.

Therefore, $\bar{u}$ is the CFP of A_0, B_0, S_0 and T_0.

For the uniqueness, let $\mathfrak{z}$ be another CFP of A_0, B_0, S_0 and T_0.

We will show that $\bar{u} = \mathfrak{z}$.

If possible, $\bar{u} \neq \mathfrak{z}$.

From (2.2), we have

$$\begin{aligned}\psi_a(\hat{d}(\bar{u},\mathfrak{z}) =&\psi_a(\hat{d}(S_0\bar{u},T_0\mathfrak{z})\\ \leq&\psi_a(m_\delta(\bar{u},\mathfrak{z}))-\phi_a(m_\delta(\bar{u},\mathfrak{z}))\\ =&\psi_a(\hat{d}(\bar{u},\mathfrak{z}))-\phi_a(\hat{d}(\bar{u},\mathfrak{z})),\end{aligned}$$

$$\begin{aligned}since\ m_\delta(\bar{u},\mathfrak{z}) =&\hat{d}(\bar{u},\mathfrak{z}).\\ <&\psi_a(\hat{d}(\bar{u},\mathfrak{z})),\ a\ contradiction.\end{aligned}$$

Thus, $\bar{u}=\mathfrak{z}$, and the uniqueness follows. □

2.2 The Fixed Point Theorem of Weakly Compatible Mapping and The Attributes of E.A. and (CLR)

Theorem 2.2. *Let A_0, B_0, S_0 and T_0 be MS $(M,\hat{d})$ self-mapped meets (2.1), (2.2) and the followings:*

$$pairs\ (A_0,S_0)\ and\ (B_0,T_0)\ are\ WC, \tag{2.21}$$

$$pair\ (A_0,S_0)\ or\ (B_0,T_0)\ comply\ with\ the\ E.A.\ property. \tag{2.22}$$

If any one of A_0M, B_0M, S_0M and T_0M is a complete subspace of M, Then there is a distinct CFP A_0, B_0, S_0 and T_0.

Proof. Presume (A_0,S_0) is gratifying the E.A. property. And there is the $\{\check{x}_{\breve{n}}\}$ sequence in M s.t. $lim_{\breve{n}\to\infty}A_0\check{x}_{\breve{n}} = lim_{\breve{n}\to\infty}S_0\check{x}_{\breve{n}} = \mathfrak{z}$, for some $\mathfrak{z}$ in M.

Since $S_0M \subseteq B_0M$, $\exists$ a sequence $\{\check{y}_{\breve{n}}\}$ in M s.t. $S_0\check{x}_{\breve{n}} = B_0\check{y}_{\breve{n}}$.

Consequently, $lim_{\breve{n}\to\infty}B_0\check{y}_{\breve{n}} = \mathfrak{z}$.

Now, to prove $lim_{\breve{n}\to\infty}T_0\check{y}_{\breve{n}} = \mathfrak{z}$.

Probably, $lim_{\breve{n}\to\infty}T_0\check{y}_{\breve{n}} = \hat{t} = \mathfrak{z}$.

By (2.2), we can write

$$\psi_a(\hat{d}(S_0\check{x}_{\breve{n}},T_0\check{y}_{\breve{n}}) \leq \psi_a(m_\delta(\check{x}_{\breve{n}},\check{y}_{\breve{n}}))-\phi_a(m_\delta(\check{x}_{\breve{n}},\check{y}_{\breve{n}})).$$

Letting limit as $\breve{n} \to \infty$, we have

$$lim_{\breve{n}\to\infty}\psi_a(\hat{d}(S_0\breve{x}_{\breve{n}}, T_0\breve{y}_{\breve{n}}) \leq lim_{\breve{n}\to\infty}\psi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y}_{\breve{n}})) - lim_{\breve{n}\to\infty}\phi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y}_{\breve{n}})),\ where \tag{2.23}$$

$$\begin{aligned}lim_{\breve{n}\to\infty}m_\delta(\breve{x}_{\breve{n}}, \breve{y}_{\breve{n}}) =& lim_{\breve{n}\to\infty}[max\{\hat{d}(A_0\breve{x}_{\breve{n}}, B_0\breve{y}_{\breve{n}}), \hat{d}(S_0\breve{x}_{\breve{n}}, A_0\breve{x}_{\breve{n}}), \hat{d}(T_0\breve{y}_{\breve{n}}, B_0\breve{y}_{\breve{n}}),\\ &\frac{1}{2}(\hat{d}(S_0\breve{x}_{\breve{n}}, B_0\breve{y}_{\breve{n}}) + \hat{d}(T_0\breve{y}_{\breve{n}}, A_0\breve{x}_{\breve{n}}))\}]\\ =& max\{\hat{d}(\mathfrak{z}, \mathfrak{z}), \hat{d}(\mathfrak{z}, \mathfrak{z}), \hat{d}(\hat{t}, \mathfrak{z}), \frac{1}{2}(\hat{d}(\mathfrak{z}, \mathfrak{z}) + \hat{d}(\hat{t}, \mathfrak{z}))\}\\ =& \hat{d}(\hat{t}, \mathfrak{z}).\end{aligned}$$

So, by (2.23), we obtain

$$\begin{aligned}\psi_a(\hat{d}(\mathfrak{z}, \hat{t}) \leq& \psi_a(\hat{d}(\mathfrak{z}, \hat{t})) - \phi_a(\hat{d}(\mathfrak{z}, \hat{t}))\\ <& \psi_a(\hat{d}(\mathfrak{z}, \hat{t})),\ \ a\ conflict.\end{aligned}$$

So, $\hat{t} = \mathfrak{z}$, that is, $lim_{\breve{n}\to\infty}T_0\breve{y}_{\breve{n}} = \mathfrak{z}$.

Now let us take B_0M be a complete subspace of M. Then $\mathfrak{z} = B_0\bar{u}$ for some $\bar{u}$ in M.

In the result, we have

$$lim_{\breve{n}\to\infty}T_0\breve{y}_{\breve{n}} = lim_{\breve{n}\to\infty}S_0\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty}A_0\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty}B_0\breve{y}_{\bar{n}} = \mathfrak{z} = B_0\bar{u}.$$

We are going to demonstrate it now $T_0\bar{u} = B_0\bar{u}$.

Probably, $T_0\bar{u} \neq B_0\bar{u}$.

From (2.2), we have

$$\psi_a(\hat{d}(S_0\breve{x}_{\breve{n}}, T_0\bar{u}) \leq \psi_a(m_\delta(\breve{x}_{\breve{n}}, \bar{u}))\phi_a(m_\delta(\breve{x}_{\breve{n}}, \bar{u})).$$

Making limit $\breve{n} \to \infty$, we have

$$lim_{\breve{n}\to\infty}\psi_a(\hat{d}(S_0\breve{x}_{\breve{n}}, \bar{u}) \leq lim_{\breve{n}\to\infty}\psi_a(m_\delta(\breve{x}_{\breve{n}}, \bar{u})) - lim_{\breve{n}\to\infty}\phi_a(m_\delta(\breve{x}_{\breve{n}}, \bar{u})),\ where \tag{2.24}$$

$$\begin{aligned}lim_{\breve{n}\to\infty}m_\delta(\breve{x}_{\breve{n}}, \bar{u}) =& lim_{\breve{n}\to\infty}[max\{\hat{d}(A_0\breve{x}_{\breve{n}}, B_0\bar{u}), \hat{d}(S_0\breve{x}_{\breve{n}}, A_0\breve{x}_{\breve{n}}), \hat{d}(T_0\bar{u}, B_0\bar{u}),\\ &\frac{1}{2}(\hat{d}(S_0\breve{x}_{\breve{n}}, B_0\bar{u}) + \hat{d}(T_0\bar{u}, A_0\breve{x}_{\breve{n}}))\}]\\ =& max\{\hat{d}(\mathfrak{z}, \mathfrak{z}), \hat{d}(\mathfrak{z}, \mathfrak{z}), \hat{d}(T_0\bar{u}, \mathfrak{z}), \frac{1}{2}(\hat{d}(\mathfrak{z}, \mathfrak{z}) + \hat{d}(T_0\bar{u}, \mathfrak{z}))\}\\ =& \hat{d}(T_0\bar{u}, \mathfrak{z}).\end{aligned}$$

Thus, from (2.24)

$$\begin{aligned}\psi_a(\hat{d}(\mathfrak{z}, T_0\bar{u}) \leq & \psi_a(\hat{d}(\mathfrak{z}, T_0\bar{u})) - \phi_a(\hat{d}(\mathfrak{z}, T_0\bar{u})) \\ < & \psi_a(\hat{d}(\mathfrak{z}, T_0\bar{u})), \ a \ conflict.\end{aligned}$$

So, $T_0\bar{u} = \mathfrak{z} = B_0\bar{u}$.

After that, B_0 and T_0 are WC, so, $B_0T_0\bar{u} = T_0B_0\bar{u}$, imply, $T_0T_0\bar{u} = T_0B_0\bar{u} = B_0T_0\bar{u} = B_0B_0\bar{u}$.

Since $M \subseteq A_0M$, there is $\bar{v} \in M$, just like, $T_0\bar{u} = A_0\bar{v}$.

Now, we are saying $A_0\bar{v} = S_0\bar{v}$.

Perhaps, $A_0\bar{v} \neq S_0\bar{v}$.

By (2.2),

$$\psi_a(\hat{d}(S_0\bar{v}, T_0\bar{u}) \leq \psi_a(m_\delta(\bar{v}, \bar{u})) - \phi_a(m_\delta(\bar{v}, \bar{u})), \ where \tag{2.25}$$

$$\begin{aligned}m_\delta(\bar{v}, \bar{u}) = & max\{\hat{d}(A_0\bar{v}, B_0\bar{u}), \hat{d}(S_0\bar{v}, A_0\bar{v}), \hat{d}(T_0\bar{u}, B_0\bar{u}), \frac{1}{2}(\hat{d}(S_0\bar{v}, B_0\bar{u}) + \hat{d}(T_0\bar{u}, A_0\bar{v}))\} \\ = & \hat{d}(S_0\bar{v}, A_0\bar{v}) = \hat{d}(S_0\bar{v}, T_0\bar{u}).\end{aligned}$$

Thus, from (2.25), we have

$$\begin{aligned}\psi_a(\hat{d}(S_0\bar{v}, T_0\bar{u}) \leq & \psi_a(\hat{d}(S_0\bar{v}, T_0\bar{u})) - \phi_a(\hat{d}(S_0\bar{v}, T_0\bar{u}) \\ < & \psi_a(\hat{d}(S_0\bar{v}, T_0\bar{u})), \ a \ contradiction.\end{aligned}$$

Therefore, $S_0\bar{v} = T_0\bar{u} = A_0\bar{v}$.

Thus, we have, $T_0\bar{u} = B_0\bar{u} = S_0\bar{v} = A_0\bar{v}$.

The weak compatibility of A_0 and S_0 implies that $A_0S_0\bar{v} = S_0A_0\bar{v} = S_0S_0\bar{v} = A_0A_0\bar{v}$.

Now, we assert that the general fixed point of A_0, B_0, S_0 and T_0 is T_0u.

Expect it, $T_0T_0u \neq T_0u$.

From (2.2), one can write

$$\psi_a(\hat{d}(T_0\bar{u}, T_0T_0\bar{u}) = \psi_a(\hat{d}(S_0\bar{v}, T_0T_0\bar{u}) \leq \psi_a(m_\delta(\bar{v}, T_0\bar{u})) - \psi_a(m_\delta(\bar{v}, T_0\bar{u})), \ where \tag{2.26}$$

$$
\begin{aligned}
m_\delta(\bar{v}, T_0\bar{u}) =& max\{\hat{d}(A_0\bar{v}, B_0T_0\bar{u}), \hat{d}(S_0\bar{v}, A_0\bar{v}), \hat{d}(B_0T_0\bar{u}, T_0T_0\bar{u}), \frac{1}{2}(\hat{d}(S_0\bar{v}, B_0T_0\bar{u}) + \hat{d}(T_0T_0\bar{u}, A_0\bar{v}))\} \\
=& max\{\hat{d}(T_0\bar{u}, T_0T_0\bar{u}), 0, 0, \hat{d}(T_0\bar{u}, T_0T_0\bar{u})\} \\
=& \hat{d}(T_0\bar{u}, T_0T_0\bar{u}).
\end{aligned}
$$

Thus, from (2.26), we have

$$
\begin{aligned}
\psi_a(\hat{d}(T_0\bar{u}, T_0T_0\bar{u}) \leq& \psi_a(\hat{d}(T_0\bar{u}, T_0T_0\bar{u})) - \phi_a(\hat{d}(T_0\bar{u}, T_0T_0\bar{u})) \\
<& \psi_a(\hat{d}(T_0\bar{u}, T_0T_0\bar{u})), \ \textit{a contradiction.}
\end{aligned}
$$

Therefore, $T_0\bar{u} = T_0T_0\bar{u} = B_0T_0\bar{u}$.

Thus, the CFP of B_0 and T_0 is $T_0\bar{u}$.

Likewise, we demonstrate that the CFP of A_0 and S_0 is $S_0\bar{v}$.

Behind, $T_0\bar{u} = S_0\bar{v}$, $T_0\bar{u}$ is the CFP of A_0, B_0, S_0, T_0.

When A_0M is presumed to be a complete subspace of M, the proof is identical.

The instances in which S_0M is a complete subsection of M are equivalent to the instances in which A_0M, B_0M is a complete subsection of M, respectively, because $T_0M \subseteq A_0M$ and $S_0M \subseteq B_0M$.

Next, we show that the CFP is unique.

If possible, let $\hat{p}$ and $\hat{q}$ be two CFP of A_0, B_0, S_0 and T_0, s.t. $\hat{p} \neq \hat{q}$.

We have with this equation (2.2),

$$
\psi_a(\hat{d}(\hat{p}, \hat{q}) = \psi_a(\hat{d}(S_0\hat{p}, T_0) \leq \psi_a(m_\delta(\hat{p}, \hat{q})) - \phi_a(m_\delta(\hat{p}, \hat{q})), \ \textit{where} \tag{2.27}
$$

$$
\begin{aligned}
m_\delta(\hat{p}, \hat{q}) =& max\{\hat{d}(A_0\hat{p}, B_0\hat{q}), \hat{d}(S_0\hat{p}, A_0\hat{q}), \hat{d}(B_0\hat{q}, T_0\hat{q}), \frac{1}{2}(\hat{d}(S_0\hat{p}, B_0\hat{q}) + \hat{d}(T_0\hat{q}, A_0\hat{p}))\} \\
=& max\{\hat{d}(\hat{p}, \hat{q}), 0, 0, \hat{d}(\hat{p}, \hat{q})\} \\
=& \hat{d}(\hat{p}, \hat{q}).
\end{aligned}
$$

Thus, from (2.27), one can write

$$
\begin{aligned}
\psi_a(\hat{d}(\hat{p}, \hat{q})) \leq& \psi_a(\hat{d}(\hat{p}, \hat{q})) - \phi_a(\hat{d}(\hat{p}, \hat{q})) \\
<& \psi_a(\hat{d}(\hat{p}, \hat{q})), \ \textit{a contradiction.}
\end{aligned}
$$

Therefore, $\hat{p} = \hat{q}$, and the uniqueness follows. □

Theorem 2.3. *Let A_0, B_0, S_0 and T_0 be four mapping of a MS $(M, \hat{d})$ fulfill the conditions (2.1), (2.21) and:*

$$S_0M \subseteq B_0M \text{ and the pair } (A_0, S_0) \text{ satisfies } (CLR_A) \text{ property, or} \qquad (2.28)$$
$$T_0M \subseteq A_0M \text{ and the pair } (B_0, T_0) \text{ satisfies } (CLR_B) \text{ property.}$$

Then A_0, B_0, S_0 and T_0 have a unique CFP.

Proof. WLOG, presume $S_0M \subseteq B_0M$ and (A_0, S_0) fulfills (CLR_A) property, then there is a sequence $\{\breve{x}_{\breve{n}}\}$ in M s.t. $lim_{\breve{n}\to\infty}A_0\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty}S_0\breve{x}_{\breve{n}} = A_0\breve{x}$; $\breve{x}$ in M.

After all, $S_0M \subseteq B_0M$, there is a sequence $\{\breve{y}_{\breve{n}}\}$ in M such that $S_0\breve{x}_{\breve{n}} = B_0\breve{y}_{\breve{n}}$.

Accordingly, $lim_{\breve{n}\to\infty}B_0\breve{y}_{\breve{n}} = A_0\breve{x}$.

Next, we show that $lim_{\breve{n}\to\infty}T_0\breve{y}_{\breve{n}} = A_0\breve{x}$.

Perhaps, $lim_{\breve{n}\to\infty}T_0\breve{y}_{\breve{n}} = \mathfrak{z} \neq A_0\breve{x}$.

By (2.2), one can write

$$\psi_a(\hat{d}(S_0\breve{x}_{\breve{n}}, T_0\breve{y}_{\breve{n}}) \leq \psi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y}_{\breve{n}})) - \phi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y}_{\breve{n}})).$$

Letting limit as $\breve{n} \to \infty$, we have

$$lim_{\breve{n}\to\infty}\psi_a(\hat{d}(S_0\breve{x}_{\breve{n}}, T_0\breve{y}_{\breve{n}})) \leq lim_{\breve{n}\to\infty}\psi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y}_{\breve{n}})) - lim_{\breve{n}\to\infty}\phi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y}_{\breve{n}})), \quad (2.29)$$

where

$$\begin{aligned} lim_{\breve{n}\to\infty}m_\delta(\breve{x}_{\breve{n}}, \breve{y}_{\breve{n}}) =& lim_{\breve{n}\to\infty}[max\{\hat{d}(A_0\breve{x}_{\breve{n}}, B_0\breve{y}_{\breve{n}}), \hat{d}(S_0\breve{x}_{\breve{n}}, A_0\breve{x}_{\breve{n}}), \hat{d}(T_0\breve{y}_{\breve{n}}, B_0\breve{y}_{\breve{n}}), \\ & \frac{1}{2}(\hat{d}(S_0\breve{x}_{\breve{n}}, B_0\breve{y}_{\breve{n}}) + \hat{d}(T_0\breve{y}_{\breve{n}}, A_0\breve{x}_{\breve{n}}))\}] \\ =& max\{\hat{d}(A_0\breve{x}, A_0\breve{x}), \hat{d}(A_0\breve{x}, A_0\breve{x}), \hat{d}(\mathfrak{z}, A_0\breve{x}), \frac{1}{2}(\hat{d}(\mathfrak{z}, \mathfrak{z}) + \hat{d}(\mathfrak{z}, A_0\breve{x}))\} = \hat{d}(\mathfrak{z}, A_0\breve{x}). \end{aligned}$$

Thus, from (2.29), we get

$$\begin{aligned} \psi_a(\hat{d}(A_0\breve{x}, \mathfrak{z})) \leq& \psi_a(\hat{d}(A_0\breve{x}, \mathfrak{z})) - \phi(\hat{d}(A_0\breve{x}, \mathfrak{z})) \\ <& \psi_a(\hat{d}(A_0\breve{x}, \mathfrak{z})), \ a \ conflict. \end{aligned}$$

Accordingly, $A_0\breve{x} = \mathfrak{z}$, that is, $lim_{\breve{n}\to\infty}T_0\breve{y}_{\breve{n}} = A_0\breve{x}$.

Subsequently, we have

$$lim_{\breve{n}\to\infty}A_0\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty}S_0\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty}B_0\breve{y}_{\breve{n}} = lim_{\breve{n}\to\infty}T_0\breve{y}_{\breve{n}} = A_0\breve{x} = \mathfrak{z}.$$

Net, we will show $S_0\breve{x} = \mathfrak{z}$.

Probably $S_0\breve{x} \neq \mathfrak{z}$.

With the help of (2.2), we have

$$\psi_a(\hat{d}(S_0\breve{x}, T_0\breve{y}_{\breve{n}}))) \neq \psi_a(m_\delta(\breve{x}, \breve{y}_{\breve{n}})) - \phi_a(m_\delta(\breve{x}, \breve{y}_{\breve{n}})).$$

Letting limit as $\breve{n} \to \infty$, we have

$$lim_{\breve{n}\to\infty}\psi_a(\hat{d}(S_0\breve{x}, T_0\breve{y}_{\breve{n}}) \leq lim_{\breve{n}\to\infty}\psi_a(m_\delta(\breve{x}, \breve{y}_{\breve{n}})) - lim_{\breve{n}\to\infty}\phi_a(m_\delta(\breve{x}, \breve{y}_{\breve{n}})), \tag{2.30}$$

where

$$\begin{aligned} lim_{\breve{n}\to\infty}m_\delta(\breve{x}, \breve{y}_{\breve{n}}) =& lim_{\breve{n}\to\infty}[max\{\hat{d}(A_0\breve{x}, B_0\breve{y}_{\breve{n}}), \hat{d}(S_0\breve{x}, A_0\breve{x}), \hat{d}(T_0\breve{y}_{\breve{n}}, B_0\breve{y}_{\breve{n}}), \\ & \frac{1}{2}\hat{d}((S_0\breve{x}, B_0\breve{y}_{\breve{n}}) + \hat{d}(T_0\breve{y}_{\breve{n}}, A_0\breve{x}))\}] \\ =& max\{\hat{d}(\mathfrak{z}, \mathfrak{z}), \hat{d}(S_0\breve{x}, \mathfrak{z}), \hat{d}(\mathfrak{z}, \mathfrak{z}), \frac{1}{2}(\hat{d}(S_0\breve{x}, \mathfrak{z}) + \hat{d}(\mathfrak{z}, \mathfrak{z}))\} = \quad \hat{d}(S_0\breve{x}, \mathfrak{z}). \end{aligned}$$

Thus, from (2.30), we get

$$\begin{aligned} \psi_a(\hat{d}(S_0\breve{x}, \mathfrak{z})) \leq& \psi_a(\hat{d}(S_0\breve{x}, \mathfrak{z})) - \phi_a(\hat{d}(S_0\breve{x}, \mathfrak{z})) \\ <& \psi_a(\hat{d}(S_0\breve{x}, \mathfrak{z})), \ a\ contradiction. \end{aligned}$$

On that account, $S_0\breve{x} = \mathfrak{z} = A_0\breve{x}$.

Next to, the pair (A_0, S_0) is WC, it follows that $A_0\mathfrak{z} = S_0\mathfrak{z}$.

Additionally, $S_0M \subseteq B_0M$, there is some $\breve{y}$ in M s.t. $S_0\breve{x} = B_0\breve{y}$, i.e. $B_0\breve{y} = \mathfrak{z}$.

Next, we show that $T_0\breve{y} = \mathfrak{z}$.

Likely, $T_0\breve{y} \neq \mathfrak{z}$.

From (2.2), we have

$$\psi_a(\hat{d}(S_0\breve{x}_{\breve{n}}, T_0\breve{y})) \leq \psi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y})) - \phi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y})).$$

Making limit as $\breve{n} \to \infty$, we have

$$lim_{\breve{n}\to\infty}\psi_a(\hat{d}(S_0\breve{x}_{\breve{n}}, T_0\breve{y})) \leq lim_{\breve{n}\to\infty}\psi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y})) - lim_{\breve{n}\to\infty}\phi_a(m_\delta(\breve{x}_{\breve{n}}, \breve{y})), \tag{2.31}$$

where

$$\begin{aligned}
lim_{\breve{n}\to\infty}m_\delta(\breve{x}_{\breve{n}},\breve{y}) =& lim_{\breve{n}\to\infty}[max\{\hat{d}(A_0\breve{x}_{\breve{n}},B_0\breve{y}),\hat{d}(S_0\breve{x}_{\breve{n}},A_0\breve{x}_{\breve{n}}),\hat{d}(_0\breve{y},B_0\breve{y}),\\
&\frac{1}{2}(\hat{d}(S_0\breve{x}_{\breve{n}},B_0\breve{y})+\hat{d}(T_0\breve{y},A_0\breve{x}_{\breve{n}}))\}]\\
=& max\{\hat{d}(\mathfrak{z},\mathfrak{z}),\hat{d}(\mathfrak{z},\mathfrak{z}),\hat{d}(\mathfrak{z},T_0\breve{y}),\frac{1}{2}(\hat{d}(\mathfrak{z},\mathfrak{z})+\hat{d}(T_0\breve{y},\mathfrak{z}))\}\\
=& \hat{d}(\mathfrak{z},T_0\breve{y}).
\end{aligned}$$

Thus, from (2.31), we can write

$$\begin{aligned}
\psi_a(\hat{d}(\mathfrak{z},T_0\breve{y})) \leq& \psi_a(\hat{d}(\mathfrak{z},T_0\breve{y}))-\phi_a(\hat{d}(\mathfrak{z},T_0\breve{y}))\\
<& \psi_a(\hat{d}(\mathfrak{z},T_0\breve{y})),\ a\ contradiction.
\end{aligned}$$

Thus, $\mathfrak{z}=T_0\breve{y}=B_0\breve{y}$.

Since the pair (B_0,T_0) is WC, it follows that $T_0\mathfrak{z}=B_0\mathfrak{z}$.

Now, we claim that $S_0\mathfrak{z}=T_0\mathfrak{z}$.

Probably, $S_0\mathfrak{z}\neq T_0\mathfrak{z}$.

From (2.2), we have

$$\psi_a(\hat{d}(S_0\mathfrak{z},T_0\mathfrak{z}))\leq\psi_a(m_\delta(\mathfrak{z},\mathfrak{z}))-\phi_a(m_\delta(\mathfrak{z},\mathfrak{z})),\ where \tag{2.32}$$

$$m_\delta(\mathfrak{z},\mathfrak{z}) = max\{\hat{d}(A_0\mathfrak{z},B_0\mathfrak{z}),\hat{d}(S_0\mathfrak{z},A_0\mathfrak{z}),\hat{d}(B_0\mathfrak{z},T_0\mathfrak{z}),\frac{1}{2}(\hat{d}(S_0\mathfrak{z},B_0\mathfrak{z})+\hat{d}(T_0\mathfrak{z},A_0\mathfrak{z})\}.$$

Thus, from (2.32), we have

$$\begin{aligned}
\psi_a(\hat{d}(S_0\mathfrak{z},T_0\mathfrak{z})) \leq& \psi_a(\hat{d}(S_0\mathfrak{z},T_0\mathfrak{z}))-\phi_a(\hat{d}(S_0\mathfrak{z},T_0\mathfrak{z}))\\
<& \psi_a(\hat{d}(S_0\mathfrak{z},T_0\mathfrak{z})),\ an\ inconsistency.
\end{aligned}$$

In consequence, $S_0\mathfrak{z}=T_0\mathfrak{z}$, that is, $A_0\mathfrak{z}=S_0\mathfrak{z}=T_0\mathfrak{z}=B_0\mathfrak{z}$.

Now, we're going to explain that $\mathfrak{z}=T_0\mathfrak{z}$.

If possible, let's do it, $\mathfrak{z}\neq T_0\mathfrak{z}$.

Taken away (2.2), we have a reference to

$$\psi_a(\hat{d}(S_0\breve{x},T_0\mathfrak{z}))\leq\psi_a(m_\delta(\breve{x},\mathfrak{z}))-\phi_a(m_\delta(\breve{x},\mathfrak{z})),\ wherein \tag{2.33}$$

$$
\begin{aligned}
m_\delta(\check{x}, \mathfrak{z}) =& max\{\hat{d}(A_0\check{x}, B_0\mathfrak{z}), \hat{d}(S_0\check{x}, A_0\check{x}), \hat{d}(B_0\mathfrak{z}, T_0\mathfrak{z}), \frac{1}{2}(\hat{d}(S_0\check{x}, B_0\mathfrak{z}) + \hat{d}(T_0\mathfrak{z}, A_0\check{x}))\} \\
=& \hat{d}(S_0\check{x}, T_0\mathfrak{z}) = \hat{d}(\mathfrak{z}, T_0\mathfrak{z}).
\end{aligned}
$$

Thus, from (2.33), we have

$$
\begin{aligned}
\psi_a(\hat{d}(\mathfrak{z}, T_0\mathfrak{z})) \leq& \psi_a(\hat{d}(\mathfrak{z}, T_0\mathfrak{z})) - \phi_a(\hat{d}(\mathfrak{z}, T_0\mathfrak{z})) \\
<& \psi_a(\hat{d}(\mathfrak{z}, T_0\mathfrak{z})), \ a \ contradiction.
\end{aligned}
$$

Therefore, $\mathfrak{z} = T_0\mathfrak{z} = B_0\mathfrak{z} = A_0\mathfrak{z} = S_0\mathfrak{z}$.

Consequently, $\mathfrak{z}$ is the CFP of A_0, B_0, S_0 and T_0.

Now we can show that the CFP is special.

Presume $\bar{u}$ be another CFP of A_0, B_0, S_0 and T_0.

Maybe, $\mathfrak{z} \neq \bar{u}$.

By using of this (2.2), we can write

$$
\begin{aligned}
\psi_a(\hat{d}(\bar{u}, \mathfrak{z})) =& \psi_a(\hat{d}(S_0\bar{u}, T_0\mathfrak{z}) \\
\leq& \psi_a(m_\delta(\bar{u}, \mathfrak{z})) - \phi_a(m_\delta(\bar{u}, \mathfrak{z})) \\
=& \psi_a(\hat{d}(\bar{u}, \mathfrak{z})) - \phi_a(\hat{d}(\bar{u}, \mathfrak{z})), since \ m_\delta(\bar{u}, \mathfrak{z}) = \hat{d}(\bar{u}, \mathfrak{z}). \\
<& \psi_a(\hat{d}(\bar{u}, \mathfrak{z})), \ a \ contradiction.
\end{aligned}
$$

Thus, $\bar{u} = \mathfrak{z}$, and hence the uniqueness follows. □

Example 2.1. *Assume that the Euclid metric is equipped with $M = [0, 1]$ and $\hat{d}(\check{x}, \check{y}) = |\check{x} - \check{y}|$. Presume the self maps A_0, B_0, S_0 and T_0 be defined by*

$$
S_0\check{x} = \frac{\check{x}}{8}, B_0\check{x} = \frac{\check{x}}{4}, T_0\check{x} = \frac{\check{x}}{2}, A_0\check{x} = \check{x}.
$$

Clearly,

$$
S_0M = [0, \frac{1}{8}] \subseteq [0, \frac{1}{4}] = B_0M,
$$

$$
T_0M = [0, \frac{1}{2}] \subseteq [0, 1] = A_0M.
$$

Also A_0M is complete subspace of M and pairs (A_0, S_0), (B_0, T_0) are WC.

Now,

$$\begin{aligned}
\hat{d}(S_0\breve{x}, T_0\breve{y}) =& |\frac{\breve{x}}{8} - \frac{\breve{y}}{2}| = \frac{\breve{x}}{8}|\breve{x} - 4\breve{y}|.\\
\hat{d}(A_0\breve{x}, B_0\breve{y}) =& |\breve{x} - \frac{\breve{y}}{4}| = \frac{1}{4}|4\breve{x} - \breve{y}|.\\
\hat{d}(S_0\breve{x}, A_0\breve{x}) =& |\frac{\breve{x}}{8} - \breve{x}| = \frac{7}{8}\breve{x}.\\
\hat{d}(B_0\breve{y}, T_0\breve{y}) =& |\frac{\breve{y}}{4} - \frac{\breve{y}}{2}| = \frac{\breve{y}}{4}.\\
\frac{(\hat{d}(S_0\breve{x}, B_0\breve{y}) + \hat{d}(T_0\breve{y}, A_0\breve{x}))}{2} =& \frac{1}{2}[|\frac{\breve{x}}{8} - \frac{\breve{y}}{4}| + |\frac{\breve{y}}{2} - \breve{x}|]\\
=& \frac{1}{16}[|\breve{x} - 2\breve{y}| + 4|\breve{y} - 2\breve{x}|].
\end{aligned}$$

Let $\psi_a(\hat{t}) = \frac{\hat{t}}{3}$ *and* $\phi_a(\hat{t}) = \frac{\hat{t}}{6}$.
Thus, we have

$$\psi_a(\hat{d}(S_0\breve{x}, T_0\breve{y})) = \frac{1}{24}|\breve{x} - 4\breve{y}|.$$

$$\begin{aligned}
m_\delta(\breve{x}, \breve{y}) =& max\{\hat{d}(A_0\breve{x}, B_0\breve{y}), \hat{d}(S_0\breve{x}, A_0\breve{x}), \hat{d}(T_0\breve{y}, B_0\breve{y}), \frac{1}{2}(\hat{d}(S_0\breve{x}, B_0\breve{y}) + \hat{d}(T_0\breve{y}, A_0\breve{x}))\}\\
=& \hat{d}(S_0\breve{x}, A_0\breve{x}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi_a(\hat{d}(S_0\breve{x}, A_0\breve{x})) =& \frac{1}{3}(\frac{7}{8}\breve{x}) = \frac{7}{24}\breve{x}.\\
\phi_a(\hat{d}(S_0\breve{x}, A_0\breve{x})) =& \frac{1}{6}(\frac{7}{8}\breve{x}) = \frac{7}{48}\breve{x}.
\end{aligned}$$

Thus, we have

$$\psi_a(m_\delta(\breve{x}, \breve{y}))\phi_a(m_\delta(\breve{x}, \breve{y})) = \frac{7}{24}\breve{y} - \frac{7}{48}\breve{x} = \frac{7}{48}\breve{x}.$$

Therefore,

$$\psi_a(\hat{d}(S_0\breve{x}, T_0\breve{y}) \leq \psi_a(m_\delta(\breve{x}, \breve{y})) - \phi_a(m_\delta(\breve{x}, \breve{y})).$$

Hence condition 2 is satisfied.

If, then the series is considered $\{\breve{x}_{\breve{n}}\} = \{\frac{1}{\breve{n}}\}$, *then*

$$lim_{\breve{n}\to\infty} A_0\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty}\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty}\frac{1}{\breve{n}} = 0.$$
$$lim_{\breve{n}\to\infty} S_0\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty}\breve{x}_{\frac{\breve{n}}{8}} = lim_{\breve{n}\to\infty}\frac{1}{8\breve{n}} = 0.$$

Therefore,

$$lim_{\breve{n}\to\infty} A_0\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty} S_0\breve{x}_{\breve{n}} = 0, \ wherein\ 0 \in M.$$

So, (A_0, S_0) *satisfies the E.A. property.*

Also,

$$lim_{\breve{n}\to\infty} A_0\breve{x}_{\breve{n}} = lim_{\breve{n}\to\infty} A_0\breve{x}_{\breve{n}} = 0 = A_0(0).$$

So, the pair (A_0, S_0) *satisfies the* (CLR_A) *property.*

All the criteria of the above theorems are therefore fulfilled.

0 *is the only CFP of* A_0, S_0, B_0 *and* T_0.

2.3 A new category of Generalized $\beta_a - \phi_a - \mathcal{Z}$-Contractive Pair of Mappings

"The well-known Banach-Caccioppoli theorem published in 1922 [8] was the first important result for contractive-type mappings on fixed points."Priya Shahi et al. [56] present the idea of α-admissible mapping.

We present the following new concepts:

Definition 2.4. Let (X_i, d_i) where $i = 1, 2, 3...n$ be a MS and A_i, C_i be self maps on X_i. The (A_i, C_i) pair is called a generalized $\beta_a - \phi_a - \mathcal{Z}$-contractive mapping pair regards to ζ whether

$$\zeta(\beta_a(C_i x, C_i y) d_i(A_i x, A_i y), \phi_a(\mathfrak{M}(C_i x, C_i y))) \geq 0 \tag{2.34}$$

$\forall\ x, y \in X_i$, where $\beta_a : X_i \times X_i \to [0, \infty]$ and $\phi_a \in \Phi_a$ and

$$\mathfrak{M}(C_i x, C_i y) = max\{d_i(C_i x, C_i y), \frac{d_i(C_i x, A_i x) + d_i(C_i y, A_i y)}{2}, \frac{d_i(C_i x, A_i y) + d_i(S_i y, A_i x)}{2}\}.$$

2.4 Fixed Point Theorems Use Simulation Function For a Pair of Mappings

Khojasteh, Shukla and Radenovic [35] introduced a new class of mappings called SF. Later, the concept of SFs was slightly changed by Argoubi, Samet and Vetro [7] by removing a condition. In the context of Argoubi et al. [7], let Z^* be a set of SFs.

As we have already define SF (1.3)

Theorem 2.5. *Presume (X_i, d_i) be a CMS and $A_i, C_i : X_i \to X_i$ be s.t. $A_i(X_i) \subseteq C_i(X_i)$. Suppose that the (A_i, C_i) pair is a generalized $\beta_a - \phi_a - \mathcal{Z}$-contractive mapping pair with the following conditions:*

1. *In relation to C_i, A_i appears to be β_a-admissible;*
2. *$\exists$ $x_0 \in X_i$ s.t. $\beta_a(C_i x_0, A_i x_0) \geq 1$;*
3. *If $\{C_i x_n\}$ be a series in X_i s.t. $\beta_a(C_i x_n, C_i x_{n+1}) \geq 1$*

$\forall$ n and $C_i x_n \to C_i z \in C_i(X_i)$ as $n \to \infty$, subsequently $\exists$ an array $\{C_i x_{n(\hat{k})}\}$ of $\{C_i x_n\}$ such that $\beta_a(C_i x_{n(\hat{k})}, C_i z) \geq 1$ $\forall$ $\hat{k}$.

Proof. In view of condition (2), let $x_0 \in X_i$ be such that $\beta_a(C_i x_0, A_i x_0) \geq 1$. Since $A_i(X_i) \subseteq C_i(X_i)$, we picked the number $x_1 \in X_i$ s.t. $A_i x_0 = C_i x_1$. To this extent, if we continue this step by selecting $x_1, x_2, ..., x_n$, we want to be x_{n+1} in X_i

$$A_i x_n = C_i x_{n+1}, \quad n = 0, 1, 2, ... \tag{2.35}$$

Since A_i is β_a-admissible w.r.t. C_i, we have

$$\beta_a(C_i x_0, A_i x_0) = \beta_a(C_i x_0, C_i x_1) \geq 1 \Rightarrow \beta_a(A_i x_0, A_i x_1) = \beta_a(C_i x_0, C_i x_2) \geq 1$$

We get by using mathematical induction,

$$\beta_a(C_i x_n, C_i x_{n+1}) \geq 1 \text{ for all } n = 0, 1, 2, ... \tag{2.36}$$

If $A_i x_{n+1} = A_i x_n$ is equal to n, then press (2.35)

$$A_i x_n = C_i x_{n+1}, \quad n = 0, 1, 2, ...$$

namely, A_i and C_i be a coincidence point at $X_i = x_{n+1}$ so we completed the evidence. For this, we're going to believe that $s(A_i x_n, A_i x_{n+1}) > 0$ $\forall$ n.

Now, by placing $x = x_n$, $y = x_{n+1}$ in (2.34), we obtain

$$\begin{aligned} 0 &\leq \zeta(\beta_a(C_ix_n, C_ix_{n+1})d_i(A_ix_n, A_ix_{n+1}), \phi_a(\mathfrak{M}(C_ix_n, C_ix_{n+1}) \\ &< \phi_a(\mathfrak{M}(C_ix_n, C_ix_{n+1})) - \beta_a(C_ix_n, C_ix_{n+1})d_i(A_ix_n, A_ix_{n+1}) \end{aligned}$$

or

$$\begin{aligned} \beta_a(C_ix_n, C_ix_{n+1})d_i(A_ix_n, A_ix_{n+1}) &< \phi_a(\mathfrak{M}(A_ix_n, C_ix_{n+1})), \\ d_i(A_ix_n, A_ix_{n+1}) &\leq \beta_a(C_ix_n, C_ix_{n+1})d_i(A_ix_n, A_ix_{n+1}) \\ &< \phi_a(\mathfrak{M}(C_ix_n, C_ix_{n+1})), where \end{aligned}$$

$$\begin{aligned} \mathfrak{M}(C_ix_n, C_ix_{n+1}) &= max\{d_i(C_ix_n, C_ix_{n+1}), \tfrac{d_i(C_ix_n, A_ix_n) + d_i(S_ix_{n+1}, A_ix_{n+1})}{2}\}, \\ &\tfrac{d_i(C_ix_n, A_ix_{n+1}) + d_i(C_ix_{n+1}, A_ix_n)}{2}\} \\ &\leq max\{d_i(A_ix_{n-1}, A_ix_n), d_i(A_ix_n, A_ix_{n+1})\}. \end{aligned} \tag{2.37}$$

Despite of the monotonicity of the ϕ_a function and the use of the inequalities (2.35) and (2.37), we have $n \geq 1$ for everything

$$d_i(A_ix_n, A_ix_{n+1}) = \phi_a(max\{d_i(A_ix_{n-1}, A_ix_n), d_i(A_ix_n, A_ix_{n+1})\}). \tag{2.38}$$

If it is $n \geq 1$, one can say $d_i(A_ix_{n-1}, A_ix_n) \leq d_i(A_ix_n, A_ix_{n+1})$, we derive that from (2.38),

$$d_i(A_ix_n, A_ix_{n+1}) \leq \phi_a(d_i(A_ix_n, A_ix_{n+1}) < d_i(A_ix_n, A_ix_{n+1}),$$

paradox.

In consequence, $\forall\ n \geq 1$, we've got to

$$max\{d_i(A_ix_{n-1}, A_ix_n), d_i(A_ix_n, A_ix_{n+1}) = d_i(A_ix_{n-1}, A_ix_n)\}. \tag{2.39}$$

Note that given (2.38) and (2.39), we obtain

$$d_i(A_ix_n, A_ix_{n+1}) \leq \phi_a(d_i(A_ix_{n-1}, A_ix_n)). \tag{2.40}$$

This method is experimentally substituted, it becomes

$$d_i(A_ix_n, A_ix_{n+1}) \leq \phi_a^n(d_i(A_ix_0, A_ix_1)), \text{for all}\, n \geq 1 \tag{2.41}$$

With this (2.41), inequality used, $\forall\ \hat{k} \geq 1$, one can say

$$\begin{aligned} d_i(A_ix_n, A_ix_{n+\hat{k}}) &\leq d_i(A_ix_n, A_ix_{n+1}) + ... + d_i(A_ix_{n+\hat{k}-1}, A_ix_{n+\hat{k}}) \\ &\leq \sum_{\bar{p}=n}^{n+\hat{k}-1} \phi_a^{\bar{p}}(d_i(A_ix_1, A_ix_0)) \\ &\leq \sum_{\bar{p}=n}^{+\infty} \phi_a^{\bar{p}}(d_i(A_ix_1, A_ix_0)) \end{aligned} \tag{2.42}$$

Assuming, $p \to \infty$ in (2.42), we display that $\{A_ix_n\}$ is a CS in (X_i, d_i).

Due to passing (2.35), we have $\{A_ix_n\} = \{C_ix_{n+1}\} \subseteq C_i(X)$ and $C_i(X)$ are closed, $\exists$ $z \in X$ s.t.

$$lim_{n\to\infty} C_ix_n = C_iz. \tag{2.43}$$

We have now seen that z is a coincidence point of A_i, C_i. Instead, please believe that $d_i(A_iz, C_iz) > 0$. Because according to the conditions (3) and (2.43), it can say $\beta_a(C_ix_{n(k)}, C_iz) \geq 1$.

Taking $x = x_{n(\hat{k})}, y = z$ in (1), it becomes

$$\begin{aligned} &0 \leq \zeta(\beta_a(C_ix_{n(\hat{k})}, C_iz)d_i(A_ix_{n(\hat{k})}, A_iz), \phi_a(\mathfrak{M}(C_ix_{n(\hat{k})}, C_iz)) \\ &\quad < \phi_a(\mathfrak{M}(C_ix_{n(\hat{k})}, C_iz) - \beta_a(C_ix_{n(\hat{k})}, C_iz))d_i(A_ix_{n(k)}, A_iz) or \end{aligned}$$

$$\beta_a(C_ix_{n(\hat{k})}, C_iz)d_i(A_ix_{n(\hat{k})}, A_iz) < \phi_a\mathfrak{M}(C_ix_{n(\hat{k})}, C_iz)$$

But $\beta_a(C_ix_{n(\hat{k})}, C_iz) \geq 1$

$$\begin{aligned} d_i(A_ix_{n(\hat{k})}, A_iz) &\leq \beta_a(C_ix_{n(\hat{k})}, C_iz)d_i(A_ix_{n(\hat{k})}, A_iz) \\ &< \phi_a(\mathfrak{M}(C_ix_{n(\hat{k})}, C_iz)), \end{aligned} \tag{2.44}$$

$$\begin{aligned} \mathfrak{M}(C_ix_{n(\hat{k})}, C_iz) = max\{d_i(C_ix_{n(\hat{k})}, C_iz), &\frac{d_i(C_ix_{n(\hat{k})}, A_ix_{n(\hat{k})}) + d_i(C_iz, A_iz)}{2}, \\ &\frac{d_i(C_ix_{n(\hat{k})}, A_iz) + d_i(C_iz, A_ix_{n(\hat{k})+1})}{2}\} \end{aligned}$$

Instead, we have

$$\begin{aligned} \mathfrak{M}(C_ix_{n(\hat{k})}, C_iz) = max\{d_i(Cx_{n(\hat{k})}, C_iz), &\frac{d_i(C_ix_{n(\hat{k})}, A_ix_{n(\hat{k})}) + d_i(C_iz, S_iz)}{2}, \\ &\frac{d_i(C_ix_{n(\hat{k})}, A_iz) + d_i(C_iz, A_ix_{n(\hat{k})})}{2}\} \end{aligned}$$

Making $\hat{k} \to \infty$ in (2.44), we obtain

$$d_i(C_iz, A_iz) \leq \phi_a lim_{\hat{k}\to\infty}(\mathfrak{M}(C_ix_{(\hat{k})}, C_iz))$$
$$\leq \phi_a(max\{d_i(C_ix_{n(\hat{k})}, C_iz), \frac{d_i(C_ix_{n(\hat{k})}, A_ix_{n(\hat{k})}) + d_i(C_iz, A_iz)}{2},$$
$$\frac{d_i(C_ix_{n(\hat{k})}, A_iz) + d_i(C_iz, A_ix_{n(\hat{k})})}{2}\}$$

Render $\hat{k} \to \infty$ in the above inequality yields $d_i(C_iz, A_iz) \leq \phi_a(\frac{d_i(A_iz,C_iz)}{2}) < \frac{d_i(A_iz,C_iz)}{2}$, this is a paradox.

Therefore, our assumption is false and $\phi_a(A_iz, C_iz) = 0$, that is, $A_iz = C_iz$.

This shows that A_i and C_i have a coincidence point. □

Theorem 2.6. *Besides the Theorem (2.5) hypothesis, Assuming for everyone $u, v \in C(C_i, A_i)$, there is $w \in X_i$ such that $\beta_a(C_iu, C_iw) \geq 1$ and $\beta_a(C_iu, C_iw) \geq 1$ and A_i, C_i turn to their points of coincidence. Then, A_i, C_i have a special CFP.*

Proof. In order to prove this theorem, we can take three steps.

First and foremost, we say that if $u, v \in C(C_i, A_i)$, then $C_iu = C_iv$. There is a hypothesis that $w \in X$ exists in such a way that

$$\beta_a(C_iu, C_iw) \geq 1, \beta_a(C_iv, C_iw) \geq 1. \tag{2.45}$$

From this fact $A_i(X) \subseteq C_i(X)$, let's describe the series in order to $\{w_n\}$ in X_i by $C_iw_{n+1} = A_iw_n \; \forall \; n \geq 0$ and $w_0 = w$. We get it from (2.45) as A_i is β_a-admissible regards to C_i

$$\beta_a(C_iu, C_iw_n) \geq 1, \beta_a(C_iv, C_iw_n) \geq 1. \tag{2.46}$$

Therefore, if we put $x = u, y = w_{n+1}$ in (2.34), we'll get

$$0 \leq \zeta(\beta_a(C_iu, C_iw_{n+1})d_i(A_iu, A_iw_{n+1}), \phi_a(\mathfrak{M}(C_iu, C_iw_{n+1}))$$
$$< \phi_a(\mathfrak{M}(C_iu, C_iw_{n+1}) - \beta_a(C_iu, C_iw_{n+1})d_i(A_iu, A_iw_{n+1})$$

or

$$\beta_a(C_iu, C_iw_{n+1})d_i(A_iu, A_iw_{n+1}) < \phi_a(\mathfrak{M}(C_iu, C_iw_{n+1})).$$

But $\beta_a(C_iu, C_iw_{n+1}) \geq 1$,

$$\begin{aligned} d_i(A_iu, A_iw_{n+1}) &\leq \beta_a(C_iu, C_iw_{n+1}) d_i(A_iu, A_iw_{n+1}) \\ &< \phi_a(\mathfrak{M}(C_iu, C_iw_{n+1})) = \phi_a(\mathfrak{M}(A_iu, A_iw_n)). \end{aligned} \tag{2.47}$$

$$\begin{aligned} \mathfrak{M}(A_iu, A_iw_n) = max\{d_i(A_iu, A_iw_n), &\frac{d_i(A_iu, C_iu) + d_i(A_iw_n, C_iw_n)}{2}, \\ &\frac{d_i(A_iu, C_iw_n) + d_i(A_iw_n, C_iu)}{2}\} \\ &\leq max\{d_i(C_iu, C_iw_n), d_i(C_iu, C_iw_{n+1})\} \\ &\leq max\{d_i(C_iu, C_iw_n), d_i(C_iu, C_iw_{n+1})\}. \end{aligned} \tag{2.48}$$

Using the above-mentioned inequality (2.47) and because of the ϕa monotone property, we get the equation.

$$d_i(C_iu, C_iw_{n+1}) \leq \phi_a(max\{d_i(C_iu, C_iw_n), d_i(C_iu, C_iw_{n+1})\}) \tag{2.49}$$

$\forall\ n$. Without limiting the generality, we can assume $d_i(C_iu, C_iw_n) \geq 0\ \forall\ n$. If $max\{d_i(C_iu, C_iw_n), d_i(C_iu, C_iw_{n+1}\} = d_i(C_iu, C_iw_{n+1})$, It can be obtained from (2.49), that

$$d_i(C_iu, C_iw_{n+1}) \leq \phi_a(d_i(C_iu, C_iw_{n+1})) < d_i(C_iu, C_iw_{n+1}), \tag{2.50}$$

And that's a conflict. We, therefore, have

$$max\{d_i(C_iu, C_iw_n), d_i(C_iu, C_iw_{n+1})\} = d_i(C_iu, C_iw_n),$$

$$d_i(C_iu, C_iw_{n+1}) \leq \phi_a(d_i(C_iu, C_iw_n)),\ for\,all\,n.$$

$$d_i(C_iu, C_iw_n) \leq \phi_a^n(d_i(C_iu, C_iw_0)),\ \text{for all } n \geq 1 \tag{2.51}$$

In the following inequality, $n \to \infty$

$$lim_{n\to\infty} d_i(C_iu, C_iw_n) = 0 \tag{2.52}$$

Likewise, we will show that

$$lim_{n\to\infty} d_i(C_iv, C_iw_n) = 0 \tag{2.53}$$

It follows from (2.52) and (2.53) that $C_iu = C_iv$.

We are now able to display the presence of a CFP in the second stage. Let $u \in C(C_i, A_i)$, that is, $C_i u = A_i u$. We get at their coincidence points because of the commutativity of A_i and C_i is

$$C_i^2 u = C_i A_i u = A_i C_i u \tag{2.54}$$

Let $C_i u = z$ be denoted, then from (2.54), $C_i z = A_i z$. Therefore, z is a coincidence of A_i and C_i from stage 1 onwards. Now, we have $C_i u = C_i z = z = A_i z$. Afterwards, z is a CFP of A_i and C_i.

We will display the uniqueness in the third step.

Assume that another CFP of A_i and C_i. Then, $z^* \in C(C_i, A_i)$.

We have $z^* = C_i z^* = C_i z = z$ for the first step. This makes the facts complete. □

Our previous results can be used to obtain the following results:

Corollary 2.7. *Let (X_i, d_i) be a CMS and $A_i, C_i : X_i \to X_i$ be such that $A_i(X_i) \subseteq C_i(X_i)$. Supposing that a function $\phi_a \in \Phi_a$ occurs in such a way that the function $\phi_a \in \Phi_a$*

1. *By selecting $\beta_a(x, y) = 1$ and $\zeta(\check{t}, \check{s}) = \bar{\lambda}\check{s} - \check{t}$, $\forall\ \check{t}, \check{s} > 0$, $\bar{\lambda} \in (0, 1)$, then the outcome retains*

$$d_i(A_i x, A_i y) \leq \bar{\lambda}(\phi_a(\mathfrak{M}(C_i x, C_i y))),\ for\, all x, y \in X. \tag{2.55}$$

   Also assume that $C_i(X_i)$ is closed. Then, there is a coincidence between A_i and C_i. In addition, if A_i and C_i move at their coincidence points, then A_i and C_i have CFP.

2. $$d_i(A_i x, A_i y) \leq \lambda(\phi_a(\mathfrak{M}(x, y))), \tag{2.56}$$

   $\forall\ x, y \in X_i$. Also, A_i has a CFP.

3. $$d_i(A_i x, A_i y) \leq \phi_a(d_i(C_i x, C_i y)),\ for\ all\, x, y \in X_i. \tag{2.57}$$

   Suppose, too, that $C_i(X)$ is closed. Then there is a point of coincidence between A_i and C_i. In addition, if A_i and C_i commute at their points of coincidence, then A_i, C_i have a CFP.

4. *By putting $M = d$*

$$d_i(A_i x, A_i y) \leq (d_i(x, y))$$

$\forall\ x, y \in X_i$. *Then A_i has a unique fixed point.*

5. *Let us suppose there is a constant $\bar{\lambda} \in (0, \frac{1}{2})$ such that*

$$d_i(A_i x, A_i y) \leq \bar{\lambda}[\frac{d_i(x, A_i x) + d_i(y, A_i y)}{2}] \times 2$$
$$d_i(A_i x, A_i y) \leq \bar{\lambda}[d_i(x, A_i x) + d_i(y, A_i y)]$$

$\forall\ x, y \in X_i$. *Then, there is a unique fixed point of A_i.*

6.

$$d_i(A_i x, A_i y) \leq \bar{\lambda}[\frac{d_i(x, A_i y)) + d_i(y, A_i x)}{2}] \times 2$$
$$d_i(A_i x, A_i y) \leq \bar{\lambda}[d_i(x, A_i y) + d_i(y, A_i x)]$$

$\forall\ x, y \in X_i$. *Instead, there is a unique fixed point in A_i.*

Definition 2.8. [19] "Suppose $(X_i, \preceq)$ is a POSET and $S_i, T_i : X_i \to X_i$ are mappings of X_i into itself. One states that S_i is T_i-non-decreasing if for $x, y \in X_i$

$$T_i(x) \preceq T_i(y) \Rightarrow S(x) \preceq S_i(y)\text{"} \tag{2.58}$$

Corollary 2.9. *Let $(X_i, \preceq)$ be a POSET and d_i be a metric on X_i s.t. (X_i, d_i) is complete. Assume that $A_i, C_i : X_i \to X_i$ be s.t. $A_i(X_i) \subseteq C_i(X_i)$ and A_i is a C_i-non-reducing mapping. Assume a function exists $\phi_a \in \Phi_a$ s.t.*

$$d_i(A_i x, A_i y) \leq \phi_a(\mathfrak{M}(C_i x, C_i y)) \tag{2.59}$$

$\forall\ x, y \in X_i$ *with $C_i x \preceq C_i y$. Assume, sometimes, the criteria are applicable:*

1. *there is $x_0 \in X_i$ s.t. $C_i x_0 \leq A_i x_0$;*

2. *$(X_i, \preceq, d_i)$ is C_i-regular.*

Also, suppose the closure of $C_i(X)$. Then there is a point of coincidence between A_i and C_i. Moreover, if for every pair $(x, y) \in C(C_i, A_i) \times C(C_i, A_i)$ $\exists$ $Z_i \in X_i$ so, $C_i x \preceq C_i z$ and $C_i y \leq C_i z$, and if A_i and C_i commute at their points of coincidence, then the CFP would be unique.

Proof. Describe it $\beta_a : X_i \times X_i \to [0,\infty)$ by

$$\beta_a(x,y) = \begin{cases} 1\,;\, either\, x \preceq y\, or\, x \succeq y \\ 0\,;\, otherwise. \end{cases} \tag{2.60}$$

Audibly, the pair (A_i, C_i) is a generalized $\beta_a - \phi_a$ contractive pair of mappings, that is,

$$\beta_a(C_i x, C_i y) d_i(A_i x, A_i y) \leq \phi_a(\mathfrak{M}(C_i x, C_i y))$$

$\forall\ x, y \in X_i$. Notice that in view of condition(1), we have $\beta_a(C_i x_0, A_i x_0) \geq 1$. Furthermore, $\forall\ x, y \in X_i$, from the C_i-monotone property of A_i,

$$\beta_a(T_i x, T_i y) \geq 1$$

$$\Rightarrow C_i x \preceq C_i y \text{ or } C_i x \succeq C_i y \Rightarrow A_i x \preceq A_i y \text{ or } A_i x \succeq A_i y \Rightarrow \beta_a(A_i x, A_i y) \geq 1.$$

In other terms, Ai is β_a-admissible. Now, let $\{C_i x_n\}$ is sequence in X_i s.t. $\beta_a(C_i x_n, C_i x_{n+1}) \geq 1\ \forall\ n$ and $C_i x_n \to C_i z \in X_i$ as $n \to \infty$. There Is a subsequence $\{C_i x_{n(k)}\}$ of $\{C_i x_n\}$ from the C_i-regularity theorem, s.t. $\{C_i x_{n(\hat{k})}\} \leq C_i z$ for all $\hat{k}$. Thus, by the way of β_a, we obtain $\beta_a(\{C_i x_{n(\hat{k})}, T_i z) \geq 1$. Now, all hypothesis (2.5) are satisfied.

We therefore infer that A_i and C_i have z coincidence stage, i.e., $S_i z = T_i z$.

The hypothesis is that $z \in X_i$ occurs in such a way that $C_i x \preceq C_i z$ and $C_i y \preceq C_i z$, which means β_a and $\beta_a(C_i x, C_i y) \geq 1$and $\beta_a(C_i y, C_i Z) \geq 1$. We therefore deduce the nature and uniqueness of the Theorem (2.6) CFP. □

Corollary 2.10. *Let $(X_i, \preceq)$ be a POSET and d_i be a metric on X_i s.t. (X_i, d_i) is complete. Presume that the mapping $A_i, C_i : X_i \to X$ is non-decreasing. Presume that $\phi_a \in \Phi_a$ is a function s.t. $d_i(A_i x, A_i y) \leq \phi_a(d_i(C_i x, C_i y))\ \forall\ x, y \in X_i$ with $C_i x \preceq C_i y$. Suppose that the following requirements apply, too;*

1. *$\exists\ x_0 \in X_i$ s.t. $C_i x_0 \preceq A_i x_0$:*
2. *$(X_i, \preceq, d_i)$ is C_i-regular.*

Assume $C_i(X_i)$ is closed. Then, there is a coincidence between A_i and C_i. Moreover, if for every pair $(x, y) \in C(C_i, A_i) \times C(C_i, A_i)\ \exists\ z \in X$ s.t. $C_i x \preceq C_i z$ and $C_i y \preceq C_i z$ and if A_i and C_i commute at their points of coincidence, we receive the CFP uniqueness after that.

Chapter 3

Best Proximity and Outcomes for Fixed Points in Metric Spaces

This chapter concerns with some FPT for best proximity results. Also, we prove some common FPT for proximal generalized contraction of Type-I and Type-II, fitted with graph and results in partially ordered MS. It consists of six sections. In first section, we introduce new notions of PC of kind-R and kind-M with $\mathcal{Z}$-contraction. In second section, we show that a pair of maps have the g_α-best proximity along with introduced PC in CMS. In third section, we implement the latest notions of updated Type-I and Type-II $\alpha-(\psi_0, g_0)$-PC. In fourth section, we prove certain FPT in MS. In fifth section, we derive some results in partially ordered MS. In sixth section, our aim is to introduce the PC of first kind and second kind which generalize several known types of contractions. Secondly, we prove certain FPT using SF in CMS and an application which derived from our main results.

3.1 New Notions of Proximal $\mathcal{Z}$-Contractions of Kind-R and Kind-M

We add the new proximal $\mathcal{Z}$-contraction notions of kind-R and kind-M. There are many works which use non-self mapping for that purpose. An estimated solution for the $Tx = x$ equation is possible. Many mathematicians have discussed the theoretical and functional implications of this theorem; we refer the reader to the [4, 3, 9, 23, 38, 54, 36, 10, 49, 60]. We introduce kind-R and Kind-M notions with $\mathcal{Z}$-contraction:

Definition 3.1. Let two non-void subsets of MS $(\mathfrak{X}, \mathfrak{d})$ be f and h. The non-self mapping of $\tilde{A} : f \to h$ is known to be a $\mathcal{Z}$-PC of type-M and type-R if there is an SF s.t.

1.

$$\mathfrak{d}(\hat{u}, \tilde{A}\gamma) = \mathfrak{d}(f, h)$$
$$\mathfrak{d}(\hat{v}, \tilde{A}\delta) = \mathfrak{d}(f, h)$$
$$\Rightarrow \zeta(\mathfrak{d}(\hat{u}, \hat{v}), \mathfrak{d}(\gamma, \tilde{A}\gamma) + \mathfrak{d}(\delta, \tilde{A}\delta)) \geq 0,$$

$\forall\ \hat{u}, \hat{v}, \gamma, \delta \in f$.

2.

$$\mathfrak{d}(\hat{u}, \tilde{A}\gamma) = \mathfrak{d}(f, h)$$
$$\mathfrak{d}(\hat{u}, \tilde{A}\delta) = \mathfrak{d}(f, h)$$
$$\Rightarrow \zeta(\mathfrak{d}(\hat{u}, \hat{v}), \mathfrak{d}(\gamma, \tilde{A}\delta) + \mathfrak{d}(\delta, \tilde{A}\hat{u}) \geq 0,$$

$\forall\ \hat{u}, \hat{v}, \gamma, \delta \in f$.

3.2 The presence of g_α-best similarity in complete metric space for a pair of maps

Theorem 3.2. *Let two non-void subsets of a CMS $(\mathfrak{X}, \mathfrak{d})$ be f and h. Assume f_0 be non-void and closed. Let $\tilde{A} : f \to h$ and $g_\alpha : f \to f$ satisfies the term below:*

1. *$\tilde{A}$ is a $\mathcal{Z}$-PC of the kind-R;*
2. *$g_\alpha \in \mathcal{G}_f$;*
3. *$\tilde{A}(f_0) \subseteq h_0$;*
4. *$f_0 \subseteq g_\alpha(h_0)$.*

Then a unique $\gamma \in f$ point occurs, so that $\mathfrak{d}(g_\alpha\gamma, \tilde{A}\gamma) = \mathfrak{d}(f, h)$ has been identified. In addition, a series $\{\gamma_{\hat{n}}\} \subseteq f$ exists for each $\gamma_0 \in f_0$ such as $\mathfrak{d}(g_\alpha\gamma_{\hat{n}+1}, \tilde{A}\gamma_{\hat{n}}) = \mathfrak{d}(f, h)$ for each $\hat{n} \in \mathbb{N} \cup \{0\}$ and $\gamma_{\hat{n}} \to \gamma$.

Proof. Presume $\gamma_0 \in f_0$. After this $\tilde{A}(f_0) \subseteq h_0$ and $f_0 \subseteq g_\alpha(f_0)$, occurs in such a way that $\gamma_1 \in f_0$ and $\mathfrak{d}(g_\alpha\gamma, \tilde{A}\gamma) = \mathfrak{d}(f, h)$. Clearly, for $\gamma_1 \in f_0$, there exists $\gamma_2 \in f_0$ such that $\mathfrak{d}(g_\alpha\gamma_2, \tilde{A}\gamma_1) = \mathfrak{d}(f, h)$. For $\gamma_{\hat{n}} \in f_0$, we can find $\gamma_{\hat{n}+1} \in f_0$ by repeating this step in such a way that $\mathfrak{d}(g_\alpha\gamma_{\hat{n}+1}, \tilde{A}\gamma_{\hat{n}}) = \mathfrak{d}(f, h)$. For some $\hat{m} > \hat{n}$, in the positive phase of $\{\gamma_{\hat{n}}\}$, if we have $\tilde{A}\gamma_{\hat{m}} = \tilde{A}\gamma_{\hat{n}}$, then we select $\gamma_{\hat{m}+1} = \gamma_{\hat{n}+1}$. Even, if $\hat{m} \in N$ exists, such as $\mathfrak{d}(g_\alpha\gamma_{\hat{m}+1},\ \mathfrak{d}(g_\alpha\gamma_{\hat{m}+1}), g_\alpha\gamma_{\hat{m}}) = 0$ then $\gamma_{\hat{m}+1} = \gamma_{\hat{m}}$, and hence $\tilde{A}\gamma_{\hat{m}+1} = \tilde{A}\gamma_{\hat{m}}$

and $\delta_{\hat{m}+2} = \gamma_{\hat{m}+1}$. It follows that $\gamma_{\hat{n}} = \gamma_{\hat{m}}\ \forall\ \hat{n} \in N$, $\hat{n} \geq \hat{m}$ and then sequence $\{\gamma_{\hat{n}}\}$ converges to $\gamma_{\hat{m}} \in f$. We also have $\mathfrak{d}(g_\alpha \gamma_{\hat{m}}, \tilde{A}\gamma_{\hat{m}}) = \mathfrak{d}f, h)$.

Then we're going to believe that $0 < \mathfrak{d}(\gamma_{\hat{n}+1}, \gamma_{\hat{n}}) \leq \mathfrak{d}(g_\alpha\gamma_{\hat{n}+1}, g_\alpha\gamma_{\hat{n}}) \neq 0\ \forall\ \hat{n} \in N$. Since $\tilde{A}$ be $\mathcal{Z}$-PC of the kind-R and $g_\alpha \in \mathcal{G}_f$, we say

$$\begin{aligned} 0\ &\leq \zeta(\mathfrak{d}(g_\alpha\gamma_{\hat{n}+1}, g_\alpha\gamma_{\hat{n}}), \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}-1}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}})) \\ &< \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}+1}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}}) - \mathfrak{d}(g_\alpha\gamma_{\hat{n}+1}, g_\alpha\gamma_{\hat{n}}) \\ &\leq \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}+1}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}}) - \mathfrak{d}(\gamma_{\hat{n}+1}, \gamma_{\hat{n}}) \end{aligned} \tag{3.1}$$

$\forall\, \hat{n} \in \mathbb{N}$. Hence, a series $\{\mathfrak{d}(\gamma_{\hat{n}}, \gamma_{\hat{n}-1})\}$ is decreasing and therefore $\exists\, \hat{r} \geq 0$ s.t. $\mathfrak{d}(\gamma_{\hat{n}}, \gamma_{\hat{n}-1}) \to \hat{r}$.

By (3.1), we obtain $\mathfrak{d}(g_\alpha\gamma_{\hat{n}+1}, g_\alpha\gamma_{\hat{n}}) \leq \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}-1}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}})$, $\forall\ \hat{n} \in \mathbb{N}$.

However, on the other side, $g_\alpha \in \mathcal{G}_f$ and hence

$$\mathfrak{d}(\gamma_{\hat{n}+1}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}+1}) \leq \mathfrak{d}(g_\alpha\gamma_{\hat{n}+1}, g_\alpha\gamma_{\hat{n}}) \leq \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}-1}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}}),$$

$\forall\ \hat{n} \in \mathbb{N}$.

As a result, $lim_{\hat{n}\to\infty}\mathfrak{d}(g_\alpha\gamma_{\hat{n}+1}, g_\alpha\gamma_{\hat{n}}) = \hat{r}$. Now, using the property of the SF, we're saying that $0 \leq lim_{\hat{n}\to\infty} sup\zeta(\mathfrak{d}(g_\alpha\gamma_{\hat{n}+1}, g_\alpha\gamma_n\hat{n}), \mathfrak{d}(\gamma_{\hat{n}}, \gamma_{\hat{n}-1})) < 0$. A inconsistency, and thus $\hat{r} = 0$.

The next move is to demonstrate $\{\gamma_{\hat{n}}\}$ is Cauchy. Suppose that $\{\gamma_{\hat{n}}\}$ is not a CS. Then, $\exists$ an $\epsilon > 0$, subsequences $\{\gamma_{\hat{n}(\hat{l})}\}$, $\{\gamma_{\hat{m}(\hat{l})}\}$ of $\{\gamma_{\hat{n}}\}$ s.t. $\hat{n}_{\hat{l}} > \hat{m}_{\hat{l}} > \hat{l}$.

$\mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(^{\hat{l}})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})}) \geq \epsilon\ \forall\ \hat{l} \in \mathbb{N}$ and

$lim_{\hat{l}\to\infty}\mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})}) = \epsilon = lim_{\hat{l}\to\infty}\mathfrak{d}(\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}\gamma_{\hat{m}(\hat{l})+1}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})+1}, \tilde{A}\gamma_{\hat{n}(\hat{l})+1})$

Then we will assume that $(\mathfrak{d}(\gamma_{\hat{n}(\hat{l})+1)}, \tilde{A}\gamma_{\hat{m}(\hat{l})+1)}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})+1)}, \tilde{A}\gamma_{\hat{n}(\hat{l})+1)})) > 0\ \forall\ \hat{l} \in \mathbb{N}$. Since $\tilde{A}$ is a $\mathcal{Z}$-PC of kind-R and

$\mathfrak{d}(g_\alpha\gamma_{\hat{n}(\hat{l})+1)}, \tilde{A}\gamma_{\hat{n}(\hat{l})}) = \mathfrak{d}(f, h) = \mathfrak{d}(g_\alpha\gamma_{\hat{m}(\hat{l})+1)}, \tilde{A}\gamma_{\hat{m}(\hat{l})})$, we get

$$\begin{aligned} 0 &\leq \zeta(\mathfrak{d}(g_\alpha\gamma_{\hat{n}(\hat{l})+1)}, g_\alpha\gamma_{\hat{m}(\hat{l})+1))}, \mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})})) \\ &< (\mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{k})}, \tilde{A}\gamma_{\hat{n}(\hat{l})})) - \mathfrak{d}(g_\alpha\gamma_{\hat{n}(\hat{l})+1)}, g_\alpha\gamma_{\hat{m}(\hat{l})+1)}) \end{aligned}$$

for all $\hat{l} \in \mathbb{N}$. The preceding inequality and $g_\alpha \in \mathcal{G}_f$ therefore ensure that

$$lim_{\hat{k}\to\infty}\mathfrak{d}(g_\alpha\gamma_{\hat{n}(\hat{l})+1)}, g_\alpha\gamma_{\hat{m}(\hat{l})+1)}) = \epsilon$$

Through the use of the SF property, with

$\bar{t}_{\hat{l}} = \mathfrak{d}(g_\alpha\gamma_{\hat{n}(\hat{l})+1)}, g_\alpha\gamma_{\hat{m}(\hat{l})+1})$ and
$\bar{s}_{\hat{l}} = (\mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})}))$,we get

$$0 \le lim_{\hat{l}\to\infty}sup\zeta(\mathfrak{d}(g_\alpha\gamma_{\hat{n}(\hat{l})+1)}, g_\alpha\gamma_{\hat{m}(\hat{l})+1)}), \mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{k})}, \tilde{A}\gamma_{\hat{n}(\hat{l})})) < 0$$

That's one contradiction. The result is that the $\{\gamma_{\hat{n}}\}$ series is CS. Behind $(\mathfrak{X}, d^*)$ is completed and f_0 is empty and therefore there is $\gamma \in f_0$ s.t. $\gamma_{\hat{n}} \to \gamma$. Moreover, by the continuity of g_α, we have $g_\alpha\gamma_{\hat{n}} \to g_\alpha\gamma$ and thus $g_\alpha\gamma \in f_0$, since $g_\alpha\gamma_{\hat{n}} \in f_0\ \forall\ \hat{n} \in \mathbb{N}$ and f_0 is closed. Since $\gamma \in f_0$ and $\tilde{T}(f_0) \subseteq h_0$, there is $\bar{z} \in f_0$ s.t. $\mathfrak{d}(\bar{z}, \tilde{T}\gamma) = \mathfrak{d}(f, h)$.

If $\bar{z} = g_\alpha\gamma_{\hat{n}}$ for $\hat{n} \in \mathbb{N}$, then $\bar{z} = g_\alpha\gamma$, therefore we can assume that $\bar{z} \neq g_\alpha\gamma_{\hat{n}}$, $\forall\ \hat{n}$. Also, $\exists$ a subsequence $\{\gamma_{\hat{n}(\hat{l})}\}$ of $\{\gamma_{\hat{n}}\}$ so that $\gamma_{\hat{n}(\hat{l})} \neq \gamma\ \forall\ \hat{l} \in \mathbb{N}$.

Once more, since $\tilde{A}$ be $\mathcal{Z}$-PC of the kind-R, we get

$$\zeta(\mathfrak{d}(\bar{z}, g_\alpha\gamma_{\hat{n}(\hat{l})+1)}), \mathfrak{d}(\gamma, \tilde{A}\gamma_{\hat{n}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma)) < \mathfrak{d}(\gamma, \tilde{A}\gamma_{\hat{n}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma) - \mathfrak{d}(\bar{z}, g_\alpha\gamma_{\hat{n}(\hat{l})+1)}),$$

accordingly

$$\mathfrak{d}(\bar{z}, g_\alpha\gamma_{\hat{n}(\hat{l})+1)}) < \mathfrak{d}(\gamma, \tilde{A}\gamma_{\hat{n}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma)$$

for all $\hat{l} \in \mathbb{N}$.

Making $\hat{l} \to \infty$, we obtain $\mathfrak{d}(\bar{z}, g_\alpha\gamma_{\hat{n}(\hat{l})+1)}) \to 0$ and then $\bar{z} = g_\alpha\gamma$. This implies that

$$\mathfrak{d}(g_\alpha\gamma, \tilde{A}\gamma) = \mathfrak{d}(f, h)$$

Let $\gamma^* \neq \gamma$ to demonstrate the singularity, be another point in f_0, s.t.

$$v(g_\alpha\gamma^*, \tilde{A}\gamma^*) = \mathfrak{d}(f, h)$$

. Since $g_\alpha \in \mathcal{G}_f$, $\tilde{A}$ is $\mathcal{Z}$-PC of the kind-R, one can say

$$\begin{aligned} 0 &\le \zeta(\mathfrak{d}(g_\alpha\gamma, g_\alpha\gamma^*), \mathfrak{d}(\gamma, \tilde{A}\gamma^*) + \mathfrak{d}(\gamma^*, \tilde{A}\gamma)) \\ &< (\mathfrak{d}(\gamma, \tilde{A}\gamma^*) + \mathfrak{d}(\gamma^*, \tilde{A}\gamma)) - \mathfrak{d}(g_\alpha\gamma, g_\alpha\gamma^*) \\ &\le (\mathfrak{d}(\gamma, \tilde{A}\gamma^*) + \mathfrak{d}(\gamma^*, \tilde{A}\gamma)) - \mathfrak{d}(\gamma, \gamma^*) \end{aligned}$$

which leads to a contradiction of $\gamma = \gamma^*$. $\square$

Here are some Corollaries as shown below:

Corollary 3.3. *Suppose that f, h are non-void subsets of a CMS $(\mathfrak{X}, \mathfrak{d})$. Suppose f is non-vacant and closed. Also, presume the mapping $\tilde{A} : f \to h$ meets the following requirements:*

1. *$\tilde{A}$ be $\mathcal{Z}$-PC of the family-R;*
2. *$\tilde{A}(f_0) \subseteq h_0$.*
3. *$\tilde{A}(f_0) \subseteq h_0$;*
4. *$f_0 \subseteq g_\alpha h_0)$.*

There is a particular point then, $\gamma \in f$ so $\mathfrak{d}(g_\alpha \gamma, \tilde{A}\gamma) = \mathfrak{d}(f, h)$. Moreover, for every $\gamma_0 \in f_0$ there is a series $\{\gamma_{\hat{n}}\} \subseteq f$ s.t. $\mathfrak{d}(g_\alpha \gamma_{\hat{n}+1}, \tilde{A}\gamma_{\hat{n}}) = \mathfrak{d}(f, h)$ for every $\hat{n} \in \mathbb{N} \cup \{0\}$ and $\gamma_{\hat{n}} \to \gamma$.

Proof. Notice that a PC of the type-R is a $\mathcal{Z}$-PC of the type-R in support of the SF $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ defined by $\zeta(\bar{t}, \bar{s}) = \hat{l}\bar{s} - \bar{t} \; \forall \; \bar{t}, \bar{s} \in [0, +\infty]$, where $\hat{l} \in [0, 1)$. $\square$

Example 3.1. *Let be as in illustration $\mathfrak{X}, f, h, \mathfrak{d}, \tilde{A}, \zeta$. Notice that $f_0 = f = h_0$ is closed and $\tilde{A}(f_0) \subseteq h_0$. So by (3.3), $\tilde{A} : f \to h$ has a unique point $\gamma \in f$ s.t. $\mathfrak{d}(\gamma, \tilde{A}\gamma) = 0 = \mathfrak{d}(f, h)$; here $\gamma = 0$.*

Theorem 3.4. *Let f, h are two non-void subsets of a CMS $(\mathfrak{X}, \mathfrak{d})$. Suppose that $\tilde{A}(f_0)$ is nonempty and closed. Also, presume that $\tilde{A} : f \to h$ and $g_\alpha : f \to f$ mappings comply with the following conditions:*

1. *$\tilde{A}$ is a $\mathcal{Z}$-PC of the family-M;*
2. *$\tilde{A}$ is injective on f_0;*
3. *$\tilde{A} \in \tilde{\mathcal{A}}_{g_\alpha}$;*
4. *$\tilde{A}(f_0) \subseteq h_0$;*
5. *$f_0 \subseteq g_\alpha(f_0)$.*

Then here a unique point $\gamma \in f$ s.t. $\mathfrak{d}(g_\alpha \gamma, \tilde{A}\gamma) = \mathfrak{d}(f, h)$. Moreover, for every $\gamma_0 \in f_0$.

Proof. By pursuing the close logic to that of the Theorem's proof (3.2), we can construct series $\{\gamma_{\hat{n}}\} \subseteq f_0$ s.t. $\mathfrak{d}(g_\alpha \gamma_{\hat{n}+1}, \tilde{A}\gamma_{^n}) = \mathfrak{d}(f,h)\ \forall\ \hat{n} \in \mathbb{N}$. In addition, in the positive process of $\{\gamma_{\hat{n}}\}$ if $\tilde{A}\gamma_{\hat{m}} = \tilde{A}\gamma_{\hat{n}}$ for some $\hat{m} > \hat{n}$.

We pick, then, $\gamma_{\hat{m}+1} = \gamma_{\hat{n}+1}$. This situation can ensure $\hat{m} \in \mathbb{N}$, we have $\gamma_{\hat{m}} = \gamma_{\hat{m}+1}$, then $\gamma_{\hat{n}} = \gamma_{\hat{m}}\ \forall\ \hat{n} \geq \hat{m}$. So, $\{\gamma_{\hat{n}}\}$ converges to $\gamma_{\hat{m}}$ and also $\mathfrak{d}(g_\alpha \gamma_{\hat{m}}, \tilde{A}\gamma_{\hat{m}}) = \mathfrak{d}(f,h)$.

Therefore, we can assume $\mathfrak{d}(\gamma_{\hat{n}+1}, \gamma_{\hat{n}}) \neq 0 \forall\ \hat{n} \in \mathbb{N} \cup \{0\}$. Because $\tilde{A}$ is a $\mathcal{Z}$-PC of the family-M, we have

$$\zeta(\mathfrak{d}(\tilde{A}g_\alpha \gamma_{\hat{n}+1}, \tilde{A}g_\alpha \gamma_{\hat{n}}), \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}-1})) \geq 0$$

$\forall\ \hat{n} \in \mathbb{N}$.

With $\tilde{\mathcal{A}} \in \tilde{\mathcal{A}}_{g_\alpha}$ and $\tilde{A}$ being injective on f_0, we infer $\mathfrak{d}(\tilde{A}\gamma_{\hat{n}+1}, \tilde{A}g_\alpha \gamma_{\hat{n}}) > 0$ and $\mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}-1}) > 0$ for all $n \in \mathbb{N}$.
Using a SFs ζ_2 (1.3), we get

$$\begin{aligned} 0 \quad &\leq \zeta(\mathfrak{d}(\tilde{A}g_\alpha \gamma_{\hat{n}+1}, \tilde{A}g_\alpha \gamma_{\hat{n}}), \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}-1})) \\ &< \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + v(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}-1}) - \mathfrak{d}(\tilde{A}g_\alpha \gamma_{\hat{n}+1}, \tilde{A}g_\alpha \gamma_{\hat{n}}) \\ &\leq \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}_{\hat{n}-1}) - \mathfrak{d}(\tilde{A}\gamma_{\hat{n}+1}, \tilde{A}\gamma_{\hat{n}}), \forall \hat{n} \in \mathbb{N} \end{aligned} \tag{3.2}$$

Thus $\mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}-1})$ is diminishing and $\hat{r} \geq 0$ persists, such that $\mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}-1}) \to \hat{r}$. If $\hat{r} > 0$, then we get $\mathfrak{d}(\tilde{A}g_\alpha \gamma_{\hat{n}+1}, \tilde{A}g_\alpha \gamma_{\hat{n}}) < \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}-1}) \forall n \in \mathbb{N}$ by (**??**).

In spite of, $\tilde{\mathcal{A}} \in \tilde{\mathcal{A}}_{g_\alpha}$, so

$$\mathfrak{d}(\gamma_{\hat{n}+1}, \tilde{A}\gamma_{\hat{n}+1}) + \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) \leq \mathfrak{d}(\tilde{A}g_\alpha \gamma_{\hat{n}+1}, \tilde{A}g_\alpha \gamma_{\hat{n}}) < \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}-1}),$$

$\forall\ \hat{n} \in \mathbb{N}$.

Thus,

$$lim_{\hat{n}\to\infty} \mathfrak{d}(\tilde{A}g_\alpha \gamma_{\hat{n}+1}, \tilde{A}g_\alpha \gamma_{\hat{n}}) = \hat{r}$$

Now,using the property (1.3) of a SF, we write

$$0 \leq lim_{\hat{n}\to\infty} sup \zeta(\mathfrak{d}(\tilde{A}g_\alpha \gamma_{\hat{n}+1}, \tilde{A}g_\alpha \gamma_{\hat{n}}), \mathfrak{d}(\gamma_{\hat{n}}, \tilde{A}\gamma_{\hat{n}}) + \mathfrak{d}(\gamma_{\hat{n}-1}, \tilde{A}\gamma_{\hat{n}-1})) < 0,$$

which is a contradiction and hence $\hat{r} = 0$.

The next move is to prove that $\{\tilde{A}\gamma_{\hat{n}}\}$ is a CS. Assume that $\{\tilde{A}\gamma_{\hat{n}}\}$ is not a CS by Contradiction. Then, there exists an $\epsilon > 0$ and the subsequences $\{\tilde{A}\gamma_{\hat{n}(\hat{l})}\}$ of $\{\tilde{A}\gamma_{\hat{n}}\}$ such that $\hat{n}_(\hat{l}) > \hat{m}_(\hat{l}) \geq \hat{l}$ and

$\mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})}) \geq \epsilon$ for all $\hat{l} \in \mathbb{N}$ and

$$lim_{\hat{l}\to\infty}\mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})})+\mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})}) = \epsilon = lim_{\hat{l}\to\infty}\mathfrak{d}(\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}\gamma_{\hat{n}(\hat{l})+1})+\mathfrak{d}(\gamma_{\hat{m}(\hat{l})+1}, \tilde{A}\gamma_{\hat{m}(\hat{l})+1})$$

Then, we can assume that
$\mathfrak{d}(\gamma_{\hat{n}(\hat{l})+1}), \tilde{A}\gamma_{\hat{n}(\hat{l})+1}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})+1}, \tilde{A}\gamma_{\hat{m}(\hat{l})+1}) > 0 \ \forall \ \hat{l} \in \mathbb{N}$. Since $\tilde{A}$ is a $\mathcal{Z}$-PC of the family-M and $\mathfrak{d}(g_\alpha\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}\gamma_{\hat{n}(\hat{l})}) = \mathfrak{d}(f, h) = \mathfrak{d}(g_\alpha\gamma_{\hat{m}(\hat{l})+1}, \tilde{A}\gamma_{\hat{m}(\hat{l})})$, we get

$$\begin{aligned}
&0 \leq \zeta(\mathfrak{d}(\tilde{A}g_\alpha\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}g_\alpha\gamma_{\hat{m}(\hat{l})+1}), \mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})}))\\
&< (\mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})})) - \mathfrak{d}(\tilde{A}g_\alpha\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}g_\alpha\gamma_{\hat{m}(\hat{l})+1})
\end{aligned}$$

for all $\hat{l} \in \mathbb{N}$.

$$lim_{\hat{l}\to\infty}(\mathfrak{d}(\tilde{A}g_\alpha\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}g_\alpha\gamma_{\hat{m}(\hat{l})+1}) = \epsilon$$

By using the property (ζ_3) (1.3) of a SF, with $\bar{t}_{\hat{l}} = \mathfrak{d}(\tilde{A}g_\alpha\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}g_\alpha\gamma_{\hat{m}(\hat{l})+1})$ and $\bar{s}_{\hat{l}} = \mathfrak{d}(\gamma_{\hat{n}(\hat{k})}, \tilde{A}\gamma_{\hat{n}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})})$, we obtain

$$0 \leq lim_{\hat{l}\to\infty}sup\zeta(\mathfrak{d}(\tilde{A}g_\alpha\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}g_\alpha\gamma_{\hat{m}(\hat{l})+1}), \mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{m}(\hat{l})})) < 0$$

And that is a contradiction. The series $\{\tilde{A}\gamma_{\hat{n}}\}$ is Cauchy, we conclude.

Being $(\mathfrak{X}, \mathfrak{d})$ be complete and $\tilde{A}(f_0)$ be closed, therefore $\tilde{A}\gamma_{\hat{n}} \to \tilde{A}\bar{u} \in f_0$. To add up, there is $\bar{z} \in f_0$ such that $d^*(\bar{z}, \tilde{A}\bar{u}) = d^*(f, h)$. Since $f_0 \subseteq g_\alpha(f_0)$, i.e. $\bar{z} = g_\alpha\gamma$ for some $\gamma \in f_0$, and $\mathfrak{d}(g_\alpha\gamma, \tilde{A}\bar{u}) = \mathfrak{d}(f, h)$. Obviously, if $\gamma = \gamma_{\hat{n}}$ for infinite $\hat{n} \in \mathbb{N}$, then $\tilde{A}\gamma = \tilde{A}\bar{u}$. Consequently, we presume that $\gamma \neq \gamma_{\hat{n}}$ $\forall \hat{n} \in \mathbb{N}$. Also, $\exists$ a subsequence $\{\gamma_{\hat{n}(\hat{l})}\}$ of $\{\gamma_{\hat{n}}\}$ s.t. $\tilde{A}\gamma_{\hat{n}(\hat{l})} \neq \tilde{A}\bar{u}$ $\forall$ $\hat{l} \in \mathbb{N}$. Again, since $\tilde{A}$ is a $\mathcal{Z}$-PC of family-M, we get

$$\begin{aligned}
&0 \leq \zeta(\mathfrak{d}(\tilde{A}g_\alpha\gamma, \tilde{A}g_\alpha\gamma_{\hat{n}(\hat{l})+1}), \mathfrak{d}(\bar{u}, \tilde{A}\bar{u}) + \mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{a}\gamma_{\hat{n}(\hat{l})}))\\
&< (\mathfrak{d}(\bar{u}, \tilde{A}\bar{u}) + \mathfrak{d}(\gamma_{\hat{m}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})})) - \mathfrak{d}(\tilde{A}g_\alpha\gamma, \tilde{A}g_\alpha\gamma_{\hat{n}(\hat{l})+1})
\end{aligned}$$

and hence

$$\mathfrak{d}(\gamma, \tilde{a}\gamma) + \mathfrak{d}(\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}\gamma_{\hat{n}(\hat{l})} + 1) \leq \mathfrak{d}(\tilde{A}g_\alpha\gamma, \tilde{a}g_\alpha\gamma_{\hat{n}(\hat{l})+1}) < \mathfrak{d}(\bar{u}, \tilde{A}\bar{u}) + \mathfrak{d}(\gamma_{\hat{n}(\hat{l})}, \tilde{A}\gamma_{\hat{n}(\hat{l})})$$

$\forall\ \hat{l} \in \mathbb{N}$, since $\tilde{\mathcal{A}} \in \tilde{\mathcal{A}}_{g_\alpha}$. Making $\hat{l} \to \infty$, we obtain $\mathfrak{d}(\gamma, \tilde{A}\gamma) + \mathfrak{d}(\gamma_{\hat{n}(\hat{l})+1}, \tilde{A}\gamma_{\hat{n}(\hat{l})+1}) \to 0$ and hence $\tilde{a}\gamma = \tilde{A}\bar{u}$, this implies that

$$\mathfrak{d}(g_\alpha\gamma, \tilde{A}\gamma) = \mathfrak{d}(f, h).$$

Let $\gamma^* \neq \gamma$ be another point in the f_0 s.t. to prove its uniqueness.

$$\mathfrak{d}(g_\alpha\gamma^*, \tilde{A}\gamma^*) = \mathfrak{d}(f, h).$$

Because $\tilde{\mathcal{A}} \in \tilde{\mathcal{A}}_{g_\alpha}$ is injective on f_0 and $\tilde{A}$ is a $\mathcal{Z}$-PC of the family-M, we may claim that

$$\begin{aligned}0 &\leq \zeta(\mathfrak{d}(\tilde{A}g_\alpha\gamma, \tilde{A}g_\alpha\gamma^*), \mathfrak{d}(\gamma, \tilde{A}\gamma) + \mathfrak{d}(\gamma^*, \tilde{A}\gamma^*)) \\ &< (\mathfrak{d}(\gamma, \tilde{A}\gamma) + \mathfrak{d}(\gamma^*, \tilde{A}\gamma^*)) - \mathfrak{d}(\tilde{A}g_\alpha\gamma, \tilde{A}g_\alpha\gamma^*) \\ &\leq (\mathfrak{d}(\gamma, \tilde{A}\gamma) + \mathfrak{d}(\gamma^*, \tilde{A}\gamma^*)) - \mathfrak{d}(\tilde{A}, \tilde{A}\gamma^*) = 0\end{aligned}$$

□

Corollary 3.5. *If f, h and $\tilde{A}(f_0)$ are nonempty subsets and closed set in CMS $(\mathfrak{X}, \mathfrak{d})$ respectively and the map $\tilde{A} : f \to h$ fulfill these conditions:*

1. *$\tilde{A}$ is a $\mathcal{Z}$-PC of the family-M*
2. *$\tilde{A}$ is injective on f_0*
3. *$\tilde{A}(f_0) \subseteq h_0$*

then there is a unique stage, s.t. it is $\gamma \in f$ $\mathfrak{d}(\gamma, \tilde{A}\gamma) = \mathfrak{d}(f, h)$. In addition, for each $\gamma_0 \in f_0$ there is $\{x_n\} \subseteq \tilde{A}$ sequence, so $\mathfrak{d}(\gamma_{\hat{n}+1}, \tilde{A}\gamma_{\hat{n}}) = \mathfrak{d}(f, h)$ for every $\hat{n} \in \mathbb{N} \cup \{0\}$ and $\gamma_{\hat{n}} \to \gamma$.

Example 3.2. *The set $\mathbb{R}$ with the usual metric $\mathfrak{d}(\gamma, \delta) = |\gamma - \delta|\ \forall\ \gamma, \delta \in \mathbb{R}$. Examine $f = [-3, -1], h = [0, 1]$ so that $\mathfrak{d}(f, h) = 1$ and determine $\tilde{A} : f \to h$ next to*

$$\tilde{A}x = \begin{cases} 3 + \gamma\ if\ \gamma \in [-3, -2], \\ -1 - \gamma\ if\ \gamma \in (-2, -1], \end{cases}$$

we have

$$f_0 = \{\gamma \in f : \mathfrak{d}(\gamma, \delta) = \mathfrak{d}(f, h) = 1,\ for\ some\ \delta \in h\} = \{-1\}$$

$$h_0 = \{\delta \in h : \mathfrak{d}(\gamma, \delta) = \mathfrak{d}(f, h) = 1,\ for\ some\ \gamma \in f\} = \{0\}$$

and hence $\tilde{A}(f_0) = 0 = h_0$. *As we know* $\tilde{A}$ *is a* $\mathcal{Z}$*-PC of the kind-*M. *Actually* $\mathfrak{d}(\bar{u}, \gamma) = \mathfrak{d}(\bar{v}, \tilde{A}\delta) = 1 = \mathfrak{d}(f, h)$, *we find* $(\bar{u}, \bar{v}) = (-1, -1)$ *for* $\gamma, \delta \in [-3, -1]$ *and therefore* $\zeta(\mathfrak{d}(\tilde{A}\bar{u}, \tilde{A}\bar{v}), \mathfrak{d}(\tilde{A}\gamma, \tilde{A}\delta)) = \zeta(\mathfrak{d}(0,0), \mathfrak{d}(0,0)) = \zeta(0,0) = 1$.

Hence, all conditions of Corollary (3.5) remains valid and $\gamma = -1$ *is the unique point s.t.* $\mathfrak{d}(-1, \tilde{A}(-1)) = \mathfrak{d}(f, h)$.

3.3 New Notions of Modified $\alpha_0 - (\psi_0, g_0)$-Proximal Contractions of Type-I and Type-II

We introduce new notions of modified $\alpha_0 - (\psi_0, g_0)$- PC of Class-I and Class-II in MS.

The α-admissible mapping description has been set by Samet et al. [53]. They proved FPT by using this definition.

Theorem 3.6. *[53] "Let* (X, d) *be a CMS and* $T : X \to X$ *be an* α*-admissible mapping. Assume that the following conditions hold:*

1. *for all* $x, y \in X$ *we have*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \tag{3.3}$$

where $\psi : [0, +\infty) \to [0, +\infty)$ *is a nondecreasing function such that* $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ *for each* $t > 0$,

2. *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$,
3. *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point."*

We introduce new notions of Class-I and Class-II shown below:

Definition 3.7. If there is a non-negative integer $\alpha_0 < 1$, then the mapping $F : M \to N$ is the PC, so for all m_1, m_2, p_1, p_2 in M,

$$d_{\mathfrak{X}}(m_1, Fp_1) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(m_2, Fp_2) \quad \Rightarrow d_{\mathfrak{X}}(m_1, m_2) \leq \alpha_0 d_{\mathfrak{X}}(p_1, p_2).$$

Definition 3.8. Let $F : M \to N$, $g_0 : M \to M$ be two maps. Let $\psi_0 : [0, \infty) \to [0, \infty)$ satisfy

$\psi_0(0) = 0, \quad \psi_0(\bar{t}) < \mathfrak{t}, \quad \text{and} \quad lim_{\mathfrak{s}\to\mathfrak{t}^+} sup\, \psi_0(\mathfrak{s}) < \mathfrak{t} \quad \text{for each} \quad \mathfrak{t} > 0.$

Then, F is said to be a (ψ_0, g_0) - PC if

$$d_{\mathfrak{X}}(m_1, Fp_1) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(m_2, Fp_2) \quad \Rightarrow d_{\mathfrak{X}}(m_1, m_2) \leq \psi_a d_{\mathfrak{X}}(gp_1, gp_2))$$

for all m_1, m_2, p_1, p_2 in M.

Definition 3.9. Presume M, N are two non-void subsets of a MS $(\mathfrak{X}, d_{\mathfrak{X}})$ and a function be α_0. We can say $F : M \to N$ is triangular α_0 - proximal admissible if, $\forall p, q, r, p_1, p_2, m_1, m_2 \in M$,

1. $$\begin{cases} \alpha_0(p_1, p_2) \geq 1, \\ d_{\mathfrak{X}}(m_1, Fp_1) = d_{\mathfrak{X}}(M, N), \implies \alpha_0(m_1, m_2) \geq 1, \\ d_{\mathfrak{X}}(m_2, Fp_2) = d_{\mathfrak{X}}(M, N) \end{cases}$$

2. $$\begin{cases} \alpha_0(p, r) \geq 1, \\ d_{\mathfrak{X}}(r, q) \geq 1 \end{cases} \implies \alpha_0(p, q) \geq 1.$$

Now, we introduce the new class of PCs.

Definition 3.10. Presume M, N be two nonempty elements of a MS $(\mathfrak{X}, d_{\mathfrak{X}})$ and $\alpha_0 : M \times M \to [0, +\infty)$ be a function. We're suggesting that $F : M \to N$ is

1. a improved $\alpha_0 - (\psi_0, g_0)$ - PC if, for all $m, n, p, q \in M$,

$$\begin{cases} \alpha_0(g_0p, g_0r) \geq 1, \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fr) = d_{\mathfrak{X}}(M, N) \end{cases}$$

$$\implies d_{\mathfrak{X}}(m, n) \leq \frac{1}{2} d_{\mathfrak{X}}(g_0p, n) + d_{\mathfrak{X}}(g_0q, m)) - \psi_0 d_{\mathfrak{X}}(g_0p, n), d_{\mathfrak{X}}(g_0q, m)), \quad (3.4)$$

2. an $\alpha_0 - (\psi_0, g_0)$ - PC of class-I if, for all $m, n, p, q \in M$,

$$\begin{cases} d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N) \end{cases}$$

$$\implies \alpha_0(p, q) d_{\mathfrak{X}}(m, n) \leq \frac{1}{2} d_{\mathfrak{X}}(g_0p, n) + d_{\mathfrak{X}}(g_0q, m)) - \psi_0(d_{\mathfrak{X}}(g_0p, n), d_{\mathfrak{X}}(g_0q, m)),$$

where $0 \leq \alpha_0(g_0p, g_0q) \leq 1$ for all $g_0p, g_0q \in M$

3. an $\alpha_0 - (\psi_0, g_0)$ - PC of class-II if, $\forall\, m, n, p, q \in M$,

$$\begin{cases} d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N) \end{cases}$$

$$\implies (\alpha_0(g_0p, g_0q) + l)^{d_{\mathfrak{X}}(m,n)} \leq (l+1)^{\frac{1}{2}d_{\mathfrak{X}}(g_0p,n)+d_{\mathfrak{X}}(g_0qm))-\psi_0(d_{\mathfrak{X}}(g_0p,n),d_{\mathfrak{X}}(g_0q,m))},$$

3.4 Certain Fixed Point Theorems for Type-I and Type-II in Metric Space

Theorem 3.11. *Let us suppose M, N be two non-void members of a MS $(\mathfrak{X}, d_{\mathfrak{X}})$ so M is complete and M_0 is nonempty. Let $F : M \to N$ is a continuous modified $\alpha_0 - (\psi_0, g_0)$ - PC $g_0 : M \to M$ satisfy the following conditions:*

1. *F is a triangular $\alpha_0 - (\psi_0, g_0)$-proximal admissible mapping and $F(M_0) \subseteq N_0$,*
2. *$\exists\ p_0, p_1 \in M_0$ s.t. $d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$ and $\alpha_0(g_0p_0, g_0p_1) \geq 1$.*

Then there is a BPP for F. Moreover, the best BPP is special, if, for each $p, q \in M$ s.t. $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0q, Fq)$, we have $\alpha_0(g_0p, g_0q) \geq 1$.

Proof. By (2), there exists $p_0, p_1 \in M_0$ such that

$$d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$$

and $\alpha_0(g_0p_0, g_0p_1) \geq 1$.

On the other hand, since $F(M_0) \subseteq N_0$, then there exists $p_2 \in M_0$ such that

$$d_{\mathfrak{X}}(g_0p_2, Fp_1) = d_{\mathfrak{X}}(M, N).$$

Since F is the allowable near end of the triangle α_0, we have $\alpha_0(g_0p_1, g_0p_2) \geq 1$. Thus

$$d_{\mathfrak{X}}(g_0p_2, Fp_1) = d_{\mathfrak{X}}(M, N).$$

and $\alpha_0(g_0p_1, g_0p_2) \geq 1$.

Since $F(M_0) \subseteq N_0$, then $\exists\ p_3 \in M_0$ s.t.

$$d_{\mathfrak{X}}(g_0p_3, Fp_2) = d_{\mathfrak{X}}(M, N).$$

Next, F is a triangular $\alpha_0 - (\psi_0, g_0)$-proximal admissible, it becomes $\alpha_0(g_0p_2, g_0p_3) \geq 1$ and hence

$$d_{\mathfrak{X}}(g_0p_3, Fp_2) = d_{\mathfrak{X}}(M, N)$$

and $\alpha_0(g_0p_2, g_0p_3) \geq 1$.

In this step, we create a $\{p_a\}$ sequence in such a way that

$$\begin{cases} \alpha_0(g_0p_{a-1}, g_0p_a) \geq 1 \\ d_{\mathfrak{X}}(g_0p, Fp_{a-1} = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(g_0p_{a+1}, Fp_a) = d_{\mathfrak{X}}(M, N), \end{cases} \tag{3.5}$$

for all $a \in \mathbb{N}$. Now, from (3.4) with $m = g_0p_a$, $n = g_0p_{a-1}$ and $g_0p = gp_a$, we get

$$\begin{aligned} d_{\mathfrak{X}}(g_0p_a, g_0p_{a+1}) &\leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}) + d_{\mathfrak{X}}(g_0p_a, g_0p_a)) \\ -\psi_0(d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}), d_{\mathfrak{X}}(g_0p_a, g_0p_a)) & \\ &= \frac{1}{2}d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}) - \psi_0(d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}, 0)) \\ &\leq \frac{1}{2}d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}) \\ &\leq (d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_a) + d_{\mathfrak{X}}(g_0p_a, g_0p_{a+1})), \qquad (3.6) \end{aligned}$$

which implies that $d_{\mathfrak{X}}(g_0p_a, g_0p_{a+1}) \leq d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_a)$. It follows that the sequence $\{\delta_a\}$, where $\delta_a = \delta(g_0p_a, g_0p_{a+1})$ is decreasing and so $\exists\ \delta \geq 0$ s.t. $\delta_a \to \delta$ while $a \to \infty$. Later, we will take limit $a \to +\infty$ in (3.6), it become

$$\delta \leq \frac{1}{2}\delta(g_0p_{a-1}, g_0p_{a+1}) \leq \frac{1}{2}(\delta + \delta) = 2\delta,$$

that is,

$$lim_{a\to+\infty} d_{\mathfrak{X}}(g_0p_{a-1}, g_0p_{a+1}) = 2\delta. \tag{3.7}$$

Again, taking the limit as $a \to +\infty$ in (3.6) and (3.7) and the continuity of ψ_0, we get

$$\delta \leq \delta - \psi_0(2\delta, 0).$$

and so $\psi_0(2\delta, 0) = 0$. Therefore, by the property of ψ_0, we get $\delta = 0$, that is,

$$lim_{a\to+\infty} d_{\mathfrak{X}}(g_0p_{a+1}, g_0p_a) = 0. \tag{3.8}$$

Next, we show g_0p_a is a CS. Then there is an $\epsilon > 0$ and two subsequences $\{u(\bar{l})\}$ and $\{v(\bar{l})\}$ s.t. $\forall$ positive integer $\bar{l}$,

$$v(\bar{l}) > u(\bar{l}) > \bar{l}, \quad d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})}) \geq \epsilon, \quad d_{\mathfrak{X}}(g_0p_{v(\bar{l})-1}, g_0p_{v(\bar{l})}) < \epsilon.$$

The smallest number reaches $u(\bar{l})$ go for $v(\bar{l})$.

This means that we get $\bar{l} \in \mathbb{N}$ for all of them.

$$\begin{aligned}\varepsilon &\leq d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})}) \leq d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{v(\bar{l})-1}) + d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})+1}) \\ &\leq d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{v(\bar{l})-1}) + \epsilon.\end{aligned}$$

Making limit as $\bar{l} \to +\infty$, we obtain and using (3.8), we get

$$lim_{\bar{l} \to +\infty} d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})}) = \epsilon. \tag{3.9}$$

Again, from

$$d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})}) \leq d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{u(\bar{l})+1}) + d_{\mathfrak{X}}(g_0p_{u(\bar{l})+1}, g_0p_{v(\bar{l})+1}) + d_{\mathfrak{X}}(g_0p_{v(\bar{l})+1}, g_0p_{v(\bar{l})})$$

and

$$d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_{u(\bar{l})+1}) \leq d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{u(\bar{l})+1}) + d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{v(\bar{l})}) + d_{\mathfrak{X}}(g_0p_{v(\bar{l})+1}, g_0p_{v(\bar{l})}),$$

Proceeding limit as $\bar{l} \to +\infty$, by (3.8) and (3.9), we deduce

$$lim_{\bar{l} \to +\infty} d_{\mathfrak{X}}(g_0p_{v(\bar{l})+1}, g_0p_{u(\bar{l})+1}) = \epsilon. \tag{3.10}$$

Similarly,

$$lim_{\bar{l} \to +\infty} d_{\mathfrak{X}}(g_0p_{v(\bar{l})}, g_0p_u(\bar{l}) + 1) = \epsilon \tag{3.11}$$

and

$$lim_{\bar{l} \to +\infty} d_{\mathfrak{X}}(g_0p_{u(\bar{l})}, g_0p_{v(\bar{l})+1}) = \epsilon. \tag{3.12}$$

We're going to explain that

$$\alpha_0(g_0p_{u(\bar{l})}, g_0p_{v(\bar{l})}) \geq 1, \ where \ v(\bar{l}) > u(\bar{l}) > \bar{l}. \tag{3.13}$$

F is a triangular $\alpha_0 - (\psi_0, g_0)$ - proximal admissible mapping and

$$\begin{cases} \alpha_0(g_0p_{u(\bar{l})}, g_0p_{u(\bar{l})+1}) \geq 1, \\ \alpha_0(g_0p_{u(\bar{l})+1}, g_0p_{u(\bar{l})+2}) \geq 1. \end{cases}$$

With condition (2) of Definition (3.9), we have

$$\alpha_0(g_0p_{u(\bar{l})+1}, g_0p_{u(\bar{l})+2}) \geq 1.$$

Again, F is $\alpha_o - (\psi_0, g_0)$ - triangular proximal map,

$$\begin{cases} \alpha_0(g_0 p_{u(\bar{l})}, g_0 p_{u(\bar{l})+2}) \geq 1, \\ \alpha_0(g_0 p_{u(\bar{l})+2}, g_0 p_{u(\bar{l})+3}) \geq 1. \end{cases}$$

With condition (2) of Definition (3.9), we have

$$\alpha_0(g_0 p_{u(\bar{l})}, g_0 p_{u(\bar{l})+3}) \geq 1.$$

Therefore, we get (3.13) through this process.

On the second side, we do know that

$$\begin{cases} \alpha_0(g_0 p_{u(\bar{l})+1}, F p_{v(\bar{l})}) = d_{\mathfrak{X}}(M, N), \\ \alpha_0(g_0 p_{v(\bar{l})+1}, F p_{u(\bar{l})}) = d_{\mathfrak{X}}(M, N). \end{cases}$$

Therefore, we have

$$\begin{aligned} d_{\mathfrak{X}}(g_0 p_{u(\bar{l})+1}, g_0 p_{v(\bar{l})+1}) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0 p_{u(\bar{l})}, g_0 p_{v(\bar{l})+1}) + d_{\mathfrak{X}}(g_0 p v(\bar{l}), g_0 p_{u(\bar{l})+1})) \\ -\psi_0(d_{\mathfrak{X}}(g_0 p_{u(\bar{l})}, g_0 p_{v(\bar{l})+1}), d_{\mathfrak{X}}(g_0 p_{v(\bar{l})}, g_0 p_{u(\bar{l})+1})). \end{aligned}$$

Picking limit as $\bar{l} \to +\infty$ and using (3.10), (3.11), (3.12), the continuity of ψ_0, one become

$$\begin{aligned} &\epsilon \leq \frac{1}{2}(\epsilon + \epsilon) - \psi_0(\epsilon, \epsilon) \\ &\epsilon \leq \epsilon - \psi_0(\epsilon, \epsilon) \end{aligned}$$

and hence $\psi_0(\epsilon, \epsilon) = 0$, which leads to the contradiction $\epsilon = 0$. Thus, $\{p_a\}$ is a CS. Ahead M has been completed, there is $z \in M$ so $p_a \to r$. Hereinafter, $d_{\mathfrak{X}}(g_0 p_{a+1}, F p_a) = d_{\mathfrak{X}}(M, N)$ for all $a \in \mathbb{N} \cup \{0\}$.

Selecting limit as $a \to +\infty$, we gather $d_{\mathfrak{X}}(r, Fr) = d_{\mathfrak{X}}(M, N)$, owing to the f consistency.

Lastly, we demonstrate the uniqueness of point $p \in f$ s.t. $d_{\mathfrak{X}}(g_0 p, Fp) = d_{\mathfrak{X}}(M, N)$. Suppose, in fact, that there is $p, q \in M$ which are BPPs, viz. $d_{\mathfrak{X}}(g_0 p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0 q, Fq)$.

Since $\alpha_0(g_0 p, g_0 q) \geq 1$, we have

$$\begin{aligned} d_{\mathfrak{X}}(g_0 p, g_0 q) &\leq \frac{1}{2}(d_{\mathfrak{X}}(g_0 p, g_0 q) + d_{\mathfrak{X}}(g_0 q, g_0 p) - \psi_0(d_{\mathfrak{X}}(g_0 p, g_0 q), d_{\mathfrak{X}}(g_0 q, g_0 p)) \\ &= d_{\mathfrak{X}}(g_0 p, g_0 q) - \psi_0(d_{\mathfrak{X}}(g_0 p, g_0 q), d_{\mathfrak{X}}(g_0 q, g_0 p)), \end{aligned}$$

which implies $d_{\mathfrak{X}}(g_0p, g_0q) = 0$, that is $g_0p = g_0q$. $\square$

Corollary 3.12. *Let M,N be non-empty subsets of a MS $(\mathfrak{X}, d_{\mathfrak{X}})$ to this extent M is complete and M_0 is non - empty. Presume $F : M \to N$ and $g_0 : M \to M$ are continuous $\alpha_0 - (\psi_0, g_0)$ - PC of Type-I or a continuous $\alpha_0 - (\psi_0, g_0)$ - PC mapping of the Class-II s.t. the following requirements satisfied:*

1. *F is a triangular $\alpha_0 - (\psi, g_0)$-proximal admissible mapping and $F(M_0) \subseteq N_0$.*
2. *there exists $p_0, p_1 \in M_0$ such that*

$$d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$$

   and $\alpha_0(g_0p_0, g_0p_1) \geq 1$.

Then the F will have a BPP. Furthermore, if, for every $p, q \in M$, $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0q, Fq)$, we have $\alpha_0(g_0p, g_0q) \geq 1$, the BPP is special.

Definition 3.13. Let M, N are two non-void subsets of a MS $(\mathfrak{X}, d_{\mathfrak{X}})$. Here (M, N) hold V-property if, $\forall$ sequence $\{q_n\}$ of N s.t. $d_{\mathfrak{X}}(p, q_n) \to d_{\mathfrak{X}}(p, N)$, $\forall p \in M$, $q \in N$ is given s.t. $d_{\mathfrak{X}}(p, q) = d_{\mathfrak{X}}(p, N)$.

Theorem 3.14. *Suppose M, N be two non - void elements of a MS $(\mathfrak{X}, d_{\mathfrak{X}})$ s.t. M is complete, the pair (M, N) hold V - property and M_0 is complete. Presume $F : M \to N$ and $g_0 : M \to M$ are modified $\alpha_0 - (\psi_0, g_0)$ - PC in such a way that the following criteria hold:*

1. *F is a triangular map of $\alpha_0 - (\psi_0, g_0)$ and $F(M_0) \subseteq N_0$.*
2. *$p_0, p_1 \in M_0$ occurs to such a degree that*

$$d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$$

   and $\alpha_0(g_0p_0, g_0p_1) \geq 1$,

3. *if $\{g_0p_n\}$ is a sequence in M such that $\alpha_0(g_0p_n, g_0p_{n+1}) \geq 1$ and $g_0p_n \to g_0p$ as $n \to \infty$, then $\alpha_0(g_0p_n, g_0p) \geq 1$ $\forall n \in \mathbb{N} \cup \{0\}$.*

Then, there is a BPP for F. Furthermore, the BPP is special if we have $\alpha_0(g_0p, g_0q) \geq 1$ for every $p, q \in M$, so that $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0q, Fq)$.

Proof. After the Theorem(3.11) is proved, there are CSs $\{g_0p_a\} \subseteq M$ and $r \in M$ such that (3.5) keep $g_0p_a \to z$ as $a \to +\infty$. On next side, $\forall a \in \mathbb{N}$, write down

$$\begin{aligned} d_{\mathfrak{X}}(r, N) &\leq d_{\mathfrak{X}}(r, Fp_a) \\ &\leq d_{\mathfrak{X}}(r, g_0p_{a+1}) + d_{\mathfrak{X}}(g_0p_{a+1}, Fp_a) \\ &= d_{\mathfrak{X}}(r, g_0p_{a+1}) + d_{\mathfrak{X}}(M, N). \end{aligned}$$

Selecting this limit $p \to +\infty$, we take

$$lim_{a\to+\infty} d_{\mathfrak{X}}(r, Fp_a) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(M, N). \tag{3.14}$$

Since (M, N) has the V-attribute, there is $c \in N$, so $d_{\mathfrak{X}}(r, c) = d_{\mathfrak{X}}(M, N)$ therefore $r\ inM_0$. Moreover, since $F(M_0) \subseteq N_0$, then there is $n \in M$

$$d_{\mathfrak{X}}(n, Fr) = d_{\mathfrak{X}}(M, N).$$

Now, by (3) and (3.5), we have $\alpha_0(g_0p_a, r) \geq 1$ and $d_{\mathfrak{X}}(g_0p_{a+1}, Fp_a) = d_{\mathfrak{X}}(M, N)$ for all $a \in \mathbb{N} \cup \{0\}$. Also, since F is a modified $\alpha_0 - (\psi_0, g_0)$ - PC, we get

$$d_{\mathfrak{X}}(g_0p_{a+1}, v) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p_a, n) + d_{\mathfrak{X}}(r, g_0p_{a+1})) - \psi_0(d_{\mathfrak{X}}(g_0p_a, n), d_{\mathfrak{X}}(r, g_0p_{a+1})$$

. Taking this as $a \to +\infty$in the equation, we have

$$d_{\mathfrak{X}}(r, n) \leq \frac{1}{2} d_{\mathfrak{X}}(r, n) - \psi_0(d_{\mathfrak{X}}(r, n), 0)$$

. This means that $d_{\mathfrak{X}}(r, n) = 0$, that is, $n = r$. Therefore, r is the BPP for F. The uniqueness of the best neighbor can easily follow the process in the Theorem (3.11). □

Corollary 3.15. *Let M and N be two non-void members of a CMS $(\mathfrak{X}, d_{\mathfrak{X}})$ s.t. M is complete, the pair (M, N) hold V - property and M_0 is non-empty. Let $F : M \to N$ and $g_0 : M \to M$ are continuous $\alpha_0 - (\psi_0, g_0)$ - PC map of Class-I or a continuous $\alpha_0 - (\psi_0, g_0)$ - PC map of Class-II in such a way that the following terms and conditions hold:*

1. *F is a triangle $\alpha_0 - (\psi_0, g_0)$ - allowable near-end mapping and $F(M_0) \subseteq N_0$),*
2. *there exists elements $p_0, p_1 \in M_0$ such that*

$$d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$$

and $\alpha_0(g_0p_0, g_0p_1) \geq 1$, *3. if* $\{g_0 p_a\}$ *is a sequence in* M *such that* $\alpha_0(g_0 p_a, g_0 p_{a+1}) \geq 1$ *and* $g_0 p_a \to g_0 p$ *as* $a \to +\infty$*, then* $\alpha_0(g_0 p_a, g_0 p) \geq 1$ *for all* $a \in \mathbb{N} \cup \{0\}$*.*

Then the F *will have a BPP. Furthermore, for every* $p, q \in M$ *s.t.* $d_{\mathfrak{X}}(g_0 p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0 q, Fq)$*, we have* $\alpha_0(g_0 p, g_0 q) \geq 1$*.*

There are some results endowed with graph.

Definition 3.16. Suppose that $(\mathfrak{X}, d_{\mathfrak{X}})$ is an MS containing a G graph. A self-mapping $F : \mathfrak{X} \to \mathfrak{X}$ is a contraction of Banach G, if F retains the contour of G, i.e. $\forall\, p, q \in \mathfrak{X}$, $(p, q) \in \mathfrak{E}(G) \Longrightarrow (Fp, Fq) \in \mathfrak{E}(G)$. And F reduces the weight of the G edges as follows:

$\exists\ \alpha_0 \in (0, 1),\ \forall\ p, q \in \mathfrak{X},\ (p, q) \in \mathfrak{E}(G) \Longrightarrow d_{\mathfrak{X}}(Fp, Fq) \leq \alpha_0 d_{\mathfrak{X}}(p, q)$.

Definition 3.17. let M and N be two non-vacant closed subsets of a MS $(\mathfrak{X}, d_{\mathfrak{X}})$ own graph G. We are suggesting that $F : M \to N$ is a non-self map, $g_0 : M \to M$ are $G - (\psi_0, g_0)$ - PC, if, $m, n, p, q \in M$

$$\begin{cases} (g_0 p, g_0 q) \in \mathfrak{E}(G), \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(N, Fq) = d_{\mathfrak{X}}(M, N). \end{cases}$$

$$\Longrightarrow d_{\mathfrak{X}}(m, n) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0 p, n) + d_{\mathfrak{X}}(g_0 q, m)) - \psi_0(d_{\mathfrak{X}}(g_0 p, n), d_{\mathfrak{X}}(g_0 q, m))$$

and

$$\begin{cases} (g_0 p, g_0 q) \in \mathfrak{E}(G), \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N). \end{cases} \Longrightarrow (m, n) \in \mathfrak{E}(G),$$

Theorem 3.18. *Let us take* M *and* N *be two non-void closed elements of a CMS* $(\mathfrak{X}, d_{\mathfrak{X}})$ *with a graph* G*. Let* M *is complete and* M_0 *is non-void and* $F : M \to N$*,* $g_0 : M \to M$ *are continuous* $G - (\psi_0, g_0)$ *- PC map in such a way that the given terms and conditions hold:*

1. $F(M_0) \subseteq N_0$*,*
2. *then there exists elements* $p_0, p_1 \in M_0$ *such that*

$$d_{\mathfrak{X}}(g_0 p_0, g_0 p_0) = d_{\mathfrak{X}}(M, N)$$

and $(g_0 p_0, g_0 p_1) \in \mathfrak{E}(G)$*,*

3. *for all* $(g_0 p, g_0 q) \in \mathfrak{E}(G)$ *and* $(g_0 q, g_0 r) \in \mathfrak{E}(G)$*, we have* $(g_0 p, g_0 r) \in \mathfrak{E}(G)$*.*

Next, F has a BPP. Additionally, the BPP is unique if, for every $p, q \in M$ such that $d_{\mathfrak{X}}(g_0p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(g_0q, Fq)$, we have $(g_0p, g_0q) \in \mathfrak{E}(G)$.

Proof. Define $\alpha_0 : \mathfrak{X} \times \mathfrak{X} \to [0, +\infty)$ by

$$\alpha_0(g_0p, g_0q) = \begin{cases} 1, & if\ (g_0p, g_0q) \in \mathfrak{E}(G), \\ 0, & otherwise. \end{cases}$$

First, we prove that F is a triangle $\alpha_0 - (\ psi_0, G)$-near-end allowable map.

$$\begin{cases} \alpha_0(g_0p, g_0q) \geq 1, \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N). \end{cases}$$

Therefore, we obtain

$$\begin{cases} (g_0p, g_0q) \in \mathfrak{E}(G), \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \\ d_{\mathfrak{X}}(n, Fq) = d_{\mathfrak{X}}(M, N). \end{cases}$$

Since F is a $G - (\psi_0, g_0)$ - PC map, we get $(m, n) \in \mathfrak{E}(G)$, that is $\alpha_0(g_0m, g_0n) \geq 1$ and

$$d_{\mathfrak{X}}(m, n) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, n) + d_{\mathfrak{X}}(g_0q, m)) - \psi_0(d_{\mathfrak{X}}(g_0p, n)d_{\mathfrak{X}}(g_0q, m)).$$

Also, let $\alpha_0(g_0p, r) \geq 1$ and $\alpha_0(r, g_0q) \geq 1$, then $\alpha_0(r, g_0q) \geq 1$, then $(r, g_0q) \geq 1$, then $(g_0p, r) \in \mathfrak{E}(G)$ and $(r, g_0q) \in \mathfrak{E}(G)$. As a result, we deduce from (3) that $(g_0p, g_0q) \in \mathfrak{E}(G)$ is $\alpha_0(g_0p, g_0q) \geq 1$.

Thus, F be $\alpha_0 - (\psi_0, g_0)$ - triangular proximal admissible mapping with $F(M_0) \subseteq N_0$. In addition, F is continuously modified $\alpha_0 - (\psi_0, g_0)$-PC. From (2), there is $p_0, p_1 \in M_0$ s.t. $d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$ and $(g_0p_0, g_0p_1) \in \mathfrak{E}(G)$, that is, $d_{\mathfrak{X}}(g_0p_1, Fp_0) = d_{\mathfrak{X}}(M, N)$ and $\alpha_0(g_0p_0, g_0p_1) \geq 1$. As a result, all of Theorem's (3.11) conditions are satisfied, and F has only one fixed point. □

Similarly, we use the Theorem (3.14) to prove the following theorem.

Theorem 3.19. *Presume M and N are non - empty closed members of a MS $(\mathfrak{X}, d_{\mathfrak{X}})$ provided with a graph G. Assume that, M is complete, the pair (M, N) hold V - property and M_0 is non - empty. Presume that $F : M \to N$ and $g_0 : M \to M$ are $G - (\psi_0, g_0)$ - PC map in a way that the following criteria hold:*

1. *$F(M_0) \subseteq N_0$,*

2. *there exists elements* $p_0, q_1 \in M_0$ *such that*

$$d_{\mathfrak{X}}(g_0 p_1, F p_0) = d_{\mathfrak{X}}(M, N)$$

and $g_0 p_0, g_0 p_1) \in \mathfrak{E}(G)$,

3. $\forall$ $(p, q) \in \mathfrak{E}(G)$ *and* $(q, r) \in \mathfrak{E}(G)$, *we get* $(p, r) \in \mathfrak{E}(G)$

4. *if* $\{p_a\}$ *is a sequence in* $\mathfrak{X}$ *s.t.* $(p_a, p_{a+1}) \in \mathfrak{E}(G)$ *for all* $a \in \mathbb{N} \cup \{0\}$ *and* $p_a \to p$ *as* $a \to +\infty$, *so* $(p_a, p) \in \mathfrak{E}(G)$ $\forall$ $a \in \mathbb{N} \cup \{0\}$.

Then, F has a BPP. Further, the BPP is unique if, for each $p, q \in M$ *just like that* $d_{\mathfrak{X}}(p, Fp) = d_{\mathfrak{X}}(M, N) = d_{\mathfrak{X}}(q, Fq)$, *we get* $(p, q) \in \mathfrak{E}(G)$.

3.5 Results in Partially Ordered Metric Space

Recently, following researchers [50], [41] and [11] work on weaker contraction by representing self-map in partially ordered MS.

Theorem 3.20. *Suppose that* M, N *be two closed members of a partially ordered CMS* $(\mathfrak{X}, d_{\mathfrak{X}}, \preceq)$, M_0 *is non - empty and* (M, N) *has the* V *- property. Presume the following conditions are met by* $F : M \to N$*:*

1. F *is ordered immediately-holding* $F(M_0) \subseteq N_0$ *in such a way that,*

2. *there exist elements* $p_0, p_1 \in M_0$ *such that* $d_{\mathfrak{X}}(g_0 p_1, F p_0) = d_{\mathfrak{X}}(M, N)$ *and* $p_0 \preceq p_1$,

3. *for all* $p, q, m, n \in M$,

$$\begin{cases} g_0 p \preceq g_0 q, \\ d_{\mathfrak{X}}(m, Fp) = d_{\mathfrak{X}}(M, N), \Longrightarrow d_{\mathfrak{X}}(m, n) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0 p, n) + d_{\mathfrak{X}}(g_0 q, m)) - \psi_0(d_{\mathfrak{X}}(g_0 p, n), d_{\mathfrak{X}}(g_0 q, m)) \\ d_{\mathfrak{X}}(g_0 q, Fq) = d_{\mathfrak{X}}(M, N) \end{cases}$$

4. *if* $\{x_p\}$ *is an increasing sequence in* M *converging to* $x \in M$, $\forall$ $p \in \mathbb{N}$. *Then* F *has a BPP.*

We are currently collecting multiple FPT in this chapter, which are consequences of the results mentioned in the important area.

Theorem 3.21. *Presume* $(\mathfrak{X}, d_{\mathfrak{X}})$ *be a CMS. Assume that* $F : \mathfrak{X} \to \mathfrak{X}$ *and* $g_0 : M \to M$ *be a continuous self - map fulfills the below requirements:*

1. (a) *F is triangular* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*
   (b) *there is* p_0 *in* $\mathfrak{X}$ *so* $\alpha_0(g_0p_0, Fp_0) \geq 1$,
   (c) *for all* $p, q \in \mathfrak{X}$,

$$\alpha_0(g_0p, g_0q)d_{\mathfrak{X}}(Fp, Fq) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, Fq)+d_{\mathfrak{X}}(g_0q, Fp))-\psi_0(d_{\mathfrak{X}}(g_0p, Fq), d_{\mathfrak{X}}(g_0q, Fp)).$$

   Then there's a fixed point of F.

2. (a) F *be* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*
   (b) $\exists\ p_0$ *in* $\mathfrak{X}$ *s.t.* $\alpha_0(g_0p_0, Fp_0) \geq 1$,
   (c) $\forall\ p, q \in \mathfrak{X}$,

$$(\alpha_0(g_0p, g_0q)+l)^{d_{\mathfrak{X}}(Fp,Fq)} \leq (u+1)^{\frac{1}{2}(d_{\mathfrak{X}}(g_0p,Fq)+d_{\mathfrak{X}}(g_0q,Fp))-\psi_0(d_{\mathfrak{X}}(g_0p,Fq),d_{\mathfrak{X}}(g_0q,Fp)}.$$

   Then, there is a fixed point F.

3. (a) F *is triangular* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*
   (b) *there is* p_0 *in* $\mathfrak{X}$ *such that* $\alpha_0(g_0p_0, Fp_0) \geq 1$,
   (c) *absolutely* $p, q \in \mathfrak{X}$,
   $\alpha_0(g_0p, g_0q)d_{\mathfrak{X}}(Fp, Fq) \leq \frac{1}{2}(d_{\mathfrak{X}}(g_0p, Fq)+d_{\mathfrak{X}}(g_0q, Fp))-\psi_0(d_{\mathfrak{X}}(g_0p, Fq), d_{\mathfrak{X}}(g_0q, Fp))$.
   Then F *has a fixed point.*
   (d) *if* $\{g_0p_a\}$ *is a sequence in* $\mathfrak{X}$ *such that* $\alpha_0(g_0p_a, g_0p_{a+1}) \geq 1$ *and* $p_a \to p$ *as* $a \to +\infty$, *then* $\alpha_0(g_0p_a, g_0p) \geq 1\ \forall\ a \in \mathbb{N}$. *Then there is a fixed point at* F.

4. (a) F *be triangular* $\alpha_0 - (\psi_0, g_0)$ *- admissible,*
   (b) *there is* p_0 *in* $\mathfrak{X}$ *such that* $\alpha_0(g_0p_0, Fp_0) \geq 1$,
   (c) $\forall\ p, q \in \mathfrak{X}$,
   $(\alpha_0(g_0p, g_0q) + 1)^{d_{\mathfrak{X}}(Fp,Fq)} \leq 2^{[\frac{1}{2}(d_{\mathfrak{X}}(g_0p,Fq)+d_{\mathfrak{X}}(g_0q,Fp))-\psi_0(d_{\mathfrak{X}}(g_0p,Fq),d_{\mathfrak{X}}(g_0q,Fp))]}$.
   Then F *has a fixed point.*
   (d) *if a sequence* $\{g_0p_a\}$ *in* M *s.t.* $\alpha_0(g_0p_a, g_0p_{a+1}) \geq 1$ *and* $p_a \to p$ *as* $a \to +\infty$, *then* $\alpha_0(g_0p_a, g_0p) \geq 1\ \forall\ a \in \mathbb{N}$. *Then there is a fixed point of* F.

3.6 The Proximal Contraction of First and Second Kind regard to Simulation Function

We present notions of the first and second forms of generalized PC mappings with simulation method that vary from another type in the writings.

Suppose M and N are non-empty sets of (X, d^*) CMS. The following remarks were accompanied by:

$$d^*(M,N) = inf\{d^*(\hat{a},\hat{b}) : \hat{a} \in M, \hat{b} \in N\}$$

$$M_0 = \{\hat{a} \in M : d^*(\hat{a},\hat{b}) = d^*(M,N) \text{ for whatever } \hat{b} \in N\}$$

$$N_0 = \{\hat{b} \in N : d^*(\hat{a},\hat{b}) = d^*(M,N) \text{ for some } \hat{a} \in M\}$$

With simulation method we add new notions of first kind and second kind.

Definition 3.22. A mapping $F : M \to N$ is first-class PC if $0 < \alpha < 1$ s.t. for all $w_1, w_2, \hat{a}_1, \hat{a}_2$ in M and $\zeta \in \mathbb{Z}$ if

$$\begin{cases} d^*(w_1, F\hat{a}_1) = d^*(M,N) \\ d^*(w_2, F\hat{a}_2) = d^*(M,N), \end{cases}$$

implies that,

$$0 \leq \zeta(d^*(w_1, w_2), \alpha d^*(\hat{a}_1, \hat{a}_2)).$$

Definition 3.23. A map of $F : M \to N$ is assumed to be a strong first-class PC if a non-negative integer exists $\alpha < 1$ and $\beta < 1$ s.t. $\forall\ w_1, w_2, \hat{a}_1, \hat{a}_2$ in M and $\zeta \in \mathbb{Z}$ if

$$\begin{cases} d^*(w_1, F\hat{a}_1) \leq \beta d^*(M,N) \\ d^*(w_2, F\hat{a}_2) \leq \beta d^*(M,N), \end{cases}$$

implies that,

$$0 \leq \zeta(d^*(w_1, u_2), (\alpha d^*(\hat{a}_1, \hat{a}_2) + (\beta - 1)d^*(M,N)).$$

Definition 3.24. A mapping $F : M \to N$ is a second class PC if a non-negative integer occurs $\alpha < 1$ such that for all $w_1, w_2, \hat{a}_1, \hat{a}_2$ in M w.r.t. ζ and $\zeta \in \mathbb{Z}$ if

$$\begin{cases} d^*(w_1, F\hat{a}_1) = d^*(M,N) \\ d^*(w_2, F\hat{a}_2) = d^*(M,N), \end{cases}$$

implies that,

$$0 \leq \zeta(d^*(u_1, u_2), \alpha d^*(\hat{a}_1, \hat{a}_2)).$$

Each time $\hat{a}_1, \hat{a}_2, w_1$ and w_2 are elements in M which satisfy the requirement that

$d^*(w_1, F\hat{a}_1) = d^*(M,N)$ and $d^*(w_2, F\hat{a}_2) = d^*(M,N)$.

The precondition for a F self-map to be a proximal second-class contraction is

$$0 \leq \zeta(d^*(F^2\hat{a}_1, F^2\hat{a}_2), \alpha d^*(F\hat{a}_1, F\hat{a}_2)),$$

for all $\hat{a}_1$ and $\hat{a}_2$ in the domain of F.

We implement and explain these findings:

Theorem 3.25. *Suppose X be a CMS w.r.t. ζ and $\zeta \in \mathbb{Z}$. Let M and N be non-empty, with X closed subsets so M is equally compact to N. Suppose M_0 and N_0 are non-empty, instead. Suppose $F : M \to N$ and $g_0 : M \to M$ satisfied this:*

1. *F is the second type of persistent PC.*
2. *g_0 reflects an isometry.*
3. *FM_0 is contained in N_0.*
4. *M_0 is contained in g_0N_0.*
5. *F retains isometric variance in addition to g_0.*

Therefore an item $\hat{a}$ exists in M to this extent

$$d^*(g_0\hat{a}, F\hat{a}) = d^*(M, N).$$

In addition, if $\hat{a}^*$ is another variable that holds the preceding assumption for, then $F\hat{a}$ and $F\hat{a}^*$ are similar.

Proof. Let $\hat{a}_0$ be a fixed point in M_0. Because FM_0 is in N_0 and M_0 is in g_0M_0, there is an item $\hat{a}_1$ in M_0 that exists

$$d^*(g_0\hat{a}_1, F\hat{a}_0) = d^*(M, N).$$

Again, because $F\hat{a}_1$ is an item of FM_0 that is contained in N_0 and M_0 in g_0M_0, it follows that $\hat{a}_2$ is contained in M_0

$$d^*(g_0\hat{a}_2, F\hat{a}_1) = d^*(M, N).$$

Will start this phase. Having selected $\hat{a}_n$ in M_0, $\hat{a}_{m+1}$ can be contained in M_0 in such a way that

$$d^*(g_0\hat{a}_{m+1}, F\hat{a}_m) = d^*(M, N).$$

For any positive integer m then FM_0 is in M_0 and N_0 is in g_0M_0. As F is a second type of PC,

$$\begin{aligned}0 \leq&\zeta(d^*(g_0\hat{a}_{m+1}, F\hat{a}_m), \alpha d^*(F\hat{a}_m, F\hat{a}_{m-1}))\\ <&\alpha d^*(F\hat{a}_m, F\hat{a}_{m-1}) - d^*(g\hat{a}_{m+1}, F\hat{a}_m)\\ d^*(g_0\hat{a}_{m+1}, F\hat{a}_m) \leq&\alpha d^*(F\hat{a}_m, F\hat{a}_{m-1}).\end{aligned}$$

Because F retains isometric distance relative to g_0,

$$\begin{aligned}0 \leq&\zeta(d^*(F\hat{a}_{m+1}, F\hat{a}_m), \alpha d^*(F\hat{a}_m, F\hat{a}_{m-1}))\\ <&\alpha d^*(F\hat{a}_m, F\hat{a}_{m-1}) - d^*(F\hat{a}_{m+1}, F\hat{a}_m)\\ d^*(F\hat{a}_{m+1}, F\hat{a}_m) \leq&\alpha d^*(F\hat{a}_m, F\hat{a}_{m-1}).\end{aligned}$$

Then, $\{F\hat{a}_m\}$ is a CS and thus converges to any $\hat{b}$ vector in N. Further,

$$\begin{aligned}d^*(\hat{b}, M) \leq d^*(\hat{b}, g_0\hat{a}_m) \leq&d^*(\hat{b}, F\hat{a}_{m-1}) + d^*(F\hat{a}_{m-1}, g_0\hat{a}_m)\\ =&d^*(\hat{b}, Fa_{m-1}) + d^*(M, N)\\ \leq&d^*(\hat{b}, F\hat{a}_{m-1}) + d^*(\hat{b}, M).\end{aligned}$$

So $d^*(\hat{b}, g_0\hat{a}_m) \to d^*(\hat{b}, M)$. Provided that In terms of N, M is roughly compact, $\{g_0\hat{a}_m\}$ has the $\{g_0\hat{a}_{m(k)}\}$ subsequence which converges to any $\hat{c}$ in M. And it can be inferred that

$$d^*(\hat{c}, \hat{b}) = lim_{k\to\infty} d^*(g_0\hat{a}_{m(k)}, F\hat{a}_{m(k)-1}) = d^*(M, N).$$

Essentially, $\hat{c}$ is a part of M_0. As M_0 is in g_0M_0, $\hat{c} = g_0\hat{a}$ is in M_0 for some $\hat{a}$. Because $g_0\hat{a}_{m(k)} \to g_0\hat{a}$ and g_0 are isometries, $\hat{a}_{m(k)} \to \hat{a}$ is an isometry. Because the mapping of F is constant, $F\hat{a}_{m(k)} \to F\hat{a}$ is the result. Therefore $\hat{b}$ and $F\hat{a}$ are similar. And it follows that

$$d^*(g_0\hat{a}, F\hat{a}) = lim_{m\to\infty} d^*(g_0\hat{a}_{m(k)}, F\hat{a}_{m(k)-1}) = d^*(M, N).$$

Suppose there is another a^* factor so

$$d^*(g_0\hat{a}^*, F\hat{a}^*) = d^*(M, N).$$

As F is a second type of PC,

$$\begin{aligned}0 \leq&\zeta(d^*(Fg_0\hat{a}, Fg_0\hat{a}^*), \alpha d^*(F\hat{a}, F\hat{a}^*))\\ <&\alpha d^*(F\hat{a}, F\hat{a}^*) - d^*(Fg_0\hat{a}, Fg_0\hat{a}^*)\\ d^*(Fg_0\hat{a}, Fg_0\hat{a}^*) \leq&\alpha d^*(F\hat{a}, F\hat{a}^*).\end{aligned}$$

F retains isometric distance regards g_0, we have

$$\begin{aligned} 0 \leq& \zeta(d^*(F\hat{a}, F\hat{a}^*), \alpha d^*(F\hat{a}, F\hat{a}^*)) \\ <& \alpha d^*(F\hat{a}, F\hat{a}^*) - d^*(F\hat{a}, F\hat{a}^*) \\ d^*(F\hat{a}, F\hat{a}^*) \leq& \alpha d^*(F\hat{a}, F\hat{a}^*) \end{aligned}$$

which implies $F\hat{a} = F\hat{a}^*$. □

The following corollary is given by the theorem if g_0 is the identity mapping.

Corollary 3.26. *Presume M, N be non-void, closed subsets of a CMS X s.t. M is about compact for N. Additionally, presume that M_0 and N_0 are non-empty. Let $F : M \to N$ follow these conditions:*

1. *F is a second type of persistent PC.*
2. *FM_0 is continuous in N_0.*

Therefore an item $\hat{a} \in M$ such that

$$d^*(\hat{a}, F\hat{a}) = d^*(M, N).$$

Furthermore, if $\hat{a}^*$ is the highest proximity point of F, $F\hat{a}^*$ is equivalent.

Theorem 3.27. *Let X be a CMS with respect to ζ. Suppose M, N be closed members of X and $\zeta \in \mathbb{Z}$. Additionally, suppose M_0 and N_0 are non-empty. Let $F : M \to N$ and $g_0 : M \to M$ fulfill the requirements of:*

1. *F is a first type of continuous PC.*
2. *g_0 reflects an isometry.*
3. *FM_0 is contained in N_0.*
4. *M_0 is contained in g_0N_0.*

Then, there's a special $\hat{a}$ factor in M that exists

$$d^*(g_0\hat{a}, F\hat{a}) = d^*(M, N).$$

Proof. As in the Theorem (3.25), a sequence of $\{\hat{a}_n\}$ exists in M which satisfies the following condition.

$$d^*(g_0\hat{a}_{m+1}, F\hat{a}_m) = d^*(M, N).$$

Because F is first kind of PC, we've

$$\begin{aligned} 0 \leq & \zeta(d^*(g_0\hat{a}_{m+1}, g_0\hat{a}_m), \alpha d^*(\hat{a}_m, \hat{a}_{m-1})) \\ < & \alpha d^*(\hat{a}_m, \hat{a}_{m-1}) - d^*(g_0\hat{a}_{m+1}, g_0\hat{a}_m) \\ d^*(g_0\hat{a}_{m+1}, g_0\hat{a}_m) \leq & \alpha d^*(\hat{a}_m, \hat{a}_{m-1}). \end{aligned}$$

Since g_0 is an isometry, we can deduce that

$$\begin{aligned} 0 \leq & \zeta(d^*(\hat{a}_{m+1}, \hat{a}_m), \alpha d^*(\hat{a}_m, \hat{a}_{m-1})) \\ < & \alpha d^*(\hat{a}_m, \hat{a}_{m-1}) - d^*(\hat{a}_{m+1}, \hat{a}_m) \\ d^*(\hat{a}_{m+1}, \hat{a}_m) \leq & \alpha d^*(\hat{a}_m, \hat{a}_{m-1}). \end{aligned}$$

So, $\{\hat{a}_m\}$ is a CS, which converges in M to any $\hat{a}$. Because g_0 and F are continuing, we also have

$$d^*(g_0\hat{a}, F\hat{a}) = lim_{m\to\infty} d^*(g_0\hat{a}_{m+1}, F\hat{a}_m) = d^*(M, N).$$

Suppose there is another element $\hat{(c)}$

$$d^*(g_0\hat{c}, F\hat{c}) = d^*(M, N).$$

Because F is a first kind of PC and g_0 is isometry, we have

$$\begin{aligned} 0 \leq & \zeta(d^*(g_0\hat{a}, g_0\hat{c}), \alpha d^*(\hat{a}, \hat{c})) \\ < & \alpha d^*(\hat{a}, \hat{c}) - d^*(g_0\hat{a}, g_0\hat{c}) \\ d^*(g_0\hat{a}, g_0\hat{c}) \leq & \alpha d^*(\hat{a}, \hat{c}). \end{aligned}$$

Therefore,

$$d^*(\hat{a}, \hat{c}) = d^*(g_0\hat{a}, g_0\hat{c}) \leq \alpha d^*(\hat{a}, \hat{c}).$$

That means $\hat{a}$ and $\hat{c}$ are the same. The proof is now complete. □

If g_0 is an identity mapping, then the Theorem (3.27) provides the next inference.

Corollary 3.28. *Suppose X be a CMS and M, N are an empty closed subsets of a MS. Further, assume that M_0 and N_0 are non-empty. Let a mapping $F : M \to N$ fulfill the following conditions:*

1. *F is the first kind of continuous PC.*

2. *FM_0 is contained in N_0.*

Then there's a special $\hat{a}$ factor in M that exists

$$d^*(\hat{a}, F\hat{a}) = d^*(M, N).$$

Theorem 3.29. *Let M, N be non-void, closed members of a MS and $\zeta \in \mathbb{Z}$ and let $g_0 : M \to M$ and $F : M \to N$ fulfill the following requirements:*

1. *In M there is a $\{\hat{a}_m\}$ sequence, such that $d^*(g_0\hat{a}_m, F\hat{a}_m) \to d^*(M, N)$.*
2. *F is the first kind of persistent, powerful PC.*
3. *g_0 reflects an isometry.*

Then there is a special $\hat{a}_0$ dimension that exists in M

$$d^*(g\hat{a}_0, F\hat{a}_0) = d^*(M, N).$$

Additionally there is a $\{\hat{a}_{m(l)}\}$ subsequence of $\{\hat{a}_m\}$ converging to the $\hat{a}_0$ element.

Proof. Let us describe l for any positive integer

$$M_l = \{\hat{a} \in M_0 : d^*(g_0\hat{a}, F\hat{a}) \leq (1 + \frac{1}{l}d^*(M, N)\}.$$

Since $d^*(g_0\hat{a}_m, Fa_m) \to d^*(M, N)$, there exists a member $\hat{a}_{m(k)}$ of the sequence $\{\hat{a}_m\}$ such that

$$\begin{aligned} 0 \leq& \zeta(d^*(g_0\hat{a}_{m(l)}, F\hat{a}_{m(l)}), (1 + \frac{1}{l})d^*(M, N)) \\ <& (1 + \frac{1}{k})d^*(M, N) - d^*(g_0\hat{a}_{m(l)}, F\hat{a}_{m(l)}) \\ d^*(g_0\hat{a}_{m(l)}, F\hat{a}_{m(l)}) \leq& (1 + \frac{1}{l})d^*(M, N). \end{aligned}$$

Hence for every l, M_l is non-empty. Because of the continuousness of g_0 and F through M_l is closed. Also, it's clear that M_{l+1} is in M_l. If $\hat{a}, \hat{c}$ are two elements of some sort in M_l, then as F is a strong first form PC, we have

$$\begin{aligned} 0 \leq& \zeta(d^*(g_0\hat{a}, g_0\hat{c}), (\alpha d^*(\hat{a}, \hat{c}) + (\frac{1}{l})d^*(M, N)) \\ <& \alpha d^*(\hat{a}, \hat{c}) + (\frac{1}{l})d^*(M, N) - d^*(g_0\hat{a}, g_0\hat{c}) \\ d^*(g_0\hat{a}, g_0\hat{c}) \leq& \alpha d^*(\hat{a}, \hat{c}) + (\frac{1}{l})d^*(M, N) \end{aligned}$$

for all $\alpha \in [0, 1]$.

Because g_0 is an isometry, the consequence is

$$\begin{aligned}0 \leq & \zeta(d^*(\hat{a},\hat{c}), \frac{1}{(1-\alpha)l} d^*(M,N)) \\ < & \frac{1}{(1-\alpha)l} d^*(M,N) - d^*(\hat{a},\hat{c}) \\ d^*(\hat{a},\hat{c}) \leq & \frac{1}{(1-\alpha)l} d^*(M,N).\end{aligned}$$

So, $diam(M_l) \to 0$. Since X is a CMS, $\bigcap M_l$ comprises just one level, claim $\hat{a}_0$, which fulfills the requirement that $d^*(g_0\hat{a}_0, F\hat{a}_0) = d^*(M,N)$ does. In addition, since g_0 is an isometry and F is a strong first-type PC, it follows that

$$\begin{aligned}0 \leq & \zeta(d^*(g_0\hat{a}_{m(l)}, g_0\hat{a}_0), (\alpha d^*(\hat{a}_{m(l)}, \hat{a}_0) + (\frac{1}{l}) d^*(M,N)) \\ < & (\alpha d^*(\hat{a}_{m(l)}, a_0) + (\frac{1}{l}) d^*(M,N)) - d^*(g_0\hat{a}_{m(l)}, g_0\hat{a}_0) \\ d^*(g_0\hat{a}_{m(l)}, g_0\hat{a}_0) \leq & \alpha d^*(\hat{a}_{m(l)}, \hat{a}_0) + (\frac{1}{l}) d^*(M,N),\end{aligned}$$

$$d^*(\hat{a}_{m(l)}, \hat{a}_0) = d^*(g_0\hat{a}_{m(l)}, g_0\hat{a}_0) \leq \alpha d^*(\hat{a}_{m(l)}, \hat{a}_0) + (\frac{1}{l}) d^*(M,N).$$

Therefore,

$$\begin{aligned}0 \leq & \zeta(d^*(\hat{a}_{m(l)}, \hat{a}_0), (\frac{1}{(1-\alpha)l} d^*(M,N)) \\ < & \frac{1}{(1-\alpha)l} d^*(M,N) - d^*(\hat{a}_{m(l)}, \hat{a}_0) \\ d^*(\hat{a}_{m(l)}, \hat{a}_0) \leq & \frac{1}{(1-\alpha)l} d^*(M,N).\end{aligned}$$

Therefore, the subsequence $\{\hat{a}_{m(l)}\}$ converges to the variable $\hat{a}_0$.
This completes the theorem argument. □

The following result provides the necessary and sufficient conditions for a contraction to have the BPP.

Example 3.3. *Presume that* $X = [0,1]$ *and defined by* $d^*(\beta,\delta) = |\beta - \delta|$. *Define* $F, g_0 : X \to X$ *as* $F\beta = \frac{\beta}{2+\beta}$, $g_0\beta = \frac{\beta}{2}$. *Then,* $d^*(g_0\beta, F\beta) \leq d_0^*(M,N)$ $\forall\ \beta,\delta \in X$. *Put* $S(\check{t},\check{s}) = \frac{\check{s}}{\check{s}+1} - \check{t}$, $G(\check{s},\check{t}) = \check{s} - \check{t}$, (X, d^*) *is a CMS for* ζ *and* $\zeta \in \mathbb{Z}$.

$$\zeta(d^*(g_0\beta, F\beta), d^*(M,N))$$

$$= \frac{d^*(M,N)}{1+d^*(M,N)} - d^*(g_0\beta, F\beta)$$

$$= \frac{\frac{1}{2}|\beta-\delta|}{1+\frac{1}{2}|\beta-\delta|} - |\frac{\beta}{\beta+2} - \frac{\delta}{\delta+2}|$$

$$= \frac{|\beta - \delta|}{2 + |\beta - \delta|} - |\frac{\beta\delta + 2\beta - \delta\beta - 2\delta}{(\beta + 2)(\delta + 2)}$$

$$= \frac{|\beta - \delta|}{2 + |\beta - \delta|} - \frac{2|\beta - \delta|}{(\beta + 2)(\delta + 2)} \geq 0$$

whenever $\beta, \delta \in X$.

However, because $\beta = \frac{1}{2}$ *and* $\delta = \frac{1}{4}$ *fulfill all the conditions indicating that the* F *and* g_0 *mappings have a specific fixed point. As such* $\frac{1}{45}$ *is single point which is unique.*

Chapter 4

Theorems of Fixed Points in G-Metric and Generalized Metric Spaces

This chapter discusses theorems of certain fixed points for $G-\upsilon-\psi$-proximal cyclic weak contractive mapping, a new method of SF fixed-point theoretical research. Also, we illustrate common FPT for $\alpha_b - \psi_b$ contractive pair of mappings in G-M.S..

It consists of six parts. In the first part, we focus on the concept of $G-\upsilon-\psi$-proximal loop weak contraction mapping in G-M.S.. The second part is about the FPT of the weak contraction mapping of the PC. In the third part, we introduce the concept of $\mathcal{Z}$-contraction. In the fourth part, we compressed $\mathcal{Z}$ using simulation methods to prove some FPT. In the fifth part, we introduced the new concept of $\alpha_b - \psi_b$ contractive mapping pair in g.m.s.. In the last section, we prove several common FPT with α_b-admissible mapping for class-(i) and class-(ii).

4.1 Notions of $G-\upsilon-\psi$-Proximal Cyclic Weak Contractive Mapping

Mustafa and Sims [44] presented the G-metric notion and studied the topology of such spaces.

We introduce the notions of $G - \upsilon - \psi$ - proximal cyclic weak contractive mapping in G-M.S.. We initially believe that

$$\begin{aligned} \upsilon &= \{\upsilon : [0,\infty) \to [0,\infty) \ s.t. \upsilon \ is \ nondecreasing \ and \ continuous\}, \\ \psi &= \{\psi : [0,\infty) \to [0,\infty) \ s.t. \psi \ is \ lower \ semicontinuous\} \end{aligned} \tag{4.1}$$

$\gamma(t_a) = \Psi(t_a) = 0$ iff $t_a = o$.

$$d_G(\hat{x}, \hat{y}) = G(\hat{x}, \hat{y}, \hat{y}) + G(\hat{y}, \hat{x}, \hat{x}), \forall \ \hat{x}, \hat{y} \in X. \tag{4.2}$$

Let (X, G) be a G-M.S.. Suppose P and Q are non-empty subsets of a G-M.S. (X, G). Sets are described as follows:

$$\begin{aligned} P_0 &= \{m \in P : d_G(m, n) = d_G(P, Q) \ for \ some \ n \in Q\}, \\ Q_0 &= \{n \in Q : d_G(m, n) = d_G(P, Q) \ for \ some \ m \in P\}. \end{aligned} \tag{4.3}$$

where $d_G(P, Q) = inf\{d_G(m, n) : m \in P, n \in Q\}$.

The definitions are listed below:

Definition 4.1. Let (X, G) be a G-M.S. and P, Q be two non-empty subsets of X.

1. With respect to P, Q is considered to be roughly compact if every sequence $\{n_r\}$ in Q, fulfill the criteria $d_G(m, n_r) \to d_G(m, Q)$ for some m in P, has a convergent subsequence.

2. Let $S : P \cup Q \to P \cup Q$ be a non - self mapping s.t. $S(P) \subseteq Q$, $S(Q) \subseteq P$. We say that S is generalized $G - \upsilon - \psi$ - proximal cyclic weak contractive mapping if for $m, v, v^* \in P$, $w, n \in Q$.

$$\begin{aligned} G(v^*, Sm) &= d_G(P, Q) \\ G(v, Sv^*) &= d_G(P, Q) \\ G(w, Sn) &= d_G(P, Q) \end{aligned} \tag{4.4}$$

$$\upsilon(G(v, v^*, w)) \leq \upsilon(M(m, w, n)) - \psi(M(m, w, n))$$

holds where $\upsilon \in \Upsilon$ and $\psi \in \Psi$.

and $M(m, w, n) = max\{G(m, w, n), G(m, Sm, Sm), G(n, Sn, Sn)\}$.

4.2 Theorems of Fixed Points for Proximal Cyclic Weak contractive Mapping

Theorem 4.2. *Let P, Q are two non-vacant subsets of a G-M.S. (X, G) such that $(P, G), (Q, G)$ are complete G-M.S., P_0 is non empty and Q is similarly compact to P. Presume $S : P \cup Q \to P \cup Q$ is a $G - \upsilon - \psi$ - proximal cyclic weak contractive mapping s.t. $S(P) \subseteq Q$, $S(Q) \subseteq P$ and $S(P_0) \subseteq Q_0$. Then, S has a BPP, there is unique $\hat{z} \in P$ s.t. $d_G(\hat{z}, S\hat{z}) = d_G(P, Q)$.*

Proof. Since P_0 is not empty, we take m_0 in P_0. Take $m_1 = Sm_0 \in S(P_0) \subseteq Q$, so $d_G(m_0, m_1) = d_G(m_0, Sm_0) = d_G(P, Q)$. Further, $m_2 = Sm_1 \in S(Q_0) \subseteq P$, it follows that $d_G(m_1, Sm_1) = d_G(m_1, m_2) = d_G(P, Q)$. Recursively, we obtain a sequence $\{m_r\}$ in $P \cup Q$ satisfying

$$d_G(m_r, m_{r+1}) = d_G(P, Q) \ for \ all \ r \in \mathbb{N} \cup \{0\} \tag{4.5}$$

This shows that

$$\begin{aligned} d_G(\upsilon^*, Sm) &= d_G(P, Q) \\ d_G(\upsilon, S\upsilon^*) &= d_G(P, Q) \\ d_G(w, Sn) &= d_G(P, Q) \end{aligned}$$

where $m = m_{r-1}$, $\upsilon = m_{r+1}$, $\upsilon^* = m_{r+1}$, $n = m_r$, $w = m_r$.

Therefore, from (4.4), we get

$$\begin{aligned} \upsilon(G(m_{r+1}, m_{r+1}, m_r)) &\leq \upsilon(M(m_{r-1}, m_r, m_r)) - \psi(M(m_{r-1}, m_r, m_r)) \\ &\leq \upsilon(M(m_{r-1}, m_r, m_r)) \end{aligned}$$

where

$$\begin{aligned} M(m_{r-1}, m_r, m_r) &= max\{G(m_{r-1}, m_r, m_r), G(m_{r-1}, Sm_r, Sm_r), G(m_r, Sm_r, Sm_r)\} \\ &= max\{G(m_{r-1}, m_r, m_r), G(m_{r-1}, m_r, m_r), G(m_r, m_{r+1}, m_{r+1})\} \\ &= max\{G(m_{r-1}, m_r, m_r), G(m_r, m_{r+1}, m_{r+1})\}. \end{aligned}$$

If

$$M(m_{r-1}, m_r, m_r) = G(m_r, m_{r+1}, m_{r+1}),$$

then, we have

$$\upsilon(G(m_r, m_{r+1}, m_{r+1}) = \upsilon(G(m_r, m_{r+1}, m_{r+1}) - \psi(G(m_r, m_{r+1}, m_{r+1}))$$

$$\Rightarrow \upsilon(G(m_r, m_{r+1}, m_{r+1})) = 0$$

$$\Rightarrow \psi(G(m_r, m_{r+1}, m_{r+1})) = 0$$

$$\Rightarrow m_r = m_{r+1},$$

which is not true, if for $r_0, mr_0 = mr_{0+1} = Sm_0$,

then mr_0 would become fixed point of S.

Then, we get

$$M(m_{r-1}, m_r, m_r) = G(m_{r-1}, m_r, m_r).$$

Therefore,

$$\begin{aligned} \upsilon(G(m_r, m_{r+1}, m_{r+1})) &\leq \upsilon(G(m_{r-1}, m_r, m_r)) - \psi(G(m_{r-1}, m_r, m_r)) \\ &\leq \upsilon(G(m_{r-1}, m_r, m_r)) \end{aligned} \tag{4.6}$$

which implies

$$G(m_r, m_{r+1}, m_{r+1}) \leq G(m_{r-1}, m_r, m_r).$$

Therefore, the $\{G(m_r, m_{r+1}, m_{r+1})\}$ series decreases in $\mathbb{R}^+$ and thus convergent to $t_a \in \mathbb{R}^+$. Then, we're saying $t_a = 0$. Conversely, assume $t_a > 0$. Putting limit as $r \to +\infty$, we get

$$\upsilon(t_a) \leq \upsilon(t_a) - \psi(t_a) \tag{4.7}$$

which denotes $\psi(t_a) = 0$. i.e., $t_a = 0$, this is the inverse. Consequently, $t_a = 0$.
That is,

$$lim_{r\to\infty} G(m_r, m_{r+1}, m_{r+1}) = 0. \tag{4.8}$$

Let's prove that $\{m_r\}\binom{\infty}{r=0}$ in (X, G) is a CS. Suppose, on the other side, there are $\epsilon > 0$ and corresponding $\{p(\hat{l})\}$ and $\{q(\hat{l})\}$ sub-sections of N that satisfy $p(\hat{l}) > q(\hat{l}) > \hat{l}$ where $q(\hat{l})$ is the smallest integer with

$$G(m_{p(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}) = 0. \tag{4.9}$$

$$G(m_{p(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}) \geq \epsilon$$

$$G(m_{p(\hat{l})}, m_{q(\hat{l})-1}, m_{q(\hat{l})-1}) < \epsilon \tag{4.10}$$

$$\begin{aligned} \epsilon &\leq G(m_{p(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}) \\ &\leq G(m_{p(\hat{l})}, m_{q(\hat{l})-1}, m_{q(\hat{l})-1}) + G(m_{q(\hat{l})-1}, m_{q(\hat{l})}, m_{q(\hat{l})}) \\ &< \epsilon + G(m_{q(\hat{l})-1}, m_{q(\hat{l})}, m_{q(\hat{l})}). \end{aligned} \tag{4.11}$$

Making $\hat{l} \to \infty$ in (4.9), we get

$$lim_{\hat{l} \to +\infty} G(m_{p(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}) = \epsilon. \tag{4.12}$$

Every $\hat{l} \in N$; there is $r(\hat{l})$ meets $0 \leq r(\hat{l}) \leq p$ such that

$$q(\hat{l}) - p(\hat{l}) + r(\hat{l}) = 1 \ mod \ m = 1(m).$$

Consequently, for every sufficiently large value of $\hat{l}$,

$o(\hat{l}) = p(\hat{l}) - r(\hat{l}) > 0$ and $m_{o(\hat{l})}$ and $m_{q(\hat{l})}$ lie in the set P and Q respectively.

Now, using $m = m_{o(\hat{l})}$, $v = m_{q(\hat{l})+1}$, $v^* = m_{l(\hat{l})}$, $n = m_{q(\hat{l})}$ and $w = m_{q(\hat{l})}$,

$$\begin{aligned} \upsilon(G(m_{o(\hat{l})}, m_{q(\hat{l})+1}, m_{q(\hat{l})}) &\leq (\upsilon(M(m_{o(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})})) - (\psi(M(m_{l(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})})) \\ &\leq (\upsilon(G(m_{o(\hat{l})}, m_{q(\hat{l})+1}, m_{q(\hat{l})}) \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} M(m_{o(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}) = max\{&G(m_{o(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}), \\ &G(m_{o(\hat{l})}, Sm_{o(\hat{l})}, Sm_{o(\hat{l})}), G(m_{q(\hat{l})}, Sm_{q(\hat{k})}, Sm_{q(\hat{k})})\}. \\ &= max\{G(m_{o(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}), G(m_{l(\hat{k})}, m_{l(\hat{k})+1}, m_{l(\hat{k})+1}), \\ &G(m_{q(\hat{l})}, m_{q(\hat{l})+1}, m_{q(\hat{l})+1})\}. \end{aligned}$$

Employing rectangle inequality repeatedly, we get

$$\begin{aligned} G(m_{o(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}) &\leq G(m_{o(\hat{l})}, m_{o(\hat{l})+1}, m_{o(\hat{l})+1}) + G(m_{o(\hat{l})+1}, m_{q(\hat{l})}, m_{q(\hat{l})}) \\ &\leq G(m_{o(\hat{l})}, m_{o(\hat{l})+1}, m_{o(\hat{l})+1}) + G(m_{o(\hat{l})+1}, m_{o(\hat{l})+2}, m_{o(\hat{l})+2}) \\ &+ G(m_{o(\hat{l})+2}, m_{q(\hat{l})}, m_{q(\hat{l})}) \\ &\leq [\sum_{i=l}^{p-1} G(m_{i(\hat{l})}, m_{i(\hat{l})+1}, m_{i(\hat{l})+1})] + G(m_{p(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{\hat{l}})}) \end{aligned}$$

or

$$G(m_{o(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}) - G(m_{p(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}) \leq [\sum_{i=l}^{p-1} G(m_{i(\hat{l})}, m_{i(\hat{l})+1}, m_{i(\hat{l})+1}]$$

Letting $\hat{l} \to \infty$, we get

$$lim_{\hat{l}\to\infty} G(m_{p(\hat{l})}, m_{q(\hat{l})}, m_{q(\hat{l})}) = \epsilon. \tag{4.14}$$

Using rectangular inequality, we get

$$G(m_{o(\hat{l})}, m_{q(\hat{l})+1}, m_{q(\hat{l})+1}) \leq G(m_{o(\hat{l})}, m_{q(\hat{l})+1}, m_{q(\hat{l})+1}) + G(m_{q(\hat{l})+1}, m_{q(\hat{l})+1}, m_{q(\hat{l})}).$$

On letting $\hat{l} \to \infty$

$$lim_{\hat{l}\to\infty} G(m_{o(\hat{l})}, m_{q(\hat{l})+1}, m_{q(\hat{l})}) = \epsilon. \tag{4.15}$$

If we move the limit as $\hat{l} \to \infty$ in (4.13) and use (4.14), (4.15), we get

$$\upsilon(\epsilon) = (\upsilon(max\{\epsilon, 0, 0\}) - (\psi(max\{\epsilon, 0, 0\})) = \upsilon(\epsilon) - \psi(\epsilon)$$

and hence $\upsilon(\epsilon) = 0$ or $\psi(\epsilon) = 0$, therefore $\epsilon = 0$ which is a conflict so that $\{m_r\}$ is not G - Cauchy. Therefore, $\{m_r\}$ is a CS.

Since, P and Q are complete, there is $\hat{z} \in P \subseteq P \cup Q$ such that $m_r \to \hat{z}$ as $r \to \hat{z}$.

On the contrary, $r \in \mathbb{N}$,

$$d_G(\hat{z}, Q) \leq d_G(\hat{z}, Sm_r) = d_G(\hat{z}, m_{r+1}) \leq d_G(\hat{z}, m_r) + d_G(m_r, m_{r+1}) \leq d_G(\hat{z}, m_r) + d_G(P, Q).$$

Putting limit as $r \to \infty$, we get

$$d_G(\hat{z}, Q) \leq limit_{r\to+\infty} d_G(\hat{z}, Sm_r) = d_G(P, Q) = d_G(\hat{z}, Q).$$

Because Q is about compact to P, so $\{Sm_r\}$ has a subsequence $\{Sm_{q(\hat{k})}\}$ converges to a certain $n^* \in Q \subset P \cup Q$.

$$d_G(\hat{z}, n^*) = lim_{r\to\infty} d_G(m_{q(\hat{l})}, Sm_{q(\hat{l})}) = d_G(P, Q)$$

and so $\hat{z} \in P_0$.

Now, since $\hat{z} \in S(P_0) \subseteq Q_0$, there exists $l \in P_0$ such that $d_G(l, S\hat{z}) = d_G(P, Q)$.

Now, we claim $o = \hat{z}$. For this, with $m = m_{r-1}$, $n = m_r$, $\hat{z} = m_r$, $\upsilon = o$, $\upsilon^* = \hat{z}$, we get

$$\begin{aligned} G(\hat{z}, o, m_r) &\leq \upsilon(M(m_{r-1}, m_r, m_r)) - \psi(M(m_{r-1}, m_r, m_r)) \\ \leq (\upsilon(max\{G(m_{r-1}, m_r, m_r), G(m_{r-1}, m_r, m_r), G(m_{r-1}, m_r, m_r)\} \\ -(max\{G(m_{r-1}, m_r, m_r), G(m_{r-1}, m_r, m_r), G(m_{r-1}, m_r, m_r)\}. \end{aligned}$$

Making $r \to \infty$, we have

$$G(\hat{z}, o, \hat{z}) \leq \upsilon(G(\hat{z}, \hat{z}, \hat{z})) - \psi(G(\hat{z}, \hat{z}, \hat{z}))$$

$$\Rightarrow (\upsilon(G(\hat{z}, o, \hat{z})) = 0$$

$$\Rightarrow \upsilon(0) = 0.$$

Then, $G(\hat{z}, o, \hat{z}) = 0$. i.e., $o = \hat{z}$. Thus, $d_G(\hat{z}, S\hat{z}) = d_G(P, Q)$.

Therefore, S has BPP. □

Theorem (4.2), we take $\upsilon(t_a) = t_a$ and $\psi(t_a) = (o - \hat{l})t_a$ where $\hat{l} \in (0, 1)$ and $0 \leq \hat{l} \leq 1$.

Corollary 4.3. *Let P, Q be two subsets of a G-M.S. (X, G) s.t. (P, G) is a complete G-M.S.. P_0 be non-empty and Q be approximately compact with respect to P.*

Let us suppose $S : P \to Q$ is a non-self mapping s.t. $S(P_0) \subseteq Q_0$ and $S(Q_0) \subseteq P_0$ and for $\upsilon, \upsilon^*, m \in P$ and $w, n \in Q$.

$$\begin{aligned} d_G(\upsilon^*, Sm) &= d_G(P, Q) \\ d_G(\upsilon, S\upsilon^*) &= d_G(P, Q) \\ d_G(w, Sn) &= d_G(P, Q) \\ \Rightarrow G(\upsilon^*, \upsilon, w) &\leq \hat{l}M(m, w, n), \end{aligned}$$

where $M(m, w, n) = max\{G(m, w, n), G(m, Sm, Sm), G(m, Sm, Sm)\}$ and $\hat{l} \in (0, 1)$.

Then, S has the BPP.

In this Theorem (4.2), we will use integral type functions to make a new Consequence.

Corollary 4.4. *Let P, Q be two non-void members of a G-M.S. (X, G) s.t. (P, G) is a full G-M.S., P_0 is non-empty and Q is essentially compact with respect to P.*

Suppose that $S : P \cup Q \to P \cup Q$ satisfying

1. *$S(P) \subseteq Q$, $S(Q) \subseteq P$,*

2.

$$d_G(v^*, Sm) = d_G(P, Q)$$
$$d_G(v, Sv^*) = d_G(P, Q)$$
$$d_G(w, Sn) = d_G(P, Q)$$
$$\Rightarrow \upsilon(\int \binom{G(v^*, v, w}{0} ds \leq \upsilon(M \int \binom{M(m, w, n)}{0} ds) - \psi(M \int \binom{M(m, w, n)}{0} ds),$$

where $\upsilon \in \Upsilon$ *and* $\psi \in \Psi$.

$M(m, w, n) = max\{G(m, w, n), G(m, Sm, Sm), G(n, Sn, Sn)\}$,

where $m, v, v^* \in P$, $w, n \in Q$. *Then,* S *has a BPP.*

One found certain fixed-point results as an application of our best results in proximity.

Note that if,

$$d_G(v^*, Sm) = d_G(P, Q)$$
$$d_G(v, Sv^*) = d_G(P, Q)$$
$$d_G(w, Sn) = d_G(P, Q)$$
$$\text{and } P = Q = X, \text{ then } v = Sm, v^* = Sv \text{ and } w = Sn.$$

That is, $v^* = S^2m$.

If we consider $P = Q = X$, in Theorem (4.2), we take below results:

Theorem 4.5. *Let* X *be a complete G-M.S. and the* S *self-map meets the conditions below;* $\forall\ m, n \in X$ *where* $\upsilon \in \Upsilon$ *and* $\psi \in \Psi$,

$$\upsilon(G(v^*, v, w) \leq \upsilon(M(m, w, n)) - \psi(M(m, w, n)).$$

$$\upsilon(G(S^2m, Sm, Sn)) \leq \upsilon(M(m, Sn, n)) - \psi(M(m, Sn, n))$$

. *Then,* S *has a unique fixed point.*

Corollary 4.6. *Let* (X, G) *be a complete G-M.S. and* S *be a map that meets the conditions* $\forall\ m, n \in X$, *where* $0 \leq s \leq 1$,

$$G(S^2m, Sm, Sn) \leq sG(m, Sn, n).$$

A unique fixed point S.

4.3 The Concept of $\mathcal{Z}$-Contract With Simulation Function

The well-known BCP [8] guarantees the presence and consistency of a fixed point of contraction on a CMS. Following that concept, many scholars extended this theory by adding the separate contractions on MS [14, 16, 26, 27, 40]. Within this work, we add a mapping method called SF and $\mathcal{Z}$-contraction notion.

Example 4.1. *Let* $(M, \mathfrak{d})$ *be the CMS then,* $\mathcal{G} : M \times M \times M \to [0, \infty)$ *set to* $\mathcal{G}(\beta, \omega, \delta) = max\{\mathfrak{d}(\beta, \omega), \mathfrak{d}(\omega, \delta), \mathfrak{d}(\delta, \beta)\}\ \forall\ \beta, \omega, \delta \in M$ *is a G-M.S..*

As we have already define SF (1.3)

Lemma 4.7. *Suppose* $(M, \mathcal{G})$ *is G-M.S., and* $f : M \to M$ *is about the contraction of* $\zeta \in \mathbb{Z}$*. Then,* f *approaches the rule at each* $\beta \in M$*.*

Proof. Let it be random, $\beta \in M$. If it is $p \in \mathbb{N}$, we have got

$f^p\beta = f^{p+1}\beta$, that is $f\omega = \omega$, where $\omega = f^{p-1}\beta$, that is $f\delta = \delta$, where $\delta = f^{p-1}\beta$

then, $f^n\omega = f^{n-1}f\omega = f^{n-1}\omega = ... = f\omega = \omega\ \forall\ n \in \mathbb{N}$. Next, for $n \in \mathbb{N}$, which is large enough, we get

$$\begin{aligned}
\mathcal{G}(f^n\beta, f^{n+1}\beta, f^{n+1}\beta) &= G(f^{n-p+1}f^{p-1}\beta, f^{n-p+2}f^{p-1}\beta, f^{n-p+2}f^{p-1}\beta) \\
&= \mathcal{G}(f^{n-p+1}\omega, f^{n-p+2}\omega, f^{n-p+2}\omega) \\
&= G(\omega, \omega, \omega) = 0.
\end{aligned}$$

Therefore, $lim_{n\to\infty}\mathcal{G}(f^n\beta, f^{n+1}\beta, f^{n+1}\beta) = 0$.

Suppose, $f^n\beta \neq f^{n-1}\beta$ for all $n \in \mathbb{N}$, then it follows from (1.19) that

$$\begin{aligned}
0 &\leq \zeta(\mathcal{G}(f^{n+1}\beta, f^n\beta, f^n\beta), \mathcal{G}(f^n\beta, f^{n-1}\beta, f^{n-1}\beta)) \\
&= \zeta(\mathcal{G}(ff^n\beta, ff^{n-1}\beta, ff^{n-1}\beta), \mathcal{G}(f^n\beta, f^{n-1}\beta, f^{n-1}\beta)) \\
&\leq \mathcal{G}(f^n\beta, f^{n-1}\beta, f^{n-1}\beta) - \mathcal{G}(f^{n+1}\beta, f^n\beta, f^n\beta).
\end{aligned}$$

This suggests that $\{\mathcal{G}(f^n\beta, f^{n-1}\beta, f^{n-1}\beta)\}$ is a sequence of non-negative real numbers that monotonically decreases, it has to be convergent.

Presume that $lim_{n\to\infty}\mathcal{G}(f^n\beta, f^{n+1}\beta, f^{n+1}\beta) = r \geq 0$. If $r > 0$ therefore f is $\mathcal{Z}$-contraction relative to ζ therefore, we have

$$0 \leq lim_{n\to\infty} sup\zeta(\mathcal{G}(f^{n+1}\beta, f^n\beta, f^n\beta), \mathcal{G}(f^n\beta, f^{n-1}\beta, f^{n-1}\beta)) < 0.$$

This contradiction shows that $r = 0$, i.e. $lim_{n\to\infty}\mathcal{G}(f^n\beta, f^{n+1}\beta, f^{n+1}\beta) = 0$. is shown by this inconsistency. Therefore, f at β is an asymptotic periodic map. □

Lemma 4.8. *Let $(M, \mathcal{G})$ be a G-M.S., $f : M \to M$ be a $\mathcal{Z}$-contraction. Then, $\{\beta_n\}$ Picard sequence with an initial value $\beta_0 \in M$ generated by f is a bounded sequence, where $\beta_n = f\beta_{n-1}$ is used for $n \in \mathbb{N}$.*

Proof. Let $\beta_0 \in M$ be random and let the Picard series be $\{\beta_n\}$, i.e. $\beta_n = f\beta_{n-1}$ $\forall$ $n \in \mathbb{N}$. Assume, on the other hand, here is no constraint to $\{\beta_n\}$. We can assume with WLOG that $\beta_{n+p} \neq \beta_n$ $\forall$ $n, p \in \mathbb{N}$. Since $\{\beta_n\}$ is not bounded, $\exists$ a subsequence $\{\beta_n\}$ occurs in such a way that $n_1 = 1$ and each $q^a \in \mathbb{N}$, n_{q^a+1} is the minimum integer s.t.

$$\mathcal{G}(\beta_{n(q^a)+1}, \beta_{n(q^a)}, \beta_{n(q^a)}) > 1$$

and

$$\mathcal{G}(\beta_m, \beta_{n(q^a)}, \beta_{n(q^a)}) \leq 1$$

for $n_{q^a} \leq m \leq n_{(q^a)+1} - 1$.

Therefore, with triangular inequality, we have

$$\begin{aligned}1 &< \mathcal{G}(\beta_{n(q^a)+1}, \beta_{n(q^a)}, \beta_{n(q^a)}) \leq \mathcal{G}(\beta_{n(q^a)+1}, \beta_{n(q^a)+1} - 1, \beta_{n(q^a)+1} - 1) + \mathcal{G}(\beta_{n(q^a)+1} - 1, \beta_{n(q^a)}, \beta_{n(q^a)})\\ &\leq \mathcal{G}(\beta_{n(q^a)+1}, \beta_{n(q^a)+1} - 1, \beta_{n(q^a)+1} - 1) + 1.\end{aligned}$$

Making $k \to \infty$ and use Lemma (4.7) we get

$$lim_{q^a\to\infty}\mathcal{G}(\beta_{n(q^a)+1}, \beta_{n(q^a)}, \beta_{n(q^a)}) = 1$$

By (1.19), we get $\mathcal{G}(\beta_{n(q^a)+1}, \beta_{n(q^a)}, \beta_{n(q^a)}) \leq \mathcal{G}(\beta_{n(q^a)+1} - 1, \beta_{n(q^a)-1}, \beta_{n(q^a)-1})$, therefore use the above triangular inequality, we obtain

$$\begin{aligned}1 &< \mathcal{G}(\beta_{n(q^a)+1}, \beta_{q^a}), \beta_{n(q^a)}) \leq \mathcal{G}(\beta_{n(q^a)+1} - 1, \beta_{n(q^a)-1}, \beta_{n(q^a)-1})\\ &\leq \mathcal{G}(\beta_{n(q^a)+1} - 1, \beta_{n(q^a)}, \beta_{n(q^a)}) + \mathcal{G}(\beta_{n(q^a)}, \beta_{n(q^a)-1}, \beta_{n(q^a)-1})\\ &\leq 1 + \mathcal{G}(\beta_{n(q^a)}, \beta_{n(q^a)-1}, \beta_{n(q^a)-1}).\end{aligned}$$

Letting $q^a \to \infty$ and using Lemma (4.7), we obtain

$$lim_{q^a \to \infty} \mathcal{G}(\beta_{n(q^a)+1} - 1, \beta_{n(q^a)-1}, \beta_{n(q^a)-1}) = 1.$$

Now, since f is a $\mathcal{Z}$-contraction, therefore, it become

$$\begin{aligned} 0 &\leq lim_{q^a \to \infty} sup \zeta(\mathcal{G}(f\beta_{n(q^a)+1} - 1, f\beta_{n(q^a)-1}, f\beta_{n(q^a)-1})) \\ &= lim_{q^a \to \infty} sup \zeta(\mathcal{G}(\beta_{n(q^a)+1}, \beta_{n(q^a)}, \beta_{n(q^a)}), \mathcal{G}(\beta_{n(q^a)+1} - 1, \beta_{n(q^a)-1}, \beta_{n(q^a)-1})) < 0. \end{aligned}$$

This contradiction proves result. □

4.4 Fixed Point Theorems Using Simulation Function for $\mathcal{Z}$-Contraction

Theorem 4.9. *Presume $(M, \mathcal{G})$ is a G-M.S. and $f : M \to M$ be a $\mathcal{Z}$-contraction. Then, f has a unique fixed point u in M and for every $\beta_0 \in \mathcal{X}$ the Picard sequence $\{\beta_n\}$ where $\beta_n = f\beta_{n-1}$ converges to the fixed point of f.*

Proof. Let $\beta_0 \in M$ be arbitrary and $\{\beta_n\}$ be the Picard sequence, that is, $\beta_n = f\beta_{n-1}$. We'll demonstrate that this sequence is a CS. Let's do it this way,

$$\mathcal{C}_n = sup\{\mathcal{G}(\beta_{\mathfrak{i}}, \beta_{\mathfrak{j}}, \beta_{\mathfrak{j}}) : \mathfrak{i}, \mathfrak{j} \geq n\}.$$

Note that $\{\beta_n\}$ is a monotonically positive real sequence and that the $\{\beta n\}$ sequence is bounded by Lemma (4.8), so $\mathcal{C}_n < \infty \; \forall \; n \in \mathbb{N}$. Thus, $\{\mathcal{C}_n\}$ is monotone bounded series, thus a convergent series, i.e. $\mathcal{C} \geq 0$ s.t. $lim_{n\to\infty} \mathcal{C}_n = \mathcal{C}$. We are expected to demonstrate that $\mathcal{C} = 0$. If $\mathcal{C} > 0$ then by the Definition $\mathcal{C}_n$, for every $q^a \in \mathbb{N} \; \exists \; m_{q^a} > n_{q^a} \geq q^a$ and

$$\mathcal{C}_{q^a} - \frac{1}{q^a} < \mathcal{G}(\beta_{m(q^a)}, \beta_{n(q^a)}, \beta_{n(q^a)}) \leq \mathcal{C}_{q^a}.$$

Hence,

$$lim_{q^a \to \infty} \mathcal{G}(\beta_{m(q^a)}, \beta_{n(q^a)}, \beta_{n(q^a)}) \leq \mathcal{C}_{q^a}. \tag{4.16}$$

Using (1.19) and the triangular inequality, we obtain

$$\begin{aligned} \mathcal{G}(\beta_{m(q^a)}, \beta_{n(q^a)}, \beta_{n(q^a)}) &\leq \mathcal{G}(\beta_{m(q^a)-1}, \beta_{n(q^a)-1}, \beta_{n(q^a)-1}) \\ &\leq \mathcal{G}(\beta_{m(q^a)-1}, \beta_{m(q^a)}, \beta_{m(q^a)}) + \mathcal{G}(\beta_{m(q^a)}, \beta_{n(q^a)}, \beta_{n(q^a)}) \\ &+ \mathcal{G}(\beta_{n(q^a)}, \beta_{n(q^a)-1}, \beta_{n(q^a)-1}). \end{aligned}$$

Using Lemma (4.7), (4.16) and letting $q^a \to \infty$ in the above inequality, it become

$$lim_{q^a \to \infty} \mathcal{G}(\beta_{m(q^a)-1}, \beta_{n(q^a)-1}, \beta_{n(q^a)-1}) = \mathcal{C}. \tag{4.17}$$

Since T is a $\mathcal{Z}$-contraction, therefore using (1.19), (4.16), (4.17) and (ζ_2), we get

$$0 \leq lim_{q^a \to \infty} sup \zeta(\mathcal{G}(\beta_{m(q^a)-1}, \beta_{n(q^a)-1}, \beta_{n(q^a)-1}), \mathcal{G}(\beta_{m(q^a)}, \beta_{n(q^a)}, \beta_{n(q^a)})) < 0.$$

This inconsistency shows that $\mathcal{C} = 0$ and thus $\{\beta_n\}$ is a CS. Since M a full $\mathcal{G}$-MS file, $u \in M$ exists, so $lim_{n \to \infty} \beta_n = u$. We can demonstrate that u is a fixed point of f. Suppose $fu \neq u$ then $\mathcal{G}(u, fu, fu) > 0$. Again, using (1.19), ζ_1, ζ_2 (1.3) already defined as in previous chapter, we have

$$\begin{aligned} 0 &\leq lim_{n \to \infty} sup \zeta(\mathcal{G}(f\beta_n, fu, fu), G(\beta_n, u, u)) \\ &\leq lim_{n \to \infty} sup \zeta[\mathcal{G}(\beta_n, u, u) - \mathcal{G}(\beta_{n=1}, fu, fu)] \\ &= -\mathcal{G}(u, fu, fu). \end{aligned}$$

This contradiction shows that $\mathcal{G}(u, fu, fu) = 0$, i.e. $fu = u$. Thus, u is a fixed point of f. □

Example 4.2. *Take $M = [0,1]$ and $\mathcal{G}$ be defined by $\mathcal{G}(\beta, \omega, \delta) = max\{|\beta - \omega|, |\omega - \delta|, |\delta - \beta|\}$. Then, $(M, \mathcal{G})$ is a complete G-M.S.. Define a mapping $f : M \to M$ as $f\beta = \frac{\beta}{\beta+1}$ for all $\beta \in M$. It is a $\mathcal{Z}$-contraction, where*

$$\zeta(\mathfrak{t}, \mathfrak{s}) = \frac{\mathfrak{s}}{\mathfrak{s}+1} - \mathfrak{t} \text{ for all } \mathfrak{t}, \mathfrak{s} \in [0, \infty).$$

Indeed, if $\beta, \omega \in M$, then by a simple calculation it can be shown that

$$\zeta(\mathcal{G}(f\beta, f\omega, f\delta), \mathcal{G}(\beta, \omega, \delta)) \geq 0.$$

Obviously, 0 is the f fixed point.

There are some consequences as follows:

Corollary 4.10. *Presume $(M, \mathcal{G})$ be a complete G-M.S. and $f : M \to M$ be a map which meets the condition: $\mathcal{G}(f\beta, f\omega, f\delta) \leq \lambda \mathcal{G}(\beta, \omega, \delta)$ for all $\beta, \omega, \delta \in M$, where λ is in the range [0, 1]. Then, in M, f has a single fixed point.*

1. *Define $\zeta_B : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by*

   $\zeta_B(\mathfrak{t}, \mathfrak{s}, \mathfrak{s}) = \lambda\mathfrak{s} - \mathfrak{t}$ $\forall$ $\mathfrak{s}, \mathfrak{t} \in [0, \infty)$. It's worth noting that the mapping f is a $\mathcal{Z}$-contraction in terms of $\zeta_B \in \mathcal{Z}$.

As a consequence of taking $\zeta = \zeta_B$ in Theorem (4.9), the outcome is as follows: .

2. *Define $\zeta_R : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by*

   $\zeta_R(\mathfrak{t}, \mathfrak{s}, \mathfrak{s}) = \mathfrak{s} - \varphi(\mathfrak{s}) - \mathfrak{t} \quad \forall \quad \mathfrak{s}, \mathfrak{t} \in [0,\infty).$

   It's important to note that the mapping f is a $\mathcal{Z}$-contraction w.r.t. $\zeta_R \in \mathcal{Z}$.

   Taking $\zeta = \zeta_R$ in Theorem (4.9) as an example, the result follows.

3. *Define $\zeta_R : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by*

   $\zeta_R(\mathfrak{t}, \mathfrak{s}, \mathfrak{s}) = \mathfrak{s}\varphi(\mathfrak{s}) - \mathfrak{t}$ *for all* $\mathfrak{s}, \mathfrak{t} \in [0,\infty).$

4. *Note that, the mapping f is a $\mathcal{Z}$-contraction w.r.t. $\zeta_R \in \mathcal{Z}$.*

   Therefore, the result follows by taking $\zeta = \zeta_R$ in Theorem (4.9).

   Define $\zeta_B W : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

   $\zeta_B W(\mathfrak{t}, \mathfrak{s}, \mathfrak{s}) = \mathfrak{s}\eta(\mathfrak{s}) - \mathfrak{t}$ *for all* $\mathfrak{s}, \mathfrak{t} \in [0,\infty).$

   Note that, the mapping f is a $\mathcal{Z}$-contraction with respect to $\zeta_B W \in \mathcal{Z}$.

   Therefore, the result follows by taking $\zeta = \zeta_B W$ in Theorem (4.9).

5. *Define $\zeta_K : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by*

   $\zeta_K(\mathfrak{t}, \mathfrak{s}, \mathfrak{s}) = \mathfrak{s} - \int \binom{\mathfrak{t}}{0} \phi(u)du$ *for all* $\mathfrak{s}, \mathfrak{t} \in [0,\infty).$

   Then, $\zeta_K \in \mathcal{Z}$.

   Therefore, the result follows by taking $\zeta = \zeta_K$ in Theorem (4.9).

4.5 New Concepts of $(\alpha_b - \psi_b)$ Contractive Pair of Mappings in Generalized Metric Space

Branciari [15] has implemented the g.m.s. definition. Thus, any MS is a g.m.s., but the converse is not valid for [29]. In such a space he proved the Banach FPT. The reader might refer to [31, 34, 37, 42] for more information. We remember the notion that Branciari [15] implemented a g.m.s..

Proposition Let $\{\bar{\gamma}_\mathfrak{n}\}$ is a CS in a g.m.s. $(M, \tilde{d})$ with $lim_{m\to\infty}\tilde{d}(\bar{\gamma}_\mathfrak{n}, \sqcap) = 0$, where $\sqcap \in X$. At that point $lim_{m\to\infty}\tilde{d}(\bar{\gamma}_\mathfrak{n}, \delta) = \tilde{d}(\sqcap, \delta)$, for all $\delta \in M$. In Particular, $\{\bar{\gamma}_\mathfrak{n}\}$ series does not converge to δ if $\delta \neq \sqcap$.

As earlier in the previous chapter, we have already defined SF (1.3). We start this by introducing the new concepts of $\alpha_b - \psi_b$ contractive pair of mappings.

Definition 4.11. Let $(\mathfrak{X}, \tilde{d})$ be a g.m.s., $\hat{S} : \mathfrak{X} \times \mathfrak{X}$ be a map. We claim that $\hat{S}$ is a generalized (α_b, ψ_b)-type-I-contractive mapping regards ζ and $\zeta \in \mathbb{Z}$ if there are $\alpha_b :_X \times \mathfrak{X} \to [0, \infty)$ and $\psi_b \in \Psi_b$ s.t.

$$\zeta(\alpha_b(k,l)\tilde{d}(\hat{S}k, \hat{S}l), \psi_b(M_1(k,l))) \geq 0,$$

$$\alpha_b(k,l)\tilde{d}(\hat{S}k, \hat{S}l) \leq \psi_b(M_1(k,l)), \text{ for all } k, l \in \mathfrak{X}, \tag{4.18}$$

where

$$M_1(k,l) = max\{\tilde{d}(k,l), \tilde{d}(k, \hat{S}k), \tilde{d}(l, \hat{S}l)\}. \tag{4.19}$$

Definition 4.12. Assume $(\mathfrak{X}, \tilde{d})$ be a g.m.s. and $\hat{S}$ be a mapping. We say that $\hat{S}$ is a generalized (α_b, ψ_b)-type-II-contractive mapping and $\zeta \in \mathbb{Z}$ if there are two functions α_b and $\psi_b \in \Psi_b$ s.t.

$$\zeta(\alpha_b(k,l)\tilde{d}(\hat{S}k, \hat{S}l), \psi_b(N_1(k,l))) \geq 0,$$

$$\alpha_b(k,l)\tilde{d}(\hat{S}k, \hat{S}l) \leq \psi_b(N_1(k,l)), \text{ for all } k, l \in \mathfrak{X}, \tag{4.20}$$

where

$$N_1(k,l) = max\{\tilde{d}(k,l), \frac{\tilde{d}(k, \hat{T}k) + \tilde{d}(l, \hat{T}l)}{2}\}. \tag{4.21}$$

4.6 Fixed Point Theorems for class-(i), class-(ii) with α_b-Admissible Mapping

Theorem 4.13. *Let the g.m.s. be $(\mathfrak{X}, \tilde{d})$, and $\hat{S} : \mathfrak{X} \times \mathfrak{X}$ be the mapping provided. We are claiming $\hat{S}$ is a (α_b, ψ_b) -class-(i)-contractive mapping generalised. Assume that the fact is*

1. *$\hat{S}$ is α_b-admissible;*
2. *there is $k_0 \in \mathfrak{X}$ s.t. $\alpha_b(k_0, \hat{S}k_0) \geq 1$ and $\alpha_b(k_0, \hat{S}^2 k_0) \geq 1$;*
3. *$\hat{S}$ is constant.*

Therefore, $v \in \mathfrak{X}$ occurs such that $\hat{S}v = v$.

Proof. There is one point, by assumption (2), $k_0 \in \mathfrak{X}$ s.t. $\alpha_b(k_0, \hat{S}k_0) \geq 1$ and $\alpha_b(k_0, \hat{S}^2 k_0) \geq 1$. We have a sequence specified as $\{k_t\}$ in $\mathfrak{X}$ by $k_{t+1} = \hat{S}k_t = \hat{S}^{t+1}k_0$, $\forall\ t \geq 0$. Expect that $k_{t_0} = k_{t_0+1}$ for some t_0. Since $v = k_{t_0} = k_{t_0+1} = \hat{S}k_{t_0} = \hat{S}v$. Therefore, all through the verification, we assume that

$$k_t \neq k_{t+1} \text{ for all } t. \tag{4.22}$$

Look out for this

$$\alpha_b(k_0, k_1) = \alpha_b(k_0, \hat{S}k_0) \geq 1 \Rightarrow \alpha_b(\hat{S}k_0, \hat{S}k_1) = \alpha_b(k_1, k_2) \geq 1,$$

Since $\hat{S}$ is α_b-admissible, we infer

$$\alpha_b(k_t, k_{t+1}) \geq 1, \text{ for all } t = 0, 1, 2, ... \tag{4.23}$$

By utilizing a similar method, we get

$$\alpha_b(k_0, k_2) = \alpha_b(k_0, \hat{S}^2k_0) \geq 1 \Rightarrow \alpha_b(\hat{S}k_0, \hat{S}k_2) = \alpha_b(k_1, k_2) \geq 1,$$

The expression above yields

$$\alpha_b(k_t, k_{t+2}) \geq 1, \text{ for all } m = 0, 1, 2, ... \tag{4.24}$$

Step I: We'll show

$$lim_{t\to\infty}\tilde{d}(k_t, k_{t+1}) = 0. \tag{4.25}$$

Combining (4.18) and (4.23), we find that

$$\begin{aligned} 0 &\leq \zeta(\alpha_b(k_{t-1}, k_m)\tilde{d}(\hat{S}k_{t-1}, \hat{S}k_m)), \psi_b(M_1(k_{t-1}, k_t))) \\ &< \psi_b(M_1(k_{t-1}, k_t)) - \alpha_b(k_{t-1}, k_t)\tilde{d}(\hat{S}k_{t-1}, \hat{S}k_t) \end{aligned}$$

$$\alpha_b(k_{t-1}, k_t)\tilde{d}(\hat{S}k_{t-1}, \hat{S}k_t) \leq \psi_b(M_1(k_{t-1}, k_t))$$

$$\tilde{d}(k_t, k_{t+1}) = \tilde{d}(\hat{S}k_{t-1}, \hat{S}k_t) \leq \alpha_b(k_{t-1}, k_t)\tilde{d}(\hat{S}k_{t-1}, \hat{S}k_t) \leq \psi_b(M_1(k_{t-1}, k_t)), \tag{4.26}$$

for all $t \geq 1$, where

$$\begin{aligned} M_1(k_{t-1}, k_t) &= max\{\tilde{d}(k_{t-1}, k_t), \tilde{d}(k_{t-1}, \hat{S}k_{t-1}), \tilde{d}(k_t, \hat{S}k_t)\} \\ &= max\{\tilde{d}(k_{t-1}, k_t), \tilde{d}(k_{t-1}, k_t), \tilde{d}(k_t, k_{t+1})\} \\ &= max\{\tilde{d}(k_{t-1}, k_t), \tilde{d}(k_t, k_{t+1})\}. \end{aligned} \tag{4.27}$$

If for some t, $M_1(k_{t-1}, k_t) = \tilde{d}(k_t, k_{t+1})(\neq 0)$, then the inequality (4.26) turns into

$$\tilde{d}(k_t, k_{t+1}) \leq \psi_b(M_1(k_{t-1}, k_t)) = \psi_b(\tilde{d}(k_t, k_{t+1}) < \tilde{d}(k_t, k_{t+1}),$$

a contradiction. Hence $M_1(k_{t-1},k_t)=\tilde{d}(k_{t-1},k_t)$, for all $t\in\mathbb{N}$, and (4.26) becomes

$$\begin{aligned}0 &\leq \zeta(\tilde{d}(k_t,k_{t+1}),\psi_b(\tilde{d}(k_{t-1},k_m)))\\ &< \psi_b(\tilde{d}(k_{t-1},k_t))-\tilde{d}(k_t,k_{t+1})\\ \tilde{d}(k_t,k_{t+1}) &\leq \psi_b(\tilde{d}(k_{t-1},k_t)),\ \text{for all } t\in\mathbb{N}. \end{aligned} \tag{4.28}$$

This yields

$$\begin{aligned}0 &\leq \zeta(\tilde{d}(k_t,k_{t+1}),\tilde{d}(k_{t-1},k_t))\\ &< \tilde{d}(k_t,k_{t+1})-\tilde{d}(k_t,k_{t+1})\\ \tilde{d}(k_t,k_{t+1}) &\leq \tilde{d}(k_{t-1},k_t),\ \text{for all } t\in\mathbb{N}. \end{aligned} \tag{4.29}$$

By (4.28), we have

$$\begin{aligned}0 &\leq \zeta(\tilde{d}(k_t,k_{t+1}),\psi_b^t(\tilde{d}(k_0,k_1)))\\ &< \psi_b^t(\tilde{d}(k_0,k_1))-\tilde{d}(k_t,k_{t+1})\\ \tilde{d}(k_t,k_{t+1}) &\leq \psi_b^t(\tilde{d}(k_0,k_1)),\ \text{for all } t\in\mathbb{N}. \end{aligned} \tag{4.30}$$

Through the ψ_b property, it is obvious that

$$lim_{m\to\infty}\tilde{d}(k_t,k_{t+1})=0.$$

Step II: We will show

$$lim_{t\to\infty}\tilde{d}(k_t,k_{t+2})=0. \tag{4.31}$$

By (4.18) and (4.24), we get

$$\begin{aligned}0 &\leq \zeta(\alpha_b(k_{t-1},k_{t+1})\tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1}),\psi_b(M_1(k_{t-1},k_{t+1})))\\ &< \psi_b(M_1(k_{t-1},k_{t+1}))-\alpha_b(k_{t-1},k_{t+1})\tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1})\\ \alpha_b(k_{t-1},k_{t+1})\tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1}) &\leq \psi_b(M_1(k_{t-1},k_{t+1})).\end{aligned}$$

$$\begin{aligned}\tilde{d}(k_t,k_{t+2}) &= \tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1})\leq \alpha_b(k_{t-1},k_{t+1})\tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1})\\ &\leq \psi_b(M_1(k_{t-1},k_{t+1})),\end{aligned} \tag{4.32}$$

for all $t\geq 1$, where

$$\begin{aligned}M_1(k_{t-1},k_t) &= max\{\tilde{d}(k_{t-1},k_{t+1}),\tilde{d}(k_{t-1},\hat{S}k_{t-1}),\tilde{d}(k_{t+1},k_{t+2})\}\\ &= max\{\tilde{d}(k_{t-1},k_{t+1}),\tilde{d}(k_{t-1},k_t),\tilde{d}(k_{t+1},k_{t+2}).\end{aligned} \tag{4.33}$$

By (4.31), we have

$$M_1(k_{t-1}, k_{t+1}) = max\{\tilde{d}(k_{t-1}, k_{t+1}), \tilde{d}(k_{t-1}, k_t)\}.$$

Thus, from (4.33)

$$b_t = \tilde{d}(k_t, k_{t+2}) \leq \psi_b(M_1(k_{t-1}, k_{t+1})) = \psi_b(max\{b_{t-1}, c_{t-1}\}), \text{ for all } t \in \mathbb{N}. \quad (4.34)$$

Again, by (4.31)

$$c_t \leq c_{t-1} \leq max\{b_{t-1}, c_{t-1}\}.$$

Therefore, the $max\{b_t, c_t\}$ sequence is non-increasing in monotony, and it converges to any $t \geq 0$. Suppose, $r > 0$.
Now, by (4.25)

$$lim_{t\to\infty} b_t = lim_{t\to\infty} sup max\{b_t, c_t\} = lim_{t\to\infty} max\{b_t, c_t\} = r.$$

Putting $m \to \infty$ in (4.34), we get

$$\begin{aligned} z = lim_{t\to\infty} b_t &\leq lim_{t\to\infty} sup \psi_b(max\{b_{t-1}, c_{t-1}\}) \\ &\leq \psi_b(lim_{t\to\infty} max\{b_{t-1}, c_{t-1}\}) \\ &= \psi_b(r) < r, \end{aligned}$$

which appeared to be a contradiction.
Step III: We'll show

$$k_t \neq k_j, \text{ every } t \neq j. \quad (4.35)$$

For all of that $t, j \in \mathbb{N}$, presume $k_t = k_j$ with $t \neq j$. Since $\tilde{d}(k_s, k_{s+1}) > 0$, for each $s \in \mathbb{N}$. without loss of consensus, we may expect that $j > t + 1$.
Examine it next,

$$\begin{aligned} 0 &\leq \zeta(\alpha_b(k_{j-1}, k_j)\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j), \psi_b(M_1(k_{j-1}, k_j))) \\ &< \psi_b(M_1(k_{j-1}, k_j)) - \alpha_b(k_{j-1}, k_j)\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \end{aligned}$$

$$\alpha_b(k_{j-1}, k_j)\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \leq \psi_b(M_1(k_{j-1}, k_j))$$

$$\begin{aligned} \tilde{d}(k_t, k_{t+1}) = \tilde{d}(k_t, \hat{S}k_t) = \tilde{d}(k_j, \hat{S}k_j) &= \tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \leq \alpha_b(k_{j-1}, k_j)\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \\ &\leq \psi_b(M_1(k_{j-1}, k_j)). \end{aligned} \quad (4.36)$$

where

$$\begin{aligned} M_1(k_{j-1},k_j) &= max\{\tilde{d}(k_{j-1},k_j),\tilde{d}(k_{j-1},\hat{S}k_{j-1}),\tilde{d}(k_j,\hat{S}k_j)\} \\ &= max\{\tilde{d}(k_{j-1},k_j),\tilde{d}(k_{j-1},k_j),\tilde{d}(k_j,\hat{S}k_j)\} \\ &= max\{\tilde{d}(k_{j-1},k_j),\tilde{d}(k_j,k_{j+1})\}. \end{aligned} \tag{4.37}$$

If $M_1(k_j,k_{j-1})=\tilde{d}(k_{j-1},k_j)$, then from (4.36), we get

$$\begin{aligned} \tilde{d}(k_t,k_{t+1}) &= \tilde{d}(k_t,\hat{S}k_t)=\tilde{d}(k_l,\hat{S}k_j) \\ &= \tilde{d}(k_j,k_{j+1})\leq \alpha_b(k_j,k_{j+1})\tilde{d}(\hat{S}k_{j-1},\hat{S}k_j) \\ &\leq \psi_b(M_1(k_{t+1},k_t))=\psi_b(\tilde{d}(k_{t+1},k_t)) \\ &\leq \psi_b^{j-t}(\tilde{d}(k_t,k_{t+1})). \end{aligned} \tag{4.38}$$

If $M_1(k_{j-1},k_j)=\tilde{d}(k_j,k_{j+1})$, (4.36) becomes

$$\begin{aligned} \tilde{d}(k_t,k_{t+1}) &= \tilde{d}(k_t,\hat{S}k_t)=\tilde{d}(k_j,\hat{S}k_j) \\ &= \tilde{d}(\hat{S}k_{j-1},\hat{S}k_j)\leq \alpha_b(k_{j-1},k_j)\tilde{d}(\hat{S}k_{j-1},\hat{S}k_j) \\ &\leq \psi_b(M_1(k_{j-1},k_j))=\psi_b(\tilde{d}(k_j,k_{j+1})) \\ &\leq \psi_b^{j-t+1}(\tilde{d}(k_t,k_{t+1})). \end{aligned} \tag{4.39}$$

Due to a property of ψ_b, (4.38) and (4.39) together yields

$$\tilde{d}(k_t,k_{t+1})\leq \psi_b^{j-t}(\tilde{d}(k_t,k_{t+1}))<\tilde{d}(k_t,k_{t+1}) \tag{4.40}$$

and

$$\tilde{d}(k_t,k_{t+1})\leq \psi_b^{j-t+1}(\tilde{d}(k_t,k_{t+1}))<\tilde{d}(k_t,k_{t+1}), \tag{4.41}$$

respectively. There is a contradiction in each case.
Step IV: We must show $\{k_t\}$ to be a CS, that is,

$$lim_{t\to\infty}\tilde{d}(k_t,k_{t+h^*})=0,\ \text{for all } h^*\in\mathbb{N}. \tag{4.42}$$

Two cases arise: $h^*=1$ and $h^*=2$ are proved by (4.25) and (4.31) respectively. Now, carry on the arbitrary $h^*\geq 3$. Two situations are plenty to look at.
Situation(I): Expect that $h^*=2l+1$, where $j\geq 1$. Next, along with Phase-III and

Quadrilateral Inequality (4.30), we consider

$$\begin{aligned}\tilde{d}(k_t, k_{t+h^*}) &= \tilde{d}(k_t, kt+2j+1) \leq \tilde{d}(k_t, k_{t+1}) + \tilde{d}(k_{t+1}, k_{t+2}) + ... + \tilde{d}(k_{t+2j}, k_{t+2j+1}) \\ &\leq \sum_{p=t+2}^{t+2j-1} \psi_b^p(\tilde{d}(k_0, k_1)) \\ &\leq \sum_{p=t}^{+\infty} \psi_b^p(\tilde{d}(k_0, k_1)) \to 0 \text{ as } t \to \infty. \end{aligned} \tag{4.43}$$

Case (II): Assume $h^* = 2j$, where $j \geq 2$ is. By the implementation of quadrilateral inequalities and step-III along with (4.30), we consider again

$$\begin{aligned}\tilde{d}(k_t, k_{t+h^*}) &= \tilde{d}(k_t, kt+2j) \leq \tilde{d}(k_t, k_{t+1}) + \tilde{d}(k_{t+1}, k_{t+2}) + ... + \tilde{d}(k_{t+2j-1}, k_{t+2j}) \\ &\leq \tilde{d}(k_t, k_{t+2}) + \sum_{p=t}^{t+2j} \psi_b^p(\tilde{d}(k_0, k_1)) \\ &\leq \tilde{d}(k_t, k_{t+2}) + \sum_{p=t}^{+\infty} \psi_b^p(\tilde{d}(k_0, k_1)) \to 0 \text{ as } t \to \infty. \end{aligned} \tag{4.44}$$

Now, from these two expressions (4.43) and (4.44), we have

$$lim_{m\to\infty}\tilde{d}(k_j, k_{j+h^*}) = 0, \text{ for all } h^* \geq 3.$$

We conclude that a CS in $(\mathfrak{X}, \tilde{d})$ is $\{k_t\}$. Due to the completeness of $(\mathfrak{X}, \tilde{d})$, it occurs in such a way that $v \in \mathfrak{X}$ occurs

$$lim_{t\to\infty}\tilde{d}(k_t, v) = 0. \tag{4.45}$$

Because $\hat{S}$ is continuous, we get that from (4.45)

$$lim_{t\to\infty}\tilde{d}(k_{t+1}, \hat{S}v) = lim_{t\to\infty}\tilde{d}(\hat{S}k_t, \hat{S}v) = 0, \tag{4.46}$$

that is, $lim_{t\to\infty}k_{t+1} = \hat{S}v$.
Considering Proposition (4.5), we infer that $\hat{S}v = v$, i.e. v be fixed point of $\hat{S}$.
The below sentence is taken from the (4.13) Theorem due to the inequality of $N_1(k, l) \leq M_1(k, l)$. □

Theorem 4.14. *Let the g.m.s. be $(\mathfrak{X}, \tilde{d})$ and $\hat{S} : \mathfrak{X} \times \mathfrak{X}$ be the mapping provided. Expect that $\hat{S}v = v$ be fixed point of $\hat{S}$. We say that $\hat{S}$ is a generalized (α_b, ψ_b)-class-(ii)-contractive mapping. Assume that*

1. *$\hat{S}$ is α_b-admissible;*
2. *there is $k_0 \in \hat{S}$ such that $\alpha_b(k_0, \hat{S}k_0) \geq 1$ and $\alpha_b(k_0, \hat{S}^2k_0) \geq 1$;*

3. $\hat{S}$ is constant.

There is then $v \in \mathfrak{X}$ such that $\hat{S}v = v$.

Theorem 4.15. *If $\hat{S}$ is a generalized (α_b, ψ_b)-class-(i)-contractive mapping on g.m.s. $(\mathfrak{X}, \tilde{d})$. Assume that*

1. *$\hat{S}$ is α_b-admissible;*
2. *there is $k_0 \in \mathfrak{X}$ s.t. $\alpha_b(k_0, \hat{S}k_0) \geq 1$ and $\alpha_b(k_0, \hat{S}^2 k_0) \geq 1$;*
3. *if $\{k_t\}$ is a $\mathfrak{X}$ sequence like $\alpha_b(k_t, k_{t+1}) \geq 1$, $\forall$ t and $k_t \to k \in \mathfrak{X}$ as $t \to \infty$, then there is a $\{k_t(h^*)\}$ subsequence of $\{k_t\}$, like $\alpha_b(k_t(h^*), x) \geq 1$, $\forall$ h^*.*

So $v \in \mathfrak{X}$ exists, such that $\hat{S}v = v$.

Proof. We know the $\{k_t\}$ series defined by $k_{t+1} = \hat{S}k_t$ $\forall$ $t \geq 0$is a CS and converges to some $v \in X$. Provided the Preposition (4.5),

$$lim_{h^* \to \infty} \tilde{d}(k_{t(h^*)+1}, \hat{S}v) = \tilde{d}(v, \hat{S}v). \tag{4.47}$$

Now, we 're going to know $\hat{S}v = v$. On the opposite, assume that $\hat{S}v \neq v$, so $\tilde{d}(\hat{S}v, v) > 0$. The subsequence $\{k_t(h^*)\}$ of $\{k_t\}$ occurs from (4.23) and (3) in such a way that $\alpha_b(k_t(h^*), v) \geq 1$, for all h^*.
By applying (4.18), we get

$$\begin{aligned} 0 &\leq \zeta((\alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, v)), \psi_b(M_1(k_{t(h^*)}, v))) \\ &< \psi_b(M_1(k_{t(h^*)}, v)) - \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, v) \\ \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, v) &\leq \psi_b(M_1(k_{t(h^*)}, v)) \end{aligned}$$

$$\tilde{d}(k_{t(h^*)+1}, \hat{S}v) \leq \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, \hat{S}v) \leq \psi_b(M_1(k_{t(h^*)}, v)), \tag{4.48}$$

where

$$\begin{aligned} M_1(k_{t(h^*)}, v) &= max\{\tilde{d}(k_{t(h^*)}, v), \tilde{d}(k_{t(h^*)}, \hat{S}k_{t(h^*)}), \tilde{d}(v, \hat{S}v)\} \\ &= max\{\tilde{d}(k_{t(h^*)}, v), \tilde{d}(k_{t(h^*)}, k_{t(h^*)+1}), \tilde{d}(v, \hat{S}v)\}. \end{aligned} \tag{4.49}$$

By (4.25) and (4.47), we have

$$lim_{h^* \to \infty} M_1(k_{t(h^*)}, v) = \tilde{d}(v, \hat{S}v). \tag{4.50}$$

Making $h^* \to \infty$ in (4.48) and regarding that ψ_b is upper semi continuous

$$\tilde{d}(v, \hat{S}v) \leq \psi_b(\tilde{d}(v, \hat{S}v)) < \tilde{d}(v, \hat{S}v), \tag{4.51}$$

That's one contradiction. But we consider v to be a fixed point of $\hat{S}$, that is, $\hat{S}v = v$. The upper semi-continuity hypothesis of ψ_b is not needed below. For the generalized class-(ii) $\alpha_b - \psi_b$ contractive mappings we have the following, which is similar to (4.15), we have the following for the generalized class-(ii) .

Theorem 4.16. *If $\hat{S}$ is generalized (α_b, ψ_b)-type-II-contractive pair of mappings on g.m.s. $(\mathfrak{X}, \tilde{d})$,*

1. *$\hat{S}$ is α_b-admissible;*
2. *$k_0 \in \mathfrak{X}$ exists s.t. $\alpha(k_0, \hat{S}k_0) \geq 1$ and $\alpha(k_0, \hat{S}^2k_0) \geq 1$ are available;*
3. *if $\{k_t\}$ is a sequence in $\mathcal{X}$ s.t. $\alpha_b(k_t, k_{t+1}) \geq 1$, $\forall\ t$ and $k_t \to \mathfrak{X} \in \mathfrak{X}$ as $t \to \infty$, then $\exists$ a subsequence $\{k_t(h^*)\}$ of $\{k_t\}$ s.t. $\alpha_b(k_t(h^*), v) \geq 1$, for all h^*.*

Then $\exists\ v \in \mathfrak{X}$ s.t. $\hat{S}v = v$.

We know that the sequence $k_{m+1} = \hat{S}k_m\ \forall\ m \geq 0$ is cauchy and converges to some $v \in \mathfrak{X}$ after proof of this theorem is the same as the Theorem (4.15). Similarly, in Proposition (4.5), we obtain

$$lim_{h^*\to\infty}\tilde{d}(k_{t(h^*)+1}, \hat{S}v) = \tilde{d}(v, \hat{S}v). \tag{4.52}$$

We will show that $\hat{S}v = v$. Assume that $\hat{S}v \neq v$. From (4.23) and condition (3), there is a $\{k_t(h^*)\}$ subsequence to $\{k_t\}$ such that $\alpha_b(k_t(h^*), v) \geq 1$, for all h^*. By applying (4.20), for all h^*, we get

$$\begin{aligned} 0 &\leq \zeta(\alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, S^*v), \psi_b(N_1(k_{t(h^*)}, v))) \\ &< \psi_b(N_1(k_{t(h^*)}, v)) - \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, \hat{S}v) \end{aligned}$$
$$\alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, \hat{S}v) \leq \psi_b(N_1(k_{t(h^*)}, v))$$

$$\tilde{d}(k_{t(h^*)+1}, \hat{S}v) \leq \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, \hat{S}v) \leq \psi_b(N_1(k_{t(h^*)}, v)), \tag{4.53}$$

where

$$N_1(k_{t(h^*)}, v) = max\{\tilde{d}(k_{t(h^*)}, v), \frac{\tilde{d}(k_{t(h^*)}, \hat{S}k_{t(h^*)}) + \tilde{d}(v, \hat{S}v)}{2}\}. \tag{4.54}$$

Letting $h^* \to \infty$ in (4.53), we have

$$lim_{h^* \to \infty} N_1(k_{t(h^*)}, v) = \frac{\tilde{d}(v, \hat{S}v)}{2}. \tag{4.55}$$

From (4.55), for a sufficiently large h^*, we have $N_1(k_{t(h^*)}, v) > 0$, which means

$$\begin{aligned} 0 &\leq \zeta(\psi_b(N_1(k_{t(h^*)}, v)), N_1(k_{t(h^*)}, v)) \\ &< N_1(k_{t(h^*)}, v) - \psi_b(N_1(k_{t(h^*)}, v)) \\ \psi_b(N_1(k_{t(h^*)}, v)) &\leq N_1(k_{t(h^*)}, v). \end{aligned}$$

We have h^* big enough from (4.55),

$$\psi_b(N_1(k_{t(h^*)}, v)) < N_1(k_{t(h^*)}, v).$$

Thus, from (4.53) and (4.55), we have

$$\tilde{d}(v, \hat{S}v) \leq \frac{\tilde{d}(v, \hat{S}v)}{2},$$

this's the fallacy.

We therefore consider v to be $\hat{S}$ as a fixed point. And that is, $\hat{S}v = v$. □

Chapter 5

Theorems on fixed points in Partial Metric Space

We are showing some common FPT in PMS with cyclic rational contraction in this section, adding some new sorts of cyclic (ψ_1, ϕ_1, I, J)-rational contraction in PMS settings, an altering distance function. There's also an illustration given to help our findings and validated implementation of the data. It is composed of two parts. In first part, we introduce the new notions of cyclic (ψ_1, ϕ_1, I, J)-rational contraction in the settings of PMS. In the second part, we prove certain theorems of fixed points in PMS. There is one example and one application of key cyclic contraction results.

5.1 New Notions of Cyclic (ψ_1, ϕ_1, I, J)-Rational Contraction

In PMS, we introduce the new notions of cyclic (ψ_1, ϕ_1, I, J)-rational contraction. Agarwal et al. [2] began the FPT analysis for complete PMS mappings that satisfy cyclically generalised contractive conditions. Matthews [39] presented the definition of PMS.

Now we implement the modern notion of cyclic (ψ_1, ϕ_1, I, J)-rational contraction in PMS, mentioned earlier:

Definition 5.1. Suppose I, J be two disjoint members of a PMS $(X, \mathfrak{p})$. An $R : X \to X$ mapping is referred to as a cyclical (ψ_1, ϕ_1, I, J)-rational contraction if

1. ψ_1 and ϕ_1 are changing the distance function.

2. Cyclic representation of $I \cup J$ is w.r.t. R;

In other terms $R(I) \subseteq J$ and $R(J) \subseteq I$.

3.

$$\begin{aligned} \psi_1(\mathfrak{p}(R\hat{x}, R\hat{y})) &\leq \psi_1(\frac{\mathfrak{p}(\hat{x}, R\hat{x})\mathfrak{p}(\hat{x}, R\hat{y}) + \mathfrak{p}(\hat{y}, R\hat{y})\mathfrak{p}(\hat{y}, R\hat{x})}{\mathfrak{p}(\hat{x}, R\hat{y}) + \mathfrak{p}(\hat{y}, R\hat{x})}) \\ &- \phi_1(\frac{\mathfrak{p}(\hat{x}, R\hat{x})\mathfrak{p}(\hat{x}, R\hat{y}) + \mathfrak{p}(\hat{y}, R\hat{y})\mathfrak{p}(\hat{y}, R\hat{x})}{\mathfrak{p}(\hat{x}, R\hat{y}) + \mathfrak{p}(\hat{y}, R\hat{x})}) \end{aligned} \tag{5.1}$$

for all $\hat{x} \in I$ and $\hat{y} \in J$.

5.2 Theorems of fixed point in PMS with cyclic Rational contraction

Theorem 5.2. *Let I and J be non-empty, with $(X, \mathfrak{p})$ PMS subsets closed. If $R : X \to X$ is a cyclic (ψ_1, ϕ_1, I, J) - rational contraction, then R has a CFP $w \in I \cap J$.*

Proof. Let us have $\hat{x}_0 \in I$. Because $RI \subseteq J$, we pick $\hat{x}_1 \in J$ to $R\hat{x}_0 = \hat{x}_1$. Also, we choose $\hat{x}_2 \in I$ since $RJ \subseteq I$ such that $R\hat{x}_1 = \hat{x}_2$. We will start with this method and create sequences $\{\hat{x}_l\}$ in X such that $\hat{x}_{2l} \in I$, $\hat{x}_{2l+1} \in J$, $\hat{x}_{2l+1} = R\hat{x}_{2l}$ and $\hat{x}_{2l+2} = R\hat{x}_{2l+1}$. If $\hat{x}_{2l_0+1} = x_{2l_0+2}$ for some $l \in N$, then $\hat{x}_{2l_0+1} = R\hat{x}_{2l_0+1}$. Therefore, $\hat{x}_{2l_0+1}$ is a fixed point of R in $I \cap J$. And we would say $\hat{x}_{2l+1} \neq \hat{x}_{2l+2}$ $\forall$ $l \in \mathbb{N}$.

Provided with $l \in N$. If l is even, then for any $l = 2\tilde{\alpha}$, $\tilde{k} \in \mathbb{N}$.

$$\begin{aligned} &\psi_1(\mathfrak{p}(\hat{x}_{l+1}, \hat{x}_{l+2})) \\ &= \psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})) \\ &= \psi_1(\mathfrak{p}(R\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}+1})) \\ &\leq \psi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}+1}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}+1})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}+1}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}})}) \\ &- \phi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}+1}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}+1})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}+1}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}})}) \\ &= \psi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+1})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+1})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+1})}) \\ &- \phi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+1})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+1})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+1})}). \end{aligned}$$

If $\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+1}) \leq \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})$, then

$$\begin{aligned}
&\psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})) \\
&\qquad = \psi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+1})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+1})}) \\
&\qquad - \phi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+1})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+1})}) \\
&\qquad = \psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})) - \phi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})) \\
&\qquad \leq (\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})) \\
&\qquad < \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2}),
\end{aligned}$$

then,

$\psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2al\tilde{p}ha+2})) \leq \psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})) - \phi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2}))$.

Therefore, $\phi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})) = 0$ and hence

$$\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2}) = 0.$$

By (1) and (2) of definition of PMS.

$\hat{x}_{2\tilde{\alpha}+1} = \hat{x}_{2\tilde{\alpha}+2}$, a contradiction.

Therefore, $\psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})) = \mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+1})$.

Hence,

$$\begin{aligned}
\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2}) &= \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2}) \\
&\leq \mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+1}) \\
&= \mathfrak{p}(\hat{x}_l, \hat{x}_{l+1}), \qquad (5.2)
\end{aligned}$$

and

$$\psi_1(\mathfrak{p}(\hat{x}_{l+1}, \hat{x}_{l+2})) \leq \psi_1(\mathfrak{p}(\hat{x}_l, \hat{x}_{l+1})) - \phi_1(\mathfrak{p}(\hat{x}_l, \hat{x}_{l+1})). \qquad (5.3)$$

If l is odd, then $l = 2\tilde{\alpha} + 1$, $\tilde{\alpha} \in N$.

From (5.3), we have reference to

$$\begin{aligned}
&\psi_1(\mathfrak{p}(\hat{x}_{n+1}, \hat{x}_{n+2}))\\
&= \psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3}))\\
&= \psi_1(\mathfrak{p}(R\hat{x}_{2\tilde{\alpha}+2}, R\hat{x}_{2\tilde{\alpha}+1}))\\
&\leq \psi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, R\hat{x}_{2\tilde{\alpha}+2})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, R\hat{x}_{2\tilde{\alpha}+1}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}+1})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}+2})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, R\hat{x}_{2\tilde{\alpha}+1}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}})})\\
&- \phi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}+1}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}+1})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, R\hat{x}_{2\tilde{\alpha}+1}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, R\hat{x}_{2\tilde{\alpha}+2})})\\
&= \psi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+3})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+3})})\\
&- \phi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+3})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+3})}).
\end{aligned}$$

If $\mathfrak{p}(\hat{x}_{2\tilde{\alpha}}, \hat{x}_{2\tilde{\alpha}+1}) \leq \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})$, then

$$\begin{aligned}
&\psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3}))\\
&= \psi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+1})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+3})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+3})})\\
&- \phi_1(\frac{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+1})\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+3})}{\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+2}) + \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+3})})\\
&= \psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3}) - \phi_1(\mathfrak{p}(\hat{x}_{2\tilde{k}+2}, \hat{x}_{2\tilde{\alpha}+3}))\\
&\leq (\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})\\
&< \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3}).
\end{aligned}$$

then,

$\psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})) \leq \psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})) - \phi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3}))$.

Therefore, $\phi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})) = 0$ and hence

$$\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3}) = 0.$$

Through (1) and (2)

$\hat{x}_{2\tilde{\alpha}+2} = \hat{x}_{2\tilde{\alpha}+3}$, a contradiction.

Therefore, $\psi_1(\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})) = \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})$. Hence,

$$\begin{aligned}
\mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3}) &= \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+2}, \hat{x}_{2\tilde{\alpha}+3})\\
&\leq \mathfrak{p}(\hat{x}_{2\tilde{\alpha}+1}, \hat{x}_{2\tilde{\alpha}+2})\\
&= \mathfrak{p}(\hat{x}_l, \hat{x}_{l+1}). \qquad (5.4)
\end{aligned}$$

and

$$\psi_1(\mathfrak{p}(\hat{x}_{l+1}, \hat{x}_{l+2})) \quad \leq \quad \psi_1(\mathfrak{p}(\hat{x}_l, \hat{x}_{l+1})) - \phi_1(\hat{x}_l, \hat{x}_{l+1})). \tag{5.5}$$

From (5.2) and (5.4), we get

$\{\mathfrak{p}(\hat{x}_{l+1}, \hat{x}_n) : l \in N\}$ is a non-decreasing number, and thus $s \geq 0$ occurs in such a way that it does not decrease.

$$lim_{n\to\infty}\mathfrak{p}(\hat{x}_l, \hat{x}_{l+1}) = s.$$

We get from (5.3) and (5.6),

$$\psi_1(\mathfrak{p}(\hat{x}_{l+1}, \hat{x}_{l+2})) \quad \leq \quad \psi_1(\mathfrak{p}(\hat{x}_l, \hat{x}_{l+1})) - \phi_1(\mathfrak{p}(\hat{x}_l, \hat{x}_{l+1})) \ for\ all\ l \in N. \tag{5.6}$$

If we put $n \to \infty$ in (5.6) and use the ψ_1 and ϕ_1 facts which are constant, we have

$$\psi_1(s) \leq \psi_1(s) - \phi_1(s).$$

Therefore, $\phi_1(s) = 0$ and hence $s = 0$. Thus

$$lim_{l\to\infty}\mathfrak{p}(\hat{x}_l, \hat{x}_{l+1}) = 0. \tag{5.7}$$

By (2), we get

$$lim_{l\to\infty}\mathfrak{p}(\hat{x}_l, \hat{x}_l) = 0. \tag{5.8}$$

Since $d^*_{\mathfrak{p}}(\hat{x}, \hat{y}) \leq 2\mathfrak{p}(\hat{x}, \hat{y})\ \forall\ \hat{x}, \hat{y} \in X$, we get

$$lim_{l\to\infty}d^*_{\mathfrak{p}}(\hat{x}_l, \hat{x}_{l+1}) = 0. \tag{5.9}$$

We then say that $\{\hat{x}_l\}$ be CS in the MS $(I \cup J, d^*_{\mathfrak{p}})$. It is enough to say that $\{\hat{x}_{2l}\}$ is a CS in $(I \cup J, d^*_{\mathfrak{p}})$. Alternatively, let's say, $\{\hat{x}_{2l}\}$ is not a CS in $(I \cup J, d^*_{\mathfrak{p}})$. So, there exists $\epsilon > 0$, we will consider two sub-sequences for this $\{\hat{x}_{2m(\breve{w})}\}$ and $\{\hat{x}_{2l(\breve{w})}\}$ of $\{\hat{x}_{2l}\}$ s.t. the index for which $l(\breve{w})$ is the smallest is

$$l(\breve{w}) > m(\breve{w}),\ \ d^*_{\mathfrak{p}}(\hat{x}_{2m(\breve{w})}, \hat{x}_{2l(\breve{w})}) \geq \epsilon. \tag{5.10}$$

This means that

$$d^*_{\mathfrak{p}}(\hat{x}_{2m(\breve{w})}, \hat{x}_{2l(\breve{w})-2}) < \epsilon. \tag{5.11}$$

We get from (5.10) and (5.11) as well as the triangular inequalities

$$\begin{aligned}\epsilon &\leq d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})}, \hat{x}_{2l(\breve{w})}) \\ &\leq d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})}, \hat{x}_{2l(\breve{w})-2}) + d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})-2}, \hat{x}_{2l(\breve{w})-1}) + d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})-1}, \hat{x}_{2l(\breve{w})}) \\ &< \epsilon + d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})}, \hat{x}_{2l(\breve{w})-1}) + d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})-1}, \hat{x}_{2l(\breve{w})}).\end{aligned}$$

Making $n \to \infty$ in (5.10) and (5.11) and using (5.9), we have

$$lim_{l\to\infty} d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})-1}, \hat{x}_{2l(\breve{w})}) = \epsilon. \tag{5.12}$$

Once again, we get from (5.10) and from that inconsistency, we get

$$\begin{aligned}\epsilon &\leq d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})}, \hat{x}_{2l(\breve{w})}) \\ &\leq d_{\mathfrak{p}}^{*}(\hat{x}_{2(\breve{w})}, \hat{x}_{2l(\breve{w})-1}) + d_{\mathfrak{p}}^{*}\hat{x}_{2l(\breve{w})}, \hat{x}_{2m(\breve{w})}) \\ &\leq d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})}, \hat{x}_{2l(\breve{w})-1}) + d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})-1}, \hat{x}_{2m(\breve{w})+1}) + d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})+1}, \hat{x}_{2m(\breve{w})}) \\ &\leq d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})}, \hat{x}_{2l(\breve{w})-1}) + d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})-1}, \hat{x}_{2m(\breve{w})}) + 2d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})+1}, \hat{x}_{2m(\breve{w})}) \\ &\leq 2d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})}, \hat{x}_{2l(\breve{w})-1}) + d_{\mathfrak{p}}^{*}(\hat{x}_{2l(\breve{w})}, \hat{x}_{2m(\breve{w})}) + 2d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})+1}, \hat{x}_{2m(\breve{w})}).\end{aligned}$$

Putting $lim_{\breve{w}\to+\infty}$ in the inequalities described above and using (5.9) and (5.12), we have

$$\begin{aligned}lim_{\breve{w}\to+\infty} d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})}, \hat{x}_{2l(\breve{w})}) &= lim_{\breve{w}\to+\infty} d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})+1}, \hat{x}_{2l(\breve{w})-1}) \\ &= lim_{\breve{w}\to+\infty} d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})+1}, \hat{x}_{2l(\breve{w})}) \\ &= lim_{\breve{w}\to+\infty} d_{\mathfrak{p}}^{*}(\hat{x}_{2m(\breve{w})}, \hat{x}_{2l(\breve{w})-1}) \\ &= \epsilon.\end{aligned}$$

Since

$$d_{\mathfrak{p}}^{*}(\hat{x}, \hat{y}) = 2\mathfrak{p}(\hat{x}, \hat{y}) - \mathfrak{p}(\hat{x}, \hat{x}) - \mathfrak{p}(\hat{y}, \hat{y})$$

$\forall\ \hat{x}, \hat{y} \in X$, then

$$\begin{aligned}lim_{\breve{w}\to+\infty} \mathfrak{p}(\hat{x}_{2m(\breve{w})}, \hat{x}_{2l(\breve{w})}) &= lim_{\breve{w}\to+\infty} \mathfrak{p}(\hat{x}_{2m(\breve{w})+1}, \hat{x}_{2l(\breve{w})-1}) \\ &= lim_{\breve{w}\to+\infty} \mathfrak{p}(\hat{x}_{2m(\breve{w})+1}, \hat{x}_{2l(\breve{w})}) \\ &= lim_{\breve{w}\to+\infty} \mathfrak{p}(\hat{x}_{2m(\breve{w})}, \hat{x}_{2l(\breve{w})-1}) \\ &= \frac{\epsilon}{2}.\end{aligned}$$

We get it by (5.1),

$$\psi_1(\mathfrak{p}(\hat{x}_{2m(\breve{w})+1},\hat{x}_{2l(\breve{w})}))=\psi_1(\mathfrak{p}(R\hat{x}_{2m(\breve{w})},R\hat{x}_{2l(\breve{w})-1}))$$
$$\leq \psi_1(\frac{\mathfrak{p}(\hat{x}_{2m(\breve{w})},R\hat{x}_{2m(\breve{w})})\mathfrak{p}(\hat{x}_{2m(\breve{w})},R\hat{x}_{2l(\breve{w})-1})+\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},R\hat{x}_{2l(\breve{w})-1})\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},R\hat{x}_{2m(\breve{w})})}{\mathfrak{p}(\hat{x}_{2m(\breve{w})},R\hat{x}_{2l(\breve{w})-1})+\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},R\hat{x}_{2m(\breve{w})})})$$
$$-\phi_1(\frac{\mathfrak{p}(\hat{x}_{2m(\breve{w})},R\hat{x}_{2m(\breve{w})})\mathfrak{p}(\hat{x}_{2m(\breve{w})},R\hat{x}_{2l(\breve{w})-1})+\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},R\hat{x}_{2l(\breve{w})-1})\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},R\hat{x}_{2m(\breve{w})})}{\mathfrak{p}(\hat{x}_2m(\breve{w}),\hat{x}_{2l(\breve{w})})+\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},R\hat{x}_{2m(\breve{w})})})$$
$$\leq \psi_1(\frac{\mathfrak{p}(\hat{x}_{2m(\breve{w})},R\hat{x}_{2m(\breve{w})+1})\mathfrak{p}(\hat{x}_{2m(\breve{w})},\hat{x}_{2l(\breve{w})})+\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},\hat{x}_{2l(\breve{w})})\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},\hat{x}_{2m(\breve{w})+1})}{\mathfrak{p}(\hat{x}_{2m(\breve{w})},\hat{x}_{2l(\breve{w})})+\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},\hat{x}_{2m(\breve{w})+1})})$$
$$-\phi_1(\frac{\mathfrak{p}(\hat{x}_{2m(\breve{w})},\hat{x}_{2m(\breve{w})+1})\mathfrak{p}(\hat{x}_{2m(\breve{w})},\hat{x}_{2l(\breve{w})})+\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},\hat{x}_{2l(\breve{w})})\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},\hat{x}_{2m(\breve{w})+1})}{\mathfrak{p}(\hat{x}_2m(\breve{w}),\hat{x}_{2l(\breve{w})})+\mathfrak{p}(\hat{x}_{2l(\breve{w})-1},\hat{x}_{2m(\breve{w})+1})}).$$

Letting $\breve{w}\to+\infty$ and using the continuity of ϕ_1 and ψ_1, we get $\psi_1(\frac{\epsilon}{2})\leq\psi_1(\frac{\epsilon}{2})-\phi_1(\frac{\epsilon}{2})$.

Hence, we get that $\phi_1(\frac{\epsilon}{2})=0$. $\epsilon=0$ is thus, a paradox. Thus $\{\hat{x}_l\}$ is a CS in $(I\cup J,d_p)$. But $I\cup J$ is a closed subset of $(\hat{x},\mathfrak{p})$ and $(\hat{x},\mathfrak{p})$ is complete. Therefore, $(I\cup J,d_p)$ is complete. The $\{\hat{x}_l\}$ sequence converges in the MS beginning with the Lemma (1.16), $(I\cup J,d^*_{\mathfrak{p}})$, say $lim_{l\to\infty}d^*_{\mathfrak{p}}(\hat{x}_l,v)=0$.
Again, by (1.16), we obtain

$$\mathfrak{p}(v,v)=lim_{l\to\infty}\mathfrak{p}(\hat{x}_l,v)=lim_{l,m\to\infty}\mathfrak{p}(\hat{x}_l,\hat{x}_m). \tag{5.13}$$

Additionally, since $\{\hat{x}_l\}$ is a CS in the MS $(I\cup J,d^*_{\mathfrak{p}})$, we have

$$lim_{l,m\to\infty}d^*_{\mathfrak{p}}(\hat{x}_l,\hat{x}_m)=0. \tag{5.14}$$

We get from the $d^*_{\mathfrak{p}}$ description,

$$d^*_{\mathfrak{p}}(\hat{x}_l,\hat{x}_m)=2\mathfrak{p}(\hat{x}_l,\hat{x}_m)-\mathfrak{p}(\hat{x}_l,\hat{x}_l)-\mathfrak{p}(\hat{x}_m,\hat{x}_m).$$

Letting, $l,m\to\infty$ and using (5.8) and (5.14), we get

$$lim_{l,m\to\infty}\mathfrak{p}(\hat{x}_l,\hat{x}_m)=0.$$

Therefore, by means of (5.13), it become

$$lim_{l\to\infty}\mathfrak{p}(\hat{x}_l,v)=\mathfrak{p}(v,v)=0. \tag{5.15}$$

Because $\mathfrak{p}(\hat{x}_{2l},v)\to 0=\mathfrak{p}(v,v)$, $\{\hat{x}_{2l}\}$ is a sequence in I, and I is closed in $(X,\mathfrak{p})$, we have $v\in I$. Equally, we've got $v\in J$, which means $v\in I\cap J$. Similarly, according to the $\mathfrak{p}$ description, we are getting

$$\begin{aligned}\mathfrak{p}(\hat{x}_l, Rv) &\leq \mathfrak{p}(\hat{x}_l, v) + \mathfrak{p}(v, Rv) - \mathfrak{p}(v, v)\\ &\leq \mathfrak{p}(\hat{x}_l, v) + \mathfrak{p}(v, \hat{x}_l) + \mathfrak{p}(\hat{x}_l, Rv) - \mathfrak{p}(\hat{x}_l, \hat{x}_l) - \mathfrak{p}(v, v).\end{aligned}$$

Making $l \to \infty$ and using (5.9) and (5.15), we get

$$lim_{l\to\infty}\mathfrak{p}(\hat{x}_l, Rv) = \mathfrak{p}(v, Rv).$$

Now, we say $Rv = v$.
Since $\hat{x}_{2l} \in I$ and $v \in J$,
From (5.1), we get

$$\begin{aligned}\psi_1(\mathfrak{p}(\hat{x}_{2l+1}, Rv) &= \psi_1(\mathfrak{p}(R\hat{x}_{2l}, Rv))\\ &\leq \psi_1(\frac{\mathfrak{p}(\hat{x}_{2l}, R\hat{x}_{2l})\mathfrak{p}(\hat{x}_{2l}, Rv) + \mathfrak{p}(v, Rv)\mathfrak{p}(v, R\hat{x}_{2l})}{\mathfrak{p}(\hat{x}_{2l}, Rv) + \mathfrak{p}(v, R\hat{x}_{2l})})\\ &- \phi_1(\frac{\mathfrak{p}(\hat{x}_{2l}, R\hat{x}_{2l})\mathfrak{p}(\hat{x}_{2l}, Rv) + \mathfrak{p}(v, Rv)\mathfrak{p}(v, R\hat{x}_{2l})}{\mathfrak{p}(\hat{x}_{2l}, Rv) + \mathfrak{p}(v, R\hat{x}_{2l})})\\ &\leq \psi_1(\frac{\mathfrak{p}(\hat{x}_{2l}, \hat{x}_{2l+1})\mathfrak{p}(\hat{x}_{2l}, Rv) + \mathfrak{p}(v, Rv)\mathfrak{p}(v, \hat{x}_{2l})}{\mathfrak{p}(x_{2l}, Rv) + \mathfrak{p}(v, x_{2l+1})})\\ &- \phi_1(\frac{\mathfrak{p}(x_{2l}, \hat{x}_{2l+1})\mathfrak{p}(\hat{x}_{2l}, Rv) + \mathfrak{p}(v, Rv)\mathfrak{p}(v, \hat{x}_{2l})}{\mathfrak{p}(\hat{x}_{2l}, Rv) + \mathfrak{p}(v, \hat{x}_{2l+1})}).\end{aligned}$$

Making $l \to \infty$, we get

$$\psi_1(\mathfrak{p}(v, Rv)) \leq \psi_1(\mathfrak{p}(v, Rv)) - \phi_1(\mathfrak{p}(v, Rv)).$$

So, $\phi_1(\mathfrak{p}(v, Rv)) = 0$.
Behind that ϕ_1 is a distance altering function, so $\mathfrak{p}(v, Rv) = 0$, i.e. $v = Rv$. v is, thus, the fixed point of R. Now, for the purpose of proving the uniqueness of the R fixed point, assume that $\lfloor$ is every other fixed point of R in $I \cap J$. Proving that $\mathfrak{p}(\lfloor, \lfloor) = 0$ is simple. We are displaying $v = \lfloor$, however. Since $v \in I \cap J \subseteq I$ and $\lfloor \in I \cap J \subseteq J$, we're

having

$$\begin{aligned}\psi_1(\mathfrak{p}(v, \lfloor)) &= \psi_1(\mathfrak{p}(Rv, R\lfloor)) \\ &\leq \psi_1(\frac{\mathfrak{p}(v, R\lfloor)\mathfrak{p}(v, R\lfloor) + \mathfrak{p}(\lfloor, R\lfloor)\mathfrak{p}(\lfloor, Rv)}{\sqrt{}(v, R\lfloor) + \mathfrak{p}(\lfloor, Rv)}) \\ &- \phi_1(\frac{\mathfrak{p}(v, Rv)\mathfrak{p}(v, R\lfloor) + \mathfrak{p}(\lfloor, R\lfloor)\mathfrak{p}(\lfloor, Rv)}{\mathfrak{p}(v, R\lfloor) + \mathfrak{p}(\lfloor, Rv)}) \\ &\leq \psi_1(\frac{\mathfrak{p}(v, v)\mathfrak{p}(v, \lfloor) + \mathfrak{p}(\lfloor, \lfloor)\mathfrak{p}(\lfloor, v)}{\mathfrak{p}(v, \lfloor) + \mathfrak{p}(\lfloor, v)}) \\ &- \phi_1(\frac{\mathfrak{p}(v, v)\mathfrak{p}(v, \lfloor) + \mathfrak{p}(\lfloor, \lfloor)\mathfrak{p}(\lfloor, v)}{\mathfrak{p}(v, \lfloor) + \mathfrak{p}(\lfloor, v)}) \\ &= \psi_1(\mathfrak{p}(v, \lfloor)) - \phi_1(\mathfrak{p}(v, \lfloor)).\end{aligned}$$

So $\phi_1(\mathfrak{p}(v, \lfloor)) = 0$ and thus $\mathfrak{p}(v, \lfloor) = 0$. Hence, $v = \lfloor$. □

Putting $\psi_1 = I_{[0,\infty)}$ (the identity function) into Theorem (5.2) would result as shown below:

Corollary 5.3. *"Let I and J be nonempty closed subsets of a $(X, \mathfrak{p})$ complete PMS. Let R be a map that gives a cyclic representation of $I \cup J$. Presume that distance function ϕ_1 is adjusted so that*

$$\mathfrak{p}(R\hat{x}, R\hat{y}) \leq \frac{\mathfrak{p}(\hat{x}, R\hat{x})\mathfrak{p}(\hat{x}, R\hat{y}) + \mathfrak{p}(\hat{y}, R\hat{y})\mathfrak{p}(\hat{y}, R\hat{x})}{\mathfrak{p}(\hat{x}, R\hat{y}) + \mathfrak{p}(\hat{y}, R\hat{x})} - \phi_1(\frac{\mathfrak{p}(\hat{x}, R\hat{x})\mathfrak{p}(\hat{x}, R\hat{y}) + \mathfrak{p}(\hat{y}, R\hat{y})\mathfrak{p}(\hat{y}, R\hat{x})}{\mathfrak{p}(\hat{x}, R\hat{y}) + \mathfrak{p}(\hat{y}, R\hat{x})})$$

for all $\hat{x} \in I$, $\hat{y} \in J$. Then R has a unique fixed point $v \in I \cap J$."

Example 5.1. *Let's assume $X = [0, 1]$. Defines the $\mathfrak{p}$ partial metric on X by χ*

$$\mathfrak{p}(\chi, \omega) = \begin{cases} 0, \ if \ \chi = \omega; \\ max\{\chi, \omega\}, \ if \chi \neq \omega. \end{cases}$$

Let $R : X \to X$ be the mapping by $R\chi = \frac{\chi}{8}$. Also, let $\psi_1, \phi_1 : [0, \infty) \to [0, \infty)$ be defined by $\psi_1(t) = \frac{t}{4}$ and $\phi_1(t) = \frac{t}{8}$. Take $I = [0, \frac{1}{2}]$ and $J = [0, 1]$. Then

1. *Complete PMS is $(X, \mathfrak{p})$.*
2. *The cyclic representation of $I \cup J$ is w.r.t. R*
3. *We have $\chi \in I$ and $\chi \in J$ for any*

$$\psi_1(\mathfrak{p}(R\chi, R\omega)) \leq \psi_1(\frac{\mathfrak{p}(\chi, R\chi)\mathfrak{p}(\chi, R\omega) + \mathfrak{p}(\omega, R\omega)\mathfrak{p}(\omega, R\chi)}{\mathfrak{p}(\chi, R\omega) + \mathfrak{p}(\omega, R\chi)})$$
$$-\phi_1(\frac{\mathfrak{p}(\chi, R\chi)\mathfrak{p}(\chi, R\omega) + \mathfrak{p}(\omega, R\omega)\mathfrak{p}(\omega, R\chi)}{\mathfrak{p}(\chi, R\omega) + \mathfrak{p}(\omega, R\chi)})$$

Proof. Note that $RI = [0, \frac{1}{6}] \subseteq J$ and $RJ = [0, \frac{1}{2}] \subseteq I$. Thus, $I \cup J$ has a cyclic representation of R. Given that $\chi \in I$, $\omega \in J$. WLOG, suppose that $\chi \geq \omega$. So,

$$\psi_1(\mathfrak{p}(R\chi, R\omega)) = \psi_1(\mathfrak{p}(\frac{\chi}{8}, \frac{\omega}{8})) = \psi_1(\frac{\chi}{8}) = \frac{\omega}{32}.$$

Now, $\mathfrak{p}(\chi, R\chi) = \mathfrak{p}(\chi, \frac{\chi}{8}) = \chi, \mathfrak{p}(\omega, R\omega) = \mathfrak{p}(\omega, \frac{\omega}{8)}) = \omega, \mathfrak{p}(\chi, R\omega) = \mathfrak{p}(\chi, \frac{\omega}{8}) = \chi, \mathfrak{p}(\omega, R\chi) = \mathfrak{p}(\omega, \frac{\chi}{8})$.

Case 1. If $\mathfrak{p}(\omega, \frac{\omega}{8}) = \omega$, then

$\psi_1(\frac{\mathfrak{p}(\chi,R\chi)\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\omega)\mathfrak{p}(\omega,R\omega)}{\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\chi)}) = \psi_1(\frac{\chi.\chi+\omega.\omega}{\chi+\omega}) = \psi_1(\frac{\chi^2+\omega^2}{4(\chi+\omega)}) \leq \frac{\chi^2+\omega^2}{4} \leq \frac{2\chi^2}{4} = \frac{\chi^2}{2}.$

and

$\phi_1(\frac{\mathfrak{p}(\chi,R\chi)\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\omega)\mathfrak{p}(\omega,R\chi)}{\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\omega)}) = \phi(\frac{\chi.\chi+\omega.\omega}{\chi+\omega}) = \phi_1(\frac{\chi^2+\omega^2}{8(\chi+\omega)}) \leq \frac{\chi^2+\omega^2}{8} \leq \frac{2\chi^2}{8} = \frac{\chi^2}{4}.$

Since

$\frac{\chi}{32} \leq \frac{\chi^2}{2} - \frac{\chi^2}{4} = \frac{\chi^2}{4}$

and

Case 2. If $\mathfrak{p}(\omega, \frac{\chi}{8}) = \frac{\chi}{8}$, then

$\psi_1(\frac{\mathfrak{p}(\chi,R\chi)\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\omega)\mathfrak{p}(\omega,R\chi)}{\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\chi)}) = \psi_1(\frac{\chi.\chi+\omega.\frac{\chi}{8}}{\chi+\frac{\chi}{8}}) = \psi_1(\frac{8\chi+\omega}{9}) \leq \frac{9\chi}{36} = \frac{\chi}{4}.$

$\phi_1(\frac{\mathfrak{p}(\chi,R\chi)\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\omega)\mathfrak{p}(\omega,R\chi)}{\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\chi)}) = \phi_1(\frac{\chi.\chi+\chi.\frac{\chi}{8}}{\chi+\frac{\chi}{8}}) = \phi_1(\frac{8(\chi+\omega)}{9}) \leq \frac{9\chi}{72} = \frac{\chi}{8}.$

$\frac{\chi}{32} \leq \frac{\chi}{4} - \frac{\chi}{8} = \frac{\chi}{4}.$ □

In order to satisfy the hypothesis, denote the set of functions $\hat{\mu} : [0, +\infty) \to [0, +\infty)$ with I:

1. Every compressed Lebesgue integrable map of $[0, +\infty)$ is $\hat{\mu}$.

2. For every $\epsilon > 0$, we have
$$\int_0^\epsilon \hat{\mu}(t)dt > 0.$$

Theorem 5.4. *Let I, J be non-void closed subsets and R be a map of a $(X, \mathfrak{p})$ complete PMS such that $I \cup J$ has a cyclic representation w.r.t. R. Presume $\chi \in I$, $\omega \in J$, it becomes*

$$\int_0^{\mathfrak{p}(R\chi,R\omega)} \hat{\mu}_1(t)dt \leq \int_0^{(\frac{\mathfrak{p}(\chi,R\chi)\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\omega)\mathfrak{p}(\omega,R\chi)}{\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\chi)})} \hat{\mu}_1(t)dt - \int_0^{(\frac{\mathfrak{p}(\chi,R\chi)\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\omega)\mathfrak{p}(\omega,R\chi)}{\mathfrak{p}(\chi,R\omega)+\mathfrak{p}(\omega,R\chi)})} \hat{\mu}_2(t)dt$$

where $\hat{\mu}_1, \hat{\mu}_2 \in I$. Since R has a fixed point that no one else has $v \in I \cap J$.

Proof. Pass (5.2) Theorem, setting $\psi_1, \phi_1 : [0, +\infty) \to [0, +\infty)$ via $\psi_1(t) = \int_0^t \hat{\mu}_1(s)ds$ and $\phi_1(t) = \int_0^t \hat{\mu}_2(s)ds$ and noticed that ψ_1, ϕ_1 are altering distance functions. □

5.3 Summary and Conclusion

It is clear that new view of proximal contractions of kind-R and kind-M in the frame of CMS upgrade and enhance the existing results, which are given in the literature. A new class of generalized $\beta - \phi - \mathcal{Z}$-contractive pair of mappings extend other well-known material of FPT within the literature. More precisely, with aid of SF and PC of first kind and second kind with respect to ζ which also generalize several known types of contractions. Several interesting results for BPP and also a new approach to the study of SF in G-M.S.. Consequences of almost $\mathcal{Z}$-contraction w.r.t. to G-M.S. and MS is beneficial to discover unique solution of the integral equations. We introduce the new notions of modified $\alpha - (\psi, g)$-PC of type-I and type-II. We also extend our results with the new notion of cyclic (ψ_1, ϕ_1, I, J)-rational contraction in the settings of partial metric spaces. Some fixed point results are proved using this notion. We also prove some FPT in g.m.s.. Some FPT are proved in the setting of new notion $(G - v - \phi)$-proximal cyclic weak contractive mapping in G-M.S..

There are some results which have been proved using property E.A. and (CLR) in MS. Using the same method introduced by Altun et al., we prove a specific FPT for a pair of maps for four WC self maps that satisfy a general condition. Extending and generalizing new fixed point findings, as well as E.A. and (CLR) properties, using this method of analysis. Moreover, we enhance our results for $\mathcal{C}$-contractive condition with the aid of SF in ordered G-M.S.. We will not to list all results due to our concern on the size of the thesis.

5.4 Future Scope

The new fixed point results during research work can be applied to obtain the solutions of linear and non-linear problems. It is the answer to world's present and future problems.The nature theory of differential and integral equations will benefit from our work. Fixed point results with SF can also be proved in PMS and in other spaces. The $\mathcal{Z}$-contractive mapping can also use in partial metric and other spaces. The property E.A. and (CLR) can also be used for WC maps to get fixed points in G-M.S., g.m.s. and PMS.

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Publications and Presentations

Publications

- R.Sharma, M.kumar. *Best proximity results for contractions of kind-R and kind-M in complete metric spaces*, AIP Conference Proceedings 1860, 020052(2017):1-7, 2017.
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Presentations

- *Best proximity results for contractions of kind-R and kind-M in complete metric spaces*, International Conference on Recent Advances in fundamental and Applied Science (RAFAS), Lovely Professional University, Phagwara, Punjab, India, November 25-26, 2016.

- *Fixed point theorems for almost generalized C-contractive mappings in ordered complete G-metric spaces* was presented in an international conference Advancements in Engineering and Technology (ICAET-2017), BGIET, Sangrur, Punjab, India, March 24-25, 2017.
- *Cyclic* (ψ, ϕ, A, B)*-rational contraction for fixed point results in partial metric spaces* was presented in an international conference Science and Technology: Trends and Challenges (ICSTTC-2018), GGN Khalsa College, Ludhiana, Punjab, India April 16-17, 2018.
- *Best Proximity Point Theorems using simulation functions* was presented for poster presentation in the Mathematical Sciences (including Statistics) Section of the 106th Indian Science Congress, Lovely Professional University, Phagwara, Punjab, India, 3-7th January, 2019.
- *Common fixed point theorems using CLCS property in complex valued metric spaces* was presented in an international conference Recent Advances in Fundamental and Applied sciences, Lovely Professional University, Phagwara, Punjab, India, Nov. 5-6th 2019.

Workshop

- *Advanced technical writing through Latex*, Starex University, Binola, Gurugram, 30th November and 1st December, 2018.