Mathematics for Economists

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MATHEMATICS FOR ECONOMISTS

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SYLLABUS

Mathematics for Economists

Objectives

- To aware of students the mathematical aspects of Economics.
- To introduce the concept of interrelation and inter dependency of mathematical Economics.
- To increase understanding of the application of the mathematical properties of Economics.

Sr. No.	Content
1	Types of Functions: constant function, polynomial functions, rational functions, non-algebraic function, exponential function, log function, Limits & Continuity
2	Differentiation : Simple, Logarithmic differentiation, Second and higher order differentiation
3	Differentiation: Partial, Homogeneous function and Euler's theorem, Economic Applications of differentiation
4	Maxima and Minima of one variable, Maxima and Minima of two variables, Constrained Maxima and Minima, Economic Applications of Maxima and Minima
5	Integration : Basic rules of integration, Methods of integration, Integration as a summation, Definite Integration, Economic Applications of Integration
6	Differential Equations: Introduction, Solution – variable separable case, homogenous case
7	Matrices : Meaning and types, Transpose, trace of a matrix, Adjoint and inverse of the matrix, Cramer's rule, Determinants: Types and properties, Rank of a matrix, Application of matrices in economics
8	Input – Output analysis, Hawkins – Simon Conditions, Closed Economic Input – Output analysis
9	Introduction to Linear Programming, Formulation of Linear programming problems, Graphic methods
10	Linear Programming - Simplex methods

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Unit 1: Functions

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Functions and Related Quantities.
- Determine the Value of Functions.
- Know the Definition of Functions by Mapping.
- Determine the Domain and Range of Functions.
- Understand the use of Linear Functions in Economics.

Introduction

In many of the questions, we have to determine the effect of increase-decrease of an independent number on a number dependent on it. For example, area of circle always depends on the radius of it, because if radius is increased or decreased, then area of the circle will also decrease or increase according to the radius. Here radius of the circle is a number and area of the circle is another number, which are related to each other. Thus volume of a cylinder depends on its radius, area of square and volume of a cuboid depend on the length of its arm. The distance covered by a running train in dynamic velocity depends on the time taken. Velocity of a falling particle depends on the distance covered by it. Atmospheric pressure of a certain place depends on the alleviation of its height from sea-coast etc.

Note

With the change of value of one number, the rate of change in the value of other number and questions related to rate of such change and analysis and study of functions is referred to as Differential Calculus.

1.1 Quantities

There are two types of quantities:

1. Variable 2. Constant

1. **Variables –** Changing quantities are referred to as variables. Quantities for which values keeps on changing viz which can be given indefinite numerical values are Variables. These are general expressed with the last characters such as x, y, z, u, v, w etc. of English Words.

2. **Constants –** Numbers for which value is unchangeable under any process of mathematics, are constants. Constant quantities are of two types:

(i) Absolute constants

(ii) Arbitrary constants

Value of which in any problem is unchanged, is referred to as absolute constants *e.g.*

$5, -3, 1, 1, \sqrt{5}, \pi, \frac{2}{5}, e$ etc.

Value of which remains constant in a problem, but get different value in different problems, they are called arbitrary constants. These are expressed with the beginning letters such as a, b, c, d, etc. of English word.

1.2 Related Quantities

We know that the area of the circle depends on its radius. In other words, circles with different radii have different areas. Similarly, square of any positive number increases or decreases with any change in it. Here radius of circle and its area or number and its square are related quantities.

Notes Any two numbers, in which change in any one number affects another number, are referred to as related quantities.

Although both these two numbers are constants, but value of any one number can be changed liberally and the value of other number will change not independently but applying any rule. For example if we assume liberally 1, 2, 3, 4 …….. for the radius of the circle (r), then its area (A) applying the rule $A = \pi r^2$ would be π, 4π, 9π, 16π,..... respectively.

There are two types of constants:

(i) Independent Variables, **(ii) Dependent variables**

If two constants x and y are related in a manner where one variable can be given any value liberally and the value of y depends on it, then x will be referred to as independent variables and y would be its dependent variables.

For example: Assume $y = 2x + 5$

Now giving x variable different values like 0, 1, 2, ….. etc. we get different results for y variable such 5, 7, 9, …… etc. respectively, which is completely depended on x.

Therefore, here x is independent variable and y is a dependent one.

Note

1.3 Functions

The principle of functions is mutually based on the related quantities.

Assume area of a circle of any radius (r) is (A), then $A = \pi r^2$

Here r is a an independent variable and A is a dependent variable, since the area (A) of the circle is based on radius (r)

In this situation we say that variable (A) is function of (r), which in the language of mathematics is expressed in the following manner: $A = f(r)$

Thus, [A is a function of variable r]

1.4 Definition of Functions

Here two variables x (independent) and y (dependent) are related with the function (f) in such a way that for each value of x, a certain and unique value of y can be obtained, then y would be the f-function of x, which can be expressed as $y = f(x)$.

The functions of x are expressed with the symbols $f(x)$, $g(x)$, $\phi(x)$.....etc.

1.5 Explanation of Functions

If y is a function of x variable then $y = f(x)$.

Then x would be an independent variable and y is a dependent varialble.

If $y = 5x + 7$, then for each value of x, there is a certain value of y. therefore, y is said to be the function of x.

1.6 Functional Notations

Many symbols are used to express this function where y variable is the function of x variable. The symbol mostly used is

$y = f(x)$.

Which is read as "y is equal to function of x"

Other symbols are: $y = F(x), y = \phi(x),\ y = \psi(x), \ldots,$ etc.

> *Did u know?* If for any value of x variable, there are more than one value, then y is not called the function of x, but the relation. For e.g. $y^2 = x$, here for each value of x, two values are obtained, such type of relations are sometimes also called multi-valued function.

1.7 Value of Functions

Assume any function of variable x is $y = f(x)$

Putting $x = a$ in $f(x)$ we get $f(a)$ which would be the value of function for $x = a$.

Value of $f(x)$ on $x = a$ would be $f(a)$
Which is obtained replacing x with a in $f(x)$

For *e.g.* assume $f(x) = 2x^2 - 3x + 5$

Note

Then $f(2) = 2(2)^2 - 3.(2) + 5,$ [putting 2 in place of x]

$$= 8 - 6 + 5 = 7$$

Self Assessment

1. Fill in the blanks:

(i) Changing numbers are called..............

(ii) Numbers, which remain same under any mathematical formula, are called..............

(iii) Any two numbers in which changing of one number affect second number are called .... numbers

(iv) The area of circle is depended on its ....

(v) Value of the numbers which remains constant, is called.................

1.8 Definition of Functions by Mapping

If correspondence of each element (x) of Non-empty set X is obtained by any rule (f) for a certain and unique element (y) of Non-empty set, then f is called Mapping of non-empty set (x) of non-empty set (Y). Mapping shows the geometrical aspect of association of the elements of non-empty set X and Y, and the function shows its analytical aspect.

In other words,

If X and Y are two non-empty sets and by any rule (f) element of X is associated with the a unique element of Y, then binary operation R is called the function of X and Y.

Thus, $R = \{(x, y): x \in X,\ y \in Y\}$ is the functions of X and Y.

For example: if $X = \{1, 2, 3\}$, $Y = \{3, 4, 5\}$, then with the function $y = x + 2$, element 1, 2, 3 of X will be associated with element 3, 4, 5 of Y. thus, binary operation would be the functions of X and Y and it can be written as:

$$f : X \rightarrow Y \text{ or } X \xrightarrow{f} Y$$

This way f is a rule by which element of X is associated with a unique element of Y.

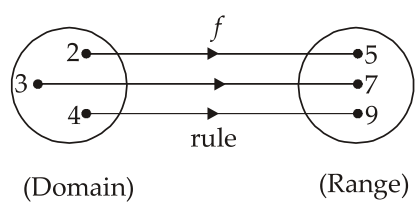
1.9 Domain and Range of Functions

If function $f : X \rightarrow Y$ is defined, then non-empty set X is called domain of this mapping or function (f) and non-element set (Y) would be its range or co-domain.

Domain of function (f) = Element of actual values of X for which function is defined.

Range of function (f) = values of elements of Y corresponding to domain of all numbers x .

For Example: in $y = 2x + 1$, y is the function of x. If x holds positive bigger value than 1 and smaller than 5, domain element of the function would be {2, 3, 4} and the range value would be {5, 7, 9}. Similarly domain of binary R = {(5, 8), (6, 9), (7, 10)} operation will be {5, 6, 7} and range value would be {8, 9, 10}



Note

If domain of element of function x is associated with the element of y, then element (y) of this domain which is associated with x, is expressed with $f(x)$, where $f(x)$ expresses the value of function (f), which also can be written as under:

$$f : x \to y \;\; f : x \longrightarrow f(x)$$

1.10 Kinds of Functions

1. **Functions of single variable –** if variable y is only depended on variable x, then $y = f(x)$ is called functions of single variable. Similarly $y = f(\theta)$, $s = f(t)$, $v = f(t)$ etc. are functions of single variable.
2. **Functions of Many variable –** if variable u is depended on variable x and y, then $u = f(x, y)$ is called functions of many variable.
3. **Explicit Function –** function $y = f(x)$ is called explicit function, if it is possible to express y as independent element of x. for *e.g.* etc.

   $y = x^2 + 2x - 5, \quad y = \cos x,\ y = \dfrac{x}{1+x^2}$
4. **Implicit Function –** function $y = f(x)$ is called implicit function, if it is not possible to express y as independent element of x.

   For example: $y = x \sin(x+y)$, and $f(x, y) = 0$, $x^2 + y^2 - xy = 0$
5. **Even Function –** assume $y = f(x)$ is function of y.
   If $f(-x) = f(x)$

   Viz. if in place of x, putting $-x$ in $f(x)$, there is no change in sign of function, it is called even function.

   For example: $f(x) = \cos x = \cos(-x) = f(-x)$
6. **Odd Function –** assume $y = f(x)$ is function of x.

   For example: $f(x) = x^3 = -(x)^3 = -f(-x)$

   If $f(-x) = -f(x)$

   Viz. if in place of x, putting $-x$ in $f(x)$, the sign of function changed then it is called odd function.
7. **Algebraic Function –** Function with different exponential value of x variable, in which factors are certain is called algebraic function.

   For example:

   $f(x) = x^3 + 5x^2 + 2x$

   $f(x) = x^{3/2} + x^{-1/3}$

   $f(x) = (x + a)^{2/3}$
8. **Rational Function -** Function expressed in the form of fraction, in which numerators and denominators are of algebraic function of exponential value, is called rational function.

   $$f(x) = \frac{2x^3 + x + 7}{x^3 + 5x^2 + x + 5}$$
9. **Transcendental Function –** these are of following types:

   (i) Trigonometrical Functions such as $\sin x$, $\cos x$, $\sec x$, $\sin 2x$, $\sec^2 x$ etc.

   (ii) Inverse Circular Functions such as $\sin^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$ etc.

   (iii) Logarithmic Functions such as $\log_e x$, $\log_a x$, $\log_e (x^2 + 4x + 3)$ etc.

   (iv) Exponential Functions such as e^x, a^x, $x^{\sin x}$, $(\sin x)^{\cos x}$ etc.

Note

10. **Defined Function On –** $x = a$, $f(x)$ is a defined function if putting $x = a$ in $f(x)$, the value of function $f(a)$ is equal to a certain and finite value, which is completely meaningful and real.
11. **Undefined function On –** $x = a$ $f(x)$ is called undefined function if the value $f(a)$ is uncertain and meaningless for example

    $\frac{0}{0}, \frac{\infty}{\infty}, \infty\times\infty, \infty-\infty, 0\times\infty, 0^0, \infty^0, 1^\infty$

    For example

    (i) $y=\frac{\sin x}{x}$; then $x=0, \left(\frac{0}{0}\right)$

    (ii) $y=\frac{x^2+3x+4}{x^2-4}$; then $x=\infty\left(\frac{\infty}{\infty}\right)$

    (iii) $y=x^{\frac{1}{1-x}}$; then $x=1, (1^\infty)$

    (iv) $y=\frac{1}{x}-\frac{1}{\sin x}$; then $x=0, (\infty-\infty)$ etc.

12. **Constant Function –** $y = f(x) = c$ where c is a constant function shows a constant function. The value of y or $f(x)$ always remains constant for the value of x under constant function.
13. **Periodic Function –** Function $y = f(x)$ is called periodic function. If for each value of x $f(x + k) = f(x)$,

    Where k is an actual number which is not equal to 0 viz k is the period of the function.
14. **Function of the Function –** Assume $f(x)$ and $g(x)$ are the functions of variable x, then $f[g(x)]$ is the function of function which is obtained putting $g(x)$ in place of x in the function $f(x)$.

    Thus, $g[f(x)]$ is also the function of function which is obtained putting $f(x)$ in place of x in $g(x)$ Viz. if in place of x, putting $-x$ in $f(x)$, the sign of function changed then it is called odd function.

1.11 Operations in the Set of Functions

Assume S is the actual function of the sets with D domain. Then if f and g are the function of non-empty set S, then $f+g, f-g, fg, \frac{f}{g}$ will have the following meaning:

(i) $(f+g)(x) = f(x) = g(x), \forall\ x \in D$

(ii) $(f-g)(x) = f(x) = -g(x), \forall\ x \in D$

(iii) $(fg)(x) = f(x)\cdot g(x), \forall\ x \in D$

(it is clear that $f+g$, $f-g$ and fg have the same domain.

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \forall\ x \in D \sim S$

Where S is the solution of equation $g(x) = 0$. In other words S is a set where $x \in S \Rightarrow g(x) = 0$ or $S = \{x : g(x) = 0\}$

Note

It is clear that $D \sim S$ is the domain of f/g viz those elements (x) are removed from D for which $g(x) = 0$.

For example: assume $f(x)=\frac{1}{x}$ and $\phi(x)=x^2$, are two function, where the algebraic total of $f(x)$ and $\phi(x)$ is expressed as $f(x)+\phi(x)$ or $(f+\phi)\ (x)$ and would be $(f+\phi)(x)=\frac{1}{x}+x^2$

Further the multiplications of functions will be expressed as $f(x)\cdot\phi(x)$ and $f(x)\cdot\phi(x)=\frac{1}{2}.x^2 = x$

$f\{(\phi(x)\}$, describes the function of functions of x, where f, $\phi(x)$ is the function and again $\phi(x)$ is the function of x.

Examples with Solution

Example 1: If followings are functions?

(i) $y=\sqrt{x}$, (ii) $y=x^3+5x$, (iii) $R=\{(2,3), (5,6), (2, 11)\}$

(i) If any positive or negative value of $\sqrt{x}$ is taken, then $y = \sqrt{x}$ would be the function. If both the values are not taken together, then there won'*t* be any function.

(ii) For each value of x, we will obtain a unique and certain value for $x^3 + 5x$. Therefore it is a function.

(iii) R is not function, because element 2 of non-empty set X is associated with the element 3 and 11 of Y, whereas each element of X should be associated with a certain and unique element of Y.

Example 2: Find out the domains of defined functions $x^2 - 1$, $\frac{1}{x^3-1}$, where x is real number.

Solution: (i) in $f(x) = x^2 - 1$, there is certain real value of function $f(x)$ for each real value of x, therefore its non-empty set is R.

(ii) In $f(x)=\frac{1}{x^3-1}$ the value of function $f(x)$ for value of x is uncertain, but if the value of x is all other balance real numbers, then value of $f(x)$ is also real. Therefore its domain is $R \sim \{1\}$.

Example 3: If x is a real number, then ascertain the domain of $f(x)=\frac{x^2-4}{x-2}$.

Solution: where $x = 2$, then the value of x is $\frac{0}{0}$, which is an undefined value. Besides for each real value of x, there is a defined real value of function $f(x)$. Therefore, domain of the function is set of all real numbers except 2 which can be expressed as under:

$D(f) = R \sim \{2\}$

Example 4: If $x \in R$, then find out the domain of $f(x)=\frac{1}{x^2-9}$.

Solution: $\frac{1}{x^2-9}=\frac{1}{(x-3)\ (x+3)}$ therefore, $x = 3$ and $x = -3$, denominator of the fraction is obtained in zero (0), therefore, the values of $f(3)$ and $f(-3)$ become uncertain. Besides these two value, we

Note

obtain a certain value of function $f(x)$ for each real value of x, therefore domain of f is the complete set of real numbers except 3 and -3, which can be read as $D(f) = R \sim \{3, -3\}$

Example 5: If $f(x) = 5x^2 + 8x + 7$, then find out the value of $f(-9)$.

Solution:

$\because$ $f(x) = 5x^2 + 8x + 7$

$\therefore$ $x = -9$

$f(-9) = 5(-9)^2 + 8(-9) + 7$

$= 405 - 72 + 7 = 340.$ **Ans.**

Example 6: If $f(x)$ = log, 'x' , then prove that $f(uv) = f(u) + f(v)$.

Solution:

$\because$ $f(x) = \log_e x$

then $f(u) = \log_e u$ ...(1)

then $f(v) = \log_e v$ ...(2)

$f(uv) = \log_e (uv)$

$= \log_e u + log_e v = f(u) + f(v)$

From equation (1) and (2) $f(uv) = f(u) + f(v)$

Example 7: If $f(x) = x^2 - x$, then find out the value of $f(y + 1) - y^2$.

Solution: here

here $f(x) = x^2 - x$

$\therefore$ $f(y + 1) = (y + 1)^2 - (y + 1)$

$= y^2 + 2y + 1 - y - 1 = y^2 + y$

$\therefore$ $f(y + 1) - y^2 = y^2 + y - y^2 = y$

Now $f(y + 1) - y^2 = y.$ **Ans.**

Example 8: If $f(x) = \dfrac{1}{(1 + \tan^2 x)}$ then find out the value of $f\left(\dfrac{\pi}{4}\right)$.

Solution:

$\because$ $f(x) = \dfrac{1}{(1 + \tan^2 x)}$

$\therefore$ $f\left(\dfrac{\pi}{4}\right) = \dfrac{1}{1 + \tan^2\left(\dfrac{\pi}{4}\right)} = \dfrac{1}{1+1} = \dfrac{1}{2}.$ **Ans.**

Task If $f(x) = 5x^2 + 8x + 7$, then find out the value of $f(-10)$ **Ans.:** 427

Example 9: If $f(\theta) = \dfrac{1 - 2\tan\theta}{1 + 2\tan\theta}$, then find out the value of $f\left(\dfrac{\pi}{4}\right)$.

Solution:

$\because$ $f(\theta) = \dfrac{1 - 2\tan\theta}{1 + 2\tan\theta}$

Note

$\therefore \qquad f\left(\frac{\pi}{4}\right) = \dfrac{1-2\tan\frac{\pi}{4}}{1+2\tan\frac{\pi}{4}} = \dfrac{1-2(1)}{1+2(1)}$

$$= \frac{1-2}{1+2} = -\frac{1}{3}.$$ **Ans.**

Example 10: If $f(x) = \log\dfrac{1+x}{1-x}$, then prove that $f\left(\dfrac{2x}{1+x^2}\right) = 2f(x)$.

Solution: $\therefore \quad f(x) = \log\dfrac{1+x}{1-x}$

Here putting $\dfrac{2x}{1+x^2}$ in place of x

$$f\left(\frac{2x}{1+x^2}\right) = \log\left[\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right] = \log\left[\frac{1+x^2+2x}{1+x^2-2x}\right]$$

$$= \log\left[\left(\frac{1+x}{1-x}\right)^2\right] = 2\log\left(\frac{1+x}{1-x}\right)$$

$$= 2f(x).$$ **Ans.**

Example 11: If $\phi(x) = \cot x$, then prove that $\phi(-x) = -\phi(x)$.

$\because \qquad \phi(x) = \cot x$

Solution: $\therefore \quad \phi(-x) = \cot(-x) = -\cot x = -\phi(x)$. **Ans.**

Example 12: If $f(x) = \dfrac{x}{x-1}$, then find out the value of $\dfrac{f(a/b)}{f(b/a)}$.

Solution: $f(a/b) = \dfrac{(a/b)}{(a/b)-1} = \dfrac{(a/b)}{(a-b)/b} = \dfrac{a}{a-b}$

And $= \dfrac{(b/a)}{(b/a)-1} = \dfrac{(b/a)}{(b-a)/a} = \dfrac{b}{b-a}$

$\therefore \qquad = \dfrac{a}{a-b}\cdot\dfrac{b-a}{b} = -\dfrac{a}{b}.$ **Ans.**

Example 13: Find out the value of $f\left(\dfrac{1}{x}\right)$ if $f(x) = x^3 - \dfrac{1}{x^3}$.

Solution: $\because \quad f(x) = x^3 - \dfrac{1}{x^3}$

Note

Putting $\frac{1}{x}$ in place of x

$$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - \frac{1}{\left(\frac{1}{x}\right)^3} = \frac{1}{x^3} - x^3 .$$ **Ans.**

Example 14: If $f(x) = \frac{x}{x^2+1}$, then find out the value of $\left\{f\left(\frac{a}{b}\right) - f\left(\frac{b}{a}\right)\right\}$.

Solution: $\because$ $f(x) = \frac{x}{x^2+1}$

$\therefore$ $$\left\{f\left(\frac{a}{b}\right) - f\left(\frac{b}{a}\right)\right\} = \frac{\frac{a}{b}}{\frac{a^2}{b^2}+1} - \frac{\frac{b}{a}}{\frac{b^2}{a^2}+1}$$

$$= \frac{ab}{a^2+b^2} - \frac{ab}{a^2+b^2} = 0 .$$ **Ans.**

Example 15: Find out the value of $f(x) + f(-x)$, if $f(x) = x^3 - \frac{1}{x^3}$.

Solution: $f(x) = x^3 - \frac{1}{x^3}$

$\therefore$ $$f(-x) = \frac{1}{(-x)^3} = -x^3 + \frac{1}{x^3}$$

$\therefore$ $$f(x) + f(-x) = x^3 - \frac{1}{x^3} - x^3 + \frac{1}{x^3} = 0 .$$ **Ans.**

Example 16: Prove that $f(x) + f(b) = f\left(\frac{a+b}{1+ab}\right)$, if $f(x) = \log_e \frac{1-x}{x^2+1}$.

Solution: $f(x) = \log_e \frac{1-x}{1+x}$

$\therefore$ putting $x = a$ $f(x) = \log_e \frac{1-a}{1+a}$

Now putting $x = b f(x) = \log_e \frac{1-b}{1+b}$

$\therefore$ $$f(a) + f(b) = \log_e \frac{1-a}{1+a} + \log_e \frac{1-b}{1-b} = \log_e \left\{\frac{1-a}{1+a} \times \frac{1-b}{1-b}\right\}$$

$$= \log_e \left\{\frac{1-a-b+ab}{1+a+b+ab}\right\} = \log_e \left\{\frac{(1+ab)-(a+b)}{(1+ab)+(a+b)}\right\}$$

$$= \log_e \left\{\frac{1-\frac{a+b}{1+ab}}{1+\frac{a+b}{1+ab}}\right\} = f\left(\frac{a+b}{1+ab}\right).$$ **Ans.**

Note

Example 17: Prove that $\dfrac{f(a)-f(b)}{1-f(a)\cdot f(b)}=\dfrac{a-b}{a+b}$ **if** $f(x)=\dfrac{x-1}{x+1}$.

Solution: $\because$ $$f(x) = \frac{x-1}{x+1}$$

$\therefore$ $$\frac{f(a)-f(b)}{1-f(a)\,f(b)} = \frac{\left(\frac{a-1}{a+1}\right)-\left(\frac{b-1}{b+1}\right)}{1-\left(\frac{a-1}{a+1}\right)\times\left(\frac{b-1}{b+1}\right)}$$

$$= \frac{\frac{(a-1)(b+1)-(b-1)(a+1)}{(a+1)\,(b+1)}}{1-\frac{(a-1)\,(b-1)}{(a+1)(b+1)}}$$

$$= \frac{(a-1)\,(b+1)-(b-1)\,(a+1)}{(a+1)\,(b+1)-(a-1)\,(b-1)}$$

$$= \frac{ab+a-b-1-ba-b+a+1}{(ab+a+b+1)-(ab-a-b+1)}$$

$$= \frac{2(a-b)}{2(a+b)} = \frac{a-b}{a+b}.$$ **Ans.**

Example 18: If domain $=\{-2, -1, 0, 1, 2\}$**, and** $f(-2)=4,\ f(-1)=1,\ f(0)=0\quad f(1)=1,\ f(2)=4,$ **then** f **expresses a function, prove and determine the range also.**

Solution: Given that domain is $=\{-2, -1, 0, 1, 2\}$

According to the question

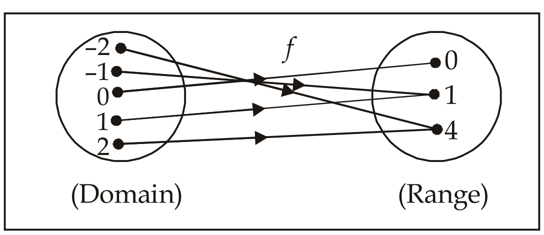
$f(-2) = 4 \Rightarrow f$-image of $-2 = 4$

$f(-1) = 1 \Rightarrow f$-image of $-1 = 1$

$f(0) = 0 \Rightarrow f$-image of $0 = 0$

$f(1) = 1 \Rightarrow f$-image of $1 = 1$

$f(2) = 4 \Rightarrow f$-image of $2 = 4$



Therefore f is expressed in the following explanation.

Note Since by f, each element of domain, is connected with the unique element of set {0, 1, 4} and any two or more than two elements of set {0, 1, 4} is not connected with the domain.

Therefore, f expresses a function, which is defined by $f(x) = x^2$ formula whose range is {0, 1, 4}

Example 19: Find out the value of $f\{g(x)\} - g\{f(x)\}$, if $f(x) = 1 + 2x$ and $g(x) = \frac{1}{2}x$.

Solution: Putting $g(x)$ in place of x in the function $f(x) = 1 + 2x$

$$f\{g(x)\} = 1 + 2g(x) = 1 + 2\cdot\left(\frac{1}{2}x\right) = 1 + x$$

Thus, putting $f(x)$ in place of x in the function $g(x) = \frac{1}{2}x$

$$g\{f(x)\} = \frac{1}{2}\cdot f(x) = \frac{1}{2}(1 + 2x) = \frac{1}{2} + x$$

$$\therefore \quad f\{g(x)\} - g\{f(x)\} = (1 + x) - \left(\frac{1}{2} + x\right) = \frac{1}{2}.$$ **Ans.**

Example 20: If function $f : R \to R$ is defined in the following way

$$f(x) = \begin{cases} 3x - 1, \text{ if } x > 3 \\ x^2 - 2, \text{ if } 2 \le x \le 3 \\ 2x + 3, \text{ if } x < -2 \end{cases}$$ **then find out the value of $f(2)$, $f(4)$, $f(-1)$ and $f(-3)$**

Solution: Since set (–2, 3) has 2 intervals, therefore here $f(x) = x^2 - 2$

$\therefore \quad f(2) = (2)^2 - 2 = 2$

Since set $(1, \infty)$ has 4 intervals, therefore $f(x) = 3x - 1$

$\therefore \quad f(4) = 3.4 - 1 = 11$

Since set (–2, 3) has – 1 intervals, therefore $f(x) = x^2 - 2$

Since $(-\infty, -2)$ has – 3 intervals, therefore $f(x) = 2x + 3$

$\therefore \quad f(-3) = 2(-3) + 3 = -3.$ **Ans.**

Example 21: Find out the range of function $\sqrt{(x-2)(4-x)}$.

Solution: Assume $f(x) = \sqrt{(x-2)(4-x)}$

Where $x > 4$, then $f(x) = \sqrt{(\text{negative value})}$ = assumed value

Where $x < 2$, then $f(x) = \sqrt{(\text{negative value})}$ = assumed value

Therefore, $f(x)$ is real for $2 \le x \le 4$

Therefore, range of function $\sqrt{(x-2)(4-x)}$ is $2 \le x \le 4$. **Ans.**

Note

Example 22: If $f: x \to x + 3$ and domain = $\{x: -2 \le x \le 2, x\}$, then find out its function (f) and range.

Solution: Range of f = $\{x: -2 \le x \le 2, x\} = \{-2, -1, 0, 1, 2\}$

Each element of domain f by f function is connected, therefore

Element –2 by function f is connected –2 + 3 = 1

Element –1 by function f is connected –1 + 3 = 2

Element 0 by function f is connected 0 + 3 = 3

Element 1 by function f is connected 1 + 3 = 4

Element 2 by function f is connected 2 + 3 = 5

Therefore the range of function = {1, 2, 3, 4, 5}

And function (f) = {(–2, 1), (–1, 2), (0, 3), (1, 4), (2, 5)}.

Questionnaire 1.1

1. Prove that $f(x)+f\left(\frac{1}{x}\right)=0$ if $f(x)=x^2-\frac{1}{x^2}$
2. Find out the value of $f(b)+f\left(\frac{1}{n}\right)$ if $f(b)=\frac{n}{1+n}$ [**Ans.:** 1]
3. If $f(x) = \log_e x$, then find out the value of $f(1)$ [**Ans.:** 0]
4. Find out the domain and range of function $f(x)=\frac{x}{1+x^2}$. If the function is single? [**Ans.:** R]
5. Find out the value of $\frac{f\left(\frac{1}{b}\right)}{f\left(\frac{b}{a}\right)}$ if $f(x)=\frac{x}{x-1}$ [**Ans.:** $(-a/b)$]
6. Prove that $x = f(y)$, if $y = f(x)=\frac{x+2}{x-1}$
7. Prove that $f(x) \cdot f(-x) = 1$ if $f(x)=\frac{2x-3}{2x+3}$
8. Prove that $f(x + y + z) = f(x) \cdot f(y) \cdot f(z)$, if $f(x) = e^x$
9. Find out the values of $f(-1)$ and $f\left(\frac{1}{x}\right)$ if $f(x)=x^2+\frac{1}{x^2}$ [**Ans.:** 2, $\left(\frac{1}{x^2}+x^2\right)$]
10. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, then find out the value of $f(6)$ [**Ans.: 30**]
11. Which of the following functions are not defined for the value of x?

    (a) $\frac{1}{x-3}$ (b) $\frac{1}{x^3-1}$ (c) $\frac{x^3-4}{x-2}$

    (d) $\tan x$ (e) $\sqrt{x}$ (f) $\frac{a^x-1}{x}$

Note

12. If $f(x) = 3x^2 + 2$ and $g(x) = 2x + 5$, then find out the value following:
    (a) $(f + g)\ (0),\ (f + g)\ (-2)$ (b) $(f - g)\ (3),\ (f - g)\ (-1)$
    (c) $(f\,g)\left(\frac{1}{2}\right),\ (fg)\ (1)$ (d) $\left(\frac{f}{g}\right)(2),\ \left(\frac{f}{g}\right)(-1)$

13. If $A = \{a, b, c, -1\}$, then explain if following relations are function of A in A. Describe the reason also:
    (a) $R_1 = \{(-1, c), (c, b), (a, b), (-1, -1)\}$
    (b) $R_2 = \{(a, b), (b, c), (c, -1), (-1, a)\}$

14. If $f: R \to R$, where $f(x) = \begin{cases} 2x+5, & x > 9 \\ x^2 - 1, & x \in (-9, 9) \\ x - 4, & x < -9 \end{cases}$ then find out the value of $f(3),\ f(12),\ f(-15)$ and $f\{f(5)\}$.

15. If $g : R \to R$, where $g(x) = \begin{cases} x^2 - x, & x \geq 2 \\ x - 2, & x < 2 \end{cases}$, find out the value of $g(5),\ g(0)$ and $g(-2)$ in 3, where $-3 \leq x < -1$ [**Ans.:** 20, –2, –4]

16. If $f(x) = \begin{cases} -6x - 3, \text{where } -1 \leq x \leq 0, \text{ then} \\ 3x - 3, \text{ where } 0 < x \leq 1 \end{cases}$
    (a) Find out the domain of function *f*.
    (b) Find out the value of $f(-2),\ f\left(\frac{1}{2}\right), f(0),\ f(-1),\ f(2)$
    (c) Solve the equation $2f(x) + 3 = 0$

Ans.: (a) $D = \{x : -3 \leq x \leq 1\}$ (b) $3, -3/2, -3, 3$ (c) $x = -1/4\ \&\ 1/2$

1.12 Graph of Functions

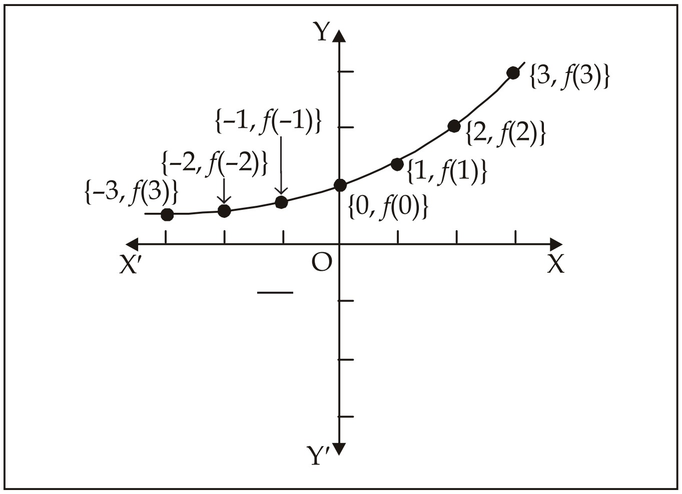
If $y = f(x)$, is the function of x, then assigning $x_1, x_2, \ldots, x_n, \ldots$ etc. to x, value of the function will become $f(x_1),\ f(x_2), \ldots, f(x_n), \ldots$ or $(y_1, y_2, \ldots, y_n, \ldots)$ respectively. If as per the symbolic geometry, putting x on x-axis and y or $f(x)$ on y-axis, marking the points as $\{x_1, f(x_1)\},\ \{x_2, f(x_2)\}, \ldots, \{x_n, f(x_n)\}, \ldots$ etc., a smooth curve is drawn, then the same curve will be call Graph of Functions.

For sketching the drawing of function $y = f(x)$, for the selected value of x, find out the value of y or $f(x)$ and draw a table

x	–3	–2	–1	0	1	2	3
$y = f(x)$	$f(-3)$	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$	$f(3)$

Now marking points $\{-3, f(-3)\},\ \{-2, f(-2)\},\ \{-1,\ f(-1)\},\ \{0, f(0)\},\ \{1, f(1)\},\ \{2, f(2)\},\ \{3,\ f(3)\}$, etc. on a square paper, draw a smooth curve. This will be called Graph of Functions.

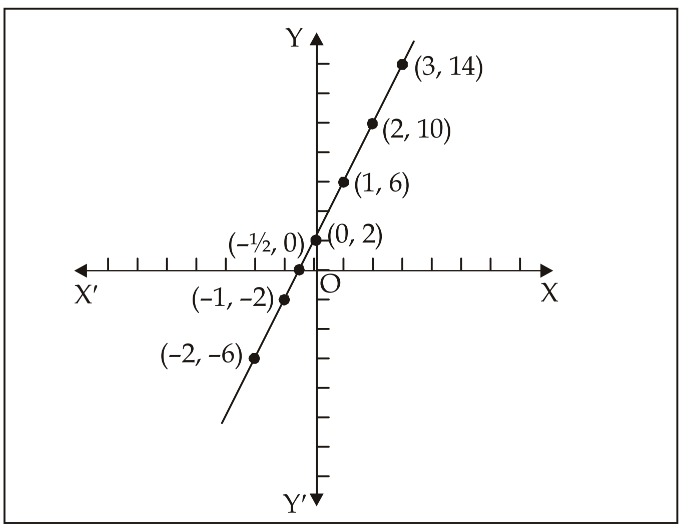
Note



Example 1: Draw a sketch of function $y = 4x + 2$.

Solution : Assigning different value to x and determining values of y, following table will be drawn

x	–2	–1	0	1	2	3	$-\frac{1}{2}$
y	–6	–2	+2	6	10	14	0



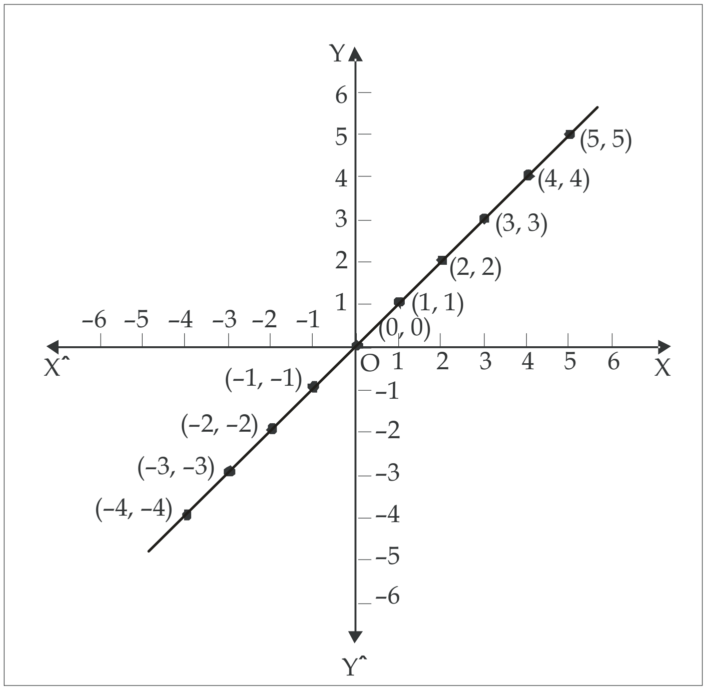
Now taking 1 box = 1 on x-axis and 2 boxes = 1 on y-axis, mark-up the points. Join them together. This is the final drawing. This is a simple line.

Example 2: Draw a sketch of function $y = x$.

Solution: Table for $y = x$

x	–4	–3	–2	–1	0	1	2	3	4	5
y	–4	–3	–2	–1	0	1	2	3	4	5

Note

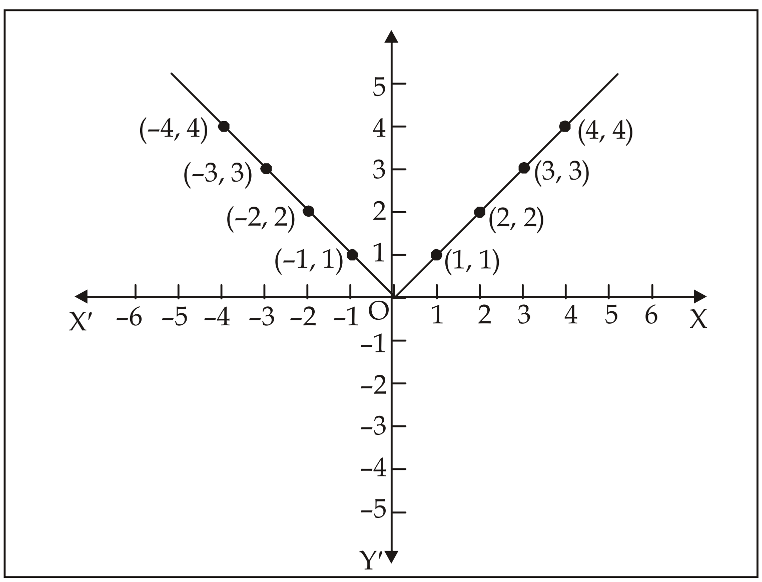


Example 3: Draw a sketch of $y = |x|$.

. . . . here $y = x$, if and $y = -x$, if $x < 0$

Table for $y = |x|$

x	4	3	−2	−1	0	1	2	3	4
y	4	3	2	1	0	1	2	3	4



Note

Example 4: Draw a sketch of function $y = \frac{|x|}{x}$ if $x \neq 0$.

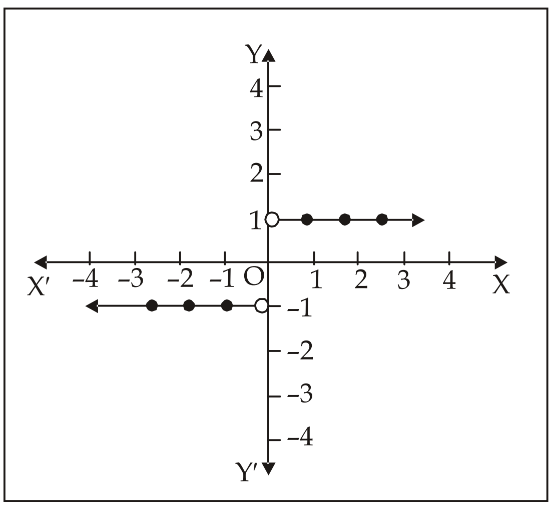
Solution: This function is called Signum function, which can be expressed as under

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$$

Table for function $y = \frac{|x|}{x}$

x	–4	–3	–2	–1	1	2	3	4
y	–1	–1	–1	–1	1	1	1	1

Drawing of this table has been reverse figure.



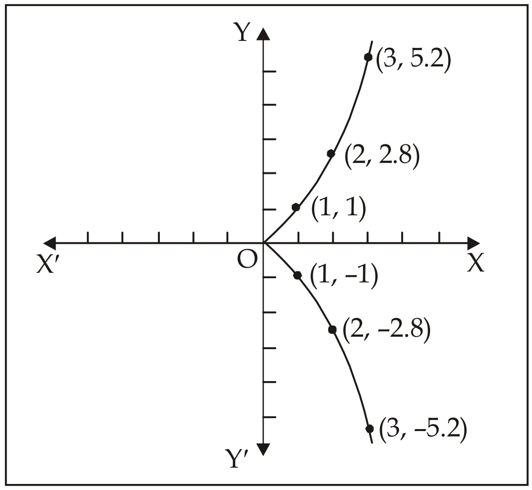
Example 5: Draw a sketch of $y^2 = x^3$.

Solution: Here if x is a negative, then value of y is imaginary. Therefore, we will take only positive value of x.

Table for $y^2 = x^3$

x	0	1	2	3	4
y	0	±1	±2.8	±5.2	±8

Note Drawing of this table has been reverse figure.

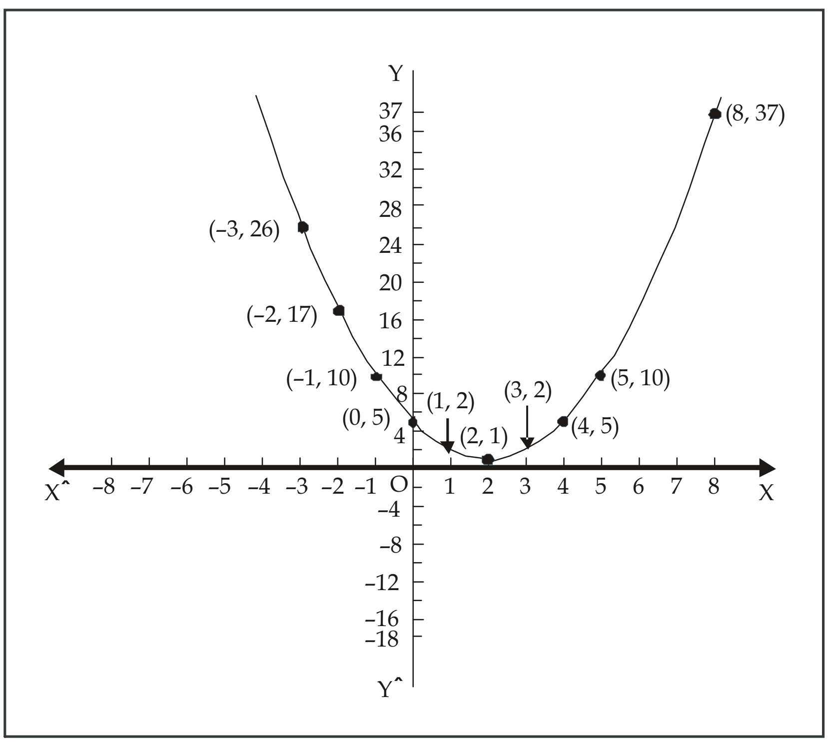


Example 6: Draw a sketch of $y = x^2 - 4x + 5$.

Solution: First of all create the following table

x	–4	–3	–2	–1	0	1	2	3	4	5	8
y	37	26	17	10	5	2	1	2	5	10	37

Here we see that as much as the value of x increases, the value of y also increases and marking $x \to \infty, y \to \infty$ when $x \to -\infty, y \to +\infty,$ we will find the following smooth curve.

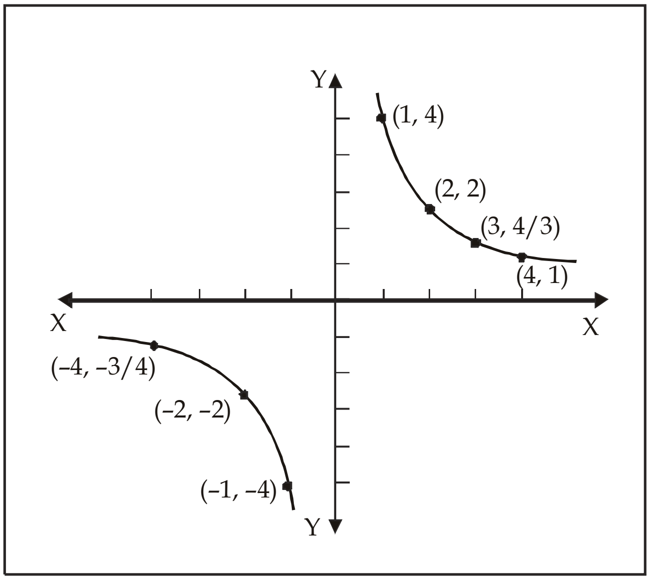


Example 7: Draw a sketch of $xy = 4$.

Solution: Here $y = \frac{4}{x}$, therefore if $x \to 0, y \to \infty$ and if $x \to \infty, y \to 0$

Table for $xy = 4$

x	$-\infty$	-4	-3	-2	-1	0	1	2	3	4	∞
x	0	-1	$-\frac{4}{3}$	-2	-4	∞	4	2	$\frac{4}{3}$	1	0



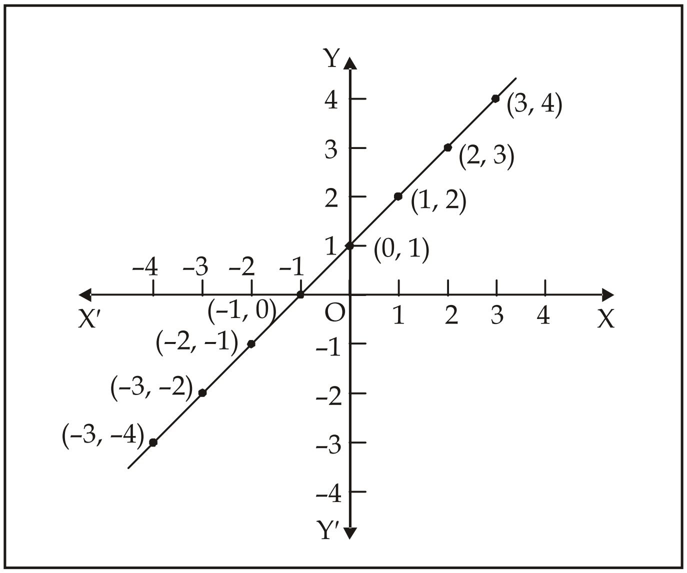
Example 8: Draw sketch of function $y = \frac{x^2 - 1}{x - 1}$.

Solution: Here function $x = 1$ is not defined. Besides for each value $y = \frac{(x-1)(x+1)}{x-1} = (x+1), (x \neq 1)$ is defined.

Table for function $y = \frac{x^2 - 1}{x - 1}$

x	0	1	2	3	-1	-2	-3	-4
y	1	2	3	4	0	-1	-2	-3

Note Drawing has been shown in the following figure



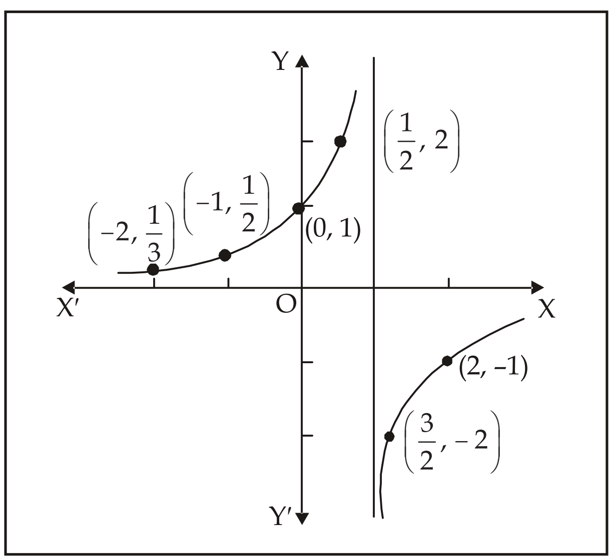
Example 9: Draw sketch of function $y = \frac{1}{1-x}$.

Solution: Assigning different value to x, and getting value of y, following table is prepared

Table for function $y = \frac{1}{1-x}$

x	–2	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1	2	∞	–2	–1

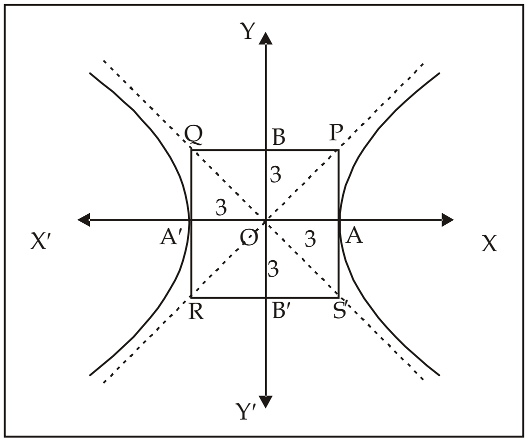
Drawing has been shown in the following figure



Note

Example 10: Draw a sketch for curve $x^2 - y^2 = 9$.

Solution: Following is the given curve $x^2 - y^2 = 9$ or $(x^2/9) - (y^2/9)$ which is a hyperbola of $(x^2/a^2) - (y^2/b^2) = 1$



Comparing them

$$a^2 = 9 \text{ and } b^2 = 9$$

$\Rightarrow$ $$a^2 = \pm 3 \text{ and } b = \pm 3$$

$\therefore$ $$= a = 3$$

And $$= b = 3$$

Now OA on x-axis = OA' = 3 and

OB on y-axis = OB' = 3

Taking distances into consideration an square $PQRS$ is drawn.

1.13 Use of Linear Functions in Economics

There are many uses of linear functions and simple lines in economics. The price of commodity in the market, is determined by demand and supply. If demand and supply function is given, then at the equilibrium level, we can determine such a level of price and quantity, where both the seller and customer would be satisfied.

Assume that demand function is $q = 16 - 4p$ …(i)

And supply function is $q = -8 + 4p$ ... (ii)

Where q shows the quantity of commodity and p is meant for price.

Now we will try to derive such a value of q and p, which would satisfy both the equations (i) and (ii). For this both the equations need to be solved. In the state of equilibrium demand and supply

$$16 - 4p = -8 + 4p$$

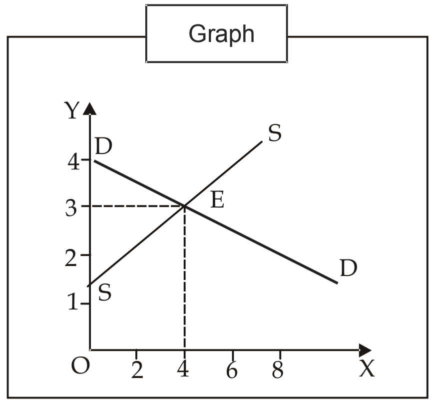
Or $$8p = 24$$

Or $$p = \frac{24}{8} = 3$$

Putting value of p in equation $q = 16 - (4 \times 3) = 16 - 12 = 4$

This way equilibrium price = 3 and quantity of demand and supply would be 4.

Note We can also show these equations into the following figure



In the figure on x-axis quantity of commodity and on y-axis price of commodity is shown. *DD* and *SS* express demand function and *SS* supply function respectively. Both of them intersect each other on point E. therefore, equilibrium will also be obtained on point *E*. at this point y-axis will show equilibrium price and x-axis will show the quantity of demand and supply.

Therefore, equilibrium price = 3 and quantity of commodity = 4.

Functions used in the economics

(a) Parabola

(b) Hyperbola

(c) Logarithmic Function

(d) Exponential Function

(a) Parabola – This is used heavily in economics. If y is dependent on x and its exponents, then y is called quardratic function. This can be expressed in the following manner:

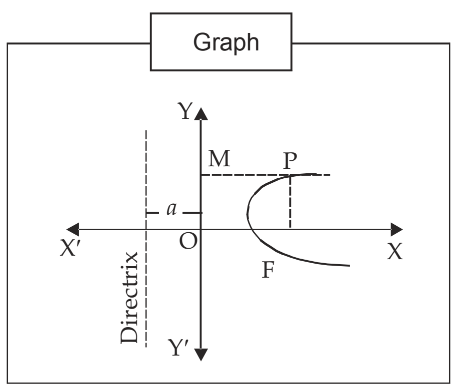
$$y = a + bx + cx^2 \qquad \text{...(i)}$$

Similarly If x is dependent on y and its exponents, then x is called quadratic function of y. This can be mathematically expressed in the following manner:

$$x = a + bx + cy^2 \qquad \text{...(ii)}$$

Both (i) and (ii) shows the general result of parabola. Here it is pertinent that in the equation (i) of parabola x is not the factor of y and (ii) it is same in both x^2 or y^2.

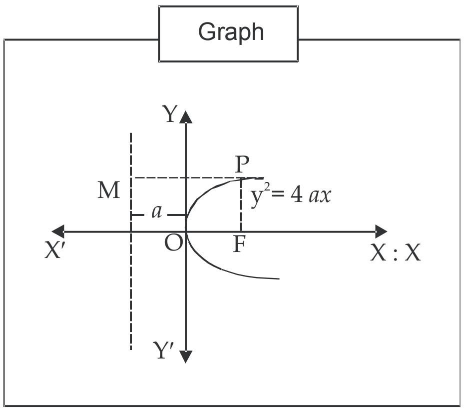
Here it is noteworthy that the center of above equations will not be on main point as shown in the figure.

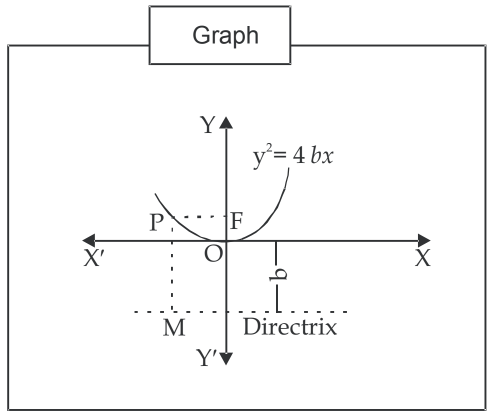


If equation of parabola is $y^2 = 4ax$, then the center of the equation will be on main point and parabola curve will be as per the figure. Similarly, if the equation of parabola is $x^2 = 4by$ then parabola curve will be as per the figure.

Note

O is called the Vertix of parabola in the above figures. For Parabola a simple line parallel to OY axis, is placed on top opposite to the other side of the curve, which is called Directrix. Similarly, for curve $x^2 = 4by$ on the vertrix b is placed parallel to directrx OX.





The simple line which goes from Vertrix and is parallel to Directrix, is called the axis of the parabola. The axis divides the parabola in two equal parts. On the axis from top at the distance of a or b on the opposite side of the directrix placed point is called focus. The special property of the parabola is that focus of any point and directrix are placed equally on it.

(b) **Hyperbola –** Similar to parabola, hyperbola is also very much used in the economics. This function can be expressed mathematically in the following manner:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where a, b are constants

Or $$\frac{x^2}{a^2} - 1 = \frac{y^2}{b^2}$$

then $$\frac{y^2}{b^2} = \frac{x^2 - a^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

$$y = \pm \frac{b}{a}\sqrt{(x^2 - a^2)}$$

If we follow the following facts, then

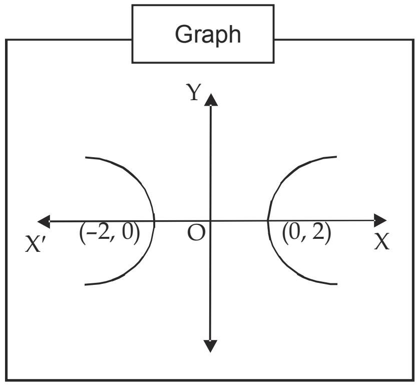
(i) Where $x = \pm a,\ y = 0$

(ii) $x > |a|$, assumed that $a = b = 3$ then $y = \pm\sqrt{(x^2 - 9)}$

(iii) If $x < 2$, then y's value would be imaginary, but for the other value of x, following values of y can be obtained

Value of x	value of y
+2 or –2	0 or 0
+3 or –3	$+\sqrt{5}$ or $-\sqrt{5}$
+4 or –4	$+\sqrt{12}$ or $-\sqrt{12}$

Note



If main point (0, 0) is the center of hyperbola, then hyperbola curve would be of following way

If center-point is (h, k), then the equation of hyperbola would be of different type

$$\frac{(x-h^2)}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Generally, hyperbola function is expressed as $xy = a$, whereas $a > 0$.

(c) **Logarithmic Function –** When y is a function of log x, but not x, then that function is called logarithmic function. The general equation for a logarithmic function is under:

$y = \alpha + \beta \log x$. Here if $\alpha = 0$ and $\beta = 1$, then the equation will be $y = \log x$.

Assuming that logarithmic equation is $y = 5 + 5 \log_{10} x$ (here we have assumed $\alpha = 5, \beta = 5$) then

$$y = -5 + 5 \times 0 = -5 \qquad (x = 1)$$
$$y = -5 + 5 \times 1 = 0 \qquad (x = 10)$$
$$y = -5 + 5 \times 2 = 5 \qquad (x = 100)$$
$$y = -5 + 5 \times 3 = 10 \qquad (x = 1000)$$

The curve for this would be as per Figure

It is noteworthy that the value of x in this equation can not be 0 or negative because it has no logs. Most of the times e is taken as base for log.

The general equation for binary-logarithmic function is as under

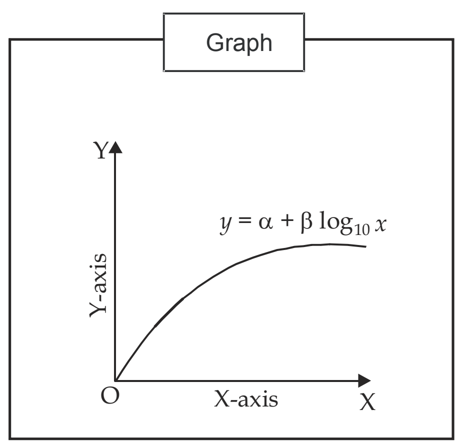
$$\log y = \alpha + \beta \log x$$

Or
$$\log y = \alpha \log x^{\beta}$$
$$\log y - x^{\beta} = \alpha$$

Or
$$\log \frac{y}{x^{\beta}} = a$$

$\therefore$
$$\frac{y}{x^{\beta}} = e^{a}$$

$\therefore$
$$y = e^{\alpha} x^{\beta} = A x^{\beta}$$



Where $A = e^{x}$

The special features of this function is as under

(i) If $\beta = 1$ or $y = \frac{A}{x}$ or $xy = A$, then it will become orthogonal hyperbola

Note

(ii) If $\beta < 1$ then y will increase at the rate of increase in x.

(iii) If $0 < \beta < 1$, then y decreases at the rate of increase in x.

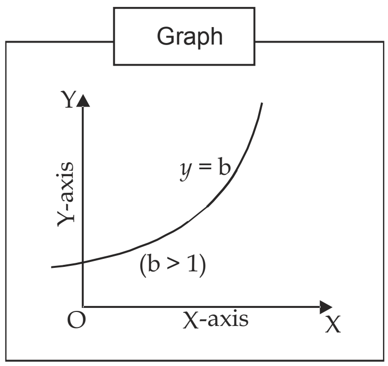
(iv) If $\beta < 1$, then y decreases when x increases.

Here it is pertinent to mention that positive exponent is production function and negative exponents are used in demand analysis. The special feature of this function is that exponent describes constant elasticity.

(d) Exponential function – These are also very much important in economics. In this function y is an exponential function of x instead of just a function. This can be expressed as

$y = \log_e x$, $y = e^x$, $y = e^{\sin x}$

If our exponential function is $y = b^x$, whereas b >, 1 then its curve will be as shown in the figure



Assume exponential function $x = AB^y$, then this equation can be expressed in logarithmic for as under:

$$\log x = \log A + y \log B$$

∴ $$y \log B = \log x - \log A$$

Or $$y = \frac{\log x}{\log B} - \frac{\log A}{\log B}$$

$$\frac{1}{\log B} = \beta \text{ and } -\frac{\log A}{\log B} = \alpha \text{ then}$$

Assume $$y = \alpha + \beta \log x, \text{ which is our logarithmic function.}$$

Example 1: Assume that demand and supply are a simple line. If at the different prices, quantity of demand and price is shows as under, then determine the demand and supply equation.

Price (P)	Demand (D)	Supply (S)
2	40	50
5	20	60

Solution: Marking various points in the table above, we can give it a linear shape

	Demand (D)	Supply (S)
1	(40, 2)	(50, 2)
2	(20, 5)	(60, 5)

Note Assume x = quantity and y = price, then equation of line joining (40, 2) and (20, 5) is

$$y \cdot y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Putting y_1, y_2, x_1, and x_2 in to the equation

$$y - 2 = \frac{5-2}{20-40}(x-40)$$

Or $$y - 2 = \frac{3}{-20}(x-40)$$

Or $$-20(y-2) = 3(x-40)$$

Or $$-20y + 40 = 3x$$

Or $$3x = 160 - 20y$$

Or $$x = \frac{160}{3} - \frac{20}{3}y = 53.33 - 6.66y$$

If $x = D$ in place of demand and $y = p$ in place of price, the demand equation will be

$$D = 53.33 - 6.66\,p$$

Similarly supply line passed through (50, 2) and (60, 5) points, and equation for joining line for these points will be

$$y - 2 = \frac{5-2}{60-5}(x-5)$$

Or $$y - 2 = \frac{3}{10}(x-50)$$

Or $$10y - 20 = 3x - 150$$

$$3x = 130 - 10y$$

$$x = 13 + 3.33y$$

If $x = S$ in place of Supply quantity, and $y = p$ in place of price, then supply equation will be

$S = 13 + 3.33p$

Example 2: Find out the focus and directrix of following parabola

(i) $y^2 = 16x$ **(ii) $x^2 = 20\,y$**

Solution: (i) comparing $y^2 = 16x$

Thus, directrix is placed at the distance of – 4 at the opposite side from Top *viz* $x = -4$ and focus is situated opposite to the directrix at (4, 0)

(ii) comparing $x^2 = 4by$ to $x^2 = -20y$

$b = 5$

thus, directrix is placed at the distance of +5 at the opposite side of the curve from Top viz $y = 5$ and focus is situated at (0, 5).

Self Assessment

2. Multiple Choice Questions:

(i) If $f : x \to y$ has been defined, then what would you call non-empty set x of this function (f)?

(a) Domain (b) range

(c) variable (d) constant

(ii) If variable number y is depended on variable number x, then what would you call $y = f(x)$?

(a) Multi-variable function (b) Single variable function

(c) Explicit function (d) Implicit Function

(iii) Function expressed in the form of fraction, in which numerators and denominators are of algebraic function of exponential value, is called

(a) algebraic function (b) Odd function

(c) rational function (d) undefined Function

1.14 Summary

- The rate of change in the value of other number and questions related to rate of such change and analysis and study of functions is referred to as Differential Calculus.
- Changing quantities are referred to as variables. Quantities for which values keep on changing viz which can be given indefinite numerical values are Variables.
- Numbers for which value is unchangeable under any process of mathematics, are constants.
- The principle of functions is mutually based on the related quantities.
- If function $f : X \rightarrow Y$ is defined, then non-empty set X is called domain of this mapping or function (f) and non-element set (Y) would be its range or codomain.
- If variable y is only dependent on variable x, then $y = f(x)$ is called functions of single variable. Similarly $y = f(q)$, $s = f(t)$, $v = f(t)$ etc. are functions of single variable.
- If variable u is dependent on variable x and y, then $u = f(x, y)$ is called functions of many variable.
- Function with different exponential value of x variable, in which factors are certain is called algebraic function.
- Function expressed in the form of fraction, in which numerators and denominators are of algebraic function of exponential value, is called rational function.
- There are many uses of linear functions and simple lines in economics. The price of commodity in the market, is determined by demand and supply. If demand and supply function is given, then at the equilibrium level, we can determine such a level of price and quantity, where both the seller and customer would be satisfied.
- The simple line which goes from Vertrix and is parallel to Directrix, is called the axis of the parabola.

1.15 Keywords

- ***Domain:*** Affected area.
- ***Range:*** Series, limits of variation.

1.16 Review Questions

1. Express the following functions with graph:

(a) $y = f(x) = 5 - 3x$; for range $x = -2$ to $x = 5$

(b) $y = f(x) = 2x^2 - 5x + 1$; for range $x = -5$ to $x = 7$

(c) $y = f(x) = \dfrac{24}{x}$ for range $x = 1$ to $x = 9$

Note

2. Determine the demand and supply function with the help of following table

Price (P)	Demand (D)	Supply (S)
1	100	50
2	50	75
3	0	100

3. Determine price and quantity with respect to equilibrium of demand and supply function of any market from the following:

   (i) $D = 15 - 3p; S = -10 + 2p$

   (ii) $D = 12 - 2p; S = -20 - 4p$

   (iii) $D = 50 - 4p; S = 2 + 10p - p^2$

4. The demand and supply function of any commodity is as under

   (i) $D = 100 - 10p \quad S = -12 + 9p$

5. Find out the vertex, focus and directrixs of parabola

   (i) $y = x^2 + 3x - 2$ **[Hints:** $-y = \left(x + \frac{3}{2}\right)^2 - \frac{17}{4}$]

   (ii) $(x - 2)^2 = 4y - 16$

6. L is Labour and K is capital in the production function $y = 5_L^{5} K^{5}$. If in short term capital is constant and $K = 100$, then draw the production function. Determine labour product (Y/L) and draw its curve.

Answers: Self Assessment

1. (i) variable, (ii) constant (iii) related, (iv) radius, (v) rational
2. (i) (a) (ii) (b) (iii) (c).

1.17 Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematical Economy – Michael Harrison, Patrik Walderon.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economics and Finance – Martin Norman.

Note

Unit 2: Limits and Continuity

CONTENTS

Objectives

After reading this unit, students will be able to :

- Easily Solve the Problems Related to Limit of Function.
- Easily Solve the Inter-related Problems of Limit and Value at $x = a$ of Function $f(x)$.
- Understand the Theorem of Limits and Related Questions.
- Know the Method of Finding the Limit of any Function.
- Know the Geometrical Definition.
- Know the Method to Finding Continuity of a Function at any Point and Solve the Related Problems.
- Understand the Theorem on Continuous Functions and Related Problems.

Introduction

1. Explain the following geometrical series

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \text{to infinite.}$$

Note

Total of 1 term = .5

Total of 2 terms = .75

Total of 3 terms = .875

Total of 4 terms = .9375

Total of 5 terms = .96875

Total of 6 terms = .984375

Total of 7 terms = .9921875

Total of 8 terms = .99609375

Total of 9 terms = .998046875

Total of 10 terms = .9990234375

……………….. = ………………

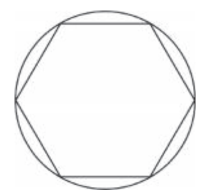
From the above it is apparent that as long as the term grows, total of the sequence moves towards 1, although the total could not be equal to 1.

Thus taking into account the sufficient number of terms, we can reduce the difference of total and 1 as much as we want.

2. Assume that in a circle with the given radius, a polygon is drawn. It is clear from the geometry that:

   (a) The area of polygon can never exceed from the area of circle, even if the polygon has more number of arms.

   (b) By increasing the number of arms indefinitely, we can reduce the difference of area between circle and polygon as much as we want.



In calculus the fact can be expressed that the limit of area of polygon drawn under a circle, where the number of arms of polygon increases increasingly (viz moves towards indefinite), is the area of circle.

$\lim_{n \to \infty}$ area of polygon =area of circle

3. Limit of a sequence.

Consider on the sequence 1, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$ for which $s_n = \frac{1}{n}$] where s_n expresses the sequence of n^{th} term. We see that as the value of n grows the value of $\frac{1}{n}$ goes down. In fact, selecting n sufficiently greater we can scale down $\frac{1}{n}$ as much as we want. Here if $n > 10,000$ then $\frac{1}{n} < 0.0001$ and if $n > 10^8$ then $\frac{1}{n} < 10^{-8}$. Thus we see that as the $m$inclines towards infinite $\frac{1}{n}$ tends to zero (0). We see that limit of sequence $\{s_n\}$ is 0 and this is taken as $\lim_{n \to \infty} s_n = 0$.

Now we will consider 0, $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...$ for which $s_n = \frac{n-1}{n}$. We see that as the n grows s_n, inclines closer to 1. In other words in fact, selecting n sufficiently greater we can scale down the statistical difference between s_n and 1 viz. $|s_n - 1|$ as much as we want. For this sequence.

$$\lim_{n \to \infty} s_n = 1.$$

Note

Generally for any sequences_1, s_1, s_2, s_3,......, choosing n sufficiently greater, if we can reduce the difference $|s_n - A|$ of s_n and any number A as much as we want, A will be called the sequence of $\{s_n\}$ and this can be expressed as

$$\lim_{n\to\infty} s_n = A$$

Now the sequence is $1^2, 2^2, 3^2, 4^2, ... n^2, ...$

And considering on this we see that as the value of n grows, accordingly s_n grows. Taking n sufficiently greater we can maximize s_n as much as we want. Here $n > 100$, $s_n > 10^8$ if $n > 10^4$. In fact S_n here can be made greater that any greatest number. This can be expressed as $S_n \to \infty$ if $n \to \infty$ is expressed.

4. Before defining the limits of function, it will be better to consider the following example. Some functions are undefined for special value of x, such as .

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

It is clear that the value of $\frac{x^2 - 1}{x - 1}$ at $x = 1$ is $\frac{0}{0}$ viz indeterminate.

If $x = 0.9$ then the value of function $\frac{(0.9)^2 - 1}{0.9 - 1} = 1.9$,

If $x = 0.99$ then the value of function $\frac{(0.99)^2 - 1}{0.99 - 1} = 1.99$,

If $x = 0.999$ then the value of function $\frac{(0.999)^2 - 1}{0.999 - 1} = 1.999$,

If $x = 0.9999$ then the value of function etc. $\frac{(0.9999)^2 - 1}{0.9999 - 1} = 1.9999$ and so on.

x	0.9	0.99	0.999	0.9999	---
$f(x)$	1.9	1.99	1.999	1.9999	---

Thus we see that as the value of x goes closer to 1, the value of function $\frac{x^2 - 1}{x - 1}$ tends toward 2.

Now if $x = 1.1$ then value of the function $= \frac{(1.1)^2 - 1}{1.1 - 1} = 2.1$

if $x = 1.01$ then value of the function $= \frac{(1.01)^2 - 1}{1.01 - 1} = 2.01$,

if $x = 1.001$ then value of the function $= \frac{(1.001)^2 - 1}{1.001 - 1} = 2.001$,

if $x = 1.0001$ then value of the function $= \frac{(1.0001)^2 - 1}{1.0001 - 1} = 2.0001$

Note

x	1.1	1.01	1.001	1.0001	---
$f(x)$	2.1	2.01	2.001	2.0001	---

Here we see that as the value of x decreases and goes down to 1, the value of function $\frac{x^2-1}{x-1}$ reaches to 2.

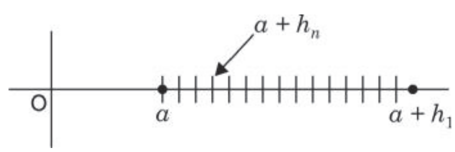
Thus, whether the value of x decreases down to 1 from a little more than $1 + \varepsilon$ (where e is smallest independent positive number) or increases to 1 from a little less than $1 + \varepsilon$, the value of the function tends to a certain value 2. This certain value 2 is called limit of the function $\frac{x^2-1}{x-1}$ at $x \to 1$. This can be expressed as under $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$

2.1 Limit of a Function

Assume $y = f(x)$ is a function and $h_1, h_2, ... h_n, ...$ is a set of positive numbers, which value is continually decreasing viz

$$h_1 > h_2 > h_3 > ...... > h_n > ...... > 0 \qquad ...(1)$$

And which, choosing n sufficiently greater, can be made smaller as desired. In this state, as the h_n goes down, the value of function decreases

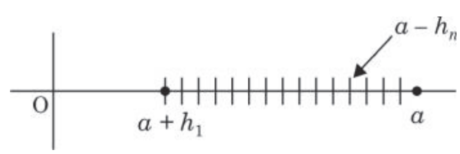


$$f(a + h_1), f(a + h_2), ..., f(a + h_n) \qquad ...(2)$$

If a number tends to A then this number is call right hand limit of function $f(x)$ at $x = a$ or this number A is called the right hand limit of function $f(x)$, when x tends to a. This can be expressed as:

$$\lim_{x\to a+0} f(x) = A = f(a+0)$$

Here we have considered only those values of x, which is greater that a. (in the figure only a at the right side)



Now we will consider those values of x, which is smaller (viz in the figure only at the left side of a.) As h_n goes down, the value of function $f(a - h_1), f(a - h_2)......, f(a - h_n),......$ tends to B. This number B is called the left hand limit of function f(x), when x = a.

This is expressed as $\lim_{x\to a-0} f(x) = B = f(a-0)$

If $A = B$ viz $\qquad \lim_{x\to a+0} f(x) = \lim_{x\to a-0} f(x)$

Then A is called limit of $f(x)$ at $x = a$

Set $h_1, h_2, ..., h_n, ...$ is a sequence, for which limit is 0. Similarly second makes sequence (2). Here is it to be specially noted that for limit to exist, like sequence (1) $f(a + h_n)$ every type of sequence should tend to A. viz the statistical difference of $f(a - h_n) - A$, choosing h_n sufficient smaller, can be reduced as desired. Assigning $a + h_n$ {or $a - h_n$} $= x$ or $| x - a | = h_n$ we can define the limit as under

Definition – At $x = a$, limit of function $f(x)$ is any number (assume A, which has the property that for each value of x for which $|x - a|$ viz $x - a$ is numerical value) sufficiently smaller (but not zero), $|f(x) - A|$ viz the numerical value of $f(x)$-A is smaller as desired.

Limit can also be defined with the following

Second definition of limit

When $x \to a$ (when x tends to a), the limit of function f(x) is any number (Assume A), which has the property that for any independent positive smallest number ε, a second number δ greatest than 0 can be obtained, for which $|f(x) - A| < \varepsilon$ for every values of x,

$$0 < | x - a | < \delta.$$

> *Notes* If at $x = a$, L is the limit (L) of $f(x)$, then this can be expressed as
>
> $$\underset{x \to a}{\text{limit}} f(x) = L \text{ or } \lim_{x \to a} f(x) = L.$$

2.2 Right Hand and Left Hand Limits

2.2.1 Right Hand Limit

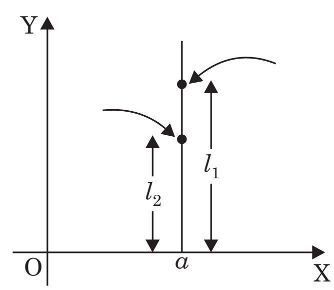
When the limit of function is obtained from the right hand of the independent variable, then it is called Right Hand Limit (R.H.L.) and applying positive (+) sign for the right side, this can be expressed as under

Right Hand Limit = $f(a + 0) = \lim_{x \to a^+} f(x) = l_1$.

2.2.2 Left Hand Limit

When the limit of function is obtained from the left hand of the independent variable, then it is called Left Hand Limit (L.H.L.) and applying negative (–) sign for the left side, this can be expressed as under

Left Hand Limit = $f(a - 0) = \lim_{x \to a^-} f(x) = l_2$.



2.3 Working Rules for Finding Right Hand Limit and Left Hand Limit

(i) To obtain the limit of right and left hand, replace x variable with $(x + h)$ ad $(x - h)$ respectively in the function

(ii) Thus, obtained function x, should be replaced with point (assume a)

(iii) Now at $h \to 0$ determine the limit of function [viz function obtained by (ii) to be put in the above, put $h = 0$].

Note

Discriptive Examples

Example 1: At x =2, find out the right hand and left hand limit of function $f(x) = \frac{1}{2+x}$.

Solution:

Right Hand Limit

I. $f(x+h) = \frac{1}{2+(x+h)}$

II. $f(2+h) = \frac{1}{2+(2+h)} = \frac{1}{4+h}$

III. $\lim_{h\to 0} f(2+h) = \lim_{h\to 0}\frac{1}{4+h}$

Or $f(2+0) = \frac{1}{4}$

Left Hand Limit

I. $f(x-h) = \frac{1}{2+(x-h)}$

II. $f(2-h) = \frac{1}{2+(2-h)} = \frac{1}{4-h}$

III. $\lim_{h\to 0} f(2-h) = \lim_{h\to 0}\frac{1}{4-h}$

Or $f(2-0) = \frac{1}{4}$.

Example 2: At x = 0, find out the right hand and left hand limit of function $f(x) = x\cos\left(\frac{1}{x}\right)$.

Solution:

Right Hand Limit

I. $f(x+h) = (x+h)\cos\frac{1}{x+h}$

II. $f(0+h) = h\cos\frac{1}{h}$

III. $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} h\cos\frac{1}{h}$

$= \lim_{h\to 0} h \times \lim_{h\to 0}\cos\frac{1}{h}$

= 0 [finite value in between –1 or 1]

Or $f(0+0) = 0$

Left Hand Limit

I. $f(x-h) = (x-h)\cos\frac{1}{x-h}$

II. $f(0-h) = -h\cos\left(-\frac{1}{h}\right)$

III. $\lim_{h\to 0} f(0-h) = \lim_{h\to 0}(-h)\cos\left(-\frac{1}{h}\right)$

$= \lim_{h\to 0}(-h)\lim_{h\to 0}\cos\left(-\frac{1}{h}\right)$

=–0 [finite value in between –1 or 1

Or $f(0-0) = 0$

2.4 Existence of Limit

If at $x = a$ both the limits of right and left hand of any function $f(x)$ exist and are equal, then at $x = a$, there is existence of limit of function $f(x)$ viz $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = l$ (Assumed)

Here l is called limit of the function and this is expressed as under $\lim_{x\to a} f(x) = l$.

Note

2.5 Distinction Between Limit and Value of a Function $f(x)$ on $x = a$

	$\lim_{x \to a} f(x) = l$	$f(a)$
1.	Limit of the function is a number, towards which function trends to respective when the value of independent variable x tends to a.	A number which is obtained putting $x = a$ in function $f(x)$
2.	To know h limit, we get to study the values of to $f(x)$ in the small neighborhood of $x = a$ and conclude	To find out the value at $x = a$, we get find out the value of $f(x)$ at only $x=a$
3.	At $x = a$, there can be existence of limit f function	At $x = a$ there can not be the value of function

2.6 Value and Limit of a Function

For the any value a of variable x, value and marginal value of function $f(x)$ should be different is not necessary. There are some kinds of function for which at $x = a$, value and marginal value of $f(x)$ are equal viz $f(a) = \lim_{x \to a} f(x)$.

2.7 Theorems on Limits

Did u know? If $f(x)$ and $\phi(x)$ are two function and then $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} f(x) = B$

then,

(*i*) $\lim_{x \to a} \{f(x) \pm f(x)\} = \lim_{x \to a} f(x) \pm \lim_{x \to a} \phi(x) = A \pm B.$

(*ii*) $\lim_{x \to a} \{k f(x)\} = k \lim_{x \to a} f(x) = kA.$

(*iii*) $\lim_{x \to a} \{f(x)\, \phi(x)\} = \lim_{x \to a} f(x). \lim_{x \to a} \phi(x) = AB$

Which has the property that limit of function f(x) is any number (A assumed)

(*iv*) $\lim_{x \to a} \dfrac{f(x)}{\phi(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} \phi(x)} = \dfrac{A}{B}$] if $B \neq 0$.

(*v*) $\lim_{x \to a} f(x) = \lim_{h \to a} f(h + a).$

2.8 Method of Finding the Limit of any Function

Although to find out the limit, we should find out the limits of right and left hand, but the intermediate level, we can often find out the limit directly.

If $y = f(x) = \dfrac{\phi(x)}{\Psi(x)}$, then to find out the limit, there would be following four types:

Type I: where $x \to a$: if $\phi(a) = 0$, $\Psi(a) = 0$, $f(a) = \dfrac{0}{0}$ then which is accessible. Limit of this be found out in the following way

Note

(i) Solving the function and assuming $x \neq a$, remove the common factors of numerator and denominator and

(ii) Then find out the value assigning a in place of x in the balance factors. This will be the limit of function $t\ x \to a$.

Type 2: Assigning $x = a + h$ to solve the limit of $f(x)$ at $x \to a$, here h is an indication of positive or negative increment. Find out the limit of $f(a+h)$. Where $h \to 0$ viz putting $h = 0$ in $f(a+h)$, get the value of function. This will be the limit of function at $x \to a$, because when

$$X = a + h \text{ and } x \to a \text{ then } h - 0$$

Self Assessment

1. Fill in the blanks:

(i) If any number tends to A, then at $x = a$ this number A of function $f(x)$ is called………..

(ii) If the limit of function is obtained from the value of left hand of independent variable, then it is called…………………

(iii) If at $x = a$ both the limits of right and left hand of any function $f(x)$ exist and are equal, then at $x = a$, there is ………………….. of limit of function $f(x)$.

(iv) For the any value a of variable x, value and …………… value of function $f(x)$ should be different is not necessary.

(v) Although to find out the limit, we should find out the limits of right and left hand, but the ………………….. level, we can often find out the limit directly.

EXAMPLES WITH SOLUTION

Sometimes it is not easy to divide with common factors. In this state applying the (ii) method, limit can be determined easily. Following examples make it clear. If it is not possible to divide the numerator and denominator with a common factor which is not zero, then after expansion in series or transformation, it can be possible.

Example 1: Find out the value of $\lim_{x \to a} \frac{x^3 - a^3}{x - a}$.

Solution:

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} \frac{(x-a)(x^2 + a^2 + ax)}{(x-a)}$$

$$= \lim_{x \to a} (x^2 + a^2 + ax), \qquad [\because x \neq a]$$

$$= a^2 + a^2 + a^2 = 3a^2.$$ **Ans.**

Example 2: Find out the value of $\lim_{x \to 0} \frac{\log_e (1 + x) - x}{x^2}$.

Solution:

$$\lim_{x \to 0} \frac{\log_e (1+x) - x}{x^2} = \lim_{x \to 0} \frac{\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right] - x}{x^2}$$

$$= \lim_{x \to 0} \frac{-\frac{x^2}{2} + \frac{x^3}{3} - \ldots}{x^2} = \lim_{x \to 0} \frac{x^2\left[-\frac{1}{2} + \frac{x}{3} - \ldots\right]}{x^2}$$

Note

$$= \lim_{x \to 0} \left[-\frac{1}{2} + \frac{x}{3} - \ldots \right]$$

$$= -\frac{1}{2} + 0 - \ldots = -\frac{1}{2}.$$ **Ans.**

Task Find the value of $\lim_{x \to b} \frac{x^3 - b^3}{x - b}$ **[Ans.:** $3b^2$**]**

Example 3: Find out the value of $\lim_{x \to a} \left(\frac{x^2 - 2ax + a^2}{x - a} \right)$.

Solution:

$$\lim_{x \to a} \left(\frac{x^2 - 2ax + a^2}{x - a} \right)^2 = \lim_{x \to a} \frac{(x-a)^2}{x - a}$$

$$= \lim_{x \to a} (x - a) = a - a = 0.$$ **Ans.**

Example 4: Find out the value of $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 + x - 6}$.

Solution:

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 + x - 6} = \lim_{x \to 2} \frac{(x-2)(x-1)}{(x-2)(x+3)} = \lim_{x \to 2} \frac{(x-1)}{(x+3)} \qquad [\because x \neq 2]$$

$$= \frac{2-1}{2+3} = \frac{1}{5}.$$ **Ans.**

Type 3: Limit of Irrational Function

If $y = f(x)$ and $f(x)$ is an irrational function, then to find out the limit of $f(x)$ first of all, the function should be made rational and then applying 1 method, find out the limit

Example 5: Find out the value of $\lim_{x \to 0} \frac{\sqrt[3]{1 + x - 1}}{x}$.

Solution:

$$\lim_{x \to 0} \frac{\sqrt[3]{1 + x - 1}}{x} = \lim_{x \to 0} \frac{(1+x)^{\frac{1}{3}} - 1}{x}$$

$$= \lim_{x \to 0} \frac{1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!} x^2 + \ldots\ldots\infty - 1}{x}$$

$$= \lim_{x \to 0} \frac{1}{3} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right) x}{2!} + \ldots\ldots \infty = \frac{1}{3} + 0 = \frac{1}{3}.$$ **Ans.**

Note

Example 6: Prove that $\lim_{x \to 0} \frac{(1+x)^{1/2} - (1-x)^{1/2}}{x} = 1$.

Solution: $\lim_{x \to 0} \frac{(1+x)^{1/2} - (1-x)^{1/2}}{x} = \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

$$= \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \lim_{x \to 0} \frac{(1+x) - (1-x)}{x\,(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \to 0} \frac{2x}{x\,(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} = \frac{2}{1+1} = 1.$$

Ans.

Type 4: To know the limit of function when $x \to \infty$.

In this condition, first of all putting $x = \frac{1}{Z}$, change the form of limit from $x \to \infty$ to $Z \to 0$. Then applying method 3, find out the limit

Example 7: Find out the value of $\lim_{x \to \infty} \frac{9x^2 + 3x + 7}{5x^2 + 2x + 1}$.

Solution: It is clear that if $x = \infty$ is put, then numerator and denominator would become $\frac{\infty}{\infty}$, therefore dividing both numerator and denominator with x^2

$$\lim_{x \to \infty} \frac{9x^2 + 3x + 7}{5x^2 + 2x + 1} = \lim_{x \to \infty} \frac{9 + \frac{3}{x} + \frac{7}{x^2}}{5 + \frac{2}{x} + \frac{1}{x^2}} = \frac{9+0+0}{5+0+0}$$

$$= \frac{9}{5}, \text{ whereas } x \to \infty$$

Since $x \to \infty$ therefore, $\frac{3}{x}, \frac{7}{x^2}, \frac{2}{5x}, \frac{1}{x^2}$ will tend to zero and function will tend to $\frac{9}{5}$

Therefore, $\lim_{x \to \infty} \frac{9x^2 + 3x + 7}{5x^2 + 2x + 1} = \frac{9}{5}$. **Ans.**

Example 8: Prove that $\lim_{x \to a} \frac{x^m - a^m}{x - a} = ma^{m-1}$ where m is a rational number.

Solution:

First method: Left Hand $= \lim_{x \to a} \frac{x^m - a^m}{x - a}$

Note

$$= \lim_{x \to a} \frac{(x-a)(x^{m-1} + x^{m-2}a + x^{m-3}a^2 + ... + a^{m-1})}{(x-a)}$$

$$= \lim_{x \to a} (x^{m-1} + x^{m-2}a + x^{m-3}a^2 + ... + a^{m-1}) \text{ ,(when } x \neq a)$$

$$= a^{m-1} + a^{m-1} + ... + a^{m-1} = ma^{m-1} = \text{Left Hand.}$$

Second Method: $\lim_{x \to a} \dfrac{x^m - a^m}{x - a}$, (if $x = a + h$ whereas $x \to a, h \to 0$)

$$= \lim_{h \to 0} \frac{(a+h)^m - a^m}{(a+h) - a} = \lim_{h \to 0} \frac{a^m \left(1 + \frac{h}{a}\right)^m - a^m}{h}$$

$$= \lim_{h \to 0} \frac{a^m \left\{1 + m\left(\frac{h}{a}\right) + \frac{m(m-1)}{2}\left(\frac{h}{a}\right)^2 + ... - 1\right\}}{h}$$

$$= \lim_{h \to 0} \frac{a^m h \left\{\frac{m}{a} + \frac{m(m-1)}{2}\frac{h}{a^2} + ...\right\}}{h}$$

$$= \lim_{h \to 0} a^m \left\{\frac{m}{a} + \frac{m(m-1)}{2!}\frac{h}{a^2} + ...\right\}, (h \neq 0)$$

$$= a^m \times \frac{m}{a} = ma^{m-1} = \text{Left Hand.}$$

Notes To determine the limit of algebraic function, this is used in the form of major formula.

Example 9: Find out the value of $\lim_{x \to 0} \left(1 + \dfrac{p}{x}\right)^x$.

Solution: $\lim_{x \to \infty} \left(1 + \dfrac{p}{x}\right)^x$

Upper limit $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$

$$\lim_{x \to \infty} \left(1 + \frac{p}{x}\right)^x = e^p, \text{ or } \lim_{x/p \to \infty} \left\{\left(1 + \frac{p}{x/p}\right)^{x/p}\right\}^p = e^p.$$

Ans.

Example 10: Find out the value of $\lim_{x \to 1} \left(\dfrac{\log x}{1 - x}\right)$.

Solution: $x = 1 + h$, where h is smallest

Note

If $$\lim_{x\to 1}\frac{\log x}{1-x} = \lim_{h\to 0}\frac{\log(1+h)}{-h} = \lim_{h\to 0}\frac{h-\frac{h^2}{2}+\frac{h^3}{3}-....}{-h}$$

$$= \lim_{h\to 0} -\left(1-\frac{h}{2}+\frac{h^2}{3}-....\right) \text{ (when } h\neq 0) = -1.$$ **Ans.**

Example 11: Find out the value of $\lim_{x\to 0}\frac{e^x - e^{-x}}{x}$.

Solution: $$\lim_{x\to 0}\frac{e^x-e^{-x}}{x} = \lim_{x\to 0}\frac{1}{x}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+....\right)-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+....\right)\right]$$

$$= \lim_{x\to 0}\frac{1}{x}\left[2\left(x+\frac{x^3}{3!}+....\right)\right]$$

$$= \lim_{x\to 0}\left[2\left(1+\frac{x^2}{3!}+....\right)\right] \qquad [\because \quad x\neq 0]$$

$= 2.$ **Ans.**

Example 12: Find out the value of $\lim_{x\to 0}\frac{e^x - 1}{x}$.

Solution: $$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}....\infty\right)-1}{x}$$

$$= \lim_{x\to 0}\frac{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+....\infty\right)}{x} = \lim_{x\to 0}\frac{x\left(1+\frac{x}{2!}+\frac{x^2}{3!}+....\infty\right)}{x}$$

$$= \lim_{x\to a}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+....\infty\right)$$

= (when $x\neq 0$) = 1. **Ans.**

Example 13: Find out the value of $\lim_{x\to\infty}\frac{1+2+3+....+x}{x^2}$.

Solution: $$\lim_{x\to\infty}\frac{1+2+3+....+x}{x^2} = \lim_{x\to\infty}\frac{x(x+1)}{2x^2}$$

$$= \lim_{x\to\infty}\frac{x^2\left(1+\frac{1}{x}\right)}{2x^2} = \lim_{x\to\infty}\frac{1+\frac{1}{x}}{2}$$

$$= \frac{1}{2}. \qquad \left[\because \lim_{x\to\infty}\frac{1}{x}=0\right]$$

Therefore, the desired value is = $\frac{1}{2}$. **Ans.**

Note

Example 14: Prove that $\lim_{x\to 0}\frac{a^x - b^x}{x} = \log_e \frac{a}{b}$. or find out the value of $\lim_{x\to 0}\frac{a^x - b^x}{x}$.

Solution: It is clear that when x = 0 then numerator and denominator become 0. Therefore expanding a^x and b^x with the help of exponential theory

$$\text{Left hand} = \lim_{x\to 0}\frac{a^x - b^x}{x} = \lim_{x\to 0}\frac{e^{x\log_e a} - e^{x\log_e b}}{x}$$

$$= \lim_{x\to 0}\frac{\left[1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + ...\right] - \left[1 + x\log_e b + \frac{(x\log_e b)^2}{2!} + ...\right]}{x}$$

$$= \lim_{x\to 0}\left\{\log_e a - \log_e b + \frac{x[(\log_e a)^2 - (\log_e b)^2]}{2!} + ...\right\}$$

$$= \log_e a - \log_e b = \log_e\left(\frac{a}{b}\right) = \text{Right hand}$$ **[Proven]**

Example 15: Function f(x) is defined as under

$$f(x) = \begin{cases} 1, & \text{when } x > 0 \\ -1, & \text{when } x < 0 \\ 0, & \text{when } x = 0 \end{cases}$$

Show that $\lim_{x\to\infty} f(x)$ is not in existence

Solution: here $f(0 + h) = 1$.

So that Right hand limit $= f(0 + 0) = \lim_{x\to 0+0} f(x) = \lim_{x\to 0+0} (1) = 1$

and $f(0 - h) = -1$

Therefore Left hand limit $= f(0 - 0) = \lim_{x\to 0-0} f(x) = \lim_{x\to 0-0} (-1) = -1$.

$\because$ $\lim_{x\to 0+0} f(x) \neq \lim_{x\to 0-0} f(x)$

$\lim_{x\to 0} f(x)$ is not in existence

Example 16: Show that $\lim_{x\to 2}\frac{|x-2|}{x-2}$ is not in existence.

Solution: Where $x > 2, |x - 2| = (x - 2)$

$\therefore$ $\text{R.H.L.} = \lim_{x\to 2+0} f(x) = \lim_{x\to 2+0}\frac{|x-2|}{x-2} = \lim_{x\to 2+0}\frac{x-2}{x-2} = 1$

$\text{L.H.L.} = \lim_{x\to 2-0} f(x) = \lim_{x\to 2-0}\frac{|x-2|}{x-2} = \lim_{x\to 2-0}\frac{-(x-2)}{x-2} = -1$

$\therefore$ R.H.L. $\neq$ L.H.L.

Therefore, $\lim_{x\to 2}\frac{|x-2|}{x-2}$ is not in existence

Note

Example 17: Function $f(x)$ for which $f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$, show that $\lim_{x \to 1} f(x) = 1$.

Solution: $\lim_{x \to 1+0} f(x) = \lim_{x \to 1+0} f(1+h) = \lim_{h \to 0} (1+h)^2 = 1$

and $\lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} f(1-h) = \lim_{h \to 0} (1-h)^2 = 1$

$$\lim_{x \to 1+0} f(x) = \lim_{x \to 1-0} f(x) = 1$$

Therefore, $\lim_{x \to 1} f(x) = 1.$ **Ans.**

Example 18: Function $f(x)$ is defined as under.

$$f(x) = \begin{cases} x, & \text{when } 0 \leq x < \frac{1}{2} \\ 0, & \text{when } x = \frac{1}{2} \\ 1 - x, & \text{when } \frac{1}{2} < x \leq 1 \end{cases}$$

Find out the value of $\lim_{x \to 1/2} f(x)$

Solution: R.H.L. $= f\left(\frac{1}{2} + 0\right) = \lim_{x \to 1/2+0} f(x)$

$$= \lim_{h \to 0} f\left(\frac{1}{2} + h\right) = \lim_{h \to 0} \left[1 - \frac{1}{2} - h\right] = \lim_{h \to 0} \left[\frac{1}{2} - h\right] = \frac{1}{2}$$

L.H.L. $= f\left(\frac{1}{2} - 0\right) = \lim_{x \to 1/2-0} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$

$$= \lim_{h \to 0} \left(\frac{1}{2} - h\right) = \frac{1}{2}$$

$\therefore$ R.H.L. = L.H.L.

Therefore $\lim_{x \to 1/2} f(x) = \frac{1}{2}.$ **Ans.**

Questionnaire 2.1

Find out the value of following:

1 $\lim_{x \to 0} (7x^2 - 5x + 1).$

2. $\lim_{x \to 0} \frac{e^x - 1}{e^x}.$

3. $\lim_{x \to 0} \frac{e^{x/2} - 1}{x}.$

4. $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}.$

5. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$

6. $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}.$

Note

7. $\lim_{x\to 0} \frac{y^2}{x}$ where $y^2 = ax + bx^2 + cx^3$.

8. $\lim_{x\to \infty} \frac{x^2}{1+x^2}$.

9. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$.

10. $\lim_{x\to a} \frac{x^4 - a^4}{x^2 - a^2}$.

11. $\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}$.

12. $\lim_{x\to 0} \frac{x^3 - 8}{x - 2}$.

13. $\lim_{x\to a} \frac{x^5 - a^5}{x^2 - a^2}$.

14. $\lim_{x\to 1} \frac{x^n - 1}{x - 1}$.

15. $\lim_{x\to 2} \frac{x^2 - 5x + 6}{x^2 - 6x + 8}$.

16. $\lim_{x\to \infty} \frac{3x^2 + 2x - 5}{x^2 + 5x + 1}$.

17. $\lim_{x\to \infty} \left(1 - \frac{1}{x}\right)^x$.

18. $\lim_{x\to 0} \frac{x}{\sqrt{1+x} - 1}$.

19. If $f(x) = |x|$, then show that $\lim_{x\to 0} f(x) = 0$.

20. If $f(x) = \frac{|x-1|}{x-1}$. then show that $\lim_{x\to 1} f(x)$ is not in existence

21. Function f(x) is defined as under:

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < 1 \\ 2, & \text{if } x = 0 \\ 2 - x, & \text{if } x \ge 1 \end{cases}$$

Show that $\lim_{x\to 1} f(x) = 1$.

22. Prove that $\lim_{x\to \infty} \frac{e^{x^2} - 1}{e^{x^2} + 1} = 1$.

23. Find out the value of $\lim_{x\to 1} \frac{x^m - 1}{x^n - 1}$.

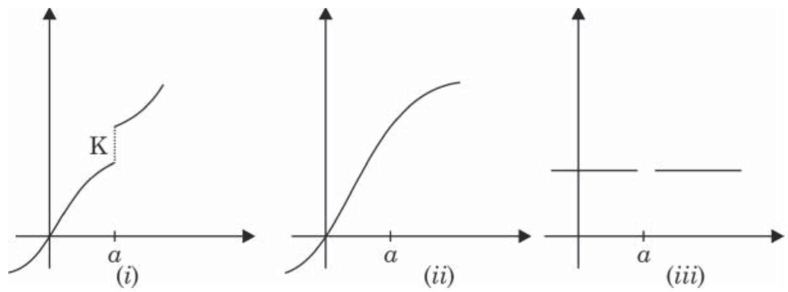
Answers

1. 1	2. 0	3. $\frac{1}{2}$	4. 3	5. 4
6. $\frac{1}{2}$	7. a	8. 1	9. 1	10. $2a^2$
11. $\frac{3}{2}$	12. 4	13. $\frac{5}{2}a^3$	14. n	15. $\frac{1}{2}$
16. 3	17. $\frac{1}{e}$	18. 2	23. $\frac{m}{n}$	

Note

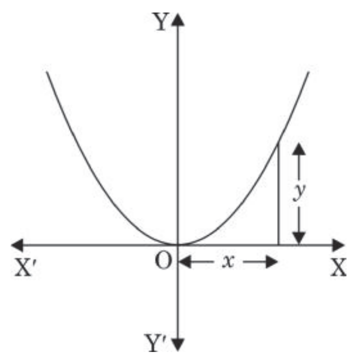
2.9 Geometrical Definition

Continuity and Discontinuity: If drawing a graph of any function $f(x)$, the curve which we obtain. And it is in such a way that it does not break on any point $x = a$, then the function is called continuous on that point.



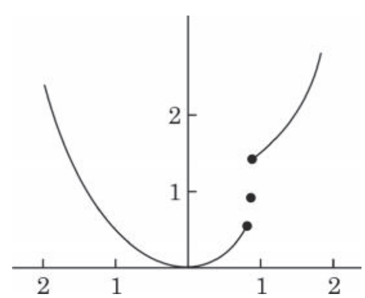
Contrary to this if at any point $x = a$, the graph of function $f(x)$ breaks, then at point $x = a$, function $f(x)$ will be called discontinuous. Looking at the graphs of the functions it is understood if in figure (i) of $f(x)$ at $x = a$, any height k is a jump that if we are moving from left hand to right hand of the graphs, then $x = a$ is a break and we have to lift the pencil from one side of $x = a$ to another side. The curve of function (iii) at $x = a$ is break. Function (ii) is not break at $x = a$ viz the curve is in such a way that it does not break. Therefore, at $x = a$, function $f(x)$ is continuous.

Example 1: If we draw the graph of function $y = x^2$, then this is a parabola as shown in the shape. On the graph of such functions if we from right to left or from left to right, then we will not observe any break and if we draw pencil on this graph from one side to another side, then in between we will not be required to lift the pencil. This is a continuous function.



Example 2: If we draw the graph of , then we will find two parabola and one point (1,1) as shown in the figure. Here we move from right to left or left to right, then at $x = 1$ there is a break in sequence. And if we start moving from one side of $x = 1$ to another side, then at $x = 1$ we have to lift the pencil. This function is discontinuous.

$$y = \begin{cases} \frac{1}{2}x^2, & \text{if } x < 1 \\ 1, & \text{if } x = 1 \\ 1 + \frac{1}{2}x^2, & \text{if } x > 1 \end{cases}$$



We see that if at any point the function is discontinuous then at the point there is a sudden jump in the value of function, whereas

Note

at the change of value of x, the value of function changes gradually. Thus, if at any point $x = a$, function is continuous then the value of $|f(x)-f(a)|$, can be scaled down at the small neighborhood of a choosing the smallest value of x.

2.10 Continuity of a Function at any point

Cauchy's definition: A function $f(x)$ is called continuous at $x = a$. If for one chosen arbitrary positive number ε, which is the smallest one, but not a zero, we can obtain a positive number δ based on ε that $|f(x) - f(a)| < \varepsilon$, for every values of x, for which $0 < |x - a| < \delta$

Viz at any point x of interval (a-δ a+δ), the positive difference of function $f(x)$ and $f(a)$ is smaller to arbitrarily defined positive number e, then at x = a, function $f(x)$ is called continuous.

Alternate definition: Function $f(x)$ is called continuous at $x = a$ if there is existence of $\lim\limits_{x \to a}$ and that is equal to the value of function at $x = a$.

Notes $f(x) = f(a)$ viz limit of [$f(x)$ where $x \to a$] = [Value of function, where $x = a$]

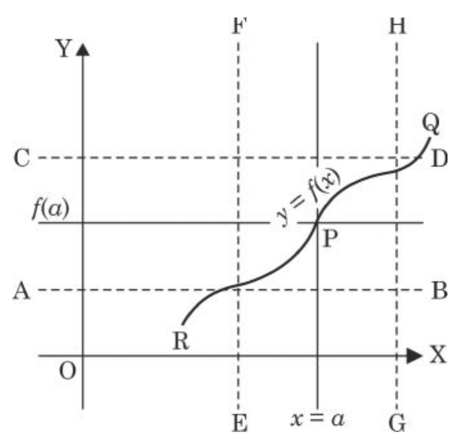
Therefore, if $f(a+0) = f(a - 0)$, then $f(x)$ at $x = a$ is called continuous otherwise it is discontinuous. Function $f(x)$ is called continuous on point $x = a$, if the function satisfies three conditions:

1. If at $x = a$, function $f(x)$ is defined viz there is a certain value of function at $x = a$
2. When the value of x tends to a, and $f(x)$ inclines towards any limit viz there is existence of $\lim\limits_{x \to a} f(x)$
3. $\lim\limits_{x \to a} f(x) = f(a)$

2.11 Geometrical Meaning of Continuity

Assume QR is the graph of function $f(x)$. In this graph consider on point $P[a,f(a)]$. If ε is any arbitrarily taken positive number, then lines $y = f(a) - \varepsilon$, $y = f(a) + \varepsilon$ will be parallel to x-axis and down and up to P.

If each points of the graph of function $f(x)$ is between the two lines $x = a - \delta$, $x = a + \delta$ is also available in between $y = a - \varepsilon = \text{a} + \varepsilon$, then function $f(x)$ at $x = a$ is continuous.



For example: (i) Constant function $f(x) = c$, for each real value of x is continuous.

(ii) function $f(x) = \sin x$ and $f(x) = \cos x$, for each real value of x is continuous.

2.12 Method to Finding Continuity of a Function at any Point

From the definition of limit it is clear that existence of $\lim\limits_{x \to a} f(x)$ can happen only

when left hand limit of $f(x)$ = right hand limit of $f(x)$ viz

Note

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Viz $\quad f(a-0) = f(a+0)$

Therefore, to show the continuity of function $f(x)$ at point $x = a$, we should show that at this point left hand limit of $f(x)$ = right hand limit of $f(x)$ = value of function viz

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Or $\quad f(a-0) = f(a+0) = f(a)$

For Right Hand Limit in $f(x)$, put $x = a + h$, where $h \to 0$ where $x \to$ a

For Left Hand Limit in $f(x)$, put $x = a - h$, where $h \to 0$ where $x \to$ a

2.13 Continuity of a Function in an Interval

A function $f(x)$ is called continuous in an open interval (a,b) if it is continuous for every values of x in this interval (a,b)

Function $f(x)$ is called continuous in any closed interval, if

(i) It is continuous for each value of x for which $a < x < b$

(ii) $\lim_{x \to a+0} f(x) = f(a)$

(iii) $\lim_{x \to b-0} f(x) = f(b)$.

Viz function is continuous in open interval (a,b) and at $x = a$ from right side and at $x = b$ from left hand side is continuous.

Discontinuity in an interval: Function $f(x)$ is called discontinuous in any interval if it is discontinuous at any or many points of interval.

2.14 Theorem on Continuous Functions

(i) If both $f(x)$ and $g(x)$ at any point $x=a$ is continuous, then $f(x) \pm g(x)$ also will be continuous on $x =$ a.

(ii) If both $f(x)$ and $g(x)$ at any point $x=a$ is continuous, then $f(x)\,g(x)$ also will be continuous on $x = a$.

(iii) If $f(x)$ at any point $x = a$ is continuous and k is a certain real number, then $kf(x)$ also will be continuous on $x = a$.

(iv) If $f(x)$ and $g(x)$ at any point $x=a$ is continuous and $g(a) \neq 0$, then $\frac{f(x)}{g(x)}$ also will be continuous on $x = a$.

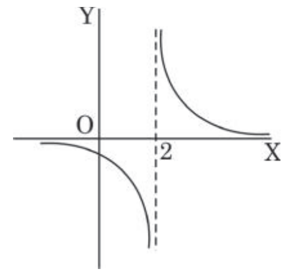
(v) If $f(x)$ at $x=a$ is continuous and $f(a) \neq 0$ then $\frac{1}{f(x)}$ also will be continuous on $x = a$.

(vi) If $f(x)$ at $x=a$ is continuous then at $x=a$ $f(x)$ is also continuous.

Example 1: Express that function $f(x) = x^2 + 1$ is continuous on $x = 2$.

Solution: $\lim_{x \to 2} f(x) = 5 = f(2)$ Therefore, function x = 2 is continuous

Example 2: At $x = 2$ function $f(x) = \dfrac{1}{x-2}$ is discontinuous. Prove.

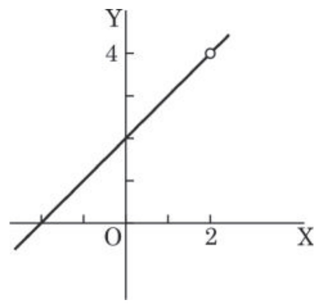


Solution: (i) $f(2)$ is not defined (denominator is 0)

(ii) $\lim_{x\to 2} f(x)$ has no existence (is equal = ∞)

leaving $x = 2$, at every point, function is continuous. Therefore at $x = 2$ function is discontinuous.

Example 3: Explain that at $x = 2$ function $f(x) = \dfrac{x^2-4}{x-2}$ is discontinuous.



Solution: (i) $f(2)$ is not defined (numerator and denominator are zero)

(ii) $\lim_{x\to 2} f(x) = 4$

Therefore function x= 2 is discontinuous

Discontinuity in example 3 can be escaped, because further $f(x) = \dfrac{x^2-4}{x-2}$

defining the function at $f(2) = 4$

The graph of and $g(x) = x+2$ are same, whereas first one has hole.

Discontinuity in example 2 can not be ignored, because there is no existence of limit.

Example 4: If function $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous on main point.

Solution: $f(x)$ is continuous on $x = a$ because $f(0) = 1$ (given)

And $$\lim_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$$

Example 5: Explain that function $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$ is discontinuous on main point.

Solution: $f(x)$ is discontinuous on x = a because $f(0) = 1$

And $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 2x}{x}$$

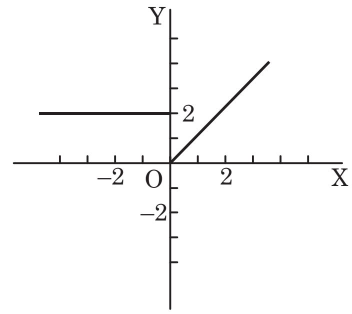
$$= \lim_{x\to 0} \frac{2\sin x}{x} \cdot \cos x = 2.1 \cos 0^0 = 2$$

Or $$f(0) \neq \lim_{x\to 0} f(x).$$

Note

Illustrations

Example 1: Draw the graph of $f(x) = \begin{cases} 2, & x < 0 \\ x, & x \geq 0 \end{cases}$. Prove that function is discontinuous on main point.



Solution: It is easy to draw the graph of $f(x)$. It is shown in the figure. It is clear that at point $x = 0$ there is a jump in the graph. Now we will verify the limits of right and left hand side.

For $x \geq 0$

$x \geq 0$

$$f(x) = x$$
$$f(x+h) = (x+h)$$
$$f(0+h) = (0+h)$$
$$\lim_{h \to 0} f(0+h) = \lim_{h \to 0} h$$

$\therefore$ $$f(0+0) = 0 \qquad ...(i)$$

Left Hand Limit at main point (for x<0)

$x < 0$

$$f(x) = 2$$
$$f(x-h) = 2$$
$$f(0-h) = 2$$
$$\lim_{h \to 0} f(0-h) = 2$$

$\therefore$ $$f(0-0) = 2 \qquad ...(ii)$$

Thus from (i) and (ii) $\quad f(0+0) \neq f(0-0)$

Therefore at x=0, function f(x) is discontinuous.

Example 2: If $f(x) = |x - 1| + |x + 2|$ where x is any real number, then prove that function $f(x)$ is continuous on $x = 1$ and $x = 2$.

Solution:

Right Hand Limit at $x = 1$

$$f(x) = |x - 1| + |x + 2|$$
$$f(x+h) = |x + h - 1| + |x + h + 2|$$
$$f(1+h) = |1 + h - 1| + |1 + h + 2|$$
$$\lim_{h \to 0} f(1+h) = \lim_{h \to 0} [\,|h| + |3 + h|\,]$$
$$= |0| + |3 + 0| = 3$$

Left Hand Limit at $x = 1$

$$f(1-h) = |1 - h - 1| + |1 - h + 2|$$
$$f(1-h) = |1 - h| + |3 - h|$$
$$\lim_{h \to 0} f(1-h) = \lim_{h \to 0} [\,|h| + |3 - h|\,]$$
$$= |0| + |3 - 0| = 3$$
$$\text{L.H.L.} = \text{R.H.L.}$$

Therefore $x = 1$ is continuous

Right Hand Limit at $x = -2$

Note

$$f(x) = |x - 1| + |x + 2|$$
$$f(-2 + h) = |-2 + h - 1| + |-2 + h + 2|$$
$$= |h - 3| + |h|$$
$$\lim_{h \to 0} f(-2 + h) = \lim_{h \to 0} [\,|h - 3| + |h|\,]$$
$$= |-3| + |0| = 3$$

Left Hand Limit at x = -2

$$f(x) = |x - 1| + |x + 2|$$
$$f(-2 - h) = |-2 - h - 1| + |-2 - h + 2|$$
$$= |-3 - h| + |-h|$$
$$= |3 + h| + |h|$$
$$\lim_{h \to 0} f(-2 - h) = \lim_{h \to 0} [\,|3 + h| + |h|\,]$$
$$= 3$$
$$\text{L.H.S.} = \text{R.H.S.}$$

Therefore $x = 1$ is continuous

Example 3: Show that at $x = 1$, function $f(x) = \dfrac{1}{x - a}$, is discontinuous.

Solution:

$$\text{L.H.L.} = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$$
$$= \lim_{h \to 0} \frac{1}{(a - h) - a} = \lim_{h \to 0} \frac{1}{-h} = -\infty$$
$$\text{R.H.L.} = \lim_{x \to a+0} f(x) = \lim_{h \to 0} f(a + h)$$
$$= \lim_{h \to 0} \frac{1}{(a + h) - a} = \lim_{h \to 0} \frac{1}{h} = \infty$$

$\therefore$ L.H.L. $\neq$ R.H.L.

Therefore at $x = a$ function is discontinuous. **Ans.**

Example 4: Express that at $x = 0$ $f(x) = |x|$ is continuous.

Solution: Here $f(0) = |0| = 0$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$$
$$= \lim_{h \to 0} |0 - h| = \lim_{h \to 0} (h) = 0$$
$$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$$
$$= \lim_{h \to 0} |0 + h| = \lim_{h \to 0} (h) = 0$$

$\therefore$ L.H.L. = R.H.L. = $f(0)$

Therefore at x = 0 function is continuous **Ans.**

Example 5: Express that at $x = 1$ $f(x) = \begin{cases} x^2, \text{ if } x \neq 1, \\ 2, \text{ if } x = 1. \end{cases}$ is discontinuous.

Solution: Given that $f(1) = 2$

Note

$$\lim_{x\to 1+0} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} (1+h)^2 = 1$$

Or $$\lim_{x\to 1-0} f(x) = \lim_{h\to 0} (1-h) = \lim_{h\to 0} (1-h)^2 = 1$$

$\therefore$ $$\lim_{x\to 1} f(x) = 1 \neq f(1)$$

Thus function is discontinuous at x = 1.

Example 6: For which value of k function $f(x) = \begin{cases} \dfrac{x^2-16}{x-4}, & x \neq 4 \\ k, & x = 4 \end{cases}$ is continuous at $x = 4$.

Solution: Since at $x = 4$, value of function $f(x)$ is k

$\therefore$ $$f(4) = k$$

$x = 4$ $$\text{R.H.L.} = \lim_{x\to 4+0} f(x) = \lim_{h\to 0} f(4+h)$$

$$= \lim_{h\to 0} \frac{(4+h)^2-16}{4+h-4} = \lim_{h\to 0} \frac{16+h^2+8h-16}{h}$$

$$= \lim_{h\to 0} \frac{h(h+8)}{h} = \lim_{h\to 0} (h+8), \quad \because h \neq 0$$

$$= 8$$

$$\text{L.H.L.} = \lim_{x\to 4-0} f(x) = \lim_{h\to 0} f(4-h)$$

$$= \lim_{h\to 0} \frac{(4-h)^2-16}{4-h-4} = \lim_{h\to 0} \frac{16+h^2-8h-16}{-h}$$

$$= \lim_{h\to 0} (-h+8), \ h \neq 0$$

$$= 8$$

Given that at $x \neq 4$ function is continuous

$$\text{L.H.L.} = \text{R.H.L.} = f(4)$$

$$8 = 8 = k$$

Therefore, $k = 8$. **Ans.**

Task For which value of k function $f(x)$ $\begin{cases} \dfrac{x^2-25}{x-5}, & x \neq 5 \\ k, & x = 5 \end{cases}$ is continuous at $x = 5$

Ans.: $k = 10$

Example 7: Examine the continuity of function $f(x)$ at point (0,0) when $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$.

Solution: Here $$f(0) = 1$$

$$\text{L.H.L.} = \lim_{x\to 0-0} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \frac{|0-h|}{0-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

Note

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} f(x)$$

$$= \lim_{h \to 0} \frac{|0-h|}{0+h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

$\therefore$ R.H.L. $\neq$ L.H.L.

Thus at x=0 function is discontinuous

Therefore, LHL = RHL = $f(1)$

Therefore at x = 1 function is continuous.

Example 8: If at $x = 1$ $f(x) = \begin{cases} 3ax + b, \text{ if } x > 1 \\ 11 \quad , \text{ if } x = 1 \\ 5ax - 2b, \text{ if } x < 1 \end{cases}$ is a continuous function, then find out the value of a and b.

Solution: Since the given function $f(x)$ at x = 1 is continuous. Therefore

$$\lim_{x \to 1} f(x) = f(1)$$

$$\Rightarrow \quad \lim_{x \to 1+0} f(x) = f(1)$$

$$\Rightarrow \quad \lim_{h \to 0} f(1+h) = 11$$

$$\lim_{h \to 0} 3a(1+h) + b = 11$$

$$3a + b = 11 \qquad \text{...(i)}$$

Therefore $\lim_{x \to 1-0} f(x) = 11$

$$\Rightarrow \quad \lim_{h \to 0} 5a(1-h) - 2b = 11$$

$$\Rightarrow \quad 5a - 2b = 11 \qquad \text{...(ii)}$$

Solving the equations (i) and (ii)

$$a = 3, b = 5.$$ **Ans.**

Example 9: Function $f(x)$ at an interval [0,1] is defined as under:

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{2} - x, & \text{if } 0 < x < \frac{1}{2} \\ \frac{1}{2}, & \text{if } x = \frac{1}{2} \\ \frac{2}{3} - x, & \text{if } \frac{1}{2} < x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

Find out the points where function is discontinuous.

Solution: (i) at x = 0 continuity $f(0) = 0$

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \left\{ \frac{1}{2} - (0+h) \right\} = \frac{1}{2} \neq f(0)$$

Therefore function is discontinuous at x = 0

Note

(ii) At $\boldsymbol{x = \frac{1}{2}}$, continuity $f\left(\frac{1}{2}\right) = \frac{1}{2}$

$$\text{L.H.L.} = \lim_{x \to 1/2-0} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$$

$$= \lim_{h \to 0} \left\{\frac{1}{2} - \left(\frac{1}{2} - h\right)\right\} = 0$$

$$\text{R.H.L.} = \lim_{x \to 1/2+0} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$$

$$= \lim_{h \to 0} \left\{\frac{2}{3} - \left(\frac{1}{2} + h\right)\right\} = \lim_{h \to 0} \left(\frac{1}{6} - h\right) = \frac{1}{6}.$$

$\therefore$ L.H.L. $\neq$ R.H.L.

Thus, function is discontinuous at $x = \frac{1}{2}$

(iii) At x = 1, continuity f(1) = 1

$$\text{L.H.L.} = \lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} f(x)$$

$$= \lim_{h \to 0} \left\{\frac{2}{3} - (1 - h)\right\} = -\frac{1}{3} \neq f(1)$$ **Ans.**

Thus function is discontinuous at x = a.

Example 10: Examine the continuity of function $f(x)$ at $x = 0,1$ when

$$f(x) = |x| + |x - 1| \text{ or}$$

$$f(x) = \begin{cases} 1 - 2x, & \text{if } x < 0 \\ 1, & \text{if } 0 \le x < 1 \\ 2x - 1, & \text{if } x \ge 1 \end{cases}$$

Solution: (i) Continuity at x = 0

$f(0) = 1$

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} (1) = 1$$

$$\text{L.H.L.} = \lim_{x \to 0-0} f(x) = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} (1 + 2h) = 1$$

$\therefore$ L.H.L. = $f(0)$ = R.H.L.

Thus LHL = $f(0)$ = RHL

Therefore function is continuous at $x = 0$

(ii) Continuity at $x = 1$

$$f(1) = (2 \times 1 - 1) = 1,$$

$$\text{L.H.L.} = \lim_{x \to 1-0} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1) = 1.$$

$$\text{R.H.L} = \lim_{x \to 1+0} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \{2(1+h) - 1\}$$

$$= \lim_{h \to 0} \{1 + 2h\} = 1$$

$\therefore$ L.H.L. = $f(1)$ = R.H.L.

Therefore function is continuous at $x = 1$.

Example 11: Show that the function $f(x)$, $\forall x \in R$ which is defined as under

$$(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$ **is discontinuous at $x = 0$.**

Solution: At $x = 0$, given that

$$f(0) = 0 \qquad \text{...(i)}$$

RHL at $x = 0$

$$f(x+h) = \frac{e^{1/(x+h)}}{1+e^{1/(x+h)}}$$

$$f(0+h) = \frac{e^{1/(0+h)}}{1+e^{1(0+h)}}$$

$$\therefore \quad \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}} = \lim_{h \to 0} \frac{1}{e^{-1/h+1}}$$

Therefore $$f(0+0) = \frac{1}{0+1} = 1, \qquad [\because \lim_{h \to 0} e^{-1/h} = 0] \qquad \text{...(ii)}$$

LHL at $x = 0$ $$f(x-h) = \frac{e^{1/(x-h)}}{1+e^{1/(x-h)}}$$

$$f(0-h) = \frac{e^{1/(0-h)}}{1+e^{1/(0-h)}} = \frac{e^{-1/h}}{1+e^{-1/h}}$$

$$\therefore \quad \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{-1/h}}$$

Therefore $$f(0-0) = \frac{0}{1+0} = 0 \qquad \text{...(iii)}$$

From (ii) and (iii) RHL $\neq$ LHL

Therefore the given function is discontinuous at $x = 0$.

Note

Example 12: A function $f(x)$ is defined as under

$$f(x) = \begin{cases} x, x < 1 \\ 2 - x, 1 \le x \le 2 \\ x - \frac{1}{2}x^2 \; x > 2. \end{cases} \quad ...(i)$$

If $f(x)$ is continuous at $x = 1$ and $x = 2$?

Solution: first of all we will examine $x = 1$

Therefore at x=1, $f(x) = 2 - x$ to $f(1) = 2 - 1 = 1$

RHL at $x = 1$, thus for x >1 $f(x) = 2 - x$

Now $$f(1 + 0) = \lim_{h \to 0} f(1 + h) = \lim_{h \to 0} 2 - (1 + h)$$

$$= \lim_{h \to 0} 1 - h = 1 \quad ...(ii)$$

LHL at $x = 1$, thus $f(x) = x$

$$f(1 - 0) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} (1 - h) = 1 \quad ...(iii)$$

Therefore, at $x = 1$ from (i), (ii) and (iii) $f(1) = f(1+0) = f(1-0)$

Therefore at $x = 1$ $f(x)$ is continuous

again $x = 2$ *to* $f(x) = 2 - x$, $f(2) = 2 - 2 = 0$ ...(iv)

RHL at x = 2 , thus for x>2 $x - \frac{1}{2}x^2$

$$f(2 + 0) = \lim_{h \to 0} f(2 + h)$$

$$= \lim_{h \to 0} (2 + h) - \frac{1}{2}(2 + h)^2$$

$$= \lim_{h \to 0} -\frac{1}{2}h^2 - h = 0; \quad ...(v)$$

LHL at x = 2 , thus for x< 2, $f(x) = 2 - x$

$$f(2 - 0) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} 2 - (2 - h) = 0. \quad ...(vi)$$

Therefore from (iv), (v), (vi) $f(2) = f(2 + 0) = f(2 - 0) = 0$.

Thus at x =2 $f(x)$ is continuous.

Questionnaire 2.2

Find out the point of discontinuity, if any from the following functions and draw the graph:

1. $f(x) = \begin{cases} -1, x \le 0 \\ +1, x > 0. \end{cases}$
2. $f(x) = |x - 1|$.
3. $f(x) = |x| + |x - 1|$.

Note

Show that following functions are continuous at the points given against them:

4. Function at x = 1, 3 $f(x) = x^2 - 7x + 3$.

5. $f(x) = \begin{cases} \dfrac{1}{x+2}, x \neq -2 \\ 0, x = -2 \end{cases}$

6. If $f(x) = \begin{cases} x-4, \text{ whereas } x \geq 5 \\ 5x-24, \text{ whereas } x < 5 \end{cases}$, then show that at $x = 5$,p $f(x)$ is continuous function

7. Prove that function $f(x)$ where $f(x) = \begin{cases} x-1, x<2 \\ 2x-3, x \geq 2 \end{cases}$ is continuous at every points. Draw the diagram of the function.

Examine the continuity of following functions

8. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{x^2-a^2}{x-a}, & x \neq a \\ 2a, & x = a \end{cases}$

10. $f(x) = \begin{cases} x^2, x \neq 1 \\ 2, x = 1 \end{cases}$

11. $f(x) = \begin{cases} x, x \geq 0 \\ x^2, x < 0. \end{cases}$

12. $f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 3, & x = 0 \end{cases}$

13. $f(x) = \begin{cases} \dfrac{|x-3|}{x-3}, & x \neq 3 \\ 0, & x = 3. \end{cases}$

14. $f(x) = \begin{cases} -x^2, x \leq 0 \\ x^2, x > 0 \end{cases}$

15. $f(x) = \begin{cases} \dfrac{1}{1-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

16. $f(x) = \begin{cases} e^{1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

17. $f(x) = \begin{cases} \dfrac{1}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

18. $f(x) = \begin{cases} \cos\dfrac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

19. $f(x) = \begin{cases} x\cos\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

20. $f(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

21. $f(x) = \begin{cases} 5x-4, & 0 < x \leq 1 \\ 4x^3-3x, & 1 < x < 2 \end{cases}$

22. $f(x) = \begin{cases} x+2, & -1 < x < 3 \\ 5, & x = 3 \\ 8-x, & x > 3 \end{cases}$

23. At $x = 1$ and $x = 2$, find out the continuity of function $f(x) = \begin{cases} 0, \text{ if } 0 < x < 1 \\ x, \text{ if } 1 \leq x < 2 \\ \dfrac{x^3}{4}, \text{ if } 2 \leq x < 3 \end{cases}$

Note

24. Find out the continuity of function $f(x)$ at $x = 0, 1, 2$ when

$$f(x) = \begin{cases} -x^2, & \text{if } x \le 0 \\ 5x-4, & \text{if } 0 < x \le 1 \\ 4x^2 - 3x, & \text{if } 1 < x < 2 \\ 3x+4, & \text{if } x \ge 2. \end{cases}$$

25. If $f(x) = x^2 + 1$ where $x \ne 1$ and $f(x) = 3$ where $x = 1$, then find out that at point $x = 1$ whether the function is continuous or discontinuous.

Answers

1. 0,	2. None	3. None	8. Continuous
9. Continuous	10. Discontinuous,	11. Continuous	12. Discontinuous
13. x = 3, Discontinuous	14. $X = 0$, Continuous	15. Discontinuous	16. Discontinuous
17. $X = 0$, Continuous	18. Discontinuous	19. Discontinuous	20. Discontinuous
21. Continuous	22. Continuous	23. Yes	

24. Continuous, $x = 1, 2$ and Discontinuous at $x = 0$25. Discontinuous

Self Assessment

2. Multiple Choice Questions:

(i) The value of $\lim\limits_{x \to a} \frac{x^3 - a^3}{x - a}$

(a) $3a^2$ (b) $3a$ (c) $3b^2$ (d) $3b$

(ii) $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = ...$

(a) $\frac{1}{e}$ (b) e (c) $-e$ (d) ∞

(iii) $\lim\limits_{x \to 0} \frac{a^x - b^x}{x} = \log e....$

(a) $\frac{b}{a}$ (b) $\frac{1}{b}$ (c) $\frac{a}{b}$ (d) $-\frac{a}{b}$

(iv) If both $f(x)$ and $g(x)$ are continuous at any point $x = a$, then at $x = a$ what would be $f(x) \pm g(x)$

(a) Continuous (b) Discontinuous (c) Relevant (d) Irrelevant

(v) $\lim\limits_{x \to \infty} \frac{e^{x^2} - 1}{e^{x^2} + 1} = ...$

(*a*) 2 (*b*) $2e$ (*c*) 1 (*d*) $\frac{1}{e}$

2.15 Summary

- When the limit of function is obtained from the right hand of the independent variable, then it is called Right Hand Limit (R.H.L.) and applying positive (+) sign for the right side, this can be expressed as under

Note

Right Hand Limit $= f(a + 0)$

$= \lim_{x \to a^+} f(x) = l_1.$

- When the limit of function is obtained from the left hand of the independent variable, then it is called Left Hand Limit (LH.L.) and applying negative (–) sign for the left side, this can be expressed as under

  Left Hand Limit $= f(a - 0)$

  $= \lim_{x \to a^-} f(x) = l_2.$

- To obtain the limit of right and left hand, replace x variable with $(x+h)$ ad $(x-h)$ respectively in the function.
- Thus, obtained function x, should be replaced with point (assume a).
- Now at $h \to 0$ determine the limit of function [viz function obtained by (ii) to be put in the above, put $h = 0$].
- If at $x = a$ both the limits of right and left hand of any function $f(x)$ exist and are equal, then at $x = a$, there is existence of limit of function $f(x)$ viz $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$ (Assumed).
- Here l is called limit of the function and this is expressed as $\lim_{x \to a} f(x) = l$.
- We should find out the limits of right and left hand, but the intermediate level, we can often find out the limit directly.
- If it is not possible to divide the numerator and denominator with a common factor which is not zero, then after expansion in series or transformation, it can be possible.
- If drawing a graph of any function $f(x)$, the curve which we obtain. And it is in such a way that it does not break on any point $x = a$, then the function is called continuous on that point.
- A function $f(x)$ is called continuous in an open interval (a,b) if it is continuous for every values of x in this interval (a,b).
- Function $f(x)$ is called continuous in any closed interval, if

  (i) It is continuous for each value of x for which $a < x < b$

  (ii) $\lim_{x \to a+0} f(x) = f(a)$

  (iii) $\lim_{x \to b-0} f(x) = f(b)$.

2.16 Keywords

- ***Sequence:*** Serial
- ***Continually:*** Continuously

2.17 Review Questions

1. Find out the value of $\lim_{x \to 0} \left(\frac{x^2 - 2ax + a^2}{x - a} \right)$ **Ans.:** –a

2. Prove that $\lim_{x \to 0} = \frac{a^x - b^x}{x} = \log_e \frac{a}{b} = \log_e$

Note

3. Find out the value of $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$ **Ans.:** 3

4. Examine the continuity of function $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0\, but. \end{cases}$ **Ans.:** Discontinuous

5. Show that at $x = 0$, $f(x) = |x|$ is continuous

Answers: Self Assessment

1. (i) Right Hand Limit, (ii) Left Hand Limit, (iii) Existence,
   (iv) Marginal, (v) Intermediate
2. (i) (a), (ii) (b), (iii) (c)
   (iv) (a), (v) (c)

2.18 Further Readings

Books

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economics – Malcom, Nicolas, U C London.

Mathematics for Economics – Carl P Simone, Lawrence Bloom.

Mathematical Economics – Michael Harrison, Patrick Walderan.

Mathematics for Economist – Yamane, Prentice Hall India.

Mathematics for Economics and Finance – Martin Norman.

Unit 3: Differentiation

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Calculate Differential Coefficient.
- Calculate Differential Coefficient of a Constant.
- Calculate Differential Coefficient of x^n with Respect to x.
- Calculate Differential Coefficient of Sum, Subtract, Multiplication and Division of two Functions.

Introduction

Assume $y = x^2$

Where $x = 2$, hence $y = 4$, where $x = 3$, $y = 9$

When x increases from 2 to 3, y also increases from 4 to 9. Any increase in the constant function is referred as Increment, which is indicated δx (delta x).

In the above example $\delta x = 3 - 2 = 1$ and $\delta y = 9 - 4 = 5$

If x, 1 changes to .8, $\delta x = 0.8 - 1 = -0.2$

Note

Notes It must be noted that δx does not mean $\delta \times x$ e.g. this δ and x are not the multiplication but a symbol. This is a single unit.

3.1 Differential Coefficient

Assume $y = x^2$, is a function of x. Assume the prime value of $x = 3$.

The table below shows how the ratio of increment δx in x and δy in y changes. Here increment as 4, 3, 2, 1 etc. in the prime value of x has been considered.

δx	$x(x + \delta x)$	$y(y + \delta y)$	δy	$\frac{\delta y}{\delta x}$
4	7	49	40	10
3	6	36	27	9
2	5	25	16	8
1	4	16	7	7
0.1	3.1	9.61	.61	6.1
0.01	3.01	9.0601	.0601	6.01
0.001	3.001	9.006001	.006001	6.001
0.0001	3.0001	9.0006001	.00060001	6.0001
h	$3 + h$	$6 + 6h + h^2$	$6h + h^2$	$6 + h$

The table makes it clear that:

1. Wherever the decreasing value of δx approaches to zero, δy also decreases and reaches to ZERO.
2. But, the ratio of these values, instead of moving to ZERO approaches to particular value, which in this example is 6.

Therefore, we may conclude that whenever δx and as a result δy decreases, $\frac{\delta y}{\delta x}$ moves to value 6, in other words

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 6$$

$$y = x^2$$

$$\therefore \quad y + \delta y = (x + \delta x)^2$$

$$\therefore \quad y + \delta y - y = (x + \delta x)^2 - x^2$$

$$\therefore \quad \delta y = x^2 + 2x, \delta x + (\delta x)^2 - x^2$$

$$\therefore \quad \delta y = 2x.\delta x + (\delta x)^2$$

$$\therefore \quad \frac{\delta y}{\delta x} = 2x + \delta x$$

$$\therefore \quad \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0} (2x + \delta x)$$

$$\therefore \quad \frac{dy}{dx} = 2x$$

Thus, $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$ is a differential coefficient or derivative of y with respect to x. To avoid any doubt with respect to independent variables, the differential coefficient or derivative with respect to x is written as $\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \frac{d}{dx}(y)\frac{dy}{dx}$. This way the process of knowing the limit is known as differentiation or in other words method of calculating the differential coefficient of any product is referred as Differentiation.

> *Did u know?* $\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x}$

Students must understand the difference of $\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x}$ very carefully. $\frac{\delta y}{\delta x}$ is a division, for which Numerator and denominator can be separated from each other, but $\frac{dy}{dx}$ is not a division, but only a symbol to show the limiting value of $\frac{\delta y}{\delta x}$. δy can not be separated from δx. It would be incorrect to read it as δy upon δx. $\frac{dy}{dx}$ is read as [δ-δx of y) and as have been told earlier this means $\frac{d}{dx}(y)$ or differential coefficient of y with respect to x.

3.2 Definition

If $f(x)$ is the function of x and the same function of $x + \delta x$ is $f(x + \delta x)$, then limiting value is $\lim_{\delta x \to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$, for the differential coefficient $f(x)$ with respect to x. Generally differential coefficient is symbolized with $\frac{dy}{dx}, y', y_1, \frac{df(x)}{dx}, \frac{d}{dx}$ $f(x)$, $f(x)$, $Df(x)$, f', etc. The method of calculating differential coefficient is referred as differentiating the function.

Thus, we can see that there are four steps for calculating differential coefficient

First step changing x into $x + \delta x$ and determining $f(x + \delta x)$

Second step determining difference of $f(x + \delta x) - f(x)$

Third step dividing difference by δx and determining $\frac{f(x+\delta x)-f(x)}{\delta x}$

Fourth step when δx approaches towards ZERO then determining the limiting value of ratio

here onwards incremental value δx of x would be replaced with h, so that students should not get confused with δ and x. Thus, if $f(x)$ is the product of x, then $\lim_{h \to 0}\frac{f(x+h)-f(x)}{h}$ is the differential coefficient of $f(x)$. It needs to be remembered that above expression is considered as the product of h i.e. h is variable and x is constant.

Note

3.3 Differential Coefficient of a Constant

Assume that c is a constant. Thus, here $f(x) = c$ now for every value of x, no change can happen in the constant value.

Therefore,

$$f(x+h) = \text{c}$$

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0 .$$

$$\boxed{\frac{d}{dx}c = 0}$$

Thus, differential coefficient of a constant value is ZERO.

3.4 Differential Coefficient of the Production of a Constant and a Function

Assume that a is a constant value and $f(x)$ is the determined product, then

$$\frac{d}{dx}\{af(x)\} = \lim_{h\to 0}\frac{af(x+h)-af(x)}{h}$$

$$= \lim_{h\to 0}\left\{a\frac{f(x+h)-f(x)}{h}\right\}$$

$$= a\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = a\frac{d}{dx}\{f(x)\}$$

$$\boxed{\frac{d}{dx}\{af(x)\} = a\frac{d}{dx}\{f(x)\}}$$

if $$y = au$$

$$\boxed{\frac{dy}{dx} = a\frac{du}{dx}}$$

Did u know? Differential coefficient of Constant and multiplication of any product is equal to differential coefficient of product and multiplication of constant value.

3.5 Differential Coefficient of x^n with Respect to x

Assume $f(x) = x^2$, then $f(x+h) = (x+h)^n$

Thus, Differential Coefficient of x^n with respect to x $\frac{d}{dx}(x^n)$

$$\frac{d}{dx}(x^n) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{(x+h/x)^n - 1}{h}, \ x^n \text{ ascertained}$$

Now, since $h \to 0$, we can assume $\frac{h}{x}$ is smaller to a unit. Therefore, expanding every value of n $(1 + h/x)^n$ by using Binomial Theorem

$$\frac{d}{dx}x^n = \lim_{h \to 0} \frac{x^n}{h}\left\{1 + n.\frac{h}{x} + \frac{n(n-1)}{1.2}\frac{h^2}{x^2} + ... - 1\right\}$$

$$= \lim_{h \to 0} \frac{x^n}{h}\left\{n.\frac{h}{x} + \frac{n(n-1)}{1.2}\cdot\frac{h^2}{x^2} + ...\right\}$$

$$= \lim_{h \to 0} x^n \left\{\frac{n}{x} + \frac{n(n-1)h}{1.2x^2} + ...\right\}$$

$$= \lim_{h \to 0} x^n \left\{\frac{n}{x} + h\times\right\}$$

$$= nx^{n-1},\ h \to 0$$

$\therefore$ $$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

The special result of this is, $$\boxed{\frac{d}{dx}x = 1}$$

Example 1: $\frac{d}{dx}(9x^7) = 9.\frac{d}{dx}(x^7) = 9.7\,x^{7-1} = 63x^6$

Example 2: $\frac{d}{dx}(-6x^2) = -6\frac{d}{dx}x^2 = -\,6.2\,x^{2-1} = -12x$

Example 3: $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}\,x^{-1/2-1} = -\frac{1}{2}x^{-3/2}$

Task Determine the value of $\frac{d}{dx}(5x^6)$ [**Ans.:** $30x^5$]

Self Assessment

1. Fill in the blanks:

(i) The method for determining the differential coefficient of any function is referred as ………………..

(ii) Differential coefficient of Constant and multiplication of any product is equal to ………………………. of product and multiplication of constant value.

(iii) $\frac{dy}{dx} = ...\frac{du}{dx}$

(iv) $\frac{dy}{dx} = x^n = nx........$

Note

3.6 Differential Coefficient of Sum and Subtract of two Functions

Assume that $f(x) = f_1 \pm f_2(x)$

$$\therefore \quad f(x+h) = f_1(x+h) \pm f_2(x+h)$$

$$\frac{d}{dx}\{f(x)\} = \lim_{h\to 0} \frac{\{f_1(x+h) \pm f_2(x+h)\} - \{f_1(x) \pm f_2(x)\}}{h}$$

$$= \lim_{h\to 0} \frac{\{f_1(x+h) - f_1(x)\} \pm \{f_2(x+h) - f_2(x)\}}{h}$$

$$= \lim_{h\to 0} \frac{f_1(x+h) - f_1(x)}{h} \pm \lim_{h\to 0} \frac{f_2(x+h) - f_2(x)}{h}$$

$$\boxed{\frac{d}{dx}\{f(x)\} = \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x)}$$

Thus, Differential coefficient of addition or subtraction of any two product would be equal to addition or subtraction of differential coefficient of them

Assume that $f(x) = f_1(x) \pm f_2(x) \pm f_3(x) \pm ..... + f_n(x)$

Then $\boxed{\frac{d}{dx} f(x) = \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x) \pm .... \pm \frac{d}{dx} f_n(x)}$

> *Notes* Differential Coefficient of Sum and Subtract of two or more than two functions would be equal to Sum and Subtract of differential coefficient of their separated value.

if $y = u + v$, and $y = u - v$

then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ in the same way $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

in expanded expression, if $y = u \pm v \pm w \pm ....$ Then $a^2 + b^2$

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{du}{dx} \pm \frac{dw}{dx} \pm ....$$

EXAMPLES WITH SOLUTION

Example 1: Determine value of $\frac{d}{dx}(5x^7 + 2x)$.

Illustration $\frac{d}{dx}(5x^7 + 2x) = \frac{d}{dx}(5x^7) + \frac{d}{dx} 2x = 5\frac{d}{dx}(x^7) + 2\frac{d}{dx}(x)$

$$= 5.7x^6 + 2 = 35x^6 + 2.$$ **Ans.**

Note

Example 2: Determine value of $\frac{d}{dx}(x^5 - 4x^3 + 8x - 7)$.

Illustration $$\frac{d}{dx}(x^5 - 4x^3 + 8x - 7) = \frac{d}{dx}x^5 + \frac{d}{dx}(-4x^3) + \frac{d}{dx}(8x) + \frac{d}{dx}(-7)$$

$$= \frac{d}{dx}(x^5) + (-4)\frac{d}{dx}x^3 + 8\frac{d}{dx}(x) + \frac{d}{dx}(-7)$$

$$= 5x^4 - 4 \times 3x^2 + 8 \times 1 - 0$$

$$= 5x^4 - 12x^2 + 8.$$ **Ans.**

Now we can determine differential coefficient of some function by using derivative limiting value. But we will notice afterwards that with the knowledge of standard form of differential coefficient, we can save time from evaluation

Example 3: Determine the differential coefficient of product $\left\{\frac{lx^2 + mx + n}{\sqrt{x}}\right\}$ with respect to x.

Illustration $$\frac{d}{dx}\left\{\frac{lx^2 + mx + n}{\sqrt{x}}\right\} = \frac{d}{dx}\{lx^{3/2} + mx^{1/2} + mx^{-1/2}\}$$

$$= l\frac{d}{dx}x^{3/2} + m\frac{d}{dx}x^{1/2} + n\frac{d}{dx}x^{-1/2}$$

$$= l\,\frac{3}{2}x^{1/2} + m\cdot\frac{1}{2}x^{1/2-1} + n\cdot\left(-\frac{1}{2}\right)x^{-1/2-1}$$

$$= \frac{3}{2}lx^{1/2} + \frac{1}{2}mx^{-1/2} - \frac{1}{2}nx^{-3/2}$$ **Ans.**

> *Task* Determine the value of $\frac{d}{dx}(2x^7 + 5x)$ **[Ans.: $14x^6 + 5$]**

Example 4: Determine the differential coefficient of product $1 + x + \left(\frac{x^2}{2!}\right) + \left(\frac{x^3}{3!}\right) + \left(\frac{x^4}{4!}\right) + ...$ with respect to x.

$$\frac{d}{dx}\left\{1 + x + \left(\frac{x^2}{2!}\right) + \left(\frac{x^3}{3!}\right) + \left(\frac{x^3}{4!}\right) + ...\right\}$$

Illustration $$= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{x^2}{2!}\right) + \frac{d}{dx}\left(\frac{x^3}{3!}\right) + \frac{d}{dx}\left(\frac{x^4}{4!}\right) + ...$$

$$= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + ...$$

Note

$$= 0 + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$ **Ans.**

Questionnaire 3.1

Determine the differential coefficient of following product with respect to x.

1. $3x^3, x^{-5}, x^9$
2. $3x^2, 6x^{-3}, \frac{x^5}{3}$
3. $x^{1/2}, x^{7/3}, x^{-5/2}$
4. $\sqrt{x^3}, \sqrt{x}, \sqrt{x^{-7}}$
5. $3x^{1/3}, 5x^{1/7}, 2x^{1/4}$
6. $\frac{7}{x^2}, \frac{5}{x^{3/2}}, \frac{1}{x}$
7. $x + \frac{2}{x}$
8. $x^m + a^n$
9. $ax^2 + bx + c$
10. $(ax)^m + (2b)^m$
11. $\frac{1}{2}\sqrt{x} + \sqrt{a}$
12. $y = (ax)^m + \left(\frac{b}{x}\right)^n$
13. $y = ax + (ax)^2 + (ax)^3 + ...$
14. $y = a + \frac{a}{x} + \left(\frac{a}{x}\right)^2 + \left(\frac{a}{x}\right)^3 + ...$
15. If $y = x^5 + 2x^4 + 7$ then determine the value of $\frac{dy}{dx}$ on $x = 0$
16. If $y = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ...,$ then determine the value of $\frac{dy}{dx}$.

Answers

1. $9x^2, -5x^{-6}, 9x^8$
2. $6x, -18x^{-4}, \frac{5}{3}x^4 3$
3. $\frac{1}{2\sqrt{x}}, \frac{4}{3}x^{\frac{4}{3}}, -\frac{5}{2}x^{\frac{-7}{2}}$
4. $\frac{3}{2}x^{\frac{1}{2}}, \frac{1}{2\sqrt{x}}, \frac{-7}{2}x^{\frac{-9}{2}}$
5. $x^{\frac{-2}{3}}, \frac{5}{6}x^{\frac{-6}{7}}, \frac{1}{2}x^{\frac{-3}{4}}$
6. $14x^{-3}, \frac{15}{2}x^{\frac{-5}{2}}, -\frac{1}{x^2}$
7. $1 - \frac{2}{x^2}$
8. mx^{m-1}

9. $2ax + b$

10. $ma^m x^{m-1}$

11. $\frac{1}{4\sqrt{x}}$

12. $a^m mx^{m-1} - nb^n x^{-n-1}$

13. $a + 2ax + 3ax^2 + ...$

14. $-\frac{a}{x^2} - \frac{2a^2}{x^3} - ......$

15. $\frac{dy}{dx} = 0$

16. $n_{c_2} + 2n_{c_2}x + ...$

3.7 Differential Coefficient of Function e^x with Respect to x when e is Exponential

Here $f(x) = e^x$

Then $f(x+h) = e^{x+h}$,

Therefore, differential coefficient of e^x with respect to x is $\frac{d}{dx} e^x$

Further
$$\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = \lim_{h \to 0} \frac{e^x (e^h - 1)}{h}$$
$$= \lim_{h \to 0} \frac{e^x \left(1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + ... - 1\right)}{h}$$
$$= \lim_{h \to 0} e^x h \frac{\left(1 + \frac{1}{1!} + \frac{h}{2!} + \frac{h^2}{3!} + ...\right)}{h}$$
$$= \lim_{h \to 0} e^x (1 + h)$$
$$= e^x \left(\frac{1}{1!} + 0\right) = e^x$$

$$\therefore \quad \boxed{\frac{d}{dx}(e^x) = e^x}$$

3.8 Differential Coefficient of Function a^x with Respect to x when a is a Non-variable

Here $f(x) = a^x$

Then $f(x+h) = a^{x+h}$

Note

Therefore, differential coefficient of a^x with respect to x is $\frac{d}{dx}a^x$

again
$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x(a^h-1)}{h}$$

$$= \lim_{h\to 0}\frac{a^x\left[1+\frac{h\log_e a}{1!}+\frac{h^2(\log_e a)^2}{2!}+....-1\right]}{h}$$

$$= \lim_{h\to 0}\frac{a^x\left[\frac{h\log_e a}{1!}+\frac{h^2(\log_e a)^2}{2!}+...\right]}{h}$$

$$= \lim_{h\to 0}\frac{a^x\cdot h\left[\frac{\log_e a}{1!}+\frac{h(\log_e a)^2}{2!}+...\right]}{h}$$

$$= \lim_{x\to\infty} a^x\left[\frac{\log_e a}{1!}+h\text{ (A convergent class)}\right]$$

$$= a^x\left[\frac{\log_e a}{1!}+a\right] = a^x\log_e a$$

$\therefore$
$$= \boxed{\frac{d}{dx}a^x = a^x\log_e a}$$

3.9 Differential Coefficient of Function $\log_e x$ with Respect to x, when base is Exponential

Here $f(x)=\log_e x$

Then $f(x+h)=\log_e(x+h)$

Further

$$\frac{d}{dx}\log_e x = \lim_{h\to 0}\frac{\log_e(x+h)-\log_e x}{h}$$

again
$$= \lim_{h\to 0}\frac{\log_e(1+h/x)-\log_e x}{h}$$

$$= \lim_{h\to 0}\frac{\log_e x+\log_e(1+h/x)-\log_e x}{h}$$

[Formula is $\log_e(mn)=\log_e m+\log_e n$]

$$= \lim_{h \to 0} \frac{\log_e (1 + h/x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \ldots}{h}$$

$$\left[\text{Formula is } \log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]$$

$$= \lim_{h \to 0} \frac{h\left[\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^3} - \frac{h^3}{4x^4} + \ldots\right]}{h}$$

$$= \lim_{h \to 0} \left\{\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^2} - \frac{h^3}{4x^4} + \ldots\right\}$$

$$= \lim_{h \to 0} \left\{\frac{1}{x} - h \times\right\} = \frac{1}{x}$$

$\therefore$ $$\boxed{\frac{d}{dx} \log_e x = \frac{1}{x}}$$

Notes $\log_e x$ can also be indicated as *lnx*

3.10 Differential Coefficient of Function $\log_a x$ with Respect to x, when base of Logarithm is a Non-variable

Here $f(x) = \log_a x = (\log_e x) \log_a e$ by formula

$$= \log_a e \cdot \log_e x$$

$$\frac{d}{dx} \log_a x = \frac{d}{dx} \log_a e \log_e x = \log_a e \frac{d}{dx} \log_e x,$$

$$= \log_a e \cdot \frac{1}{x} \qquad \left[\because \frac{d}{dx} \log_e x = \frac{1}{x}\right]$$

$\therefore$ $$\boxed{\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e}$$

Note

EXAMPLES WITH SOLUTION

Example 1: Determine the differential coefficient of $6x^{1/3} + 2e^x$.

Solution:

$$\frac{d}{dx}(6x^{1/3} + 2e^x) = \frac{d}{dx}(6x^{1/3}) + \frac{d}{dx}(2e^x)$$

$$= 6.\frac{d}{dx}(x^{1/3}) + 2\frac{d}{dx}(e^x)$$

$$= 6.\frac{1}{3}x^{-2/3} + 2e^x = 2x^{-3/2} + 2e^x$$ **Ans.**

Example 2: Determine the differential coefficient of $6 \log x - \sqrt{x} - 7$.

Solution: $$\frac{d}{dx}(6\log x - \sqrt{x} - 7) = 6\cdot\frac{d}{dx}\log x - \frac{d}{dx}(x^{1/2}) - \frac{d}{dx}(7)$$

$$= 6.\frac{1}{x} - \frac{1}{2}x^{-1/2} - 0 = \frac{6}{x} - \frac{1}{2}x^{-1/2}$$ **Ans.**

Example 3: Determine the differential coefficient of product $5\sqrt{x} + 7\log_e x - 11\log_a x$ with respect to *x*.

Solution: $$\frac{d}{dx}(5\sqrt{x} + 7\log_e x - 11\log_a x) = \frac{d}{dx}(5x^{1/2}) + \frac{d}{dx}(7\log_e x) - \frac{d}{dx}(11 \log_e x)$$

$$= 5\frac{d}{dx}x^{1/2} + 7\cdot\frac{d}{dx}\log_e x - 11\cdot\frac{d}{dx}\log_e x$$

$$= \frac{5}{2}x^{-1/2} + \frac{7}{x} - 11 \log_a e \cdot \frac{1}{x}$$ **Ans.**

Questionnaire 3.2

Determine the differential coefficient of following product with respect to x.

1. $e^x + \log_e x + l^x$
2. $\frac{1}{2x} + 7e^x$
3. $\frac{x^3 \log x + x - x^3 e^x}{x^3}$
4. $5\log_{10} x + 3$
5. $3\log_e x + x^{3/2} + 3$
6. $x(1 + x^2) + a^x$
7. $e^x + a^x + 1$
8. $\log_a x + \log_e x^2$
9. $\log_{10} x$
10. $7x^{-2/7} + \log_2 x$
11. $\frac{xe^x - 1}{x}$
12. $\sqrt{a + 2a^2e^x + a^3e^{2x}}$
13. $\sqrt[3]{1 + 3\log_a x + 3(\log_a x)^2 + (\log_a x)^3}$
14. $\log_e \sqrt{x} + a^x + 3$
15. $\frac{1}{a^x} + \log_a x$

Note

Answers

1. $e^x + \frac{1}{x} + l^x \log_e l$
2. $-\frac{1}{2x^2} + 7e^x$
3. $\frac{1}{x} - 2x^{-2} - e^x$
4. $\frac{5}{x}\log_{10} e$
5. $\frac{3}{x} + \frac{3}{2}x^{1/2}$
6. $1 + 3x^2 + a^x \log_e a$
7. $e^x + a^x \log_e a$
8. $\frac{1}{x}\log_e a + \frac{2}{x}$
9. $\frac{1}{x}\log_e e$
10. $-2x^{-9/7} + \frac{1}{x}\log_2 e$
11. $e^x + x^{-2}$
12. $a^{3/2}e^x$
13. $\frac{1}{x}\log_a e$
14. $\frac{1}{2x} + a^x \log_e a$
15. $-a^{-x}\log_e a + \frac{1}{x}\log_a e$

3.11 Differential Coefficient of the Quotient of Two Functions

Assume $F(x) = \frac{f_1(x)}{f_2(x)}\ f(x+h) = \frac{f_1(x+h)}{f_2(x+h)}$

$$\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{\frac{f_1(x+h)}{f_2(x+h)} - \frac{f_1(x)}{f_2(x)}}{h}$$

$$= \lim_{h\to 0}\frac{f_1(x+h)f_2(x) - f_1(x)f_2(x+h)}{hf_2(x+h)f_2(x)}$$

Adding and subtracting the product $f_1(x)f_2(x)$ in fraction

$$= \lim_{h\to 0}\frac{f_2(x)\{f_1(x+h) - f_1(x)\} - f_1(x)\{f_2(x+h) - f_2(x)\}}{hf_2(x+h)f_2(x)}$$

$$= \lim_{h\to 0}\frac{f_2(x)\left\{\frac{f_1(x+h) - f_1(x)}{h}\right\} - f_1(x)\left\{\frac{f_2(x+h) - f_2(x)}{h}\right\}}{f_2(x+h)f_2(x)}$$

$$= \frac{f_2(x)\cdot\frac{d}{dx}f_1(x) - f_1(x)\frac{d}{dx}f_2(x)}{[f_2(x)]^2}$$

Note Viz differential coefficient of division of two functions

$$= \frac{\text{Denr. (Diff. coett. of Numr.)} - \text{Numr. (Diff. coeff. of Denr.)}}{\text{Square of Denominator}}$$

Did u know?

$$\left[\begin{array}{l} \text{If } y = \dfrac{u}{v} \\ \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \end{array}\right]$$

EXAMPLES WITH SOLUTION

Example 1: Calculate the value of $\frac{dy}{dx}$ if $y = \frac{e^x}{x}$.

Solution: $\frac{dy}{dx} = \frac{x - \frac{d}{dx}e^x - e^x \frac{d}{dx}x}{x^2} = \frac{xe^x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$. **Ans.**

Example 2: Calculate the value of $\frac{d}{dx}\left(\frac{\sin x}{\log_e x}\right)$.

Solution:

$$\frac{d}{dx}\left(\frac{\sin x}{\log_e x}\right) = \frac{\log_e x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\log_e x)}{(\log_e x)^2}$$

$$= \frac{(\log_e x) \cdot \cos x - (\sin x) \cdot \frac{1}{x}}{(\log_e x)^2}$$

$$= \frac{x \log_e x \cos x - \sin x}{x(\log_e x)^2}.$$ **Ans.**

Example 3: Determine the differential coefficient of product $\frac{x^n}{\log_e x}$ with respect to x.

Solution:

$$\frac{d}{dx}\left\{\frac{x^n}{\log_e x}\right\} = \frac{\log_e x \frac{d}{dx}x^n - x^n \frac{d}{dx}\log_e x}{(\log_e x)^2}$$

$$= \frac{\log_e x \cdot nx^{n-1} \cdot \frac{1}{x}}{(\log_e x)^2} = \frac{nx^{-1}\log_e x - x^n - 1}{(\log_e x)^2}$$

$$= \frac{x^{n-1}(n \log_e x - 1)}{(\log_e x)^2}$$ **Ans.**

Note

Example 4: Prove that $x\frac{dy}{dx} = y(1-y)$ **if** $y = \frac{x}{x+5}$.

Solution: given that $y = \frac{x}{x+5}$

$$\therefore \quad \frac{dy}{dx} = \frac{(x+5)\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(x+5)}{(x+5)^2} = \frac{(x+5)\cdot 1 - x\cdot 1}{(x+5)^2}$$

$$= \frac{x+5-x}{(x+5)^2} = \frac{5}{(x+5)^2} \Rightarrow x\frac{dy}{dx} = \frac{5x}{(x+5)^2} \quad ...(1)$$

And

$$= y(1-x) = \frac{x}{x+5}\left(1-\frac{x}{x+5}\right) = \frac{x}{x+5}\left(\frac{5}{x+5}\right) = \frac{5x}{(x+5)^2} \quad ...(2)$$

Therefore from (1) and (2)

$$x\frac{dy}{dx} = y(1-y)$$

Questionnaire 3.3

Determine the differential coefficient of the following with respect to x.

1. $\frac{1}{x^{1/4}}$
2. $\frac{x^n}{\log_e x}$
3. $\frac{x}{a^2+x^2}$
4. $\frac{x^2}{e^2+x^2}$
5. $\frac{e^x}{1+x^2}$
6. $\frac{e^x}{1+e^x}$
7. If $f(x) = \frac{x^3}{a^2-x^2}$, then evaluate $f'\left(\frac{a}{2}\right)$
8. If $y = \frac{x-4}{2\sqrt{x}}$, then evaluate the value of $\frac{dy}{dx}$ on $x = 4$. Can we find the value of $\frac{dy}{dx}$ on $x = 0$?
9. If $y = \frac{5x^2+6x+7}{2x^2+3x+4}$, then find the value of $\frac{dy}{dx}$
10. If $y = \frac{a^x}{x^n}$, then find the value of $\frac{dy}{dx}$

Determine the differential coefficient of following product with respect to x

11. (i) $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$ (ii) $\frac{e^x+e^{-x}}{e^x-e^{-x}}$
12. (i) $\frac{xe^x-1}{x}$ (ii) $\frac{x}{\sqrt{1-x^2}}$
13. $\frac{1}{(x+a)(x+b)(x+c)}$

Note

Answers

1. $-\frac{1}{4}x^{-5/4}$
2. $\frac{nx^{n-1}\log_e x - x^{n-1}}{(\log_e 2)^2}$
3. $\frac{a^2 - x^2}{(a^2 + x^2)^2}$
4. $\frac{2xe^2}{(e^2 + x^2)^2}$
5. $\frac{e^x(1-x)^2}{(1+x^2)^2}$
6. $\frac{e^x}{(1+e^x)^2}$
7. $\frac{11}{19}$
9. $\frac{3(x^2+4x+1)}{(2x^2+2x+4)^2}$
10. $\frac{a^2}{x^n}[\log_e a - \frac{n}{x}]$
11. (i) $\frac{\sqrt{a}}{\sqrt{x}(\sqrt{x}-\sqrt{x})^2}$ (ii) $\frac{-4}{[e^x - e^{-x}]^2}$
12. (i) $e^x + \frac{1}{x^2}$ (ii) $\frac{1}{(1-x^2)^{3/2}}$
13. $\frac{3x^2 + 2(a+b+c)x + (ab+bc+ca)}{(x+a)^2(x+b)^2(x+c)^2}$

Self Assessment

2. Multiple Choice Questions:

(i) $\frac{d}{dx}\{f(x)\} = \frac{d}{dx} f_1(x) \pm$.....

(a) $\frac{d}{dx}f_2$ (b) $\frac{d}{dx}f_1$ (c) $\frac{d}{dx}f(x)$ (d) $\frac{d}{dx}\{f(x)\}$

(ii) What will be the value of $\frac{d}{d(x)}\left(\frac{1}{\sqrt{x}}\right)$?

(a) $\frac{1}{2}x^{-\frac{3}{2}}$ (b) $-\frac{1}{2}x^{-\frac{3}{2}}$ (c) $-\frac{1}{2}x^{\frac{3}{2}}$ (d) $\frac{1}{2}x^{\frac{3}{2}}$

(iii) What will be differential coefficient of $3x^3$ with respect to x?

(a) $6x^2$ (b) $3x^2$ (c) $9x^2$ (d) $9x$

(iv) $\frac{d}{dx}(e^x) =$ .....

(a) e (b) 1 (c) $e^{\frac{1}{x}}$ (d) e^x

(v) $\frac{d}{dx}\log_a x =$ ....$\log_a e$

(a) $\frac{1}{x}$ (b) x (c) $\log_a$ (d) $\log x$

Note

3.12 Summary

- $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is known as Derivative for the differential coefficient of y function with respect to x. To avoid any doubt with respect to independent variables, the differential coefficient or derivative $\lim_{x \to 0} \frac{y}{x}$ with respect to x is written as $\frac{d}{dx}(y)$ or $\frac{dy}{dx}$. This way the process of knowing the limit is known as differentiation or in other words method of calculating the differential coefficient of any product is referred as Differentiation.
- If $f(x)$ is the function of x and the same function of $x + \delta x$ is $f(x + \delta x)$, then limiting value is $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$, for the differential coefficient $f(x)$ with respect to x.
- $\frac{d}{dx}\{f(x)\} = \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x)$
- Differential coefficient of Constant and multiplication of any product is equal to differential coefficient of product and multiplication of constant value.
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx} a^x = a^x \log_e a$ *i.e.* differential coefficient of division of two function

3.13 Keywords

- ***Differential coefficient:*** Differentiation
- ***Growth:*** Increment

3.14 Review Questions

1. Find the value of $\frac{d}{dx}(-6x^2)$ **[Ans.:** = $-12x$**]**
2. Find the value of $\frac{d}{dx}(5x^6 + 2x)$ **[Ans.:** = $30x^5 + 2$**]**
3. Prove that $\frac{d}{dx} a^x = a^x \log_e a$
4. Find the differential coefficient of $6\log x - \sqrt{x} - 7$ **[Ans.:** = $\frac{6}{x} - \frac{1}{2} x^{-\frac{1}{2}}$**]**
5. If $y = \frac{x}{x+5}$, then prove that $x\frac{dy}{dx} = y(1-y)$

Answers: Self Assessment

1.	(i) Differentiation	(ii) Multiplication	(iii) *a*	(iv) $n - 1$	(v) $63x^6$
2.	(i) *a*	(ii) (b)	(iii) *(c)*	(iv) *(d)*	(v) *(a)*

Note

3.15 Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Mathematical Economy – Michael Harrison, Patrick Walderan.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economics and Finance – Martin Norman.

Unit 4: Logarithmic Differentiation

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Find Logarithmic Differentiation.
- Find Sum of Infinite Tables.
- Calculate Implicit Function.

Introduction

If you need to differentiate such functions in which exponent is also a function of that variable and a function that needs to be differentiated is the product or division of many functions, then we first need to find logarithm of those functions and then differentiate it. This process is known as logarithmic differentiation.

4.1 Logarithmic Differentiation

Let a function of x be equal to y. Then either take log of both the sides such that the exponent is the product or sum form, etc.

Now differentiate both sides wrt x and find $\frac{dy}{dx}$

Remember that to to differentiate the function of y wrt x, function is difrentiated wrt y and multiplied with $\frac{dy}{dx}$.

Caution log (a + b) ≠ log a + log b

Therefore if then $y = x^x + (\sin x)^{\cos x}$ then $\log y \neq \log x^x + \log(\sin x)^{\cos x}$

Note In such questions, differentiation of every element needs to be find separately and then only combined differentiation can be found out.

Notes Some useful formulae

(i) $\log(m.n) = \log m + \log n$

(ii) $\log\left(\frac{m}{n}\right) = \log m - \log n$

(iii) $\log(m)^n = n\log m.$

EXAMPLES WITH SOLUTION

Example 1: Find the differentiation of $e^x.\log_e x.\tan x$ with respect to x.

Solution: Let $y = e^x.\log_e \tan x$

By taking log of both sides

$\log y = \log e^x + \log(\log_e x) + \log\tan x$

By differentiating both sides

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{e^x}(e^x) + \frac{1}{\log_e x}\left(\frac{1}{x}\right) + \frac{1}{\tan x}(\sec^2 x)$$

$$= 1 + \frac{1}{x\log_e x} + \frac{\cos x}{\sin x}.\frac{1}{\cos^2 x}$$

$$= 1 + \frac{1}{x\log_e x} + \frac{1}{\sin x\cos x}$$

$$= 1 + \frac{1}{x\log_e x} + 2\text{cosec } 2x$$

$\therefore$
$$\frac{dy}{dx} = y\left[1 + \frac{1}{x\log_e x} + 2\text{cosec } 2x\right]$$

Therefore
$$\frac{dy}{dx} = e^x\log_e x\tan x\left[1 + \frac{1}{x\log_e x} + 2\text{cosec } 2x\right].$$ **Ans.**

Example 2: Find the value of $y = x^{\sin^{-1}x}\ \frac{dy}{dx}$.

Solution: $y = x^{\sin^{-1}x}$

By taking log on both sides,

Note

$\log y = \log x^{\sin^{-1} x} = \sin^{-1} x . \log x$

By differentiating wrt x

$$\frac{1}{y}\frac{dy}{dx} = \sin^{-1} x . \frac{1}{x} + \log x . \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = y\left[\frac{1}{x}.\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}.\log x\right]$$

therefore $$\frac{dy}{dx} = x^{\sin^{-1} x}\left[\frac{1}{x}\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}.\log x\right].$$ **Ans.**

Example 3: Find the differentiation of 10^x with respect to x, where x is a constant.

Solution: If $y = 10^x$

By taking log of both sides

$$\log y = \log 10^x$$

$$\log y = x \log 10$$

$$\log y = x$$

By differentiating both sides

$$\frac{1}{y}.\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y$$

By putting $$y = 10^x$$

$$\frac{dy}{dx} = 10^x$$ **Ans.**

Example 4: Find differentiation of $\frac{1}{(x+a)(x+b)(x+c)}$ wrt x.

Solution: Let us assume $y = \frac{1}{(x+a)(x+b)(x+c)}$

By taking log of both sides

$$\log y = \log \frac{1}{(x+a)(x+b)(x+c)}$$

Or $$\log y = \log 1 - \log(x+a) - \log(x+b) - \log(x-c)$$

By differentiating both sides

$$\frac{1}{y}\frac{dy}{dx} = 0 - \frac{1}{x+a} - \frac{1}{x+b} - \frac{1}{x+c}$$

Or $$\frac{dy}{dx} = -y\left[\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c}\right]$$

Or $$\frac{dy}{dx} = -\frac{1}{(x+a)(x+b)(x+c)}\left[\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c}\right].$$ **Ans.**

Note

Example 5: If $(\sin y)^x = a.$ then find the value of $\frac{dy}{dx}$.

Solution: $(\sin y)^x = a.$

By taking log of both sides

$$\log(\sin y)^x = \log a$$

Or
$$x \log \sin y = \log a$$

By differentiating both sides wrt x

$$x.\frac{1}{\sin y}.\cos y \frac{dy}{dx} + \log \sin y = 0$$

$$x \cot y \frac{dy}{dx} = -\log \sin y$$

Or
$$\frac{dy}{dx} = -\frac{\log \sin y}{x \cot y}.$$

Ans.

Example 6: If $(\cos x)^y = (\sin y)^x$ then find the value of $\frac{dy}{dx}$.

Solution: $(\cos x)^y = (\sin y)^x$

By taking log of both sides

$$y \log \cos x = x \log \sin y$$

By differentiating both sides wrt x

$$y\frac{1}{\cos x} x^{-\sin x} + \log \cos x.\frac{dy}{dx} = x.\frac{1}{\sin y}.\cos y.\frac{dy}{dx} + \log \sin y.1$$

$$-y \tan x + \log \cos x \frac{dy}{dx} = x \cot y \frac{dy}{dx} + \log \sin y$$

$$\frac{dy}{dx}(\log \cos x - x \cot y) = (\log \sin y + y \tan x)$$

$$\frac{dy}{dx} = \frac{\log \sin y + y \tan x}{\log \cos x - x \cot y}.$$

Ans.

Example 7: Find $\frac{dy}{dx}$ of $\log (xy) = x^2 + y^2$ wrt x.

$$\log(xy) = x^2 + y^2$$

$$\log x + \log y = x^2 + y^2$$

Solution: By differentiating both sides wrt x

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx} = = 2x+2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{1}{y}-2y\right) = 2x-\frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1-2y^2}{y}\right) = \frac{2x^2-1}{x}$$

$$\frac{dy}{dx} = \frac{y(2x^2-1)}{x(1-2y^2)}.$$ **Ans.**

Questionnaire 4.1

Short Answer questions:

Find the differentiation of following functions wrt x:

1. x^x
2. $x^{\sin x}$
3. $(\log x)^x$
4. $(1+x)^x$
5. $(x-1)(x-2)(x-3)$
6. $\dfrac{x}{\sqrt{1+x^2}}$
7. $\sqrt{\dfrac{1-x}{1+x}}$
8. $x^x+a^x+x^a$
9. $\dfrac{x\sqrt{1+x}}{(1+x^2)^{3/2}}$
10. e^{x^x}
11. $\sqrt{\dfrac{x^2+x+1}{x^2-x+1}}$
12. $\dfrac{(x-a)(x-b)}{\sqrt{x-c}}$
13. $(x\log x)^{\log\log x}$
14. If $y=\sqrt{\dfrac{1-x}{1+x}}$ then prove that $(1-x^2)\dfrac{dy}{dx}+y=0$
15. If $y=10^{10^x}$ then find $\dfrac{dy}{dx}$
16. If $y=\dfrac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}$ then find $\dfrac{dy}{dx}$

Note

Answers

1. $[x^x(1+\log x)]$
2. $[\sin x(\cos x \log x + \frac{\sin x}{x})]$
3. $(\log x)^x(\log\log x + \frac{1}{\log x})$
4. $(1+x)^x\left[\frac{x}{1+x} + \log(1+x)\right]$
5. $(x-2)(x-3)+(x-11)(x-3)+x-1)(x-2)$
6. $\frac{1}{(1+x^2)^{3/2}}$
7. $\frac{-1}{(1+x)^{3/2}(1-x)^{1/2}}$
8. $x^x \log_e x + a^x \log_e a + ax^{a-1}$
9. $\frac{2+3x-4x^2-3x^3}{2\sqrt{(1+x)}(1+x^2)^{5/2}}$
10. $e^{2x}x^x(1+\log_e x)$
11. $\sqrt{\frac{x^2+x+1}{x^2-x+1}}\left(\frac{1-x^2}{x^4+x^2+1}\right)$
12. $\frac{(x-a)(x-b)}{\sqrt{x-c}}\left[\frac{1}{x-a} + \frac{1}{x-b)} - \frac{1}{2(x-c)}\right]$
13. $(x\log x)(\log\log x - 1)$
15. $10^x.10^x(\log e\, 10)^2$
16. $\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3 e^x}\left[\frac{2}{x+1} + \frac{1}{2(x-1)} - \frac{3}{x+4} - 1\right]$

4.2 Sum of Infinite Tables

Example 1: If $y = x^{x^{x\ldots\infty}}$ then prove that $x\frac{dy}{dx} = \frac{y^2}{1-y\log x}$.

Solution:

$$y = x^{x^{x\ldots\infty}} = x^y \text{ because } x^{x\ldots\infty} = y.$$

By taking log $\log y = y\log x$

$$\frac{1}{y}.\frac{dy}{dx} = y.\frac{d}{dx}\log x + \log x\frac{dy}{dx}$$

$$= y.\frac{1}{y} + \log x\frac{dy}{dx}$$

$$\therefore \quad \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}.$$

So that

$$x\frac{dy}{dx} = \frac{y^2}{1-y\log x}.$$

Hence Proved.

Note

Example 2: If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ... \infty}}}$, **then prove that** $\frac{dy}{dx} = \frac{\cos x}{2y - 1}$.

Solution: Given that $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ...\infty}}}$

$$\Rightarrow \quad y = \sqrt{\sin x + y}$$

$$y^2 = \sin x + y$$

By differentiating both sides wrt x

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$(2y-1)\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Hence Proved

Example 3: If $y = e^{x+e^{x+e^{x+....\infty}}}$ **then prove that** $\frac{dy}{dx} = \frac{y}{1-y}$.

Solution: Given that $y = e^{x+e^{x+e^{x+.....\infty}}}$

$$\Rightarrow \quad y = e^{x+y}$$

By taking log of both sides,

$$\log y = \log\{e^{x+y}\}$$

$$= (x+y)\log e$$

$$= x + y, \qquad [\because \log e = 1]$$

By differentiating both sides wrt x

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\therefore \quad \left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$$

$$\therefore \quad \left(\frac{1-y}{y}\right)\frac{dy}{dx} = 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{1-y}.$$

Hence Proved

Example 4: If $y = a^{x^{a^{x....\infty}}}$, **then prove that** $\frac{dy}{dx} = \frac{y^2 \log y}{x(1 - y\log x \,.\, \log y)}$.

Solution: Here $y = a^{x^{a^{x....\infty}}} = a^{x^y}$

Note By taking log of both sides,

$\log y = x^y \log a$

By again taking log of both sides

$\log(\log y) = y\log x + \log(\log a)$

By differentiating wrt x

$$\frac{1}{\log y}.\frac{1}{y}\frac{dy}{dx} = y.\frac{1}{x} + \frac{dy}{dx}(\log x) + 0$$

$$\therefore \left(\frac{1}{y\log y} \log x\right)\frac{dy}{dx} = \Rightarrow \left(\frac{1 - y\log x.\log y}{y\log y}\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y^2 \log y}{x(1 - y\log x.\log y)}.$$

Hence Proved

Self Assessment

1. Fill in the blanks:

(i) Such functions which are differentiated by taking logarithm are known as ………………………. Differentiation.

(ii) $\log\left(\frac{m}{n}\right) = \log m - ....$

(iii) $\log(m)^n ......\log m$

(iv) $......... = \log m + \log n$

(v) $\log\left(\frac{25}{12}\right) = \log 25 - \log ....$

Questionnaire 4.2

Short Answer Questions:

1. If $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$ then prove that $(2y-1)\frac{dy}{dx} = 1$.

2. If $y = \sqrt{\tan x + \sqrt{\tan x} + \sqrt{\tan x + ...\infty}}$ then prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$

3. If $y = -\sqrt{x}^{\sqrt{x}^{\sqrt{x}...\infty}}$ then prove that $x\frac{dy}{dx} = 2\frac{y^2}{2 - y\log x}$

4. If $y = (\sin x)^{(\sin x)^{(\sin x)...\infty}}$ then prove that $\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y\log(\sin x)}$

5. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + ... + \infty}}}$ then prove that $\frac{dy}{dx} = \frac{1}{x(2y-1)}$

6. If $y = x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + ...}}$ then prove that $\frac{dy}{dx} = \frac{2xy^2}{1+y^2}$

Note

4.3 Implicit Function

If any such equation exists between x and y such that cannot be solved for y instantaneously then y is said to be the implicit function of x. In contrast if value of y can be found out in terms of x then y is said to be explicit function of x.

Did u know? Differentiation of implicit function : To dy/dx find of implicit function, differentiate each element of the equation wrt x then by bringing the value of dy/dx to find one side, find its value.

EXAMPLES WITH SOLUTION

Example 1: If $ax^2 + 2hxy + by^2 = 0$ **, find the value of** dy/dx**.**

Solution: Given that : $ax^2 + 2hxy + by^2 = 0$

By differentiating wrt x

$$a.\frac{d}{dx}(x^2) + 2h\frac{d}{dx}(xy) + b\cdot\frac{d}{dx}(y^2) = 0$$

Or
$$2ax + 2h\left(x\cdot\frac{dy}{dx} + y\cdot 1\right) + 2by\cdot\frac{dy}{dx} = 0$$

$$2(hx + by)\frac{dy}{dx} = -2(ax + hy)$$

Or
$$\frac{dy}{dx} = -\left(\frac{ax + hy}{hx + by}\right)$$
Ans.

Example 2: If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ **, then find the value of** dy/dx**.**

Solution: Given that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

By differentiating wrt x

$$a\frac{d}{dx}(x^2) + 2h\cdot\frac{d}{dx}(xy) + b\cdot\frac{d}{dx}(y^2) + 2g\cdot\frac{d}{dx}(x) + 2f\cdot\frac{d}{dx}(y) + \frac{d}{dx}(c) = 0$$

Or
$$a(2x) + 2h\left(x\frac{dy}{dx} + y\right) + b\left(2y\frac{dy}{dx}\right) + 2g\cdot 1 + 2f\cdot\frac{dy}{dx} + 0 = 0$$

$$2(hx + by + f)\frac{dy}{dx} = -2(ax + hy + g)$$

Or
$$\frac{dy}{dx} = -\frac{(ax + hy + g)}{(hx + by + f)}$$
Ans.

Note

Example 3: If $y = x^y$ prove that $x\frac{dy}{dx} = \frac{y^2}{1 - y\log x}$.

Solution: Given that $y = x^y$, by taking $\log y = y\log x$

By differentiating wrt x

$$\frac{1}{y}\frac{dy}{dx} = y\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{dy}{dx}$$

$$= y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

Or $$\left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

Therefore $$x\frac{dy}{dx} = \frac{y^2}{1 - y\log x}$$ **Ans.**

Example 4: If $\sin y = x\sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Solution: Given that $\sin y = x\sin(a+y)$, or $x = \frac{\sin y}{\sin(a+y)}$

By differentiating wrt x

$$1 = \frac{\sin(a+y)\cdot\frac{d}{dx}\sin y - \sin y\cdot\frac{d}{dx}\sin(a+y)}{\{\sin(a+y)\}^2}$$

Or $$\sin^2(a+y) = \sin(a+y)\cdot\cos y\frac{dy}{dx} - \sin y\cdot\cos(a+y)\frac{dy}{dx}$$

Or $$\sin^2(a+y) = \{\sin(a+y)\cos y - \sin y\cos(a+y)\}\frac{dy}{dx}$$

Or $$\sin^2(a+y) = \sin(a+y-y)\frac{dy}{dx}$$

Or $$\sin a\frac{dy}{dx} = \sin^2(a+y)$$

Therefore $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$. Hence Proved.

Example 5: If $x^y = e^{x-y}$ then prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Solution: Given that $x^y = e^{x-y}$, by taking log of both sides

$y\log_e x = (x-y)\log e$ or $y\log_e x = x - y$

Or $y(1+\log x) = x$ or $y = \frac{x}{1+\log x}$

$\therefore$ $$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x(1/x)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

Note

Task Find differentiation of $x^{\sin x}$ wrt x. **Ans.:** $[\sin x(\cos x \log x + \frac{\sin x}{x})]$

Questionnaire 4.3

Find the value of $\frac{dy}{dx}$

1. $xy = c$
2. $x^2 + y^2 = a^2$
3. $3x^2 + y^2 = 5$
4. $5x^2 + 5y^2 - 11x - 9y - 12 = 0$
5. $x^n + y^n = a^n$
6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
7. $y^2 = 4ax$
8. $x^{2/3} + y^{2/3} = a^{2/3}$
9. If $x^p y^q = (x+y)^{p+q}$ then prove $\frac{dy}{dx} = \frac{y}{x}$
10. $x^y + y^x = a^b$
11. $y^x = x^y$
12. $x^x + y^y = 1$
13. If $y\sqrt{(1-x^2)} + x\sqrt{(1-y^2)} = 1$ then prove $\frac{dy}{dx} = \sqrt{\frac{(1-y^2)}{(1-x^2)}} = 0$
14. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then prove that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$
15. If $x\sqrt{1+y} + y\sqrt{1+x} = 0,$ then prove that $\frac{dy}{dx} = -(1+x)^{-2}$

Answers

1. $-\frac{y}{x}$
2. $-\frac{x}{y}$
3. $\frac{-3x}{y}$
4. $-\frac{(10x-11)}{10y-9}$
5. $-\left(\frac{x}{y}\right)^{n-1}$
6. $\frac{-b^2x}{a^2y}$
7. $\frac{2a}{y}$
8. $\frac{-y^{1/3}}{x^{1/3}}$
10. $\frac{-xy^{y-1} + y^x \log y}{xy^{x-1} + x^y \log x}$
11. $\frac{y(y - x\log x)}{x(x - y\log x)}$
12. $-\frac{x^x[1+\log x]}{y^y(1+\log y)}$

Note

4.4 Parametric Function

Sometimes x and y are given in the form of function of a third variable. This third element is known as Parameter and this type of equation is known as parametric function. In this way we can find the $\frac{dy}{dx}$ without elimination of parameter.

Example 1: If $x = f_1(t)$ and $y = f_2(t)$ where t is independent variable and a and y are dependent variables.

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

Therefore differentiation of y wrt x is the multiple of differentiation of x and y wrt to the parameter.

EXAMPLES WITH SOLUTION

Example 1: If $x = at^2$ and y = 2at, find the value of $\frac{dy}{dx}$.

Solution: Here $x = at^2$

Differentiating wrt parameter

$$\frac{dx}{dt} = 2at$$

And $y = 2at$

Differentiating both wrt t

$$\frac{dy}{dt} = 2a$$

$$\frac{dy}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{2a}{2at} = \frac{1}{t}$$ **Ans.**

Example 2: If $x = a\cos\theta$ and $y = b\sin\theta$ find value of $\frac{dy}{dx}$.

Solution: Here,

$$\frac{dx}{d\theta} = -a\sin\theta \text{ or } \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$ **Ans.**

Note

Questionnaire 4.4

Find $\frac{dy}{dx}$ of the following:

1. $x = a\sec\theta,\ y = b\tan\theta$
2. $x = \cos t,\ y = \sin t$
3. $x = \log t,\ y = e^t + \cos t$
4. If $x = \frac{3at}{1+t^3},\ y = \frac{3at^2}{1+t^3}$ then prove that

   (i) $\frac{dy}{dx} = \frac{t(2-t^3)}{1-2t^3}$ (ii) find $\frac{dy}{dx}$ when $t = \frac{1}{2}$

Answers

1. $\frac{b}{a}\operatorname{cosec}\theta$ 2. $-\cot t$ 3. $t(e^t - \sin t)$ 4. (ii) $-3/2$

Self Assessment

2. Multiple Choice Questions:

(i) If there exists an equation between x and y such that cannot be solved for y instantaneously then y will be said to be what function of x?

(a) Implicit (b) Explicit

(c) Equal (d) group

(ii) If y can be calculated in terms of x, then y will be said to be what funtion of x?

(a) Implicit (b) Explicit

(c) Equal (d) group

(iii) What will be the value of $xy = c$

(a) $-\frac{x}{y}$ (b) $\frac{x}{y}$

(c) $-\frac{y}{x}$ (d) $\frac{y}{x}$.

4.5 Summary

- If you need to differentiate such functions in which exponent is also a function of that variable and a function that needs to be differentiated is the product or division of many functions, then we first need to find logarithm of those functions and then differentiate it.
- If any such equation exists between x and y such that cannot be solved for y instantaneously then y is said to be the implicit function of x. In contrast if value of y can be found out in terms of x then y is said to be explicit function of x.

  $\log(m\cdot n) = \log m + \log n$

Note

$$\log\left(\frac{m}{n}\right) = \log m - \log n$$

$$\log (m)^n = n \log m$$

- Sometimes x and y are given in the form of function of a third variable. This third element is known as Parameter and this type of equation is known as parametric function. In this way we can find the $\frac{dy}{dx}$ without elimination of parameter.

4.6 Keywords

- ***Function:*** Work
- ***Infinite:*** Which does not have an end

4.7 Review Questions

1. If $y = x^{\sin^{-1} x}$ then find $\frac{dy}{dx}$. (**Ans.**: $x^{\sin^{-1} x}\left[\frac{1}{x}\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}\log x\right]$)

2. If $(\cos x)^y = (\sin y)^x$ find $\frac{dy}{dx}$ (**Ans.**: $\frac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$)

3. If $y = x^y$, Prove that $x\frac{dy}{dx} = \frac{y^2}{1 - y \log x}$

4. If $x = a \cos\theta$ and $y = b \sin\theta$ find $\frac{dy}{dx}$ (**Ans.**: $-\frac{b}{a}\cot\theta$)

Answers: Self Assessment

1. (i) Logarithmic (ii) log n (iii n (iv) log(m, n) (v) 12
2. (i) (a) (ii) (b) (iii) (c)

4.8 Further Readings

Books

Mathematics for Economics- Council for Economic Education.

Mathematical Economics-Michael Harrison, Patrick Waldren.

Mathematics for Economics- Malcom, Nicholas, U.P. London

Mathematics for Economics-Carl P. Simon, Lawrence Bloom.

Mathematics for Economist- Mehta and Madnani-Sultan Chand and Sons.

Essential Mathematics for Economics-Nut Sedester, Peter Hamand, Prentice Hall Publication.

Mathematics for Economist-Simon and Bloom-Viva Publications.

Mathematics for Economist-Yamane-Prentice Hall India.

Mathematics for Economics and Finance-Martin Norman.

Unit 5: Second and Higher Order Differentiation

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Find out Successive Differentiation.

Introduction

If y is the product of x then $\frac{dy}{dx}$ would also be the product of x. The differential coefficient of this would be the second differential coefficient and then this will be referred as third differential coefficient.

5.1 Successive Differentiation

If y is the product of x then $\frac{dy}{dx}$ would also be the differential coefficient of y with respect to x, which further can be differentiated. The $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ differential coefficient of $\frac{dy}{dx}$ would be referred to as second differential coefficient. Thus, second differential coefficient of y, would be referred to as third differential coefficient. Thus, the different coefficient can be shown as $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, ...$.

Prime product $y = f(x)$	Simple Indication	Other Indication
First coefficient	$\frac{dy}{dx}$	$f'(x), Dy, y_1$
Second coefficient	$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$	$f''(x), D^2y, y_2$
Third coefficient	$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^2y}{dx^2}$	$f'''(x), D^3y, y_3$
....	....	....
n^{th} coefficient	$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=\frac{d^ny}{dx^n}$	$f^n(x), D^n y, y_n$

Note

Thus, if $y = x^7$, then $\frac{dy}{dx} = 7x^6$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}(7x^6) = 42x^5$$

And $$\frac{d^3y}{dx^3} = \frac{d}{dx}(42x^5) = 210x^4$$

EXAMPLES WITH SOLUTION

Example 1: If $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, then find out the value of $f''''(x)$.

Solution: given then

$$f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

$$f'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e + 0$$

$$f''(x) = 20ax^3 + 12bx^2 + 6cx + 2d + 0$$

$$f'''(x) = 60ax^2 + 24bx + 6c = 0$$

$$f''''(x) = 120ax + 24b$$

Ans.

Example 2: If $y = A \sin mx + B \cos mx$, then prove that $\frac{d^2y}{dx^2} + m^2 y = 0$.

Solution: given then $y = A \sin mx + B \cos mx$

$$\frac{dy}{dx} = A \cos mx . \frac{d}{dx}(mx) + B(-\sin mx) \cdot \frac{d}{dx}(mx)$$

$$= Am \cos mx - Bm \sin mx$$

Further evaluating with respect to x

$$\frac{d^2y}{dx^2} = Am(-\sin mx). \frac{d}{dx}(mx) - Bm \cos mx. \frac{d}{dx}(mx)$$

$$= -Am^2 \sin mx - Bm^2 \cos mx$$

$$= -m^2 (A \sin mx + B \cos mx) = -m^2y$$

Therefore, $\frac{d^2y}{dx^2} + m^2y = 0.$

Example 3: If $y = \sin(\sin x)$, then prove that $y_2 + y_1 \tan x + y \cos^2 x = 0$.

Solution: given that

$$y = \sin(\sin x)$$

$$y_1 = [\cos(\sin x)] \cdot \cos x \qquad ...(1)$$

On Differentiation $$y_2 = [\cos(\sin x)] \cdot (-\sin x) + \cos x \cdot [-\sin(\sin x)] \cos x$$

$$= -\sin x \cos(\sin x) - \cos^2 x \sin(\sin x)$$

Note

$$= -\sin x \cos(\sin x) - y\cos^2 x, \qquad [\because y = \sin(\sin x)]$$

$$= -\sin x \cdot \frac{y_1}{\cos x} - y\cos^2 x, \qquad \text{[From (1)]}$$

Therefore, $y_2 + y_1 \tan x + y\cos^2 x = 0$.

Example 4: If $y = e^{ax} \sin bx$, , then prove that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$.

Solution: given that $y = e^{ax} \sin bx$ ...(1)

Evaluating with respect to x

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(\sin bx) + \sin bx.\frac{d}{dx}(e^{ax})$$

$$= e^{ax}\cos bx.\frac{d}{dx}(bx) + \sin bx.e^{ax}.\frac{d}{dx}(ax)$$

$$= be^{ax}\cos bx + ae^{ax}\sin bx$$

$$= be^{ax}\cos bx + ay, \quad \text{[From equation (1)]} \qquad ...(2)$$

Further evaluating with respect to x

$$\frac{d^2y}{dx^2} = b\left[e^{ax}.\frac{d}{dx}(\cos bx) + \cos bx.\frac{d}{dx}(e^{ax})\right] + a\frac{dy}{dx}$$

$$= b\left[e^{ax}.(-\sin bx).\frac{d}{dx}(bx) + \cos bx.e^{ax}.\frac{d}{dx}(ax)\right] + a\frac{dy}{dx}$$

$$= b[-be^{ax}\sin bx + ae^{ax}\cos bx] + a\frac{dy}{dx}$$

$$= -b^2 e^{ax}\sin bx + a(be^{ax}\cos bx) + a\frac{d}{dy}$$

$$= -b^2 y + a\left(\frac{dy}{dx} - ay\right) + a\frac{dy}{dx} = 2a\frac{dy}{dx} - (a^2 + b^2)y,$$

$$= -b^2 y + a\left(\frac{dy}{dx} - ay\right) + a\frac{dy}{dx} = 2a\frac{dy}{dx} - (a^2 + b^2)y,$$

from equation (1) and (2)

Therefore, $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$

Note

Self Assessment

1. Fill in the blanks:

(i). $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the .... differential coefficient of y.

(ii). $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$.............

(iii). $\frac{d}{dx}(..........)=\frac{d^2y}{dx^2}$

Questionnaire 5.1

1. If $y = 8x^3 + 4x^2 + 3x + 11$ then find out the value of $\frac{d^3y}{dx^3}$

2. If $y = ax^3 + bx^2 + cx + d$ then find out the value of $\frac{d^3y}{dx^3}$

3. If $y = x^2 \log x$, then prove that $\frac{d^3y}{dx^3} = \frac{2}{x}$

Determine the second differential coefficient of following product

4. (i) $x^3 \log x$ (ii) $x \log x$

5. $\sin(\cos x)$

6. $x^3 e^{4x}$

7. $\tan e^x$

8. Find out the n^{th} differential coefficient of e^{nx}

9. If $y = A \sin px + B \cos px$, then prove that $\frac{d^2y}{dx^2} + p^2y = 0$

10. If $x^3 + y^3 - 3axy = 0$, then prove that $\frac{d^2y}{dx^2} = \frac{2a^2xy}{(ax - y^2)^3}$

11. If y is the product of z and $z = ax$ then prove that $\frac{d^2y}{dx^2} = a^2\frac{d^2y}{dz^2}$

12. If $y=(\sin^{-1} x)^2$, then prove that $(1 - x^2)y_2 - xy_1 = 2$

13. If $y = e^{\tan^{-1} x}$, then prove that $(1 + x^2)y_2 + (2x - 1)y_1 = 0$

Note

14. If $\sqrt{x+y}+\sqrt{y-x}=c$, then prove that $\frac{d^2y}{dx^2}=\frac{2}{c^2}$

15. If $x^2+xy+y^2=a^2$, then prove that $\frac{d^2y}{dx^2}+\frac{6a^2}{(x+2y)}=0$

16. If $y=\tan^{-1}\frac{1-2\log x}{1+2\log x}+\tan^{-1}\frac{3+2\log x}{1-6\log x}$, then prove that $\frac{d^2y}{dx^2}=0$

17. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$, then prove that $\frac{d^2p}{d\theta^2}+p=\frac{a^2b^2}{p^3}$

18. If $y=e^{a\sin^{-1}x}$, then prove that $(1-x^2)y_2-xy_1-a^2y=0$

19. If $y=\sin(m\sin^{-1}x)$, then prove that $(1-x^2)y_2-xy_1+m^2y=0$.

5.2 Summary

- If y is the product of x then $\frac{dy}{dx}$ would also be the differential coefficient of y with respect to x, which further can be differentiated. The $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ differential coefficient of $\frac{dy}{dx}$ would be referred to as **second differential coefficient.**

5.3 Keywords

- ***Successive:*** In a series
- ***Miscellaneous:*** Mixed

5.4 Review Questions

1. Find the value of $\frac{d^2y}{dx^2}$ if $y=x^3-\frac{1}{x^3}$ [**Ans.:** = $6x-12x^{-5}$]

2. If $y=x^3\log x$, then prove that $\frac{d^4y}{dx^4}=\frac{6}{x}$

3. If $y=e^{ax}\sin bx$ then prove that $\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+(a^2+b^2)y=0$

Answers: Self Assessment

1. (i) 2nd (ii) $\frac{d^3y}{dx^3}$ (iii) $\frac{dy}{dx}$

Note

5.5 Further Readings

Books

Mathematical Economy – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

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Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economics and Finance – Martin Norman.

Note

Unit 6: Differentiation : Partial

CONTENTS

Objectives

After reading this unit students will be able to:

- Understand the Method of Differentiation of a Function in Respect to other Function.

Introduction

Differential coefficient of the first function relative to some other function is the ratio of the differential coefficient of the first function with respect to x to the differential coefficient of the second function with respect to x.

6.1 Differentiation of a Function in Respect to Other Function

Suppose $y_1 = f_1(x)$ and $y_2 = f_2(x)$

That is y_1 and y_2 are the functions of x, on differentiating both with respect to x

$$\frac{dy_1}{dx} = f_1'(x) \text{ and } \frac{dy_2}{dx} = f_1'(x)$$

Now the differential coffiecient of y_1 with respect to y_2 is $\frac{dy_1}{dy_2}$

Notes

$$\frac{dy_1}{dy_2} = \frac{\frac{dy_1}{dx}}{\frac{dy_2}{dx}} = \frac{f_1'(x)}{f_2'(x)}$$

Therefore $\frac{dy_1}{dy_2} = \frac{\text{Differential coefficient of } y_1 \text{ with respect to } x}{\text{Differential coefficient of } y_2 \text{ with respect to } x}$

Did u know? Differential coefficient of the first function relative to some other function is the ratio of the Differential coefficient of the first function with respect to x to the Differential coefficient of the second function with respect to x.

Note

EXAMPLES WITH SOLUTION

Example 1: Find the differential coefficient of $\tan^{-1}x$ with respect to $\sin^{-1}x$ at $x = \frac{1}{2}$.

Solution : $$\frac{d(\tan^{-1}x)}{d(\sin^{-1}x)} = \frac{\frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}} = \frac{\sqrt{1-x^2}}{1+x^2}$$

At $$x = \frac{1}{2} = \frac{\sqrt{1-\frac{1}{4}}}{1+\frac{1}{4}} = \frac{\frac{\sqrt{3}}{2}}{\frac{5}{4}} = \frac{2\sqrt{3}}{5}$$ **Ans.**

Example 2: Find the differential coefficient of $e^{\tan x}$ with respect to $\sin x$.

Solution : Let us suppose $y_1 = e^{\tan x}$ and $y_2 = \sin x$

Here $$dy_1 = \frac{d}{dx} e^{\tan x} = e^{\tan x}.\sec^2 x$$

And $$dy_2 = \frac{d}{dx}\sin x = \cos x$$

Therefore: $$\frac{dy_1}{dy_2} = \frac{de^{\tan x}}{d\sin x} = \frac{e^{\tan x}.\sec^2 x}{\cos x} = \frac{e^{\tan x}}{\cos^3 x}$$ **Ans.**

Example 3: Find the differential coefficient of $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$.

Solution : Suppose $y_1 = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ and $y_2 = \tan^{-1}x$

On putting, $x = \tan\theta$

$$y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$= \tan^{-1}\frac{\sec\theta-1}{\tan\theta} = \tan^{-1}\frac{1-\cos\theta}{\sin\theta}$$

$$= \tan^{-1}\frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{\tan^{-1}x}{2}$$

Note

$$\frac{dy_1}{dx} = \frac{d}{dx}\left[\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}\right]$$

$$\therefore \quad = \frac{d}{dx}\left(\frac{\tan^{-1}x}{2}\right) = \frac{1}{2(1+x^2)}$$

And
$$\frac{dy_2}{dx} = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{dy_1}{dy_2} = \frac{d[\tan^{-1}\{\sqrt{1-x^2}-1\}/x}{d(\tan^{-1}x)}$$

$$\therefore \quad = \frac{\dfrac{1}{2(1+x^2)}}{\dfrac{1}{1+x^2}} = \frac{1}{2}.$$

Ans.

> *Task* Find the differential coefficient of $\tan^{-1}x$ with respect to $\sin^{-1}x$ at $x = \frac{1}{3}$
>
> (**Ans.** : $\frac{3\sqrt{2}}{5}$)

Example 4: Find the differential coefficient of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Solution: Suppose $y_1 = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $y_2 = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

On putting $x = \tan\theta$

$$y_1 = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}x$$

Then
$$\frac{dy_1}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Also
$$y_2 = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$$

Then
$$\frac{dy_2}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

$$\frac{dy_1}{dy_2} = \frac{d[\tan^{-1}\{2x/(1-x^2)\}]}{d[\sin^{-1}\{2x/(1+x^2)\}]}$$

Note

$$= \frac{\frac{d}{dx}[\tan^{-1}\{2x/(1-x^2)\}]}{\frac{d}{dx}\sin^{-1}\{2x/(1+x^2)\}}$$

$$\therefore \quad = \frac{2}{1+x^2} \times \frac{1+x^2}{2} = 1 .$$ **Ans.**

Example 5: Find the differential coefficient of function $f(x) = \sin^{-1}(2x\sqrt{1-x^2})$ with respect to $\sin^{-1}x$.

Solution : Suppose

$$y_1 = \sin^{-1} 2x\sqrt{1-x^2}$$

$$x = \sin\theta$$

$$y_1 = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}) = \sin^{-1}(2\sin\theta\cos\theta)$$

$$y_1 = \sin^{-1}(\sin 2\theta)$$

$$y_1 = 2\theta = 2\sin^{-1} x$$

$$\frac{dy_1}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$y_2 = \sin^{-1} x$$

Again $$\frac{dy_2}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Then $$\frac{d\sin^{-1}(2x\sqrt{1-x^2})}{d\sin^{-1} x} = \frac{\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}} = 2 .$$ **Ans.**

Example 6: If $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$ then prove that $\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$.

Solution: Suppose $x^3 = \sin\theta$, $y^3 = \sin\phi$, then $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$

Or $$\sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a^3(\sin\theta - \sin\phi)$$

$$\cos\theta + \cos\phi = a^3(\sin\theta - \sin\phi)$$

$$2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = a^3.2\cos\frac{\theta-\phi}{2}\sin\frac{\theta-\phi}{2}$$

$$\Rightarrow \quad \cot\frac{\theta-\phi}{2} = a^3$$

$$\Rightarrow \quad \theta - \phi = 2\cot^{-1} a^3$$

$$\Rightarrow \quad \sin^{-1} x^3 - \sin^{-1} y^3 = 2\cot^{-1} a^3$$

On differentiating with respect to x **Note**

$$\Rightarrow \quad \frac{1}{\sqrt{1-x^6}}.3x^2 - \frac{1}{\sqrt{1-y^6}}.3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}.$$ **Ans.**

Example 7: If $y = \dfrac{1}{1+x+x^2+x^3}$, then find the value of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $x = 0$.

Solution:

$$y = \frac{1}{1+x+x^2+x^3}$$

$$= (1+x+x^2+x^3)^{-1}$$

$$\frac{dy}{dx} = -(1+x+x^2+x^3)^{-2}(1+2x+3x^2)$$

At $x = 0$

$$\frac{dy}{dx} = -1$$

$$\frac{d^2y}{dx^2} = 2(1+x+x^2+x^3)^{-3}(1+2x+3x^2)$$

$$-(1+x+x^2+x^3)^{-2}(2+6x)$$

At $x = 0$ $\quad \dfrac{dy}{dx} = 0.$ **Ans.**

Example 8: If $y = x\cos(a+y)$, then prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\cos a}$ and at $x = 0$, $\dfrac{dy}{dx} = \cos a$.

Solution: $\sin y = x\cos(a+y)$

$$\Rightarrow \quad x = \frac{\sin y}{\cos(a+y)}$$

On differentiating with respect to x

$$1 = \frac{\cos(a+y)\cos y\dfrac{dy}{dx} - \sin y\cdot\{-\sin(a+y)\}\dfrac{dy}{dx}}{\cos^2(a+y)}$$

$$1 = \frac{\{\cos(a+y).\cos y + \sin(a+y).\sin y\}\dfrac{dy}{dx}}{\cos^2(a+y)}$$

$$1 = \frac{\cos(a+y-y)\dfrac{dy}{dx}}{\cos^2(a+y)}$$

$$1 = \frac{\cos a}{\cos^2(a+y)}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\cos a}$$

Note

Again when $x = 0$

Then $\sin y = 0 \Rightarrow y = n\pi$

At, $x = 0$

$$\frac{dy}{dx} = \frac{\cos^2(a + n\pi)}{\cos a}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos^2 a}{\cos a} = \cos a$$

Example 9: If $y = (\sin^{-1}x)^2 + (\cos^{-1}x)^2$, then prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 4$$

Solution :

$$y = (\sin^{-1}x)^2 + (\cos^{-1}x)^2$$

$$\frac{dy}{dx} = \frac{2(\sin^{-1}x - \cos^{-1}x)}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = 2(\sin^{-1}x - \cos^{-1}x)$$

On differentiating with respect to x,

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{1}{2\sqrt{1-x^2}}(-2x).\frac{dy}{dx} = 2\left[\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right]$$

$$\frac{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}}{\sqrt{1-x^2}} = 2 \cdot \frac{2}{\sqrt{1-x^2}}$$

$$\therefore \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 4$$

Self Assessment

1. Multiple Choice Questions:

(i) What will be the differential coefficient of ax^7 with respect to x^7?

(a) a (b) x (c) x^7 (d) a^2

(ii) What will be the differential coefficient of log x with respect to tan x?

(a) $\frac{\sin^2 x}{x}$ (b) $\frac{\cos^2 x}{x}$ (c) $\frac{x}{\cos^2 x}$ (d) $\frac{x}{\sin^2 x}$

(iii) Differential coefficient of $\tan^{-1}x$ with respect to $\sin^{-1}x$ at $x = \frac{1}{2}$ will be

(a) $\frac{3\sqrt{2}}{5}$ (b) $\frac{5\sqrt{2}}{3}$ (c) $\frac{2\sqrt{3}}{5}$ (d) $\frac{5}{2\sqrt{3}}$

Note

Questionnaire 6.1

Find the Differential coefficients of the following:

1. x^5 with respect to x^2
2. e^x with respect to $\sqrt{x}$
3. $x \sin^{-1} x$ with respect to $\sin^{-1} x$.
4. $\sin^{-1}\left(\frac{1-x}{1+x}\right)$ with respect to $\sqrt{x}$
5. $(\log \sin x)^{\sin x}$ with respect to sin x
6. $\log (x^2+2x+1)$ with respect to (x^2+2x) .
7. $\sec^{-1}\frac{1}{2x^2-1}$ with respect to $\sqrt{1-x^2}$ at $x=\frac{1}{2}$

Answers

1. $\frac{5}{2}x^3$
2. $2\sqrt{x}\ e^x$
3. $x+\sqrt{1-x^2}\cdot\sin^{-1}x$
4. $\frac{-2}{1+x}$
5. $(\log \sin x)^{\sin x}$. [log (log sin x + 1 log sin x)]
6. $\frac{1}{x^2+2x+1}$
7. 4

6.2 Summary

- Suppose $y_1 = f_1(x)$ and $y_2 = f_2(x)$ i.e. y_1 and y_2 are the two functions of x. On differentiating both with respect to x.

$$\frac{dy_1}{dx} = f_1'(x)\ \frac{dy_2}{dx} = f_2'(x)$$

6.3 Keywords

- ***Partial:*** Unfair, divided

Note

6.4 Review Questions

1. Find the differential coefficient of $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1} x$ [**Ans.:** $\frac{1}{2}$]
2. If $\sqrt{1-x^6}+\sqrt{1-y^6}=a^3(x^3-y^3)$ then prove that $\frac{dy}{dx}=\frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$.

Answers: Self Assessment

(i) a (ii) b (iii) c

6.5 Further Readings

Books

Mathematics for Economics – Malcom, Nicolas, U.C.London.
Mathematics for Economics – Karl P. Simon, Laurence Bloom.
Mathematics for Economics – Council for Economic Education.
Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.
Mathematics for Economist – Yamane – Prentice Hall India.
Mathematics for Economics and Finance – Martin Norman.
Mathematics for Economics – Simon and Bloom – Viva Publlications.

Unit 7: Homogeneous Function and Euler's Theorem

Note

CONTENTS

Objectives

After reading this unit students will be able to :

- Know the Definition of Homogeneous Function.
- Understand Euler's Theorem.
- Understand Cobb-Douglas Production Function.
- The C.E.S. (Constant Elasticity Substitution) Production Function.

Introduction

Maximum use of special production is referred to as Homogeneous.

Euler's Theorem describes when the every means of production are increased in a proportion, as a result production will increase in the same ratio.

Euler's Theorem has an important place in economic area especially in marketing area. Production is made in conjugation with many means.

Cobb-Douglas Production Function also keeps an important place in economic area. In today's era economists are using Cobb-Douglas Production Function in various economical areas.

7.1 Homogeneous Function: Definition

Maximum use of special production is referred to as Homogeneous function.

For *e.g.* $f(x, y) = x^2 - y^2$

If $x \to tx$ and $y \to ty$ is put in the above production where t is positive constant, then

$$f(tx, ty) = (tx)^2 - (ty)^2 = t^2 (x^2 - y^2)$$
$$= t^2 f(x, y)$$

If production is $f(x, y) = x^2 - y^2$

$$f(tx, ty) = t^3 (x, y)$$
$$f(x, y) = x^n - y^n$$

Note Then $f(tx, ty) = t^n f(x, y)$

Another example of homogeneous function

$$f(x, y) = x^2 + xy - 3y^2 \text{ then}$$

$$f(x, y) = x^3 + 3x^2y + y^3$$

And $$f(tx, ty) = t^3 f(x, y)$$

Production can be written as $$f(tx, ty) = t^n f(x, y)$$

> *Notes* The above production is n category of $f(x, y)$. In economics majorly production of Zero Category is used.

Example 1: $$f(x, y, z) = \frac{x}{z} + \frac{y}{z}$$

Then, $$f(tx, ty, tz) = \frac{tx}{tz} + \frac{ty}{tz} = \frac{x}{z} + \frac{y}{z} = f(x, y, z) = t^0 f(x, y, z)$$

Example 2: $$f(x, y, z) = \frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy}$$

$$f(tx, ty, tz) = \frac{t^2x^2}{t^2yz} + \frac{t^2y^2}{t^2xz} + \frac{t^2z^2}{t^2xy}$$

$$= t^0 f(x, y, z)$$

Example 3: If q quantity, p price and y is income, then demand function is as under

$$q = f(p, y) = \frac{y}{kp} \text{ where } k \text{ is constant value, then}$$

$$f(tp, ty) = \frac{ty}{kp} = \frac{y}{kp} t^0 f(p, y)$$

$$q = f(p, y) = f(tp, ty)$$

Therefore, when price (p) and income (y) changes in same ratio, then there would be no change in demand (q).

7.2 Euler's Theorem

Euler's Theorem states that all factors of production are increased in a given proportion resulting output will also increase in the same proportion each factor of production (input) is paid the value of its marginal product, and the total output is just exhausted. If every means of production is credited equal to its marginal productivity and total production is liquidated completely. In mathematical formula Euler's Theorem can be indicated. If production, $P = f(L, K)$ is Linear Homogeneous Function:

$$P = L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} \text{ in other words } P = LMP_L + KMP_K$$

Note

Where P = Total Production, L = Unit of labor means, K = Unit of capital, or

MP_L = Marginal productivity of Labor and $\frac{\partial P}{\partial K}$ = or MP_K = Marginal productivity of Capital

Assumptions

The theorem is based on following important assumptions:

1. The Law of Constant Returns of Scale is functional. This is possible only when production is linear homogeneous and is of a degree.
2. Market is fully competitive
3. When it considers the division of means of production
4. For the given period, techniques are constant

7.2.1 Diagrammatical Presentation of Euler's Theorem

Keeping all the above assumptions into consideration, Euler's Theorem can be shown in Figure. Assume that production is $P = f(L, K)$ then both the factors L and K affect total production P. First, We will see the effect of L factor over P where K is constant. Thereafter we will see the effect of K factor over P where L is constant

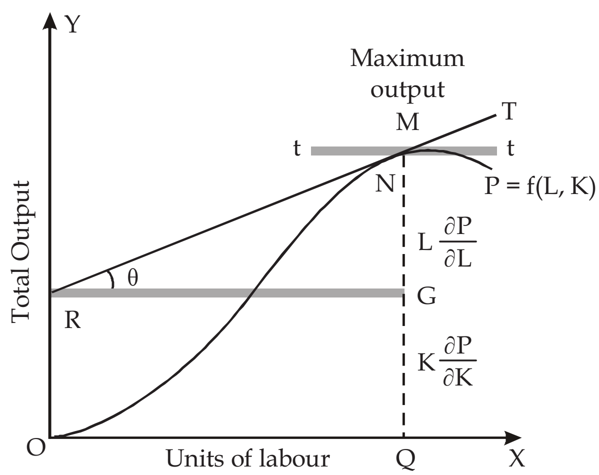


Figure 7.1

In Figure 7.1, X-axis contains the unit factor of L and Y-axis shows Total Production. TP is Total Production. At point M total production will be maximum. Assume at N point of total production curve TP, TI touching line is drawn. From N-point vertical line is drawn on X-axis, which meets at Q-point on X-axis.

Now, Slope on N-point $\frac{\partial P}{\partial K} = \tan\theta$

$$= \frac{NG}{RG} = \frac{NG}{OQ}$$

Now $$L\frac{\partial P}{\partial L} = OQ\frac{NG}{OQ} = NG \qquad \text{...(i)}$$

Note

Euler's Theorem $$P = L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K}$$

Or $$K\frac{\partial P}{\partial K} = P - L\frac{\partial P}{\partial L} = QN - NG = QG \qquad \text{...(ii)}$$

Where P = Total production level $$P = L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K}$$

Or $$QN = NG + QG$$

Thus, total production QN is divisible in NG and OG. Where NG and QG are L factor is means and K factor is paid the value respectively. This way if marginal productivity of every factors of any firm is paid the value of its marginal product, then total production will be exhausted.

7.2.2 Mathematical Solution of Euler's Theorem

With the help of homogeneous production Euler's Theorem establishes a special relation in the principle of marginal productivity. If $u = f(x, y)$ h a homogeneous production of h degree, then theory will be shown as $= xf_{tx} + xf_{ty}$

$$\frac{\partial}{\partial t} \text{ (R.H.S.)} = \frac{\partial t^h}{\partial t} f(x,y) + t^h \frac{\partial f(x,y)}{\partial t} \text{ (L.H.S.)}$$

$$= ht^{h-1} f(x, y) - 0$$

$$= ht^{h-1} f(x, y)$$

Where L.H.S. means Left Hand side and R.H.S. means Right, then L.H.S. = R.H.S.

$$xf_{tx} + yf_{ty} = ht^{h-1} f(x, y)$$

Now if $t = 1$, then t may be any number

$$xf_x + yf_y = hf(x, y)$$

h is the degree of equation. If now we consider linear homogeneous function, then $h = 1$

then $$xf_x + yf_y = f(x, y)$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

or if we consider second equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} + \dots = 2u$$

Generalizing for more than two constants

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} + \dots = ku\ (x, y, z)$$

Where k is degree of equation $f(x, y, z) = 3x + 2y - 4z$

Is a linear homogeneous function

Note

Then as per prime

$$xf_x + yf_y + zf_z = 1 \times (3x + 2y - 4z)$$

$$f_x = 3, f_y = 2, f_z = -4$$

Therefore, left side would be as under

$$\text{L.H.S.} = 3x + 2y - 4z = \text{R.H.S.}$$

7.2.3 Importance of Euler's Theorem

Euler's Theorem has an important place in economic area especially in marketing area. Production is made in conjugation with many means. Now the question arises how all factors of production should be distributed in proportion to the total productivity so that total production is exhausted. In addition to that this helps in resolving short-term problems of production – like distribution of all factors of production and distribution of total production into all factors etc. with the help of this theory firm can ascertain how factors should be used. Factors of production should be utilized to the level where its price is equal to its marginal productivity. This way it also helps in determining the price factors of production.

Self Assessment

1. Fill in the blanks:

1. Maximum use of special production in economics is referred to as ……………………
2. ……….. Theorem establishes a special relation in the principle of marginal productivity.
3. Euler's Theorem has an important place in ……………….. area.
4. …………….. is made in conjugation with many means.
5. If production is made in …………….. with many means, it describes how all factors of production should be distributed in proportion to the total productivity so that total production is exhausted

7.3 Cobb-Douglas Production Function

Cobb-Douglas Production Function is used widely in economic area. This production function was developed by *C W* Cobb and D *H* Douglas. They studied various industries of the world to devise this function. This way, it is used as a Universal Law for production.

Did u know? Cobb-Douglas Production Function is indicated as

$$P = AL^{\alpha} K^{\beta} u$$

Where P = Production, L = Labor, K = Capital u = Disturbance Terms and A is a constant value.

α and β are positive parameters. With these $\alpha > 0$, $\beta > 0$, $L > 0$ and $\alpha + \beta = 1$.

Some economists show this production function with u

$$P = AL^a K^b \ (\alpha > 0, \beta > 0)$$

If $\alpha + \beta = 1$

In Cobb-Douglas Production Function, there are two factors of production viz. $P = f(L, K)$

If both the sides are multiplied by λ (Lemda) then $\lambda P = f(\lambda L, \lambda K)$

Note

Then $$f(\lambda L, \lambda K) = A(\lambda L)^{\alpha}(\lambda K)^{\beta} u$$

$$= \lambda^{\alpha+\beta} A L^{\alpha} K^{\beta} u$$

$$= \lambda\alpha+\beta P \qquad (\therefore\ P = AL^{\alpha} K^{\beta} u)$$

If $\alpha + \beta = 1$, then production would be under the constant result.

If $\alpha + \beta > 1$, then increase in production will happen

If $\alpha + \beta < 1$, then decrease in production will happen

7.3.1 Characteristics of Cobb Douglas Production Function

1. If factor of production is increased by some constant value *i.e.* λ, then productivity will also grow by λ. Cobb Douglas Production Function $P = L^{\alpha}K^{\beta} u$ is if Labor L and Capital K is increased by λ.

$$p^t = A(\lambda L)^{\alpha} (\lambda K)^{\beta} u$$

$$= \lambda^{\alpha+\beta} AL^{\alpha} K^{\beta} u$$

$$= \lambda AL^{\alpha} K^{\beta} u \qquad (\text{if } \alpha + \beta = 1)$$

$$= \lambda P$$

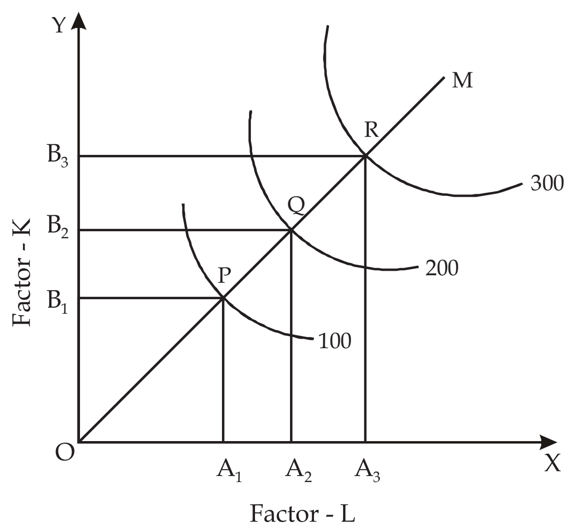


Figure 7.2

This way productivity will increase in the ratio o increase in factors. This can be seen in Figure 7.2. X-axis shows L (Labor) and Y-axis shows K (Capital). To produce 100 units OA_1 units of factor L and OB_1 units of factor K is taken. If the productivity is taken to 200 units, similarly L and K would also required to be doubled. Here OA_2 is just double of OA_1 units. Similarly OB_2 is also twice of OB_1. In a similar fashion for production of 300 units, triple unit of A and B would be required as can be seen in the figure 7.2.

This can be understood in other way. OM is the expansion path of production where P, Q and R meet the balance point. These balance point will be on equal distance viz $OP = PQ = QR$, which described that production will increase in the ratio of factors increased.

Note

2. Production function is homogeneous and of a degree:

If Production function is homogeneous and of a degree, it means that production will come under constant formula.

Production function = $P = AL^{\alpha} K^{\beta} u$

Taking log of two sides log $P = \log A + \alpha \log L + \beta \log K + \log u$

Partially differentiating with respect to L and K separately

$$\frac{1\partial P}{P\partial L} = \frac{\alpha}{L} \quad \text{...(i)}$$

And

$$\frac{1\partial P}{P\partial K} = \frac{\beta}{L} \quad \text{(ii)}$$

Writing (i) and (ii) further

$$L\frac{\partial P}{\partial L} = \alpha P \quad \text{...(iii)}$$

$$K\frac{\partial P}{\partial L} = \beta P \quad \text{...(iv)}$$

Adding equation (iii) and (iv)

$$\frac{L\partial P}{\partial P} + K\frac{\partial P}{\partial L} = P\,\alpha + P\,\beta = P(\alpha + \beta)$$

3. If Production function is homogeneous and of a degree, then elasticity of substitution will always be equal to unit. If production function is $P = AL^{\alpha}K^{\beta}\, u$, where $\alpha + \beta = 1$ we know that elasticity of substitution

= σ = Change in ration of factor's quantity/ % change in price ratio of factor

$$\therefore \quad \alpha = \frac{\partial(K/L)/(K/L)}{\partial(P_L/P_K)/P_L/P_K} = \frac{\partial(K/L)/K/L}{\partial R/R}$$

Where K/L = ratio of factor quantity

$R = P_L\, P_K$ = Price ratio of factor

We know that rate of marginal substitute technique = $\frac{\partial K}{\partial L}$

$$\therefore \quad \frac{\partial K}{\partial L} = \frac{MP_L}{MP_K} = \frac{P_L}{P_K} = R$$

In other words

$$R = \frac{\partial P/\partial L}{\partial P/\partial K}$$

Our production function $P = AL^{\alpha}L^{\beta}u$

Differentiating with respect to L and K separately

$$\frac{\partial P}{\partial L} = \beta AL^{\alpha}K^{\beta}u$$

Note

Therefore

$$\frac{\partial P}{\partial L} = \alpha A L^{\alpha-1} K^{\beta-1} u$$

$$R = \frac{\partial P / \partial L}{\partial P / \partial K} = \frac{\alpha A L^{\alpha-1} K^{\beta} u}{\beta A l^{\alpha} K^{\beta} u} = \frac{\alpha}{L} \frac{K}{\beta}$$

$$= \frac{\alpha}{\beta}\left(-\frac{K}{L}\right)$$

$$\partial R = \alpha / \beta \partial (K / L)$$

Therefore,

$$\sigma = \frac{\partial(K/L)/K/L}{\partial(P_L/P_K)/P_L/P_K} = \frac{\partial(K/L)/K/L}{\partial R/R}$$

$$s = \frac{\partial(K/L)/K/L}{\frac{\partial(K/L)\cdot\alpha/b}{(K/L)\cdot\alpha/b}} = 1$$

It's proved

4. For production capital and labour are important requirements – If capital becomes Zero, production will get into Zero

   Production function = $P = AL^{\beta} K^{\beta} u$

$$L = 0$$

Assume

$$P = A.B.\ K^{\beta}\ u = 0$$

$$K = 0$$

Assume

$$P = AL^{\alpha} \cdot O \cdot u = 0$$

Thus, for production both the factors are essential.

5. If Production function $P = AL^{\alpha} K^{\beta} u$ is homogeneous and of a degree, then α and β reflects the position of labour and capital in the production.

   Production function is $P = AL^{\alpha} K^{\beta} u$

   Differentiating with respect to L and K separately

$$\frac{1}{P} = \frac{\partial P}{\partial L} \alpha \frac{1}{L}$$

$$\frac{1}{P}\frac{\partial P}{\partial K} = \beta . \frac{1}{K}$$

$\therefore$

$$\alpha = \frac{L}{P}\frac{\partial R}{\partial L} = \frac{\text{Labour}}{\text{product}} \times \text{Marginal product of labour}$$

$$= \frac{\text{Wage of labour}}{\text{Product}}$$

And

$$\beta = \frac{K}{P}\frac{\partial P}{\partial K} = \frac{\text{Capital}}{\text{Product}} \times \text{Marginal product of capital}$$

= Labour share of total production

= Share of capital in production.

Note

6. Production function displays the elasticity of Labour and Capital. By the characteristics of production function

$$\alpha = \frac{L}{P}\frac{\partial P}{\partial L}$$

$$= \frac{\partial P / P}{\partial L / L} = \text{ elasticity of labour}$$

$$\beta = \frac{K}{P}\frac{\partial P}{\partial K}$$

$$= \frac{\partial P / P}{\partial K / K} = \text{elasticity of capital}$$

The expansion path of Cobb Douglas Production Function is linear homogeneous and its passed through main point.

Cobb Douglas Production Function $P = AL^{\alpha}K^{\beta}u$

Taking log into both sides

$\log P = \log A + \alpha \log L + \beta \log K + \log u$

Differentiating with respect to L and K separately

$$\frac{1}{P}\frac{\partial P}{\partial L} = \frac{\alpha}{L} \qquad ...(A)$$

$$\frac{1}{P}\frac{\partial P}{\partial K} = \frac{\beta}{K} \qquad ...(B)$$

Writing both the equation (A) and (B) again

$$MP_L = \frac{\partial P}{\partial L} = P.\frac{\alpha}{L}$$

and

$$MP_K = \frac{\partial P}{\partial L} = P.\frac{\beta}{K}$$

$\therefore$

$$\frac{MP_L}{MP_K} = \frac{P_L}{P_K}$$

$$\frac{P\alpha / L}{\beta P / K} = \frac{P_L}{P_K}$$

Or

$$\frac{\alpha}{\beta}.\frac{K}{L} = \frac{P_L}{P_K}$$

Or

$$\alpha.KP_K = \beta.LP_L$$

Or

$$\alpha.KP_K - \beta LP_L = 0$$

Thus, Production Function is linear homogeneous and its passes through main point.

Note

7.3.2 Economic significance of Cobb Douglas Production Function

Cobb Douglas Production Function has a very importance role in economic area. At present many economists are using Cobb Douglas Production Function in various economic areas. The use of this function is day-by-day is increasing especially in various industries and agriculture. This bring important information for these sectors. This also helps in framing various policies.

With the help of this function, we can also determine the Marginal Productivity and similarly it helps in determining principle of wages. Production function describes production technique. With the help of this function we can also determine whether any factor is paid the value with respect to its equality with the marginal productivity. In a same fashion it helps in agriculture to find the elasticity of economy. By this function we also display elasticity coefficients. These elasticity coefficients help us in comparing the international and internal areas.

As has already be described when function is linear and homogeneous and $\alpha + \beta = 1$, then production would be under the constant result, when $\alpha + \beta > 1$, then increase in production happens, and if $\alpha + \beta < 1$, then decrease in production happens. This way this function helps us in studying the rules of various results. Besides these it also fetches important information related to substitutability of various factors of production.

In short, this function plays an important role especially in agriculture and industries. This is used in determining the labour policies, inter-area comparison, substitutability of factors and degree of homogeneity.

7.3.3 Limitation of Cobb-Douglas Production Functions

Although Cobb Douglas Production Function is used widely in economic areas and its use is increasing in especially in various industries and agriculture, but some economists criticize this production function. Among them are Prof K.J. Arrow, H.B. Chenery, B.S. Minhas and R.M. Salow. Their main criticizes are:

1. The main demerit of this function is this that it considers only two factors of production i.e. Capital and Labour, whereas in reality other factors also have important role in production. In other words, this function does not apply to more than two factors. Besides it can be used only in construction industries. This way its use becomes narrow.
2. This function works under the constant result of formula. Rule of increase and decrease in result also apply to production function. But this function does not work under these rules.
3. Function is based on the assumptions that technical knowledge remains constant and no change in techniques happen in production. But the same can change in production. This way assumption of constant technique is irrelevant.
4. Cobb-Douglas Production Functions assume that all inputs are homogeneous. In reality all units of a factors are not homogeneous. For example some people are skilled and others are not in a labour population.
5. This does not determine any maximum level of production. Prof M. Chand says "Since, this does not ascertain the maximum level of P (Production), it would be practical and convenient not to use this function beyond a certain limit for statistical measurement of its values.
6. α and β of the function reflects the proportion of labour and capital in production. This becomes true only when market has a complete competition. But in case economy has a incomplete competition or monopoly, then above relation can not be obtained.
7. It takes into account only positive marginal productivity of factors and ignores the negative marginal productivity. Whereas marginal productivity of any factor can be zero or negative.
8. Last, the function is unable to produce information related to inter-relation of factors.

Note

7.4 The Constant Elasticity Substitution (C.E.S.) Production Function

In the Cobb-Douglas Production Functions it has already been discussed that elasticity of substitution is always a unit in it. Here we will discuss a function where elasticity of substitution is not required. This is known as Constant Elasticity Substitution (C.E.S.) Production Function. This was devised by two groups of economists. First was K J Arrow, Chenery and B S Minhas and R M Salow, whereas second group consists M Brown, De Cani. Although they devised this function in other forms, but result were same. First group has shown the production function as:

$$P = \gamma[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v-\alpha}$$

$$(\gamma > 0, 0 < \delta < 1, \alpha > -1)$$

Where P = Production, C = Capital, N = Labour α = substitution parameter; γ = technical efficiency coefficient or efficiency parameter (this is considered as A of Cobb-Douglas Production Functions in C. E.S. function); δ = coefficiency of capital intensity (this is considered as α of Cobb-Douglas Production Functions in C. E.S. function)

1–δ = Labour Intensity Coefficient

v = Degree of Homogeneity

7.4.1 Properties of C.E.S. Production Function

1. If Production Function is linear homogeneous then substitution parameter α would be equal to constant $\left(\frac{1}{1+\alpha}\right)$ whereas production function is $P = \gamma(\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha}$ provided $\gamma > 0, 0 < \delta$ and $\alpha > -1$

   Rational: According to definition elasticity of substitution

$$\sigma = \frac{\partial \log(N/C)}{\partial \log R} = \frac{\partial(N/C)/N/C}{\partial R/R}$$

Here, $\frac{N}{C} =$ ratio of production factors and $R = \frac{P_C}{P_N} =$ Price Ratio

Now production function

$$P = \gamma[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha} \qquad ...(7.1)$$

Partially differentiating with respect to N

$$\frac{\partial P}{\partial N} = \gamma[-v/\alpha][\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha} - 1 \times [-\alpha(1-\delta)N^{-\alpha-1}]$$

$$= \frac{\gamma v}{\alpha}[\delta C^{-\alpha} + (1-\alpha)N^{-\alpha}]^{-v/\alpha-1}[\alpha(1-\delta)N^{-(\alpha+1)}] \qquad ...(7.2)$$

From equation 7.1

$$\left[\frac{P}{\gamma}\right] = [\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha}$$

Note

Or $$\left[\frac{P}{\gamma}\right]^{-\alpha/v} = [\delta C^{-\alpha} + (1-\alpha)N^{-\alpha}]$$

Or $$\left[\frac{P}{\gamma}\right]^{-(\alpha/v)(-v/\alpha-1)} = [\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha-1} \quad ...(7.3)$$

Putting equation 7.3 into equation 7.2

Or $$\frac{\partial P}{\partial N} = \gamma v\left[\frac{P}{\gamma}\right]^{1+\alpha/v} = (1-\delta)N^{-(1+\alpha)} \quad ...(7.4)$$

Differentiating equation 7.1 further with respect to C

$$\frac{\partial P}{\partial C} = \gamma v[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha 1}\,\delta.C^{-1+\alpha}$$

$$= \gamma v\left[\frac{P}{\gamma}\right]^{1+\alpha/v} \delta C^{-(1+\alpha)} \quad \text{[from equation (7.3)]}$$

We know that

$$MTRS = \frac{\partial P/\partial C}{\delta P/\partial N} = \frac{\partial N}{\partial C} = R$$

$$R = \frac{\gamma v\left[\frac{P}{\gamma}\right]^{1+\alpha/v} \delta C^{-(1+\alpha)}}{\gamma v\left[\frac{P}{\gamma}\right]^{1+a/v} (1-\delta)N^{-(1+a)}}$$

Or $$= \frac{d}{1-d}\left(\frac{C}{N}\right)^{(1+\alpha)}$$

Or $$R = \frac{\delta}{1-\delta}\left(\frac{N}{C}\right)^{1+\alpha}$$

Taking log of both sides

$$\log R = \log\left(\frac{\delta}{1-\delta}\right) + (1+\alpha)\log\left[\frac{N}{C}\right]$$

$$= \log \delta' + (1+\alpha)\log G$$

Here, $$\delta' = \frac{\delta}{1-\delta} \text{ and } G = \frac{N}{C}$$

Differentiating with respect to G

$$\frac{1}{R}\frac{\partial R}{\partial G} = \frac{1+\alpha}{G}$$

Or $$\frac{G}{R}\cdot\frac{\partial R}{\partial G} = 1 + \alpha$$

Note

Or $$\frac{\partial G}{G} / \partial R / R = \frac{1}{1+\alpha}$$

Or $$\frac{\partial (NC) / N / C}{\partial R / R} = \frac{1}{1+\alpha}$$

$\therefore$ Elasticity of substitution $= \sigma = \dfrac{1}{1+\alpha}$ that's it

2. Marginal Product of CES will always be positive, viz more than zero and never is negative.

   Rationale: Production function $P = \gamma[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha}$

   Differentiating with respect to N

$$\frac{\partial P}{\partial N} = \gamma^{\alpha/v} \cdot \gamma P^{1+\alpha/v}(1-\delta)N^{-\alpha-1} \text{ [from first characteristics]}$$

Or $$MP_N = \gamma^{\alpha/v} \cdot v(1-\delta)P^{1+\alpha}N^{-(1+\alpha)}$$

$$= R_1 \frac{P^{1+\alpha/v}}{N^{1+\alpha/v}} \text{ here } R_1 = \gamma^{-\alpha/v}(1-\delta)$$

If production happens under constant result then $v = 1$

$\therefore$ $$MP_N = R_1 \frac{P^{1+\alpha}}{N^{1+\alpha}} = R_1 \left[\frac{P}{N}\right]^{1+\alpha}$$

Or $$MP_N = R_1 \left(\frac{P}{N}\right)^{1/\sigma} \qquad \left[\text{here, } \sigma = \frac{1}{1+\alpha}\right]$$

$\therefore$ $$MP_N = \text{Positive or } MP_N > 0$$

$$P = \gamma[\delta C^{-\alpha} + (1-\delta)N^{-\alpha}]^{-v/\alpha}$$

Differentiating with respect to C

$$MP_C = \frac{\partial P}{\partial C} = \gamma^{-\alpha/v}.vp^1 + \alpha\delta C^{-\alpha-1}$$

$$= \gamma^{-\alpha/v} \cdot v\delta P^{-1+\alpha/v} C^{-(1+\alpha)} \text{ [from first characteristics]}$$

Or $$MP_C = R_2 \frac{P^{1+\alpha/v}}{C^{1+\alpha/v}} \quad \text{here, } R_2 = \gamma^{-\alpha/w} v\delta$$

If production function is working under constant result then $v = 1$

Therefore $$MP_C = R_2 \frac{P^{1+\alpha}}{C^{1+\alpha}} = R_2 \left(\frac{P}{C}\right)^{1+\alpha}$$

$$= R_2 \left(\frac{P}{C}\right)^{1/\alpha} \qquad \left[\text{here, } \sigma = \frac{1}{1+\alpha}\right]$$

$$= \text{Positive}$$

Now $MP_C > O$

Note

3. Marginal production of production function is always moves downward viz

$$\frac{\partial^2 P}{\partial M^2} < 0 \text{ and } \frac{\partial^2 P}{\partial C^2} < 0$$

Task What is the merit of C.E.S. Production function?

7.4.2 Advantages of CES Production Function over Cobb-Douglas Production Function

1. Compared to Cobb-Douglas Production Function, CES Production Function brings more general technique. Under CES Production Function, elasticity of substitution is constant and it is not necessary that this elasticity is equal to a unit.
2. Compared to Cobb-Douglas Production Function, CES Production Function keeps more important parameters. This way it has a wide area in substitutability and efficiency.
3. Cobb-Douglas Production Function and CES Production Function have a special form. If in CES function $a = 0$ then we will get Cobb-Douglas Production Function.
4. It is easy to find out the parameters under CES Production Function. Besides this function has removed all the problems and unrealistic assumptions of Cobb-Douglas Production Functions.

7.4.3 Limitation of CES Production Function

Although CES production function has removed all the problems and unrealistic assumptions of Cobb-Douglas Production Function and is widely used in economics, but yet it is criticized:

1. Like Cobb-Douglas Production Function, this function also considers only two factors of production (Labor and Capital). It does not apply to other factors of production. Prof H. Uzawa says it is difficult to apply this function on n^{th} factor of production.

   For example: if $A = \sqrt{ab}$ is production function, find out the demand of factor a and b, where their prices are constant at P_a and P_b. If demand curve $x = \beta - \alpha p$, what is the factor demand in term s of prices and constants.

   Solution: Given production function

$$x = A\sqrt{ab}$$

$$= Aa^{1/2}b^{1/2} \qquad \text{...(i)}$$

Differentiating a and b from equation (i) separately

$$\frac{\partial x}{\partial a} = \frac{1}{2}Aa^{-1/2}b^{1/2}$$

And

$$\frac{\partial x}{\partial b} = \frac{1}{2}Aa^{1/2}b^{-1/2} \qquad \text{...(ii)}$$

From equation (i) and (ii)

$$MP_a = \frac{1}{2}\frac{Aa^{1/2}b^{1/2}}{a} = \frac{x}{2a} \qquad \left(x = A\sqrt{ab}\right)$$

$$MP_b = \frac{1}{2}\frac{Aa^{1/2}b^{1/2}}{b} = \frac{x}{2b}$$

But we know that **Note**

$$\frac{MP_a}{MP_b} = \frac{P_a}{P_b}$$

$$\frac{x/2a}{x/2b} = \frac{P_a}{P_b}$$

Or $$\frac{b}{a} = \frac{P_a}{P_b}$$

Or $$b = \frac{P_a}{P_b} \text{ and } a = b\,\frac{P_b}{P_a}$$

Putting the value of *a* and *b* in equation (i)

$$\frac{x}{A} = a\sqrt{P_a / P_b}$$

Or $$a = \frac{x}{A} = \sqrt{P_a / P_b} \qquad \text{...(iii)}$$

Thus $$b = \frac{x}{A} = \sqrt{P_a / P_b} \qquad \text{...(iv)}$$

equation (iii) and (iv) displays the factor demand in form of factor price of *a* and *b*

Now, demand curve $x = \beta - \alpha p$

Since $$TC = ap_a + a\frac{P_a}{P_b}.pb = 2ap_a$$

$$= 2p_a \frac{x}{A}\sqrt{P_b / P_a} \qquad \text{[from equation (iii)]}$$

Differentiating with respect to *x*

$$\frac{\partial(TC)}{\partial x} = \frac{2p_a}{A}\sqrt{p_b / p_a} = \frac{2}{A}\sqrt{(P_a P_b)}$$

$$MC = \frac{2}{A}\sqrt{(P_a P_b)}$$

Since $MC = p$ (In case of full competition)

$$x = \beta - \alpha p$$

Further $$= \beta - \alpha\frac{2}{A}\sqrt{(p_a\,p_b)}$$

Putting the value of *x* in equation (i)

$$= \beta - \alpha\frac{2}{A}\sqrt{(p_a\,p_b)} = Aa^{1/2}b^{1/2} = Aa^{1/2}b^{1/2}\sqrt{P_a / P_b}$$

Or $$= \frac{1}{A}\left(\beta - \frac{2\alpha}{A}\sqrt{(p_a p_b)}\right)\sqrt{p_a / p_b} = a$$

Note

Or $$= \frac{1}{A}\left(\beta - \frac{2\alpha}{A}\sqrt{(p_a p_b)}\right)\sqrt{p_a / p_b} = a \quad \text{...(v)}$$

Thus $$b = \frac{1}{A}\left(\beta - \frac{2\alpha}{A}\sqrt{(p_a p_b)}\right)\sqrt{p_a / p_b} \quad \text{...(vi)}$$

Equation (v) and (vi) displays the factor demand of *a* and *b*

Example 1: If production is in form of

$$Q = A\, K^{\alpha} L^{\beta}$$

then (A) Find out the marginal productivity of Capital (*K*) and Labour (*L*).

(B) Prove that there is elasticity of capital and labour in production function

Solution: Given production function

$$Q = AK^{\alpha} L^{\beta} \quad \text{...(i)}$$

(A) Partially differentiating from equation with respect to *K* and *L* separately

$$\frac{\partial Q}{\partial K} = A\alpha K^{\alpha-1} L^{\beta} \quad \text{...(ii)}$$

$$\frac{\partial Q}{\partial L} = A\beta K^{\alpha} L^{\beta-1} \quad \text{...(iii)}$$

Function (ii) and (iii) displays marginal productivity of Capital (*K*) and Labour (*L*). writing this in simple form

$$MP_K = \frac{\partial Q}{\partial K} = \frac{\alpha}{K} \qquad (\therefore \quad Q = AK^{\alpha} L^{\beta})$$

And $$MP_L = \frac{\partial Q}{\partial K} = \frac{\beta}{K} \cdot Q$$

(B) Production Elasticity of Capital

$$= \frac{K}{Q}\frac{\partial Q}{\partial K} = \frac{K}{Q} \cdot A\alpha K^{\alpha-1} L^{\beta}$$

$$= \frac{K}{AL^{\alpha} L^{\beta}} A\alpha K^{\alpha-1} L^{\beta} = \alpha \frac{A\alpha K^{\alpha-1} L^{\beta}}{A\alpha K^{\alpha-1} L^{\beta}} = \alpha$$

Production Elasticity of Labour

$$= \frac{L}{Q}\frac{\partial Q}{\partial L} = \frac{L}{Q} \cdot A\beta K^{\alpha} L^{\beta-1}$$

$$= \frac{L}{AK^{\alpha} L^{\beta}} A\beta K^{\alpha} L^{\beta-1}$$

$$= \frac{\beta}{AK^{\alpha} L^{\beta-1}} AK^{\alpha} L^{\beta-1} = \beta$$

This way in the given production function α and β shows the production elasticity of capital and labor.

Self Assessment

Note

2. Multiple Choice Quesitons:

6. Who has been credited to devise the Cobb-Douglas Production Function?
   (a) C W Cobb and D H Douglas (b) Cobb and Marshal
   (c) Douglas and Arastu (d) Above all
7. Production function is
   (a) $P = AL^{\beta}K^{\alpha}u$ (b) $P = AL^{\alpha}K^{\beta}u$
   (c) $P = L^{\alpha}K^{\beta}$ (d) $P = AK^{\beta}u$
8. Marginal production of CES is always –
   (a) negative (b) cubic
   (c) positive (d) None of them

7.5 Summary

- Maximum use of special production is referred to as Homogeneous function.
- Euler's Theorem states that all factors of production are increased in a given proportion resulting output will also increase in the same proportion each factor of production (input) is paid the value of its marginal product, and the total output is just exhausted.
- Euler's Theorem has an important place in economic area especially in marketing area. Production is made in conjugation with many means.
- Cobb-Douglas Production Function is used widely in economic area. This production function was developed by C W Cobb and D H Douglas.
- If Production function is homogeneous and of a degree, it means that production will come under constant formula.
- Cobb Douglas Production Function has a very important role in economic area. At present many economists are using Cobb Douglas Production Function in various economic areas.
- Although Cobb Douglas Production Function is used widely in economic areas and its use is increasing in especially in various industries and agriculture, but some economists criticize this production function.
- In the Cobb-Douglas Production Functions it has already been discussed that elasticity of substitution is always a unit in it.
- Compared to Cobb-Douglas Production Function, CES Production Function brings more general technique.

7.6 Keywords

- ***Homogeneous:*** Undifferentiated, similar
- ***Theorem:*** practically which can be proved

7.7 Review Questions

1. Define homogeneous function with example.
2. Explain Euler's Theorem with realistic example.
3. Write down the mathematical solution of Euler's Theorem.

Note

4. Explain Cobb-Douglas Production Function.
5. Describe the economic importance of Cobb-Douglas Production Function.
6. Write down the limitation of Cobb-Douglas Production Function.
7. Explain the Constant Elasticity Substitution.

Answers: Self Assessment

1. Homogeneous	2. Euler's	3. Economic	4. Production
5. Homogeneous	6. (a)	7. (b)	8. (c)

7.8 Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematical Economy – Michael Harrison, Patrick Walderan.

Note

Unit 8: Use of Differentiation in Economics

CONTENTS

Objectives

After reading this unit students will be able to :

- Understand the usage of Differentiation in Economics.
- Explain Marginal Revenue and Elasticity of Demand.

Introduction

The use of differentiation in economics is growing day-by-day. The presented lesson discusses the use of differentiation in economics.

8.1 Use of Differentiation in Economics

The use of differentiation in economics is growing day-by-day. Based on the following points use of differentiation in economics can be cleared-

8.1.1 Elasticity

To calculate elasticity, differentiation is used in economics. If any product $y = f(x)$ is there, then with y, elasticity of x can be found in the following way:

$$E_x = \underset{\Delta x \to 0}{Lt} \frac{\Delta y / y}{\Delta x / x} = \frac{x}{y}\left(\frac{dy}{dx}\right)$$

Notes If demand is $q = f(p)$, then elasticity of demand (E_d) can be calculated in the following manner:

$$E_d = \underset{\Delta p \to 0}{\text{Lt}} \frac{\Delta q / q}{\Delta p / p} = \frac{p}{q}\left(\frac{dy}{dx}\right)$$

Note

Example 1: If demand is q = 300 -4.p^2, and price becomes p = 2, calculate the elasticity of demand.

Solution: Elasticity of demand $(E_d) = \frac{p\,dq}{q\,dp}$, here $q = 300 - 4p^2, \frac{dq}{dp} = -8p$

$$\therefore \qquad E_d = \left(\frac{p}{300 - 4p^2}\right) \times (-8p) = \frac{-8p^2}{300 - 4p^2}$$

If p = 2 then

$$E_d = \left(\frac{-8 \times 4}{300 - (4 \times 2)}\right) = \frac{-32}{292} = \frac{-8}{73}.$$

8.1.2 Marginal Revenue and Elasticity of demand

We know that Total Revenue (TR) = Price (p) x Quantity (q)

Differentiating with respect to q Or $TR = p \times q$

$$MR = \frac{d(TR)}{dq} = \frac{d}{dq}(d \times q)$$

$$MR = \left[p + q\frac{dp}{dq}\right] = p\left[1 + \frac{p}{q}\frac{dp}{dq}\right]$$

Or

$$MR = p\left[1 + \frac{1}{E_d}\right] \qquad \therefore \left[E_d = \frac{q}{p}\frac{dq}{dp}\right]$$

Or

$$MR = AR\left[1 + \frac{1}{E_d}\right]$$

Did u know? Since elasticity of demand is always negative, therefore, the above relation can be expressed in the following manner

$$MR = AR\left[1 - \frac{1}{E_d}\right]$$

8.1.3 Finding Marginal Cost from Total Cost and Marginal output from Total output

Assume Total Cost (C) = $f(q)$, q shows the quantity of total production

Then Marginal Cost (MC) = $\frac{d}{dq}(C)$

This way, Total Revenue (R) = $f(p,q)$

Then Marginal Revenue = $\frac{d}{dq}(R)$

Note

Example 2: Assuming Total Cost is C = $15 + 10q - 9q^2 + q^3$, then find out the Marginal Cost (*MC*).

Solution: Total Cost = $15 + 10q - 9q^2 + q^3$

Differentiating with respect to *q*

$$MC = \frac{dc}{dq} = 10 - 18q + 3q^2.$$

Example 3: Assuming Total Revenue is *R* = $6q - 9q^2$ then find out the Marginal Revenue (*MR*) and taking total Cost, calculate the total time.

Solution: Total Cost = $R = 6q - 9q^2$,

Differentiating with respect to *q*

$$MR = \frac{dR}{dq} = 6 - 18q$$

In case of equilibrium *MR* = *MC*, therefore

$$6 - 18q = 10 - 18q + 3q^2$$

Or $3q^2 = 4$

$$q^2 = \frac{4}{3} \text{ or } q = \pm\sqrt{4/3}.$$

Self Assessment

1. Fill in the blanks:

1. Use of differentiation ………………… is growing day-by-day.
2. To calculate elasticity, ………………….. is used in economics.
3. Revenue = Price x …………….
4. Price = ……………….. / Quantity
5. $MR = \ldots\ldots\left[1 - \frac{1}{E_d}\right]$,

8.1.4 Equilibrium in Monopoly: Finding Maximum Profit

With the help of differentiation, find out the equilibrium in Monopoly. We can easily show the maximum benefit in the following manner:

Equilibrium in Monopoly ***MR = MC*** viz

$$\frac{d(R)}{dq} = \frac{d(C)}{dq}$$

Total profit $\pi = R - C$

In case of maximum profit, following two conditions are essential-

$$\frac{d\pi}{dq} = 0 \qquad \text{...(i)}$$

And $$\frac{d^2\pi}{dq^2} < 0 \quad \text{(negative)} \qquad \text{...(ii)}$$

Note

Example 4: If demand is $p = 20 - 5q$ and average cost $AC = q$, then find out the equilibrium of monopoly and maximum profit.

Solution: (i) Given that demand $p = 20 - 5q$

And $AC = q$

Total Revenue $= p \times q$

$= (20 - 5q)\, q = 20q - 5q^2$

Differentiating with respect to q

$$MR = \frac{d(R)}{dq} = 20 - 10q$$

Total cost $(C) = AC \times q = q \times q = q^2$

Differentiating with respect to q

$$MC = \frac{DC}{dq} = 2q$$

In equilibrium MR = MC

$$20 - 10q = 2q$$

viz $8q = 20$

or $q = \frac{20}{8} = 2.5$

(i) Total profit $(\pi) = R - C$

$= 20q - 5q^2 - q^2$

$= 20q - 6q^2$

(ii) for maximum profit $\frac{d\pi}{dq} = 0$ and $\frac{d^2\pi}{dq^2} < 0$

then $\frac{d\pi}{dq} = 20 - 12q = 0$

or $12q = 20$

or $q = \frac{20}{12} = \frac{5}{3}$

$$\frac{d^2\pi}{dq^2} = -12$$

Therefore, at $q = \frac{5}{3}$ monopolized profit would be maximum.

Establish relation between $\frac{d(R)}{dq}$ and $\frac{d(C)}{dq}$.

Note

Example 5: Given than Demand $Q = 23 - 4p + p^2$, where p and q are price and quantity respectively. What would be the elasticity of demand when the value of commodity is (i) ₹ 8 and (ii) ₹ 5.

Solution: Product of Demand given

$$Q = 25 - 4p + p^2 \left[\frac{p}{q}\alpha - \frac{dq}{p}\right]$$

Partially differentiating with respect to p

$$\frac{\partial Q}{\partial p} = -4 + 2p$$

We know that

$$\text{Elasticity of Demand} = \frac{\text{\% change in demanded quality}}{\text{\% change in price of the commodity}}$$

$$E_d = \frac{\partial Q / Q}{\partial p / p} = \frac{P}{Q}\frac{\partial Q}{\partial p}$$

$$= \frac{P}{Q}(-4 + 2p) = \frac{-4p + 2p^2}{25 - 4p + p^2}$$

$$Ed_{(p=8)} = \frac{-32 + 128}{25 - 32 + 64} = \frac{96}{57} = +1.6$$

$$Ed_{(p=5)} = \frac{-20 + 50}{25 - 20 + 25} = \frac{30}{360} = +1$$

Thus, when price is ₹ 8, elasticity of demand would be 1.6 and when price is ₹ 5, elasticity of demand would become a unit.

Example 6: When cross demand $Q = 150 - 15p$, then find out the elasticity of demand when $p = 4$.

Solution: Product of Demand given $Q = 150 - 15p$,

Differentiating with respect to p

$$\frac{\partial Q}{\partial p} = -15$$

We know that

Elasticity of Demand

$$(E_d) = \frac{P\partial Q}{Q\partial p} = \frac{-15p}{150 - 15p}$$

$$E_{d(p=4)} = \frac{-15 \times 4}{150 - (15 \times 4)} = \frac{-60}{150 - 60} = \frac{-60}{90} = -0.66$$

Question: Write short notes on (I) Cross elasticity of demand.

Answer

Cross elasticity of demand

Demand product shows that demand of any commodity is the product of price of that commodity. But demand of any commodity is also related to price of other related commodity. Cross demand tells that if the price of related commodity changes, in that case demand of that commodity also changes. Assume, here there are two commodities X and Y. Here because of the price of Y commodity, demand of X commodity changes, viz

$$E_c = \frac{\text{\% Change in quality of X commodity}}{\text{\% Change in price of Y commodity}}$$

Note

Example 7: If the demand and supply of any fully competitive firm is following

Demand $P = 30 - x$

Supply $C = x^2 + 6x + 7$

Then at which level of production maximum profit can be earned and what would be the corresponding value of Price, Profit and Total Revenue?

Solution: In case of firm earning maximum profit, following condition should essentially be met with

$$\text{MR} = \text{MC viz } \frac{\partial \pi}{\partial x} = 0 \quad \text{...(i)}$$

And $$\frac{\partial^2 \pi}{\partial x^2} < 0 \quad \text{...(ii)}$$

Here Total Profit π = Total Revenue (*R*) = Total Cost (*C*)

Total Revenue (*R*) = Price x quantity of the commodity

$$= P.x = (30 - x)\cdot x = 30x - x^2$$

$$MR = \frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(30x - x^2) = 30 - 2x$$

Thus, $$C = x^2 + 6x + 7$$

$$MC = \frac{\partial C}{\partial x} = 2x + 6$$

In case of equilibrium *MR* = *MC*

$$30 - 2x = 2x + 6$$

Or $$-4x = -30 + 6 = -24$$

Or $$x = \frac{24}{4} = 6$$

The value of *x* can be found out in other way

$$\pi = R - c = (30x - x^2) - (x^2 + 6x + 7)$$

$$= 30x - x^2 - x^2 - 6x - 7$$

$$= 24x - 2x^2 - 7$$

Differentiating with respect to *x*

$$\frac{\partial \pi}{\partial x} = 24 - 4x$$

For maximum profit $\frac{\partial \pi}{\partial x} = 0$ or $24 - 4x = 0$ or $-4x = 24$ or $x = 6$

Second condition for maximum profit $\frac{\partial^2 \pi}{\partial x^2} < 0$

Thus, $\frac{\partial^2 \pi}{\partial x^2} = 4 < 0$, this way the condition is met with

Putting the value of x = 4 in Demand

$$p = 30 - x = 30 - 4 = 26$$

Total Revenue $(R) = 30x - x^2$

Putting x = 6

$$R = (30 \times 6) - (6 \times 6) = 180 - 36 = 144$$

Thus,

$$\begin{aligned} \pi &= 24x - 2x^2 - 7\,(x = 6) \\ &= (24 \times 6) - (2 \times 6 \times 6) - 7 \\ &= 144 - 72 - 7 = 65\,. \end{aligned}$$

Example 8: In case of full competition, if the total cost of any firm is $C = 0.3x^3 - 3x^2 + 20x + 15$, then find out its supply.

We know that for supply

$P \geq AVC$ here P = Price, AVC = Average variable cost

We have that $AVC = \dfrac{TVC}{x} = \dfrac{TC - TFC}{x}$

Here TVC = Total Variable Cost, TC = Total Cost, TFC = Total Fixed Cost and x = quantity of the commodity. Thus

$$AVC = \frac{(0.3x^3 - 3x^2 + 20x + 15) - 15}{x}$$

$(\because TFC = 15)$

$$= 0.3x^2 - 3x + 20$$

Minimizing $AVC \dfrac{\partial(AVC)}{\partial x} = 0$ and $\dfrac{\partial^2(AVC)}{\partial x^2} > 0$

Thus, $$\frac{\partial(AVC)}{\partial x} = \frac{\partial(0.3x^2 - 3x + 20)}{\partial x} = 0.6x - 3 = 0$$

Or $0.6x$ = 3 thus, x = 5

thus, AVC will be minimum at x = 5

To calculate minimum AVC, put the value of x = 5

$$\begin{aligned} AVC &= (3 \times 5 \times 5) - (3 \times 5) + 20 \\ &= 7.5 - 15 + 20 = 12.5 \end{aligned}$$

If, $P < AVC = 12.5$, production level will be ZERO

If, $P > AVC = 12.5$, then, supply level would be positive.

To calculate supply

$$\begin{aligned} MC &= \frac{\partial C}{\partial x} = 3 \times 0.3x^2 - 6x + 20 \\ &= 0.9x^2 - 6x + 20 \end{aligned}$$

Note Since in case of equilibrium in full competition

$P = MC$

Thus $P = 0.9x^2 - 6x + 20$

Or $0.9x^2 - 6x + 20 - P = 0$

Using binomial equation to know the value of X,

$$x = \frac{6 \pm \sqrt{36 - 4 \times 0.9(20 - P)}}{2 \times .6}$$

$$= \frac{6 \pm \sqrt{36 - 3.6(20 - P)}}{1.2} = \frac{6 \pm \sqrt{36 - 72 + 3.6P}}{1.2}$$

$$= \frac{6 \pm \sqrt{-36\ 2\ 3.6P}}{1.2}$$

Thus, $P \geq 12.5$ supply $x = \frac{6 \pm \sqrt{3.6P - 36}}{1.2}$

And $P < 12.5$, then supply $x = 0$

Example 9: Demand and Cost of any monopoly are $P = 50 - 6q$, $x = 60 + 14q$ respectively.

What would be equilibrium level of issue price and profit? Prove the second condition of profit maximization.

Solution: Demand and cost is following

$$P = 50 - 6q \quad \text{...(i)}$$

$$x = 60 + 14q \quad \text{...(ii)}$$

Total Revenue

$$(R) = p \times q$$

$$= (50 - 6q)\, q = 50q - 6q^2 \quad \text{...(iii)}$$

Profit

$$\pi = R - C$$

$$= 50q - 6q^2 - 60 - 14q$$

$$= 36q - 6q^2 - 60 \quad \text{...(iv)}$$

For profit maximization

Differentiating equation (iv) with respect to q

$$\frac{d\pi}{dq} = 36 - 12q = 0$$

Or $12q = 36$

Or $q = \frac{36}{12} = 3$

Putting the value of q in equation (i)

Price = $p = 50 - (6 \times 3) = 50 - 18 = 32$

Putting the value of q in equation (iv)

Note

Profit = $\pi = 36 \times 3 - 6 \times 3 \times 3 - 60$

= 108 – 54 – 60

= 108 – 114 = – 6

Second condition of profit maximization

$$\frac{d^2\pi}{dq^2} < 0$$

Here, $\frac{d^2\pi}{dq^2} = -12 < 0$

Example 10: Linear product of any monopoly firm is $p = 15 - 0.5q$ and Total cost is $C = 0.5q^2 + 5q + 10$, then find out the optimum value of issue (q), price (p), gross profit (π) and Total Revenue.

(a) Under profit maximization

(b) Sales or revenue maximization

Solution: (a) we know that

$$p = 15 - 0.5q \quad \text{...(i)}$$

$$C = 0.5q^2 + 5q + 10 \quad \text{...(ii)}$$

Thus Total Revenue

$$(R) = pq = (15 - 0.5q)\, q$$

$$= 15q - 0.5q^2 \quad \text{...(iii)}$$

Total profit

$$(\pi) = R - C$$

$$= 15q - 0.5q^2 - 0.6q^2 - 5q - 10$$

$$= 10q - q^2 - 10 \quad \text{...(iv)}$$

For profit maximization

$$\frac{d\pi}{dq} = 10 - 2q = 0$$

Or $2q = 10$

Or $q = \frac{10}{2} = 5$

Putting the value of q in equation (i), (iii) and (iv)

$$p = 15 - (0.5 \times 5) = 15 - 2.5$$

$$= 12.5$$

$$R = 15 \times 5 - (0.5 \times 5 \times 5)$$

$$= 75 - 12.5 = 62.5$$

$$\pi = (10 \times 5) - (5 \times 5) - 10$$

$$= 50 - 35 = 15$$

Note

(B) Total Revenue $R = 15q - 0.5q^2$

R will become maximum, if

$$\frac{dR}{dq} = 15q - (0.5 \times 2)q = 0$$

Or $15 - q = 0$

Or $q = 15$

Putting the value of q in equation (i), (ii) and (iv)

$$\pi = 15 - (0.5 \times 15)$$
$$= 15 - 7.5 = 7.5$$
$$R = 15 \times 15 - 0.5 \times 15 \times 15$$
$$= 225 - 112.5$$
$$= 112.5$$
$$\pi = (10 \times 15) - (15 \times 15) - 10$$
$$= 150 - 225 - 10$$
$$= -85.$$

Example 11: Linear demand of any monopoly is $p = 12 - 0.4q$ and Demand $C = 0.6q^2 + 4q + 5$, then calculate issue (q), price (p) and gross profit (π) under profit maximization method.

Solution: We know that

$$p = 12 - 0.4q \quad \text{...(i)}$$
$$C = 0.6q^2 + 4q + 5 \quad \text{...(ii)}$$

Total Revenue

$$R = p \times q$$
$$= (12 - 0.4q) \times q$$
$$= 12q - 0.4q^2 \quad \text{...(iii)}$$

Total Profit

$$\pi = R - c$$
$$= (12q - 0.4q^2) - (0.6q^2 + 4q + 5)$$
$$= 12q - 0.4q^2 - 0.6q^2 - 4q - 5$$
$$= 8q - q^2 - 5 \quad \text{...(iv)}$$

For profit maximization

$$\frac{d\pi}{dq} = 8 - 2q = 0$$

Or $2q = 8$

$$q = \frac{8}{2} = 4$$

For second condition, $\frac{d^2\pi}{dq^2} = -2 < 0$

Putting the value of q in equation (i) and (iv)

$$p = 12 - (0.4 \times 4)$$
$$= 12 - 1.6 = 10.4$$

$$\pi = (8 \times 4) - (4 \times 4) - 5$$
$$= 32 - 16 - 5$$
$$= 16 - 5 = 11$$

Example 12: Demand of any monopoly is $p = 14 - 6q$ and Total cost is $C = 60 + 20q$ then calculate the optimum level of issue (q), price (p), Total Revenue (R) and profit (π). Tell us the second condition of profit maximization.

If monopoly becomes full competition, then what would be value of price (p), issue (q) and profit?

Solution: given values are

$$P = 140 - 6q \qquad ...(i)$$
$$C = 60 + 20q \qquad ...(ii)$$

Therefore, Total Revenue

$$R = P \times q$$
$$= (140q - 6q)q = 140q - 6q^2 \qquad ...(iii)$$

Total Profit

$$\pi = R - C$$
$$= 140q - 6q^2 - 60 - 20q$$
$$= 120q - 6q^2 - 60 \quad ...(iv)$$

For profit maximization

$$\frac{d\pi}{dq} = 120 - 12q = 0$$

Or $\quad 12q = 120$

Or $\quad q = \frac{120}{12} = 10$

Putting the value of q in equation (i), (iii) and (iv)

$$P = 140 - (6 \times 10) = 140 - 60 = 80$$
$$R = (140 \times 10) - (6 \times 10 \times 10)$$
$$= 1400 - 600 = 800$$
$$\pi = (120 \times 10) - (6 \times 10 \times 10) - 60$$
$$= 1200 - 600 - 60$$
$$= 1200 - 660 = 540$$

Second condition of profit maximization

$$\frac{d^2\pi}{dq^2} = -12 < 0$$

If monopoly becomes full competition, then in this condition of equilibrium would be $MC = P$ as in Full Competition $P = MR$

Now $\quad C = 60 + 20q$

Differentiating with respect to q $\quad MC = \frac{dC}{dq} = 20$

Note

Therefore, $20 = 140 - 6q$

Or $6q = 140 - 20 = 120$

$q = 20$ or $p = 20$

And $\pi = -60$

Profit would be positive.

Example 13: Following is the demand and cost function of two separate markets

$$P_1 = 80 - 5q_1, P_2 = 180 - 29q_2 \text{ and } C = 50 + 20(q_1 + q_2)$$

In case of price difference, determine the price, production, marginal revenue and total profit of production of both the two markets.

Solution: We know that

$$P_1 = 80 - 5q_1 \quad ...(i)$$

$$P_2 = 180 - 29q_2 \quad ...(ii)$$

$$C = 50 + 20(q_1 + q_2) \quad ...(iii)$$

Total revenue of first market

$$R_1 = P_1q_1 = (80 - 5q_1)q_1$$

$$= 80q_1 - 5q_1^2 \quad ...(iv)$$

Total revenue of second market

$$R_2 = p_2q_2 = (180 - 29q_2)q_2$$

$$= 180q_2 - 29q^2{}_2 \quad ...(v)$$

By partially differentiating equation (iv) with respect to q_1, equaling it to Zero

$$MR_1 = \frac{\partial R_1}{\partial q_1} = 80 - 10q_2 = 0 \quad ...(vi)$$

Or $10q_1 = 80$ or $q_1 = 8$

Similarly by partially differentiating equation (v), equaling it to Zero

$$MR_2 = \frac{\partial R_2}{\partial q_2} = 180 - 58q_1 = 0 \quad ...(vii)$$

Or $58q_2 = 180$ or $q_2 = \frac{180}{58} = \frac{90}{29}$

Total Profit $\pi = R_1 + R_2 - C$

$$= 80q_1 - 5q_1^2 + 180q_2 - 29q_2^2 - 50 - 20q_1 - 20q_2$$

$$= 60q_1 - 5q_1^2 + 160q_2 - 29q_2^2 - 50 \quad ...(viii)$$

Putting the value of q_1 and q_2 in equations (i), (ii), (vi), (vii) and (viii)

$$P_1 = 80 - 5 \times 8 = 80 - 40 = 40$$

$$P_2 = 180 - 29 \times \frac{90}{29} = 180 - 90 = 90$$

Note

$$MR_1 = 80 - 10 \times 8 = 0$$

$$MR_2 = 180 - 58 \times \frac{90}{29} = 0$$

$$\pi = (60 \times 8) - (5 \times 8 \times 8) + 160 \times \frac{90}{29} - 29 \times \frac{90 \times 90}{29 \times 29} - 50$$

$$= 240 - 320 + \frac{14400}{29} - \frac{8100}{29} - 50$$

$$= -130 + \frac{14400 - 8100}{29}$$

$$= -130 + \frac{6300}{29} = -130 + 217\frac{7}{29}$$

$$= 87\frac{7}{29}$$

Example 14: Demand and Total Cost of a monopoly in two markets are as under

$$P_1 = 2 - q_1$$
$$P_2 = 9 - 6q_2$$
$$C = q_1 + q_2$$

In case of price difference in two markets determine the price, production (sale quantity), marginal revenue and profit of monopoly. Also find out the elasticity of demand for Market A and market B.

Solution: We know that

$$P_1 = 2 - q_1 \quad \text{...(i)}$$
$$P_2 = 9 - 6q_2 \quad \text{...(ii)}$$
$$C = q_1 + q_2 \quad \text{...(iii)}$$

Total Revenue for Market A $\quad R_1 = P_1q_1 = 2q_1 - q_1^2 \quad$ ...(iv)

Total Revenue for Market B $\quad R_2 = P_2q_2 = 9q_2 - 6q_2^2 \quad$ ...(v)

Separately differentiating equation (iv) and (v) separately with respect to q_1 and q_2:

$$MR_1 = \frac{\partial R_1}{\partial q_1} = 2 - 2q_1 \quad \text{...(vi)}$$

$$MR_2 = \frac{\partial R_2}{\partial q_2} = 9 - 12q_2 \quad \text{...(vii)}$$

Total profit

$$\pi = R_1 + R_2 - C$$

$$= q_1 + q_1^2 + 9q_2 - 6q_2^2 - q_1 - q_2$$

$$= q_1 - q_1^2 = 8q_2 - q_2^2 \quad \text{...(viii)}$$

Note

For profit maximization

By separately differentiating equation (viii) with respect to q_1 and q_2 equaling it to Zero

$$\frac{\partial \pi}{\partial q_1} = 1 - 2q_1 = 0 \text{ and } q_1 = \frac{1}{2}$$

$$\frac{\partial \pi}{\partial q_2} = 8 - 12q_2 = 0 \text{ and } q_2 = \frac{2}{3}$$

Putting the value of q_1 and q_2

$$P_1 = 2 - \frac{1}{2} = \frac{3}{2} = 1.5$$

$$P_2 = 9 - 6 \times \frac{2}{3} = 9 - 4 = 5$$

$$MR_2 = 2 - 2 \times \frac{1}{2} = 1$$

$$MR_2 = 9 - 12 \times \frac{2}{3} = 1$$

Here $MR_1 = MR_2$, but $P_1 \neq P_2$

Now
$$\begin{bmatrix} MR_1 = P_1\left(1 - \frac{1}{e_1}\right), e_1 = A & \text{here } e_1 = \text{elasticity of market } A \\ MR_2 = P_2\left(1 - \frac{1}{e_2}\right), e_2 = B & \text{here } e_2 = \text{elasticity of market } B \end{bmatrix}$$

Putting value of MR_1, MR_2, P_1 and P_2

$$\begin{bmatrix} 1 = 1.5\left(1 - \frac{1}{e_1}\right) \text{ and } e_1 = 3 \\ 1 = 5\left(1 - \frac{1}{e_2}\right) \text{ and } e_2 = \frac{4}{5} \end{bmatrix}$$

Thus elasticity of demand in Market A is more than the market, therefore the price of the commodity in Market A is less that market B.

Example 15: Demand and Total Cost of a monopolistic competitive firm are as under:

$$\boldsymbol{P = 36 - 5q}$$

$$\boldsymbol{C = q^2 + 6q + 5}$$

Determine price, production (q) and profit of the firm. Evaluate the monopolistic capacity of this firm? Determine its elasticity of demand.

Solution: We know that

$$P = 36 - 5q \quad \text{...(i)}$$

$$C = q^2 + 6q + 5 \quad \text{...(ii)}$$

$$\therefore \quad R = Pq = 36q - 5q^2 \quad \text{...(iii)}$$

$$\therefore \quad MR = \frac{dR}{dq} = 36 - 10q \quad \text{...(iv)}$$

Note

And

$$\pi = R - C$$
$$= 36q - 5q^2 - q^2 - 6q - 5$$
$$= 30q - 6q^2 - 5 \quad ...(v)$$

For profit maximization

$$\frac{d\pi}{dq} = 30 - 12q = 0$$

$$12q = 30$$

Or $$q = \frac{30}{12} = \frac{5}{2} = 2.5$$

Putting the value of q in equation (i), (iv) and (v)

$$P = 36 - 5 \times \frac{5}{2} = 36 - \frac{25}{2} = 36 - 12.5 = 23.5$$

$$MR = 36 - 10q$$

$$= 36 - 10 \times \frac{5}{2} = 36 - 25 = 11$$

$$\pi = 30q - 6q^2 - 5$$

$$= 30 \times \frac{5}{2} - 6 \times \frac{5}{2} \times \frac{5}{2} - 5$$

$$= 75 - \frac{75}{2} - 5$$

$$= 37.5 - 5 = 32.5$$

Since here $P > MR$, therefore monopolistic capacity of the firm is $= \frac{P - MR}{MR} \times 100$

$$= \frac{23.5 - 11}{11} \times 100$$

$$= \frac{12.5 \times 100}{11} = 113.6\%$$

In this condition, firm has the monopolistic capacity to increase the price by 113.6%

Thus, elasticity of demand (e) = $\frac{P}{P - MR}$

$$= \frac{23.5}{23.5 - 11} = \frac{23.5}{12.5}$$

$$= 1.8.$$

Example 16: If following is the Demand and Cost:

$$P = 100 - 0.5(q_1 + q_2)$$
$$C_1 = 5q_1$$
$$C_2 = 0.5q_2^2$$

Note

Then find out the value of q_1, q_2, P, π and π_2.

Find out the second level condition for both firms in case of profit maximization. Whether increase in production of a firm will affect the second firm to less productivity?

Solution: We know that

$$P = 100 - 0.5\,(q_1 + q_2)$$
$$C_1 = 5q_1$$
$$C_2 = 0.5q_2^{\,2}$$

Total revenue of first firm

$$\therefore \quad R_1 = Pq_1 = [100 - 0.5(q_1 + q_2)]q_1$$

$$\text{Second} = 100q_1 - 0.5q_1^2 - 0.5q_2$$

And total revenue of second firm

$$R_2 = Pq_2 = [100 - 0.5(q_1 + q_2)]q_2$$
$$= 100q_2 - 0.5q_1q_2 - 0.5q_2^2$$

Profit of first firm

$$\pi_1 = R_1 - C_1$$
$$= 100q_1 - 0.5q_1^2 - 0.5q_1q_2 - 5q_1$$
$$= 95q_1 - 0.5q_1^2 - 0.5q_1q_2$$

Profit of second firm

$$\pi_1 = R_2 - C_2$$
$$= 100q_2 - 0.5q_1q_2 - 0.5q_2^2 - 0.5q_2^2$$
$$= 100q_2 - 0.5q_1q_2 - q_2^2$$

Profit maximization

$$\frac{\partial \pi_1}{\partial q_1} = 95 - q_1 - 0.5q_2 = 0$$

Or $$q_1 + 0.5q_2 = 95 \quad ...(i)$$

$$\frac{\partial \pi_2}{\partial q_2} = 100 - 0.5q_2 - 2q_1 = 0$$

Or $$0.5q_1 + 2q_2 = 100 \quad ...(ii)$$

Multiplying equation (i) by 4

Or $$4q_1 + 2q_2 = 300 \quad ...(iii)$$

Subtracting equation (iii) from equation (iv)

$$3.5q_1 = 280$$

Or $$q_1 = \frac{280}{3.5} = 80$$

Putting the value of q_1 in equation (i)

Note

$$80 + 0.5q_2 = 95$$

Or $$0.5q_2 = 95 - 80 = 15 \text{ or } q_2 = \frac{15}{0.5} = 7.5$$

Putting the value of q_1 and q_2 in demand and profit

$$\begin{aligned} P &= 100 - (q_1 + q_2) \\ &= 100 - 0.5\,(80 + 7.5) \\ &= 100 - 0.5 \times 87.5 = 100 - 43.75 = 56.25 \\ \pi_1 &= 95 \times 80 - (0.5 \times 80 \times 80) - (0.5 \times 7.5 \times 80) \\ &= 7600 - 3200 - 300 = 4100 \\ \pi_2 &= 100 \times 7.5 - 0.5 \times 7.5 \times 80 - 7.5 \times 7.5 \\ &= 75 - 300 - 56.25 = -281.25 \end{aligned}$$

Second condition

$$\frac{\partial \pi_1}{\partial q_1^2} = -1 < 0$$

$$\frac{\partial^2 \pi_2}{\partial q_2^2} = -2 < 0$$

By reciprocal method (i) and (ii)

$$q_1 = 95 - 0.5q_2$$

and $$q_2 = \frac{100 - 0.5q_1}{2} = 50 - 0.25q_1$$

Since the slope of these curves is negative, therefore in case of increase in productivity of a firm, productivity of second firm will go down.

Self Assessment

2. State whether the following statements are True or False:

6. With the help of differentiation, level of equilibrium in monopoly is assessed.
7. $\frac{d(R)}{dq} \neq \frac{d(c)}{dq}$
8. $MR = MC$
9. $AVC = \frac{TC - TVC}{x}$
10. $AVC = \frac{TC - TFC}{x}$

8.2 Summary

- Differentiation is used in economics to determine elasticity.
- Demand product shows that demand of any commodity is the product of price of that commodity. But demand of any commodity is also related to price of other related commodity. Cross demand tells that if the price of related commodity changes, in that case demand of that commodity also changes.

Note

8.3 Keywords

- ***Use:*** Application
- ***Elasticity:*** Resilience as like spring.

8.4 Review Questions

1. Explain the method of determining elasticity.
2. Establish the relation between Marginal Revenue and Elasticity of Demand.
3. How do we determine marginal cost from total cost and marginal revenue from total revenue?
4. If demand is $P = 20 - 5q$ and average cost is $AC = q$, then brief the equilibrium value of monopoly and maximum profit. (**Ans.:** Equilibrium value = 2.5, maximum profit = 5/3).

Answers: Self Assessment

1. Economics	2. Differentiation	3. Quantity	4. Revenue
5. AR	6. True	7. False	8. True
9. False	10. True		

8.5 Further Readings

Books

Mathematics for Economics – Carl P Simone, Lawrence Bloom.

Mathematics for Economist– Yamane, Prentice Hall India.

Mathematics for Economist–Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economics – Council for Economic Education.

Essential Mathematics for Economics- Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist– Simone and Bloom, Viva Publication.

Mathematics for economics – Malcom, Nicolas, U C London.

Mathematical Economics – Michael Harrison, Patrick Walderan.

Unit 9: Maxima and Minima: One Variable

Note

CONTENTS

Objectives

After reading this unit students will be able to :

- Understand the Concept of Maxima and Minima.
- Understand the Definition of Maxima and Minima.
- Know the Condition to Finding Maxima and Minima.
- Determine the Steps of Finding Maxima and Minima of the Function $y = f(x)$.

Introduction

If the height of your house is more than the houses situated in neighborhood (right or left), then the height of your house will be called maximum and contrary to this if the height is less, then it will be called Minimum.

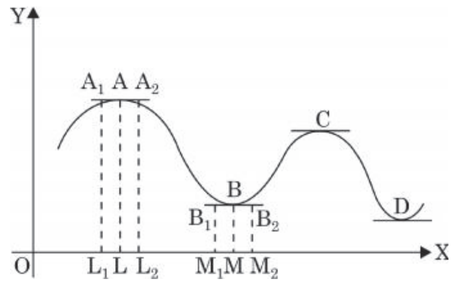
9.1 Concept of Maxima and Minima

One of the main uses of mathematics is to determine the maxima and minima of any function. If any function grows to a certain value of its independent variable and decreases for the maximum value of its independent variable, then from its increasing state arriving at the state of decreasing function receives the maximum value. Similarly when function decreases to some certain point of independent variable and grows towards the next values, then arriving from the state of decreasing to an increasing state, the function obtains minimum value. This clearly describes that at its maximum point, the value of the function is maximum from the value of its immediate and small neighborhood and at its minimum point, the value of the function is minimum from the value of its immediate and small neighborhood.

For example: Assume that for each value of x, which is less than 1, function $y = x^3 - 6x^2 - 2$, x viz ($x < 1$), the value of y increases and for $y = 1 < x < 3$, the value of y decreases. Thus, among it, y obtains a maximum value.

Note

Assume $y = f(x)$ is a function. Drawing of the function $y = f(x)$ is given. At this curve four points A, B, C and D are there, at which touching line is parallel to x-axis.



Now the value of $f(x)$ is maximum at A and C and when A and C increases, the value of function stops increasing. Therefore at A and C, the value of function is called Maximum. Similarly at B and D, the value of function stops increasing viz increasing further, $f(x)$ starts increasing. Therefore at B and D, whatever value $f(x)$ obtains, they are called Minimum.

Maxima

Assume that for A point located at the curve $x = a = OL$

$\therefore \quad x = a \; y = f(a) = AL$

Now at the left side of point A, from small neighborhood take any point A_1, for which $x = a - h = OL_1$, here h is small. Therefore,

$\therefore \quad x = a - h$ but $y = f(a - h) = A_1L_1 < AL$

Similarly at the left side of point A, from small neighborhood take any point A_2, for which

$$x = a + h = OL_2.$$

Therefore, at $x = a + h \qquad y = f(a + h) = A_2L_2 < AL$

Now since A_1L_1 is smaller than AL, viz $A_1L_1 < AL$, therefore $f(a - h) < f(a)$ or $f(a) > f(a - h)$

And since A_2L_2 is smaller than AL, viz $A_2L_2 < AL$, therefore $f(a + h) < f(a)$ or $f(a) > f(a + h)$

Thus, at point A (for which $x = a$) the value of $f(x)$ viz. at each point on right or left of $f(a)$ A, corresponding value of f(x) is greater than $f(a - h)$ or $f(a + h)$.

9.2 Definition of Maxima and Minima

Maxima

Assume $y = f(x)$ is any given function, where $x = a$ is any given point.

Assume at the L.H.S. of point x=A, closest point is $x = a - h$, and at the R.H.S. of point x=a, closest point is $x = a + h$, where h is minimum number.

At point x=a, the value of $f(x) = f(a)$

At point x = $(a - h)$, the value of $f(x) = f(a - h)$

And at point $x = (a + h)$, the value of $f(x) = f(a + h)$

At point $x = a$, the value of function of $f(x)$ is called maximum, if

$$f(a - \text{h}) < f(a) > f(a+h)$$

viz at $x = a$, the value of $f(x)$, on the closest point at left and right, will be maximum fro $f(a\text{-}h)$ and f(a+h)

At $x = a$, any function f(x) is called maximum where $f(a)$ is greater that all the given values which in the short neighborhood of each value of x, can accept $f(x)$

Minima

Assume that $x = a = OM$ for point B at the curve

Note

Thus, at $x = a$, $y = f(a) = BM$

Now in small neighborhood at LHS of point B, take a point B_1 for which $x = a - h = OM_1$, where h is smallest.

Thus, at $x = a - h$, $y = f(a\text{-}h) = B_1M_1 > BM$

Now in small neighborhood at LHS of point B, take a point B_2 for which $x = a + h = OM_2$.

Thus, at $x = a + h$, $y = f(a+h) = B_2M_2 > BM$

Now since B_1M_1 is greater than BM, viz $B_1M_1 > BM$

Thus, $f(a\text{-}h) > f(a)$ or $f(a) < f(a\text{-}h)$

And since B_2M_2 is greater than BM, viz $B_2M_2 > BM$

Thus, $f(a+h) > f(a)$ or $f(a) < f(a+h)$

Thus at point B for which $x = a$, value of $f(x)$ viz the corresponding value of $f(x)$ at the left or right side of point B, $f(a)$ viz $f(a\text{-}h)$ and $f(a+h)$ is smaller

At the point $x = a$, the function $f(x)$ is called minimum if

$$\boxed{f(a - h) > f(a) < f(a + h)}$$

viz At $x=a$, the value $f(a)$ of $f(x)$ is smaller than both the value $f(a\text{-}h)$ and $f(a+h)$ in its small neighborhood.

The maximum value of any function does not mean that it is the biggest value and similarly minimum value of does not mean that it's the smallest value. There can be many maximum and minimum value of any function and it is possible that a maximum value is smaller than minimum value. At A, maximum value of function or degree is there, it only means that in the small neighborhood of this point, its value is maximum and similarly in the small neighborhood of this point, its value is minimum.

9.3 Conditions for Finding Maxima and Minima

Following are the conditions to find maximum and minimum of function $y = f(x)$ at point $x = a$:

(i) Necessary condition – the essential condition for both maximum and minimum is as under:

$$\boxed{f'(x) = 0 \text{ or } \frac{dy}{dx} = 0}$$

(ii) Sufficient condition - the sufficient condition for both maximum and minimum is as under:

For maximum

At x = a, the value of $\frac{d^2y}{dx^2}$ = negative value

For minimum

At $x = a$, the value of = $\frac{d^2y}{dx^2}$ positive value

Self Assessment

1. Fill in the blanks:

1. One of the main uses of mathematics is to determine the maxima and minima of any ....
2. There can be many maximum and .... value of any function.

Note

3. For …………………., at $x = a$, the value of $\frac{d^2y}{dx^2}$ = negative value.

4. For minimum at $x = a$, the value of $\frac{d^2y}{dx^2}$ = is …………….. value.

9.4 Conditions to Absence of Maxima or Minima

At point $x = a$, the value of function $y = f(x)$ will neither be maximum nor minimum if

The value of $\frac{d^2y}{dx^2} = 0$ and value of $\frac{d^3y}{dx^3} \neq 0$

Properties of Maximum and Minimum value

1. Maximum value comes after minimum value and minimum comes after maximum viz maximum and minimum comes in a sequence.
2. There will be a certain maximum or minimum value between the two equal values of function.
3. At point touching lines are parallel to x-axis where the maximum and minimum of function are there. Therefore, at such points value of $\frac{dy}{dx}$ will be 0, solving the equation after putting $\frac{dy}{dx} = 0$, value of x can be obtained, over which the value of the function is maximum or minimum.
4. At the maximum or minimum point of function the sign of $\frac{dy}{dx}$ changes. At maximum point it becomes negative from positive and contrary to this it becomes positive from negative.

Task Define the minimum.

9.5 Steps for Finding Maxima and Minima of the Function $y = f(x)$

(i) Calculating $\frac{dy}{dx}$ of $y = f(x)$

(ii) Determining various values of x from the equation obtain by assigning $\frac{dy}{dx} = 0$

(iii) Assume the different values of x are a_1, a_2, a_3 etc.

(iv) Obtaining, d^2y/dx^2 finding the value of $\frac{d^2y}{dx^2}$ on a_1, a_2, a_3 etc.

If for any value of x value of $\frac{d^2y}{dx^2}$ is positive, then function for that value of x is minimum and if it is negative, then value would be maximum.

(v) If for the value of x, $\frac{d^2y}{dx^2}$ =0, then for that value of x determining $\frac{d^3y}{dx^3}$. If for the value of x, $\frac{d^3y}{dx^3} \neq 0$, then for that value of x, value of function would neither be maximum nor minimum.

(vi) If for the value of x, $\frac{d^3y}{dx^3}$ = 0, then for that value of x determining $\frac{d^4y}{dx^4}$. If for the value of x, is negative, $\frac{d^4y}{dx^4}$ then for that value of x, value of function would be maximum and if it positive, then value would be minimum. And if it is also zero, then the same process has to be repeated.

EXAMPLES WITH SOLUTION

Example 1: Prove that the maximum value is $\sqrt{2}$ for sin x + cos x .

Solution: Assume y = sin x + cos x

Differentiating both the sides with respect to x $\frac{dy}{dx}$ = cos x – sin x

Further differentiating both the sides with respect to x $\frac{d^2y}{dx^2}$ = – sin x – cos x

For the maximum and minimum value of y, $\frac{dy}{dx}$ = 0.

Thus,

$\therefore$ $\cos x - \sin x = 0$

Or $\sin x = \cos x$

Or $\tan x = 1$

Or $x = \frac{\pi}{4}$

$x = \frac{\pi}{4}$ but the value $\frac{d^2y}{dx^2} = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= \frac{-2}{\sqrt{2}} = -\sqrt{2} = \text{negative value}$$

Therefore, at , $x = \frac{\pi}{4}$ the maximum value of y

the maximum value of $y = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$.

Note

Example 2: Find out the value of x for the maximum and minimum value of function $x^4 - 8x^3 + 22x^2 - 24x$.

Solution: Assume $y = x^4 - 8x^3 + 22x^2 - 24x$

Differentiating both the sides with respect to x

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24 \quad ...(i)$$

For the maximum and minimum value of the function

$$\frac{dy}{dx} = 0$$

$\therefore$ $4x^3 - 24x^2 + 44x - 24 = 0$

Or $4(x^3 - 6x^2 + 11x - 6) = 0$

Or $x^3 - 6x^2 + 11x - 6 = 0$

Or $(x-1)(x-2)(x-3) = 0$

$\therefore$ $x = 1, 2, 3$

Differentiating both the sides of equation 1 with respect to x

$$\frac{d^2y}{dx^2} = 12x^2 - 48x + 44$$

$x = 1$ but the value $\frac{d^2y}{dx^2} = 12(1)^2 - 48(1) + 44$

$= 12 - 48 + 44$

$= 8$ = positive value

Thus, the value of function is minimum at $x = 1$

$x = 2$ but the value $\frac{d^2y}{dx^2} = 12(2)^2 - 48(2) + 44$

$= 48 - 96 + 44$

$= -4$ = negative value

Therefore, the value of function is maximum at x = 2

$x = 3$ but the value $\frac{d^2y}{dx^2} = 12(3)^2 - 48(3) + 44$

$= 108 - 144 + 44 = 8$ = positive value

Therefore, at $x = 3$, the value of function is 0.

Example 3: Find out the maximum and minimum value of $y = x^3 - 2x^2 + x + 6$.

Solution: Assume $y = x^3 - 2x^2 + x + 6$

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

Or $$\frac{d^2y}{dx^2} = 6x - 4$$

Now the value of $\frac{dy}{dx} = 0$

$3x^2 - 4x + 1 = 0$ Or $(3x - 1)(x - 1) = 0$, then $x = \frac{1}{3}$ or 1

At these points, function would be maximum or minimum

Now $x = \frac{1}{3}$ but the value $\frac{d^2y}{dx^2}$ of $6.\frac{1}{3} - 4 = -2 < 0$ (negative value)

$\therefore$ Function would be maximum $x = \frac{1}{3}$.

Or $x = 1$ but $\frac{d^2y}{dx^2} = 6.1 - 4 = 2 > 0$ (positive value)

Function would be minimum $x = 1$.

Maximum value $= f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 2.\left(\frac{1}{3}\right)^2 + \frac{1}{3} + 6 = \frac{166}{27}$ and

Minimum value $= f(1) = 1^3 - 2.1^2 + 1 + 6 = 6$.

Example 4: At which value of x, the value of function $2x^3 - 9x^2 + 12x - 3$, x is maximum or minimum?

Solution: Assume that $y = 2x^3 - 9x^2 + 12x - 3$

Differentiating the function with respect to x

$$\frac{dy}{dx} = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) \qquad ...(1)$$

For the maximum or minimum value of function

$$\frac{dy}{dx} = 0$$

$\therefore$ $6(x^2 - 3x + 2) = 0$ Or $x^2 - 3x + 2 = 0$

Or $(x - 1)(x - 2) = 0$ $\therefore$ $x = 1, 2$

Note

Now differentiating (1)

$$\frac{d^2y}{dx^2} = 6\,(2x - 3)$$

And for $x = 1$ $\frac{d^2y}{dx^2} = 6(2.1 - 3) = -6$ which is negative

Therefore, at $x = 1$ function is maximum. This maximum value

$= 2.1^3 - 9.1^2 + 12.1 - 3 = 2 - 9 + 12 - 3 = 2$

For for $x = 2$ $\frac{d^2y}{dx^2} = 6\,(2.2 - 3) = 6$ which is positive

Therefore at $x = 2$ function is maximum, and the minimum value of function is

$= 2.2^3 - 9.2^2 + 12.2 - 3 = 1.$

Example 5: For which value of x, the value of function $f(x) = x^5 - 5x^4 + 5x^3 - 1$, is maximum or minimum? Prove that at $x = 0$, the function is neither maximum nor minimum.

Solution: Assume that $y = x^5 - 5x^4 + 5x^3 - 1$

Differentiating both the sides with respect to x

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2$$

$$= 5x^2(x^2 - 4x + 3) = 5x^2\,(x - 1)\,(x - 3)$$

For maximum and minimum value of function $\frac{dy}{dx} = 0$

$\therefore$ $5x^2\,(x - 1)\,(x - 3) = 0$ $\therefore$ $x = 0, 1, 3$

Now $\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$

Replacing $x = 1$

$$\frac{d^2y}{dx^2} = 10\,(2 - 6 + 3) = -10 \text{ negative}$$

Therefore At $x = 1$, function is maximum

Replacing $x = 3$

$$\frac{d^2y}{dx^2} = 10 \times 3\,(2 \times 3^2 - 6 \times 3 + 3) = 30(18 - 18 + 3) = 90 \text{ positive}$$

Therefore At $x = 3$, function is minimum

Replacing $x = 0$

$$\frac{d^2y}{dx^2} = 0, \text{ function is maximum and can not be said negative}$$

Now $\frac{d^3y}{dx^3} = 60x^2 - 120x + 30$

Replacing with $x = 0$

$$\frac{d^3y}{dx^3} = 0 - 0 + 30 = 30 \text{ which is not } 0$$

Therefore At $x = 0$, function is neither maximum nor minimum

Example 6: Find out the maximum value of function $(x - 1)(x - 2)(x - 3)$.

Solution: Assume that $y = (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$

Differentiating both the sides with respect to x

$$\frac{dy}{dx} = 3x^2 - 12x + 11$$

For the maximum or minimum value

$$\frac{dy}{dx} = 0$$

$\therefore$ $3x^2 - 12x + 11 = 0$

Or $$x = \frac{12 \pm \sqrt{144 - 4 \times 3 \times 11}}{6} = \frac{12 \pm 2\sqrt{3}}{6}$$

$\Rightarrow$ $$x = 2 \pm \frac{1}{\sqrt{3}}$$

$\therefore$ $$x = 2 + \frac{1}{\sqrt{3}}$$

Or $$x = 2 - \frac{1}{\sqrt{3}}$$

Now $$\frac{d^2y}{dx^2} = 6x - 12$$

$$x = 2 + \frac{1}{\sqrt{3}}$$

$$\frac{d^2y}{dx^2} = 6\left(2 + \frac{1}{\sqrt{3}}\right) - 12 = 2\sqrt{3} \text{ positive}$$

At $x = 2 + \frac{1}{\sqrt{3}}$, function is minimum

And for $x = 2 - \frac{1}{\sqrt{3}}$

$$\frac{d^2y}{dx^2} = 6\left(2 - \frac{1}{\sqrt{3}}\right) - 12 = -2\sqrt{3} \text{ negative}$$

Therefore at $x = 2 - \frac{2}{\sqrt{3}}$, function is maximum

Note

Now in the given function replacing $x = 2 - \frac{1}{\sqrt{3}}$

Maximum value

$$= \left(2 - \frac{1}{\sqrt{3}} - 1\right)\left(2 - \frac{1}{\sqrt{3}} - 2\right)\left(2 - \frac{1}{\sqrt{3}} - 3\right)$$

$$= \left(1 - \frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)\left(-1 - \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}.$$

Example 7: Prove that the maximum value of $\left(\frac{1}{x}\right)^x$ is $(e)^{1/e}$.

Solution: Assume $y = \left(\frac{1}{x}\right)^x$

Taking logarithm of both sides based on 'e'

$$\log y = \log\left(\frac{1}{x}\right)^x = x \log \frac{1}{x} = x \log x^{-1} = -x \log x$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = -\left[x \cdot \frac{1}{x} + 1.\log x\right]$$

$$\therefore \quad \frac{dy}{dx} = -(1 + \log x)\, y$$

For the maximum and minimum value of y

$$\frac{dy}{dx} = 0$$

$\therefore \quad -(1 + \log x)\, y = 0 \quad$ Or $\quad \log x = -1$

$\because \quad y \neq 0$

Or $\quad -\log x = 1$

Or $\quad \log \frac{1}{x} = \log e \quad \Rightarrow \quad x = \frac{1}{e}.$

Now $\quad \frac{d^2y}{dx^2} = -\left[y \cdot \frac{1}{x} + (1 + \log x)\frac{dy}{dx}\right]$

Replacing with $x = \frac{1}{e}$

$$\frac{d^2y}{dx^2} = -[ey + 0], \quad \because \quad \frac{dy}{dx} = 0$$

Note

$\frac{d^2y}{dx^2} = -ey$ is negative, therefore for the function $x = \frac{1}{e}$ is maximum

Thus, in the referred function replacing with $x = \frac{1}{e}$

Maximum value = $\left(\frac{1}{\frac{1}{e}}\right)^{1/e} = (e)^{1/e}$ is proven.

Example 8: Find out the maximum value of $\frac{\log x}{x}$ where $0 < x < \infty$

Solution: Assume that $y = \frac{\log x}{x}$

$$\frac{dy}{dx} = \frac{x.(1/x) - (\log x).1}{x^2} = \frac{1 - \log x}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x^2(-1/x) - (1 - \log x).2x}{x^4}$$

$$= \frac{-x - 2x + 2x \log x}{x^4} = \frac{2 \log x - 3}{x^3}$$

Putting , $\frac{dy}{dx} = 0$, $\frac{1 - \log x}{x^2} = 0$ or $1 - \log x = 0$ or $\log x = 1 = \log e$, $\therefore x = e$

At $x = e$ $\frac{d^2y}{dx^2} = \frac{2 \log e - 3}{e^3} = \frac{2-3}{e^3} = -\frac{1}{e^3}$ is negative

Therefore, at $x = +e$, function is maximum and its maximum value is = $\frac{\log e}{e} = \frac{1}{e}$.

Example 9: If at extremum values of $x = -1$ and $x = 2$ are $y = a \log x + bx^2 + x$ then find out the value of a and b.

Solution: $y = f(x) = a \log x + bx^2 + x \Rightarrow \frac{dy}{dx} = a.\frac{1}{x} + 2bx + 1$

For extremum $\frac{dy}{dx} = 0$,

$$\left(\frac{dy}{dx}\right)_{-1} = 0, \left(\frac{dy}{dx}\right)_2 = 0$$

$\Rightarrow$ $-a - 2b + 1 = 0$ ...(*i*)

Or $\left(\frac{a}{2}\right) + 4b + 1 = 0$ ...(*ii*)

Note The value of example (i) and (ii)

$$a = -2, b = -\frac{1}{2}.$$ **Ans.**

Example 10: Find out the minimum or maximum values of function $x + \sin 2x$, $(0 < x < 2\pi)$.

Solution: Assume $y = x + \sin 2x$

$\therefore$ $\frac{dy}{dx} = 1 + 2\cos 2x$

Replacing with $\frac{dy}{dx} = 0$, $1 + 2\cos 2x = 0$ or $\cos 2x = -\frac{1}{2}$

Therefore $2x = \frac{2\pi}{3}, \frac{4\pi}{3}$ $[\because 0 < x < 2\pi] \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$

Now $\frac{d^2y}{dx^2} = -4\sin 2x$

(1) when $x = \frac{\pi}{3}$

$$\frac{d^2y}{dx^2} = -4\sin\frac{2\pi}{3} = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3} \text{ negative}$$

Therefore at $x = \frac{\pi}{3}$ function is maximum

And in the given function replacing with $x = \frac{\pi}{3}$, maximum value of function

$$= \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} = \frac{2\pi + 3\sqrt{3}}{6}.$$

(2) where $x = \frac{2\pi}{3}$ then $\frac{d^2y}{dx^2} = -4\sin\frac{4\pi}{3} = -4\left(-\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$ positive

Therefore, at $x = \frac{2\pi}{3}$, function is minimum

And in the given function replacing with $x = \frac{2\pi}{3}$, minimum value of function

$$= \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} = \frac{4\pi - 3\sqrt{3}}{6}.$$ **Ans.**

Self Assessment

Note

2. State whether the following statements are True or False:

5. At $x = a$ for maximum, the value of $\frac{d^2y}{dx^2}$ is positive.
6. At $x = a$ for minimum, the value of $\frac{d^2y}{dx^2}$ is negative.
7. There will be a certain maximum or minimum value between the two equal values of function.
8. At the maximum or minimum point of function the sign of $\frac{dy}{dx}$ changes.

9.6 Summary

- If the height of your house is more than the houses situated in neighbourhood (right or left), then the height of your house will be called maximum and contrary to this if the height is less, then it will be called Minimum.
- Function decreases to some certain point of independent variable and grows towards the next values, then arriving from the state of decreasing to an increasing state, the function obtains minimum value.
- Maximum value comes after minimum value and minimum comes after maximum viz maximum and minimum comes in a sequence.
- If for any value of x value of $\frac{d^2y}{dx^2}$ is positive, then function for that value of x is minimum and if it is negative, then value would be maximum.

9.7 Keywords

- ***Maximum:*** more value
- ***Minimum:*** less value

9.8 Review Questions

1. Find out the maximum and minimum value of $x^3 - 2x^2 + x + 6$

   [**Ans.:** Maximum $\frac{166}{27}$, Minimum = 6]

2. Find out the maximum value of function $(x - 1)(x - 2)(x - 3)$. [**Ans.:** Maximum $\frac{2}{3\sqrt{3}}$]
3. Prove that the maximum value of $\left(\frac{1}{x}\right)^x$ is $(e)^{1/e}$
4. At what values of x, function $2x^3 - 9x^2 + 12x - 3$, x is maximum or minimum

   [**Ans.:** Maximum = 2, Minimum = 1]

Note

Answers: Self Assessment

1. Function	2. Minima	3. Maxima
4. Positive	5. False	6. True
7. True	8. False	

9.9 Further Readings

Books

Mathematics for Economics – Carl P Simone, Lawrence Bloom.

Mathematics for Economics and Finance – Martin Norman.

Essential Mathematics for Economics- Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist– Yamane, Prentice Hall India.

Mathematics for Economist–Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist– Simone and Bloom, Viva Publication.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics – Malcom, Nicolas, U C London.

Mathematical Economics – Michael Harrison, Patrick Walderan.

Unit 10: Maxima and Minima: Two Variables and Constrained Maxima and Minima with Lagrange's Multiplier

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Understand the Conditions to finding Maxima and Minima.
- Understand Lagrange's Method.
- Know the Slutsky Equation.
- Understand the Slutsky Equation in Elasticity form.

Introduction

Suppose $U = f(xy)$ is any given function, and (x_0, y_0) are two given points on which Maxima and Minima has to be found and e < n are two positive constants then the function at the maxima (x_0, y_0) will be of the following form–

$f(x_0 - e, y_0 - n) < f(x_0, y_0)$ and $f(x_0, y_0) > f(x_0 + e, y_0 + n)$ and Minima $f(x_0 - e, y_0 - n) > f(x_0, y_0)$ and $f(x_0, y_0) < f(x_0 + e, y_0 + x)$ where $f(x_0, y_0)$ are the maxima and minima values of the function.

10.1 Conditions to finding Maxima and Minima

(A) Necessary Condition: If $u = f(x,y)$

Then $\partial u/\partial x = \partial u/\partial y = 0$

(B) Sufficient Condition:

For maxima, $u = f(x, y)$ if and Necessary Condition $fx = 0$, and $fy = 0$

$$\partial^2 u/\partial x^2 < 0 \text{ and } \partial^2 u/\partial y^2 > 0$$

$$\therefore \qquad \underset{\downarrow A}{\frac{\partial^2 u}{\partial x^2}} \cdot \underset{(B)}{\frac{\partial^2 u}{\partial y^2}} > \underset{C}{\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2} \quad \text{or} \quad AB > C^2 \quad \text{or} \quad \boxed{f_{xx} \cdot f_{yy} > f_{xy}}$$

Note For Minima,

$$u = f(x, y), fx = 0 \text{ and } f_y = 0$$

Then

$$\partial^2u/\partial x^2 > 0 \text{ and } \partial^2u/\partial yz > 0$$

$$\therefore \qquad \partial^2u/\partial x^2 . \frac{\partial^2 u}{\partial y^2} < \left(\frac{\partial^2 u}{\partial x \cdot \partial y}\right)^2 \quad \text{or} \quad \boxed{A.B < C^2}$$

or

$$f_{xx} \cdot f_{yy} < fxy$$

Example 1: Find the Maxima and Minima of $u = x^3 + x^2 + xy + y^2 + 4$.

Solution:

$$\frac{\partial u}{\partial x} = 3x^2 + 2x - y = 0 \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = -x + 2y = 0 \qquad ...(2)$$

On solving (1) and (2)

$$3x^2 + 2x - y = 0$$

$$-x + 2y = 0$$

If $x = 2y$ then putting it in the first equation

$$3(2y)^2 + 2(2y) - y = 0$$

$$y(12y + 3) = 0$$

When $\qquad y = 0$ or $y = -\frac{1}{4}$

When $\qquad y = 0$

$x = 2y = 0$

When $\qquad y = -\frac{1}{4}$

Then $\qquad x = 2y = -\frac{1}{2}$

In this way we have two points (0, 0) and (–1/2, –1/4)to find out Necessary condition. Now we will find that whether the above values satisfy the condition of maxima or minima or not.

$$\frac{\partial^2 u}{\partial x^2} = 6x + 2, \cdot \frac{\partial^2 u}{\partial y^2} = 2, \frac{\partial^2 u}{\partial x \partial y} = -1 \qquad ...(3)$$

For (0, 0) values;

$$\frac{\partial^2 u}{\partial x^2} = 6(0) + 2 = 2 > 0, \frac{\partial^2 u}{\partial y^2} = 2 > 0$$

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} = 2 \cdot 2 = 4 > \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2 = (-1)^2 = 1$$

In this way (0, 0) value satisfies the minima condition.

$$u = x^3 + x^2 - xy + y^2 + 4 = 4 \quad (0, 0)$$

will be minima value at (0 , 0) basic point.

For the points (–1/2, –1/4),

Note

$$\frac{\partial^2 u}{\partial x^2} = 6\left(-\frac{1}{2}\right) + 2 = -1, \frac{\partial^2 u}{\partial y^2} = 2$$

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \, \partial y}\right)^2 = (-1)2 - (-1) = -3 < 0$$

Point (–1/2, –1/4) gives the Saddle point.

Results can briefly be written as:

(Maximum)	(Minimum)
$f_x = 0, \quad f_y = 0$	$f_x = 0, \quad f_y = 0$
$f_{xx} < 0, \quad f_{yy} < 0$	$f_{xx} > 0, \quad f_{yy} > 0$
$f_{xx}f_{yy} - (f_{xy})^2 > 0$	$f_{xx}f_{yy} - (f_{xy})^2 > 0$
(Saddle Point)	(No Information)
$f_x = 0, \quad f_y = 0$	$f_x = 0, \quad f_y = 0$
$f_{xx}f_{yy} - (f_{xy})^2 < 0$	$f_{xx}f_{yy} - (f_{xy})^2 = 0$

Example 2: Find the maximum value of *u* with the help of the given function $y = x^3 + y^3 - 3x - 27y + 24$.

Solution: Conditions of first class

$$f_x = 3x^2 - 3 = 0, x^2 - 1 = 0$$

$$f_y = 3y^2 - 27 = 0, y^2 - 9 = 0$$

$\therefore$ (1, 3), (1, –3), (–1, 3), (–1, –3)

Conditions of second class

$$f_{xx} = 9x$$

$$f_{yy} = 9y$$

$$f_{xy} = 0$$

In context of (1 , 3)

$$f_{xx} = 9x = 9 > 0$$

$$f_{yy} = 9y = 27 > 0$$

$$f_{xx}f_{yy} - (f_{xy})^2 = 243 - 0 = 243 > 0$$

In this way, u is minima at point (1, 3)

In context of (1, – 3)

$$f_{xx} = 9x = 9 > 0$$

$$f_{yy} = 9y = -27 < 0$$

$$f_{xx}f_{yy} - (f_{xy})^2 = 9(-27) - 0 < 0$$

In this way, saddle point (neither minima nor maxima) will be at (1, –3)

In context of point (–1, 3)

$$f_{xx} = 9x = -9 < 0$$

$$f_{yy} = 9y = 27 > 0$$

$$f_{xx}f_{yy} - (f_{xy})^2 = (-9)\,(27) - 0 < 0$$

Saddle solution will be get at the point (1, –3)

Note In context of (–1, –3)

$$f_{xx} = 9x = -9 < 0, \quad f_{yy} = 9y = -27 < 0$$

$$f_{xx}f_{yy} - (f_{xy})^2 = (-9)(-27) - 0 > 0$$

Value of u will be maxima at point (–1, –3)

Maxima and Minima with Constraint: Lagrange's Multiplier Method: Suppose utility function and income constraint is given as - $u = f(x,y)$ $P_x\, x + P_y\, y = M$

Here $u \rightarrow$ utility, x, $y \rightarrow$ things, M $\rightarrow$ Income, $p\, x$ and $p\, y \rightarrow$ costs of things

Here the consumer wants to maximize his utility, applying Lagrange's Multiplier on the given income constraint

$$v = f(x, y) + \lambda(M - P_x \,.\, X - P_y \,.\, y)$$

10.2 Constrained Maxima and Minima with Lagrange's Multiplier

Same results will also be found from Lagrange's method as those found from the following method. On taking gratification function and Budget Line.

$$V = f(q_1 q_2) + \lambda(y - p_1 q_1 - p_2 q_2)$$

Here V is the function of V, λq_1 and q_2 and λ is a Lagrange's Multiplier. Here our purpose is to maximize V. Therefore on partial differentiating V with respect to q_1, q_2 and λ and equating it to zero –

$$\frac{\partial V}{\partial q_1} = f_1 - \lambda p_1 = 0 \qquad ...(4)$$

$$\frac{\partial V}{\partial q_2} = f_2 - \lambda p_2 = 0 \qquad ...(5)$$

$$\frac{\partial V}{\partial \lambda} = y - p_1\, q_1 - p_2\, q_2 = 0 \qquad ...(6)$$

On taking equations (4) and (5)

$$f_1 = \lambda p_1 \text{ and } f_2 = \lambda p_2$$

On dividing both

$$\frac{f_1}{f_2} = \frac{p_1}{p_2} \text{ or } \frac{f_1}{f_2} = \frac{p_1}{p_2}$$

Here same result has been achieved as got from equation A in the first method.

On Total differentiating equation

$$f_{11}dq_1 + f_{12}dq_2 - p_1\, d\lambda = \lambda dp_1$$

$$f_{21}dq_1 + f_{22}dq_2 - p_2\, d\lambda' = dp_2$$

$$-p_1 dq_1 - p_2 dq_2 = -dy + q_1 dp_1 + q_2 dp_2$$

For Second Order Conditions, Bordered Hession determinant of Second Order should be there. Therefore

$$\begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix} > 0 \qquad ...(7)$$

On expanding equation (7)

Note

$f_{11}\,(0.f_{22} - p_2^2) - f_{12}(0.f_{21} - p_1p_2) - p_1\{f_{21}(-p_2) - f_{22}\,(-p_1)\} > 0$

Or $\quad f_{11}p_2^2 - f_{12}p_1p_2 + f_{21}p_1p_2\,(-f_{22})\,(-p_1^2) > 0$

Or $\quad f_{11}p_2^2 - 2f_{12}p_1p_2 + f_{22}\,p_1^2 > 0$

Here same result has been achieved as got from equation in the first method.

Notes We can say that same results are received from both methods of Utility Maximization.

Example: Discuss mathematically the theory of consumer's behavior with the help of indifference curve technique.

Solution*:* Income of a consumer remains constant and a judgmatic consumer wants to attain maximum satisfaction from that income. That point at which consumer gets maximum satisfaction is called the Consumer Balance**.**

Consumer balance is in the case when, Budget line or Price line touches the indifference curve. In other words, Consumer is in the condition of balance when slope of Budget line and the slope of indifference curve are equal. Mathematically if Marginal rate of substitution (slope of indifference curve) and the value ratio of both things (slope of Budget line) are same then consumer remains in the balanced condition.

In the indifference curve technique slope of indifference curve is defined in terms of the ratio of marginal utilities and the slope of Budget line is defined in terms of ratio of prices. That is,

Slope of indifference curve = ratio of marginal utilities

$$= \frac{MU_1}{MU_2}$$

And slope of Budget line = ratio of prices

$$= \frac{p_1}{p_2}$$

Indifference Curve predication clarifies that similar property is received at every point of the curve. That is

$$u = f(q_1 \cdot q_2)$$

On total differentiating and equating to zero

$$du = f_1dq_1 + f_2dq_2 = 0$$

(du = 0 means that Marginal satisfaction is always zero) that is there is no difference in the utility.

Or $\quad f_1dq_1 = -f_2dq_2$

Or $\quad -\dfrac{dq_2}{dq_1} = -\dfrac{f_1}{f_2}$

$-\dfrac{dq_2}{dq_1}$ represents the slope of indifference curve that is called the Marginal rate of Substitution (MRS). Marginal rate of Substitution is that quantity of thing Q_2 which the consumer remains ready to sacrifice in order to achieve extra unit of Q_1. Negative sign of show that the slope of $-\dfrac{dq_2}{dq_1}$ indifference curve is downwards.

Note

Therefore f_1 and f_2 in the form of Cardinal is first partial differential or Marginal gratification. That is

$$f_1 = \frac{\partial U}{\partial q_1} = MU_1$$

$$f_2 = \frac{\partial U}{\partial q_2} = MU_2$$

And $$\frac{f_1}{f_2} = \frac{MU_1}{MU_2}$$

Therefore $$MRSq_1q_2 = \frac{MU_1}{MU_2} = \frac{\text{Marginal Utility of } Q_1}{\text{Marginal Utility of } Q_2}$$

Budget line can be defined by the following equation-

$$y = p_1q_1 + p_2q_2$$

Here y = income, p_1 = q_1 –value of thing and p_2 = q_2– value of thing.

On differentiating the above equation

$$dy = p_1dq_1 + p_2dq_2$$

dy = 0 since the income of consumer remains constant, there is no change in it. Therefore

$$p_1dq_1 + p_2dq_2 = 0$$

Or $$p_1dq_1 = -p_2dq_2$$

Or $$-\frac{dq_2}{dq_1} = \frac{p_1}{p_2}$$

Or $$MRSq_1q_2 = \frac{p_1}{p_2}$$

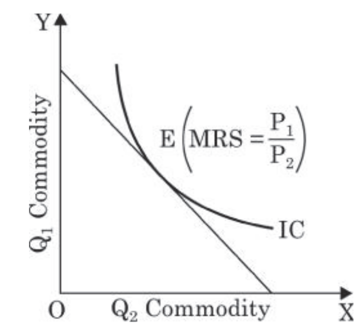


Diagram 10.1: Consumer Balance

That is slope of indifference curve = slope of budget line consumer balance predication can be represented by the following-

In the diagram 10.1 IC is a Indifference curve and Budget line is given that touches each other at point E. Tha is slope of both are equal at point E. $MRSq_1q_2 = \frac{p_1}{p_2}$ will be received at this point where consumer will get maximum satisfaction.

Self Assessment

1. Fill in the blanks:

1. Income of a consumer remains ....
2. That point at which consumer gets maximum satisfaction is called .... Balance.
3. Consumer balance is in the case when, Budget line or Price line touches the .... curve.
4. In the indifference curve technique slope of indifference curve is defined in terms of the ratio of .... utilities.
5. slope of indifference curve = ratio of marginal utilities ....

10.3 Slutsky Equation

Note

If there is a change in price of anything then what will be the changes in the consumption of a consumer? Following effects are produced due to price effect – 1. Income Effect and 2. Substitution Effect.

If there is a decrease in the price of anything then the real income of a consumer increases. And contrast to this if there is an increase in the price of anything then the real income of a consumer decreases. In this way real income of the consumer changes with the price and so the consumption of a consumer will also change. This type of situation is called Income Effect.

If there is a price increase relative of a thing then the consumer uses more the substituted thing. And contrast to that if there is a price decrease relative of a thing then the consumer lessens the use of substituted thing. In this way there is a change in prices and income remains constant, and then more or less quantity of thing is used. This type of effect is called Substitution Effect.

Slutsky Equation is used to study these types of effects -

Utility Equation

$$U = f(q_1 \cdot q_2)$$

And Budget Constraint

$$y = p_1 q_1 + p_2 q_2$$

By using Lagrang's Multiplier undetermined equation –

$$Y = f(q_1, q_2) + \lambda(y - p_1 q_1 - p_2 q_2)$$

On differentiating equation with respect to q_1, q_2 and λ and equating to zero,

$$\frac{\partial V}{\partial q_1} = f_1 - \lambda p_1 = 0$$

$$\frac{\partial V}{\partial q_2} = f_2 - \lambda p_2 = 0 \qquad \text{...(8)}$$

$$\frac{\partial V}{\partial \lambda} = y - p_1 q_1 - p_2 q_2 = 0$$

Now to calculate the price of consumer, income and substitution effect, differentiating complete equation –

$$f_{11}\, dq_1 + f_{12}\, dq_2 - p_1\, d\lambda = \lambda\, dp_1$$

$$f_{21}\, dq_1 + f_{22}\, dq_2 - p_2\, d\lambda = \lambda\, dp_2$$

$$-p_1\, dp_1 - p_2\, dq_2 = -dy + q_1\, dp_1 + q_2\, dp_2 \qquad \text{...(9)}$$

We use Cramer's rule to solve this equation. According to this rule if $AB = C$ where A, B and C are matrix then-

$$B = A^{-1}C \text{ Where, } A^{-1} = \frac{adjA}{|A|}$$

Presenting equation (9) in the matrix form

$$\begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & p_2 & 0 \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1 dp_1 + q_2 dp_2 \end{bmatrix} \qquad \text{...(10)}$$

Note Using the cramer's rule

$$\begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1 dp_1 + q_2 dp_2 \end{bmatrix}$$

Here, $$\left[A^{-1} = \frac{[D_{ij}]}{|D|} \right]$$

Therefore $$dq_1 = \frac{D_{11}\lambda dp_1 + D_{21}\lambda dp_2 + D_{31}(-dy + q_1 dp_1 + q_1 dp_2)}{|D|} \quad ...(11)$$

$$dq_2 = \frac{D_{12}\lambda dp_1 + D_{22}\lambda dp_2 + D_{32}(-dy + q_1 dp_2 + q_2 dp_2)}{|D|} \quad ...(12)$$

and $$d\lambda = \frac{D_{13}\lambda dp_1 + D_{23}\lambda dp_2 + D_{33}(-dy + q_1 dp_1 + q_2 dp_2)}{|D|} \quad ...(13)$$

dividing both sides of equation (11) by dp_1 and on considering p_2 and y constant dy = and $dp_2 = 0$

$$\frac{\partial q_1}{\partial p_1} = \frac{D_{11}\lambda}{|D|} + q_1 \frac{D_{31}}{|D|} \quad ...(14)$$

Here, $\frac{\partial q_1}{\partial p_1}$ = price effect since there is a change in q_1 because of the change in p_2 as p_2 and y remains constant.

In the same way $$\frac{\partial q_1}{\partial p_2} = \frac{D_2\lambda}{|D|} + p_2 \frac{D_{32}}{|D|} \quad ...(15)$$

dividing both sides of equation (11) by dy and considering p_1 and p_2 constant (that is)

Therefore $$\frac{\partial q_1}{\partial y} = -\frac{D_{33}}{|D|} \quad ...(16)$$

Here $\frac{\partial q_1}{\partial p_1}$, shows income effect since there is a change in q_1 because of the change in y as p_1 and p_2 remains constant. When there is any change in the price of anything then the satisfaction level of the consumer also changes. Therefore consumer balance also shifts on the other indifference curve. Suppose price change is compensated by income change and consumer remains on the same indifference curve, that is there is no change in his satisfaction, means $dU = 0$, therefore

$$U = f(q_1, q_2)$$

That means $$dU = f_1 dq_1 + f_2 dq_2 = 0$$

But we know that $\frac{f_1}{f_2} = \frac{p_1}{p_2}$, therefore

$$p_1 dq_1 + p_2 dq_2 = 0$$

Putting this value in the third part of equation (9)

$$-dy + q_1 dp_1 + q_2 dp_2 = 0$$

Therefore from equation (11)

$$\left(\frac{\partial q_1}{\partial p_1}\right) \text{(Constant)} = \frac{D_{11}\lambda}{|D|}$$

Rewriting equation (14)

$$\frac{\partial q_1}{\partial p_1} = \left(\frac{\partial q_1}{\partial q_1}\right) U \text{ Constant } q_1 \left(\frac{\partial q_1}{\partial y}\right) \text{ price = constant} \qquad \ldots (17)$$

Price effect   Substitution effect   income effect

Equation (17) is called Slutsky Equation.

Task Write the Utility Equation

10.4 Slutsky Equation in Elasticity Form

We know that demand of elasticity,

$$EP = \frac{p}{q} \cdot \frac{\Delta q}{\Delta p} = \frac{p}{q} \cdot \frac{dq}{dp}$$

Slutsky equation

$$\frac{\partial q_1}{\partial p_1} = \left(\frac{\partial q_1}{\partial p_1}\right) U = \text{Const.} - q_1\left(\frac{\partial q_1}{\partial y}\right) p = \text{Constant}$$

Multiplying both sides by p_1/q_1

$$\frac{p_1}{q_2}\frac{\partial q_1}{\partial p_1} = \frac{p_1}{q_2}\left(\frac{\partial q_1}{\partial p_1}\right) U = \text{Const.} - \frac{q_1 p_1}{q_1}\left(\frac{\partial q_1}{\partial y}\right) p = \text{Constant}$$

Or $$\eta_{1p} = \eta_1 - \frac{p_1 q_1}{y}\eta_{1y}$$

Here η_{1p} = price Elasticity, η_{1s} is substitution elasticity and η_{1y} is income elasticity.

Generalization: Predication of the above two things can be expanded for n things. Suppose n consumer consumes things. In this case his utility function will be-

$$U = f(q_1, q_2 \ldots, q_n)$$

And budget line will be

$$y = p_1 q_1 + p_2 q_2 + \ldots + p_n q_n$$

Or $$= y - \sum_{i=1}^{n} p_i q_i = 0$$

Where $i = 1, 2 \ldots, n$

Using the Lagrange's Multiplier overall Utility function will be –

$$V = f(q_1, q_2, \ldots, q_n) + \left(y - \sum_{i=1}^{n} p_i q_i\right)$$

Note

For maximum satisfaction first partial differential should be zero means-

$$\frac{\partial V}{\partial q_i} = f_i - \lambda p_i = 0 \text{ same way } \frac{\partial V}{\partial p_j} = f_i - \lambda p_j = 0$$

Or $$f_i = \lambda p_i \text{ and } f_j = \lambda p_j$$

Thus $$\frac{f_i}{f_j} = \frac{p_i}{p_j}$$

Did u know? Substitution rate of j thing for I goods is goods to their price ratio.

Bordered Hesison Determinant of Second order differentiation of maximum utilization should be negative.

Means $$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} & -p_1 \\ f_{21} & f_{22} & \cdots & f_{2n} & -p_2 \\ \vdots & \vdots & & \vdots & \vdots \\ fn_1 & fn_2 & \cdots & fnn & -p_n \\ -p_1 & -p_2 & \cdots & -p_n & 0 \end{vmatrix} > 0$$

Slutsky equation in generalization will be

$$\frac{\partial q_i}{\partial p_i} = \left(\frac{\partial q_i}{\partial p_i}\right) u = \text{const.} - q\left(\frac{\partial q_i}{\partial y}\right) p = \text{const.}$$

In Elasticity form – Multiplying Slutsky equation both sides by

$$\frac{p_i}{q_i}\left(\frac{\partial q_i}{\partial p_i}\right) = \frac{p_i}{q_i}\left(\frac{\partial q_i}{\partial p_i}\right)_{u\,=\,\text{const.}} - q_i\,\frac{p_i}{q_i}\left(\frac{\partial q_i}{\partial y}\right)_{p\,=\,\text{const.}}$$

Note $$\eta_{ip} = \eta_{is} - \frac{p_i q_i}{y}\eta_{1y}$$

Here η_{ip} = Price elasticity

η_{is} = Substitution elasticity

η_{iy} = Common elasticity

Substitute Goods and Complementary Goods: If two things are substituted then substitution effect will be negative. In contrast, If two things are complementary then substitution effect will be positive. That is,

$$\left(\frac{\partial q_2}{\partial p_1}\right) U = \text{const.} > 1, \text{ Substituted goods}$$

$$\left(\frac{\partial q_2}{\partial p_1}\right) U = \text{const.} < 1, \text{ Complementary goods}$$

$$\left(\frac{\partial q_2}{\partial p_1}\right) U = \text{const.} = 0, \text{ Independent goods}$$

Normal Goods, Inferior Goods and Giffen Goods

Note

We know that

Price effect = Substitution effect + Income effect

Income effect is negative for general things, that is

$$\left(\frac{\partial q_1}{\partial y}\right)_{\text{price = constant}} < 0$$

Because of the decrease in the prices negative income effect strengthens negative substitution effect. In contrast, Real income of consumer decreases with increase in price and substitution effect changes effectively to lessen the quantity q_1. That is,

$$\left(\frac{\partial q_1}{\partial p_1}\right) < 0, \text{ for general goods}$$

In contrast if income effect is negative income effect does not strengthens substitution effect then things will be cheap.

Those things whose income effect is more than substitution effect, that is, price of things drops with demand are called Giffen Goods.

Giffen goods and Inferior goods, in both situations income effect is negative, so if both things are equal, answer is no, since for Inferior goods substitution effect is more than income effect. That means inferior things follow the demand rule, in contrast, Giffen Goods do not follow demand rule.

Example 1: Is the $u = x^a y^b$ gratified function of two things where x and y are the quantities of these things. M is the quantity spend on things x and y then prove that demand of things –

$$x = \frac{aM}{(a+b)p_x} \text{ and } y = \frac{bM}{(a+b)p_y}$$

Where P_x and P_{yx} and y are the prices of things.

Solution: Given gratifed function is

$$U = x^a y^b$$

Separately partial differentiating with respect to 'x' and 'y'

$$MU_x = \frac{\partial U}{\partial x} = ax^{a-1}y^{b-1} \; MU_y = \frac{\partial U}{\partial y} = bx^a y^{b-1}$$

In the condition of consumer's maximum satisfaction –

$$\frac{MU_x}{p_x} = \frac{MU_y}{p_y}$$

Or $$\frac{MU_x}{MU_y} = \frac{p_x}{p_y} \quad ...(i)$$

And $$x \cdot p_x + y \cdot p_y = M \quad ...(ii)$$

Putting the value of MU_x and MU_y in equation (i)

$$\frac{ax^{a-1}b}{bx^a y^{b-1}} = \frac{p_x}{p_y} \text{ or } \frac{a}{b} \cdot \frac{y}{x} = \frac{p_x}{p_y}$$

Or $$y = \frac{p_x}{p_y}\frac{b}{a} \cdot x$$

Note

Putting the value of y in equation (ii)

$$M = x \cdot px + \frac{p_x\, b}{p_y\, a} x \cdot Py \quad \text{or } aM = (a+b)\, p_x \,.\, x$$

Or

$$x = \frac{aM}{(a+b)px} \text{ and } y = \frac{bM}{(a+b)\, py}$$

Example 2: If Utility function $u = xy$ and Budget constraint is $2x + y = 6$ then find the value of x, y and Utility.

Solution: Using the Langrange's Multiplier

$$z = xy + \lambda(2x + y - 6)$$

$$\frac{\partial z}{\partial x} = y + 2\lambda = 0$$

$$\frac{\partial z}{\partial y} = x + \lambda = 0$$

$$\frac{\partial z}{\partial \lambda} = 2x + y - 6 = 0$$

Solving all for x, y and λ, $x = 2/3$, $y = 4/3$ and $\lambda = -2/3$

Then, Maximum Utility $u = xy = \frac{2}{3}\left(\frac{4}{3}\right) = \frac{8}{9}$

It represents Equi Marginal Utility rule.

Self Assessment

2. Multiple Choice Question:

6. If there is a change in the price of anything then the change in the consumption of a consumer tells whose effect?

   (a) Price (b) Consumption (c) Consumer (d) Thing

7. If there is a decrease in the price of anything then what will be the change in the real income of a consumer?

   (a) Decrease (b) Increase (c) equal (d) None of these

8. What will be the change in the real income of a consumer if the price of goods increases?

   (a) Same (b) Increase (c) Decrease (d) None of these

9. $E_p = \ldots\ldots\ \frac{\Delta q}{\Delta p}$.

   (a) $\frac{q}{p}$ (b) $\frac{d}{q}$ (c) $\frac{q}{d}$ (d) $\frac{p}{q}$

10. Income Effect = Substitution Effect + …….

    (a) Income effect (b) Price effect (c) Consumer Effect (d) All

10.5 Summary

Note

- $\underset{\downarrow_A}{\frac{\partial^2 u}{\partial x^2}} \cdot \underset{(B)}{\frac{\partial^2 u}{\partial y^2}} < \underset{C}{\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2}$ or $AB < C^2$ or $f_{xx} \cdot f_{yy} < f_{xy}$

  For Minima,

  $$u = f(x, y), fx = 0 \quad \text{or} \quad f_y = 0$$

  Then,

  $$\partial^2 u/\partial x^2 > 0 \quad \text{and } \partial^2 u/\partial yz > 0$$

  $$\therefore \quad \partial^2 u/\partial x^2 \frac{\partial^2 u}{\partial y^2} < \left(\frac{\partial^2 u}{\partial x \cdot \partial y}\right)^2 \quad \text{or} \quad A.B < C^2$$

  or $$f_{xx} \cdot f_{yy} < fxy$$

- Consumer balance is in the case when, Budget line or Price line touches the indifference curve. In other words, Consumer is in the condition of balance when slope of Budget line and the slope of indifference curve are equal.
- In the indifference curve technique slope of indifference curve is defined in terms of the ratio of marginal utilities and the slope of Budget line is defined in terms of ratio of prices.
- Indifference Curve analysis clarifies that similar properties is received at every point of the curve.
- If there is a change in price of anything then what will be the changes in the consumption of a consumer? This is calculated by Income Effect.
- If there is a decrease in the price of anything then the real income of a consumer increases.
- If there is a price increase relative of a thing then the consumer uses more the substituted thing.
- When there is any change in the price of anything then the satisfaction level of the consumer also changes. Therefore consumer balance also shifts on the other indifference curve.
- If two things are substituted then substitution effect will be positive.
- Because of the decrease in the prices negative income effect strengthens negative substitution effect. In contrast, Real income of consumer decreases with increase in price and substitution effect changes.
- Those things whose income effect is more than substitution effect, that is, price of things drops with demand are called Giffen Goods.
- Giffen Goods and Inferior goods, in both situations income effect is negative.

10.6 Keywords

- ***Complementary:*** Supplementary
- ***Income:*** Earnings

10.7 Review Questions

1. Write Cramer's rule.
2. Explain Slutsky Equation.

Note

3. What is Lagrange's Method?
4. Prove for minima

   $f_{xx} \cdot f_{yy} < f_{xy}$.
5. Find the maxima and minima values of $u = x^3 + x^2 - xy + y^2 + 4$.

Answers: Self Assessment

1. Constant	2. Consumer	3. Indifferent
4. Marginal	5. $\frac{MU_1}{MU_2}$	6. (a)
7. (b)	8. (c)	9. (d)
10. (a)		

10.8 Further Readings

Books

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economics – Simon and Bloom – Viva Publlications.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics – Council for Economic Education.

Unit 11: Constrained Maxima and Minima

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Understand Profit Maximization.
- Understand Compensated Demand Function.

Introduction

Suppose a Firm purchase two products Q_1 and $Q_2$of quantity q_1 and q_2 at the rates p_1 and p_2, then in this condition total proceeds will be

$$R = p_1q_1 = p_2q_3 \qquad ...(1)$$

Suppose a firm manufactures two outputs θ_1 and θ_2 from an input. In this condition the cost of two products (x) of firm in the context X will be the function of q_1 and q_2, i.e.,

$$x = h(q_1, q_2) \qquad ...(2)$$

In this condition to maximize the proceeds of a farm, on taking total proceeds and functions

$$V = p_1q_1 = p_2q_2 + \mu[x - h(q_1q_2)\,\sigma] \qquad ...(3)$$

Here μ is a Lagrange Coefficient. By partially differentiating eq. (1) and putting it equal to zero

$$\frac{\partial V}{\partial q_1} = p_1 - \mu h_1 = 0$$

$$\frac{\partial V}{\partial q_2} = p_2 - \mu h_2 = 0 \qquad ...(4)$$

$$\frac{\partial V}{\partial \mu} = x - h(q_1, q_2) = 0$$

On taking equations (1) and (2)

$$\frac{p_1}{p_2} = \frac{h_1}{h_2} = RPT$$

(here, $RPT = -\frac{dq_2}{dq_1}$ The conversion rate of production)

Note

Therefore $$\frac{p_1}{p_2} = \frac{\partial q_2 / \partial x}{\partial q_1 / \partial x} = RPT \qquad ...(5)$$

Equation (5) shows that the conversion rate of production is equal to the price ratio which tells that the proceeds line touches the product conversion curve.

First order conditions can be represented as –

$$\mu = \frac{p_1}{h_2} = \frac{p_2}{h_2}$$

Or $$\mu = p\,\frac{\partial q_1}{\partial x} = p\,\frac{\partial q_2}{\partial x}$$

Since RPT in the Marginal Product can be represented in the following way

$$\frac{\partial q_1}{\partial x} = \frac{1}{h_1},\ \frac{\partial q_2}{\partial x} = \frac{1}{h_2}$$

Notes For RPT second order derivative, the condition is that Bordered HessionDeterminant for maximum profit should be positive.

Therefore

$$\begin{vmatrix} -\mu h_{11} & -\mu h_{12} & -h_1 \\ -\mu h_{21} & -\mu h_{22} & -h_2 \\ -h_1 & -h_2 & 0 \end{vmatrix} > 0 \qquad ...(6)$$

On expanding equation (6)

$$\mu\,(h_{11} h_2^2 - 2h_{12} h_1 h_2 + h_{22} h_1^2) > 0 \qquad ...(7)$$

Or $$h_{11} h_2^2 - 2h_{12} h_1 h_2 + h_{22} h_1^2) > 0 \qquad ...\ (8)\ (\text{Since } \mu > 0)$$

This way Firm satisfy equations (5) and (6) then firm will definitely be able to maximize its proceeds.

11.1 Profit Maximization

Profit θ_1 and θ_2 are the function of production quantity q_1 and q_2-

$$\pi = p_1 q_1 + p_2 q_2 - rh\,(q_1 q_2) \qquad ...(8)$$

By Partial derivative of equation (8) and placing them equal to zero

$$\frac{\partial \pi}{\partial q_1} = p_1 - rh_1 = 0 \qquad ...(i)$$

$$\frac{\partial \pi}{\partial q_2} = p_2 - rh_2 = 0 \qquad ...(ii)$$

On taking equations (1) and (2) we get –

$$r = \frac{p_1}{h_1} = \frac{p_2}{h_2}$$

Or $$r = p_1\,\frac{\partial q_1}{\partial x} = p_2\,\frac{\partial q_2}{\partial x} \qquad ...(9)$$

According to Second-order conditions **Note**

$$\begin{vmatrix} -rh_{11} & -rh_{12} \\ -rh_{21} & -rh_{22} \end{vmatrix} \qquad (\text{here, } -rh_{11} < 0)$$

On expanding Determinant

$$r^2 (h_{11}h_{12} - h_{12}^2) > 0$$

Since $r > 0$, second order conditions can be represented as –

$$h_{11} > 0 \; h_{11}\, h_{22} - h_{12}^2 > 0 \qquad \text{...(10)}$$

Both shows that $h_{22} > 0$. In this condition the Marginal costs of every product increases. Conditions of equation (10) tells that Product Possibility curve roaches back from that point towards basic point where first order conditions of equation (9) gets satisfied. Firm will be in the condition of maximum profit.

Task If profits are θ_1 and θ_2 and the quantity of production are θ_1 and θ_2 then what is its function.

11.2 Compensated Demand Function

Compensated demand shows that quantity of thing that a customer purchase within the following fixed conditions (like Tax, Financial help).

If $U = q_1\, q_2$ then expression

$$Z = p_1\, q_1 + p_2 q_2 + \lambda\ (\overline{u})\ - q_1\, q_2$$

On partially differentiating and putting it equal to zero,

$$\frac{\partial Z}{\partial q_1} = p_1 - \lambda\, q_2 = 0$$

$$\frac{\partial Z}{\partial q_2} = p_2 - \lambda\, q_1 = 0$$

$$\frac{\partial Z}{\partial \lambda} = \overline{U}\ - q_1\, q_2 = 0$$

On solving for q_1 and q_2 following will be the compensated Demand –

$$q_1 = \sqrt{\left(\frac{\overline{U}.p_2}{p_1}\right)},\ q_2 = \sqrt{\left(\frac{\overline{U}.p_1}{p_2}\right)}$$

Example 1: Following are the Demand and Supply function

$$q = \frac{5}{2} - \frac{1}{2}p \ \text{ and } q = 2p - 3$$

Find the balanced price and quantity.

Solution: In the balanced condition the demand and supply of thing will be same, i.e.

Note

$$\frac{5}{2} - \frac{1}{2}p = 2p - 3 \text{ or } \frac{5}{2} + 3 = 2p + \frac{1}{2}p$$

Or $$\frac{5}{2} + 3 = \frac{5}{2}p \text{ or } \frac{5}{2}p = \frac{11}{2}$$

Or $$p = \frac{11}{5} = 2.2$$

Again $$q = 2p - 3$$

Putting the value of P

$$q = \frac{22}{5} - 3 = \frac{22 - 15}{5} = \frac{7}{5} = 1.4.$$

Example 2: If the utility function and budget constraint for two things are $U = 3x^2 y^2$ and $2x + 3y = 18$ then find the quantities sought of things X and Y.

Solution: Given Utility function is

$$U = 3x^2y^2$$

And Budget Constraint is

$$2x + 3y = 18$$

Or $$2x + 3y - 18 = 0$$

Using the Lagrange coefficient, Demand function will be –

$$V = 3x^2 y^2 + \lambda (2x + 3y - 18) \qquad ...(i)$$

Separate-Separate partial differentiation of utility function will be done in terms of 'x' 'y' and 'λ'. After that to maximize utility on putting first derivative equal to zero –

$$\frac{\partial V}{\partial x} = 6x y^2 + 2\lambda = 0 \qquad ...(ii)$$

$$\frac{\partial V}{\partial x} = 6x^2 y + 3\lambda = 0 \qquad ...(iii)$$

$$\frac{\partial V}{\partial x} = 2x + 3y - 18 = 0 \qquad ...(iv)$$

On taking equations (ii) and (iii)

$$\frac{-2\lambda}{-3\lambda} = \frac{6xy^2}{6x^2y} \text{ or } \frac{2}{3} = \frac{y}{x} \text{ or } 2x = 3y$$

Or $$x = (3/2)\, y$$

On putting the value x of in equation (iv)

$$2 \times (3/2)\, y + 3y - 18 = 0$$

Or $6\,y = 18$ or $y = 18/6 = 3$

Again putting the value of y in equation (4)

$$2x + (3 \times 3) - 18 = 0$$

Or $2x = 9$ or $x = 9/2 = 4.5$

Therefore the quantities sought of things x and y will be 4.5 and 3.

Note

Example 3: Find the Firm's expansion paths expressed in terms of its total expenditure on its inputs in the given production function is $P = 12\ log\ L + 30\ log\ K$ and the input prices are $P_L = 2$ and $P_K = 5$.

Solution: Here main purpose of Manufacturer is to maximize his production (P) whereas his conditions are following –

$$TC = 2L + 5K \quad ...(1)$$

By using the Lagrange coefficient

$$Z = 12 \log L + 30 \log K + \gamma (TC - 2L - 5K) \quad ...(2)$$

By Separate-Separate partial differentiation of Z with respect to K and λ

$$\frac{\partial Z}{\partial L} = \frac{12}{L} - 2\lambda = 0 \quad ...(i)$$

$$\frac{\partial Z}{\partial K} = \frac{30}{K} - 5\lambda = 0 \quad ...(ii)$$

$$\frac{\partial Z}{\partial \lambda} = TC - 2L - 5K = 0 \quad ...(iii)$$

On taking equations (i) and (ii)

$$\frac{12}{L} = 2\lambda \text{ or } \frac{30}{K} = 5\lambda$$

On splitting both

$$\frac{12/L}{30/K} = \frac{2}{5}$$

Or $$\frac{2/L}{5/K} = \frac{2}{5}$$

Or $$\frac{2}{L} = \frac{2}{K}$$

Or $$L = K$$

On putting the value of L in equation (1)

$$TC = 2K + 5K = 7K$$

Or $$K = \frac{TC}{7}, \text{ in the same way } L = \frac{TC}{7}$$

$$\lambda = \frac{12}{2L} = \frac{6}{TC/7} = \frac{42}{TC}$$

(A) The parameter v of this function is influenced by the scale of operation and technical changes. Both of these influence the rate of return of the parameter but they cannot be differentiated themselves.

(B) Here we assume that replacement rate is influenced by technical changes and the ratio of instruments do not affect it but with behavioural studies it has been found that even ratio of instruments also influence it. In this way this function leaves an important fact.

(C) Lastly, the parameter δ of this function cannot be calculated. Apart from this, there is a great difficulty to verify the figures in this function.

Note

Example 4: The production function is $P = AL^{\alpha} C^{\beta}$ where A, α and β are constant. Show that if the factors L, C are increased in the same proportion, the product increases in greater $\alpha + \beta$, equal or less proportion, according as is $\alpha + \beta$ greater than, equal to, or less than unity.

Solution: Production function is

$$P = AL^{\alpha} C^{\beta}$$

Consider if there is an increase in the instrument L and C then

$$P' = A\,(\psi L)^{\alpha}\,(\psi\, C)^{\beta} \qquad \text{(here } \psi > 0)$$

$$= \psi^{\alpha + \beta} AL^{\alpha} C^{\beta}$$

$$= \psi^{\alpha + \beta} P \qquad (\therefore\ P = AL^{\alpha} C^{\beta})$$

If $\alpha + \beta = 1$ then

$$P' = \psi\, P$$

Therefore production will increase in the same ratio as sources will be increased. In this condition production will be according to the rule of constant by product of scale.

If $\alpha + \beta < 12\text{-}3$ i.e. $\alpha + \beta = \frac{1}{2}$ suppose then

$$P' = \psi^{1/2} P$$

In this condition production will increase lesser as sources will be increased. This condition will be according to the rule of decreasing by product of scale.

If, $\alpha + \beta > 1$ i.e. suppose then $\alpha + \beta = 2$ then

$$P' = \psi^{2} P$$

Did u know? Production will be more than the increase in the sources and production will be operative within the rule of increasing by product of scale.

$$L = \frac{TC}{7},\ K = \frac{TC}{7} \text{ and } \lambda = \frac{42}{TC}$$

These all will form the price desirable expansion path.

Example 5: If the profit rate (price of capital) remain unaltered , the ratio of amount of capital employed per unit of labor shifts from 10 : 10 to 12 : 11, given that rise in wages is 25% . Determine the elasticity of substitution.

Solution: We know that

$$\text{Elasticity of substitution} = \sigma = \frac{\partial\left(\frac{L}{K}\right) / L / K}{\frac{\partial\,(p_K / P_L)}{P_K / P_L}}$$

Here L and K represents labour and capital and P_L/P_K is the ratio of prices of source.

It is given that

Initially, $\frac{L}{K} = \frac{10}{10}$ and finally, $\frac{L}{K} = \frac{12}{11}$

Note

$$\text{Change in } \frac{L}{K} = \frac{12}{11} - \frac{10}{10} = \frac{12-11}{11} = \frac{1}{11}$$

$$\therefore \quad \frac{\partial\left(\frac{L}{K}\right)}{L/K} = \frac{\frac{1}{11}}{\frac{10}{10}} = \frac{1}{11}$$

Initially, $\frac{P_K}{P_L} = 1:1$ Now the wages increases to 25% then the ratio become 1.25:1 = 5/4

$$\text{Change in } \frac{P_K}{P_L} = \partial\left(\frac{P_K}{P_L}\right) = \frac{5}{4} - \frac{1}{1} = \frac{5-4}{4} = \frac{1}{4}$$

$$\frac{\partial\left(\frac{P_K}{P_L}\right)}{\frac{P_K}{P_L}} = \frac{+\frac{1}{4}}{\frac{1}{1}} = +\frac{1}{4}$$

$$\sigma = \frac{1/11}{1/4} = \frac{4}{11} = 0.36$$

Self Assessment

1. Fill in the blanks:

1. x = ... $(q_1 q_2)$
2. $V = p_1 q_1 = p_2\, q_2 + \mu\, [x - h\, (.........)\sigma]$
3. π = ... $+ p_2\, q_2 - rh\, (q_1 q_2)$
4. Elasticity of Substitution = $\sigma = \dfrac{\partial\left(\frac{L}{K}\right)/L/K}{.........}$

Solve the following Numericals:

1. It is known

   Utility Function, $U = q_1 q_2$

   Price of first thing, P_1 = ₹ 4

   Price of Second thing, p_2 = ₹ 10

   Income of customer, Y = ₹100

   Find the balanced level of consumption of things Q_1 and Q_2 and verify the condition of Maximization.

   Ans.: $\left[q_1 = \frac{25}{2}, q_2 = 5\right]$

2. If Utility function is $U = 4q_1\, q_2 + 3q_1$ and income constraint is $60 = 2q_2 + 6q_1$ then for maximum utilization find the quantities of Q_1, Q_2

   Ans.: $[q_1 = 16.125, q_2 = -4.625]$

3. If desired property function is $U = (x_1, x_2) = 2 \log x_2$ then what will be the demand curve for first thing? If total income is W and the prices of two things are p_1 and p_2.

   Ans.: $\left[x_1 = \frac{2W}{p_1}, x_2 = \frac{W}{3p_2}\right]$

Note

11.3 Summary

- Suppose a Firm purchase two products Q_1 and Q_2 of quantity q_1 and q_2 at the rates p_1 and p_2, then in this condition total proceeds will be

  $R = p_1q_1 = p_2q_2$
- Suppose a firm manufactures two outputs θ_1 and θ_2 from an input. In this condition the cost of two products (x) of firm in the context X will be the function of q_1 and q_2, i.e., $x = h\ (q_1, q_2)$
- Profit θ_1 and θ_2 are the function of production quantity q_1 and q_2-

  $\pi = p_1q_1 + p_2q_2 - rh\ (q_1q_2)$
- Compensated demand shows that quantity of thing that a customer purchases within the following fixed conditions (like Tax, Financial help).

11.4 Keywords

- ***Operation:*** Process
- ***Condition:*** Bond

11.5 Review Questions

1. Describe Profit Maximization.
2. What is Compensated Demand Function? Describe.
3. If the Utility function of two things x and Y is $U = 3x^2y^2$ and Budget constraint is $2x + 3y = 18$ then find the quantities sought of things X and Y?

   Ans.: $x = 4.5$, $y = 3$)

Answer: Self Assessment

1. h
2. q_1q_2
3. p_1q_1
4. $\frac{\partial(P_k / P_L)}{P_K / P_L}$

11.6 Further Readings

Books

Mathematical Economics – Michael Harrison, PatricWalderen.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economics – Simon and Bloom – Viva Publlications.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics and Finance – Martin Norman.

Unit 12: Integration : Basic Rules of Integration

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Solve Comprehensive Integrals.
- Understand Standard Integral.
- Solve Integration of x^n relative to x, where $n \neq 1$.
- Find out the Integration of Multiplication of a Constant and a Function.
- Do the Integration of the Sum and Subtract of the Function.

Introduction

Integration of the Function— The Inverse process of finding the differentiation of a function is called an Integration. In differential Maths we find the differential coefficient of a function. But in Integral Mathematics we have to find those functions whose differential coffecient is the given function.

For example,the differential coefficient of sin x with respect to x, is cos x, then on integrating function cos x with respect to x integral will be sin x.

Suppose $f(x)$ is some function of x, whose differential coefficient is $f'(x)$, i.e.

$$\frac{d}{dx}\{f(x)\} = f'(x)$$

Then we say that integral of f′(x) is, f (x).

It can be written in the symbol as :

$$\int f'(x)\, dx = f(x).$$

Sign "$\int$" is the symbol of integration which is called the sign of integration. It represents the integration of a function. X in dx shows that integration is done with respect to variable x. If integration has to be done relative to some other variable, then that variable is kept in place of x.

Note

Signal $\int$ is the distorted form of English letter S. Basically integration is a special method of addition and the signal $\int$ is formed from the first letter of English word SUM.

> *Notes* The process of integration of any function is called integration.The function to be integrated is called **integrand** and the function obtained after integration is called **integral**.

In $\int f'(x)\,dx = f(x)$ is $f'(x)$ integrand and $f(x)$ is integral

From the above interpretation it is clear that

$$\int \cos x\,dx = \sin x, \qquad \because \quad \frac{d}{dx}\sin x = \cos x$$

$$\int \sin x\,dx = -\cos x, \qquad \because \quad \frac{d}{dx}(-\cos x) = \sin x$$

$$\int e^x dx = e^x, \qquad \because \quad \frac{d}{dx}e^x = e^x$$

And $$\int \frac{1}{t}dt = \log t, \qquad \because \quad \frac{d}{dt}(\log t) = \frac{1}{t}$$

In the above functions $\cos x$, $\sin x$, e^x and $\frac{1}{t}$ are integrands and $\sin x$, $-\cos x$, e^x and $\log t$ are integrals.

If on any, function both integration and differentiation would be done then that function remains unaltered.

For example, $\frac{d}{dx}\left[\int \cos x\,dx\right] = \frac{d}{dx}\sin x = \cos x$

i.e. $\frac{d}{dx}\left[\int f'(x)\,dx\right] = \frac{d}{dx}f(x) = f'(x)$.

It is clear from this example that integration and differentiation are opposite activities.

12.1 Comprehensive Integrals

From Differential Mathematics we know that, if the differential coefficient of $f(x)$ is $F(x)$, then the differential coefficient of f $(x) + c$ will also $F(x)$, where there is any arbitrary constant c.

i.e., if $$\frac{d}{dx}f(x) = F(x)$$

then $$\frac{d}{dx}\{f(x)+c\} = \frac{d}{dx}f(x) + \frac{d}{dx}(c) = \frac{d}{dx}f(x) + 0 = F(x)$$

$\therefore$ $$\int F(x)\,dx = f(x) + c,$$

Where c is an arbitrary constant which can have indefinite values.

Therefore the integral of any function is not unique, since many integrals of F (x) were obtained on giving different values to c, therefore if $f(x)$ is any integral of $F(x)$, then $f(x) + c$ will be its Comprehensive Integral.

Note

Did u know? Arbitrary Constant c is called **Constant of Integration.**

12.2 Standard Integral

On the basis of Differential Coefficients of standard functions we will find the differentials of some functions, which are the following:

Integration	Differentiation
1- $\int x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$	$\because \frac{1}{n+1} \cdot \frac{d}{dx}(x^{n+1}) = x^n$
2- $\int \frac{1}{x} dx = \log\|x\| + c$	$\because \frac{d}{dx}(\log\|x\|) = \frac{1}{x}$
3- $\int e^x = e^x + c$	$\because \frac{d}{dx}(e^x) = e^x$
4- $\int a^x dx = \frac{a^x}{\log_e a} + c$	$\because \frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x$
5- $\int \sin x\, dx = -\cos x + c$	$\because \frac{d}{dx}(-\cos x) = \sin x$
6- $\int \cos x\, dx = \sin x + c$	$\because \frac{d}{dx}(\sin x) = \cos x$
7- $\int \sec^2 x\, dx = \tan x + c$	$\because \frac{d}{dx}(\tan x) = \sec^2 x$
8- $\int \text{cosec}^2 x\, dx = -\cot x + c$	$\because \frac{d}{dx}(-\cot x) = \text{cosec}^2 x$
9- $\int \sec x \tan x\, dx = \sec x + c$	$\because \frac{d}{dx}(\sec x) = \sec x \tan x$
10- $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$	$\because \frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x$
11- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$	$\because \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
12- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$	$\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
13- $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$	$\because \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ -

Note

12.3 Integration of x^n relative to X, where $n \neq -1$

$\because$ $$\frac{d}{dx}x^{n+1} = (n+1)x^n$$

Or $$\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n$$

$\therefore$ $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \text{, Where } n \neq -1.$$

Example 1: $\int dx = \int 1\,dx = \int x^0 dx = \dfrac{x^{0+1}}{0+1} + c = x + c$.

Example 2: $\int x^5 dx = \dfrac{x^{5+1}}{5+1} + c = \dfrac{x^6}{6} + c$.

Example 3: $\int x^{-10}\,dx = \dfrac{x^{-10+1}}{-10+1} + c = \dfrac{x^{-9}}{-9} + c = -\dfrac{1}{9}x^{-9} + c$.

Example 4: $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \dfrac{x^{1/2+1}}{\frac{1}{2}+1} + c = \dfrac{2}{3}x^{3/2} + c$.

Example 5: $\int x^{2/3} dx = \dfrac{x^{2/3+1}}{\frac{2}{3}+1} + c = \dfrac{3}{5}x^{5/3} + c$.

Self Assessment

1. Fill in the blanks:

1. Opposite process of finding the differential of any function is called ....
2. The function to be integrated is called ....
3. $\int a^x\,dx$ = ... + c
4. $\int$ .... dx .... = – cos x + c
5. $\dfrac{d}{dx}x^{n+1}$ = (.........) x^n

Questionnaire 12.1

Very short answer questions.

Find the values of the following Integrals:

1. (a) $\int x\,dx$ (b) $\int x^4\,dx$

Note

(c) $\int x^a \, dx$ (d) $\int t^3 \, dt$

2. (a) $\int x^{-7} \, dx$ (b) $\int x^{-1/2} \, dx$

(c) $\int x^{-3/2} \, dx$ (d) $\int z^{-1/3} \, dz$

3. (a) $\int \frac{dx}{x^2}$ (b) $\int \frac{dt}{t}$

(c) $\int \frac{dx}{\sqrt{x}}$ (d) $\int \frac{dx}{x\sqrt{x}}$

4. (a) $\int 2^x \, dx$ (b) $\int 3^x \, dx$

(c) $\int b^x dx$ (d) $\int b^{x+a} dx$

5. (a) $\int \frac{dt}{\sqrt{1-t^2}}$ (b) $\int \frac{dz}{z\sqrt{z^2-1}}$

(c) $\int \frac{dy}{1+y^2}$ (d) $\int e^x dx$

6. (a) $\int \sec^2 t \, dt$ (b) $\int \text{cosec}^2 z \, dz$

(c) $\int \sqrt{1-\cos^2 x \, dx}$ (d) $\int (1+\cot^2 x) \, dx$

7. (a) $\int \frac{dx}{\text{cosec}\, x}$ (b) $\int \frac{dx}{\cos x \cot x}$

(c) $\int \frac{dx}{5^{-x}}$ (d) $\int \frac{dx}{\sec x}$.

8. Prove that $\int dx = x,$ when $x = 0$.

Answers

1. (a) $\frac{x^2}{2}+c$ (b) $\frac{x^5}{5}+c$ (c) $\frac{x^{a+1}}{a+1}+c$ (d) $\frac{t^4}{4}+c$

2. (a) $-\frac{1}{6x^6}+c$ (b) $2x^{1/2}+c$ (c) $x^{-1/2}+c$ (d) $\frac{3}{2}z^{2/3}+c$

3. (a) $\frac{-1}{x}+c$ (b) $\log 1 + 1 + c$ (c) $2x^{\frac{1}{2}}+c$ (d) $2x^{1/2}+c$

Note

4.	(*a*) $\frac{2^x}{\log e^2} + c$	(*b*) $\frac{3^x}{\log e^3} + c$	(*c*) $\frac{b^x}{\log e^b} + c$	(*d*) $\frac{b^{x+a}}{\log e^b} + c$
5.	(*a*) $\sin^{-1} t + c$	(*b*) $\sec^{-1} z + 1\, c$	(*c*) $\tan^{-1} x + c$	(*d*) $e^x + c$
6.	(*a*) $\tan t + c$	(*b*) $-\cot z + c$	(*c*) $-\cos x + c$	(*d*) $-\cot x + c$
7.	(*a*) $-\cos x + c$	(*b*) $\sec x + c$	(*c*) $\frac{5^x}{\log_e 5} + c$	(*d*) $\sin x + c$

12.4 Integration of the Multiplication of a Constant and a Function

If $$\int f'(x)\, dx = f(x) \text{ if } \frac{d}{dx} f(x) = f'(x)$$

$\because$ $$\frac{d}{dx}\{a \,.\, f(x)\} = a\frac{d}{dx} f(x)$$

$$= af'(x)$$

$\therefore$ $$\int af'(x)\, dx = a \,.\, f(x) = a\int f'(x)\, dx$$

$$\int af'(x)\, dx = a\int f'(x)\, dx$$

That is Integration of the Multiplication of a Constant and a Function is equal to the multiplication of the constant and integration of that function.

EXAMPLES WITH SOLUTION

Example 1: Integrate function $15x^4$ with respect to x.

Solution: $$\int 15x^4\, dx = 15\int x^4\, dx$$

$$= \frac{15}{5}x^5 + c = 3x^5 + c.$$ **Ans.**

Example 2: Find the value of $\int 7\frac{dx}{x}$.

Solution: $$\int 7\frac{dx}{x} = 7\int \frac{dx}{x} = 7\log|x| + c$$ **Ans.**

Example 3: Find the value of $\int 6 \sin x\, dx$.

Solution: $$\int 6\sin x\, dx = 6\int \sin x\, dx$$

$$= 6(-\cos x) + c = -6\cos x + c.$$ **Ans.**

Example 4: Find the value of $\int 3e^x dx$.

Solution: $$\int 3e^x dx = 3\int e^x dx = 3e^x + c$$ **Ans.**

Example 5: Integrate function e^{x+a} with respect to x.

Solution:
$$\int e^{x+a} dx = \int e^x . e^a dx$$
$$= e^a \int e^x dx = e^a . e^x + c$$
$$= e^{x+a} + c.$$ **Ans.**

> *Task* Integrate the function $20x^4$ with respect to x. (**Ans.:** $4x^5 + c$)

12.5 Integration of the Sum and Subtract of the Function

If $$\int f_1'(x)\,dx = f_1(x) \text{ and } \frac{d}{dx}f_1(x) = f_1'(x)$$

And $$\int f_2'(x)\,dx = f_2(x) \text{ and } \frac{d}{dx}f_2(x) = f_2'(x)$$

From the differential method of addition and subtraction of functions

$$\boxed{\frac{d}{dx}\{f_1(x) \pm f_2(x)\} = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x)}$$
$$= f_1'(x) \pm f_2'(x)$$

$\therefore$ $$\int \{f_1'(x) \pm f_2'(x)\}\,dx = f_1(x) \pm f_2(x) = \int f_1'(x)\,dx \pm \int f_2'(x)\,dx$$

Therefore $$\boxed{\int \{f_1'(x) \pm f_2'(x)\}\,dx = \int f_1'(x)dx \pm \int f_2'(x)\,dx}$$

This method in the same way is true for two or more functions.

> *Did u know?* Integration of addition or subtraction of any functions is equal to the addition or subtraction of the integration of those functions.

Therefore $$\int [f_1'(x) \pm f_2'(x) \pm f_3'(x) \pm ....]\,dx$$
$$= \int f_1'(x)\,dx \pm \int f_2'(x)\,dx \pm \int f_3'(x)\,dx \pm .....$$

Note

EXAMPLES WITH SOLUTION

Example 1: Integrate the function $x^6 + \frac{1}{x} - e^x + 1$ with respect to x.

Solution: $\int \left(x^6 + \frac{1}{x} - e^x + 1\right) dx = \int x^6 dx + \int \frac{1}{x} dx - \int e^x dx + \int dx$

$$= \frac{1}{7}x^7 + \log|x| - e^x + x + c$$ **Ans.**

Example 2: Find the value of $\int \frac{1}{\sin^2 x \cos^2 x} dx$.

Solution: $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x . \cos^2 x} dx$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x \; dx + \int \text{cosec}^2 x \, dx$$

$= \tan x - \cot x + c$. **Ans.**

Example 3: Find the value of $\int \left(3e^x - \frac{1}{5x} + \sec x \tan x\right) dx$.

Solution: $\int \left(3e^x - \frac{1}{5x} + \sec x \tan x\right) dx$

$$= 3\int e^x dx - \frac{1}{5}\int \frac{1}{x} dx + \int \sec x \tan x \, dx$$

$$= 3e^x - \frac{1}{5}\log|x| + \sec x + c.$$ **Ans.**

Example 4: Find the value of $\int (ax^2 + bx + c)\, dx$.

Solution: $\int (ax^2 + bx + c)\, dx = a\int x^2 \, dx + b\int x \, dx + c\int 1 \, dx$

$$= a\left(\frac{1}{3}x^3\right) + b\left(\frac{1}{2}x^2\right) + cx + d$$

$$= \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx + d,$$

where d is integration constant. **Ans.**

Example 5: Find the value of $\int \left(x - \frac{1}{x}\right)^2 dx$ **.**

Solution: $\int \left(x - \frac{1}{x}\right)^2 dx = \int \left(x^2 - 2 + \frac{1}{x^2}\right) dx$

$$= \int x^2\,dx - 2\int 1\,dx + \int x^{-2}\,dx$$

$$= \frac{x^3}{3} - 2x + \left(\frac{x^{-1}}{-1}\right) + c = \frac{x^3}{3} - 2x - \frac{1}{x} + c.$$ **Ans.**

Example 6: Find the value of $\int (5x-4)^3\,dx$ **.**

Solution: $\int (5x-4)^3 dx = \int (125x^3 - 300x^2 + 240x - 64)\,dx$

$$= 125\int x^3 dx - 300\int x^2 dx + 240\int x\,dx - 64\int 1\,dx$$

$$= 125\left(\frac{1}{4}x^4\right) - 300\left(\frac{1}{3}x^3\right) + 240\left(\frac{1}{2}x^2\right) - 64x + c$$

$$= \frac{125}{4}x^4 - 100x^3 + 120x^2 - 64x + c\,.$$ **Ans.**

Example 7: Integrate $\sin^2 \frac{x}{2}$ **with respect to** x**.**

Solution: $\int \sin^2 \frac{x}{2} dx = \int \frac{1-\cos x}{2} dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos x\,dx$

$$= \frac{1}{2}x - \frac{1}{2}\sin x + c = \frac{1}{2}(x - \sin x) + c\,.$$ **Ans.**

Example 8: Find the value of $\int (\cos^4 x - \sin^4 x)\,dx$ **.**

Solution: $\int (\cos^4 x - \sin^4 x)\,dx = \int (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)\,dx$

$$= \int 1 \,.\, \cos 2x\,dx$$

$$= \int \cos 2x$$

$$= \frac{\sin 2x}{2} + c.$$ **Ans.**

Note

Example 9: Find the value of $\int \frac{x^4}{x^2+1} dx$.

Solution:

$$\int \frac{x^4}{x^2+1} dx = \int \left(x^2 - 1 + \frac{1}{x^2+1}\right) dx$$

$$= \frac{x^3}{3} - x + \tan^{-1} x + c$$ **Ans.**

Example 10: Find the value of $\int \tan^2 x \, dx$.

Solution:

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$$

$$= \tan x - x + c.$$ **Ans.**

Example 11: Find the value of $\int \left(\frac{5x+7}{x} + e^x\right) dx$.

Solution:

$$\int \left(\frac{5x+7}{x} + e^x\right) dx = \int \left(5 + \frac{7}{x} + e^x\right) dx$$

$$= 5x + 7 \log |x| + e^x + c.$$ **Ans.**

Example 12: Find the value of $\int \left(\frac{ax^4 + bx^2 + c}{x^4}\right) dx$.

Solution:

$$\int \left(\frac{ax^4 + bx^2 + c}{x^4}\right) dx = \int \left(a + \frac{b}{x^2} + \frac{c}{x^4}\right) dx$$

$$= \int (a + bx^{-2} + cx^{-4}) \, dx$$

$$= \int a \, dx + b \int x^{-2} \, dx + c \int x^{-4} dx$$

$$= ax + b \frac{x^{-1}}{-1} + c \, . \frac{x^{-3}}{-3} \, . + d = ax - \frac{b}{x} - \frac{c}{3x^3} + d$$ **Ans.**

Example 13: Find the value of $\int \frac{x^2}{x^2+1} dx$.

Solution:

$$\int \frac{x^2}{x^2+1} dx = \int \frac{(x^2+1) - 1}{x^2+1} dx$$

$$= \int \frac{x^2+1}{x^2+1} dx - \int \frac{1}{x^2+1} dx$$

Note

$$= \int dx - \int \frac{dx}{x^2 + 1}$$

$$= x - \tan^{-1} x + c.$$

Ans.

Example 14: Find the value of $\int \frac{1}{1 + \cos x} dx$.

Solution: $\int \frac{1}{1 + \cos x} dx = \int \frac{1 - \cos x}{1 - \cos^2 x} dx$, multiplying both numerator and denominator by $1 - \cos x$

$$= \int \frac{1 - \cos x}{\sin^2 x} dx = \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx$$

$$= \int (\text{cosec}^2 x - \cot x \,.\, \text{cosec } x)\, dx$$

$$= -\cot x + \text{cosec } x + c.$$

Ans.

Example 15: Find the integration of function $\frac{\sec x + \tan x}{\sec x - \tan x}$ **with respect to** x.

Solution:

$$\int \frac{\sec x + \tan x}{\sec x - \tan x} dx = \int \frac{(\sec x + \tan x)(\sec x + \tan x)}{(\sec x - \tan x)(\sec x + \tan x)} dx$$

$$= \int \frac{(\sec x + \tan x)^2}{\sec^2 x - \tan^2 x} dx$$

$$= \int (\sec^2 x + \tan^2 x + 2 \sec x \tan x)\, dx$$

$$= \int (2 \sec^2 x - 1 + 2 \sec x \tan x)\, dx$$

$$= \int 2 \sec^2 x\, dx - \int dx + \int 2 \sec x \tan x\, dx$$

$$= 2\int \sec^2 x\, dx - \int dx + 2\int \sec x \tan x\, dx$$

$$= 2 \tan x - x + 2 \sec x + c.$$

Ans.

Example 16: Find the integration of function $\frac{(x + 2)(4x^2 - 5)}{x}$ **with respect to** x.

Solution:

$$\int \frac{(x + 2)(4x^2 - 5)}{x} dx = \int \frac{(4x^3 + 8x^2 - 5x - 10)}{x} dx$$

$$= 4\int x^2 dx + 8\int x\, dx - 5\int dx - 10\int \frac{1}{x} dx$$

Note

$$= \frac{4}{3}x^3 + \frac{8}{2}x^2 - 5x - 10\log|x| + c$$

$$= \frac{4}{3}x^3 + 4x^2 - 5x - 10\log|x| + c\,.$$ **Ans.**

Questionnaire 12.2

Find the values of the following:

1. $\int (e^x + 2\sin x - 3\cos x)dx$.

2. $\int (x^2 - \cos x + \sec x \tan x)\,dx$.

3. $\int (7x^6 - 8x^3 + 5)\,dx$.

4. $\int (x+2)(2x+6)\,dx$.

5. $\int \frac{ax^3 + bx + c}{x^2}\,dx$.

6, $\int \frac{(1+x)^3}{\sqrt{x}}\,dx$.

7. $\int \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)dx$.

8. $\int \left(\frac{3x^2 + 4x + 5}{\sqrt{x}}\right)dx$.

9. $\int \frac{x}{a+x}\,dx$.

10. $\int \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)dx$.

11. $\int (1-x^2)\sqrt{x}\,dx$.

12. $\int \frac{e^{\log x}}{x}\,dx$.

13. $\int \left(\cos x - \frac{5}{x} + e^x\right)dx$.

14. $\int \frac{x^4+1}{x^2+1}\,dx$.

Answers

1. $e^x - 2\cos x - 3\sin x + c$
2. $\frac{x^3}{3} - \sin x + \sec x + c$
3. $x^7 - 2x^4 + 5x + c$
4. $\frac{2}{3}x^3 + 5x^2 + 12x + c$
5. $\frac{a}{2}x^2 + b\log_e|x|\frac{-c}{x} + c_1$
6. $\frac{2}{7}x^{7/2} + \frac{6x^{3/2}}{5} + 2x^{3/2} + 2\sqrt{x} + c$
7. $2x^{1/2} - 2/3\,x^{3/2} + c$
8. $2\sqrt{x}\left[\frac{3}{5}x^5 + \frac{4}{3}x + 5\right] + c$
9. $x\,a\log_e|a+x| + c$
10. $x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots.$
11. $\frac{2}{3}x^{3/2} - \frac{2}{7}x^{7/2} + c$
12. $x + c$
13. $\sin x - 5\log|x| + e^x + c$
14. $-\frac{2}{3}\operatorname{cosec} x + \sec x + c$.

Self Assessment

Note

2. State whether the following statements are True or False:

6. Integration of addition or subtraction of any functions is equal to the addition or subtraction of their integrations.
7. Integration of multiplication of a constant and function is equal to the addition of that constant and the integration of the function.
8. $\int af'(x)\,dx = a\int f'(x)\,dx$
9. $\frac{d}{dx}\{f_1(x) \pm f_2(x)\} = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x)$

12.6 Summary

- The Inverse process of finding the differentiation of a function is called Integration. In differential Math's we find the differential coefficient of a function. But in Integral Mathematics we have to find those functions whose differential coefficient is the given function.
- Sign "$\int$" is the symbol of integration which is called the sign of integration. It represents the integration of a function. X in dx shows that integration is done with respect to variable x. If integration has to be done relative to some other variable, then that variable is kept in place of x.
- If on any function both integration and differentiation would be done then that function remains unaltered.
- Therefore the integral of any function is not unique, since many integrals of $F(x)$ were obtained on giving different values to c, therefore if $f(x)$ is any integral of $F(x)$, then $f(x) + c$ will be its Comprehensive Integral.

12.7 Keywords

- ***Inverse:*** Antipod
- ***Symbol:*** Icons

12.8 Review Questions

1. Find the integration of $\int x^5\,dx$ with respect to x where $x \neq -1$. (**Ans.:** $\frac{x^6}{6} + c$)
2. Find the value of the integral $\int e^x\,dx$ (**Ans.:** $e^x + c$)
3. Find the value of the integral $\int 6\sin x\,dx$ (**Ans.:** $-6\cos x + c$)
4. Find the value of the integral $\int x - \frac{1}{x}\,dx$ (**Ans.:** $\frac{x^3}{3} - 2x - \frac{1}{x} + c$)
5. Find the value of the integral $\int \tan^2 x\,dx$ (**Ans.:** $\tan x - x + c$)

Note

Answers: Self Assessment

1. Integration
2. Integrand
3. $\dfrac{a^x}{\log_e a}$
4. $\sin x$
5. $n + 1$.
6. True
7. False
8. True
9. True

12.9 Further Readings

Books

Mathematical Economics – Michael Harrison, Patric Walderen.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economics – Simon and Bloom – Viva Publications.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics and Finance – Martin Norman.

Unit 13: Methods of Integration

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Understand Substitution.
- Understand Some results Gained from Substitution.
- Take out the Integration of Function of x^n.
- Aware of Integrand.
- Take out the Integration of some Standard Functions by Substitution.

Introduction

Below mentioned are the two methods for knowing the integration of any function:

1. Integration by Substitution
2. Integration by Parts

13.1 Substitution

In this operation integration is done by changing the given integrand in standard formula. For this any integration function of variable amount is put equal to another variable amount and substituting in integration, integration is changed in form of function of new variable amount in such a way that use of formula may be done easily.

13.2 Some Results Gain from Substitution

(i) For the function sin ($ax + b$), find relative integration relative of x.

Solution: assume $ax + b = t$

Note

$$\frac{d}{dx}(ax+b) = dt \quad \therefore \quad a\,dx = dt$$

Or $$dx = \frac{1}{a}\,dt$$

$\therefore$ $$\int \sin(ax+b)\,dx = \int (\sin t)\frac{1}{a}\,dt = \frac{1}{a}\int \sin t\,dt$$

$$= -\frac{1}{a}\cos t + c = -\frac{1}{a}\cos(ax+b) + c$$

Notes $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

From the above given example, relative integration of x may be known for the below mentioned functions:

$$\int \cos ax\,dx = \frac{\sin ax}{a} + c$$

$$\int \sin ax\,dx = -\frac{\cos ax}{a} + c$$

$$\int \sec^2 ax\,dx = \frac{\tan ax}{a} + c$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b| + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

$$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$$

$$\int \text{cosec}\,(ax+b)\tan(ax+b) = -\frac{1}{a}\text{cosec}\,(ax+b) + c$$

$$\boxed{\int f(ax+b)\,dx = \frac{1}{a}f(ax+b) + c}$$

(ii) Find relative integration of x for the function $\frac{1}{a^2 + x^2}$

Solution: Assume, $x = at \quad \therefore \quad dx = a\,dt$

$$\int \frac{1}{a^2 + x^2}\,dx = \int \frac{1}{a^2 + a^2t^2} \cdot a\,dt$$

$$= \frac{1}{a}\int \frac{1}{1 + t^2}\,dt = \frac{1}{a}\tan^{-1} t + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Hence,

$$\boxed{\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c}$$

Similarly,

$$\boxed{\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + c}$$

$$\boxed{\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c}$$

Above mentioned results may be used in integration without any substitution.

EXAMPLES WITH SOLUTION

Very Short Answer Questions

Example 1: $\int (\sqrt{1 - 5x} + 9e^{3x})\ dx.$

Solution:

$$\int (\sqrt{1 - 5x} + 9e^{3x})\ dx = \int (1 - 5x)^{1/2}\,dx + 9\int e^{3x}\,dx$$

$$= \left(-\frac{1}{5}\right)\frac{(1 - 5x)^{3/2}}{3/2} + 9\,\frac{e^{3x}}{3} + c$$

$$= \frac{-2}{15}(1 - 5x)^{3/2} + 3e^{3x} + c.$$ **Ans.**

Example 2: Find the Value $\int \sin 4x \cos 6x\,dx.$

Solution:

$$\int \sin 4x \cos 6x\,dx = \frac{1}{2}\int (2\sin 4x \cos 6x)\ dx$$

$$= \frac{1}{2}\int \sin 10x - \sin 2x)\ dx$$

Note

$$= \frac{1}{2}\int \sin 10x\, dx - \frac{1}{2}\int \sin 2x\, dx$$

$$= -\frac{1}{2}\frac{\cos 10x}{10} + \frac{1}{2}\frac{\cos 2x}{2} + c$$

$$= -\frac{1}{20}\cos 10x + \frac{1}{4}\cos 2x + c.$$ **Ans.**

Example 3: Find the Value of $\int \sin^3\theta\, d\theta$.

Solution:

$$\int \sin^3\theta\, d\theta = \frac{1}{4}\int (3\sin\theta - \sin 3\theta)\, d\theta$$

$$= \frac{3}{4}\int \sin\theta\, d\theta - \frac{1}{4}\int \sin 3\theta\, d\theta$$

$$= \frac{3}{4}(-\cos\theta) - \frac{1}{4}\left(-\frac{1}{3}\right)\cos 3\theta + c$$

$$= -\frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta + c.$$ **Ans.**

Example 4: Find the Value of $\int \sqrt{1 - \sin x}\; dx$.

Solution: Here,

$$\sqrt{1-\sin x} = \sqrt{\sin^2\frac{1}{2}x + \cos^2\frac{1}{2}x - 2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$

$$= \sqrt{\left(\sin\frac{1}{2}x - \cos\frac{1}{2}x\right)^2} = \pm\left(\sin\frac{1}{2}x - \cos\frac{1}{2}x\right)$$

$$\therefore \quad \int \sqrt{1-\sin x}\; dx = \pm\int \left(\sin\frac{1}{2}x - \cos\frac{1}{2}x\right) dx$$

$$= \pm\left[-2\cos\frac{1}{2}x - 2\sin\frac{1}{2}x\right] + c$$

$$= \pm 2\left(\cos\frac{1}{2}x + \sin\frac{1}{2}x\right) + c.$$ **Ans.**

Example 5: Find the Value of $\int \left(7e^{2x} + \frac{5}{2x+7} + \sec^2 3x - 2\sqrt{1-5x}\right) dx$.

Solution: $\int \left(7e^{2x} + \frac{5}{2x+7} + \sec^2 3x - 2\sqrt{1-5x}\right) dx$

$$= 7\int e^{2x}\,dx + 5\int \frac{1}{2x+7}\,dx + \int \sec^2 3x\,dx - 2\int (1-5x)^{1/2}\,dx$$

$$= \frac{7}{2}e^{2x} + \frac{5}{2}\log(2x+7) + \frac{1}{3}\tan 3x - 2\,\frac{(1-5x)^{3/2}}{(-5)\left(\frac{3}{2}\right)} + c$$

$$= \frac{7}{2}e^{2x} + \frac{5}{2}\log(2x+7) + \frac{1}{3}\tan 3x + \frac{4}{15}(1-5x)^{3/2} + c. \qquad \textbf{Ans.}$$

Questionnaire 13.1

Find the value of the following integrations with respect to x:

1. (*a*) $(x+2)^3$ (*b*) $(7x-2)^3$
   (*c*) $(ax+b)^4$ (*d*) $(3+4x)^5$
2. (*a*) $\sqrt{4-5x}$ (*b*) $\sqrt{4x+3}$
   (*c*) $\sqrt{ax+b}$ (*d*) $\sqrt{2x+\frac{5}{3}}$
3. (*a*) $\frac{1}{(5x+4)^2}$ (*b*) $\frac{1}{(a+bx)^4}$ (*c*) $\frac{1}{\left(\frac{c}{2}+bx\right)^3}$
4. (*a*) $\frac{1}{3x+1}$ (*b*) $\frac{1}{a-bx}$
5. (*a*) $\frac{1}{\sqrt{25-x^2}}$ (*b*) $\frac{1}{\sqrt{1-(3x-2)^2}}$
6. (*a*) $\frac{1}{5+(2-3x)^2}$ (*b*) $\frac{1}{1+9x^2}$
7. (*a*) $\sin 3x$ (*b*) $\cos(4x+5)$
8. $\sqrt{1+\sin\frac{1}{2}x}$
9. (*a*) e^{3x+4} (*b*) $e^{x/2}$
10. $\text{COS}\left(\frac{2}{5}x-2\right) + A^{3x+2}$.

Answers

1. (*a*) $\frac{1}{4}(x+2)^4 + c$ (*b*) $\frac{1}{28}(7x-2)^4 + c$ (*c*) $\frac{1}{5a}(ax+b)^5 + c$
   (*d*) $\frac{1}{24}(3+4x)^6 + c$

Note

2. (a) $-\frac{2}{15}(4-5x)^{3/2}+c$ (b) $\frac{1}{6}(4x+3)^{3/2}+c$ (c) $\frac{2}{3a}(ax+b)^{3/2}+c$

(d) $\frac{1}{3}\left(2x+\frac{5}{3}\right)^{3/2}+c$

3. (a) $-\frac{1}{5(5x+4)}+c$ (b) $-\frac{1}{3b(a+bx)^3}+c$ (c) $-\frac{1}{2b(c/2+bx)^2}+c$

4. (a) $\frac{1}{3}\log 3x+1+c$ (b) $-\frac{1}{b}\log|9-bx|+c$

5. (a) $\sin^{-1}\frac{x}{5}+c$ (b) $\frac{1}{3}\sin^{-1}(3x-2)+c$

6. $-\frac{1}{3\sqrt{5}}\tan^{-1}\frac{(2-3x)}{\sqrt{5}}+c$ (b) $\frac{1}{3}\tan^{-1}3x+c$

7. $-\frac{1}{3}\cos 3x+c$ (b) $\frac{1}{4}\sin(4x+5)+c$

8. $(\sin\frac{1}{4}x-\cos\frac{1}{4}x)+c$ **9.** (a) $\frac{1}{3}e^{3x+4}+c$ (b) $2e^{x/2}+c$

10. $\frac{5}{2}\sin\left(\frac{2}{5}x-2\right)+\frac{1}{3}\frac{a^{3x+2}}{\log_e a}+c$

13.3 Integration of x^n Function

If in the function of x^n product of x^{n-1} is given, then integration of that function may be done, assuming $x^n = t$.

We know that,

$$\int f\{\phi(x)\}\,\phi'(x)\,dx = \int f(t)dt, \text{ where, } \phi(x) = t$$

Now if in the same formula,

$$\phi(x) = x^n$$

$\therefore$ $\phi'(x) = nx^{n-1}$ is kept, then on dividing by n,

$$\int f(x^n)\,x^{n-1}\,dx = \frac{1}{n}\int f(t)\,dt$$

Did u know? If integration is of the form $\int f(x^n)\,x^{n-1}\,dx$ then by putting value $x^n = t$ integration will become easy.

Note

EXAMPLES WITH SOLUTION

Example 1: Find the relative integration of *x* for function $\cos^2 x \sin x$.

Solution: $\int \cos^2 x \sin x \, dx$ assumed $\cos x = t$

$$\sin x \, dx = -dt = \int -t^2 \, dt$$

$$= -\frac{t^3}{3} + c = -\frac{\cos^3 x}{3} + c.$$ **Ans.**

Example 2: Find the value of $\int \frac{x^8 \, dx}{(1-x^3)^{1/3}}$.

Solution: Assumed $1 - x^3 = t$

$\therefore$ $-3x^2 \, dx = dt$

$$x^2 \, dx = -\frac{dt}{3}$$

$$\int \frac{x^8 \, dx}{(1-x^3)^{1/3}} = \int \frac{(x^3)^2 \, x^2 \, dx}{(1-x^3)^{1/3}}$$

$$= -\frac{1}{3} \int \frac{(1-t)^2 \, dt}{t^{1/3}}$$

$$= -\frac{1}{3} \int \frac{(t^2 - 2t + 1) \, dt}{t^{1/3}}$$

$$= -\frac{1}{3} \int (t^{5/3} - 2t^{2/3} + t^{-1/3}) \, dt$$

$$= -\frac{1}{3} \left[\frac{t^{8/3}}{\frac{8}{3}} - \frac{2t^{5/3}}{\frac{5}{3}} + \frac{t^{2/3}}{\frac{2}{3}} \right] + c$$

$$= -\frac{1}{3} \left[\frac{3}{8}(1-x^3)^{8/3} - \frac{6}{5}(1-x^3)^{5/3} + \frac{3}{2}(1-x^2)^{2/3} \right] + c$$

$$= -\frac{1}{8}(1-x^3)^{8/3} + \frac{2}{5}(1-x^3)^{5/3} - \frac{1}{2}(1-x^2)^{2/3} + c.$$ **Ans.**

Example 3: Find the value of $\int \frac{dx}{x + \sqrt{x}}$.

Solution: Considered, $\sqrt{x} + 1 = t$

$\therefore$ $\frac{1}{2\sqrt{x}} \, dx = dt$

Note

$$\therefore \quad \frac{dx}{\sqrt{x}} = 2\,dt$$

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)}$$

$$= \int \frac{2\,dt}{t} = 2\log t + c$$

$$= 2\log|\sqrt{x}+1| + c.$$

Ans.

Example 4: Find the value of $\int \sin\left(2x+\frac{\pi}{2}\right) dx$.

Solution: $$\int \sin\left(2x+\frac{\pi}{2}\right) dx = \int \cos 2x\,dx$$

$$= \frac{\sin 2x}{2} + c.$$

Ans.

Example 5: Find the value of $\int \frac{\sin x}{\sqrt{9-\cos^2 x}}\,dx$.

Solution: Considered $\cos x = t$

$$\therefore \quad -\sin x\,dx = dt$$

$$\therefore \quad \sin x\,dx = -dt$$

$$\int \frac{\sin x\,dx}{\sqrt{9-x^2}} = \int \frac{-dt}{\sqrt{9-t^2}}$$

$$= \int \frac{-dt}{\sqrt{3^2-t^2}}, \text{ formula } \int \frac{-dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$= -\sin^{-1}\frac{t}{3} + c$$

$$= -\sin^{-1}\left(\frac{\cos x}{3}\right) + c.$$

Ans.

Example 6: Find the value of $\int \frac{\log x}{x}\,dx$.

Solution: Consider, $\int \frac{\log x}{x}\,dx$

$$\log x = t$$

$$\frac{1}{x}\,dx = dt \qquad \text{(by relative differentation of } x\text{)}$$

$$= \int t\,dt = \frac{t^2}{2} + c = \frac{1}{2}(\log x)^2 + c.$$

Ans.

Note

Example 7: Find the value of $\int \frac{x^7}{1+x^{16}} dx$.

Solution: Considered that $x^8 = t$ to $8x^7\, dx = dt$

$$\int \frac{x^7}{1+x^{16}} dx = \int \frac{x^7}{1+(x^8)^2} dx = \frac{1}{8}\int \frac{1}{1+t^2} dt$$

By the use of the formula, $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$

$$= \frac{1}{8} \tan^{-1} t + c = \frac{1}{8} \tan^{-1} x^8 + c.$$ **Ans.**

Example 8: Find the value of $\int x \sin^3 x^2 \cos x^2\, dx$.

Solution: Assume that, $\sin x^2 = t \Rightarrow (\cos x^2)\, 2x\, dx = dt$

Or $x \cos x^2\, dx = \frac{1}{2} dt$

$$\int x \sin^3 x^2 \cos x^2\, dx = \int (\sin x^2)^3 . x \cos x^2\, dx = \frac{1}{2} \int t^3\, dt$$

$$= \frac{1}{2}\frac{t^4}{4} + c = \frac{1}{8} (\sin x^2)^4 + c = \frac{1}{8} \sin^4 x^2 + c.$$ **Ans.**

Example 9: Find the value of $\int \frac{ax^2}{1-2x^3} dx$.

Solution: Considered $\int \frac{ax^2}{1-2x^3} dx$, $1 - 2x^3 = t$

$$-6x^2\, dx = dt$$

$$x^2\, dx = -\frac{1}{6}\, dt = a \int \frac{-\frac{1}{6} dt}{t}$$

$$= -\frac{a}{6} \log t + c = -\frac{a}{6} \log (1 - 2x^3) + c.$$ **Ans.**

Example 10: Find the value of $\int \frac{(1+\log x)^2}{x} dx$.

Solution: Consider, $1 + \log x = t$ then $\frac{1}{x} dx = dt$

Note

$$\therefore \quad \int \frac{(1+\log x)^2}{x}\, dx = \int t^2\, dt = \frac{t^{2+1}}{2+1} + c \Rightarrow \frac{t^3}{3} + c$$

$$= \frac{(1+\log x)^3}{3} + c. \qquad \textbf{Ans.}$$

Task Find the value of $\int \frac{1}{x\,(1+\log x)^n}$. [**Ans. :** $\frac{1}{1-n}$ $(1 + \log x)^{-(n-1)} + c$.]

Questionnaire 13.2

Find the value of the following integrations:

1. $\int x^3 \cos x^4\, dx.$

2. $\int \frac{x^2}{\sqrt{1+x^3}}\, dx.$

3. $\int \frac{3x^2}{9^2 + x^3}\, dx.$

4. $\int nx^{n-1} \cos x^n\, dx.$

5. $\int \frac{x^2}{1+x^6}\, dx.$

6. $\int \frac{x}{\sqrt{1-x^4}}\, dx.$

7. $\int \frac{x}{1+x^4}\, dx.$

8. $\int \frac{x\, dx}{\sqrt{1+x^2}}\, dx.$

9. $\int \frac{x^{n-1}}{a+bx^n}\, dx.$

10. $\int \frac{x\, dx}{(1+x^2)^{3/2}}\, dx.$

11. $\int \frac{x^{m-1}}{\sqrt{1-x^m}}\, dx.$

12. $\int \frac{2x^3}{4+x^8}\, dx.$

13. $\int \frac{x^3\, dx}{(4-x^4)^2}.$

14. $\int x^{-1/2} \operatorname{cosec}^2 \sqrt{x}\, dx.$

15. $\int \operatorname{cosec}^2 x \sqrt{\cot x}\, dx.$

16. $\int \frac{e^{\sqrt{x}} \cos e^{\sqrt{x}}}{\sqrt{x}} = dx.$

17. $\int \frac{x^2}{16+25x^6}\, dx.$

18. $\int e^x (1+x) \operatorname{cosec}^2 (xe^x)\, dx.$

19. $\int \frac{x^4+1}{x^6+1}\, dx.$

20. $\int x \cos^3 x^2 \sin x^2\, dx.$

Note

Answers

1. $\frac{1}{4}\sin x^4 + c$

2. $\frac{2}{3}\sqrt{1+x^3} + c$

3. $\log|9^2 + x^3| + c$

4. $\sin x^n + c$

5. $\frac{1}{3}\tan^{-1} x^2 + c$

6. $\frac{1}{2}\sin^{-1} x^2 + c$

7. $\frac{1}{2}\tan^{-1} x^2 + c$

8. $\sqrt{1+x^2} + c$

9. $\frac{1}{nb}\log|a + bx^n| + c$

10. $-\frac{1}{\sqrt{1+x^2}} + c$

11. $-\frac{2}{m}\sqrt{1-x^m} + c$

12. $\frac{1}{4}\tan^{-1}\frac{1}{2}x^4 + c$

13. $\frac{1}{4}.\frac{1}{4-x^4} + c$

14. $-2\cot\sqrt{x} + c$

15. $-\frac{2}{3}(\cot x)^{3/2} + c$

16. $2\sin e^{\sqrt{x}} + c$

17. $\frac{1}{60}\tan^{-1}\frac{5x^3}{4} + c$

18. $-\cot(xe^x) + c$

19. $\tan^{-1} x + \frac{1}{3}\tan^{-1} x^3 + c$

20. $-\frac{1}{8}\cos^4 x^2 + c.$

Self Assessment

1. Fill in the blanks:

1. In substitution operation .... is done by changing the given integrand in standard formula.
2. $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(.........) + c$
3. $\int \sec^2 ax + dx = \frac{.........}{a} + c$
4. If in the function of x^n product of .... is given, then integration of that function may be done by assuming $x^n = t$.

13.4 Integrand

If integrand is a form of $\phi\,[f(x)]\,f'(x)$ i.e., it is a function of any amount $f(x)$ and product of integral coefficient $f(x)$ of this same amount $f'(x)$ or may be written in this form, then we do integration considering this amount $f(x)$ equal to t.

Note

Notes If $f(x) = t$ then $f'(x)\, dx = dt$

$$\therefore \quad \int \phi\,[f(x)]\, f'(x)\, dx = \int \phi(t)\, dt = \psi(t) \text{ (Assume)}$$

$$= \psi\,[\phi(x)],\ t \text{ (on putting the value of } t)$$

EXAMPLES WITH SOLUTION

Example 1: Find the Value of $\int \sin^4 x \cos x\, dx$.

Solution: Consider, $\sin x = t \Rightarrow \cos x\, dx = dt$.

Hence, $$\int \sin^4 x . \cos x\, dx = \int t^4\, dt$$

$$= \frac{t^5}{5} + c = \frac{1}{5} \sin^5 x + c.$$ **Ans.**

Example 2: Find the Value of $\int \cot^3 \theta . \operatorname{cosec}^2 \theta\, d\theta$.

Solution: Consider, $\cot \theta = x$ then, $- \operatorname{cosec}^2 \theta\, d\theta = dx$

$\Rightarrow$ $\operatorname{cosec}^2 \theta\, d\theta = - dx$

$$\therefore \quad \int \cot^3 \theta \operatorname{cosec}^2 \theta\, d\theta = - \int x^3\, dx = - \frac{x^4}{4} + c$$

$$= - \frac{\cot^4 \theta}{4} + c.$$ **Ans.**

Example 3: Find the Value of $\int \frac{4 \sin^{-1} x}{(1 - x^2)}\, dx$.

Solution: Consider, $\sin^{-1} x = t$, then, $\frac{dx}{\sqrt{(1 - x^2)}} = dt$

$$\therefore \quad \int \frac{4 \sin^{-1} x\, dx}{\sqrt{(1 - x^2)}} = \int 4t\, dt = 4 \times \frac{1}{2} t^2 + c = 2\ (\sin^{-1} x)^2 + c.$$ **Ans.**

Example 4: Find the Value of $\int \frac{\cos^2 (\log x)}{x}\, dx$.

Solution: Consider, $\log x = t \Rightarrow \frac{1}{x}\, dx = dt$

Note

$$\therefore \quad \int \frac{\cos^2 (\log x)}{x}\, dx = \int \cos^2 t\, dt = \frac{1}{2} \int (1 + \cos 2t)\, dt$$

$$= \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) + c$$

$$= \frac{1}{2}\left[\log x + \frac{\sin 2 (\log x)}{2}\right] + c.$$ **Ans.**

Questionnaire 13.3

Very Short Answer Questions:

Find the Value of the Following:

1. (*a*) $\int \sin x \cos x\, dx$ (*b*) $\int \sin^2 x \cos x\, dx$

(*c*) $\int \cos^2 x \sin x\, dx$ (*d*) $\int \cot^2 x \operatorname{cosec}^2 x\, dx$

(*e*) $\int \sec^p x \tan x\, dx$

2. (*a*) $\int \frac{(\log_e x)^2}{x}\, dx$ (*b*) $\int \frac{\cos (\log_e x)}{x}\, dx$

(*c*) $\int \frac{1 + \log_e x}{x}\, dx$ (*d*) $\int \frac{1}{x \cos^2 (\log_e x)}\, dx$

3. (*a*) $\int e^x \cos e^x\, dx$ (*b*) $\int e^x (a + be^x)^n\, dx$

(*c*) $\int \frac{e^{m \sin^{-1} x}}{\sqrt{1 - x^2}}\, dx$ (*d*) $\int e^{\tan x} \sec^2 x\, dx$

4. (*a*) $\int \frac{\tan^{-1} x}{(1 + x^2)^{3/2}}\, dx$ (*b*) $\int \frac{\tan^{-1} x}{1 + x^2}\, dx$

(*c*) $\int \frac{(\sin^{-1} x)^2}{\sqrt{1 - x^2}}\, dx$

5. $\int x \sqrt{x^2 - 1}\, dx$ **6.** $\int \frac{x \tan^{-1} x^2}{1 + x^4}\, dx$

7. $\int \frac{1}{\sqrt{x}\, [\sqrt{x} + 1]}\, dx.$

Note

Answers

1. (a) $\frac{1}{2}\sin^2 x + c$ (b) $\frac{1}{3}\sin^3 x + c$

   (c) $-\frac{1}{3}\cos^3 x + c$ (d) $-\frac{1}{3}\cot^3 x + c$

   (e) $\frac{1}{p}\sec^p x + c$

2. (a) $\frac{1}{3}(\log_e x)^3 + c$ (b) $\sin(\log_e x) + c$

   (c) $\frac{1}{2}(1 + \log_e x)^2 + c$ (d) $\tan(\log_e x) + c$

3. (a) $\sin e^x + c$ (b) $\frac{1}{b(n+1)}(a + be^x)^{n+1} + c$

   (c) $\frac{1}{m}e^{m\sin^{-1}x} + c$ (d) $e^{\tan x} + c$

4. (a) $\frac{1 + x\tan^{-1}x}{\sqrt{1+x^2}} + c$ (b) $\frac{1}{2}(\tan^{-1}x)^2 + c$

   (c) $\frac{1}{3}(\sin^{-1}x)^3 + c$

5. $\frac{(x^2-1)^{3/2}}{3} + c$ 6. $\frac{1}{4}\left(\tan^{-1}x^2\right)^2 + c$

7. $2\log|1 + \sqrt{x}| + c$

13.5 Integration of the Fraction whose Numerator is Integral Co-efficient of the Denominator

Consider that we have to find the value of $\int \frac{f'(x)}{f(x)}$ where, $f'(x)$, is integral co-efficient of $f(x)$

On putting $f(x) = t$, $f'(x)\,dx = dt$

$$\therefore \qquad \int \frac{f'(x)}{f(x)}\,dx = \int \frac{dt}{t} = \log t = \log[f(x)]$$

Hence,

$$\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c$$

Did u know? If numerator (N^r) of any fraction is integral co-efficient of denominator (D^r), then integration of that fraction will be equal to Log of the denominator.

$$\therefore \qquad \int \frac{d(\text{denominator})}{(\text{denominator})}\,dx = \log\ (\text{denominator})$$

Note

EXAMPLES WITH SOLUTION

Example 1: Find the Value of $\int \frac{e^x}{1+e^x}\,dx$.

Solution: Consider, $1 + e^x = t$ then $e^x\,dx = dt$

$$\therefore \quad \int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{t}\,dt = \log|t| + c$$

$$= \log|1 + e^x| + c. \qquad \textbf{Ans.}$$

Example 2: Find the Value of $\int \frac{x^4}{x^5+4}\,dx$.

Solution: Consider, $x^5 + 4 = t$

$\therefore \quad 5x^4\,dx = dt$

Or $\quad x^4\,dx = \frac{1}{5}\,dt$

$$\therefore \quad \int \frac{x^4}{x^5+4}\,dx = \frac{1}{5}\int \frac{1}{t}\,dt = \frac{1}{5}\log|t| + c$$

$$= \frac{1}{5}\log|x^5 + 4| + c. \qquad \textbf{Ans.}$$

Example 3: Find the Value of $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$.

Solution: Consider, $e^x + e^{-x} = t \Rightarrow (e^x - e^{-x})\,dx = dt$

so that
$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{dt}{t} = \log|t| + c = \log|e^x + e^{-x}| + c. \qquad \textbf{Ans.}$$

Example 4: Find the Value of $\int \frac{dx}{e^x - 1}$.

Solution:
$$\int \frac{1}{e^x - 1}\,dx = \int \frac{e^{-x}}{1 - e^{-x}}\,dx$$

[on multiplying e^{-x} with the numerator and denominator]

$$= \int \frac{1}{t}\,dt, \text{ Assumed, } 1 - e^{-x} = t \Rightarrow e^{-x}\,dx = dt$$

$$= \log|t| + c = \log|1 - e^{-x}| + c. \qquad \textbf{Ans.}$$

Task Find the Value of $\int \frac{a}{b + ce^x}\,dx$. (**Ans.:** $\frac{a}{b}\log|be^{-x} + c| + c'$.)

Note

13.6 Integration of Some Standard Function By Substitution

(i) Integration of tan x and cot x

(1) $$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

Consider $\cos x = t$, on differentiation, $-\sin x\, dx = dt$

$\therefore$ $$\int \tan x\, dx = -\int \frac{dt}{t} = -\log|t| + c$$

$$= -\log|\cos x| + c = \log|\sec x| + c$$

$\therefore$ $$\boxed{\int \tan x\, dx = -\log|\cos x| + c \text{ or } \log|\sec x| + c}$$

(2) $$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$$

Consider $\sin x = t$, on differentiation, $\cos x\, dx = dt$

$\therefore$ $$\int \cot x\, dx = \int \frac{dt}{t} = \log|t| + c = \log|\sin x| + c$$

$\therefore$ $$\boxed{\int \cot x\, dx = \log|\sin x| + c.}$$

Questionnaire 13.4

Very Short Answer Questions.

Find the value of the following:

1. (*a*) $\int \frac{ax^{n-1}}{x^n + b}\, dx$ (*b*) $\int \frac{3x^2}{(x^3+4)^5}\, dx$

(*c*) $\int \frac{x^2}{1-2x^3}\, dx$ (*d*) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx.$

Answers

1. (*a*) $\frac{a}{n}\log|x^n + b| + c$ (*b*) $-\frac{1}{4}(x^3+4)^{-4} + c$

(*c*) $-\frac{1}{6}\log_e|1-2x^3| + c$ (*d*) $2e^{\sqrt{x}} + c$

Self Assessment

2. Multiple Choice Questions:

5. Value of $\int \sin x \cos x\, dx$ is :

(*a*) $\frac{1}{2}\sin^2 x + c$ (*b*) $\frac{1}{2}\cos^2 x + c$

(*c*) $\frac{1}{2}\text{cosec}^2 x + c$ (*d*) None of these.

Note

6. Value of $\int \cot^3 \theta . \text{cosec}^2 \theta \, d\theta$ will be,

(*a*) $\frac{\cot^4 \theta}{4} + c$ (*b*) $-\frac{\cot^4 \theta}{4} + c$

(*c*) $\frac{\text{cosec}^4 \theta}{4} + c$ (*d*) None of these.

7. $\sin(\log_e x) + c$ is solution for which of the following:

(*a*) $\int \sin^2 x \cos^2 x \, dx$ (*b*) $\int \cot^2 x \, \text{cosec}^2 \, x \, dx$

(*c*) $\int \frac{\cos(\log_e x)}{x} dx$ (*d*) None of these.

8. $\log|1 + e^x| + c$ is solution for which of the following:

(*a*) $\int \frac{e^x}{1+e^x} dx$ (*b*) $\int \frac{1+e^x}{e^x} dx$

(*c*) $\int e^x \, dx$ (*d*) $\int \frac{-e^x}{1+e^x} dx$.

13.7 Summary

- There are two main methods to find the integration of any function:
  - (i) Integration by Substitution
  - (ii) Integration by Parts
- In this operation integration is done by changing the given integrand in standard formula.
- If integrand is a form of $\phi\,[f(x)]\, f'\,(x)$ i.e., it is a function of any amount $f(x)$ and product of integral coefficient $f(x)$ of this same amount $f\,\phi\,(x)$ or may be written in this form, then we do integration considering this amount equal to t.

13.8 Keywords

- ***Substitution:*** Replacement
- ***Methods****:* Manner, process

13.9 Review Questions

1. Find the Value of $\int \sec^2 3x \, dx$ (**Ans.:** $\frac{\tan 3x}{3} + c$)

2. Calculate the value of $\int \sin^3 \theta \, d\theta$ (**Ans.:** $-\frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta + c$)

3. Find the relative integration of x for the function $\cos^2 x \sin x$ (**Ans.:** $-\frac{\cos^3 x}{3} + c$)

Note

4. Find the value of $\int \sin\left(2x + \frac{\pi}{2}\right) dx$ (**Ans.:** $\frac{\sin 2x}{2} + c$)

5. Find the Value of $\int \frac{4 \sin^{-1} x}{(1 - x^2)} dx$ (**Ans.:** $2 (\sin^{-1} x)^2 + c$)

Answers: Self Assessment

1. Integration	2. $ax + b$	3. tan ax	4. x^{n-1}
5. (a)	6. (b)	7. (c)	8. (a)

13.10 Further Readings

Books

Mathematical Economics – Michael Harrison, Patric Walderen.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economics – Simon and Bloom – Viva Publications.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics and Finance – Martin Norman.

Unit 14: Integration as a Summation

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Integral of Multiplication of two Functions.
- Find out the Integration by Partial Fraction.
- Get the Information about two Standard Integrals.

Introduction

We should be careful before using the integration rule and we should ensure that by doing the integration operation once or twice it takes the form of regular integration or is modify such that it could be integrated. If one function of integral is known then this known function will always be considerd as first function. If integrals becomes undo form with negative sign in right hand side then equation should be solved with integration rule.

14.1 Finding Integral of Multiplication of two Functions (Divided Integrals)

if the multiplication of $f(x)$ and $\phi(x)$ is $f(x).\ \phi(x)$ the differentiation is–

$$\frac{d}{dx}\{f(x)\,.\,\phi(x)\} = f(x)\,\phi'(x) + f'(x)\,\phi(x)$$

Integration of both side with respect of x

$$f(x).\,\phi(x) = \int f(x)\,\phi'(x)\,dx + \int f'(x)\,\phi(x)\,dx$$

Or

$$\int f(x)\,\phi'(x)\,dx = f(x).\,\phi(x) - \int f'(x)\,\phi(x)\,dx$$

$$= f(x)\int \phi'(x)\,dx - \int\{f'(x)\int \phi'(x)\,dx\}\,dx$$

By the definition of $\phi(x)$, $\phi'(x)$ is the integral with respect to x, so $\int \phi'(x)\,dx$ is put at the place of $\phi'(x)$.

Note For expressing the above conclusion in easier form, on putting $f_1(x)$ and $f_2(x)$ instead of $f(x)$ and $\phi'(x)$ respectively, $\because$

$$\int f_1(x).\, f_2(x)\, dx = f_1(x) \int f_2(x)\, dx - \int \left\{ \frac{d}{dx} f_1(x). \int f_2(x)\, dx \right\} dx$$

Did u know? Integral of Multiplication of two functions = First function x integral of second function–[differentiation coefficient of first function × integral of integration of second function].

It should be seen before using the above rule that on using above process once or two times the integrant would take the form of any authentic integrant or come in such form which could be integrated. The success of this function much more depends on the study of first and second function. The first function is so selected that the second part of RHS could be easily and successfully integrated. Therefore, there should take care in the selection of first function.

In the use of this method, few important things are as following:

(1) If between two function there is a function whose integration we don't know, then that function must be considered as the first function.

In $\int x.(\log x)\, dx$, we don't know the integral of log x. Therefore log x should be considered as the first function.

(2) if both the functions were so that the integral of both is known then among those if there is any function of x^n form, then always consider it as the first function.

For example, in $\int \log x\, dx, \int \tan^{-1} x\, dx$, x^2 must be considered as the first function.

(3) To know the integral of $\int \log x\, dx, \int \tan^{-1} x\, dx$, etc. integration is done with considering '1' as the second function. As $\int \log x\, dx = \int (\log x).\, 1\, dx$ etc.

(4) The divided integration rule should be used more than once according to the use.

(5) If the integral in RHS comes in its initial form with the negative sign then questions should be solved with using the rules of solving the equations.

Notes We do divided integration with considering the first coming and second coming function, in the word 'ILATE', as the first and second function respectively.

Where I – is for the Inverse trigonometrical functions (as ($\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc.).

L – is for the Logarithmic functions (as log x, log ($x^2 \pm a^2$ etc.).

A – is for the algebraic functions (as $x, x+1, 2x, \sqrt{x}$, etc.).

T – is for the Trigonometrical functions. (as sin x, cos x, tan x etc.).

E – is for the Exponential functions (as a^x, e^x, 2^x, 10^x, 3^{-x} , etc.).

As there is no any authentic method to select the functions during the divided integration, but the students can do the divided integration on selecting the function with the help of above mentioned 'ILATE'.

EXAMPLES WITH SOLUTION

Example 1: Find the Value $\int \log_e x\, dx$.

Solution: Let's consider that here the second function is unit because if $\log_e x$ is considered as second function then it can't be integrated.

$$\int \log_e x\, dx = \int \log_e x \,.\, 1\, dx$$

$$= \log_e x \,.\, \int 1\, dx - \int \left[\frac{1}{x}\int 1\, dx\right] \quad \text{(On doing divided inegration)}$$

$$= \log_e x\, x - \int \frac{1}{x} x\, dx = x \log_e x - \int 1\, dx$$

$$= x \log_e x - x + c.$$ **Ans.**

We can also write $x \log \frac{x}{e}$ of $x \log_e x - x$ if want because $\log_e e = 1$.

Example 2: Find the Value $\int x \sin x\, dx$.

Solution: There are two factors of integrant, the integrant of both x and $\sin x$ is known but among these the function x is of x^n type. Therefore it should be considered as the first function.

Therefore,
$$\int x \sin x\, dx = x\left[\int \sin x\, dx\right] - \int \left[\left(\frac{d}{dx}x\right)\int \sin x\, dx\right] dx$$

$$= x(-\cos x) - \int 1.(-\cos x)\, dx$$

$$= -\, x \cos x + \int \cos x\, dx$$

$$= -x \cos x + \sin x + c.$$ **Ans.**

Example 3: Find the Value $\int x^2 \log x\, dx$.

Solution:
$$\int x^2 \log x\, dx = \int (\log x)\; x^2\, dx$$

$$= \log x \,.\, \frac{x^3}{3} - \int \frac{1}{x}\,.\,\frac{x^3}{3}\, dx$$

$$= \frac{1}{3}x^3 \log x - \frac{1}{3}\int x^2\, dx$$

$$= \frac{1}{3}x^3 \log x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + c$$

$$= \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + c.$$ **Ans.**

Note

Example 4: Find the Value $\int x^n \log x$.

Solution: $$\int x^n \log x\, dx = \int \log x\, x^n\, dx$$

[log x, the integration of which is unknown, consider is as the first function]

$$= \log x \cdot \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{n+1}\, dx$$

(On doing divided inegration)

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \int x^n\, dx = \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \frac{x^{n+1}}{n+1}$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2} = \frac{x^{n+1}}{n+1}\left[\log x - \frac{1}{n+1}\right] + c.$$ **Ans.**

Example 5: Find the Value $\int x^2 \cos x\, dx$.

Solution: Let's assume that x^2 is the first function.

$$\therefore \quad \int x^2 . \cos x\, dx = x^2 . \sin x - \int 2x . \sin x\, dx.$$

Take x as the first function in the second integral then integrate by factors.

$$\therefore \quad \int x^2 \cos x\, dx = x^2 \sin x - 2[x(-\cos x) - \int 1.(-\cos x)\, dx]$$

$$= x^2 \sin x + 2x \cos x - 2\int \cos x\, dx$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

$$= (x^2 - 2) \sin x + 2x \cos x + c.$$ **Ans.**

Example 6: Find the Value $\int e^x \sin x\, dx$.

Solution: Assume that e^x is the first function.

$$\therefore \quad \int e^x \sin x\, dx = e^x (-\cos x) - \int e^x (-\cos x)\, dx$$

$$= -e^x \cos x + \int e^x \cos x\, dx.$$

Again by the integration by factors,

$$\int e^x \sin x\, dx = -e^x \cos x + [e^x \sin x - \int e^x \sin x\, dx]$$

Shift the last term towards left. Then on dividing by 2,

$$\int e^x \sin x\, dx = \frac{1}{2} e^x (\sin x - \cos x) + c.$$ **Ans.**

Note

Example 7: Find the Value $\int \frac{1}{x^2} \log x \, dx$.

Solution: $$\int \frac{1}{x^2} \log x \, dx = \int \log x \cdot \frac{1}{x^2} \, dx$$

$$= \log x \left(\frac{-1}{x}\right) - \int \left(\frac{1}{x}\right)\left(\frac{-1}{x}\right) dx$$

$$= \frac{-\log x}{x} + \int \frac{1}{x^2} \, dx$$

$$= \frac{-\log x}{x} - \frac{1}{x} + c = \frac{-(1+\log x)}{x} + c.$$ **Ans.**

Example 8: Integrate the function $x^2 a^x$ with respect to x.

Solution: Assume that = $\int x^2 a^x \, dx$

Here, the integrant is the multiplication of two functions x^2 and a^x. where x^2 is an algebraic function and a^x is an exponential function and firstly A comes in **ILATE**. Therefore we'll do divided integration on taking x^2 and a^x as the first and second function respectively.

$$I = x \int a^x \, dx - \int \left\{ \frac{d}{dx}(x). \int a^x \, dx \right\} dx$$

$$= x\left(\frac{a^x}{\log a}\right) - \int 1.\left(\frac{a^x}{\log a}\right) dx$$

$$= x \frac{a^x}{\log a} - \frac{1}{\log a} \int a^x \, dx$$

$\therefore$ $$I = x.\frac{a^x}{\log a} - \frac{a^x}{(\log a)^2} + c.$$ **Ans.**

Example 9: Find the Value $\int \log \{x + \sqrt{x^2 + a^2}\} \, dx$.

Solution: $\int \log \{x + \sqrt{x^2 + a^2}\} \, dx$

$$= \int \log \{x + \sqrt{x^2 + a^2}\}. 1 \, dx$$

$$= \log \{x + \sqrt{x^2 + a^2}\} \, x - \int \frac{1}{(x + \sqrt{x^2 + a^2})} . \left\{1 + \frac{1(2x)}{2\sqrt{x^2 + a^2}}\right\} x \, dx$$

$$= \log \{x + \sqrt{x^2 + a^2}\} \, x - \int \frac{1}{(x + \sqrt{x^2 + a^2})} \left\{\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right\} x \, dx$$

$$= \log \{x + \sqrt{x^2 + a^2}\} \, x - \int \frac{x \, dx}{\sqrt{x^2 + a^2}}$$

Note

$$= \log\{x + \sqrt{x^2 + a^2}\}\, x - \frac{1}{2}\int \frac{2x\, dx}{\sqrt{x^2 + a^2}}$$

$$= \log\{x + \sqrt{x^2 + a^2}\}\, x - \frac{1}{2}\{2\sqrt{x^2 + a^2}\} + c$$

$$= \log\{x + \sqrt{x^2 + a^2}\}\, x - \sqrt{x^2 + a^2} + c$$

$$= x \log\{x + \sqrt{x^2 + a^2}\} - \sqrt{x^2 + a^2} + c.$$ **Ans.**

Example 10: Find the Value $\int \tan^{-1} x\, dx$.

Solution: $\int \tan^{-1} x\, dx = \int (\tan^{-1} x).1\, dx$

$$= (\tan^{-1} x)\, x - \int \frac{1}{1 + x^2}.\, x\, dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1 + x^2}\, dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{dt}{t},$$ (on putting $1 + x^2 = t$ and $2x\, dx = dt$)

$$= x\tan^{-1} x - \frac{1}{2}\log|t| + c$$

$$= x\tan^{-1} x - \frac{1}{2}\log|1 + x^2| + c.$$ (On putting the value of t) **Ans.**

> *Task* Find the value of $\int x^2 \cos x\, dx$ [**Ans:** $(x^2 - 2)\sin x + 2x\cos x + c$]

Example 11: Find the Value $\int x \tan^{-1} x\, dx$.

Solution: $\int x\tan^{-1} x\, dx = \int (\tan^{-1} x).\, x\, dx$

$$= (\tan^{-1} x).\frac{x^2}{2} - \int \frac{1}{1 + x^2}.\frac{x^2}{2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \int \frac{x^2}{1 + x^2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int\left(1 - \frac{1}{1 + x^2}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + c$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + c$$

$$= \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{x}{2} + c.$$ **Ans.**

Example 12: Find the Value $\int \frac{x}{1+\cos x}\,dx$ **.**

Solution:

$$\int \frac{x}{1+\cos x}\,dx = \int \frac{x(1-\cos x)}{(1+\cos x)(1-\cos x)}\,dx$$

$$= \int \frac{x - x\cos x}{\sin^2 x}\,dx$$

$$= \int \frac{x}{\sin^2 x}\,dx - \int \frac{x\cos x}{\sin^2 x}\,dx$$

$$= \int x\,\text{cosec}^2\, x\,dx - \int x\cot x\,\text{cosec}\, x\,dx$$

$$= x(-\cot x) - \int 1.(-\cot x)\,dx$$

$$-[x(-\text{cosec}\, x) - \int 1.(-\text{cosec}\, x)\,dx] + c)$$

$$= -x\cot x + \log|\sin x| + x\,\text{cosec}\, x - \log\left|\tan\frac{x}{2}\right| + c.$$ **Ans.**

Example 13: Find the Value $\int \cos^{-1}\frac{1}{x}\,dx$ **.**

Solution: Assuming that

$$I = \int \cos^{-1}\frac{1}{x}\,dx, \qquad \left[\because \cos^{-1}\frac{1}{x} = \sec^{-1} x\right]$$

$$= \int \sec^{-1} x\,dx$$

$$= \int (\sec^{-1} x).1\,dx$$

Integration on taking $\sec^{-1} x$ and 1 as the first and second function respectively

$$= \sec^{-1} x\int dx - \int\left\{\frac{d}{dx}(\sec^{-1} x).\int dx\right\}dx$$

$$= x\sec^{-1} x - \int \frac{1}{x\sqrt{x^2-1}}.x\,dx$$

$$= x\sec^{-1} x - \int \frac{1}{\sqrt{x^2-1}}\,dx$$

$$= x\sec^{-1} x - \log|x + \sqrt{x^2-1}| + c.$$ **Ans.**

Note

Questionnaire 14.1

Long Answer Question.

Find out the value of following integrals:

1. (*a*) $\int x \log x \, dx$ (*b*) $\int (\log x)^2 \, dx$

(*c*) $\int x^3 \log x \, dx$ (*d*) $\int \log (1 + x^2) \, dx$

(*e*) $\int \sec x \log (\sec x + \tan x) \, dx$

2. (*a*) $\int x e^{ax} \, dx$ (*b*) $\int \frac{e^{1/x}}{x^3} \, dx$

3. (*a*) $\int x \operatorname{cosec}^2 ax \, dx$ (*b*) $\int x \sec^2 x \, dx$

(*c*) $\int x \sec^2 2x \, dx$ (*d*) $\int x^2 \sin x \, dx$

(*e*) $\int \sec^3 x \, dx$ (*f*) $\int x^2 \sin 2x \, dx$

4. (*a*) $\int x \sin nx \, dx$ (*b*) $\int \sin^2 x \, dx$

(*c*) $\int x \tan^2 x \, dx$ (*d*) $\int \sin \sqrt{x} \, dx$

5. (*a*) $\int \sin^{-1} x \, dx$ (*b*) $\int \cot^{-1} x \, dx$

6. (*a*) $\int x^3 \tan^{-1} x \, dx$ (*b*) $\int \frac{\log x}{(1+x)^2} \, dx$

7. $\int \frac{x^2}{(x \sin x + \cos x)^2} \, dx$ **8.** $\int \sqrt{x} (\log x)^2 \, dx$

9. (*a*) $\int x \log (1 + x) \, dx$ (*b*) $\int \log_{10} x \, dx$

10. $\int \cot^{-1} (1 + x^2 - x) \, dx$ **11.** $\int x^3 e^{x^2} dx$

12. $\int \sin x \log (\sec x + \tan x) \, dx$.

Answers

1. (*a*) $\frac{1}{2} x^2 \log x - \frac{x^2}{4} + c$ (*b*) $x (\log (x))^2 - 2x \log x + 2x + c$

(*c*) $\frac{1}{4} x^4 \log x - \frac{x^2}{16} + c$ (*d*) $x \log (1 + x^2) - 2 (x - \tan^{-1} x) + c$

2. (*a*) $\frac{1}{a} x e^{ax} - \frac{1}{a^2} e^{ax} + c$ (*b*) $e^{1/x}\left(1 - \frac{1}{x}\right) + c$

Note

3. (*a*) $\frac{1}{a^2}[\log \sin ax - ax \cot x] + c$ (*b*) $x \tan x + \log|\cos x| + c$

(*c*) $\frac{1}{2} x \tan 2x - \frac{1}{4} \log|\sec 2x| + c$ (*d*) $-x^2 \cos x + 2x \sin x + 2 \cos x + c$

(*e*) $\frac{1}{2} \sec x \,.\, \tan x + \frac{1}{4} \log|\sec x + \tan x| + c$

(*f*) $-\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c$

4. (*a*) $\frac{1}{n} x \cos nx + \frac{1}{n^2} \sin nx + c$ (*b*) $\frac{x^2}{4} - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + c$

(*c*) $x \tan x + \log \cos x - \frac{x^2}{2} + c$ (*d*) $2[-\sqrt{x} \cos \sqrt{x} + \sin \sqrt{x}] + c$

5. (*a*) $x \sin^{-1} x + \sqrt{1 - x^2} + c$ (*b*) $x \cot^{-1} x + \frac{1}{2} \log|1 + x^2| + c$

6. (*a*) $\frac{1}{4} x^4 \tan^{-1} x - \frac{1}{4}\left(\frac{x^3}{3} - x + \tan^{-1} x\right) + c$ (*b*) $\frac{-\log x}{1 + x} + \log\left|\frac{x}{x + 1}\right| + c$

7. $\frac{-x}{\cos x\,(x \sin x + \cos x)} + \tan x + c$

8. $\frac{2}{3} x^{3/2}[(\log x)^2 - \frac{4}{3} \log x + \frac{8}{9}] + c$

9. (*a*) $\frac{x^2 - 1}{2} \log(1 + x) - \frac{x^2}{4} + \frac{x}{2} + c$ (*b*) $x(\log_e x - 1) \log_{10} e + c$

10. $2[x \tan^{-1} x - \frac{1}{2} \log(1 + x^2)] + c$

11. $\frac{1}{2} e^{x^2} (x^2 - 1) + c$

12. $x - \cos x \log(\sec x + \tan x) + c$

14.2 Integration by Partial Fraction

We proved that the integral of difference between one or more functions is equals to the addition or difference of integrals of that function. So the important method of integration is used the above principle to break the additional and difference form of given function. So the methods of partial fraction are used when necessary.

You already have study about the method of break partial fraction in algebra book.

You know by algebra that exponential and quadratic factors of every polynomial can be possible- it may be possible some factors are repeated. So Partial fraction of rational fraction will be like that-

(i) The related partial fraction of non-reaped denominator of single power factor *x-a* is of $\frac{A}{x - a}$, where $A \neq a$.

(ii) *R* partial fraction are following with respect to $(x - a)$ factor of repeated.

$$\frac{B_1}{x - b} + \frac{B_2}{(x - b)^2} + \ldots\ldots + \frac{B_r}{(x - b)^r}$$

Note Here $B \neq 0$.

(iii) The function of $x^2 + px + q$, which are not repeated then respected partial fraction is like that-

$$\frac{Cx + D}{x^2 + px + q}$$

(iv) The factors of $x^2 + px + q$ is repeated r times and the factors are- $(x^2 + px + q)^r$

Then, partial fraction is-

$$\frac{C_1x + D_1}{x^2 + px + q} + \frac{C_2x + D_2}{(x^2 + px + q)^2} + ... + \frac{C_rx + D_r}{(x^2 + px + q)^r}$$

Self Assessment

1. Fill in the blanks:

1. If two functions are same and the …………….. of both function known, then if there is any x^n type function, then it always considers first function.
2. If necessity then the formula of ……………………… integration used many times.
3. There is no any…………………… method for the selection of function during divided integration.
4. The integral of addition or differentiation of one function is ………………… to the addition or difference of that function.
5. Exponential an …………………. of every polynomial can be possible.

14.3 Two Standard Integrals

1. Find out the value of $\int \frac{1}{x^2 - a^2} dx$ **.**

When $x > a$,

Solution: $\because$
$$\frac{1}{x^2 - a^2} = \frac{1}{(x + a)(x - a)}$$

Let,
$$\frac{1}{x^2 - a^2} = \frac{A}{x + a} + \frac{B}{x - a} = \frac{A(x - a) + B(x + a)}{(x + a)(x - a)}$$

$\therefore$ $A(x - a) + B(x + a) = 1$

$\Rightarrow$ $(A + B)x + (B - A)a = 1$

comparing the factors of both sides

$$A + B = 0 \text{ and } (B - A)a = 1$$

$\therefore$
$$B = \frac{1}{2a} \text{ and } A = -\frac{1}{2a}$$

$\therefore$
$$\frac{1}{x^2 - a^2} = -\frac{1}{2a(x + a)} + \frac{1}{2a(x - a)} = \frac{1}{2a}\left\{\frac{1}{x - a} - \frac{1}{x + a}\right\}$$

Note

$$\therefore \quad \int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \int \left\{ \frac{1}{x-a} - \frac{1}{x+a} \right\} dx$$

$$= \frac{1}{2a} \int \frac{1}{x-a}\, dx - \frac{1}{2a} \int \frac{1}{x+a}\, dx$$

$$= \frac{1}{2a} \log |x - a| - \frac{1}{2a} \log |x + a| + c$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$\boxed{\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c, \text{ when } x > a}$$

2. Find out the value of $\int \frac{dx}{a^2 - x^2}$ **.**

When $x < a$.

Solution:

$$\int \frac{dx}{a^2 - x^2} = \int \frac{dx}{(a+x)(a-x)} = \frac{1}{2a} \int \left\{ \frac{1}{a+x} + \frac{1}{a-x} \right\} dx$$

$$= \frac{1}{2a} [\log |a + x| - \log |a - x|] + c = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$\therefore \quad \boxed{\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c}$$

EXAMPLES WITH SOLUTION

Example 1: Find the Value $\int \frac{dx}{a^2x^2 - b^2}$ **.**

Solution:

$$\int \frac{dx}{a^2x^2 - b^2}\, dx = \frac{1}{a^2} \int \frac{1}{x^2 - (b/a)^2}\, dx$$

$$= \frac{1}{a^2} \cdot \frac{1}{2(b/a)} \log \left| \frac{x - (b/a)}{x + (b/a)} \right| + c = \frac{1}{2ab} \log \left| \frac{ax - b}{ax + b} \right| + c.$$ **Ans.**

Example 2: Find the Value $\int \frac{dx}{24 - 6x^2}$ **.**

Solution:

$$\int \frac{1}{24 - 6x^2} dx = \frac{1}{6} \int \frac{1}{4 - x^2}\, dx = \frac{1}{6} \int \frac{1}{2^2 - x^2}\, dx$$

$$= \frac{1}{6} \frac{1}{2.2} \log \left| \frac{2+x}{2-x} \right| + c = \frac{1}{24} \log \left| \frac{2+x}{2-x} \right| + c.$$ **Ans.**

Note

Example 3: Find the Value $\int \frac{1}{5-2x-x^2}\,dx$.

Solution:
$$\int \frac{dx}{5-2x-x^2} = \int \frac{dx}{5+1-1-2x-x^2} = \int \frac{dx}{6-(1+2x+x^2)}$$
$$= \int \frac{dx}{6-(x+1)^2} = \int \frac{dx}{(\sqrt{6})^2-(x+1)^2}$$

$\because$ by formula
$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$$

$\therefore$
$$\int \frac{dx}{5-2x-x^2} = \frac{1}{2\sqrt{6}}\log\left|\frac{\sqrt{6}+(x+1)}{\sqrt{6}-(x+1)}\right| + c.$$ **Ans.**

Example 4: Find the Value $\int \frac{3x}{(x-1)(x-2)(x-3)}\,dx$.

Solution: $\frac{3x}{(x-1)(x-2)(x-3)} = \frac{3}{2}\frac{1}{(x-1)} - 6.\frac{1}{(x-2)} + \frac{9}{2}.\frac{1}{(x-3)}$ devided by partial fraction

$$\int \frac{3x}{(x-1)(x-2)(x-3)}\,dx$$
$$= \frac{3}{2}\int\frac{dx}{x-1} - 6\int\frac{dx}{x-2} + \frac{9}{2}\int\frac{dx}{x-3}$$
$$= \frac{3}{2}\log|(x-1)| - 6\log|(x-2)| + \frac{9}{2}\log|(x-3)| + c.$$ **Ans.**

Example 5: Find the Value $\int \frac{dx}{2x^2+x-1}$.

Solution: $\because \frac{1}{2x^2+x-1} = \frac{1}{(x+1)(2x-1)}$

Let,
$$\frac{1}{2x^2+x-1} = \frac{A}{x+1} + \frac{B}{2x-1} = \frac{A(2x-1)+B(x+1)}{(x+1)(2x-1)}$$

$\therefore$ $A(2x-1) + B(x+1) = 1$

Or $(2A + B)x + (B - A) = 1$

on comparing the function of both side,

$$2A + B = 0 \text{ and } B - A = 1$$

$\therefore A = -\frac{1}{3}$ and $B = \frac{2}{3}$

$$\frac{1}{2x^2+x-1} = \frac{2}{3}\frac{1}{2x-1} - \frac{1}{3}\frac{1}{x+1}$$

$\therefore$
$$\int \frac{1}{2x^2+x-1}\,dx = \frac{2}{3}\int\frac{1}{2x-1}\,dx - \frac{1}{3}\int\frac{1}{x+1}\,dx$$

Note

$$= \frac{2}{3}\cdot\frac{1}{2}\log|(2x-1)| - \frac{1}{3}\log|(x+1)| + C$$

$$= \frac{1}{3}\log\left|\frac{2x-1}{x+1}\right| + C.$$ **Ans.**

Example 6: Find the Value $\int \frac{dx}{x - x^3}$.

Solution: $$\frac{1}{x-x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1-x)(1+x)}$$

Let, $$\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

$\therefore$ $$1 = A(1-x^2) + Bx(1+x) + Cx(2-x) \quad ...(1)$$

Putting $x = 0$, 1 and –1 in equation (1)

$$A = 1, B = \frac{1}{2}, C = -\frac{1}{2}$$

$\therefore$ $$\int\frac{dx}{x-x^3} = \int\left[\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right]dx$$

$$= \log|x| - \frac{1}{2}\log|1-x| - \frac{1}{2}\log|1+x| + c.$$ **Ans.**

Example 7: Find the Value $\int \frac{dx}{4 - x^2}$.

Solution: $$\int\frac{dx}{4-x^2} = \int\frac{dx}{(2)^2 - x^2}$$

$$= \frac{1}{2.2}\log\left|\frac{2+x}{2-x}\right| + c, \qquad \left[\because \int\frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$= \frac{1}{4}\log\left|\frac{2+x}{2-x}\right| + c.$$ **Ans.**

Example 8: Find the Value $\int \frac{1}{(x+b)(x^2+a^2)}dx$.

Solution: On adding the integrant in partial fraction

$$\frac{1}{(x+b)(x^2+a^2)} = \frac{A}{x+b} + \frac{Bx+C}{x^2+a^2}$$

on solving, $$\frac{1}{(x+b)(x^2+a^2)} = \frac{1}{a^2+b^2}\left(\frac{1}{x+b} + \frac{b-x}{(a^2+b^2)(x^2+a^2)}\right)$$

Note

$\therefore$
$$I = \int \frac{1}{(x+b)(x^2+a^2)}\,dx$$

$$= \frac{1}{a^2+b^2}\int \frac{dx}{x+a} + \frac{1}{a^2+b^2}\int \frac{b-x}{x^2+a^2}\,dx$$

$$= \frac{1}{a^2+b^2}\log|x+b| + \frac{b}{a^2+b^2}\int \frac{dx}{x^2+a^2} - \frac{1}{2(a^2+b^2)}\int \frac{2x}{x^2+a^2}\,dx$$

$$= \frac{1}{a^2+b^2}\left[\log|x+b| + \frac{b}{a}\tan^{-1}\frac{x}{a} - \frac{1}{2}\log|x^2+a^2|\right] + c$$

$$= \frac{1}{a^2+b^2}\left[\log\left|\frac{x+b}{\sqrt{x^2+b^2}}\right| + \frac{b}{a}\tan^{-1}\frac{x}{a}\right] + c.$$ **Ans.**

Example 9: Find the Value $\int \frac{dx}{1+x^3}$.

Solution:
$$\frac{1}{1+x^3} = \frac{1}{(1+x)(1-x+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2};$$

on solving,
$$\frac{1}{1+x^3} = \frac{1}{3}\cdot\frac{1}{1+x} + \frac{2-x}{3(1-x+x^2)}$$

$\therefore$
$$\int \frac{1}{1+x^3}\,dx = \frac{1}{3}\int \frac{dx}{1+x} + \frac{1}{3}\int \frac{(2-x)\,dx}{1-x+x^2}$$

$$= \frac{1}{3}\log|1+x| + \frac{1}{2}\cdot\frac{1}{3}\int \frac{(4-2x)\,dx}{1-x+x^2}$$

$$= \frac{1}{3}\log|1+x| + \frac{1}{6}\int \frac{3-(2x-1)}{1-x+x^2}\,dx$$

$$= \frac{1}{3}\log|1+x| + \frac{1}{2}\int \frac{dx}{1-x+x^2} - \frac{1}{6}\int \frac{2x-1}{1-x+x^2}\,dx$$

$$= \frac{1}{3}\log|1+x| + \frac{1}{2}\int \frac{dx}{\left(x^2-x+\frac{1}{4}\right)+\frac{3}{4}} - \frac{1}{6}\int \frac{2x-1}{1-x+x^2}\,dx$$

$$= \frac{1}{3}\log|1+x| + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + (\sqrt{3}/2)^2} - \frac{1}{6}\log|1+x+x^2| + c$$

$$= \frac{1}{6}\log|1+x| + \frac{1}{2}\,\frac{1}{\sqrt{3}/2}\tan^{-1}\frac{x-\frac{1}{2}}{\sqrt{3}/2} - \frac{1}{6}\log|1-x+x^2| + c$$

$$= \frac{1}{6}\log\left|\frac{(1+x)^2}{1-x+x^2}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + c.$$ **Ans.**

Note

Example 10: Find the Value $\int \frac{2x-3}{x^2+3x-18}dx$.

Solution: $\frac{2x-3}{x^2+3x-18} = \frac{2x-3}{(x-3)(x+6)} = \frac{A}{x-3}+\frac{B}{x+6}$

or $x-3 = A(x+6)+B(x-3)$

when, $x-3=0$ or $x=3$

$3 = 9A$ or $A = \frac{1}{3}$

when, $x+6=0$ or $x=-6$

$-15 = B(-9)$ or $B = \frac{5}{3}$

then, $\int \frac{2x-3}{x^2+3x-18}dx = \frac{1}{3}\int\frac{dx}{x-3}+\frac{5}{3}\int\frac{dx}{x+6}$

$= \frac{1}{3}\log|x-3|+\frac{5}{3}\log|x+6|+c.$ **Ans.**

Example 11: Find the Value $\int \frac{dx}{e^x-1}$.

Solution: $\int\frac{dx}{e^x-1} = \int\frac{e^x\,dx}{e^x(e^x-1)} = \int\frac{dt}{t(t-1)}$ (when $e^x=t, e^x\,dx=dt$)

$= \int\left(\frac{1}{t-1}-\frac{1}{t}\right)dt = \log\left|\frac{t-1}{t}\right|+c$

$= \log\left|\frac{e^x-1}{e^x}\right|+c = \log|1-e^{-x}|+c.$ **Ans.**

Example 12: Find the Value $\int \frac{x}{(x-2)(x-1)^2}dx$.

Solution: Let, $\frac{x}{(x-2)(x-1)^2} = \frac{A}{x-2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$

$x = A(x-1)^2+B(x-1)(x-2)+C(x-2)$

putting equally the factor of same power

$A+B=0,\ -2A-3B+C=1,\ A+2B-2C=0$

$\therefore$ $A=2, B=-2, C=-1.$

$\therefore$ $\int\frac{x\,dx}{(x-2)(x-1)^2} = \int\frac{x\,dx}{(x-2)(x-1)^2}$

$= 2\log|x-2|-2\log|x-1|-\frac{(x-1)^{-1}}{-1}+c$

Note

$$= 2\{\log|x-2| - \log|x-1|\} + \frac{1}{x-1} + c$$

$$= 2\log\left|\frac{x-2}{x-1}\right| + \frac{1}{x-1} + c.$$ **Ans.**

Questionnaire 14.2

Short Answer Questions

Find out the value of following integrals:

1. (*a*) $\int \frac{dx}{x^2-4}$ (*b*) $\int \frac{x^2\,dx}{x^2-4^2}$

(*c*) $\int \frac{dx}{x^2-5}$ (*d*) $\int \frac{dx}{(x+1)^2-4}$

(*e*) $\int \frac{x^2}{x^6-a^6}\,dx$

2. (*a*) $\int \frac{3x\,dx}{(x-2)(x+1)}$ (*b*) $\int \frac{(x+5)}{(x-1)(x-4)}\,dx$

(*c*) $\int \frac{(2x+3)\,dx}{(x+2)(x-2)}$ (*d*) $\int \frac{dx}{(x-1)(x^2-4)}$

Answers

1. (*a*) $\frac{1}{4}\log\left|\frac{x-2}{x+2}\right| + c$ (*b*) $x + 2\log\left|\frac{x-4}{x+4}\right| + c$

(*c*) $\frac{1}{2\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + c$ (*d*) $\frac{1}{4}\log\left|\frac{x-1}{x+3}\right| + c$

(*e*) $\frac{1}{6a^3}\log\left|\frac{x^3-a^3}{x^3+a^3}\right| + c$

2. (*a*) $\log 1|(x-2)^2\,(x+1)| + c$ (*b*) $-2\log|x-1| + 3\log|x-4| + c$

(*c*) $\frac{1}{4}\log|x+2| + \frac{7}{4}\log|x-2| + c$

(*d*) $-\frac{1}{3}\log|x-1| + \frac{1}{4}\log|x-2| + \frac{1}{12}\log|x+2|$

Self Assessment

Note

2. Multiple Choice Questions:

6. What will be value of $\int \frac{dx}{a^2 - x^2}$

   When $x < a$,

   (*a*) $\frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + c$ (*b*) $\frac{1}{a} \log \left|\frac{a+x}{a-x}\right| + c$

   (*c*) $\frac{1}{2a} \left|\frac{a+x}{a-x}\right| + c$ (*d*) $\frac{1}{2a} \log \left|\frac{a-x}{a+x}\right| + c$

7. What will be value of $\int \frac{dx}{4 - x^2}$

   (*a*) $\frac{1}{4} \log \left|\frac{2-x}{2+x}\right| + c$ (*b*) $\frac{1}{4} \log \left|\frac{2+x}{2-x}\right| + c$

   (*c*) $\frac{1}{2} \log \left|\frac{4-x}{4+x}\right| + c$ (*d*) $\frac{1}{2} \log \left|\frac{4+x}{4-x}\right| + c$

8. What will be value of $\int \frac{dx}{x^2 - 4}$

   (*a*) $\frac{1}{4} \log \left|\frac{x+2}{x-2}\right| + c$ (*b*) $\frac{1}{2} \log \left|\frac{x-2}{x+2}\right| + c$

   (*c*) $\frac{1}{4} \log \left|\frac{x-2}{x+2}\right| + c$ (*d*) None of these.

14.4 Summary

- If the multiplication of $f(x)$ and $\phi(x)$ is $f(x). f(x)$, the differentiation is –

  $$\frac{d}{dx}\{f(x)\,.\,\phi(x)\} = f(x)\,\phi'(x) + f'(x)\,\phi(x)$$

- The first function is so selected that the second part of RHS could be easily and successfully integrated. Therefore, there should take care in the selection of first function.
- If between two functions there is a function whose integration we don't know, then that function must be considered as the first function.
- If two functions are same and the integral of both functions known, then if there is any x^n type function, then it always considers first function.
- There is no any authentic method to select the functions during the divided integration, but the students can do the divided integration on selecting the function with the help of above-mentioned 'ILATE'.
- The important method of integration is used the above principle to break the additional and difference form of given function. So the methods of partial fraction are used when necessary.

Note

14.5 Keywords

- ***Summation:*** addition
- ***Integral:*** whole

14.6 Review Questions

1. Find out the value of- $\int x \sin x \, dx$ (**Ans.:** $-x \cos x + \sin x + c$)
2. Find out the value of- $\int x^2 \cos x \, dx$ (**Ans.:** $(x^2 - 2) \sin x + 2x \cos x + c$))
3. Find out the value of- $\int x \tan^{-1} x \, dx$ (**Ans.:** $\frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{x}{2} + c$)
4. Find out the value of- $\int \frac{dx}{2x^2 + x - 1}$ (**Ans.:** $\frac{1}{3} \log \left| \frac{2x-1}{x+1} \right| + c$)
5. Find out the value of- $\int \frac{dx}{x - x^3}$ (**Ans.:** $\log |x| - \frac{1}{2} \log |1 - x| - \frac{1}{2} \log |1 + x| + c$)

Answers: Self Assessment

1. Integral	2. Divided	3. Proved	4. Equals
5. Quardatic	6. (a)	7. (b)	8. (c)

14.7 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India.

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Blum-Viva Publications.

Mathematics for Economist – Mehta and Madnani-Sultan Chand and sons.

Mathematics Economist – Makcal Harrison, Patrick Waldron.

Mathematics for Economist – Karl P. Simon, Laurence Blum.

Mathematics for Economist and Finance – Martin Norman

Mathematics for Economist – Council for Economic Education.

Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 15: Definite Integration

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Meaning, Limitation and Methods of Finding Definite Integration.
- Understand the Substitution of Definite Integration.
- Know the General Properties of Definite Integrals.
- Get Information of Integrals of Infinity Limits.

Introduction

Let the integral of $f(x)$ relative to x is $F(x)$.

If two values of independent variables x, assume, for a and b, difference between integral $F(x)$ of function $f(x)$ is $F(a) - F(b)$, then its difference are known as Definite Integrals of $f(x)$ for interval $[a, b]$ and it will show below:

$$\int_a^b f(x)\ dx$$

The meaining of definite integration- when integral of any function is find out for any two certain limitations then it's called definite integration.

$$\text{If} \int f(x)\,dx = F(x)$$

$$\text{Then} \int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

The value of any definite integrals is unique, since,

If $\int f(x)\,dx = F(x) + c$, then

$$\int_a^b f(x)\,dx = \left[F(x) + c\right]_a^b = \{F(b) + c\} - \{F(a) + c\}$$

$$= F(b) + c - F(a) - c = F(b) - F(a).$$

Note

Notes The value of definite integrals are not effected by **invariable integration.**

15.1 Limitations of Integration

Digit a, that is written below in integration sign called lower limit and digit b that is written up side of integration sign is called the upper limit of integration.

15.2 Method of Finding Definite Integration

To find out the given integrals and keep it in big bracket and written the limitation of integration at right hand side of bracket. Now put the upper limit at the place of variable digit in integrals and keep lower limit in same integrals then both are differentiate, the desired integrals will find out.

EXAMPLES WITH SOLUTIONS

Example 1: Find out the Value of $\int_a^b \frac{\log x}{x}\, dx$**.**

Solution: $\int_a^b \frac{\log x}{x}\, dx$ Let $\log x = t$

$$\frac{1}{x}\, dx = dt$$

$$= \int t\, dt$$

$$= \frac{t^2}{2} = \frac{1}{2}\left[(\log x)^2\right]_a^b$$

$$= \frac{1}{2}[(\log b)^2 - (\log a)^2]$$

$$= \frac{1}{2}\ [\log b + \log a]\ [\log b - \log a]$$

$$= \frac{1}{2}\ \log ab \times \log \frac{b}{a}.$$ **Ans.**

Example 2: Find out the Value of $\int_0^{\pi/4} \tan x \sec x\ dx$**.**

Solution: $\int_0^{\pi/4} \tan x \sec x\ dx = \left[\sec x\right]_0^{\pi/4} = \left[\sec \frac{\pi}{4} - \sec 0\right]$

$$= \sqrt{2}\ - 1.$$ **Ans.**

Example 3: Find out the Value of $\int_1^3 \frac{dx}{x}$**.**

Solution: $= \left[\log x\right]_1^3 = \log 3 - \log 1 = \log 3.$ **Ans.**

Example 4: Find out the Value of $\int_0^{\pi/2} \cos^2 x \; dx$.

Solution: $\int_0^{\pi/2} \cos^2 x \; dx = \int_0^{\pi/2} \left\{\frac{1+\cos 2x}{2}\right\} dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\sin 2 \cdot \frac{\pi}{2}}{2} - 0 - \frac{\sin 2 \times 0}{2}\right) = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$. **Ans.**

Example 5: Find out the Value of $\int_0^{\pi/4} \tan^2 x \, dx$.

Solution: $\int_0^{\pi/4} \tan^2 x \, dx = \int_0^{\pi/4} (\sec^2 x - 1) \; dx$

$= \left[\tan x - x\right]_0^{\pi/4} = \left[\tan \frac{\pi}{4} - \frac{\pi}{4} - (\tan 0 - 0)\right]$

$= 1 - \frac{\pi}{4}$. **Ans.**

Example 6: Find out the Value of $\int_0^{a} y^2$, $x^2 + y^2 = a^2$.

Solution: $\because$ $x^2 + y^2 = a^2 \quad \therefore \quad y^2 = a^2 - x^2$

$\therefore$ $\int_0^{a} y^2 \; dx = \int_0^{a} (a^2 - x^2) \; dx$

$= \left[a^2 x - \frac{x^3}{3}\right]_0^a = a^3 - \frac{a^3}{3} - 0 = \frac{2}{3} a^3$. **Ans.**

Example 7: Find out the Value of $\int_0^{a} \frac{dx}{\sqrt{a^2 - x^2}}$.

Solution: $\int_0^{a} \frac{dx}{\sqrt{a^2 - x^2}} = \left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= \sin^{-1}\left(\frac{a}{a}\right) - \sin^{-1}\left(\frac{0}{a}\right)$

$= \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$. **Ans.**

Example 8: Find out the Value of $\int_0^{\pi/2} x \cos x \; dx$.

Solution: $\int_0^{\pi/2} x \cos x \; dx = \left[x \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx$

Note

$$= \left[\frac{\pi}{2}\sin\frac{\pi}{2} - 0\right] - \left[-\cos x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \left[\cos\frac{\pi}{2} - \cos 0\right] = \frac{\pi}{2} + 0 - 1 = \frac{\pi}{2} - 1.$$ **Ans.**

Example 9: Find out the Value of $\int_0^{\pi/4}\sqrt{1-\sin 2x}\; dx$.

Solution: $\int_0^{\pi/4}\sqrt{1-\sin 2x}\; dx = \int_0^{\pi/4}\sqrt{\cos^2 x + \sin^2 x - 2\sin x\cos x}\; dx$

$$= \int_0^{\pi/4}\sqrt{(\cos x - \sin x)^2}\; dx$$

$$= \int_0^{\pi/4}(\cos x - \sin x)\; dx$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4}$$

$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0° - \cos 0°)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 - 1)$$

$$= \frac{2}{\sqrt{2}} + 1 = (\sqrt{2} + 1).$$ **Ans.**

Example 10: Find out the Value of $\int_1^2 x\log x\; dx$.

Solution: $\int_1^2 x\log x\; dx = \left[\frac{x^2}{2}\log x\right]_1^2 - \int_1^2 \frac{1}{x}\cdot\frac{x^2}{2}\; dx$

$$= \left[\frac{x^2}{2}\log x\right]_1^2 - \frac{1}{2}\int_1^2 x\; dx$$

$$= \left[\frac{x^2}{2}\log x\right]_1^2 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2$$

$$= 2\log 2 - \frac{1}{4}\left[x^2\right]_1^2$$

$$= 2\log 2 - \frac{1}{4}[4 - 1]$$

$$= 2\log 2 - \frac{3}{4}.$$ **Ans.**

Example 11: Find out the Value of $\int_0^{\pi/4}\tan x\; dx$

Solution: $\int_0^{\pi/4}\tan x\; dx = \left[\log\sec x\right]_0^{\pi/4}$

Note

$= \log \sec \frac{\pi}{4} \quad \log \sec 0$

$= \log \sqrt{2} - \log 1, \qquad [\because \quad \log 1 = 0]$

$= \log \sqrt{2}$. **Ans.**

Example 12: Find out the Value of $\int_0^{\pi/6} \sqrt{1 - \sin 2x}\ dx$.

Solution: $\int_0^{\pi/6} \sqrt{1-\sin 2x}\ dx = \int_0^{\pi/6} \sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}\ dx$

$= \int_0^{\pi/6} \sqrt{(\cos x - \sin x)^2}\ dx$

$= \int_0^{\pi/6} (\cos x - \sin x)\, dx = [\sin x + \cos x]_0^{\pi/6}$

$= \left[\sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right] - [\sin 0 + \cos 0]$

$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) - (0 + 1) = \frac{1}{2} + \frac{\sqrt{3}}{2} - 1$

$= \frac{1 + \sqrt{3} - 2}{2} = \frac{\sqrt{3} - 1}{2}$.

Questionnaire 15.1

Find out the values of following integrals:

1. $\int_1^2 x^4\ dx$.
2. $\int_1^4 \frac{dx}{\sqrt{x}}$.
3. $\int_0^3 e^{x/3}\ dx$.
4. $\int_0^{\pi/2} \cos x\ dx$.
5. $\int_1^2 \frac{dx}{x}$.
6. $\int_0^{\pi/4} \sec^2 x\ dx$.
7. $\int_0^{\pi/2} \sin x\ dx$.
8. $\int_0^{\pi} \sin 3x\ dx$.
9. $\int_{\pi/6}^{\pi/2} \cos x\ dx$.
10. $\int_4^9 \sqrt{x}\ dx$.
11. $\int_0^1 \frac{dx}{1+x^2}$.
12. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx$.
13. $\int_{-\pi/4}^{\pi/4} \text{cosec}^2 x\ dx$.

Answers

1. $\frac{31}{5}$
2. 2
3. $3(e-1)$
4. 1
5. log 2
6. 1

Note

7. 1 **8.** $\frac{2}{3}$ **9.** $\frac{1}{2}$

10. $\frac{38}{3}$ **11.** $\frac{\pi}{4}$ **12.** $\frac{\pi}{2}$

13. – 2

15.3 Substitution in Definite Integration

Some time it is necessary to change the variable in definite integration like indefinite integration. Limits changed with the substitution for making the method easy.

Example, let we put $\psi(x) = t$, then

$$\int f[\psi(x)\,\psi'(x)]\;dx = \int (t)\;dt.$$

Did u know? If integrals limit are a to b for variables x, then limit relation for t is $t = \psi(x)$.

Since, When $x = a$, then $t = \psi(x)$ and when $x = b$, then $t = \psi(x)$,

So,

$$\therefore \quad \int_a^b f[\phi(x)]\,\phi'(x)\,dx = \int_{\psi(a)}^{\psi(b)} f(t)\,dt.$$

EXAMPLES WITH SOLUTIONS

Example 1: Find out the value of- $\int_1^3 \frac{\cos(\log x)}{x}\,dx.$

Solution: $\int_1^3 \frac{\cos(\log x)}{x}$ Let, $\log x = t$ then $(1/x)\,dx = dt$

When $x = 1,\ t = \log(1) = 0$

and when, $x = 3,\ t = \log 3$

$$\therefore \quad \int_1^3 \frac{\cos(\log x)}{x}\,dx = \int_0^{\log 3} \cos t\;dt = [\sin t]_0^{\log 3}$$

$$= \sin(\log 3) - \sin(0) = \sin(\log 3).$$ **Ans.**

Example 2: Find out the value of- $\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx$.

Solution: Let, $x = \tan\theta$; $\therefore\ dx = \sec^2\theta\,d\theta$

$x = 0$, and $\tan\theta = 0$ so, $\theta = 0$

and when, $x = 1$, then $\tan\theta = 1$ so, $\theta = \frac{1}{4}\pi$

$$\therefore \quad \int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx = \int_0^{\pi/4} \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\,.\sec^2\theta\,d\theta$$

Note

$$= \int_0^{\pi/4} \frac{\tan\theta}{\sec\theta} \,.\, \sec^2\theta \, d\theta = \int_0^{\pi/4} \sec\theta \tan\theta \; d\theta$$

$$= \left[\sec\theta\right]_0^{\pi/4} = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1.$$ **Ans.**

Example 3: Find out the value of- $\int_0^a \frac{x}{\sqrt{a^2 - x^2}}\, dx.$

Solution: Let, $a^2 - x^2 = t$, then, $-2x\,dx = dt$ or $x\,dx = -\frac{1}{2}\,dt$

when, tc $x = 0$, $t = a^2$ and when, $x = a$, $t = a^2 - a^2 = 0$

$$\int_0^a \frac{x}{\sqrt{a^2 - x^2}}\, dx = -\frac{1}{2}\int_{a^2}^0 \frac{dt}{\sqrt{t}} = -\frac{1}{2}\int_{a^2}^0 t^{-1/2}\, dt$$

$$= -\frac{1}{2}\left[2t^{1/2}\right]_{a^2}^0$$

$$= -[0 - (a^2)^{1/2}] = (a^2)^{1/2} = a.$$ **Ans.**

Example 4: Find out the value of- $\int_0^{\pi/3} \frac{\cos x}{3 + 4\sin x}\, dx.$

Solution: Let, $3 + 4\sin x = t$, then, $4\cos x\, dx = dt$

and when, $x = 0$, $t = 3 + 4\sin(0) = 3 + 4(0) = 3$, $\because \sin(0) = 0$

or when, $x = \frac{1}{3}\pi$, $t = 3 + 4\sin\left(\frac{1}{3}\pi\right) = 3 + 4\,(\sqrt{3}/2) = 3 + 2\sqrt{3}$

$$\therefore \quad \int_0^{\pi/3} \frac{\cos x}{3 + 4\sin x}\, dx = \frac{1}{4}\int_3^{(3+2\sqrt{3})} \frac{dt}{t}$$

$$= \frac{1}{4}\left[\log t\right]_3^{3+2\sqrt{3}} = \frac{1}{4}[\log(3 + 2\sqrt{3}) - \log 3]$$

$$= \frac{1}{4}\left[\log\left(\frac{3+2\sqrt{3}}{3}\right)\right].$$ **Ans.**

Example 5: Find out the value of- $\int_0^{\pi} x\sin^2 x\, dx$ **.**

Solution: $\int x\sin^2 x\, dx = \frac{1}{2}\int x\,(2\sin^2 x)\, dx$

$$= \frac{1}{2}\int x\,(1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\int x\, dx - \frac{1}{2}\int x\cos 2x\, dx$$

Note

$$= \frac{1}{2} \cdot \frac{1}{2} x^2 - \frac{1}{2} \left[\frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x\right]$$

$$= \frac{1}{4} x^2 - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x,$$

$$\therefore \quad \int_0^{\pi} x \sin^2 x \, dx = \left[\frac{1}{4} x^2 - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x\right]_0^{\pi}$$

$$= \frac{1}{4} [\pi^2] - \frac{1}{4} [\pi \sin 2\pi] - \frac{1}{8} [\cos 2\pi] - \left[0 - 0 - \frac{1}{8} \cos 0\right]$$

$$= \frac{1}{4} \pi^2 - \frac{1}{4} [0] - \frac{1}{8} [1] + \frac{1}{8}$$

$$= \frac{1}{4} \pi^2.$$ **Ans.**

Example 6: Find out the value of $\int_1^2 \frac{\log_e x}{x^2} \, dx$.

Solution:
$$\int \frac{\log_e x}{x^2} \, dx = \int (\log_e x)(x^{-2}) \, dx$$

$$= (\log_e x)(-x^{-1}) - \int (1/x)(-x^{-1}) \, dx,$$

$$= -(\log_e x)(x^{-1}) + \int x^{-2} \, dx$$

$$= -\frac{\log_e x}{x} - \frac{1}{x} = -\left[\frac{\log_e x + 1}{x}\right]$$

$$\therefore \quad \int_1^2 \frac{\log_e x}{x^2} \, dx = -\left[\frac{\log_e x + 1}{x}\right]_1^2$$

$$= -\left[\left(\frac{\log_e 2 + 1}{2}\right) - \left(\frac{\log_e 1 + 1}{1}\right)\right]$$

$$= -\left[\frac{1}{2} \log_e 2 + \frac{1}{2} - 1\right] \qquad [\because \log_e 1 = 0]$$

$$= -\left[\frac{1}{2} \log_e 2 - \frac{1}{2}\right] = \frac{1}{2} [1 - \log_e 2].$$ **Ans.**

Example 7: Find out the value of $\int_0^{\pi/2} \frac{\cos x}{(1 + \sin x)(2 + \sin x)} \, dx$.

Solution: Put $\sin x = t$. then solve by partial method

Note

$$\int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)} = \left[\log\left(\frac{1+\sin x}{2+\sin x}\right)\right]_0^{\pi/2}$$

$$= \log\left(\frac{1+\sin\frac{1}{2}\pi}{2+\sin\frac{1}{2}\pi}\right) - \log\left(\frac{1+\sin 0}{2+\sin 0}\right)$$

$$= \log\left(\frac{2}{3}\right) - \log\left(\frac{1}{2}\right) = \log\left(\frac{\frac{2}{3}}{\frac{1}{2}}\right) = \log\frac{4}{3}. \qquad \textbf{Ans.}$$

Example 8: Find out the value of $\int_0^{\pi/2} x \sin x\, dx$.

Solution: $$\int_0^{\pi/2} x\sin x\; dx = [x(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} 1.(-\cos x)\; dx$$

$$= [-x\cos x]_0^{\pi/2} + [\sin x]_0^{\pi/2}$$

$$= -[x\cos x]_0^{\pi/2} + [\sin x]_0^{\pi/2}$$

$$= -\left[\frac{\pi}{2}\cos\frac{\pi}{2} - 0\right] + \left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= -0 + 1$$

$$= 1. \qquad \textbf{Ans.}$$

Questionnaire 15.2

Short Answer Questions.

Find out the value of following integrals:

1. $\int_1^3 \frac{\log x}{x}\; dx$
2. $\int_1^2 \frac{\cos(\log x)}{x}\; dx$
3. $\int_0^1 x\log\left(1+\frac{x}{2}\right)\; dx$
4. $\int_1^e \frac{e^x}{x}(1 + x\log x)\, dx$
5. $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\; dx$
6. $\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2}\; dx$
7. $\int_0^1 \frac{x\sin^{-1} x}{\sqrt{1-x^2}}\; dx$
8. $\int_0^\infty \frac{\sin\tan^{-1} x}{1+x^2}\; dx$
9. $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$; $\beta > \alpha$
10. $\int_0^1 \frac{x}{\sqrt{1-x^2}}\; dx$

Note

11. $\int_0^a \sqrt{a^2 - x^2}\ dx$

12. $\int_0^1 x\sqrt{\frac{1 - x^2}{1 + x^2}}\ dx$

13. $\int_0^1 \frac{x^2\ dx}{1 + x^6}$

14. $\int_0^1 x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\ dx$

15. $\int_0^1 \frac{dx}{3 + 2x + x^2}$

16. $\int_0^1 \frac{2x}{1 + x^4}\ dx$

17. $\int_0^2 \sqrt{\frac{2 + x}{2 - x}}\ dx.$

Answers

1. $\frac{(\log 3)^2}{2}$

2. $\sin(\log 2)$

3. $\frac{3}{4} - \frac{3}{2}\log\frac{3}{2}$

4. e^e

5. $\frac{1}{8}\pi^2$

6. $\frac{\pi^3}{192}$

7. 1

8. 1

9. π

10. 1

11. $\frac{\pi a^2}{4}$

12. $\frac{\pi - 2}{4}$

13. $\frac{\pi}{12}$

14. $\frac{a^2(\pi - 2)}{4}$

15. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{2\sqrt{2}}$

16. $\frac{\pi}{4}$

17. $\pi + 2$

Self Assessment

1. Fill in the blanks:

1. When integral of any function is find out for any two certain limitations then it's called …………… …..
2. The value of any definite integrals is ………………….
3. Some time it is………….. ………….. to change the variable in definite integration like indefinite integration
4. The value of definite integration are not affected by the integration of…………………….
5. Digit a that are written below of integration sign, called the……………………..of integration.

15.4 General Properties of Definite Integrals

Property: 1. $\int_a^b f(x)\ dx = -\int_a^b f(x)\ dx.$

Proof: Let, $\int f(x)\ dx = F(x)$

Note

$\therefore$ Left hand side = $\int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a)$

Right hand side = $-\int_b^a f(x)\ dx = -\left[F(x)\right]_b^a$

$= -[F(a) - F(b)] = F(b) - F(a)$

$\therefore$ $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$

Property: 2. $\int_a^b f(x)\ dx = \int_a^b f(t)\ dt.$

Proof: Left hand side = $\int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a)$

Right hand side = $\int_a^b f(t)\ dt = \left[F(t)\right]_a^b = F(b) - F(a)$

$\therefore$ $\int_a^b f(x)\ dx = \int_a^b f(t)\ dt.$

Property: 3. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$, **when** $a < c < b$.

Proof: Let, $\int f(x)\ dx = F(x)$

Left hand side = $\int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a)$

Right hand side = $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

$= \left[F(x)\right]_a^c + \left[F(x)\right]_c^b$

$[F(c) - F(a)] + [F(b) - F(c)] = F(b) - F(a)$

On detailed from

$$\int_a^b f(x)\ dx = \int_a^{c_1} f(x)\, dx + \int_{c_1}^{c_2} f(x)\, dx + \ldots.. + \int_{c_{n-1}}^{c_n} f(x)\, dx + \int_{c_n}^{b} f(x)\, dx$$

when, $a < c_1 < c_2 < c_3 < \ldots..\ c_n < b.$

Property: 4. $\int_0^a f(x)\ dx = \int_0^a f(a - x)\ dx.$

Proof: Let, $a - x = t$ $\therefore\ -dx = dt$;k $dx = -dt$

and when $x = 0$, then, $t = a$ and when, tc $x = a$, then, $t = 0$

$\therefore$ Right hand side = $\int_0^a f(a - x)\ dx = \int_a^0 f(t)\ (-dt) = -\int_a^0 f(t)\ dt$

$= \int_0^a f(t)\ dt$ By property 1

$= \int_0^a f(x)\ dx$ By property 2

= Right hand side

Note

Property: 5. $\int_{-a}^{a} f(x)\, dx = 0$, **if** $f(x)$**, is the odd function of** x

mean $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$$

if $f(x)$ is the even function of x mean $f(-x) = f(x)$.

Proof: $\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx$ ...(*i*)

$\because -a < 0 < a$

Now, $\int_{-a}^{0} f(x)\, dx = \int_{a}^{0} f(-t)\, (-dt)$, Put $x = -t \Rightarrow dx = -dt$

$$= -\int_{a}^{0} f(-t)\, dt$$

$$= \int_{0}^{a} f(-t)\, dt \quad \text{(By property 1)}$$

$$= \int_{0}^{a} f(-x)\, dx \quad \text{(By property 2)}$$

Condition I: If $f(x)$ is the odd function of x.

then, $f(-x) = -f(x)$ (by definition)

$$\int_{-a}^{0} f(x)\, dx = -\int_{0}^{a} f(x)\, dx$$

By equation (i),

$$\int_{-a}^{a} f(x)\, dx = -\int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(x)\, dx = 0.$$

Condition II: If $f(x)$ is the even function of x.

then, $f(-x) = f(x)$ (By definition)

$\therefore$ $\int_{-a}^{0} f(x)\, dx = \int_{0}^{a} f(x)\, dx$

Bt equation (i)

$$\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$

$$= 2\int_{0}^{a} f(x)\, dx.$$

EXAMPLES WITH SOLUTIONS

Example 1: Prove that-

$$\int_{0}^{\pi/2} \sin^2 x\, dx = \int_{0}^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right) dx.$$

Solution: Let hand side = $\int_{0}^{\pi/2} \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_{0}^{\pi/2}$

Note

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\sin \pi - \sin 0}{2}\right] = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} \qquad \text{...(i)}$$

$$\text{Right hand side} = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \cos^2 x = \int_0^{\pi/2} \frac{1 + \cos 2x}{2} dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) + \frac{\sin \pi - \sin 0}{2}\right]$$

$$= \frac{\pi}{4} \qquad \text{...(ii)}$$

$\therefore$ Left hand side = Right hand side

Example 2: Prove that

$$\int_{-\pi/2}^{\pi/2} x \sin x \, dx = 2 \int_0^{\pi/2} x \sin x \, dx$$

and, $\int_{-\pi/2}^{\pi/2} x \cos x \, dx = 0.$

Solution: Here function is, $f(x) = x \sin x$

$\therefore \quad f(-x) = (-x) \sin(-x) = x \sin x = f(x)$

Function is even

$$\therefore \quad \int_{-\pi/2}^{\pi/2} x \sin x \, dx = 2 \int_0^{\pi/2} x \sin x \, dx$$

If, $f(x) = x \cos x$

$\therefore \quad f(-x) = (-x) \cos(-x) = -x \cos x = -f(x)$

function is odd

$$\therefore \quad \int_{-\pi/2}^{\pi/2} x \cos x \, dx = 0.$$

Example 3: Prove that

$$\int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0.$$

Solution:

Let, $$I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx \qquad \text{...}(i)$$

$$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx, \text{ (By property) (4) is}$$

Note

$$= \int_0^{\pi/2} \sin(\pi - 2x) \log \cot x \, dx$$

$$= \int_0^{\pi/2} \sin 2x \log \cot x \, dx \qquad ...(ii)$$

Add equation (i) and (ii),

$$2I = \int_0^{\pi/2} (\sin 2x \log \tan x + \sin 2x \log \cot x) \, dx$$

$$= \int_0^{\pi/2} \sin 2x \, (\log \tan x + \log \cot x) \, dx$$

$$= \int_0^{\pi/2} \sin 2x . \log (\tan x . \cot x) \, dx$$

$$= \int_0^{\pi/2} \sin 2x . \log 1 \, dx$$

$$= \int_0^{\pi/2} \sin 2x . 0 \, dx = 0$$

$\therefore \qquad I = 0.$

Example 4: Prove that-

$$\int_{\pi/8}^{3\pi/8} \frac{\tan^2 x}{\tan^2 x + \cot^2 x} \, dx = \frac{\pi}{8}.$$

Solution: Let, $I = \int_{\pi/8}^{3\pi/8} \frac{\tan^2 x}{\tan^2 x + \cot^2 x}$ ...(i)

$$x = \frac{\pi}{2} - t \quad \text{because;} \quad a + b = \frac{\pi}{8} + \frac{3\pi}{8} = \frac{\pi}{2}$$

$$I = \int_{3\pi/8}^{\pi/8} \frac{\cot^2 t}{\cot^2 t + \tan^2 t} (-dt)$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{\cot^2 t}{\cot^2 t + \tan^2 t} \, dt$$

or $$I = \int_{\pi/8}^{3\pi/8} \frac{\cot^2 t}{\cot^2 x + \tan^2 x} \, dx \qquad ...(ii)$$

Adding equation (i) and (ii),

$$2I = \int_{\pi/8}^{3\pi/8} dx = [x]_{\pi/8}^{3\pi/8}$$

$$2I = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{\pi}{4}$$

or $$I = \frac{\pi}{8}.$$ **Ans.**

Note

Questionnaire 15.3

Long Answer Questions:

1. Prove that $\int_0^{\pi/2} \log\ (\tan x)\ dx = 0.$

2. Prove that $\int_1^4 \frac{\sqrt{x}\ dx}{\sqrt{5-x}+\sqrt{x}} = \frac{3}{2}.$

3. Prove that $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{4}.$

4. Prove that $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\ dx = \frac{\pi}{4}.$

15.5 Integral of Infinite Limit

If high limit of any definite integrals is ∞, then

$$\int_a^{\infty} f(x)\ dx \text{ it means } \lim_{b\to\infty} \int_a^b f(x)\ dx,$$

Condition is that limit is any finite variable.

If the limit of any definite integration is (-∞)

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{a\to-\infty} \int_a^b f(x)\ dx,$$

Condition is that limit is any finite variable.

EXAMPLES WITH SOLUTIONS

Example 1: Find the value of $\int_1^{\infty} \frac{1}{x^2}\ dx.$

Solution: $$\int_1^{\infty} \frac{1}{x^2}\ dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1.$$ **Ans.**

Example 2: Find the value of $\int_0^{\infty} \frac{1}{1+x^2}\ dx.$

Solution: $$\int_0^{\infty} \frac{1}{1+x^2}\ dx = \lim_{b\to\infty}\left[\tan^{-1} x\right]_0^b$$

$$= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{1}{2}\pi.$$ **Ans.**

Example 3: Find the Value of $\int_0^{\infty} \frac{dx}{(1+x^2)^2}.$

Solution: Firstly we find the value of $I = \int_0^b \frac{dx}{(1+x^2)^2}$

Note Let $x = \tan\theta$ $\therefore$ $dx = \sec^2\theta\, d\theta$

Limits of θ are, 0, $\tan^{-1} b$

$$I = \int_0^{\tan^{-1} b} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int_0^{\tan^{-1} b} \cos^2\theta\, d\theta = \int_0^{\tan^{-1} b} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\tan^{-1} b} = \frac{\tan^{-1} b}{2} + \frac{2b}{4(1+b^2)}.$$

$\therefore$ Require integration $= \lim_{b\to\infty}\left[\frac{\tan^{-1} b}{2} + \frac{\frac{1}{2b}}{1+\frac{1}{b^2}}\right] = \frac{\pi}{4} + 0 = \frac{\pi}{4}.$ **Ans.**

Questionnaire 15.4

Short Answer Questions:

Find the values of following integration:

1. $\int_0^\infty \frac{dx}{a^2+x^2}.$ **2.** $\int_0^\infty \frac{dx}{x^2\sqrt{a^2+x^2}}.$ **3.** $\int_0^\infty e^{-x/2}\, dx.$

4. $\int_0^\infty xe^{-x}\, dx.$ **5.** $\int_0^\infty \frac{e^x\, dx}{1+e^{2x}}.$ **6.** $\int_1^\infty \frac{(x^2+3)\, dx}{x^6(x^2+1)}.$

Answers

1. $\frac{\pi}{2a}$ **2.** $\left(\frac{\sqrt{2}-1}{a^2}\right)$ **3.** 2

4. 1 **5.** $\frac{\pi}{4}$ **6.** $\frac{1}{30}(58-15\pi)$

Self Assessment

2. Multiple Choice Question:

6. $\int_a^\infty f(x)\, dx$ it means

(a) $\lim_{b\to\infty}\int_a^b f(x)\, dx$ (b) $\lim_{b\to\infty}\int_b^a f(x)\, dx$

(c) $\lim_{b\to 0}\int_a^b f(x)\, dx$ (d) $\lim_{a\to\infty}\int_a^b f(x)\, dx$

7. If lower limit of any definite integration is (-∞), then $\int_{-\infty}^b f(x)\, dx$

(a) $\lim_{a\to\infty}\int_a^b f(x)\, dx$ (b) $\lim_{a\to-\infty}\int_a^b f(x)\, dx$

(c) $\lim_{a\to-\infty}\int_b^a f(x)\, dx$ (d) $\lim_{a\to 0}\int_a^b f(x)\, dx$

Note

8. Value of $\int_{1}^{\infty} \frac{1}{x^2}\, dx$

   (*a*) 0 (*b*) ∞

   (*c*) 1 (*d*) –1

15.6 Summary

- If two values of independent variables x, assume, for a and b, difference between integral $F(x)$ of function $f(x)$ is $F(a)$-$F(b)$, then its difference are known as Definite Integrals of $f(x)$ for interval $[a, b]$
- The value of any definite integrals is unique.
- To find out the given integrals and keep it in big bracket and written the limitation of integration at right hand side of bracket. Now put the upper limit at the place of variable digit in integrals and keep lower limit in same integrals then both are differentiate, the desired integrals will find out.
- Some time it is necessary to change the variable in definite integration like indefinite integration. Limits changed with the substitution for making the method easy.

15.7 Keywords

- ***Function:*** A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.
- ***Lower limit:*** inferior limit
- ***Upper limit:*** higher limit.

15.8 Review Questions

1. Find out the value of- $\int_{0}^{\pi/4} \tan^2 x\, dx$ (**Ans.:** $1 - \frac{\pi}{4}$)
2. Find out the value of- $\int_{0}^{a} y^2\, dx$ Where, $x^2 + y^2 = a^2$ (**Ans.:** $\frac{2}{3}a^2$)
3. Find out the value of- $\int_{0}^{\pi} x \sin^2 x\, dx$ (**Ans.:** $\frac{1}{4}\pi^2$)
4. Prove that- $\int_{0}^{a} f(x)\, dx = \int_{0}^{a} f(a-x)\, dx$
5. Prove that- $\int_{0}^{\pi/2} \sin 2x \log \tan x\, dx = 0$

Answers: Self Assessment

1. Definite Integration	2. Unique	3. Necessary	4. Non-variable
5. Lower limit	6. (a)	7. (b)	8. (c)

Note

15.9 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India.

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Bloom – Viva Publications

Mathematics for Economist – Makcal Harrison, Patrick Waldron.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and sons.

Mathematics for Economist – Karl P. Simon, Laurence Blum.

Mathematics for Economist and Finance – Martin Norman.

Mathematics for Economist – Council for Economic Education.

Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 16: Economic Applications of Integration

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Find out the Area under Plane Curves.
- Understand the Use of Integration in Economics.

Introduction

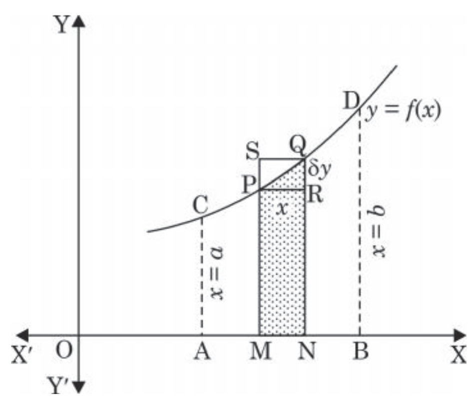
Now-a-day's use of integration in economics has increased fast. Definite integral is the range of addition. Use of integration in economics can be explained from the following facts -

1. Marginal Costs, Average Costs and Total Costs since Marginal Costs = $\frac{d(c)}{dq}$, therefore total costs $\int \frac{d(c)}{dq} . dq$
2. ***Consumer Saving:*** Consumer saving can also be found out from integration.

16.1 Area Under Plane Curves

Suppose *CPQD* represents curve $y = f(x)$, where $f(x)$ is a continuous function of x *in prant* $[a, b]$ and suppose like the value of x increases from a to b, y also increases. Suppose *CA* and *DB* are the ordinates at $x = a$ and $x = b$.

Take any point $P(x, y)$ located on the curve and assume that *PM* is its grade. Take another point $Q(x + \delta x, y + \delta y)$ on the curve close at point *P* and suppose its ordinate is *QN*.



Draw perpendicular *PR*, *QS* on *QN* and *MP* then

$$OM = x, ON = x + \delta x;$$

$\therefore$ $MN = \delta x$

And $PM = y,$

$$QN = y + \delta y; \quad \therefore \quad QR = \delta y.$$

Note If $AMPC$ and $ANQC$ are serially presented by $S + \delta S$ then

δS = Area $ANQC$ – area $AMPC$ = Area $MNQP$

Now $MNQP$ in area result is located between the rectangles $MNRP$ and $MNQS$, i.e.,

$$y\delta x < \delta S < (y + \delta y)\,\delta x$$

Or
$$y < \frac{\delta S}{\delta x} < (y + \delta y) \quad \text{(dividing by } \delta x\text{)} \qquad ...(1)$$

Or
$$\frac{dS}{dx} = y = f(x), \qquad \text{(when } \delta x \to 0 \text{ and } dy \to 0^o\text{)}$$

Therefore
$$\int_a^b f(x)\,dx = \int_a^b \frac{dS}{dx}.\,dx = \int_a^b dS = [S]_a^b$$

$$= \text{(Value of } S \text{ when } x = b) - \text{(Value of } S \text{ when } x = a)$$

$$= \text{Area ABDC}$$

Definite integral is a limit of addition. Suppose any function is continued in any interval $[a,b]$ its article is a curve in which C and D are two points. CA and DA are the perpendiculars on the X-axis. 'a' and 'b' are the abscissa of points C and D. Interval $[a,b]$ i.e. AB whose value is $(b\text{-}a)$, is divided into n equal small parts. Suppose the abscissa of points P and Q which are on the curve are $(a + rh)$ and $(a + r + 1h)$. PM and QN are their ordinates. This way under the marginal condition when $h \to 0$, $PRNM$ is a very small rectangle. Area of this rectangle will be $f\ (a + rh) \times h$. This way definite Integral in the words of Addition,

$$\lim_{n \to \infty} h \cdot [f(a+h) + f(a+2h) + ... + f(a+nh)]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} h\,.\,f(a + rh), \text{ where } b - a = nh$$

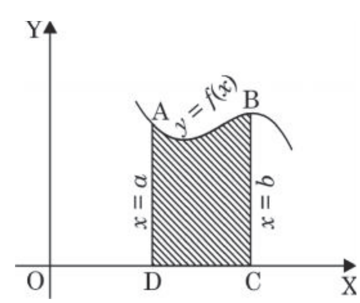
$$= \int_a^b f(x)\,dx \ \text{ is as shown.}$$

> *Notes* $\int_a^b f(x)\,dx$ is the complete area which is formed within the limits of curve $y = f(x)$, x-axis, and $x = a$ to $x = b$.

Area of the space surrounded by the curve $y = f(x)$, x-axis, and $x = a$ to $x = b$ is calculated from the following formula:

$$\boxed{A = \int_{x=a}^{x=b} y\,dx}$$

Value of x $(x = a)$ from the left side at end D is considered as the lower limit of integration and value of x $(x = b)$ from the right side at end C is considered as the higher limit of integration.



In the same way, Area of the space surrounded by the curve $x = f(y)$, y-axis, and $y = a$ to $y = b$ s calculated from the following formula:

$$\boxed{A = \int_{y=a}^{y=b} x\,dy}$$

In this case limits of integration are taken from lower to upper side.

Note

EXAMPLES WITH SOLUTION

Example 1: Find the area between $y = mx$, x-axis and ordinate $x = 2$.

Solution: Equation of the given curve is

$$Y = mx \quad \ldots (1)$$

On putting $y = 0$ in eq. 1, $x = 0$ $\quad (\therefore \text{ at } x\text{-axis}, y = 0)$

$$\therefore \quad \text{Desired area} = \int_0^2 y\,dx = \int_0^2 mx\,dx = m\int_0^2 x\,dx$$

$$= m\left[\frac{x^2}{2}\right]_0^2 = \frac{m}{2}(4-0) = 2m \text{ square unit.}$$ **Ans.**

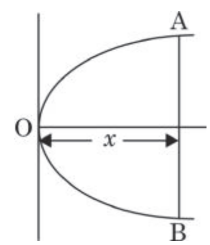
Example 2: Find the area between Parabolic line $y^2 = 4x$ and simple line $x = 4$.

Solution: Equation of Parabola

$$y^2 = 4x$$

Line $x = 4$ meet the parabola at points A and B.

Desired area is $OBAO$.



Note therefore

$$\text{Area} = 2\int_0^4 y\,dx$$

$$= 2\int_0^4 \sqrt{4x}\,dx = 4\int_0^4 x^{\frac{1}{2}}\,dx$$

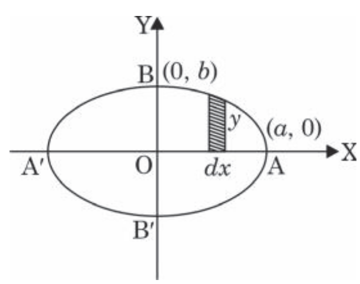
$$= 4 \times \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$= \frac{8}{3}\left[4^{\frac{3}{2}} - 0\right]$$

$$= \frac{8}{3}(8-0) = \frac{64}{3} \text{ square unit}$$ **Ans.**

Example 3: Find the area of Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: Area of Ellipse $ABA'B'$ has to calculate. First we will find the area of space OAB and on four folding it we will find the desired area.



$$\because \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \quad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\Rightarrow \quad y = \frac{b}{a}\sqrt{a^2 - x^2}$$

On putting $x = 0$ for point b and $y = 0$ for point A, coordinates of points B and A will be (a, b) and $(a, 0)$.

Note

$$\text{Desired Area} = 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= 4\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a = \pi ab \text{ square unit}$$ **Ans.**

Example 4: Find the area of Parabola $y^2 = 4ax$ and the space surrounded by its Latus Rectum.

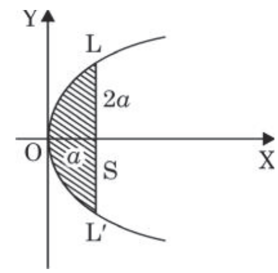
Solution: Equation of Parabola

$$y^2 = 4ax$$

Latus Rectum meets the parabola at the points L and L'.

Desired Area is $OL'SLO$

If Area is A, then



$$A = 2\int_0^a y\,dx$$

$$= 2\int_0^a \sqrt{4ax}\,dx$$

$$= 2\int_0^a 2a^{1/2}x^{1/2}\,dx = 2 \times 2a^{1/2}\int_0^a x^{1/2}\,dx$$

$$= 4a^{1/2}\left[\frac{x^{3/2}}{3/2}\right]_0^a = \frac{2.4}{3}a^{1/2}\left[x^{3/2}\right]_0^a$$

$$= \frac{8}{3}a^2 \text{ Square unit}$$ **Ans.**

> *Task* Find the area between the Parabola $y^2 = 4x$ and simple line $x = 4$
>
> (**Ans.:** $\frac{8}{3}a^2$ square unit)

Questionnaire 16.1

Short Answer Questions:

Find the area surrounded by the following curves, X-axis and given Abscissa:

1. $y = e^x$ from $x = 0$ to $x = 2$
2. $y = x \sin x$, from $x = 0$ to 2π

   a. [**Hint:** Desired Area $= \int_0^{2\pi} y\,dx = \int_0^{2\pi} x\sin x\,dx = \left[-x\cos x + \sin x\right]_0^{2\pi}$ on integrating]

3. $y = \left(1 + \frac{8}{x^2}\right)$, from $x = 2$ to $x = 10$

4. $y = 5+\frac{1}{10}x^2$, from $x = 2$ to $x = 10$
5. $xy = b^2$ from $x = a$ to $x = b$
6. $y = x\,e^{x^2}$ from $x = 0$ to $x = b$
7. Find the area between simple line $y = mx$, x-axis, and abscissa $x = 3$
8. Find the ratio of areas of parts of curve $x^2 + y^2 = a^2$ divided by the simple line $x = \frac{1}{2}a$.

[**Hint:** let us assume the ratio of areas of the desired parts $x = a \sin \theta$].

$$= \int_{a/2}^{a} y\,dx : \int_{-a}^{a/2} y\,dx$$

$$= \int_{a/2}^{a} \sqrt{a^2-x^2}\,dx : \int_{-a}^{a/2} \sqrt{a^2-x^2}\,dx$$

Answers

1. $e^2 - 1$
2. 2π square unit
3. $\frac{56}{5}$ Square unit
4. $73\frac{1}{15}$ square unit
5. $b^2\left[\log\frac{b}{a}\right]$
6. $\frac{1}{2}\left[e^{b^2}-1\right]$
7. $\frac{9}{2}m$ Square unit
8. $(4\pi - 3\sqrt{3}) : (8\pi + 3\sqrt{3})$

Self Assessment

1. Fill in the blanks:

1. Now a day's use integration in economics has increased ....
2. Definite integral is a limit of ....
3. Consumer saving can also be found out from ....

16.2 Use of Integration in Economics

Now-a-day's use integration in economics has increased very fast. This we can prove from the following facts–

1. **Marginal Costs, Average Costs and Total Costs**

   Since Marginal Costs = $\frac{d(c)}{dq}$, therefore

   Total Costs = $\int \frac{d(c)}{dq} \cdot dq$

Here, $\frac{d(c)}{dq}$ = Marginal Costs, C = Total Costs dq = Change in production.

Note

Example 1: Marginal Costs = $3 + 150q - 9q^2$ of any firm is given, then find the total costs when On manufacturing 3 units the total costs will be ₹ 300 .

Solution: We know that Total costs,

$$TC = \int (3 + 150q - 9q^2)\, dq$$

$$= 3q + 150\frac{q^2}{2} - \frac{q^3}{3} + c$$

$$= 3q + 75q^2 - 3q^3 + c$$

And Average costs $$AC = \frac{TC}{Q} = 3 + 75q - 3q^2 + \frac{c}{Q}$$

Since $q = 3$ and $TC = 300$ given, therefore

$$300 = (3 \times 3) + 75\,(3)^2 - 3 \times 3 \times 3 + c$$

$$= 9 + (75 \times 9) - 27 + c$$

$$= 9 + 675 - 27 + c$$

Or $$c = -675 + 300 = -357$$

$$TC = 3q + 75q^2 - 3q^3 - 357$$

And $$AC = 3 + 75q - 3q^2 - \frac{357}{q}.$$

Example 2: If Constant costs is ₹ 100 and Marginal function is in the form ($6x + 3$)Then find the Total costs function.

Solution: We know that Total Costs = $\int$ Marginal Costs

Therefore $$TC = \int (6x + 3)\, dx$$

$$= \frac{6x^2}{2} + 3x + c$$

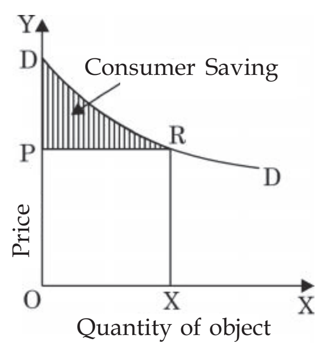
Here constant costs is ₹ 100 then in this condition if the production zero (i.e. $x = 0$) then also constant costs will remain ₹ 100 and Total costs will also be ₹ 100. In this condition

$$TC = (6 \times 0) + (3 \times 0) + c = 100$$

Therefore total costs function will be $3x^2 + 3x + 100$.

Consumer Saving

Consumer saving can also be found out from integration. This can be explained from the following.



In this diagram *DD* is a demand. Consumer give costs *OP* of anything and suppose he does *OX* purchase. Consumer is ready to give total costs of *OX* quantity of thing equal to the *OXRD* part in the diagram, but actually he is ready to give *OXRP* costs. Consumer Saving is the difference between the both.

Did u know? consumer saving = Area *OXRD* - Area *OXRP*

$\int_0^x$ (Demand curve $dx - p\,.\,x$

Example 3: Find the consumer saving when demand curve is $p = 100 - 5x + 3x^2$ and $x = 5$ unit.

Solution: Here demand curve is $p = 100 - 5x + 3x^2$ and quantity of demand is $x = 5$ then here

$$\text{Consumer Demand} = \int_0^x \text{(Demand Curve)}\, dx - p.x$$

$$= \int_0^3 (100 - 5x + 3x^2)\, dx - (p \times 3)$$

$$= \left[100x - \frac{5x^2}{2} + \frac{3x^3}{3}\right]_0^3 - 30$$

$$= 100 \times 3 - \frac{5}{2}(3 \times 3) + (3 \times 3 \times 3) - 30$$

$$= 300 - 22.5 + 27 - 30 = 274.5.$$

Example 4: Find the consumer saving when demand function is $p = 50 - 5x - x^2$ and p = 0.

Solution: Demand Function; $p = 50 - 5x - x^2$

If $p = 0$ then

$0 = 50 - 5x - x^2$

or $x^2 + 5x - 50 = 0$

$x^2 + 10x - 5x - 50 = 0$

or $x(x - 10) - 5(x + 10) = 0$

or $(x - 5)(x + 10) = 0$

$x - 5 = 0$ or $x = 5$; in the same way, $x + 10 = 0$ or $x = -10$

If we ignore $x = -10$ since $x < 0$ therefore

$$\text{Consumer Saving} = \int_2^5 (50 - 5x + x^2)\, dx - p \times x$$

$$= \int_0^5 (50 - 5x + x^2)\, dx - 0 \times 5$$

$$= \left[50x - \frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5$$

$$= \left[50 \times 5 - \frac{5 \times 5 \times 5}{2} - \frac{5 \times 5 \times 5}{3}\right]_0^5$$

$$= 250 - \frac{125}{2} - \frac{125}{3}$$

$$= \frac{1500 - 375 - 250}{6}$$

$$= \frac{825}{6} = \frac{275}{2} = 137.5$$ **Ans.**

Note

Questionnaire 16.2

Solve Yourself :

1. Find : $\int x^2 \cos x \, dx$
2. Find the Integration : $\int (8+7x)^7 \, dx$
3. Find the values of the following: $\int_0^{\frac{1}{2}} \cos^2 x \, dx$
4. Find :

   (*i*) $\int \frac{dx}{a^2 + x^2}$ (*ii*) $\int \frac{\sin x}{\sin(x-\alpha)} dx$

5. Integrate:

   (*i*) $\int 4x^2\sqrt{x^3+3}$ (*ii*) $\int \frac{dx}{4x^2 - 9}$

   (*iii*) $\int \frac{x^2 - 4}{x^2 + 3x + 2} dx$ (*iv*) $\int x^3 e^x \, dx$

6. Find the value of the followings:

   (*i*) $\frac{e^{3x}}{1+e^{3x}}$ (*ii*) $\frac{3x}{(3x+1)(x-3)}$

   (*iii*) $x^3 \log x$, (*iv*) $\sin (ax + b)$.

7. Find total costs function $f(x)$ if x = Total production.

   (i) ₹ 100 Total costs at $3x^3 - 4x + 5$, $x = 0$.

   (ii) ₹ 100 Total costs at $\frac{100}{\sqrt{x}}, x = 0$.

   (iii) $6.75 - 0.0006x$, $x = 0$, total costs = ₹ 10,485

8. Find:

   (*i*) $\int_1^5 x^3 x \, dx$ (*ii*) $\int_1^{14} \sqrt{x} \, dx$

   (*iii*) $\int_1^3 (1 + 5x + x^3) \, dx$ (*iv*) $\int_1^3 (e^{2x} + e^x) \, dx$

9. Find the consumer saving if Demand function curve is $p = 33 - 3x - 2x^2$ and $x = 3$ unit.
10. If p = ₹ 5 then How much will be the consumer saving on Demand Function $p = 100 - 2x^3$. If $p = 0$ then what will be the saving?

Self Assessment

2. Multiple Choice Questions:

4. Total Costs = ...?

   (a) $\int$ Marginal Costs (b) $\int$ Total Costs (c) $\int$ Total Profit (d) $\int$ loss

Note

5. Consumer Saving can be calculated by which method?
   (a) Differentiation (b) Differential
   (c) Integration (d) Cramer
6. Marginal Costs is ....
   (a) $\frac{d}{dq}$ (b) $\frac{dq}{d(c)}$
   (c) $\frac{d(c)}{dq}$ (d) None of these

16.3 Summary

- Suppose *CPQD* represents curve $y = f(x)$, where $f(x)$ is a continuous function of x *in prant* $[a, b]$ and suppose like the value of x increases from a to b, y also increases. Suppose *CA* and *DB* are the ordinates at $x = a$ and $x = b$.
- Take any point $P(x, y)$ located on the curve and assume that *PM* is its grade. Take another point $Q(x + \delta x, y + \delta y)$ on the curve close tom point *P* and suppose its ordinate is *QN*.
- Now a day's use integration in economics has increased very fast This we can prove from the following facts – Marginal Costs, Average Costs and Total Costs

  Since Marginal Costs = $\frac{d(c)}{dq}$, therefore

  Total Costs = $\int \frac{d(c)}{dq} \cdot dq$
- Here, $\frac{d(c)}{dq}$ = Marginal Costs, C = Total Costs, = Change in production.
- Consumer Saving can be calculated by Integration.

16.4 Keywords

- ***Applications:*** Experiments
- ***Ordinate:*** Grade

16.5 Review Questions

1. Find the area of Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (**Ans.:** πab square unit)
2. If a constant cost is ₹ 100 and Marginal Function is in the form $(6x + 3)$, then find the Total costs function.
   (**Ans.:** $3x^2 + 3x + 100$)
3. Find the area under simple line $Y = mx$, X-axis, and Ordinate $x = 3$. (**Ans.:** square $\frac{9}{2}m$ unit)

Answers: Self Assessment

1. Fast	2. Addition	3. Integration
4. (a)	5. (c)	6. (c)

Note

16.6 Further Readings

Books

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economist – Yamane – Prentice Hall India.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economics – Simon and Bloom – Viva Publlications.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications

Mathematics for Economics – Council for Economic Education.

Note

Unit 17: Introduction to Differential Equations and Solutions: Variable Separable Case and Homogeneous Equation

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know about Differential Equation of First Order and First Degree.
- Know about Exact Differential Equations.

Introduction

Differential Equations are the equations in which dependent, independent variable exists, and there are different derivatives of dependent variables with respect to one or more independent variables. The order of a differential equation is the highest order of different derivatives included in that equation. An equation will be called as linear if the derivatives of dependent variables are of first degree, otherwise it will be called as non-linear.

Function $f(x)$ will be called as the solution of different equations if it is replaced in any equation then it reduces the equation to the identity and the method of finding all the solutions will be called as the solution of differential equation.

Normal Solution: The solution of a differential equation in which the independent imaginary constant is equal to the order of differential equation, then it is called as Normal Solution.

Particular Solution: In normal solution, if any particular value is given to constants then it is called the Particular Solution of that equation.

Example: Find the differential equation of curve $y = Ae^x + B/e^x$ for different values of A and B.

Given value $$y = Ae^x + B/e^x \quad \ldots(1)$$

To get the differential equation for the values of A and B of above equation, we'll differentiate equation (1) twice. On differentiating equation (1)

$$\frac{dy}{dx} = Ae^x - Be^{-x} \quad \ldots(2)$$

On differentiating equation (2) again

$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x} \quad \ldots(3)$$

Note

On eliminating A and B from equation (2) and (3), we get –

$$\frac{d^2y}{dx^2} = y$$

This is our differential equation.

17.1 Differential Equation of First Order and First Degree

We represent the differential equation of first order and first degree in the following form –

$$M + N\,(dy/dx) = 0 \quad \text{or} \quad M\,dx + N\,dy = 0$$

Where M and N are constant and x and y are few functions. Though all the differential equations of first order can't be solved always, then also if they are available in the following form then their solution can be found out by few methods –

17.1.1 Variable Separable Case

$$f_1(x)\,dx = f_2(y)\,dy$$

If the differential equation is shown in the following form –

Where $f_1(x)$ and $f_2(y)$ are the functions of x and y respectively.

In such situation, we differentiate the both sides of the equation and then add an imaginary constant on any one side of the equation. Therefore

$$\int f_1(x)\,dx = \int f_2(y)\,dy + c$$

Notes Here 'c' is an imaginary constant.

Example 1: Find the solution of $\left[x/\sqrt{1+x^2}\right]dx = -\left[y/\sqrt{1+y^2}\right]dy$ **.**

Solution: On differentiating both sides

$$\int (1+x^2)^{-1/2}.\,x\,dx = -\int (1+y^2)^{-1/2}.\,y\,dy + c$$

Or $$\frac{1}{2}\int (1+x^2)^{-1/2}.(2x)\,dx = -\frac{1}{2}\int (1+y^2)^{-1/2}.(2y)\,dy + c$$

Or $\sqrt{1+x^2} + \sqrt{1+y^2} = c$, This is the solution of the given equation.

Example 2: Find the solution of $(1 + e^x)\,y\,dy = (1 + y)\,e^x\,dx$.

Solution: We can also write this equation as the following form –

$$\left(\frac{y}{1+y}\right)dy = \frac{e^x}{1+e^x}\,dx$$

On differentiating both sides

$$\int \frac{y}{1+y}\,dy = \int \frac{e^x}{1+e^x}\,dx + \text{constant}$$

Note

Or $y - \log(1 + y) = \log(1 + e^x) + \log c$

Or $y = \log[c(1 + y)(1 + e^x)]$

Or $c(1 + y)(1 + e^x) = e^y$ This will be the solution of above equation.

17.1.2 Homogeneous Equation

If an equation is represented in the form of $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$, then it is called as Homogeneous equation. Here $f_1(x, y)$ and $f_2(x, y)$; are the functions of x similar degree of x and y. To solve such equations, we

Put $y = vx$, where $\frac{dy}{dx} = v + x\frac{dv}{dx}$

In such situation, the given equation is represented as following written form –

$$v + x\frac{dv}{dx} = f(v)$$

i.e.,

$$x\frac{dv}{dx} = f(v) - v$$

The variables can be separated here and then integrate them on the both sides –

$$\int \frac{dv}{f(v) - v} = \log x + c$$

Here, c is an imaginary value.

> *Did u know?* We put y/x on the place of v after differentiating the equation and that will be the solution of that equation.

Example 3: Find the solution of $(x^3 - 3xy^2)\,dx = (y^3 - 3x^3 y)\,dy$.

Solution: The given equation can also be written as the followng form –

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2 y} \quad \ldots\text{(i)}$$

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in equation (i)

$$v + x\frac{dv}{dx} = \frac{x^3 - 3xv^2x^2}{v^3x^3 - 3x^2vx} = \frac{x^3(1 - 3v^2)}{x^3(v^3 - 3v)} = \frac{1 - 3v^2}{v^3 - 3v}$$

Or

$$x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v = \frac{1 - 3v^2 - 3v^4 + 3v^2}{v^3 - 3v} = \frac{1 - v^4}{v^3 - 3v}$$

Or

$$\frac{dx}{x} = \frac{v^3 - 3v}{1 - v^4}\,dv = \left[\frac{1}{2(v+1)} + \frac{1}{2(v-1)} - \frac{2v}{v^2 + 1}\right]$$

Note (By Partial Function)

On differentiating both side

$$\log x + \log c = \frac{1}{2}\log(v+1) + \frac{1}{2}\log(v-1) - \log(v^2+1)$$

Or $$\log(cx) = \frac{1}{2}\log[(v+1)^{1/2}(v-1)^{1/2}/v^2+1]$$

Or $$cx = \frac{(v^2-1)^{1/2}}{v^2+1} \text{ or } c^2x^2 = \frac{v^2-1}{(v^2+1)} \text{ or } c^2x^2(v^2+1)^2 = (v^2-1)$$

Or $$c^2x^2[y^2/x^2+1]^2 = [y^2/x^2-1] \qquad [v = y/x \text{ On putting}]$$

$$c^2(y^2+x^2) = (y^2-x^2)$$ This is the solution of above equation.

17.1.3 Equations Changeable to Homogeneous Form

If the differential is in the following form –

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}, \text{ where } \frac{a}{a_1} \neq \frac{b}{b_1}$$

Then we change it in the homogeneous equation on putting $x = x + h$ and $y = y + k$. (Here h and K are constant). Except it, also put $dy = DY$ and $dx = DX$ –

$$\frac{DY}{DX} = \frac{a(x+h)+b(y+k)+c}{a_1(x+h)+b_1(y+k)+c_1}$$

$$= \frac{ax+by+(ah+bk+c)}{a_1x+b_1y+(a_1h+b_1k+c_1)}$$

Above equation will be homogeneous if $ah + bk + c = 0$ and $a_1h + b_1k + c_1 = 0$

$$\frac{DY}{DX} = \frac{ax+by}{a_1x+b_1y}$$

Now we'll solve the equation on putting $Y = vX$ according the last example. At the last, on putting $X = x - h$ and $Y = y - k$, the solution of given equation would be fould.

If there is $\frac{a}{a_1} = \frac{b}{b_1} = m$ then the above given equation will be as following –

$$\frac{dy}{dx} = \frac{m(a_1x+b_1y)+c}{a_1x+b_1y+c_1}$$

To solve such differential equations, $v = a_1x + b_1y$ is considered.

Example 4: Find the solution of $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$.

Here, the situation is of , therefore, on putting $x = X + h$ and $y = Y + k$

$$\frac{DY}{DX} = \frac{(Y+k)-(X+h)+1}{(Y+k)-(X+h)+5} = \frac{Y-X+(k-h+1)}{Y-X\,(k-h+5)} \qquad \ldots(i)$$

On putting $k - h + 1 = 0$ and $k + h + 5 = 0$, we get that $k = -3$, $h = -2k$ and the equation (i) will be as following on putting the value of h –

$$\frac{dY}{dX} = \frac{Y - X}{Y + X}$$

On putting $Y = vX$, we get

Or $$v + X\frac{dv}{dX} = \frac{v-1}{v+1} \text{ or } X\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{-1-v^2}{v+1}$$

Or $$\frac{v+1}{v^2+1}dv = -\frac{dx}{X} \text{ or } \frac{v}{v^2+1}dv + \frac{dv}{v^2+1} = -\frac{X}{dx}$$

On multiplying with 2 on both sides,

Or $$\frac{2v}{v^2+1}dv + \frac{2v}{v^2+1}dv = \frac{-2X}{dx}$$

On integrating both sides

$$\log(v^2+1) = 2\tan^{-1} v = -2\log X + c$$

Or $$\log(v^2+1) + 2\log X = -2\tan^{-1} v + c$$

Or $$\log[(v^2+1)X^2] = -2\tan^{-1} v + c$$

On putting thevalue of v

$$\log(Y^2 + X^2) = -2\tan^{-1}(Y/X) = c$$

On putting the values of Y and X

$\log[(y+3)^2 + (x+2)^2] + 2\tan^{-1}\{(y+3)/(x+2)\} = c$

This is the solution of given differential equation.

17.1.4 Linear Differential Equation

The Linear Differential Equation is represented in the following form –

$$\frac{dy}{dx} + PY = Q$$

Here P and Q are the functions of x only and y is a dependent variable. To solve this equation, on multiplying both sides with $e^{\int p\,dx}$

$$e^{\int p\,dx}\left(\frac{dy}{dx}\right) + e^{\int p\,dx} Py = e^{\int p\,dx}.Q$$

Or $$\frac{d}{dx}\left\{ye^{\int p\,dx}\right\} = Qe^{\int p\,dx}$$

On differentiating both sides with respect to 'x'

$$ye^{\int p\,dx} = \int Q\,e^{\int p\,dx}\,dx + c$$

This will be the solution of given equation.

Note

Example 5: Find the solution of $\frac{dy}{dx} + 2xy - e^{-x^2}$.

On comparing the above equation with $\frac{dy}{dx} + PY = Q$, we get

$$P = 2x,\ Q = e^{-x^2}$$

On integrating both separately with respect to 'x'

$$\int P\,dx = 2\int x\,dx, \int Q\,dx = \int e^{-x^2}$$

$$\int P\,dx = 2\int \frac{x^2}{2} = x^2$$

Therefore, $$\int_e p\,dx = e^{x^2}$$

The solution of the given differential equation will be –

$$y(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$$

[Here I.F. = Integrating Factor]

$$y\,e^{x^2} = \int e^{-x^2}\,e^{x^2}\,dx + c$$

Or $$y\,e^{x^2} = x + c$$

This is the solution of above equation.

17.1.5 Change in linear Form

If any differential equation can be converted in linear form then that can be solved by two methods –

(I) Barnauli Equation: If any equation is given in the following form –

$$\frac{dy}{dx} + PY = Q\,y^n$$

In this situation, we multiply with y^{-n} on the both sides.

$$y^{-n}\frac{dy}{dx} + PY^{-n+1} = Q \qquad \text{...(i)}$$

We put $y^{-n+1} = v$ Then there will be $\frac{dv}{dx}(1-n)y^{-n}\frac{dy}{dx}$

Now equation (i) will be in the following form –

$$\frac{1}{1-n}\frac{dv}{dx} + pv = Q$$

Or $$\frac{dv}{dx} + (1-n)\,pv = (1-n)\,Q \qquad \text{...(ii)}$$

This equation is a linear equation, where v is a dependent variable. The solution of this differential equation will be according to 17.1.4.

(II) If an equation is shown in the following form –

$$\frac{dv}{dx} + P\phi(y) = Q\,f(y)$$

Note

Where, P and Q are the functions of x only.

Then, for convering the above equation into linear equation, on dividing with $f(y)$ on the both sides of equation

$$\frac{1}{f(y)}\frac{dv}{dx} + P\frac{\phi(y)}{f(y)} = Q$$

In equation $$\frac{\phi(y)}{f(y)} = v \text{ take it } \frac{dv}{dx} = \frac{d}{dx}\left\{\frac{\phi(y)}{f(y)}\right\}$$

$$= k\frac{1}{f(y)}\frac{dy}{dx}, \text{ where } k \text{ is a constant.}$$

Therefore, we can write equation (ii) in the following form –

$$\frac{1}{k}\frac{dv}{dx} + Pv = Q$$

Or $$\frac{dv}{dx} = k\,Pv = kQ \text{ which is a linear equation.}$$

Example 6: Find the solution of $\frac{dy}{dx} + \frac{1}{x}y = x^2y^6$ **.**

On dividing given equation by y^6

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}.\frac{1}{y^5} = x^2 \qquad \text{...(i)}$$

Assume $$\frac{1}{y^5} = v \text{ then } \frac{dv}{dx} = -\frac{5}{y^6}\frac{dy}{dx}$$

Or $$\frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dy}{dx}$$

Putting in equation (i)

$$-\frac{1}{5}\frac{dv}{dx} + \frac{1}{x}v = x^2 \text{ or } \frac{dv}{dx} - \frac{1}{5}v = -5x^2$$

This is a linear equation in which v is a dependent variable.

Here $P = -5/x$ and $Q = -5x^2$

Therefore $$\int P dx = \int\left(-\frac{5}{x}\right)dx = -5\log x = \log x^5 = \log\left(\frac{1}{x^5}\right)$$

$\therefore$ $$\int_e P\,dx = e\log^{(1/5)} = \frac{1}{x^5}$$

$$v\text{ (I.F)} = \int Q\text{ (I.F.) } dx + c$$

Or $$v(1/x^5) = \int -5x^2(1/x^5)\,dx + c = -5\int x^{-3}dx + c$$

Note

$$\left(\frac{1}{y^5}\right)\left(\frac{1}{x^5}\right) = -\frac{5}{2}\left(\frac{1}{x^2}\right)+c \qquad [\because v = 1/y^5]$$

Or $$\frac{1}{x^5y^5}+\frac{5}{2}\left(\frac{1}{x^2}\right)+c$$

This is the solution of differential equation.

17.2 Exact Differential Equations

The differential Equation $M\,dx + N\,dy = 0$ will be exact if there is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. If the given equation is exact then we'll use following steps after it –

i. Integrate M with respect to x when y is constant.

ii. Integrate N with respect to y only and we'll integrate only the terms which don't have x.

iii. Add above both the integrations.

Therefore if the differential equation $M\,dx + N\,dy = 0$ is exact the its solution will be as following –

$\int M\,dx$ (Considering y as a constant) + $\int M\,dy$ (Only the terms which dont have x) $= c$

For example: Find the solution of $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$.

Here, $M = x^2 - ay$ and $N = (ax - y^2)$.

$$\frac{\partial M}{\partial y} = -a \text{ and } \frac{\partial N}{\partial x} = -a$$

Therefore, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Therefore the given equation is exact.

Now $\int M\,dx$ (Considering y as a constant) $= \int (x^2 - ay)\,dx = \frac{1}{3}x^3 - ay\,x$ …(i)

And $\int N\,dy$ (Only the terms which don't have x) $= \int y^2\,dy = \frac{1}{3}y^3$ …(ii)

Therefore, we'll get –

(i) + (ii) = c(imaginary constant)

Or $\frac{1}{3}x^3 - ayx + \frac{1}{3}y^3 = c$

Or $x^3 - 3\,ayx + y^3 = 3c$..

Task Find the solution of $\frac{dy}{dx}+\frac{1}{x}y = x^2\,y^6$. (**Ans.**: $\frac{1}{x^5y^5}+\frac{5}{2}\left(\frac{1}{x^2}\right)+c$)

Self Assessment

Note

1. Fill in the blanks:

1. Differential Equations are the equations in which ……………, independent variable exists.
2. An equation will be called as linear if the …………….. of dependent variables are of first degree.
3. The solution of a differential equation in which the independent imaginary constant is equal to the order of differential equation, then it is called as ……….. Solution.
4. In normal solution, if any particular value is given to constants then it is called the …………………. solution of that equation.
5. If an equation is represented in the form of $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$, then it is called as Homogeneous equation.

17.3 Summary

- Differential Equations are the equations in which dependent, independent variable exists, and there are different derivatives of dependent variables with respect to one or more independent variables. The order of a differential equation is the highest order of different derivatives included in that equation. An equation will be called as linear if the derivatives of dependent variables are of first degree, otherwise it will be called as non-linear.
- Normal Solution: The solution of a differential equation in which the independent imaginary constant is equal to the order of differential equation, then it is called as Normal Solution.
- Particular Solution: In normal solution, if any particular value is given to constants then it is called the Particular Solution of that equation.
- If the differential equation is shown in the following form –

  $$f_1(x)\,dx = f_2(y)\,dy$$

  Where $f_1(x)$ and $f_2(x)$ are the functions of x and y respectively.
- In such situation, we differentiate the both sides of the equation and then add an imaginary constant on any one side of the equation.

17.4 Keywords

- ***Homogeneous:*** Similar
- ***Separable:*** Able to be separate

17.5 Review Questions

1. Find the differential equation for the different values of A and B of $y = A\,e^x + B/e^x$.

   (**Ans.:** $\frac{d^2y}{dx^2} = y$)

2. Find the solution of $\left[\frac{x}{\sqrt{1+x^2}}\right] dx = -1\left[\frac{y}{\sqrt{1+y^2}}\right] dy$. (**Ans.:** $\sqrt{1+x^2} + \sqrt{1+y^2} = c$)

3. Find the solution of $\frac{dy}{dx} + 2xy - e^{-x2}$. (**Ans.:** $ye^{x2} = x + c$)

Note

Answers: Self Assessment

1. Dependent
2. Differentiation
3. General
4. Special
5. Homogeneous

17.6 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Bloom – Viva Publications

Mathematics for Economist – Makcal Harrison, Patrick Waldron.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and sons.

Mathematics for Economist – Karl P. Simon, Laurence Blum.

Mathematics for Economist and Finance – Martin Norman

Mathematics for Economist – Council for Economic Education

Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 18: Matrices: Meaning and Types

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Learn the Order and Notation of a Matrix.
- Get the Knowledge about Kinds of Matrix.
- Understand the Important Properties of Matrices.
- Know the Addition and Subtraction of Matrices.
- Solve the Questions related with Matrix Multiplication.

Introduction

A Rectangular Arromy, which is arranged in rows and columns, is called Matrix.

The Permutation of mn numbers in m rows and n columns is called the matrix of $m \times n$ order.

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{matrix}$$

Notes The numbers a_{11}, a_{12}, a_{13},............ are called the Elements or Constituents of Matrix.

The rows and columns are made from the horizontal constituents and vertical constituents respectively. As -

Note

$A = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 3 \end{bmatrix}$ is a Rectangular Permutation in which 2 rows and 3 columns and the numbers of constituents are 2 × 3 = 6.

Similarly $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is also a Rectangular Permutation in which there are 3 rows and 1 column

And the number of entries are 3 × 1 = 3.

18.1 Order of Matrix

As there are two dimensions length and breadth in a rectangle similarly the dimension or order of matrix is 'number of rows × number of columns'. We indicate the matrices given in definition as *A*, *B*, *C*, etc. As Am *x n* , $A_{m \times n}$. $B_{2 \times 3}$. $C_{1 \times 3}$, etc.

18.2 Notation of a Matrix

If the number of rows and columns in any matrix are *m* and *n* respectively, then there will be *m* × *n* number of constituents in the rectangular permutation. If the place of constituents is a_{ij} in the i^{th} row and j^{th} column, then matrix is written as following -

$$A_{m \times n} = \; A = [a_{ij}]_{m \times n}$$

Where *i* = 1, 2, 3, ………., *m* and *j* = 1, 2, 3, …………., *n*.

Did u know? Here, it is remarkable that Matrix is only the rectangular permutation (Written in well organized and brief form) of numbers. It have no numerical value.

18.3 Kinds of Matrix

1. **Square Matrix**

   If the number of rows and columns are equal in a matrix then it is called as Square Matrix. i.e., if *m* = *n* then the arromy is called as the matrix of order *n*. if there is other matrices then they are called as rectangular matrices. For example,

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix} B = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \\ 7 & 2 & 5 \end{bmatrix}$$

   Both of above matrices are Square Matrices of order (2 × 2) and (3 × 3) respectively.

2. **Row Matrix**

   If *m* = 1, then there will be only one row in the matrix, it will be called as Row Matrix. For example -

$$A = [2\ 3]_{1 \times 2}$$

$$B = [-5\ 2\ 0]_{1 \times 3}$$

$$C = [1\ 3\ 0\ -7]_{1 \times 4}$$

Note

3. **Column Matrix**

   If $n = 1$, there will be only one column and many numbers of rows, it will be called as Column Matrix. For example -

$$A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{2 \times 1}, B = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}_{3 \times 1}, C = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 7 \end{bmatrix}_{4 \times 1}$$

4. **Null or Zero Matrix**

   If all the elements in a matrix are zero, then it is called Zero or Null Matrix. For example -

$$A = [0\ 0], B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. **Identity or Unit Matrix**

   It is a Square Matrix in which all the diagonal elements are unit and others are zero. It is called as Identity or Unit Matrix. For example -

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. **Diagonal Matrix**

   If all the elements except the elements situated on the diagonal of a square matrix, are zero then it is called as Diagonal Matrix. For example -

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}; B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}; C = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

   In short it is indicated as $[d_1, d_2, d_3, d_4]$.

7. **Scalar Matrix**

   The diagonal Matrix in which all the diagonal elements are same, is called Scalar Matrix. For example -

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$A = [a_{ij}]$ will be Scalar if $a_{ij} = 0$,when $i \neq j$; and $a_{ij} = 2$ when $i = j$

$$B = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

$B = [a_{ij}]_{3 \times 3}$, where $a_{ij} = 0$ when $i \neq j$; and $a_{ij} = k$, when $i = j$, Then B is a Scalar Matrix.

Note

8. **Lower Triangular Matrix**

The Square Matrix in which all the elements above the diagonal are zero, is called as Lower Triangular Matrix. For example -

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ -2 & -1 & -3 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}, \text{ here } a_{ij} = 0; \text{ when } i < j$$

Task Write an example of identity matrix.

9. **Upper Triangular Matrix**

The Square Matrix in which all the elements of diagonal are zero, is called the Upper Triangular Matrix. For example -

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & -1 \\ 0 & 0 & 5 \end{bmatrix}; B = \begin{bmatrix} a_{11} & a_{12} & a_{13}\dots & a_{1n} \\ 0 & a_{21} & a_{23}\dots & a_{2n} \\ 0 & 0 & a_{33}\dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & a_{mn} \end{bmatrix}$$

Square Matrix A (order $n \times n$); where element $a_{ij} = 0$, when $i < j$

10. **Trace of a Matrix**

The addition of all the elements on the principal diagonal of a Square Matrix, is called the Trace of a Matrix. For example -

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

The above Matrix, Trace of Matrix = $a + b + c$, in the square matrix A of order $n \times n$, is called as the Trace of Matrix = $\sum_{i=1}^{n} a_{ij}$, where $j = i$, and written as Tr.

11. **Symmetric Matrix**

The Square Matrix A (order $n \times n$) in which there is $a_{ij} = a_{ji}$ for each $(i - j)^{\text{th}}$ element, is called Symmetric Matrix. For example -

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

The above matrix is a Symmetric Matrix because $a_{ij} = a_{ji}$ *i.e.* $a_{21} = h = a_{12}$; $a_{32} = f = a_{23}$; $a_{22} = b = a_{22}$.

12. **Skew - Symmetric Matrix**

The Square Matrix A (order $n \times n$) in which there is $a_{ji} = a_{ij}$ for each $(i - j)^{\text{th}}$ element, is called Symmetric Matrix. For example -

$$A = \begin{bmatrix} 0 & g \\ -h & f \\ -g & 0 \end{bmatrix}$$

The diagonal elements are $a_{11}, a_{22}, a_{33}, ..., a_{ij}$ and for all the values os the condition $a_{ij} = -a_{ji}$ and

Note

$\therefore$ $2a_{ij} = 0$ or $-a_{ij} = 0$

So all the diagonal elements of Skew - Symmetric Matrix are zero.

Self Assessment

1. Fill in the blanks:

1. A Rectangular Arromy, which is arranged in rows and columns, is called …………….
2. If the number of rows and columns in any matrix are m and n respectively, then there will be …………………… number of constituents in the rectangular permutation.
3. If the number of rows and columns are equal in a matrix then it is called as ………… Matrix.
4. If all the elements in a matrix are zero, then it is called …………………………. Matrix.
5. If all the elements except the elements situated on the diagonal of a square matrix, are zero then it is called as …………………. Matrix.

18.4 Important Properties of Matrices

1. **Equality of Matrices**

   Two matrices $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{p \times q}$ will be equal if the order of both are the same i.e., $m = p$; $n = q$ and $a_{ij} = b_{ij}$.

2. **Equivalance Relations of Matrices**

   If Matrices A, B, C are confirmable i.e., having similar order; then

   (i) $A = A$ (Reflexive Relation)

   (ii) $A = B$?$\Leftrightarrow$ $B = A$ (Symmetric Relation)

   (iii) $A = B$ and $B = A \Leftrightarrow A = C$; Transitive Relation

Where, notation $\Rightarrow$ (if ………… then) and $\Leftrightarrow$ (iff = if and only if)

(a) $A = A$ means $a_{ij} = a_{ij}$ where $i = 1,2,3,\ldots\ldots\ldots\ldots m$ and $j = 1,2,3,\ldots\ldots\ldots n$. It is called Reflexive Relation.

(b) If there are $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then $A = B$ means $a_{ij} = b_{ij}$ or $b_{ij} = a_{ij}$., then $B = A$.

$\therefore$ $A = B \Rightarrow B = A$ it is called Symmetric Relation

(c) If $A = B$, it is called $B = C$ Symmetric Relation

$a_{ij} = b_{ij}$ and $b_{ij} = c_{ij}$

therefore $a_{ij} = c_{ij}$ $\therefore$ $A = C$

The relationship in Matrices A, B, C which is Reflexive, Symmetric and Transitive, is called the Equivalance Relationship. For example -

From the definitions of equal matrices, the value of x, y and z are

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Therefore, the relative elements of two equal matrices are equal.

$\therefore$ $x + 3 = 0$; $2y + x = -7$; $z - 1 = 3$ and $4a - 6 = 2a$

From these equations, $x = -3$; $y = -2$; $z = 4$ and $a = 3$.

Note

18.5 Addition and Subtraction of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of order $m \times n$, then we'll show their addition $A + B$ from $C = [c_{ij}]$ of $m \times n$ order.

Where $c_{ij} = b_{ij}$,, $i = 1,2,3,\ldots m$; $j = 1,2,3,\ldots n$ for all values.

Here it is remarkable that if matrices are not in same order then their addition is not possible.

Example 1: Evaluate -

(*i*) [1 2 3] + [4 5 6];

(*ii*) $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$; then $A + B = ?$

Solution: (i) Here $A = [1\ 2\ 3]$; $B = [4\ 5\ 6]$

$\therefore \quad A + B = [1 + 4\ 2 + 5\ 3 + 6] = [5\ 7\ 9]$

(ii) Here $A + B = C$; where $c_{ij} = a_{ij} + b_{ij}$

$$\therefore \quad C = \begin{bmatrix} 1+7 & 2+3 & 4+2 \\ 0+5 & 5+1 & 3+9 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 6 \\ 5 & 6 & 12 \end{bmatrix}$$

Field Properties of Matrix Addition

(i) If two matrices are of same order and conformable then the matrix from these addition is also of the similar order.

If $A = [a_{ij}]_{m \times n}$; $B = [b_{ij}]_{m \times n}$ then

$A + B = C \quad \Rightarrow \quad [a_{ij}\, b_{ij}]_{m \times n}$; (Closure Rule for Addition)

(ii) If two matrices are conformable (of similar order) then

$A + B = B + A$ (Commutative Rule For Addition)

If $A + B = [a_{ij} + b_{ij}]_{m \times n} = [b_{ij} + a_{ij}]_{m \times n} = B + A$ then the Addition of Conformable Matrices follow the Commutative Rule.

(iii) if three matrices A, B, C are conformable, then

$(A + B) + C = A + (B + C)$ (Associative Law for Addition)

Here $A = [a_{ij}]_{m \times n}$; $B = [b_{ij}]_{m \times n}$; $C = [c_{ij}]_{m \times n}$, then

$\therefore \quad (A + B) + C = [a_{ij} + b_{ij}]_{m \times n} + [c_{ij}]_{m \times n} = [a_{ij} + b_{ij} + c_{ij}]_{m \times n}$

And $\quad (A + B) + C = [a_{ij}]_{m \times n} + [b_{ij} + c_{ij}]_{m \times n} = [a_{ij} + b_{ij} + c_{ij}]_{m \times n}$

$\therefore \quad (A + B) + C = A + (B + C)$

(iv) If matrix A is of $m \times n$ order and Zero Matrix is also of same order then $A + 0 = A$; where the addition of zero matrix is identity element.

(v) If matrix A is of $m \times n$ order and $A + (-A) = 0$, then $-A$ Matrix is the additive inverse of matrix A.

Subtraction of Two Matrices

If $A = [a_{ij}]_{m \times n}$; $B = [b_{ij}]_{m \times n}$ then Subtraction of two matrices, is that matrix C in which $C = [c_{ij}]_{m \times n}$ and if $c_{ij} = a_{ij} = b_{ij}$

i.e., $c_{ij} = a_{ij} + (-\, b_{ij})$

Note

The Subtraction of two matrices is found from the addition of the additive inverse of *A* and *B* i.e., *A* – *B*.

Example 2: If $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix}$ **and** $\begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$ **then evaluate *A* – *B*.**

Solution: $A - B = \begin{bmatrix} 1-7 & 2-3 & 4-2 \\ 0-5 & 5-1 & 3-9 \end{bmatrix} = \begin{bmatrix} -6 & -1 & 2 \\ -5 & 4 & -6 \end{bmatrix}$

18.6 Matrix Multiplication

If $A = [a_{ij}]$, $m \times n$ is the matrix of $m \times n$ order and $B = [b_{ik}]$, $n \times p$ is the matrix of $n \times p$ order, then the Multiplication of Matrices *A* and *B* is $AB = C$; where, $C = [c_{ik}]$, $m \times n$ is a matrix *A* of $m \times n$ order. i.e., if two matrices *A* and *B* are conformable in which the number of columns *A* is equal to the number of rows B, then we can multiply both the matrices and denote the product from *AB*.

Thus, the method to get the common element c_{ik} of Matrix Multiplication *AB* is -

To multiply and add the elements of *i*th row of *A* and *k*th column of *B*. it is called Row Multiplied Column Method.

Assume the $A = [a_{i1}, a_{i2}....a_{in}]$ and $B = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$ then the multiplication of *AB* will be -

$$[a_{i1}\, a_{i2}....a_{in}] \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix} = \rightarrow \begin{bmatrix} \vdots \\ \vdots \\ c_{ik} \\ \vdots \end{bmatrix}$$

Where $c_{ik} = a_{i1}\, b_{1k} + a_{i2} b_{2k} .... + a_{ij} b_{ik} + ... + a_{in} b_{nk}$.

It is attentable that The Multiplication *AB* is definite only then if the number of columns in matrix *A* is equal to the numbers of rows in Matrix *B*. Matrix *A* and *B* are comfortable for multiplication. Therefore we can multiply the one row matrix and column if the number of elements in both are equal.

If Matrix *A* is of $1 \times m$ order and *B* is of $n \times 1$ order then Matrix Multiplication *A.B* will be of 1×1 order but the product of matrices *B* and *A* will be of $n \times n$ order.

$$A \,.\, B = [a_1, a_2....a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [a_1 b_1 + a_2 b_2 + ... + a_n b_n]$$

$$B \,.\, A = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{1 \times n} [a_1 a_2....a_n]_{1 \times n} = \begin{vmatrix} b_1 a_1 & b_1 a_2 ..... & b_1 a_n \\ b_2 a_1 & b_2 a_2 ..... & b_2 a_n \\ ..... & .... & ..... \\ ..... & .... & ..... \\ b_n a_1 & b_n a_2 ..... & b_n a_n \end{vmatrix}.$$

Note

Example 3: If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}_{2\times3}$ **and** $B = \begin{vmatrix} 4 & 1 & 2 \\ 3 & -1 & 5 \\ 2 & 3 & 1 \end{vmatrix}_{3\times3}$ **then evaluate** AB **-**

Solution: $AB = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 2 \\ 3 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2.4 + 1.3 + 3.2 & 2.1 + (-1) + 3.3 & 2.2 + 1.5 + 3.1 \\ (-1)4 + 3.3 + 1.2 & (-1)\,.\,1 + 3(-1) + 1.3 & (-1)2 + 3.5 + 11 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 10 & 22 \\ 7 & 1 & 14 \end{bmatrix}_{2\times3}$$

Important Note - If Matrices A and B are the Square Matrices of same order then AB and BA both will be defined and both will be the square matrices of that order, then also in general $AB \neq BA$ i.e., Matrix Multiplication is not commutative.

Example 4: If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$**, then evaluate** AB**.**

Solution: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

It proves that Matrix Multiplication is a Zero Matrix but $A \neq 0$ and $B \neq 0$.

Associative Law of Matrix Multiplication

(i) **Associative Law of Matrix Multiplication**

If $A = [a_{ij}]_{, m\times n}$ is of $m \times n$ order, $B = [b_{ik}]$, $n \times p$ is of $n \times p$ order and $C = [c_{ik}]$, $p \times q$ is of $p \times q$ order, then $A\,.\,(B\,.\,C) = (A\,.\,B)\,.\,C$, both sides are the matrices of $m \times q$ order, i.e., Matrix Multiplication is Associative.

It can be proved very easily.

Assume that $B.C = [d_{ji}]$, where

$$d_{ji} = b_{ji}\,c_{i1} + b_{j2}c_{2i} + ... + b_{ip}C\pi = \sum_{k=1}^{p} b_{ik}c_{ki}$$

Therefore $A\,.\,(B\,.\,C) = [c_{ij}]$, where $c_{ji} = a_{i1}d_{1i} + ...\, a_{in}d_{ni}$

$$= \sum_{i=1}^{n} a_{ij}d_{ji} = \sum_{j=1}^{n} n_{ij} \sum_{k=1}^{P} B_{jk}c_{ki}$$

Because the number of elements is symmetric, so on changing the order of addition

$$= \sum_{k=1}^{p}\left(\sum a_{ji}b_{jk}\right)c_{ki} = (A.B)\,.\,C$$

(ii) **Distributive Law of Matrix Multiplication**

If $A = [a_{ij}]$, $m \times n$ is of $m \times n$ order, and $B = [b_{ik}]$, $C = [c_{jk}]$, $n \times p$, are of $n \times p$ order,

Note

Then $A(B + C) = AB + AC$

Because $B + C = [b_{ij}] + [c_{jk}] = [b_{jk}] + [c_{jk}]$

Therefore $A(B + C) = [d_{ik}]$,

Where $d_{ik} = a_{i1}(b_{1k} + c_{1k}) + ... + a_{in}(b_{nk} + c_{nk})$

$$= \sum_{j=1}^{p} a_{ij}(b_{jk} + c_{jk})$$

$$= \sum_{i=1}^{n} (a_{ij}b_{jk} + a_{jj}c_{jk}) = \sum_{i=1}^{n} a_{ij}b_{jk} + \sum_{i=1}^{n} a_{ij}c_{jk}$$

Therefore $[d_{ik}] = AB + AC$

Therefore $A(B + C) = AB + AC$

Similarly, if $B = [b_{ik}]$] $C = [c_{jk}]$, are of $n \times p$ order and $A = [a_{ij}]$, is of $p \times n$ order, then $(B + C)A = BA + CA$.

Example 5: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ **then prove that** $AB \neq BA$**:**

Solution: $AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\times 0 + 2\times 2 + 3\times -1 & 0\times 0 + 1\times 2 + 1\times -1 & 1\times 0 + -1\times 2 + 0\times -1 \\ 1\times 1 + 2\times 1 + 3\times 0 & 0\times 1 + 1\times 1 + 1\times 0 & 1\times 1 + -1\times 1 + 0\times 0 \\ 1\times 2 + 2\times -1 + 3\times 1 & 0\times 2 + 1\times -1 + 1\times 1 & 1\times 2 + -1\times -1 + 0\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -2 \\ 3 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

Now, $BA = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0\times 1 + 1\times 0 + 2\times 1 & 2\times 1 + 1\times 0 + -1 - 1\times 1 & -1\times 1 + 0\times 0 + 1\times 1 \\ 0\times 2 + 1\times 1 + 2\times -1 & 2\times 2 + 1\times 1 + -1\times -1 & -1\times 2 + 0\times 1 + 1\times -1 \\ 0\times 3 + 1\times 1 + 2\times 0 & 2\times 3 + 1\times 1 + -1\times 0 & -1\times 3 + 0\times 1 + 1\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & -3 \\ 1 & 7 & -3 \end{bmatrix}$$

Therefore, $AB \neq BA$.

Note

Example 6: If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ **then evaluate** AB**.**

Solution:

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0\times 0+1\times 2 & 1\times 0+0\times 2 & 0\times 0+1\times 2 \\ 0\times 1+1\times 1 & 1\times 1+0\times 1 & 0\times 1+1\times 1 \\ 0\times 2+1\times 0 & 1\times 2+0\times 0 & 0\times 2+1\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Self Assessment

2. State whether the following statements are True or False:

6. $A = A$; is Reflexive Relationship when matrix A is conformable.
7. $A = B \Leftrightarrow B = A$ is symmetric relationship, when A, B are conformable.
8. When Matrices A, B, C are conformable then $A = B$ and $B = C \Leftrightarrow A = C$ shows the Reflexive relationship.
9. If the three matrices A, B, C are conformable then $(A + B) + C = A + (B + C)$ shows the associative law of addition.
10. The Subtraction of two matrices is found from the addition of additive inverses of A and B i.e., $A + B$.

18.7 Summary

- A Rectangular Arromy, which is arranged in rows and columns, is called Matrix.
- The Permutation of mn numbers in m rows and n columns is called the matrix of $m \times n$ order.
- As there are two dimensions length and breadth in a rectangle similarly the dimension or order of matrix is 'number of rows × number of columns'. We indicate the matrices given in definition as A, B, C, etc. As $A_{m \times n}$. $B_{2 \times 3}$. $C_{1 \times 3}$, etc.
- If the number of rows and columns in any matrix are m and n respectively, then there will be $m \times n$ number of constituents in the rectangular permutation.
- If $n = 1$, there will be only one column and many numbers of rows, it will be called as Column Matrix.
- If the number of rows and columns are equal in a matrix then it is called as Square Matrix.
- If all the elements in a matrix are zero, then it is called Zero or Null Matrix.

Note

- It is a Square Matrix in which all the diagonal elements are unit and others are zero. It is called as Identity or Unit Matrix.
- If all the elements except the elements situated on the diagonal of a square matrix, are zero then it is called as Diagonal Matrix.
- The diagonal Matrix in which all the diagonal elements are same, is called Scalar Matrix.
- The Square Matrix in which all the elements above the diagonal are zero, is called as Lower Triangular Matrix.
- The Square Matrix in which all the elements of diagonal are zero, is called the Upper Triangular Matrix.
- The addition of all the elements on the principal diagonal of a Square Matrix, is called the Trace of a Matrix.
- The Square Matrix A (order $n \times n$) in which there is $a_{ij} = a_{ji}$ for each $(i - j)^{th}$ element, is called Symmetric Matrix.
- The Square Matrix A (order $n \times n$) in which there is $a_{ji} = a_{ij}$ for each $(i - j)^{th}$ element, is called Symmetric Matrix.

18.8 Keywords

- ***Matrix:*** Rectangular arromy
- ***Elements:*** Constituents

18.9 Review Questions

1. If $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$, then find A+B. (**Ans.:** $\begin{bmatrix} 8 & 5 & 6 \\ 5 & 6 & 12 \end{bmatrix}$)
2. If $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, then find A+B. (**Ans.:** $\begin{bmatrix} 3 & 7 & 11 \\ 5 & 9 & 13 \end{bmatrix}$)
3. If $A = \begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, then find A–B. (**Ans.:** $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$)
4. If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}_{2 \times 3}$ and $B = \begin{bmatrix} 4 & 1 & 2 \\ 3 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix}_{3 \times 3}$. Find AB. (**Ans.:** $\begin{bmatrix} 17 & 10 & 22 \\ 7 & 1 & 14 \end{bmatrix}_{2 \times 3}$)

Answers: Self Assessment

1. Matrix	2. m x n	3. square	4. Zero
5. diagonal	6. True	7. True	8. False
9. True	10. False		

Note

18.10 Further Readings

Books

Mathematics for Economist – Yamane - Prentice Hall India

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Bloom – Viva Publications

Mathematics for Economist – Makcal Harrison, Patrick Waldron.

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Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 19: Transpose and Inverse of Matrix

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Transpose of Matrix.
- Get the Knowledge about Transpose of Product of Two Matrices.
- Understand the Regular Matrix.
- Solve the Questions Related to Inverse or Opposite of Matrix.
- Know the Things Related to Orthogonal Matrix.

Introduction

A Rectangular Arromy, which is arranged in rows and columns, is called Matrix.

There is the discussion on inverse or opposite of matrix. Under it, if there is a matrix of $m \times n$ order, then the $n \times m$ matrix that will be found on exchanging the rows and columns of it, will be called as Transpose of A.

If matrix A and B are adaptable for product then the transpose of AB will be equal to the product of transposes in inverse.

19.1 Transpose of Matrix

If A is a matrix of order m x n, then the matrix that will be found on exchanging the rows and columns of it, will be called as Transpose of A and will be denoted as A' or A^t or A^T.

Example: If

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 5 & 2 & 4 \end{bmatrix} \text{ then } A' = \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 7 & 4 \end{bmatrix}$$

Note It is clear that element A_{ij}, which was in the i^{th} row and j^{th} column, will be on j^{th} row and i^{th} column of transpose matrix A'. Therefore if

$$A = [a_{ij}] \text{ and } A' = [a_{ij}] \Rightarrow a_{ij}' = a_{ij}$$

Note – If we'll transpose *A'* then we'll get *A* again.

$$\therefore \quad (A')' = (A^T)^T = [A^t]^t = A$$

Similarly [*KA*]' = *KA'*; here *K* is a scaler.

Theorem 1 – If A and B both are the matrix of order $m \times n$, then

$$(A + B)' = A' + B'$$

Proof – We are known that A and B will be adaptable to addition if both are of the same order. So assume that

$A = [a_{ij}]$ and $B = [b_{ij}]$ then, $C = A + B = [c_{ij}]$, where $c_{ji} = a_{ji} + b_{ij}$

Now $A' = [a_{ji}]$, $B' = [b_{ji}]$, where $a_{ji} = a_{ij}$ and $b_{ji} = b_{ij}$

$$(A + B)' = C' = [c_{ji}] = [a_{ij} + b_{ij}] = [a_{ji}] + [b_{ji}] = A' + B'.$$

19.2 Transpose of Product of two Matrices

Theorem 2 – If matrix A and B are adaptable for product then the transpose of AB will be equal to the product of transposes in inverse i.e., if A and B are of order $m \times n$ and $n \times p$, then $(AB)' = A'B'$.

Proof – The order of AB is $m \times p$, then $(AB)'$ will be of order $p \times m$, the order of B' is $p \times n$ and that of A' is n x m, therefore $B'A'$ will be of order $p \times m$. It means both $(AB)'$ and $B'.A'$ are of same order.

Assume then $A = [a_{ij}]$, $B = [b_{jk}]$, then

The $(k - \text{i})^{th}$ element of $(AB)'$ = The $(i - k)^{th}$ element of (AB)

$$= \sum_{j=1}^{n} a_{ij} b_{jk}$$

$(B)' = (b_{kj})$, $A' = (a_{ji})$ where $b_{jk}' = b_{jk}'$, $a_{ji}' = a_{ij}'$

The $(k - i)^{th}$ element of $B'A'$ will be

$$= \sum_{j=1}^{n} a_{jk} b_{ij}$$

$$= \sum_{j=1}^{n} a_{jk} b_{ij} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

i.e., The $(k - i)^{th}$ element of $B'A$ = The $(k - i)^{th}$ element of $(AB)'$

$$(AB)' = B'.A'$$

This result can be used till the matrices of adaptable order of any number, i.e.,

$(ABC........L.M)' = M'\,L'........C'\,B'\,A'$

Notes If $A = B$ then $(A^2)' = A' \,.\, A' = (A')^2$ similarly, on taking any positive integer power of A, $(A^k)' = (A')^k$.

19.3 Regular Matrix

Note

The square – matrix A will be called as Regular if there is $BA = I$, then there will also be $AB = I$. It will be proved as matrix – derivative if

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \delta_{up}; i, j = 1, 2, \ldots, n$$

Then $$\sum_{k=1}^{n} a_{ik} b_{ij} = \delta_{ij}; i, j = 1, 2, \ldots, n$$

Which is not only true but is clear also.

Self Assessment

1. Fill in the blanks:

1. If matrix A and B are adaptable for product then the transpose of AB will be equal to the …………… of transposes in inverse.
2. The square – matrix A will be called as Regular if B is in such matrix existence when $BA = I$, where I is the ……………….. matrix of same order.
3. The inverse of a transpose matrix is ………………
4. If A is square matrix, then $AB = 0 \Rightarrow B = 0$
5. If $AA' = I$ then A is called ………………….. matrix.

19.4 Inverse or Opposite of Matrix

If the square – matrix is regular then there exist such square – matrix B that $BA = AB = I$.

Therefore we'll call the regular matrix A as Invertible and B as the inverse of A and denote it as A^{-1}. In this situation, the inverse of matrix is found out by following technique –

(i) Form a transpose of matrix on replacing the a_{ij} elements of matrix A from the co – factors c_{ji}. So made changed matrix is called Adjoint of A so that

$$\begin{bmatrix} c_{11} & c_{21} \ldots\ldots & c_{n1} \\ c_{12} & c_{22} \ldots\ldots & c_{n2} \\ \vdots & & \\ c_{1n} & c_{2n} \ldots\ldots & c_{nn} \end{bmatrix}$$

(ii) Divide the Adj. (A) from the matrix of A, $|A|$. (If $\neq 0$)

$$\frac{\text{Adj}(A)}{|A|} = \begin{bmatrix} \frac{c_{11}}{|A|} & \frac{c_{21}}{|A|} \cdots & \frac{c_{n1}}{|A|} \\ \frac{c_{12}}{|A|} & \frac{c_{22}}{|A|} \cdots & \frac{c_{n2}}{|A|} \\ \vdots & & \\ \frac{c_{1n}}{|A|} & \frac{c_{2n}}{|A|} \cdots & \frac{c_{nn}}{|A|} \end{bmatrix}$$

$\left|\frac{(A)}{|A|}\right|$ is the required A^{-1}.

Note

Example 1: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 12 \end{bmatrix}$ **then find** A^{-1}**.**

Solution: We know that $A^{-1} = \frac{\text{Adj.}A}{|A|}$

On making the Transpose of given Matrix

$$A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 12 \end{bmatrix}$$

To find Adjoint, we'll find the co–factors of matrix A (a_{ij}) as following –

$$\text{Co - factors 1} = + \begin{bmatrix} 3 & 5 \\ 5 & 12 \end{bmatrix} = 36 - 25 = 11$$

$$\text{Co - factors 2} = - \begin{bmatrix} 2 & 5 \\ 3 & 12 \end{bmatrix} = (24 - 15) = -9$$

$$\text{Co - factors 3} = + \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = 10 - 9 = 1$$

$$\text{Co - factors 2} = - \begin{bmatrix} 2 & 3 \\ 5 & 12 \end{bmatrix} = 24 - 15 = -9$$

$$\text{Co - factors 3} = + \begin{bmatrix} 1 & 3 \\ 3 & 12 \end{bmatrix} = 12 - 9 = 3$$

$$\text{Co - factors 5} = - \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = -(5 - 6) = -(-1) = +1$$

$$\text{Co - factors 3} = + \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = (10 - 9) = 1$$

$$\text{Co - factors 5} = - \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = -(5 - 6) = +1$$

$$\text{Co - factors 12} = + \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = (3 - 4) = -1$$

$$Adj.\ A = \begin{bmatrix} 11 & -9 & 1 \\ -9 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 12 \end{bmatrix} = 1\begin{bmatrix} 3 & 5 \\ 5 & 12 \end{bmatrix} - 2\begin{bmatrix} 2 & 5 \\ 3 & 12 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

Note

$= 1\,(36 - 25) - 2\,(24 - 15) + 3\,(10 - 9)$

$= (1 \times 11) - (2 \times 9) + (3 \times 1) = 11 - 18 + 3 = -4$

$$A^{-1} = \frac{Adj.\ A}{|A|} = -\frac{1}{4}\begin{bmatrix} 11 & -9 & 1 \\ -9 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{11}{4} & \frac{9}{4} & -\frac{1}{4} \\ \frac{9}{4} & -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Properties of Inverse Matrices

The inverse of an Invertible Matrix is unique. Unique means – if the inverse of A is B, then there is no other inverse of matrix A except B.

Proof – If it is possible, assume that B and C both are the inverse of matrix A. then

$$A\,B = BA = I$$

And $$AC = CA = I$$

$$AB = AC = \Rightarrow B(AB) = B(AC)$$

$\Rightarrow$ $$(BA)B = (BA)C$$

$\Rightarrow$ $$IB = IC$$

$\Rightarrow$ $$B = C$$

(ii) If A and B are the Invertible Matrices and their order is n, then their product is inverse of AB and $(AB)^{-1} = B^{-1}\,A^{-1}$.

Proof – Because A and B are invertible, therefore there exist A^{-1} and B^{-1}. Consequently

$(B^{-1}\,A^{-1})\,(AB) = B^{-1}\,(A^{-1}\,A)B$ Therefore the product is Homogeneous

$$= B^{-1}\,(IB)\,A^{-1}\,A = 1$$

$$= B^{-1}\,B = I$$

Again $$(AB)\,(B^{-1}\,A^{-1}) = A\,(BB^{-1})\,A^{-1}$$

$$= A(IA^{-1})$$

$$= AA^{-1} = I$$

Therefore AC is Invertible and $(AB)^{-1} = B^{-1}\,A^{-1}$.

Comment – (i) If A, B, C,…………AM are the invertible matrices of the same order, then $(A.B.C.....M) = M^{-1}\,...C^{-1}\,B^{-1}\,A^{-1}$

(ii) If A is a regular square matrix, then $AB = 0 \Rightarrow B = 0$;

If B is a regular square matrix, then $AB = 0 \Rightarrow A = 0$

(iii) The Transpose and inverse process in matrix is commutative.

$$(A')^{-1} = (A^{-1})^{1}.$$

19.5 Orthogonal Matrix

If $AA' = I$ then A is called Orthogonal Matrix.

Involuntary Matrix

If A is a square matrix and $A^2 = I$ (Corresponding or identity), then A is called Involuntary Matrix.

Note

Example 2: Show that the product for the following 2 × 2 matrices is commutative –

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

Solution: Here

$$AB = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

$$= \begin{bmatrix} ax - by & ay + bx \\ -bx - ay & -by + ax \end{bmatrix}$$

And

$$BA = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} xa - yb & xb + ya \\ -ya - xb & -yb + xa \end{bmatrix}.$$

Self Assessment

2. State whether the following statements are True or False:

6. Form a transpose of matrix on replacing the a_{ij} elements of matrix A from the co – factors c_{ji}. So made changed matrix is called Adjoint of A.
7. The inverse of an Invertible Matrix is not unique.
8. If B is a regular square matrix, then $AB = 0 \Rightarrow A = 0$.
9. If $AA' = I$ then A is called Zero Matrix.
10. If A is a square matrix and $A^2 = I$ (Corresponding or identity), then A is called Involuntary Matrix.

19.6 Summary

- If A is a matrix of order $m \times n$, then the matrix that will be found on exchanging the rows and columns of it, will be called as Transpose of A and will be denoted as A' or A^t or A^T.
- If matrix A and B are adaptable for product then the transpose of AB will be equal to the product of transposes in inverse i.e., if A and B are of order $m \times n$ and $n \times p$, then $(AB)'=A'B'$.
- The square – matrix A will be called as Regular if there is $BA = I$ Where I is the identity matrix of same order.
- We'll call the regular matrix A as Invertible and B as the inverse of A and denote it as A^{-1}.
- The inverse of an Invertible Matrix is unique. Unique means – if the inverse of A is B, then there is no other inverse of matrix A except B.
- If A and B are the Invertible Matrices and their order is n, then their product is inverse of AB and $(AB)^{-1} = B^{-1} A^{-1}$.
- If A, B, C,…………AM are the invertible matrices of the same order, then $(A.B.C…………M) = M^{-1}.....C^{-1}B^{-1}A^{-1}$. If A is a regular square matrix, then $AB = 0 \Rightarrow B = 0$;
- If B is a regular square matrix, then $AB = 0 \Rightarrow A = 0$. The Transpose and inverse process in matrix is commutative.
- If $AA' = I$ then A is called Orthogonal Matrix.
- If A is a square matrix and $A^2 = I$ (Corresponding or identity), then A is called Involuntary Matrix.

Note

19.7 Keywords

- ***Scalar:*** Which has magnitude, but don't have direction
- ***Inverse:*** Opposite

19.8 Review Questions

1. Prove that $(A + B)' = A' + B'$
2. Prove that $(AB)' = B'.A'$
3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 12 \end{bmatrix}$ Then find A^{-1}. **(Ans.:** $\begin{bmatrix} \frac{-11}{4} & \frac{9}{4} & \frac{-10}{4} \\ \frac{9}{4} & \frac{-3}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{1}{4} \end{bmatrix}$**)**

Answers: Self Assessment

1. Product	2. Identity	3. Unique	4. Regular
5. Orthogonal	6. True	7. False	8. True
9. False	10. True		

19.9 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India

Mathematics for Economist Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Blum- Viva Publications

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Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Note

Unit 20: Cramer's Rule

CONTENTS

Objectives

After reading this unit, students will be able to :

- Learn the Method to Solve the Simultaneous Equations.
- Known with Cramer's Rule.

Introduction

Cramer searches the easy method to solve the simultaneous equations that is known as Determinant Method. Equations can be solving easily by this methods.

20.1 Solution to Simultaneous Equations (Cramer's Rule)

Simultaneous equations can be solved easily by determinant method following:

Firstly we will solve the two variable simultaneous equations-

The determinant of the variable of x and y is-

$$a_1x + b_1y = c_1 \quad \text{...(i)}$$

$$a_2x + b_2y = c_2 \quad \text{...(ii)}$$

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, for finding the value of x we put the invariable column in place of the column of coefficient. So determinant will be-

$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, same for find out the value of y we put the invariable column in place of the column of coefficient. We can write it like that,

$$\frac{x}{\text{the determinant in which the column of invariables in place of the column of the } x \text{ coefficient}} = \frac{y}{\text{the determinant in which the column of invariable in place of the column of the } y \text{ coefficient}} = \frac{1}{\text{the determinant of } x \text{ and } y}$$

Here shown that all invariable written at right hand side in the equations.

It can be solved by the Cramer rule like that-

$$x = \frac{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}},\ y = \frac{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$$ It can also written like that-

$$\frac{x}{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}} = \frac{y}{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}} = \frac{1}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$$

So we can also solve more than two variable equations

Example 1: Solve the following equation by determinant method.

$$\mathbf{3x + 4y = 5}$$

$$\mathbf{3x - 4y = 2}$$

Solution: The determinant of coefficient is-

$\begin{bmatrix} 3 & 4 \\ 3 & -4 \end{bmatrix}$ and column of invariable is- $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ so,

$$\frac{x}{\begin{bmatrix} 5 & 4 \\ 2 & -4 \end{bmatrix}} = \frac{y}{\begin{bmatrix} 3 & 5 \\ 3 & 2 \end{bmatrix}} = \frac{1}{\begin{bmatrix} 3 & 4 \\ 3 & -4 \end{bmatrix}}$$

or, $$\frac{x}{(-20-8)} = \frac{y}{(6-15)} = \frac{1}{(-12-12)}$$

and $$\frac{x}{-28} = \frac{y}{-9} = \frac{1}{-24}$$

so, $$x = \frac{-28}{-24} = \frac{7}{6}$$

$$y = \frac{-9}{-24} = \frac{3}{8}.$$

Task Write the two variable simultaneous equations.

Now, lets discuss about the set of following simultaneous equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

Above equation can be written as array according to Cramer's rule.

$$AX = z \qquad ...(i)$$

Note

Here A= set of a_{ij} coefficient,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

X = vector column of variable

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } C = \text{vector columnof variables}$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Equation (i) again multiplied by A^{-1}.

$$A^{-1} AX = A^{-1} Z$$

or $$X = A^{-1} Z \qquad [\therefore AA^{-1} = I]$$

or $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Here C_{ij} is the intersection with respect to a_{ij}.

So, $$X = \frac{z_1 c_{11} + z_2 c_{21} + z_3 c_{31}}{|A|}$$

So the value of x_2 and x_3 can be find out.

Example 2: Solve the following equation by determinant method

$$4x + 2y = 2 \qquad \text{...}(i)$$

$$3x + 5y = 21 \qquad \text{...}(ii)$$

Solution: On writing the above equations in matrix form

$$AX = Z$$

Here, $$A = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } Z = \begin{bmatrix} 2 \\ 21 \end{bmatrix}$$

On using the Cramer's Rule

$$X = A^{-1}Z$$

$$= \frac{\text{Adjoint } A}{|A|} . Z$$

To get A^{-1}, firstly we would have to get the Transpose of given matrix –

$$A' = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$\text{Adjoint } A = \begin{vmatrix} 5 & -2 \\ -3 & 4 \end{vmatrix}$$

Note

Co-factor of 4 = + 5

Co-factor of 3 = – 2

Co-factor of 2 = – 3

Co-factor of 5 = + 4

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{14}\begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} \frac{5}{14} & \frac{-2}{14} \\ \frac{-3}{14} & \frac{4}{14} \end{bmatrix}$$

Now $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & \frac{-2}{14} \\ \frac{-3}{14} & \frac{4}{14} \end{bmatrix}\begin{bmatrix} 2 \\ 21 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{5}{14} \times 2\right) + \left(\frac{-2}{14} \times 21\right) \\ \left(\frac{-3}{14} \times 2\right) + \left(\frac{4}{14} \times 21\right) \end{bmatrix} = \begin{bmatrix} \frac{5}{7} - 3 \\ \frac{-3}{7} + 6 \end{bmatrix} = \begin{bmatrix} \frac{-16}{7} \\ \frac{39}{7} \end{bmatrix}$$

Therefore $x = \frac{-16}{7}$ and $y = \frac{39}{7}$.

Example 3: Solve by Matrix Method

$x - 2y + 3z = 1$ ...(*i*)

$3x - y + 4z = 3$ ...(*ii*)

$2x + y - 2z = -1$ ...(*iii*)

Solution: On writing the above equation in matrix form

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

On using Cramer Rule

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\text{Co-factor } A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -1 & 1 \\ 3 & 4 & -2 \end{bmatrix}$$

$$\text{Co-factor } 1 = +\begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix} = +(-2-4) = -6$$

$$\text{Co-factor } 3 = -\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} = -(4-3) = -1$$

Note

$$\text{Co-factor } 2 = +\begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix} = +(-8+3) = -5$$

$$\text{Co-factor } -2 = -\begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix} = -(-6-8) = +14$$

$$\text{Co-factor } -1 = +\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} = +(-2-6) = -8$$

$$\text{Co-factor } 1 = -\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} = -(4-9) = +5$$

$$\text{Co-factor } 3 = +\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = +(3+2) = +5$$

$$\text{Co-factor } 4 = -\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = -(1+4) = -5$$

$$\text{Co-factor } -2 = +\begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix} = +(-1+6) = +5$$

$$\text{Adjoint } A = \begin{bmatrix} -6 & -1 & -5 \\ +14 & -8 & +5 \\ +5 & -5 & +5 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} = 1\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix} + 2\begin{bmatrix} 3 & 4 \\ 2 & -2 \end{bmatrix} + 3\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= 1 \times (2-4) + 2(-6-8) + 3(3+2)$$

$$= -26 - 28 + 15 = -15$$

Therefore

$$A^{-1} = \frac{\text{Adj.}A}{|A|} = \frac{1}{-15}\begin{bmatrix} -6 & -1 & -5 \\ 14 & -8 & 5 \\ 5 & -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{-15} & \frac{1}{15} & \frac{5}{15} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{5}{15} \\ -\frac{5}{15} & \frac{5}{15} & -\frac{5}{15} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Now,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Note

$$= \begin{bmatrix} -\frac{2}{5} \times 1 + \frac{3}{15} - \frac{1}{3} \\ -\frac{14}{15} + \frac{24}{15} + \frac{1}{3} \\ -\frac{1}{3} + \frac{3}{3} + \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{8}{15} \\ 1 \\ 1 \end{bmatrix}$$

Therefore, on solving further we'll get

$$x = -\frac{8}{15},\ y = 1,\ z = 1$$

Self Assessment

1. Fill in the blanks:

1. Simultaneous equations can be solved easily by .... Method.

2. $\dfrac{x}{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}} = \dfrac{y}{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}$ = ---------

3. To get A^{-1}, firstly we would have to get the .... of given matrix.

20.2 Summary

- Simultaneous equation can be solved easily by following determinant method

  Firstly we will solved the two variable simultaneous equations-

  $a_1x + b_1y = c_1$

  $a_2x + b_2y = c_2$

  The determinant of the variable of x and y is- $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

- for finding the value of x we put the invariable column in place of the column of coefficient.

20.3 Keywords

- ***Simultaneous:*** Together

20.4 Review Questions

1. Solve the following equations:

(*i*) $4x + 2y = 2$; $3x - 5y = 21$

(*ii*) $2x - 3y + 4z = 8$

$3x - 4y + 5z = -4$

$4x - 5y + 6z = 12$ (**Ans.:** (*i*) $x = 2, y = 3$ (*ii*) $x = 1, y = 2, z = 3$)

Note

2. Prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = a+b+c.$$

3. Solve the following equation by determinant method –

$$3x + 4y = 5$$

$$3x - 4y = 2$$

(**Ans.:** $x = \frac{7}{6}$, $y = \frac{3}{8}$)

Answers: Self Assessment

1. Determinant
2. $\dfrac{1}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$
3. Transpose

20.5 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall India

Mathematics for Economist – Malkam, Nikolas, U. C. Landon.

Mathematics for Economist – Simon and Bloom – Viva Publications

Mathematics Economist – Makcal Harrison, Patrick Waldron.

Mathematics for Economist – Mehta and Madnani- Sultan Chand and sons.

Mathematics for Economist – Karl P. Simon, Laurence Bloom.

Mathematics for Economist and Finance – Martin Norman

Mathematics for Economist – Council for Economic Education

Essentials Mathematics for Economist – Nut Sedestor, Pitter Hammond, Prentice Hall Publications.

Unit 21: Determinant: Types and Properties

CONTENTS

Objectives

After reading this unit students will be able to :

- Understand Definition of Determinant.
- Know the Rows and Columns of a Determinant.
- Know Shape and Constituents of a Determinant.
- Expansion of a Determinant.
- Know the Questions Related to Cofactors.
- Understand Properties of Determinants.
- Calculate the Multiplications of Two Determinants.

Introduction

It is difficult to solve the parallel equations because as much as number of variables exist in these equations, the same amount of equations are available. To solve these equations in algebra, we do use a very special which is known to as Determinant.

21.1 Definition of Determinant

Think on the following exponential equations-

$$a_1x + b_1y = 0 \quad \text{...(i)}$$

$$a_2x + b_2y = 0 \quad \text{...(ii)}$$

From the above equations, for eliminations of x and y, equation (i) and equation (ii) are to be subtracted from a_2 and a_1 respectively.

Note

$$(b_1a_2 - b_2a_1)y = 0$$

$$b_1a_2 - b_2a_1 = 0$$

$$a_1b_2 - a_2b_1 = 0$$

Expression $(a_1b_2 - a_2b_1)$ has been expressed in the following manner:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Which is referred to as determinant

$\therefore$
$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_2 \\ a_2 & b_2 \end{vmatrix}$$

The expression $(a_1b_2 - a_2b_1)$ is referred as expansion or value of this determinant.

In the above determinant there are two Rows and two Columns. Thus the above determinant is known as second order determinant. The expression $(a_1b_2 - a_2b_1)$ is known as the expansion of this determinant. $a_1.a_2b_1.b_2$ are the constituents of determinant and a_1, b_2 and a_2, b_1 are its elements.

Now consider on the following three exponential equation

$$a_1 + b_1y + c_1z = 0$$

$$a_2 + b_2y + c_2z = 0$$

$$a_3 + b_3y + c_3z = 0$$

Eliminating x, y, z from these three equations, we find following result

$$a_1(b_2c_3 - b_1(a_2c_3 - a_3c_2) + c_1(a_3b_2 - a_3b_2) = 0$$

or

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) = 0$$

Expression of obtained value in the result can be shown as under:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This is called third order determinant. In third order determinant there are three rows and three columns.

Notes Expression on the left side is called expansion or value of this determinant.

21.2 Rows and Columns of a Determinant

In a determinant, horizontal lines from top to bottom are called its first, second and third rows, which is expressed as R_1, R_2, R_3...... respectively and vertical lines from left side to right side are called its first, second and third columns, which are expressed as C_1, C_2, C_3......

Note

21.3 Shape and Constituents of a Determinant

Each determinant has a square shape. Therefore, the more the number of determinant, the more will be the rows and columns.

For example: In a third order determinant there are 3 rows and 3 columns and the number of constituents is 3^2 or 9. Therefore in an nth order determinant there would be n rows and n columns and the number of constituents is n^2.

> *Did u know?* Number of constituents in a determinant = (order of determinant)^2

21.4 Expansion of a Determinant

A second order determinant can be expressed as under

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 \diagdown & \\ & b_2 \end{vmatrix} - \begin{vmatrix} & \diagup b_2 \\ a_2 & \end{vmatrix} = a_1 b_2 - b_1 a_2$$

For example:

$$\begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = \begin{vmatrix} 4 \diagdown & \\ & 5 \end{vmatrix} - \begin{vmatrix} & \diagup 2 \\ -3 & \end{vmatrix} = (4 \times 5) - (-2 \times -3) = 20 - 6 = 14$$

Expansion of third order determinant

Assume a third order determinant

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expansion of determinant is generally relative to constituents of first row or constituents of first column

(i) Expansion in terms of First Row

Imagine $\Delta = a_1 A - b_1 B + c_1 C$ ...(i)

Where *A, B, C* express determinants respectively

While expansion, signs of steps are taken in +, - , +, - , …….. order,

A, B, C can be found in the following manner:

$A = a_1$ comes in which row and column, leaving that obtained determinant

$B = b_1$ comes in which row and column, leaving that obtained determinant

$C = c_1$ comes in which row and column, leaving that obtained determinant

Here, a_1 comes in first row and first column, thus leaving them obtained determinant is

$$|A| = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

Note

Here, b_1 comes in first row and second column, thus leaving them obtained determinant is

$$|B| = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

And c_1 comes in first row and third column, thus leaving them obtained determinant is

$$|C| = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

From equation (i), expanding the given determinant we get

$$a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$a_1(b_2c_3 - c_2b_3) - b_1(a_1c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

$$a_1b_2c_3 - a_1c_2b_3 - b_1a_1c_3 - b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3$$

Task What would be the number of rows and columns of a second order determinants?

(ii) Expansion in Terms of First Column

Imagine $\Delta = a_1P - a_2Q + a_3R$ ...(ii)

Where *P, Q, R* displays determinants respectively

a_1, a_2 and a_3 falls in which rows and columns, leaving them and writing them as above

$$P = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, Q = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, R = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

From equation (ii)

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

$$= a_1b_2c_3 - a_1c_2b_3 - a_2b_1c_3 + a_2c_1b_3 + a_3b_1c_2 - a_3c_1b_2$$

Thus it is clear that with the application of both the methods expanding separately, we get the similar result. Here, it is noteworthy that there is a certain value of each determinant which is obtained after their expansion.

Method of finding co-determinant of some element

Suppose we have to find the co-determinant of constituent c_2 of above discussed third order determinant then in which row and column c_2 falls, leaving them together whatever determinant is left, that would be the co-determinant of constituent c_2.

Thus, co-determinant of constituent c_2 is as under

$$\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

Note

In the same fashion co-determinant of all other constituents can be found. Note that while determining the co-determinant we don't need to consider the sign. Therefore each co-determinant has a (+) sign.

Self Assessment

1. Fill in the blanks:

1. In ……………. order determinant there are three rows and three columns.
2. In a determinant, …………….. lines from top to bottom are called its first, second and third rows.
3. …………….. lines from left side to right side are called its first, second and third columns.
4. Each determinant has a …………… shape.
5. There is a ……….. value of each determinant which is obtained after their expansion.

21.5 Cofactors

As has been told earlier that while determining the co-determinant, we don't need to consider the sign. Now even if sign of co-determinants of constituents are taken into consideration, they become the co-factors of those constituents, which are expressed in C_1, C_2, C_3,.......

It is clear that there is only difference of sign between co-determinant and co-factor, whereas value of both are same.

Following is the definition of co-factors

Cofactors of constituent a_{ij} = $(-1)^{+} \times A_{ij}$

Where a_{ij} is the constituent of i^{th} row and j_{th} column and is the A_{ij} co-determinant of constituent a_{ij}

$$a_1 \text{ Cofactor } = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = C_1$$

$$a_2 \text{ Cofactor } = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = C_2$$

$$a_3 \text{ Cofactor } = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = C_3$$

Thus from equation (i), given determinant is = $a_1C_1 + a_2C_2 + a_3C_3$

21.6 Properties of Determinants

Properties of determinants are given below. With the help of these properties problems related to determinants can easily be solved. Following are the properties of determinants:

1. If all the rows and columns of a certain determinants are interchanged, there would be no change in the value of the determinants.
2. If the two adjacent rows or columns are mutually interchanged, the numerical value of the determinants remain same, but sign changes.
3. If all elements of any two rows or columns of any determinants are the multiple of a single number, then main determinants is multiplied by the same figure.

Note

4. Special attention is required on two rules related to this property:
   (i) If in a certain row or column of the determinants any common figure exists, then that is taken out of the determinants as a factor

   For example

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

   (ii) If row or column of the determinants is multiplied by any figure, then whole determinant should be divided by that figure.

   For example

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{k} \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

5. If element of any row or column of determinants are total of two numbers, then the same determinants can be expressed as a total of two determinants of same order.

   For example

$$\begin{vmatrix} a_1+p & b_1 & c_1 \\ a_2+q & b_2 & c_2 \\ a_3+r & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix}$$

6. If all elements of rows or columns of determinants are multiplied by a certain number and added to or subtracted from corresponding element of any other row and column, then the value of determinants remain unchanged.

   For example $\Delta' = \Delta$, whereas

$$\begin{vmatrix} a_1+mb_1-nc_1 & b_1 & c_1 \\ a_2+mb_2-nc_2 & b_2 & c_2 \\ a_3+mb_3-nc_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This property is of special importance while solving the problems related to determinants. According to this property, in determinants, in the elements of any row or rows, number of times corresponding value of any other row or column can be added or subtracted. The same process can be exercised for the columns.

21.7 Multiplication of Two Determinants

Multiplication of Two Determinants can be found only when both of them are of same order, else not.

Assume $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $B = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

Two determinants of third order are there. Their multiplication can be expressed with AB, there will be a third order determinants which can be found in the following way

Note

First of all, keeping first row (a_1 b_1 c_1) of determinants A at constant, multiply its elements with corresponding elements of first, second and third row of determinants B and find out their total separately, as shown below:

	$(a_1b_1c_1)$	$(a_1b_1c_1)$	$(a_1b_1c_1)$
	×	×	×
	$(x_1y_1z_1)$	$(x_2y_2z_2)$	$(x_3y_3z_3)$
Total =	$a_1x_1 + b_1y_1 + c_1z_1$	$a_1x_2 + b_1y_2 + c_1z_2$	$a_1x_3 + b_1y_3 + c_1z_3$

All the above three totals will be the first, second and third element of first row of determinants AB.

This way, keeping second row ($a_2b_2c_2$) of determinants A at constant, multiply its elements with corresponding elements of first, second and third row of determinants B and find out their total separately, which will be the elements of second row of determinant AB

The same process can be adopted for third row ($a_3b_3c_3$) of A and all rows of B.

Thus, multiplication of A and B = AB

$$= \begin{vmatrix} a_1x_1 + b_1y_1 + c_1z_1 & a_1x_2 + b_1y_2 + c_1z_2 & a_1x_3 + b_1y_3 + c_1z_3 \\ a_2x_1 + b_2y_1 + c_2z_1 & a_2x_2 + b_2y_2 + c_2z_2 & a_2x_3 + b_2y_3 + c_2z_3 \\ a_3x_1 + b_3y_1 + c_3z_1 & a_3x_2 + b_3y_2 + c_3z_2 & a_3x_3 + b_3y_3 + c_3z_3 \end{vmatrix}$$

Working rule

With the help of above property, create Zero in any row or a column to the extent possible and expand the determinants relative to elements of the same row or column

In solving the problems first, second and third …… etc. rows are expressed as R_1, R_2, R_3,....... etc and first, second, third………….. etc columns are expressed as C_1, C_2, C_3,....... respectively and whatever working is done between the rows and columns of determinants, the same is written at the right side of the determinants

Example 1: Find out the value of the following determinants:

$$\begin{vmatrix} \mathbf{13} & \mathbf{16} & \mathbf{19} \\ \mathbf{14} & \mathbf{17} & \mathbf{20} \\ \mathbf{14} & \mathbf{18} & \mathbf{21} \end{vmatrix} \mathbf{= 0}$$

Solution: Assume determinats = Δ

$$\therefore \quad \Delta \begin{vmatrix} 13 & 16 & 3 \\ 14 & 17 & 3 \\ 14 & 18 & 3 \end{vmatrix} (C_3 - C_2) = \begin{vmatrix} 13 & 3 & 3 \\ 14 & 3 & 3 \\ 14 & 3 & 3 \end{vmatrix} (C_2 - C_1) = 0 \quad (\because C_2 = C_3)$$

Example 2: Prove that:

$$\begin{vmatrix} \mathbf{23} & \mathbf{12} & \mathbf{11} \\ \mathbf{36} & \mathbf{10} & \mathbf{26} \\ \mathbf{63} & \mathbf{26} & \mathbf{37} \end{vmatrix} \mathbf{= 0.}$$

Solution: Given is the determinant

$$\begin{vmatrix} 11 & 12 & 11 \\ 26 & 10 & 26 \\ 37 & 26 & 37 \end{vmatrix} \qquad \text{(Subtracting } C_2 \text{ from } C_1\text{)}$$

= 0 $\because$ C_1 and C_3 are equal

Note

Example 3: Find out the value of:

$$\begin{vmatrix} \mathbf{13} & \mathbf{18} & \mathbf{23} \\ \mathbf{14} & \mathbf{19} & \mathbf{24} \\ \mathbf{14} & \mathbf{20} & \mathbf{25} \end{vmatrix}$$

Solution: Given is the determinants

$$= \begin{vmatrix} 13 & 18 & 5 \\ 14 & 19 & 5 \\ 14 & 20 & 5 \end{vmatrix} \text{ (Subtracting } C_3 \text{ from } C_2)$$

$= 0 \therefore C_2$ and C_3 are equal

Example 4: Find out the value:

$$\begin{vmatrix} \mathbf{29} & \mathbf{26} & \mathbf{22} \\ \mathbf{25} & \mathbf{31} & \mathbf{27} \\ \mathbf{65} & \mathbf{54} & \mathbf{46} \end{vmatrix}.$$

Solution: Given is the determinants

$$= \begin{vmatrix} 3 & 26 & 22 \\ -6 & 31 & 27 \\ 9 & 8 & 46 \end{vmatrix}. \quad \text{(Subtracting } C_2 \text{ from } C_1)$$

$$= \begin{vmatrix} 3 & 4 & 22 \\ -6 & 4 & 27 \\ 9 & 8 & 46 \end{vmatrix} \quad \text{(Subtracting } C_2 \text{ from } C_1)$$

$$= 3 \times 4 \begin{vmatrix} 1 & 1 & 22 \\ -2 & 1 & 27 \\ 3 & 2 & 46 \end{vmatrix}. \quad \therefore 3 \text{ from } C_1 \text{ and } 4 \text{ from } C_2 \text{ is common}$$

$= 0 \therefore C_2$ and C_3 are equal

$$= 12 \begin{vmatrix} 1 & 1 & 22 \\ 0 & 3 & 71 \\ 0 & -1 & -20 \end{vmatrix}.$$

Adding $2R_1$ to R_2 and Subtracting $3R_1$ to R_3

$$= 12[3(-20) - 71(-1)] = 12[-60 + 71]$$

$$= 12 \times 11 = 132.$$

Example 5: Find out the value of:

$$\frac{\mathbf{1}}{\mathbf{(x + y)}} \begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & x^2 & \mathbf{1} \\ \mathbf{3} & y^2 & \mathbf{1} \end{vmatrix}.$$

Solution: Given is the determinants

Note

$$\frac{1}{(x+y)}\begin{vmatrix} x^2 & 1 \\ y^2 & 1 \end{vmatrix}$$

Expanding with respect to first row

$$\frac{1}{(x+y)}.(x^2-y^2)=\frac{(x-y)(x+y)}{(x+y)}=x-y$$

Example 6: Prove that:

$$\begin{vmatrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{y+z} \\ \mathbf{1} & \boldsymbol{y} & \boldsymbol{z+x} \\ \mathbf{1} & \boldsymbol{z} & \boldsymbol{x+y} \end{vmatrix} = \mathbf{0.}$$

Solution:

Determinants $= \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0.$ (Adding C_2 to C_3)

$$=(x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix}$$

$$=(x+y+z)\times 0 = 0.$$

$$= 0$$

$$[\because\ C_1 = C_3 \therefore \Delta = 0]$$

Example 7: Prove that:

$$\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a^2} & \boldsymbol{b^2} & \boldsymbol{c^2} \end{vmatrix} = \boldsymbol{(a-b)(b-c)(c-a).}$$

Solution: Given determinants

$$=-\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$ (Subtracting C_1 from C_2) and (Subtracting C_1 from C_3)

$$=\begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}$$ (Expanding with respect to R_1)

$$=(b-a)(c^2-a^2)-(c-a)(b^2-a^2)$$

$$=(b-a)(c-a)[(c+a)-(b+a)]$$

Note

$$= (b-a)(c-a)(c-b)$$

$$= (a-b)(b-c)(c-a) \qquad \text{(Expanding with respect to } R_1\text{)}$$

Example 8: Prove that:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b).$$

Solution: Apply processes $R_2 - R_1$ and $R_3 - R_1$ as in example 6

Example 9: Prove that:

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(b-c)(c-a)(a-b).$$

Solution : In C_1 a, C_2 b and C_3 c is common

$$\therefore \text{ Determinants } = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Further follow solution of example 7.

Example 10: Prove that:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

Solution: Given determinants

$$= \begin{vmatrix} 0 & 0 & 0 \\ a-b & b-c & c \\ a^3-a^3 & b^3-c^3 & c^3 \end{vmatrix}, (C_1 - C_2) \text{ and } (C_2 - C_3)$$

$$= \begin{vmatrix} a-b & b-c \\ a^2-b^2 & b^2-c^2 \end{vmatrix} \qquad \text{(Expanding it with respect to } R_1\text{)}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & 1 \\ a^2ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$

$$= (a-b)(b-c)[b^2+bc+c^2] - (a^2+ab+b^2)]$$

Note

$$= (a-b)(b-c)[bc + c^2 - a^2 - ab]$$

$$= (a-b)(b-c)[b(c-a) + (c-a)(c+a)]$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

Example 11: Prove that:

$$\begin{vmatrix} \mathbf{1} & \boldsymbol{a} & \boldsymbol{a^2 - bc} \\ \mathbf{1} & \boldsymbol{b} & \boldsymbol{b^2 - ca} \\ \mathbf{1} & \boldsymbol{c} & \boldsymbol{c^2 - ab} \end{vmatrix} = \mathbf{0} \quad \text{or} \quad \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a^2 - bc} & \boldsymbol{b^2 - ca} & \boldsymbol{c^2 - ab} \end{vmatrix} = \mathbf{0}.$$

Solution: L.H.S. $= \begin{vmatrix} 1 & a & a - bc \\ 0 & b-a & b^2 - a^2 - ca + bc \\ 0 & c-a & c^2 - a^2 - ab + bc \end{vmatrix}$, $(R_1 - R_2 \text{ or } R_3 - R_1)$

$$= \begin{vmatrix} b-a & (b-a)(a+b+c) \\ c-a & (c-a)(a+b+c) \end{vmatrix} \qquad \text{(Expanding with respect to } C_1\text{)}$$

$$= (b-a)(a+b+c) - (b-a)(c-a)(a+b+c)$$

= 0 L.H.S

Example 12: Prove that:

$$\begin{vmatrix} \boldsymbol{a+b+2c} & \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{b+c+2a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{c+a+2b} \end{vmatrix} = \mathbf{2}\ (\boldsymbol{a+b+c})^{\mathbf{3}}.$$

Solution: Given determinants

$$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix} \qquad (C_1 + C_2 + C_3 \text{ at add})$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix} \qquad C_1 + 2(a+b+c) \text{ of common}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} \qquad (R_2 - R_1 \text{ or } R_3 - R_1)$$

$$= 2(a+b+c)\,[(b+c+a)\,(c+a+b) - 0] = 2(a+b+c)^3$$

Note

Example 13: Prove that

$$\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz.$$

Solution: From $R_1 - (R_2 + R_3)$, given determinants

$$= \begin{vmatrix} (y+z)-(y+z) & x-(x+2z) & x-(x+2y) \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2z & -2y \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 2z\begin{vmatrix} y & y \\ z & x+y \end{vmatrix} - 2y\begin{vmatrix} y & z+x \\ z & z \end{vmatrix} \qquad \text{(Expanding with respect to } R_1\text{)}$$

$$= 2z[y(x+y) - yz] - 2y\,[yz - z(z+x)]$$

$$= 2yz[(x+y-z)] - 2yz\,(y-z-x)$$

$$= 2yz(x+y-z-y+z-x) = 4xyz$$

Self Assessment

2. State whether the following statements are True or False:

6. There is only difference of sign between co-determinant and co-factor, whereas values of both are same.
7. If all the rows and columns of a certain determinants are interchanged, then value of the determinants change.
8. If the two adjacent rows or columns are mutually interchanged, the numerical value of the determinants remain same, but sign changes.
9. If row or column of the determinants is multiplied by any figure, then whole determinant should be multiplied by that figure.
10. Multiplication of Two Determinants can be found only when both of them are of same order, else not.

21.8 Summary

- It is difficult to solve the parallel equations because as much as number of variables exist in these equations, the same amount of equations are available. To solve these equations in algebra, we do use a very special which is known to as Determinant.
- The expression is referred as expansion or value of this determinant.
- In a determinant, horizontal lines from top to bottom are called its first, second and third rows, which is expressed as respectively and vertical lines from left side to right side are called its first, second and third columns, which are expressed as C_1, C_2, C_3.....
- Expansion of determinant is generally relative to constituents of first row or constituents of first column.
- There is a certain value of each determinant which is obtained after their expansion.

Note

- There is only difference of sign between co-determinant and co-factor, whereas values of both are same.
- If all the rows and columns of a certain determinants are interchanged, there would be no change in the value of the determinants.
- If the two adjacent rows or columns are mutually interchanged, the numerical value of the determinants remain same, but sign changes.
- If all elements of any two rows or columns of any determinants are the multiple of a single number, then main determinants is multiplied by the same figure.
- Multiplication of Two Determinants can be found only when both of them are of same order, else not.

21.9 Keywords

- ***Expansion:*** Detailed
- ***Value:*** Price

21.10 Review Questions

1. Define determinants with example.
2. Write down the properties of determinants.
3. Find out the value of following determinants.

(a) $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 65 & 54 & 46 \end{vmatrix}$ **[Ans.: = 132]**

(b) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ **[Ans.: =** $(a-b)(b-c)(c-a)$ **]**

4. Prove that: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

5. Prove that: $\begin{vmatrix} 23 & 12 & 11 \\ 36 & 10 & 26 \\ 63 & 26 & 37 \end{vmatrix} = 0$

Answers: Self Assessment

1. Third,	2. Horizontal,	3. Vertical,
4. Square,	5. Certain	6. True,
7. False,	8. True,	9. False
10. True		

Note

21.11 Further Readings

Books

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economics – Council for Economic Education.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematical Economy – Michael Harrison, Patrick Walderan.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Unit 22: Rank of Matrix

Note

CONTENTS

Objectives

After reading this unit students will be able to :

- Know the Merits of the Rank.
- Understand the Merits of the Rank with Example.

Introduction

An important number related to matrices is referred as Rank. In a matrix A, m x n, n is column vector and m is a factor, then in that case linear independent set maximum column vector would be the Rank of the matrix, this can be shown as $r(A)$.

In other words, it can be said that maximum number of independent column is Rank of the matrix. An m x n matrix becomes non-singular when its rank reaches n.

22.1 Characteristics of Rank

- Only Zero matrix has a zero rank
- rank $(A) \le \min(m, n)$ $\{(A) \le (m, n)\}$
- If A is a square matrix ($m = n$), A will be plural only when A's rank becomes n.
- Rank (AB) $\le$ minimum (Rank A, Rank B)
- Rank (A) + Rank (B) – $n \le$ Rank (AB)
- Rank (CA) = Rank (A) {C is constant figure}

Did u know? Maximum number of independent columns is known as Rank of the Matrix.

Example 1: Find out the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Note

Solution: Given that

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Applying row operation

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_{21}(1-2),\, R_{22}(-1)]{} \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_2(-1),\, R_3(-1)]{}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_{22}(-2)]{R_2(1/2)}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_{2-3}(-1),\, R_{12}(1)]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ here Matrix A keeps three non-zero rows

Therefore, Rank (A) is 3.

Example 2: If $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ the find out the Rank of the matrix.

Solution: Given that A = $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to - R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

Here non-zero rows are two, hence Rank (A) will be 2.

Example 3: If $A = \begin{bmatrix} 5-x & 2 & 1 \\ 2 & 1-x & 0 \\ 1 & 0 & 1-x \end{bmatrix}$, then determine the rank for each value of x.

Solution: Given that

$$A = \begin{bmatrix} 5-x & 2 & 1 \\ 2 & 1-x & 0 \\ 1 & 0 & 1-x \end{bmatrix}$$

Based on three-column expanding the above table

$$= x(x-1)\,(x-6)\,|\,A\,| = 1\{0-1\,(1-x)\} - 0\,\{0-2\} + (1-x)\,\{(5-x)\,(1-x) - 4\}$$

Note

Or, if $x \neq 0, 1$ and 6 would be the Rank 3 of Matrix A

And if x = 0, 1, 6 then $\begin{bmatrix} 5-x & 2 \\ 1 & 0 \end{bmatrix} = -2 \neq 0$, Rank will be 2

Linear Dependence and Rank of Matrix:

Linear dependency is found in a rows (columns) of a matrix only when linear conjugation of those rows (columns) equal to zero vector, viz

$$\text{If } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

$$K_1 a_{11} + K_2 a_{12} + K_3 a_{13} = 0$$

$$K_1 a_{12} + K_2 a_{22} + K_3 a_{32} = 0$$

$$K_1 a_{13} + K_2 a_{23} + K_3 a_{33} = 0$$

Here among K_1, K_2 and K_3 at least value of one should be Zero.

Example 1: Find out the linear dependency and rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix}.$$

Solution: While multiplying Row 1 (R_1) and deducting -1 from 2(R_2) and adding 3(R_3) then

$$-1R_1 - 1R_2 + R_3 = 0 \quad \text{...(i)}$$

$$-2R_1 + R_2 = 0 \quad \text{...(ii)}$$

$$-2R_1 + R_3 = 0 \quad \text{...(iii)}$$

R_1, R_2 and R_3 in equation (i) is not linear independence. Equation (ii) R_1 and R_2 and Equation (iii) R_1 and R_2 show the linear dependency. For Rank

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ viz Rank } (A) = 1 .$$

Example 2: If $A = \begin{bmatrix} 6 & 3 & 5 \\ -10 & 2 & 8 \\ 5 & 2 & 3 \end{bmatrix}$**, then examine the linear dependency and find out the Rank of Matrix (A).**

Solution: $C_1R_1 + C_2R_2 + C_3R_3 = 0$

$$6C_1 - 10C_2 + 5C_3 = 0 \quad \text{...(i)}$$

$$3C_1 + 2C_2 + 2C_3 = 0 \quad \text{...(ii)}$$

$$5C_1 + 8C_2 + 3C_3 = 0 \quad \text{...(iii)}$$

Note

Solving equation (i), (ii) and (iii)

$C_1 = -10$, C_2 1 and $C_3 = 14$, then linear dependency will be found

Viz and will be linear dependency because none of the linear are Zero viz Rank (A) = 2

22.2 Summary

- Only Zero matrix has a zero rank.
- Rank $(A) \leq \min(m, n)\ \{(A) \leq (m, n)\}$.
- If A is a square matrix ($m = n$), A will be plural only when A's rank becomes n.
- Rank (AB) $\leq$ minimum (Rank A, Rank B).
- Rank (A) + Rank (B) – $n \leq$ Rank (AB).
- Rank (A) = Rank (A) {C is constant figure}.

22.3 Keywords

- ***Matrix Rank:*** Maximum number of independent columns.

22.4 Review Questions

1. Find out the rank of following matrix.

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 4 & 6 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

2. Verify the linear independence

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 3 & 8 & 4 \\ 5 & 6 & 6 \end{bmatrix}.$$

22.5 Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematics for Economics and Finance – Martin Norman.

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Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Unit 23: Application of Matrices in Economics

Note

CONTENTS

Objectives

After reading this unit students will be able to :

- Understand the Application of Matrices in Economics.

Introduction

To solve the economical problems matrices are useful mathematical technique. With the help of it we can solve various economical problems. Whatever widest economical problem is linear, we can solve them with the help of matrices. Under this, we by using Cramer Rule (A^{-1}) we can solve the problems related to production, demand, supply, incoming and outgoing and national income is determined.

23.1 Application of Matrices in Economics

Example 1: Assume an equation for tri-regional economical model is given

$$Y = C + A_0$$
$$C = a + b\ (Y - T)$$
$$T = d + ty$$

Where $Y \rightarrow$ Income, C – Consumption, $T \rightarrow$ Tax Revenue, $t \rightarrow$ Rate of tax

$A_{0,}$ a, b and c are constant, then calculate national income, consumption and tax revenue.

Solution: Given that $Y = C + A_0$

Or
$$y - C = A_0 \qquad \text{...(i)}$$
$$-by + c + bT = a \qquad \text{...(ii)}$$
$$-ty + T = d \qquad \text{...(iii)}$$

Writing the equation (i), (ii) and (iii) in the following form

$$\begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} A_0 \\ a \\ d \end{bmatrix}$$

Note Here

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix}$$

Or $|A| = 1\,(1 + 0) + 1\,(-b + tb) = 1 - b + tb = 1 - b\,[1 - t]$

Applying Cramer Rule

$$Y = \frac{|A_1|}{|A|} \frac{\begin{vmatrix} A_0 & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{A_0(1) + 1(a - bd)}{1 - b(1 - t)} = \frac{a - bd + A_0}{1 - b[1 - t]}$$

$$C = \frac{|A_2|}{|A|} \frac{\begin{vmatrix} 1 & A_0 & 0 \\ -b & a & b \\ -t & d & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{a - bd - A_0(-b + bt)}{1 - b(1 - t)}$$

$$T = \frac{|A_3|}{|A|} \frac{\begin{vmatrix} 1 & -1 & A_0 \\ -b & 1 & a \\ -t & 0 & d \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{d + (a - bd) + A_0(t)}{1 - b(1 - t)}.$$ **Ans.**

Example 2: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 2 & 5 & 12 \end{bmatrix}$, then determine the value of A^{-1}

Transposing the given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 2 & 5 & 12 \end{bmatrix},$$

To find the adjoints, adjoints of matrix A (*aij*) will be determined in this way

$$\text{Adjoint 1} = \begin{bmatrix} 3 & 5 \\ 5 & 12 \end{bmatrix} = 36 - 25 = 11$$

$$\text{Adjoint 2} = -\begin{bmatrix} 2 & 5 \\ 3 & 12 \end{bmatrix} = (12 - 15) = -9$$

$$\text{Adjoint 3} = +\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = 10 - 9 = 1$$

Note

$$\text{Adjoint } 2 = -\begin{bmatrix} 2 & 3 \\ 5 & 12 \end{bmatrix} = 24 - 15 = -9$$

$$\text{Adjoint } 3 = +\begin{bmatrix} 1 & 3 \\ 3 & 12 \end{bmatrix} = 12 - 9 = 3$$

$$\text{Adjoint } 5 = -\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = -(5 - 6) = (-1) = +1$$

$$\text{Adjoint } 3 = +\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = -(10 - 9) = 1$$

$$\text{Adjoint } 5 = -\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = -(5 - 6) = 1$$

$$\text{Adjoint } 12 = +\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = (3 - 4) = -1$$

$$Adj.\ A = \begin{bmatrix} 11 & -9 & 1 \\ -9 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 12 \end{bmatrix} = 1\begin{bmatrix} 3 & 5 \\ 5 & 12 \end{bmatrix} - 2\begin{bmatrix} 2 & 5 \\ 3 & 12 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= 1(36 - 25) - 2(24 - 15) + 3\,(10 - 9)$$

$$= (1 \times 11) - (2 \times 9) + (3 \times 1) = 11 - 18 + 3 = -4$$

$$A^{-1} = \frac{Adj.\ A}{|A|} = -\frac{1}{4}\begin{bmatrix} 11 & -9 & 1 \\ -9 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{11}{4} & \frac{9}{4} & -\frac{1}{4} \\ \frac{9}{4} & -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Ans.

Example 3: Find out the optimal price of x, y and z by using Matrix Method

$$x - 2y + 3z = 1 \quad \text{...(i)}$$

$$3x - y + 4z = 3 \quad \text{...(ii)}$$

$$2x + y - 2z = -1 \quad \text{...(iii)}$$

Solution: Writing all the above equations in form of matrix

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Note

Applying Cramer's Rule

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\text{Transpose of } A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -1 & 1 \\ 3 & 4 & -2 \end{bmatrix}$$

$$\text{Adjoint } 1 = +\begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix} = +(-2-4) = -6$$

$$\text{Adjoint } 3 = -\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} = -(4-3) = -1$$

$$\text{Adjoint } 2 = +\begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix} = +(-8-3) = -5$$

$$\text{Adjoint } -2 = -\begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix} = +(-6-8) = +14$$

$$\text{Adjoint } -1 = +\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} = +(-2-6) = -8$$

$$\text{Adjoint } 1 = -\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} = -(4-9) = +5$$

$$\text{Adjoint } 3 = +\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = +(3+2) = +5$$

$$\text{Adjoint } 4 = -\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = -(1+4) = -5$$

$$\text{Adjoint } -2 = +\begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix} = +(-1-6) = +5$$

$$\text{Adjoint A} = \begin{bmatrix} -6 & -1 & -5 \\ +14 & -8 & +5 \\ +5 & -5 & +5 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} = 1\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix} - 2\begin{bmatrix} 3 & 4 \\ 2 & -2 \end{bmatrix} + 3\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

$= 1 \times (2 - 4) + 2 (-6 - 8) + 3 (3 + 2)$

$= -2b - 28 = 15$

$= -15$

Therefore, **Note**

$$A^{-1} = \frac{Adj.A}{|A|} = \frac{1}{-15}\begin{bmatrix} -6 & -1 & -5 \\ 14 & -8 & 5 \\ 5 & -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{-15} & \frac{1}{15} & \frac{5}{15} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{5}{15} \\ -\frac{5}{15} & \frac{5}{15} & -\frac{5}{15} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Now

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{2}{5}\times 1 & +\frac{3}{15} & -\frac{1}{3} \\ -\frac{14}{15} & +\frac{25}{15} & +\frac{1}{3} \\ -\frac{1}{3} & +\frac{3}{3} & +\frac{3}{3} \end{bmatrix} = \begin{bmatrix} \frac{-8}{15} \\ 1 \\ 1 \end{bmatrix}$$

Therefore, after solving we will get $x = -\frac{8}{15}, y = 1, z = 1$.

23.2 Summary

- If the economical problem is widest and linear, we can solve them with the help of matrices. Under this, we by using Cramer Rule (A^{-1}) we can solve the problems related to production, demand, supply, incoming and outgoing and national income is determined.

23.3 Keywords

- ***Matrix:*** Frame, Structure, Texture, Sequence.

23.4 Review Questions

1. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 6 \\ 3 & 6 & 13 \end{bmatrix}$, then find out A^{-1}.
2. Find out the optimal price of a, b and c by using Matrix Method.

   $a + 2b + 6c = 1$

   $3a - 4b - 2c = 1$

   $2a - b - 5c = -2$

Note

23.5 Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Mathematical Economy – Michael Harrison, Patrick Walderan.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Unit 24: Input-Output Analysis

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Understand the Assumptions of Input-Output Analysis.
- Know the Leontief's Input-Output Closed Model.
- Get the Information Related to First set for Equilibrium Equation of Closed Model.
- Get the Information related to the Second Set of Equilibrium Equation of Closed Model.

Introduction

Before we get to know the explanation of Input-Output Analysis, it would be better for us to understand the meaning of Input and Output. Input means the demand from the producer for material to produce the product, whereas output means the result of efforts made towards production. According to **Prof J R Hicks** "Input means materials which are purchased by the producer for production, contrary to this output stands for what is sold by the producer". We can say that input is the cost for the firm and output is revenue.

With the help of Input-Output Analysis, we get inter-trade relation and inter-dependence of entire economy because output of one industry could be input for another industry. Similarly output of other industry could be the input for first industry. For example the output of coal industry could be the input of steel industry.

Notes Output of steel industry is input of coal industry.

24.1 Assumptions of Input-Output Analysis

Followings are the assumptions of Input-Output Analysis:

1. Economy is in balanced condition.
2. Economy is divided in two parts – inter-trade part and last demand part and each part can further be bifurcated.

Note

3. Each industry produces only one product and there is no joint production of two products.
4. Total output of an industry is used as input for any other industry.
5. Production is done under rule of constant return.
6. Technical development is constant, it means that input coefficients are constant
7. Here in production no external austerity or improvidences are created

Leontief's Static Input-Output Model – Open Model

Leontief's Static Input-Output Model is based on the above assumptions. This can be understood with an example. Suppose the economy is divided into three parts. Out of which agriculture and industry are inter-trades and domestic part is expressed as last demand part.

With the help of Input-Output Analysis we can understand this model. In the given table, output of all the three parts is shown in horizontal rows, whereas inputs are shown in vertical columns. Total of the first row is 300 units, which shows the total output of agriculture. Out of which 50 units are of agriculture, 200 units are related to industry and balance 50 units are utilized as input for domestic part. The second row of the table shows total production of industry. Production is done equal to 150 units in the industry, out of which 55 units for agriculture, 25 units are related to industry and 70 units are used in domestic part.

Similarly columns show the cost of these areas. First column shows that for total production of 300 units in the industry, cost of 125 units comes, out of which 5 units are related to agriculture, 55 units are for industry and 20 units related to domestic area. Second column shows for total production of 150 units in the industry, total cost comes equal to 255 units, out of which 200 units relate to agriculture, 25 for industry and 30 units for domestic part. Zero in third columns shows depicts that domestic part is a consuming area, where not sells are made

Table 24.1: Input-Output Table

		Purchase area			Total output or total receipt
	Areas	Agriculture(1)	Industry(2)	Last Demand (3)	
Sell area ↑	(1) Agriculture	50	200	50	300
	(2) Industry	55	25	70	150
	(3) Last Demand	20	30	0	50
	Total cost or Total input	125	255	120	500

With the help of above table general Transaction Matrix can be created

Table 24.2: Transaction Matrix

		Purchase area			Total output
	Areas	Agriculture(1)	Industry(2)	Last Demand (3)	
Sell area ↑	(1) Agriculture	x_{11}	x_{12}	D_1	X_1
	(2) Industry	x_{21}	x_{22}	D_2	X_2
	(3) Last Demand	x_{31}	x_{32}	D_0	X_3

If we take columns of above table, then we will get following production function

Note

$X_1 = f_1 (x_{11}, x_{21}, x_{31})$ or $300 = f_1 (50, 55, 20)$

$X_2 = f_2 (x_{12}, x_{22}, x_{33})$ or $150 = f_2 (200, 25, 30)$

This way total production can be divided in various parts in the following way (adding division of all parts in the rows)

$X_1 = x_{11} + x_{12} + D_1$

$X_2 = x_{21} + x_{22} + D_2$

$X_3 = x_{31} + x_{32}$

Here we assume that total production of i industry is utilized as input in *n* industries, in this codition

$X_1 = x_{11} + x_{12} + ... + x_{in} + D_1$

In Leontifs, the concept of constant coefficient has also a value. In this situation technical coefficient will be

$$a_{ij} = \frac{x_{ij}}{X_j}$$

Here, x_{11} = production of i^{th} industry which is utilized by j^{th} industry

X_1 = Total production of i^{th} industry

In the above Table -1 technical coefficient can be found in the following manner

Table 24.3: Technical Matrix

		Working area		Input-Output Coefficient	
		Agriculture(1)	Industry(2)	Last Demand (3)	Total production
Sell area ↑	Agriculture	0.16	1.33	50	600
	Industry	0.18	0.16	70	150
	Sell area	0.06	0.20	0	50

Method of finding technical coefficient is very simple. Here we divide input of desired are by total production of that area. For example, total production of agriculture area is 300 units and inputs are 50, 55 and 20 units, in this condition technical coefficient would be $\frac{50}{300} = 0.16$, $\frac{55}{300} = 0.18$ and $\frac{20}{300} = 0.06$. In the similar way it can be calculated for other areas.

Leontief's Input-Output Matrix can be shows in algebraic expression in the following manner:

Assume our general model is following:

$X_1 = x_{11} + x_{12} + ... x_{in} + D_1$ ...(24.1)

Here X_1 = total production of i^{th} area, where $i = 1, 2, \ldots\ldots n$

x_{ij} = production of i^{th} industry which is utilized by j^{th} industry

Model 24.1 can be divided for n^{th} areas in the following manner:

$X_1 = x_{11} + x_{12} + ... + x_{1n} + D_1$

Note

$$X_2 = x_{21} + x_{22} + ... + x_{2n} + D_2$$
$$X_n = x_{n1} + x_{n2} + ... + x_{nn} + D_n$$

We know the technical coefficient

$a_{ij} = \frac{x_{ij}}{X_j}$ here x_1 = total production of i^{th} area

Or $$x_{ij} = a_{ij} X_j$$

If $x_{11} = a_{11} X_1, X_{12} = a_{12}X_2$.

Now our binomial equation would be

$$X_1 = a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n + D_1$$
$$X_2 = a_{21}X_1 + a_{22}X_2 + ... + a_{2n} X_n + D_2$$
$$X_n = a_{n1}X_1 + a_{n2}X_2 + ... + a_{nn}X_n + D_n$$

General matrix can be written in the following way

$$X_1 = \sum_{j=1}^{n} a_{ij} X_j + D_n$$

Or $X = AX + D$ ...(24.2)

Here $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} A = \begin{bmatrix} a_{12} & ... & a_{1n} \\ a_{n1} & ... & a_{nn} \end{bmatrix} D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix}$

From equation (24.2)

$$D = X - AX = (I - A) X$$

Here I means Idently Matrix, therefore

$$X = (I - A)^{-1}D \qquad ...(24.3)$$

Did u know? Here $(I - A)$ is an Inverse Matrix.

24.1.2 Limitations of Input-Output Model

Although Leontief Input-Output Analysis plays an important role in economic analysis, but there are some limitation which are as under:

1. *Impracticability of the Assumptions* – Assumptions of Leontief model are impractical. It assumes that technological coefficients are constant, which means technology will be constant or change in production will result to change in means devoted for production. Similarly it is also assumed that capital requirement for all parts of the economy would be that, but capital requirement of each part varies, thus their requirement would also be different.
2. *Neglecting Certain Factors* – because of the rigid nature of this model, solution to increasing cost and different other problems are not possible.
3. *One Sided Analysis* – This model is considered to be one-sided analysis, because it takes into account only productive sector of economy. Thus it ignores unproductive sectors completely.

Note

4. *Neglecting Factors of Substitution* – assumption of constant technical coefficient neglects the possibilities of factors of substitution. But in reality it is seen that such type of possibility of factors of substitution exist for short term also. For long term its possibility grows much more.
5. *Absence of Linear Relations* – The model assumes that input of one part is output for others viz there is linear relation in parts, which is contrary to the fact, as because of indivisibility of factors, increase in outputs are always not equal to increase in inputs.
6. *Use of Physical Units* – In practical commodities and services are expressed in monetary form only. Therefore, forecasting of economy in composite form is difficult with the balanced equation by Input-Output Analysis. Besides, measurements of physical units for various commodities and services are different. Therefore to create an Input-Output Table and Technological Coefficient, one has got to face many difficulties.
7. *Complexity* – This analysis takes help of various constructed equations and mathematical techniques, for which knowledge of advanced maths and statistical methods is required. Thus, the technique becomes more complex.

Task Describe assumptions of Input-Output Analysis.

24.2 Leontief's Input-Output Closed Model

In Leontief's Input-Output Open Model we had considered domestic demand as a separate area. If it is also merged in the economy, then no other area will remain left which has relation to external area. In this condition, each good will have a nature of intermediate goods because under this arrangement (n + 1), produced outputs are used for production of these outputs. This condition is known as closed model.

In this situation, full competition is found in the whole economy and there is no government interference. Here each of the industries (n + 1) produces a different quantity x_i (i = 1, 2.... n + 1 of output. The last area is domestic area for which output is X_{n+1}. Suppose x_{ij} = output of i^{th} industry, which is sent to j^{th} industry.

Self Assessment

1. Fill in the blanks:

1. Input means the demand from the producer for material to …………………….. the product.
2. …………….. means the result of efforts made towards production.
3. Total output of an industry is used as …………… for any other industry.
4. Assumptions of ………………. model are impractical.
5. With the help of Input-Output Analysis, we get inter-trade relation and inter-…………….. of entire economy.

24.3 First Set for Equilibrium Equation of Closed Model

Mathematical form under the closed model

$$X_i = \sum_{j=1}^{n+1} X_{ij} \qquad \text{...(24.4)}$$

Note

$$(i = 1, 2, ... n + 1)$$

Under constant technological coefficient

$$x_{ij} = a_{ij} X_i \quad ...(24.5)$$

Putting the value of equation 24.5 in equation 26.4

$$X_i = \sum_{j=1}^{n+1} a_{ij} X_j$$

Or

$$X_i = \sum_{j=1}^{n+1} a_{ij} X_j = 0 \quad ...(24.6)$$

Equation (24.6) is the first set for equilibrium equation.

24.4 The Second Set of Equilibrium Equation of Closed Model

Here we will concentrate on P_i (i = 1, 2 .... n + 1). Here P_i is the labour rate of domestic area.

Here assuming that Receipts and Costs are equal in Industry, equilibrium equation can be derived in the following manner:

For i^{th} industry

$$\text{Receipts} = X_i P_i$$

$$\text{And costs} = \sum_{j=1}^{n+1} P_i X_{ij} = \sum_{j=1}^{n+1} a_{ij} X_i \; P_i$$

Where

$$x_{ij} = a_{ij} X_i$$

In equilibrium condition

$$X_i P_i = \sum_{j=1}^{n+1} a_{ij} X_i \; P_i$$

or

$$X_i P_i = \sum_{j=1}^{n+1} a_{ij} X_i \; P_i$$

or

$$X_i P_i = \sum_{j=1}^{n+1} a_{ij} X_i \; P_i = 0 \quad ...(24.7)$$

$$i = 1, 2, ...n$$

Equation (24.7) is the second set for equilibrium equation.

24.4.1 Short note on Leontief's Dynamic Model

In constant model, with the help of technical coefficient of various areas of economy, mutual dependency is studied. But these coefficients do not throw light on the actual requirement of stock in economy. They are even to unable to enlighten us about how much capital is required for the required factors which is to be consumed by the industry. These capital is required to maintain constant capital appropriation such as construction, machine etc or to maintain raw material stock for production. If under constant input-output model (open model) effect of capital is involved, then this arrangement becomes dynamic. Thus, capital investment is a unique characteristic of dynamic input-output model.

Thus, constructive equations which are used under Input-Output Constant Model (Open Model), involve capital requirement including time-interval for each areas. Therefore, this model is just expansion of Constant Input-Output Model (Leontief's Open Model).

Assumptions

Input-Output Dynamic Model is generalized form of constant model. Therefore its assumptions would be similar to constant model. As **(1)** each industry produces homogenous products and **(2)** for production of each product only one technology is available, where conjugation of factors of products are constant.

Assume $C_i(t)$ is the consumption for the current year which is produced by i^{th} industry. $X_i(t)$ is the total production of i^{th} industry during t time-period, which is used for three purposes **(i)** consumption $\{C_i\ (t + 1)\}$, for the next period, **(ii)** net stock $S_i(t + 1) - S_i\ (t)$ of capital products for n industries and **(iii)** current flow of industrial production in the economy. Under these situations, following will be the equilibrium equation:

$X_i(t) = C_i\ (t + 1) + S_i\ (t + 1) - S_i\ (t) + x_{11}\ (t) + x_{12}\ (t)$ here $i = 1, 2$

Now the question arises how do current productions $X_1\ (t)$ and $X_2\ (t)$ happen? In Leontief production function, total production is the product of two factors – **(i)** Raw material for the current year and **(ii)** stock of capital products

$$X_1\ (t) = f[x_{11}\ (t), x_{21}(t), S_{11}(t), S_{21}(t)]$$

$$X_2\ (t) = f[x_{12}\ (t), x_{22}(t), S_{12}(t), S_{22}(t)]$$

In an economy total capital stock would be the total of capital stock of all industries

$$S_i\ (t) = S_{i1}\ (t) + S_{i2}\ (t)$$

And we will obtain change rate in capital stock.

$$\Delta S_i(t) = S_i\ (t + 1) - S_i\ (t)$$

Capital stock coefficient can be inversed in following manner:

$$b_{ij} = \frac{S_{ij}}{X_j}$$

Here b_{ij} = capital stock coefficient, x_i = total production of j^{th} industry and refinement of production of j^{th} industry by i^{th} industry.

24.4.2 The Importance of Input-Output Models for a Planned Economy

The procedures of input-output analysis play a very important role in economic area. Discussions on economic principles, drawing of national papers, planning of economic plans, study of inter-relation and inter-dependence of industries, study of trade cycles etc are the areas, which has been possible because of application of this procedure. Today almost in every area of economy, this technique is being used, but at present this technique is especially being used in planned economy.

Today many socialist and other countries have adopted the basic concept of input-output technique to accomplish their programs related to economic development. Most of the socialist and communist economy has a structural thought that economy should be molded under convenient integrated principle. Therefore this is the liability of this central agency that it ascertains the requirements of the economy and according to the requirement make available different production factors so that the objective of maximum social welfare can be met. Therefore, this is essential that industrial activities be instructed in a planned manner.

It can be ascertained with the help of this technique that how much of the total production of one industry can be used as input by the other industry. With the help of input-output matrix, statisticians and planning officers get to understand the transactional relation of whole economy.

Note

Input-Output Analysis helps us understanding the relation among the various areas of the economy. Thus, this analysis is convenient to the planners to understand the internal structure of the economy. Unplanned economy is based on the system of Test and Error, but under the planned economy, mitigation of such type of errors are done through this analysis.

This is noteworthy fact that in national economic program concept of dynamic input-output is more important that constant input-output as because due to fast change in the development of economy the flow of economic structure can not remain constant. In this context, this technique will be more suitable to the economic programs for developing economies. The reason for this is that under the linear identical input-output model constant technical coefficient, it can be applied even in the absence of reliable even numbers.

Input-Output model is not only used for studying the mutual relation in different production areas, but also is used for accomplishment of different purposes. For study of the administrative part of national and international, this model is also used. Under this model we can get scheme for easy study of input-output flow relation between the different areas of any country with that of different country.

This analysis is also used for the study of railway freight. For the very years, in railway trade tables contribution of production areas has been taken by the various railway stations. Inter-area flow is replaced by the division of tonnes, which is sent from one station to another station.

From the above analysis it is clear that input-output matrix is used for the analysis of economic factors. Tables of this analysis can be elected in different ways in the following way, which different areas of economy such as mutual relation and dependence of trades, study of inter-flow of different countries can be done easily. Under the socialist conditions, input-output analysis is an essential tool (Oskar Lang: Introduction to Economics, 1959) for the investigation of internal adjustment of national plans.

Example 1: Find out the input-out coefficient from the given transfer table

Purchase area → / Production area ↓	Agriculture	Industry	Last Demand	Total output
Agriculture	300	600	100	1000
Industry	400	1200	400	2000

If the obtained demand changes to 200 and 800 respectively, then find out total production which will be equivalent to new demand

Solution: Applying the formula of technical coefficient

$$a_{ij} = \frac{x_{ij}}{X_j}$$

Here a_{ij} = Technical coefficient,

X_i = Total production of j^{th} area

x_{ij} = Total production of j^{th} area which is absorbed by the i^{th} area in its production

Table to determine Technical Coefficient

Sector	**Agriculture**	**Industry**	**Total Output**
Agriculture	300/ 1000 =0.3	600/2000 = 0.3	1000
Industry	400/1000 = 0.4	1200/2000 = 0.6	2000

If the last demand becomes 200 and 800 respectively for the agriculture and industry, then

Sector	Agriculture	Industry	Last demand	Total Output
Agriculture	300	600	200	1100
Industry	400	1200	800	2400

Thus, Gross Ouput = 1100 + 2400 = 3500

Example 2: Table given below shows the inter-trade transactions of many crores of three sectors S_1, S_2 or S_3 of economy

	S_1	S_2	S_3	Last Demand	Total output
S_1	50	25	25	100	200
S_2	40	50	10	200	300
S_3	100	50	150	300	600

Calculate the coefficient matrix

Solution: We know that [technical coefficient = $a_{ij} = \frac{x_{ij}}{X_j}$]

Table to determine Technical Coefficient

	S_1	S_2	S_3	Total output
S_1	50/200 = 0.25	25/300 = 0.08	25/ 600 = 0.04	200
S_2	40/200 = 0.20	50/300 = 0.16	10/600 = 0.016	300
S_3	100/200 = 0.50	50/300 = 0.16	150/600 =0.25	600

Example 3: Technological matrix under Input-Output Model is given below:

Sector 1	Sector 2	Last demand	
Sector I	0.1	0.3	F_1
Sector II	0	0.2	F_2
Labor	0.9	0.8	-

If last demand = $F_1 = 0.5y + 100$

$F_2 = 0.3y + 204$

Then find out the equilibrium of revenue and output of different sectors. Compare the results if

Solution: Under equilibrium situation

$$F_1 = F_2$$

$\therefore$ $0.5y + 100 = 0.3y + 200$

or $5y + 1000 = 3y + 2000$

or $2y = 1000$

or $y = 500$

Again $F_1 = 0.5y + 100 = (0.5 \times 500) + 100 = 350$

$F_2 = 0.3y + 200 = (0.3 \times 500) + 200 = 350$

We know that

$$X = AX + D$$

or $X = (I - A)^{-1} D$

Note

Here

$X =$ area and I area II is total production

$I =$ Unit Matrix

$D =$ Last Demand

$A =$ Coefficient of Matrix

Now

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & -0.3 \\ 0 & 0.8 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{\text{Adjoint}}{\text{Determinant}}$$

$$\text{Det. of } (I - A) = \begin{bmatrix} 0.9 & -0.3 \\ 0 & 0.8 \end{bmatrix}$$

$$= 0.72$$

Transferred matrix $(I - A) = \begin{bmatrix} 0.9 & 0 \\ -0.3 & 0.8 \end{bmatrix}$

$$\left.\begin{array}{l} \text{Co - factor of } 0.9 = 0.8 \\ \text{Co - factor of } 0.3 = 0 \\ \text{Co - factor } 0 \quad = 0.3 \\ \text{Co - factor } 0.8 = 0.9 \end{array}\right\} \qquad \left[\begin{array}{l} \therefore (-1)^{i+j} \\ \text{for cofactor sign} \end{array}\right]$$

$$\text{Adjoint of } (I - A) = \begin{bmatrix} 0.8 & 0.3 \\ 0 & 0.9 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{1}{0.72}\begin{bmatrix} 0.8 & 0.3 \\ 0 & 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0.8}{0.72} & \frac{0.3}{0.72} \\ 0 & \frac{0.9}{0.72} \end{bmatrix} = \begin{bmatrix} 1.11 & 0.41 \\ 0 & 1.25 \end{bmatrix}$$

Now $(I - A)^{-1} D = \begin{bmatrix} 1.11 & 0.41 \\ 0 & 1.15 \end{bmatrix} = \begin{bmatrix} 350 \\ 350 \end{bmatrix} = \begin{bmatrix} 532.0 \\ 437.5 \end{bmatrix}$

Thus, total production of first sector = 532 units and

Total production of second sector = 437.5 units

Note

if F_1 = 100 and F_0 = 200, then

$$X\,(I - A)^{-1}\,D = \begin{bmatrix} 1.11 & 0.41 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$$= \begin{bmatrix} 192 \\ 250 \end{bmatrix}$$

Under this situation, total production of first sector = 192 units and

Total production of second sector = 250 units

Self Assessment

2. State whether the following statements are True or False:

6. $X_i = \sum_{j=1}^{n+1} X_{ij}$

7. $X_i P_i = \sum_{j=1}^{n+1} a_{ij}\ X_i\ P_i = 0$ equation is the second set of equilibrium equations.

8. Procedures of Input-Output Analysis does not play an important role in economic sector

9. Input-Output Analysis does not help us to understand the relation among the various sectors of the economy.

10. Technical coefficient = $a_{ij} = \dfrac{x_{ij}}{x_j}$.

24.5 Summary

- Input means the demand from the producer for material to produce the product, whereas output means the result of efforts made towards production.
- With the help of Input-Output Analysis, we get inter-trade relation and inter-dependence of entire economy because output of one industry could be input for another industry.
- Leontief's Static Input-Output Model is based on the above assumptions. This can be understood with an example.
- With the help of Input-Output Analysis we can understand this model. In the given table, output of all the three parts is shown in horizontal rows, whereas inputs are shown in vertical columns.
- Method of finding technical coefficient is very simple. Here we divide input of desired area by total production of that area.
- Assumptions of Leontief model are impractical. It assumes that technical coefficient are constant, which means technology will be constant or change in production will result to change in means devoted for production.
- In practical, commodities and services are expressed in monetary form only. Therefore, forecasting of economy in composite form is difficult with the balanced equation by Input-Output Analysis.
- In Leontief's Input-Output Open Model we had considered domestic demand as a separate area. If it is also merged in the economy, then no other area will remain left which has relation to external area.

Note

- In constant model, with the help of technological coefficient of various areas of economy, mutual dependency is studied.
- Input-Output Dynamic Model is generalized form of constant model. Therefore its assumptions would be similar to constant model.
- The procedures of input-output analysis play a very important role in economic area.
- Discussions on economic principles, drawing of national papers, planning of economic plans, study of inter-relation and inter-dependence of industries, study of trade cycles etc are the areas, which has been possible because of application of this procedure.
- Input-Output Analysis helps us understanding the relation among the various areas of the economy. Thus, this analysis is convenient to the planners to understand the internal structure of the economy.

24.6 Keywords

- ***Input:*** Receipts
- ***Output:*** Issuance Final product

24.7 Review Questions

1. Mathematically explain the first set of equilibrium equations of closed model.
2. What do you mean by the Leontief's Input-Output closed model?
3. Give a short note on Leontief's dynamic model.
4. Explain the importance of input-output model for a planned economy.
5. Mathematically explain the second set of equilibrium equations of closed model.

Answers: Self Assessment

1. Production	2. Output	3. Input	4. Leontief
5. Dependency	6. True	7. True	8. False
9. False	10. True		

24.8 Further Readings

Books

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economist – Malcom, Nicolas, U C London.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Mathematics for Economics – Council for Economic Education.

Mathematical Economy – Michael Harrison, Patrick Walderan.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Mathematics for Economist – Yamane, Prentice Hall Publication.

Unit 25: Conditions of Hawkins and Simon

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Conditions of Hawkins and Simon.
- Find the Technical Multiplications of Matrix.

Introduction

Sometime solution of in-out matrix comes to negative. If our out-result come to negative, it means that we are using more in than out, which is an unrealistic situation. Thus we can say that the system is not viable.

25.1 Conditions of Hawkins and Simon

Conditions of Hawkins and Simon can get us out from this situation. Our primary equation is $X = (I - A)^{-1}F$, then $(I - A)$can be written in following way

$$\begin{vmatrix} (1-a_{11}) & -a_{+2} & -a_{+3} & \dots & a_{1n} \\ -a_{21} & (1-a_{22}) & -a_{23} & \dots & a_{2n} \\ -a_{31} & -a_{32} & (1-a_{33}) & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & -a_{n3} & \dots & (1-a_{nn}) \end{vmatrix}$$

Then there would be two conditions of H.S.

(1) Table of matrix should always be positive

(2) Defacement element *i.e.* $(1 - a_{11})$, $(1 - a_{22})$, $(1 - a_{33})$ ...... $(1 - a_{nn})$ should be positive viz $a_{11}, a_{22}, a_{33}, \dots a_{nn}$ should always be less than 1

Note

Did u know? The above two conditions are called the conditions of Hawkins-Simon.

Example 1: If $(A) = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$

$$\therefore \qquad (I - A) = \begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$$

And if value of table is $(I - A)$, then 0.06 – 0.18 = (–) 8.12 which is less than zero. Here H.S. condition does not fulfill.

Then there is no solution possible.

Task Write down the conditions of Hawkins-Simon.

Example 2: For the year 1990, following is the transaction of inter-trade

Given table of inter-trade transaction for the year 1990 has been created for an economy

Trade	1	2	Last consumption	Total
1	500	1600	400	2500
2	1750	1600	4650	8000
Labor	250	4800	-	5050
Total	**2500**	**8000**	**5050**	**15550**

25.2 To Find the Technical Multiplications of Matrix

Showing direct requirement, prepare the technical multiplication of matrix. Is there any solution available for this method?

Dividing all outputs of the sector from all inputs, showing direct requirement of each unit of output technical multiplication of matrix can be found.

Solution:

$$a_{11} = \frac{500}{2,500} = 0.20 \left(= \frac{X_{11}}{X_1} \right)$$

$$a_{12} = \frac{1,600}{8,000} = 0.20 \left(= \frac{X_{12}}{X_2} \right)$$

$$a_{21} = \frac{1,750}{2,500} = 0.70 \left(= \frac{X_{21}}{X_1} \right)$$

$$a_{22} = \frac{1,600}{8,000} = 0.20 \left(= \frac{X_{22}}{X_2} \right)$$

Note

Therefore, given technical matrix

$\therefore \quad A =$

Industry	1	2
1	0.20	0.20
2	0.70	0.20
labour	0.10	0.60

and $$(I - A) = \begin{pmatrix} 1 & -0.20 & -0.20 \\ & -0.70 & 1-0.20 \end{pmatrix} = \begin{pmatrix} 0.80 & -.20 \\ -0.70 & 0.80 \end{pmatrix}$$

$\therefore$ $$(I - A) = \begin{vmatrix} 0.80 & -0.20 \\ -0.70 & 0.80 \end{vmatrix}$$

$$= 0.80 \times 0.80 - 0.20 \times 0.70 = 0.50$$

Since $|I - A|$ is positive and each element of basic diagonal of $(I - A)$ will be positive, therefore conditions of Hawkins-Simon is fulfilled, thus there is a solution for practical oriented method.

Example: Given

$$A = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.3 \end{bmatrix}$$

And last demand are F_1, F_2, F_3 then find out the level of output?

And last demand is F_1, F_2 and F_3. With the regularity of the model, find out the level of output. What would be the level of output if $F_1 = 20$, $F_2 = 0$ and $F_3 = 100$?

Solution: $$= \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = (I - A)^{-1} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

Now $$(I - A) = \begin{bmatrix} 0.9 & -0.3 & -0.1 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Thus co-divisive is

$$A_{11} = \begin{vmatrix} 0.8 & -0.2 \\ 0 & 0.7 \end{vmatrix} = 0.56$$

$$A_{12} = -\begin{vmatrix} 0 & -0.2 \\ 0 & 0.7 \end{vmatrix} = 0$$

$$A_{13} = +\begin{vmatrix} 0 & -0.8 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{21} = -\begin{vmatrix} -0.3 & -0.1 \\ 0 & 0.7 \end{vmatrix} = 0.21$$

$$A_{22} = \begin{vmatrix} 0.9 & -0.1 \\ 0 & 0.7 \end{vmatrix} = 0.63$$

Note

$$A_{23} = -\begin{vmatrix} -0.9 & -0.3 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} -0.3 & -0.1 \\ 0.8 & -0.2 \end{vmatrix} = 0.14$$

$$A_{32} = -\begin{vmatrix} -0.9 & -0.1 \\ 0 & -0.2 \end{vmatrix} = 0.18$$

$$A_{33} = \begin{vmatrix} 0.9 & -0.3 \\ 0 & 0.8 \end{vmatrix} = 0.72$$

Therefore the value of table is $0.9 \times 0.56 = 0.504$

Therefore, $$(I - A)^{-1} = \frac{1}{0.504}\begin{bmatrix} 0.56 & 0.21 & 0.14 \\ 0 & 0.63 & 0.18 \\ 0 & 0 & 0.72 \end{bmatrix}$$

$$= \begin{bmatrix} 1.11 & 0.42 & 0.28 \\ 0 & 1.25 & 0.36 \\ 0 & 0 & 1.43 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1.11 & 0.42 & 0.28 \\ 0 & 1.25 & 0.36 \\ 0 & 0 & 1.43 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1.11F_1 & + & 0.42F_2 & + & 0.28F_2 \\ 0 & + & 1.25F_2 & + & 0.36F_3 \\ 0 & + & 0 & + & 1.43F_3 \end{bmatrix}$$

From the given value of F_1, F_2 and F_3, we get

$$X_1 = 1.11F_1 + 0.42F_2 + 0.28F_3$$
$$= 1.11 \times 20 + 0 + 0.28 \times 100$$
$$= 50.2$$
$$X_2 = 1.25F_2 + 0.36F_3$$
$$0 + 0.36 \times 100 = 36 \text{ or}$$
$$X_3 = 1.43F_3 = 143.$$

Notes This is worth notable that if technical matrix is highly triangular, if all elements of main diagonal is zero or near to zero, then $(I - A)^{-1}$ matrix would be triangular. In this condition X_3 output will be totally dependent on the last demand of sector 3 and X_3 will be dependent on the last demand of sector 2 and sector 3.

Note

Example: In the above example, if final demands change by 10, 10, 10 then what will be the change in sector output?

We have $X = (I - A)^{-1} F$

$\therefore$ $\Delta X = (I - A)^{-1} \Delta F$

Where ΔX and ΔF are the conveyor of change in output and last demand, therefore

$$\Delta X = \begin{bmatrix} 1.11 & 0.42 & 0.28 \\ 0 & 1.25 & 0.36 \\ 0 & 0 & 1.43 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

$$\Delta X_1 = 1.11 \times 10 + 0.42 \times 28 \times 10 = 18.1$$

$$\Delta X_2 = 0 + 1.25 \times 10 + 0.36 \times 10 = 16.1$$

$$\Delta X_3 = 14.3$$

Self Assessment

1. Fill in the blanks:

1. Sometime solution of in-out matrix comes to ………………..
2. If our out-result come to negative, it means that we are using …………. more than out.
3. Table of matrix should always be ………….

25.3 Summary

- Sometime solution of in-out matrix comes to negative. If our out-result come to negative, it means that we are using in more than out, which is an unrealistic situation. Thus we can say that the system is not viable.
- Our primary equation is $X = (I - A)^{-1}F$, then $(I - A)$ can be written in following way

$$\begin{bmatrix} (1 - a_{11}) & -a_{+2} & -a_{+3} & a_{1n} \\ -a_{21} & (1 - a_{22}) & -a_{23} & a_{2n} \\ -a_{31} & -a_{32} & (1 - a_{33}) & a_{3n} \\ -a_{n1} & -a_{n2} & -a_{n3} & (1 - a_{nn}) \end{bmatrix}$$

25.4 Keywords

- ***Condition:*** Constraints

25.5 Review Questions

1. Find out output, when following technical multiplication (A) and last demand is given

$$A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.4 \\ 0.1 & 0.3 & 0.3 \end{bmatrix}_3 \quad F = \begin{bmatrix} 100 \\ 50 \\ 60 \end{bmatrix}$$

Note

2. With the help of following table calculate technical multiplication

↓ Purchase area → / Production area	Agriculture	Industries	Last Demand	Total production
(1) Agriculture	500	1000	200	1700
(2) Industries	700	1500	600	2800

Answers: Self Assessment

1. Negative   2. Input   3. Positive

25.6 Further Readings

Books

Mathematics for Economist – Yamane, Prentice Hall Publication.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Carl P Simone, Lawrence Bloom.

Mathematics for Economist – Malcom, Nicolas, U C London.

Mathematical Economy – Michael Harrison, Patrick Walderan.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economist – Mehta and Madnani, Sultan Chand and Sons.

Mathematics for Economist – Simone and Bloom, Viva Publication.

Essential Mathematics for Economics – Nutt Sedester, Peter Hawmond, Prentice Hall Publication.

Unit 26: Closed Economy: Input-Output Model

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Prime Elements of Input-output Analysis.
- Understand closed Input-Output Model.

Introduction

Meaning of the word input is Producer's demand of material for production. Similarly, meaning of the word output is associated with the result of productivity. By the word cost is meant that material which the producer purchases for production whereas by the word output is meant that which the producer sells. Hence for any firm, input is cost and output is receivable.

Input-Output analysis was searched by W.W. Leontief in 1951. This analysis studies that on a given technique, how much production should be done in various areas which is completely taken for use in form of consumption by consumers and industries and here the level of satisfaction is the maximum.

26.1 Prime Elements of Input- Output Analysis

1. This analysis is applicable in balanced economy.
2. This analysis studies production activities and problems related to technique.
3. This analysis is based on universal searches.

Input Output Model:

For making its technique clear, two types of equations are used:

(A) **Comparative equation:** this equation clarifies that complete production of any industry is either consumed by itself or is consumed by other industries and outside areas. Assumed that number of other industries from other areas is n and industry is *I*, whose total production is x_i, then following comparative equation will be there:

$$X_1 = X_{11} + X_{12} + X_{13} + \ldots + X_{1n} + F_1$$

$$X_2 = X_{21} + X_{22} + X_{23} + \ldots + X_{2n} + F_2$$

$$\therefore \quad X_i = X_{i1} + X_{i2} + X_{i3} + \ldots + X_{in} + F_i$$

here F $\Rightarrow$ Demand of outside area.

Note

(B) ***Structural Equation:*** for construction of this equation, support of technical co-efficient is taken. Assumed that i and j are two industries:

Technical co-efficient of industry *i* in industry *j* is

$$j = \frac{\text{Quantity of consumption of product of industry } i \text{ in industry } j}{\text{Total production of industry } i}$$

i.e.,

$$a_{ij} = \frac{X_{ij}}{X_j}$$

or

$$X_{iy} = a_{ij} \times X_j \qquad \ldots(A)$$

Here, a_{ij} is technical coefficient

The above give equation (A) is called structural equation. Now on converting comparative equation to structural equation:

$$X_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \ldots\ldots + a_{in}X_n + F_1$$

$$X_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \ldots\ldots + a_{2n}X_n + F_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_i = a_{i1}X_1 + a_{i2}X_2 + a_{i3}X_3 + \ldots\ldots + a_{in}X_n + F_i$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_n = a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 + \ldots\ldots + a_{nn}X_n + F_n$$

On writing in tabular form:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_i \\ X_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots & a_{in} \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ : \\ X_i \\ : \\ X_n \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ : \\ F_i \\ : \\ F_n \end{bmatrix}$$

or

$$X = AX + F \qquad \ldots(i)$$

It is the general equation of input-output model.

26.2 Closed Input- Output Model

If demand of outside area i.e. F, works like internal area, then that model is called closed input-output model. Just like final consumption area, supplies labour to other industries then it works like an industry. In this state, number of industries will be $n + 1$, if outside areas are represented by X_0 then,

$$X_0 = a_{00}X_0 + a_{01}X_1 + a_{02}X_2 + \ldots\ldots + a_{0n}X_n$$

$$X_1 = a_{10}X_0 + a_{11}X_1 + a_{12}X_2 + \ldots\ldots + a_{1n}X_n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_n = a_{n0}X_0 + a_{n1}X_1 + a_{n2}X_2 + \ldots\ldots + a_{nn}X_n$$

or

$$X = AX$$

Note

or $X - AX = 0$

or $[I - A]X = 0$, here $I \Rightarrow$ identity Matrix

For example, consumption function analysis (closed model) assumed that two industries and a final demand is given:

–	1	2	Final Demand
1	0.1	0.3	F_1
2	0	0.2	F_2
Labour (L)	0.9	0.5	–

Now assumed that final demand is also operating like an industry

$$F_1 = 0.5\,y + \bar{C}_1$$
$$F_2 = 0.3\,y + \bar{C}_2$$

Here 0.5 and 0.3 MPC or $\bar{C}_1$ and $\bar{C}_2$ constant.

Then coefficient $k = \dfrac{1}{1-(0.5+0.3)} = \dfrac{1}{0.2} = 5$

Then new input-output model;

$$X_1 = 0.1X_1 + 0.3X_2 + 0.5\,y + \bar{C}_1$$
$$X_2 = 0.0X_1 + 0.2X_2 + 0.3\,y + \bar{C}_2$$

And labour rate is given to be 0.9 and 0.5 then $.9x_1 + 0.5x_2 = y$

Then, $X_1(1-0.1) - 0.3\,X_2 - 0.5y = \bar{C}_1$

$$-0.0X_1 + X_2(1-0.2) - 0.3\,y = \bar{C}_2$$
$$0.9X_1 + 0.5X_2 + y = 0$$

Then in tabular form:

$$\begin{bmatrix} X_1 \\ X_2 \\ y \end{bmatrix} = \frac{1}{0.144}\begin{bmatrix} 0.65 & 0.55 & 0.49 \\ 0.27 & 0.45 & 0.27 \\ 0.72 & 0.72 & 0.72 \end{bmatrix}\begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ 0 \end{bmatrix}$$

Or
$$\begin{bmatrix} X_1 \\ X_2 \\ y \end{bmatrix} = \begin{bmatrix} 4.51 & 3.82 & 3.34 \\ 1.87 & 3.12 & 1.87 \\ 5.0 & 5.0 & 5.0 \end{bmatrix} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ 0 \end{bmatrix}$$

Then balanced income $y = 5X\bar{C}_1 + 5X\bar{C}_2 = 5(\bar{C}_1 - \bar{C}_2)$. **Ans.**

26.3 Summary

- Meaning of the word input is Producer's demand of material for production. Similarly, meaning of the word output is associated with the result of productivity. By the word cost is meant that material which the producer purchases for production whereas by the word output is meant that which the producer sells. Hence for any firm, input is cost and output is receivable.

Note

26.4 Keywords

- ***Input:*** Material for production for any firm

26.5 Review Questions

1. Clarify the closed input-output model.
2. What will be the effect on production in increase of its cost?

26.6 Further Readings

Books

Mathematics for Economist – Yamane – Prentice Hall, India.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics – Malcolm, Nicholas, U.C. London.

Mathematics for Economics – Karl P. Simone, Lawrence Bloom.

Mathematical Economics – Micheal Harrison, Patrick Waldaron.

Mathematics for Economist – Mehta and Madnani- Sultan Chand and Sons.

Mathematics for Economics and Finance – Martin Norman .

Mathematics for Economist – Simone and Bloom – Viva Publication.

Essential Mathematics for Economics – Nut Sedester, Peter Hamond, Prentice Hall Publication.

Unit 27: Linear Programming

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Obtain Knowledge related to Meaning of Linear Programming, its Conditions and Generalisation.
- Obtain Knowledge of Application of Firm Theory.
- Know the Limitations of Linear Programming.

Introduction

Linear programming is a mathematical method which mathematician **George Dantzig** had developed in 1947 for making the plan of various activities of American air force related to the problem of providing supplies to the army. It was also developed for use in economic theory of firm, administrative economics, inter-state trade, general balance analysis, welfare economics and development planning. In this unit, linear programming relating to firm is being described.

27.1 Meaning of Linear Programming

Problems of maximisation and minimisation are also called problems of optimisation. The techniques that are adopted by the economists for solving these problems, they are known as linear programming. With some constraints remaining in form of linear inequalities, it is a mathematical technique for analysis of optimum decisions. In mathematical language, it is applicable on all those problems in which, despite of arrangements of linear inequalities expressed in form of some variable; there is need for solution of maximisation and minimisation. If two variables x and y are functions of z, then value of z will be maximum when value of z is less than any movement done from that point. Value of z is minimum when value of z is more than any movement. When there is change in per unit cost and price along with the size of output then problem is not linear and if there is no change in it with the output, then the problem is linear. In this way programming may be defined like this that it is that method which is optimum combination for sources of production of given output or is produced from the given plant and machinery decides the optimum combination of goods. It is also used for deciding technical diversity.

Note

27.2 Conditions and Generalisation

Use of linear programming technique depends upon some conditions and generalisation.

First, a definite objective is there. This objective may be to maximise profit or income or to minimise costs. It is called **objective function** or **tenterion function.** If one quantity is maximised, its negative quantity is minimised. Dual of each maximization problem is minimisation problem. Original problem is primal, which always has a dual. If primal problem is related to maximisation, then dual problem will be minimisation and vice-versa.

Second, alternative production process must be there for fulfilling the objective. Thought of process or activity is very important in linear programming. Process, "is a specific method of doing any economic work" it is "a physical activity of some kind like, consuming something, collecting something, purchasing something, throwing away something and producing something in a special way."

Notes Linear Programming technique helps the agency taking the decision in the thing that for fulfilling the objective, they may select the most efficient and economical process.

Third, some constraints or restraints of the problem are also important. It is the limitations or barriers related to the problem which tell that what cannot be done and doing what is important. These are also known as inequalities. In production they are often given quantities of land, labour and capital, which are used in most efficient process for fulfilling a definite objective.

Fourth, choice variables are also there. These are those institutions which are selected so that the objective function may be made maximum or minimum and all restraints may be satisfied.

Last, feasible and optimum solutions are there. on income of the consumer and price of good being given, all possible combination of goods, which he may purchase from feasibility, will be feasible solution. For consumer feasible solution of two goods is all those combinations which are located on the budget line or to its left while on isocost line they are either located on it or to its right.

In other words we may say that feasible solution is that, which satisfies all restraints. Best from all feasible solutions is the optimum solution. If a feasible solution makes the objective function maximum or minimum, then it is optimum solution. Best available method for searching optimum solution from all possible solutions is simplex method. This process famous by the name simplex method is extremely mathematical and technical. Main objective of linear programming is to find optimum solutions and studying its specialities.

Task What is Linear Programming?

27.3 Application to the Theory of the Firm

Till the neo classic theory of firm, did the analysis of the problem of decision maker, taking one or two variable at time. It was related to one production process at a time. In linear programming, production function goes beyond these limited areas of economic theory. It thinks over various capacities and restrictions that are created in the production process. It select between various complex production process for doing the maximisation of profits and minimisation of costs.

Assumptions: linear programming analysis of the firm is based on the following assumption:

i. Institutions taking decision have to face some constraints or restrictions. It may happen that there is borrowing, raw material or space constraint on its activities. Type of constraint

Note

actually depends on the nature of the problem. Mostly they are constant sources of production process.

ii. It moves assuming number of optional production process to be limited.

iii. One assumption of it is there are linear relations among different variables which mean that under a process there is stable proportionality between input-output.

iv. Prices of inputs and coefficients are given and stable. They are known definitely.

v. Concept of additivity is also stable in the core of linear programming which means that total resources used by all firms will be equal to the sum resources used by each personal firm.

vi. Linear programming techniques also consider continuity and divisibility in goods and resources.

vii. Institutional resources considered as stable.

viii. For programming a definite duration is assumed. For convenience and more accurate results duration is generally small though possibility of comparatively longer duration is not ended.

On these assumptions being given application of linear programming for theory of the firm is done for solution to three problems:

(1) ***Maximisation of Output:*** We assume that a firm is made to produce a good *Z* with the use of input *X* and *Y*. Its objective is to **maximise the production.** It has two optional production processes *C* (Capital intensive) and *L* (Labour intensive). Restraint, cost-expense is line *MP*. As has been shown in figure 27.1. Rest all assumptions related to linear programming technique (as told above) are applicable. The problem is being described in language of figure 27.1.

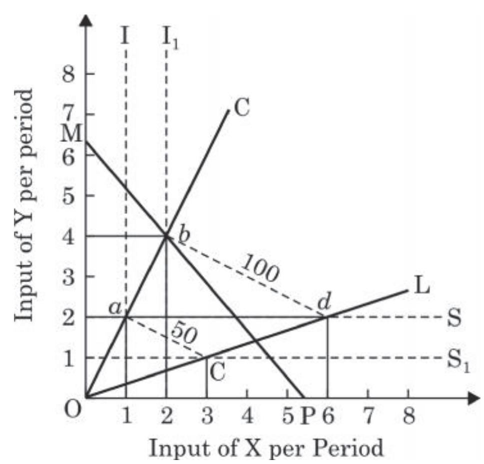

Figure 27.1

Units of cost (resources) per period have been measured on vertical *Y* axis and units of input *X* per period have been shown on the horizontal axis. If process *C* needs 2 units of *Y* with each unit of input *X*, then it will produce 50 units of good *Z*. If by doubling the inflow of *X and Y* units of *X* and units of *Y* are made 4, then outflow will also be doubled to 100 units of *Z*. these combinations of *X* and *Y* expressed by *a* and *b* establish production parameter scale **capital intensive process** line *OC*. At the other side original unit of *goods X* (50) may be produced through process *L* with the combination of one unit *of X* and *Y* and 100 units of *Z* may be produced by doubling *X and Y* with 2 units of *X* and 6 units of *Y*. these production parameters are established on line *OL* of labour intensive process which are expressed by combinations *c* and *d* of the prices. If at unit level of 50, points *a* and *c* are joined on linear rays *OC* and *OL*, they make isoquant (which is shown by dotted line). *t* in isoquant according to production level of 100 units is *bds*. Isocost *MP* represents cost-expenditure restraint and determines a limit of production capacity of the firm. Inside the area expressed by the triangle *obd*, firm may produce from any of the two available techniques *C* and *L*. firm will not be able to produce outside the "area of these feasible solutions." Optimum solution maximising the production of the firm will be at the point where isocost curve touches the isoquant curve of maximum production. In the figure above isocost curve *MP* touches the isoquant $I\text{-}_1bds$ at point b of process ray *OC*. It shows that firm, using 4 units of input *Y* and 2 units if input *X*, will use the capital intensive technique and produce 100 units of good *Z*.

Note

(2) ***Maximisation of Revenue:*** Take the other firm whose objective function is to maximise its revenue with restraints of limited capacity remaining. Consider that planning produces two goods *X* and *Y*. It has four departments in which capacity of each is fixed. Consider that these four departments are associated with making, compilation, polishing and packing which we name *A, B, C; D*. problem has been shown in figure 27.2

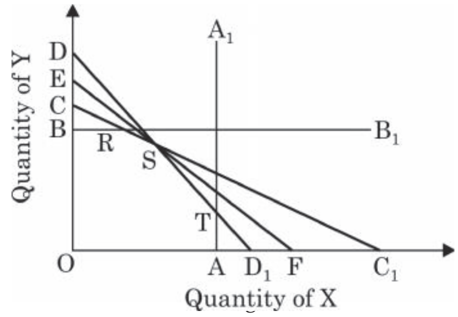

Figure 27.2

With restraints *A, B, C, D* remaining, *X* and *Y* are produced. Restraint A limits the production of good *X* to *OA*. Restraint *B* limits the production of good *Y* to *OB*. Restraint *C* limits production of both goods, *X* and *Y* to OC_1 and *OC* respectively whereas; restraint *D* limits their production to OD_1 and *OD* respectively. Area OATSRB expresses all those combinations of X and Y which may be produced without violating any constraint. It is the area of feasible production inside which X and Y may be produced but there is no possibility of production of any combination on any point outside this area.

Optimum solution may be looked for inside the feasible area by taking isoprofit line. Isoprofit line expresses all those combinations of X and Y which provide equal profit to the firm. Optimum solution is located on point S of highest isoprofit line EF inside the polygon OATSRB. Any point other than S will be situated outside the feasible production area.

(3) ***Minimisation of Cost:*** Food problem was first economic problem whose solution through linear programming was done through minimisation of cost. Consider that a consumer buys bread and butter at market price. Problem is that from various quantities of both foods cost of receiving total nutrients is made minimum.

Dotted linear solution of food problem is shown in figure 27.3. Bread (x_1) and butter (x_2) have been both measured respectively on both axis. Line AB expresses combination of less bread and more butter and line CD expresses combination of more bread and less butter. Feasible solution is located on deep line AZD or above it. Optimum solution is on point Z, where isocost line (dotted) RK is there which passes through the intersection point of AB and CD. If bread is expensive, possible solution may be on A and if butter is comparatively expensive, it can be on D. but in this problem, this solution will be on Z because here only cost is minimised.

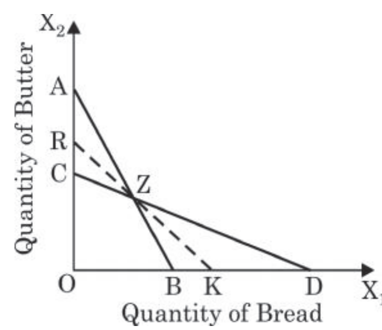

Figure 27.3

Self Assessment

1. Fill in the blanks:

1. Linear programming was developed in ....
2. .... was the father of linear programming.
3. Problems of maximisation and minimisation are also known as problems of ....
4. Optimum solution may be looked for inside the feasible area by taking .... line.
5. Food problem was first economic problem whose solution through .... programming was done through minimisation of cost.

Note

27.4 Limitations of Linear Programming

Linear programming proved to be a very profitable resource in economics. But it has its own limitations. In reality, because of many restraints, actual problems cannot be solved through linear programming. **First**, it is not easy to define a specific objective function. **Secondly**, even if a specific objective function is defined, knowing various social, institutional financial and other constraints popular in the path of the view of the objective is not an easy task. **Thirdly,** on a set of specific objectives and constraints being given, it is important that constraints may not be expressed directly in form of linear inequalities. **Fourthly**, even if described problems are worth crossing over also, main problem is of estimation in relation to various constant coefficients, which enter a linear programming problem like prices. **Fifth**, Main shortcoming of this technique is it is based on establishment of linear relation between inputs and outputs which means that relations of sum, multiplication and divisibility of found between various inputs and outputs. But these relations are not in each linear programming problem because in many problems non-linear relations are found. **Sixth**, technical goods and resources are based on assumption of free competition in the market. But in state of free competition is not found. **Seventh**, linear programming moves with the assumption of consideration in the economy, but in reality consideration are either decreasing or increasing. **Lastly**, it is a very complex mathematical and complex technique. Solution of a problem with linear programming expects maximisation and minimisation of a clear designated variable. Solution of this linear programming problem may also be found through complex methods like Simplex Method, in which many mathematical enumerations have to be done. For it special computational technique like electric computer of desk calculator is needed. Such calculators are not only expensive, but for operating them, specialists are also needed.

Did u know? Linear Programming model mostly presents trial and error solutions. Searching optimum solution of various economic problems in reality is difficult.

27.5 Summary

- Linear programming is a mathematical method which mathematician George Dantzig had developed in 1947 for making the plan of various activities of American air force related to the problem of providing supplies to the army. It was also developed for use in economic theory of firm, administrative economics, inter-state trade, general balance analysis, welfare economics and development planning.
- Problems of maximisation and minimisation are also called problems of optimisation. The techniques that are adopted by the economists for solving these problems, they are known as linear programming.

27.6 Keywords

- ***Optimise:*** As per requiement
- ***Primal:*** Main, chief

27.7 Review Questions

1. What is linear programming? Describe its conditions.
2. On which assumptions is linear programming analysis of firm based upon?
3. Describe maximisation of production.

Note

Answers: Self Assessment

1. 1947
2. George Dantzig
3. Optimisation
4. Isoprofit
5. Linear

27.8 Further Readings

Books

Mathematics for Economics – Malcolm, Nicholas, U.C. London.

Mathematics for Economics – Karl P. Simone, Lawrence Bloom.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Simone and Bloom – Viva Publication.

Essential Mathematics for Economics – Nut Sedester, Peter Hamond, Prentice Hall Publication.

Mathematical Economics – Micheal Harrison, Patrick Waldaron.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Unit 28: Formulation of Linear Programming

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know Linear Programming Formulation

Introduction

Technique adopted by economists for solving the problems of maximisation and minimisation is called Linear Programming. It is a mathematical method. It was developed by a mathematician George Denzing in 1947.

Technique of knowing maximum or minimum value of any problem, on constraints being given, is known as Linear Programming Formulation. Here one is objective functional while the other is given with condition.

28.1 Linear Programming Formulation

Steps of Linear Programming Formulation may be understood through example. As we know that LPP (Linear Programming Problem) has three parts:

1. Objective Function: which we maximise or minimise
2. Structural constraint
3. Non- Negative Constraint

For e.g., Assumed that two things are X and Y whose price are ₹ 2 and ₹ 5, objective function

$$\text{Max} : f = 2x + 5y \qquad \text{(A)}$$

And structural constraint:

$$X + 4y \leq 24$$
$$3X + y \leq 21 \qquad \text{(B)}$$
$$X + y \leq 9$$

And non-negetive constraint:

$$X \geq 0 \qquad \text{(C)}$$

And, $y \geq 0$

In the above problem, equation A represents objective function whose objective is to maximise and minimise the problem. While equation B shows that how much production of x and y is possible by combinations of different resources of production. It means it shows the constraint of resources of production and equation. C shows non-negetive constraint. Above process is known as Formulation of Linear Programming.

Note

Example: Convert the below given problem in form of Linear Programming

Vitamin (Type)	Vitamin in Per Kg food item		Minimum daily vitamin requirement
	I	II	
A_1	10	4	20
A_2	5	5	20
A_3	52	6	12
Per Kg Price of Food Item	₹ 0.60	₹ 1.00	

Solution: $\text{Min } f = 0.6\,x_1 + x_2$

$x_2 = 0.6\,x_1 + x_2$

Constraint $10x_1 + 4x_2 \geq 20$

$5x_1 + 5x_2 \geq 20$

$2x_1 + 6x_2 \geq 12$

And, $x_1 \geq 0$

$x_2 \geq 0$

Non-negative constraint

28.2 Summary

- Technique adopted by economists for solving the problems of maximisation and minimisation is called Linear Programming. It is a mathematical method. It was developed by a mathematician George Denzing in 1947.

28.3 Keywords

- ***Linear Programming:*** Economical method for solving the problem associated with maximisation and minimisation.

28.4 Review Questions

Convert the below given table in linear programming:

Calcium (Type)	Calcium in Per Kg food item		Minimum daily calcium requirement
	I	II	
Z_1	20	8	40
Z_2	10	5	40
Z_3	4	6	24
Per Kg Price of Food Item	₹ 1.20	₹ 2.00	

28.5 Further Readings

Books

Mathematics for Economics – Council for Economic Education.

Mathematics for Economics – Karl P. Simone, Lawrence Bloom.

Mathematical Economics – Michael Harrison, Patrick Waldran.

Mathematics for Economics – Malcolm, Nicholas, U.C. London.

Essential Mathematics for Economics – Nut Sedester, Peter Hamond, Prentice Hall Publication.

Mathematics for Economics and Finance – Martin Norman.

Unit 29: Graphic Method

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Know the Graphic Solution of Economic Problems.
- Understand the Minimisation of Cost.

Introduction

Take a firm which at given prices ₹ 12 and ₹ 15 respectively produce two goods *X* and *Y* per unit. For producing goods *X* firm need 12 units of input *A*, 6 units of input B and 14 units of input *C*. but for goods *Y*, 4 units of input *A*, 12 unit of input *B*, and 12 units of input *C*. Total available units of input *A* are 48, of *B* is 7 and of *C* is 84 units. Input-output data of this linear programming problem has been shown in table 29.1.

Table 29.1: Input-Output Data

Input	Number of units for producing goods		Units of total receivable costs
	Goods X	Goods Y	
A	12	4	48
B	66	12	72
C	14	12	84
Cost Per unit	₹ 12	₹ 15	-

Each Linear Programming problem has three parts. They are of the following type of the above given problem:

(i) ***Objective Function:*** Objective function tells this that if two goods X and Y bring ₹ 12 and ₹ 15 per unit cost then how much quantity of these goods be produced so the firm may acquire maximum cost or income. It may be written like this:

Maximise $R = 12X + 15Y$

(ii) ***The Constraints:*** The above table may now be converted in form of equations which express the constraints under which a firm works. These are called structural constraints.

Let us first take input A. Maximum available quantity of input *A* is 48 units. But quantity of both goods *X* and *Y* cannot be more than 48 units. Mathematically, because $12X + 4Y$ units

Note

cannot be more than 48, that is why constraint of input A will be $12X + 4Y \leq 48$. Through similar logic, inequalities of constraints of inputs B and C may be written. Hence there are three structural constraints of our problem:

$$12X + 4Y \leq 48 \quad ...(1)$$

$$6X + 12Y \leq 72 \quad ...(2)$$

$$14X + 12Y \leq 84 \quad ...(3)$$

(iii) ***Non Negative Constraint:*** In Linear Programming problem, non-negative constraints are also there which are dependent on this assumption that in solution to the problem, many variables cannot have negative values. It means that production of goods X and Y may be zero or positive but it cannot be negative. Hence non negative constraint of our problem is $X \geq 0$ and $Y \geq 0$.

29.1 The Graphic Solution of the Problem

For graphic solution, let us write the above describes problem again:

Maximise $R = 12X + 15Y$

Subject to (*i*)

$$12X + 4Y \leq 48 \quad ...(1)$$

$$6X + 12Y \leq 72 \quad ...(2)$$

$$14X + 12Y \leq 84 \quad ...(3)$$

$$X \geq 0,\ Y \geq 0$$

(ii) For expressing each inequality through a graph, we will leave inequality sign (≤) in each equation and take = sign. Hence equation (1) will be written as follows:

$$12X + 4Y = 48.$$

Or $X = 4$ (when $Y = 0$)

Similarly assuming that from all 48 units only goods Y is produced:

$$0 + 4Y = 48$$

Or $Y = 12$ (when $X = 0$)

Equation $12X + 4Y= 48$ has been shown by line AB in figure 29.1, where $OA = Y$ and $OB = 4X$. Any point of line AB, like T, satisfies equation $12X + 4Y= 48$ because area below and towards the left of this line AB satisfies the inequality equation $12X + 4Y \leq 48$.

Similarly on solving 6X + 12Y = 72: $X = 12$ and $Y = 6$ are obtained. Which have been indicated by line CD in figure 29.1 where $OC = 6Y$ and $OD = 12X$ and on solving equation $14X + 12Y = 84$ we find, $X = 6$, $Y = 7$ which has been shown in figure 29.1 through line EF, where $OE = 7Y$ and $OF = 6X$.

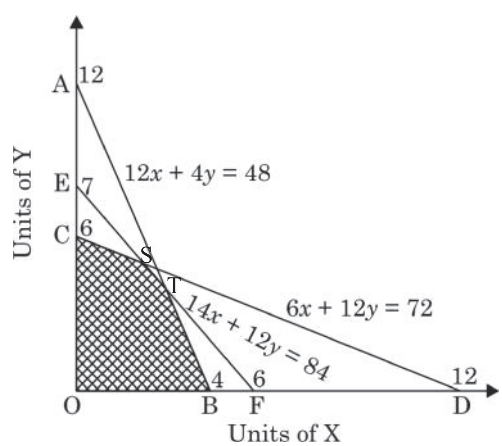

Figure 29.1

Feasible Region: Figure 29.1 shows that in shaded region all points that are surrounded by the three intersecting lines will satisfy each of the three inequalities. At point S line EF intersects line CD and at point T line CD intersects Line AB. In this way area OBTSC which is situated below the points intersecting these three line, S and T at the left side, satisfies the inequalities of all the three equations. This shaded region is called feasible area of production and each point which is inside it or on its boundary expresses feasible solution of the problem.

Optimum Solution: which point various points B, T, S, and C which express feasible solution is optimum point which will maximise the cost of the firm? How can this point be known through algebra?

Through equations (1) and (2) we know the co-ordinates of point B and C according to which $OB = 4X$ and $OC = 6Y$. For determining the coordinates of point T, let us take equations (1) and (2) in form of simultaneous equation (Because lines AB and EF intersect at point T) and solve them:

$$12X + 4Y = 48 \quad ...(1)$$

$$14X + 12Y = 84 \quad ...(3)$$

Multiplying equation (1) with 3 and subtracting equation (3) from it:

$$36X + 12Y = 144$$

$$14X + 12Y = 84$$

$$22X = 60$$

$$X = 2.73$$

Applying the value of $X = 2.73$ in equation 1,

$$12 \times 2.73 + 4Y = 48$$

$$32.76 + 4Y = 48$$

$$4Y = 48 - 32.76$$

Or $$4Y = 15.24$$

$$Y = 3.81$$

Hence, coordinates of point T are $X = 2.73$ and $Y = 3.81$. Similarly on solving equations (3) and (2), coordinates of Point S are, X=1.5 and Y= 5.25.

For searching the optimum combination of X and Y, let us substitute prices of X and Y (₹ 12 and ₹ 15 respectively) in values of points of these co-ordinates which have been calculated above. At point B, $X = 4$ and $Y = 0$. Substituting them in objective function $f = 12X + 15Y$:

$$(₹\ 12)\ (4) + (₹\ 15)\ (0) = ₹\ 48 \quad ...(4)$$

At point T, $X = 2.73$ and $Y = 3.81$, let us obtain in the same way:

$$(₹\ 12)\ (2.73) + (₹\ 15)\ (3.81) = ₹\ 89\text{-}91 \quad ...(5)$$

At point S, X = 1.5 and $Y = 5.25$, we obtain,

$$(₹\ 12)\ (1.5) + (₹\ 15)\ (5.25) = 96.75 \quad ...(6)$$

At point C, $X = 0$, $Y = 6$:

$$(₹\ 12)\ (0) + (₹\ 15)\ (6) = ₹\ 90 \quad ...(7)$$

$$14X + 12Y = 84$$

$$6X + 12Y = 72$$

	(Primal Problem)		**(Dual Problem)**
Maximise Revenue	$R = 12X + 15Y$	Minimise Cost	$C = 48A + 72B + 84C$
Object to	$12X + 4Y \leq 48$	Subject to	$12A + 6B + 14C \geq 12$
	$6X + 12Y \leq 72$		$4A + 12B + 12C \geq 15$
	$14X + 12Y \leq 84$		$A \geq 0, B \geq 0, C \geq 0$
	$X \geq 0, Y \geq 0.$		

Students solve this dual problem themselves like the solution for the food problem.

29.2 Minimisation of Cost- Solution of the Food Problem

Food problem was the first economic problem whose solution through linear programming was done through cost equation. Consider that consumer buys bread and butter at market price. Problem is that from various quantities of both goods cost of receiving their net material is made minimum.

Note

Table 29.2: Data of Food problem

Nutrition element	Nutrient Material per unit		Minimum ideal
	Bread X_1	Butter X_2	
Calorie (1000)	1	2	3
Protein (25 g)	2	8	8
Cost (₹ Per unit)	2	6	(?)

Consider that X_1 and X_2 express Bread and butter respectively of which amount of Calories and grams of proteins is given in table 29.2. Nutrition material of bread per half kilogram is 1000 calorie and quantity of protein is 50 g and of butter is 2000 calories and 200 g protein is per half kilogram. In ideal diet, per day 3000 calories and 200 g protein is needed. Market price of 500 g bread is ₹ 2 and price of per 500 g butter is ₹ 6.

Problem is that according to minimum diet ideal given in the last column of the above table what will be the best diet and what will be the minimum cost expressed by (?).

Total cost of the food

Minimise

Subject to

$$\left.\begin{aligned} C &= 2x_1 = 6x_2 \\ x_1 + 2x_2 &\geq 3 \\ 2x_1 + 8x_2 &\geq 8 \\ x_1 \geq 0&, x_2 \geq 0 \end{aligned}\right\} \quad \text{...(1)}$$

And the cost being minimised is C, which is the linear function of both variables x_1 and x_2. Side relations 3 and 8 are inequalities which express the minimum ideal food diet obtained by the given food. Problem is linear because despite of linear inequalities, non- variable are to be minimised. Out of the three, solution to any two situations may be obtained. For e.g. with one side relation remaining, cost C can be minimised: $x_1 + 2x_2 = 3$ on solving it, $x_1=3$ and $x_2=3/2=1.5$. In figure 29.2, it has been expressed through line *AB* where *OA*= $1.5x_2$ and *OB* $=3x_1$

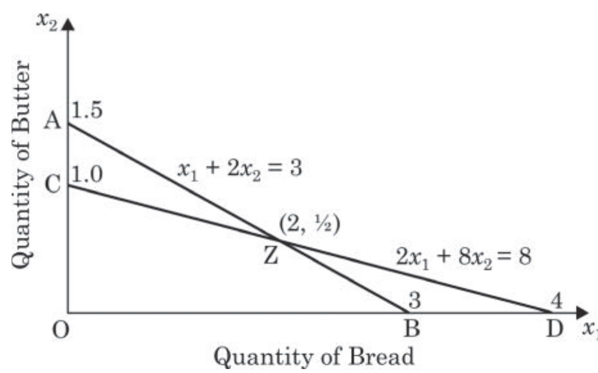

Figure 29.2

Second side relation is $2x_1 + 8x_2 = 8$ and on solving it, $x_1 = 4$ and $x_2 = 1$are obtained. It has been drawn through line *CD* in figure 29.2, which satisfies this equation where $OC = 1x_2$ $OD = 4x_1$.

Hence in the figure, x_1, (Bread) has been measured on horizontal axis and x_2, (butter) has been measured on vertical axis. Line AB expresses equation $x_1 + 2x_2 = 3$

And line CD expresses equation $2x_1 + 8x_2 = 8$. Feasible solution will be on thick line AZD or above it. In our problem, it happen on point Z where both lines AB and CD intersect.

$$x_1 + 2x_2 = 3 \quad \text{...(1)}$$

$$2x_1 + 8x_2 = 8 \quad \text{...(2)}$$

To find out that feasible solution is on Z itself or on points A or B, we will solve both equations of the problem in form of simultaneous equation:

Note

Example: Assumed that a producer wants to maximise its revenue under given constraints. Considered that a firm wants to produce two products X_1 andX_2 and for it three resources a, b, and c of the following type are given:

$$a = 40, b = 50, c = 42$$

We also assumed that for producing one unit of X_1 it will need each resource of following type-

$$a = 4, b = 10, c = 6$$

Similarly we also assumed that for producing unit of X_2 it will need each resource in following way:

$$a = 10, b = 5, c = 7$$

We also assumed that per unit cost of X_1 and X_2 is ₹ 5 and ₹ 7 respectively. In this way that producer will like to maximise his total revenue from the given resources. Mathematically,

$$\text{Max } TR = 5X_1 + 7X_2 = Z$$

Constraints on the firm have been given and firm will not use more than those resources.

$$4X_1 + 10X_2 \leq 40 \quad ...(i)$$

$$10X_1 + 5X_1 \leq 50 \quad ...(ii)$$

$$6X_1 + 7X_2 \leq 42 \quad ...(iii)$$

Here, $X_1, X_2 \leq 0$

These constraints of the firm may be shown in the following manner:

First constraint tells this that resource 'a' which has been brought in production of X_1 and X_2; it cannot be more than 'total supply' of resource 'a'. This resource can be less than or equal to supply, not more. Similar will be in relation to other constraints. Here we will show a linear object with the help of graph:

First we will take out the coordinates of X_1 and X_2, for which we will remove inequalities of constraint equations and in this way we will obtain coordinates of all the three techniques. These techniques will express feasible solution of object problem. In the figure, on axis X, product X_1 and on Y axis Product X_2 has been measured. Here with the help of coordinates of constraints adopted by the firm has been marked in form of an easy line through graph. Z_1 is objective function.

Considered that producer brings a and b constraints in use then area *OMAG* will be obtained as feasible area for the firm. If firm use 'a' and 'c' constraints then the possible area will be OATK. Similarly if firm brings constraints b and c in use then area *ONLG* will be obtained.

If he takes all the three constraints a, b and c together in use, then he will obtain feasible area equal to OAMG. Feasible solution of firm will be under this area only.

If producer will want to search the feasible solution of product X_1 and X_2 in area *OAMG* (which has been shown by shaded area in the figure), it will not be correct because he must make maximum use of each technique. Hence feasible solution area will be on the boundary of area *OAMG*.

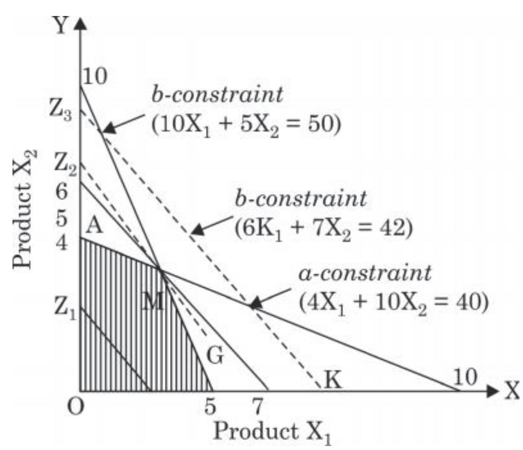

Figure 29.3

Now the question arises that, if he takes the objective function at the boundary of feasible area then for obtaining **optimum solution** of X_1 and X_2 firm should get only one point which will give only one solution of X_1 and X_2. For this, it will have to search corner solution. Hence in such situation, firm will change the objective function Z_1 equally. That point of the corner of feasible solution which the feasible solution touches, it will be an optimum solution of X_1 and X_2. Here in figure 29.3, at point M, optimum solution of X_1 and X_2 will be obtained.

Note

Self Assessment

1. Multiple Choice Questions:

1. In which field linear programming has proved very useful?
   (a) In Economics (b) In Science
   (c) In Mathematics (d) In Politics
2. Real problems cannot be solved through which technique due to many constraints?
   (a) Differentiation (b) Linear Programming
   (c) Integration (d) None of these
3. Which problem was the first economic problem whose solution through linear programming was done through cost equation?
   (a) Money (b) Residence
   (c) Food (d) Water

29.3 Summary

- Till Neo-classic theory of the firm in time, taking one or two variable did the analysis of the problem of decision making.
- Food problem was the first economic problem whose solution through linear programming was done through cost equation.
- Linear programming proved to a very profitable resource in economics, but it had its own limitations. In reality, because of many constraints real problems cannot be solved through linear programming technique.

29.4 Keywords

- ***Optimise:*** Required
- ***Primal:*** Main

29.5 Review Questions

1. Interpret the minimisation of cost.
2. Write down the limitations of linear Programming.
3. Present the minimisation of cost: Solution of the food problem.

Answers: Self Assessment

1. (a) 2. (b) 3. (c)

29.6 Further Readings

Books

Mathematics for Economist – Yamane- Prentice Hall, India.

Mathematical Economics – Micheal Harrison, Patrick Waldaron.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Simone and Bloom- Viva Publication.

Essential Mathematics for Economics – Nut Sedester, Peter Hamond, Prentice Hall Publication.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Unit 30: Simplex Method

Note

CONTENTS

Objectives

After reading this unit, students will be able to :

- Understand clearly Simplex Method.
- Understand the Steps of Method.

Introduction

Because of more numbers of inequations in linear programming method, Graphical method becomes more complicated. This method can be used for two, three or more inequations. Therefore because of large number of equations another mathematical method is used for a paired equation which is called Simplex Method.

30.1 Simplex Method

Suppose following is an Objective function which is to be maximized (or minimized)-

$$Z = c_1x_1 + c_2x_2 + \text{.........} + c_nx_n \qquad \text{...(i)}$$

Here the values of all constants (C_i) are known.

Here we also supposed that m is a linear inequation in which there are n variables in each equation.

Thus Constraints are the following -

$$a_{i1} + a_{i2} + \text{...}\ a_{in}x_n \{\le = \ge\}\ b_i,\ i = 1, 2 \text{.........}, m \qquad (30.1)$$

Notes Here one and only one sign (≤ = ≥) will be for each constraint. Value of variable will always be positive, i.e. $x_i \ge 0$, $j = 1, 2 \text{....} n$.

Each group of x_j which satisfies constraints will be called a solution. Any solution which satisfies Non-negativity restrictions will be called a feasible solution. In the same way any feasible solution which minimizes or Maximizes Objective Function Z will be called Optimal Feasible Solution.

Here we will try to find out the Optimal Feasible Solution of an Objective function within the given constraints. To find the Optimal Feasible Solution firstly we will find all the feasible Solutions, after

Note

that we will find one feasible solution which satisfies maximum (or minimum) of Objective function, that will be our Optimal Feasible Solution.

Like the number of variables will increase the same way number of Feasible solutions will also increase. If there are 6 variables in our problem then number of feasible solutions will be $|\underline{6}\,||\underline{3}\,||\underline{3} = 20$, where there were 3 yugmpad equations and three unknown values.

Task What is Simplex Method?

Equation (i) can be rewritten as –

Objective Function $\sum_{j=1}^{n} c_j x_j$ (Maximum or Minimum)

Within the following constraints

$$\sum_{j=1}^{n} a_i x_j = b_j \qquad \text{...(30.2)}$$

$i = 1, 2$ .... m

$j = 1, 2$ ...... n (Making equations same here)

$j = 1$

Knowledge of Basic Theorems is necessary related to the following problems –

1. If the possible solution of problems of Linear programming is a Vector $X = [x_1, x_2, ... x_n]$, which satisfies equation 30.1. Vector a_j associated with x_j can be defined as -

   $$a_j = [a_{1j} a_{2j} .... a_{mj}]$$

2. Quantities $b_1\ b_2 .... b_m$ are the elements of a column vector which is called requirement vector. Here

   $$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ and } b \geq 0, \text{ where 0 is a zero vector.}$$

3. Coefficients $C_1\ C_2 .... C_n$ under Objective function are called the Prices Associated of variables $x_1, x_2 ...... x_n$ and the vector formed from them is called Price Vector represented by C. Thus

   $$C = [c_1, c_2, ... c_n]$$

4. Set of Values of $x_1, x_2 ...... x_n$ which satisfies **Equation** 30.1 and the Non-negativity condition, is called feasible Solution.

5. That Feasible Solution, which optimizes Objective function (30.1), is called Optimal Solution. That is if feasible solution minimizes Objective function then it is called Minimum Feasible Solution and if it maximizes Objective function then it is called Maximum Feasible solution.

Note

6. If feasible solution which does not have positive more then m, then it is called Basic feasible Solution. Therefore if feasible solution is to be converted into Basic feasible Solution then it is necessary to vanish at least $(n - m)$ variables.
7. In the Basic feasible Solution if m is positive then it is called Non-degenerate Basic feasible Solution. Non-Zero variables are called Basic Variables. When at least one Basic Feasible Solution vanishes then Basic feasible Solution is called degenerate Basic feasible Solution.
8. Solution which does not satisfies constraints Equation (30.1) and the Non-negativity condition is called Non-feasible solution.
9. If Equation has the sign (≤) then the variables used to make changes in its balanced equations, are called Slack-Variable.

Did u know? Those variables which are used to change dissimilation into equation, are called a Surplus Variables. These both types of variables in the combined form is called a Dummy variable. Whose numbering is equal to the number of variables in the Objective Function.

30.2 Calculating Steps of Method

1. Slack Variables should be used according to the need for changing dissimilation into equations.
2. If needed Artificial variables should be included. After that write constraints as $AX = b$, where $b \geq 0$. If any b is negative then make it positive by multiplying its equation by (– 1).
3. Calculate X_j by solving Initial Basic Feasible Solution and the value of $Z_j - C_j$ should be calculated for every column of A.
4. For Authorized Equation every $Z_j - C_j \geq 0$, then this Iteration is the Optimum Solution. Again if $Z_j - C_j > 0$ (for every non-based Variables) then proper solution is Unique otherwise optional Solution can exists.
5. Entering Vector and departing Vector should be selected.
6. Artificial Vector should be separated.

Example 1: Solve the following by Simplex Method –

Of which $\quad Z = 2X_1 - 3X_2 + 7X_2$ ...(1)

Minimum $\quad 3X_1 - 4X_2 - 6X_3 \leq 2$ ...(2)

$2X_1 - X_2 - 2X_3 \geq 11$ ...(3)

$X_1 - 3X_2 - 3X_3 \leq 5$ ...(4)

So that $\quad X_1, X_2, X_3 \geq 0$

Solution: First of all Slack and Surplus Variables will be used according to the need for changing dissimilation into equations. Since there are 3 Equation equations, therefore 3 Slack variables and 3 Surplus Variables will be used. Therefore including X_4, X_5 and X_6 given problem can be represented as the following –

$2X_1 - 3X_2 + 6X_3 + 0X_4 + 0X_5 + 0X_6$ ...(5)

When equations are the following –

$3X_1 - 4X_2 - 6X_3 + X_4 + 0X_5 + 0X_6 = 2$ ...(6)

$2X_1 - X_2 + 2X_3 + 0X_4 - X_5 + 0X_6 = 11$ ...(7)

$X_1 + 3X_2 - 2X_4 + 0X_4 + 0X_5 + X_6 = 5$ ...(8)

Note

For finding the feasible solution data is represented in the following Matrix form –

$$AX = b$$

Here $A = [a_{ij}] = [P_{ij}]$

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_6 \end{bmatrix} \text{ and } b = P_0 = \begin{bmatrix} 2 \\ 11 \\ 5 \end{bmatrix}$$

Or

$$\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ 3 & -4 & -6 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & -1 & 0 \\ 1 & 3 & -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_6 \end{bmatrix}$$

Since the Basis Matrix must be a Unit Matrix but it is not becoming possible from the known facts. Therefore the inclusion of a New Vector P_7 is necessary. Therefore Primary basis is P_4, P_7 and P_6. Therefore writing the following table –

Table I

(Basis Vector)	$\downarrow C_j$	$C_j \rightarrow$ P_0	2 P_1	–3 P_2	6 P_3	0 P_4	0 P_5	0 P_6	0 P_7
P_4	0	2	3	–4	–6	1	0	0	0
P_7	0	11	2	–1	2	0	–1	0	1
P_6	0	5	1	3	–2	0	0	1	0
Z_j		0	0	0	0	0	0	0	0
$Z_j - C_j$			–2	+3	–6	0	0	0	0

Here the value of Objective Function

$$Z_j = Z_o = 0$$

But it is not minimized, since $Z_j - C_j > 0$ in context to vector P_2. Therefore we will use θ trick to minimize P_2.

$$P_o = 2P_4 + 11P_7 + 5p_6 \quad \text{...(i)}$$

$$P_2 = -4P_4 + P_7 - 3P_6 \quad \text{...(ii)}$$

Multiplying eq. (ii) by θ

$$\theta P_2 = -4\theta\, P_4 + 5P_7 = 3\theta\, P_6 \quad \text{...(iii)}$$

Subtracting eq. (iii) from (i)

$$P_o = \theta P_2 + (2 - 4\theta)\, P_4 + (11 - \theta)\, P_7 + (5 - 3\theta)\, P_6 \quad \text{...(iv)}$$

From the three values of θ, $\frac{1}{2}$, 11 and $\frac{5}{3}$, $\theta = \frac{1}{2}$ is the related value. Therefore putting $\theta = \frac{1}{2}$ in eq. (iv)

Note

$$P_0 = \frac{1}{2}P_2 + 0P_4 + \frac{21}{2}P_7 + \frac{7}{2}P_6$$

$$= \frac{1}{2}P_2 = \frac{21}{2}P_7 + \frac{7}{2}P_6 \quad \text{...(v)}$$

Equation (v) becomes new basis 11 in which P_2, P_7 and P_6 are included. Therefore putting P_2 in place of P_4 can be written in the following table form.

Here the value of Objective function $Z_j = Z_o = -\frac{3}{2}$

This is the desired value since $Z_j - C_j < 0$

$j = 1, 2, \ldots\ldots 7$

Table II

(Basis Vector)	$\downarrow C_j$	$C_j \rightarrow$	2	–3	6	0	0	0	0
		P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
P_2	–3	$\frac{1}{2}$	$\frac{3}{8}$	1	$\frac{6}{3}$	0	0	0	0
P_7	0	$\frac{21}{3}$	5	0	8	0	–3	0	1
P_6	0	$\frac{7}{2}$	$\frac{13}{12}$	0	$-\frac{1}{6}$	$\frac{1}{4}$	0	1	0
Z_j		$-\frac{3}{2}$	$\frac{9}{8}$	–3	$-\frac{9}{4}$	0	0	0	0
$Z_j - C_j$			$-\frac{7}{8}$	0	$-\frac{33}{4}$	0	0	0	0

Example 2: Solve the following Linear Programming equation –

Maximum $Z = 5X_1 + 4X_2$

Which $X_1 + 2X_2 \leq 8000$

$3X_1 + 2X_2 \leq 9000$

From which $X_1, X_2 > 0$

Solution: First of all dissimilarities of equations will be removed by the inclusion of Slack Variables. Now we can show the constraints in the following way –

$$X_1 + 2X_2 + 0X_3 = 8000$$

$$3X_2 + 2X_2 + 0X_4 = 9000$$

Note

Now on the basis of above facts following table will be made –

Table III

			Cost	**5**	**4**	**0**	**0**	
C_β	Solution	X_β	β	Y_1	Y_2	Y_3	Y_4	$\theta = \frac{\text{Sol.}}{Y_k}$
0	8000	X_3	Y_3	1	2	1	0	8000
0	9000	X_4	Y_4	3	2	1	0	$\frac{9000}{3} = 3000$
$\sum Z_j - C_j = \sum C_\beta\,(Y_j - C_j)$				–5	–4	0	0	

Here firstly we will include row Y_k. For this will calculate the value of 0. Value of q can be calculated by division of the value of Y_k.

$$\theta = \frac{8000}{1} = 8000$$

And

$$0 = \frac{9000}{3} = 3000$$

After that the row will include the minimum value. Here l th row (second row) will be removed by k th (i.e. 3) vector solution.

In second table, first of all we have to divide kth row with the pivot of second row of first table. Multiply new row of second table with rest rows of table. Then subtract first row from ith row. You will get the result of ith row. Repeat the all steps till you get the value for $Z_j - Cj$

Table IV

			Cost	**5**	**4**	**0**	**0**	
C_β	Solution	X_β	β	Y_1	Y_2	Y_3	Y_4	θ
0	00	X_3	Y_3	0	$\frac{4}{3}$	1	$-\frac{1}{3}$	$\frac{15000}{4}$
0	3000	X_1	Y_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{9000}{2}$
	$Z_j - C_j$			$5 - 5$ $= 0$	$\frac{10}{3} - 4$ $= -\frac{2}{3}$	0	$\frac{5}{3}$	

For first row

Note

8000	1	2	1	0
–3000	1	$\frac{2}{3}$	0	$\frac{1}{2}$
5000	0	$\frac{3}{4}-\frac{1}{3}$ on Subtracting		

This method will be repeated until then value of $Z_j - C_j$ comes negative. Since β is minimum in the first row and $Z_j - C_j$; is negative for Y_2. Therefore first row will be deleted from the table. Here the value of Pivot is 4/3 and when we repeat the method –

Table V

			Cost	**5**	**4**	**0**	**0**	
C_β	Solution	X_β	β	Y_1	Y_2	Y_3	Y_4	θ
4	$\frac{1500}{4}$	X_2	Y_2	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	
5		X_1	1	0	1	$-\frac{1}{2}$	$\frac{1}{6}$	
		$Z_j - C_j$		0	0	$\frac{1}{2}$	$\frac{1}{6}$	

For second row

3000	1	$\frac{2}{3}$	0	$\frac{1}{3}$
2500	1	0	$+\frac{1}{2}$	$-\frac{1}{6}$
500	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$

on Subtracting

Since here the value of $Z_j - C_j$ is positive. Column $X_2 = \frac{15000}{4}$ of solution and $X_1 = 500$ provides maximum value.

Now the maximum value of 2 will be found –

$$2 = 5X_1 + 4X_2 = (5 \times 500) + \frac{4 \times 15000}{5}$$

$$= 2500 + 15000 = 17500$$

Example 3: Maximize the profit. $Z = 4X + 3Y$.

Of which $x + \frac{7}{2}y \leq 9$

$$2x + y \leq 8$$

$$x + y \leq 6$$

Note

Like $x \overline{<} 0, y \leq 0$

Solution: First of all Slack Variables will be used to balance the equation. Slack Variables are s_1, s_2, s_3 etc. Multiply these variables by zero, these are added just to balance the equations. Like

$$Z = 4x + 3y + os_1 = os_2 + os_3 = R$$

Primary table by Simplex Method

Table VI

$C_j \to$ ↓			4	3	0	0	0	
	Profit	**Qty**	**x**	**y**	**s_1**	**s_2**	**s_3**	**Ratio**
0	s_1	9	1	$\frac{7}{2}$	1	0	0	9
0	s_2	8	2	1	0	1	0	←4
0	s_3	6	1	1	0	0	1	6
	Z_j		0	0	0	0	0	
	$C_j - Z_j$		4	3	0	0	0	

Key Column ↑ Here Key Factor = 4

Note: Key factor = 4 since the value of this ratio is least.

Subject to $x + \frac{7}{2} y + s_1 + os_2 + os_3 = 9$

$$2x + y + os_1 + s_2 + os_3 = 8$$

$$x + y + os_1 + os_2 + s_3 = 6$$

Table VII

$C_j \to$ ↓			**4**	**3**	**0**	**0**	**0**	
	Profit	**Qty**	x	y	s_1	s_2	s_3	**Ratio**
4	x	4	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	8
0	s_1	5	0	3	1	$-\frac{1}{2}$	0	$\frac{5}{3}$
0	s_3	2	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	4
	Z_j		16	4	2	0	2	
	$C_j - Z_j$			0	1	0	–2	

Note

Now the value of $C_j - Z_j$ is either Positive, zero or less. We will continue to make new table until the value of $C_j - Z_j$ becomes negative or zero.

Table VIII

	$C_j \rightarrow$ ↓		4	3	0	0	0	
	Profit	**Qty**	x	y	s_1	s_2	s_3	**Ratio**
3	y	$\frac{5}{3}$	0	1	$\frac{1}{3}$	$-\frac{1}{6}$	0	
4	x	$\frac{19}{6}$	1	0	$-\frac{1}{6}$	$\frac{7}{12}$	0	
0	s_3	$\frac{7}{6}$	0	0	$-\frac{1}{6}$	$\frac{5}{12}$	0	
	Z_j	$\frac{53}{3}$	4	3	$\frac{1}{3}$	$\frac{11}{6}$	0	
	$C_j - Z_j$		0	0	$-\frac{1}{3}$	$\frac{11}{6}$	0	

In the above table all values of $C_j - Z_j$ are either Negative or equal to Zero.

$x = 19/6$

$y = 5/3$

Therefore $s_3 = 7/6$

Working Notes
Table IX

	Qty	x	y	s_1	s_2	s_3
O.V.s_1	9	1	$\frac{7}{2}$	1	0	0
N.T. × 1	–4	–1	$-\frac{1}{2}$	–0	$-\frac{1}{2}$	–0
	5	0	3	1	$-\frac{1}{2}$	0
D.V.s_3	6	1	1	0	0	1
N.T. × 1	–4	–1	$-\frac{1}{2}$	–0	$-\frac{1}{2}$	–0
	2	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1

Note

Table X

O.V.	4	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
N.T. × $\frac{1}{2}$	$+\frac{5}{6}$	+0	$+\frac{1}{2}$	$+\frac{1}{6}$	$-\frac{1}{12}$	0
	–1	–1	–1	–1	+1	+1
	19/6	1	0	–1/6	7/12	0
O.V.s_3	2	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1
N.T. × $\frac{1}{2}$	+5/6	+0	+1/2	+1/6	–1/12	–0
	7/6	–0	0	–1/6	5/12	1
	5	0	3	1	$-\frac{1}{2}$	0
	38/3	4	0	–2/3	7/3	0
Z_j	53/3	4	3	1/2	+11/6	0

Note: O.V. = Original value,

N.T. = New table value.

Self Assessment

1. Fill in the blanks:

1. Because of more numbers of inequation in linear programming .... method becomes more complicated.
2. Any Solution which satisfies Non-Negativity constraints is called .... solution.
3. Any feasible solution which minimizes or Maximizes Objective Function will be called a .... Feasible Solution.
4. .... and Surplus Variables should beused according to the need for changing dissimilation into equations.
5. Solution which does not satisfies Non–negativity condition is called .... solution.

30.3 Summary

Because of more numbers of inequations in linear programming method, Graphical method becomes more complicated. This method can be used for two, three or more inequations. Therefore, because of large number of equations another mathematical method is used for Yugmpad equations which is called Simplex Method.

Note

30.4 Keywords

- ***Linear :*** Lined, in a line
- ***Method :*** Process

30.5 Review Questions

1. What do you understand by Feasible and Optimal Feasible Solution?
2. Write the different calculating steps of method.
3. Solve the following Linear Programming problem:

   Maximum $z = 5X_1 + 4X_2$

   Subject $X_1 + 2X_2 \leq 8000$

   $3X_1 + 2X_2 \leq 9000$

   Thus $X_1, X_2 > 0$ (**Ans.:** Maximum value: 17500)

Answers: Self Assessment

1.	Graphical	2.	Feasible	3.	Optimal
4.	Slack	5.	Non-feasible		

30.6 Further Readings

Books

Mathematical Economics – Michael Harrison, Patric Walderen.

Mathematics for Economics – Karl P. Simon, Laurence Bloom.

Mathematics for Economics and Finance – Martin Norman.

Mathematics for Economics – Malcom, Nicolas, U.C.Londan.

Mathematics for Economist – Yamane – Prentice Hall India.

Essential Mathematics for Economics – Nut Sedestor, Peter Hamond, Prentice Hall Publications.

Mathematics for Economics – Council for Economic Education.

Mathematics for Economist – Mehta and Madnani – Sultan Chand and Sons.

Mathematics for Economics – Simon and Bloom – Viva Publications.

LOVELY PROFESSIONAL UNIVERSITY
Jalandhar-Delhi G.T. Road (NH-1)
Phagwara, Punjab (India)-144411
For Enquiry: +91-1824-300360
Fax.: +91-1824-506111
Email: odl@lpu.co.in