A STUDY ON PRIME LABELING AND PRIME DISTANCE LABELING OF GRAPHS

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in

(Mathematics)

By

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Abstract

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by Ram Dayal

"Let $G = (V, E)$ be a graph with V as non- empty set of vertices or nodes or points and E as the set of edges or lines or arcs. If the nodes or arcs or both of G are assigned labels (mostly integers) subject to certain constraints, then the graph G is said to have vertex labeling, edge labeling or total labeling, respectively. One of the important areas in graph theory is graph labeling used in many applications like coding theory, $x-$ray crystallography, cryptography, astronomy, circuit design, communication networking, data base management, etc. There are many types of graph labeling studied by many Mathematicians like prime labeling, prime distance labeling, vertex prime labeling, cordial labeling, k-equitable labeling, etc. Prime numbers have always fascinated Mathematicians and the concept of relatively prime numbers has also been extensively used in many areas of Mathematics. So, the prime labeling and prime distance labeling have also been extensively studied by many Mathematicians. But the origin of most of the graph labeling can be traced when $\beta-$valuation of a graph was introduced. A one-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ is said to be a $\beta-$valuation of a graph $G(V, E)$, where $|E| = m$, if each edge $e = st$ is assigned the label $|f(s) - f(t)|$ and all the labels are distinct. A bijection $f : V(G) \to \{1, 2, \ldots, |V|\}$ is called a prime labeling of G if for each edge $e = st$, $GCD(f(s),\ f(t)) = 1$, where GCD denotes the greatest common divisor. G is a prime graph if it admits a prime labeling. A graph $G(V, E)$, is a prime distance graph if there exists a one-one labeling of its vertices $f : V(G) \to Z$ such that for any two adjacent vertices u and v, the integer $|f(u) - f(v)|$ is a prime and f is called a prime distance labeling of G. So, a graph G is a prime distance graph if and only if there exists a prime distance labeling of G. Though a significant work has been done in the area of prime labeling and prime distance labeling, complete characterization of both these labeling is pending and have attracted many Mathematicians for research in this direction. In an effort to achieve complete characterization of both of these labeling, researchers across the globe have ingrained prime labeling and prime distance labeling for various classes and families of graphs. Moreover both of these labeling have been extensively studied in the context of various graph operations such as

extension, duplication, join, disjoint union, barycentric subdivision, Cartesian product, vertex switching, etc.

In the present proposed work, an effort has been made to derive prime labeling and prime distance labeling of some classes and families of graphs.
Sufficient conditions for the degree splitting graph of a bipartite graph to exhibit prime distance labeling and barycentric subdivision of a graph to exhibit prime labeling have been obtained. The complete characterization of prime distance labeling in the context of barycentric subdivision has also been established.
Besides this some interesting conjectures and open problems have also been formulated for future work. Thus the thesis titled: **A Study on Prime Labeling and Prime Distance Labeling of Graphs** deals with the following objectives:

1 Deriving the prime labeling of some classes of graphs.

2 Establishing prime distance labeling of certain families of graphs.

3 Obtaining the prime and prime distance labeling in the context of extension of vertices of some graphs.

4 Obtaining the prime and prime distance labeling in the context of barycentric subdivision of some graphs.

The thesis *A Study on Prime Labeling and Prime Distance Labeling of Graphs* has been divided into five chapters of which the first is introduction. Apart from giving a historical background of graph theory, some basic notions of graph theory and graph labeling with some established results in prime labeling and prime distance labeling are also included in this introductory chapter to make this thesis self-contained. On the grounds of literature review, research gap has been identified and some realistic objectives are proposed.

In the second chapter *'Results on Prime Labeling'*, prime labeling of some classes of graphs has been investigated. The constraints on the orders of two prime graphs so that their disjoint union exhibits prime labeling are also investigated. The prime labeling of path graphs and complete graphs in the context of degree splitting of a graph has been studied. Besides this, prime labeling of the complement of gear graph and middle graph of a path graph has also been studied.

In the third chapter, *'Prime Labeling in the context of Extension and Barycentric subdivision'*, we study prime labeling of star, bistar and tree graphs in the context of barycentric subdivision besides giving the sufficient condition for a graph to exhibit prime labeling in the context of barycentric subdivision. Prime labeling of path and complete graphs in the context of extension of vertices has also been investigated.

In the fourth chapter, *'Results on Prime Distance Labeling'*, the prime distance labeling of the non-commuting graphs of some non-abelian groups like dihedral group D_{2n} and Quartenion group Q_8 are investigated. Prime distance labeling of the degree splitting graph of complete graph, complete bipartite graph, Jelly fish graph $J(m, n)$, Flower graph $F(C_m, K_n)$, Pizza graph, Diamond graph, Jewel graph, Super subdivision of a graph, one point union of $2-$regular graphs and sufficient condition for the degree splitting graph of a bipartite graph to exhibit PDL, have been investigated. Besides this some general results on prime distance graphs have also been studied.

The last chapter, *'Prime Distance Labeling in the context of Extension and Barycentric subdivision'* deals with obtaining complete characterization of prime distance labeling of simple graphs in the context of barycentric subdivision and prime distance labeling of path and complete graphs in the context of duplication and extension of vertices. The characterisation of family of graphs that exhibits PDL if Goldbach's conjecture is true has also been obtained.

In the end, a detailed bibliography has been included to justify the present study".

Acknowledgements

I express my gratitude to my supervisor Dr. Arunava Majumder, Assistant Professor, Department of Mathematics, Lovely Professional University, Punjab and Co-supervisor Dr. A. Parthiban, Assistant Professor, Department of Mathematics, Vellore Institute of Technology (VIT), Vellore, Tamil Nadu, who engineered me throughout the present work. Without their readily inspiring guidance, learned suggestions and valuable assistance the present work could have not been possible.

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I acknowledge with a deep sense of reverence, my gratitude towards my mother Smt Rattno Devi, Parents-in-law Dr. G. L. Dogra (Paediatrician) and Smt. Kailash Kumari whose constant encouragement and blessings helped me a lot during this work.

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It will be injudicious if mention is not made of Dr. Sumit Dubey, Tr. Suram Chand, Dr. Kavita, Er. Dheeraj Kant, Dr. Mukesh Kant, Er. Amit Kumar, my siblings and all family members for their constant encouragement.

Finally, I want to pay my thanks to the Almighty God Who blessed me with this opportunity.

December 2023 Ram Dayal

Contents

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List of Symbols and Abbreviations

$R-$ The set of reals

$Z-$ The set of integers

$N-$ The set of naturals

$|V(G)|-$ Cardinality of vertex set of G

$|E(G)|-$ Cardinality of Edge set of G

$\delta(G)-$ Minimum degree of G

$\Delta(G)-$ Maximum degree of G

$\lceil x \rceil-$ Ceiling function of x

$\lfloor x \rfloor-$ Floor function of x

G^2-Square graph of G

$K_{m,n}-$ Complete bipartite graph

K_n- Complete graph

$M(G)-$ Middle graph of G

$DS(G)-$ Degree splitting graph of G

$S(G)-$ Subdivision graph ofG

$\mu(G)-$ Myceilskian graph ofG

$N(v)-$ Neighborhood of a vertex v

$N[v]-$ Closedneighborhood of a vertex v

$PG-$ Prime grpah

$PDG-$ Prime distance graph

$SS(G)-$ Super subdivision of a graph G

$ASS(G)-$ Arbitrary Super subdivision of a graph G

Chapter 1

Introduction

In this chapter an incisive introduction to graph theory and graph labeling has been given along with a few applications of graph labeling. Apart from these basic notions of graph theory and graph labeling, some established results in PL and PDL in this introductory chapter are given to make this dissertation self-contained. The aim of the thesis is presented along with the detailed review of literature. On the grounds of literature review, research gap has been identified and some realistic objectives of the present work are proposed.

1.1 Introduction to Graph Theory

In many different domains, graph theory is very significant. Graph theory first appeared in the 1700s, when Swiss mathematician L. Euler published a paper on the Königsberg bridge problem in 1736. Many people view this as the birth of the discipline of graph theory. After Euler's work, the area of graph theory virtually stagnated for the following 100 years. Euler's theories on graphs were initially intended to be used for puzzle-solving and entertainment. Later, around the middle of the 1800s, it was discovered that graphs can be used to model a variety of real-world issues that are very important to society. The fact is that practically every real-world issue involving a discrete arrangement of things may be represented as a graph, provided that only the relationships between the objects are considered and not their intrinsic attributes. G.R. Kirchoff introduced the "theory of trees" in 1847, and it was applied in electrical networks. Arthur Cayley made the discovery of trees in 1857 while attempting to enumerate the structural isomers of saturated hydrocarbons represented by the formula C_kH_{2k+2}. In 1852, *A. De Morgan* introduced the *"Four-Color Conjecture"*, that defines 4 colors are sufficient to color any map on a plane such that the bordering regions are colored differently. The problem was first published by Cayley in 1879.This famous problem in graph theory has stimulated a large volume of developments in "graph theory". This result now has been established and requires the help of computer for verification and hence we call it now *Four-Color Theorem*. In the year 1859 Sir *W. R. Hamilton* invented a puzzle consisting of a wooden, regular dodecahedron. The goal of the task was to identify a path along the borders of the dodecahedron that starts from any city, travels through each city precisely once, and ends at the city from which it started. A dodecahedron has twenty corners, each of which was marked with the names of 20 cities. Hamiltonian circuits are what such pathways are known as in graph theory. Finding a characterisation theorem for a Hamiltonian circuit to exist in any given graph has not yet been resolved, even though this problem can be addressed quite quickly. The field of graph theory remained largely dormant for roughly 50 years. The interest in graphs again returned in the 1920s, and one of the most well-known graph theorists of the time, D. König, compiled both his own work and that of other mathematicians, publishing the first book in this field in 1936. Numerous books and thousands of papers have been written in the past 70 years as a result of the extensive study that has been done in graph theory. Graph theory has a wide range of applications in many different scientific domains, even though it was initially intended to solve amusing puzzles. A Computer network can be depicted as a graph with nodes denoting web pages and directed lines denoting links connecting those sites. Like how a graph is a natural depiction of a molecule in chemistry, where nodes stand in for atoms and lines for bonds, Computer-assisted molecular structure processing, which includes anything from chemical editors to database searches, benefits from this technology. It's interesting to note that chemical graph theory [9], which uses graph theory to the mathematical modelling of chemical events, is given special attention even though mathematical modelling in Organic Chemistry stems from several fields of Mathematics. Graphs can also depict the dynamics of a physical process on those systems as well as the local relationships between interacting system components in statistical physics. The concept of graphs find application in describing micro-scale links in porous media, with nodes denoting holes and lines denoting the smaller links connecting pores [55]. The analysis of high symmetry graphs led to the development of algebraic graph theory [8]. It examines many classes of graphs with relation to specific characteristics of automorphism groups, including semi-symmetric graphs, line-transitive graphs, node-transitive graphs, and distance transitive graphs. Additionally, graph theory has plenty of applications in many fields of mathematics, science, engineering, & technology, as well as communication networks and practical issues [6, 10]. GT and GL, respectively denote graph theory and graph labeling.

"GL is a vital concept in GT that finds countless uses in different domains. GL is a map that allocates integers to the links or points, or both under a few restrictions. The importance of GL can be witnessed by its variety of uses in many domains like circuit design, radar, etc". For a detailed study, see [7, 22, 36, 40, 74].

1.1.1 Preliminaries

In this subsection some basic ideas of "graph theory" and results relevant to the study undertaken have been recalled.

Definition 1.1.1. "A simple graph H consists of a non-null set V of finite number of points/nodes & a set E of links/lines in a way that each line of G is a pair $\{y_i,\ y_j\}$ for some y_i, y_j in V with $y_i \neq y_j$ (See Figure 1.1).

Definition 1.1.2. The degree of $x \in V(H)$, denoted by $d_H(x)$, is the count of lines of H incident with x (see Figure 1.1).

Definition 1.1.3. Two nodes are adjacent in G if they are joined by a line in G.

Definition 1.1.4. Two adjacent nodes of G_1=(V, E) are said to be neighbors. Open & closed neighborhood of x represented by $N(x)$ or $N[x]$ respectively(see Figure 1.1).

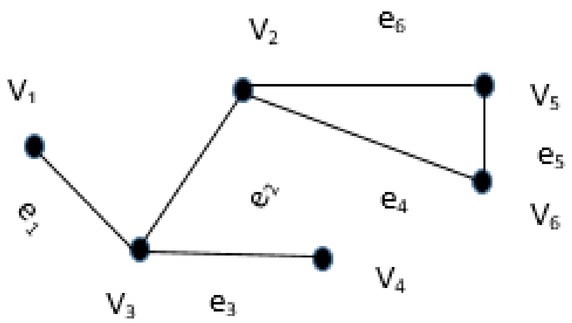

FIGURE 1.1: A simple graph H

Definition 1.1.5. Two or more lines of G_1 are known to be incident if they have a common node.

Definition 1.1.6. A *walk* in G is a finite alternating sequence of nodes & lines that begins and ends with nodes.

Definition 1.1.7. A walk in G without repetition of nodes is said to be a *path*, denoted by P_n.

Definition 1.1.8. A closed path in G=(V,E) is a *cycle*, C_n.

Definition 1.1.9. G_1 is connected if $\exists$ a P_n between each pair of nodes in G_1, otherwise disconnected.

Definition 1.1.10. G=(V,E) without any cycle is said to be *acyclic*.

Definition 1.1.11. A tree is connected & has no cycles (see Figure 1.2).

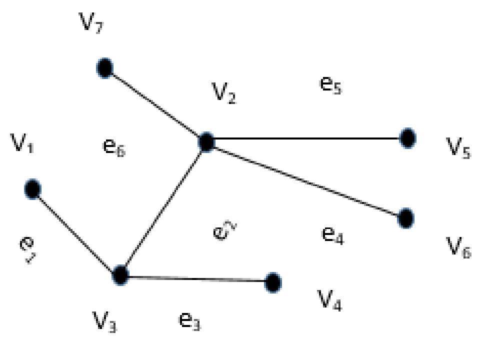

FIGURE 1.2: A tree

Definition 1.1.12. A graph denoted by K_n is said to be *complete* if every pair of nodes is connected by a line.

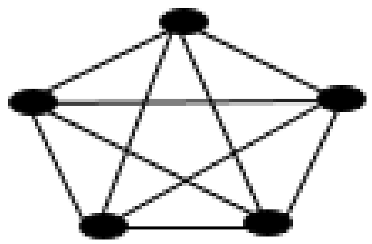

FIGURE 1.3: K_5

Definition 1.1.13. H=(V ,E) is *bipartite* when V shall be divided into 2 sets Y_1 & Y_2 so that each line of H is having one end node in Y_1 & the other in Y_2.

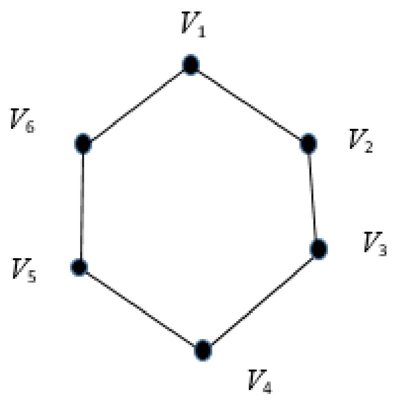

FIGURE 1.4: C_6

Definition 1.1.14. A bipartite graph G_1 with decomposition as $V = Y_1 \cup Y_2$ is said to be *complete bipartite* whenever every node in Y_1 is connected to every node in Y_2, denoted by $K_{m,n}$ (see Figure 1.5).

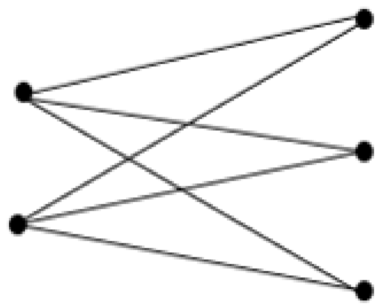

FIGURE 1.5: $K_{2,3}$

Definition 1.1.15. A star with n nodes is a complete bipartite graph $K_{1,n-1}$.

Definition 1.1.16. A *banana tree* is a family of stars with a new node adjoined to one end node of each star (see Figure 1.6).

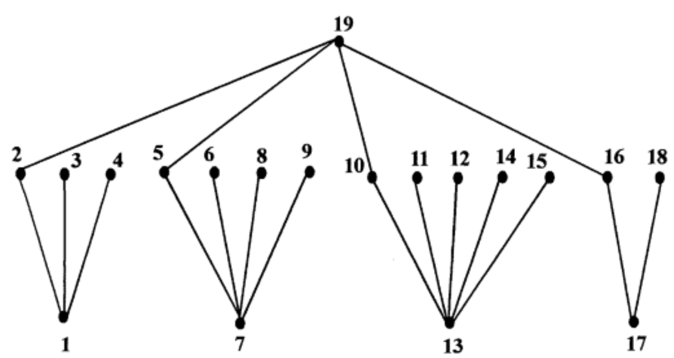

FIGURE 1.6: Banana tree

Definition 1.1.17. A *wheel graph* is $W_n = C_n \wedge K_1$. (see Figure 1.7).

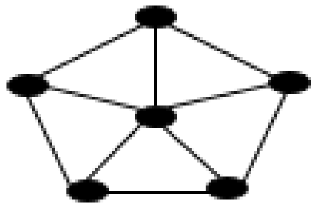

FIGURE 1.7: W_5

Definition 1.1.18. The *helm*, H_n can be derived out of W_n by connecting a pendant point to every rim point (See Figure 1.8).

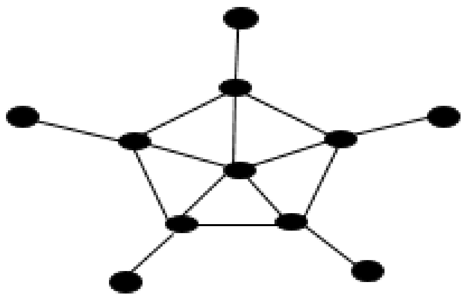

FIGURE 1.8: H_5

Definition 1.1.19. The Fl_n is formed from H_n by connecting each pendant node to the centre of H_n (See Figure 1.9).

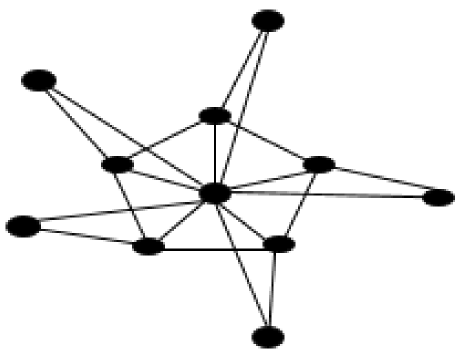

FIGURE 1.9: Fl_5

Definition 1.1.20. The *gear graph* G_n is formed by inserting a node between adjacent nodes on the perimeter of W_n (see Figure 1.10).

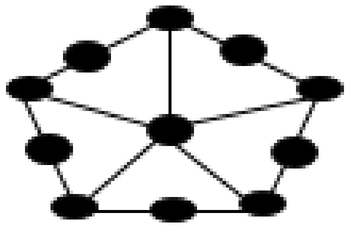

FIGURE 1.10: Gear graph G_5

Definition 1.1.21. A line $e = xy$ is known to be subdivided when xy is replaced by lines $e_1 = xw$ & $e_2 = wy$. When every line of H is subdivided, the resulting graph is said to be *barycentric subdivision* of H. (see Figure 1.11).

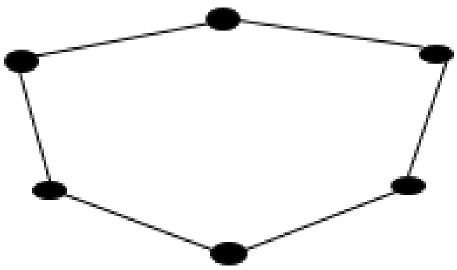

FIGURE 1.11: $S(C_3)$

Definition 1.1.22. The extension of any arbitrary node in G, say r, is obtained by introducing a new node u_1 in G which produces a new graph G_1 & $N(u_1)=\{r\} \cup N(r)$". (see Figure 1.12).

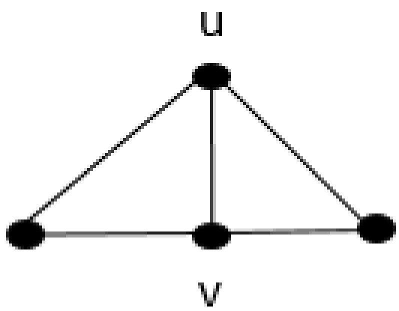

FIGURE 1.12: Extension of v by u in P_3

Definition 1.1.23. "The shadow graph $D_2(H)$ of H is formed by having 2 copies of H, say H' and H'' and joining every node $x' \in H'$ to the neighbors of the respective node $y' \in H''$(see Figure 1.13).

Example 1.1.1. *G and $D_2(G)$ (see Figure 1.13).*

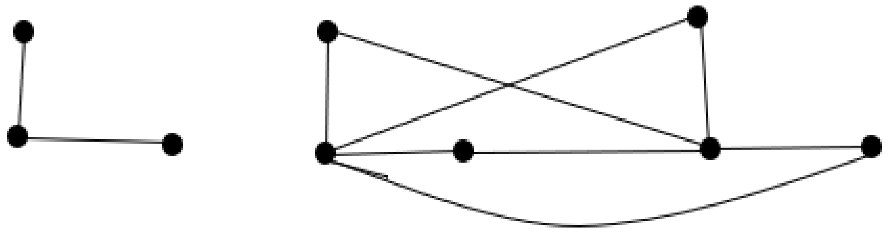

FIGURE 1.13: G and $D_2(G)$

Definition 1.1.24. The web graph $W_{n,r}$ can be obtained as a Cartesian product of C_n and P_r. Note that $W_{n,1}$ is the same as C_n, and $W_{n,2}$ is a prism.

Definition 1.1.25. [41] A triangular snake T_s is derived from P_s: $x_1,x_{2,} \ldots x_s$ by connecting x_i and x_{i+1} to a new node u_i: $1 \leq i \leq s-1$.

Definition 1.1.26. [41] A double triangular snake $D(T_s)$ contains of 2 T_s with a common P_s.

Definition 1.1.27. [41] The duplication of

(i) a point x of G_1 is obtained by adding a new point $x^{'}$ to G_1 & placing links so that $N(x) = N(x')$.

(ii) a node v_k of G_1 by a line $e = v_k{'}v_k{"}$is formed by introducing 2 new nodes $v_k{'}$, $v_k{"}$ & a line $e = v_k{'}v_k{"}$ so that $N\left(v_k^{'}\right) = \{v_k, v_k{"}\}, N\left(v_k{"}\right) = \left\{v_k, v_k{'}\right\}$.

(iii) $e_1 = cd \in G_1$ by a point w is obtained by introducing a new point w to G_1 so that $N\left(w\right) = \{c,\ d\}$.

(iv) a line $e_1 = x_1y_1$ in G_1 produces $G_1{'}$ by introducing a line $e^{'} = x_1{'}y_1{'}$, where $x_1{'}, y_1{'}$ are newly added points to G_1 so that $N\left(x_1{'}\right) = N\left(x_1\right) \cup \left\{y_1{'}\right\} - \{y_1\}$ & $N\left(y_1{'}\right) = N\left(y_1\right) \cup \left\{x_1{'}\right\} - \{x_1\}$.

Definition 1.1.28. [41] The total graph $T(H)$ of H with node set, $V(H) \cup E(H)$ and 2 nodes are adjacent when they are either adjacent or incident in H".

Definition 1.1.29. [41] "The $k-th$ power G^k of G has the same node set as G and two distinct nodes u and v of G are adjacent in G^k if and only if their distance in G is at most k. The graphs G^2 and G^3 are also known as square and cube of G respectively (Parthiban, 2018)".

Definition 1.1.30. [44] "DW_n of size n is composed of $2C_n + K_1$.

Definition 1.1.31. [44] $F_{1,n}$ is defined as $F_{1,n} = K_1 + P_n$.

Definition 1.1.32. [44] Umberlla graph $G = U(r, s)$ is defined as follows: The node set $V(U(r, s)) = \{v_1, v_2, \ldots, v_{r+s}\}$ & $E(U(r, s)) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{v_i v_{i+1} : 1 \le i \le s-1\}$,
$E_2 = \{v_s v_{s+j} : 1 \le j \le r\}$,
$E_3 = \{v_k v_{k+1} : s+1 \le k \le r+s-1\}$.

Definition 1.1.33. The join of two disjoint graphs H_1 and H_2, $H_1 + H_2$ has node set V$(H_1 + H_2) = V_1 \cup V_2$ and $E(H_1 + H_2) = E_1 \cup E_2 \cup \{uv : u \in V_1,\ v \in V_2\}$ (see Figure 1.14).

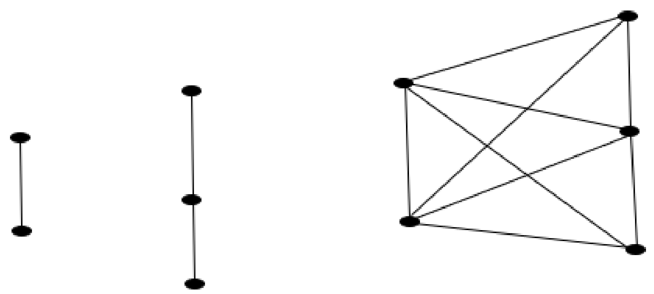

FIGURE 1.14: G_1, G_2 and $G_1 + G_2$

Definition 1.1.34. Two graphs H_1 & H_2 are known to be isomorphic if $\exists$ a one-one correspondence between their nodes & lines such that the incidence relationship is preserved" (see Figures 1.15, 1.16).

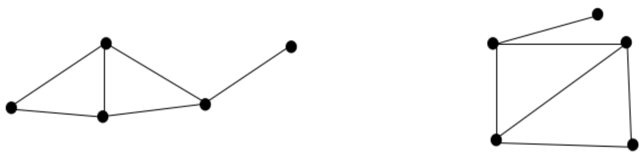

FIGURE 1.15: Isomorphic Graphs

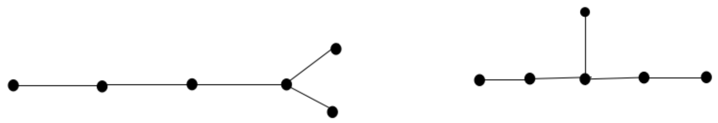

FIGURE 1.16: Non - isomorphic Graphs

1.2 Graph Labeling

"An allocation of numbers to V or E or both of H subject to a few restrictions is called a GL. There are many types of GL studied by many Mathematicians like PL, PDL, node PL, cordial labeling, k-equitable labeling, etc. Prime numbers have always fascinated mathematicians, and the concept of relatively prime numbers has also been extensively used in many areas of Mathematics. So, the PL and PDL have also been extensively studied by many mathematicians. But the origin of GL can be traced when Rosa [57] introduced β-valuation of a graph in 1967. A one-one function g: $V(H) \to \{0,1,2,\ldots,m\}$ is a β-valuation of $H(V,E)$, where $|E|=m$, if each line $e=st$ is given the label $|g(s)-g(t)|$ and every label is unique". The present study is focused on PL and PDL of simple and undirected graphs.

Definition 1.2.1. If the nodes or lines or both of G are assigned labels (mostly integers) subject to certain conditions, then the graph is said to have node labeling, line labeling or total labeling respectively.

Definition 1.2.2. "$H=(m,\ n)$ is graceful when $\exists$ a one-one mapping $g\colon V(H) \to \{0, 1,\ldots,n\}$ such that the resulting absolute difference of the node labels of all the lines is $\{1,\ 2,\ \cdots\ ,\ n\}$" (see Figure 1.17).

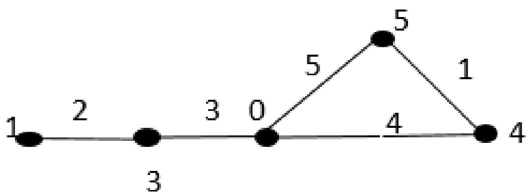

FIGURE 1.17: Graceful labeling of G

1.3 Prime Labeling and Prime Distance Labeling

Some number theory results are recalled here:

Conjecture 1.3.1. *"(**The Goldbach's Conjecture (GC)**): Every* $2k > 2, 2k = p_1 + p_2, p_i \in P$ *(Burton, 2011)".*

Conjecture 1.3.2. *"(**The Twin Prime Conjecture (TPC)**)* $\exists$ *infinitely many primes* p_i, p_j *such that* $|p_i - p_j| = 2$ *(Burton, 2011)".*

Conjecture 1.3.3. *"**(de Polignac's)** For any positive integer $2s$, $\exists$ infinitely many pairs of consecutive primes that difference is $2k$ (Burton, 2011)".*

Theorem 1.3.1. *"**(Ramare's)** Every $2k$ is the sum of at most 6 primes (Burton, 2011)".*

Theorem 1.3.2. *"**(The Green-Tao)** For any positive integer l, $\exists$ a prime A.P of length l (Burton, 2011)".*

Theorem 1.3.3. *"Every sufficiently large odd number, $2k+1, 2k+1 = p_1+p_2+p_3, p_i \in P$. is the sum of three primes (Burton, 2011)".*

Lemma 1.3.1. ***(Euclid's Lemma)*** *When q is a prime and c, d are any two integers, then $q\,|cd \Rightarrow q|\,c$ or $q|d$.*

Theorem 1.3.4. *Two integers c, d are co-prime if and only if $\exists$ two integers x and y such that $cx + dy = 1$.*

Theorem 1.3.5. *If two integers r and s are co-prime, then $r\,|sc \Rightarrow r|\,c$.*

Theorem 1.3.6. *The number of positive primes is infinite.*

Definition 1.3.1. The Euler $\varphi-$ function is the function $\varphi : Z^+ \to Z^+$ and defined as follows: (i) : $\varphi(1) = 1$, (ii) for $k > 1, \varphi(k)$= the count of positive integers $< k$ and co-prime to k.

Theorem 1.3.7. *If r and s are co-primes, then $\varphi(rs) = \varphi(r)\,\varphi(s)$.*

Theorem 1.3.8. *If q is a prime, then $\varphi(q^n) = q^n - q^{n-1} = q^n\left(1 - \frac{1}{q}\right)$, where n is any positive integer.*

Theorem 1.3.9. *If $n > 1$ and $p_i; i = 1, 2, 3...m$ are the distinct prime factors of n, then $\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_m}\right)$.*

Theorem 1.3.10. *If q is a prime and x is any integer such that q is not a divisor of x so that $(x, q) = 1$, then $x^{q-1} \equiv 1\,(mod\ q)$.*

Theorem 1.3.11. *If m is a positive integer & a is any integer such that $(a, m) = 1$, then $a^{\varphi(m)} \equiv 1\,(mod\ m)$.*

Theorem 1.3.12. *If q is a prime, then $(q-1)! + 1 \equiv 0\,(mod\ q)$ i.e. $(q-1)! + 1$ is a multiple of q.*

Theorem 1.3.13. ***(Converse of Wilson's Theorem)*** *If $(p-1)! + 1 \equiv 0\,(mod\ p)$, then p must be a prime.*

Definition 1.3.2. A graph H with $k-$ nodes is said to allow prime labeling (PL) if $\exists$ a bijective map $g\ :V(H)\ \rightarrow\{1,\ 2,...,k\}$ so that for each $e\ =\ kl$, $GCD(g\ (k),\ g\ (l))=1$. Any G admitting PL is a prime graph (PG) (see Figure 1.18).

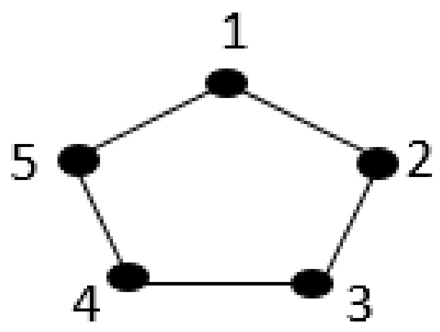

FIGURE 1.18: A prime graph

Definition 1.3.3. H is a PDG if $\exists$ a one-one labeling $h:V\ (H)\rightarrow$ Z so that given 2 adjacent nodes k & l, the integer $|h(k)-h(l)|$ is prime. (see Figure 1.19).

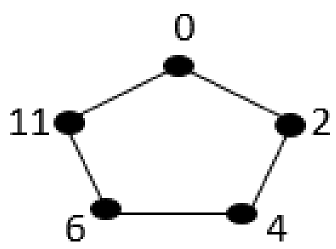

FIGURE 1.19: A PDG

Interestingly, the establishment of PDL of a few classes of graphs is very much related to some important results in Number Theory. In fact the PDL of bipartite graphs can be shown by using the Green-Tao theorem; the PDL of cycles can be done by using GC, TPC, etc".

1.4 Review of Literature

1.4.1 Some Important Results on Prime Labeling

Some important results on PL are recalled in this subsection.

"In [26] the given results are established:

- If H is a prime graph of order s, then $\alpha_0\left(H\right) \geq \left\lfloor \frac{s}{2} \right\rfloor$, where $\alpha_0(H)$ is the independence number of H.
- A complete bipartite graph $G = (A,\ B)$ of order n with $|A| \leq |B|$, is prime iff $|A| \leq \left|P\left(\frac{n}{2},\ n\right)\right| + 1$.
- If $H = (A,\ B)$ is bipartite of order p with $|A| \leq |B|$ & $|A| \leq \left|P\left(\frac{p}{2},\ p\right)\right| + 1$, then H is prime.
- If $T = (A,\ B)$ is a connected acyclic graph of order s with $|A| \leq |B|$ & $|A| \leq \left|P\left(\frac{s}{3},\ s\right)\right| + 1$. Then T is prime.
- All trees of order less than 16 admit PL.
- If $\alpha_0\left(G\right) \geq n - \left|P\left(\frac{n}{2},\ n\right)\right| - 1$, then G *is prime.*

P. Haxell et al [25] established the following:

- $\exists\ k'$ such that every tree of order k with $k \geq k'$ is prime.
- For every $d \geq 1\ \exists\ n''$ such that every $s\left(n\right)-$ separable bipartite $d-$ degenerate graph F of order $n \geq n''$ is prime, where $s\left(n\right) = n^{1-\frac{10^6.d}{\ln \ln n}}$."

"S. K. Patel and Jayesh Vasava [51] established the following:

- If n and m both are odd, then $C_n^{(j)} \cup C_m^{(k)}$ is not a prime graph.
- $C_{2n}^{(j)} \cup C_m^{(k)}$ is prime $\forall$ n and m.
- $C_{2n}^{(2)} \cup C_{2m}^{(2)} \cup C_k^{(2)}$ is prime $\forall$ n, m and k.
- $C_{2n} \cup C_{2n} \cup C_{2n} \cup C_{2n} \cup C_{2m} \cup C_k$ is prime $\forall$ n, m and k"

"S. K. Vaidya et al. [72] proved the following:
The graph formed by performing duplication of

- $x \in P_k$ is a PG.
- a node by a line in P_k is a PG.
- every node by a line in P_k is not a PG.

- a line by a node in P_k is a PG.
- a line in P_k is a PG.
- every node by a line in C_k is not a PG.
- a line by a node in C_k is a PG.
- a line in C_k is a PG if $k \geq 3$.

S. Meena and P. Kavitha [37] proved the following:
The graph formed by performing duplication of

- each node by a line in subdivision of $S_{1,n}$ does not admit PL.
- a node in subdivision of $S_{1,n}$ admits PL.
- all pendant nodes in subdivision of $S_{1,n}$ admits PL.
- nodes of degree 2 in subdivision of $S_{1,n}$ admits PL.
- the nodes of the subdivision of $S_{1,n}$ except the apex node admits PL.
- the lines by a node which are incident with pendant nodes in subdivision of $S_{1,n}$ does not admit PL.

S. Meena and K. Vaithilingam [38] established the following:
The graph obtained by

- fusing any 2 consecutive nodes in H_n is a PG.
- duplicating a node v_k in the rim of H_n is a PG.
- switching of any node v_k in the rim of H_n is a PG.
- the path P_k union of 2 pieces of H_r is a PG if $r \neq 5k + 1$.
- H_n admits PL.

S. Ashokkumar and S. Maragathavalli [37] proved the following:

- Flower graph Fl_n admits a PL.
- Splitting graph of star graph admits a PL.
- The bistar $B_{n,n}$ admits a PL.

- The friendship graph F_n admits a PL.

The authors in [69] proved the following:
The graph obtained by

- identifying any 2 nodes x_i & x_j ($d(x_i, x_j) \geq 3$) of C_k is a PG.
- duplicating arbitrary node of C_k is a PG.
- path union of finite copies of C_r is a PG except for odd r.
- joining two copies of C_r by P_s is a PG except r & s both odd".

1.4.2 Some Important Results on Prime Distance Labeling

Joshua D. Laison, Colin Starr, Andrea Walker [33] proved the following:

- "Every bipartite graph is PDG.
- If GC is true, then each C_n is PDG.
- Every cycle is a PDG.
- All Dutch windmill graphs are PDG iff the TPC is true.

A. Parthiban and N. Gnanamalar David [45] established the following:

- "$\mu(G)$ of any non-PDG G is again a non-PDG.
- Any k-th power $k \geq 2$ of $\mu(P_n)$, $n \geq 2$, does not admit a PDL.
- Any k-th power ($k \geq 2$) of $\mu(C_n)$, $n \geq 3$, does not admit a PDL.
- The shadow graph $D_2(P_n)$ of P_n admits a PDL.
- The middle graph $M(P_n)$ of P_n admits a PDL.
- The total graph $T(P_n)$ of P_n admits a PDL.
- The double triangular snake $D(T_n)$ admits a PDL.
- The triangular snake T_n admits a PDL".

A. Parthiban and N. Gnanamalar David [41] established the following:
"The graphs formed by

- (i) switching a pendant node in P_k, (ii) switching a neighbor of a pendant node in P_n and (iii) switching a middle node in P_n admit no PDL for $n \geq 14$".
- switching a central node in W_n admits a PDL for $n \geq 4$.
- performing duplication of a point by a line at all points in P_n admit a PDL for $n \geq 1$.
- performing (i) duplication of a node in P_n or (ii) duplication of a node by an line in P_n admits PDL for $n \geq 1$.
- performing duplication of a line by a node at all lines in P_n admits PDL.
- performing extension at all nodes in P_n admits PDL.
- taking union of k-copies of PDG H admits PDL.
- gluing at each node of a PDG H finite number of node disjoint paths permits PDL.
- gluing 2 copies of C_k by P_l admits PDL".

A. Parthiban et al. [44] proved the following:

- "$G_1 = H_1 \times K_2$ admits PDL".
- "$W_n = C_{n-1} + K_1,\ n \geq 10$ admits no PDL".
- "$H_n,\ n \geq 10$ admits no PDL".
- "$DW_n,\ n \geq 5$ admits no PDL".
- "$F_{1,n},\ n \geq 11$ admits no PDL".
- "$U\left(m,n\right),\ m \geq 11$ admits no PDL".

A. Parthiban & N. Gnanamalar David [42] proved the following:

- "$W_{n,r}$ for any $n,\ r \geq 3$ admits a PDL".
- "$DSG(P_n)$ does not admit a PDL $\forall\ n \geq 13$".
- "$DSG(K_n),\ n \geq 4$ does not admit a PDL".
- "The braid graph $B\left(n\right)$ permits PDL $\forall\ n \geq 3$".
- "The triangular ladder TL_n admits a PDL $\forall\ n \geq 2$".

1.5 Research Gap and Objectives

Though plenty of study has been pursued in the section of PL and PDL of graphs, still there are many interesting conjectures and open problems to work on. For instance, Entringer-Tout Conjecture: Every tree is prime graph has not been settled till date, though a partial success has been achieved wherein it has been proved that all large trees are prime. Complete characterizations of prime graphs and PDGs are still pending. Researchers across the globe have done an enormous amount of work in this direction. Moreover both of these labeling have been studied by many mathematicians for many classes and "families of graphs" in the context of many "graph operations" like barycentric sub-division, extension of nodes, join, Cartesian product, corona, duplication, node switching, etc. But there are many families and classes of graphs for which the exhibition of these labeling in the context of barycentric sub-division and extension of nodes is pending. Moreover an attempt has been made to narrow the gap of establishing the complete characterization of these labeling. Based on these research gaps, the following objectives are framed:

1. Deriving the PL of some classes of graphs.
2. Establishing PDL of certain families of graphs.
3. Obtaining the prime and PDL in the context of extension of nodes of some graphs.
4. Obtaining the prime and PDL in the context of barycentric subdivision of some graphs.

1.6 Contributions of the thesis

The present research study enriches the field of graph theory, specifically the area of PL and PDL of graphs. The contributions advance the possibility of attaining the complete characterizations of PL and PDL. The study also provides the future directions for the readers and research community.

1.7 Conclusion

In this chapter, an introduction to GT and GL has been given for references in the exposition. In addition to PL and PDL, a few other graph labeling techniques have also been recalled. Finally, a comprehensive review of literature along with research gap followed by the proposed objectives are also presented.

Chapter 2

Results on Prime Labeling

2.1 Introduction

In this chapter PL of some classes of graphs has been studied. The constraints on the orders of two prime graphs so that their disjoint union exhibits PL have been investigated. Also studied PL of P_n and K_n in the context of degree splitting graph of given graph. Besides this, prime lableling of the complement of gear graph & middle graph of a path have also been derived.

2.2 Certain Results on PL of Graphs

First we recall some established results:

Theorem 2.2.1. *"Assume that T_k is a connected acyclic graph on k nodes, where $k \geq 3$. Let $A = \{x \in V | deg\,(x) \neq 1\}$ and $B = \{y \in A|\ y$ is adjacent to pendant verices$\}$. If $|A| \leq |P(1,\ k)| + 1$ and $|B| \leq \left|P\left(\frac{k}{2}, k\right)\right| + 1$, then T_k has PL, where $P\,(x,\ y)$ is the set of all primes t such that $x < t \leq y$".*

Example to illustrate the above theorem: Consider T in Figure 2.1. See that $n = 31$, "$A = \{v_i : i = 1, 2, 3, 4, 5, 6, 7\}$, $B = \{v_1, v_2, v_4,\ v_6, v_7\}$", $|A| = 7$, $|B| = 5$. Also, $|P\,(1, 31)| = 11$, $|P\,(15.5, 31)| = 5$. Since theorem 2.2.1 is satisfied, T has PL as shown in the Figure 2.1.

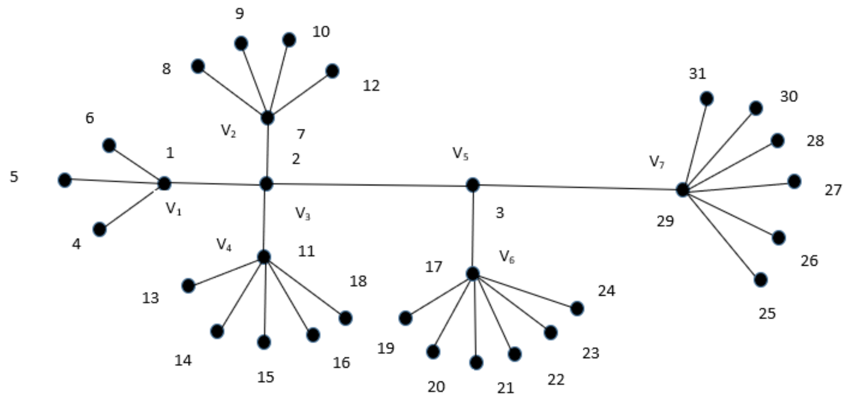

FIGURE 2.1: PL of a tree with 31 nodes

Theorem 2.2.2. *[35] "W_{2n} is switching invariant.*

Theorem 2.2.3. *[39] $U(m,n)$ is a PG.*

Theorem 2.2.4. *[39] Shell graphs are prime graphs for $n \geq 5$.*

Theorem 2.2.5. *[39] 2 copies of C_k sharing a common node is prime graph $k \geq 3$, where k is any positive integer.*

Theorem 2.2.6. *[38] A graph G obtained by the path union of two pieces of H_n is a PG if $n \neq 5k+1$.*

Definition 2.2.1. The Jahangir graph $J_{c,d}$, $c \geq 1$, $d \geq 3$, is on $cd+1$ nodes consisting of C_{cd} with an extra central node u which is adjacent to cyclically labeled nodes $u_1, u_2, u_3, ..., u_d$ such that $d(u_i, u_{i+1}) = c, 1 \leq i \leq d-1$ in C_{cd}.

Theorem 2.2.7. *[1] If cd is even, then the $J_{c,d}$ for $c \geq 2,\ d \geq 3$ is prime.*

Theorem 2.2.8. *[1] When nm is an odd number then $J_{n,m}$ ceases to be a PG.*

Theorem 2.2.9. *[1] Lilly graph $I_n,\ n \geq 2$ admits PL.*

Theorem 2.2.10. *If H_1 and H_2 are two prime graphs with orders m & n respectively (m, an odd positive integer ≥ 3 and $n = 2 * LCM\ [1,\ 3,\ 5, \ldots,\ m-2]$), then the disjoint union of G_1 and G_2 admits PL.*

Proof. If G_1 be a prime graph with m-nodes and $V(G_1) = \{w_1,\ w_2,\ \ldots,\ w_m\}$, m is an odd ≥ 3, then $\exists$ a bijective map $h: V(G_1) \rightarrow \{1,\ 2,\ 3, \ldots,\ m\}$ so that if $e = w_i w_j \in E(G_1)$ then GCD $\Big(h(w_i),\ h(w_j)\Big) = 1$. Similarly, let G_2 be a PG with n-points & $V(G_2) = \{v_1, v_2, \ldots, v_n\}$, where $n = 2$. LCM $[1,\ 3,\ 5, ...,\ m-2]$, $m \geq 3$. Then $\exists$ a bijective map $g:\ V(G_2) \rightarrow \{1,\ 2,\ 3, ...,\ n\}$ so that if $e' = v_i v_j \in E(G_2)$ then $GCD\,(g(v_i), g(v)_j)) = 1$. Obtain G as $G_1 \cup G_2$ and consider a new function

$h : \{w_1,\ w_2,\ ...,\ w_r, v_1,\ v_2, ...,\ v_s\} \rightarrow \{1,\ 2,\ 3, \ldots, r, r+1,\ r+2, \ldots,\ r+s\}$ defined as follows: $h(x) = \begin{cases} g(x), & x = v_i,\ i = 1,\ 2,\ 3, \ldots, s \\ s + f(x), & x = w_j,\ j = 1,\ 2,\ 3, \ldots, r \end{cases}$.

Claim: h is one-one.

Let $x,\ y \in V(G_1) \cup V(G_2)$. Then three cases arise:

Case 1: Let $x,\ y \in\ V(G_1)$

Take $h(x) = h(y)$. This implies that $n + f(x) = n + f(y) \Rightarrow x = y$ as f is one-one.

Case 2: Let $x,\ y \in\ V(G_2)$

Take h $(x) = h(y)$. This again implies $g(x) = g(y) \Rightarrow x = y$ as g is one-one.

Case 3: Let $x\ in\ V(G_1)$ & $y\ in\ V(G_2)$ or $x \in\ V(G_2)$ and $y \in\ V(G_1)$

Let $x\ in\ V(G_1)$ & $y\ in\ V(G_2)$. Then clearly $x \neq y$. Here $n + 1 \leq h(x) \leq n + m$, $1 \leq h(y) \leq n$ implies $h(x) \neq h(y)$ showing that h is one-one. The other possibilities can be dealt in a similar manner.

Further, if $e = v_i v_j\ \in E(G_2)$, then $GCD(h(v_i),\ h(v_j)) = GCD\left(g(v_i),\ g(v_j)\right) = 1$. If $e' = w_i w_j \in E(G_1)$, then let $GCD(h(w_i),\ h(w_j)) = d$ implies $GCD\left(n + f(w_i),\ n + f(w_j)\right) = d$. Then since GCD $(f(w_i),\ f(w_j)) = 1$, therefore one of $f(w_i)$ or $f(w_j)$ is odd. Also n is even implies one of $n + f(w_i)$ or$n + f(w_j)$ is odd which implies d is odd. As $d \mid n + f(w_i)$ and $d \mid n + f(w_j)$ implies $d \mid f(w_i) - f(w_j)$ i.e., $d \mid \left|f(w_i) - f(w_j)\right|$. But possible values of $\left|f(w_i) - f(w_j)\right|$ are $1,\ 2,\ 3,\ \ldots,\ m-1$ as $w_i \neq w_j$. Now $m \geq 3$ is odd implies $m-1$ is even and $m-2$ is odd. As d is odd, therefore, $d = 1,\ 3,\ 5, ...,\ m-2$. If $d = 3$, then $3 \mid n + f(w_i)$ and $3 \mid n + f(w_j)$. Also $3 \mid n$ implies $3\ \mid \mathrm{f}(w_i)$ and $3 \mid \mathrm{f}(w_j)$, which is impossible as $GCD(f(w_i),\ f(w_j)) = 1$. Similarly, one can get a contradiction if $d = 5,\ 7, \ldots,\ m-2$. Therefore, $d = 1$ which proves that the disjoint union of G_1 and G_2 admits PL". □

Theorem 2.2.11. *The complement graph* $\overline{G_n}$ *of* $G_n,\ n \geq 3$, *does not admit PL.*

Proof. For $n = 3$, one can see that $\alpha_0\left(\overline{G_3}\right) < \left\lfloor \frac{|V(\overline{G_3})|}{2} \right\rfloor$ and so by lemma 2.2.2, $\overline{G_3}$ is not a PG. Note that $|V(G_n)| = \left|V\left(\overline{G_n}\right)\right| = 2n+1$. For $n > 3, (2n-3) \geq 5$ and one can easily see that every rim node of $\overline{G_n}$ is adjacent to $(2n-3)$ other rim nodes. Further, there are $2n+1$ labels out of which $\left[\frac{2n+1}{2}\right]$ are even labels, where [ ] denotes the greatest integer function. Thus, there are $\left[\frac{2n+1}{2}\right]$ even labels for $2n+1$ nodes. Without loss of generality, label any rim node of $\overline{G_n}$ with an even number and other $(2n-3)$ rim nodes cannot be assigned the even label. Moreover, the remaining two rim nodes are adjacent in $\overline{G_n}$ which implies that only one of them can be assigned the even label. The apex node can also be assigned the even label. Thus, at the most only three nodes can be assigned even labels in $\overline{G_n}$. But for $n > 3$, there are even labels greater than or equal to 4. Hence, $\overline{G_n}$ is not a PG $\forall\ n > 3$. □

Theorem 2.2.12. *$DS(P_n)$ of P_n, $n > 1$ (an odd integer) admits PL.*

Proof. If P_n be the given path on n–nodes with $V = \{v_1, v_2, \ldots, v_n\}$, $n > 1$ is an odd positive integer, then $V = S_1 \cup S_2$, where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_2, \ldots, v_{n-1}\}$. Let $V(DS(P_n)) - V(P_n) = \{w_1, w_2\}$. Here, $|V(\text{DS}(P_n))| = n + 2$. A function $f : \{v_1\ , v_2, \ldots, v_n, w_1, w_2\} \to \{1, 2, \ldots, n+2\}$ is given as below:"$f(w_1) = 2,\ f(w_2) = 1,\ f(v_k) = 2 + k,\ 1 \le k \le n$. Then, clearly f is a bijective function and also $E(\text{DS}(P_n)) = \bigcup E_i : 1 \le i \le 3$, where $E_1 = \{v_i v_{i+1} : 1 \le i \le n-1\}$, $E_2 = \{w_1 v_1,\ w_1 v_n\}$, $E_3 = \{w_2 v_i : 2 \le i \le n-1\}$. Now $GCD(f(v_i),\ f(v_{i+1})) = GCD(\ 2+i,\ 3+i) = 1$", being consecutive integer, $1 \le i \le n-1$. Also $GCD(f(w_1),\ f(v_1)) = GCD(2,\ 3) = 1$, $GCD(f(w_1),\ f(v_n)) = GCD(2,\ 2+n\ \ ) is 1$, as n an odd, $GCD(f(w_2),\ f(v_k)) = GCD(1,\ \ 2+k) is 1,\ 2 \le k \le n-1$. Hence, f induces PL of $\text{DS}(P_n)$, for an odd $n > 1$ (see Figure 2.2 for the illustration when $n = 11$).

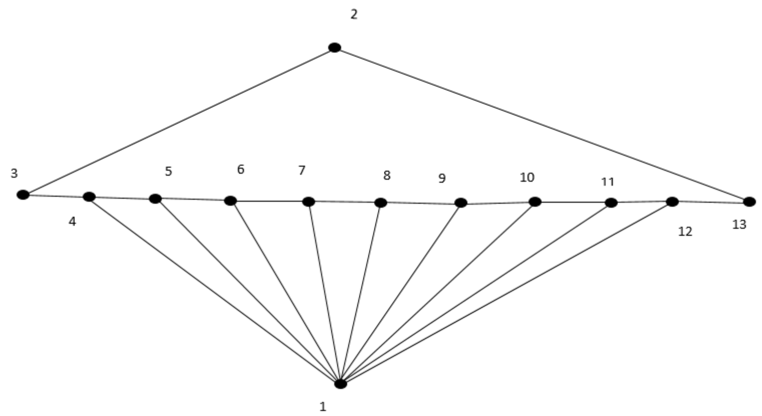

FIGURE 2.2: A PL of $\text{DS}(P_{11})$

□

Conjecture 2.2.1. *$DS(P_n)$ of P_n (n an even integer ≥ 4) does not admit PL.*

Theorem 2.2.13. *[35] K_n, $n \ge 4$ does not admit PL.*

Lemma 2.2.1. *[42] $DS(K_n) = K_{n+1}$.*

Theorem 2.2.14. *$DS(K_n)$, $n \ge 3$ does not admit PL.*

Proof. The result is direct from Lemma 2.2.1 and Theorem 2.2.13. □

Theorem 2.2.15. *"$M(P_n)$ admits PL.*

Proof. If P_n is on $n \geq 2$ nodes $s_1, s_2, \ldots, s_n$ and $n-1$ lines $t_1, t_2, \ldots, t_{n-1}$, then $M(P_n)$ having $2n-1$ nodes and $3n-4$ lines is defined as follows: $V(M(P_n))$= $V(P_n) \cup E(P_n)$= $\{u_1, u_2, \ldots, u_{2n-1}\}$ where the nodes u_i, $1 \leq i \leq n$ correspond to $s_i's$ and u_{n+j}, $1 \leq j \leq n-1$ correspond to t_j's and $E(M(P_n)) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{u_a u_{n+a}, 1 \leq a \leq n-1\}$, $E_2 = \{u_{b+1} u_{n+b}, 1 \leq b \leq n-1\}$ and $E_3 = \{u_{n+c} u_{n+c+1}, 1 \leq c \leq n-2\}$. For $n = 2$, the case becomes trivial, so take $n \geq 3$. Now $h : V(M(P_n)) \to \{1, 2, 3, \ldots, 2n-1\}$ is given as below:$h(u_1) = 1$, $h(u_2) = 3$, $h(u_{n)} = 4$, $h(u_{n+1}) = 2$, $h(u_{n+2}) = 5$, $h(u_k) = 2k$, $3 \leq k \leq n-1$, $h(u_{n+k}) = h(u_{n+k-1}) + 2$, $3 \leq k \leq n-1$. Then f induces PL for $M(P_n)$". □

Example 2.2.1. *The PL of $M(P_{12})$ is shown below in Figure 2.3.*

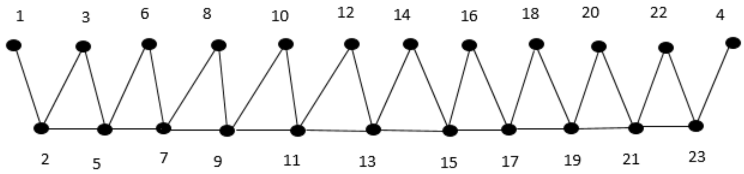

FIGURE 2.3: The PL of $M(P_{12})$

Definition 2.2.2. "A graph H of order k with $V(H) = \{1, 2, ..., k\}$ is a maximal PG if and only if for any two adjacent nodes i and j, GCD $(i, j) = 1$".

Example 2.2.2. *A maximal prime graph of order 6 is shown in Figure 2.4.*

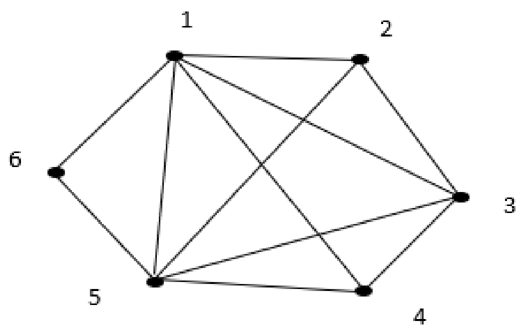

FIGURE 2.4: A maximal prime graph of order 6

Theorem 2.2.16. *If H_α is a maximal PG on k nodes, then $\alpha_0(H_\alpha) = \left\lfloor \frac{k}{2} \right\rfloor$.*

Proof. Consider the set $\{1, 2, 3, \ldots, m\}$ of first m natural numbers, then see that number of even natural numbers is equal to $\left\lfloor \frac{m}{2} \right\rfloor$. Moreover, in a maximal prime graph the nodes labeled with even integers form a maximal independent set which proves that $\alpha_0(H_\alpha) = \lfloor \frac{m}{2} \rfloor$. □

Lemma 2.2.2. *[26] If $\alpha_0(H) < \left\lfloor \frac{|V(H)|}{2} \right\rfloor$, then H is not a prime.*

Theorem 2.2.17. *Let H be a maximal PG with order k and C_m be a cycle of order m, where m, k being odd. Then their disjoint union is not prime.*

Proof. Since G is a maximal prime graph and therefore in view of Theorem 2.2.16, $\alpha_0(G) = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ as n is odd. Also $\alpha_0(C_m) = \frac{m-1}{2}$ as m being odd. Therefore, $\alpha_0(G \cup C_m) = \frac{n-1}{2} + \frac{m-1}{2} = \frac{m+n}{2} - 1 < \left\lfloor \frac{|V(G \cup C_m)|}{2} \right\rfloor$. Hence according to Lemma 2.2.2, the result follows. □

Definition 2.2.3. [51] "$C_m^{(k)}$ is obtained by identifying only one node of every k copies of C_m. Clearly, $\left|V\left(C_m^{(k)}\right)\right| = k(m-1) + 1$ and $\left|E\left(C_m^{(k)}\right)\right| = km$".

For example, $C_3^{(4)}$ is shown in Figure 2.5.

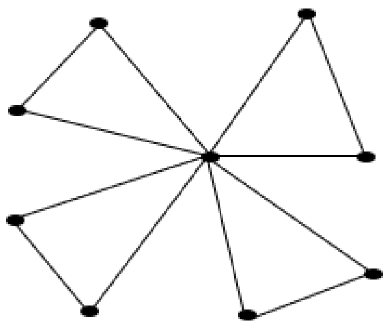

FIGURE 2.5: $C_3^{(4)}$

Note: If $\lceil x \rceil$ denotes the ceiling function and $\lfloor x \rfloor$ denotes the floor function, then $\lceil x \rceil = -\lfloor -x \rfloor$.

Theorem 2.2.18. *Let $G = \left(\bigcup_{k=1}^{n} C_{n_k}^{(2)}\right) \cup \left(\bigcup_{j=1}^{m} C_{m_j}^{(2)}\right)$, where both m, n being odd, each n_k an odd integer and each m_j is an even integer. If $n > m$, then G is not prime.*

Proof. Since each n_k is an odd integer and each m_j is an even integer, therefore $\alpha\left(C_{n_k}^{(2)}\right) = n_k - 1$ and $\alpha\left(C_{m_j}^{(2)}\right) = m_j$. Now $\alpha(G) = \sum_{k=1}^{n}(n_k - 1) + \sum_{j=1}^{m} m_j = \sum_{k=1}^{n} n_k + \sum_{j=1}^{m} m_j - n$. Further $|V(G)| = \left(\sum_{k=1}^{n} 2n_k\right) - n + \left(\sum_{j=1}^{m} 2m_j\right) - m = 2\left(\sum_{k=1}^{n} n_k\right) + 2\left(\sum_{j=1}^{m} m_j\right) - (n+m)$.

Now $\left\lfloor \frac{|V(G)|}{2} \right\rfloor = \sum_{k=1}^{n} n_k + \sum_{j=1}^{m} m_j - \left\lceil \frac{n+m}{2} \right\rceil = \sum_{k=1}^{n} n_k + \sum_{j=1}^{m} m_j - \frac{n+m}{2}$ as $n,\ m$ being odd. Also, it is given that $n > m$, therefore $2n > n+m$, *i.e.*, $\frac{n+m}{2} < n$ implies $-n < -\frac{n+m}{2}$. This further implies that $\alpha_0\left(G\right) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor$. Hence by Lemma 2.2.2, G is not a PG. □

2.3 Conclusion

In this chapter, the sufficient condition for the existence of PL of disjoint union of prime graphs has been established. Further, prime labeling of P_n and K_n in the context of degree splitting graph and complement of gear graph have been investigated. Also, PL of middle graph of path graph & disjoint union of graphs has been established in addition to formulating some interesting conjectures.

Chapter 3

PL in the Context of Extension and Barycentric Sub-division

3.1 Introduction

In this chapter, the sufficient condition for PL of barycentric subdivision of a graph has been obtained. Also PL of star graph, bistar graph and tree in the context of barycentric sub-division have been investigated. Further, the PL of complete graph and path graph in the context of extension of nodes besides formulating some open problems have also been investigated.

3.2 Certain Results on PL of Graphs in the Context of Extension and Barycentric Sub-division

Recall some important results.
A.N. Kansagara et al. [31] showed the following:

Theorem 3.2.1. *"$S(W_n)$ is a prime graph $\forall n \geq 3$".*

Example 3.2.1. *PL of $S(W_9)$ is shown in Figure 3.1.*

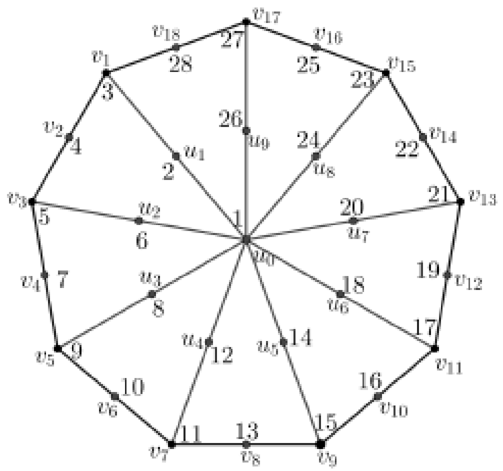

FIGURE 3.1: PL of $S\,(W_9)$

Theorem 3.2.2. *"$S(Fl_n)$ is a prime graph $\forall n \geq 3$".*

Example 3.2.2. *PL of $S(Fl_6)$ is given in Figure 3.2.*

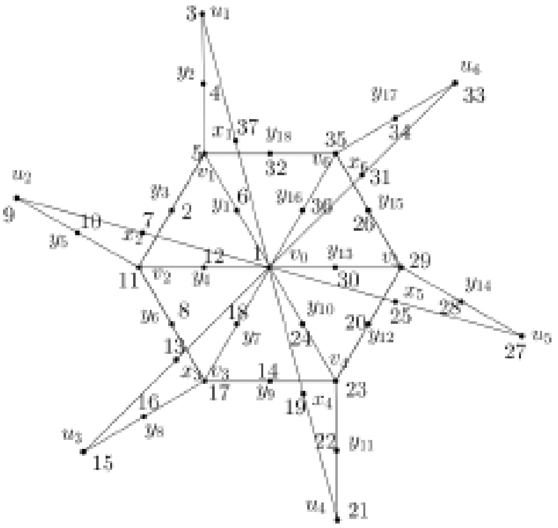

FIGURE 3.2: PL of $S\,(\mathrm{Fl}_6)$

Theorem 3.2.3. *"$S((C_8 \bigodot K_n) \bigcup (C_8 \bigodot K_n))$ is a prime graph $\forall n$".*

Example 3.2.3. *PL of $S((C_8 \bigodot K_1) \bigcup (C_8 \bigodot K_1))$ is shown in Figure 3.3.*

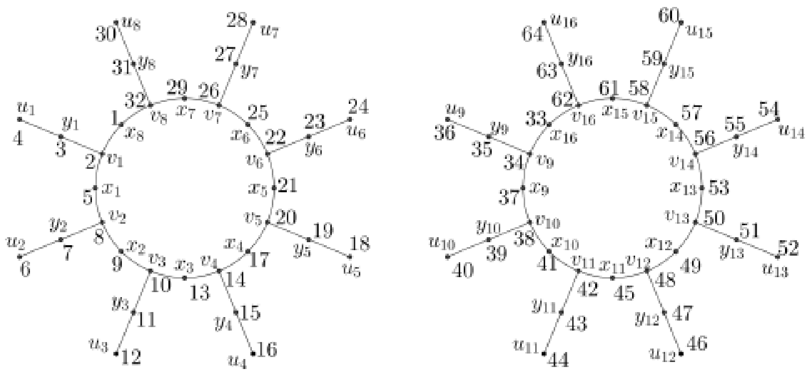

FIGURE 3.3: PL of $S((C_8 \bigodot K_1) \bigcup (C_8 \bigodot K_1))$

Theorem 3.2.4. *"$S(K_{1,n})$ admits PL.*

Proof. If $G_1 = K_{1,n}$ is a star & $S(G_1)$ is its barycentric subdivision. Let $V(G_1) = \{y_0,\ y_1, y_2, \ldots,\ y_n\}$ and $V(S(G_1)) = \{y_0,\ y_1, y_2, \ldots,\ y_n,\ w_1, w_2, \ldots,\ w_n\}$. $|V(S(G_1))| = 2n+1$. Now, $g : V(S(G_1)) \to \{1, 2, 3, \ldots, n,\ n+1, \ldots, 2n, 2n+1\}$ is defined as $g(y_0) = 1,\ g(w_r) = 2r,\ 1 \le r \le n,\ g(y_j) = 2j+1,\ 1 \le j \le n$. Then f induces PL of $S(G_1)$. □

Theorem 3.2.5. *$S(B_{n,n})$ admits PL if $4n+3$ is a prime number.*

Proof. Let $G = B_{n,n}$ be a bistar and $S(G)$ be its barycentric subdivision. Let $V(G) = \{u_0,\ v_0 u_1, u_2, \ldots,\ u_n, v_1,\ v_2,\ \ldots, v_n\}$ and $V(S(G)) = \{u_0,\ v_0,\ w_0,\ u_1, u_2, \ldots,\ u_n,\ w_1, w_2,\ \ldots, w_n,\ v_1$ where w_0 is obtained by subdividing $u_0 v_0$, w_i by subdividing $u_0 u_i$, $1 \le i \le n$ and t_i by subdividing $v_0 v_i$, $1 \le i \le n$. Then $|V(S(G))| = 4n+3$. Now $f : V(S(G)) \to \{1, 2, 3, \ldots, 4n+3\}$is given by $f(u_0) = 1,\ f(v_0) = 4n+3,\ f(w_0) = 4n+2,\ f(w_i) = 2i,\ 1 \le i \le n,\ f(u_j) = 2j+1,\ 1 \le j \le n,\ f(t_k) = 2(n+k),\ 1 \le k \le n,\ f(v_l) = 2(n+l)+1,\ 1 \le l \le n$. Then f induces PL of $S(G)$". □

Conjecture 3.2.1. ***(Entringer-Tout Conjecture)*** *[68] Every tree is a prime graph.*

Theorem 3.2.6. *The barycentric subdivision of a tree is prime if Prime Tree Conjecture is true.*

Proof. Since the barycentric subdivision of a tree results in a tree, therefore it is a PG if conjecture 3.2.1 is true. □

Notation: $P(x,\ y)= \{t\!:\ t$ is a prime and $x < t \le y\}$.

Theorem 3.2.7. *[26] If $G = (V_1, V_2)$ is bipartite with $|V_1| \le |V_2|$ & $|V_1| \le \left|P\left(\frac{n}{2},\ n\right)\right| + 1$, where $n = |V(G)|$, then G is prime.*

Theorem 3.2.8. *Barycentric sub-division of all simple graphs is bipartite.*

Proof. Since the barycentric sub-division makes all the odd cycles, if any, even and hence a bipartite graph. ☐

Theorem 3.2.9. *If $S(\mathrm{G}) = (V_1, V_2)$ is the barycentric sub-division of G with $|V_1| \leq |V_2|$ and $|V_1| \leq \left|P\left(\frac{n}{2},\ n\right)\right| + 1$, then $S(\mathrm{G})$ is prime.*

Proof. Proof follows from Theorem 3.2.7 and Theorem 3.2.8. ☐

Theorem 3.2.10. *[35] K_n does not have PL for $n \geq 4$.*

Theorem 3.2.11. *The graph obtained by taking extension of any arbitrary node in K_n is not a PG.*

Proof. Since the extension of a node in K_n gives rise to K_{n+1} and therefore in view of Theorem 2.2.14 the graph formed by taking extension of any arbitrary node in $K_n,\ n \geq 3$ is not a PG. ☐

Theorem 3.2.12. *Graph formed by taking extension of any arbitrary node of P_n is prime.*

Proof. If P_n is on n-nodes, say $x_1, x_2, x_3, \ldots, x_n$ and extension of x_k in P_n is obtained by adding new node $x_k^{'}$ such that $N\left(x_k^{'}\right) = N[x_k]$, then the node set of new graph having $(n+1)$–nodes, namely, is $\left\{x_1, x_2, \ldots x_{k-1}, x_k, x_k^{'}, x_{k+1}, \ldots,\ x_n\right\}$. "$y : \left\{x_1, x_2, \ldots x_{k-1}, x_k, x_k^{'}, x_{k+1}, \ldots,\ x_n\right\} \rightarrow \{1,\ 2,\ 3,\ 4, \ldots k-1, k, k+1, \ldots,\ n-1, n, n+1\}$ is given by $y\left(x_k^{'}\right) = 1$ and $y\left(x_l\right) = l+1,\ 1 \leq l \leq n$". Then y induces the required PL. ☐

Theorem 3.2.13. *The graph obtained by taking extension of pendant nodes of P_n, is prime $\forall$ odd $n \geq 1$.*

Proof. Let P_n be on n-nodes, say $v_1, v_2, v_3, \ldots, v_n$ and extensions of the pendant nodes v_1 and v_n are respectively taken by adding new nodes $v_1^{'}$ and $v_n^{'}$ so that "$N\left(v_1^{'}\right) = N[v_1]$ and $N\left(v_n^{'}\right) = N[v_n]$". Then the node set of new graph having $(n+2)$-nodes is $\left\{v_1, v_1^{'}, v_2, \ldots, v_n, v_n^{'}\right\}$. A function "$y : \left\{v_1, v_1^{'}, v_2, \ldots, v_n, v_n^{'}\right\} \rightarrow \{1, 2, 3, 4, \ldots,\ n-1, n, n+1, n+2\}$ is defined as $y\left(v_1^{'}\right) = 1,\ y\left(v_n^{'}\right) = n+2$ & $y\left(v_k\right) = k+1,\ 1 \leq k \leq n$". Then y induces the required PL. ☐

3.3 Open Problems

Open Problems 3.3.1. Is theorem 3.2.13 true if n is even?

Open Problem 3.3.2. Is the graph formed by taking extension of all the nodes of P_n a prime graph?

3.4 Conclusion

In this chapter, the PL of barycentric sub-division of star graph, bistar graph and sufficient condition for PL of barycentric sub-division of a graph have been established. Further PL of path graph and complete graph in the context of extension of nodes has been investigated besides formulating some open problems. Similarly, one can study the PL for barycentric subdivision of some other subclasses of trees; this is the future work.

Chapter 4

Results on Prime Distance Labeling

4.1 Introduction

In this chapter, the PDL of the non-commuting graphs of some non-abelian groups like dihedral group D_{2n} and Quartenion group Q_8 has been investigated. PDL of the degree splitting graph of complete graph, complete bipartite graph, Jellyfish graph $J\,(m, n)$, Flower graph $F\,(C_m, K_n)$, super subdivision of a graph, one point union of 2-regular graphs and sufficient condition for the degree splitting graph of a bipartite graph to exhibit PDL, have been investigated. Besides this, some general results on PDG have also been studied. By W.L.G, it means that "without loss of generality".

4.2 Some Results on Prime Distance Graphs

The PDL of some named graphs like diamond, bull, net, dart, house, house x, R, cricket, banner, paw, kite, and butterfly graphs are given along with definition (see Figures 4.1 to 4.14).

Definition 4.2.1. " A flower $F(C_s,\ K_t)$ is constructed by having a copy of C_s & s copies of K_t & connecting the i^{th} copy of K_t at the i^{th} lines of C_s".

"If s & t be two positive integers where $s\ \geq\ 3$ & $t\ \geq\ 3$, then the node set & the line set of $F(C_s, K_t)$ are defined as given below.

$V(F(C_s, K_t)\ )\ =\ \{u_i \mid i\ in\ [1, s]\} \cup \{u_j^{i\ ,i+1} \mid i\ in\ [1, s],\ j\ \in\ [1,\ t\ -\ 2],\ u_j^{s,s+1}\ =\ u_j^{s,1}\}$. $E(F(C_s, K_t))\ =\ \{u_i u_{i+1} \mid i\ \in\ [1, s],\ u_{s+1}\ =\ u_1\}\ \cup\ \{u_i u_j^{i,i+1} \mid \text{i}\ \in [1, s],\ j\ \in$

$[1,\ t-2]$, $u_j^{s\,s+1} = u_j^{s,1}\} \cup \{u_{i+1}u_j^{i,i+1}|$ i $in[1,s]$, $j\ in[1,\ t-2]$, $u_{s+1} = u_1$, $u_j^{s,s+1} = u_j^{s,1}\}$ $\cup \{u_j^{i,i+1}u_k^{i,i+1}|$ i $in[1,s]$, $j\ in[1,\ t\ -\ 2]$, $k\ in[1,\ t\ -\ 2]$, $j \neq k$, $u_j^{s,s+1} = u_j^{s,1}, u_k^{s,s+1} = u_k^{s,1}\}$".

Definition 4.2.2. The n-pan graph is formed by connecting C_n to K_1 with a cut edge.

Definition 4.2.3. The bull graph is a graph on 5 nodes and 5 lines as given in figure 4.3).

Definition 4.2.4. The butterfly graph is with 5 nodes and 6 lines and formed by connecting two copies of C_3 with a common node.(see Figure 4.13).

Definition 4.2.5. The cricket graph is the 5 -node graph (see Figure 4.10).

Definition 4.2.6. "The diamond graph is a planar graph with 4 nodes and 5 lines. It consists of K_4 minus one lines" (see Figure 4.2).

Definition 4.2.7. "H is a $(k,\ s)-$dart if every node of H of degree $\geq\ k+2$ is a central node of some $(k,\ i)-$diamond D as an induced subgraph of H with $2\ \leq\ i\ \leq\ s$, for which (1) $d_D(x)\ \geq\ d_H(x)\ -\ 1$ for each $x\ in\ V(D)$;(2) no 2 nodes of $C(D)$ have a common neighbor in $H\ -\ D$" (see Figure 4.5).

Definition 4.2.8. The house graph is on 5 nodes and 6 lines as given in figure 4.6.

Definition 4.2.9. "The house X-graph is the house graph plus the two lines connecting diagonally opposite nodes of the square base".(see Figure 4.7).

Definition 4.2.10. The net graph is the graph on 6-nodes (see Figure 4.4).

Definition 4.2.11. "The Jelly fish graph $J(a, b)$(see Figure 4.1) is derived from a C_4-x_1, x_2, x_3, x_4 by connecting x_1 and x_3 with a line & joining a pendent lines to x_2 and b pendent lines to x_4".

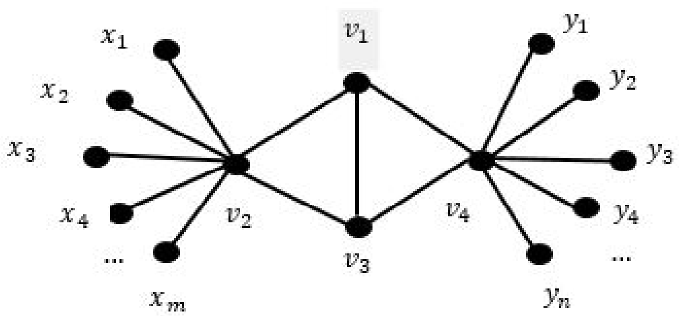

FIGURE 4.1: $J(m, n)$

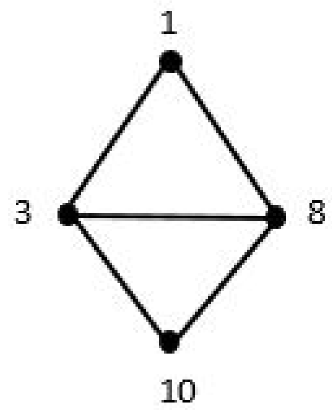

Figure 4.2: Diamond graph

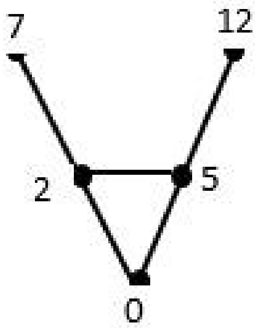

Figure 4.3: Bull graph

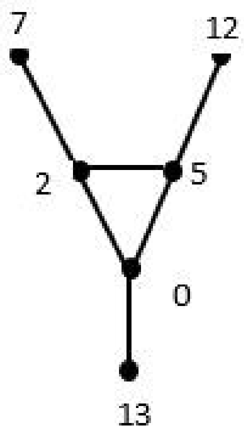

Figure 4.4: Net graph

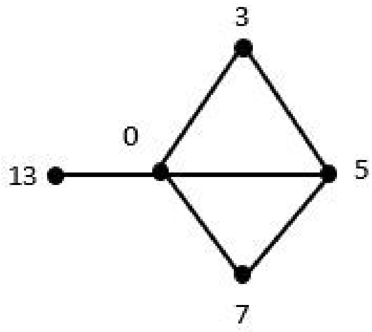

Figure 4.5: Dart graph

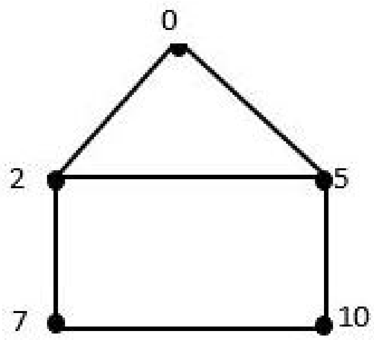

Figure 4.6: House graph

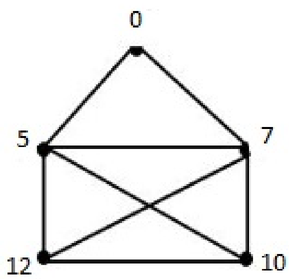

Figure 4.7: House X graph

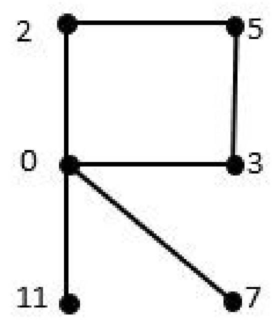

FIGURE 4.8: *R* graph

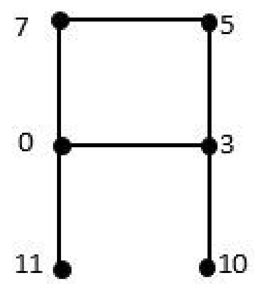

FIGURE 4.9: *A* graph

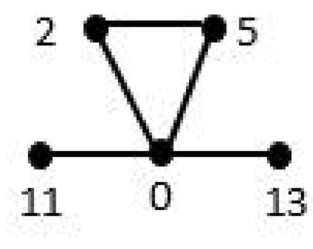

FIGURE 4.10: Cricket graph

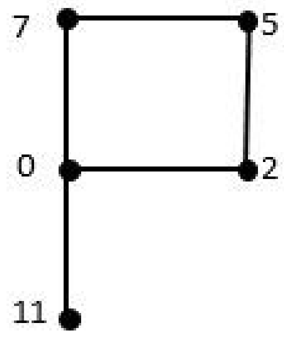

FIGURE 4.11: Banner graph

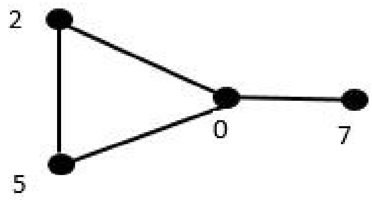

FIGURE 4.12: Paw graph

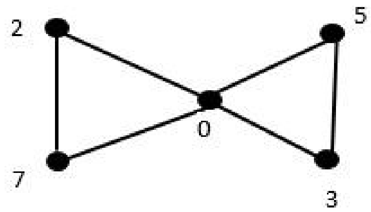

FIGURE 4.13: Butterfly graph

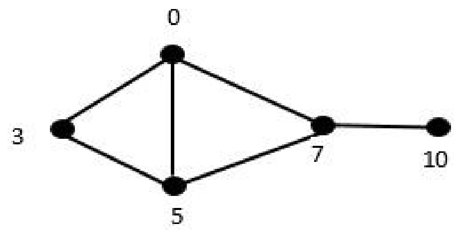

FIGURE 4.14: A kite graph

Theorem 4.2.1. *[33] Every bipartite graph admits PDL.*

Theorem 4.2.2. *G with $\Delta(G) > 2$ cannot have PDL with all its nodes labeled odd (even).*

Proof. Since $\Delta(G) > 2$, therefore $\exists\ v \in V$ such that $d(v) > 2$. If possible, let $f : V(G) \rightarrow Z$ be PDL of G. W.L.G, let $f(v) = m$, where$m$ is an odd integer. Let v_1, v_2, v_3 be adjacent to v. Then $|f(v_1) - m| = p_{1,}$ $|f(v_2) - m| = p_{2,}$ $|f(v_3) - m| = p_{3,}$ where p_1, p_2, p_3 are primes. Now one can observe that p_1, p_2, p_3 are all even primes and hence equal to 2. But there can be at most two odd numbers at a distance 2 from one odd number, a contradiction. So, therefore any graph G with $\Delta(G) > 2$ cannot have a prime distance labeling with all its nodes labelled odd. Similar argument holds good for other nodes of G. The possibility of assigning even labels to all the nodes of G is ruled out in a similar fashion. □

Observations 1: Only P_n can have PDL with all its nodes labelled odd or even.

Example 4.2.1. *The PDL of P_n is given in Figure 4.15.*

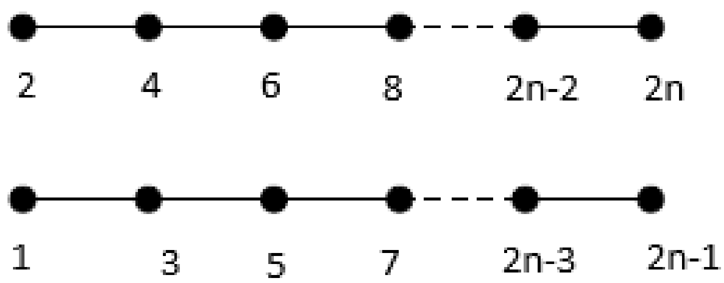

FIGURE 4.15: Prime distance labeling of P_n

Observation 2: C_n cannot have PDL with all its nodes labeled only odd (even).

Definition 4.2.12. [28] "The super subdivision of H_1, SS(H_1) is constructed from H_1 by replacing each link of H_1 by $K_{2,m}$, m a positive integer.

(For example see figure 4.16)

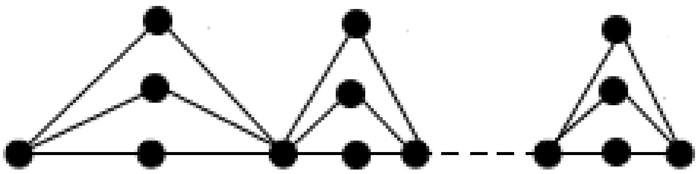

FIGURE 4.16: $\mathrm{SS}(P_n)$ by $K_{2,3}$

Definition 4.2.13. [28] Arbitrary super division $\mathrm{ASS}(H_1)$ is derived from H_1 by replacing each line kl of H_1 by K_{2,m_i}(m_i is any positive integer) by identifying k and l with the nodes x and y, respectively, where $\{x,\ y\}$ is a partition of K_{2,m_i}" (see Figure 4.17).

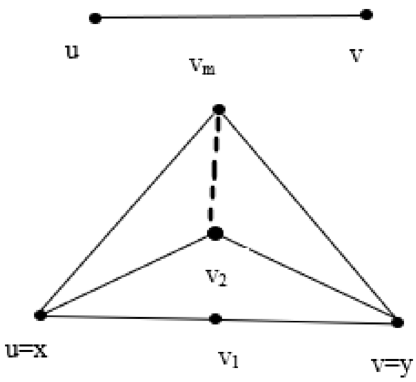

FIGURE 4.17: $\mathrm{ASS}(P_2)$

Lemma 4.2.1. *$SS(G)$ is bipartite.*

Proof. "Let the super subdivision of G is constructed by replacing each lines of G by $K_{2,m}$ and $H_1 = SS(G)$. Now, let $V(G) = \{x_1, x_2, \ldots,\ x_p\}$ & $\left\{w_i^1, w_i^2, \ldots,\ w_i^m\right\}$ be the nodes of H_1 corresponding to the i^{th} lines of G. Then $|V(H_1)| = p + mq$ & $|E(H_1)| = 2mq$. Further, if $W_1 = V(G), W_2 = \left\{w_i^j : 1 \le i \le q,\ \ 1 \le j \le m\ \right\}$, then $V(H) = W_1 \cup W_2$ is the required bipartition". □

Theorem 4.2.3. *$SS(G)$ is a PDG.*

Proof. The result follows from Lemma 4.2.1.and Theorem 4.2.1 □

Remark 4.1. Theorem 4.2.3 is true for arbitrary super subdivision of G also.

Plane Coloring Problem (PCP): "PCP looks for the least number of colours required to colour the plane (P) so that no 2 points at a unit distance from each other get the same colour. The answer to this problem is not known but $\chi(P)$ has been narrowed down to either one of 5, 6 or 7."

Theorem 4.2.4. *[45] G with $V(G) \subseteq Z$ and $\chi(G) \geq 5$ does not admit PDL.*

Theorem 4.2.5. *[45] The graph G whose node set consists of all points in the plane and there exists a line between two points if they are at a unit distance, cannot have PDL.*

Proof. The proof follows from the truth that $5 \leq \chi(G) \leq 7$ and Theorem 4.2.4. □

Remark 4.2. There may be some finite subgraphs of G defined in Theorem 4.2.5 which may or not admit PDL whose $\chi \leq 4$.

Theorem 4.2.6. *G obtained from taking a finite copies of a PDG, H and joining i^{th}-node of each copy of H by a line admits PDL.*

Proof. Since H is a PDG, there exists PDL $f : V(H) \rightarrow Z$ of H. Obtain G by taking m copies of H and connecting i^{th}-node of each copy of H by an lines. "Define a map $h_1 : V(G) \rightarrow Z$ as given below: $h_1(H_1) = f(H)$". Let k be the greatest label assigned by h_1 to H_1 and p_1 be a prime such that $p_1 > k$. Now assign labels to nodes of H_2 by adding p_1 to the labels of corresponding nodes of H_1. Let l be the greatest label assigned to H_2 and p_2 be a prime such that $p_2 > l$. Again, assign labels to nodes of H_3 by adding p_2 to the labels of corresponding nodes of H_2. Continuing in this way, one can get primes $p_3, p_4, \ldots, p_{m-1}$ such that each succeeding copy H_r of H is assigned the labels by adding p_{r-1} to the labels of corresponding nodes of preceding H_{r-1} of H. This induces PDL of G (see Figure 4.18).

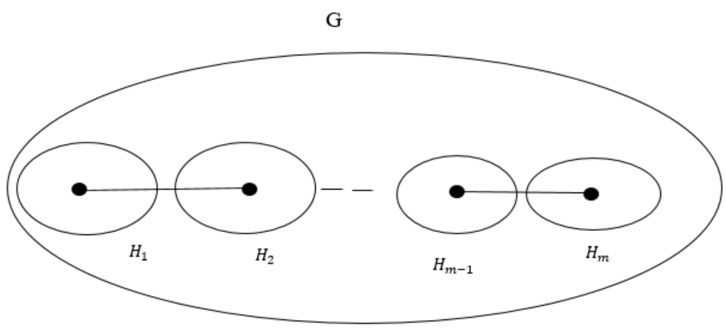

FIGURE 4.18: A graph obtained from taking a finite copies of H & connecting i^{th}-node of every copy of H by an lines

□

Lemma 4.2.2. *"One vertex union of finite copies of bipartite graph results in bipartite.*

Proof. Since the graphs are bipartite, therefore they do not contain any odd cycle. Moreover, one point union of these graphs again does not give rise to any odd cycle in them and so a bipartite graph. □

Theorem 4.2.7. *G formed by applying one point union of two 2-regular graphs admits PDL if Goldbach's Conjecture is true.*

Proof. If G_1 and G_2 are two 2-regular graphs, then there are three cases:

Case (1): When G_1 and G_2 both are bipartite.

The result follows from Lemma 4.2.2 and Theorem 4.2.1.

Case (2): When none of G_1 or G_2 is bipartite.

Let G_1 and G_2 be the given 2-regular graphs with node sets $V_1 = \{u_1, u_2, \ldots,\ u_m\}$ & $V_2 = \{v_1, v_2, \ldots, v_n\}$, respectively & G be the one point union of G_1 and G_2. W.L.G, let the nodes v_1 and u_1 be identified to obtain G. Since G_1 is a 2-regular graph and by using Theorem 4.2.1, there exists PDL, say f. Similarly, since G_2 is also a 2-regular graph, there exists PDL, say g. "A 1-1 map $h : V(G) \to Z$ is defined as below: W.L.G, let $h(u_1) = h(v_1) = 0$. Then $h(u_k) = 2(k-1)$, for $2 \leq k \leq m-1$". Now by using GC, $h(u_{m-1}) = p_1\ + p_2$ and $h(u_m) = p_1$. Similarly, let $h\left(v_j\right) = -2(j-1)$, for $2 \leq j \leq n-1$. Now again by Goldbach's Conjecture, $h\left(v_{n-1}\right) = -2\left(n-1\right) = -(p_3 + p_4)$ and $h(v_n) = -p_3$. Clearly, h is the PDL of G.

Case (3): When either G_1 or G_2 is not bipartite.

This case can be dealt with that of case 2. □

Definition 4.2.14. A unicyclic graph is a graph containing only one cycle.

Conjecture 4.2.1. *Any unicyclic graph admits PDL.*

Conjecture 4.2.2. *The flower graph $F\left(C_m,\ K_n\right)$ admits PDL for $m \geq 3$ and $n = 3$.*

Conjecture 4.2.3. *The flower graph $F\left(C_m,\ K_n\right)$ does not admit PDL for $m \geq 3$ and $n = 4$.*

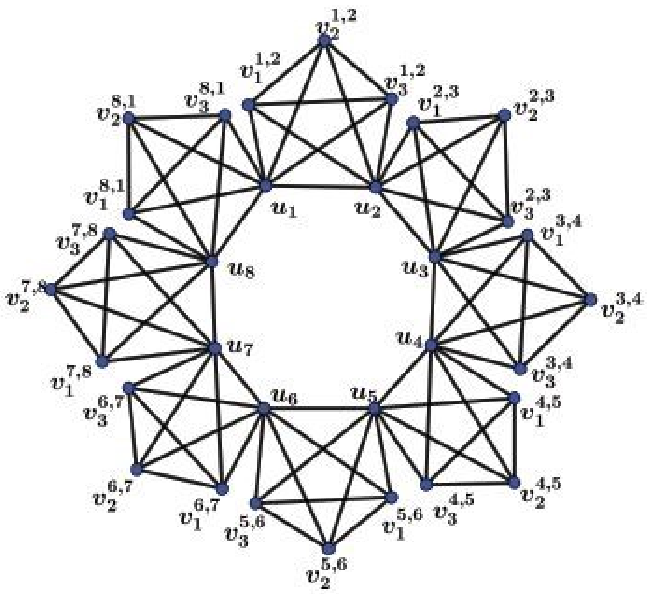

FIGURE 4.19: $F(C_8, K_5)$

Theorem 4.2.8. *[44] "If a subgraph H_1 of G_1 does not admit PDL, then so does G_1".*

Theorem 4.2.9. *[44] $K_n,\ n \geq 5$ does not permit PDL.*

Theorem 4.2.10. *The flower graph $F(C_m,\ K_n)$ does not admit PDL for $n \geq 5$ and $m \geq 3$.*

Proof. The result is clearly from Theorem 4.2.8 and Theorem 4.2.9. □

The PDL of $F(C_m,\ K_3)$ is given in support of Conjecture 4.2.2 (see Figure 4.20).

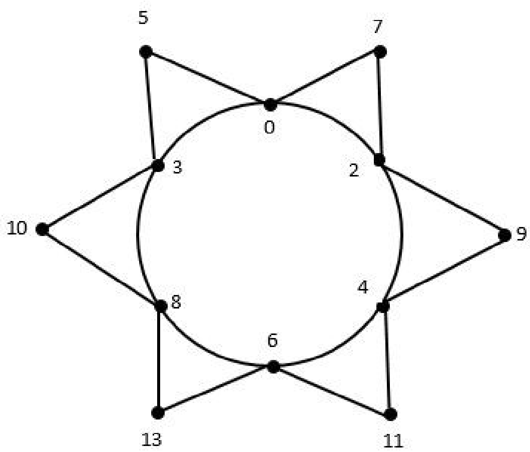

FIGURE 4.20: PDL of $F(C_6,\ K_3$

Theorem 4.2.11. *[44] "$J(m,n)$ admits PDL for any $m, n \geq 1$".*

Proof. Let $J(m,n)$ be on m pendent lines joining to v_2 and n pendent lines joining to v_4. "A 1-1 map $f : V(J(m,n)) \to Z$, is defined as given below: W.L.G, let $f(v_1) = -3$, $f(v_2) = 0$, $f(v_3) = 2$ and $f(v_4) = -1$". Let x_i be the pendant nodes of v_2 and y_i be the pendant nodes of v_4. Let p_1, p_2, p_3, $\cdots p_m$ be the distinct prime numbers other than the used primes. Label the pendant nodes which are joined with the node v_2 with those prime numbers. One can easily see that $|f(x_i) - f(v_2)| = p \in P$, for $1 \leq i \leq m$. Similarly, label the pendant nodes which are joined to the node v_4. Label the nodes y_i, with sufficiently suitable even number such that $|f(y_i) - f(v_4)| = p_i \in P$. This is possible as there are infinitely many even numbers whose difference with -1 is a prime number. Clearly f is the required PDL of $J(m,n)$ (see Figure 4.21).

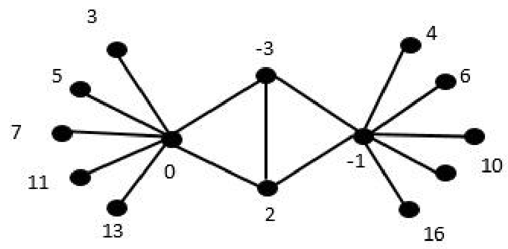

FIGURE 4.21: PDL of $J(5\ ,4)$

□

Definition 4.2.15. A Theta graph having 7- nodes and 8-lines is a block with 2 non-adjacent nodes of degree 3 and the remaining nodes of degree 2 (see Figure 4.22).

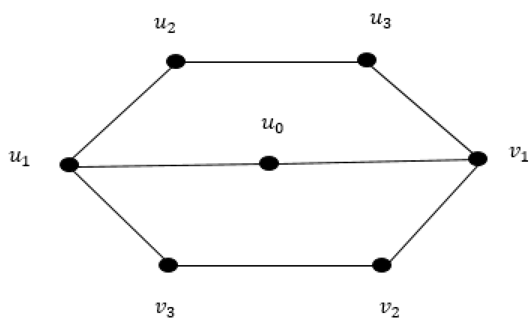

FIGURE 4.22: Theta graph

Theorem 4.2.12. *The Theta graph admits PDL.*

Proof. Let $V = \{u_0, u_1, u_2, u_3, v_1, v_2, v_3\}$ denote the node set of Theta graph with $d(v_1) = d(u_1) = 3$ and $d(v_i) = d(u_j) = 2,\ i = 2,\ 3$ and $j = 0,\ 2,\ 3$. "$g : V \to Z$ is defined as below: $g(v_i) = 2i + 1,\ i = 1,\ 2,\ 3$ and $g(u_j) = 2j,\ j = 0,\ 1,\ 2,\ 3$". Then f induces PDL of Theta graph. □

Theorem 4.2.13. *The sufficient condition for $DS(G)$ of a bipartite graph G to admit PDL is that there exists no pair of nodes x_i, y_j such that $d_G(x_i) \neq d_G(y_j)$, where $V = X_1 \cup X_2$ is the decomposition of V & $x_i \in X_1,\ y_j \in X_2$.*

Proof. Take a bipartite graph G with two partitions V_1 and V_2 of its node set. Take $W = V(\mathrm{DS}(G)) - V(G) = \{w_1,\ w_2, \ldots, w_t\}$. Since $d_G(x_i) \neq d_G(y_j),\ \forall\ x_i \in V_1,\ y_j \in V_2$, therefore it is obvious that no node from W is adjacent to nodes both from V_1 & V_2. Thus, assume that nodes from the set $W_1 = \{w_1, w_2, \ldots, w_s\}$ are adjacent to nodes in V_1 and nodes from the set $W_2 = \{w_{s+1}, w_{s+2}, \ldots, w_t\}$ are adjacent to nodes in V_2. This shows that $\mathrm{DS}(G)$ is bipartite with bipartition $(V_1 \cup W_1,\ \ V_2 \cup W_2)$. Hence in view of Theorem 4.2.1, $\mathrm{DS}(G)$ admits PDL. □

Corollary 4.2.1. *If $m \neq n$, then $DS(K_{m,n})$ is a PDG.*

Proof. Since $m \neq n$, therefore applying Theorem 4.2.13, one can find that $DS(K_{m,n})$ is bipartite & hence the proof from Theorem 4.2.1. □

4.3 PDL of Non-Commuting Graphs of Some Finite Non-Abelian Groups

Recall some important definitions and results to make this sub-section self explanatory.

Definition 4.3.1. "The centre of $(G,\ \ *)$, $Z(G) = \{a \in G : a * b = b * a, \forall b \in G\}$.

Definition 4.3.2. The non-commuting graph of $(G,\ \ *)$, $\Gamma(G)$ having $V = G - Z(G)$ & $E = \{e\ : e$ is lines between node pair (l, m) so that $l * m \neq m * l$, for $l, m \in V\}$.

Definition 4.3.3. Let $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. Define product on Q_8 by usual multiplication together with $i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ \ jk = -kj = i,\ ki = -ik = j$. Then Q_8 forms a non-abelian group called the Quaternion group.

Lemma 4.3.1. *[46] No triangular graph can possess PDL with all nodes labeled either odd integers or even integers.*

Definition 4.3.4. The set of all permutations on finite set $X = \{1,\ 2,\ 3, \ldots, n\},\ n \in N$ forms a group with respect to operation $'o'$ of composition of functions, known as symmetric group and denoted by $(S_n,\ o)$, for $n \in N$.

Theorem 4.3.1. *[46]* $\Gamma(S_3)$ *of* $(S_3,\ o)$ *does not permit PDL.*

Theorem 4.3.2. *[46]* $\Gamma(S_n),\ n \geq 4$ *of* $(S_n,\ o)$ *does not permit PDL.*

Definition 4.3.5. $D_{2n},\ n \in N$ is the group of symmetries of polygon of $n-$sides and is defined as $D_{2n} = \{< x, y > : x^n = e = y^2,\ \text{xy} = yx^{-1}\}$, where e is the identity element.

Corollary 4.3.1. *PDL of non-commuting graph of* D_6 *of order 6 does not exist.*

Proof. It is known that Dihedral group D_6 can be represented by: D_6 $=\{< x, y > : x^3 = e = y^2,\ \text{xy} = yx^{-1}\}$, and D_6 is isomorphic to symmetric group $(S_3,\ o)$. So, $\Gamma(D_6)$ is same as that of S_3. Hence, Theorem 4.3.1 implies that $\Gamma(D_6)$ does not admit PDL. □

Theorem 4.3.3. *PDL of* $\Gamma(D_8)$ *does not exist.*

Proof. "D_8 is also known as Octic group, which is represented by: $D_8 = \{ < x, y > : x^4 = e = y^2,\ xy = yx^{-1}\}$, where e is the identity element of $D_8 = \{e, x, x^2, x^3, y,\ xy,\ x^2y, x^3y\}$ and the centre of D_8, $Z(D_8) = \{e,\ x^2\}$ which imply that e and x^2 commute with each of the remaining elements of group D_8. Firstly, construct $\Gamma(D_8)$. Now, the node set V of graph $\Gamma(D_8)$ is given by $V = D_8 - Z(D_8) = \{x, x^3, y, xy, x^2y, x^3y\}$. Taking $v_1 = x,\ v_2 = x^3,\ v_3 = y,\ v_4 = xy,\ v_5 = x^2y,\ v_6 = x^3y,\ V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. To find the lines set E of $\Gamma(D_8)$, one needs to find the pair of elements of V which don't commute with each other. Since, $ea = ae, \forall a \in D_8,\ \ xy = yx^{-1} \neq yx \Rightarrow xy \neq yx,\ x^3y = yx^{-3} = yx^{-1} = xy \Rightarrow x^3y \neq yx^3,\ \ x(xy) = x^2y$ and $(xy)x = yx^{-1}x = y \neq x^2y \Rightarrow x(xy) \neq (xy)x,\ x^3(xy) = x^4y = ey = y$ and $(xy)x^3 = yx^{-1}x^3 = yx^2 = x^2y \neq y \Rightarrow x^3(xy) \neq (xy)x^3,\ x(x^2y) = x^3y$ and $(x^2y)x = yx^{-2}x = yx^{-1} = xy \neq x^3y \Rightarrow x(x^2y) \neq (x^2y)x,\ x^3(x^2y) = x^5y = exy = xy$ and $(x^2y)x^3 = yx^{-2}x^3 = yx = x^{-1}y \neq xy \Rightarrow x^3(x^2y) \neq (x^2y)x^3,\ \ x(x^3y) = x^4y = ey = y$ and $(x^3y)x = yx^{-3}x = yx^{-2} = x^2y \neq y \Rightarrow x(x^3y) \neq (x^3y)x,\ x^3(x^3y) = x^6y = ex^2y = x^2y$ and $(x^3y)x^3 = yx^{-3}x^3 = ye = y \neq x^2y \Rightarrow x^3(x^3y) \neq (x^3\ y)x^3$. Hence, the following are the pair of adjacent nodes: $(x, y), (x, xy), (x, x^2y), (x, x^3y), (x^3, y), (x^3, xy), (x^3, x^2y), (x^3, x^3y), (xy, y),$
$(x^3y, y),$
(x^2y, xy), and (x^3y, x^2y). Hence, $\Gamma(D_8)$ is shown in Figure 4.23:

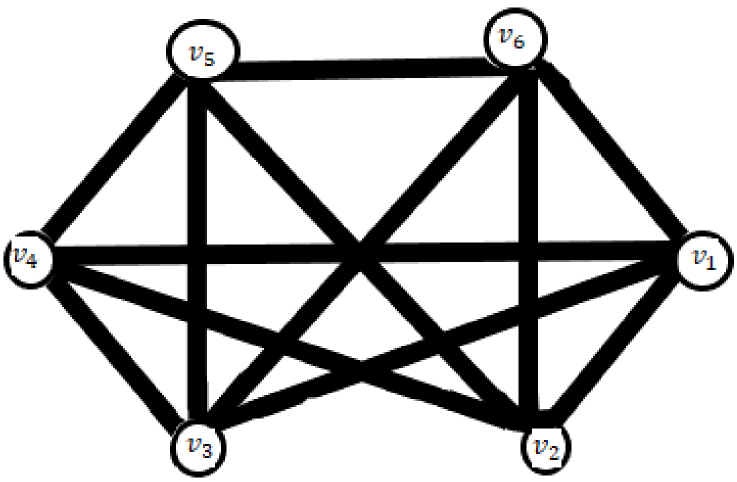

FIGURE 4.23: The Non-Commuting Graph

Suppose that there exists a function $\varphi : V(\Gamma(D_8)) \rightarrow \mathbb{Z}$ which determines PDL of $\Gamma(D_8)$, i.e. $|\varphi(u) - \varphi(v)|$ is prime for each pair of adjacent nodes (u, v) in $\Gamma(D_8)$. Applying Lemma 4.2.1 on $\Delta v_1v_2v_6$, if $\varphi(v_1), \varphi(v_2)$ is even, then $\varphi(v_6)$ must be odd. Say, $\varphi(v_1) = 2m$, $\varphi(v_6) = 2n + 1$, for $n, m \in \mathbb{Z}$ .....(I). Then, $\varphi(v_2) = 2m + 2$.....(II).

Also, $\Delta v_1v_2v_4$ implies that $\varphi(v_4)$ must be an odd integer, say $\varphi(v_4) = 2r + 1$, $r \in \mathbb{Z}$. Now in $\Delta v_5v_2v_4$, $\varphi(v_4)$ is odd and $\varphi(v_2)$ is even. So by Lemma 4.2.1, $\varphi(v_5)$ can be either even or odd. Thus the following 2 cases arise:

Case 1: When $\varphi(\mathbf{v_5})$ an even

Since (v_5, v_2) is a pair of adjacent nodes and $\varphi(v_2) = 2m+2$, so, $\varphi(v_5) = 2m+4$.....(III)

Now as v_3 is the common node of $\Delta v_3v_4v_5$, $\Delta v_1v_3v_6$, and$\Delta v_5v_3v_6$, it is obvious that $\varphi(v_3)$ can either be odd or even. If $\varphi(v_3)$ is even and from Figure 4.23, it is clear that the pairs of nodes (v_3, v_5)
textand (v_3, v_1) are connected by an lines. Since, $\varphi(v_5) = 2m+4$ and $\varphi(v_1) = 2m$, then $\varphi(v_3)$ must be selected in such a way that both terms $|\varphi(v_3) - \varphi(v_5)|$ and $|\varphi(v_3) - \varphi(v_1)|$ are prime numbers, in particular, the even prime 2. So, if $\varphi(v_5) = 2m + 4$, then $\varphi(v_3)$ should be the consecutive even integer i.e. $\varphi(v_3) = 2m + 2$ or $2m + 6$, but equation (II) implies that $\varphi(v_3) = 2m + 2$ and the PDL of nodes with integers is unique. So $\varphi(v_3) = 2m + 6$, which implies $|\varphi(v_3) - \varphi(v_1)| = 6$, which is not a prime. So, $\varphi(v_3)$ cannot be even and hence, it must be an odd integer. Also Figure 4.23 clears that (v_3, v_4) and (v_3, v_6) are pair of adjacent nodes. Combining the fact that $|\varphi(v_3) - \varphi(v_6)|$ and $|\varphi(v_3) - \varphi(v_4)|$ are primes, particularly even primes with the assumption that $\varphi(v_3)$, $\varphi(v_6)$ and $\varphi(v_4)$ all are odd numbers. So $(\phi(v_3), \phi(v_6))$ and $(\phi(v_3), \phi(v_4))$ must

be pair of consecutive odd numbers as the difference of any two consecutive odd integers is 2. So, one can re-write their values as $\phi(v_6) = 2n+1, \phi(v_3) = 2n+3$, and $\phi(v_4) = 2n+5$. Hence, the following labeling is attained in this case: $\phi(v_1) = 2m$, $\phi(v_2) = 2m+2$, $\phi(v_3) = 2n+3$, $\phi(v_4) = 2n+5$, $\phi(v_5) = 2m+4$ and $\phi(v_6) = 2n+1$. Since $\varphi : V(\Gamma(D_8)) \to \mathbb{Z}$ determines PDL on graph shown in Figure 4.23, so the following interpretations can be made:

$(\mathbf{v_i},\mathbf{v_j}) \to$	(v_1,v_3)	(v_1,v_4)	(v_1,v_6)	(v_2,v_4)
$\lvert\varphi(\mathbf{v_i})-\varphi(\mathbf{v_j})\rvert$	$P_1 = \lvert 2(m-n)-3\rvert$	$P_2 = \lvert 2(m-n)-5\rvert$	$P_3 = \lvert 2(m-n)-1\rvert$	$P_1 = \lvert 2(m-n)-3\rvert$

$(\mathbf{v_i},\mathbf{v_j}) \to$	(v_2,v_6)	(v_5,v_3)	(v_5,v_4)	(v_5,v_6)
$\lvert\varphi(\mathbf{v_i})-\varphi(\mathbf{v_j})\rvert$	$P_4 = \lvert 2(m-n)+1\rvert$	$P_4 = \lvert 2(m-n)+1\rvert$	$P_3 = \lvert 2(m-n)-1\rvert$	$P_5 = \lvert 2(m-n)+3\rvert$

Here, each P_i $(i = 1, 2, 3, 4, 5)$ is an odd prime. Let $2(m-n) = x \Rightarrow x$ is an even integer. So, the purpose is to find the even integer x for which each of $P_1 = |x-3|$, $P_2 = |x-5|$, $P_3 = |x-1|$, $P_4 = |x+1|$, $P_5 = |x+3|$ is an odd prime ..... (IV) It is obvious that $x \neq \pm 2, \pm 4, \pm 6$ otherwise at least one of $P_i = 1$ as defined in equation (IV), which contradicts the fact that each P_i is an odd prime. So, either $x > 6$ or $x < -6$. W.L.G, assume that $x > 6$, so the equation (IV) reduces to $P_1 = x-3$, $P_2 = x-5$, $P_3 = x-1$, $P_4 = x+1$, $P_5 = x+3$ ..... (V)

Firstly, the claim is that none of P_i can be 3 ..... (VI).

If $P_1 = 3$, then $x = 6$, which is a contradiction as $x > 6$. If $P_2 = 3$, then $x = 8 \Rightarrow P_4 = 9$, which is not a prime number. If $P_3 = 3$, then $x = 4$, which is a contradiction as $x > 6$. If $P_4 = 3$, then $x = 2$, which is again a contradiction as $x > 6$. If $P_5 = 3$, then $x = 0$ which implies that $P_4 = 1$, which is not a prime. So, it is clear that none of the P_i is 3. Further, equation (V) implies that $P_1 - P_2 = 2$, $P_3 - P_1 = 2$, $P_4 - P_3 = 2$, $P_5 - P_4 = 2$ which means that the pairs $(P_1, P_2), (P_1, P_3)$ are twin odd primes with P_1 as common and same way $(P_4, P_3), (P_4, P_5)$ are twin odd primes with P_4 as common. It is known that each odd prime is of the form either $4k+1$ or $4k+3$, k being a natural number. So, if $P_1 = 4k+1$, then $P_2 = 4k-1$, $P_3 = 4k+3$, $P_4 = 4k+1$, $P_5 = 4k+3$ or if $P_1 = 4k+3$, then $P_2 = 4k+1$, $P_3 = 4k+5$, $P_4 = 4k+7$, $P_5 = 4k+9$, for a fixed natural number k. So, either P_2, P_1, P_3 or P_2, P_1, P_3, P_4, P_5 will be the consecutive twin odd primes. Since every third odd number is divisible by 3, which means that no three successive odd numbers can be prime unless one of them is 3, which is impossible by assertion (VI). Hence, the supposition is wrong. So, $\varphi(v_5)$ cannot be an even integer and this case is rejected.

Case 2: When $\varphi(\mathbf{v_5})$ is odd

Assume that $\varphi(v_5)$ is an odd integer. Since (v_6, v_5) and (v_4, v_5) are adjacent nodes in Figure 4.23, so as proved in Case 1, $(\varphi(v_5), \varphi(v_6))$ and $(\varphi(v_5), \varphi(v_4))$ must be pair of consecutive odd numbers as the difference of any two consecutive odd integers is 2. So, one can write their values as $\varphi(v_6) = 2n+1, \varphi(v_5) = 2n+3$ and $\varphi(v_4) = 2n+5$, where n is any integer. Hence, $\Delta v_5 v_4 v_3$ implies that $\varphi(v_3)$ must be an even integer by Lemma 4.2.1. But (v_3, v_1) is an adjacent pair of nodes and $\varphi(v_1) = 2m$. So, $\varphi(v_3) = 2m-2$. Hence, the labeling of graph in Figure 4.23 is as follows: $\varphi(v_1) = 2m, \varphi(v_2) = 2m+2, \varphi(v_3) = 2m-2, \varphi(v_4) = 2n+5, \varphi(v_5) = 2n+3$, and $\varphi(v_6) = 2n+1$ since $\varphi : V(\Gamma(D_8)) \to \mathbb{Z}$ determines the PDL on graph shown in Figure 4.23. So, the following interpretations are made:

$(\mathbf{v_i},\mathbf{v_j}) \to$	(v_1,v_4)	(v_1,v_6)	(v_2,v_4)	(v_2,v_5)
$\lvert\varphi(\mathbf{v_i})-\varphi(\mathbf{v_j})\rvert$	$P_1= \lvert 2(m-n)-5\rvert$	$P_2= \lvert 2(m-n)-1\rvert$	$P_3= \lvert 2(m-n)-3\rvert$	$P_1= \lvert 2(m-n)-1\rvert$

$(\mathbf{v_i},\mathbf{v_j}) \to$	(v_2,v_6)	(v_3,v_4)	(v_3,v_5)	(v_3,v_6)
$\lvert\varphi(\mathbf{v_i})-\varphi(\mathbf{v_j})\rvert$	$P_4= \lvert 2(m-n)+1\rvert$	$P_4= \lvert 2(m-n)-7\rvert$	$P_3= \lvert 2(m-n)-5\rvert$	$P_5= \lvert 2(m-n)-3\rvert$

Here, each P_i $(i = 1, 2, 3, 4, 5)$ is an odd prime. Let $2(m-n) = x \Rightarrow x$ be an even integer. So, the purpose is to find that even integer x for which each of $P_1 = |x-5|, P_2 = |x-1|, P_3 = |x-3|, P_4 = |x+1|, P_5 = |x-7|$ is an odd prime. It is obvious that $x \neq \pm 2, \pm 4, \pm 6, \pm 8$ otherwise at least one of $P_i = 1$ as defined in equation (IV), which contradicts the fact that each P_i is an odd prime. So, either $x > 8$ or$x < -8$. W.L.G, assume that $x > 8$. So the equation (IV) reduces to $P_1 = x-5, P_2 = x-1, P_3 = x-3, P_4 = x+1, P_5 = x-7$.... (VII)

Again the claim is that none of P_i can be 3 or 5... (VIII)

If P_1= 3, then $x = 8$, which is a contradiction as $x > 8$. Also, if $P_1 = 5$, then x=10 and so P_2= 9, which is not a prime. If P_2=3 or 5, then $x = 4$ or 6 which is impossible as $x > 8$. If P_3=3 or 5, then $x = 6$ or 8, which is a contradiction as $x > 8$. If P_4= 3 or 5, then x=2 or 4, which is again a contradiction as x> 8. If P_5=3, then x= 10 or 12 which implies that P_2= 9 or P_3=9, which is not a prime. So, it is clear that none of P_i

can be 3 or 5. Further, the equation (VII) implies that P_1-P_5= 2, P_3-P_1= 2, P_2-P_3= 2, P_4-P_2= 2 which further imply that $(P_1, P_5), (P_1, P_3)$ and $(P_2, P_3), (P_4, P_2)$ are pair of twin odd primes with P_1 and P_2 as common odd primes, respectively. This condition is to the one obtained in Case 1. So, by the same argument as applied in Case 1, this case is also rejected. Hence, $\varphi(v_5)$ can neither be an odd nor an even integer. Hence, the supposition is wrong and so the conclusion is that the PDL of $\Gamma(D_8)$ does not exist". □

Theorem 4.3.4. $\Gamma(D_{2n})$ *does not permit PDL for* $n \geq 5$.

Proof. "Consider D_{2n}, where $n \geq 5$ given by: $D_{2n} = \{< x, y >: \ x^n = e = y^2,\ xy = yx^{-1}\}$ & $Z(D_{2n}) = \begin{cases} \{e\} & \text{, if } n \text{ is odd natural number} \\ \left\{e, x^{\frac{n}{2}}\right\} & \text{, if } n \text{ is even natural number} \end{cases}$

So if V denote the node set for $\Gamma(D_{2n})$, then $V = D_{2n} - Z(D_{2n})$, then V consists of non-commuting elements of group D_{2n}. Consider S = $\{ x,\ y\ ,\ xy\ ,\ x^2y\ , x^3y\}$. Note that order of $x = n$ is atleast 5. So, $xy = yx^{-1} \neq yx$

$x(xy) = x^2y$, $(xy)x = yxx^{-1} = ye = y \Rightarrow x(xy) \neq (xy)x$

$x(x^2y) = x^3y$ and $(x^2y)x = yx^{-2}x = yx^{-1} = \text{xy} \neq x^3y \Rightarrow x(x^2y) \neq (x^2y)x$

$x(x^3y) = x^4y$ and $(x^3y)x = yx^{-3}x = yx^{-2} = x^2y \neq x^4y \Rightarrow x(x^3y) \neq (x^3y)x$

$y(xy) = x^{-1}yy = x^{-1}e = x^{-1}$ and $(xy)y = x \neq x^{-1} \Rightarrow y(xy) \neq (xy)y$

$y(x^2y) = yyx^{-2} = y^2x^{-2} = ex^{-2} = x^{-2}$ and $(x^2y)y = x^2y^2 = x^2e = x^2 \neq x^{-2} \Rightarrow y(x^2y) \neq (x^2y)y$

$(\text{xy})(x^2y) = x(yy)x^{-2} = xex^{-2} = x^{-1}$ and $(x^2y)(xy) = x^2yyx^{-1} = x \neq x^{-1} \Rightarrow (xy)(x^2y) \neq (x^2y)(xy)$

$(xy)(x^3y) = xyyx^{-3} = x^{-2}$ and $(x^3y)(xy) = x^3yyx^{-1} = x^2 \neq x^{-2} \Rightarrow (xy)(x^3y) \neq (x^3y)(xy)$

$(x^2y)(x^3y) = x^2yyx^{-3} = x^{-1}$ and $(x^3y)(x^2y) = x^3yyx^{-2} = x \neq x^{-1} \Rightarrow (x^2y)(x^3y) \neq (x^3y)(x^2y)$

But $y(x^3y) = yyx^{-3} = y^2x^{-3} = ex^{-3} = x^{-3}$ and $(x^3y)y = x^3y^2 = x^3e = x^3 = x^{-3}$ if $n = 6$ ...(I)

So, (x^3y) commute with y D_{12} whereas the above equations show that the pairs $(x, y), (x, xy), (x, x^2y), (x, x^3y), (y, xy), (y, x^2y), (xy, x^2y), (xy\, x^3y), (x^2y,\ x^3y)$ are non-commuting in each D_{2n}, for $n \geq 5$. The proof is divided into the following two cases:

Case 1: When n an odd

Since $n \geq 5$ & n is odd, so $n \neq 6$. By (I), $x^3 \neq x^{-3}$ and $\left(x^3y\right)$ does not commute with y.

So, in $\Gamma\left(D_{2n}\right)$, the following are the minimum pairs of adjacent nodes:

$(x,y)\,,(x,xy)\,,(x,x^2y),(x,x^3y)\,,(y,xy)\,,\ (y,x^2y)\,,(xy,x^2y)\,,(xy\ ,x^3y)\,,(x^2y,x^3y)\,,(y,x^3y)\,.$
And S $= \{\ x,\ y,\ xy,\ x^2y,\ x^3y\} \subseteq V$. Set $x = v_1,\ y = v_2,\ xy = v_3,\ x^2y = v_4,\ x^3y = v_5$. Hence, in this case $\Gamma\left(D_{2n}\right)$ has the following form:

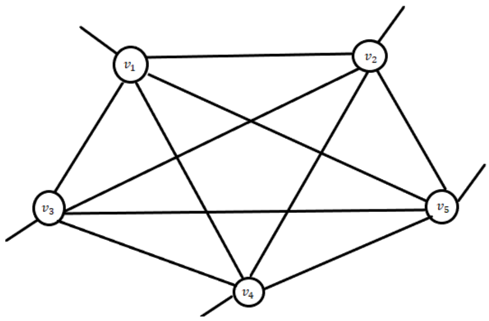

FIGURE 4.24: Subgraph of Non-Commuting graph $\Gamma\left(D_{2n}\right)$

Clearly, the χ of the graph shown in Figure 4.24 is 5. Since it is subgraph of $\Gamma\left(D_{2n}\right)$, it implies that $\chi(\Gamma\left(D_{2n}\right))$ is ≥ 5. As it is known that, the PDL of any graph with chromatic number 5 is not possible, the conclusion is that for any odd number $n \geq 5$, the PDL of non-commuting graph of D_{2n}, is not possible.

Case 2: When n an even

Since $n \geq 5$ & even, so n is at least 6. Also, equation (I) implies that $x^3 = x^{-3}$ if $n = 6$, so $\left(x^3y\right)$ commutes with y in group D_{12}. So, the prove is for D_{12} separately. It is known that, $D_{12} = \{e,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ y,\ xy,\ x^2y,\ x^3y,\ x^4y,\ x^5y\}$. Consider S $= \left\{x,\ y,\ xy,\ x^2y,\ x^3y,\ x^4y,\ x^5y\right\} \subseteq V$, where V is set of nodes of $\Gamma(D_{12})$. Then,
$x\left(x^4y\right) = x^5y$ and $\left(x^4y\right)x = x^4x^{-1}y = x^3y \neq x^5y \Rightarrow x(x^4y) \neq \left(x^4y\right)x$

$x\left(x^5y\right) = x^6y = ey = y$ and $\left(x^5y\right)x = x^5x^{-1}y = x^4y \neq y \Rightarrow x(x^5y) \neq \left(x^5y\right)x$

$y\left(x^4y\right) = x^{-4}yy = x^{-4} = x^2$ and $\left(x^4y\right)y = x^4 \neq x^2 \Rightarrow y(x^4y) \neq \left(x^4y\right)y$

$y\left(x^5y\right) = x^{-5}yy = x^{-5}e = x^{-5} = x$ and $\left(x^5y\right)y = x^5 \neq x \Rightarrow y(x^5y) \neq \left(x^5y\right)y$

$\left(x^2y\right)\left(x^4y\right) = x^2yyx^{-4} = x^{-2} = x^4$ and $\left(x^4y\right)\left(x^2y\right) = x^4yyx^{-2} = x^2 \neq x^4 \Rightarrow \left(x^2y\right)\left(x^4y\right) \neq \left(x^4y\right)\left(x^2y\right)$

$(xy)\left(x^5y\right) = xyyx^{-5} = x^{-4} = x^2$ and $\left(x^5y\right)(xy) = x^5yyx^{-1} = x^4 \neq x^2 \Rightarrow (xy)\left(x^5y\right) \neq \left(x^5y\right)(xy)$

$(x^5y)(x^4y) = x^5yyx^{-4} = x$ and $(x^4y)(x^5y) = x^4yyx^{-5} = x^{-1} = x^5 \neq x \Rightarrow (x^5y)(x^4y) \neq (x^4y)(x^5y)$.

Hence, in $\Gamma(D_{12})$, the following are minimum pairs of adjacent nodes: "$(x,y), (x,xy), (x,x^2y)$, $(x,x^4y), (x,x^5y), (y,xy), (y,x^2y), (y,x^4y), (y,x^5y)$,
$(xy,x^2y), (xy, x^5y), (x^2y,x^4y), (x^5y, x^4y)$". By setting $x = v_1$, $y = v_2$, $xy = v_3$, $x^2y = v_4$, $x^4y = v_5$, $x^5y = v_6$, the following graph is obtained as a subgraph of $\Gamma(D_{12})$.

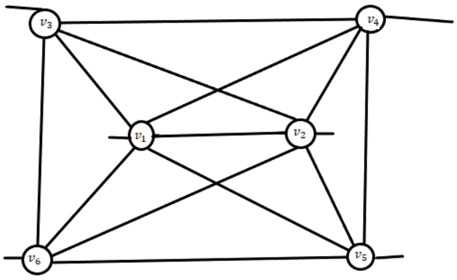

FIGURE 4.25: A subrgraph of non-Commmuting graph $\Gamma(D_{12})$

Suppose that there exist $\varphi : V(\Gamma(D_{12})) \to \mathbb{Z}$ which determines PDL so that $|\varphi(u) - \varphi(v)|$ is an odd prime for each pair (u,v) of adjacent nodes in $\Gamma(D_{12})$. Then it is clear from Figure 4.25 and by Lemma 4.2.1, that in $\Delta v_1v_2v_3$, for any two of three nodes should be labeled with either an odd or even integer. Assuming $\varphi(v_1) = 2n$, $\varphi(v_2) = 2n+2$, $n \in \mathbb{Z}$, then $\varphi(v_3)$ must be odd integer, say $\varphi(v_3) = 2m+1$, $m \in \mathbb{Z}$ ... (II)

So, from $\Delta v_1v_2v_4$, one can get that $\varphi(v_4)$ must be an odd integer and also (v_3,v_4) are adjacent nodes which imply that $\varphi(v_4) = 2m+3$ ... (III)

Proceeding in a similar way, from $\Delta v_1v_2v_6$ and $\Delta v_1v_2v_5$, using the fact that $(v_3,v_6), (v_5,v_4)$ are adjacent nodes, one can get $\varphi(v_6)$ and $\varphi(v_5)$ must be odd integers and so, $|\varphi(v_6) - \varphi(v_3)|$ and $|\varphi(v_4) - \varphi(v_5)|$ are even primes. It implies that $(\varphi(v_6), \varphi(v_3))$ and $(\varphi(v_4), \varphi(v_5))$ must be pair of consecutive odd integers ... (IV)

But (v_5,v_6) is a pair of adjacent nodes and $|\varphi(v_6) - \varphi(v_5)|$ is an even prime (difference of any two odd integers is even) if and only if $\varphi(v_6)$ and $\varphi(v_5)$ are consecutive odd integers ... (V)

Combining statements (II), (III), (IV) and (V), one can see that $(\varphi(v_6), \varphi(v_3))$, $(\varphi(v_4), \varphi(v_5))$, $((\varphi(v_6$ are pairs of consecutive odd integers. Again (II) and (III) imply that, $\varphi(v_5) = 2m+5$, $\varphi(v_6) = 2m-1$ or $2m+7$. So, $|\varphi(v_6) - \varphi(v_5)| = 6$ or $|\varphi(v_6) - \varphi(v_3)| = 6$, a contradiction. Hence, it is not possible to label the subgraph as shown in Figure 4.25 with $\varphi : V(\Gamma(D_{12})) \to \mathbb{Z}$ so that $|\varphi(u) - \varphi(v)|$ is an odd prime for each pair (u,v)

of adjacent verices in Figure 4.25. So, if a subgraph of $\Gamma(D_{12})$ does not admit PDL, then $\Gamma(D_{12})$ does not possess PDL. Hence, the PDL of the non-commuting graph of D_{2n}, for $n = 6$ is not possible. Further, the left thing is to prove the statement for even integers $n > 6$. Thus $n \geq 10$ and n is even integer. So equation (I) implies that $y(x^3y) = yyx^{-3} = y^2x^{-3} = ex^{-3} = x^{-3}$ and $(x^3y)y = x^3y^2 = x^3e = x^3 \neq x^{-3}$ "for $n = 10, 12 \ldots$. So, (x^3y) commute with y in D_{2n} for even integers $n \geq 10$. Hence, $(x,y), (x,xy), (x,x^2y), (x,x^3y), (y,xy), (y,x^2y), (xy,x^2y), (xy,x^3y), (x^2y,x^3y), (y,x^3y)$ are non-commuting in each D_{2n}, for even integers $n \geq 10$" and $S = \{x,\ y,\ xy,\ x^2y,\ x^3y\} \subseteq V$. Set $x = v_1, y = v_2, xy = v_3, x^2y = v_4, x^3y = v_5$. Hence, in this case the non-commuting graph $\Gamma(D_{2n})$ for even integers $n \geq 10$ has the following form (see Figure 4.26):

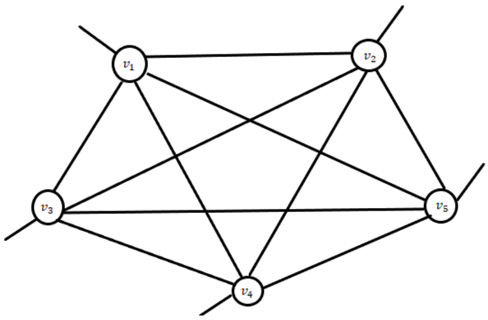

FIGURE 4.26: A subgraph of $\Gamma(D_{2n})$ for even integers $n \geq 10$

This is similar to Case 1. As stated in Case 1, the chromatic number of the graph shown in Figure 4.26 is 5. Since it is a subgraph of $\Gamma(D_{2n})$, for even integers $n \geq 10$, it implies that $\chi(\Gamma(D_{2n}))$ is ≥ 5. As it is known that, the PDL of any graph with chromatic number 5 is not possible. So, one can conclude that for any even integers $n \geq 10$, the PDL of $\Gamma(D_{2n})$, is not possible. Hence, combining both Cases 1 and 2, one can conclude the PDL of the $\Gamma(D_{2n})$, for $n \geq 5$, does not exist". □

Definition 4.3.6. Define product on Q_8 by usual multiplication together with $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Then Q_8 forms a non-abelian group called the Quaternion group.

Lemma 4.3.2. *$\Gamma(D_8)$ and $\Gamma(Q_8)$ are isomorphic.*

Proof. $D_8 = \{< x,\ y >:\ x^4 = e = y^2,\ xy = yx^{-1}\}$, where "$e$ is the identity element" of D_8 i.e. $D_8 = \{e,\ x,\ x^2, x^3,\ y,\ xy,\ x^2y,\ x^3y\}$ and centre of D_8 is $Z(D_8) = \{e,\ x^2\}$. Therefore $V(\Gamma(D_8)) = \{x,\ x^3,\ y,\ xy,\ x^2y,\ x^3y\}$. Let $v_1 = x, v_2 = x^3, v_3 = y,\ v_4 = xy,\ v_5 = x^2y,\ v_6 = x^3y$ so that $(\Gamma(D_8)) = \{v_1, v_2,\ v_3,\ v_4, v_5,\ v_6\}$. It is easy to check that (v_1, v_2), $(v_1,\ v_6)$, (v_1, v_3), $(v_1,\ v_4)$, $(v_2,\ v_6)$,

(v_2, v_4), (v_2, v_5), (v_3, v_4), (v_3, v_5), (v_3, v_6), (v_4, v_5), (v_5, v_6) are the pairs of adjacent nodes. Therefore, $\Gamma(D_8)$ is as given below (see Figure 4.27):

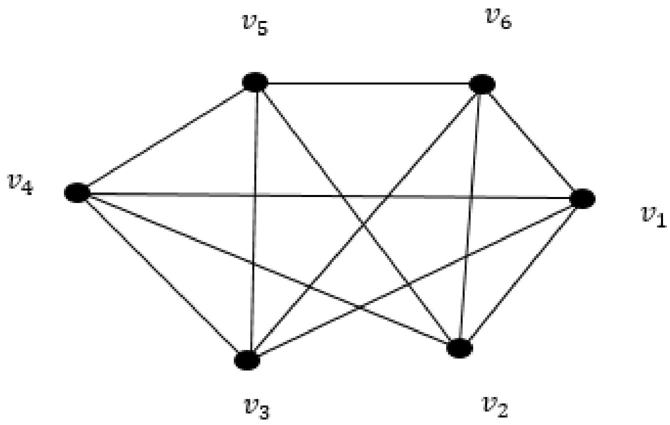

FIGURE 4.27: $\Gamma(D_8)$

Now the Quaternion group is given by $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and $Z(Q_8) = \{1, -1\}$. Therefore, $V(\Gamma(Q_8)) = \{i, -i, j, -j, k, -k\}$. Let$u_1 = i$, $u_2 = j$, $u_3 = -j$, $u_4 = k$, $u_5 = -i$, $u_6 = -k$ so that "$V(\Gamma(Q_8)) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Then (u_1, u_2), (u_1, u_3), (u_1, u_4), (u_1, u_6), (u_2, u_4), (u_2, u_5), (u_2, u_6), (u_3, u_4), (u_3, u_5), (u_3, u_6), (u_4, u_5), (u_5, u_4) are the pairs of adjacent nodes". Therefore, the non-commuting graph of Q_8 is as given below (see Figure 4.28):

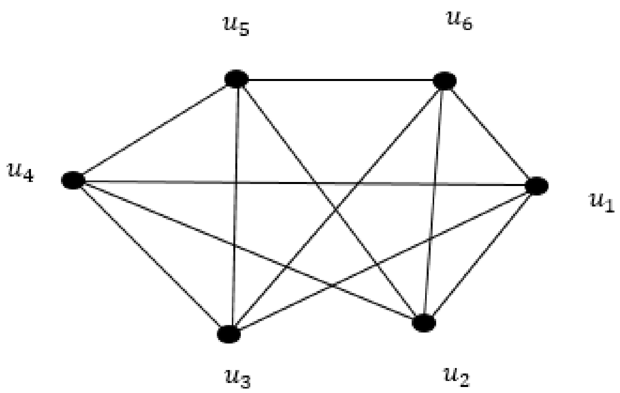

FIGURE 4.28: $\Gamma(Q_8)$

Now, the correspondence $\Psi(v_i) = u_i,\ i = 1,\ 2,\ 3,\ 4,\ 5,\ 6$ induces an isomorphism between $\Gamma(D_8)$ and $\Gamma(Q_8)$. □

Theorem 4.3.5. *PDL of* $\Gamma(Q_8)$ *does not exist.*

Proof. Since $\Gamma(D_8)$ and $\Gamma(Q_8)$ are isomorphic by Lemma 4.3.2 and the PDL of $\Gamma(D_8)$ does not exist in view of the Theorem 4.3.3. So one can conclude that the PDL of $\Gamma(Q_8)$ also does not exist. □

Definition 4.3.7. A pizza graph on $2n+1$ nodes, Pz_n, is formed from a $S(W_n)$ in each of its spokes.

The PDL of Pz_{11} is given in Figure 4.29.

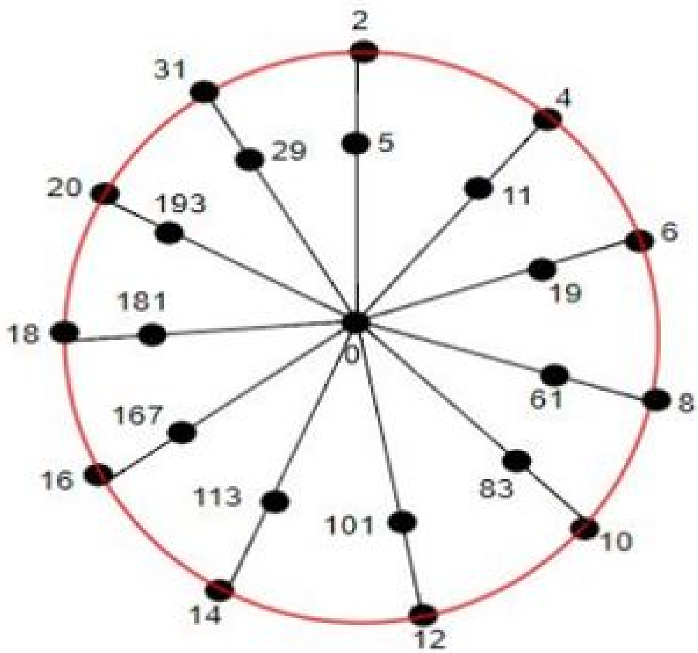

FIGURE 4.29: PDL of pizza graph Pz_{11}

Definition 4.3.8. The jewel graph J_k is defined by $V(J_k) = \{x, y, s, t,\ w_j : 1 \le j \le k\}$ and $E(J_k) = \{xy_i, yy_i, xs, ys, yt,\ xt, st : 1 \le j \le k\}$ (see Figure 4.30).

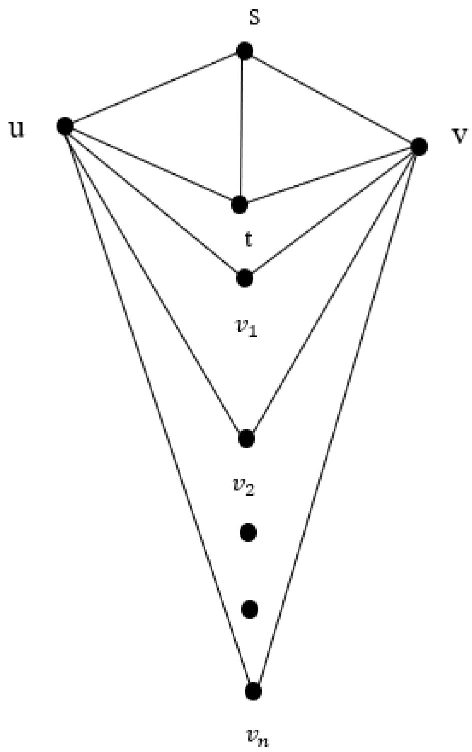

FIGURE 4.30: Jewel graph J_k

Definition 4.3.9. "A fan $F_{(1,k)} = K_1 \Lambda P_k$."

Definition 4.3.10. "If L_r is a ladder with $V(L_r) = \{v_j, u_j : j = 1, 2, 3, \ldots, r\}$ and one can add links $u_i v_{i+1}$, $i = 1, 2, 3, \ldots, r-1$ to L_r & delete u_r which is incident to both links $u_{r-1}u_r$ and $u_r v_r$, then by deleting the node, one gets a TL_r. The diamond graph, Br_k, $k \geq 3$, is formed by connecting a single point y to all the points v_j, $j = 1, 2, \ldots, k$ of TL_k".

Example 4.3.1. *Br_5 is given in Figure 4.31.*

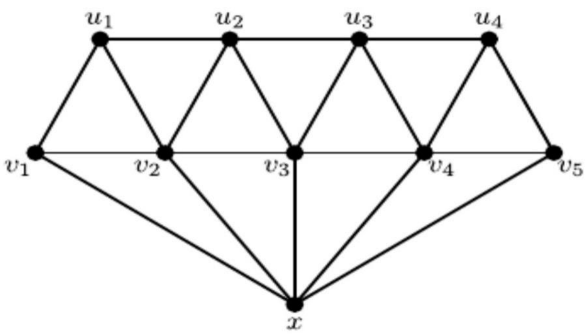

FIGURE 4.31: Diamond graph Br_5

Theorem 4.3.6. *The pizza graph Pz_n permits PDL $\forall n \geq 3$ if TPC is true.*

Proof. Take Pz_n be with $V(Pz_n) = \{u, v_k, w_k : 1 \le k \le n\}$, and $E(Pz_n) = \{uv_k,\ v_kw_k,\ w_kw_{k+1},\ 1 \le k \le n\}$, so $V(Pz_n) = 2n+1$ and $v_{n+1} = v_1$, $w_{n+1} = w_1$. "A 1-1 function $w : V(Pz_n) \to Z$ is defined as below: W.L.G, let $w(u) = 0$, $w(w_i) = 2i;\ 1 \le i \le n - 1$". Now choose sufficiently large prime p_α and twin primes (p_1, p_2) (with $p_1 < p_2$) so that $w(w_n) = w(w_{n-1}) + p_\alpha = p_2$ with a condition that $|w(w_{n-1}) - w(w_n)|$ is a prime. Then $w(v_n) = p_1$. Also, let $w(v_i) = w(w_i) + p_i$, where $p_i^{'}s$ are sufficiently large primes. One can check that Pz_n permits PDL $\forall n \ge 3$. □

Theorem 4.3.7. *J_n admits PDL, $\forall n \ge 1$.*

Proof. Let J_n be on $n + 4$ nodes with $V(J_n) = \{u, v, s, t,\ v_i : 1 \le i \le n\}$. Let $E(J_n) = E_1 \cup E_2$, where $E_1 = \{e_p,\ e_q,\ e_r,\ e_s,\ e_t\}$ and $E_2 = \{e_{i1},\ e_{i2} : 1 \le \text{i} \le n\}$. Now define an injective function $y : V(J_n) \to Z$ as follows: W.L.G, let $(y(u) = 1,\ y(v) = -1,\ y(s) = y(u)+3$, and $y(t) = y(u)+5$. Next $y(v_i) = 2r$; $1 \le i \le n,\ r \in N,\ 2r > y(t)$ in such a way that $|y(u) - y(v_i)| = p_{k+1}$ and $|y(v) - y(v_i)| = p_{k+2};\ 1 \le i \le n$, p_{k+1} and p_{k+2} are twin primes. Similarly, $y(v_i) = -y(v_i) : n + 1 \le \text{i} \le 2n$. Clearly, y is PDL of J_n. □

Proposition 4.3.1. *$F(1, n)$ for $n \ge 11$ admits no PDL.*

Theorem 4.3.8. *The diamond graph Br_n does not admit PDL for $n \ge 11$.*

Proof. The proof is obvious from Proposition 4.3.1. □

Definition 4.3.11. " $B_k^{(3)}$ for $k \ge 1$ is a planar graph on $k + 2$ points $x,\ y,\ y_1, y_2, \ldots y_k$ & $2k + 1$ links obtained by k times $C_3's$ sharing a common link $(y,\ x)$"

Definition 4.3.12. The triangular book with book marks is a triangular book $B_n^{(3)}$ with a finite number of pendant lines attached at any one of the end nodes of the spine.

Theorem 4.3.9. *$B_n^{(3)}$ admits PDL for all $n \in Z^+$ iff the TPC is true.*

Corollary 4.3.2. *The triangular book graph $B_n^{(3)}$ with any finite number of book marks admits PDL iff TPC is true.*

Proof. The labeling for the triangular book graph is done as it is given in Theorem 4.3.9. Now W.L.G, the book mark nodes, say $w_i\ i \ge 1$ are attached with u. The nodes w_i can be labeled with unused sufficiently large even numbers, say $2t_i$ where $t_i \in N$ such that $2t_i - 1$ is a prime. Hence the proof. □

Definition 4.3.13. [29] "The generalized Jahangir graph $J_{m,k}$, for $m \ge 3,\ k \ge 1$, consisting of $C_{m(k+1)}$ with an extra node which is adjacent to m nodes of $C_{m(k+1)}$ at distance $k + 1$ on $C_{m(k+1)}$".

Theorem 4.3.10. *The Jahangir graph $J_{m,k}$ permits PDL $\forall m \geq 3,\ k \geq 1$.*

Proof. If $J_{m,k}$ is with $V\ (J_{m,k}) = \{u\} \cup \{u_1, \ldots, u_m\} \cup \{v_1^i\ , v_i^2\ , \ldots, v_i^m\ ; i = 1,\ ...,\ m\}$, then there arises two cases:

Case 1: If k an even.

Here $J_{m,k}$ is bipartite and hence the proof.

Case 2: If k an odd.

A 1-1 map $f\ :\ V\ (J_{m,k}) \rightarrow Z$ is given below: W.L.G, let $f(u) = 0,\ f(u_1) = p_1,\ f(v_1^1\ ) = f(u_1) + 2$ and $f(v_j^1\ ) = f(v_{j-1}^1\ ) + 2$, where $2 \leq j \leq k - 1$ and p_1 is a prime. Next, $f(u_2) = p_2$ so that $|f(v_{k-1}^1\ ) - f(u_2)| = p$, where $p \in P$. Again, $f\,(u_2) = p_2,\ f(v_1^2\ ) = f(u_2) + 2$ and $f(v_j^2\ ) = f(v_{j-1}^2\ ) + 2$, where $2 \leq j \leq k - 1$ and p_2 is prime. Next, $f(u_3) = p_3$ such that $|f(v_{k-1}^2\ ) - f(u_3)| = p$, where $p \in P$. Thus continuing the process up to the node u_m one can verify that f is the required PDL of $J_{m,k}$.

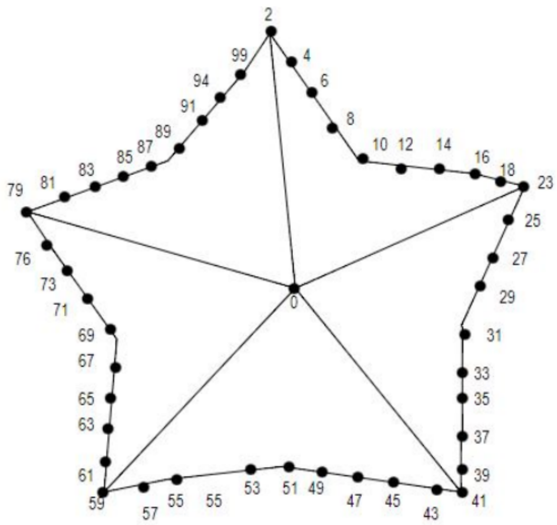

FIGURE 4.32: PDL of $J_{5,9}$

□

4.4 Conclusion

In this chapter some general results on PDL are obtained and the non-existence of PDL of non-commuting graphs of Dihedral groups $D_{2n},\ n \geq 3$ and Quaternion group Q_8 are also established. Besides this, the sufficient condition for the existence of PDL of degree splitting graph of a bipartite graph has also been obtained.

Chapter 5

Prime Distance Labeling in the Context of Extension and Barycentric Subdivision

5.1 Introduction

In this chapter, certain results on PDL in the context of duplication and extension of nodes have been investigated. Further PDL of simple graphs in the context of barycentric sub-division has been completely characterized.

5.2 Results on PDL in the Context of Duplication and Extension Operations

Theorem 5.2.1. *[43] If G is obtained from $C_n = \{u_1, u_2, \ldots, u_n, u_1\}$, $n \geq 6$ by duplicating an arbitrary node by a node, then G admits PDL if GC is true.*

Theorem 5.2.2. *Let G_n be obtained from C_n by duplicating an arbitrary node by a node. Then G_n admits PDL for all $n \geq 6$ if and only if GC is true.*

Proof. Let C_n be with $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $n \geq 6$. Obtain G_n by performing duplication of node v_n by a node $v_n^{'}$ as given in Figure 5.1. So, $V(G_n) = V(C_n) \cup \{v_n^{'}\}$, where $N(v_n^{'}) = N(v_n) = \{v_{n-1}, v_1\}$. First assume that G_n has a PDL for a +ve integer $n \geq 6$, & take one such PDL of G_n. W.L.G, assume that $v_r = 2(r-1)$ for $1 \leq r \leq n-1$. Note that the remaining nodes v_n and $v_n^{'}$ cannot be of even labels (If v_n or $v_n^{'}$ is assigned -2, then it gives a contradiction). Moreover, one cannot give an arbitrary odd number to v_n or $v_n^{'}$ as they must be of odd primes only. So the labels of v_n and $v_n^{'}$ must be odd primes and whose sum is equal to the label of v_{n-1}. Therefore, if all $G_n,\ n \geq 6$ are PDGs, then the GC is true. Conversely, if the GC is true, then "define $g : V(G_n) \rightarrow Z^+$ as given below: $g(v_k) = 2(k-1)$ for $1 \leq k \leq n-1$". Now $g(v_{n-1})$ can be expressed as $g(v_{n-1}) = p_1 + p_2$. Now letting $g(v_n) = p_1$ and $g(v_n^{'}) = p_2$ implies that g is a PDL of G_n.

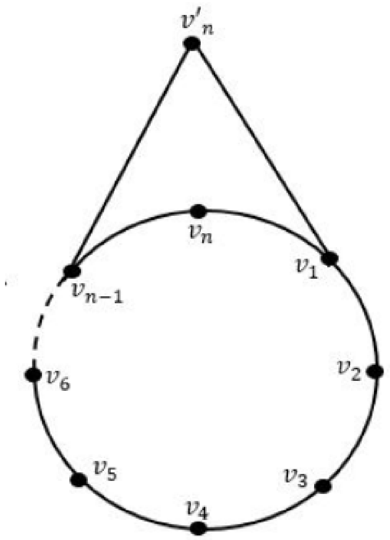

FIGURE 5.1: G derived from C_n by duplicating v_n by $v_{n'}$.

□

Theorem 5.2.3. *"The graph formed by applying duplication of a point by a line at all points in P_k allows PDL $\forall\ n \geq 1$.*

Proof. If $v_1,\ v_2,\ \ldots,\ v_n$ are the successive nodes of P_n & G is formed by performing duplication of v_k by $v_k^{'}v_k^{"}$ for $k = 1,\ 2,\ \ldots,\ n$, then it is interesting to note that G has $3n$ nodes and n node disjoint cycles each of length three. Thus $V(G) = V_1 \cup V_2$, where $V_1 = V(P_n)$ and $V_2 = \{v_i^{'}, v_i^{"}\}$ for all i and $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$, where $E_1 = E(P_n)$, $E_2 = (v_i v_i^{'})$, $E_3 = (v_i v_i^{"})$ and $E_4 = (v_i^{'} v_i^{"})$ for all i. Now define an injective map $f : V(G) \rightarrow Z$ as given below: W.L.G, $f(v_1) = 0$, $f(v_1^{'}) = f(v_1) + 2$ and $f(v_1^{"}) = f(v_1) + 5$. Now let p_1 be a prime larger than the utilized labels. Then $f(v_2) = f(v_1) + p_1$, $f(v_2^{'}) = f(v_2) + 2$, and $f(v_2^{"}) = f(v_2) + 5$. Proceeding in this way, let p_n be a prime larger than the utilized labels. Then $f(v_n) = f(v_{n-1}) + p_n$, $f(v_n^{'}) = f(v_n) + 2$, and $f(v_n^{"}) = f(v_n) + 5$.

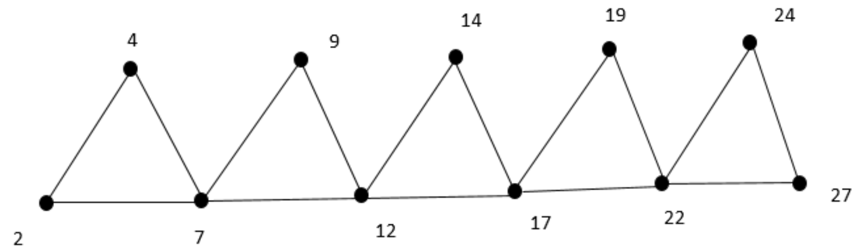

FIGURE 5.2: PDL of duplication of an line by a node at all lines in P_6

□

Theorem 5.2.4. *The graph formed by applying duplication of a link by a point at all links in P_n admits a PDL.*

Proof. Let P_n be on n nodes say v_i, $1 \leq i \leq n$ and m lines. Then G_1 is formed by applying duplication of a link by a node at all lines can be defined as below: $V(G_1) = Y_1 \cup Y_2$, where $Y_1 = V(P_n)$ & $Y_2 = \{u_j : 1 \leq j \leq m\}$ and $E(G_1) = E_1 \cup E_2$, where $E_1 = E(P_n) = \{e_i : 1 \leq i \leq m\}$ and $E_2 = \{e_j : 1 \leq j \leq 2m\}$. Now an injection is $g_1 : V(G_1) \rightarrow Z$ is defined as below: W.L.G, $g_1(v_1) = 2$ and $g_1(u_1) = 4$. Next $g_1(v_l) = g_1(v_{l-1}) + 5, \forall\, 2 \leq l \leq n$ and $g_1(u_j) = g_1(u_{j-1}) + 5$, for $2 \leq j \leq m$ (see Figure 5.2). □

Theorem 5.2.5. *The graph obtained by performing extension at all nodes in P_n admits a PDL.*

Proof. Let P_n be on n-nodes, where $n \geq 3$, say, u_k, $1 \leq k \leq n$ & $E(P_n) = \{e_k : 1 \leq k \leq m\}$. Let G be obtained by performing extension at all nodes in P_n. Then $V(G) = V_1 \cup V_2$, where $V_1 = V(P_n)$, $V_2 = \{v_k : 1 \leq k \leq n\}$ & $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$, where $E_1 = E(P_n), E_2 = \{u_i v_i : 1 \leq i \leq n\}$, $E_3 = \{u_{i+1} v_i\}$ and $E_4 = \{u_i v_{i+1}\}$. Define $h : V(G) \rightarrow Z$ as $h(u_1) = 0$, $h(v_1) = 2$, $h(u_k) = h(u_{k-1}) + 5$, $2 \leq k \leq n$ & $h(v_j) = h(v_{j-1}) + 5$, $2 \leq j \leq n$. Then h gives a PDL of G (see Figure 5.3).

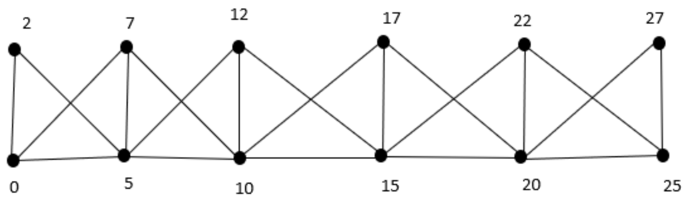

FIGURE 5.3: PDL of the graph obtained by performing extension at all the nodes of P_5

□

Theorem 5.2.6. *G derived by taking extension of $v \in K_n$, $n \geq 4$ is not a PDG.*

Proof. Since G_1 is a PDG iff all of its subgraphs are PDGs & G formed by taking extension of $v \in K_n$, $n \geq 4$ contains K_5 as a subgraph. Therefore in view of theorem 4.2.4 the proof follows". □

5.3 The Characterization of PDL in the Context of Barycentric Subdivision(BS)

In this section, the proof of a result which completely characterizes PDL of simple graphs in the context of BS.

Theorem 5.3.1. *The barycentric subdivision of any simple graph G admits PDL.*

Proof. Let G be any simple graph. There arises 2 possibilities.

Case 1. G a bipartite

Here G contains no odd cycles. The barycentric subdivision of G also a bipartite graph, which is by Theorem 4.2.1, admits PDL.

Case 2. G a non-bipartite

Here G contains at least an odd cycle. The barycentric subdivision of G becomes a bipartite graph as every odd C_n changes to even C_n and hence admits PDL by Theorem 4.2.10. □

5.4 Conclusion

In this chapter, the PDL in the context of duplication and extension operations of certain graphs has been investigated. Further, the complete characterization of PDL of simple graphs in the context of barycentric subdivision has also been done.

Conclusion

In this thesis, prime PL and PDL of various graphs are discussed such as P_n, K_n, $\Gamma(G)$, pizza graph, diamond graph, jelly fish graph, flower graph, generalized Jahangir graph. The complete characterization of PDL of simple graphs in the context of BS has also been established. The sufficient condition for barycentric subdivision of graph to admit PL & sufficient condition for $DS(G)$ of a bipartite graph G to admit PDL have also been established. Moreover, a few open problems and conjectures have also been formulated paving a way for furtherance of research in this direction. Both labeling have also been investigated for various classes of graphs in the context of extension of vertices, duplication and barycentric subdivision operations. A complete characterization of both the labeling is still unsettled but the present work may help to open a gateway to achieve the characterization of both labeling either fully or partially. One can investigate both of these labeling for some new families of graphs that are not studied in the present work as a future task. Though, GL is widely applicable in countless domains, exploring the immense uses of PL and PDL in various fields is a fascinating area of research.

Publications and Presentations

Papers Published/Accepted from the Thesis

1. Ram Dayal and A. Parthiban, "Results on Prime Labeling and Prime Distance Labeling of Graphs", Advances and Applications in Mathematical Sciences, 21 (11) (2022) 6197-6204 (**WoS**).

2. A. Parthiban and Ram Dayal, "A Comprehensive Survey on Prime Graphs", Journal of Physics, Conference Series, 1531 (012077) (2020), 1-9 (**Scopus**).

3. A. Parthiban, Ram Dayal and Swati Sharma, "Prime Distance Labeling of the Non-Commuting Graph of Some Non-Abelian Groups", European Journal of Molecular & Clinical Medicine, 07 (07) (2020), 3919-3926 (**Scopus**).

4. Ram Dayal and A. Parthiban, "On Questions Concerning Finite Prime Distance Graphs" has been accepted for publication in Mathematics and Statistics (2023) (**Scopus**).

5. Ram Dayal and A. Parthiban, "Prime Labeling and Prime Distance Labeling of Some Simple Graphs", has been accepted for publication in AIP: Conference proceedings (2022) (**Scopus and WoS**).

6. Ram Dayal and A. Parthiban, "Prime Labeling and Prime Distance Labeling of Some Classes of Graphs" has been accepted for publication in AIP: Conference Proceedings (2022) (**Scopus and WoS**).

7. Ram Dayal et al., "Recent Advances in Manufacturing and Processing Technologies through Graph Theoretical Approach: A Survey" has been accepted for publication in AIP: Conference proceedings (2022) (**Scopus and WoS).**

8. Ram Dayal, Arunava Majumder and A. Parthiban, "Recent Advancements in Prime Labeling and Prime Distance Labeling of Graphs" has been accepted for publication in TWMS J. App. And Eng. Math. (2022) **(Scopus).**

Papers Presented in Conferences

1. Presented a poster presentation titled "A Comprehensive Survey on Prime Graphs" in RAFAS-2019, Lovely Professional University, Punjab.

2. Presented the paper titled "Prime Distance Labeling of the Non-Commuting Graph of Some Non-Abelian Groups" in Two Day National Multidisciplinary Conference

on "Recent Innovations in Sciences, Social Sciences, Humanities & Arts" in March 2021, Govt Degree College Poonch, J&K.

3. Presented a poster presentation titled "Prime Labeling and Prime Distance Labeling of Some Simple Graphs" in RAFAS-2021, Lovely Professional University, Punjab.

4. Presented the paper titled "Prime Labeling and Prime Distance Labeling of Some Classes of Graphs" in Two Day International Conference on Recent Advances in Mathematics and Computational Engineering, SSNCE, Kalavakkam, Tamil Nadu during 06-07, January 2022.

5. Presented the paper titled "Results on Prime Labeling and Prime Distance Labeling of Graphs" in Two Day International Conference on Recent Trends in Applied Mathematics-ICRTAM 2022, Loyola College, Chennai during 03-04, March 2022.

6. Presented the paper titled "Recent Advancements in Prime Labeling and Prime Distance Labeling of Graphs" in the International Conference on Wavelet Analysis and Graph Theory(ICWAGT-2022) organized by the School of Arts, Sciences, Humanities & Education(SASHE), SASTRA Deemed University, Thanjavur during September 15-16, 2022.

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