Some Strategical Methods for Solving Decision Making Problems Using Type-2 Fuzzy Sets

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In

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By

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This is to certify that Mr. **SUHAIL AHMAD GANAI,** has completed the thesis entitled **Some Strategical Methods for Solving Decision Making Problems Using Type-2 Fuzzy Sets** under my guidance and supervision. As far as I'm aware, his original research and study led to the creation of the current work. Nothing from this thesis has ever been used as part of a submission for another degree, either at another university or elsewhere.

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Abstract

The current thesis, entitled **Some Strategical Methods for Solving Decision Making Problems Using Type-2 Fuzzy Sets** is the result of research outcomes conducted by me under the esteemed guidance and supervision of **Dr. NITIN BHARDWAJ**, Professor, Department of Mathematics, Lovely Professional University, Phagwara, Punjab. The research work is now being submitted to the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara-144411, Punjab, India, for the award of a Doctor of Philosophy in Mathematics.

Metacriterion group decision-making ($MCGDM$) problems are an integral part of contemporary decision theory. These problems involve evaluating a set of alternatives against multiple influential criteria to determine the optimal choice. In our daily lives, we often face the dilemma of whether or not to take action, pondering the best course of action before making a decision. The decision-making ($D - MG$) process heavily relies on having the right information available to the relevant individuals at the appropriate times. In general, decision-makers ($D-MRs$) establish specific characteristics or criteria that need to be satisfied in order to select the best alternatives when solving problems.

The complexity of today's socioeconomic environment and the limited knowledge available pose significant challenges for decision-makers, making it difficult to arrive at precise decisions. Uncertainty, imprecision, and vagueness are common features of the information used in decision-making processes. To mitigate these issues, researchers have extensively employed the theory of fuzzy sets (FSs) and its extensions, including intuitionistic fuzzy sets ($IFSs$), type-2 fuzzy sets ($T2FSs$), type-2 intuitionistic fuzzy sets ($T2IFSs$), and Soft sets. These approaches effectively minimize the level of uncertainty inherent in decision-making.

In recent decades, substantial research efforts have been dedicated to addressing multi-criteria decision making ($MC - DM$) or multi group-decision making problems across various fields. However, an essential factor in determining the best alternatives is the environment in which D-MRs evaluate the available options. This environment can exhibit both quantitative and qualitative characteristics, depending on the nature of the real-life problem at hand. To tackle this challenge, researchers have developed the concept of linguistic variables (LV) and corresponding analytical approaches that

employ various information measures. Following these groundbreaking contributions, researchers have been actively involved in expanding and applying these concepts to various disciplines. Nonetheless, the primary objective for decision-makers remains the ranking of objects to achieve their desired outcomes.

The primary objective of this research is to introduce innovative methodologies using intuitionistic fuzzy sets, type-2 fuzzy sets, and type-2 intuitionistic fuzzy sets to effectively address decision making problems that involve uncertainty. To achieve this goal, we define a range of measures tailored for solving both multi-criterion decision making and MCGDM problems. These measures allow the expression of information about each alternative using fuzzy numbers derived from intuitionistic fuzzy sets, type-2 intuitionistic fuzzy sets, and type-2 Fermatean fuzzy sets ($T2FFSs$). Furthermore, we extensively examine the desirable relationships between the proposed measures and operators.

Leveraging these measures, we develop an efficient method to solve decision making problems by incorporating the expertise of a group of experts. Our approach comprehensively considers the information associated with each alternative to provide robust solutions. To demonstrate the effectiveness of our method, we apply it to various real-life practical examples and compare its performance against existing studies in the field.

The thesis is structured into five chapters, each of which is summarized below:

The first chapter provides a brief overview of the related work conducted by various authors in the evaluation of decision making approaches using different methodologies. The fundamentals and introductory concepts pertaining to fuzzy sets, type-2 fuzzy sets, intuitionistic fuzzy sets, and type-2 intuitionistic fuzzy sets are presented.

In chapter 2, we explores the significance and practical applications of fuzzy set extensions, including intuitionistic fuzzy set, Pythagorean Fuzzy Sets (PFS), and Fermatean Fuzzy Sets (FFS), among others, which overcome these limitations and enable more complex analysis. We also discuss operators on intuitionistic fuzzy sets, establish theorems on their relations, and introduce a new distance measure (dmr) which consider both membership function ($M - F$) and non-membership functions ($N - MF$), highlighting its importance through a pattern recognition ($P - R$) problem. The results showcase the potential of fuzzy set extensions, operators, and distance measures in gaining deeper insights into complex real-world systems and making informed decisions in various fields.

In chapter 3, we focused on decision making issues as decision making can be challenging, especially when dealing with imprecise or uncertain information. In recent years, type-2 fuzzy sets have been proposed as an improvement over traditional fuzzy sets, allowing decision makers to express their preferences with greater flexibility. However, even with type-2 fuzzy sets, decision making can still be difficult, especially in group decision making scenarios. To address this issue, a novel approach based on type-2 fermatean fuzzy sets has been proposed, along with a set of distance measures based on Hamming and Euclidean metrics. This approach was evaluated in a group decision making process using a numerical example, demonstrating its effectiveness in improving decision outcomes. This study offers a promising new perspective on decision making that can lead to better outcomes and improved satisfaction among decision makers.

In chapter 4, we made comparison between different fuzzy sets and type-2 fuzzy sets. Fuzzy sets have revolutionized decision making by providing a mathematical tool for modeling uncertainty and imprecision. However, traditional fuzzy sets may not be sufficient in certain situations, leading to the development of extensions such as Type-2 fuzzy sets, Intuitionistic fuzzy sets, and Type-2 intuitionistic fuzzy sets. This paper provides an overview of these sets, comparing and contrasting them using operations of union, intersection, and distance measures. Additionally, a new distance measure is proposed for type-2 intuitionistic fuzzy sets, which is demonstrated with a numerical example. By understanding the properties and applications of these sets, informed decisions can be made in real-world situations with uncertainty and imprecision.

In chapter 5, we focussed on the extension of fuzzy sets, specifically intuitionistic fuzzy sets and type-2 intuitionistic fuzzy sets, with the use of illustrative examples. The paper highlights the significance of type-2 intuitionistic fuzzy sets in decision making. The study also discusses the challenges faced in decision making situations and how type-2 intuitionistic fuzzy sets can address them. Additionally, the paper introduces a novel distance measure for type-2 intuitionistic fuzzy sets that considers the uncertainty in the membership function and non-membership function. A numerical example is provided to demonstrate the practical application of the proposed distance measure. Also, a comprehensive analysis is conducted to compare the proposed distance measure with existing measures like Euclidean distance $(E-D)$ and Hamming distance $(H-D)$ to determine its accuracy and reliability in representing uncertainty and vagueness in decision making.

A bibliography is included at the end of the thesis, which is by no means comprehensive, but does identify all of the research articles and books that were mentioned in the main text.

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With sincere appreciation and deep gratitude

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Contents

List of Symbols and Abbreviations

$D - MG$- Decision making.

$D - MR$- Decision maker.

$MC - DM$- Multi criteria decision making.

FS- Fuzzy set.

IFS- Intuitionistic fuzzy set.

PFS- Pythagorean fuzzy set.

FFS- Fermatean fuzzy set.

$IVFS$- Interval valued fuzzy set.

$IVIFS$- Interval valued intuitionistic fuzzy set.

$T2FS$- Type-2 fuzzy set

$T1FS$- Type-1 fuzzy set.

$T2IFS$- Type-2 intuitionistic fuzzy set.

$T2FFS$- Type-2 fermatean fuzzy set.

MV- Membership value.

$N - MV$- Non membership value.

dmr- Distance measure.

smr- Similarity measure.

$M - F$- Membership function .

$N - MF$- Non membership function.

$M - G$- Membership grade.

$N - MG$- Non membership grade.

$IT2FS$- Interval type-2 fuzzy set.

$M - D$- Membership degree.

$N - MD$- Non membership degree.

$MCGDM$- Multicriteria group decision making.

LV- Linguishtic variable.

$P - R$- Pattern recognition.

$H - D$- Hamming distance.

$E - D$- Euclidean distance.

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Chapter 1

Introduction

People like engineers, surgeons, lawyers, scientists, or hr managers deal with a variety of issues every day in the real world in order to properly execute their tasks. To choose the best one(s) among them, which is an essential element of everyday life, is one of the difficult decisions that must be made in order to reach the optimal points with the desired goal. A decision making ($D-MG$) theory is crucial to this goal's accomplishment in the area of the $D-MG$ process. To do or not to do is, in fact, one of the most important decisions one must make in everyday life. The proper data being available to the right people at the right times is a prerequisite for the entire $D-MG$ process. By identifying the decision makers ($D-MRs$) and stakeholder(s) in the decision, Baker et al. [11] state that the likelihood of dispute over the problem definition, requirements, goals, and criteria is reduced. According to Campling [18]."The process of $D-MG$ involves choosing between possible courses of action and entails a cycle of activities and events that begins with the identification of a problem and continues with the evaluation of implemented solutions". In a nutshell, $D-MG$ is the experimental process of picking the best option(s) among a variety of possibilities. The process of conducting experiments is a mental process of knowing, which includes consciousness, perception, reasoning, and judgement.

Typically, when choosing the best option(s) to solve an issue, a $D-MR$ will define some characteristics or criteria that must be met in order to evaluate the offered items. The criteria are used to categorise $D-MG$ situations into two categories: (1) decisions based on a single criterion; and (2) decisions based on two or more criterion, also known as multicriterion decision making ($MC-DM$). For instance, the selection committee will always use a certain criterion when choosing a marketing manager for a particular company, such as their past record, communication skills, experience, and motivation power.

1.0.1 Multicriterion Decision Making $(MC - DM)$

Undoubtedly, one of the most significant things humans can do is make decisions, from the numerous situations that we face on a daily basis to very complex systems. A logical $D - MG$ process is used to determine and select the best options depending upon the preferences and the values of $D - MR$ with relation to its criteria. From a mathematical perspective, there should be a methodology and an algorithm that one can use to arrive at a sensible and correct decision. $D - MG$ procedures have recently gained popularity across industries and at various administrative levels within the relevant departments of many organisations due to their increased global competitiveness, ability to make sound plans, and need to thrive in their particular markets. Hence, For lowering material prices, reducing manufacturing time, and improving product or service quality, $D - MG$ is crucial, especially in the procuring department. Choices happen when variations of alternatives are present in front of us associated with diverse criteria. The $D - MG$ process explains how choices are really framed as well as how they may be framed more successfully or effectively. Some other areas of management like as inventory control, investment, manpower activity, new-product development, allocation of resources, medical diagnosis and also including plenty of others, $D - MG$ process has a great importance. There are various categories of $D - MG$. Broadly speaking, it falls into one of four categories: individual $D - MG$, group $D - MG$, $MC - DM$, and multi-stage $D - MG$. $MC - DM$ is a modelling and methodological approach used in decision sciences to construct different types of $D - MG$ problems. The basic objective of decision analysis is to lessen ambiguity. Preferences and information $D - MG$ challenges are modelled in $MC - DM$ and pertinent alternatives are assessed in the presence of numerous competing criteria. Qualitative benefits, quantitative benefits, and cost benefits are only a few examples of numerous criteria. Creating a decision environment, which is a collection of values, alternatives, attributes, and preferences that are available to the connected problems, is how decisions are formed in decision sciences. Depending on the nature of the issues, $D - MRs$ must consider a wide range of factors when selecting the best choice, including technological, economic, ethical, political, legal, and social considerations. Certain types of information can be quantified numerically, while others can only be described verbally or subjectively. By studying the challenges $D - MRs$ can create $MC - DM$ problems and recommend more effective $MC - DM$ techniques. The traditional techniques of making decisions work well for issues where the performance criterion can be accurately represented by a single crisp value, making it possible to rate and rank the alternatives with no issues. Because the criteria in most real-world $D - MG$ situations in various fields often contain imprecision or ambiguity, it may be more appropriate to describe the information with the use of some language variable. Fuzzy set (FS) theorey can be used as a technique for problem modelling and solution when the information that is accessible in relation to a problem is ambiguous, imprecise, or incomplete. FSs were first used in the $MC-DM$ area by Bellman and Zadeh [12] and Zimmermann [171]. To address issues that could not be addressed or solved using the conventional, classical $MC-DM$ procedures, they introduced a new family of methodologies. Fuzzifications of the traditional $D-MG$ theories have been used as applications of FSs in the $D-MG$ domain. $MC-DM$ problems frequently have ambiguous, imprecise, or insufficient parameters given by the $D-MR$. So, it is preferable to treat the expertise of specialists on the parameter as fuzzy data. Yet, there are circumstances in which the perception of membership values (MV) may not always be possible and the evaluation of non-membership values (NMV) may not always be possible due to the lack of information. As a result, there is still an element of uncertainty on which reluctance persists. Intutionistic fuzzy set (IFS) theory can undoubtedly be used to manage this scenario better. As a result, since its inception, IFS has drawn increasing attention, and as a result, scholars have given $MC-DM$ theory more consideration. Here, a straightforward $MC-DM$ framework is provided. Think of R as the set of alternatives $(R_1, R_2...R_m)$ and K as the set of criteria (K_1, K_2, K_n) for the $D-MG$ situation. The following matrix can now be used to present the $MC-DM$ situation:

$$\mathbf{A} = \begin{array}{c} \\ R_1 \\ R_2 \\ . \\ . \\ R_m \end{array} \begin{array}{c} \begin{array}{cccccc} K_1 & K_2 & . & . & . & K_n \end{array} \\ \begin{pmatrix} S_{11} & S_{12} & . & . & . & S_{1n} \\ S_{21} & . & . & . & . & S_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ S_{m1} & . & . & . & . & S_{mn} \end{pmatrix} \end{array}$$

where S_{ij} represents evaluation of alternatives R_i under criteria K_j For a decision making problem, we have
A: Objective.
B: Criteria (K_j).
C: Alternatives (R_i).

1.0.2 Fuzzy Sets (FS)

A binary function known as the characteristic function of a crisp set gives each item in a set a value of either 1 or 0, indicating whether the object is a member or non-member of the set. However, this approach is not always sufficient for representing complex real-world phenomena where objects can have partial membership in a set. To address this issue, a generalization of the characteristic function was introduced that allows for values between 0 and 1 to be assigned to elements of a set, indicating the level of membership in the set that each element has. This function is known as a $M-F$, and the set it defines is referred to as a FS. Unlike crisp sets, FSs are capable of capturing the gradations of membership in a set, ranging from full membership (1) to complete non-membership (0), with all possible degrees of partial membership in between. This makes FSs ideal for representing concepts that are vague, ambiguous or ill-defined, such as good, very good, poor, intelligent, large, and medium-large. In 1965, Zadeh was the first to present the idea of FSs and has since given rise to a new section of mathematics which deals in characterizing and analyzing uncertainty. Fuzzy logic and FSs have been widely employed in a wide range of areas, including artificial intelligence, control systems, $D-MG$, and expert systems. In summary, while crisp sets and logic are suitable for representing binary concepts, FSs and logic provide a more flexible and realistic approach to modeling complex real-world systems by allowing for gradations of membership and capturing the inherent uncertainty in many real-world phenomena. J. A. GOGUEN (1967) [58] extends the foundational work of Zadeh, introducing new perspectives and generalizations. Notably, it explores order structures that go beyond the conventional unit interval. This broader consideration of order structures has led to the development of a fresh outlook on optimization problems. The significance of this research may lie primarily in its unique perspective rather than specific findings. Throughout the evolution of FS theory, pattern classification has played a crucial role, serving as a significant influence. The 1973 work of Richard Bellman and Magnus Giertz [13], it is not simple to apply the basic set theory ideas, such as union and intersection, to the world of FSs. Various approaches and strategies have been proposed to address this challenge. However, it has been observed that the operations of maximum and minimum are particularly significant and play an essential part in the FSs arithmetic. Ronald Yager (1975) [152] when goals and constraints are not precisely defined, $D-MG$ problems become increasingly significant, especially when dealing with complex and social systems. Yager provides a summary of Zadeh's FS theory methodology and its use in making fuzzy decisions.

1.0.3 Histry of Fuzzy Sets

Zadeh (1965a) [159] made the remark that most object classes in the physical world lack clearly specified membership criteria, which gave rise to the idea of a FS. This insight emphasises the discrepancy between traditional mathematical representations and mental representations of reality, which rely on binary logic, exact integers, and differential equations. Classes of objects that zadeh mentioned, like “big size”, “bird”, “chair”, etc., only exist in our minds as mental representations that employ nouns or phrases from natural language. For such categories, where membership seems to be a developing idea rather than a simple issue of being in or out, classical logic is not appropriate. Mental representations, which are useful summaries of perceptual experiences that explain the complexity of the universe, cannot match the precise level of actual numbers. Analytical models of physical events can properly reflect reality, but they can be difficult to understand because they don’t provide much justification on their own and might be unintelligible to non-experts. On the other hand, the vagueness that plagues mental representations is caused by the vagueness of linguistic expressions and the absence of definite bounds for the categories of objects they refer to. As a result, we can refer to these linguistical concepts as gradul qualities or fuzzy predicates. It might have appeared futile and even insane a century ago to attempt to express human knowledge in a way that is both user-friendly and scientifically accurate. However, the development of computers has fundamentally changed the field of science, ushering in the era of information management. Given that so many people rely on computers to acquire information that aids in $D - MG$, It is essential that solid theories and innovative technologies be developed for knowledge representation and automated reasoning. How to preserve and use human knowledge in many sectors where little unbiased and accurate data is accessible is a crucial concern in this regard. Due to its significance in this development Dubois, et al. [46], FS theory is closely related to artificial intelligence.

There have been numerous recent attempts to improve logic’s capacity for representation and to put forth non-additive models of uncertainty. Lotfi Zadeh started the more prominent and successful of these endeavours in 1965 when he released his article titled “Fuzzy Sets”. A logic of gradualness in attributes has been developed using zadeh’s methodology, whose foundation is the concept of progressive membership. As a result, “Possibility Theory” Zadeh 1978 [164], a novel and extremely successful uncertainty calculus, has been developed. It treats the ideas of possibility and certainty (or necessity) as incremental modalities. This idea has proven to be especially straightforward and useful in practise. FSs were first put forth by zadeh in 1973 [160] with the intention of making a contribution to the fields of abstraction and summarization, information processing and communication, and pattern categorization. When they were first put forth in the early 1960s, the claims regarding the applicability of FSs in these fields may have seemed speculative, but subsequent advancements in the fields of data science and engineering have demonstrated the fact that these intutions were accurate and far exceeded expectations. The word “fuzzy” frequently alludes to the idea of vagueness when discussing FSs. It is important to talk about the connection between fuzziness and obscurity. When used in daily speech, the adjective "fuzzy" can describe an object's lack of solidity or firmness or its fringe's loose fibres. It may also imply that anything is covered in loose, volatile material or has leaks. Similar to how descriptions of objects might be ambiguous or imprecise, objects themselves are not vague.

1.0.4 Type-2 Fuzzy sets ($T2FSs$)

Making decisions under uncertain conditions, which involve evaluating or selecting from a range of available options, is a common challenge in real-life scenarios. Such problems are difficult to model and handle due to the presence of uncertainty. While probability theory is a useful tool in many cases, uncertainty is often imprecise or vague in nature and cannot be described using traditional probabilistic approaches. To address these situations,$T2FSs$ emerged as a development of traditional FSs. [161]. $T2FSs$ offer a more flexible framework for dealing with uncertainty, enabling a more accurate representation of the underlying imprecision or vagueness in the problem. As a result, $T2FSs$ have emerged as a crucial tool for making decisions, under uncertain conditions, particularly in situations where traditional probabilistic approaches are inadequate. A $T2FS$ is a FS where the MVs are themselves Type-1 fuzzy sets ($T1FSs$) defined on the unit interval [0, 1]. This gives you more room to directly model the uncertainty that exists in a problem. $T2FSs$ are more complex than $T1FSs$ and can be difficult to understand and clarify. However, they provide an effective means of expressing uncertainty and handling imprecision in information. $T2FSs$ are three-dimensional and are particularly useful at interfaces where increasing levels of imprecision, uncertainty, and fuzziness are present. Several studies have focused on developing operations on $T2FSs$ [40, 75, 110]. $T2FSs$ have a representation theorem stated by Mendel and John [99], It dispenses with the "Extension Principle" and allows the development of formulas for the union, intersection, and complement of $T2FSs$. This theorem provides a valuable tool for working with $T2FSs$ and has facilitated their use in a wide range of applications.

Mizumoto and Tanaka [111, 112] proposed operations on $T2FSs$ and associated properties. While Nieminen [117] revealed the algebraic structure of $T2FSs$. Fuzzy-valued logic was studied by Dubois and Prade [41, 42, 44], who also expanded "type-1 fuzzy sup-star composition to type-2 fuzzy relations". Karnik and Mendel [72–76] developed a generic formula for the "extended sup-star composition of type-2 fuzzy relations and operations" on $T2FSs$. Mendel [102] expanded on the more sophisticated characteristics of $T2FSs$. Mendel [103] evaluated a plane representation of $T2FSs$ that is consistent with the ideas of a cutting of $T1FSs$, while Castillo and Melin [20] discussed theories of type-2 fuzzy logic. Mendel [104] offered a high-level history of $T2FSs$ and fuzzy logic systems, whereas Ling and Zhang [91] established operations on triangular $T2FSs$. A brand-new parameterization technique for universal type-2 fuzzy membership functions was created by Castillo et al. [21]. While Shahparast and Mansoori [124] used evolving type-1 rules to produce an online broad type-2 fuzzy classifier, Xing et al. [147] utilised interval $T2FSs$ for the categorization of remote sensing data. In the area of medical sciences, Ontiveros et al. [118] compared interval type-2 and general type-2 fuzzy systems. By use of an interval type-2 fuzzy logic system-based similarity measure, Ashraf et al. [2] calculated the similarity between the pixels in a digital picture.

1.0.5 Intuitionistic Fuzzy sets ($IFSs$)

Atanassov presented $IFSs$ [4, 6, 8], as a type of higher-order FSs that have proven to be effective in handling vagueness. In some situations, it may not be possible to evaluate MVs to our satisfaction due to a lack of information, in addition to the presence of vagueness. Similarly, evaluating $NMVs$ may also not be possible in such cases, leaving a part of the problem indeterminate and uncertain. In situations where there is insufficient information to define an imprecise concept using a conventional FSs, $IFSs$ offer an alternative approach. It is important to note that while FSs are a type of $IFSs$, the reverse is not true. This theory is a FS extension, giving it a more flexible tool for replicating human $D-MG$ procedures and activities requiring human skill and expertise [83, 84]. Since such activities are inherently imprecise and often not entirely reliable, $IFSs$ are a valuable tool for addressing these challenges.

In recent years, academics have paid a lot of attention to distance measure (dmr) and simmillarity measure (smr), which are crucial mathematical tools used in $D-MG$ and pattern recognition ($P-R$) tasks [39]. To date, $IFSs$ have been subjected to a variety of distance or similarity measurements [131, 144]. A $M-F$ and a $N-MF$ make up $IFSs$ two-dimensional representation. While Grzegorzewski [64] suggested using the Hausdorff metric to create a distance measurement, Szmidt and Kacprzyk [134] introduced $d-mr$ for $IFSs$ using the hamming distance ($H-D$) and euclidean distance ($E-D$). A generalised $d-mr$ for $IFSs$ was suggested by Wang and Xin and is effective for $P-R$ tasks [64]. Song and Wang [132] developed a similarity metric based on the similarity matrix's positive definiteness, whereas Hatzimichailidis et al. [66] proposed a dmr for $IFSs$ constructed using a matrix norm and a fuzzy consequence. The hesitation function in $IFSs$ was not taken into account by these methods, which produced erroneous findings. Researchers have looked into a three-dimensional model of $IFSs$, encompassing the $M-F$, $N-MF$, and hesitancy function, to get around these restrictions. Wang and Xin's approach from Wang and Xin [140] was expanded by Park et al. [120], who also provided a distance metric for $IFSs$

used in $P - R$. Yang and Chiclana adjusted grzegorzewski's method [64] to determine the separation between $IFSs$. Yang and Chiclana's study established a brand-new spherical dmr afterwards used in decision analysis for $IFSs$ in 3-D space [59]. The approach suggested by hatzimichailidis et al. was expanded upon by Luo and Zhao [93]. We found that, although most distances are linear in nature, several of the current approaches do not entirely satisfy the axiomatic definition of a dmr after analysing the existing dmr methods for $IFSs$. Some of the distance or similarity measurements for $IFSs$ that are currently in use may not be sufficient to explain judgements or may result in surprising outcomes. As a result, the issue of creating distance or similarity measurements for $IFSs$ is still unresolved and intriguing.

1.0.6 Type-2 Intuitionistic Fuzzy Sets ($T2IFSs$)

$T2FS$ is a more advanced version of the $T1FS$, which is an extension of the classical FS. $T2FS$ allows for more accurate handling of uncertainty and ambiguity in $D - MG$ and reasoning problems, which is a fundamental superiority over $T1FS$. The $T1FSs$ MV, a real number between [0,1], reflects the level of belongingness. When compared to $T2FS$ MV, which are FSs themselves, providing a more flexible and nuanced representation of uncertainty. Zadeh was the one who first suggested the $T2FS$ idea. [161, 162, 165] and extensively explored by Mendel [102]. $T2FS$ encompasses both ordinary FSs and interval valued fuzzy sets ($IVFS$) as special cases, they are particularly useful in situations with higher degree of uncertainty. Researchers have investigated $T2FSs$ in many domains, including theoretical studies [30, 63, 76, 77] and various application areas [53, 65, 67, 81, 121]. Singh and Garg [130] proposed a novel approach called the symmetric triangular intuitionistic $T2FSs$ that combines both $IFSs$ and $T2FS$ environments. Using this method, novel interval type-2 intuitionistic fuzzy aggregation operators were created that can account for various relationships between input arguments. Building on this work, Garg and Singh [57] introduced triangular interval $T2IFSs$ and developed three new aggregation operators for triangular interval T2IFSs. In addition to this, Garg and Singh [129] proposed $T2IFS$, which is another extension of $T2FSs$. The use $T2IFS$ offers significant advantages in modeling complex $D-MG$ problems that involve high levels of uncertainty and ambiguity. By combining intuitionistic fuzzy and T2F environments, these techniques can provide more flexible and nuanced representations of uncertainty, leading to more accurate and reliable $D - MG$. A $T2IFS$ is an extension of the $T1IFS$, which itself is an extension of the classical FS. In situations involving $D - MG$ and reasoning, it enables greater flexibility in the expression of uncertainty. In $T2IFS$, the membership and non-membership of an element in a set are FSs known as upper and lower $M - Fs$, and upper and lower $N - MFs$, respectively. The upper and lower $M - Fs$ quantify the degree of belongingness, while the upper and lower $N - MFs$ quantify the degree of non-belongingness of an element to the set. Unlike a $T1IFS$, which has a fixed degree of uncertainty associated with each element, a $T2IFS$ allows for a varying degree of uncertainty based on the context of the decision problem. This makes it a more powerful tool for modeling complex $D-MG$ problems where there is a high degree of uncertainty and ambiguity. $T2IFSs$ have been effectively applied to a lot of different fields, including $D-MG$, $P-R$, and image processing. However, their increased complexity also makes them more computationally demanding than $T1IFSs$, which can be a challenge in some applications.

1.0.7 Review of Distance or Similarity Measures (dmr and smr)

Measurements of distance and similarity are crucial ideas in data analysis and $D-MG$. A smr establishes the degree of similarity between two sets, whereas a dmr establishes the degree of difference. When there is greater closeness between two objects, the value of a dmr decreases and the value of a smr increases. These two metrics can be normalised so that distance = 1 minus similarity, and vice versa, as they are dual concepts. Measures of distance and similarity have drawn a lot of interest in helping people make decisions in the real world. For both $IFSs$ and $IVFS$, researchers have put forth a variety of distance measurements. Szmidt and Kacprzyk, for instance, [134] proposed four different $dmrs$ for $IFSs$, including Hamming, Euclidean, normalised Hamming, and normalised $E - D$. Grzegorzewski [64] proposed novel dmr for $IFSs$ and $IVFSs$ depending upon the Hausdorff metric, while Wang and Xin [140] introduced an axiom definition for $dmrs$ of $IFSs$ to handle pattern recognition difficulties. $IVFSs$ were given distance and similarity measurements by Xu [149], and the $dmrs$ for $IVFSs$ were expanded by Park et al. [119] by including the amplitude margin. The inequalities of Euclidean or normalised Euclidean $dmrs$, however, are not valid, according to Chen [25], who demonstrated faults in the current $dmrs$ put out by grzegorzewski [64]. Szmidt and Kacprzyk [136] presented new dmr depending on the Hausdorff metric to tackle these problems, while Zhang and Yu [168] provided new $dmrs$ for $IFSs$ and $IVFSs$ that do away with the shortcomings of current $dmrs$. A three-dimensional Hausdorff dmr was also proposed by Yang and Chiclana, and its consistency was compared to that of its two-dimensional equivalent [154]. The normalised Euclidean dmr was used by Vasanti and Viswanadham to assess student performance [139]. In general, the creation of new distance and similarity metrics for $IFSs$ and $IVFSs$ has important ramifications for a variety of industries, including finance, medicine, and engineering, where precise comparison of objects' similarities and differences is essential for making decisions. A fresh distance metric was proposed by Ejegwa and Modom and its use in a medical diagnosis issue [48]. A numerical method was devised by Gupta and Mohanty [62] to assess the degree of compensation for $MC-DM$ issues in fuzzy contexts. By employing fuzzy numbers to reflect the choices and weights of the $D-MRs$, the technique accounts for the uncertainties in the $D-MG$ process. Chen et al. [24] developed the idea of similarity degree between two FSs, but Dengfeng and Chuntian [37] expanded this idea to include $IFSs$ and used it to solve pattern recognition issues. Dengfeng and Chuntian's *smr*, however, had several flaws in its axiom qualities, and Mitchell [107] suggested a more suitable modification. Another *smr* was provided by Liang and Shi in 1998, and it was contrasted with other methods already in use. By using numerical illustrations, Hung and Yang [68] created a *smr* based on the Hausdorff distance and demonstrated its efficacy. Dengfeng and Chuntian's method has some drawbacks, and Liu [92] presented a *smr* for $IFSs$ that does not have those drawbacks. Three distinct forms of *smrs* for $IFSs$ based on geometric distance, set theory, and matching function were defined by Xu [149] and used to address multi attribute decision making (MADM) issues. A number of similarity measurements for $IFSs$ were put forth by Xia and Xu and used in group $D-MG$. A cosine similarity metric was created by Ye [157] for $IFSs$ and medical diagnosis issues. According to entropy measurements, Wei, Wang, and Zhang [143] developed *smrs* for $IVIFSs$. A cosine similarity metric for $IVIFSs$ and pattern identification issues was proposed by Singh [126]. Ye [158], which expanded the cosine *smr* by including a degree of hesitation and used it to solve MADM issues. Wu et al. [145] found flaws in the similarity metrics outlined by Wei, Wang, and Zhang [143] and presented a new measure that takes the $IVIFSs$ degree of reluctance into account and gets over pattern recognition's constraints. By offering numerical counterexamples, Boran and Akay [15] proposed parametric distance and *dmrs* for $IFSs$ and carried out a comparison with other *smrs*. [37, 60, 61, 68, 87, 107, 157]. By generalising the *dmr* suggested by Boran and Akay [15], Dugenci cite 33 produced a fresh *dmr* between two $IVIFSs$ and provided counterexamples of an existing measure established by Xu [149]. In order to improve the limitations of the *smr* provided by Chen and Chang [28], Nguyen [116] developed a knowledge measure of $IFSs$ and built a similarity or dissimilarity measure on its foundation. In order to prove the validity of their suggested measure, Chen, Cheng, and Lan [29] conducted a comparison between the suggested metric and current measurements [15, 28, 37, 68, 92, 107, 157, 168] and provided a measure of similarity that satisfies the triangular property. Garg gave similarity and distance metrics for intuitionistic multiplicatives and used them to solve $D-MG$ issues in his citation [55]. In order to evaluate credit risk, Shen et al. [125] introduced a *dmr* for $IFSs$ that addresses the shortcomings of the one previously proposed by Chen, Cheng, and Lan [29](19). With an application to medical diagnosis issues, Luo and Zhao [93] analysed the current *dmr* [64, 125, 134, 140, 154] and presented a *dmr* based on the binary function and matrix norm. By converting $IFSs$ into $IVFSs$, Ke et al. [78] suggested an efficient *dmr* based on interval values for $IFSs$ and performed a comparative study based on numeric demonstrations to show the viability of the proposed dmr. In order to calculate the entropy measure, Rashid et al. [122] built a dmr between $IVIFSs$ and used it to address MADM issues. The basic formulation and features of these measures were established by Hung and Yang in their [68] paper, which also gave a method for determing the similarity between $T2FSs$. By adding a similarity metric to assess the similarity between two $T2FSs$, Mitchell [108] expanded on this work and used the suggested approach to address the issue of automatic evaluation of welded structures. Similarity and inclusion metrics between $T2FSs$ were defined by Yang and Lin in [155], and their characteristics and interactions were studied. They developed a clustering approach for type-2 fuzzy data by fusing Yang and Shih's algorithm from [156] with the suggested similarity metrics. They also contrasted their findings with Hung and Yang's [68] work. Based on the Sugeno integral, Hwang et al. [69] provided similarity, inclusion, and entropy measurements for $T2FSs$. For clustering the patterns of $T2FSs$, they combined a reliable clustering algorithm with the proposed smr. Overall, these investigations aid in the creation of techniques for $T2FS$ clustering and similarity analysis. To gauge the degree of similarity between $T2IFSs$, Garg and Singh [129] established similarity metrics. Numerous professions, like data mining and pattern recognition, can use these measurements in the real world.

1.1 Prelimanaries and Basic Concepts

1.1.1 Fuzzy Sets [159]

Assuming S is a universal set, let s represent any one of its elements. Then, an ordered pair collection can be used to represent a FS D specified on S.

$$D = \big\{(s, \mu_D(s))|s \in S\big\}. \tag{1.1}$$

1.1.2 Equality of Fuzzy Sets [159]

If D and E are FSs on a universal set S, they are considered equal (denoted by $D = E$) if they contain the same number of elements and have the same membership function for every element s ∈ S.

$$\mu_D(s) = \mu_E(s). \tag{1.2}$$

1.1.3 Union of Fuzzy Sets [159]

Considering two FSs D and E that are specified on the same universe S, their union is defined as

$$\mu_{D\cup E}(s) = max\{\mu_D(s), \mu_E(s)\}\forall s \in S. \tag{1.3}$$

1.1.4 Intersection of Fuzzy Sets [159]

Let D and E be two FSs stated on the same universe S, their intersection is defined as

$$\mu_{D\cap E}(s) = min\{\mu_D(s), \mu_E(s)\}\forall s \in S. \tag{1.4}$$

1.1.5 Fuzzy Set Compliment [159]

$$\mu_{\bar{D}}(s) = 1 - \mu_D(s)\forall s \in S. \tag{1.5}$$

1.1.6 α- Level Set [159]

Let D be a FS in S then α- Level set of D is defined as

$$D_\alpha = \{s \in S : \mu_D(s) \geq \alpha\}, \tag{1.6}$$

where α be any real number such that $\alpha \in$ [0,1].

1.1.7 Strong α- Level cut [159]

Let D be a FS in S then strong α- cut of D is defined as

$$D_{\alpha^+} = \{s \in S : \mu_D(s) > \alpha\}, \tag{1.7}$$

where α be any real number such that $\alpha \in$ [0,1].

1.1.8 Support of a Fuzzy Set [159]

The support of a FS D, which is defined on a set S, "refers to a crisp set that contains all the elements in S whose $M - D$ in D is greater than zero". In simple terms, the support of a FS consists of the specific elements from the original set that have some level of membership in the FS.

$$Support(D) = \{s \in S : \mu_D(s) > 0\}. \tag{1.8}$$

1.1.9 Core of a Fuzzy Set [159]

The core of $FS\ D$ consists of all the elements in S that are completely and unambiguously represented by D, without any fuzziness or uncertainty.

$$Core(D) = \{s \in S : \mu_D(s) = 1\}. \tag{1.9}$$

1.1.10 Height of a Fuzzy Set [159]

The height of a $FS\ D$, which is represented by $h(D)$, is expressed as the highest $M - D$ that any element in the set D can obtain. In other words, $h(D)$ is the maximum membership grade $(M - G)$ that is attained by any element in the $FS\ D$.

$$h(D) = \sup_{s \in S} \mu_D(s). \tag{1.10}$$

1.1.11 Normal of a Fuzzy Set [159]

A $FS\ D$ defined on set S is called Normal if and only if

$$h(D) = \sup_{s \in S} \mu_D(s) = 1, \tag{1.11}$$

for at least one $s \in S$ and is called subnormal otherwise.

1.1.12 Cardinality of a Fuzzy Set [159]

The scale cardinality of a $FS\ D$ defined on a finite set S is a measure of the effective size of the FS and is defined as the sum of the $M - Gs$ of all the elements in the set. Mathematically, the scale cardinality of D is given by:

$$|D| = \sum_{s \in S} \mu_D(s). \tag{1.12}$$

1.1.13 Convex Fuzzy Set [159]

A FS D defined on a set S is convex if

$$\mu_D\left\{(\lambda s_1 + (1-\lambda s_2)) \leq \min\left(\mu_D(s_1), \mu_D(s_2)\right) \forall s_1, s_2 \in S, \lambda \in [0,1]\right\}. \tag{1.13}$$

1.1.14 Type-2 Fuzzy Set [161]

Let S be the universe of discourse (UOD). Then we define structure of $T2FS$ D on S as

$$D = \big(s, u, \mu_D(s, u_D)\big) \quad |s \in S, u_D \in j_s \subseteq [0,1], \tag{1.14}$$

in which

$$0 \leq \mu_D(s, u_D) \leq 1, \tag{1.15}$$

where u_D is a primary $M-F$ ($P-MF$) and $\mu_D(s, u_D)$ is a fuzzy $M-F$, $\mu_D : S \to [0,1]$. is said to be secondary $M-F$ ($S-MF$). It can also be written as

$$D = \int_{s\in S} \mu_D(s)/s \quad |s \in S, u \in j_s \subseteq [0,1] = \int_{s\in S} [\int_{u\in j_s} (f_s(u_D)/u_D)]/s,$$

where $\mu_D(s) = \int_{u\in j_s}(f_s(u_D)/u_D)$ is the grade of membership,$f_s(u_D) = \mu_D(s, u_D)$ is named as $S-MF$ where u_D is $P-MF$ of D and j_s is called $P-MF$ of s.

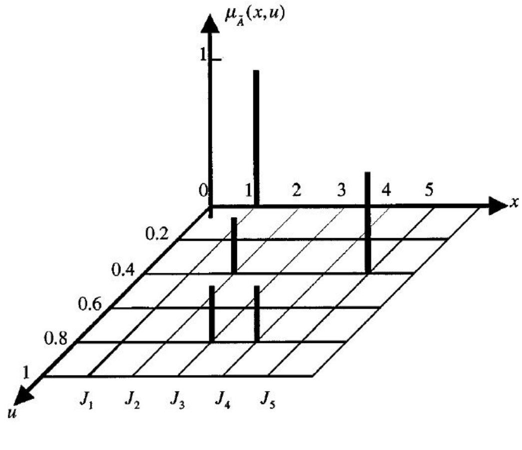

Figure 1.1: Type-2 fuzzy set

1.1.15 Footprint of Uncertainity (FOU) [113]

The uncertainty associated with the $P-MF$ of a $T2FS$, is represented by a defined and limited area known as the FOU. The FOU encompasses the entirety of the $P-MF$

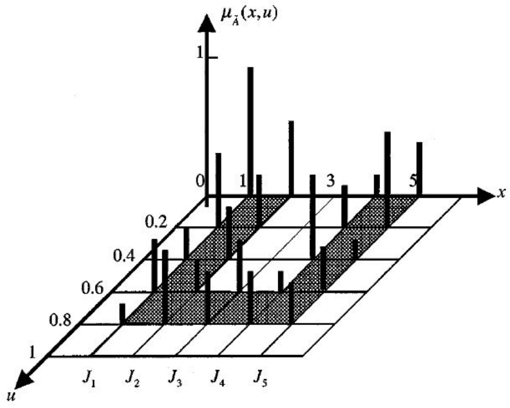

Figure 1.2: Type-2 membership function and the shaded area is FOU

.

by forming their logical union represented as

$$\text{FOU}(\tilde{D}) = \cup_{s \in S} J_s. \tag{1.16}$$

The region that is marked with shading in Figure 3 represents the FOU.

1.1.1. Mendel and John [99]. A type-2 fuzzy set ($T2FS$), labeled $\tilde{D}$, is describes by a type-2 $M - F$ $\mu_{\tilde{D}}(s, u)$, where $s \in$ S and $u \in J_s \subseteq$ [0,1], i.e,

$$\tilde{D} = \big(s, u, \mu_{\tilde{D}}(s, u)\big) \quad |s \in S, u \in j_s \subseteq [0, 1], \tag{1.17}$$

where $0 \leq \mu_{\tilde{D}}(s, u) \leq 1$. $\tilde{D}$ can also be expressed in the form of

$$\tilde{D} = \int_{s \in S} \int_{u \in J_s} \mu_{\tilde{D}}(s, u)/(s, u) \quad |s \in S, u \in j_s \subseteq [0, 1], \tag{1.18}$$

where $\int\int$ represents the combination of all permissible s and u and for discrete UOD we replace $\int$ by $\sum$.

1.1.2. Mendel and John [99] for every s (say s=s') the 2-dimentional plane whose axis are u and $\mu_{\tilde{D}}(s', u)$ is said to be the vertical slice of $\mu_{\tilde{D}}(s, u)$. $S - MF$ is a vertical slice of $\mu_{\tilde{D}}(s, u)$ defined as

$$\mu_{\tilde{D}}(s = s', u) \equiv \mu_{\tilde{D}} s' = \int_{u \in J_{s'}} f_{s'}(u)/u, J_{s'} \subseteq [0, 1], \tag{1.19}$$

where $0 \leq f_{s'}(u) \leq 1$. As $\forall$ $s' \in S$, we drop prime notation $\mu_{\tilde{D}}(s')$ and refer $\mu_{\tilde{D}}$ as $S - MF$.

1.1.3. Mendel and John [99]. A type-1 fuzzy set ($T1FS$) can be written in terms of $T2FS$ as $(1/\mu_F(s))/s$ or $1/\mu_F(s)$ $\forall$ s $\in$ S. $1/\mu_F(s)$ implies $S - MF$ has only one value in its domain called PMV $\mu_F(s)$ at which the secondary grade is equall to one.

1.1.16 Operations on Type-2 Fuzzy Sets ($T2FSs$)

Let $\tilde{D}$ and $\tilde{E}$ be two $T2FSs$ in UOD S. Let $\mu_{\tilde{D}}(s)$ and $\mu_{\tilde{E}}(s)$ be the coresponding $M-Gs$ of these two sets, represented as $\mu_{\tilde{D}}(s) = \int_u f_s(u)/u$ and $\mu_{\tilde{E}}(s) = \int_w g_s(w)/w$, where u, $w \in j_s$ represent $P-MF$ of s and $f_s(u)$,$g_s(w) \in$ [0,1] represent $S-MF$ of s. By extension principle of zadeh [40, 72, 161], the $M-Gs$ for union, intersection and compliment of $T2FSs$ $\tilde{D}$ and $\tilde{E}$ are defined as

Union

$$\tilde{D} \cup \tilde{E} \Leftrightarrow \mu_{\tilde{D}\cup\tilde{E}}(s) = \mu_{\tilde{D}}(s) \sqcup \mu_{\tilde{E}}(s) = \int_u \int_w (f_s(u) \star g_s(w))/(u \vee w). \tag{1.20}$$

Intersection

$$\tilde{D} \cap \tilde{E} \Leftrightarrow \mu_{\tilde{D}\cap\tilde{E}}(s) = \mu_{\tilde{D}}(s) \sqcap \mu_{\tilde{E}}(s) = \int_u \int_w (f_s(u) \star g_s(w))/(u \star w). \tag{1.21}$$

Compliment

$$\bar{\tilde{D}} = \mu_{\bar{\tilde{D}}}(s) = \neg\mu_{\tilde{D}}(s) = \int_u (f_s(u))/(1-u), \tag{1.22}$$

"where $\vee$ denotes the max t-conorm and $\star$ denotes a t-norm. The integrals denotes logical union and the operations $\sqcup$, $\sqcap$ and $\neg$ refer as join, meet and negation respectively".

Where a t-norm is represented by $\star$ and the maximum t-conorm is represented by $\vee$. The procedures $\sqcup$, $\sqcap$, and $\neg$ relate to join, meet, and negation, respectively, while integrals indicate logical union.

1.1.17 Extension Principle

One of the most fundamental notions in FS theory that may be utilised to apply simple mathematical concepts is the extension principle. It was already suggested in Initial input by zadeh in its simplest form. Adjustments have been suggested in the interim. Zadeh, Dubois, and Prade [43, 160–162] provided the following definition of the extension principle.

Let $E_1, E_2..., E_r$ be r fuzzy sets in $S_1, S_2..., S_r$ and S be the Cartesian product of universes S $= S_1 \times ..., \times S_r$, respectively. , where f is a mapping from S to a universe T. t $= f(s_1, ..., s_r)$. We can then define a fuzzy set F in T by using the extension principle concept

$$\bar{F} = \{t, \mu_{\bar{F}}(t) | t = f(s_1, ..., s_r), (s_1, ..., s_r) \in S\}, \quad (1.23)$$

$$\mu_{\bar{F}}(t) = \begin{cases} sup_{(s_1,...,s_r)\in f^{-1}(t)} min\{\mu_{\bar{E}_1(s_1)}, ..., \mu_{\bar{E}_r(s_r)}\} \\ if \quad f^{-1}(t) \neq 0 \\ 0 & otherwise\}, \end{cases} \quad (1.24)$$

where f^{-1} is the inverse of f.

If we put r=1, then the extension principle is reduced to

$$\bar{F} = \{f(\bar{E}) = \{(t, \mu_{\bar{F}}(t)) | t = f(s), s \in S\}, \quad (1.25)$$

where

$$\mu_{\bar{F}}(t) = \begin{cases} \{sup_{(s)\in f^{-1}(t)} min\{\mu_{\bar{E}(s)}\} & if \quad f^{-1}(t) \neq 0 \\ 0 & otherwise\} \end{cases}. \quad (1.26)$$

1.1.18 Type-2 Intuitionistic Fuzzy Set ($T2IFS$) [129]

A $T2IFS$ D in the UOD S is set $\{s, \mu_D(s), \nu_D(s)\}$ where s is the element of $T2IFS$, $\mu_D(s)$ and $\nu_D(s)$ are called $M-G$ and $N-MG$ respectively defined in the closed interval [0,1] as

$$\mu_D(s) = \int_{s\in j_s^1} (f_s(u_D)/u_D), \quad \nu_D(s) = \int_{s\in j_s^2} (g_s(v_D)/v_D), \quad (1.27)$$

where $f_s(u_D)/u_D$ and $g_s(v_D)/v_D$ are termed as $S-MF$ and secondary non membership function ($S-NMF$). In addition u_D, v_D denotes the $P-MF$ and primary non-membership functions ($P-NMF$) and j_{s^1} and j_{s^2} are named as the $P-MF$ and $P-NMF$ of s, respectively. In other words, $T2IFS$ D is defined in the UOD as

$$D = \{(s, u_D, v_D), f_s(u_D), g_s(v_D) | s \in S, u_D \in j_{s^1}, v_D \in j_{s^2}\}, \quad (1.28)$$

where the element of the domain $(s, (u_D, v_D))$ called as $P-MF$ (u_D) and $P-NMF$ (v_D) of $s \in S$ where $f_s(u_D)$ and $g_s(v_D)$ $S-MF$ and $S-NMF$ respectively.

$$D = \{s, (u_D, f_s(u_D)), (v_D, g_s(v_D))\}, \quad (1.29)$$

and is called type-2 intuitionistic fuzzy number ($T2IFN$).

1.1.19 Operation on $T2IFSs$ [32]

Let's consider two $T2IFS$ D and E

$$D = \int_{s\in S} \left(\int_{u\in i_s^u} (\mu_D(s,u), \nu_D(s,u))/u \right) /S$$

and

$$E = \int_{s\in S} \left(\int_{v\in i_s^v} (\mu_E(s,v), \nu_E(s,v))/v \right) /S,$$

where $i_s^u \subseteq [0,1]$ and $i_s^v \subseteq [0,1]$ are domains for $S - MF$ respectively. Then we define

Union

$$D \cup E = \int_{s\in S} \frac{\frac{\left(\int_{v\in i_s^w} (\mu_{D\cup E}(s,w), \nu_{D\cup E}(s,w)\right)}{w}}{S}, i_s^u \cup i_s^v = i_s^w \subseteq [0,1],$$

where

$$\mu_{D\cup E}(s) = \phi\left(\int_{u\in i_s^u} (\mu_D(s,u))/u, \int_{v\in i_s^v} (\mu_E(s,v))/v \right).$$

By making use of extension principle, we obtain

$$\mu_{D\cup E}(s,w) = \int_{u\in i_s^u} \int_{v\in i_s^v} (\mu_D(s,u) \wedge \mu_E(s,u))/\phi(u,v),$$

where $\phi(u,v)$ is t-conorm of u and v

$$\mu_{D\cup E}(s,w) = \int_{u\in i_s^u} \int_{v\in i_s^v} (\mu_D(s,u) \wedge \mu_E(s,u))/(u \vee v).$$

Similarly

$$\nu_{D\cup E}(s,w) = \int_{u\in i_s^u} \int_{v\in i_s^v} (\nu_D(s,u) \vee \nu_E(s,u))/(u \vee v).$$

Intersection

$$D \cap E = \int_{s\in S} \frac{\frac{\left(\int_{v\in i_s^w} (\mu_{D\cap E}(s,w), \nu_{D\cap E}(s,w))\right)}{w}}{S}, \quad i_s^u \cup i_s^v = i_s^w \subseteq [0,1],$$

where

$$\mu_{D\cap E}(s,w) = \int_{u\in i_s^u} \int_{v\in i_s^v} (\mu_D(s,u) \wedge \mu_E(s,u))/(u \wedge v),$$

and

$$\nu_{D\cap E}(s,w)=\int_{u\in i_s^u}\int_{v\in i_s^v}(\nu_D(s,u)\vee\nu_E(s,u))/(u\wedge v).$$

1.1.20 Distance Measures

Distance measure for $T2FSs$ [127]

Let $G_2\tilde{D}$ be the set of all $T2FSs$, a real function $d_m\colon G_2\tilde{D}\times G_2\tilde{D}\rightarrow[0,1]$ is called dmr if d_m satisfies following axioms:

$$(p1)\quad 0\leq d_m(\tilde{D}_1,\tilde{D}_2)\leq 1,\forall(\tilde{D}_1,\tilde{D}_2)\in G_2(\tilde{D})\quad\text{(Boundedness)}. \tag{1.30}$$

$$(p2)\quad d_m(\tilde{D}_1,\tilde{D}_2)=d_m(\tilde{D}_2,\tilde{D}_1)\quad\text{(Symetric)}. \tag{1.31}$$

$$(p3)\quad d_m(\tilde{D}_1,\tilde{D}_2)=0,IF\quad\tilde{D}_1=\tilde{D}_2\quad\text{(Reflexive)}. \tag{1.32}$$

$$(p4)\quad d_m(\tilde{D}_1,\tilde{D}_2)=0,d_m(\tilde{D}_1,\tilde{D}_3)=0,\tilde{D}_3\in G_2(\tilde{D})\quad\text{then}\quad d_m(\tilde{D}_2,\tilde{D}_3)=0\quad\text{(Transitive)}. \tag{1.33}$$

Due of convenience, two $T2FS$ $\tilde{D}_1$ and $\tilde{D}_2$ in S are denoted by $\tilde{D}_1=\{s,u,g_s(u_{\tilde{D}_1})|s\in S\}$ and $\tilde{D}_2=\{s,u,g_s(u_{\tilde{D}_2})|\text{s}\in S\}$. Based on these notations [127] has developed several dmr for $T2FS$ $\tilde{D}_1$ and $\tilde{D}_2$.

- **Normalised Hamming Distance**

$$\begin{aligned}h_2(\tilde{D}_1,\tilde{D}_2)=&1/2n\sum_{j=1}^{n}\{|u_{\tilde{D}_1}(sj)-u_{\tilde{D}_2}(sj)|+|g_{sj}(u_{\tilde{D}_1})-g_{sj}(u_{\tilde{D}_2})|+\\&|\phi_{\tilde{D}_1}(sj)-\phi_{\tilde{D}_2}(sj)|\},\end{aligned} \tag{1.34}$$

where ϕ is defined as the distinction between the $P-MF$ and the $S-MF$ in a $T2FS$.

- **Normalised Weighted Hamming Distance**

$$\begin{aligned}h_{2W}(\tilde{D}_1,\tilde{D}_2)=&1/2n\sum_{j=1}^{n}W_j\{|u_{\tilde{D}_1}(sj)-u_{\tilde{D}_2}(sj)|+|g_{sj}(u_{\tilde{D}_1})-g_{sj}(u_{\tilde{D}_2})|+\\&|\phi_{\tilde{D}_1}(sj)-\phi_{\tilde{D}_2}(sj)|\}.\end{aligned} \tag{1.35}$$

- **Normalised Euclidean Distance**

$$\begin{aligned}e_2(\tilde{D}_1,\tilde{D}_2)=&\{1/2n\sum_{j=1}^{n}\{|u_{\tilde{D}_1}(sj)-u_{\tilde{D}_2}(sj)|^2+|g_{sj}(u_{\tilde{D}_1})-g_{sj}(u_{\tilde{D}_2})|^2+\\&|\phi_{\tilde{D}_1}(sj)-\phi_{\tilde{D}_2}(sj)|^2\}\}^{1/2}.\end{aligned} \tag{1.36}$$

▪ **Normalised Weighted Euclidean Distance**

$$\begin{aligned}e_{2W}(\tilde{D}_1, \tilde{D}_2) =& \{1/2n \sum_{j=1}^{n} W_j \{|u_{\tilde{D}_1}(sj) - u_{\tilde{D}_2}(sj)|^2 + \\ & |g_{sj}(u_{\tilde{D}_1}) - g_{sj}(u_{\tilde{D}_2})|^2 + |\phi_{\tilde{D}_1}(sj) - \phi_{\tilde{D}_2}(sj)|^2\}\}^{1/2}.\end{aligned} \tag{1.37}$$

Distance Measures Between $T2IFS$ [55]

Garg presented the $H - D$ and the $E - Ds$ between $T2IFNs$. Let $G_2^I(s)$ be the class of $T2IFSs$ over the universal set S. A real function $d_2\colon G_2^I(s) \times G_2^I(s) \to [0, 1]$ is called dmr, where d_2 satisfies the following postulates.

The $H - D$ and $E - D$ between T2IFNs were provided in [129]. The family of $T2IFSs$ over the UOD S is denoted by the symbol $G_2^I(s)$.A real function is defined as $d_2\colon G_2^I(s) \times G_2^I(s) \to [0, 1]$. d_2 is referred to as the dmr, and it must satisfy the following axioms:

$$(p1) \quad 0 \leq d_2(\tilde{D}_1, \tilde{D}_2) \leq 1, \forall(\tilde{D}_1, \tilde{D}_2) \in G_2^I(t), \tag{1.38}$$

$$(p2) \quad d_2(\tilde{D}_1, \tilde{D}_2) = 0, IF \quad \tilde{D}_1 = \tilde{D}_2, \tag{1.39}$$

$$(p3) \quad d_2(\tilde{D}_1, \tilde{D}_2) = d_2(\tilde{D}_2, \tilde{D}_1), \tag{1.40}$$

$$(p4) \quad d_2(\tilde{D}_1, \tilde{D}_2) = 0, d_2(\tilde{D}_1, \tilde{D}_3) = 0, \tilde{D}_3 \in G_2^I(t) \;\; then \;\; d_2(\tilde{D}_2, \tilde{D}_3) = 0. \tag{1.41}$$

For convenience, two $T2IFSs$ $\tilde{D}_1$ and $\tilde{D}_2$ in S are expressed by $\tilde{D}_1 = \{s, u, f_{sj}(u_{\tilde{D}_1}), (v, g_{sj}(v_{\tilde{D}_1}))|s \in S\}$ and $\tilde{D}_2 = \{s, u, f_{sj}(u_{\tilde{D}_2}), (v, g_{sj}(v_{\tilde{D}_2}))|s \in S\}$ then following distances for $\tilde{D}_1$ and $\tilde{D}_2$ are defined by considering the $P - MF$, $S - MF$, $P - NMF$, $S - NMF$, FOU and VMF.

▪ **Hamming Distance**

$$\begin{aligned}d_1(\tilde{D}_1, \tilde{D}_2) =& 1/4 \sum_{j=1}^{n} \{|u_{\tilde{D}_1}(sj) - u_{\tilde{D}_2}(sj)| + |g_{sj}(u_{\tilde{D}_1}) - g_{sj}(u_{\tilde{D}_2})| + \\ & |\phi_{\tilde{D}_1}(sj) - \phi_{\tilde{D}_2}(sj)| + |v_{\tilde{D}_1}(sj) - v_{\tilde{D}_2}(sj)| + |h_{sj}(v_{\tilde{D}_1}) - \\ & h_{sj}(v_{\tilde{D}_2})| + |\omega_{\tilde{D}_1}(sj) - \omega_{\tilde{D}_2}(sj)|\}.\end{aligned} \tag{1.42}$$

- **Normalised Hamming Distance**

$$\begin{aligned} d_2(\tilde{D}_1,\tilde{D}_2) =& 1/4n\sum_{j=1}^{n}\{|u_{\tilde{D}_1}(sj)-u_{\tilde{D}_2}(sj)|+|g_{sj}(u_{\tilde{D}_1})-g_{sj}(u_{\tilde{D}_2})|+ \\ & |\phi_{\tilde{D}_1}(sj)-\phi_{\tilde{D}_2}(sj)|+|v_{\tilde{D}_1}(sj)-v_{\tilde{D}_2}(sj)|+|h_{sj}(v_{\tilde{D}_1})- \\ & h_{sj}(v_{\tilde{D}_2})|+|\omega_{\tilde{D}_1}(sj)-\omega_{\tilde{D}_2}(sj)|\}. \end{aligned} \tag{1.43}$$

- **Euclidean Distance**

$$\begin{aligned} d_3(\tilde{D}_1,\tilde{D}_2) =& \{1/4\sum_{j=1}^{n}\{|u_{\tilde{D}_1}(sj)-u_{\tilde{D}_2}(sj)|^2+|g_{sj}(u_{\tilde{D}_1})-g_{sj}(u_{\tilde{D}_2})|^2+ \\ & |\phi_{\tilde{D}_1}(sj)-\phi_{\tilde{D}_2}(sj)|^2+|v_{\tilde{D}_1}(sj)-v_{\tilde{D}_2}(sj)|^2+|h_{sj}(v_{\tilde{D}_1})- \\ & h_{sj}(v_{\tilde{D}_2})|^2+|\omega_{\tilde{D}_1}(sj)-\omega_{\tilde{D}_2}(sj)|^2\}\}^{1/2}. \end{aligned} \tag{1.44}$$

- **Normalized Euclidean Distance**

$$\begin{aligned} d_4(\tilde{D}_1,\tilde{D}_2) =& \{1/4n\sum_{j=1}^{n}\{|u_{\tilde{D}_1}(sj)-u_{\tilde{D}_2}(sj)|^2+|g_{sj}(u_{\tilde{D}_1})-g_{sj}(u_{\tilde{D}_2})|^2+ \\ & |\phi_{\tilde{D}_1}(sj)-\phi_{\tilde{D}_2}(sj)|^2+|v_{\tilde{D}_1}(sj)-v_{\tilde{D}_2}(sj)|^2+|h_{sj}(v_{\tilde{D}_1})- \\ & h_{sj}(v_{\tilde{D}_2})|^2+|\omega_{\tilde{D}_1}(sj)-\omega_{\tilde{D}_2}(sj)|^2\}\}^{1/2}. \end{aligned} \tag{1.45}$$

1.2 Literature Review

L.A. Zadeh (1965) [159]. In this paper zadeh introduced concept of FSs. A FS is a group of objects characterized by a range of $M-Gs$. Each object within the set is assigned a $M-G$ between zero and one, which is determined by a $M-F$. The $M-F$ captures the degree of membership or similarity of an object to the set. These sets are given the concepts of inclusion, union, intersection, complement, relation, convexity, etc., and different features of these concepts are determined with relation to FSs.

J. A. GOGUEN (1967) [58]. This paper builds upon and extends the foundational work of zadeh, introducing new perspectives and generalizations. Notably, it explores order structures that go beyond the conventional unit interval. This broader consideration of order structures has led to the development of a fresh outlook on optimization problems. The significance of this research may lie primarily in its unique perspective rather than specific findings. Throughout the evolution of FS theory, pattern classification has played a crucial role, serving as a significant influence. One of the reasons for this is the intuition that probability theory may not be suitable for addressing the specific type of uncertainty encountered in pattern classification. The uncertainty in this context is often perceived as ambiguity rather than statistical variation.

Richard Bellman and Magnus Giertz (1973) [13]. In this article, author extended the broad set theory notions, such as union and intersection, to the world of FSs is a difficult process. Various approaches and strategies have been proposed to address this challenge. However, it has been observed that the operations of maximum and minimum are particularly significant and are crucial to the FS arthematics.

Ronald Yager (1975) [152]. This article summarises Zadeh's FS theory approach and how it may be used for fuzzy $D-MG$. when objectives and restrictions are not clearly stated, $D-MG$ problems become increasingly significant, especially when dealing with complex and social systems.

L.A. Zadeh (1975) [161] presented the idea of a $T2FS$. The concept of linguistic variables allows for a more intuitive and human-like representation of information. It enables the incorporation of imprecision and ambiguity by using linguistic terms to describe the values of a variable.

Krassimir T. ATANASSOV (1986) [4]. In this paper, author presented the term IFS, which is a generalisation of the term FS is defined and an example is shown. A variety of modal and topological operator properties specified throughout the set of $IFSs$, in addition to operations and relations among sets, are illustrated.

Eulalia Szmidt and Janusz Kacprzyk (1996) [133]. In this paper, It is taken into consideration to use $IFSs$ to determine solutions in group $D-MG$. A set of unique intuitionistic fuzzy preference relations serves as the starting point. We also assume that a fuzzy linguistic quantifier is equivalent to a conventional fuzzy majority. Either immediately after developing a social intuitionistic fuzzy preference connection or after starting with individual intuitionistic fuzzy preference relations, a solution is obtained. The consensus winner and the intuitionistic fuzzy core are the two proposed solution concepts.

Ranjit Biswas (1997) [14]. In this article, author stated The circumstances where IFS theory is better suited to handle than FS theory are examined. Author consider an i-v FS to be an IFS and an IFS to be a collection of an unlimited number of i-v FSs.

N. N. Karnik and J. M. Mendel (2001) [75]. The author explores the concept of type-2 relations and their properties, including compositions, M-Gs of type-2 relations, set operations on $T2FSs$, and algebraic operations. $T2FS$ allow for set operations such as join and meet using either the minimum or product t-norm.

J.M. Mendel and R.I.B. John (2002) [99]. To tackle these concerns, the author of the paper suggests a novel representation for $T2FSs$, which enables the derivation of formulas for union, intersection, and complement operations without relying on the extension Principle. This approach to defining $T2FSs$ enhances clarity and facilitates effective communication when discussing them.

Jerry M. Mendel (2003) [100]. It is not scientifically valid to model words using $T1FSs$. We can model the inherent uncertainties in words as well as other uncertainties using $T2FSs$. Through a series of questions and answers, this article serves as an introductory resource on $T2FSs$, perhaps inspiring the reader to study more and apply them.

Hung, W. L., & Yang, M. S. (2004) [68]. In this article, the author provides axiom definitions, characteristics, and similarity metrics between $T2FSs$. The author presents a practical method for calculating the similarities between Gaussian $T2FSs$.

Wang, W., & Xin, X. (2005) [140]. The distance measure between $IFSs$ is defined by an axiom in this study. The proposed distance measurements are supported by the accompanying evidence. Analysis is done on the relationships between the $IFSs$ similarity and $dmrs$. Finally, pattern recognition is applied to the distance measurements of $IFSs$.

Li, D. F. (2005) [85]. This work explores MADM using $IFSs$. To create the ideal weights for the traits, a number of linear programming models are constructed, and the appropriate $D-MG$ strategies are also recommended. The practicality and effectiveness of the suggested approach are demonstrated through the use of a numerical example.

Lin etal.(2007) [90]. This paper employs $IFSs$ as a novel approach to address fuzzy $MC-DM$ problems. The proposed method enables the description of the degrees of satisfiability and non-satisfiability of each alternative with respect to a specific set of criteria using $IFSs$. Additionally, the method allows the $D-MR$ to aggregate the degrees of membership and non-membership of the criteria using the broad term "importance". By utilizing this recommended strategy, $D-MRs$ can make informed choices in a more realistic manner.

Kahraman, C. (2008) [71]. In this study, the dissemination of the FS theory into the crisp MADM and fuzzy multi-objective decision making (MODM) approaches is first briefly summarised. Here are a few instances of recently released studies on fuzzy MADM and MODM.

Ashtiani, B. etal (2009) [3]. The interval-valued fuzzy TOPSIS approach is described in this study with the goal of resolving $MC - DM$ issues where the weights of the criteria are not equal.

Zhang, Q. S. etal (2010) [169]. The paper introduces a novel measure of information entropy for $IVIFSs$. This measure utilizes the membership interval and non-membership interval of the $IVIFS$.

Torra, V. (2010) [138]. The author of this work suggests reluctant FSs. Although they may be seen as fuzzy multisets from a formal perspective, the author demonstrated that their perception is different from the two current techniques for fuzzy multisets. As a result, in addition to their definition, they also covered several fundamental operations and looked at how these related to $IFSs$. The author also demonstrated that the hesitant FSs envelopes are $IFSs$.

Dubois, D. (2011) [47]. This essay provides a heuristic evaluation of the role of s in decision analysis. It discusses various aspects including linguistic variables, $M - Fs$, aggregation processes, fuzzy intervals, and the valuable preference connections they offer. The essay also highlights the importance of bipolarity and explores the potential of qualitative evaluation techniques. The author adopts a critical stance on the contemporary in order to emphasise the real accomplishments and cast doubt on what is frequently thought to be arguable by decision scientists who study the literature on fuzzy $D - MG$.

Zhu, B. etal (2012) [170]. In this study, the author introduces a concept called $DHFSs$. The author then examines the essential characteristics and functions of $DHFSs$. Additionally, they analyse the connections between the aforementioned sets, utilise the concept of nested intervals to highlight their shared characteristics, and then suggest an extension principle for $DHFSs$.

Chen, S. M., & Wang, C. Y. (2013) [27]. The authors of this study offer an innovative method for making decisions using fuzzy multiple characteristics that is built upon $IT2FS$. They initially created a novel fuzzy ranking method based on the $alpha$-cuts of $IT2FSs$. Then, using the $IT2FSs$ recommended fuzzy ranking method, they present a novel way for making decisions with numerous fuzzy qualities.

Cuong, B. C., & Kreinovich, V. (2014) [31]. The paper introduces picture-FSs, which are extensions of both FSs and $IFSs$. The author then discusses several picture-FSs procedures that possess specific properties.

Bustince, H. etal (2015) [17]. The definition and fundamental characteristics of the many forms of FSs that have so far surfaced in the literature are reviewed by the author in this work. They list some of the applications they have been employed in and analyse the connections between them.

Celik, E. etal (2015) [22]. This study examines 82 distinct publications that employ various $MC-DM$ strategies grounded on $IT2FSs$ that are divided into 35 categories. All studies pertaining to single and hybrid techniques are examined, highlighting their practical uses, empirical findings, and shortcomings. The articles are also statistically examined to reveal fresh developments in the field of $IT2FSs$.

Mendel, J. M. etal (2016) [105]. The concepts and notations $T2FSs$ have seen some important changes in the past 16 years, which are explained in this paper. The article investigates issues related to the notation of the $S-MFs$ and provides an explanation of when and why it is important to differentiate between the FOU and the DOU (Domain of Uncertainty). It also discusses why the notational concerns have not resulted in errors in $T2FS$ calculations and offers advice on notation in this context.

Mendel, J. M., & Mendel, J. M. (2017) [106]. The $T2FSs$ are explicitly introduced in this chapter, which also serves as the book's foundation. There are several brand-new terminology in it. The paper covers several topics including the concept of a $T2FS$, definitions of general $T2FS$ and their associated concepts, definitions of interval $T2FS$ and their associated concepts.

Singh, S., & Garg, H. (2017) [129]. A family of $dmrs$ utilising Hamming, Euclidean, and Hausdorff metrics are described in this study since a notion known as $T2IFS$ has been introduced. Its advantageous characteristics have also been thoroughly studied. Finally, a strategy for rating the options based on group $D-MG$ has been provided and is based on these metrics. A numerical example has been used to demonstrate the recommended measures.

Deveci, M. etal (2018) [36]. In this article, a brand-new model is put out to offer a quick method for assessing probable vehicle sharing stations for the site selection issue. The paper proposes a method that combines the TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) approach with the Weighted Aggregate Sum

Product Assessment (WASPAS) technique and an interval type-2 fuzzy $MC - DM$ model. The use of $IT2FS$ is suggested to better handle the uncertainty in expressing $M - F$ and $N - MF$. This approach aims to enhance the effectiveness of $D - MG$ in uncertain situations.

Luo, M., & Zhao, R. (2018) [93]. In this study, a new distance metric for $IFSs$ is introduced, which relies on a matrix norm and a strictly binary, increasing (or decreasing) function. The novel *dmr* effectively resolves counterintuitive scenarios while also meeting the axiomatic requirements of a *dmr*. By using numerical examples, it is demonstrated that the new distance measurement is valid. Additionally, they provide the pattern recognition algorithms and employ them to resolve diagnostic medical issues.

Dan, S. etal (2019) [32]. The author of this paper put out the $T2IFSs$ notion. The algebraic properties of $T2IFSs$ and a number of mathematical operations on $T2IFSs$, "such as union, intersection, complement, containment", etc., are discussed that are connected to these operations are also investigated. They then discussed some of their fundamental aspects after defining two new operators, the necessity operator and the possibility operator, to transform an $T2IFSs$ into a regular $T2FS$. Additionally, two distance metrics—the Euclidian distance of $T2IFS$ and the $H - D$ are introduced in this paper, and an example of one of their applications is provided.

Senapati, T., & Yager, R. R. (2020) [123]. The author of this study suggests $FFSs$. They contrasted $IFSs$, $PFSs$, and $FFSs$. They identify the basic set of operations for $FFSs$ and concentrate on the complement operator of these sets. In order to rank $FFSs$, defined a scoring function and an accuracy function. They also looked at the Euclidean separation between two $FFSs$. Finally, a Fermata fuzzy TOPSIS approach was developed to address the issue of $MC - DM$.

McCulloch, J., & Wagner, C. (2020) [98]. The author examined all of the available *smr* on $T2FSs$ to ascertain which metrics share common similarities and which do not. For those who do not, they addressed the reasons why the characteristics differ, demonstrated if and what effects this has in applications, and spoke about how a precaution may prevent forgetting to include an essential attribute. Additionally, they examined current metrics in the context of word-based computation employing a vast array of data-driven FSs.

Wang, H. etal (2021) [142]. In this paper, an interval type-2 fuzzy set-based multi-attribute assessment model is created and used to assess service quality. In order to determine how similar two trapezoidal $IT2FSs$ are to one another, an area similarity measure algorithm is first presented. The TOPSIS technique is adapted to serve as the assessment strategy using the area similarity metric. The evaluation model is then used to arrange each evaluation dimension into the established classes in a challenge evaluating a public transport service.

Jiang, W. etal (2021) [70]. In this study, interval similarity and generic $T2FS$ cosine similarity are proposed. The suggested smr for the generic $T2FSs$ are based on vector similarity; as a result, they are independent of any particular representation. Additionally, weighted dice and cosine similarity metrics are suggested in this study to cope with unique circumstances. To demonstrate that the offered similarities are in fact $smrs$ and may produce acceptable similarity results, a number of features and a discussion are shown. In the end, a $MC - DM$ procedure is suggested based on the $smrs$ provided in the scenario where the weights of the criterion are fully unknown.

De, A. K. etal (2022) [35]. This study presents a thorough overview of the literature on $T2FS$. It is thoroughly demonstrated through graphical illustrations why $T2FS$ have been drawing academics' attention for years on end since they were first developed. This article investigates the topics where $T2FSs$ have previously shown that they can deal with incomplete information. Additionally, numerous $T2FS$ advances and expansions have been systematically reported.

Chapter 2

From Fuzzy Sets to Deep Learning: Exploring the Evolution of Pattern Recognition Techniques

In this chapter, we deeply explores the significance and practical applications of FS extensions, including $IFSs$, $PFSs$ and $FFSs$, among others. We also discuss operators on $IFSs$, establish theorems on their relations and introduce a new distance measure which consider both membership and non-membership functions, highlighting its importance through a pattern recognition problem.

Various extension of FSs have been discussed on the basis of their need and importance. Some important results regarding the operation of $IFSs$ has been obtained. As we know different $dmrs$ have been discussed by numerous researchers for different types of FSs. These distance measurements undoubtedly meet the metric's requirements, and the normalised Euclidean distance has certain desirable geometric characteristics. Yet it might not fit as well in practise. For instance consider three IFS J, K and L in the equation $\{X = x_1\}$, where $J = (1, 0, 0)$, $K = (0, 1, 0)$, and $L = (0, 0, 1)$. If we interpret using the ten-person deciding model, $J = (1, 0, 0)$ represents ten people who all are in favour of a candidate; $K = (0, 1, 0)$ denotes ten people who all are against him; and $L = (0, 0, 1)$ denotes ten people who all hesitate. So, it makes sense for us to assume that J and L differ less from one another than J and K do. But, for the above-described Euclidean distance, the distance between J and L is nearly identical to the distance between J and K, which does not seem to make sense to us As a result, We offer a broader definition of the distance between $IFSs$. in this study based on the definition of *smr* provided by Li and Cheng [37] our offered distance was proved more reasonable than Li and cheng.

The remaining portion of the chapter is structured as follows: Section 2.1 contains the introduction. Preliminaries and fundamental ideas are contained in Section 2.2. Extension of FSs is specified in Section 2.3 in terms of their politeness. Section 2.4 contains proerties of $IFSs$ and theorem proofs. A *dmr* between IFS is introduced in Section 2.5, including new *dmr* with a numerical example.

2.1 Introduction

L.A. Zadeh created FS theory in 1965 [159] to resolve ambiguous and inaccurate information. Each entry in a FS has a MV, which indicates the degree of an event and has a value between [0,1]. Numerous $D-MG$ issues can be solved with FSs, including medical diagnosis, pattern identification, cluster analysis [115, 141], and many others. Atanasov thought up the IFS [4]. Each IFS element has a $M-D$ (membership degree) and a $N-MD$ (non-membership degree) in the range [0,1] having sum less than or equal to 1. This limit on the total of $M-D$ limits the application of $IFSs$. Yager [153] proposed the concept of PFS as an extension of $IFSs$. Every element in a PFS has a membership grade ($M-G$) of $h_A(x)$ and a non-membership grade ($N-MG$) of $g_A(x)$, with the square sum of these two grades being no more than one, $(h_A(x))^2 + (g_A(x))^2 \leq 1$. $PFSs$ have numerous uses across many different fields, yet they are unable to manage situations where $(h_A(x))^2 + (g_A(x))^2 \geq 1$ for instance, if $(h_A(x) = 0.8$ and $(g_A(x) = 0.7$, then $(h_A(x))^2 + (g_A(x))^2 = 1.13 > 1$ Senapati and Yager [123] then put out the idea of $FFSs$. A FFS has the following properties: $(r_f(x))^3 + (s_f(x))^3 \leq 1$. This suggests that $FFSs$ are more powerful than FSs, $IFSs$, and $PFSs$. Since they are all confined within the space of $FFSs$. Torra [138] $HFSs$ are described as a function that generates a set of MVs for each domain element. $IVFS$ [163], presented by Zadeh and modified the specific number of the $M-D$ to an interval number. $IVIFS$, which combines IFS and $IVFS$, was first introduced by Atanasov.

2.2 Basic Definitions

Definition 2.2.1. [159] A FS E in S is an ordered pair set if s is group of elements denoted generally by

$$E = \{(s, \mu_E(s))|s \in S\}, \tag{2.1}$$

is called $M - F$ and its value lies in closed interval [0,1].

Definition 2.2.2. $T2FS$ [99] is defined as the extension of ordinary FS that is $T1FS$ and is characterised by Type-2 membership function $\mu_{\bar{Z}}(s,u)$. Let S be a fixed universe a $T2FS$ $\bar{Z} \subseteq S$ is defined mathematically as

$$\bar{Z} = (s, u, \mu_{\bar{Z}}(s,u)) \quad |s \in S, u \in j_s \subseteq [0,1],$$

in which $0 \leq \mu_{\bar{Z}}(s,u) \leq 1$. It can also be written as

$$\bar{Z} = \int_{s \in S} \mu_{\bar{Z}}(s)/s \quad |s \in S, u \in j_t \subseteq [0,1] = \int_{s \in S} [\int_{u \in j_s} (g_s(u)/u)]/s,$$

where $\mu_{\bar{Z}}(s) = \int_{u \in j_s} (g_s(u)/u)$ is the $M - G$, $g_s(u) = \mu_{\bar{Z}}(s,u)$ is named as $S - MF$, where u is $P - MF$ of $\bar{Z}$ and j_s is called $P - MF$ of s.

Definition 2.2.3. FOU (Footprint of Uncertainty) [113] actually for $T2FS$ we are having 3-D structure which becomes very difficult for calculation so we take the base of 3rd dimension to calculate the values which is called FOU. It can be defined as the union of all $P - MFs$ that is

$$FOU(Z) = \cup_{s \in S}(j_s). \tag{2.2}$$

Distance Measure Between $T2FSs$

[127] Examine the following factors in order to calculate the distance measure for $T2FSs$. $P - MF$, $S - MF$ and FOU in the currently used dmr the following dmr is defined for $T2FSs$ J and K.

$$d_{2h}(J,K) = \frac{1}{2n} \sum_{j=1}^{n} |u_J(s_j) - u_K(s_j)| + |f_{sj}(u_J) - f_{sj}(u_k)| + |\xi_J(s_j) - \xi_K(s_j)|. \tag{2.3}$$

2.2.1 Numerical Example

Let's consider four types of metal fields and each field is featured by 5 metals . We can express these four fields by $T2FSs$ $\{c_1, c_2, c_3, c_4\}$ in space $\{S = s_1, s_2, s_3, s_4, s_5\}$.See table 4.3.8.1. There is another kind of special metal $\{n\}$ so we have to find which metal field this metal belongs.

Table 4.3.8.1

	s_1	s_2	s_3	s_4	s_5
$u_{c_1}(s)$	1	0.7	0.5	0.7	1
$f_s(u_{c_1})$	0.7	0.9	0.2	0.5	0.9
$u_{c_2}(s)$	1.0	0.7	0.9	0.9	0.9
$f_s(u_{c_2})$	0.9	0.7	1.0	0.7	0.7
$u_{c_3}(s)$	1.0	0.9	1.0	0.9	0.9
$f_s(u_{c_3})$	0.7	1.0	0.9	0.9	0.4
$u_{c_4}(s)$	0.9	0.9	0.9	0.2	0.7
$f_s(u_{c_4})$	1.0	0.7	0.5	0.0	0.4
$u_n(s)$	0.9	0.2	0.2	0.2	0.9
$f_s(u_n)$	0.4	0.5	0.4	0.0	0.7

we have

$$d_{2h}(J,K) = \frac{1}{2n}\sum_{j=1}^{n} |u_J(s_j) - u_K(s_j)| + |f_{sj}(u_J) - f_{sj}(u_k)| + |\xi_J(s_j) - \xi_K(s_j)|, \quad (2.4)$$

since from the table 4 and using $d_{2h}(J,K)$ we get following result

$d_{2h}(c_1, n) = 0.44, d_{2h}(c_2, n) = 0.48, d_{2h}(c_3, n) = 0.6, d_{2h}(c_4, n) = 0.46,$

which implies special metal n is produced from metal field c_1.

2.3 Extension of Fuzzy Sets

$$\mathbf{A} = \begin{array}{c} \\ A_1 \\ A_2 \\ . \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} K_1 & K_2 & . & . & . & K_n \end{array} \\ \begin{pmatrix} S_{11} & S_{12} & . & . & . & S_{1n} \\ S_{21} & . & . & . & . & S_{2n} \\ . & . & . & . & . & . \\ S_{m1} & . & . & . & . & S_{mn} \end{pmatrix} \end{array}$$

,

where S_{ij} represents evaluation of alternatives A_i under criteria k_j. For a $D - MG$ problem, we have

A: Objective,

B: Criteria (K_j),

C: Alternatives (A_i).

Definition 2.3.1. Intuitionistic Fuzzy set (IFS) [4]; If a person is representing the ratio of S_{ij} in terms of $M-D$ and $N-MD$. An object of the following form is what Atanassov defines as an IFS J in S as

$$J = \{s, \mu_J(s), \nu_J(s) : s \in S, \mu_J(s) \in [0,1], \nu_J(s) \in [0,1]\}, \tag{2.5}$$

where as $\mu_J(s) : S \to [0,1]$ and $\nu_J(s) : S \to [0,1]$ is called as $M-D$ and $N-MD$ respectively, such that $0 \leq \mu_J(s) + \nu_J(s) \leq 1 \forall s \in S$.

2.3.1 Intuitionistic Fuzzy Set Operations [5]

Let D and E be two $IFSs$ on S then some operations are defined as

$$(a) \quad D \subseteq E \Leftrightarrow \quad \mu_D(s) \leq \mu_E(s), \nu_D(s) \geq \nu_E(s) \ \ \forall \ \ s \in S. \tag{2.6}$$

$$(b) \quad D = E \Leftrightarrow \quad \mu_D(s) = \mu_E(s), \nu_D(s) = \nu_E(s) \ \ \forall \ \ s \in S. \tag{2.7}$$

$$(c) \quad D^C = \{s, \nu_D(s), \mu_D(s)\}, \text{where } D^C \text{is the compliment of D.} \tag{2.8}$$

$$(d) \quad \cap D_i = \{(s, \min \mu_{D_i}(s), \max \nu_{D_i}(s)) : s \in S\}. \tag{2.9}$$

$$(e) \quad \cup D_i = \{(s, \max \mu_{D_i}(s), \min \nu_{D_i}(s)) : s \in S\}. \tag{2.10}$$

$$(f) \quad D + E = \{s, \mu_D(s) + \mu_E(s) - \mu_D(s)\mu_E(s), \nu_D(s)\nu_E(s) : s \in S\}. \tag{2.11}$$

$$(g) \quad D \cdot E = \{s, \mu_D(s) \cdot \mu_E(s), \nu_D(s) + \nu_E(s) - \nu_D(s) \cdot \nu_E(s) : s \in S\}. \tag{2.12}$$

2.3.2 Intuitionistic Fuzy Number (IFN) [4]

An IFS D is called an IFN if D is

- Intuitionistic fuzzy sub-set of real line.
- Normal that is there is an $s_0 \in R$ such that $\mu_D(s_0) = 1, \nu_D(s_0) = 0$.
- Convex for $M-F$ $\mu_D(s_0)$, that is

$$\mu_D \{(\lambda s_1 + (1 - \lambda s_2)) \geq \min(\mu_D(s_1), \mu_D(s_2)) \forall s_1, s_2 \in R, \lambda \in [0,1]\}.$$

- Concave for $N-MF$ $\nu_D(s_0)$, that is

$$\nu_D \{(\lambda s_1 + (1 - \lambda s_2)) \geq \min(\nu_D(s_1), \nu_D(s_2)) \forall s_1, s_2 \in R, \lambda \in [0,1]\}.$$

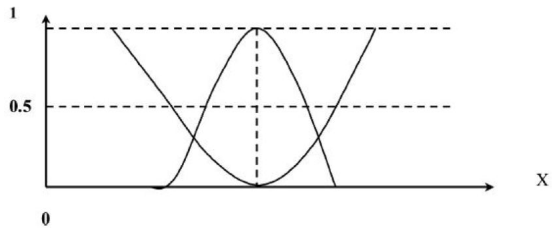

FIGURE 2.1: The $M-F$ and the $N-MF$.

Definition 2.3.2. Interval valued intuitionistic fuzzy set ($IVIFS$): If one provides the value of S_{ij} in terms of interval [L.U] [6] introduced IVIF. Let a set S be fixed, an $IVIFS$ J over S is an object having the form

$$J = \{S, (\mu_J^l(s), \mu_J^u(s), (\nu_J^l(s), \nu_J^u(s)\}, \tag{2.13}$$

where $\mu_J^l(s), \mu_J^u(s) \subset [0,1]$ and $\nu_J^l(s), \nu_J^u(s) \subset [0,1]$ under the constraint $\mu_J^u(s) + \nu_J^u(s) \leq 1$.

Definition 2.3.3. Hesitant fuzzy set (HFS); Tora.v [138] extended the concept of IFS to HFS which permits the $M-D$ a discrete set of [0, 1]. If a person rate the value of S_{ij} as $\{0.5, 0.6, 0.55\}$. Let P be a reference set , then we describe HFS on S in terms of function h that when applied to S yields a subset of [0,1]

$$E = \{S, h_E(s); s \in S\}. \tag{2.14}$$

They consider only agreenance that is why we feel need of dual hesitant fuzzy set.

Definition 2.3.4. A dual hesitant FS [170] is a type of FS that is defined using two different functions to determine the $M-D$ and $N-MD$ for every set's element. These functions provide two sets of values, one as $M-D$ and another as $N-MD$, which can be used to represent the degree of uncertainty or hesitation associated with each element's membership in the set. Given a fixed set P, a dual hesitant FS α on P is interpreted as

$$\alpha = \{(p, h(p), g(p)); p \in P\}, \tag{2.15}$$

in which " $h(p)$ and $g(p)$ are some values in [0,1] signifying the possible $M-D$ and $N-MD$ of the element $p \in P$ to the set α, respectively, under the constraint $0 \leq \gamma, \theta \leq 1 : 0 \leq \gamma^+ + \theta^+ \leq 1$. Where γ^+ and θ^+ denotes the maximum of degree of agree Nance and degree of disagree Nance".

Definition 2.3.5. Pythagorean Fuzzy set (PFS): If someone provides rating of S_{ij} as (0.7, 0.4) whose sum is not less than 1 then we use PFS introduced by [153]. Let S be a UOD, a PFS in S is given by

$$E = \{(S, h_E(s), g_E(s); s \in S)\}, \tag{2.16}$$

where $h_E, g_E : s \to [0,1]$ are $M-D$ and $N-MD$ with condition $(h_E(s))^2+(g_E(s))^2 \leq 1$ for all s in S, The degree of indeterminacy is given by $\gamma_E(s) = \sqrt{1-(h_E(s))^2-(g_E(s))^2}$ For connivance yager [153] called $h_E(s), g_E(s)$ a Pythagorean fuzzy number and denoted as $E = (h_E, g_E)$.

Definition 2.3.6. Hesitant Pythagorean fuzzy set $(HPFS)$ was introduced by [86] defined as

$$E = \{(s, h(s), g(s))\}; s \in S, \tag{2.17}$$

with condition $0 \leq \gamma, \theta \leq 1 : 0 \leq (\gamma^+)^2 + (\theta^+)^2 \leq 1$ for all $s \in S$ $\gamma \in h(s), \theta \in g(s)$.

Definition 2.3.7. Linguistic Pythagorean fuzzy set $(LPFS)$ [56]. If someone has to say about linguistic behavior for example beauty we can't say 70 percent or 80 percent beautiful here we use terms like more beautiful very beautiful etc. $LPFS$ is defined as

$$E = \{(S, (h_E(s), (g_E(s)); s \in S\}, \tag{2.18}$$

where (h_E, g_E) represents linguistic $M-D$ and $N-MD$ respectively with condition $(h^2+g^2 \leq t^2)$.

Definition 2.3.8. Single valued neutrosophic fuzzy set $(SVNFS)$ [33]. In this set, we have indeterminacy factor as well and is defined as

$$E = (S, h_E(s), g_E(s), i_E(s); s \in S, \tag{2.19}$$

with condition $h_E, g_E, i_E \in [0,1]$ and $0 \leq h_E + g_E + I_E \leq 3$ for each s in S. Here $h_E(s), g_E(s), i_E(s)$ represents $M-D$, $N-MD$, and indeterminacy If a person says 0.5% is true, 0.7% not true and 0.2% is not sure here not sure part is only taken into consideration in neutrosophic set.

Definition 2.3.9. Fermatean fuzzy set (FFS) [123]. When someone provides a pair $(r_f(s), s_f(s))$ as the $M-D$ and $N-MD$ like (0.9, 0.6) then the condition of IFS and PFS are not satisfied $(0.9) + (0.6) > 1$. $(0.9)^2 + (.6)^2 > 1$. However, it satisfies the condition $(0.9)^3 + (.6)^3 \leq 1$. So $FFSs$ are here good to control it. Let S be the UOD and F be the FFS defined as

$$F = \{(S, r_F(s), s_F(s)); s \in S\}, \tag{2.20}$$

with condition $0 \leq (r_F(s))^3 + (s_F(s))^3 \leq 1$. Also $i_F(s) = \sqrt[3]{1-(r_F(s))^3-(s_F(s))^3}$ is identified as degree indeterminacy.

2.4 Properties of Intuitionistic Fuzzy Set Operators

Definition 2.4.1. Operators of $IFSs$ [38, 45, 79, 94] For every two $IFSs$ U and V. The following operations and relations are defined. Let $\mu_U(s), \mu_V(s)$ be the $M-D$ and $\nu_U(s), \nu_V(s)$ be $N-MD$ of FS U and V respectively.

Max Operator

$$U + V = \{max(\mu_U(s), \mu_V(s)), min(\nu_U(s), \nu_V(s))\}. \tag{2.21}$$

$$U \cdot V = \{min(\mu_U(s), \mu_V(s)), max(\nu_U(s), \nu_V(s))\}. \tag{2.22}$$

Algebraic Operator

$$U \oplus V = (\mu_U(s) + \mu_V(s) - \mu_U(s) \cdot \mu_V(s), \nu_U(s) \cdot \nu_V(s)). \tag{2.23}$$

$$U \ominus V = (\mu_U(s) \cdot \mu_V(s), \nu_U(s) + \nu_V(s) - \nu_U(s) \cdot \nu_V(s)). \tag{2.24}$$

Einstein Operator

$$U \oslash V = \frac{(\mu_U(s) + \mu_V(s))}{(1 + \mu_U(s)\mu_V(s))}, \frac{(2\nu_U(s)\nu_V(s))}{((2 - \nu_U(s))(2 - \nu_V(s)) + (\nu_U(s)\nu_V(s)))}. \tag{2.25}$$

$$U \times V = \frac{(2\mu_U(s)\mu_V(s))}{((2 - \mu_U(s))(2 - \mu_V(s)) + \mu_U(s)\mu_V(s))}, \frac{(\nu_U(s) + \nu_V(s))}{(1 + \nu_U(s)\nu_V(s))}. \tag{2.26}$$

Proof of Theorems

Let U, V and W be three $IFSs$, $\mu_U(s), \mu_V(s), \mu_W(s)$ and $\nu_U(s), \nu_V(s), \nu_W(s)$ be the $M-D$ and $N-MD$ respectively.

Theorem 2.4.1.

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W).$$

Proof.

$$Let \;\; U \cup (V \cap W) = \{(\mu_U(s), \nu_U(s)) \cup (min(\mu_V(s), \mu_W(s))\}, max(\nu_V(s), \nu_W(s)).$$

Let $\mu_U(s) < \mu_V(s) < \mu_W(s)$ and $\nu_U(s) < \nu_V(s) < \nu_W(s)$, then

$$\mu_U(s), \nu_U(s)) \cup (\mu_V(s), \nu_W(s))$$

$$= max(\mu_U(s), \mu_V(s)), min(\nu_U(s), \nu_W(s)),$$

$$= (\mu_V(s), \nu_U(s)). \tag{2.27}$$

Now

$$(U \cup V) \cap (U \cup W)$$

$$= \{max(\mu_U(s), \mu_V(s)), min(\nu_U(s), \nu_V(s))\} \cap \{max(\mu_U(s), \mu_W(s)), min(\nu_U(s), \nu_W(s))\},$$

$$= (\mu_V(s), \nu_U(s)) \cap (\mu_W(s), \nu_U(s)),$$

$$= \{min(\mu_V(s), \mu_W(s)), max(\nu_U(s), \nu_U(s))\},$$

$$= (\mu_V(s), \nu_U(s)). \tag{2.28}$$

From equations (2.27) and (2.28) we proved $IFSs$ are distributive in nature. □

Theorem 2.4.2. $U \cap (V \cup W) = (U \cap V) \cup (U \cap W)$

Proof. Similarly, we can prove the result as proved in theorem.2.4.1 □

Theorem 2.4.3. $U \ominus V \subseteq U \oplus V$.

Proof.

$$U \ominus V = \mu_U(s), \mu_V(s), \nu_U(s) + \nu_V(s) - \nu_U(s), \nu_V(s),$$

$$U \oplus V = (\mu_U(s) + \mu_V(s) - \mu_U(s)\mu_V(s), \nu_U(s)\nu_V(s)).$$

Assume that

$$\mu_U(s)\mu_V(s) \leq \mu_U(s) + \mu_V(s) - \mu_U(s)\mu_V(s),$$

$$\implies \mu_U(s)\mu_V(s) - \mu_U(s) - \mu_V(s) + \mu_U(s)\mu_V(s) \leq 0,$$

$$\implies \mu_U(s) + \mu_V(s) - \mu_U(s)\mu_V(s) - \mu_U(s)\mu_V(s) \geq 0,$$

$$\implies \mu_U(s)(1 - \mu_V(s)) + \mu_V(s)(1 - \mu_U(s)) \geq 0.$$

Which is true as $0 \leq \mu_U(s) \leq 1$ *and* $0 \leq \mu_V(s) \leq 1$.

Similarly

$$\nu_U(s)\nu_V(s) \leq \nu_U(s) + \nu_V(s) - \nu_U(s)\nu_V(s),$$

$$\implies \nu_U(s) + \nu_V(s) - \nu_U(s)\nu_V(s) - \nu_U(s)\nu_V(s) \geq 0,$$

$$\implies \nu_U(s)(1 - \nu_V(s)) + \nu_V(s)(1 - \nu_U(s)) \geq 0,$$

which is true as $0 \leq \nu_U(s) \leq 1$ and $0 \leq \nu_V(s) \leq 1$.

Hence

$$U \ominus V \subseteq U \oplus V.$$

□

Theorem 2.4.4.

$$U \oplus U \supseteq U.$$

Proof.

$$\mu_U(s) + \mu_U(s) - \mu_U(s)\mu_U(s), \nu_U(s)\nu_U(s),$$

$$\implies 2\mu_U(s) - (\mu_U(s))^2, (\nu_U(s))^2,$$

$$\implies 2\mu_U(s) - (\mu_U(s))^2 = \mu_U(s) + \mu_U(s)(1 - \mu_U(s)) \geq \mu_U(s),$$

and $(\nu_U(s))^2 \leq \nu_U(s)$.

Hence

$$U \oplus U \supseteq U.$$

□

Theorem 2.4.5.

$$U \ominus U \subseteq U.$$

Proof. Similarly we can prove the result as proved in theorem 2.4.4. □

Theorem 2.4.6.

$$((U)^C)^C = U.$$

Proof.

$$U = (\mu_U(s), \nu_U(s)),$$

$$U^C = (\nu_U(s), \mu_U(s)),$$

$$((U)^C)^C = (\mu_U(s), \nu_U(s)).$$

□

Theorem 2.4.7.

$$(U \cup V)^c = (U^c \cap V^c).$$

Proof.

$$(U \cup V)^c = \{(max(\mu_U(s), \mu_V(s)), min(\nu_U(s), \nu_V(s))\}^C,$$

$$= min(\nu_U(s), \nu_V(s)), max(\mu_U(s), \mu_V(s)). \tag{2.29}$$

$$(U^c \cap V^c) = (\nu_U(s), \mu_U(s)) \cap (\nu_V(s), \mu_V(s)),$$

$$= min(\nu_U(s), \nu_V(s)), max(\mu_U(s), \mu_V(s)). \tag{2.30}$$

Hence from (2.29) and (2.30) we proved the result. □

Theorem 2.4.8.

$$(U \cap V)^C = U^C \cup V^C.$$

Proof. Similarly, We can prove the result by theorem 2.4.7 □

Theorem 2.4.9.

$$U \oplus (V \cup W) = (U \oplus V) \cup (U \oplus W).$$

Proof.

$$U \oplus (V \cup W) = (\mu_U(s), \nu_U(s)) \oplus (\mu_V(s), \nu_V(s)) \cup (\mu_W(s), \nu_W(s)),$$

$$= \{(\mu_U(s), \nu_U(s)) \oplus (max(\mu_V(s), \mu_W(s)), min(\nu_V(s), \nu_W(s)))\}$$

$$= (\mu_U(s), \nu_U(s)) \oplus (\mu_W(s), \nu_V(s)),$$

$$= \mu_U(s) + \mu_W(s) - \mu_U(s)\mu_W(s), \nu_U(s)\nu_V(s). \tag{2.31}$$

Now

$$(U \oplus V) \cup (U \oplus W)$$

$$\begin{aligned} =&\mu_U(s) + \mu_V(s) - \mu_U(s)\mu_V(s), \nu_U(s)\nu_V(s) \\ &\mu_U(s) + \mu_W(s) - \mu_U(s)\mu_W(s), \nu_U(s)\nu_W(s). \end{aligned} \tag{2.32}$$

Assume that $\mu_U(s) < \mu_V(s) < \mu_W(s) and \nu_U(s) < \nu_V(s) < \nu_W(s)$, then

$$\begin{aligned} &max(\mu_U(s) + \mu_V(s) - \mu_U(s)\mu_V(s), \mu_U(s) + \mu_W(s) - \mu_U(s)\mu_W(s)), \\ &min(\nu_U(s)\nu_V(s)), (\nu_U(s)\nu_W(s)) \end{aligned} \tag{2.33}$$

$$= \mu_U(s) + \mu_W(s) - \mu_U(s)\mu_W(s), \nu_U(s)\nu_V(s). \tag{2.34}$$

From (2.31) and (2.34), we proved the result. □

Theorem 2.4.10.

$$U \cup (V \oplus W) = (U \cup V) \oplus (U \cup W).$$

Proof. Similarly we can prove the result by theorem 2.4.9 □

Theorem 2.4.11.

$$U \oslash (V \cup W) = (U \oslash V) \cup (U \oslash W).$$

Proof.

$$U \oslash (V \cup W) = (\mu_U(s), \nu_U(s)) \oslash (max(\mu_V(s), \mu_W(s)), min(\nu_V(s), \nu_W(s))).$$

Assume that $\mu_U(s) < \mu_V(s) < \mu_W(s)$ and $\nu_U(s) < \nu_V(s) < \nu_W(s)$, then

$$(\mu_U(s), \nu_U(s)) \oslash (\mu_W(s), \nu_V(s)),$$

$$= \frac{(\mu_U(s) + \mu_W(s))}{(1 + \mu_U(s)\mu_W(s))}, \frac{(2\nu_U(s)\nu_V(s))}{((2 - \nu_U(s))(2 - \nu_V(s)) + \nu_U(s)\nu_V(s))}. \tag{2.35}$$

Now

$$(U \oslash V) \cup (U \oslash W) = (\mu_U(s), \nu_U(s)) \oslash (\mu_V(s), \nu_V(s)) \cup (\mu_U(s), \nu_U(s)) \oslash (\mu_W(s), \nu_W(s)),$$

$$= \{\frac{(\mu_U(s) + \mu_V(s))}{(1 + \mu_U(s)\mu_V(s))}, \frac{(2\nu_U(s)\nu_V(s))}{(2 - \nu_U(s))(2 - \nu_V(s)) + \nu_U(s)\nu_V(s))}\}$$
$$\cup \{\frac{(\mu_U(s) + \mu_W(s))}{(1 + \mu_U(s)\mu_W(s))}, \frac{(2\nu_U(s)\nu_W(s))}{(2 - \nu_U(s))(2 - \nu_W(s)) + \nu_U(s)\nu_W(s))}\},$$

$$= max\{\frac{((\mu_U(s) + \mu_V(s))}{(1 + \mu_U(s)\mu_V(s))}, \frac{(\mu_U(s) + \mu_W(s))}{(1 + \mu_U(s)\mu_W(s))}\},$$
$$min\{\frac{(2\nu_U(s)\nu_V(s))}{((2 - \nu_U(s))(2 - \nu_V(s)) + \nu_U(s)\nu_V(s))}, \frac{(2\nu_U(s)\nu_W(s))}{((2 - \nu_U(s))(2 - \nu_W(s)) + \nu_U(s)\nu_W(s)))}\}.$$

Let $\mu_U(s) < \mu_V(s) < \mu_W(s)$ and $\nu_U(s) < \nu_V(s) < \nu_W(s)$, then

$$= \frac{(\mu_U(s) + \mu_W(s))}{(1 + \mu_U(s)\mu_W(s))}, \frac{(2\nu_U(s)\nu_V(s))}{((2 - \nu_U(s))(2 - \nu_V(s)) + \nu_U(s)\nu_V(s))}. \tag{2.36}$$

From (2.35) and (2.36) result is proved □

Theorem 2.4.12.

$$U \cup (V \oplus W) = (U \cup V) \oplus (U \cup W) \tag{2.37}$$

Proof. Similarly we can prove we can prove the above result by theorem 2.4.11. □

2.5 Distance Measure Between $IFSs$

Due to the fact that dmr refers to the distinction between $IFSs$, it is conceivable to consider it as a parallel concept to smr. Due to the wide range of real-world applications they provide, such as pattern identification, machine learning, $D - MG$, and market forecasting, distance measurements between IFS, a key notion in fuzzy mathematics, are also attracting a lot of attention. Many distance measurements between $IFSs$ have been presented and researched in recent years. The following $dmrs$ were put out by Szmidt and Kacprzyk [134] between J and K

- **Hamming Distance**

$$d_H(J,K) = 1/2\sum_{j=1}^{n}\{|\mu_J(tj) - \mu_K(tj)| + |\nu_J(tj) - \nu_K(tj)| + |\phi_J(tj) - \phi_K(tj)|. \tag{2.38}$$

- **Normalised Hamming Distance**

$$d_{NH}(J,K) = 1/2n\sum_{j=1}^{n}\{|\mu_J(tj) - \mu_K(tj)| + |\nu_J(tj) - \nu_K(tj)| + |\phi_J(tj) - \phi_K(tj)|. \tag{2.39}$$

- **Euclidean Distance**

$$d_E(J,K) = \{1/2\sum_{j=1}^{n}|\mu_J(tj) - \mu_K(tj)|^2 + |\nu_J(tj) - \nu_K(tj)|^2 + |\phi_J(tj) - \phi_K(tj)|^2\}^{1/2}. \tag{2.40}$$

- **Normalized Euclidean Distance**

$$d_{NE}(J,K) = \{1/2n\sum_{j=1}^{n}|\mu_J(tj) - \mu_K(tj)|^2 + |\nu_J(tj) - \nu_K(tj)|^2 + |\phi_J(tj) - \phi_K(tj)|^2\}^{1/2}. \tag{2.41}$$

These distance measurements undoubtedly meet the metric's requirements, and the normalised Euclidean distance has certain desirable geometric characteristics. Yet it might not fit as well in practise. For instance consider three IFS J, K and L in the equation $\{X = x_1\}$, where $J = (1, 0, 0)$, K = (0, 1, 0) and $L = (0, 0, 1)$. If we interpret using the ten-person deciding model, $J = (1, 0, 0)$ represents ten people who are in favour of a candidate; $K = (0, 1, 0)$ denotes ten people who all are against him and $L = (0, 0, 1)$ represents ten people who all hesitate. So, it makes sense for us to assume that J and L differ less from one another than J and K do. But, for the above-described Euclidean distance, the distance between J and L is nearly identical to the distance between J and K, which does not seem to make sense to us. As a result, we provide a more broad definition of *dmr* between $IFSs$ in this study based on the definition of *smr* provided by Li and Cheng [37] and was proved more reasonable than Li and cheng.

2.5.1 New Distance Measure Between Intuitionistic Fuzzy Sets

For convenience, two $IFSs$ J and K in S are denoted by $J = \{s, \mu_J(s), \nu_J(s)|s \in S\}$ and $K = \{s, \mu_K(s), \nu_K(s)|s \in S\}$, then we defined new distance for J and K by considering

$M - F$ and $N - MF$.

$$d_1(J,K) = \frac{1}{n}\sum_{i=1}^{n}\frac{|\mu_J(s_i)-\mu_K(s_i)|+|\nu_J(s_i)-\nu_K(s_i)|}{4} + \frac{min(|\mu_J(s_i)-\mu_K(s_i)|,|\nu_J(s_i)-\nu_K(s_i)|)}{2}. \tag{2.42}$$

Definition 2.5.1. A real function d: $F^I(s) \times F^I(s) \rightarrow [0,1]$ is said to be a *dmr*, if d meets the following axioms:

(A_1) $0 \leq d(J,K) \leq 1, \forall (J,K) \in F^I(s)$,

(A_2) $d(J,K) = 0$, if $J = K$,

(A_3) $d(J,K) = d(K,J)$,

(A_4) If $E \subseteq K \subseteq L$, where $J,K,L \in F^I(s)$, then $d(J,L) \geq d(J,K)$ and $d(J,L) \geq d(K,L)$.

Now, we will prove the above defined measure is a valid *dmr* for IFS.

$$(A_1) = 0 \leq d_1(J,K) \leq 1.$$

Let J and K be two IFS then, we have $|\mu_J(s_i) - \mu_K(s_i)| \geq 0$,

$|\nu_J(s_i) - \nu_K(s_i)| \geq 0$,

$d_2(J,K) \geq 0$.

Then we have $|\mu_J(s_i) - \mu_K(s_i)| \leq 1$,

$|\nu_J(s_i) - \nu_K(s_i)| \leq 1$,

$\implies d_1(J,K) \leq 1$,

hence

$$0 \leq d_1(J,K) \leq 1.$$

A_2 holds trivialy, now we will prove for A_3 and A_4.

$$(A_3) \implies d_1(J,K) = d_1(K,J).$$

We have

$$d_1(J,K) = \frac{1}{n}\sum_{i=1}^{n}\frac{|\mu_J(s_i)-\mu_K(s_i)|+|\nu_J(s_i)-\nu_K(s_i)|}{4} + \frac{min|\mu_J(s_i)-\mu_K(s_i)|,|\nu_J(s_i)-\nu_K(s_i)|}{2}, \tag{2.43}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{|\mu_K(s_i)-\mu_J(s_i)|+|\nu_K(s_i)-\nu_J(s_i)|}{4}+ \frac{min|\mu_K(s_i)-\mu_J(s_i)|,|\nu_K(s_i)-\nu_J(s_i)|}{2}, \tag{2.44}$$

$$= d_1(K,J),$$

$$\implies d_1(J,K) = d_1(K,J).$$

Now to prove (A_4)

$$d_1(J,L) \geq d_1(J,K), \tag{2.45}$$

it can be easily seen that $|\mu_J(s_i)-\mu_L(s_i)| \geq |\mu_J(s_i)-\mu_K(s_i)|$ and $|\nu_J(s_i)-\nu_L(s_i)| \geq |\nu_J(s_i)-\nu_K(s_i)|$ so, we have

$$d_1(J,L) = \frac{1}{n}\sum_{i=1}^{n}\frac{|\mu_J(s_i)-\mu_L(s_i)|+|\nu_J(s_i)-\nu_L(s_i)|}{4}+ \frac{min|\mu_J(s_i)-\mu_L(s_i)|,|\nu_J(s_i)-\nu_L(s_i)|}{2} \tag{2.46}$$

$$\geq \frac{1}{n}\sum_{i=1}^{n}\frac{|\mu_J(s_i)-\mu_K(s_i)|+|\nu_J(s_i)-\nu_K(s_i)|}{4}+ \frac{min|\mu_J(s_i)-\mu_K(t_i)|,|\nu_J(s_i)-\nu_K(s_i)|}{2} = d_1(J,K) \tag{2.47}$$

then we get inequality $d_1(J,L) \geq d_1(J,K)$. Similarly, we can prove $d_1(J,L) \geq d_1(K,L)$. Hence satisfies condition (A_4), so we proved this is a valid distance measure for $IFSs$.

2.5.2 Advantages of New Distance Measure

As new distance measure is based on inclusion principle rather than triangle inequality. The inclusion principle and the triangle inequality are both concepts related to distance measures, but they serve different purposes and are applied in different contexts. Let's discuss the advantages of using the inclusion principle in the context of distance measures for fuzzy sets.

Reflects Set Inclusion:

The inclusion principle is particularly suitable for fuzzy sets as it directly reflects the concept of set inclusion. In fuzzy sets, elements can have varying degrees of membership, and the inclusion principle accounts for this variability.

Considers Degrees of Membership:

Fuzzy sets allow for the representation of degrees of membership, indicating the extent to which an element belongs to a set. The inclusion principle naturally incorporates these degrees, making it more aligned with the nature of fuzzy sets.

Applicability in Fuzzy Logic:

Fuzzy logic is based on the idea of degrees of truth and degrees of membership. The inclusion principle is more consistent with fuzzy logic principles, making it a preferred choice for measuring distances in fuzzy sets.

Flexibility in Representation:

Fuzzy sets offer a flexible way to represent uncertainty and vagueness. The inclusion principle allows for a more nuanced representation of this uncertainty by considering the partial membership of elements in sets.

Conformance to Fuzzy Set Operations:

The inclusion principle aligns well with fuzzy set operations such as union and intersection. This makes it easier to integrate distance measures into broader fuzzy set-based algorithms and computations.

2.5.3 Numerical Example for Pattern Recognition

Example 2.5.1. *Let's consider a pattern recognition problem regarding the classification of industrial materials. Every material is represented by intuitionistic fuzzy sets* I_1, I_2, I_3, I_4, I_5 *in the feature space* $T = \{t_1, t_2, ..., t_6\}$ *(see table 1). We have one unknown industrial material M. Our purpose is to clarify to which class this unknown material belongs. From the data given in table 2.5.1 we have following results for* $d_1(P, Q)$.

Table 2.5.1

	t_1	t_2	t_3	t_4	t_5	t_5
$\mu_{I_1}(t)$	*0.739*	*0.033*	*0.188*	*0.492*	*.020*	*0.739*
$\nu_{I_1}(t)$	*0.125*	*0.818*	*0.626*	*0.358*	*0.628*	*0.125*
$\mu_{I_2}(t)$	*0.124*	*0.030*	*0.048*	*0.136*	*0.019*	*0.393*
$\nu_{I_2}(t)$	*0.665*	*0.825*	*0.800*	*0.648*	*0.823*	*0.653*
$\mu_{I_3}(t)$	*0.449*	*0.662*	*1.000*	*1.000*	*1.000*	*1.000*
$\nu_{I_3}(t)$	*0.387*	*0.298*	*0.000*	*0.000*	*0.000*	*0.000*
$\mu_{I_4}(t)$	*0.280*	*0.521*	*0.470*	*0.295*	*0.188*	*0.735*
$\nu_{I_4}(t)$	*0.715*	*0.368*	*0.423*	*0.658*	*0.806*	*0.118*
$\mu_{I_5}(t)$	*0.326*	*1.000*	*0.182*	*0.156*	*0.049*	*0.675*
$\nu_{I_5}(t)$	*0.452*	*0.000*	*0.725*	*0.765*	*0.896*	*0.263*
$\mu_M(t)$	*0.629*	*0.524*	*0.210*	*0.218*	*0.069*	*0.658*
$\nu_M(t)$	*0.303*	*0.356*	*0.689*	*0.753*	*0.876*	*0.256*

$d_1(I_1, M) = 0.199$, $d_1(I_2, M) = 0.238$, $d_1(I_3, M) = 0.470$, $d_1(I_4, M) = 0.147$, $d_1(I_5, M) = 0.109$. *It is clear that the material M belongs to I_5 because it has least difference from M. Naturally, this conclusion agrees with Liang and Shi's findings.[134] But our approach is far better as it contains inclusion relation which is failed for many existing measures.*

Chapter 3

Type-2 Fermatean Fuzzy Sets: A Novel Approach for Enhancing Group Decision-Making

The objective of this chapter is to study type-2 fermatean fuzzy sets in decision making. Even with Type-2 fuzzy sets, decision-making can still be difficult, especially in group decision-making scenarios. To address this issue, a novel approach based on Type-2 Fermatean fuzzy sets has been proposed, along with a set of distance measures based on Hamming and Euclidean metrics. This approach was evaluated in a group decision-making process using a numerical example, demonstrating its effectiveness in improving decision outcomes. This study offers a promising new perspective on decision-making that can lead to better outcomes and improved satisfaction among decision-makers.

The following sections make up the remaining text: Section 3.1 gives the introductry part. The core descriptions of $T2FS$ and $T2FFS$ are covered in Section 3.2, along with distance measurements. New normalised and weighted normalised distance measures are suggested in Section 3.3. We developed a ranking method based on these metrics for group decision-making problems in Section 3.4 and supported it with numerical examples.

3.1 Introduction

Most mathematics issues in everyday life lack accurate or comprehensive information, which can make it challenging for $D-MRs$ to handle them without thoroughly examining the problem. Zadeh's [159] theory of FSs, along with its relevant extensions, has been utilized to manage imperfect information. Examples of extensions to the theory of FSs include IFS [4], $T2FS$ [99], among others. Many literary sources include The $D-MG$ issue has been examined by researchers [54, 88, 89] in both FS and IFS environments. Xu [149] and Xu and Yager [148] introduced techniques for merging information from multiple $IFSs$ using geometric and arithmetic aggregation operations. The $IVIFS$ information was aggregated using several averaging and geometric aggregation techniques devised by Xu and Chen [150] and Xu [151]. Yager [153] proposed the PFS as a development of the IFS with the limitation that the square sum of its $M-D$ and $N-MD$ be less than or equal to 1. Although the FSs and or IFS environments have been used to explore the aforementioned work, they have certain limitations. For instance, in some situations, it can be challenging for the $D-M$ to pinpoint the precise $M-F$ of a FS that corresponds to an element. An extension of FS known as $T2FSs$ has been employed as a solution, which consists of three components: $P-MF$, $S-MF$, and FOU. This approach has been implemented to overcome the issue. However, because the $T2FS$ is so sophisticated, it is challenging for $D-MR$ to use it in actual circumstances. An interval type-2 fuzzy sets ($IT2FS$) [101] with $M-D$ ranging from zero to one has been taken into consideration for this. Several articles have utilized the $IT2FS$ theory in addressing $D-MG$ challenges through different techniques such as linguistic weighted average [146], as well as ranking and arithmetic operations as discussed in [26]. Some t-conorm-based $dmrs$ and knowledge measures for $PFSs$ with their application in $D-MG$ was given by Ganai. A. H.[51]. A $MC-DM$ based on dmr and knowledge measures of $FFSs$ given by Ganie. A. H. [50]. A Generalized hesitant fuzzy knowledge measure with its application to $MC-DM$ is given by Singh, S. and Ganie, A. H. [128]. "Almulhim, T. and Barahona, I. [1] gave an extended picture fuzzy $MC-DM$, provided a case study on COVID-19 vaccine allocation".

3.1.1 Motivation and Advantages

In the realm of fuzzy logic, Type-2 Fermatean fuzzy sets (T2FFSs) emerge as a beacon of innovation and resilience. As we navigate the complexities of real-world uncertainties, the conventional Type-1 fuzzy sets often fall short in capturing the nuanced and dynamic nature of imprecise information. Enter T2FFSs, a paradigm that transcends the limitations of its predecessors. T2FFSs provide a sophisticated framework for modeling uncertainty, allowing us to delve deeper into the intricacies of imprecision and vagueness. By incorporating higher-order uncertainty, these fuzzy sets empower decision-makers to confront ambiguity with a more refined and robust tool.

In various applications such as decision-making, control systems, and artificial intelligence, where uncertainties are inherent, T2FFSs serve as a promising avenue for enhancing the precision and reliability of systems. This novel approach enables us to not only acknowledge uncertainty but to embrace it, transforming it into a valuable asset for informed decision-making. The motivation behind delving into Type-2 Fermatean fuzzy sets lies in the recognition that the real world is inherently uncertain, and our ability to navigate this uncertainty defines the success of our models and systems. By embracing the richness and depth offered by T2FFSs, we embark on a journey to elevate the field of fuzzy logic, pushing the boundaries of what is possible in the representation and manipulation of uncertain information.

In embracing the paradigm of Type-2 Fermatean Fuzzy Sets, we embark on a quest for precision in uncertainty, a journey that extends beyond the conventional boundaries of fuzzy set theory. It is a call to researchers, a beckoning to explore the uncharted territories of dynamic uncertainty modeling, and a promise of enhanced accuracy in the ever-changing landscape of real-world applications. As we delve into this frontier, we unlock new possibilities for advancing the state-of-the-art in fuzzy set theory, where the interplay of Fermatean principles and Type-2 Fuzzy Sets paves the way for a more nuanced and adaptive understanding of uncertainty.

The use of $T2FFS$ in this study is a significant contribution to the field of FS theory. By incorporating the concept of $N-MD$ or rejection degree, $T2FFS$ enables $D-MRs$ to consider not only the acceptance degree but also the rejection of an object. This provides a more complete picture of the $D-MG$ process in real-life situations. Furthermore, the use of $T2FFS$ is particularly important because it has not been widely explored in previous research. This means that the findings from this study could pave the way for further investigations into the applications of $T2FFS$ in other areas of $D-MG$.

In this research, the concept of $T2FFS$ is presented, which is capable of effectively handling uncertain and imprecise information in various practical scenarios. Additionally, the study proposes a new dmr to complement the $T2FFS$ approach. This is necessary because $T2FFS$ has significant capabilities in modeling and dealing with vague or ambiguous information. A number of distance measurements based on Hamming, Euclidean, and maximum metrics have been suggested as a result. The proposed measures and various desired features have all been carefully examined. Lastly, a ranking technique has been suggested for ordering the $T2FFS$ based on these metrics.

3.2 Basic Concepts

3.2.1 Type-2 Fermatean Fuzzy Set ($T2FFS$)

Definition 3.2.1. A $T2FFS$ Z in the UOD T is a set of pairs $\{t, \mu_Z(t), \nu_Z(t)\}$ where t is the element of $T2FFS$, $\mu_Z(t)$ and $\nu_Z(t)$ are $M-G$ and $N-MG$ respectively defined in [0,1] as

$$\mu_Z(t) = \int_{t\in j_t^1} (g_t(u)/u), \quad \nu_Z(t) = \int_{t\in j_t^2} (h_t(v)/v), \tag{3.1}$$

where $g_t(u)$ and $h_t(v)$ are termed as $S-MF$ and $S-NMF$ respectively and j_{t^1} and j_{t^2} are said to be $P-MF$ and $P-NMF$ of t respectively, where

$$0 \leq (u_Z(t))^3 + (v_Z(t))^3 \leq 1 \;\; and \;\; 0 \leq (g_t(u_Z))^3 + (h_t(v_Z))^3 \leq 1. \tag{3.2}$$

$T2FFS$ is also defined in UOD T as

$$\big\{((t, u_Z, v_Z), g_t(u_Z), h_t(v_Z))| \;\; t \in T, u_Z \in j_{t^1} \;\; and \;\; v_Z \in j_{t^2}\big\}, \tag{3.3}$$

where $(t, u_Z, v_Z)$are called as $P-MF$ and $P-NMF$ of $t \in T$ and $g_t(u_Z), h_t(v_Z)$ are termed as $S-MF$ and $S-NF$ respectively. We denote this pair as $(t, u_Z, g_t(u_Z), v_Z, h_t(v_Z))$ are said to be type 2 fermatean fuzzy number ($T2FFN$).

Definition 3.2.2. Variance margin function ($V-MF$) of $T2FFS$ is defined as the difference between $P-MF$ and $S-MF$, $P-NMF$ and $S-NMF$. It is denoted by ϕ and ω respectively.

3.3 Distance Measure Between $T2FFS$

Here we introduce Hamming and Euclidean distances between $T2FFNs$. Suppose $F_2^f(t)$ class of $T2FFSs$ over the universal set T.

Definition 3.3.1. A real function d: $F_2^f(t) \times F_2^f(t) \to [0,1]$ is said to be dmr when following axioms are being satisfied.

$$(p1) \quad 0 \leq d(Z_1, Z_2) \leq 1, \forall (Z_1, Z_2) \in F_2^f(t), \tag{3.4}$$

$$(p2) \quad d(Z_1, Z_2) = 0, \; IF \;\; Z_1 = Z_2, \tag{3.5}$$

$$(p3) \quad d(Z_1, Z_2) = d(Z_2, Z_1), \tag{3.6}$$

$$(p4) \quad d(Z_1, Z_2) = 0, d(Z_1, Z_3) = 0, Z_3 \in F_2^f(t) \;\; \text{then} \;\; d(Z_2, Z_3) = 0. \tag{3.7}$$

For simplicity, let Z_1 and Z_2 are two $T2FFSs$ in T denoted by

$$Z_1 = \{t(u, g_{tj}(u_{z1}), (v, h_{tj}(v_{z1}))|t \in T\} \quad \text{and} \quad Z_2 = \{t(u, g_{tj}(u_{z2}), (v, h_{tj}(v_{z2}))|t \in T\}. \tag{3.8}$$

Then, we define different distances for Z_1 and Z_2 taking into account $P-MF$, $S-MF$, $P-NMF$, $S-NMF$, FOU and VMF.

- **Hamming Distance**

$$\begin{aligned} d_H(Z_1, Z_2) =& \frac{1}{4}\sum_{j=1}^{n}\{|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3| + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3| \\ &+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3| + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3| + |(h_{tj}(v_{Z_1}))^3 \\ &- (h_{tj}(v_{Z_2}))^3| + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|\}. \end{aligned} \tag{3.9}$$

- **Normalized Hamming Distance**

$$\begin{aligned} d_{NH}(Z_1, Z_2) =& \frac{1}{4n}\sum_{j=1}^{n}\{|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3| + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3| \\ &+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3| + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3| + |(h_{tj}(v_{Z_1}))^3 \\ &- (h_{tj}(v_{Z_2}))^3| + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|\}. \end{aligned} \tag{3.10}$$

- **Euclidean Distance**

$$\begin{aligned} d_E(Z_1, Z_2) =& \{\frac{1}{4}\sum_{j=1}^{n}|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3|^2 + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^2 \\ &+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3|^2 + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3|^2 + |(h_{tj}(v_{Z_1}))^3 \\ &- (h_{tj}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|^2\}^{1/2}. \end{aligned} \tag{3.11}$$

- **Normalized Euclidean Distance**

$$\begin{aligned} d_{NE}(Z_1, Z_2) =& \{\frac{1}{4n}\sum_{j=1}^{n}|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3|^2 + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^2 \\ &+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3|^2 + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3|^2 + |(h_{tj}(v_{Z_1}))^3 \\ &- (h_{tj}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|^2\}^{1/2}. \end{aligned} \tag{3.12}$$

We are going to obtain following properties on the basis of above defined distances: First we prove the above defined distances are valid for $T2FFSs$.

Proposition 3.3.1. *The above defined distances $d_S(Z_1, Z_2)$ for $S = NH, NE$ between $T2FFSs$ Z_1 and Z_2 satisfies following properties ($P1$, $P2$, $P3$ and $P4$).*

$$(p1)\quad 0 \leq d_S(Z_1, Z_2) \leq 1, \forall (Z_1, Z_2) \in F_2^f(t), \tag{3.13}$$

$$(p2)\quad d_S(Z_1, Z_2) = 0, IF \quad Z_1 = Z_2, \tag{3.14}$$

$$(p3)\quad d_S(Z_1, Z_2) = d_S(Z_2, Z_1), \tag{3.15}$$

$$(p4)\quad d_S(Z_1, Z_2) = 0, d_S(Z_1, Z_3) = 0, Z_3 \in F_2^f(t) \quad then \quad d_S(Z_2, Z_3) = 0. \tag{3.16}$$

Proof. For $L = 1, 2$ we have
($P1$) Because Z_1 and Z_2 are $T2FFSs$, we have

$$\begin{aligned}
&|(u_{z_1})(tj))^3 - (u_{Z_2})(tj))^3|^L \geq 0, \quad |(g_{tj}(u_{z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^L \geq 0 \\
&|(\phi_{z_1}(tj))^3 - (\phi_{z_2}(tj))^3|^L \geq 0, \quad |(v_{z_1})(tj))^3 - (v_{Z_2})(tj))^3|^L \geq 0 \\
&|(h_{tj}(v_{z_1}))^3 - (h_{tj}(v_{Z_2}))^3|^L \geq 0, \quad |(\omega_{z_1}(tj))^3 - (\omega_{z_2}(tj))^3|^L \geq 0,
\end{aligned} \tag{3.17}$$

then we can say

$$\begin{aligned}
&\{|((u_{z_1})(tj))^3 - (u_{Z_2})(tj))^3|^L + |(g_{tj}(u_{z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^L + \\
&|(\phi_{z_1}(tj))^3 - (\phi_{z_2}(tj))^3|^L + |(v_{z_1})(tj))^3 - (v_{Z_2})(tj))^3|^L + \\
&|(h_{tj}(v_{z_1}))^3 - (h_{tj}(v_{Z_2}))^3|^L + |(\omega_{z_1}(tj))^3 - (\omega_{z_2}(tj))^3|^L\} \geq 0.
\end{aligned} \tag{3.18}$$

Which implies $d_S(Z_1, Z_2) \geq 0$, also

$$\begin{aligned}
&|(u_{z_1})(tj))^3 - (u_{Z_2})(tj))^3|^L \leq 1, \quad |(g_{tj}(u_{z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^L \leq 1 \\
&|(\phi_{z_1}(tj))^3 - (\phi_{z_2}(tj))^3|^L \leq 1, \quad |(v_{z_1})(tj))^3 - (v_{Z_2})(tj))^3|^L \leq 1 \\
&|(h_{tj}(v_{z_1}))^3 - (h_{tj}(v_{Z_2}))^3|^L \leq 1, \quad |(\omega_{z_1}(tj))^3 - (\omega_{z_2}(tj))^3|^L \leq 1,
\end{aligned} \tag{3.19}$$

therefore

$$\begin{aligned}
&\sum_{j=1}^{n}\{|((u_{z_1})(tj))^3 - (u_{Z_2})(tj))^3|^2 + |(g_{tj}(u_{z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^2 + \\
&|(\phi_{z_1}(tj))^3 - (\phi_{z_2}(tj))^3|^2 + |(v_{z_1})(tj))^3 - (v_{Z_2})(tj))^3|^2 + \\
&|(h_{tj}(v_{z_1}))^3 - (h_{tj}(v_{Z_2}))^3|^2 + |(\omega_{z_1}(tj))^3 - (\omega_{z_2}(tj))^3|^2\} \leq 4,
\end{aligned} \tag{3.20}$$

which means $d_S(Z_1, Z_2) \leq 1$, therefore $0 \leq d_S(Z_1, Z_2) \leq 1$.
(P2) Let $d_S(Z_1, Z_2) = 0$, which implies

$$\begin{aligned} &\{|((u_{z_1})(tj))^3 - (u_{Z_2})(tj))^3|^L + |(g_{tj}(u_{z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^L + \\ &|(\phi_{z_1}(tj))^3 - (\phi_{z_2}(tj))^3|^L + |(v_{z_1})(tj))^3 - (v_{Z_2})(tj))^3|^L + \\ &|(h_{tj}(v_{z_1}))^3 - (h_{tj}(v_{Z_2}))^3|^L + |(\omega_{z_1}(tj))^3 - (\omega_{z_2}(tj))^3|^L\} = 0, \end{aligned} \tag{3.21}$$

$\Longrightarrow$

$$\begin{aligned} &((u_{z_1})(tj))^3 = (u_{Z_2})(tj))^3, \quad (g_{tj}(u_{z_1}))^3 = (g_{tj}(u_{Z_2}))^3 \\ &(\phi_{z_1}(tj))^3 = (\phi_{z_2}(tj))^3, \quad (v_{z_1})(tj))^3 = (v_{Z_2})(tj))^3 \\ &(h_{tj}(v_{z_1}))^3 = (h_{tj}(v_{Z_2}))^3, \quad (\omega_{z_1}(tj))^3 = (\omega_{z_2}(tj))^3, \end{aligned} \tag{3.22}$$

therefore $Z_1 = Z_2$.

$$\begin{aligned} (P3) \;\; d_S(Z_1, Z_2) = &\frac{1}{4n}\sum_{j=1}^{n}\{|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3|^L + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^L \\ &+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3|^L + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3|^L + \\ &|(h_{tj}(v_{Z_1}))^3 - (h_{tj}(v_{Z_2}))^3|^L + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|^L\} \\ &= \frac{1}{4n}\sum_{j=1}^{n}\{|(u_{Z_2}((tj))^3 - (u_{Z_1}(tj))^3|^L + |(g_{tj}(u_{Z_2}))^3 - (g_{tj}(u_{Z_1}))^3|^L \\ &+ |(\phi_{Z_2}(tj))^3 - (\phi_{Z_1}(tj))^3|^L + |(v_{Z_2}(tj))^3 - (v_{Z_1}(tj))^3|^L + \\ &|(h_{tj}(v_{Z_2}))^3 - (h_{tj}(v_{Z_1}))^3|^L + |(\omega_{Z_2}(tj))^3 - (\omega_{Z_1}(tj))^3|^L\}. \end{aligned} \tag{3.23}$$

$$= d_S(Z_2, Z_1). \tag{3.24}$$

(P4) $d_S(Z_1, Z_2) = 0$, which implies

$$\begin{aligned} &((u_{z_1})(tj))^3 = (u_{Z_2})(tj))^3, \quad (g_{tj}(u_{z_1}))^3 = (g_{tj}(u_{Z_2}))^3 \\ &(\phi_{z_1}(tj))^3 = (\phi_{z_2}(tj))^3, \quad (v_{z_1})(tj))^3 = (v_{Z_2})(tj))^3 \\ &(h_{tj}(v_{z_1}))^3 = (h_{tj}(v_{Z_2}))^3, \quad (\omega_{z_1}(tj))^3 = (\omega_{z_2}(tj))^3, \end{aligned} \tag{3.25}$$

and $d_S(Z_1, Z_3) = 0$, implies that

$$\begin{aligned} &((u_{z_1})(tj))^3 = (u_{Z_3})(tj))^3, \quad (g_{tj}(u_{z_1}))^3 = (g_{tj}(u_{Z_3}))^3 \\ &(\phi_{z_1}(tj))^3 = (\phi_{z_3}(tj))^3, \quad (v_{z_1})(tj))^3 = (v_{Z_3})(tj))^3 \\ &(h_{tj}(v_{z_1}))^3 = (h_{tj}(v_{Z_3}))^3, \quad (\omega_{z_1}(tj))^3 = (\omega_{z_3}(tj))^3, \end{aligned} \tag{3.26}$$

therefore

$$\begin{aligned}&((u_{z_2})(tj))^3=(u_{Z_3})(tj))^3,\quad (g_{tj}(u_{z_2}))^3=(g_{tj}(u_{Z_3}))^3\\&(\phi_{z_2}(tj))^3=(\phi_{z_3}(tj))^3,\quad (v_{z_2})(tj))^3=(v_{Z_3})(tj))^3\\&(h_{tj}(v_{z_2}))^3=(h_{tj}(v_{Z_3}))^3,\quad (\omega_{z_2}(tj))^3=(\omega_{z_3}(tj))^3,\end{aligned}\tag{3.27}$$

implies $d_S(Z_2, Z_3) = 0$. Therefore $d_S(Z_1, Z_2)$ for $(S = NH, NE)$ are valid distance measure for $T2FFSs$. □

Proposition 3.3.2. *d_H and d_E dmr satisfies following properties*
(a) $(0 \leq d_H \leq n)$.

Proof. we know

$$\begin{aligned}d_H(Z_1,Z_2)=&\frac{1}{4}\sum_{j=1}^{n}\{|(u_{Z_1}((tj))^3-(u_{Z_2}(tj))^3|+|(g_{tj}(u_{Z_1}))^3-(g_{tj}(u_{Z_2}))^3|\\&+|(\phi_{Z_1}(tj))^3-(\phi_{Z_2}(tj))^3|+|(v_{Z_1}(tj))^3-(v_{Z_2}(tj))^3|+|(h_{tj}(v_{Z_1}))^3\\&-(h_{tj}(v_{Z_2}))^3|+|(\omega_{Z_1}(tj))^3-(\omega_{Z_2}(tj))^3|\},\end{aligned}\tag{3.28}$$

and

$$\begin{aligned}d_{NH}(Z_1,Z_2)=&\frac{1}{4n}\sum_{j=1}^{n}\{|(u_{Z_1}((tj))^3-(u_{Z_2}(tj))^3|+|(g_{tj}(u_{Z_1}))^3-(g_{tj}(u_{Z_2}))^3|\\&+|(\phi_{Z_1}(tj))^3-(\phi_{Z_2}(tj))^3|+|(v_{Z_1}(tj))^3-(v_{Z_2}(tj))^3|+|(h_{tj}(v_{Z_1}))^3\\&-(h_{tj}(v_{Z_2}))^3|+|(\omega_{Z_1}(tj))^3-(\omega_{Z_2}(tj))^3|\}.\end{aligned}\tag{3.29}$$

Which implies $d_H(Z_1, Z_2) = nd_NH(Z_1, Z_2)$ thus, we can say $0 \leq d_H \leq n$. □

(b) $0 \leq d_E \leq n^{1/2}$.

Proof.

$$\begin{aligned}d_E(Z_1,Z_2)=&\{\frac{1}{4}\sum_{j=1}^{n}|(u_{Z_1}((tj))^3-(u_{Z_2}(tj))^3|^2+|(g_{tj}(u_{Z_1}))^3-(g_{tj}(u_{Z_2}))^3|^2\\&+|(\phi_{Z_1}(tj))^3-(\phi_{Z_2}(tj))^3|^2+|(v_{Z_1}(tj))^3-(v_{Z_2}(tj))^3|^2+|(h_{tj}(v_{Z_1}))^3\\&-(h_{tj}(v_{Z_2}))^3|^2+|(\omega_{Z_1}(tj))^3-(\omega_{Z_2}(tj))^3|^2\}^{1/2}\leq d_H(Z_1,Z_2)\leq n^{1/2},\end{aligned}\tag{3.30}$$

which implies $0 \leq d_E \leq n^{1/2}$.

□

We have different practical situations where we take different weights to different sets hence $w_j(j = 1, 2, 3 \ldots n)$ with $w_j \geq 0, \sum_{j=1}^{n} w_j = 1$ of element $t_j \in T$ to be taken into account. Here we proposed normalised weighted Hamming distance and normalised weighted Euclidean distances between $T2FFSs$.

- **Normalized Weighted Hamming Distance**

$$\begin{aligned} d_{NWH}(Z_1, Z_2) = & \frac{1}{4n} \sum_{j=1}^{n} W_j \{ |(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3| + |(g_{tj}(u_{Z_1}))^3 \\ & - (g_{tj}(u_{Z_2}))^3| + |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3| \\ & + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3| + |(h_{tj}(v_{Z_1}))^3 \\ & - (h_{tj}(v_{Z_2}))^3| + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3| \}. \end{aligned} \tag{3.31}$$

- **Normalized Weighted Euclidean Distance**

$$\begin{aligned} d_{NWE}(Z_1, Z_2) = & \{ \frac{1}{4n} \sum_{j=1}^{n} W_j |(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3|^2 + |(g_{tj}(u_{Z_1}))^3 \\ & - (g_{tj}(u_{Z_2}))^3|^2 + |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3|^2 \\ & + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3|^2 + |(h_{tj}(v_{Z_1}))^3 \\ & - (h_{tj}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|^2 \}^{1/2}. \end{aligned} \tag{3.32}$$

Proposition 3.3.3. *Let the weight vector of element $t_j \in T$ be w_j then weighted distance $d_S(Z_1, Z_2), (S = NWH, NWE)$ satisfies properties of (P1,P2,P3 and P4) As $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$ then we can obtain $0 \leq d_{NWH}(Z_1, Z_2) \leq d_{NH}(Z_1, Z_2)$. Hence satisfies (P1) and explanation for (P2,P3 and P4) are similar to preposition 3.1, hence same for d_{NWE}.*

Proposition 3.3.4. ***Relation between** d_{NH} **and** d_{NWH} **as** $d_{NWH} \leq d_{NH}$.*

Proof. Let Z_1 and Z_2 are $T2FFSs$ also $w_j \geq 0, \sum_{j=1}^{n} w_j = 1$, so

$$\begin{aligned} d_{NWH}(Z_1, Z_2) =& \frac{1}{4n} \sum_{j=1}^{n} W_j \{|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3| + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3| \\ &+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3| + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3| + |(h_{tj}(v_{Z_1}))^3 \\ &- (h_{tj}(v_{Z_2}))^3| + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|\} \\ &= \frac{1}{4}\{[W_1\{|(u_{Z_1}((t1))^3 - (u_{Z_2}(t1))^3| + |(g_{t1}(u_{Z_1}))^3 - (g_{t1}(u_{Z_2}))^3| \\ &+ |(\phi_{Z_1}(t1))^3 - (\phi_{Z_2}(t1))^3| + |(v_{Z_1}(t1))^3 - (v_{Z_2}(t1))^3| + |(h_{t1}(v_{Z_1}))^3 \\ &- (h_{t1}(v_{Z_2}))^3| + |(\omega_{Z_1}(t1))^3 - (\omega_{Z_2}(t1))^3|] + [W_2\{|(u_{Z_1}((t2))^3 \\ &- (u_{Z_2}(t2))^3| + |(g_{t2}(u_{Z_1}))^3 - (g_{t2}(u_{Z_2}))^3| + |(\phi_{Z_1}(t2))^3 - (\phi_{Z_2}(t2))^3| \\ &+ |(v_{Z_1}(t2))^3 - (v_{Z_2}(t2))^3| + |(h_{t2}(v_{Z_1}))^3 - (h_{t2}(v_{Z_2}))^3| + |(\omega_{Z_1}(t2))^3 \\ &- (\omega_{Z_2}(t2))^3|] + ...[W_n\{|(u_{Z_1}((tn))^3 - (u_{Z_2}(tn))^3| + |(g_{tn}(u_{Z_1}))^3 \\ &- (g_{tn}(u_{Z_2}))^3| + |(\phi_{Z_1}(tn))^3 - (\phi_{Z_2}(tn))^3| + |(v_{Z_1}(tn))^3 - (v_{Z_2}(tn))^3| \\ &+ |(h_{tn}(v_{Z_1}))^3 - (h_{tn}(v_{Z_2}))^3| + |(\omega_{Z_1}(tn))^3 - (\omega_{Z_2}(tn))^3|]\}, \\ &\text{as} \quad w_j \in [0, 1], \quad thus \end{aligned} \tag{3.33}$$

$$\begin{aligned} d_{NWH}(Z_1, Z_2) \leq& \frac{1}{4n}\{[|(u_{Z_1}((t1))^3 - (u_{Z_2}(t1))^3| + |(g_{t1}(u_{Z_1}))^3 - (g_{t1}(u_{Z_2}))^3| + |(\phi_{Z_1}(t1))^3 \\ &- (\phi_{Z_2}(t1))^3| + |(v_{Z_1}(t1))^3 - (v_{Z_2}(t1))^3| + |(h_{t1}(v_{Z_1}))^3 - (h_{t1}(v_{Z_2}))^3| \\ &+ |(\omega_{Z_1}(t1))^3 - (\omega_{Z_2}(t1))^3|] + [|(u_{Z_1}((t2))^3 - (u_{Z_2}(t2))^3| + |(g_{t2}(u_{Z_1}))^3 \\ &- (g_{t2}(u_{Z_2}))^3| + |(\phi_{Z_1}(t2))^3 - (\phi_{Z_2}(t2))^3| + |(v_{Z_1}(t2))^3 - (v_{Z_2}(t2))^3| \\ &+ |(h_{t2}(v_{Z_1}))^3 - (h_{t2}(v_{Z_2}))^3| + |(\omega_{Z_1}(t2))^3 - (\omega_{Z_2}(t2))^3|] + ...[|(u_{Z_1}((tn))^3 \\ &- (u_{Z_2}(tn))^3| + |(g_{tn}(u_{Z_1}))^3 - (g_{tn}(u_{Z_2}))^3| + |(\phi_{Z_1}(tn))^3 - (\phi_{Z_2}(tn))^3| \\ &+ |(v_{Z_1}(tn))^3 - (v_{Z_2}(tn))^3| + |(h_{tn}(v_{Z_1}))^3 - (h_{tn}(v_{Z_2}))^3| + |(\omega_{Z_1}(tn))^3 \\ &- (\omega_{Z_2}(tn))^3|]\} \leq \{\frac{1}{4n} \sum_{j=1}^{n} \{|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3| + |(g_{tj}(u_{Z_1}))^3 \\ &- (g_{tj}(u_{Z_2}))^3| + |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3| + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3| \\ &+ |(h_{tj}(v_{Z_1}))^3 - (h_{tj}(v_{Z_2}))^3| + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|\} = d_{NH}(Z_1, Z_2). \end{aligned} \tag{3.34}$$

But $Z_1$and Z_2 are arbitrary $T2FFSs$ hence proves $d_{NWH} \leq d_{NH}$. □

Proposition 3.3.5. ***Relation between*** d_{NE} ***and*** d_{NWE} ***as*** $d_{NWE} \leq d_{NE}$.

Proof. Let Z_1 and Z_2 are $T2FFSs$ also $W_j \geq 0, \sum_{j=1}^{n} W_j = 1$, so

$$\begin{aligned}
d_{NWE}(Z_1, Z_2) =&\{\frac{1}{4n}\sum_{j=1}^{n} W_j|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3|^2 + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^2 \\
&+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3|^2 + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3|^2 + |(h_{tj}(v_{Z_1}))^3 \\
&- (h_{tj}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|^2\}^{1/2} \\
&= \frac{1}{4}\{W_1|(u_{Z_1}((t1))^3 - (u_{Z_2}(t1))^3|^2 + |(g_{t1}(u_{Z_1}))^3 - (g_{t1}(u_{Z_2}))^3|^2 \\
&+ |(\phi_{Z_1}(t1))^3 - (\phi_{Z_2}(t1))^3|^2 + |(v_{Z_1}(t1))^3 - (v_{Z_2}(t1))^3|^2 + |(h_{t1}(v_{Z_1}))^3 \\
&- (h_{t1}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(t1))^3 - (\omega_{Z_2}(t1))^3|^2\}^{1/2} + W_2|(u_{Z_1}((t2))^3 \\
&- (u_{Z_2}(t2))^3|^2 + |(g_{t2}(u_{Z_1}))^3 - (g_{t2}(u_{Z_2}))^3|^2 + |(\phi_{Z_1}(t2))^3 - (\phi_{Z_2}(t2))^3|^2 \\
&+ |(v_{Z_1}(t2))^3 - (v_{Z_2}(t2))^3|^2 + |(h_{t2}(v_{Z_1}))^3 - (h_{t2}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(t2))^3 \\
&- (\omega_{Z_2}(t2))^3|^2\}^{1/2} + ...W_n|(u_{Z_1}((tn))^3 - (u_{Z_2}(tn))^3|^2 + |(g_{tn}(u_{Z_1}))^3 \\
&- (g_{tn}(u_{Z_2}))^3|^2 + |(\phi_{Z_1}(tn))^3 - (\phi_{Z_2}(tn))^3|^2 + |(v_{Z_1}(tn))^3 - (v_{Z_2}(tn))^3|^2 \\
&+ |(h_{tn}(v_{Z_1}))^3 - (h_{tn}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(tn))^3 - (\omega_{Z_2}(tn))^3|^2\}^{1/2},
\end{aligned} \tag{3.35}$$

as $w_j \in [0, 1]$, thus

$$\begin{aligned}
d_{NWE}(Z_1, Z_2) \leq& \frac{1}{4}\{|(u_{Z_1}((t1))^3 - (u_{Z_2}(t1))^3|^2 + |(g_{t1}(u_{Z_1}))^3 - (g_{t1}(u_{Z_2}))^3|^2 + |(\phi_{Z_1}(t1))^3 \\
&- (\phi_{Z_2}(t1))^3|^2 + |(v_{Z_1}(t1))^3 - (v_{Z_2}(t1))^3|^2 + |(h_{t1}(v_{Z_1}))^3 - (h_{t1}(v_{Z_2}))^3|^2 \\
&+ |(\omega_{Z_1}(t1))^3 - (\omega_{Z_2}(t1))^3|^2\}^{1/2} + |(u_{Z_1}((t2))^3 - (u_{Z_2}(t2))^3|^2 + |(g_{t2}(u_{Z_1}))^3 \\
&- (g_{t2}(u_{Z_2}))^3|^2 + |(\phi_{Z_1}(t2))^3 - (\phi_{Z_2}(t2))^3|^2 + |(v_{Z_1}(t2))^3 - (v_{Z_2}(t2))^3|^2 \\
&+ |(h_{t2}(v_{Z_1}))^3 - (h_{t2}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(t2))^3 - (\omega_{Z_2}(t2))^3|^2\}^{1/2} + ... \\
&|(u_{Z_1}((tn))^3 - (u_{Z_2}(tn))^3|^2 + |(g_{tn}(u_{Z_1}))^3 - (g_{tn}(u_{Z_2}))^3|^2 + |(\phi_{Z_1}(tn))^3 \\
&- (\phi_{Z_2}(tn))^3|^2 + |(v_{Z_1}(tn))^3 - (v_{Z_2}(tn))^3|^2 + |(h_{tn}(v_{Z_1}))^3 - (h_{tn}(v_{Z_2}))^3|^2 \\
&+ |(\omega_{Z_1}(tn))^3 - (\omega_{Z_2}(tn))^3|^2\}^{1/2}\}. \\
&\leq \{\frac{1}{4n}\sum_{j=1}^{n}|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3|^2 + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^2 \\
&+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3|^2 + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3|^2 + |(h_{tj}(v_{Z_1}))^3 \\
&- (h_{tj}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|^2\}^{1/2} = d_{NE}(Z_1, Z_2).
\end{aligned} \tag{3.36}$$

Because Z_1 and Z_2 are arbitrary $T2FFSs$, hence $d_{NWE} \leq d_{NE}$.

□

Proposition 3.3.6. ***Relation between*** d_{NH} ***and*** d_{NWE} ***as*** $d_{NWE} \leq \sqrt{d_{NH}}$.

Proof.

$$
\begin{aligned}
d_{NWE}(Z_1, Z_2) =&\{\frac{1}{4n}\sum_{j=1}^{n} W_j|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3|^2 + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^2 \\
&+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3|^2 + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3|^2 + |(h_{tj}(v_{Z_1}))^3 \\
&- (h_{tj}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|^2\}^{1/2}. \\
&\leq \{\frac{1}{4n}\sum_{j=1}^{n} |(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3|^2 + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3|^2 \\
&+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3|^2 + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3|^2 + |(h_{tj}(v_{Z_1}))^3 \\
&- (h_{tj}(v_{Z_2}))^3|^2 + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|^2\}^{1/2}. \\
&\leq \{\frac{1}{4n}\sum_{j=1}^{n} \{|(u_{Z_1}((tj))^3 - (u_{Z_2}(tj))^3| + |(g_{tj}(u_{Z_1}))^3 - (g_{tj}(u_{Z_2}))^3| \\
&+ |(\phi_{Z_1}(tj))^3 - (\phi_{Z_2}(tj))^3| + |(v_{Z_1}(tj))^3 - (v_{Z_2}(tj))^3| + |(h_{tj}(v_{Z_1}))^3 \\
&- (h_{tj}(v_{Z_2}))^3| + |(\omega_{Z_1}(tj))^3 - (\omega_{Z_2}(tj))^3|\}^{1/2} = (d_{NH}(Z_1, Z_2))^{1/2}.
\end{aligned}
\tag{3.37}
$$

□

3.4 Group Decision Making with $T2FFSs$ Based on Distance Measures

Here, we suggest a strategy for rating the various $T2FFSs$ using the suggested distance metrics for group $D-MG$ issues.

3.4.1 Approach for Distance Measure

Consider a limited number of m criteria like $\{K = k_1, k_2, k_3 \ldots k_m\}$ and n alternatives $\{R = R_1, R_2, R_3 ... R_n\}$ which are being evaluated by r $D-MRs$ $\{DM = Dm1, Dm2, Dm3 ... Dmr\}$ having weight vector $\{W = W_1, W_2, W_3 ...\}$ where $w_j \geq 0, j = 1, 2, 3 ... n$ and $\sum_{j=1}^{n} w_j = 1$.Consider the rating of $D-MRs$ as $P-MF$,$S-MF$, $P-NMF$ and $S-NMF$.

In order to determine the best alternative, the following processes have been explained.

1. Sort the overall data for each alternative R_i according to the criteria K_j for $P-MF$, $S-MF$, $P-NMF$, and $S-NMF$.

2. Calculate the distance between the $D-MRs$ $D-Mr$ and the null decision N, where N is the decision of the $D-MR$ and has one $P-NMF$ and $S-NMF$ for each alternative R_j that meets each of the criteria K_i and zero $P-MF$ and $S-MF$.

3. Consider all the alternative R_i, criteria K_j, and their associated maximum value of dmr to determine the maximum value of the $dmrs$ corresponding to $D-MR$, and then create the type-2 fermatean fuzzy alternative $R_i, (i = 1, 2, ..., n)$.

4. Determine the distance in units of d between the alternative R_i and the null decision N.

5. Rank the options $R_i, i = 1, 2, ..., t$ to see which is best.

Table 3.4.1.1 Linguistic grade and corresponding $P-MF$ and $P-NMF$ value

Grades	$P-MFV$	Grades	$P-NMFV$
Extremely faint(E-F)	0	Extremely strong(E-S)	1
Faint (F)	0.5	Strong(S)	0.9
Moderately Faint($m-f$)	0.6	Moderately Strong(M-S)	0.8
Moderately Strong(M-S)	0.8	Moderately Faint(m-f)	0.7
Strong(S)	0.9	Faint(F)	0.6
Extremely Strong(E-S)	1	Extremely Faint(E-F)	0

Table 3.4.1.2 Linguistic grade and corresponding $S-MF$ and $S-NMF$ value

Grades	S-MFV	Grades	S-NMFV
Extremely Faint	0	Extremely strong	1
Faint	0.4	Strong	0.9
Moderately Faint	0.5	Moderately Strong	0.8
Moderately Strong	0.8	Moderately Faint	0.7
Strong	0.9	Faint	0.6
Extremely Strong	1	Extremely Faint	0

Table 3.4.1.3 Graded values of the alternative corresponding to each attribute (criteria)

		DM1	DM1	DM1	DM1	DM2	DM2	DM2	DM2	DM3	DM3	DM3	DM3
		P-MF	S-MF	P-NMF	S-NMF	P-MF	S-MF	P-NMF	S-NMF	P-MF	S-MF	P-NMF	S-NMF
K1	R1	E-S	M-S	E-F	m-f	S	E-S	F	E-F	S	m-f	F	M-S
K1	R2	M-S	S	F	F	S	M-S	E-F	F	F	M-S	S	M-S
K1	R3	M-S	m-f	S	M-S	S	m-f	F	M-S	F	m-f	S	M-S
K1	R4	M-S	S	m-f	m-f	S	M-S	F	m-f	F	E-F	S	E-S
K2	R1	E-S	S	F	F	S	m-f	F	M-S	S	M-S	F	m-f
K2	R2	M-S	M-S	m-f	m-f	S	M-S	E-F	F	m-f	M-S	M-S	m-f
K2	R3	S	E-S	F	E-F	M-S	m-f	m-f	M-S	m-f	M-S	M-S	m-f
K2	R4	E-S	S	E-F	F	M-S	m-f	m-f	M-S	m-f	S	M-S	F
K3	R1	E-S	M-S	E-F	m-f	S	E-S	F	E-F	S	S	F	F
K3	R2	S	m-f	F	M-S	M-S	m-f	m-f	M-S	E-S	S	E-F	F
K3	R3	E-S	S	E-F	F	S	S	F	F	m-f	M-S	M-S	m-f
K3	R4	E-S	S	E-F	F	S	M-S	F	m-f	m-f	M-S	M-S	m-f
K4	R1	E-S	S	E-F	F	S	E-S	F	E-F	M-S	M-S	m-f	m-f
K4	R2	E-S	S	E-F	F	S	E-S	F	E-F	E-S	E-S	E-F	E-F
K4	R3	E-S	m-f	E-F	M-S	m-f	E-S	M-S	E-F	m-f	M-S	M-S	m-f
K4	R4	S	m-f	F	m-f	M-S	m-f	m-f	M-S	m-f	M-S	M-S	m-f

Table 3.4.1.4 Distance Measure Between d_{NH} and N

K1	R1	1	0.94	0.75
K1	R2	0.75	0.86	0.5
K1	R3	0.5	0.75	0.30
K1	R4	0.69	0.76	0.19
K2	R1	1	0.75	0.75
K2	R2	0.58	0.75	0.58
K2	R3	1	0.58	0.58
K2	R4	1	0.58	0.75
K3	R1	1	1	0.75
K3	R2	0.75	0.58	1
K3	R3	1	0.75	0.58
K3	R4	1	0.75	0.58
K4	R1	1	1	0.58
K4	R2	1	1	1
K4	R3	1	1	0.58
K4	R4	0.75	0.58	0.58

3.4.2 Mathematical Illustration

Take the case of a person who is trying to decide how much money to put into the market. There are five possible answers (I) $R1$ is an automobile firm, (ii) $R2$ is a pesticides company, (iii) $R3$ is a multinational enterprise, (iv) $R4$ is an armaments company, and (v) $R5$ is a tyre company. For this, they paid a specified panel of experts $(DM1, DM2, andDM3)$ whose weight vector is $(0.40, 0.35, 0.25)^T$. Under the $T2FFS$ set, the investor makes a choice based on a number of factors, including the project risk

K1, the revenue analysis K2, the social effect analysis K3, and the allocated space K4. Tables 3.4.1.1 and 3.4.1.2 display the $P-MF$, $P-NMF$, and $S-MF$, $S-NMF$ linguistic grades necessary for this purpose.

1. Table 3.4.1.3 provides the accumulated data of each alternative that corresponds to each criterion, ordered in terms of the linguistic grades based on the knowledge and experience of the $D-MRs$.

2. Determine the value of d(DMk,N) (k = 1, 2, 3) for each possible solution. Table 3.4.1.4 summarises the numbers we use for $d_{NH}(DMk, N)$ in our calculations.

3. Find the highest value of $d_{NH}(DMk, N)$ in Table 4 for all options Rj, (j = 1, 2,..., 4) for each criterion Ki, (i = 1, 2, 3, 4). And hence build the $T2FFS$ alternative, $Rj = (K_i((u_{Rj}), g_{Ki}(R_j)), v_{Rj}, h_{Ki}(R_j))$, as
$R_1 = K_1(1, 0.8, 0, 0.7), K_2(1, 0.9, 0, 0.6), K_3(1, 0.8, 0, 0.7), K_4(0.9, 1, 0.6, 0).$
$R_2 = K_1(0.9, 0.8, 0, 0.6), K_2(0.9, 0.8, 0.6, 0.7), K_3(1, 0.9, 0, 0.6), K_4(0.9, 1, 0.6, 0).$
$R_3 = K_1(0.9, 0.5, 0.6, 0.8), K_2(0.9, 1, 0.1, 0), K_3(1, 0.9, 0, 0.6), K_4(1, 0.5, 0, 0.8).$
$R_4 = K_1(0.9, 0.8, 0.6, 0.7), K_2(1, 0.9, 0, 0.6), K_3(1, 0.9, 0, 0.6), K_4(0.9, 0.8, 0.6, 0.7).$

4. Now, we have computed the recommended distance measurements, d_{NH} from N to Rj (j = 1, 2,..., 4) and the results are presented below. The values for $d_{NH}(R1, N)$ are 1.00, $d_{NH}(R2, N)$ are 0.9025, $d_{NH}(R3, N)$ are 0.9375 and $d_{NH}(R4, N)$ are 0.8775.

5. Our research has led us to the conclusion that R_1 is the most deserving of our investment capital.

3.4.3 Comparative Analysis

Comparative studies based on interval-valued and $T2FS$ and $T2IFS$ as suggested by the authors [16, 68, 135, 143, 155, 166, 167], To assess how well the proposed methods perform in comparison to existing methods, and their related findings are given in Table 3.4.3.1. This table shows that $A1$ is the best company to put money into compared to the others, and this result overlaps with the suggested outcomes. Therefore, compared to other existing approaches, the suggested technique can be used effectively to address the problem of $D-MG$.

Table 3.4.3.1 comparative analysis

Existing approach	score	values	score	values	Order of alternatives
	R1	R2	R3	R4	
[[16]]	0.800	0.800	0.7500	0.7400	$R1 \geq R2 \geq R4 \geq R3$
[[135]]	0.833	0.604	0.733	0.506	$R1 \geq R3 \geq R2 \geq R4$
[[68]]	0.676	0.727	0.372	0.471	$R2 \geq R1 \geq R4 \geq R3$
[[166]]	0.800	0.700	0.650	0.525	$R1 \geq R2 \geq R3 \geq R4$
[[167]]	0.400	0.400	0.375	0.387	$R1 \geq R2 \geq R4 \geq R3$
[[155]]	0.181	0.144	0.090	0.117	$R1 \geq R2 \geq R4 \geq R3$
[[143]]	0.784	0.555	0.470	0.352	$R1 \geq R2 \geq R3 \geq R4$
[d_{NH}]	1.000	0.902	0.9375	0.8775	$R1 \geq R3 \geq R2 \geq R4$

Chapter 4

Improving Decision-Making Under Uncertainty: A Comparative Study of Fuzzy Set Extensions

In this chapter we studied different fuzy sets like type-2 fuzzy sets, intuitionistic fuzzy sets and type-2 intuitionistic fuzzy sets. This chapter provides an overview of these sets, comparing and contrasting them using operations of union, intersection, and distance measures. Additionally, a new distance measure is proposed for Type-2 intuitionistic fuzzy sets.

This chapter is divided into several sections to help you understand and compare different existing FSs. Section 4.1 contains the introduction. In section 4.2, we'll cover the preliminaries and basic concepts to give you a solid foundation. Then in section 4.3 , we compare different FSs using the operations of union and intersection. We explore their similarities and differences, helping you make informed decisions for your specific needs. Section 4.4 proposes a new distance measure for $T2IFS$, accompanied by a numerical example to compare the results.

4.1 Introduction

L.A. Zadeh [159] developed FS theory in response to the requirement to represent the activity of modelling in the human mind, which must take into account subjective and imprecise elements. Its key idea is $M - G$, a member is either in or out of a subset according to conventional set theory. A proposition is either true or false in boolean logic. Information by its nature contains uncertainty, we make decisions in environments with various types of uncertainty in many scientific and industrial applications. Currently, the majority of $D - MG$ procedures involve acquiring and processing information, much of which is noisy, fragmented, inconsistent, or all of the above. As a result, the models that explain the real world must be supplemented by appropriate ambigious representations. "With the introduction of soft computing approaches, many strong tools in the field of computational intelligence, such as type-1 fuzzy logic, evolutionary algorithms, hybrid intelligent systems, and neural networks", were produced. [19, 114].

An extension of the ordinary FS, or $T1FS$ is the $T2FS$. $T2FSs$ could be referred to as a "fuzzy-fuzzy set" because the $M - Gs$ are ambiguous and the domain of $T2FSs$ is $T1FS$ instead of crisp value. Zadeh [161, 162] introduced the idea of $T2FS$. Mendel [102] provided overviews of $T2FSs$. Since $T2FSs$ are a specific case of ordinary FSs and $IVFS$, Takac [137] suggested that $T2FSs$ are very useful in situations where there are more uncertainties. From the perspectives of type reduction and the centroid, Kundu et al [80] gave a fixed charge transportation problem with type-2 fuzzy parameters. Both Dubois, Prade [42] and Mizumoto, Tanaka [77, 111] looked at the logical behaviour of $T2FS$. Later, a large number of scholars conducted extensive research on $T2FS$, theoretical and numerous application areas [53, 67, 76, 77].

The $IFSs$ developed by Atanassov [9] that can be expressed in terms of the $M - D$, and $N - MD$, a more generalised variant of the FS. The study of problems like $D - MG$ by utilising $IFSs$, however has attracted more attention [95]. In order to address the issue of students satisfaction with university instruction, Marasini et al. [96] used an IFS technique that may take into consideration two sources of uncertainty: one connected to items and the other to subjects. Dan et al.[32] Present the generalised $T2IFS$, whose type-1 membership is the conventional fuzzy membership and whose type-2 comprises both $M - F$ and $N - MF$ as the IFS. Singh.S and Garg.H [129] proposed a $MC - DM$ problem by providing a dmr for $T2IFS$.

FSs have transformed $D - MG$ by providing a mathematical tool for modeling uncertainty and imprecision. However, traditional fuzzy sets may not be adequate in certain situations, leading to the development of $T2FSs$, which introduce a third dimension $M - Fs$ to allow for more precise definitions of uncertainty. Different extensions of FSs exist to make them more manageable, and understanding their properties is crucial for selecting the most suitable set for specific conditions. $T1FS$, $T2FS$, IFS, and $T2IFS$ are sets examined for their properties, with numerical examples provided for comparison. Furthermore, a new dmr is proposed for $T2IFSs$, demonstrating its significance with an example. By grasping the diverse properties and applications of these FSs, informed decisions can be made in real-world situations with uncertainty and imprecision.

4.2 Preliminaries and Basic Concepts

4.2.1 Fuzzy set (FS)

1.1.1

4.2.2 Operation on Fuzzy Sets

The following operations for FSs are defined by [159] as generalisations of crisp sets and crisp statements in his first paper.

Definition 4.2.1. Intersection [logical and]: The following $M - F$ is used to describe the intersection of the FSs J and K

$$\mu_{J\cap K}(s) = Min\{(\mu_J(s), \mu_K(s))\forall s \in S. \tag{4.1}$$

Definition 4.2.2. Union [exclusive or]: The union's $M - F$ is described as

$$\mu_{J\cup K}(s) = Max\{(\mu_J(s), \mu_K(s)) \quad \forall s \in S. \tag{4.2}$$

Definition 4.2.3. Complement (negation): The following is a definition of the complement's $M - F$:

$$\mu_J(s) = 1 - \mu_J(s) \quad \forall s \in S. \tag{4.3}$$

Later, the above defined definitions were expanded. Both the intersection and the union can be modelled as t-norms [10, 38, 45, 49, 79, 94, 172]. Both kinds are associative, commutative, and monotonic. Below is a compilation of typical dual pairs of non-parameterized t-norms and t-conorms:

Definition 4.2.4. Drastic Product:

$$t_W(\mu_J(s), \mu_K(s)) = \begin{cases} Min\{(\mu_J(s), \mu_K(s))\} & if \quad Max\{(\mu_J(s), \mu_K(s))\} = 1 \\ 0 & otherwise\}. \end{cases} \tag{4.4}$$

Definition 4.2.5. Drastic Sum:

$$S_W(\mu_J(s), \mu_K(s)) = \begin{cases} Max\{(\mu_J(s), \mu_K(s))\} & if \quad Min\{(\mu_J(s), \mu_K(s))\} = 0 \\ 1 & otherwise\}. \end{cases} \tag{4.5}$$

Definition 4.2.6. Bounded Difference:

$$t_1(\mu_J(s), \mu_K(s)) = Max\{0, \mu_J(s) + \mu_K(s) - 1\}. \quad (4.6)$$

Definition 4.2.7. Bounded sum:

$$s_1(\mu_J(s), \mu_K(s)) = Min\{1, \mu_J(s) + \mu_K(s)\}. \quad (4.7)$$

Definition 4.2.8. Einstein Product:

$$t_{1.5}(\mu_J(s), \mu_K(s)) = \frac{\mu_J(s) \cdot \mu_K(s)\}}{2 - [\mu_J(s) + \mu_K(s) - \mu_J(s) \cdot \mu_K(s)]}. \quad (4.8)$$

Definition 4.2.9. Einstein Sum:

$$s_{1.5}(\mu_J(s), \mu_K(s)) = \frac{\mu_J(s) + \mu_K(s)\}}{1 + \mu_J(s) + \mu_K(s)}. \quad (4.9)$$

Definition 4.2.10. Hamachar Product:

$$t_{2.5}(\mu_J(s), \mu_K(s)) = \frac{\mu_J(s) \cdot \mu_K(s)\}}{\mu_J(s) + \mu_K(s) - \mu_J(s) \cdot \mu_K(s)}. \quad (4.10)$$

Definition 4.2.11. Hamachar Sum:

$$s_{2.5}(\mu_J(s), \mu_K(s)) = \frac{\mu_J(s) + \mu_K(s) - 2\mu_J(s) \cdot \mu_K(s)\}}{1 - \mu_J(s) \cdot \mu_K(s)}. \quad (4.11)$$

Definition 4.2.12. Minimum:

$$t_3(\mu_J(s), \mu_K(s)) = min\{\mu_J(s), \mu_K(s)\}. \quad (4.12)$$

Definition 4.2.13. Maximum:

$$s_3(\mu_J(s), \mu_K(s)) = max\{\mu_J(s), \mu_K(s)\}. \quad (4.13)$$

The above defined operators have been ordered as follows:

$$t_w \leq t_1 \leq t_{1.5} \leq t_2 \leq t_{2.5} \leq t_3. \quad (4.14)$$

$$s_3 \leq s_{2.5} \leq s_2 \leq s_{1.5} \leq s_1 \leq s_w. \quad (4.15)$$

The operations defined above are not valid for $T2FSs$ because $T2FSs$ contain type-2 $M - F$ so, extension principle is defined to deal with the operations for $T2FSs$.

4.2.3 $(T2FS)$

(1.1.14)

Definition 4.2.14. Footprint of Uncertainty(FOU) (1.1.15)

Example 4.2.1. *Let "Young" people be the set defined by* $T2FS$ $\bar{E}$ *and the* $P-MF$ *of* $\bar{E}$ *be "Youthness" and* $S-MF$ *be degree of "Youthness". Let* $T = \{7, 9, 13\}$ *be the car set having primary membership at point* T *respectively.* $j_7 = \{0.7, 0.8, 0.9\}, j_9 = \{0.5, 0.6, 0.7\}$ *and* $j_{13} = \{0.3, 0.4, 0.5\}$ *then* $S-MF$ *of point* 7 *is* $\bar{\mu}_{\bar{E}}(7, u) = \{(0.8/0.7) + (0.6/0.8) + (0.5/0.9)\}$ *that is* $\bar{\mu}_{\bar{E}}(7, 0.7) = 0.8$ *is the* $S-MG$ *of* 7 *with respect to* 0.7*, similarlaly* $\bar{\mu}_{\bar{E}}(9, u) = \{(0.7/0.5) + (0.6/0.6) + (0.5/0.7)\}$ *and* $\bar{\mu}_{\bar{E}}(13, u) = \{(0.8/0.3) + (0.7/0.4) + (0.4/0.5)\}$ *then discrete* $T2FS$ *can be defined accordingly* $\bar{E} = \{(0.8/0.7) + (0.6/0.8) + (0.5/0.9)\}/7 + \{(0.7/0.5) + (0.6/0.6) + (0.5/0.7)\}/9 + \{(0.8/0.3) + (0.7/0.4) + (0.4/0.5)\}/13.$

Definition 4.2.15. Extension Principle (1.1.17)

4.2.4 Intuitionistic Fuzzy set

(2.3.1)

Example 4.2.2. *Let "Young"persons be the set defined by* IFS J*. The degree of "Youthness"and "Adultness"represents* MV *and* NMV *respectively. Let* $T = \{11, 14, 16\}$ *and the* $M-G$ *of the point* 11 *be* $\mu_P(12) = \{0.7, 0.8, 0.9\}$ *and the* $N-MG$ *of point* 11 *is* $\nu_P(11) = \{0.1, 0.2, 0.0\}$ *similarly* $\mu_P(14) = \{0.5, 0.6, 0.7\}, \nu_P(14) = \{0.4, 0.3, 0.1\}$ *and* $\mu_P(16) = \{0.4, 0.5, 0.6\}, \nu_P(16) = \{0.5, 0.4, 0.2\}$.

4.2.5 Type 2 intuitionistic Fuzzy set$(T2IFS)$

Definition 4.2.16. [129] A $T2IFS$ J in the UOD S is set of pairs $\{s, \mu_J(s), \nu_J(s)\}$ where s is the element of $T2IFS$, $\mu_J(s)$ and $\nu_J(s)$ are called $M-G$ and $N-MG$ respectively defined in the interval [0,1] as

$$\mu_J(s) = \int_{s \in j_s^1} (g_s(u)/u), \quad \nu_J(s) = \int_{s \in j_s^2} (h_s(v)/v). \tag{4.16}$$

Where $g_s(u)/u$ and $h_s(v)/v$ are termed as $S-MF$ and $S-NMF$. In addition μ_J,ν_J denotes the $P-MF$ and $P-NMF$ and j_{s^1} and j_{s^2} are named as the $P-MF$ and $P-NMF$ of S, respectively. In other words, $T2IFS$ J is defined in the UOD as

$$J = \{(s, u_J, v_J), g_{sj}(u_J), h_{sj}(v_J) | s \in S, u_J \in j_{s^1}, v_J \in j_{s^2}\}. \tag{4.17}$$

Where the element of the domain $(s, (u_J, v_J))$ called as $P - MF$ (u_J) and $P - NMF$ (v_J) of $s \in S$ where $g_{sj}(u_J)$ and $h_{sj}(v_J)$ $S - MF$ and $S - NMF$ respectively.

4.3 Comperative Analysis on Different Types of FSs

4.3.1 Comparison on the Basis of Operation

In order to make comparison we take few FSs into account, ordinary FS or $T1FS$, $T2FS$, IFS and $T2IFS$ we define union and intersection for these defined sets.

4.3.2 Union and Intersection for $T1FS$ [159]

let J and K be two FSs then their union and intersection is defined as follows

Union:

$J \cup K = max\{\mu_J(s), \mu_K(s)\}$,

where $\mu_J(s)$ and $\mu_K(s)$ are the MVs of FS J and K.

Example 4.3.1. *Let J={s,0.8} and K={s,0.7}, then* $J \cup K = max\{0.8, 0.7\} \implies J \cup K = 0.8$.

Intersection:

$J \cap K = min\{\mu_J(s), \mu_K(s)\}$,

where $\mu_J(s)$ and $\mu_K(s)$ are the MVs of FS J and K.

Example 4.3.2. *Let J={s,0.8} and K={s,0.7} then* $J \cup K = min\{0.8, 0.7\} \implies J \cap K = 0.7$.

4.3.3 Union and Intersection for $T2FS$ [161]

Let μ_J and μ_K are two $T2FS$.

Intersection:

$\mu_J = \{s, \mu_J(s)\}$ and $\mu_K = \{s, \mu_K(s)\}$,

where $\mu_J(s) = \{u_i, \mu_{ui}(s)\}$,

$\mu_K(s) = \{v_j, \mu_{vj}(s)\}$,

by extension principle intersection is defined as

$$\mu_{J\cap K}(s) = \{z, \mu_{J\cap K}(z)| \ \ z = \min\{u_i, v_j\}\}, \tag{4.18}$$

where $\mu_{J\cap K}(z) = \sup_{z=\min(u_i,v_j)} \min\{\mu_{ui}, \mu_{vj}\}$.

Union:

$$\mu_{J\cup K}(s) = \{z, \mu_{J\cup K}(z)| \ \ z = \max\{u_i, v_j\}\}, \tag{4.19}$$

where $\mu_{J\cup K}(z) = \sup_{z=\max(u_i,v_j)} \min\{\mu_{ui}, \mu_{vj}\}$.

Example 4.3.3. *Let J be a small integer and K be an integer. Find $\mu_{J\cap K}(s)$ at s=3*

Table 4.3.3.1

i	u_i	μ_{ui}	v_j	μ_{vj}
1	0.8	1	1	1
2	0.7	0.5	0.8	0.5
3	0.6	0.4	0.7	0.3

$J = \{s, \mu_J(s)\}$ *at s=3*

$\mu_J(s) = \{(u_1, \mu_{u1}), (u_2, \mu_{u2}), (u_3, \mu_{u3})\}$.

={(0.8,1), (0.7,0.5), (0.6,0.4)},

similarly

$\mu_K(s) = \{(v_1, \mu_{v1}), (v_2, \mu_{v2}), (v_3, \mu_{v3})\}$.

={(1,1), (0.8,0.5), (0.7,0.3)}.

Table 4.3.3.2

u_i	v_j	$\min(u_i, v_j)$	$\mu_{ui}(3)$	$\mu_{vj}(3)$	$\min(\mu_{ui}(3), \mu_{vj}(3))$
0.8	1	0.8	1	1	1
0.8	0.8	0.8	1	0.5	0.5
0.8	0.7	0.7	1	0.3	0.3
0.7	1	0.7	0.5	1	0.5
0.7	0.8	0.7	0.5	0.5	0.5
0.7	0.7	0.7	0.5	0.3	0.3
0.6	1	0.6	0.4	1	0.4
0.6	0.8	0.6	0.4	0.5	0.4
0.6	0.7	0.6	0.4	0.3	0.3

$\mu_{J\cap K}(s) = \sup_{z=0.8}\{1, 0.5\} = 1,$

$\sup_{z=0.7}\{0.3, 0.5, 0.5, 0.3\} = 0.5.$

$\sup_{z=0.6}\{0.4, 0.4, 0.3\} = 0.4.$

4.3.4 Union and Intersection for IFS [52, 95]

let J and K be two $IFSs$ then we define

Union:

$$J \cup K = \max\{\mu_J(s), \mu_K(s)\}, \min\{\nu_J(s), \nu_K(s)\}. \tag{4.20}$$

Intersection:

$$J \cap K = \min\{\mu_J(s), \mu_K(s)\}, \max\{\nu_J(s), \nu_K(s)\}. \tag{4.21}$$

Example 4.3.4. *Let we have two IFS defined as*

J={s,0.6,0.4} and K={s,0.7,0.2}, then

$$J \cup K = \max\{0.6, 0.7\}, \min\{0.4, 0.2\}. = \{0.7, 0.2\}.$$

4.3.5 Union and Intersection for $T2IFSs$ [32]

lets consider two $T2IFS$ J and K

$$J = \int_{s\in S}\left(\int_{u\in i_s^u}(\mu_J(s,u),\nu_J(s,u))/u\right)/S.$$

And

$$K = \int_{s\in S}\left(\int_{v\in i_s^v}(\mu_K(s,v),\nu_K(s,v))/v\right)/S.$$

Where $i_s^u \subseteq [0,1]$ and $i_s^v \subseteq [0,1]$ are domains for $S-MF$ respectively. Then we define union for J and K as:

$$J\cup K = \int_{s\in S}\frac{\frac{\left(\int_{v\in i_s^w}(\mu_{J\cup K}(s,w),\nu_{J\cup K}(s,w)\right)}{w}}{S}, i_s^u \cup i_s^v = i_s^w \subseteq [0,1],$$

where

$$\mu_{J\cup K}(s) = \phi\left(\int_{u\in i_s^u}(\mu_J(s,u))/u, \int_{v\in i_s^v}(\mu_K(s,v))/v\right),$$

by using extension principle, we obtain

$$\mu_{J\cup K}(s,w) = \int_{u\in i_s^u}\int_{v\in i_s^v}\left(\mu_J(s,u)\wedge\mu_K(s,u)\right)/\phi(u,v),$$

where $\phi(u,v)$ is t-conorm of u and v,

$$\mu_{J\cup K}(s,w) = \int_{u\in i_s^u}\int_{v\in i_s^v}\left(\mu_J(s,u)\wedge\mu_K(s,u)\right)/(u\vee v),$$

similarly

$$\nu_{J\cup K}(s,w) = \int_{u\in i_s^u}\int_{v\in i_s^v}(\nu_J(s,u)\vee\nu_K(s,u))/(u\vee v).$$

Intersection for J and K is defined as:

$$J\cap K = \int_{s\in S}\frac{\frac{\left(\int_{v\in i_s^w}(\mu_{J\cap K}(s,w),\nu_{J\cap K}(s,w))\right)}{w}}{S}, i_s^u \cup i_s^v = i_s^w \subseteq [0,1],$$

where

$$\mu_{J\cap K}(s,w) = \int_{u\in i_s^u}\int_{v\in i_s^v}(\mu_J(s,u)\wedge\mu_K(s,u))/(u\wedge v).$$

And

$$\nu_{J\cap K}(s,w) = \int_{u\in i_s^u}\int_{v\in i_s^v} (\nu_J(s,u) \vee \nu_K(s,u))/(u \wedge v).$$

Example 4.3.5. *Let J and K be two $T2IFSs$ representing the set "Young" persons. The "Youthness" is $P-MF$ of J and K. Then the degree of "Youthness" and "Adultness" are the $S-MF$ and $S-NMF$ respectively. We consider both J and K to be defined on S={7,9,13} which are eventyualy represented as:*

J=((0.8,0.1)/0.7 +(0.6,0.2)/0.8 + (0.5,0.4)/0.9) /7 + ((0.7,0.2)/0.5 + (0.6,0.3) /0.6 +(0.5,0.4)/0.7) /9 +((0.8,0.2)/0.3 +(0.7,0.3)/0.4 +(0.4,0.5)/0.5)/13.

K= ((0.7,0.2)/0.6 +(0.5,0.4)/0.7 +(0.5,0.5)/0.8)/7 + ((0.8,0.2)/0.4 +(0.8,0.1)/0.5 +(0.4,0.5)/0.6)/9 +((0.7,0.3)/0.2 +(0.6,0.3)/0.3 +(0.4,0.4)/0.4)13.

Now for 7, $S-MF$ *and* $S-NMF$ *of* J *and* K *are*

$((0.8,0.1)/0.7 + (0.6,0.2)/0.8 + (0.5,0.4)/0.9)/7,$

and

$((0.7,0.2)/0.6 + (0.5,0.4)/0.7 + (0.5,0.5)/0.8)/7.$

For $S = 7$, *the union of* J *and* K *is* $(\mu_{J\cup K}(7), \nu_{J\cup K}(7))$

$= ((0.8,0.1)/0.7 + (0.6,0.2)/0.8 + (0.5,0.4)/0.9)/7 \vee ((0.7,0.2)/0.6 + (0.5,0.4)/0.7 + (0.5,0.5)/0.8)/7,$

$((0.8\wedge 0.7),(0.1\vee 0.2))/(0.7\vee 0.6) + ((0.8\wedge 0.5),(0.1\vee 0.4))/(0.7\vee 0.7) + ((0.8\wedge 0.5),(0.1\vee 0.5))/(0.7\vee 0.8) + + ((0.6\wedge 0.7),(0.2\vee 0.2))/(0.8\vee 0.6) + ((0.6\wedge 0.5),(0.2\vee 0.4))/(0.8\vee 0.7) + ((0.6\wedge 0.5),(0.2\vee 0.5))/(0.8\vee 0.8) + ((0.5\wedge 0.7),(0.4\vee 0.2))/(0.9\vee 0.6) + ((0.5\wedge 0.5),(0.4\vee 0.4))/(0.9\vee 0.7) + ((0.5\wedge 0.5),(0.4\vee 0.5))/(0.9\vee 0.8).$

$= (0.5,0.4)/0.7 + (0.5,0.5)/0.8 + (0.6,0.2)/0.8 + (0.5,0.4)/0.8 + (0.5,0.5)/0.8 + (0.5,0.4)/0.9 + (0.5,0.4)/0.9 + (0.5,0.5)/0.9.$

$= (0.5,0.4)/0.7 + (\max(0.5,0.6,0.5,0.5), \min(0.5,0.2,0.4,0.5))\ 0.8 + (\max(0.5,0.5,0.5), \min(0.4,0.4,0.5))/0.9.$

$= (0.5,0.4)/0.7 + (0.6,0.2)/0.8 + (0.5,0.4)/0.9.$

Analysis on Operations of Union and Intersection for Different Fuzzy Sets

FSs use a $M-F$ to assign a degree of membership to each element of a set. This allows for a more flexible and nuanced representation of uncertainty than the binary membership characteristic of classical sets. The union and intersection operations of FSs are defined by taking the maximum and minimum of the $M-Fs$, respectively.

$T2FSs$ take this idea one step further, by allowing the $M-F$ itself to be a FS. This enables an even more sophisticated representation of uncertainty, but also makes the union and intersection operations more complex.

IFS go beyond the binary membership characteristic of FSs and also incorporate a $N-MD$. This allows for a more nuanced representation of uncertainty, particularly when dealing with vague or ambiguous information. The union and intersection operations of $IFSs$ take into account both $M-D$ and $N-MD$.

$T2IFSs$ combine the concepts of $T2FS$ and IFS, allowing for an even more sophisticated representation of uncertainty. The union and intersection operations of $T2IFSs$ also take into account both $M-D$ and $N-MD$, making them particularly useful for handling uncertain or ambiguous information.

Overall, these set types offer a rich and powerful toolbox for dealing with uncertainty and imprecision in a wide range of applications, including $D-MG$, data analysis, and control systems.

Results of Comparison

As we compared different FSs on the basis of union and intersection every FS has their importance, but we found that $T2IFSs$ offer a best tool for solving $D-MG$ problems. In terms of operations, $T2IFSs$ exhibit differences compared to other FSs. The union and intersection operations for $T2IFSs$ involve considering the lower and upper MV and NMV separately. This allows for a more flexible and granular manipulation of FSs, enabling $D-MR$ to capture the various degrees of uncertainty and ambiguity inherent in complex decision problems.

4.3.6 Comparison on the Basis of Distance Measures

Distance Measure Between FSs and $T2FSs$ [97, 127]

Definition 4.3.1. Distance measure plays an important role in $D-MG$. Let $F_1(S)$ be the class of all $T1FS$ of S. $\mu_J(s) \rightarrow [0,1]$is the $M-F$ of S in $F_1(S)$. Let's consider two

FSs J and K in $F_1(S)$. Then d(J,K) is said to be a *dmr* between J and K if

$$d\colon F_1(S) \times F_1(S) \to [0,1]. \tag{4.22}$$

satisfies following axioms:

$$(p1) \quad 0 \leq d(J,K) \leq 1 \;\forall\; J,K \in F_1(S). \tag{4.23}$$

$$(p2) \quad d(J,K) = d(K,J). \tag{4.24}$$

$$(p3) \quad d(J,K) = 0 \quad if\;\; J = K. \tag{4.25}$$

$$(p4) \quad d(J,K) = 0, d(J,L) = 0, L \in F_1(S) \;\text{ then }\; d(K,L) = 0. \tag{4.26}$$

For two FSs J and K, the following *dmr* is provided.

Hamming distance

$$d_{1h}(J,K) = \frac{1}{n}\sum_{j=1}^{n} |\mu_J(s_j) - \mu_K(s_j)|. \tag{4.27}$$

Euclidian distance

$$d_{1e}(J,K) = \{\frac{1}{n}\sum_{j=1}^{n} |\mu_J(s_j) - \mu_K(s_j)|^2\}^{1/2}. \tag{4.28}$$

4.3.7 Numerical Example

Lets consider four types of metal fields and each fieldis characterized by five different metals. We can express these four fields by FSs $\{c_1, c_2, c_3, c_4\}$ in space $\{S = s_1, s_2, s_3, s_4, s_5\}$. See table 4.3.7.1, there is another kind of special metal n, so we have to find which metal field this metal belongs.

Table 4.3.7.1

	s_1	s_2	s_3	s_4	s_5
$u_{c_1}(s)$	1	0.7	0.5	0.7	1
$u_{c_2}(s)$	1.0	0.7	0.9	0.9	0.9
$u_{c_3}(s)$	1.0	0.9	1.0	0.9	0.9
$u_{c_4}(s)$	0.9	0.9	0.9	0.2	0.7
$u_n(s)$	0.9	0.2	0.2	0.2	0.9

we have

$$d_{1h}(J,K) = \frac{1}{5}\sum_{j=1}^{5} |\mu_J(s_j) - \mu_K(s_j)|, \tag{4.29}$$

since from the table 3 and using $d_{1h}(J,K)$ we get following result

$d_{1h}(c_1,n) = 0.3, d_{1h}(c_2,n) = 0.4, d_{1h}(c_3,n) = 0.575, d_{1h}(c_4,n) = 0.32,$

which implies special metal n is produced from metal field c_1

for $T1FS$, we have only $M-F$ but for $T2FS$ we have $P-MF$, $S-MF$ and FOU.

Distance Measure Between $T2FSs$

[127] Examine the following factors in order to calculate the distance measure for $T2FSs$. $P-MF$, $S-MF$ and FOU in the currently used dmr the following dmr is defined for $T2FSs$ J and K.

$$d_{2h}(J,K) = \frac{1}{2n}\sum_{j=1}^{n} |u_J(s_j) - u_K(s_j)| + |f_{sj}(u_J) - f_{sj}(u_k)| + |\xi_J(s_j) - \xi_K(s_j)|. \tag{4.30}$$

4.3.8 Numerical Example

Let's consider four types of metal fields and each field is featured by 5 metals . We can express these four fields by $T2FSs$ $\{c_1, c_2, c_3, c_4\}$ in space $\{S = s_1, s_2, s_3, s_4, s_5\}$.See table 4.3.8.1. There is another kind of special metal $\{n\}$ so we have to find which metal field this metal belongs.

Table 4.3.8.1

	s_1	s_2	s_3	s_4	s_5
$u_{c_1}(s)$	1	0.7	0.5	0.7	1
$f_s(u_{c_1})$	0.7	0.9	0.2	0.5	0.9
$u_{c_2}(s)$	1.0	0.7	0.9	0.9	0.9
$f_s(u_{c_2})$	0.9	0.7	1.0	0.7	0.7
$u_{c_3}(s)$	1.0	0.9	1.0	0.9	0.9
$f_s(u_{c_3})$	0.7	1.0	0.9	0.9	0.4
$u_{c_4}(s)$	0.9	0.9	0.9	0.2	0.7
$f_s(u_{c_4})$	1.0	0.7	0.5	0.0	0.4
$u_n(s)$	0.9	0.2	0.2	0.2	0.9
$f_s(u_n)$	0.4	0.5	0.4	0.0	0.7

we have

$$d_{2h}(J,K) = \frac{1}{2n}\sum_{j=1}^{n}|u_J(s_j) - u_K(s_j)| + |f_{sj}(u_J) - f_{sj}(u_k)| + |\xi_J(s_j) - \xi_K(s_j)|, \quad (4.31)$$

since from the table 4 and using $d_{2h}(J,K)$ we get following result

$d_{2h}(c_1,n) = 0.44, d_{2h}(c_2,n) = 0.48, d_{2h}(c_3,n) = 0.6, d_{2h}(c_4,n) = 0.46,$

which implies special metal n is produced from metal field c_1.

Distance Measures Between IFS [140]

Definition 4.3.2. Let J and K be two IFS in $S = \{s_1, s_2, ...s_n\}$

$$d_3(J,K) = \frac{1}{n}\sum_{i=1}^{n}\frac{|\mu_J(s_i) - \mu_K(s_i)| + |\nu_J(s_i) - \nu_K(s_i)|}{4} + \frac{max(|\mu_J(s_i) - \mu_K(s_i)|, |\nu_J(s_i) - \nu_K(s_i)|)}{2}, \quad (4.32)$$

where$J = \{s_i, \mu_J(s_i), \nu_J(s_i)|s_i \in S\}, K = \{s_i, \mu_K(s_i), \nu_K(s_i)|s_i \in S\}$.

4.3.9 Numerical Example

Lets consider four kinds of metal fields and each field is featured by five metals . We can express these four fields by $T2IFSs$ $\{c_1, c_2, c_3, c_4\}$ in space $\{S = s_1, s_2, s_3, s_4, s_5\}$.

See table 4.3.9.1, there is another kind of special metal $\{n\}$ so, we have to find which metal field this metal belongs.

Table 4.3.9.1

	x_1	x_2	x_3	x_4	x_5
$u_{c_1}(x)$	1	0.7	0.5	0.7	1
$v_{c_1}(x)$	0	0.1	0.4	0.2	0
$u_{c_2}(x)$	1.0	0.7	0.9	0.9	0.9
$v_{c_2}(x)$	0	0.4	0.1	0.1	0.1
$u_{c_3}(x)$	1.0	0.9	1.0	0.9	0.9
$v_{c_3}(x)$	0.0	0.1	0.0	0.1	0.1
$u_{c_4}(x)$	0.9	0.9	0.9	0.2	0.7
$v_{c_4}(x)$	0.1	0.0	0.1	0.7	0.2
$u_n(x)$	0.9	0.2	0.2	0.2	0.9
$v_n(x)$	0.1	0.7	0.7	0.7	0.0

we have

$$d_3(J,K) = \frac{1}{n}\sum_{i=1}^{n} \frac{|\mu_J(s_i) - \mu_K(s_i)| + |\nu_J(s_i) - \nu_K(s_i)|}{4} + \frac{max(|\mu_J(s_i) - \mu_K(s_i)|, |\nu_J(s_i) - \nu_K(s_i)|)}{2}, \tag{4.33}$$

since from the table 5 and using $d_2(P,Q)$ we get following result

$$d_3(c_1,n) = 0.305, d_3(c_2,n) = 0.285, d_3(c_3,n) = 0.460, d_3(c_4,n) = 0.315,$$

which implies special metal n is produced from metal field c_2.

Definition 4.3.3. [129] The variance margin function $(V - MF)$ of $T2IFS$ is defined as the difference between $P-MF$ and $S-MF$, $P-NMF$ and $S-NMF$. It is denoted by η and ξ respectively.

Now we extended this new distance measure for $T2IFSs$ and provided the comparison between this dmr with exixting dmr with a numerical example.

4.4 New Distance Measures Between $T2IFS$

Firstly we analyse the definition of "dmr for $T2IFS$". Singh, S., & Garg, H. [129] defined the concept for $T2IFS$ where they used triangle inequality and we defined the inclusion relation between $T2IFS$ which is not satisfied by euclidean dmr. It is necessary to establish the inclusion relation between $T2IFS$, so we introduced a new dmr which satisfies inclusion relation in $T2IFS$.

For convenience, two $T2IFSs$ P and Q in T are denoted by $P = \{t(u, f_{tj}(u_P), (v, g_{tj}(v_P))|t \in T\}$ and $Q = \{t(u, f_{tj}(u_Q), (v, g_{tj}(v_Q))|t \in T\}$, then we defined new distance for P and Q by considering the $P-MF$, $S-MF$, $P-NMF$ and $S-NMF$.

$$\begin{aligned} d_4(P,Q) = \frac{1}{2n}\sum_{i=1}^{n}\frac{1}{4}\{&|u_P(t_i) - u_Q(t_i)| + |v_P(t_i) - v_Q(t_i)| + \\ &|f_{ti}(u_P) - f_{ti}(u_Q)| + |g_{ti}(u_P) - g_{ti}(u_Q)|\} \\ &+ \frac{1}{2}\{max|u_P(t_i) - u_Q(t_i)|, |v_P(t_i) - v_Q(t_i)|, \\ &|f_{ti}(u_P) - f_{ti}(u_Q)|, |g_{ti}(u_P) - g_{ti}(u_Q)|\}. \end{aligned} \tag{4.34}$$

Definition 4.4.1. A real function $d_4\colon F_2^I(t) \times F_2^I(t) \to [0, 1]$ is called dmr, where d_4 defines the following axioms

(p1). $0 \leq d_4(P, Q) \leq 1, \ \forall \ (P, Q) \in F_2^I(t)$.

(p2). $d_4(P, Q) = 0$, If $P = Q$.

(p3). $d_4(P, Q) = d_4(Q, P)$.

(p4). $P \subseteq Q \subseteq R$, where $P, Q, R \in F_2^I(t)$, then $d_4(P, R) \geq d_4(P, Q)$ and $d_4(P, R) \geq d_4(Q, R)$.

Now we will prove the above defined measure is a valid dmr for $T2IFS$. Condition (P_1)

$$(P_1) \implies 0 \leq d_4(P, Q) \leq 1.$$

Let P and Q be two $T2IFS$ then we have

$|u_P(t_i) - u_Q(t_i)| \geq 0, |f_{ti}(u_P) - f_{ti}(u_Q)| \geq 0,$

$|v_P(t_i) - v_Q(t_i)| \geq 0, |g_{ti}(u_P) - g_{ti}(u_Q)| \geq 0,$

this implies $d_2(P, Q) \geq 0$,
then we have $|u_P(t_i) - u_Q(t_i)| \leq 1, |f_{ti}(u_P) - f_{ti}(u_Q)| \leq 1,$

$|v_P(t_i) - v_Q(t_i)| \leq 1, |g_{ti}(u_P) - g_{ti}(u_Q)| \leq 1,$

$\implies d_4(P,Q) \leq 1$, hence

$$0 \leq d_4(P,Q) \leq 1.$$

Condition (P_2) follows trivialy so we prove for (P_3) and (P_4).

$$(P_3) \implies d_4(P,Q) = d_4(Q,P),$$

we have

$$\begin{aligned} d_4(P,Q) = \frac{1}{2n}\sum_{i=1}^{n}\frac{1}{4}\{&|u_P(t_i) - u_Q(t_i)| + |v_P(t_i) - v_Q(t_i)| + \\ &|f_{ti}(u_P) - f_{ti}(u_Q)| + |g_{ti}(u_P) - g_{ti}(u_Q)|\} \\ &+ \frac{1}{2}\{max|u_P(t_i) - u_Q(t_i)|, |v_P(t_i) - v_Q(t_i)|, \\ &|f_{ti}(u_P) - f_{ti}(u_Q)|, |g_{ti}(u_P) - g_{ti}(u_Q)|\}, \end{aligned} \tag{4.35}$$

$$\begin{aligned} = \frac{1}{2n}\sum_{i=1}^{n}\frac{1}{4}\{&|u_Q(t_i) - u_P(t_i)| + |v_Q(t_i) - v_P(t_i)| + \\ &|f_{ti}(u_Q) - f_{ti}(u_P)| + |g_{ti}(u_Q) - g_{ti}(u_P)|\} \\ &+ \frac{1}{2}\{max|u_Q(t_i) - u_P(t_i)|, |v_Q(t_i) - v_P(t_i)|, \\ &|f_{ti}(u_Q) - f_{ti}(u_P)|, |g_{ti}(u_Q) - g_{ti}(u_P)|\}. \end{aligned} \tag{4.36}$$

$$= d_4(Q,P).$$

$$\implies d_4(P,Q) = d_4(Q,P).$$

Now to prove (P_4)

$$(P_4) \implies d_4(P,R) \geq d_4(P,Q), \tag{4.37}$$

it is easy to see that $|u_P(t_i) - u_R(t_i)| \geq |u_P(t_i) - u_Q(t_i)|, |f_{ti}(u_P) - f_{ti}(u_R)| \geq |f_{ti}(u_P) - f_{ti}(u_Q)|$

$|v_P(t_i) - v_R(t_i)| \geq |v_P(t_i) - v_Q(t_i)|, |g_{ti}(u_P) - g_{ti}(u_R)| \geq |g_{ti}(u_P) - g_{ti}(u_Q)|$, so we have

$$\begin{aligned} &\frac{1}{2n}\sum_{i=1}^{n}\frac{|u_P(t_i) - u_R(t_i)| + |v_P(t_i) - v_R(t_i)| + |f_{ti}(u_P) - f_{ti}(u_R)| + |g_{ti}(u_P) - g_{ti}(u_R)|}{4} \\ &\quad + \frac{max|u_P(t_i) - u_R(t_i)|, |v_P(t_i) - v_R(t_i)|, |f_{ti}(u_P) - f_{ti}(u_R)|, |g_{ti}(u_P) - g_{ti}(u_R)|}{2}, \end{aligned} \tag{4.38}$$

$$\geq \frac{1}{2n}\sum_{i=1}^{n}\frac{|u_P(t_i)-u_Q(t_i)|+|v_P(t_i)-v_Q(t_i)|+|f_{ti}(u_P)-f_{ti}(u_Q)|+|g_{ti}(u_P)-g_{ti}(u_Q)|}{4}$$
$$+\frac{max|u_P(t_i)-u_Q(t_i)|,|v_P(t_i)-v_Q(t_i)|,|f_{ti}(u_P)-f_{ti}(u_Q)|,|g_{ti}(u_P)-g_{ti}(u_Q)|}{2}, \tag{4.39}$$

hence we obtained $d_4(P,R) \geq d_4(P,Q)$, similarly we can also prove for $d_4(P,R) \geq d_4(Q,R)$, hence satisfies condition (P_4) so we proved this is a valid distance measure for $T2IFS$.

4.4.1 Numerical Example

Let's explore four categories of metal fields, where each field is represented by five distinct metals. We can express these four fields by $T2IFSs$ $\{c_1, c_2, c_3, c_4\}$ in space $\{T = t_1, t_2, t_3, t_4, t_5\}$. See table 4.4.1.1. There is another kind of special metal $\{n\}$ so we have to find which metal field this metal belongs.

Table 4.4.1.1

	t_1	t_2	t_3	t_4	t_5
$u_{c_1}(t)$	1	0.7	0.5	0.7	1
$f_t(u_{c_1})$	0.7	0.9	0.2	0.5	0.9
$v_{c_1}(t)$	0	0.1	0.4	0.2	0
$g_t(u_{c_1})$	0.2	0.1	0.5	0.4	0.1
$u_{c_2}(t)$	1.0	0.7	0.9	0.9	0.9
$f_t(u_{c_2})$	0.9	0.7	1.0	0.7	0.7
$v_{c_2}(t)$	0	0.4	0.1	0.1	0.1
$g_t(u_{c_2})$	0.1	0.4	0	0.2	0.2
$u_{c_3}(t)$	1.0	0.9	1.0	0.9	0.9
$f_t(u_{c_3})$	0.7	1.0	0.9	0.9	0.4
$v_{c_3}(t)$	0.0	0.1	0.0	0.1	0.1
$g_t(u_{c_3})$	0.2	0	0.1	0.1	0.5
$u_{c_4}(t)$	0.9	0.9	0.9	0.2	0.7
$f_t(u_{c_4})$	1.0	0.7	0.5	0.0	0.4
$v_{c_4}(t)$	0.1	0.0	0.1	0.7	0.2
$g_t(u_{c_4})$	0	0.1	0.4	1.0	0.5
$u_n(t)$	0.9	0.2	0.2	0.2	0.9
$f_t(u_n)$	0.4	0.5	0.4	0.0	0.7
$v_n(t)$	0.1	0.7	0.7	0.7	0.0
$g_t(u_n)$	0.5	0.4	0.5	1.0	0.1

we have

$$d_4(P,Q) = \frac{1}{2n}\sum_{i=1}^{n}\frac{|u_P(t_i)-u_Q(t_i)|+|v_P(t_i)-v_Q(t_i)|+|f_{ti}(u_P)-f_{ti}(u_Q)|+|g_{ti}(u_P)-g_{ti}(u_Q)|}{4} + \frac{max|u_P(t_i)-u_Q(t_i)|,|v_P(t_i)-v_Q(t_i)|,|f_{ti}(u_P)-f_{ti}(u_Q)|,|g_{ti}(u_P)-g_{ti}(u_Q)|}{2}, \tag{4.40}$$

since from the table 6 and using $d_2(P,Q)$ we get following result

$$d_2(c_1,n) = 0.275, d_2(c_2,n) = 0.312, d_2(c_3,n) = 0.385, d_2(c_4,n) = 0.259.$$

Which implies special metal n is produced from metal field c_4 obviously this coincides with the result of Sukhveer Singh and Harish Garg [129] but there approach is not valid for some calculations as it gives value beyond 1.0 which means our approach is better and also our approach includes inclusion relation which is stronger than triangle inequality.

Analysis on the Basis of Distance measure for Different Fuzzy Sets

$T1FSs$ are distinguished by $M-Fs$ that are created using the degree of membership between each element, set in the range [0, 1]. Yet, a wide variety of recent publications on $D-MG$ issues have taken $IFSs$ into account to handle the ambiguity. $IFSs$ are the generalised version of FSs proposed by Atanassov [4], which gives the freedom to also model the reluctance in the $D-MG$. They are specified by a $M-D$, $N-MD$, and the hesitation margin is obtained by subtracting both from unity. Yet, as these traditional $T1FSs$ or $IFSs$ still have crisp membership values, they are frequently linked to interpretability problems. There is a membership and a non-membership in type-1 when dealing with these classical $IFSs$, and it is thought that the uncertainty in the evaluation can be seen of as dissipating. There may still be some confusion close to the membership and non-membership boundaries, though. Moreover, confusing and imprecise information tends to be more prevalent in real-world application contexts. Type-2 $M-Fs$ can be used to solve this issue, as type-2 $M-F$ demonstrate $T2FSs$. It can be easily seen from the above defined two examples for $T1IFS$ and $T2IFS$ respectively. In first example we use only MVs and NMVs but in 2nd example we take secondary membership and secondary non-membership values into consideration, so it better to use $T2IFS$ instead of $T1IFS$ when the uncertainty is so high. We analysed different FSs and calculated the $dmrs$ between these sets by using numerical examples to check out the comparison and we found that $T2IFS$ are better.

Results of Comparison

To understand their importance, a comparison based on distance measures was conducted, using examples for each type of fuzzy set. Distance measures provide a quantitative assessment of similarity or dissimilarity between fuzzy sets. Through these examples, it becomes apparent that $T2IFSs$ outperform the other fuzzy sets when faced with ambiguous or uncertain information.

Chapter 5

Exploring the Power of Type-2 Intuitionistic Fuzzy Sets in Multicriteria Decision Making with a Novel Distance Measure

The main focus of this chapter is to study type-2 intuitionistic fuzzy sets and highlights the significance of type-2 intuitionistic fuzzy sets in decision-making. The study also discusses the challenges faced in decision-making situations and how type-2 intuitionistic fuzzy sets can address them. Additionally, the chapter introduces a novel distance measure for type-2 intuitionistic fuzzy sets that considers the uncertainty in the membership and non-membership functions. The importance of new distance measure is defined with the aid of numerical illustration.

This chapter is structured into several sections, each covering an important aspect of the proposed dmr for $T2IFSs$. Section 5.1 provides the introduction. Section 5.2 provides preliminaries and basic concepts to help readers understand the foundation of the study. In section 5.3, the new dmr based on three-dimensional representations is introduced, and its advantages over previous methods are discussed. This section is the heart of the study, and it provides a detailed description of the proposed dmr. Section 5.4 explores the application of the proposed dmr in group $D-MG$ with $T2IFS$. This section shows how the new dmr can be used to make better decisions in a group setting.

5.1 Introduction

Professionals such as engineers, surgeons, lawyers, scientists, and HR managers face diverse challenges daily to perform their duties effectively. Selecting the most suitable professional for a task is a crucial element of everyday life, as it can significantly impact the outcome and success of a project or goal. However, making this decision can be difficult as it requires careful consideration of various factors. A robust $D-MG$ theory can help facilitate the $D-MG$ process by providing a systematic framework for analysing and evaluating alternatives. This enables $D-MRs$ to make informed decisions based on objective criteria and reduces the risk of making poor or irrational choices.

[159] Zadeh's pioneering theory of FSs has demonstrated significant accomplishments across various fields. According to this theory, an element's belongingness to a FS is denoted by a solitary number within the range of 0 to 1, encompassing both endpoints. This number is commonly referred to as the $M-G$, and it communicates the degree to which an element pertains to a specific FS. Although the $N-MD$ in a FS is commonly understood as the complement of the $M-D$, this is not always the case. In other words, the value of $N-MD$ may not always be equal to 1 minus $M-D$, indicating that there may be some ambiguity or hesitation in membership determination. The concept of an IFS was introduced by Attnassov [4–6, 9]. IFS offers an even more precise, realistic, and practical representation of the objective world than traditional FSs. IFS have gained widespread popularity and are more frequently utilised than FSs, mainly because they have been extensively researched and utilised in various fields, including $D-MG$ [23, 134], pattern recognition [68, 82], and medical diagnosis [34]. Yager [153] proposed the $PFSs$ as a development of the IFS with the limitation that the square sum of its $M-D$ and $N-MD$ be less than or equal to 1. Some t-conorm-based $dmrs$ for $PFSs$ applid to $D-MG$ was given by Ganai. A. H.[51]. A $MC-DM$ based on $dmrs$ and knowledge measures of $FFSs$ given by Ganie. A. H. [50]. A Generalized hesitant fuzzy knowledge measure with its application to $MC-DM$ is given by Singh, S. and Ganie, A. H. [126]. "Almulhim, T. and Barahona, I. [1] gave an extended picture fuzzy $MC-DM$. Gave a case study of COVID-19 vaccine allocation".

Sing.S and Garg.H [129] proposed $T2IFS$, and in order to get around it, they took into account a $M-D$, an $N-MD$, and their related FOU and referred to the theory as a $T2IFS$. They introduced the idea of the $T2IFS$ and also defined $dmrs$ for $T2IFSs$ as a result, taking into account the fact that the $T2IFS$ is better equipped to handle imprecise and uncertain data in practical situations. A number of $dmrs$ based on Hamming, Euclidean, and maximum metrics have been suggested. The study introduces a new approach to measuring the distance between $T2IFSs$ that is based on their three-dimensional representations and satisfies the axiomatic definition of a distance. Unlike previous methods that rely on the triangle inequality property, the proposed dmr uses the inclusion relation, which makes it more accurate and reliable. The Euclidean distance fails to satisfy the inclusion relation, which highlights the importance of developing new distance measurement techniques. The study evaluates the relationships between the $dmrs$ of various $T2IFSs$ and demonstrates that the novel dmr can distinguish between them more effectively. The numerical example presented in the study showcases how the new distance measure gives more logical findings than other distance measurement techniques, which can have important implications in fields such as computer graphics and image processing. Overall, the study presents a significant contribution to the field of $T2IFS$ distance measurement and provides a solid foundation for further research in this area.

5.2 Preliminaries and Basic Concepts

Definition 5.2.1. Type-2 fuzzy set (1.1.14)

Definition 5.2.2. Footprint of Uncertainty (1.1.15)

Definition 5.2.3. Type-2 intuitionistic fuzzy set ($T2IFS$) (4.2.16)

5.2.1 Union and Intersection for $T2IFSs$ [32]

(4.3.5)

Definition 5.2.4. [127] Variance margin function ($V-MF$) of $T2IFS$ is defined as the difference between $P-MF$ and $S-MF$, $P-NMF$ and $S-NMF$. It is denoted by η and ξ respectively.

Example 5.2.1. *Let "young" be the set defined by T2IFS $\bar{J}$. "youthness" is the $P-MF$ of $\bar{J}$ then the degree of "youthness" and "adultness" are the $S-MF$ and $S-NMF$ respectively. Let S={7,9,13} be the set and $P-MF$ of the points of S is* $j_7 = \{0.7, 0.8, 0.9\}, j_9 = \{0.5, 0.6, 0.7\}$ *and* $j_{13} = \{0.3, 0.4, 0.5\}$ *respectively.*

5.2.2 Distance Measures Between ($T2IFS$)

Distance Measure between $T2IFS$ has been defined by [129] presented the $H-D$ and the $E-D$ between $T2IFNs$. Let $F_2^I(s)$ be the class of $T2IFSs$ over the universal set S.

Definition 5.2.5. A real function $D\colon F_2^I(s) \times F_2^I(s) \to [0,1]$ is said to be a dmr, if D satisfies the following properties:

$$0 \leq D(R_1, R_2) \leq 1, \forall (R_1, R_2) \in F_2^I(s). \tag{5.1}$$

$$D(R_1, R_2) = 0, IF \quad R_1 = R_2. \tag{5.2}$$

$$D(R_1, R_2) = D(R_2, R_1). \tag{5.3}$$

$$D(R_1, R_2) = 0, D(R_1, R_3) = 0, R_3 \in F_2^I(s) \;\; then \;\; D(R_2, R_3) = 0. \tag{5.4}$$

For convenience, two $T2IFSs$ R_1 and R_2 in T are denoted by

$R_1 = \{s(u, f_{sj}(u_{R_1}), (v, g_{tj}(v_{R_1}))|s \in S\}$ and $R_2 = \{s(u, f_{tj}(u_{R_2}), (v, g_{tj}(v_{R_2}))|s \in S\}$ then following distances for R_1 and R_2 are defined by considering the $P-MF$, $S-MF$, $P-NMF$, $S-NMF$, FOU and $V-MF$.

- **Hamming Distance**

$$d_1(R_1, R_2) = 1/4 \sum_{j=1}^{n} \{|u_{R_1}(sj) - u_{R_2}(sj)| + |g_{sj}(u_{R_1}) - g_{sj}(u_{R_2})| + |\phi_{R_1}(sj) - \phi_{R_2}(sj)| + |v_{R_1}(sj) - v_{R_2}(sj)| + |h_{sj}(v_{R_1}) - h_{sj}(v_{R_2})| + |\omega_{R_1}(sj) - \omega_{R_2}(sj)|\}. \tag{5.5}$$

- **Normalised Hamming Distance**

$$d_2(R_1,R_2) = 1/4n\sum_{j=1}^{n}\{|u_{R_1}(sj)-u_{R_2}(sj)|+|g_{sj}(u_{R_1})-g_{sj}(u_{R_2})|+|\phi_{R_1}(sj)-\phi_{R_2}(sj)| + |v_{R_1}(sj)-v_{R_2}(sj)|+|h_{sj}(v_{R_1})-h_{sj}(v_{R_2})|+|\omega_{R_1}(sj)-\omega_{R_2}(sj)|\}. \quad (5.6)$$

- **Euclidean Distance**

$$d_3(R_1,R_2) = \{1/4\sum_{j=1}^{n}\{|u_{R_1}(sj)-u_{R_2}(sj)|^2+|g_{sj}(u_{R_1})-g_{sj}(u_{R_2})|^2+|\phi_{R_1}(sj)-\phi_{R_2}(sj)|^2 + |v_{R_1}(sj)-v_{R_2}(sj)|^2+|h_{sj}(v_{R_1})-h_{sj}(v_{R_2})|^2+|\omega_{R_1}(sj)-\omega_{R_2}(sj)|^2\}\}^{1/2}. \quad (5.7)$$

- **Normalized Euclidean distance**

$$d_4(R_1,R_2) = \{1/4n\sum_{j=1}^{n}\{|u_{R_1}(sj)-u_{R_2}(sj)|^2+|g_{sj}(u_{R_1})-g_{sj}(u_{R_2})|^2+|\phi_{R_1}(sj)-\phi_{R_2}(sj)|^2 + |v_{R_1}(sj)-v_{R_2}(sj)|^2+|h_{sj}(v_{R_1})-h_{sj}(v_{R_2})|^2+|\omega_{R_1}(sj)-\omega_{R_2}(sj)|^2\}\}^{1/2}. \quad (5.8)$$

5.3 New Distance Measures Between $T2IFS$

In this part, we suggest a new technique to compute the distance between $T2IFSs$ by replacing the axiom of triangular inequality from Sing.S and Garg.H [129] *dmr* with an inclusion relation based on the 3-D representation of $T2IFSs$. For two $T2IFSs$ G and H in S denoted by $G = \{s(u, f_{sj}(u_G),(v, g_{sj}(v_G))|s \in S\}$ and $H = \{s(u, f_{sj}(u_H),(v, g_{sj}(v_H))|s \in S\}$. A new *dmr* for G and H by considering the $P-MF$, $S-MF$, $P-NMF$, $S-NMF$ and $V-MF$.

$$\begin{aligned} d_2(G,H) = & 1/8n\sum_{j=1}^{n}\{|u_G(sj)-u_H(sj)|+|g_{sj}(u_G)-g_{sj}(u_H)|+|\phi_G(sj)-\phi_H(sj)| \\ & + |v_G(sj)-v_H(sj)|+|h_{sj}(v_G)-h_{sj}(v_H)|+|\omega_G(sj)-\omega_H(sj)|+ \\ & 4max(|u_G(sj)-u_H(sj)|,|g_{sj}(u_G)-g_{sj}(u_H)|,|\phi_G(sj)-\phi_H(sj)|, \\ & |v_G(sj)-v_H(sj)|,|h_{sj}(v_G)-h_{sj}(v_H)|,|\omega_G(sj)-\omega_H(sj)|)\}. \end{aligned} \quad (5.9)$$

The chapter presents the definition of the axiom, which is outlined as follows.

Definition 5.3.1. A real function d: $F_2^I(s) \times F_2^I(s) \to [0,1]$ is referred to as a *dmr* if it fulfills the following axioms.

$$(A_1) \quad 0 \le d(G,H) \le 1, \forall (G,H) \in F_2^I(s). \quad (5.10)$$

$$(A_2) \quad d(G,H) = 0, IF \quad G = H. \quad (5.11)$$

$$(A_3) \quad d(G,H) = d(H,G). \quad (5.12)$$

$$(A_4) \quad (G \subseteq H \subseteq I) \;\; where \;\; G, H, I \in F_2^I(s), \;\; then \;\; d(G, I) \geq d(G, H) andd(G, I) \geq d(H, I). \tag{5.13}$$

Now we will prove the above defined measure is a valid dmr for $T2IFS$. condition A_1 and A_2 holds trivialy so we will prove for A_3 and A_4.

$$(A_3) \implies d_2(G, H) = d_2(H, G),$$

we have

$$\begin{aligned} d_2(G, H) =& 1/8n \sum_{j=1}^{n} \{|u_G(sj) - u_H(sj)| + |g_{sj}(u_G) - g_{sj}(u_H)| + |\phi_G(sj) - \phi_H(sj)| \\ & + |v_G(sj) - v_H(sj)| + |h_{sj}(v_G) - h_{sj}(v_H)| + |\omega_G(sj) - \omega_H(sj)| + 4max \\ & (|u_G(sj) - u_H(sj)|, |g_{sj}(u_G) - g_{sj}(u_H)|, |\phi_G(sj) - \phi_H(sj)|, |v_G(sj) - v_H(sj)|, \\ & |h_{sj}(v_G) - h_{sj}(v_H)|, |\omega_G(sj) - \omega_H(sj)|)\}, \end{aligned}$$

$$\begin{aligned} =& 1/8n \sum_{j=1}^{n} \{|u_H(sj) - u_G(sj)| + |g_{sj}(u_H) - g_{sj}(u_G)| + |\phi_H(sj) - \phi_G(sj)| + |v_H(sj) - v_G(sj)| \\ & + |h_{sj}(v_H) - h_{sj}(v_G)| + |\omega_H(sj) - \omega_G(sj)| + 4max(|u_H(sj) - u_G(sj)|, |g_{sj}(u_H) - g_{sj}(u_G)|, \\ & |\phi_H(sj) - \phi_G(sj)|, |v_H(sj) - v_G(sj)|, |h_{sj}(v_H) - h_{sj}(v_G)|, |\omega_H(sj) - \omega_G(sj)|)\}, \\ & = d_2(H, G). \\ & \implies d_2(G, H) = d_2(H, G), \end{aligned}$$

Now to prove (A_4)

$$(A_4) \implies d_2(G, I) \geq d_2(G, H), \tag{5.14}$$

it is easy to see that $|u_G(s_i) - u_I(s_i)| \geq |u_G(s_i) - u_H(s_i)|, |f_{si}(u_G) - f_{si}(u_I)| \geq |f_{si}(u_G) - f_{si}(u_H)|$

$|v_G(s_i) - v_I(s_i)| \geq |v_G(s_i) - v_H(s_i)|, |g_{si}(u_G) - g_{si}(u_I)| \geq |g_{si}(u_p) - g_{si}(u_H)|$, so we have

$$\begin{aligned} d_2(G, I) =& 1/8n \sum_{j=1}^{n} \{|u_G(sj) - u_I(sj)| + |g_{sj}(u_G) - g_{sj}(u_I)| + |\phi_G(sj) - \phi_I(sj)| \\ & + |v_G(sj) - v_I(sj)| + |h_{sj}(v_G) - h_{sj}(v_I)| + |\omega_G(sj) - \omega_I(sj)| + 4max \\ & (|u_G(sj) - u_I(sj)|, |g_{sj}(u_G) - g_{sj}(u_I)|, |\phi_G(sj) - \phi_I(sj)|, |v_G(sj) - v_I(sj)|, \\ & |h_{sj}(v_G) - h_{sj}(v_I)|, |\omega_G(sj) - \omega_I(sj)|)\}, \end{aligned}$$

$$\begin{aligned} \geq& 1/8n \sum_{j=1}^{n} \{|u_G(sj) - u_H(sj)| + |g_{sj}(u_G) - g_{sj}(u_H)| + |\phi_G(sj) - \phi_H(sj)| \\ & + |v_G(sj) - v_H(sj)| + |h_{sj}(v_G) - h_{sj}(v_H)| + |\omega_G(sj) - \omega_H(sj)| + 4max \\ & (|u_G(sj) - u_H(sj)|, |g_{sj}(u_G) - g_{sj}(u_H)|, |\phi_G(sj) - \phi_H(sj)|, |v_G(sj) - v_H(sj)|, \\ & |h_{sj}(v_G) - h_{sj}(v_H)|, |\omega_G(sj) - \omega_H(sj)|)\}, \end{aligned}$$

then we get inequality $d_2(G, I) \geq d_2(G, H)$. Similarly we can prove $d_2(G, I) \geq d_2(H, I)$, hence satisfies condition (A_4) so we proved this is a valid dmr for $T2IFS$.

5.4 Group-Decision-Making with $T2IFSs$ Based on New Distance Measure

We present an approach to assess various $T2IFSs$ for group $D-MG$ issues using the proposed distance measurements.

5.4.1 Approach for Distance Measure

Let's Consider m criteria like $\{K = k_1, k_2, k_3 \dots k_m\}$ and n alternatives $\{A = a_1, a_2, a_3 ... a_n\}$ are being evaluated by r $D-MRs$ $\{DM = Dm_1, Dm_2, Dm_3 ... Dm_r\}$ having weight vector $\{W = W_1, W_2, W_3 ...\}$ where $w_j \geq 0, j = 1, 2, 3 ... n$ and $\sum_{j=1}^{n} w_j = 1$.Consider the rating of $D-MRs$ as $P-MF$, $S-MF$, $P-NMF$ and $S-NMF$.
Then we describe the following steps for finding best alternatives.

1. Order the information collected for every alternative with respect to the criterion in the form of $P-MF$, $S-MF$, $P-NMF$, and $S-NMF$.
2. Calculate the dmr coresponding to $D-MRs$ and the void decision (V), $d(DM_r, V)$, where V is a decision with $P-MF$ and $S-MF$ as zero and $P-NMF$ and $S-NMF$ as one.
3. To find the maximum value of the $dmrs$ that correspond to the $D-MRs$ preferences, evaluate the highest $dmrs$ among all alternatives a_i, criteria K_j, and their respective maximum values of the $dmrs$, and then create the type-2 fermatean fuzzy alternative $a_i, (i = 1, 2, ..., n)$.
4. Compute the dmr for alternatives and void decision $d(P_j, V)$.
5. Provide ranking to alternatives and obtain the best one.

Table 5.4.1.1 Linguistic rating and corresponding $P - MF$ and $P - NMF$ value

Grades	P-MFV	Grades	P-NMFV
Extremely week (Ex-We)	0.0	Extremely strong (Ex-St)	1.0
Week (Wk)	0.2	Strong (St)	0.7
Little week (L-Wk)	0.3	Little strong (L-St)	0.6
Average (A-V)	0.4	Average (A-V)	0.5
Little strong (L-St)	0.6	Little week (L-Wk)	0.3
Strong (St)	0.8	Week (Wk)	0.1
Extremely strong (Ex-St)	1.0	Extremely week (Ex-We)	0.0

Table 5.4.1.2 Linguistic rating and corresponding $S - MF$ and $S - NMF$ value

Grades	P-MFV	Grades	P-NMFV
Extremely week (Ex-Wk)	0.0	Extremely strong (Ex-St)	1.0
Week (Wk)	0.2	Strong (St)	0.7
Little week (L-Wk)	0.3	Little strong (L-St)	0.6
Average (A-V)	0.4	Average (A-V)	0.5
Little strong (L-St)	0.6	Little week (L-Wk)	0.3
Strong (St)	0.8	Week (Wk)	0.1
Extremely strong (Ex-St)	1.0	Extremely week (Ex-We)	0.0

Table 5.4.1.4 Distance measure between d_2 and N

k_1	a_1	1.0	1.0	0.875
k_1	a_2	0.875	0.950	0.0.475
k_1	a_3	0.475	0.875	0.375
k_1	a_4	0.675	0.875	0.275
k_2	a_1	1.0	0.875	0.875
k_2	a_2	0.575	0.875	0.475
k_2	a_3	1.0	0.675	0.675
k_2	a_4	0.875	0.675	0.475
k_3	a_1	1.0	1.0	0.875
k_3	a_2	1.0	1.0	1.0
k_3	a_3	1.0	0.875	0.675
k_3	a_4	1.0	0.875	0.675
k_4	a_1	1.0	1.0	0.875
k_4	a_2	1.0	1.0	1.0
k_4	a_3	1.0	0.875	0.375
k_4	a_4	0.675	0.875	0.675

Table 5.4.1.3 Graded values of the alternative corresponding to each attribute (criteria)

		DM_1	DM_1	DM_1	DM_1	DM_2	DM_2	DM_2	DM_2	DM_3	DM_3	DM_3	DM_3
		P-MF	S-MF	P-NMF	S-NMF	P-MF	S-MF	P-NMF	S-NMF	P-MF	S-MF	P-NMF	S-NMF
k_1	a_1	Ex-St	L-St	Ex-Wk	L-Wk	S	Ex-St	Wk	Ex-Wk	St	L-Wk	W	L-St
k_1	a_2	L-St	St	Wk	Wk	St	L-St	E-Wk	Wk	Wk	A-V	St	A-V
k_1	a_3	A-V	L-Wk	A-V	L-St	St	A-V	Wk	A-V	Wk	L-Wk	St	L-St
k_1	a_4	L-St	A-V	L-Wk	A-V	S	A-V	Wk	A-V	Wk	Ex-Wk	St	Ex-St
k_2	a_1	Ex-St	St	Ex-Wk	Wk	St	L-Wk	Wk	L-St	St	L-St	Wk	L-Wk
k_2	a_2	L-St	L-St	A-V	A-V	St	L-St	Wk	L-Wk	Wk	A-V	St	A-V
k_2	a_3	St	Ex-St	Wk	Ex-Wk	L-St	L-Wk	L-Wk	L-St	L-Wk	L-St	L-St	L-Wk
k_2	a_4	St	L-St	Wk	L-Wk	L-St	L-Wk	L-Wk	L-St	L-Wk	A-V	L-St	A-V
k_3	a_1	Ex-St	L-St	Ex-Wk	L-Wk	St	Ex-St	Wk	Ex-Wk	St	St	Wk	Wk
k_3	a_2	Ex-St	L-St	Ex-Wk	L-Wk	St	Ex-St	Wk	Ex-Wk	Ex-St	St	Ex-Wk	Wk
k_3	a_3	Ex-St	St	Ex-Wk	Wk	St	St	Wk	Wk	L-Wk	L-St	L-St	L-Wk
k_3	a_4	Ex-St	St	Ex-Wk	Wk	St	A-V	Wk	A-V	L-Wk	L-St	L-St	L-Wk
k_4	a_1	Ex-St	St	Ex-Wk	Wk	St	Ex-St	Wk	Ex-Wk	St	A-V	Wk	A-V
k_4	a_2	Ex-St	St	Ex-Wk	Wk	St	Ex-St	Wk	Ex-Wk	Ex-St	L-St	E-Wk	L-Wk
k_4	a_3	Ex-St	L-Wk	Ex-Wk	L-St	L-St	St	L-Wk	Wk	L-Wk	L-Wk	L-St	L-St
k_4	a_4	L-St	A-V	L-Wk	A-V	St	A-V	Wk	A-V	A-V	L-St	A-V	L-Wk

5.4.2 Mathematical illustration

Take the case of a person who is trying to decide how much money to put into the market. There are five possible answers (I) a_1 is lithium battery firm, (ii) a_2 is a pesticides company, (iii) a_3 is a multinational enterprise, (iv) a_4 is an armaments company, and (v) a_5 is a tyre company. For this, they arranged a specified panel of experts (DM1,DM2, and DM3) whose weight vector is $(0.40, 0.35, 0.25)^T$. Under the $T2IFS$ set, the investor makes a choice based on a number of factors, including the project risk K_1, the revenue analysis K_2, the social effect analysis K_3, and the allocated space K_4. Tables 5.4.1.1 and 5.4.1.2 display the $P-MF$, $P-NMF$, and $S-MF$, $S-NMF$ linguistic grades necessary for this purpose.

1. Table 5.4.1.3 provides the accumulated data of each alternative that corresponds to each criterion, ordered in terms of the linguistic grades based on the knowledge and experience of the $D-MRs$.

2. Determine the value of d(DMk,N) (k = 1, 2, 3) for each possible solution. Table 5.4.1.4 summarises the numbers we use for $d_2(DMk, V)$ in our calculations.

3. Find the highest value of $d_2(DMk, V)$ in Table 4 for all options a_j, (j = 1, 2,..., 4) for each criterion Ki, (i = 1, 2, 3, 4). And hence build the $T2FFS$ alternative, $a_j = (K_i((u_{a_j}), g_{Ki}(a_j)), v_{a_j}, h_{Ki}(a_j))$ as
   $a_1 = K_1(0.8, 1.0, 0.1, 0.0), K_2(1.0, 0.8, 0.0, 0.1), K_3(1.0, 0.6, 0.0, 0.3), K_4(1.0, 0.8, 0.0, 0.1).$
   $a_2 = K_1(0.8, 0.6, 0.0, 0.1), K_2(0.8, 0.6, 0.1, 0.3), K_3(0.8, 1.0, 0.1, 0.0), K_4(0.8, 1.0, 0.1, 0.0).$
   $a_3 = K_1(0.8, 0.4, 0.1, 0.5), K_2(0.8, 1.0, 0.1, 0.0), K_3(1.0, 0.8, 0.0, 0.1), K_4(1.0, 0.3, 0.0, 0.6).$
   $a_4 = K_1(0.8, 0.4, 0.1, 0.5), K_2(0.8, 0.6, 0.1, 0.3), K_3(1.0, 0.8, 0.0, 0.1), K_4(0.8, 0.4, 0.1, 0.5).$

4. Now, we have computed the recommended distance measurements, d_2 from V to a_j (j = 1, 2,..., 4) and the results are presented below.The values for $d_2(a_1, V)$ are 1.00, $d_2(a_2, V)$ are 0.900, $d_2(a_3, V)$ are 0.950 and $d_2(a_4, V)$ are 0.850.

5. Our research has led us to the conclusion that a_1 is the most deserving of our investment capital. which is coinciding with [129] but our approach is stronger than the previous exixting because we use inclusion relation in our defined distance measure.

5.4.3 Comparative analysis

To evaluate the effectiveness of the proposed method in comparison to other existing methods, the authors carried out comparative studies using interval-valued and $T2FSs$, along $T2IFSs$ [16, 68, 129, 135, 143, 155, 166, 167]. Their related findings are presented in Table 5.4.3.1. This table demonstrates that business a_1 is the best to invest in relative to the others, and this finding aligns with the predicted results. As a result, the recommended technique can be employed more successfully to address the $D - MG$ problem than other existing methods.

Table 5.4.3.1 comparative analysis

Existing approach	score	values	score	values	Order of alternatives
	a_1	a_2	a_3	a_4	
$[d_2]$	1.0	0.900	0.950	0.850	$a_1 \geq a_3 \geq a_2 \geq a_4$
[[16]]	0.800	0.800	0.7500	0.7400	$a_1 \geq a_2 \geq a_4 \geq a_3$
[[135]]	0.833	0.604	0.733	0.506	$a_1 \geq a_3 \geq a_2 \geq a_4$
[[68]]	0.676	0.727	0.372	0.471	$a_2 \geq a_1 \geq a_4 \geq a_3$
[[166]]	0.800	0.700	0.650	0.525	$a_1 \geq a_2 \geq a_3 \geq a_4$
[[167]]	0.400	0.400	0.375	0.387	$a_1 \geq a_2 \geq a_4 \geq a_3$
[[155]]	0.181	0.144	0.090	0.117	$a_1 \geq a_2 \geq a_4 \geq a_3$
[[143]]	0.784	0.555	0.470	0.352	$a_1 \geq a_2 \geq a_3 \geq a_4$
[[129]]	1.000	0.962	0.975	0.887	$a_1 \geq a_3 \geq a_2 \geq a_4$

5.4.4 Limitations of the Proposed Method

Type-2 Intuitionistic Fuzzy Sets (T2IFS) extend the concept of Intuitionistic Fuzzy Sets (IFS) by introducing an additional dimension of uncertainty, providing a more flexible framework for handling vagueness and ambiguity. However, like any mathematical model, T2IFS have their limitations. Here are some potential limitations

Increased Complexity:

T2IFS introduce an additional level of complexity compared to traditional IFS, making the mathematical operations and interpretations more intricate. Handling this increased complexity may require more computational resources and more sophisticated algorithms.

Data Requirement:

T2IFS may demand a larger amount of data for accurate modeling, especially when considering the uncertainties associated with both membership and non-membership functions at two levels. In situations where data is limited, constructing and validating T2IFS models might be challenging.

Computational Intensity:

Operations involving T2IFS, such as intersection and union, can be computationally intensive. This can be a limitation in real-time applications or scenarios where quick decision-making is required.

Interpretability Challenges:

Interpreting and communicating the meaning of T2IFS can be more challenging due to the added dimension of uncertainty. This may hinder the practical adoption of T2IFS in fields where a straightforward and intuitive understanding of fuzzy sets is essential.

It's important to note that the limitations mentioned above do not necessarily make T2IFS unsuitable for all applications. They highlight areas where researchers and practitioners should exercise caution and carefully consider the trade-offs between the added expressiveness of T2IFS and the associated challenges. Future research and advancements may address some of these limitations and further enhance the applicability of T2IFS in various domains.

Chapter 6

Conclusion

The study of FS extensions and $IFSs$ has significantly expanded our ability to accurately model and analyze real-world systems that exhibit uncertainty and imprecision. By defining various operators and distance metrics, we can manipulate and compare $IFSs$, enabling more nuanced and comprehensive analysis. The practical applications of these extensions and metrics are vast, from $D-MG$ processes to image recognition and data compression. With the continued development and implementation of these tools, we can gain deeper insights and make more informed decisions in various fields.

This study further proposes a family of $H-D$ and $E-D$ for $T2FFSs$ by considering the $P-MF$, $S-MF$, $P-NMF$, $S-NMF$, FOU, and $V-MF$. The favourable features of these measurements have been carefully explored. A ranking method based on these measures has also been advocated for overcoming problems with group $D-MG$, and it is demonstrated using a numerical example. The suggested method has more fuzziness and uncertainty since $T2FFSs$ are used instead of already-existing FSs. A different approach to addressing $D-MG$ concerns has been placed by the studies, and here, we may also extend the domain that is constrained in intuitionistic fuzzy sets. Therefore, compared to other existing approaches, the suggested technique can be used effectively to address the problem of $D-MG$.

Operation of union and intersection between T1FS, T2FS, IFS and T2IFS is discussed with the help of examples, to understand the importance of these FSs a comparison is made on the basis of *dmrs* by the aid of examples on each above defined FSs. However, it is worth noting that the existing *dmrs* for T2IFSs have limitations. To address this, a new *dmr* is proposed specifically tailored for T2IFSs. This measure overcomes the limitations of the existing one, enabling a more accurate and reliable comparison of T2IFSs. In conclusion, when faced with decision-making scenarios where information is ambiguous or uncertain, it is better to utilize T2IFSs. Their ability to consider both membership and non-membership values, along with the proposed improved dmr, allows for a more comprehensive and effective analysis of fuzzy information. By employing T2IFSs in such conditions, decision-makers can obtain more reliable and informed outcomes, leading to better decision-making overall.

The study defines the role of $T2IFSs$ in $D-MG$ and introduces a new dmr to enhance the $D-MG$ process. The practical application of the new dmr is illustrated through a numerical example. By utilizing the newly proposed dmr, the study found significantly better results in addressing the $D-MG$ problem compared to the existing $dmrs$.

In conclusion, the study of FS extensions and $IFSs$ has greatly expanded our ability to model and analyze real-world systems with uncertainty and imprecision. By defining various operators and distance metrics, we can manipulate and compare $IFSs$, enabling more comprehensive analysis. This has practical applications in $D-MG$ processes, image recognition, and data compression, among others. This thesis proposes a family of Hamming and Euclidean distances for $T2FFSs$, considering different types of $M-Fs$. The favorable features of these measurements have been carefully explored, and a ranking method based on these measures has been advocated for group $D-MG$. The use of $T2FFSs$ adds more fuzziness and uncertainty to the $D-MG$ process, extending the domain compared to existing approaches. The thesis also addresses the operation of union and intersection between different types of FSs, namely $T1FS$, $T2FS$, IFS, and $T2IFS$, highlighting their importance. Furthermore, a new dmr is introduced specifically for $T2IFSs$ to overcome the limitations of existing measures. The role of $T2IFSs$ in $D-MG$ is emphasized, and the newly proposed dmr is applied through a numerical example. The results demonstrate that the suggested technique significantly outperforms existing measures in addressing $D-MG$ problems.

Overall, this research contributes to the advancement of FS theory by providing new tools for modeling uncertainty and imprecision. The proposed dmr enhances the $D-MG$ process and shows promising results. These findings can be valuable in various fields that require accurate modeling and analysis of complex systems.

Future Scope

Despite the fact that our research met its objectives, there are some substantial unresolved problems that we hope to address in future work. More specifically, we propose to look at the following research problems:

1. To generate a generalised type-n fuzzy set.
2. To provide a distance measure for generalised type-n fuzzy set.
3. To use general type-n fuzzy set in decision making problems.
4. To investigate the geometry type-n fuzzy sets.

Despite the study of various types of fuzzy sets and decision making, there are many results yet to be formulated.

6.0.1 Advantages, Limitations and Scope of Future Work

Type-n fuzzy sets generalize traditional fuzzy sets by allowing for multiple membership grades, providing a more flexible framework for capturing complex and nuanced uncertainties. Here are the advantages, limitations, and scope of type-n fuzzy sets:

Advantages:

1. Enhanced Representation of Uncertainty: Type-n fuzzy sets allow for a richer representation of uncertainty by accommodating multiple membership grades. This provides a more nuanced description of the varying degrees of membership and non-membership.
2. Increased Expressiveness: The additional parameters in type-n fuzzy sets offer increased expressiveness in capturing and modeling complex relationships, especially in situations where the concept under consideration exhibits multiple aspects or dimensions.

3. Flexibility in Modeling: The flexibility of type-n fuzzy sets makes them suitable for a wide range of applications, from decision-making and control systems to pattern recognition and information retrieval.

Limitations:

1. Computational Complexity: The increased number of parameters in type-n fuzzy sets can lead to higher computational complexity, especially in terms of implementing fuzzy set operations and decision-making processes.
2. Data Requirements: Constructing accurate type-n fuzzy sets may require a significant amount of data, and obtaining such data might be challenging in some applications. Insufficient data can affect the reliability of the model.

Scope:

1. Multi-Dimensional Uncertainty Modeling: Type-n fuzzy sets are particularly well-suited for applications where uncertainties have multiple dimensions or facets. This makes them applicable in fields such as risk assessment, decision support systems, and complex system modeling.
2. Dynamic Systems: In systems where uncertainties evolve or change over time, type-n fuzzy sets can offer a flexible modeling approach that adapts to dynamic conditions.
3. Control Systems: In control systems, especially those dealing with complex processes, type-n fuzzy sets can provide a more accurate representation of uncertainty, leading to improved control strategies.

It's important to note that the suitability of type-n fuzzy sets depends on the specific characteristics of the problem at hand, and their advantages and limitations should be carefully considered in the context of the application.

Paper Publications

Papers Published from the Thesis

1. Suhail Ahmad Ganai, Nitin Bhardwaj, Riyaz Ahmad Padder, From Fuzzy Sets to Deep Learning: Exploring the Evolution of Pattern Recognition Techniques, International Journal of Science, Mathematics and Technology Learning, **31(1)**, (2023), 250-264. (**Scopus**)
2. Suhail Ahmad Ganai, Nitin Bhardwaj, Riyaz Ahmad Padder, Improving Decision-Making Under Uncertainty: A Comparative Study of Fuzzy Set Extensions , Multidisciplinary Journal of Engineering Sciences , **2**, (2023), 01-21. (**Google Scholar**)

Papers Accepted from the Thesis

1. Nitin Bhardwaj and Suhail Ahmad ganai, Exploring the Power of Type-2 Intuitionistic Fuzzy Sets in Multicriteria Decision Making with a Novel Distance Measure , Educational Studies in Mathematics, (2023).(**Scopus**)
2. Nitin Bhardwaj and Suhail Ahmad ganai, Type-2 intuitionistic fuzzy set and a multicriteria decision making problem based on new distance measure , Mathematics and Statistics, (2023).(**Scopus**)

Papers Communicated from the Thesis

1. Nitin Bhardwaj and Suhail Ahmad ganai, Type-2 Fermatean Fuzzy Sets: A Novel Approach for Enhancing Group Decision-Making, International Journal of Fuzzy Systems, (2023).(**Scopus**)
2. Nitin Bhardwaj and Suhail Ahmad ganai, NAVIGATING AMBIGUITY: Smart Decision-Making with Pythagorean Fuzzy Sets in Granular Uncertainty ,JAEM, (2023).(**Scopus**)

Paper Presentations

Papers Presented in Conferences

1. Suhail Ahmad Ganai and Nitin Bhardwaj, Stratigical method for solving decision making problem using type-2 fuzzy sets for assesment of students answer scripts under high level of uncertainty, "International Conference on Mathematical science and its Recent Advansements (ICMSRA-2022), May 5-7, 2022, department of mathematics, Rathinam college of arts and science coimbatore-21".

2. Suhail Ahmad Ganai and Nitin Bhardwaj, Intuitionistic fuzzy sets and properties of intuitionistic fuzzy set operators, "International Virtual Conference on Recent Trends in Applied Mathematics (ICRTAM-2022), oct 07, 2022, Department of science and Humanities (Mathematics) Sri Ramakrishna Institute of Techenlogy, Pachapalayam, Coimbatore".

Conferences & Workshops attended

1. Participated in the International Conference on Mathematical science and its Recent Advansements (ICMSRA-2022), May 5-7, 2022, department of mathematics, Rathinam college of arts and science coimbatore-21.

2. Participated in International Virtual Conference on Recent Trends in Applied Mathematics (ICRTAM-2022), oct 07, 2022, Department of science and Humanities (Mathematics) Sri Ramakrishna Institute of Techenlogy, Pachapalayam, Coimbatore.

3. Participated in International Seminar on Indian Mathematicians and their Contributions. (ISIMC-2022). during 21-22, December 2022.

4. Participated in the National Conference on Applied Mathematics and Numerics (NCAMN 2022) held on 8th to 10th March 2023 at Mar Ivanios College, Trivandrum.

5. Participated in two day workshop on "Foundations of Deep Learning and its Applications"organised by the Division of Mathematics, School of Advanced Sciences, Vellore institute of techenlogy, chennai on 19 & 20 june 2023.

6. Participated in one day international conference on Astronomy, Cosmology and Space technology Exploration (ACSTEX). Jointly organised by Institute of Astronomy Space Science (IASES) Department of Physics and Department of Mathematics held on 16 oct 2023.

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