STUDY OF GENERALIZATION OF CLOSED SETS IN ALEXANDROFF TOPOLOGICAL SPACES AND BITOPOLOGICAL SPACES WITH FUZZY APPROACH

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By

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I, Pallvi Sharma, declare that this thesis entitled, "Study of Generalization of Closed Sets in Alexandroff Topological Spaces and Bitopological Spaces with Fuzzy Approach" and the work presented in it is my own. I confirm that:

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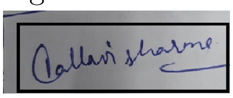

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Certificate

This is to certify that Pallvi Sharma, has completed the thesis entitled “Study of Generalization of Closed Sets in Alexandroff Topological Spaces and Bitopological Spaces with Fuzzy Approach” under my guidance and supervision. To the best of my knowledge, the present work is the result of her original investigation and study. No part of this thesis has ever been submitted for any other degree at any University.

The thesis is fit for the submission and the partial fulfilment of the conditions for the award of Doctor of Philosophy, in Mathematics.

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Abstract

This thesis concentrated on the emergence of new type of topological spaces and analysing their properties with the practical applications. Also, this brought a new insight to the field of study of generalization of closed sets in newly developed spaces. First and foremost we merged two types of topologies namely Alexandroff spaces and Soft topological spaces to develop a new kind of space known as Alexandroff Soft Topological Spaces(ASTS). This space satisfies a more grounded condition that an arbitrary intersection of open sets are open. We have likewise examined different ideas like basis of a topology, sub base, subspace, closure of a space etc. Also, different separation axioms known as Alexo T_i-spaces have been presented along with their properties. This space is also the parametrized type of general topology. Further, we have studied the concept of generalization of closed sets and introduced new two classes of generalized closed sets namely rw^*-closed sets in Alexandroff spaces and $g_{\overset{\circ}{A}_s}$-soft closed sets in ASTS. Also, we presented some new results with examples. These are the new generalization of closed sets in topological spaces with new outcomes. Then, we introduced another form of topological spaces known as Fuzzy Alexandroff Soft Topological Spaces(FASTS) by using fuzzy soft sets. We explored various topological properties like base, sub base etc of these spaces. Also, we investigated two major topological properties viz connectedness and compactness property by giving the definitions of $c_{f_{\mathcal{A}}}$-connectedness, c_{f_i}-connectedness and $c_{f_{\mathcal{A}}}$-compactness. We also contemplated various separation axioms in these spaces. Besides this, we utilized these fuzzy soft sets to solve the problems of decision-making. We proposed a method which uses fuzzified evidence theory along with D-S theory to calculate total degree of fuzziness of the parameters and reduce uncertainty of data. Subsequently, we used Dempster's rule of combination to fuse independent parameters into integrated one. An experiment has been performed to validate our proposed method. We also compared our results with grey relational analysis

method to show the efficiency and correctness of our results. In addition to this, we gave a practical medical diagnosis application in reference to COVID 19 which helps a doctor to take decision on patient's condition easily. This also supports the fidelity of our method. Next, we took two spaces namely Alexandroff spaces and Soft spaces into bitopological view. There, we introduced one more new type of topological spaces known as Alexandroff Soft Bitopological Spaces(ASBS). Then, we looked into various topological properties of these spaces and defined new separation axioms as well. After that, we presented a new class of generalized closed sets known as $(1,2)-\breve{g}$- soft closed sets in these spaces. We have studied various properties of these closed sets and compared our results with existing generalized closed sets along with examples. Additionally, we used this new class of closed sets in attribute reduction problem. For this, we used Pawlak's rough approximation theory to produce new set approximations in bitopology using $(1,2)-\breve{g}$ soft closed sets. Then, we defined Alexandroff Soft Bitopological Approximation Spaces(ASBAS) by using topological interior and closure concept. We also studied various properties of rough sets in ASBAS. Finally, we gave an application of these approximations in data reduction in multi-valued information systems. This thesis is divided into nine chapters. Following is the brief core of chapters given in the thesis. This thesis is divided into nine chapters.

Chapter-I is the introductory chapter which contains general presentation of topic with some important results. It also contains some preliminaries which describes some basic results and basic properties of spaces which are useful for the accomplishment of our work.

Chapter-II is the review of literature. It contains the brief summary of work done by many researchers since 1960 till date. It establishes our in-depth understanding and knowledge of our study. This review of literature leads us to find out research gap and then formulate our objectives of thesis.

Chapter-III is the foundation of new type of topology known as Alexandroff Soft Topological Spaces. These spaces are defined with their general topological properties and some new separation axioms.

Chapter-IV is the generalization of closed sets in Alexandroff Soft Topological Spaces. A new notion of $g_{\mathring{A}_s}$- soft closed sets has been given. We also investigated various results related to this class of closed sets with the help of examples.

Chapter-V also revealed the concept of generalization of closed sets in Alexandroff Spaces. A new class of closed sets known as Regular-Weakly Star closed sets in Alexandroff Spaces was given. We have also studied our properties of these closed sets and compared some results with previously established generalized closed sets.

Chapter-VI describes a topological space known as Fuzzy Alexandroff Soft Topological Spaces by using fuzzy soft sets with the study of properties like connectedness and compactness and new separation axioms.

Chapter-VII elaborates the application of fuzzy soft sets in solving decision-making problems. A new methodology has been given to solve decision-making problems and reduce uncertainty of data by using fuzzy soft sets. A practical example (in view of COVID-19) has been given to show the efficiency and correctness of our method.

Chapter-VIII revolves around the topic Alexandroff Soft Bitopological spaces and its applications in data reduction field. Firstly, an introduction to ASBS has been given. Also, topological properties of these spaces have been analysed. After that, a new generalization of closed sets in these spaces are given along with some basic properties and examples. Furthermore, these new generalized closed sets are used in the field of data reduction in information systems. Firstly, it gives a brief introduction to rough set theory and approximation spaces. Then, we defines Alexandroff Soft Bitoplogical Approximation Spaces and give its various properties. After that, we give an example to reduce data in information systems by using generalized closed sets in these spaces.

Chapter-IX is the conclusion of whole thesis and future scope for many researchers.

Acknowledgement

Firstly, I raise my heart in gratitude to the Almighty who walked with me throughout my endeavour, gave me strength and courage to overcome all the hurdles for the successful completion of my thesis. He has been the guiding force behind all my efforts. Needless to say, this accomplishment is not just of my own. No work can be an outcome of individual effort, unless and until it is supported, helped and encouraged by many individuals. This work of mine is an effort of many well beings who have contributed to my research at different levels. At this moment of accomplishment, firstly I would like to express my sincere and deep gratitude to my esteemed supervisor Dr.Nitin Bhardwaj, (Associate Professor and Deputy Dean), Department of Mathematics, Lovely Professional University, Phagwara, Punjab, for his commitment, dedication, enthusiastic interest and professional expertise that led me to pursue this pioneering study. It has been an absolute honour to work with him. I owe him a lot for his deep interest, motivation, enthusiasm, scholarly guidance, prompt help, availability at all times and persistent encouragement. He has helped me in identifying my strengths and weaknesses, besides boosting my self-confidence and I successfully overcame many difficulties that I faced during the research work.

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Symbols

τ	tau
$\mathbb{R}$	Real line
$\mathcal{Q}$	Rational Numbers
$\mathcal{R}$	Real Numbers
(X, τ)	Topological space
$\mathring{A}_s$	Arbitrary set of parameters
$(\mathcal{X}, \tau_{\mathcal{A}})$	Alexandroff Spaces
$(\mathcal{X}, \tau_{\mathring{A}_s})$	Alexandroff Soft Topological Spaces
$(X, \tau_f, \mu_{f_\varepsilon})$	Fuzzy Alexandroff Topological Spaces
$(\mathcal{X}, \tau_1^s, \tau_2^s)$	Alexandroff Soft Bitopological Spaces

Chapter 1

Introduction

1.0.1 General Introduction

The word Topology was coined by Johanne Benedict Listing in the 19th century and it originated as a well-defined mathematical discipline since 20th century. It can be formally characterized as the investigation of subjective characterization of specific articles that remain unchanged under a particular sort of change, particularly the properties which remain invariant under a particular sort of invertible change. There are disparate forms of topology namely point-set topology, bitopological spaces, fuzzy topology, soft topology, Alexandroff spaces and nano topology. These all kinds of topology has their own importance. Point-set topology is the base for all types of topologies. There is a brief description of different types of topologies established so far.

Given, X is a set and τ is a family of subsets of X. Then τ is called a topology on X if it satisfies the following conditions:

1) Any union of sets in τ is itself in τ.

2) Any finite intersection of sets in τ is itself in τ.

3) The empty set and X itself belong to τ.

We say (X, τ) is a topological space, sometimes abbreviated as X is a topological space.

J.C Kelley [1] in 1963 defined the notion of bitopological spaces (a set equipped with two topologies). He used the concept of quasi-metric spaces to define bitopological spaces. A new class of topology was introduced to deal with uncertainties, called as fuzzy topology. Fuzzy set theory was introduced by Zadeh [2] as a generalization of crisp or classical set theory. It has many applications in the field of engineering, sciences, economics, computer science, decision making problems, etc. Alexandroff spaces were introduced by P.Alexandrov in 1937. It was named after his name and known by the name of Discrete Raume, where he produced the characterization in context of sets and neighbourhood. By an Alexandroff space, we mean a topological space in which the intersection of every family of open sets is open. D. Molodstov in 1999 introduced the concept of soft sets as a new mathematical tool to solve the problems of uncertainty. After that, P.K Maji in 2003 studied the concept of soft set theory with some important definitions and results. Shabir [3] defined soft topology with a fixed set of parameters and showed that it is a parametrized form of topological spaces. Also, soft separation axioms for soft topological spaces have been defined with some important results.

Open sets and closed sets both have their own importance in general topological spaces. The notion of closeness helps to generate new concepts like continuity, convergence, connectedness etc. Closed sets give a useful characterization of compactness property, separation axioms and various covering lemmas. The concept of generalization of closed sets played an important role in topology. N.Levine [4] was the first to define g-closed sets in general topology. After that, many authors extend this work to different classes of generalized closed set. Similarly, this concept of generalization of closed sets gained its importance in bitopology, soft topology, fuzzy topology etc as well.

1.0.2 Brief Preface of our Research Work

As topological space is very useful for the study of mathematical problems and closed sets played an important role in these spaces, our work mainly focussed on introducing new types of topological spaces and study the concept of generalization of closed sets in those newly developed spaces with some practical applications. In the first place, we have studied two types of spaces specifically Soft spaces and Alexandroff spaces and then amalgamate them to develop a new type of topological spaces known as Alexandroff Soft Topological Spaces.
It is defined as-
An Alexandroff Soft Topological Space is a set $\mathcal{X}$ with an arbitrary set of parameters $\mathcal{A}$ together with a system $(\mathcal{K}_i, \mathcal{A})$, where $\mathcal{K}_i \colon \mathcal{A} \longrightarrow \wp(X)$ with the property that an arbitrary intersection of open sets is open [5].

We have studied various topological properties of these spaces like basis, subbase, continuity, homeomorphism etc. Afterwards, we investigated and generated new separation axioms known as Alexo T_i, Alexo-regular, Alexo-normal etc and also talked about their general properties with examples.

Now, we discussed the concept of generalization of closed sets in topological spaces. Firstly, we introduced a new class of closed sets in ASTS known as $g_{\mathring{A}_s}$ -Soft closed sets where $\mathring{A}_s$ defines the arbitrary set of parameters. We also incorporated its various properties as well as their characteristics and compared it with already existing classes of generalized closed sets.

It is defined as-

In $(\mathcal{X}, \tau_{\mathring{A}_s})$, a soft closed set $(\mathcal{P}', \mathring{A}_s)$ is called Alexandroff Soft Generalized closed set (in short, $g_{\mathring{A}_s}$ -closed set) if there is a soft closed set $(\mathcal{A}', \mathring{A}_s)$ containing $(\mathcal{P}', \mathring{A}_s)$ such that $(\mathcal{A}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ when $(\mathcal{P}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{V}, \mathring{A}_s)$ is soft open in $\mathcal{X}$.

Further, we introduced another new class of generalized closed sets which is named as Regular Weakly-Star Closed (briefly known as rw^*-closed) sets in Alexandroff spaces. We have also explored its various properties like union, intersection etc along with the concept of rw^*-open sets.

It is explicated as-

A set $\mathcal{P} \sqsubset (\mathcal{X}, \tau_{\mathcal{A}})$ is known as Regular Weakly Star-closed (rw^*-closed) if there exist $\mathcal{S}$, a closed set such that $\mathcal{P} \sqsubset \mathcal{S} \sqsubset \mathcal{V}$ wherever $\mathcal{P} \sqsubset \mathcal{V}$ and $\mathcal{V}$ is regular semi-open in $(\mathcal{X}, \tau_{\mathcal{A}})$. RW^*C $(\mathcal{X})$ represents the collection of all rw^*-closed sets [6].

In addition to, we established another type of topological space known as Fuzzy Alexandroff Soft Topological Spaces which uses fuzzy soft sets as a basic need. This space is a mixture of Alexandroff Soft Topological Spaces and Fuzzy Soft Topological Spaces.

It is defined as-

A set X together with a topology τ_{f_A} containing fuzzy soft closed sets satisfying following three conditions:

1) An arbitrary intersection of any number of members of τ_{f_A} belongs to τ_{f_A}.

2) Finite union of members of τ_{f_A} belongs to τ_{f_A}.

3) 0_{f_A} and $1_{f_A} \in \tau_{f_A}$.

Thus, $(X, \tau_{f_A}, \mu_{f_\mathcal{E}})$ is said to be FASTS where $\mu_{f_\mathcal{E}}$ is the membership function of the fuzzy soft sets with respect to an arbitrary set of parameters $\mathcal{E}$. Members of topology

τ_{f_A} are fuzzy soft closed sets and their complements are known as fuzzy soft open sets respectively [7].
We also explored different topological properties of these spaces like base, subbase etc. We have also investigated two major properties of topology namely connectedness and compactness by giving the definitions of $c_{f_{\mathcal{A}}}$-connectedness, c_{f_i}-connectedness and $c_{f_{\mathcal{A}}}$-compactness. Our work revealed various results related to these properties with examples.

Further, we used fuzzy soft sets with Dempster-Shafer theory to solve the problem of decision-making. The method we proposed uses fuzzified evidence theory to calculate total degree of fuzziness of the parameters. Firstly, we measured the uncertainties (fuzziness) of parameters and then modulated the uncertainties calculated. Next, we used the fuzzy preference relation analysis to produce the consistency matrix. After that, a while later, a suitable fundamental basic probability assignment (BPA) in terms of each parameter was produced. In the last, we utilized Dempster's rule of combination to blend the independent parameters into integrated one. Unavoidably, the best ideal decision can be gotten dependent on the positioning of choices. A brief description of steps have been given to calculate fuzziness of parameters followed by an experiment. Also, we compared our outcomes with grey relational technique to show the effectiveness of our proposed method. At last, we solved a real-life decision-making problem in medical diagnosis field (specifically in reference to COVID-19) based on our proposed work. Finally, we also performed the practical with other method (grey relational analysis method given by Li *et al* [8]) which showed that our technique can reduce uncertainty to a greater extent and this is more accurate and efficient in solving decision-making problems [9].

Now, we defined one new type of topological spaces known as Alexandroff Soft Bitopological Spaces.
It is designed as-
An Alexandroff Soft Bitopology is a non-empty set endowed with two soft topologies having arbitrary intersection of open sets is open. Thus, a triplet $(\mathcal{X}, \tau_1^s, \tau_2^s)$ is known as Alexandroff Soft Bitopological Space. Elements of τ_1^s and τ_2^s are closed sets and their complements are open sets in $\mathcal{X}$.
We have also explored its several topological properties. After that, we generated different separation axioms like Alexo soft bi- $\mathcal{T}_0$, Alexo soft bi- $\mathcal{T}_1$, Alexo soft bi- regular etc. Then, we considered the concept of generalization of closed sets in these spaces

and defined a new class of soft closed sets known as $(1,2)-\breve{g}$ soft closed sets. We also explored its various properties like union, intersection etc and proved some results with examples. With this, we utilized this new class of closed sets in data reduction problems in multi-esteemed information systems. For this, we developed Alexandroff Soft Bitopological Approximation spaces and then used $(1,2)-\breve{g}$ soft closed sets to reduce attributes in information systems.

1.1 Important Prerequisites of our work

1.1.1 Topological Spaces

Definition 1.1. [10] A set X along with a family τ of subsets of X is known as a topological space if it satisfies the following conditions:
1) Any union of sets in τ is itself in τ.
2) Any finite intersection of sets in τ is itself in τ.
3) The empty set and X itself belong to τ.

Definition 1.2. [1] A triplet $(X,\mathcal{U},\mathcal{V})$ is defined to be a bitopological space by Kelley in 1963, where X is a non-empty set and $\mathcal{U}$ and $\mathcal{V}$ are two arbitrary topologies defined on X.

Definition 1.3. [3] Suppose an initial universal set be X and $\mathcal{A}$ be a set of parameters, and $\tau \sqsubseteq \mathrm{SS}(X,\mathcal{E})$. A family τ describes a soft topology on X if the below given statements are satisfied:
a. $0_{\mathcal{A}}, 1_{\mathcal{A}}$ belongs to τ.
b. For $(P,\mathcal{A}),(Q,\mathcal{A}) \in \tau$, $(P,\mathcal{A}) \sqcap (Q,\mathcal{A}) \in \tau$.
c. For $(P_i,\mathcal{A}) \in \tau \ \forall\ i \in I$, $\sqcup\{(P_i,\mathcal{A})\colon i \in I\} \in \tau$.
A soft topological space is denoted by $(X,\tau,\mathcal{A})$. The complement of soft open sets are known as soft closed if they belongs to τ.

Definition 1.4. [11] A set X along with a system $\mathcal{F}$ of subsets is said to be an Alexandroff space (or σ- space)if an arbitrary intersection of an open set is open.

1.1.2 Important definitions and results

Definition 1.5. [10] A basis for general topological space is defined as a collection of $\mathbb{B}$ of subsets of X which satisfies below mentioned two conditions:
(1) There is at least one base component B for each point containing that point $x \in X$.
(2) For $x \in B_1 \sqcap B_2$, there is B_3 base component containing x such that $B_3 \sqsubseteq B_1 \sqcap B_2$.
The collection $\mathbb{B}$ which satiates these two axioms can generate the topology τ brought by $\mathbb{B}$.

Definition 1.6. [12] Let X be an Alexandroff space and $\mathcal{U}$ be a family of open sets. Then $\mathcal{U}$ is the minimal base for the topology of X if and only if:
1) $\mathcal{U}$ covers X.
2) If $\mathcal{A}, \mathcal{B} \in \mathcal{U}$, there exists a subfamily $\{\mathcal{U}_i : i \in I\}$ of $\mathcal{U}$ such that $\mathcal{A} \sqcap \mathcal{B} = \sqcup_{i \in I} \mathcal{U}_i$.
3) If a subfamily $\{\mathcal{U}_i : i \in I\}$ of $\mathcal{U}$ verifies $\sqcup_{i \in I} \mathcal{U}_i$, then there exists $i_0 \in I$ such that $\sqcup_{i \in I} \mathcal{U}_i = \mathcal{U}_{i_0}$.

Definition 1.7. [13] A soft set is said to be a soft closed set if its relative complement $\in \tau$. The relative complement of a soft set $(\mathcal{F}, \mathcal{A})$ is defined by $(\mathcal{F}, \mathcal{A})' = (\mathcal{F}', \mathcal{A})$ where $\mathcal{F}'$ is a mapping from $\mathcal{A}$ to $\wp(X)$ given by $\mathcal{F}'(a) = X \setminus \mathcal{F}(a)\ \forall\ a \in \mathcal{A}$.

Definition 1.8. [13] The soft closed set $(\mathcal{K}, \mathcal{A}) \in$ SCS $(X, \mathring{A}_s)$, where $\mathcal{K}(p) = \emptyset$, for every $p \in \mathcal{A}$ is known as null soft set of (X, τ) and symbolised as 0. The soft closed set $(\mathcal{K}, \mathcal{A})$, where $\mathcal{K}(p) = X$, $\forall\ p \in \mathcal{A}$ is known to be absolute soft of (X, τ) and symbolised as 1.

Definition 1.9. [3] Let (X, τ) be soft topological space and $Y \subseteq X$ be a non-empty set. Then, $\tau_Y = \{(\mathcal{F}_Y, \mathcal{A}) \mid (\mathcal{F}, \mathcal{A}) \in \tau\}$ is said to be relative topology on Y and (Y, τ_Y) is called soft subspace of (X, τ).

Theorem 1.10. *[3] Let (Y, τ_Y) be a soft subspace of soft topological space (X, τ) and $(\mathcal{F}, \mathcal{A})$ be a soft set over X, then-*
1) $(\mathcal{F}, \mathcal{A})$ is soft open in Y iff $(\mathcal{F}, \mathcal{A}) = Y \cap (\mathcal{G}, \mathcal{A})$ for some $(\mathcal{G}, \mathcal{A}) \in \tau$.
2) $(\mathcal{F}, \mathcal{A})$ is soft closed in Y iff $(\mathcal{F}, \mathcal{A}) = Y \cap (\mathcal{G}, \mathcal{A})$ for some $(\mathcal{G}, \mathcal{A})$ in X.

Definition 1.11. [3] For any two soft sets $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$, soft set $(\mathcal{F}, \mathcal{A})$ is said to be soft subset of $(\mathcal{G}, \mathcal{B})$ if-
1) $\mathcal{A} \subseteq \mathcal{B}$ and

2) for all $a \in \mathcal{A}, \mathcal{F}(a)$ and $\mathcal{G}(a)$ are identical approximations.

We can write it as $(\mathcal{F}, \mathcal{A}) \subseteq (\mathcal{G}, \mathcal{B})$.

Definition 1.12. [3] The intersection $(\mathcal{H}, C)$ of two soft sets $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ over a common universe X, denoted by $(\mathcal{F}, \mathcal{A}) \cap (\mathcal{G}, \mathcal{B})$ is defined as-

$C = \mathcal{A} \cap \mathcal{B}$ and $\mathcal{H}(e) = \mathcal{F}(e) \cap \mathcal{G}(e)$ for all $e \in C$.

1.1.3 Fuzzy topological spaces

Definition 1.13. [2] (Fuzzy set): Let $\mathcal{X}$ be a non-empty set and $\mathcal{A} \sqsubseteq \mathcal{X}$. A fuzzy set $\mathcal{A}$ is determined by its membership function $\mu_{\mathcal{A}} : \mathcal{X} \longrightarrow [0, 1]$ whose value determines the grade of membership of point x in $\mathcal{A}$ for x belongs to $\mathcal{X}$.

Definition 1.14. [14] Let X be an initial universe set and $\mathcal{E}$ be the set of parameters. The pair $(F_{\mathcal{E}}, \mathcal{A})$ is a fuzzy soft set over X where $\mathcal{A} \sqsubseteq \mathcal{E}$ and $F_{\mathcal{E}}$ is a mapping defined as $F_{\mathcal{E}} \colon \mathcal{A} \to I^X$, where I^X is a set of all fuzzy subsets of X.

It is clear that every soft set can be considered as a fuzzy soft set. Also, when both X and $\mathcal{A}$ is finite, fuzzy soft sets are either represented by matrices or in tabular form.

Example 1.1. *Let $X = \{g_1, g_2, g_3, g_4\}$ be the universal set and $\mathcal{A} = \{e_1, e_2, e_3\}$ be the set of parameters.*

Then, $(F_\epsilon, \mathcal{A})$ is a fuzzy soft set over X described as follows :

$F(e_1) = \{g_1/.5, g_2/.2, g_3/.2, g_4/.1\}$

$F(e_2) = \{g_1/.6, g_2/.1, g_3/.1, g_4/.2\}$

$F(e_3) = \{g_1/.4, g_2/.3, g_3/.2, g_4/.1\}$

$\mathcal{A}$	g_1	g_2	g_3	g_4
e_1	0.5	0.2	0.2	0.1
e_2	0.6	0.1	0.1	0.2
e_3	0.4	0.3	0.2	0.1

Table 1.1: Fuzzy soft set $(F_{\mathcal{E}}, A)$

Definition 1.15. A fuzzy topology is a family τ_f of fuzzy sets in X which satisfies the following conditions:

1) $\emptyset_f$ and $X_f \in \tau_f$.

2) If $\mathcal{A}_f$ and $\mathcal{B}_f \in \tau_f$, then $\mathcal{A}_f \sqcap \mathcal{B}_f \in \tau_f$.

3) If $\mathcal{A}_{f_i} \in \tau_f$ for each $i \in I$, $\sqcup \mathcal{A}_{f_i} \in \tau_f$.

Definition 1.16. [15] A fuzzy soft topology τ on $(U, \mathcal{E})$ is a family of fuzzy soft sets over $(U, \mathcal{E})$ satisfying the following properties :

1) $\emptyset, \mathcal{E} \in \tau$.

2) If $F_A, G_B \in \tau$, then $F_A \sqcap G_B \in \tau$.

3) If $F^{\alpha}_{A_\alpha} \in \tau \forall \alpha \in \Lambda$, an indexed set, then $\sqcup_{\alpha \in \Lambda} F^{\alpha}_{A_\alpha} \in \tau$.

Definition 1.17. [16] In a fuzzy topological space, two sets $\mathcal{A}$ and $\mathcal{B}$ are said to be weakly separated iff $\exists$ $\mathcal{P}$ and $\mathcal{Q} \in \tau_f$ such that $\mathcal{A} \subseteq \mathcal{P}$ and is not quasi- coincident to $\mathcal{Q}$, that is, there exists $x \in X$ such that $\mathcal{A}(x) + \mathcal{Q}(x) > 1$ and $\mathcal{B} \subseteq \mathcal{Q}$ and is not quasi-coincident to $\mathcal{P}$ in the same manner.

Theorem 1.18. *[17] Two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{Q}$-separated or strongly separated in (Y, τ_Y) iff these two sets are $\mathcal{Q}$-separated in (X, τ_X) where Y is the subset of X.*

Definition 1.19. [18] An entropy measure is a sequence of mappings $\mathcal{E}_n : \mathcal{X}_n * \mathcal{P}_n * \mathcal{W}_n \longrightarrow \mathcal{R}+$ satisfying several properties (symmetry, monotonicity, additivity etc).

Definition 1.20. [19] (Shannon Entropy): Shannon in 1948 introduced the concept of Shannon entropy to handle basic probability problem.
Shannon entropy $(\mathcal{H})$ is derived as –
$\mathcal{H} = -\sum_i^N p_i \log_2 p_i$.
Where p_i is the probability of state i satisfying $\sum_i^N p_i = 1$ and N is the number of basic states in a system.

Definition 1.21. [20] (Deng entropy): This novel belief entropy was introduced by Deng in 2016. It also measures the uncertainty conveyed by basic probability assignment.
It is defined as –

$$\mathcal{E}_d = \frac{\sum_i m(\mathcal{A}_i) \log m(\mathcal{A}_i)}{2^{|\mathcal{A}_i|} - 1}.$$

Where m is the belief function and $\mathcal{A}_i$ is the hypothesis of belief function. Deng entropy is degenerated into Shannon entropy when the belief value is allocated to one single element.

Definition 1.22. [21] (W-entropy): This type of entropy was given by Dan Wang et al in 2019. It is the unified form about belief entropy based on deng entropy which considers the scale of frame of discernment and the relative scale of focal element with respect to Frame of Discernment.

W-entropy is calculated as below-

$\mathcal{E}_W(m) = \sum m(\mathcal{A}) \log_2(\frac{m(\mathcal{A})}{2^{|\mathcal{A}|-1}}(1+\epsilon)^{f|\mathcal{X}|})$.

Where ϵ is a constant and $\epsilon \geq 0$ and $f \mid \mathcal{X} \mid$ is the function determines the cardinality of $\mathcal{X}$.

The function $f \mid \mathcal{X} \mid = \sum_{\mathcal{B} \sqsubseteq \mathcal{X}, \mathcal{B} \neq \mathcal{A}} \frac{|\mathcal{A} \sqcap \mathcal{B}|}{2^{|\mathcal{X}|}-1}$.

Definition 1.23. [22] (Fuzziness): A measure of fuzziness is a function from the set of all fuzzy subsets of $\mathcal{X}$ to the set of all positive real numbers. The function $f(\mathcal{A})$ expressed the degree to which boundary of $\mathcal{A}$ is not sharp.

The measure of fuzziness is calculated as-

$$f(\mathcal{A}) = \sum_{x \in \mathcal{X}} (1- \mid 2\mathcal{A}(x) - 1 \mid). \tag{1.1}$$

The range of function f is $[0, \mid \mathcal{X} \mid]$; $f(\mathcal{A}) = 0$ iff $\mathcal{A}$ is a crisp set; $f(\mathcal{A}) = \mid \mathcal{X} \mid$ when $\mathcal{A}(x) = 0.5 \ \forall x \in \mathcal{X}$.

Definition 1.24. [22] (Fuzziness in evidence theory): Total degree of fuzziness $\mathcal{F}(m)$, of the body of evidence $< m, \mathcal{F} >$ is calculated as follows-

$\mathcal{F}(m) = \sum_{\mathcal{A} \in \mathcal{F}} m(\mathcal{A}) f(\mathcal{A})$ Where $f(\mathcal{A})$ is given by eq (1.1).

Definition 1.25. [23, 24] (Performance measure): The performance measure of a method satisfies the optimal criteria for resolving decision making problem. It is denoted by γ_S.

Mathematically, $\gamma_S = \frac{1}{\sum_i^n \sum_j^n |\mathcal{F}(e_i)(\mathcal{O}_p) - \mathcal{F}(e_j)(\mathcal{O}_p)|} + \sum_{i=1}^n \mathcal{F}(e_i)(\mathcal{O}_p)$.

Here, n is the number of choice parameters and $\mathcal{F}(e_i)(\mathcal{O}_p)$ depicts the membership value of the ideal object $\mathcal{O}_p$ for the choice parameter e_i.

If the performance measure of one method is greater than other, then that method is much finer than other and vice-versa.

Dempster -Shafer theory is proposed by Dempster [25] and Shafer [26]. This theory deals with the uncertain information and applied to uncertainty modelling [27, 28], decision making [29, 30] and information fusion [31, 32, 33] etc. This theory does not need prior information in modelling uncertainty and also able to fuse multiple evidences into integrated one.

Definition 1.26. [26] (Frame of discernment): A frame of discernment is a finite non-empty set of mutually exclusive and exhaustive hypotheses denoted by $\Theta = \{\mathcal{A}_1, \mathcal{A}_2 \ldots \mathcal{A}_n \ldots \mathcal{A}_t\}$ and $\mathcal{A}_i \sqcap \mathcal{A}_j = \emptyset$ and 2^Θ represents the set of all subsets of Θ.

Definition 1.27. [26] (Basic Probability assignment (BPA)): It is also known as mass function. A mass function is a mapping m from 2^Θ to $[0, 1]$ satiates the following conditions-
$m(\emptyset) = 0$ and $\sum_{\mathcal{A} \in 2^\Theta} m(\mathcal{A}) = 1$.
If $m(\mathcal{A}) > 0$, $\mathcal{A}$ is called a focal element and its union is known as the core of the mass function.

Definition 1.28. [26] (Belief function): It can be defined as a mapping $Bel : 2^\Theta \longrightarrow [0, 1]$ satisfying following conditions:
$Bel(\emptyset) = 0, Bel(\Theta) = 1$ and $Bel(\mathcal{A}) = \sum_{\mathcal{B} \sqsubseteq \mathcal{A}} m(\mathcal{B}), \forall \mathcal{A} \in \Theta$.
$Bel(\mathcal{A})$ exemplify the imprecision and uncertainty in decision making problems. When there is single element, then, $Bel(\mathcal{A}) = m(\mathcal{A})$.

Definition 1.29. [25] (Dempster's rule of combination): This rule computes an integrated set of combined evidences. Suppose m_1 and m_2 are two independent BPAs in Θ, then rule of combination is defined as -

$$m(\mathcal{A}) = \begin{cases} \frac{1}{1-\mathcal{K}} \sum_{\mathcal{B} \sqcap \mathcal{C} = \mathcal{A}} m_1(\mathcal{B}) m_2(\mathcal{C}), & \mathcal{A} \neq \emptyset \\ 0, \quad \mathcal{A} = \emptyset \end{cases} \tag{1.2}$$

and

$$\mathcal{K} = \sum_{\mathcal{B} \sqcap \mathcal{C} = \emptyset} m_1(\mathcal{B}) m_2(\mathcal{C}) < 1 \tag{1.3}$$

where $\mathcal{B} \in 2^\Theta$ and $\mathcal{C} \in 2^\Theta \& \mathcal{K} \in [0, 1]$ represents the coefficient for confliction between two BPAs.

Zhaowen Li *et al* [8] utilized grey relational analysis with Dempster Shafer theory to solve the problem of decision making. They calculated grey relational degree and then calculated uncertainty degree of various parameters. Further, BPA of each independent alternative can be obtained on the basis of this degree and used Dempster's rule of combination to fused different alternatives into collective alternative. Finally, the best alternative based on the ranking of these fused alternatives can be obtained.

Definition 1.30. [8] (Grey mean relational degree): The grey means relational degree between d_{ij} and $\tilde{d}_i$ can be computed as-

$$r_{ij} = \frac{\min_{1\leq i\leq s} \Delta d_{ij} + 0.5 \max_{1\leq i\leq s \Delta d_{ij}}}{\Delta d_{ij} + 0.5 \max_{1\leq i\leq s \Delta d_{ij}}} \tag{1.4}$$

$(i = 1, 2 \ldots, m, j = 1, 2 \ldots n)$

Where d_{ij} denotes the membership value of x_i with e_j , $\tilde{d}_i$ is the mean of all parameters with respect to each alternatives and Δd_{ij} is the difference information between d_{ij} and $\tilde{d}_i$.

Definition 1.31. [34] (Fuzzy preference relation): Fuzzy preference orderings can be defined as fuzzy binary relations related to reciprocity and maximum and minimum transitivity. Mathematically, it is denoted by-

$\mathcal{P} = (p_{jk})_{n\times n}$.

where $p_{jk} \in [0, 1]$ represents the preference value of alternative e_j over e_k.

Also, $p_{jk} + p_{kj} = 1$, $p_{jj} = 0.5, 1 \leq j \leq n$ and $1 \leq k \leq n$.

Definition 1.32. [35] (Consistency matrix): The consistency matrix can be developed on the basis of fuzzy preference relation as follows:

$$p = \overline{(p_{jl})}_{n\times n} = \left(\frac{1}{n} \sum_{k=1}^{n} (p_{jk} + p_{kl}) - 0.5 \right)_{n\times n}. \tag{1.5}$$

1.1.4 Generalized closed sets

Definition 1.33. [4] A non-empty subset $\mathcal{P}$ of X is known as g -closed if and only if $\bar{P} \sqsubseteq \mathcal{V}$ when $\mathcal{P} \sqsubseteq \mathcal{V}$ and $\mathcal{V}$ is open.

Definition 1.34. [36] Soft generalized closed set (in brief, soft g -closed) is defined when the closure of soft set $(\mathcal{P}, E)$ is contained in $(\mathcal{V}, E)$ whenever $(\mathcal{P}, E) \sqsubseteq (\mathcal{V}, E)$ and $(\mathcal{P}, E)$ is soft open in X.

Definition 1.35. A subset $\mathcal{A}$ of a topological space (X, τ) is called

- Regular open if $\mathcal{A} = (\overline{\mathcal{A}})^\circ$ and regular closed if $\mathcal{A} = \overline{(\mathcal{A}^\circ)}$.
- Semi-open if $A \sqsubseteq \overline{(\mathcal{A}^\circ)}$ and semi-closed if $(\overline{\mathcal{A}})^\circ \sqsubseteq A$.
- Regular semi-open if there is a regular open set $\mathcal{U}$ such that $\mathcal{U} \sqsubset \mathcal{A} \sqsubset \overline{\mathcal{U}}$.
- Weakly closed sets (w -closed) [37] if $\overline{\mathcal{A}} \sqsubseteq \mathcal{U}$ whenever $\mathcal{A} \sqsubseteq \mathcal{U}$ and $\mathcal{U}$ is semiopen in X.

• Regular w-closed(rw-closed sets) [38] if $\overline{\mathcal{A}} \sqsubset \mathcal{U}$ whenever $\mathcal{A} \sqsubset \mathcal{U}$ and $\mathcal{U}$ is regular semi-open in (X, τ).

Lemma 1.36. *[38] Every regular semi-open set in (X, τ) is semi-open but the converse is not true.*

Lemma 1.37. *[38] If $\mathcal{A}$ is regular semiopen in (X, τ), then $X \setminus \mathcal{A}$ is also regular semi-open.*

Chapter 2

Literature Review

This chapter contains review of literature since 1940s till date. These reviews of research papers identifies, evaluates and synthesis the relevant literature within our particular field. It discusses published work and gives us an overview of research done by other researchers.

2.0.1 Overview of previous research work done by researchers till date

- J.C Kelly (1963): He was the first to introduce the concept of bitopological spaces (equipped by two topologies). He used the concept of quasi-metric spaces which has been studied before by Wilson [39] to define bitopological spaces. A triplet $(\mathcal{X}, \tau_1, \tau_2)$, was said to be a bitopological space which helped to obtain systematic generalizations of standard results. This paper gave very important results on quasi metrics also. This paper also included the counter examples for the various results related to metrization space [1].

- L.A Zadeh (1965): He was the first to present the concept of fuzzy sets to manage the problem of uncertainty. A fuzzy set is portrayed by a membership function which assigns to each individual of a set. The notion of unions, intersections, complement, and inclusion were also established in this paper. Its membership function can take only two values 0 and 1, with $f_a(x) = 1$ or 0 according as x does or does not belong to $\mathcal{A}$. He also defined the algebraic operations like algebraic sum, algebraic product of fuzzy sets and studied their properties [2].

- A.P Dempster (1967): Dempster introduced the renowned theory known as Dempster Shafer theory which can show flexibility and effectiveness in modelling both uncertainty and imprecision of data without prior information. This theory is considered as the generalization of Bayesian probability theory. Here, he used the concept of lower and upper probablities over subsets of a space S. Further, in the last, he talked about a mechanism of combining multiple information sources to fuse them into integrated one [25].

- N.Levine (1970): This paper defined the generalised closed sets in topology and then determined the behaviour of sets with respect to unions, intersections, subspaces etc. It also defined the generalised open and studied the Cartesian products of g-closed, g-open sets. The basic properties of g-closed and g-open had also been discussed. The main result of the paper was to introduce the $\mathcal{T}_{1/2}$ space in which closed sets and g-closed sets coincided. A space $(\mathcal{X}, \tau)$ is said to be symmetric if and only if $\forall$ x and y in $\mathcal{X}$, $x \in c(y)$ implies that $y \in c(x)$ where c denotes the closure operator [4].

- R. Lowen (1976): The main purpose of this paper was to go deeper inside the idea of fuzzy topological spaces and then redefined fuzzy topological spaces given by C.L Chang(1968) [40]. Two functors $\tilde{\omega}$ and $\tilde{\tau}$ were also introduced which clearly explained the connection between fuzzy topological spaces and topological spaces. It also gave the mathematical reasons to redefine the definition given by C.L Chang and then gave the new definition as under:
  $\delta \in \mathcal{I}^{\mathcal{E}}$ is a fuzzy topology on $\mathcal{E}$ iff
  1) $\forall \alpha$ constant, $\alpha \in \delta$,
  2) $\forall \mu$, ν, $\mu \wedge \nu \in \delta$,
  3) $\forall (\mu_j)_{j \in J} \subset \delta \implies sup_{j \in J}\ \mu_j \in \delta$.
  Further, the properties like union, intersection, complement, compactness had been studied in the paper [41].

- G. Shafer (1976): Shafer along with Dempster developed the renowned theory known as Dempster Shafer theory which can show flexibility and effectiveness in modelling both uncertainty and imprecision of data without prior information. Shafer used the framework of belief and plausibility and then, defined various new concepts like basic probability assignment, mass function etc. Also, their theory has the ability to fuse multiple evidences into integrated one [26].

- N. Ajmal *et al.* (1989): This paper explained the concept of connectedness in different way. He gave a new definition of connectedness in the form $\mathcal{C}_i$ - connectedness. Four types of connectedness in fuzzy topological spaces has been given. He further discussed all the implications that exist among them. It has been shown that a fuzzy space is disjoint union of its components. Various counter examples are given to prove the relevance of results. The concept of compactness of fuzzy sets has also been discussed [42].

- K.C Chattopadhyay *et al.* (1993): This paper redefined fuzzy topology and explained various properties of it. It followed the definition of connectedness given by Ajmal to redefine connectedness in the context of fuzzy sets in some new way. He gave the notion of fuzzy closure operator, product theorems for fuzzy connectedness, fuzzy compactness etc with various results [43].

- F.G Arenas (1999): This paper was all about Alexandroff spaces which were first defined by P. Alexandrov with name of Diskrete Raume in 1937. These spaces were not studied systematically, so this paper studied these spaces systematically from several point of view and its properties including quasi-uniform spaces. It meant by an Alexandrov space a topological space such that every point has a minimal neighbourhood. The main focus of the paper was on studying the topological properties of these spaces systematically which is relevant to the application of these spaces in digital topology also [12].

- P.K Maji (2001): D.Molodstov [44] initiated the concept of soft sets as a new mathematical tool to deal with uncertainties. Following his concept, Maji gave an application of soft sets with the approach of Pawlak's rough theory. He solved a decision-making problem for which he used an analogous representation of soft sets in the form of a binary information table. One numerical example related to real-life has been solved by defining an algorithm and find the optimum value of choice function which can easily show the use of soft sets with arbitrary sets of parameters [14].

- P. Das *et al.* (2003): Levine [4] in 1970 defined g closed sets in general topology. This paper is the generalization of g closed sets in Alexandroff spaces which they called g^* closed sets. They produced those results which shows g^* closed sets not always behave like g closed sets in topology. They also gave various examples to show where these sets behave like g closed and where they were not. By the use of g^* closed sets, they deduced a new separation axiom, called as, $\mathcal{T}_w-$ axiom which was similar to that of $\mathcal{T}_{(1/2)}$ axiom defined by Levine. They compared the results of these new generaisation and axiom with the previous ones by the help of examples [45].

- O.A El Tantawy *et al.* (2005): In this paper, new classes of sets denoted by $ij-\Omega$-closed sets are defined in bitopological spaces. These sets are used to define new bitopological properties and new kind of continuous functions between bitopological spaces. This paper also showed that some bitopological separation properties are preserved under some types of continuous functions. He also introduced new four bitopological separation axioms and investigated their relations between the properties [46].

- S.S Benchalli *et al.* (2007): He introduced a new class which lies between the class of all w-closed sets and all regular g-closed sets, known as regular w-closed sets in topological spaces. It also investigated the relation among closed sets, πg closed sets, w closed sets etc. Also studied the properties of unions, intersections and subspaces of rw- closed sets . It also gave the various implications which results in different relations among other definitions of different closed sets [38].

- F. Jinming (2007): He established a very new notion of I-fuzzy Alexandrov topology which was instigated by a fuzzy preordered relation and also this new kind of topology induced a fuzzy preordered relation in return. This paper explained connections between certain generalized topological structures and fuzzy order structures on a universe set X in details. It introduced a notion of a I- fuzzified set of all upper sets of a fuzzy preordered set with the residuation operation and proved several theorems over it. The representation theorem of fuzzy preorders by I-fuzzy topologies had also been obtained [47].

- T. Speer (2007): The basic properties of Alexandroff spaces along with several examples have been studied. He also explained how we can construct new Alexandroff spaces from given ones. He talked about the concept of minimal open neighbourhoods, continuous maps etc for these spaces. He additionally introduced Hausdorff Alexandroff Spaces and proved some theorems related to this. Also, two invariants for compact Alexandroff spaces are defined and elaborated with the help of given examples [48].

- K. Chandrasekhara Rao *et al.* (2009): He along with K. Joseph [49] introduced the concepts of semi-star generalised open and closed sets in topological spaces. This paper defined the notion of semi-star gw- closed sets in bitopological spaces. It also studied the properties of unions, intersections, closure and interior operations. This paper also studied the concept of pairwise semi-star generalised $w-T_{1/2}$ spaces and their properties with respect to open sets and closed sets [50].

- A.S Salama (2010): This paper used the concept of lower and upper approximations of Pawlak rough sets to introduce new generalization concepts. It investigated the basic concepts of generalized rough sets generated by bitopological structures. A measure of roughness with new approximations was defined in this paper with various results. Also, a medical application for data reduction was done based on medical data. He also elucidated a new concept of rough membership function which can helps to analyze and make decision according to a conditional attribute in decision table [51].

- M.S Sarsak *et al.* (2010): He placed a new class of sets strictly in between πgp closed sets and $gspr$- closed sets which is known as πgsp- closed sets in topological spaces and then studied the properties of unions, intersections, closure operation etc. He also gave the notion of πgsp- open sets and demonstrated a figure of implications which shows relation among different classes of closed sets. Further, he showed that the converse of πgsp- closed to πgp closed is not true in general with the help of an example [52].

- M. Shabir *et al.* (2011): Classical set theory is not fully suitable in handling the problems of uncertainties. D. Molodstov in 1999 [44] introduced the concept of soft sets which were free from all the problems of certainties and vagueness. In this paper, the author introduced the new notion of soft topological spaces and also showed that it gives a parametrized family of topological spaces but the converse for the same is not true. Also, the notion of soft open sets, soft closed sets, soft interior and soft closure had been introduced. Also, the concept of soft subspaces was introduced [3].

- K. Kannan *et al.* (2012): It introduced a new class of $\hat{\beta}$ generalized closed sets and $\hat{\beta}g$ open sets in topology and studied the properties of closure operation, unions and intersections on them. It also gave the important results related to a new set and with the help of an examples, it also proved the converses of those theorems need not to be true wherever necessary [36].

- S. Roy *et al.* (2012): They constructed a topology on fuzzy soft set. D. Molodtsov in 1999 was the first who defined a new theory known as soft set theory which can solve complicated problems in the field of economics, engineering, and environment etc. This paper established a topology on fuzzy soft set and studied their properties [15].

- A. Mukherjee (2013): The new notions of strong separation axioms including strongly pairwise hasdorffness, strongly pairwise regularity, strongly pairwise normality etc in bitopological spaces are introduced. With this, new notion of strongly pairwise compactness is also defined. Several examples are given to explain the given new notions [53].

- T. Simsekler *et al.* (2013): This paper introduced fuzzy soft topology which is a combination of soft topology and fuzzy topology. Soft set theory and fuzzy theory provides a way to solve the problems of certainties etc. Fuzzy soft topology is a generalization of soft topology, over a fuzzy soft set with a fixed set of parameter. It showed the belongness of fuzzy point to fuzzy soft set and then introducing fuzzy soft interior, fuzzy soft closure, fuzzy soft neighbourhood as well as fuzzy soft Q- neighbourhood of a fuzzy point. It has also an application to decision making problem [54].

- D.N Georgiou *et al.* (2014): He discussed the concept of soft set theory and then investigated soft topological spaces by giving new characterizations, new notions like soft closure, soft interior, soft boundary, soft continuity, soft open and closed maps and soft homeomorphism. Various theorems and results related to these properties were represented in the paper [13].

- S. Guzide *et al.* (2014): This paper is the general study of soft closed sets in soft bitopological spaces. Different notions like soft closed sets, soft α-closed, soft semi-closed, soft pre-closed, soft sg-closed sets were defined in the paper. The figure explaining the relationship among these different closed sets was explained. Various results along with the examples were established [37].

- Y. Chan Kim (2014): In this paper, the author made use of join preserving maps and studied the properties of them in complete residuated lattices and then defined join approximation operators as a generalization of fuzzy rough sets in complete residuated lattices. It used the concepts of fuzzy complete lattices to define or generalise upper approximation operators without fuzzy relations in fuzzy complete lattices. It defined Alexandrov topology in terms of operators and further investigated the relationships between join preserving maps, fuzzy preorders and

Alexandrov topologies and also gave examples [55].

- S. husain *et al.* (2015): In this paper, the authors readdress several soft separation axioms like soft T_i axioms, soft normal, soft regular and soft T_4 etc by using soft points. They also explore the concept of soft invariance properties for example soft topological property and soft hereditary property with the hope that these concepts might be useful in practical applications and in some general man-made machine systems [56].

- G. Shafer (2015): The author defined a rule for combining two or more belief functions where the belief functions are from distinct or independent sources of evidence and named this rule as Dempster's rule of combination. This rule also preserves the regularity conditions of continuity and condensability in the theory of belief functions [57].

- L. Zhaowen (2015): In this paper, an approach to fuzzy soft sets in decision making to avoid selecting a suitable level soft set and to solve the problem of medical diagnosis is presented. This approach combines grey relational analysis with the Dempster Shafer theory of evidence. The advantages of this approach have been shown by comparing this approach with mean potentiality approach.

  Various examples are solved and also medical diagnosis problem has been solved as a numerical problem with this approach [8].

- Deng (2016): In this paper, he proposes novel belief entropy which is more efficient to measure the uncertain information as it can also measure the uncertainty expressed by a basic probability assignment. It is the generalization of Shannon entropy as it degenerates into Shannon entropy when allocated to the single elements [20].

- N. Bhardwaj *et al.* (2016): They defined a new class of closed sets known as Regular $\hat{\beta}$- generalized closed sets which lies between $\hat{\beta}$- generalized and regular generalized closed sets in bitopological spaces. Also the characterizations of these sets have been studied. They also defined the notion of $r\hat{\beta}$- generalized open sets and $r\hat{\beta}$- generalized neighbourhoods in bitopological spaces [58].

- B.K Tripathy *et al.* (2016): This paper gave an application of fuzzy soft sets in decision-making problems and showed that their approach is more realistic than the algorithm given by Maji [59] in 2002. It also defined membership function for fuzzy soft sets and various concepts related to fuzzy soft theory are redefined [60].

- M. Lellis Thivagar *et al.* (2016): This is a very interesting application of nano topology. In this, they computed the technique for recruitment process via nano topology. In nano topological granular computing, only single granular had been used but this study developed a new multi granular nano topological model which was based on multi indiscernibility relations on the universe. It also discussed the relationship between multi granular nano topological spaces and multi* granular model of it. It provides a new pathway to solve decision making problems based on nano topological theory [61].

- A.K Banerjee *et al.* (2016): In this paper, a new generalisation of closed sets in Alexandroff spaces had done, named as, λ^*- closed sets, $\hat{g}$- closed sets in Alexandroff spaces. The authors' also studied the previous properties of $g*$ closed sets, explained in [45] and gave some properties which were not previously found out. They introduced a new class of closed sets, $\hat{g}$ and λ^* closed sets and studied their properties. They also studied various separation axioms $T_{w/4}, T_{3w/8}, T_w$ and introduced a new separation axiom $T_{5w/8}$ in the Alexandroff spaces which lies between $T_{3w/8}$ and T_w spaces [62].

- C.K Raman (2016): The new generalisation of closed sets known as b- closed sets was introduced by M. Ganster and M. Steiner for the general topology and then he introduced the concept of gb- closed sets in bitopological spaces, symbolized by $ij-gb$-closed sets. He also introduced $ij-gbr$ closed sets and studied the important properties of named sets and two new bitopological spaces namely $ij-T^*b_{1/2}$

and $ij - *Tb_{1/2}$ were introduced. The fundamental characteristics of such spaces have also been analyzed and correlated with $ij - Tb_{1/2}$ space [63].

- B. Meera Devi (2017): This paper defined a new class of closed sets known as $(i, j) - g^{**}b$- closed sets in bitopological spaces. Also, some of its basic properties have been studied with the investigation of relationship of these sets with other existing closed sets. An implicated figure is also given to investigate the relationship among different classes of closed sets in bitopological spaces [64]

- S. Chandrasekar *et al.* (2017): Lellis Thivagar was the first to define a new kind of topology known as nano topological space, in terms of approximations and boundary region of a subset of a universe using an equivalence relation. Nano g closed and nano sg closed sets were already introduced by K. Bhuvaneswari *et al.* in 2014 and 2015 respectively. In this paper, the new notions of nano sg –interior & nano sg- closure has been introduced which are very important in point of view of generalization of closed sets in nano topological spaces. The properties of Nsg interior & Nsg closure had also been studied with the help of examples [65].

- S. Acharjee *et al.* (2017): This paper studied the properties under soft bitopology and defined various notions like soft nowhere dense set, soft boundary and first category etc. It specifically focussed on fundamental structures for soft bitopology and leaves the new point of view for utilizing these new notions in different aspects of science for better mankind. It also investigated the results given by Cagman and Semen in 2014 with more interest [66].

- S. Lazaar *et al.* (2017): This was the very recent work on Alexandroff spaces. In this paper, the author characterised the new type of topology known as homogeneous functionally Alexandroff spaces. As indicated by this paper, a function $f : \mathcal{X} \longrightarrow X$ determines a topology $P(f)$ on X by taking the closed sets to be those sets $A \sqsubseteq X$ with $f(A) \sqsubseteq A$. The topological space $(X, P(f))$ is called functionally Alexandroff spaces. It also elaborated the concept of homogeneity of functionally Alexandroff spaces and proposed very important propositions and theorems with examples which may help other scholars for future results on it [67].

- S. Al Ghour *et al.* (2018): The paper explained the concepts of minimality and homogeneity in fuzzy respect and defined two notions of minimality in fuzzy bitopological spaces. Two new notions of homogeneity known as homogeneous and pairwise homogeneous fuzzy bitopological spces were introduced. Also, the connection between minimality and homogeneity has been given in this paper. Various important results and theorems in fuzzy homogeneous bitopological spaces have been given [68].

- M. Ferri (2018): This paper was a fascinating case of utilization of topology to artificial intelligence and learning extraction. It gave a solid base to both the age of novel hypothetical instruments and finding forefront common applications. Both machine learning and knowledge extraction need to comprehend the data shape on minute scale as well as major scale for which it require brilliant utilization of geometry for example topology. This paper manages a typical issue that data can be depicted by a high number of factors; however the dataset $\mathcal{X}$ can be inherently low-dimensional. It utilized topological summary of the dataset which makes it a lot simpler to manage it [69].

- K. Vithyasangaran *et al.* (2018): This paper generalised g- closed sets of topological space to bitopological space. They named it as $\tau_1\tau_2g-$ closed set in bitopological space $(\mathcal{X}, \tau_1, \tau_2)$. It studied the properties of this new set and compared them with the properties of $g-$ closed sets in general topological spaces. It defined $\tau_1\tau_2g$-continuous functions and also studied its properties. It also gave the counter examples for the converses of given theorems [70].

- M.Kameshwari *et al.* (2018): The authors extended the concept of clopen sets in fuzzy topological spaces to fuzzy bitopological space in this paper. It also defined many notions like fuzzy pairwise slightly precontinuous functions, defined fuzzy pairwise pre-T_0, fuzzy pairwise pre-T_1, fuzzy pairwise pre T_2, fuzzy pairwise $Co-T_0$, fuzzy pairwise $Co-T_1$, fuzzy pairwise $Co-T_2$. The main work of this paper was to investigate the separation axioms like hausdorffness, normality etc using fuzzy pairwise slightly precontinuous functions and gave important results with examples [71].

- R. Tripathi *et al.* (2018): In this research, the new concept of regular fuzzy biclosure space was introduced. It compared the results of this concept with the other existing definitions and results as well. It also explained the properties of sum, product, and subspaces of the defined space. It also defined the fuzzy pre continuous map fuzzy biclosure spaces. It studied the various possible concepts of regularity using fuzzy pre open sets and obtained the interrelation among them. And also concepts of regularity using fuzzy pre continuous map and found various interrelation among them [72].

- E. Akin (2018): This paper named fuzzy topological spaces with a new name Induced fuzzy topological spaces with a simple property affine invariance. And also studied some simple notions of compactness of such spaces. It made use of definition of lower semi continuity of a function to define affine invariance and also gave the definition of laminated fuzzy topology by changing the first condition in the definition of basic fuzzy topology. It defined weakly induced fuzzy topological spaces also and further, in the last, it explained the notion of compactness and its properties with respect to induced fuzzy topological spaces [73].

- J. Pandey *et al.* (2018): In this paper, a new class of intuitionistic fuzzy closed sets known as intuitionistic $g*p$ closed sets are introduced which lies between pre closed sets and gp closed sets. Intuitionistic fuzzy g- closed concept was introduced by Thakur and Chaturvedi in 2008 and then this paper further generalises them and introduced $g*p$ closed sets in intuitionistic fuzzy space. The author also worked on the continuity of intuitionistic fuzzy closed sets and defined intuitionistic fuzzy $g*p$ continuous mappings. Further, they introduced new kinds of spaces T^*p, $\alpha T^{**}p$ and αT^*p as an application of intuitionistic fuzzy $g*p$ closed sets and explained them with the help of examples [74].

- S.E Abbas *et al.* (2018): The concept of fuzzy soft grills are used to define the notion of connectedness in fuzzy soft topological spaces. The fuzzy soft operator $\emptyset$ is constructed from a fuzzy soft grill $\mathcal{G}_{\mathcal{E}}$ and a fuzzy soft topological space. The extension of concept of α-connectedness related to fuzzy soft operator α on the set $\mathcal{X}$ is also given [75].

- X. Fuyuan (2018): This paper gives a hybrid method for utilizing fuzzy soft sets in decision-making problems by integrating a fuzzy preference relation analysis based on belief entropy with the Dempster Shafer evidence theory. This approach can reduce the uncertainty level to greater extent and improve the quality of solving decision-making problems [24].

- D. Wang (2019): Different entropies are introduced to measure the uncertainty of data. D. Wang introduced new belief entropy known as w-entropy to measure uncertainty. It is the unified form about belief entropy based on deng entropy by considering the scale of frame of discernment and the relative scale of focal element with respect to FOD. Also, various examples have been solved with different belief entropies to compare the results [21]

- A. Emre Eysen *et al.* (2019): The paper explained the relationship between weaker forms of properties like Menger, Alster and Lindelof in bitopological spaces. It introduced weak version of the Alster property in terms of selection principles and gave counter examples with the results. Also, the properties of (i, j)- Almost Alster bitopological spaces have been examined. The paper also explained the difference between these properties with the help of examples [76].

- T.Y Ozturk *et al.* (2019): A new type of bitopological spaces defined on neutrosophic sets was introduced. This space is known as Neutrosophic Bitopological Spaces. Various notions like pairwise neutrosophic open sets, closed sets, closures and interior were presented. The paper also explained the relationship between these concepts with general topological structure. The basic properties of neutrosophic bitopological spaces along with examples were given in the paper [77].

- B. Bhattacharya *et al.* (2019): They introduced a new class of fuzzy sets called as fuzzy Λ_γ-sets and defined a completely different structure known as Fuzzy independent Alexandorff spaces. A new notion of fuzzy $\Lambda\gamma$- continuity was defined and relationship of this notion with already existing functions has been established. Further, the definition of fuzzy $\Lambda\gamma$- generalized closed sets was given and various properties in the form of results have been studied [78].

- G. Priscilla Pacifica *et al.* (2019): After the bitopological spaces [1], tri topological space [79], quad topological space [80] and penta topological spaces were introduced as a generalization of general topology. The main focus of this paper was to investigate the general properties of penta topology and analysed the nature of generalized closed sets in penta topological spaces. Some examples were also given to easily understand the outcomes [81].

- Y. Chan Kim *et al.* (2019): This paper presented another thought of Alexandroff L - fuzzy pre vicinities on complete residuated lattices which give a bound together pathway to three spaces. This paper additionally explored relations among Alexandrov L-fuzzy pre-proximities, Alexandrov L-fuzzy topologies, L - fuzzy lower approximate operators, and L-fuzzy lower approximate operators. And used all these terms in the paper to study their corresponding relations. Also the relationship between Alexandroff $L-$ Fuzzy pre proximities and Alexandov topological structure was established. Examples of the respective terms and relations were also contained in the paper [82].

- P.L Meenakshi (2019): The paper defined a new class of generalized closed sets in bitopological spaces known as $i, j - \delta$ semi-generalized star closed sets (briefly called as $i, j - \delta sg^(*)$-closed sets). Some interrelations between this kind of closed sets and already existing closed sets were given. Various properties of newly developed closed sets have been studied in this paper [83].

- Sandhiya *et al.* (2019): This paper is an example of very interesting and important application of fuzzy soft set theory. We can't make right decisions in our real life easily, so fuzzy soft set theory provides us the right guidelines to make right decisions. The problem of decision making can be solved by using this concept in the real world. In this paper, the author illustrated one numerical problem as well to deal with the problem of job requirement. They gave the algorithm to solve the problem as well. The numerical example of the paper solved job allocation problem in Indian industrial scenario [84].

- Y. Zhao *et al.* (2019): This paper put forward the concept of an improved belief entropy to measure uncertainty which is based on Deng entropy and the belief interval. More particularly the span and the center of belief interval are considered to define the total uncertainty degree. It also showed that this improved measure can be degenerated into Shannon entropy. A case study is solved to show the efficiency and flexibility of new approach as compared to previous uncertainty measures [85].

- B. Kang *et al.* (2019): There are various belief entropy functions used to measure uncertainty of data. This paper discussed the condition of the maximum of Deng entropy which is used to measure the uncertainty degree of basic probability assignment in evidence theory. Some numerical properties are also used to illustrate the basic probability assignment with the maximum Deng entropy [86].

- L. Boxer (2019): Digital topology is one of the recent notions in the real world. This paper examined the fixed point set in digital topology. Here, the author examined the properties of fixed point set and found some new results. In this paper, the author gave a complete computation of $F(Cn)$ where Cn is the digital cycle of n points. It also studied how fixed point sets in digital images can be arranged. It also explained the concept of homotopy fixed point spectrum using digital topology. Also the notion of pull indices of the digital image was introduced. All those factors which affected fixed point sets were explained separately in the paper with their respective examples and important results [87].

- I. Bukhatwa *et al.* (2020): The authors generalized the concept of (i, j)-semi I open sets and (i, j)-βI open sets and defined γ_{ij}-semi I open sets and γ_{ij}- βI open sets in ideal bitopological spaces which was defined by K. Kuraowski [88] in 1966. Also, the notions of γ_{ij} - semi I continuous functions and γ_{ij}-βI functions were introduced. Various results based on these concepts have been given with examples and counter examples as well [89].

- R. Roshmi *et al.* (2020): This paper defined one notion of regularity and two notions of normality. She made use of some separation axioms to produce new notions. A notion of completely normal space was also given. It also explained relationship of these two properties between topological spaces and bitopological spaces. Some of the features of these properties in bitopological spaces have been explained in this paper [90].

- B.M Afsan (2020): This paper introduced a new type of covering property $\beta^t_{\omega_r,s}$-closedness in bitopological spaces and gave several characterizations via filter bases and grills. He used the idea of closure operators and closed spaces modulo grills to define new results. Various grills generalizations have been introduced in these spaces [91].

- A.S Salama (2020): The paper presented the generalization of classical approximation space and named it as bitopological approximation space. He defined new membership functions and inclusion functions and used these for redefining rough approximations. He used topological approaches for information systems. Also, some properties of rough sets on bitopological spaces have been studied. Further, a real-life application related to data reduction in multi-valued information system has been elaborated [92].

2.0.2 Research Gap

This research gap helped us to achieve our objectives.

- In general topology, τ is a family of subsets of $\mathcal{X}$ and the elements in $(\mathcal{X}, \tau)$ are open sets and three conditions have been satisfied. Alexandroff topology is a space in which an arbitrary intersection of open sets is an open set. Likewise, there are other topological spaces like soft topological spaces in which τ is the family of soft sets over $\mathcal{X}$, fuzzy topological spaces where τ is the family of fuzzy soft sets over $\mathcal{X}$, intuitionistic fuzzy topology where τ is the family of Ifs in $\mathcal{X}$ and nano topology in which $\tau_R(\mathcal{X}) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X)\}$ satisfies three conditions of general topology.There is no such relation among them. So, we will try to relate our space with one of them or may produce new kind of space. Further, we will work on generalization of closed sets over them and study their properties and produce new results as well.

- In Alexandroff topology, there is generalisation of g^*- closed sets and λ^*- closed sets (till now) and in which they defined them and generalize them to study their properties. In the same way, we intend to work on generalization of rw- closed sets and other classes of closed sets and study the properties with results by giving examples also. Further, comparing the results with previous results of generalization of closed sets in general topology. Also, the work on functions, continuity behaviour of closed sets has not been discussed yet in Alexandroff topology, so we will also work on functions in this topology.

- Previous researches introduced bitopological spaces, which are prepared with two topologies on a non–empty set $\mathcal{X}$. Different generalisation of closed sets like semi-star gw- closed sets, $ij - gb$ closed sets, $\tau_1\tau_2$ g- closed sets were given and also the properties had been discussed in them. But, Alexandroff bitopological space has not been introduced yet and that's why we intend to introduce it and work on generalisation of closed sets taking infinite number of elements in account. And also study their properties and work on functions, mappings part as well.

- Fang Jinming, in 2017, [47] focussed at introducing I-fuzzy topological spaces through a fuzzy relation on a set and also explored the connection between certain generalised topological structures and fuzzy order structures. Similarly, we will try to find out relation between Alexandroff space and fuzzy orders specifically we will consider total ordering on sets instead of preordered sets and also using biresiduation operators to generate new results and further using closure operator, work on generalisation of closed sets on them. Kim in 2014 introduced new concept Alexandroff L-fuzzy on complete residuated lattices (algebraic structure) and Kim *et al.* (2019) defined Alexandroff fuzzy topology on X in terms of operators and studied the properties of join-meet preserving maps and Alexandroff fuzzy topologies. So, we intend to merge fuzzy topology with Alexandroff in general way and investigate their mappings behaviour, their properties etc and extend all these concepts using closure (interior) operator in the same way and produce new results.

- There are interesting applications of topologies in this digital world. One of them is the world of digital topology. Digital topology is used to analyse image algorithms, counting of components etc. The use of Alexandroff topology in digital way has not been found yet. So, we will use our Alexandroff discrete topology which is finitely generated topological space in digital way and make use of closed sets and their generalisation in solving the problems of digital topology. Also, the important application of fuzzy concept in real world is decision making. Fuzzy concept was used to solve the basic problems of decision making. We will use Alexandroff soft topological concept or Alexandroff fuzzy concept in solving many more problems of real life in the same way.

2.0.3 Objectives of our proposed work

After the extensive survey of literature, the main objectives of the research work are enlisted below:

✓ Develop the new type of topological space using Alexandroff Space and Soft Topological space to study the behaviour of generalized closed sets on it.

- ✓ Examine the concept of Compactness, Connectedness, Continuity using generalized closed sets in new developed topological space to study the behaviour of different kind of functions.
- ✓ To introduce Alexandroff Bitopological Spaces and then study the concept of generalization of closed sets along with their properties. Also, study and generate new separation axioms and comparing the new results with the previous ones.
- ✓ Define Fuzzy Alexandroff Topological Spaces and generalize the concept of closed sets on it to solve problems of decision making, Engineering, digital topology etc.

Chapter 3

Alexandroff Soft Topological Spaces

This chapter illustrates the emergence of new type of topological spaces known as Alexandroff Soft Topological Spaces which is a combination of Soft topological spaces and Alexandroff spaces. Also, various important topological properties have been studied under this chapter. Further, different separation axioms known as Alexo T_i-spaces have been presented alongside their properties.
Here, $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ denotes an Alexandroff Soft Topological Spaces(ASTS), $\mathcal{A}$ denotes the arbitrary set of parameters and Alexo T_i denotes various separation axioms in this section.

3.0.1 Definition of Alexandroff Soft Topological Spaces with example

Definition 3.1. An Alexandroff Soft Topological Space is a set $\mathcal{X}$ with an arbitrary set of parameters $\mathcal{A}$ together with a system $(\mathcal{K}_i, \mathcal{A})$, where $\mathcal{K}_i \colon \mathcal{A} \longrightarrow \wp(X)$ fulfilling the axioms below:
1. An arbitrary intersection of number of elements of $\tau_{\overset{\circ}{A_s}}$ is a set in $\tau_{\overset{\circ}{A_s}}$.
2. Finite union of members of sets from $\tau_{\overset{\circ}{A_s}}$ is a set in $\tau_{\overset{\circ}{A_s}}$.
3. $0_{\overset{\circ}{A_s}}$ and $1_{\overset{\circ}{A_s}}$ are in $\tau_{\overset{\circ}{A_s}}$.

Example 3.1. *Take $\mathcal{X} = \mathbb{R}$. $\mathcal{A}$ be a set of parameters defined as $\mathcal{A} = \{0, 1\}$. Let $\tau_{\overset{\circ}{A_s}}$ $= \{(\mathcal{P}_i, \mathcal{A}), i = 1, 2 \ldots\} \sqcup \{0_{\overset{\circ}{A_s}}, 1_{\overset{\circ}{A_s}}\}$. A mapping $\mathcal{P}_i : \mathcal{A} \to \wp(X)$ defined as:*

$$\mathcal{P}_i = \begin{cases} i, i+1, \dots \quad if \quad a = 0, \\ \emptyset \quad if \quad a = 1 \end{cases}$$

Clearly, $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is an Alexandroff Soft Topological Spaces.

3.0.2 Properties of Alexandroff Soft Topological Spaces

Definition 3.2. A basis for an ASTS is a collection $\mathcal{P}$ of soft closed subsets of $\mathcal{X}$ which fulfill the following given statements:

1) $\sqcap \mathcal{P} = \emptyset$.

2) For $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{P}, \mathcal{P}_1 \sqcup \mathcal{P}_2 = \sqcap \mathcal{P}_i$, for $i \in I$.

Any collection of soft closed subsets of $\mathcal{X}$ which satiates these conditions can define topology generated by $\mathcal{P}$.

Theorem 3.3. *Suppose $(\mathcal{U}, \mathcal{A})$ be a family of soft closed sets in ASTS. Then, $(\mathcal{U}, \mathcal{A})$ is known as the minimal base for the topology $\tau_{\overset{\circ}{A_s}}$ of $\mathcal{X}$ if and only if the following axioms are satisfied:*

(1) $(\mathcal{U}, \mathcal{A})$ covers $\mathcal{X}$.

(2) For $(\mathcal{P}, \mathcal{A}), (\mathcal{Q}, \mathcal{A}) \in (\mathcal{U}, \mathcal{A})$, $\exists$ a subfamily $\{(\mathcal{U}_i, \mathcal{A})\colon i \in I\}$ of $(\mathcal{U}, \mathcal{A})$ such that $(\mathcal{P}, \mathcal{A}) \cup (\mathcal{Q}, \mathcal{A}) = \cap (\mathcal{U}_i, \mathcal{A})$, for $i \in I$.

(3) If a subfamily $\{(\mathcal{U}_i, \mathcal{A})\colon i \in I\}$ of $(\mathcal{U}, \mathcal{A})$ verifies $\cup (\mathcal{U}_i, \mathcal{A}) \in (\mathcal{U}, \mathcal{A})$, for $i \in I$, then $\exists$ $i_o \in I$ such that $\cap\ (\mathcal{U}_i, \mathcal{A}) = \mathcal{U}_{i_o}$.

Corollary 3.4. *Suppose $\mathcal{U}$ be a non-empty set and $\mathbb{B}$ be a basis for topology $\tau_{\overset{\circ}{A_s}}$ on $\mathcal{U}$. Then, the collection $\tau_{\overset{\circ}{A_s}}$ equals to the set of all intersections of basis elements.*

Proof. A family of elements of $\mathbb{B}$ are also the members of $\tau_{\overset{\circ}{A_s}}$. Since $\tau_{\overset{\circ}{A_s}}$ is a topology, their intersection must be in $\tau_{\overset{\circ}{A_s}}$. Conversely, for $\mathcal{U} \in \tau_{\overset{\circ}{A_s}}$, an element B_x of $\mathbb{B}$ be such that $x \in B_x \subset \mathcal{U}$, for each $x \in \mathcal{U}$. Then, $\mathcal{U} = \sqcap_{x \in \mathcal{U}} B_x$, so $\mathcal{U}$ equals to the intersection of members of $\mathbb{B}$. □

Definition 3.5. Let $\mathcal{Y} \subseteq \mathcal{X}$ be a non-empty set in ASTS. Then, $\tau_{\mathcal{Y}_{\overset{\circ}{A_s}}} = \{(\mathcal{P}_{\mathcal{Y}}, \mathcal{A})) \mid (\mathcal{P}, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}\}$ is known as relative topology on $\mathcal{Y}$ and $(\mathcal{Y}, \tau_{\mathcal{Y}_{\overset{\circ}{A_s}}})$ is known as Alexandroff Soft subspace of $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$.

Example 3.2. *Any Alexandroff Soft subspace of an Alexandroff discrete topological space is also discrete topological space.*

Proposition 3.6. *Let $\mathcal{Y} \subseteq \mathcal{X}$ be a non-empty set in ASTS. Then, $(\mathcal{Y}, \tau_{\beta_{\mathcal{Y}}})$ is a subspace of $(\mathcal{X}, \tau_{\beta_{\overset{\circ}{A_s}}})$ for each $\beta \in \mathcal{A}$.*

Proof. Since $(\mathcal{Y}, \tau_{\mathcal{Y}_{\overset{\circ}{A_s}}})$ is an Alexandroff Soft Topological Spaces in $\mathcal{Y}$, so $(\mathcal{Y}, \tau_{\beta_{\mathcal{Y}}})$ is also a topological space for each $\beta \in \mathcal{A}$. Now, for each $\beta \in \mathcal{A}, \tau_{\beta_{\mathcal{Y}}} = \{\mathcal{P}_{\mathcal{Y}}(\beta) \mid (\mathcal{P}, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}\}$.
$= \{\mathcal{Y} \sqcap \mathcal{P}(\beta) \mid (\mathcal{P}, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}\}$.
$= \{\mathcal{Y} \sqcap \mathcal{P}(\beta) \mid (\mathcal{P}, \mathcal{A}) \in \tau_{\beta_{\overset{\circ}{A_s}}}\}$.
Thus, $(\mathcal{Y}, \tau_{\beta_{\mathcal{Y}}})$ is a subspace of $(\mathcal{X}, \tau_{\beta_{\overset{\circ}{A_s}}})$. □

Definition 3.7. Let $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}})$ be a soft closed subset of $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$. Then, $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}}) \subseteq (\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is known as an Alexandroff Soft Closed subspace of $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ when $(\mathcal{X}, \tau_{\overset{\circ}{A_s}}) \setminus (\mathcal{Y}, \tau_{\overset{\circ}{A_s}}) \in \tau_{\overset{\circ}{A_s}}$, or equivalently, An Alexandroff Soft subspace $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}})$ of $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is known to be closed if the injection map $i\colon (\mathcal{Y}, \tau_{\overset{\circ}{A_s}}) \to (\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is closed.

Theorem 3.8. *Suppose $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ and $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}})$ are ASTS with minimal bases $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{Q}, \mathcal{A})$. If $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is an Alexandroff Soft closed subspace of $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}})$, then we have $\mathcal{P} = \{\mathcal{Q} \cap \mathcal{X} \colon \mathcal{Q} \in \mathcal{Q}\}$.*

Proof. The proof is taken directly from [93]. □

Theorem 3.9. *Suppose $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}})$ be an Alexandroff Soft closed subspace of $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$. Then, $(\mathcal{P}, \mathcal{A})$ is an Alexandroff Soft closed in $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}})$ where $\mathcal{A}$ is the set of parameters iff it equals to the intersection of an Alexandroff Soft closed set of $\mathcal{X}$ with $\mathcal{Y}$.*

Proof. Let us suppose that $(\mathcal{P}, \mathcal{A}) = \mathcal{Q} \sqcap \mathcal{Y}$, where $\mathcal{Q}$ is an Alexandroff Soft closed in $\mathcal{X}$. Then, $\mathcal{X} \setminus \mathcal{Q}$ is Alexandroff Soft Open in $\mathcal{X}$, so $(\mathcal{X} - \mathcal{Q}) \sqcap \mathcal{Y}$ is Alexandroff Soft Open in $\mathcal{Y}$.
Because $(\mathcal{X} - \mathcal{Q}) \sqcap \mathcal{Y} = \mathcal{Y} \setminus \mathcal{P} \Rightarrow \mathcal{Y} \setminus \mathcal{P}$ is Alexandroff Soft Open in $\mathcal{Y}$, we have $\mathcal{P}$ is Alexandroff Soft closed in $\mathcal{Y}$.
Conversely, $\mathcal{P}$ is Alexandroff Soft closed in $\mathcal{Y}$ which implies $\mathcal{Y} - \mathcal{P}$ is open in $\mathcal{Y}$, thus, $\mathcal{Y} - \mathcal{P} = \mathcal{V} \sqcap \mathcal{Y}$ where $\mathcal{V}$ is an Alexandroff Soft Open set of $\mathcal{X}$. The set $\mathcal{X} - \mathcal{V}$ is Alexandroff Soft Closed in $\mathcal{X}$ and $\mathcal{P}$ can be written as $\mathcal{Y} \sqcap (\mathcal{X} - \mathcal{V})$ which is the required result. □

Corollary 3.10. *Let $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}})$ be an Alexandroff Soft closed subspace of $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$. Then, $\mathcal{B} \sqsubseteq \mathcal{Y}$ is Alexandroff Soft closed in $\mathcal{Y}$ iff $\mathcal{B}$ is Alexandroff Soft closed in $\mathcal{X}$.*

Definition 3.11. The Alexandroff Soft Product Topology is the topology with basis as the collection $\mathbb{B}$ of all sets of the form $\mathcal{P} * \mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ are the Alexandroff soft subsets of $\mathcal{X}$ and $\mathcal{Y}$ respectively.

Theorem 3.12. *In Alexandroff Soft Product Topology, the product of Alexandroff Soft Closed sets is closed .*

Proof. Suppose $\mathcal{A}_i$ be Alexandroff Soft closed in $\mathcal{X}_i$, $\forall\ i = 1, 2,$
Then, $\mathcal{X}_i \setminus \mathcal{A}_i$ is an Alexandroff soft open set in $\mathcal{X}_i$ $\forall i$, where $\mathcal{X}_i$ is the collection of Alexandroff Soft Topological Spaces.
So, $(\prod_{i=1}^{n} \mathcal{X}_i) \setminus (\prod_{i=1}^{n} \mathcal{A}_i) = [(\mathcal{X}_1 \setminus \mathcal{A}_1 * \mathcal{X}_2 * \ldots * \mathcal{X}_n)] \sqcup [\mathcal{X}_1 * (\mathcal{X}_2 \setminus \mathcal{A}_2) * \mathcal{X}_3 \ldots * \mathcal{X}_n] \sqcup \ldots \sqcup [\mathcal{X}_1 * \mathcal{X}_2 \ldots * (\mathcal{X}_n \setminus \mathcal{A}_n)]$
$\Rightarrow$ L.H.S is a union of Alexandroff Soft Open sets in Alexandroff Soft Product Spaces.
Therefore, L.H.S is an Alexandroff Soft open set in $\prod_{i=1}^{n} \Rightarrow \prod_{i=1}^{n} \mathcal{A}_i$ is closed in $\prod_{i=1}^{n} \mathcal{X}_i$. □

Remark 3.13. If $\mathcal{G}$ and $\mathcal{F}$ are Alexandroff Soft closed sets in $\mathcal{X}$ and $\mathcal{Y}$ respt., they can't form a basis for Alexandroff Soft closed sets in the product topology.
Example:- Cofinite topology on $\mathcal{X} = \mathcal{Y} = \mathbb{N}$.

Proposition 3.14. *Suppose $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ be an Alexandroff Soft Topological Space over $\mathcal{X}$. The collection- $\tau_{\beta_{\overset{\circ}{A_s}}} = \{\mathcal{P}(\beta) | (\mathcal{P}, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}\}$ defines a topology on $\mathcal{X}$, where $\beta \in \mathcal{A}$.*

Proof. Since $\tau_{\beta_{\overset{\circ}{A_s}}} = \{\mathcal{P}(\beta) | (\mathcal{P}, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}\}$ for $\beta \in \mathcal{A}$.
Clearly, (1) $\emptyset, \mathcal{X} \in \tau_{\overset{\circ}{A_s}}$ implies $\emptyset, \mathcal{X} \in \tau_{\beta_{\overset{\circ}{A_s}}}$.
(2) Let $\{\mathcal{P}_i(\beta) \mid i \in I\}$ be a collection in $\tau_{\beta_{\overset{\circ}{A_s}}}$. Since $(\mathcal{P}_i, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}$ $\forall i \in I$, we have $\sqcap_{i \in I}$ $(\mathcal{P}_i, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}$, thus $\sqcap_{i \in I}$ $(\mathcal{P}_i, \mathcal{A}) \in \tau_{\beta_{\overset{\circ}{A_s}}}$.
(3) Since $(\mathcal{P}, \mathcal{A}) \sqcup (\mathcal{Q}, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}$, so $\mathcal{P}(\beta) \sqcup \mathcal{Q}(\beta) \in \tau_{\beta_{\overset{\circ}{A_s}}}$, for $\mathcal{P}(\beta), \mathcal{Q}(\beta) \in \tau_{\overset{\circ}{A_s}}$.
Thus, $\tau_{\beta_{\overset{\circ}{A_s}}}$ also defines a topology on $\mathcal{X}$, for each $\beta \in \mathcal{A}$. Also, this results implies that we have a topology $\tau_{\beta_{\overset{\circ}{A_s}}}$ on $\mathcal{X}$ corresponding to each parameter. But conversely, it doesn't holds. □

Example 3.3. *Let $\mathcal{X} = \{x_1, x_2, x_3\}, \mathcal{A} = \{a_1, a_2\}$ and $\tau_{\overset{\circ}{A_s}} = \{\emptyset, \mathcal{X}, (\mathcal{P}_1, \mathcal{A}), (\mathcal{P}_2, \mathcal{A}), (\mathcal{P}_3, \mathcal{A})\}$ where $(\mathcal{P}_i, \mathcal{A})$ are soft sets over $\mathcal{X}$, defined as follows:*

$\mathcal{P}_1(a_1) = \{x_1\} \qquad \mathcal{P}_1(a_2) = \{x_2\}$

$\mathcal{P}_2(a_1) = \{x_1, x_2\} \quad \mathcal{P}_2(a_2) = \{x_1, x_3\}$

$\mathcal{P}_3(a_1) = \{x_1, x_3\} \quad \mathcal{P}_3(a_2) = \{x_1\}.$

Then, $\tau_{\overset{\circ}{A_s}}$ is not a topology on $\mathcal{X}$ as $(\mathcal{P}_1, \mathcal{A}) \sqcup (\mathcal{P}_2, \mathcal{A}) = (\mathcal{Q}, \mathcal{A})$ where $\mathcal{Q}(a_1) = \{x_1, x_2\}$ and $\mathcal{Q}(a_2) = \mathcal{X}$ and so $(\mathcal{Q}, \mathcal{A}) \notin \tau_{\overset{\circ}{A_s}}$.Also, $\tau_{a_1} = \{\emptyset, \mathcal{X}, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}\}$ and $\tau_{a_2} = \{\emptyset, \mathcal{X}, \{x_1\}, \{x_2\}, \{x_1, x_3\}\}$ are topologies on $\mathcal{X}$. This can be shown that even if the collection corresponding to each parameter defines a topology on $\mathcal{X}$, that collection need not be an Alexandroff Soft Topology on $\mathcal{X}$.

Definition 3.15. Let $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ be an Alexandroff Soft Topological Space over $\mathcal{X}$ and $(\mathcal{P}, \mathcal{A})$ be an alexandroff soft set over $\mathcal{X}$. Then, the Alexandroff Soft closure of $(\mathcal{P}, \mathcal{A})$ denoted by $cl_{\mathcal{X}}(\mathcal{P}, \mathcal{A})$ is the intersection of all soft closed supersets of $(\mathcal{P}, \mathcal{A})$.
Clearly, $cl_{\mathcal{X}}(\mathcal{P}, \mathcal{A})$ is the smallest Alexandroff Soft closed set over $\mathcal{X}$ containing $(\mathcal{P}, \mathcal{A})$.

Definition 3.16. Let $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ be an Alexandroff Soft Topological Space over $\mathcal{X}$ and $(\mathcal{P}, \mathcal{A})$ be an Alexandroff Soft set over $\mathcal{X}$. Then, an Alexandroff Soft set $(cl_{\mathcal{X}}\mathcal{P}, \mathcal{A})$ or $(\overline{\mathcal{P}}, \mathcal{A})$ associated with $(\mathcal{P}, \mathcal{A})$ is defined as-
$\overline{\mathcal{P}}(\beta) = \overline{\mathcal{P}(\beta)}$ where $\overline{\mathcal{P}(\beta)}$ is the closure of $\mathcal{P}(\beta)$ in $\tau_{\beta_{\overset{\circ}{A_s}}}$, for each $\beta \in \mathcal{A}$.

Proposition 3.17. *If $(\mathcal{P}, \mathcal{A})$ is an Alexandroff Soft set over $\mathcal{X}$.Then, $(\overline{\mathcal{P}}, \mathcal{A}) \subset \overline{(\mathcal{P}, \mathcal{A})}$.*

Proof. Since $\overline{\mathcal{P}}(\beta)$ is the smallest soft closed set in $(\mathcal{X}, \tau_{\beta_{\overset{\circ}{A_s}}})$, which contains $\mathcal{P}(\beta)$, for any $\beta \in \mathcal{A}$. Also, $\mathcal{H}(\beta)$ is also soft closed set in $(\mathcal{X}, \tau_{\beta_{\overset{\circ}{A_s}}})$ containing $\mathcal{P}(\beta)$, if $\overline{(\mathcal{P}, \mathcal{A})} = (\mathcal{H}, \mathcal{A})$. This implies that $\overline{\mathcal{P}}(\beta) = \overline{\mathcal{P}(\beta)} \subseteq \mathcal{H}(\beta)$. Thus, $(\overline{\mathcal{P}}, \mathcal{A}) \subset \overline{(\mathcal{P}, \mathcal{A})}$. □

Definition 3.18. A function $g : \mathcal{X} \to \mathcal{Y}$ is known as Alexandroff Soft continuous if for each Alexandroff Soft closed subset $\mathcal{P}$ of $\mathcal{Y}$, $g^{-1}(\mathcal{P})$ is an Alexandroff Soft closed subset of $\mathcal{X}$ where $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ and $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}})$ are two Alexandroff Soft Topological Spaces.

Theorem 3.19. *If $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$, $(\mathcal{Y}, \tau_{\overset{\circ}{A_s}})$ and $(\mathcal{Z}, \tau_{\overset{\circ}{A_s}})$ are three Alexandroff Soft Topological Spaces and $g : \mathcal{X} \to \mathcal{Y}$ and $f : \mathcal{Y} \to \mathcal{Z}$ are Alexandroff Soft continuous maps. Then, the composition of these maps is also Alexandroff Soft continuous. That is, $fog : \mathcal{X} \to \mathcal{Z}$ is also Alexandroff Soft closed.*

Proof. Let $\mathcal{V}$ be Alexandroff Soft closed in $\mathcal{Z}$. Then, $f^{-1}(\mathcal{V})$ is Alexandroff Soft closed in $\mathcal{Y}$. By the continuity of g, $g^{-1}[f^{-1}(\mathcal{V})] = (fog)^{-1}(\mathcal{V})$ is Alexandroff Soft closed in $\mathcal{X}$. Thus, fog is Alexandroff Soft continuous. □

Definition 3.20. [13] A mapping g is known as Alexandroff Soft homeomorphism if $g\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is a bijection and both g and g^{-1} are Alexandroff Soft continuous.

Proposition 3.21. *Suppose $(\mathcal{K},\mathcal{A}) \in SCS(\mathcal{X},\mathcal{A})$, h is injective map of $\mathcal{A}$ onto $\mathcal{B}_s$ and g be a one-one map of $\mathcal{X}$ onto $\mathcal{Y}$. We have to prove the following:*
1) $\rho_{gh}(\mathcal{K},\mathcal{A}) = \rho^{-1}_{g^{-1}h^{-1}}(\mathcal{K},\mathcal{A})$.
2) $\rho_{gh}((\mathcal{K},\mathcal{A})) = (\rho_{gh}((\mathcal{K},\mathcal{A}))$.

Proof. 1) Let $\rho_{gh}(\mathcal{K},\mathcal{A}) = (\mathcal{L},\mathcal{B}_s)$, $\rho^{-1}_{g^{-1}h^{-1}}(\mathcal{K},\mathcal{A}) = (\mathcal{M},\mathcal{B}_s)$ and $q_{\mathcal{Y}} \in \mathcal{B}_s$. We must prove that $\mathcal{L}(q_{\mathcal{Y}}) = \mathcal{M}(q_{\mathcal{Y}})$.
Let $h^{-1}(q_{\mathcal{Y}}) = q_{\mathcal{X}}$. Since the map $h\colon \mathcal{A} \longrightarrow \mathcal{B}_s$ is $1-1$, $\mathcal{L}(q_{\mathcal{Y}}) = g(\mathcal{K}(q_{\mathcal{X}}))$.
On the other hand, $\mathcal{M}(q_{\mathcal{Y}}) = (g^{-1})(\mathcal{K}(h^{-1}(q_{\mathcal{Y}})) = g(\mathcal{K}(q_{\mathcal{X}})$.
Thus, $\mathcal{L}(q_{\mathcal{Y}}) = \mathcal{M}(q_{\mathcal{Y}})$.
2) Let $\rho_{gh}((\mathcal{K},\mathcal{A})) = (\mathcal{L},\mathcal{A})$, $(\rho_{gh}(\mathcal{K},\mathcal{A})) = (\mathcal{M},\mathcal{B}_s)$, $q_{\mathcal{Y}} \in \mathcal{B}_s$. We must prove that $\mathcal{L}(q_{\mathcal{Y}})$ $= \mathcal{M}(q_{\mathcal{Y}})$.
Let $h^{-1}(q_{\mathcal{Y}}) = q_{\mathcal{X}}$.
Since the map $h\colon \mathcal{A} \longrightarrow \mathcal{B}_s$ is $1-1$, $\mathcal{L}(q_{\mathcal{Y}}) = g(\mathcal{X} \setminus \mathcal{K}(q_{\mathcal{X}}))$. It is given that g mapping is $1-1$ and onto, $g(\mathcal{X} \setminus \mathcal{K}(q_{\mathcal{X}})) = \mathcal{Y} \setminus g(\mathcal{K}(q_{\mathcal{X}})$.
Therefore,$\mathcal{L}(q_{\mathcal{Y}}) = \mathcal{Y} \setminus g(\mathcal{K}(q_{\mathcal{X}}))$.
On the other hand, $\mathcal{M}(q_{\mathcal{Y}}) = Y \setminus g(\mathcal{K}(q_{\mathcal{X}}))$.
Thus, $\mathcal{L}(q_{\mathcal{Y}}) = \mathcal{M}(q_{\mathcal{Y}})$. □

3.0.3 Separation axioms in Alexandroff Soft Topological Spaces

Definition 3.22. A space $(\mathcal{X},\tau_{\mathring{A}_s})$ is called Alexo T_0- space if for points $x, y \in \mathcal{X}$ and $x \neq y$, $\exists$ soft closed sets $(\mathcal{P},\mathcal{A})$ and $(\mathcal{Q},\mathcal{A})$ such that $x \in (\mathcal{P},\mathcal{A})$ and $y \notin (\mathcal{P},\mathcal{A})$ or $y \in (\mathcal{Q},\mathcal{A})$ and $x \notin (\mathcal{Q},\mathcal{A})$.

Definition 3.23. A space $(\mathcal{X},\tau_{\mathring{A}_s})$ is called Alexo T_1- space if there exists soft closed sets $(\mathcal{P},\mathcal{A})$ and $(\mathcal{Q},\mathcal{A})$ such that $x \in (\mathcal{P},\mathcal{A})$ and $y \notin (\mathcal{P},\mathcal{A})$ and $y \in (\mathcal{Q},\mathcal{A})$ and $x \notin (\mathcal{Q},\mathcal{A})$, for points $x, y \in \mathcal{X}$ and $x \neq y$.

Theorem 3.24. *Suppose that $(\mathcal{X},\tau_{\mathring{A}_s})$ be an Alexandroff Soft Topological Space. If $(x,\mathcal{A})$ is a soft open set in $\tau_{\mathring{A}_s}$ for each $x \in \mathcal{X}$, then, $(\mathcal{X},\tau_{\mathring{A}_s})$ is Alexo T_1- space.*

Proof. Let $(x, \mathcal{A})$ is a soft open set in $\tau_{\overset{\circ}{A_s}}$, for each $x \in \mathcal{X}$. So, $(x, \mathcal{A})'$ is soft closed in $\tau_{\overset{\circ}{A_s}}$ and $y \in (x, \mathcal{A})'$ and $x \notin (x, \mathcal{A})'$ for $x \neq y$, $x, y \in \mathcal{X}$. Similarly, $(y, \mathcal{A})' \in \tau_{\overset{\circ}{A_s}}$ such that $x \in (y, \mathcal{A})'$ and $y \notin (y, \mathcal{A})'$. Thus, $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is Alexo T_1- space.
But the converse does not holds. □

Example 3.4. *Let $\mathcal{X} = \{x_1, x_2\}, \mathcal{A} = \{a_1, a_2\}$ and $\tau_{\overset{\circ}{A_s}} = \{\emptyset, \mathcal{X}, (\mathcal{P}_1, \mathcal{A}), (\mathcal{P}_2, \mathcal{A})\}$ where*
$\mathcal{P}_1(a_1) = \mathcal{X} \qquad \mathcal{P}_1(a_2) = \{x_1\}$
$\mathcal{P}_2(a_1) = \{x_2\} \qquad \mathcal{P}_2(a_2) = \mathcal{X}.$
Then, $\tau_{\overset{\circ}{A_s}}$ is a topology over $\mathcal{X}$.
We have, $\tau_{a_1} = \{\emptyset, \mathcal{X}, \{x_2\}$ and $\tau_{a_2} = \{\emptyset, \mathcal{X}, \{x_1\}$.
Neither $(\mathcal{X}, \tau_{a_1})$ nor $(\mathcal{X}, \tau_{a_2})$ is Alexo T_1- space but $x_1, x_2 \in \mathcal{X}$ with $x_1 \neq x_2$, and also $x_1 \in (\mathcal{P}_1, \mathcal{A})$ and $x_2 \notin (\mathcal{P}_1, \mathcal{A})$ and $x_2 \in (\mathcal{P}_2, \mathcal{A})$ and $x_1 \notin (\mathcal{P}_2, \mathcal{A})$. Therefore, $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is Alexo T_1- space.
Also, for $(x_1, \mathcal{A})$ and $(x_2, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}$ over $\mathcal{X}$, defined as
$x_1(a_1) = x_1 \qquad x_1(a_2) = x_1$
$x_2(a_1) = x_2 \qquad x_2(a_2) = x_2.$
Their relative complements are
$x_1'(a_1) = x_2 \quad x_1'(a_2) = x_2$
$x_2'(a_1) = x_1 \quad x_2'(a_2) = x_1.$
Neither $(x_1, \mathcal{A})'$ nor $(x_2, \mathcal{A})' \in \tau_{A_s}$. Hence, the converse of the result.

Proposition 3.25. *A non-empy subset of Alexo T_1- space is also an Alexo T_1- space.*

Proof. Since $\mathcal{X}$ is Alexo T_1- space, then $\exists$ soft closed sets $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$ and $y \notin (\mathcal{P}, \mathcal{A})$ and $y \in (\mathcal{Q}, \mathcal{A})$ and $x \notin (\mathcal{Q}, \mathcal{A})$ for $x, y \in \mathcal{X}$ and $x \neq y$.
Now, $x \in \mathcal{Y}$ and $x \in (\mathcal{P}, \mathcal{A})$ implies $x \in \mathcal{Y} \sqcap (\mathcal{P}, \mathcal{A}) = (\mathcal{P}_{\mathcal{Y}}, \mathcal{A})$ where $(\mathcal{P}, \mathcal{A}) \in \tau_{\overset{\circ}{A_s}}$.
Suppose $y \notin (\mathcal{P}, \mathcal{A})$ which implies $y \notin \mathcal{P}(\beta)$ for some $\beta \in \mathcal{A}$.
Now, $y \notin \mathcal{Y} \sqcap \mathcal{P}(\beta) \Rightarrow \mathcal{Y}(\beta) \sqcap \mathcal{P}(\beta)$. Therefore, $y \notin \mathcal{Y} \sqcap (\mathcal{P}, \mathcal{A}) = (\mathcal{P}_{\mathcal{Y}}, \mathcal{A})$.
In the same way, we will show that $y \in (\mathcal{Q}_{\mathcal{Y}}, \mathcal{A})$ and $x \notin (\mathcal{Q}_{\mathcal{Y}}, \mathcal{A})$.
Hence proved. □

Definition 3.26. A space $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is known as Alexo T_2- space if there exists soft closed sets $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$, $y \in (\mathcal{Q}, \mathcal{A})$ and $(\mathcal{P}, \mathcal{A}) \sqcap (\mathcal{Q}, \mathcal{A}) = \emptyset$, for $x, y \in \mathcal{X}$ such that $x \neq y$

Proposition 3.27. *If $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is Alexo T_2- space over $\mathcal{X}$, then $(\mathcal{X}, \tau_{\beta_{\overset{\circ}{A_s}}})$ is also Alexo T_2- space for each $\beta \in \mathcal{A}$.*

Proof. For any $\beta \in \mathcal{A}$,

$\tau_{\beta_{A_s}^{\circ}} = \{\mathcal{P}(\beta) \mid (\mathcal{P}, \mathcal{A}) \in \tau_{A_s}^{\circ}\}$.

Let $x, y \in \mathcal{X}$ such that $x \neq y$, $\exists\ (\mathcal{P}, \mathcal{A}), (\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$ and $y \in (\mathcal{Q}, \mathcal{A})$ with $(\mathcal{P}, \mathcal{A}) \sqcap (\mathcal{Q}, \mathcal{A}) = \emptyset$.

This means $x \in \mathcal{P}(\beta)$ and $y \in \mathcal{Q}(\beta)$ and $\mathcal{P}(\beta) \sqcap \mathcal{Q}(\beta) = \emptyset$.

Hence, $(\mathcal{X}, \tau_{\beta_{A_s}^{\circ}})$ is also Alexo T_2- space for each $\beta \in \mathcal{A}$. □

Remark 3.28. 1) Every Alexo T_1- space is Alexo T_0.

2) Every Alexo T_2 space is Alexo T_1.

Proof. 1) If $(\mathcal{X}, \tau_{A_s}^{\circ})$ is Alexo T_1- space, then $\exists$ two soft closed sets $(\mathcal{P}, \mathcal{A}), (\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$ and $y \notin (\mathcal{P}, \mathcal{A})$ and $y \in (\mathcal{Q}, \mathcal{A})$ and $x \notin (\mathcal{Q}, \mathcal{A})$. It is obvious to say that $x \in (\mathcal{P}, \mathcal{A})$ and $y \notin (\mathcal{P}, \mathcal{A})$ or $y \in (\mathcal{Q}, \mathcal{A})$ and $x \notin (\mathcal{Q}, \mathcal{A})$. This implies that $(\mathcal{X}, \tau_{A_s}^{\circ})$ is Alexo T_0- space.

2) Let $(\mathcal{X}, \tau_{A_s}^{\circ})$ is Alexo T_1- space, then for $x \neq y$, $\exists\ (\mathcal{P}, \mathcal{A}), (\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$ and $y \in (\mathcal{Q}, \mathcal{A})$ with $(\mathcal{P}, \mathcal{A}) \sqcap (\mathcal{Q}, \mathcal{A}) = \emptyset$. Since $(\mathcal{P}, \mathcal{A}) \sqcap (\mathcal{Q}, \mathcal{A}) = \emptyset$, so $x \notin (\mathcal{Q}, \mathcal{A})$ and $y \notin (\mathcal{Q}, \mathcal{A})$. Thus,$x \in (\mathcal{P}, \mathcal{A})$ and $y \notin (\mathcal{Q}, \mathcal{A})$ and $y \in (\mathcal{Q}, \mathcal{A})$ and $x \notin (\mathcal{Q}, \mathcal{A})$. Hence, Every Alexo T_2 space is Alexo T_1.

But not every Alexo T_0 is T_1 and not every Alexo T_1 is T_2. □

Example 3.5. *Let $\mathcal{X} = \{x_1, x_2\}, \mathcal{A} = \{a_1, a_2\}$ and $\tau_{A_s}^{\circ} = \{\emptyset, \mathcal{X}, (\mathcal{P}_1, \mathcal{A}), (\mathcal{P}_2, \mathcal{A})\}$ where*

$\mathcal{P}_1(a_1) = \mathcal{X} \qquad \mathcal{P}_2(a_1) = \{x_1\}$

$\mathcal{P}_1(a_2) = \{x_2\} \qquad \mathcal{P}_2(a_2) = \mathcal{X}$.

Thus, $(\mathcal{X}, \tau_{A_s}^{\circ})$ is Alexo T_1 but not Alexo T_2 because x_1 and $x_2 \in \mathcal{X}$, there does not exist any $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{Q}, \mathcal{A})$ soft closed sets such that $x_1 \in (\mathcal{P}, \mathcal{A})$ and $x_2 \in (\mathcal{Q}, \mathcal{A})$ with $(\mathcal{P}, \mathcal{A}) \sqcap (\mathcal{Q}, \mathcal{A}) = \emptyset$.

Now, consider $\tau_{A_s}^{\circ} = \{\emptyset, \mathcal{X}, (\mathcal{P}_1, \mathcal{A})$ where $\mathcal{P}_1(a_1) = x_2$ and $\mathcal{P}_1(a_1) = \mathcal{X}$. Then, clearly $\mathcal{X}$ with this topology is Alexo T_0 but not Alexo T_1 as there does not exist soft closed sets $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{Q}, \mathcal{A})$ such that $x_1 \in (\mathcal{P}, \mathcal{A})$ and $x_2 \notin (\mathcal{P}, \mathcal{A})$ and $x_2 \in (\mathcal{Q}, \mathcal{A})$ and $x_1 \notin (\mathcal{Q}, \mathcal{A})$.

Proposition 3.29. *Let $(\mathcal{X}, \tau_{A_s}^{\circ})$ be an Alexandroff Soft Topological Space and $\mathcal{Y} \subseteq \mathcal{X}$. Then, $(\mathcal{Y}, \tau_{A_s}^{\circ})$ is also Alexo T_2 if $(\mathcal{X}, \tau_{A_s}^{\circ})$ is Alexo T_2- space.*

Proof. Suppose points $x, y \in \mathcal{Y}$ be such that x not equal to y. Since $(\mathcal{X}, \tau_{A_s}^{\circ})$ is Alexo T_2- space, then there exists $(\mathcal{P}, \mathcal{A}), (\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$ and $y \in (\mathcal{Q}, \mathcal{A})$ with

$(\mathcal{P},\mathcal{A}) \sqcap (\mathcal{Q},\mathcal{A}) = \emptyset$. So, for each $\beta \in \mathcal{A}$, $x \in \mathcal{P}(\beta)$, $y \in \mathcal{Q}(\beta)$ and $\mathcal{P}(\beta) \sqcap \mathcal{Q}(\beta) = \emptyset$.
This implies that $x \in \mathcal{Y} \sqcap \mathcal{P}(\beta)$, $y \in \mathcal{Y} \sqcap \mathcal{Q}(\beta)$ and $\mathcal{P}(\beta) \sqcap \mathcal{Q}(\beta) = \emptyset$.
Hence, $x \in (\mathcal{P}_{\mathcal{Y}},\mathcal{A})$ and $y \in (\mathcal{Q}_{\mathcal{Y}},\mathcal{A})$ and $(\mathcal{P}_{\mathcal{Y}},\mathcal{A}) \sqcap (\mathcal{Q}_{\mathcal{Y}},\mathcal{A}) = \emptyset$,
where $(\mathcal{P}_{\mathcal{Y}},\mathcal{A}), (\mathcal{Q}_{\mathcal{Y}},\mathcal{A}) \in \tau_{\mathcal{Y}^{\circ}_{A_s}}$.
Thus, $(\mathcal{Y}, \tau^{\circ}_{A_s})$ is also Alexo T_2- space. □

Definition 3.30. (Alexo Regular space) A space $\mathcal{X}$ is said to be a Alexo Regular space if $\exists$ soft closed sets $(\mathcal{P}_1,\mathcal{A}), (\mathcal{P}_2,\mathcal{A})$ such that $x \in (\mathcal{P}_1,\mathcal{A}), (\mathcal{Q},\mathcal{A}) \subseteq (\mathcal{P}_2,\mathcal{A})$ and $(\mathcal{P}_1,\mathcal{A}) \sqcap (\mathcal{P}_2,\mathcal{A}) = \emptyset$. for $(\mathcal{Q},\mathcal{A})$ a soft open set and $x \in \mathcal{X}$ such that $x \notin (\mathcal{Q},\mathcal{A})$.

Definition 3.31. A space $(\mathcal{X}, \tau^{\circ}_{A_s})$ is known as Alexo T_3 if it is Alexo regular as well as Alexo T_1- space.

Remark 3.32. 1) An Alexo T_3- space may or may not be Alexo T_2-space.
2) A space $\mathcal{X}$ with topology $\tau_{\beta^{\circ}_{A_s}}$ corresponding to each $\beta \in \mathcal{A}$ may not be Alexo T_3-space even if $(\mathcal{X}, \tau^{\circ}_{A_s})$ is Alexo T_3-space.

Proposition 3.33. *Let $\mathcal{Y} \subseteq \mathcal{X}$ be a non-empty set in ASTS. Then, $(\mathcal{Y}, \tau_{\beta^{\circ}_{A_s}})$ is also Alexo T_3- space if $\mathcal{X}$ is Alexo T_3.*

Proof. Since $\mathcal{X}$ is Alexo T_3, it means that $\mathcal{X}$ is also Alexo T_1 as well as Regular space.
By Proposition 3.25, $\mathcal{Y}$ is also T_1.
Suppose that $(\mathcal{Q},\mathcal{A})$ is a soft open set in $\mathcal{Y}$ such that $y \notin (\mathcal{Q},\mathcal{A})$, for $y \in \mathcal{Y}$.
By using Theorem 1.10, $y \notin ((\mathcal{Y},\mathcal{A}) \sqcap (\mathcal{P},\mathcal{A})$, for $(\mathcal{Q},\mathcal{A}) = ((\mathcal{Y},\mathcal{A}) \sqcap (\mathcal{P},\mathcal{A}))$, for $(\mathcal{P},\mathcal{A})$ in $\mathcal{X}$.
Now, $y \notin (\mathcal{P},\mathcal{A})$ as $y \in (\mathcal{Y},\mathcal{A})$. Also, $\mathcal{X}$ is T_3-space, so $\exists$ $(\mathcal{P}_1,\mathcal{A}), (\mathcal{P}_2,\mathcal{A})$ such that $y \in (\mathcal{P}_1,\mathcal{A}), (\mathcal{P},\mathcal{A}) \subset (\mathcal{P}_2,\mathcal{A})$ and $(\mathcal{P}_1,\mathcal{A}) \sqcap (\mathcal{P}_2,\mathcal{A}) = \emptyset$.
Now, take $(\mathcal{Q}_1,\mathcal{A}) = (\mathcal{Y},\mathcal{A}) \sqcap (\mathcal{P}_1,\mathcal{A})$ and $(\mathcal{Q}_2,\mathcal{A}) = (\mathcal{Y},\mathcal{A}) \sqcap (\mathcal{P}_2,\mathcal{A})$, then, $(\mathcal{Q}_1,\mathcal{A}), (\mathcal{G}_2,\mathcal{A}) \in \tau_{\mathcal{Y}^{\circ}_{A_s}}$ such that $y \in (\mathcal{Q}_1,\mathcal{A})$ and $(\mathcal{Q},\mathcal{A}) \subset (\mathcal{Y},\mathcal{A}) \sqcap (\mathcal{P}_2,\mathcal{A}) = (\mathcal{Q}_2,\mathcal{A})$ and also $(\mathcal{Q}_1,\mathcal{A}) \sqcap (\mathcal{Q}_2,\mathcal{A}) \subseteq (\mathcal{P}_1,\mathcal{A}) \sqcap (\mathcal{P}_2,\mathcal{A}) = \emptyset$ implies $(\mathcal{Q}_1,\mathcal{A}) \sqcap (\mathcal{Q}_2,\mathcal{A}) = \emptyset$.
Hence, $(\mathcal{Y}, \tau_{\beta^{\circ}_{A_s}})$ is Alexo T_3- space. □

Definition 3.34. A space $(\mathcal{X}, \tau^{\circ}_{A_s})$ is known as Alexo Normal space if $\exists$ soft closed sets $(\mathcal{P}_1,\mathcal{A})$ and $(\mathcal{P}_2,\mathcal{A})$ such that $(\mathcal{P},\mathcal{A}) \sqsubset (\mathcal{P}_1,\mathcal{A})$, $(\mathcal{Q},\mathcal{A}) \sqsubset (\mathcal{P}_2,\mathcal{A})$ and $(\mathcal{P}_1,\mathcal{A}) \sqcap (\mathcal{P}_2,\mathcal{A}) = \emptyset$ for $(\mathcal{P},\mathcal{A})$ and $(\mathcal{Q},\mathcal{A})$ soft open sets over $\mathcal{X}$ and $(\mathcal{P},\mathcal{A}) \sqcap (\mathcal{Q},\mathcal{A}) = \emptyset$.

Definition 3.35. An Alexandroff Soft Topological Space $(\mathcal{X}, \tau^{\circ}_{A_s})$ is known as Alexo T_4 if it is Alexo T_1 and Alexo Normal space.

Remark 3.36. 1) An Alexo T_4- space need not be Alexo T_3-space.

2) A space with topology $\tau_{\beta_{\overset{\circ}{A_s}}}$ corresponding to each parameter $\beta \in \mathcal{A}$ need not be Alexo T_4- space if $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is Alexo T_4.

3) If $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ is Alexo T_4-space, then $(\mathcal{Y}, \tau_{\mathcal{Y}_{\overset{\circ}{A_s}}})$ is not necessary a Alexo T_4-space being a non-void subset of $\mathcal{X}$.

Example 3.6. *[3] Let* $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$, $\mathcal{A} = \{a_1, a_2\}$ *and* $\tau_{\overset{\circ}{A_s}} = \{\emptyset, \mathcal{X}, (\mathcal{P}_1, \mathcal{A}), (\mathcal{P}_2, \mathcal{A}), (\mathcal{P}_3, \mathcal{A}) \ldots ($
where $(\mathcal{P}_i, \mathcal{A})$ *are defined as-*

$\mathcal{P}_1(a_1) = \{x_1, x_2, x_4\}$ $\quad$ $\mathcal{P}_1(a_2) = \{x_1, x_2, x_3\}$

$\mathcal{P}_2(a_1) = \{x_1, x_3, x_4\}$ $\quad$ $\mathcal{P}_2(a_2) = \{x_1, x_2, x_3\}$

$\mathcal{P}_3(a_1) = \{x_1, x_4\}$ $\quad$ $\mathcal{P}_3(a_2) = \{x_1, x_2, x_3\}$

$\mathcal{P}_4(a_1) = \{x_2, x_3\}$ $\quad$ $\mathcal{P}_4(a_2) = \{x_1, x_2, x_3\}$

$\mathcal{P}_5(a_1) = \{x_2\}$ $\quad$ $\mathcal{P}_5(a_2) = \{x_1, x_2, x_3\}$

$\mathcal{P}_6(a_1) = \{x_3\}$ $\quad$ $\mathcal{P}_6(a_2) = \{x_1, x_2, x_3\}$

$\mathcal{P}_7(a_1) = \emptyset$ $\quad$ $\mathcal{P}_7(a_2) = \{x_1, x_2, x_3\}$

$\mathcal{P}_8(a_1) = \mathcal{X}$ $\quad$ $\mathcal{P}_8(a_2) = \{x_1, x_2, x_3\}$.

Then, $(\mathcal{X}, \tau_{\overset{\circ}{A_s}})$ *is Alexo* T_4*- space but not Alexo* T_3*- space.*

Now, consider $\tau_{a_1} = \{\emptyset, \mathcal{X}, \{x_1, x_2\}, \{x_1, x_3, x_4\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2\}, \{x_3\}\}$ *and* $\tau_{a_2} = \{\emptyset, \mathcal{X}, \{x_1, x_2, x_3\}\}$.

Here, τ_{a_1} *and* τ_{a_2} *are not Alexo* T_3*- spaces This shows that a space with topology* $\tau_{\beta_{\overset{\circ}{A_s}}}$ *corresponding to each parameter* $\beta \in \mathcal{A}$ *need not be Alexo* T_4*- spaces for each* $\beta \in \mathcal{A}$.

Now, take $\mathcal{Y} = \{x_1, x_2, x_3\}$. *Then* $\tau_{\mathcal{Y}_{\overset{\circ}{A_s}}} = \{\emptyset, \mathcal{X}, (\mathcal{P}_{\mathcal{Y}_1}, \mathcal{A}), (\mathcal{P}_{\mathcal{Y}_2}, \mathcal{A}), (\mathcal{P}_{\mathcal{Y}_3}, \mathcal{A}) \ldots (\mathcal{P}_{\mathcal{Y}_8}, \mathcal{A})\}$,
where $(\mathcal{P}_{\mathcal{Y}_i}, \mathcal{A})$ *are defined as-*

$\mathcal{P}_{\mathcal{Y}_1}(a_1) = \{x_1, x_2\}$ $\quad$ $\mathcal{P}_{\mathcal{Y}_1}(a_2) = \mathcal{Y}$

$\mathcal{P}_{\mathcal{Y}_2}(a_1) = \{x_1, x_3\}$ $\quad$ $\mathcal{P}_{\mathcal{Y}_2}(a_2) = \mathcal{Y}$

$\mathcal{P}_{\mathcal{Y}_3}(a_1) = \{x_1\}$ $\quad$ $\mathcal{P}_{\mathcal{Y}_3}(a_2) = \mathcal{Y}$

$\mathcal{P}_{\mathcal{Y}_4}(a_1) = \{x_2, x_3\}$ $\quad$ $\mathcal{P}_{\mathcal{Y}_4}(a_2) = \mathcal{Y}$

$\mathcal{P}_{\mathcal{Y}_5}(a_1) = \{x_2\}$ $\quad$ $\mathcal{P}_{\mathcal{Y}_5}(a_2) = \mathcal{Y}$

$\mathcal{P}_{\mathcal{Y}_6}(a_1) = \{x_3\}$ $\quad$ $\mathcal{P}_{\mathcal{Y}_6}(a_2) = \mathcal{Y}$

$\mathcal{P}_{\mathcal{Y}_7}(a_1) = \emptyset$ $\quad$ $\mathcal{P}_{\mathcal{Y}_7}(a_2) = \mathcal{Y}$

$\mathcal{P}_{\mathcal{Y}_8}(a_1) = \mathcal{Y}$ $\quad$ $\mathcal{P}_{\mathcal{Y}_8}(a_2) = \mathcal{Y}$.

It has been seen that $(\mathcal{Y}, \tau_{\mathcal{Y}_{\overset{\circ}{A_s}}})$ *is not Alexo* T_4*- space. Thus, a subspace of Alexo* T_4 *need not be Alexo* T_4*- space.*

Thus, it concludes the definition of Alexandroff Soft Topological Spaces with some important properties and the generation of new separation axioms with examples.

Chapter 4

$g_{\mathring{A}_s}$-soft closed sets in Alexandroff Soft Topological Spaces

Closed sets have their own importance in topological spaces from years and the generalization of them plays important role in the study of various topological spaces. It can produces various separation axioms, covering lemmas, different types of closed sets etc. N. levine [4] firstly initiated the concept of generalization in topological spaces. Likewise, in this chapter, we define $g_{\mathring{A}_s}$-soft closed sets and investigated its various properties in Alexandroff Soft Topological Spaces.

Throughout the chapter, we refer to Alexandroff soft topological spaces as $(\mathcal{X}, \tau_{\mathring{A}_s})$ or $\mathcal{X}$ and $\mathring{A}_s$ denotes an arbitrary set of parameters.

4.0.1 Definition of $g_{\mathring{A}_s}$-soft closed set

Definition 4.1. In $(\mathcal{X}, \tau_{\mathring{A}_s})$, a soft closed set $(\mathcal{P}', \mathring{A}_s)$ is called Alexandroff Soft Generalized Closed Set(in short, $g_{\mathring{A}_s}$-soft closed set) if there is a soft closed set $(\mathcal{A}', \mathring{A}_s)$ containing $(\mathcal{P}', \mathring{A}_s)$ such that $(\mathcal{A}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ when $(\mathcal{P}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{V}, \mathring{A}_s)$ is soft open in $\mathcal{X}$.

Example 4.1. *Suppose $\mathcal{X} = \mathbb{R}$ and $\mathring{A}_s = \{e_1, e_2, \dots e_i\}$ be an arbitrary set of parameters where $i \in I$. Let $\tau_{\mathring{A}_s}$ have the family of soft closed sets in $\mathcal{X}$. Let $(P_1', \mathring{A}_s)$, $(P_2', \mathring{A}_s)$ and $(P_3', \mathring{A}_s)$ be mappings from set of parameters to power set of X defined as-*

$$P_1^{'}(e_i) = \begin{cases} 2^i, & if \quad e_i = e_1, \\ \emptyset, & otherwise \end{cases}$$

$$P_2^{'}(e_i) = \begin{cases} 3i, & if \quad e_i = e_1, \\ \emptyset, & otherwise \end{cases}$$

$$P_3^{'}(e_i) = \begin{cases} (5+i, 6+i) & if \quad e_i = e_1, \\ \emptyset, & otherwise \end{cases}$$

Clearly, $(\mathcal{X}, \tau_{\mathring{A}_s})$ is Alexandroff Soft Topological Spaces and $(P_2^{'}, \mathring{A}_s)$ and $(P_2^{'}, \mathring{A}_s)$ are $g_{\mathring{A}_s}$- soft closed sets in $\mathcal{X}$. But $(P_3^{'}, \mathring{A}_s)$ is not $g_{\mathring{A}_s}$-soft closed sets in $\mathcal{X}$.

4.0.2 Characteristics of $g_{\mathring{A}_s}$-soft closed set

Theorem 4.2. *Suppose $(\mathcal{P}^{'}, \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft closed in $\mathcal{X}$ and $(\mathcal{P}^{'}, \mathring{A}_s) \sqsubset (\mathcal{Q}^{'}, \mathring{A}_s) \sqsubset (\mathcal{R}^{'}, \mathring{A}_s)$ where $(\mathcal{R}^{'}, \mathring{A}_s)$ is a soft closed set contained in $(\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{V}, \mathring{A}_s)$ is soft open. Then, $(\mathcal{Q}^{'}, \mathring{A}_s)$ is also $g_{\mathring{A}_s}$-soft closed set in $\mathcal{X}$.*

Proof. Assume that $(\mathcal{Q}^{'}, \mathring{A}_s)$ contained in $(\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{V}, \mathring{A}_s)$ is soft open. Since $(\mathcal{P}^{'}, \mathring{A}_s) \sqsubset (\mathcal{Q}^{'}, \mathring{A}_s)$ and $(\mathcal{Q}^{'}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$, thus $(\mathcal{P}^{'}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$. Hence, there is a soft closed set $(\mathcal{R}^{'}, \mathring{A}_s)$ such that $(\mathcal{R}^{'}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ whenever $(\mathcal{P}^{'}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{V}, \mathring{A}_s)$ is soft open (by the defn of $g_{\mathring{A}_s}$-soft closed set).
Since $(\mathcal{Q}^{'}, \mathring{A}_s) \sqsubset (\mathcal{R}^{'}, \mathring{A}_s)$, this implies $(\mathcal{R}^{'}, \mathring{A}_s)$ is a soft closed set containing $(\mathcal{Q}^{'}, \mathring{A}_s)$ such that $(\mathcal{R}^{'}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ whenever $(\mathcal{Q}^{'}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{V}, \mathring{A}_s)$ is soft open.
Therefore, $(\mathcal{Q}^{'}, \mathring{A}_s)$ is also $g_{\mathring{A}_s}$-soft closed set. □

Theorem 4.3. *A soft closed set $(\mathcal{P}^{'}, \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft closed set in $\mathcal{X}$ iff there is a soft closed set $(\mathcal{R}^{'}, \mathring{A}_s) \sqsupset (\mathcal{P}^{'}, \mathring{A}_s)$ such that $(\mathcal{R}^{'}, \mathring{A}_s) \setminus (\mathcal{P}^{'}, \mathring{A}_s)$ does n't have any non-empty soft closed sets.*

Proof. It is given that $(\mathcal{P}^{'}, \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft closed set in $\mathcal{X}$. Then, there exists soft closed set $(\mathcal{R}^{'}, \mathring{A}_s)$ containing $(\mathcal{P}^{'}, \mathring{A}_s)$ such that $(\mathcal{R}^{'}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ whenever $(\mathcal{P}^{'}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{V}, \mathring{A}_s)$ is soft open.
Now, assume that $(\mathcal{R}_1^{'}, \mathring{A}_s) \sqsubseteq (\mathcal{R}^{'}, \mathring{A}_s) \setminus (\mathcal{P}^{'}, \mathring{A}_s)$ and $(\mathcal{R}_1^{'}, \mathring{A}_s)$ is non-empty soft closed set. Since $(\mathcal{R}_1^{'}, \mathring{A}_s)$ is soft closed , $(\mathcal{R}_1, \mathring{A}_s)$ is soft open and $(\mathcal{P}^{'}, \mathring{A}_s) \sqsubseteq (\mathcal{R}_1, \mathring{A}_s)$, it follows that $(\mathcal{R}^{'}, \mathring{A}_s) \sqsubseteq (\mathcal{R}_1, \mathring{A}_s)$ and so $(\mathcal{R}_1^{'}, \mathring{A}_s) \sqsubseteq (\mathcal{R}, \mathring{A}_s)$ and thus $(\mathcal{R}_1^{'}, \mathring{A}_s) \sqsubseteq$

$(\mathcal{R}, \mathring{A}_s) \sqcap (\mathcal{R}', \mathring{A}_s) = \emptyset$, this implies $(\mathcal{R}_1', \mathring{A}_s)$ is empty , which is a contradiction to our assumption.

Conversely, $(\mathcal{P}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{V}, \mathring{A}_s)$ is soft open in $\mathcal{X}$. If $(\mathcal{R}', \mathring{A}_s) \not\sqsubset (\mathcal{V}, \mathring{A}_s)$, then,$(\mathcal{R}', \mathring{A}_s) \sqcap (\mathcal{V}, \mathring{A}_s)$ is a non-empty soft closed set in $(\mathcal{R}', \mathring{A}_s) \setminus (\mathcal{P}', \mathring{A}_s)$, which contradicts the given fact.

Hence, $(\mathcal{P}', \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft closed set in $\mathcal{X}$. □

Corollary 4.4. *A $g_{\mathring{A}_s}$-soft closed set $(\mathcal{P}', \mathring{A}_s)$ is soft closed iff $cl(\mathcal{P}', \mathring{A}_s) \setminus (\mathcal{P}', \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft closed .*

Proof. It is given that $(\mathcal{P}', \mathring{A}_s)$ is both soft closed and $g_{\mathring{A}_s}$-soft closed set, then it is evident that $cl(\mathcal{P}', \mathring{A}_s)$ and $cl(\mathcal{P}', \mathring{A}_s) \setminus (\mathcal{P}', \mathring{A}_s)$ are both empty and soft closed in nature.

Conversely, Let $(\mathcal{P}', \mathring{A}_s)$ be a $g_{\mathring{A}_s}$-soft closed such that $cl(\mathcal{P}', \mathring{A}_s) \setminus (\mathcal{P}', \mathring{A}_s)$ is soft closed. Since $(\mathcal{P}', \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft closed set, using theorem 4.3, there exists soft closed set $(\mathcal{A}', \mathring{A}_s)$ containing $(\mathcal{P}', \mathring{A}_s)$ such that $(\mathcal{A}', \mathring{A}_s) \setminus (\mathcal{P}', \mathring{A}_s)$ does n't contain any non-void soft closed set. Since $cl(\mathcal{P}', \mathring{A}_s) \setminus (\mathcal{P}', \mathring{A}_s)$ is closed and $cl(\mathcal{P}', \mathring{A}_s) \setminus (\mathcal{P}', \mathring{A}_s) \sqsubset (\mathcal{A}', \mathring{A}_s) \setminus (\mathcal{P}', \mathring{A}_s)$ and $cl(\mathcal{P}', \mathring{A}_s) \setminus (\mathcal{P}', \mathring{A}_s) = \emptyset$, it means $cl(\mathcal{P}', \mathring{A}_s) = (\mathcal{P}', \mathring{A}_s)$ and so $(\mathcal{P}', \mathring{A}_s)$ is soft closed. □

Theorem 4.5. *Suppose $(\mathcal{P}', \mathring{A}_s)$ and $(\mathcal{Q}', \mathring{A}_s)$ are two $g_{\mathring{A}_s}$-soft closed sets, the union of these two is also $g_{\mathring{A}_s}$-soft closed set in $\mathcal{X}$.*

Proof. Suppose that $(\mathcal{P}', \mathring{A}_s)$ and $(\mathcal{Q}', \mathring{A}_s)$ both are $g_{\mathring{A}_s}$-soft closed sets. Let $(\mathcal{P}', \mathring{A}_s) \sqcup (\mathcal{Q}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ where $(\mathcal{V}, \mathring{A}_s)$ is soft open in $\mathcal{X}$. Since $(\mathcal{P}', \mathring{A}_s) \sqcup (\mathcal{Q}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$, we have $(\mathcal{P}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{Q}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$. Now, $(\mathcal{P}', \mathring{A}_s)$ and $(\mathcal{Q}', \mathring{A}_s)$ are $g_{\mathring{A}_s}$-soft closed sets, by definition, there are closed sets $(\mathcal{S}', \mathring{A}_s)$ and $(\mathcal{S}_1', \mathring{A}_s)$ containing $(\mathcal{P}', \mathring{A}_s)$ and $(\mathcal{Q}', \mathring{A}_s)$ respt. such that $(\mathcal{S}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{S}_1', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ and $(\mathcal{V}, \mathring{A}_s)$ is soft open. Therefore, $(\mathcal{S}', \mathring{A}_s) \sqcup (\mathcal{S}_1', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ whenever $(\mathcal{P}', \mathring{A}_s) \sqcup (\mathcal{Q}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ where $(\mathcal{V}, \mathring{A}_s)$ is soft open in $\mathcal{X}$. Thus, the union of two $g_{\mathring{A}_s}$-soft closed sets is also $g_{\mathring{A}_s}$-soft closed set in $\mathcal{X}$. □

Remark 4.6. If $(\mathcal{P}', \mathring{A}_s)$ and $(\mathcal{Q}', \mathring{A}_s)$ are two $g_{\mathring{A}_s}$-soft closed sets, then their intersection $(\mathcal{P}', \mathring{A}_s) \sqcap (\mathcal{Q}', \mathring{A}_s)$ is not necessary to be $g_{\mathring{A}_s}$-soft closed set.

Follows from example 2.5 [4]

4.0.3 $g_{\mathring{A}_s}$-soft open sets in Alexandroff Soft Topological Spaces

Definition 4.7. A soft closed set $(\mathcal{P}', \mathring{A}_s)$ is known as $g_{\mathring{A}_s}$-soft open in an Alexandroff Soft Topological Spaces if the relative complement of $(\mathcal{P}, \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft closed in $\mathcal{X}$.
Or We can say that a set $(\mathcal{P}', \mathring{A}_s)$ is called $g_{\mathring{A}_s}$-soft open in $\mathcal{X}$ iff $\exists$ an open set $(\mathcal{R}, \mathring{A}_s)$ contained in $(\mathcal{P}', \mathring{A}_s)$ such that $(\mathcal{U}', \mathring{A}_s) \sqsubset (\mathcal{R}, \mathring{A}_s)$ when $(\mathcal{U}', \mathring{A}_s)$ is soft closed and $(\mathcal{U}', \mathring{A}_s) \sqsubset (\mathcal{P}', \mathring{A}_s)$.

Theorem 4.8. *If $(\mathcal{P}', \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft open in $\mathcal{X}$ and $int_{\mathring{A}_s}(\mathcal{P}', \mathring{A}_s) \sqsubset (\mathcal{B}', \mathring{A}_s) \sqsubset (\mathcal{P}', \mathring{A}_s)$, then $(\mathcal{B}', \mathring{A}_s)$ is also $g_{\mathring{A}_s}$-soft open.*

Proof. Let $(\mathcal{R}', \mathring{A}_s) \sqsubset (\mathcal{B}', \mathring{A}_s)$ and $(\mathcal{R}', \mathring{A}_s)$ is soft closed in $\mathcal{X}$. Since $(\mathcal{B}', \mathring{A}_s) \sqsubset (\mathcal{P}', \mathring{A}_s)$ and $(\mathcal{R}', \mathring{A}_s) \sqsubset (\mathcal{B}', \mathring{A}_s)$, so $(\mathcal{R}', \mathring{A}_s) \sqsubset (\mathcal{P}', \mathring{A}_s)$. Now, there exists open set $(\mathcal{V}, \mathring{A}_s) \sqsubset (\mathcal{P}', \mathring{A}_s)$ such that $(\mathcal{R}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ when $(\mathcal{R}', \mathring{A}_s)$ is soft closed in $\mathcal{X}$ and $(\mathcal{R}', \mathring{A}_s) \sqsubset (\mathcal{P}', \mathring{A}_s)$.
Since $int_{\mathring{A}_s}(\mathcal{P}', \mathring{A}_s) \sqsubset (\mathcal{B}', \mathring{A}_s)$, this implies that there is an open set $(\mathcal{V}, \mathring{A}_s)$ contained in $(\mathcal{B}', \mathring{A}_s)$ such that $(\mathcal{R}', \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ whenever $(\mathcal{R}', \mathring{A}_s)$ is soft closed and $(\mathcal{R}', \mathring{A}_s) \sqsubset (\mathcal{B}', \mathring{A}_s)$.
Hence, $(\mathcal{B}', \mathring{A}_s)$ is also $g_{\mathring{A}_s}$-soft open in $\mathcal{X}$. □

Theorem 4.9. *In $(\mathcal{X}, \tau_{\mathring{A}_s})$, a soft closed set $(\mathcal{P}', \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft open iff there is a soft open $(\mathcal{U}, \mathring{A}_s)$ contained in $(\mathcal{P}', \mathring{A}_s)$ such that $(\mathcal{U}, \mathring{A}_s) \sqcup (\mathcal{P}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ where $(\mathcal{V}, \mathring{A}_s)$ is soft open implies $\mathcal{V} = \mathcal{X}$.*

Proof. Since $(\mathcal{P}', \mathring{A}_s)$ is $g_{\mathring{A}_s}$-soft open, thus there exists soft open set $(\mathcal{U}, \mathring{A}_s)$ contained in $(\mathcal{P}', \mathring{A}_s)$ which satisfies the definition of $g_{\mathring{A}_s}$-soft open set. Now, let $(\mathcal{V}, \mathring{A}_s)$ be soft open such that $(\mathcal{U}, \mathring{A}_s) \sqcup (\mathcal{P}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$.Then, $(\mathcal{V}', \mathring{A}_s) \sqsubset (\mathcal{U}', \mathring{A}_s) \sqcap (\mathcal{P}', \mathring{A}_s)$. Since $(\mathcal{V}', \mathring{A}_s)$ is soft closed and $(\mathcal{V}', \mathring{A}_s) \sqsubset (\mathcal{U}', \mathring{A}_s), (\mathcal{V}', \mathring{A}_s) \sqsubset (\mathcal{P}', \mathring{A}_s)$. So, $(\mathcal{V}', \mathring{A}_s) \sqsubset (\mathcal{U}', \mathring{A}_s) \sqcap (\mathcal{U}, \mathring{A}_s) = \emptyset$ which implies $\mathcal{V} = \mathcal{X}$.
Conversely, let there is a soft open set $(\mathcal{U}, \mathring{A}_s) \sqsubseteq (\mathcal{P}', \mathring{A}_s)$ such that $(\mathcal{U}, \mathring{A}_s) \sqcup (\mathcal{P}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ where $(\mathcal{V}, \mathring{A}_s)$ is soft open set and thus implies $\mathcal{V} = \mathcal{X}$.
Let $(\mathcal{A}', \mathring{A}_s)$ be a soft closed set that is contained in $(\mathcal{P}', \mathring{A}_s)$. Now, $(\mathcal{U}, \mathring{A}_s) \sqcup (\mathcal{P}, \mathring{A}_s) \sqsubset (\mathcal{U}, \mathring{A}_s) \sqcup (\mathcal{A}, \mathring{A}_s)$ which is open and so, $(\mathcal{U}, \mathring{A}_s) \sqcup (\mathcal{A}, \mathring{A}_s) = \mathcal{X}$ which implies $(\mathcal{A}', \mathring{A}_s) \sqsubset (\mathcal{U}, \mathring{A}_s)$. Hence proved. □

Theorem 4.10. *In $(\mathcal{X}, \tau_{\mathring{A}_s})$, if $(\mathcal{P}^{'}, \mathring{A}_s)$ and $(\mathcal{Q}^{'}, \mathring{A}_s)$ are $g_{\mathring{A}_s}$-soft open sets, their intersection is also $g_{\mathring{A}_s}$-soft open.*

Proof. Supose that $(\mathcal{P}^{'}, \mathring{A}_s)$ and $(\mathcal{Q}^{'}, \mathring{A}_s)$ be $g_{\mathring{A}_s}$-soft open sets. Let $(\mathcal{P}^{'}, \mathring{A}_s) \sqcap (\mathcal{Q}^{'}, \mathring{A}_s)$ $= (\mathcal{H}^{'}, \mathring{A}_s)$. Now, suppose $(\mathcal{A}^{'}, \mathring{A}_s)$ be a soft closed set contained in $(\mathcal{H}^{'}, \mathring{A}_s)$. Since, $(\mathcal{P}^{'}, \mathring{A}_s)$ and $(\mathcal{Q}^{'}, \mathring{A}_s)$ are $g_{\mathring{A}_s}$-soft open sets, then there exists open sets $(\mathcal{U}, \mathring{A}_s)$ and $(\mathcal{U}_1, \mathring{A}_s)$ respt. such that $(\mathcal{A}^{'}, \mathring{A}_s) \sqsubset (\mathcal{U}, \mathring{A}_s)$ and $(\mathcal{A}^{'}, \mathring{A}_s) \sqsubset (\mathcal{U}_1, \mathring{A}_s)$. Thus, $(\mathcal{A}^{'}, \mathring{A}_s) \sqsubset$ $(\mathcal{U}, \mathring{A}_s) \sqcap (\mathcal{U}_1, \mathring{A}_s) = (\mathcal{V}, \mathring{A}_s)$. Now, for $(\mathcal{H}^{'}, \mathring{A}_s)$, there is a soft open set $(\mathcal{V}, \mathring{A}_s)$ contained in $(\mathcal{H}^{'}, \mathring{A}_s)$ s. t $(\mathcal{A}^{'}, \mathring{A}_s) \sqsubset (\mathcal{V}, \mathring{A}_s)$ whenever $(\mathcal{A}^{'}, \mathring{A}_s)$ is soft closed and $(\mathcal{A}^{'}, \mathring{A}_s)$ contained in $(\mathcal{H}^{'}, \mathring{A}_s)$.
Hence the proof. □

Remark 4.11. Generally, the union of two $g_{\mathring{A}_s}$-soft open sets need not to be $g_{\mathring{A}_s}$-soft open. But if $g_{\mathring{A}_s}$-soft open sets are weakly separated, their union is $g_{\mathring{A}_s}$-soft open. Follows from theorem 4.3 [4]

Thus, here, we gave a new generalization of closed sets in Alexandroff Soft Topological Spaces and investigated their properties along with the notion of $g_{\mathring{A}_s}$-soft open sets.

Chapter 5

rw^*-closed sets in Alexandroff spaces

This chapter explained and defined the notion of Regular Weakly Star closed (briefly known as rw^*-closed) sets in Alexandroff spaces in which every point has a minimal neighbourhood. We discuss the characterizations and study their properties based on set theory along with the notion of rw^*-open sets.

In this work, a space $(\mathcal{X}, \tau_{\mathcal{A}})$ or simply $\mathcal{X}$ represents Alexandroff spaces and $\mathcal{R}$ and $\mathcal{Q}$ denotes the set of real numbers and rational numbers respectively.

Definition 5.1. [11] A system $\mathcal{E}$ of subsets together with a set is said to be an Alexandroff space (or σ), if the given below conditions have been fulfilled:

1) An arbitrary intersection of number of sets of $\mathcal{E} \in \mathcal{E}$.

2) Finite union of number of sets of $\mathcal{E} \in \mathcal{E}$.

3) $\emptyset$ and $\mathcal{X} \in \mathcal{E}$.

Components of $\mathcal{E}$ are known as closed sets. complement of these sets are known to be open.

One can take open sets instead of closed sets with the conditions of finite intersectability, countable summability and the whole set and non-empty set must be open.

Remark 5.2. $\tau_{\mathcal{A}}$ is not a topology, in general, which can easily be seen when we take $\mathcal{X} = \mathcal{R}$ with $\tau_{\mathcal{A}}$ as a family of all $\mathcal{E}_\sigma$ in $\mathcal{R}$.

Definition 5.3. [58] In $(\mathcal{X}, \tau_{\mathcal{A}})$, a subset $\mathcal{P}$ is known as Regular generalized (rg-closed) if $\bar{\mathcal{P}} \sqsubseteq \mathcal{V}$ wherever $\mathcal{P} \sqsubseteq \mathcal{V}$ and $\mathcal{V}$ is regular open in $\mathcal{X}$.

Definition 5.4. [38] If a regular open set $\mathcal{V}$ satisfy the condition $\mathcal{V} \sqsubset \mathcal{P} \sqsubset cl(\mathcal{V})$ where $\mathcal{P} \subseteq \mathcal{X}$, then $\mathcal{P}$ is said to be regular semi-open. The family of regular semi-open is denoted by $RSO(\mathcal{X})$.

5.0.1 Definition of rw^*-closed sets

Definition 5.5. A set $\mathcal{P} \sqsubset (\mathcal{X}, \tau_{\mathcal{A}})$ is known as Regular Weakly Star - closed (rw^*-closed) if there exist $\mathcal{S}$, a closed set such that $\mathcal{P} \sqsubset \mathcal{S} \sqsubset \mathcal{V}$ wherever $\mathcal{P} \sqsubset \mathcal{V}$ and $\mathcal{V}$ is regular semi-open in $(\mathcal{X}, \tau_{\mathcal{A}})$.
RW^*C $(\mathcal{X})$ represents the collection of all rw^*-closed sets.

5.0.2 Various results regarding rw^*-closed sets in Alexandroff spaces

Theorem 5.6. *[38] Every w-closed set in $\mathcal{X}$ always implies rw^*-closed but not conversely.*

Proof. From the point that each regular semi-open set is semi-open.
Conversely, it is not true as seen in the following illustration:

Example 5.1. *Suppose $\mathcal{X} = \mathcal{R} \setminus \mathcal{Q}$ and $\tau_{\mathcal{A}} = \{\mathcal{X}, \emptyset, H_i\}$. Clearly, $(\mathcal{X}, \tau_{\mathcal{A}})$ is not a space in general topology. Let $\mathcal{P}$ be the set containing all irrational numbers of the interval $[0, 1]$. Then $\mathcal{P}$ is not open since the set of irrational numbers neither closed nor open and hence not semi-open which implies it is not w-closed but it is rw^*-closed set as the only regular semi-open as well as closed set is $\mathcal{X}$ that contains $\mathcal{P}$.*

□

Remark 5.7. Every closed set is rw^*-closed but the converse of this is not true which can be seen by the given illustration :

Example 5.2. *Suppose $\mathcal{X} = \mathcal{R} \setminus \mathcal{Q}$ and $\tau_{\mathcal{A}} = \{\mathcal{X}, \emptyset, H_i\}$, where H_i is the collection of all countable subsets of $\mathcal{X}$. So, $(\mathcal{X}, \tau_{\mathcal{A}})$ is not a space in general topology. Now, let $\mathcal{P}$ be the set containing all irrational numbers of interval $[0, 1]$ and clearly, it is rw^*-closed set as the only regular semi-open and closed set that contains $\mathcal{P}$ is $\mathcal{X}$ but $\mathcal{P}$ is not closed.*

Theorem 5.8. *The subset $\mathcal{P}$ is rw^*-closed in $\mathcal{X}$ if it is regular generalized closed and regular open.*

Proof. Let $\mathcal{V}$ be any regular semi-open set such that $\mathcal{P} \sqsubseteq \mathcal{V}$. Since $\mathcal{P}$ is regular open and regular generalized closed, $\bar{\mathcal{P}} \sqsubseteq \mathcal{P}$. Thus, there exist closed set $\mathcal{S}$ such that $\mathcal{P} \sqsubseteq \mathcal{S} \sqsubseteq \mathcal{V}$ wherever $\mathcal{P} \sqsubseteq \mathcal{V}$ and $\mathcal{V}$ is regular semi-open.
Hence, $\mathcal{P}$ is rw^*-closed set in $\mathcal{X}$. □

Remark 5.9. In an Alexandroff Topological Spaces, every regular semi-open sets is not semi-open.

Example 5.3. *Let $\mathcal{X} = \mathcal{R} \setminus \mathcal{Q}$. And topology $\tau_{\mathcal{A}} = \{\mathcal{X}, \emptyset, H_i\}$. Let $\mathcal{P}$ be the set of all irrationals in interval $(0, 1)$. Since $\mathcal{P}$ is uncountable, so it is not open and hence not semi-open but clearly, it is regular semi-open.*

Theorem 5.10. *If $\mathcal{P}$ and $\mathcal{Q}$ are rw^*-closed sets, then $\mathcal{P} \sqcup \mathcal{Q}$ is also rw^*-closed set in $\mathcal{X}$.*

Proof. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are rw^*-closed sets in $\mathcal{X}$.
Let $\mathcal{V}$ be a regular semi-open such that $\mathcal{P} \sqcup \mathcal{Q} \sqsubset \mathcal{V}$.
Then, $\mathcal{P} \sqsubseteq \mathcal{V}$ and $\mathcal{Q} \sqsubseteq \mathcal{V}$. Since $\mathcal{P}$ and $\mathcal{Q}$ are rw^*-closed sets, there exist closed set $\mathcal{S}$ such that $\mathcal{P} \sqsubseteq \mathcal{S} \sqsubseteq \mathcal{V}$ and $\mathcal{Q} \sqsubseteq \mathcal{L} \sqsubseteq \mathcal{V}$.
Hence, $\mathcal{P} \sqcup \mathcal{Q} \sqsubseteq \mathcal{S} \sqcup \mathcal{L}$. That is, there exist closed set $\mathcal{W}$ such that $\mathcal{P} \sqcup \mathcal{Q} \sqsubseteq \mathcal{S} \sqcup \mathcal{L} = \mathcal{W} \sqsubseteq \mathcal{V}$ wherever $\mathcal{P} \sqcup \mathcal{Q} \sqsubset \mathcal{V}$ and $\mathcal{V}$ is regular semi-open in $(\mathcal{X}, \tau_A)$. □

Remark 5.11. Generally, the intersection of two rw^*-closed sets is not rw^* in Alexandroff spaces.

Theorem 5.12. *A subset $\mathcal{P}$ of $\mathcal{X}$ is rw^*-closed set, if there exist $\mathcal{S}$ containing $\mathcal{P}$ such that $\mathcal{S} \setminus \mathcal{P}$ doesn't contain any non-void regular semi-open set in $\mathcal{X}$.*

Proof. Suppose $\mathcal{P}$ be an rw^*-closed set. Then, by definition, there exists $\mathcal{S}$ such that $\mathcal{P} \sqsubset \mathcal{S} \sqsubset \mathcal{V}$ whenever $\mathcal{P} \sqsubset \mathcal{V}$ and $\mathcal{V}$ is regular semi-open in $\mathcal{X}$. Let $\mathcal{U}$ be a regular semi-open set contained in $\mathcal{S} \setminus \mathcal{P}$ and $\mathcal{U}$ is non-empty. Now, $\mathcal{U} \sqsubset \mathcal{S} \setminus \mathcal{P}$ implies $\mathcal{U} \sqsubset \mathcal{X} \setminus \mathcal{P}$. Thus, $\mathcal{P} \sqsubset \mathcal{X} \setminus \mathcal{U}$.Since $\mathcal{U}$ is regular semi-open, then $\mathcal{X} \setminus \mathcal{U}$ is also regular semi-open (by lemma 1.37). Also, $\mathcal{P}$ is rw^*-closed set,there exist $\mathcal{S}$ such that $\mathcal{S} \sqsubset \mathcal{X} \setminus \mathcal{U}$ whenever $\mathcal{P} \sqsubset \mathcal{X} \setminus \mathcal{U}$ and $\mathcal{X} \setminus \mathcal{U}$ is regular semi-open. So, $\mathcal{U} \sqsubset \mathcal{X} \setminus \mathcal{S}$. Also, $\mathcal{U} \sqsubset \mathcal{X} \setminus \mathcal{P}$. Thus, $\mathcal{U} \sqsubset (\mathcal{X} \setminus \mathcal{S}) \sqcap (\mathcal{S} \setminus \mathcal{P}) = \emptyset$, which is a contradiction to the fact that $\mathcal{U}$ is non-empty. Hence, the proof. □

Theorem 5.13. *In $(\mathcal{X}, \tau_{\mathcal{A}})$, $\mathcal{X} \setminus p$ is regular semi-open or rw^*-closed, for an element $p \in \mathcal{X}$.*

Proof. Suppose $\mathcal{X} \setminus p$ is not regular semi-open. Thus, the only regular semi-open containing $\mathcal{X} \setminus p$ is $\mathcal{X}$.
And there exist a closed set $\mathcal{S}$ such that $\mathcal{X} \setminus p \sqsubset \mathcal{S} \sqsubset \mathcal{X}$.
Hence, $\mathcal{X} \setminus p$ is an rw^*-closed set in $\mathcal{X}$. □

Theorem 5.14. *In $(\mathcal{X}, \tau_{\mathcal{A}})$, a subset $\mathcal{P}$ is regular closed if $\mathcal{P}$ is regular open and rw^*-closed and hence it is clopen.*

Proof. It is given that $\mathcal{P}$ is regular open and rw^* in $\mathcal{X}$. Since $\mathcal{P} \sqsubseteq \mathcal{P}$ and each regular open set is regular semi-open. Hence, there exist $\mathcal{S}$ such that $\mathcal{S} \sqsubseteq \mathcal{P}$ wherever $\mathcal{P} \sqsubseteq \mathcal{P}$ and $\mathcal{P}$ is regular semi-open.
Also, $\mathcal{S}$ containing $\mathcal{P}$. Thus, $\mathcal{P} = \mathcal{S}$ which means that $\mathcal{P}$ is closed.
Since $\mathcal{P}$ is regular open, then $\mathcal{P}$ is open.
Now, $\overline{((\mathcal{P})^\circ)} = \overline{\mathcal{P}} = \mathcal{P}$.
Therefore, $\mathcal{P}$ is clopen. □

Theorem 5.15. *$\mathcal{Q}$ is an rw^*-closed set in $\mathcal{X}$ if $\mathcal{P}$ is an rw^*-closed subset of $\mathcal{X}$ with the condition $\mathcal{P} \sqsubseteq \mathcal{Q} \sqsubset \overline{\mathcal{P}}$.*

Proof. Let $\mathcal{Q} \sqsubseteq \mathcal{V}$ and $\mathcal{V}$ is open. Then, $\mathcal{P} \sqsubseteq \mathcal{V}$. Since, $\mathcal{P}$ is rw^*-closed, there exist $\mathcal{S}$, is a closed set contaning $\mathcal{P}$ such that $\mathcal{S} \sqsubseteq \mathcal{V}$.
Now, $\overline{\mathcal{Q}} \sqsubseteq \overline{\{\bar{\mathcal{P}}\}} = \overline{\mathcal{P}} \sqsubseteq \mathcal{V}$ and this shows that $\mathcal{Q}$ is rw^*-closed set in $\mathcal{X}$. □

Remark 5.16. Conversely the above result is not true. (Remark 3.4 [38])

Theorem 5.17. *Suppose $\mathcal{P}$ is rw^*-closed in $(\mathcal{X}, \tau_{\mathcal{A}})$. Then, $\mathcal{P}$ is closed iff there exist closed set $\mathcal{S}$ containing $\mathcal{P}$ such that $\mathcal{S} \setminus \mathcal{P}$ is regular semi-open.*

Proof. Suppose $\mathcal{P}$ is closed in $\mathcal{X}$. Then, the closure of $\mathcal{P}$ is $\mathcal{P}$ itself and so $\overline{\mathcal{P}} \setminus \mathcal{P} = \emptyset$, which is regular semi-open in $\mathcal{X}$.
On the other part, suppose there exist closed set $\mathcal{S}$ containing $\mathcal{P}$ such that $\mathcal{S} \setminus \mathcal{P}$ is regular semi-open .
Since $\mathcal{P}$ is rw^*-closed set, then, $\mathcal{S} \setminus \mathcal{P}$ doesn't contain any non-empty regular semi-open

set, it follows from theorem 5.12.

Hence, $\mathcal{S} \setminus \mathcal{P} = \emptyset$, thus $\mathcal{P}$ is closed in $\mathcal{X}$. □

Theorem 5.18. *If $\mathcal{P} \sqsubseteq (\mathcal{X}, \tau_{\mathcal{A}})$ is regular semi-open and rw^*-closed, thus $\mathcal{P}$ is closed.*

Proof. The proof is directly from theorem 3.11 [38]. □

Corollary 5.19. *Let $\mathcal{P}$ is regular semi-open and rw^*-closed set and $\mathcal{S}$ be closed in $\mathcal{X}$. Then, $\mathcal{P} \sqcap \mathcal{S}$ is an rw^*-closed set in $\mathcal{X}$.*

Proof. Suppose $\mathcal{P}$ be a regular semi-open and rw^*-closed set and $\mathcal{S}$ be closed in $\mathcal{X}$. By above theorem, $\mathcal{P}$ is also closed and so $\mathcal{P} \sqcap \mathcal{S}$ is closed and hence $\mathcal{P} \sqcap \mathcal{S}$ is rw^*-closed set. □

Theorem 5.20. *Suppose $\mathcal{Q} \sqsubseteq \mathcal{P}$ where $\mathcal{P}$ is rw^*-closed as well as regular semi-open. Thus, $\mathcal{Q}$ is rw^*-closed relative to $\mathcal{P}$ iff $\mathcal{Q}$ is rw^*-closed.*

Proof. Since $\mathcal{P}$ is rw^*-closed set, then there exist $\mathcal{S}$, closed set, containing $\mathcal{P}$ such that $\mathcal{S} \sqsubseteq \mathcal{V}$ wherever $\mathcal{P} \sqsubseteq \mathcal{V}$ and $\mathcal{V}$ is regular semi-open in $\mathcal{X}$.

Now, $\mathcal{P} \sqsubseteq \mathcal{P}$ and $\mathcal{P}$ is regular semi-open , so $\mathcal{S} \sqsubseteq \mathcal{P}$. That is, $\mathcal{P} = \mathcal{S}$ and so $\mathcal{P}$ is closed. Further, suppose $\mathcal{Q}$ is rw^*-closed closed. Then, there exist $\mathcal{S}_1$ which shows the rw^*-closeness of $\mathcal{Q}$. Since $\mathcal{P}$ is regular semi-open and $\mathcal{Q} \sqsubseteq \mathcal{V}'$ where $\mathcal{V}'$ is regular semi-open in $\mathcal{P}$, so $\mathcal{V}'$ is regular semi-open in $\mathcal{X}$ and hence $\mathcal{S}_1 \sqsubseteq \mathcal{V}'$ which implies $\mathcal{Q}$ is rw^*-closed in $\mathcal{P}$.

On the other hand, let $\mathcal{Q}$ be a rw^*-closed in $\mathcal{P}$. Then, there exist closed set $\mathcal{S}_2$ in $\mathcal{P}$ which shows $\mathcal{Q}$ is rw^* in $\mathcal{P}$. Since, $\mathcal{P}$ is closed, $\mathcal{S}_2$ is closed in $\mathcal{X}$. Next, $\mathcal{Q} \sqsubset \mathcal{V}_1$, $\mathcal{V}_1$ is regular semi-open in $\mathcal{X}$, so $\mathcal{Q} \sqsubseteq \mathcal{V}_1 \sqcap \mathcal{P}$ where $\mathcal{V}_1 \sqcap \mathcal{P}$ is regular semi-open and thus, $\mathcal{S}_2 \sqsubseteq \mathcal{V}_1 \sqcap \mathcal{P} \sqsubseteq \mathcal{V}_1$.

Hence proved. □

Theorem 5.21. *In an Alexandroff space $(\mathcal{X}, \tau_{\mathcal{A}})$, the family of regular semi-open sets $RSO(\mathcal{X}, \tau_{\mathcal{A}}) \sqsubseteq \{\mathcal{S} \sqsubset \mathcal{X} : \mathcal{S}^c \in \tau_{\mathcal{A}}\}$ iff every subset of $(\mathcal{X}, \tau_{\mathcal{A}})$ is rw∗-closed.*

Proof. Let $RSO(\mathcal{X}, \tau_{\mathcal{A}}) \sqsubseteq \{\mathcal{S} \sqsubset \mathcal{X} : \mathcal{S}^c \in \tau_{\mathcal{A}}\}$.

Suppose $\mathcal{P}$ be any subset of $(\mathcal{X}, \tau_{\mathcal{A}})$ such that $\mathcal{P} \sqsubseteq \mathcal{V}$ and $\mathcal{V}$ is regular semiopen. Thus, $\mathcal{V} \in RSO(\mathcal{X}, \tau_{\mathcal{A}}) \sqsubseteq \{\mathcal{S} \sqsubset \mathcal{X} : \mathcal{S}^c \in \tau_{\mathcal{A}}\}$ and hence, $\mathcal{V} \in \{\mathcal{S} \sqsubset \mathcal{X} : \mathcal{S}^c \in \tau_{\mathcal{A}}\}$.

This implies $\mathcal{V}$ is closed. Thus, $\mathcal{V}$ is the closed set as well as regular semiopen set such that $\mathcal{P} \sqsubset \mathcal{V}$.
Hence, $\mathcal{P}$ is rw^*-closed set in $\mathcal{X}$.
Conversely, suppose each subset in $(\mathcal{X}, \tau_{\mathcal{A}})$ is rw^*-closed. Now, let $\mathcal{V} \in RSO(\mathcal{X}, \tau_{\mathcal{A}})$. Since $\mathcal{P} \sqsubset \mathcal{P}$ and $\mathcal{P}$ is rw^*-closed, then there exist closed set $\mathcal{V}$ such that $\mathcal{P} \sqsubset \mathcal{V}$ where $\mathcal{V}$ is regular semi-open in $\mathcal{X}$ and it is closed also. Thus, $\mathcal{V} \in \{\mathcal{S} \sqsubset \mathcal{X} : \mathcal{S}^c \in \tau_{\mathcal{A}}\}$. Therefore, $RSO(\mathcal{X}, \tau_{\mathcal{A}}) \sqsubseteq \{\mathcal{S} \sqsubset \mathcal{X} : \mathcal{S}^c \in \tau_{\mathcal{A}}\}$. □

Definition 5.22. Regular Star semi-kernel is defined as the intersection of every regular semi-open subsets that contains $\mathcal{P}$. It is denoted by $r^*sker(\mathcal{P})$.

Theorem 5.23. *In $(\mathcal{X}, \tau_{\mathcal{A}})$ $\mathcal{P}$ is rw^*-closed iff $\mathcal{P} \sqsubseteq \mathcal{S}$, a closed set such that $\mathcal{S} \sqsubseteq \mathcal{V}$, wherever $\mathcal{P} \sqsubseteq \mathcal{V}$, and $\mathcal{V}$ is regular semi-open, that is, $\mathcal{S} \sqsubseteq r^*sker(\mathcal{P})$.*

Proof. Firstly, suppose that $\mathcal{P}$ is rw^*-closed. Then, $\mathcal{P} \supseteq \mathcal{S}$, a closed set such that $\mathcal{S} \sqsubseteq \mathcal{V}$, wherever $\mathcal{P} \sqsubseteq \mathcal{V}$ where $\mathcal{V}$ is regular semi-open. Let $x \in \mathcal{S} \sqsubset \mathcal{V}$.
Let $x \notin r^*sker(\mathcal{P})$, then there exist regular semi-open set $\mathcal{V}$ containing $\mathcal{P}$ such that $x \notin \mathcal{V}$. Since $\mathcal{P}$ is rw^*-closed , $\mathcal{S} \sqsubseteq \mathcal{V}$, it implies $x \notin \mathcal{S} \sqsubseteq \mathcal{V}$, which is a contradiction. Hence, $x \in r^*sker(\mathcal{P})$ and thus $\mathcal{S} \sqsubseteq r^*sker(\mathcal{P})$.
Conversely, suppose $\mathcal{S} \sqsubseteq \mathcal{V}$ and $\mathcal{S} \sqsubseteq r^*sker(\mathcal{P})$. Suppose $\mathcal{V}$ is regular semi-open set containing $\mathcal{P}$, then, $r^*sker(\mathcal{P}) \sqsubseteq \mathcal{V}$. Then, $\mathcal{S} \sqsubset r^*sker(\mathcal{P}) \sqsubseteq \mathcal{V}$ which implies $\mathcal{P}$ is rw^*-closed. □

5.0.3 Definition of rw^*-open sets

Definition 5.24. A subset $\mathcal{P}$ in $(\mathcal{X}, \tau_{\mathcal{A}})$ is known to be regular w star-open (rw^*-open) in X if complement of $\mathcal{P}$ is rw^*-closed in $(\mathcal{X}, \tau_{A})$.

Theorem 5.25. *A set is rw^*-open iff there exist regular semi-open set $\mathcal{V}$ contained in $\mathcal{P}$ such that $\mathcal{S} \sqsubseteq \mathcal{V}$ and $\mathcal{S} \sqsubseteq \mathcal{P}$ wherever $\mathcal{S}$ is closed.*

Thus, this chapter demonstrates a new class of class of closed sets (rw^*-closed sets) in Alexandroff spaces and give some important results with some new notions.

Chapter 6

Fuzzy Alexandroff Soft Topological Spaces

The main purpose of this chapter is to introduce a new kind of topology using the concept of fuzzy soft sets and Alexandroff spaces. This kind of topology is known as Fuzzy Alexandroff Soft Topological Spaces(FASTS). We have also studied various topological properties of it. We further explored the concept of connectedness and compactness and gave the definition of $c_{f_{\mathcal{A}}}$- connectedness, c_{f_i}-connectedness and $c_{f_{\mathcal{A}}}$-compactness in FASTS along with their results and examples.

Throughout the chapter, $(X, \tau_f, \mu_{f_{\mathcal{E}}})$ denotes the Fuzzy Alexandroff Soft Topological Spaces and $\mathcal{E}$ is the arbitrary set of parameters.

Definition 6.1. [15] A fuzzy soft topology τ on $(U, \mathcal{E})$ is a family of fuzzy soft sets over $(U, \mathcal{E})$ satisfying the following properties :

1) $\emptyset, \tilde{\mathcal{E}} \in \tau$.

2) If $F_A, G_B \in \tau$, then $F_A \sqcap G_B \in \tau$.

3) If $F^{\alpha}_{A_\alpha} \in \tau \forall \alpha \in \Lambda$, an indexed set, then $\sqcup_{\alpha \in \Lambda} F^{\alpha}_{A_\alpha} \in \tau$.

Definition 6.2. [16] In a fuzzy topological space, two sets $\mathcal{A}$ and $\mathcal{B}$ are said to be weakly separated iff $\exists$ $\mathcal{P}$ and $\mathcal{Q} \in \tau_f$ such that $\mathcal{A} \subseteq \mathcal{P}$ and is not quasi-coincident to $\mathcal{Q}$, that is, there exists $x \in X$ such that $\mathcal{A}(x) + \mathcal{Q}(x) > 1$ and $\mathcal{B} \subseteq \mathcal{Q}$ and is not quasi-coincident to $\mathcal{P}$ in the same manner.

Theorem 6.3. *[17] Two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{Q}$-separated or strongly separated in (Y, τ_Y) iff these two sets are $\mathcal{Q}$-separated in (X, τ_X) where Y is the subset of X.*

6.0.1 Definition of Fuzzy Alexandroff Soft Topological Spaces

Definition 6.4. A set X together with a topology τ_{f_A} containing fuzzy soft closed sets satisfying following three conditions:

1) An arbitrary intersection of any number of members of τ_{f_A} belongs to τ_{f_A}.

2) Finite union of members of τ_{f_A} belongs to τ_{f_A}.

3) 0_{f_A} and $1_{f_A} \in \tau_{f_A}$.

Thus, $(X, \tau_{f_A}, \mu_{f_\mathcal{E}})$ is said to be Fuzzy Alexandroff Soft Topological spaces where $\mu_{f_\mathcal{E}}$ is the membership function of the fuzzy soft sets with respect to an arbitrary set of parameters $\mathcal{E}$. Members of topology τ_{f_A} are fuzzy soft closed sets and their complements are known as fuzzy soft open sets respectively.

Example 6.1. *Let $(F_i, \mathcal{E})$ be a fuzzy soft closed sets defined on X as follows:*

$$F_i(a) = \begin{cases} 0, \quad if \quad 0 \leq a \leq 1/2, \\ 2x-1, \quad if \quad 1/2 \leq a \leq 1 \end{cases}$$

where F_i is a mapping from set of parameters $\mathcal{E}$ to powerset of X, $a \in \mathcal{E}$ and $i \in I$.

Then, $\tau_{f_A} = \{\tilde{0}_{f_A}, \tilde{1}_{f_A}, (F_i, \mathcal{E})\}$ is a topology defined on X.

Clearly, τ_{f_A} is a Fuzzy Alexandroff Soft Topological Space.

6.0.2 Properties of Fuzzy Alexandroff Soft Spaces

Definition 6.5. A Fuzzy Alexandroff Soft Base for a topology τ_{f_A} on $(X, \mathcal{E})$ is a collection of some fuzzy soft closed subsets satisfying the following axioms:

1) $f_\mathcal{E} \in \alpha_f$ i.e $(F_A, \mathcal{E}) \in \alpha_f$.

2) $\sqcap \alpha_f \in \mathcal{E}$, which means for each $e \in \mathcal{E}$ and $x \in X$, $\exists\ F_A \in \alpha_f$ such that $\mu^e_{f_A}(x) = 0$.

3) If $P_A, Q_B \in \alpha_f$, then $\forall\ e \in \mathcal{E}$ and $x \in X$, $\exists\ R_C \in \alpha_f$, such that $P_A \sqcup Q_B \sqsubseteq R_C$ and $\mu^e_{R_C}(x) = max\{\mu^e_{p_A}(x), \mu^e_{q_B}(x)\}$, where $A \sqcup B \sqsubseteq C$.

Example 6.2. *Let $\mathcal{X}$ be the set of different machines and $\mathcal{E}$ a set of arbitrary parameters on which the quality of machines depend. Suppose $\mathcal{E} = \{e_1 = efficiency, e_2 = temperature, e_3 = pressure\}$ and $\mathcal{A} = \{e_1 = efficiency, e_2 = temperature\}$ be a subset of $\mathcal{E}$.*

Suppose $f_\mathcal{A} \colon \mathcal{E} \to I^X$ be a mapping defined as-

$f_{\mathcal{A}}(e) = \mu^e_{f_{\mathcal{A}}}$ where $\mu^e_{f_{\mathcal{A}}} = 0$ if $e \in \mathcal{E} - \mathcal{A}$ and $\mu^e_{f_{\mathcal{A}}} \neq 0$ if $e \in \mathcal{A}$.

Now, $(F_{\mathcal{A}}, \mathcal{E}) = \begin{cases} e_1 = \{a_0, b_{0.4}, c_{0.5}, d_{0.9}\} \\ e_2 = \{a_{0.2}, b_{0.4}, c_{0.8}, d_{0.6}\} \end{cases}$

$$(F_{\mathcal{A}_1}, \mathcal{E}) = \begin{cases} e_1 = \{a_{0.5}, b_{0.6}, c_{0.2}, d_{0.1}\} \\ e_2 = \{a_{0.3}, b_{0.2}, c_{0.6}, d_0\} \end{cases}$$

$$(F_{\mathcal{A}_2}, \mathcal{E}) = \begin{cases} e_1 = \{a_{0.3}, b_{0.5}, c_{0.7}, d_{0.9}\} \\ e_2 = \{a_{0.2}, b_{0.4}, c_0, d_{0.6}\} \end{cases}$$

$$(F_{\mathcal{A}_3}, \mathcal{E}) = \begin{cases} e_1 = \{a_{0.3}, b_{0.6}, c_{0.3}, d_{0.5}\} \\ e_2 = \{a_0, b_{0.3}, c_{0.6}, d_{0.5}\} \end{cases}$$

Then, $\alpha_f = \{\emptyset, (F_{\mathcal{A}}, \mathcal{E}), (F_{\mathcal{A}_1}, \mathcal{E}), (F_{\mathcal{A}_2}, \mathcal{E}), (F_{\mathcal{A}_3}, \mathcal{E})\}$ is a base for τ_{f_A}.

Theorem 6.6. *Let α_f be a Fuzzy Alexandroff Soft base for a Fuzzy Alexandroff Soft topology on $(X, \mathcal{E})$. Suppose τ_{α_f} consists of those fuzzy soft closed sets G_A over $(X, \mathcal{E})$ for each $e \in \mathcal{E}$ and $x \in \mathcal{U}$, $\exists\ F_B \in \alpha_f$ such that $G_A \sqsubseteq F_B$ and $\mu^e_{F_B} = \mu^e_{G_A}$, where $A \sqsubseteq B$. Then, τ_{f_A} is a Fuzzy Alexandroff Soft topology on $(X, \mathcal{E})$.*

Proof. Since α_f is a base for topology on $(X, \mathcal{E})$. Then, $(F_A, \epsilon) \in \mathcal{E}$ which implies $(F_A, \epsilon) \in \tau_{\alpha_f}$. Also, $\emptyset \in \tau_{\alpha_f}$.

Now, by the given condition, for $F_A, G_B \in \tau_{\alpha_f}$ and for each $e \in \mathcal{E}, x \in X$, $\exists\ H_C, I_D \in \alpha_f$, where $A \sqsubseteq C$ and $B \sqsubseteq D$ such that $F_A \sqsubseteq H_C, G_B \sqsubseteq I_D$ with $\mu^e_{F_A} = \mu^e_{H_C}$ and $\mu^e_{G_B} = \mu^e_{I_D}$.

Let $F_A \sqcup G_B = J_{A \sqcup B}$.

By using third property of base, for $H_C, I_D \in \alpha_f$ and $e \in \mathcal{E}, x \in X$, $\exists\ K_p \in \alpha_f$ such that $H_C \sqcup I_D \sqsubseteq K_p$ and $\mu^e_{K_p}(x) = max\{\mu^e_{H_C}(x), \mu^e_{I_D}(x)\}$.

Now, $H_C(a) \sqcup I_D(a) \sqsubseteq K_p(a) \Rightarrow F_A(a) \sqcup G_B(a) = J_{A \sqcup B} \sqsubseteq K_p(a)$, for $a \in \mathcal{E}$.

Thus, $J_{A \sqcup B} \sqsubseteq K_p$.

Also, $\mu^e_{K_p}(x) = max\{\mu^e_{H_C}(x), \mu^e_{I_D}(x)\} = max\{\mu^e_{F_A}(x), \mu^e_{G_B}(x) = J_{A \sqcup B}$.

Hence, $F_A \sqcup G_B \in \tau_{\alpha_f}$.

Now, suppose $\sqcap_{\alpha' \in \Lambda} F_A^{\alpha'} = J_C$ where $C = \sqcap_{\alpha' \in \Lambda} A_{\alpha'}$ and $F_A^{\alpha'} \in \tau_{\alpha_f}$, for all $\alpha' \in \Lambda$. Therefore, $\mu^e_{J_C}(x) = min\{\mu^e_{F_A^{\alpha'}}(x) \colon \alpha' \in \Lambda\} \Rightarrow \mu^e_{J_C}(x) = \mu^e_{F_A^{\alpha'}}(x)$, for some $\alpha' \in \Lambda$.

Since $F_A^{\alpha'} \in \tau_{\alpha_f}$, $\exists\ G_B \in \alpha_f$ such that $F_A^{\alpha'} \sqsubseteq G_B$ and $\mu^e_{F_A^{\alpha'}}(x) = \mu^e_{G_B}(x)$.

Therefore, $J_C \sqsubseteq G_B$ and $\mu^e_{J_C}(x) = \mu^e_{G_B}(x)$.

This imples $J_C \in \tau_{\alpha_f}$.

Hence, τ_{α_f} is a Fuzzy Alexandroff Soft topology on $(X, \mathcal{E})$. □

Remark 6.7. A topology generated by a fuzzy alexandroff soft base is known as Fuzzy Alexandroff Soft topology on $(X, \mathcal{E})$ and it is denoted by τ_{α_f}.

Theorem 6.8. *Let α_f be a fuzzy alexandroff soft base for a Fuzzy Alexandroff Soft topology τ_{α_f} on $(X, \mathcal{E})$. Then-*
$F_A \in \tau_{\alpha_f}$ iff $F_A = \sqcap_{\alpha' \in \Lambda} B_A^{\alpha'}$ where $B_A^{\alpha'} \in \alpha_f \ \forall \ \alpha' \in \Lambda$, an indexed set.

Proof. Firstly, since $F_A = \sqcap_{\alpha' \in \Lambda} B_A^{\alpha'}$. Then, $F_A \in \tau_{\alpha_f}$ as every member of α_f is also a member of τ_{α_f}.

Conversely, for each $e \in \mathcal{E}$ and $x \in X$, $\exists\ U^e_{B^e_x} \in \alpha_f$ such that $F_A \sqsubseteq U^e_{B^e_x}$ and $\mu^e_{F_A} = \mu^e_{U^e_{B^e_x}}$ for $F_A \in \tau_{\alpha_f}$ and $B^e_x \sqsubseteq A$ (by using theorem 6.6)

Now, let $B = \sqcup_{e \in \mathcal{E}, x \in X} B^e_x$ and $G_B = \sqcap_{e \in \mathcal{E}, x \in X}(U^e_{B^e_x})$.

We shall prove that $G_B = F_A$. Let $a \in \mathcal{E}$ and $y \in X$.

Then, $\mu^a_{G_B}(y) = min\{\mu^a_{U^e_{B^e_x}}(y) \colon e \in \epsilon, x \in X\}$

$\leq \mu^a_{Y^a_{B^y_a}}(y)$ [corresponding to each $a \in \mathcal{E}$ and $y \in X, Y^a_{B^y_a} \in \alpha_f$]

$= \mu^a_{F_A}(y)$.

Therefore, $\mu^a_{G_B}(y) \leq \mu^a_{G_B}(y)$, for each $a \in \epsilon$ and $y \in X$.

Thus, $G_B \sqsubseteq F_A$ and $F_A \sqsubseteq G_B$ also.

So, $F_A = G_B$ which means that F_A can be represented as the intersection of some members of α_f. □

Theorem 6.9. *Let $(X, \tau_{f_A}, \mu_{f_{\mathcal{E}}})$ be a Fuzzy Alexandroff Soft Space and α_f be a sub collection of τ_{f_A} such that every member of τ_{f_A} is the intersection of some members of α_f. Then, α_f is a fuzzy alexandroff soft base for τ_{f_A} on $(X, \mathcal{E})$.*

Proof. Since $(F_A, \mathcal{E}) \in \tau_{f_A}$, $(F_A, \mathcal{E}) \in \alpha_f$.

Again, since $\mathcal{E} \in \tau_{f_A}$, then $\mathcal{E} = \sqcap \alpha_f$.

Suppose, $F_{A_1}, F_{A_2} \in \alpha_f$. Then, $F_{A_1}, F_{A_2} \in \tau_{f_A}$ and so their union.

Thus, $\exists\ B^e_{C_{\alpha'}} \in \alpha_f$ s. t $F_{A_1} \sqcup F_{A_2} = \sqcap\{B^e_{C_{\alpha'}} \ : \ \alpha' \in \Lambda\}$.

Therefore, $F_{A_1}(e) \sqcup F_{A_2}(e) = \sqcap\{B^e_{C_{\alpha'}}(e) \colon \alpha' \in \Lambda\}$ for each $e \in \mathcal{E}$.

That is, max $\{\mu^e_{F_{A_1}}(x), \mu^e_{F_{A_2}}(x)\} =$ min $\{\mu^e_{B^e_{C_{\alpha'}}}\}$ for each $e \in \mathcal{E}$ and $x \in X$.

Therefore, we have max $\{\mu^e_{F_{A_1}}(x), \mu^e_{F_{A_2}}(x)\} = \mu^e_{B^e_{C_{\alpha'}}}$, for $\alpha' \in \Lambda$.

Thus, for each $e \in \epsilon$ and $x \in X$, we get $B^e_{C_{\alpha'}} \in \alpha_f$ such that $F_{A_1} \sqcup F_{A_2} \sqsubseteq B^e_{C_{\alpha'}}$ and $\max\{\mu^e_{F_{A_1}}(x), \mu^e_{F_{A_2}}(x)\} = \mu^e_{B^e_{C_{\alpha'}}}$.
Hence the proof. □

Definition 6.10. A collection ρ of some members of FASTS $(X, \tau_{f_A}, \mu_{f_\mathcal{E}})$ is known as sub base for τ_{f_A} iff the group made by all the arbitrary intersection of members of ρ forms a base for τ_{f_A}.

Theorem 6.11. *A collection ρ is said to be sub base for τ_{f_A} over $(X, \mathcal{E})$ if and only if the following conditions satisfied:*
1) $(F_A, \mathcal{E}) \in \rho$ or $(F_A, \mathcal{E})$ is basically the intersection of arbitrary no. of elements of ρ.
2) $\mathcal{E} = \sqcap\rho$.

Proof. Suppose that ρ is the sub base for τ_{α_f} and α_f be a base generated by ρ.
Since $(F_A, \mathcal{E}) \in \alpha_f$, so either $(F_A, \mathcal{E}) \in \rho$ or it can be written as the intersection of arbitrary number of elements of ρ.
Now, Suppose $x \in X$ and $e \in \mathcal{E}$. Since $\sqcap\, \alpha_f = \mathcal{E}$, $\exists\, B_A \in \alpha_f$ such that $\mu^e_{B_A}(x) = 0$.
Since $B_A \in \alpha_f$, $\exists\, S_{A_i} \in \rho$, $i = 1, 2 \ldots, n$ such that $B_A = \sqcup^n_{i=1} S_{A_i}$.
Therefore, $\mu^e_{B_A} = max^n_{i=1}\ \mu^e_{S_{A_i}}(x)$ and so $\mu^e_{B_A} = \mu^e_{S_{A_i}}(x)$, for some $i \in \{1, 2 \ldots, n\} \Rightarrow \mu^e_{S_{A_i}} = 0$.
Hence, $\mathcal{E} = \sqcap\rho$.
Now,on the converse way, suppose ρ be the family of some fuzzy alexandroff soft sets over $(X, \mathcal{E})$ satiates the given situations.
Let α_f be the family of arbitrary intersection of members of ρ. Now, we have to prove that α_f forms a base for fuzzy alexandroff soft topology.
Since α_f is the collection of arb. intersection of members of ρ, by the given conditions $(F_A, \mathcal{E}) \in \alpha_f$ and $\mathcal{E} = \sqcap\, \alpha_f$.
Now, let $F_A, G_B \in \alpha_f$ and $x \in X$, $e \in \mathcal{E}$.
Since $F_A \in \alpha_f$, $\exists\, F_{A_i} \in \rho$ for $i = 1, 2 \ldots, n$ such that $F_A = \sqcup^n_{i=1} F_{A_i}$
where $A = \sqcup^n_{i=1} A_i$ and $\exists\, G_{B_j} \in \rho$ such that $G_B = \sqcup^m_{j=1} G_{B_i}$ where $B = \sqcup^m_{j=1} B_i$,
for $G_B \in \alpha_f$.
Therefore, $F_A \sqcup G_B = (\sqcup^n_{i=1} F_{A_i}) \sqcup (\sqcup^m_{j=1} G_{B_i}) \in \alpha_f$.
That is, $F_A \sqcup G_B \in \alpha_f$. □

6.0.3 Connectedness in Fuzzy Alexandroff Soft Topological Spaces

One of the key topological characteristics used to distinguish topological spaces is connectedness. The notion of connectedness has been studied for so many years. This concept has been investigated by lots of authors [94, 16, 95, 42]. It has been observed that connectedness given by Ajmal [42] is the strongest form and that given by Zheng Chong [95] is the weakest one.

Definition 6.12. [94] Suppose (X, τ_f) be a fuzzy topological space. A fuzzy set $\mathcal{C}$ is said to be be disconnected (briefly $\mathcal{C}_{\mathcal{M}}$ - disconnected) if there exists two non-null fuzzy sets $(\mathcal{A}, \mu_{\mathcal{A}})$ and $(\mathcal{B}, \mu_{\mathcal{B})}$ in the subspace of $\mathcal{C}$ such that $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{Q}$-separated and $\mathcal{C}$ can be represented as the union of $\mathcal{A}$ and $\mathcal{B}$ with $\mu_{\mathcal{C}} = \max\{\mu_{\mathcal{A}}, \mu_{\mathcal{B}}\}$.

Definition 6.13. [16] Suppose (X, τ_f) be a fuzzy topological space. A fuzzy set $\mathcal{C}$ is said to be be connected (briefly $\mathcal{C}_{\mathcal{S}}$ - connected) if there $\nexists$ two non-null fuzzy weakly separated sets $(\mathcal{A}, \mu_{\mathcal{A}})$ and $(\mathcal{B}, \mu_{\mathcal{B})}$ in the subspace of $\mathcal{C}$ such that $\mathcal{C} = \mathcal{A} \sqcup \mathcal{B}$. If $\mathcal{C}$ is not connected, then it is said to be $\mathcal{C}_{\mathcal{S}}$-disconnected.

Definition 6.14. [42] A fuzzy set $\mathcal{C}$ has $\mathcal{C}_i$-disconnection $(i = 1, 2, 3, 4)$ if $\exists$ two fuzzy sets $\mathcal{P}$ and $\mathcal{Q}$ satisfying following conditions:

c_1 : $\mathcal{C} \subseteq \mathcal{P} \sqcup \mathcal{Q}$, $\mathcal{P} \sqcap \mathcal{Q} \subseteq \bar{\mathcal{C}}$, $\mathcal{C} \sqcap \mathcal{P} \neq \tilde{0}$, $\mathcal{C} \sqcap \mathcal{Q} \neq \tilde{0}$;

c_2 : $\mathcal{C} \subseteq \mathcal{P} \sqcup \mathcal{Q}$, $\mathcal{C} \sqcap \mathcal{P} \sqcap \mathcal{Q} = \tilde{0}$, $\mathcal{C} \sqcap \mathcal{P} \neq \tilde{0}$, $\mathcal{C} \sqcap \mathcal{Q} \neq \tilde{0}$;

c_3 : $\mathcal{C} \subseteq \mathcal{P} \sqcup \mathcal{Q}$, $\mathcal{P} \sqcap \mathcal{Q} \subseteq \bar{\mathcal{C}}$, $\mathcal{P} \nsubseteq \bar{\mathcal{C}}$, $\mathcal{Q} \nsubseteq \bar{\mathcal{C}}$;

c_4 : $\mathcal{C} \subseteq \mathcal{P} \sqcup \mathcal{Q}$, $\mathcal{C} \sqcap \mathcal{P} \sqcap \mathcal{Q} = \tilde{0}$, $\mathcal{P} \nsubseteq \bar{\mathcal{C}}$, $\mathcal{Q} \nsubseteq \bar{\mathcal{C}}$.

If there does not exist any $\mathcal{C}_i$-disconnection, then $\mathcal{C}$ is known as $\mathcal{C}_i$- connected for $i = 1, 2, 3, 4$.

Definition 6.15. [95] A fuzzy set $\mathcal{C}$ is known as connected (briefly $\mathcal{O}_q$ - connected) if there does not exists two non-null fuzzy separated (strongly separated) sets $(\mathcal{P}, \mu_{\mathcal{P}})$ and $(\mathcal{Q}, \mu_{\mathcal{Q})}$ in the subspace of $\mathcal{C}$ such that $\mathcal{C} = \mathcal{P} \sqcup \mathcal{Q}$. If $\mathcal{C}$ is not connected, then it is said to be $\mathcal{O}_q$-disconnected.

Theorem 6.16. *A $\mathcal{C}_S$ connectedness in fuzzy topological space implies $\mathcal{C}_M$- connectedness.*

This result is the consequence of theorem [6.3], definitions [6.2], [6.12] and [6.13].

Definition 6.17. Let $(X, \tau_{f_A}, \mu_{f_\mathcal{E}})$ be a fuzzy alexandroff soft space. A fuzzy soft set $\mathcal{C}$ is known as $c_{f_\mathcal{A}}$-connected if there doesn't any proper clopen fuzzy soft set in $\mathcal{C}$. Otherwise, a fuzzy soft set is $c_{f_\mathcal{A}}$-disconnected.

Remark 6.18. A Fuzzy Alexandroff Soft $c_{f_\mathcal{A}}$-connectedness may not implies $\mathcal{O}_q$-connectedness.

Example 6.3. *Let us suppose that* $X = \{x_1, x_2, x_3\}$ *and* $\tau_{f_\mathcal{A}} = \{\tilde{0}, \tilde{1}, (F_1, \mathcal{A}), (F_2, \mathcal{A}), (F_3, \mathcal{A})\}$ *where* $(F_i, \mathcal{A})$ *are fuzzy soft sets defined as below:*

$F_1(x_1) = 0.8 \quad F_1(x_2) = 0 \quad F_1(x_3) = 0$

$F_2(x_1) = 0 \quad F_2(x_2) = 0.9 \quad F_2(x_3) = 0.9$

$F_3(x_1) = 0.8 \quad F_3(x_2) = 0.9 \quad F_3(x_3) = 0.9.$

Now, let $\mathcal{C}$ *be a fuzzy soft set in* X *such that* $\mathcal{C} = (0.6, 0.7, 0.8)$.

Then $\mathcal{C}$ *is* $c_{f_\mathcal{A}}$ *but not* $\mathcal{O}_q$ *as* $\mathcal{C} = (0.6, 0, 0) \sqcup (0, 0.7, 0.8)$ *and two sets* $(0.6, 0, 0)$ *&* $(0, 0.7, 0.8)$ *are strongly separated.*

Definition 6.19. [42] (Fuzzy Alexandroff Soft $c_{f_\mathcal{A}}$-connectedness in terms of Ajmal c_i-connectedness):

Suppose $(F_\mathcal{A}, \mathcal{E})$ is a fuzzy soft set in $(X, \tau_{f_\mathcal{A}}, \mu_{f_\mathcal{E}})$. If $\exists$ two fuzzy soft sets $\mathcal{G}_\mathcal{A}$ and $\mathcal{H}_\mathcal{A}$ satisfying follwing conditions:

$c_1 \,:\, \mathcal{F}_\mathcal{A} \subseteq \mathcal{G}_\mathcal{A} \sqcup \mathcal{H}_\mathcal{A}, \quad \mathcal{G}_\mathcal{A} \sqcap \mathcal{H}_\mathcal{A} \subseteq \overline{\mathcal{F}_\mathcal{A}}, \quad \mathcal{F}_\mathcal{A} \sqcap \mathcal{G}_\mathcal{A} \neq \tilde{0}, \mathcal{F}_\mathcal{A} \sqcap \mathcal{H}_\mathcal{A} \neq \tilde{0};$

$c_2 \,:\, \mathcal{F}_\mathcal{A} \subseteq \mathcal{G}_\mathcal{A} \sqcup \mathcal{H}_\mathcal{A}, \quad \mathcal{F}_\mathcal{A} \sqcap \mathcal{G}_\mathcal{A} \sqcap \mathcal{H}_\mathcal{A} = \tilde{0}, \quad \mathcal{F}_\mathcal{A} \sqcap \mathcal{G}_\mathcal{A} \neq \tilde{0}, \mathcal{F}_\mathcal{A} \sqcap \mathcal{H}_\mathcal{A} \neq \tilde{0};$

$c_3 \,:\, \mathcal{F}_\mathcal{A} \subseteq \mathcal{G}_\mathcal{A} \sqcup \mathcal{H}_\mathcal{A}, \quad \mathcal{G}_\mathcal{A} \sqcap \mathcal{H}_\mathcal{A} \subseteq \overline{\mathcal{F}_\mathcal{A}}, \quad \mathcal{G}_\mathcal{A} \not\subseteq \overline{\mathcal{F}_\mathcal{A}}, \quad \mathcal{H}_\mathcal{A} \not\subseteq \overline{\mathcal{F}_\mathcal{A}};$

$c_4 \,:\, \mathcal{F}_\mathcal{A} \subseteq \mathcal{G}_\mathcal{A} \sqcup \mathcal{H}_\mathcal{A}, \quad \mathcal{F}_\mathcal{A} \sqcap \mathcal{G}_\mathcal{A} \sqcap \mathcal{H}_\mathcal{A} = \tilde{0}, \quad \mathcal{G}_\mathcal{A} \not\subseteq \overline{\mathcal{F}_\mathcal{A}}, \quad \mathcal{H}_\mathcal{A} \not\subseteq \overline{\mathcal{F}_\mathcal{A}}.$

Then, $\mathcal{F}_\mathcal{A}$ is c_{f_i}- disconnected and if there does not exist any c_{f_i}-disconnection, then $\mathcal{F}_\mathcal{A}$ is said to be c_{f_i}- connected for $i = 1, 2, 3, 4$.

Theorem 6.20. *Image of* $c_{f_\mathcal{A}}$*-connected spaces under continuous map is* $c_{f_\mathcal{A}}$*-connected in FASTS.*

Proof. Let X and Y be two fuzzy alexandroff soft topological spaces and $g \colon X \to Y$ be a continuous map with bijection.

We have to prove that if $\mathcal{C}$ is $c_{f_\mathcal{A}}$-connected in X, then its image $g(\mathcal{C})$ is also connected in Y.

We shall prove this by contradiction. So, suppose that $g(\mathcal{C})$ is not $c_{f_\mathcal{A}}$-connected, then $\exists$ non-null proper clopen fuzzy soft set $\mathcal{B}$.

Thus, $\exists\ \mathcal{P} \in \tau_{f_\mathcal{A}}$ and $\mathcal{Q} \in C(\tau_{f_\mathcal{A}})$ such that $\mathcal{B} = g(\mathcal{C}) \sqcap \mathcal{P} = \mathcal{Q} \sqcap g(\mathcal{C})$.

Since the map g is bijective, so $g^{-1}(\mathcal{B}) = \mathcal{C} \sqcap g^{-1}(\mathcal{P}) = g^{-1}(\mathcal{Q}) \sqcap \mathcal{C}$.

Also, since g is a continuous map, $g^{-1}(\mathcal{P}) \in \tau_{f_\mathcal{A}}$ and $g^{-1}(\mathcal{Q}) \in C(\tau_{f_\mathcal{A}})$.

Thus, $g^{-1}(\mathcal{B})$ is a non-null proper clopen fuzzy soft set in $\mathcal{C}$ which contradicts the fact that $\mathcal{C}$ is $c_{f_{\mathcal{A}}}$-connected.

Hence the proof. □

6.0.4 Compactness Property

Compactness is a property that generalizes the notion of a subset of Euclidean space being closed. The notion of compactness is based on important property of [a, b] that every infinite subset has a limit point. Later on, mathematicians formulated the term of compactness in terms of open coverings of the space. Fuzzy compactness was studied by R. Lowen [41] in 1976. After this, many authors explored this concept in fuzzy topology. In this section, we defined open coverings, compactness in Fuzzy Alexandroff Soft Topological Spaces and gave various results related to it.

Definition 6.21. [41] (Open covering of a space): Let $(X, \tau_{f_A}, \mu_{f_{\mathcal{E}}})$ be a Fuzzy Alexandroff Soft Topological Spaces. Then, a family β of subsets of X is known as covering of X if $X = \sqcup_{B\in\beta} B$. Also, if the members of β are open, then, it is called open covering of X.

Definition 6.22. [41] A space $(X, \tau_{f_A}, \mu_{f_{\mathcal{E}}})$ is quasi $c_{f_{\mathcal{A}}}$-compact if and only if for all families $\alpha \in \tau_{f_A}$ such that $sup_{\mu\in\alpha}\mu = \tilde{1}$, $\exists\ \alpha_o \in 2^{\alpha}$ such that $sup_{\mu\in\alpha}\mu = \tilde{1}$.

Definition 6.23. A subset $(G_{\mathcal{A}}, \mathcal{E})$ of a family of fuzzy soft closed sets is said to have $c_{f_{\mathcal{A}}}$- compactness if every fuzzy soft open cover of $(G_{\mathcal{A}}, \mathcal{E})$ has a finite subcover. Also, $(X, \tau_{f_A}, \mu_{f_{\mathcal{E}}})$ is itself called $c_{f_{\mathcal{A}}}$-compact if each fuzzy soft open cover of $\tilde{1_{\mathcal{E}}}$ has a finite sub cover.

Theorem 6.24. *Any fuzzy soft closed subset of $c_{f_{\mathcal{A}}}$-compact is $c_{f_{\mathcal{A}}}$-compact.*

Proof. Let $(X, \tau_{f_A}, \mu_{f_{\mathcal{E}}})$ be a fuzzy alexandroff soft topological space and $(F_{\mathcal{A}}, \mathcal{E})$ be a fuzzy soft closed subset in X.

Suppose that $\mathcal{U} = \{(\mathcal{P}_i, \mathcal{E})\}$ is open covering of $(F_{\mathcal{A}}, \mathcal{E})$ in X.

Since $(F_{\mathcal{A}}, \mathcal{E})$ is fuzzy soft closed, then its complement $(F_{\mathcal{A}}, \mathcal{E})'$ is fuzzy soft open set.

So, $\mathcal{U} \sqcup (F_{\mathcal{A}}, \mathcal{E})'$ is the fuzzy soft open covering for X.

Since X is $c_{f_{\mathcal{A}}}$-compact, so every open covering possessses a finite subcover. Thus, $X = \sqcup_{j=1}^{n} (\mathcal{P}_{i_j}, \mathcal{E}) \sqcup (F_{\mathcal{A}}, \mathcal{E})'$ where $\{(\mathcal{P}_{i_j}, \mathcal{E})\}$ is a finite subcover.

Thus, $(F_{\mathcal{A}}, \mathcal{E}) \sqsubseteq \sqcup_{j=1}^{n} (\mathcal{P}_{i_j}, \mathcal{E})$ which implies $(F_{\mathcal{A}}, \mathcal{E})$ is also $c_{f_{\mathcal{A}}}$. □

Theorem 6.25. *Under a continuous map, the image of a $c_{f_{\mathcal{A}}}$-compact space is also $c_{f_{\mathcal{A}}}$-compact.*

Proof. Let $(X, \tau_{f_A}, \mu_{f_{\mathcal{E}}})$ and $(Y, \tau_{f_A}, \mu_{f_{\mathcal{E}}})$ be two FASTS and X is $c_{f_{\mathcal{A}}}$-compact.
Let $g\colon X \to Y$ be a continuous map. Suppose α be a covering of $g(X)$. The collection $\{g^{-1}(\alpha_o)\colon \alpha_o \in \alpha\}$ covers X; these sets are open as g is a continuous map.
Since X is $c_{f_{\mathcal{A}}}$-compact, therefore, finitely many of them, $g^{-1}(\alpha_1), \ldots, g^{-1}(\alpha_n)$, covers X. Then, the sets $\alpha_1, \ldots, \alpha_n$ covers $g(X)$.
Hence, $g(X)$ is also $c_{f_{\mathcal{A}}}$-compact. □

Thus, we can conclude that this chapter elucidated a new type of topological spaces known as Fuzzy Alexandroff Soft Topological Spaces. We investigated various general topological properties of this spaces along with their results. We had also studied two important properties of topological spaces namely connectedness and compactness.

Chapter 7

An Advanced Uncertainty Measure using Fuzzy Soft Sets: Application to Decision-Making Problems

The Fuzzy logics have emerged as a very important and useful topic in past recent years. It has aroused as an important mathematical tool to deal with uncertainties and vagueness of data. L.A Zadeh [4] presented the concept of fuzzy set theory in 1965 as a transformation of classical set theory. In this chapter, we proposed a method which uses fuzzified evidence theory to calculate total degree of fuzziness of the parameters. Also, a medical diagnosis problem in respect of COVID-19 has been solved which help a doctor to take decision on patient's condition easily. We have also compared our proposed method with Li [8] method to show the effectiveness of our method.
Let us recall some of the important definitions which helped us to proceed in our future task.

7.0.1 Few prerequisites

Definition 7.1. (Fuzzy soft sets) [4, 14]: Let $\mathcal{X}$ be an initial universe set with E as the set of parameters. The pair $(\mathcal{F}, \mathcal{A})$ is a fuzzy soft set over $\mathcal{X}$ where $\mathcal{A} \sqsubseteq \mathcal{E}$ and $\mathcal{F}$

is a mapping defined as $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{I}^{\mathcal{X}}$, where $\mathcal{I}^{\mathcal{X}}$ is the power set of $\mathcal{X}$. It is evident that every soft set can be contemplated as a fuzzy soft set. Also, when both $\mathcal{X}$ and $\mathcal{A}$ is finite, fuzzy soft sets are either represented by matrices or in tabular form.

Definition 7.2. (Shannon Entropy) [19]: Shannon in 1948 introduced the concept of Shannon entropy to handle basic probability problem.
Shannon entropy ($\mathcal{H}$) is derived as –
$\mathcal{H} = -\sum_{i=1}^{N} p_i \log_2 p_i$.
Where p_i is the probability of state i satisfying $\sum_i^N p_i = 1$ and N is the number of basic states in a system.

Definition 7.3. (Deng entropy) [20]: This novel belief entropy was introduced by Deng in 2016. It also measures the uncertainty conveyed by basic probability assignment.
It is defined as –

$\mathcal{E}_d = \frac{\sum_i m(\mathcal{A}_i) log m(\mathcal{A}_i)}{2^{|\mathcal{A}_i|}-1}$.
Where m is the belief function and $\mathcal{A}_i$ is the hypothesis of belief function. Deng entropy is degenerated into Shannon entropy when the belief value is allocated to one single element.

Definition 7.4. (W-entropy) [21]: This type of entropy was given by Dan Wang et al in 2019. It is the unified form about belief entropy based on deng entropy which considers the scale of frame of discernment and the relative scale of focal element with respect to Frame of Discernment.
W-entropy is calculated as below-
$\mathcal{E}_W(m) = \sum m(\mathcal{A}) \log_2(\frac{m(\mathcal{A})}{2^{|\mathcal{A}|-1}}(1+\epsilon)^{f|\mathcal{X}|})$.
Where $\mathcal{A}$ is a proposition in mass function m, $\epsilon \geq 0$ is a constant and $f \mid \mathcal{X} \mid$ is the function determines the cardinality of $\mathcal{X}$.
The function $f \mid \mathcal{X} \mid = \sum_{\mathcal{B} \sqsubseteq \mathcal{X}, \mathcal{B} \neq \mathcal{A}} \frac{|\mathcal{A} \sqcap \mathcal{B}|}{2^{|\mathcal{X}|}-1}$.

Definition 7.5. (Fuzziness in evidence theory) [22]: Total degree of fuzziness $\mathcal{F}(m)$, of the body of evidence $< m, \mathcal{F} >$ is calculated as follows-
$\mathcal{F}(m) = \sum_{\mathcal{A} \in \mathcal{F}} m(\mathcal{A}) f(\mathcal{A})$ Where $f(\mathcal{A})$ is given by eq (1.1).

Definition 7.6. (Performance measure) [23, 24]: The performance measure of a method satisfies the optimal criteria for resolving decision making problem. It is denoted by γ_S.

Mathematically, $\gamma_S = \frac{1}{\sum_i^n \sum_j^n |\mathcal{F}(e_i)(\mathcal{O}_p) - \mathcal{F}(e_j)(\mathcal{O}_p)|} + \sum_{i=1}^n \mathcal{F}(e_i)(\mathcal{O}_p)$.
Here, n is the number of choice parameters and $\mathcal{F}(e_i)(\mathcal{O}_p)$ depicts the membership value of the ideal object $\mathcal{O}_p$ for the choice parameter e_i.
If the performance measure of one method is greater than other, then that method is much finer than other and vice-versa.

Definition 7.7. (Frame of discernment) [26]: A frame of discernment is a finite non-empty set of mutually exclusive and exhaustive hypotheses denoted by $\Theta = \{\mathcal{A}_1, \mathcal{A}_2 \dots \mathcal{A}_n \dots \mathcal{A}_t\}$ and $\mathcal{A}_i \sqcap \mathcal{A}_j = \emptyset$ and 2^Θ represents the set of all subsets of Θ.

Definition 7.8. (Basic Probability assignment(BPA)) [26]: It is also known as mass function. A mass function is a mapping m from 2^Θ to $[0,1]$ satiates the following conditions-
$m(\emptyset) = 0$ and $\sum_{\mathcal{A} \in 2^\Theta} m(\mathcal{A}) = 1$.
If $m(\mathcal{A}) > 0$, $\mathcal{A}$ is called a focal element and its union is known as the core of the mass function.

Definition 7.9. (Belief function) [26]: It can be defined as a mapping $Bel : 2^\Theta \longrightarrow [0,1]$ satisfying following conditions:
$Bel(\emptyset) = 0, Bel(\Theta) = 1$ and $Bel(\mathcal{A}) = \sum_{\mathcal{B} \sqsubseteq \mathcal{A}} m(\mathcal{B}), \forall \mathcal{A} \in \Theta$.
$Bel(\mathcal{A})$ exemplify the imprecision and uncertainty in decision making problems. When there is single element, then, $Bel(\mathcal{A}) = m(\mathcal{A})$.

Definition 7.10. (Dempster's rule of combination) [25]: This rule computes an integrated set of combined evidences. Suppose m_1 and m_2 are two independent BPAs in Θ, then rule of combination is defined as -

$$m(\mathcal{A}) = \begin{cases} \frac{1}{1-\mathcal{K}} \sum_{\mathcal{B} \sqcap \mathcal{C} = \mathcal{A}} m_1(\mathcal{B}) m_2(\mathcal{C}), & \mathcal{A} \neq \emptyset \\ 0, & \mathcal{A} = \emptyset \end{cases} \tag{7.1}$$

and

$$\mathcal{K} = \sum_{\mathcal{B} \sqcap \mathcal{C} = \emptyset} m_1(\mathcal{B}) m_2(\mathcal{C}) < 1 \tag{7.2}$$

where $\mathcal{B} \in 2^\Theta$ and $\mathcal{C} \in 2^\Theta \& \mathcal{K} \in [0,1]$ represents the coefficient for confliction between two BPAs.

Definition 7.11. (Grey mean relational degree) [8]: The grey means relational degree between d_{ij} and $\tilde{d}_i$ can be computed as-

$$r_{ij} = \frac{min_{1\leq i\leq s}\Delta d_{ij} + 0.5max_{1\leq i\leq s\Delta d_{ij}}}{\Delta d_{ij} + 0.5max_{1\leq i\leq s\Delta d_{ij}}} \tag{7.3}$$

$(i = 1, 2 \ldots, m, j = 1, 2 \ldots n)$

Where d_{ij} denotes the membership value of x_i with e_j , $\tilde{d}_i$ is the mean of all parameters with respect to each alternatives and Δd_{ij} is the difference information between d_{ij} and $\tilde{d}_i$.

Definition 7.12. (Fuzzy preference relation) [34]: Fuzzy preference orderings can be defined as fuzzy binary relations related to reciprocity and maximum and minimum transitivity. Mathematically, it is denoted by-

$\mathcal{P} = (p_{jk})_{n\times n}$.

where $p_{jk} \in [0, 1]$ represents the preference value of alternative e_j over e_k.

Also, $p_{jk} + p_{kj} = 1$, $p_{jj} = 0.5, 1 \leq j \leq n$ and $1 \leq k \leq n$.

Definition 7.13. (Consistency matrix) [35]: The consistency matrix can be developed on the basis of fuzzy preference relation as follows:

$$p = \overline{(p_{jl})}_{n\times n} = (\frac{1}{n}\sum_{k=1}^{n}(p_{jk} + p_{kl}) - 0.5)_{n\times n}. \tag{7.4}$$

7.0.2 Our Proposed Methodology

Uncertainty can be exhibited in extraordinary ways. One of the forms of uncertainty is fuzziness. Fuzziness (vagueness) results from imprecise boundaries of fuzzy sets. In this section, fuzzified evidence theory along with DS-theory and Dempster's rule of combination has been used. First, we measure the uncertainties (fuzziness) of parameters taking the scale of frame of discernment and relative scale of focal element with respect to FOD into consideration. Next, we use the fuzzy preference relation analysis to produce the consistency matrix. At that point, the vulnerabilities of parameters are adjusted and a while later, a suitable fundamental basic probability assignment (BPA) in terms of each parameter is produced. In the last, we utilize the Dempster's rule of combination to blend the independent parameters into integrated one. Inevitably, the best ideal decision can be gotten dependent on the positioning of choices. The flowchart of the proposed technique has been appeared in FIGURE 7.1

A. Measurement of uncertainty of parameters $e_j(j = 1, 2 \ldots n)$:

Total degree of fuzziness of the parameters with respect to alternatives can be calculated as under:

$$\mathcal{F}_d(\mathcal{A}) = \sum_{\mathcal{A}\in f} m(\mathcal{A})log_2 m(\mathcal{A}) f(\mathcal{A})(1+\epsilon)^{f|\mathcal{X}|}. \tag{7.5}$$

Where $m(\mathcal{A})$ denotes the mass function for hypothesis $\mathcal{A}$ and $f(\mathcal{A}))$ is the degree of fuzziness and is calculated by using equation (1.1). The factor $(1+\epsilon)^{f|\mathcal{X}|}$ considers the scale of FOD and also the relative scale of focal elements with respect to FOD. Also, ϵ is the constant greater than 0 and an appropriate number can be given to it based on practical example and $f \mid \mathcal{X} \mid$ represents the cardinality of $\mathcal{X}$ defined as-
$f \mid \mathcal{X} \mid= \sum_{\mathcal{B}\sqsubset\mathcal{X},\mathcal{B}\neq\mathcal{A}} \frac{|\mathcal{A}\sqcap\mathcal{B}|}{2^{|\mathcal{X}|}-1}$.

Example 7.1. *Let us suppose that the frame of discernment is $\mathcal{X} = \{a_1, a_2, \ldots, a_5\}$. A body of evidence $< m, \mathcal{F} >$ is listed as-*
$m_1 : m_1 = (\{a_1, a_2, a_3\}) = 0.3,\ m_1 = (\{a_4, a_5\}) = 0.7$
$m_2 : m_2 = (\{a_1, a_2, a_3\}) = 0.3,\ m_2 = (\{a_1, a_2, a_4, a_5\}) = 0.7$
The total degree of fuzziness of m_1 and m_2 are calculated as below:
$\mathcal{F}_d(m_1) = \sum_{\mathcal{A}\in f} m(m_1)log_2 m(m_1)f(m_1)(1+\epsilon)^{\sum_{\mathcal{B}\sqsubset\mathcal{X},\mathcal{B}\neq\mathcal{A}} \frac{|\mathcal{A}\sqcap\mathcal{B}|}{2^{|\mathcal{X}|}-1}}$
$= 0.3log_2(0.3) \times 1.2 \times 2^0 + 0.7log_2(0.7) \times 1.2 \times 2^0$
$= -0.62531 - 0.43224$
$= -1.0575$
$\mathcal{F}_d(m_2) = \sum_{\mathcal{A}\in f} m(m_2)log_2 m(m_2)f(m_2)(1+\epsilon)^{\sum_{\mathcal{B}\sqsubset\mathcal{X},\mathcal{B}\neq\mathcal{A}} \frac{|\mathcal{A}\sqcap\mathcal{B}|}{2^{|\mathcal{X}|}-1}}$
$= 0.3log_2(0.3) \times 1.2 \times 2^{\frac{2}{21}} + 0.7log_2(0.7) \times 1.2 \times 2^{\frac{2}{21}}$
$= -0.65391 - 0.45201$
$= -1.10592$

7.0.3 Brief description of Steps for the proposed method

Let $\Theta = \{x_1, x_2, \ldots x_i \ldots, x_t\}$ be the FOD and $\mathcal{B} = \{e_1, e_2, \ldots, e_j, \ldots en\}$ be the set of parameters.
Consider $\mathcal{F} : \mathcal{B} \longrightarrow 2^{\Theta}$ is defined as $\mathcal{F}(e_j)(x_i) = d_{ij}$.
1. Evolve the matrix $D = (d_{ij})_{n\times n}$ by the use of fuzzy soft set $(\mathcal{F}, \mathcal{B})$ over Θ and d_{ij} is

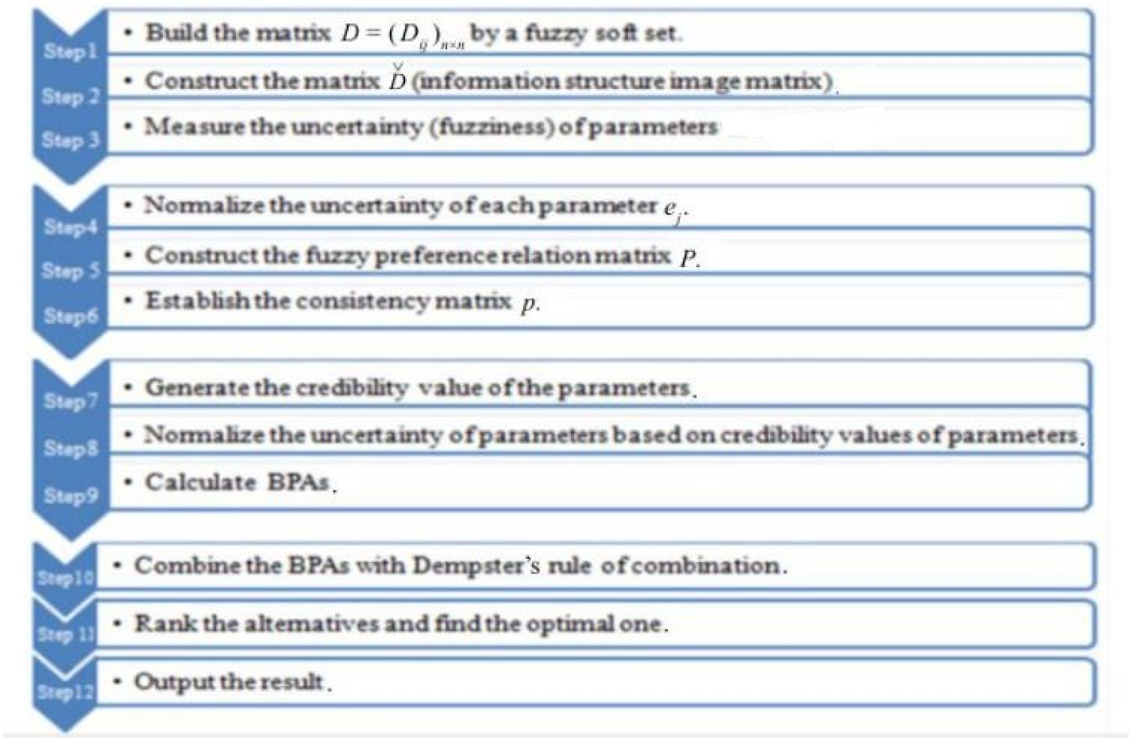

FIGURE 7.1: Flowchart of our Proposed Method

the membership value of x_i with respect to e_j.

$$\breve{D} = (d_{ij})_{n\times n} = \begin{bmatrix} d_{11} & \dots & d_{1j} & \dots & d_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{i1} & \dots & d_{ij} & \dots & d_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{t1} & \dots & d_{tj} & \dots & d_{tn} \end{bmatrix} \tag{7.6}$$

2. Construct the information structure image sequence with respect to each parameter e_j using formula $\tilde{d_{ij}} = \frac{d_{ij}}{\sum_{i=1}^{t} d_{ij}}$.

$$Thus, D = \begin{bmatrix} \tilde{d}_{11} & \dots & \tilde{d}_{1j} & \dots & \tilde{d}_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{d}_{i1} & \dots & \tilde{d}_{ij} & \dots & \tilde{d}_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{d}_{t1} & \dots & \tilde{d}_{tj} & \dots & \tilde{d}_{tn} \end{bmatrix} \tag{7.7}$$

3. Total degree of fuzziness of the parameters may be zero in some cases, so the proposed formula is used to measure the uncertainty of the parameter, denoted by $\mathcal{V}(e_j)$:

$$\mathcal{V}(e_j) = exp^{\mathcal{F}_d(e_j)} = exp^{\sum_{i=1}^{t} d_{ij}(log_2 d_{ij})f(d_{ij})(1+\epsilon)^{f|\mathcal{X}|}}. \tag{7.8}$$

4. Normalize the uncertainty of the parameter e_j as follows:

$$\overline{\mathcal{V}(e_j)} = \frac{\mathcal{V}(e_j)}{\sum_{h=1}^{n} \mathcal{V}(e_h)}, \quad 1 \leq j \leq n. \tag{7.9}$$

5. Construct the fuzzy preference relation matrix based on the variance of uncertainties of parameters. The diagonal elements of the matrix are allocated to 0.5 according to definition 7.10. When there are only two parameters, the off- diagonal elements are allocated to 0.5 as none other parameters are there to judge which one parameter is preferred to other. When there are more than two parameters,$n > 2$, the variance for the parameter $e_j(1 \leq j \leq n)$ is computed as-

$$Var(e_j) = Var(\{\bar{\mathcal{V}}(e_1), \bar{\mathcal{V}}(e_2), \ldots, \bar{\mathcal{V}}(e_{j-1}), \bar{\mathcal{V}}(e_{j+1}), \ldots, \bar{\mathcal{V}}(e_n)\}). \tag{7.10}$$

And the off-diagonal elements p_{jk} and p_{kj} is calculated as follows:

$$p_{jk} = \frac{Var(e_j)}{Var(e_j) + Var(e_k)}. \tag{7.11}$$

$$p_{kj} = \frac{Var(e_k)}{Var(e_k) + Var(e_j)}. \tag{7.12}$$

where $1 \leq j \leq n$ and $1 \leq k \leq n$.

6. Based on above fuzzy preference matrix obtained, built the consistency matrix p utilizing equation (7.4)

7. Based on the consistency matrix p, the credibility value of the parameter e_j is calculated as-

$$Cred(e_j) = \frac{2}{n^2} \sum_{k=1}^{n} \overline{p_{jk}}, \quad 1 \leq j \leq n; 1 \leq k \leq n. \tag{7.13}$$

where $\sum_{j=1}^{n} Cred(e_j) = 1$, these values will be taken as the loads to show the relative reliability preference of parameters.

8. On the basis of credibility values of parameters, normalized uncertainty can be modulated as-

$$\mathcal{MV}(e_j) = Cred(e_j) \times \overline{\mathcal{V}(e_j)}, \quad 1 \leq j \leq n. \tag{7.14}$$

9. Now, normalized the modulated uncertainty of parameters as the final degree of fuzziness as-

$$\overline{\mathcal{MV}(e_j)} = \frac{\mathcal{MV}(e_j)}{\sum_{h=1}^{n} \mathcal{MV}(e_k)}, \quad 1 \leq j \leq n. \tag{7.15}$$

10. The basic probability assignment of the alternative x_i and Θ with respect to e_j is calculated as-

$$m_{e_j}(\emptyset) = 0. \tag{7.16}$$

$$m_{e_j}(x_i) = \tilde{d_{ij}} \times (1 - \overline{\mathcal{MV}(e_j)}). \tag{7.17}$$

$$m_{e_j}(\Theta) = 1 - \sum_{i=1}^{t} m_{e_j}(x_i). \tag{7.18}$$

where $1 \leq j \leq t; 1 \leq k \leq n$ and $\sum_{\mathcal{A} \sqsubseteq \Theta} m_{e_j}(\mathcal{A}) = 1$, for $j = 1, 2, .., n$.
Hence, m_{e_j} is the basic probability assignment on Θ.

11. There are independent parameters which we have to fuse into integrated one; we make use of Dempster's rule of combination based on definition (7.10). Then, the final BPA of the alternative x_i obtained is viewed as alternative's belief measure. In the end, the candidate alternatives are positioned dependent upon the final BPAs of the alternatives x_i and the ideal one can be acquired.

7.0.3.1 Experiment

Example 7.2. *Suppose there is decision-making problem for which $(\mathcal{F}, \mathcal{D})$ represents fuzzy soft set and $\Theta = \{x_1, x_2, x_3\}$ is the frame of discernment along with $\mathcal{D} = \{e_1, e_2, e_3, e_4, e_5\}$ as the set of parameters. Following steps are followed to solve this experiment.*

1. Form the matrix $\mathcal{D} = (d_{ij})_{n \times n}$ bring about by fuzzy soft set over Θ:

$$\mathcal{D} = \begin{bmatrix} 0.85 & 0.73 & 0.26 & 0.32 & 0.75 \\ 0.56 & 0.82 & 0.76 & 0.64 & 0.43 \\ 0.84 & 0.55 & 0.82 & 0.53 & 0.47 \end{bmatrix}$$

2. Formulate $\bar{\mathcal{D}}$ the information structure image matrix:

$$\bar{\mathcal{D}} = \begin{bmatrix} 0.3778 & 0.3476 & 0.1413 & 0.2148 & 0.4545 \\ 0.2489 & 0.3905 & 0.4130 & 0.4295 & 0.2606 \\ 0.3773 & 0.2619 & 0.4457 & 0.3557 & 0.2848 \end{bmatrix}$$

3. The uncertainty measurement of the parameters $e_j (j = 1, 2, 3, 4, 5)$ is calculated using eq (7.8) as under:

$\mathcal{V}(e_1) = 0.2675 \quad \mathcal{V}(e_2) = 0.1530$

$\mathcal{V}(e_3) = 0.2428 \quad \mathcal{V}(e_4) = 0.0378$

$\mathcal{V}(e_5) = 0.0452$

4. Normalize the above uncertainty of the parameters using eq (7.9):

$\overline{\mathcal{V}(e_1)} = 0.3582 \quad \overline{\mathcal{V}(e_2)} = 0.2057$

$\overline{\mathcal{V}(e_3)} = 0.3250 \quad \overline{\mathcal{V}(e_4)} = 0.0507$

$\overline{\mathcal{V}(e_5)} = 0.0605$

5. Establish $\mathcal{P} = (p_{jk})_{n\times n}$, the fuzzy preference relation matrix:

$$\mathcal{P} = \begin{bmatrix} 0.5 & 0.3831 & 0.4486 & 0.4839 & 0.4683 \\ 0.6169 & 0.5 & 0.5671 & 0.6016 & 0.5865 \\ 0.5514 & 0.4329 & 0.5 & 0.5355 & 0.5199 \\ 0.5161 & 0.3984 & 0.4645 & 0.5 & 0.4843 \\ 0.5317 & 0.4135 & 0.4801 & 0.5157 & 0.5 \end{bmatrix}$$

6. Construct the consistency matrix $p = (p_{jl})_{n\times n}$ as-

$$p = \begin{bmatrix} 0.5 & 0.3824 & 0.4488 & 0.4841 & 0.4686 \\ 0.6176 & 0.5 & 0.5665 & 0.6017 & 0.5862 \\ 0.5512 & 0.4335 & 0.5 & 0.5353 & 0.5197 \\ 0.5159 & 0.3983 & 0.4647 & 0.5 & 0.4845 \\ 0.5314 & 0.4138 & 0.4803 & 0.5155 & 0.5 \end{bmatrix}$$

7. Produce the credibility value of parameter $e_j (j = 1, 2, 3, 4, 5)$ *by using eq (7.13) as under-*

$Cred(e_1) = 0.2173 \quad Cred(e_2) = 0.1702$

$Cred(e_3) = 0.1968 \quad Cred(e_4) = 0.2109$

$Cred(e_5) = 0.2047$

8. On the basis of consistency matrix, modulated the normalised uncertainty of parameter e_j *using eq (7.14)* $(j = 1, 2, 3, 4, 5)$ *as below-*

$\mathcal{MV}(e_1) = 0.077824 \quad \mathcal{MV}(e_2) = 0.03502$

$\mathcal{MV}(e_3) = 0.063967 \quad \mathcal{MV}(e_4) = 0.010689$

$\mathcal{MV}(e_5) = 0.012370$

9. Normalize the modulated uncertainty calculated above as under:

$\overline{\mathcal{MV}(e_1)} = 0.3893 \quad \overline{\mathcal{MV}(e_2)} = 0.175209$

$\overline{\mathcal{MV}(e_3)} = 0.320033 \quad \overline{\mathcal{MV}(e_4)} = 0.05347$

$\overline{\mathcal{MV}(e_5)} = 0.061919$

10. Now, compute the basic probability assignments of alternatives with respect to e_j *using equations (7.16), (7.17) and (7.18) which can be seen from table 7.1.*

$BPA's$	e_1	e_2	e_3	e_4	e_5
$m(x_1)$	0.2307	0.2867	0.0961	0.2033	0.4264
$m(x_2)$	0.1520	0.3221	0.2808	0.4065	0.2444
$m(x_3)$	0.2280	0.2160	0.3031	0.3367	0.2672
$m(\Theta)$	0.3893	0.1752	0.3200	0.0535	0.0620

TABLE 7.1: BPAs of x_i with respect to e_j

11. Merge the BPAs of alternatives by the use of definition (7.10) to get the fusing results which are going to be known as the belief measures of alternatives exhibited by table 7.2 and Fig.7.2.

Methods	$Bel(x_1)$	$Bel(x_2)$	$Bel(x_3)$
Grey relational Analysis Method	0.0745	0.1013	0.0990
Proposed Method	0.0212	0.0325	0.0275

TABLE 7.2: Alternatives belief measures in two unlike ways

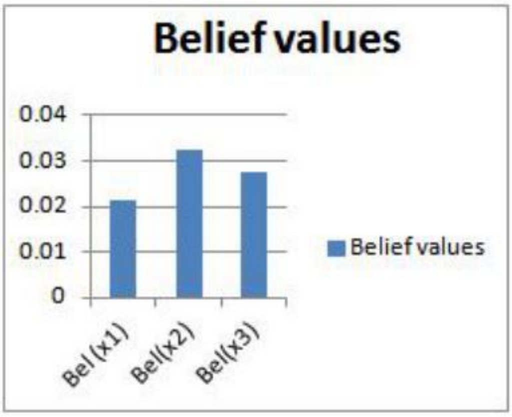

FIGURE 7.2: Interpretation of belief values for Experiment

12. On the basis of belief values of alternatives, their final ranking can be obtained. It has been observed that $x_2 > x_3 > x_1$. Hence, the maximum value showed that ideal choice is x_2 which can be easily seen through table 7.3 and Fig. 7.2 also.
Also, we compare our proposed method with the grey relational approach by comparing the belief values of alternatives along with the performance measure. It has been shown in table 7.3. The uncertainty's belief measure fell to 0.000104 attained from suggested method. It has also been observed that our proposed method can reduce the uncertainty and decision-making level as compared to grey relational method. We likewise compute the measure of performance which indicates that our technique is more exact and efficient than the other method.

Methods	Ranking	Optimal Value	$m(\Theta)$	γ(performance measure)
Grey relational Approach	$x_2 > x_3 > x_1$	x_2	0.0223	1.631
The Proposed Method	$x_2 > x_3 > x_1$	x_2	0.0001035	1.832

TABLE 7.3: Comparison of different methods in example 7.2

7.0.3.2 Application with real-life examplein reference to COVID-19

As we all know, the concept of uncertainty plays an important role in taking decisions in real-life problems. It is very difficult for human beings to take decisions with accuracy and efficiency in real-life problems. Fuzzy soft sets handle this problem efficiently with more accuracy. Hence, considering the real-life decision making problem, it can easily be shown that the given method is more efficient and accurate. We also compare our experimental result with grey relational analysis method. Fuzzy soft sets are extensively used in medical diagnosis field. Nowadays, the whole world is suffering from severe disease named corona virus. It becomes very difficult for doctors to detect that which type of disease a patient is suffering from. By using this proposed method, the ideal choice can be made out.

Example 7.3. *Suppose that the universal set consists of three types of diseases, namely, {dengue, corona virus, cholera} represented as* $\{x_1, x_2, x_3\}$ *and* $\mathcal{G}$ = *{high fever, cough, shortness of breath, nausea, vomiting, watery diarrhoea, rapid heart rate, physical examination, laboratory, rest}* $= \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, h_8, h_9, h_{10}\}$ *represents the set of parameters.*
Let $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *be the two subsets of* $\mathcal{G}$ *given by* $\mathcal{I}_1 = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ *and* $\mathcal{I}_2 = \{h_8, h_9, h_{10}\}$ *where* $(\mathcal{F}, \mathcal{I}_1)$ *is the fuzzy soft set representing "symptoms of diseases" and* $(\mathcal{F}, \mathcal{I}_2)$ *defines "decision making tools". Tables 7.4 and 7.5 represent these two fuzzy soft sets.*

Alternatives	g_1	g_2	g_3	g_4	g_5	g_6	g_7
x_1	0.50	0.70	0.00	0.30	0.20	0.80	0.9
x_2	0.40	0.60	0.90	0.00	0.90	0.70	0.00
x_3	0.60	0.00	0.10	0.40	0.00	0.70	0.00

TABLE 7.4: Fuzzy soft set $(\mathcal{F}, \mathcal{I}_1)$

Alternatives	h_8	h_9	h_{10}
x_1	0.40	0.70	0.50
x_2	0.20	0.10	0.90
x_3	0.10	0.60	0.30

TABLE 7.5: Fuzzy soft set $(\mathcal{F}, \mathcal{I}_2)$

Let us take an example of a patient put up with a disease having two symptoms-high fevers, shortening of breathe. A doctor needs to make the most suitable diagnosis regarding symptoms namely physical examination, lab investigation, history. To find out *the exact solution,* $(\mathcal{F}, \mathcal{I}_1)) \sqcap (\mathcal{F}, \mathcal{I}_2)$ *is constructed in table 7.6. There are three diseases* $\{x_1, x_2, x_3\}$ *and* $k_1 = (g_1, h_1), k_2 = (g_1, h_2), k_3 = (g_1, h_3), k_4 = (g_3, h_1), k_5 = (g_3, h_2), k_6 = (g_3, h_3)$ *represents pair of one symptom and one decision-making tool. Here,* Θ *is FOD defined by definition (2.12) and* $\mathcal{E} = \{k_1, k_2, k_3, k_4, k_5, k_6\}$ *is the set of parameters.*

Alternatives	k_1	k_2	k_3	k_4	k_5	k_6
x_1	0.40	0.50	0.50	0.00	0.00	0.00
x_2	0.20	0.10	0.40	0.20	0.10	0.90
x_3	0.10	0.60	0.30	0.10	0.10	0.10

TABLE 7.6: Fuzzy soft set $(\mathcal{F}, \mathcal{I})$

Following steps are to be followed to solve this numerical problem:

1. Form the matrix $\mathcal{D} = (d_{ij})_{n \times n}$ *bring about by* $(\mathcal{F}, \mathcal{I})$ *over* Θ *as below:*

$$\mathcal{D} = \begin{bmatrix} 0.40 & 0.50 & 0.50 & 0.00 & 0.00 & 0.00 \\ 0.20 & 0.10 & 0.40 & 0.20 & 0.10 & 0.90 \\ 0.10 & 0.60 & 0.30 & 0.10 & 0.10 & 0.10 \end{bmatrix}$$

2. Formulate $\bar{\mathcal{D}}$ *the information structure image matrix:*

$$\bar{\mathcal{D}} = \begin{bmatrix} 0.5714 & 0.4167 & 0.4167 & 0.00 & 0.00 & 0.00 \\ 0.2857 & 0.0833 & 0.3333 & 0.6667 & 0.5 & 0.90 \\ 0.1429 & 0.5 & 0.25 & 0.3333 & 0.5 & 0.10 \end{bmatrix}$$

3. The uncertainty measurement of the parameters $k_j (j = 1, 2, 3, 4, 5, 6)$ *using eq (7.8) is as under:*

$\mathcal{V}(k_1) = 0.15638 \quad \mathcal{V}(k_2) = 0.07818$

$\mathcal{V}(k_3) = 0.02424 \quad \mathcal{V}(k_4) = 0.62005$

$\mathcal{V}(k_5) = 0.76663 \quad \mathcal{V}(K_6) = 0.82895$

4. Normalize the above uncertainty of the parameters using eq (7.9):

$\overline{\mathcal{V}(k_1)} = 0.063198 \quad \overline{\mathcal{V}(k_2)} = 0.031595$

$\overline{\mathcal{V}(k_3)} = 0.009796 \quad \overline{\mathcal{V}(k_4)} = 0.2505$

$\overline{\mathcal{V}(k_5)} = 0.309821 \quad \overline{\mathcal{V}(k_6)} = 0.335006$

5. Establish $\mathcal{P} = (p_{jk})_{n \times n}$ *, the fuzzy preference relation matrix:*

$$\mathcal{P} = \begin{bmatrix} 0.5 & 0.5246 & 0.5473 & 0.4889 & 0.5324 & 0.5615 \\ 0.4754 & 0.5 & 0.5228 & 0.4643 & 0.5078 & 0.5372 \\ 0.4527 & 0.4772 & 0.5 & 0.4417 & 0.4850 & 0.5144 \\ 0.5111 & 0.5357 & 0.5583 & 0.5 & 0.5434 & 0.5724 \\ 0.4676 & 0.4922 & 0.5150 & 0.4566 & 0.5 & 0.5294 \\ 0.4385 & 0.4628 & 0.4856 & 0.4276 & 0.4706 & 0.5 \end{bmatrix}$$

6. Construct the consistency matrix $p = (p_{jl})_{n \times n}$ *as-*

$$p = \begin{bmatrix} 0.5 & 0.5245 & 0.5472 & 0.4889 & 0.5323 & 0.5616 \\ 0.4755 & 0.5 & 0.5228 & 0.4644 & 0.5078 & 0.5371 \\ 0.4528 & 0.4773 & 0.5 & 0.4417 & 0.4851 & 0.5143 \\ 0.5111 & 0.5356 & 0.5583 & 0.5 & 0.5434 & 0.5726 \\ 0.4677 & 0.4922 & 0.5149 & 0.4566 & 0.5 & 0.5293 \\ 0.4384 & 0.4629 & 0.4857 & 0.4274 & 0.4707 & 0.5 \end{bmatrix}$$

7. Produce the credibility value of parameter $k_j (j = 1, 2, 3, 4, 5, 6)$ *by using eq (7.13) as under-*

$Cred(k_1) = 0.1581 \quad Cred(k_2) = 0.1663$

$Cred(k_3) = 0.1738 \quad Cred(k_4) = 0.1544$

$Cred(k_5) = 0.1688 \quad Cred(k_6) = 0.1786$

8. On the basis of consistency matrix, modulated the normalized uncertainty of parameter k_j *using eq (7.14)* $(j = 1, 2, 3, 4, 5, 6)$ *as below-*

$\mathcal{MV}(k_1) = 0.0099 \quad \mathcal{MV}(k_2) = 0.005253$

$\mathcal{MV}(k_3) = 0.01703 \quad \mathcal{MV}(k_4) = 0.03868$

$\mathcal{MV}(k_5) = 0.052313 \quad \mathcal{MV}(k_6) = 0.059834$

9. Normalize the modulated uncertainty calculated above as under:

$\overline{\mathcal{MV}(k_1)} = 0.059544 \quad \overline{\mathcal{MV}(k_2)} = 0.031306$

$\overline{\mathcal{MV}(k_3)} = 0.010149 \quad \overline{\mathcal{MV}(k_4)} = 0.23059$

$\overline{\mathcal{MV}(e_5)} = 0.0311793 \quad \overline{\mathcal{MV}(k_6)} = 0.356619$

10. Now, compute the basic probability assignments of alternatives with respect to k_j using equations (7.16), (7.17) and (7.18) which can be seen from table 7.7.

$BPA's$	k_1	k_2	k_3	k_4	k_5	k_6
$m(x_1)$	0.5374	0.4037	0.4125	0.00	0.00	0.00
$m(x_2)$	0.2687	0.0807	0.3299	0.5130	0.3441	0.5790
$m(x_3)$	0.1344	0.4843	0.2475	0.2564	0.3441	0.0644
$m(\Theta)$	0.0595	0.0313	0.0101	0.2306	0.3118	0.3566

TABLE 7.7: BPAs of x_i with respect to k_j

11. By the use of definition 7.10, we combine BPAs of alternatives to get the fusing results which are known as the belief measures of alternatives. This is conveyed by table 7.8 and Fig. 7.3.

Methods	$Bel(x_1)$	$Bel(x_2)$	$Bel(x_3)$
Grey relational Analysis Method	0.0295	0.1260	0.0578
Proposed Method	0.004058	0.008227	0.004996

TABLE 7.8: Alternatives' belief measures in two unlike ways

12. On the basis of belief values of alternatives, their final ranking can be obtained. It has been observed that $x_2 > x_3 > x_1$.Hence, the maximum value showed that ideal choice is x_2 which can be easily seen through table 7.9 and figure 7.3 also.

Additionally, when we solved this example with grey relational analysis given by Li et al [20], it has been observed that our method can decrease the uncertainty to greater level which can be seen by comparing the uncertainty's belief measures through table 7.9. We

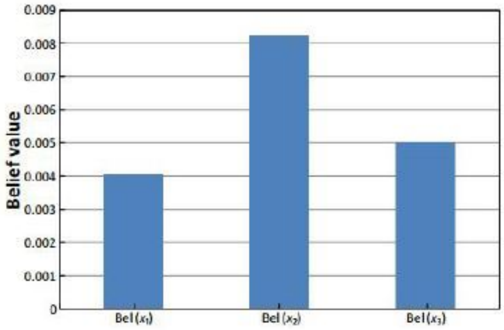

FIGURE 7.3: Belief values of alternatives for the proposed method

also calculated the performance measure γ for both methods. It has been found that our method is more accurate and efficient as compared to grey relational approach.

Methods	Ranking	Optimal Value	$m(\Theta)$	γ(performance measure)
Grey relational Approach	$x_2 > x_3 > x_1$	x_2	0.01468	1.5919
The Proposed Method	$x_2 > x_3 > x_1$	x_2	6.9578×10^{-7}	2.2698

TABLE 7.9: Comparison of different methods in example 7.3

Thus, by using this method, we can show that the belief measure of uncertainty fell to 6.9578×10^{-7} from 0.014683 in our proposed method. Hence, it can easily be deduced that the proposed method was progressively productive and reduced the level of uncertainty of the parameters and it is much more accurate to evaluate the symptoms of corona within a patient.

This chapter demonstrates the practical application of fuzzy soft sets in solving decision-making problems.

Chapter 8

Data devaluation in Multi-esteemed information frameworks using Alexandroff Soft Bitopological Approximation Spaces

This whole chapter is all about the foundation of Alexandroff Soft Bitopological Spaces(ASBS) along with their properties starting with the foundation of Alexandroff bitopological spaces and further defines new class of generalized soft closed sets known as $(1,2)-\breve{g}$-soft closed sets. Also, their properties will be discussed. Finally, we show the application of these spaces using newly developed generalized closed sets in data reduction problems in information systems and a practical example has been given to prove the relevance of our results.
Throughout the chapter, $(\mathcal{X}, \tau_1^s, \tau_2^s)$ denotes Alexandroff Soft bitopological spaces.

Following are the prerequisites for our important results:

Definition 8.1. [1] A triplet $(X, \mathcal{U}, \mathcal{V})$ is defined to be a bitopological space by Kelley in 1963, where X is a non-empty set and $\mathcal{U}$ and $\mathcal{V}$ are two arbitrary topologies defined on X.

Definition 8.2. [13] A family τ defines a soft topology on $\mathcal{X}$ if the following given conditions are satisfied:

a. $0_{\mathcal{H}}, 1_{\mathcal{H}}$ belongs to τ .

b. For $(R, \mathcal{H}), (S, \mathcal{H}) \in \tau$, $(R, \mathcal{H}) \sqcap (S, \mathcal{H}) \in \tau$.

c. For $(R_i, \mathcal{H}) \in \tau \ \forall \ i \in I$, $\sqcup\{(R_i, \mathcal{H}) \colon i \in I\} \in \tau$.

$(\mathcal{X}, \tau, \mathcal{H})$ denotes a soft topological space.

Definition 8.3. [3] Let us consider $(\mathcal{X}, \tau)$ be soft topological space and $\mathcal{G} \subseteq \mathcal{X}$ be a non-empty set. Then, $\tau_{\mathcal{G}} = \{(\mathcal{F}_{\mathcal{G}}, \mathcal{P}) \mid (\mathcal{F}, \mathcal{P}) \in \tau\}$ is considered as relative topology on $\mathcal{G}$ and $(\mathcal{G}, \tau_{\mathcal{G}})$ is called soft subspace of $(\mathcal{X}, \tau)$.

Theorem 8.4. *[3] Let us consider $(\mathcal{G}, \tau_{\mathcal{G}})$ be a soft subspace of $(\mathcal{X}, \tau)$ and $(\mathcal{H}, \mathcal{P})$ be a soft set over $\mathcal{X}$, thus-*

a) $(\mathcal{H}, \mathcal{P})$ is soft open in $\mathcal{G}$ if and only if $(\mathcal{H}, \mathcal{P}) = \mathcal{G} \cap (\mathcal{I}, \mathcal{P})$ for some $(\mathcal{I}, \mathcal{P}) \in \tau$.

b) $(\mathcal{H}, \mathcal{P})$ is soft closed in $\mathcal{G}$ if and only if $(\mathcal{H}, \mathcal{P}) = \mathcal{G} \cap (\mathcal{I}, \mathcal{P})$ for some $(\mathcal{I}, \mathcal{P})$ in $\mathcal{X}$.

Definition 8.5. [3] A soft set $(\mathcal{H}, \mathcal{P})$ is said to be soft subset of $(\mathcal{G}, \mathcal{Q})$ if-

a) $\mathcal{P} \subseteq \mathcal{Q}$ and

b) for all $p \in \mathcal{P}, \mathcal{H}(p)$ and $\mathcal{G}(p)$ are identical approximations.

We can write it as $(\mathcal{H}, \mathcal{P}) \subseteq (\mathcal{G}, \mathcal{Q})$.

8.0.1 Alexandroff Bitopological Spaces

Definition 8.6. A non-empty set $\mathcal{X}$ endowed with two arbitrary topologies (τ_1, τ_2) which is closed under arbitrary intersection is said to be an Alexandroff Bitopological Space.

Thus, a triplet $(\mathcal{X}, \tau_1, \tau_2)$ is known as Alexandroff Bitopological Space.

Members of τ_1 and τ_2 are closed sets and their complements are open sets in $\mathcal{X}$.

Example : Discrete bitopology is an Alexandroff Bitopological space.

Definition 8.7. Let $(\mathcal{X}, \tau_1, \tau_2)$ be an Alexandroff Bitopological spaces and $\mathcal{A}$ be a subset of $\mathcal{X}$. Then, $(\mathcal{X}, \tau_1^{\mathcal{A}}, \tau_2^{\mathcal{A}})$ is a subspace in $\mathcal{X}$ where $\tau_1^{\mathcal{A}} = \{\mathcal{U} \cap \mathcal{A} \colon \mathcal{U} \in \tau_1\}$ and $\tau_2^{\mathcal{A}} = \{\mathcal{V} \cap \mathcal{A} \colon \mathcal{V} \in \tau_2\}$.

Definition 8.8. Suppose $\mathcal{X}$ be an Alexandroff bitopological spaces.$(\mathcal{U}, \mathcal{A}, \tau_1, \tau_2)$ be a family of closed sets. Then, $(\mathcal{U}, \mathcal{A}, \tau_1, \tau_2)$ is known as minimal base if and only if :

1) $(\mathcal{U}, \mathcal{A}, \tau_1, \tau_2)$ covers $\mathcal{X}$.

2) For $(\mathcal{P},\mathcal{A},\tau_1,\tau_2),(\mathcal{Q},\mathcal{A},\tau_1,\tau_2)\in$(U, A,$\tau_1,\tau_2$), $\exists$ a sub family $\{(\mathcal{U}_i,\mathcal{A},\tau_1,\tau_2\}$ such that $(\mathcal{P},\mathcal{A},\tau_1,\tau_2)\cup(\mathcal{Q},\mathcal{A},\tau_1,\tau_2)=(\mathcal{U}_i,\mathcal{A},\tau_1,\tau_2)$.

(3) If a subfamily $\{(\mathcal{U}_i,\mathcal{A},\tau_1,\tau_2)\colon i\in I\}$ of $(\mathcal{U},\mathcal{A},\tau_1,\tau_2)$ verifies $\cup(\mathcal{U}_i,\mathcal{A},\tau_1,\tau_2)\in(\mathcal{U},\mathcal{A},\tau_1,\tau_2)$, for $i\in I$, then $\exists\ i_o\in I$ such that $\cap\ (\mathcal{U}_i,\mathcal{A},\tau_1,\tau_2)=\mathcal{U}_{i_o}$.

Definition 8.9. A family α_{τ_1} of sets is called a subbase for a topology τ_2 if and only if the family of finite intersections of members of α_{τ_1} is a base for τ_2.

Definition 8.10. [96] An Alexandroff Bitopological space (briefly known as AL-BI Space) is said to be connected if there doesn't exist any non-void proper subset of $\mathcal{X}$ which is both τ_1-open and τ_2 closed. It is to be noted that connectedness for a bitopology may not be equivalent to connectedness of two topologies.

Theorem 8.11. *If $\mathcal{Y}$ is a connected subset of $\mathcal{X}$ and $\mathcal{Z}$ is a set such that $\mathcal{Y}\subset\mathcal{Z}\subset\mathcal{X}$, then $\mathcal{Z}$ is also connected. In particular, the closure of a connected is connected.*

Proof. Let $\mathcal{Y}$ be a connected subset of $\mathcal{X}$ and $\mathcal{Y}'=\mathcal{A}\cup\mathcal{B}$, where $\mathcal{A}$ is clopen in τ_1 and $\mathcal{B}$ is clopen in τ_2.Then, each of $\mathcal{A}\cap\mathcal{Y}$ and $\mathcal{B}\cap\mathcal{Y}$ are clopen in $\mathcal{Y}$ and since $\mathcal{Y}$ is connected, $\mathcal{A}\cap\mathcal{Y}$ or $\mathcal{B}\cap\mathcal{Y}$ must be void.

Suppose that $\mathcal{A}\cap\mathcal{Y}$ is void. Then, $\mathcal{Y}$ is a subset of $\mathcal{B}$ and consequently, $\mathcal{Y}'$ is a subset of $\mathcal{B}$.

Hence $\mathcal{B}$ is void and thus, it follows that $\mathcal{Y}'$ is connected. □

Theorem 8.12. *If f is a continuous onto mapping from a connected space $(\mathcal{X},\tau_1,\tau_2)$ to $(\mathcal{Y},\tau_{1^*},\tau_{2^*})$, then $\mathcal{Y}$ is also connected.*

Proof. On the contrary, suppose that $\mathcal{Y}$ is not connected or disconnected. Then, there exists two non-empty disjoint open sets such that $\mathcal{Y}=\mathcal{A}\cup\mathcal{B}$. Since f is a continuous onto map, $f^{-1}(\mathcal{A})$ and $f^{-1}(\mathcal{B})$ are two disjoint non-empty sets in $\mathcal{X}$. Thus, $\mathcal{X}$ can be written as $\mathcal{A}\cup\mathcal{B}$ which implies $\mathcal{X}$ is disconnected which is again a contradiction.

Hence, $\mathcal{Y}$ is also connected. □

Definition 8.13. The component of a bitopological space $\mathcal{X}$ is the maximal connected subset of $\mathcal{X}$.

Proposition 8.14. *In $(\mathcal{X},\tau_1,\tau_2)$, $\mathcal{C}$ is a connected subset, then either $\mathcal{C}\subset\mathcal{A}$ or $\mathcal{C}\subset\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are two non-empty disjoint open sets in $\mathcal{X}$.*

Definition 8.15. An bi- Alexo $\mathcal{T}_0$ space is defined as -

For points $x, y \in \mathcal{X}$ and $x \neq y$, $\exists$ τ_1- closed set $(\mathcal{P}, \mathcal{A})$ and τ_2-closed set $(\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$ and $y \notin (\mathcal{P}, \mathcal{A})$ or $y \in (\mathcal{Q}, \mathcal{A})$ and $x \notin (\mathcal{Q}, \mathcal{A})$.

Definition 8.16. A space $(\mathcal{X}, \tau_1 \tau_2)$ is called bi- Alexo $\mathcal{T}_1$- space if there exists τ_1-closed set $(\mathcal{P}, \mathcal{A})$ and τ_2- closed set $(\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$ and $y \notin (\mathcal{P}, \mathcal{A})$ and $y \in (\mathcal{Q}, \mathcal{A})$ and $x \notin (\mathcal{Q}, \mathcal{A})$, for points $x, y \in \mathcal{X}$ and $x \neq y$.

Proposition 8.17. *A non-empy subset of bi- Alexo $\mathcal{T}_1$- space is also an bi-Alexo $\mathcal{T}_1$-space.*

Proof. Since $\mathcal{X}$ is bi-Alexo $\mathcal{T}_1$- space, then $\exists$ τ_1 closed set $(\mathcal{P}, \mathcal{A})$ and τ_2 closed set $(\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$ and $y \notin (\mathcal{P}, \mathcal{A})$ and $y \in (\mathcal{Q}, \mathcal{A})$ and $x \notin (\mathcal{Q}, \mathcal{A})$ for $x, y \in \mathcal{X}$ and $x \neq y$.

Now, $x \in \mathcal{Y}$ and $x \in (\mathcal{P}, \mathcal{A})$ implies $x \in \mathcal{Y} \sqcap (\mathcal{P}, \mathcal{A}) = (\mathcal{P}_{\mathcal{Y}}, \mathcal{A})$ where $(\mathcal{P}, \mathcal{A}) \in \tau_1$.

Suppose $y \notin (\mathcal{P}, \mathcal{A})$ which implies $y \notin \mathcal{P}(\beta)$ for some $\beta \in \mathcal{A}$.

Now, $y \notin \mathcal{Y} \sqcap \mathcal{P}(\beta) \Rightarrow \mathcal{Y}(\beta) \sqcap \mathcal{P}(\beta)$. Therefore, $y \notin \mathcal{Y} \sqcap (\mathcal{P}, \mathcal{A}) = (\mathcal{P}_{\mathcal{Y}}, \mathcal{A})$.

In the same way, we will show that $y \in (\mathcal{Q}_{\mathcal{Y}}, \mathcal{A})$ and $x \notin (\mathcal{Q}_{\mathcal{Y}}, \mathcal{A})$.

Hence proved. □

Definition 8.18. A space $(\mathcal{X}, \tau_1, \tau_2)$ is known as bi- Alexo $\mathcal{T}_2$- space if there exists τ_1 closed set $(\mathcal{P}, \mathcal{A})$ and τ_2 closed set $(\mathcal{Q}, \mathcal{A})$ such that $x \in (\mathcal{P}, \mathcal{A})$, $y \in (\mathcal{Q}, \mathcal{A})$ and $(\mathcal{P}, \mathcal{A})$ $\sqcap$ $(\mathcal{Q}, \mathcal{A}) = \emptyset$, for $x, y \in \mathcal{X}$ such that $x \neq y$

Remark 8.19. [44] 1) Every bi- Alexo $\mathcal{T}_1$- space is bi- Alexo $\mathcal{T}_0$.

2) Every bi- Alexo $\mathcal{T}_2$ space is bi- Alexo $\mathcal{T}_1$.

3) A non-void subset of $\mathcal{X}$ is also bi-Alexo $\mathcal{T}_2$-space if $\mathcal{X}$ is bi-Alexo $\mathcal{T}_2$.

Definition 8.20. A space $\mathcal{X}$ is said to be a bi- Alexo Regular space if for each $x \in \mathcal{X}$ and a τ_2 open set $(\mathcal{Q}, \mathcal{A})$ such that $x \notin (\mathcal{Q}, \mathcal{A})$, then $\exists$ τ_1- closed set $(\mathcal{P}_1, \mathcal{A})$ and τ_2-closed set $(\mathcal{P}_2, \mathcal{A})$ such that $x \in (\mathcal{P}_2, \mathcal{A}), (\mathcal{Q}, \mathcal{A}) \subseteq (\mathcal{P}_1, \mathcal{A})$ and $(\mathcal{P}_1, \mathcal{A}) \sqcap (\mathcal{P}_2, \mathcal{A}) = \emptyset$.

Definition 8.21. [51] A space $\mathcal{X}$ is said to be a bi- Alexo strongly pairwise regular space if for each $x \in \mathcal{X}$ and a τ_1 open set $(\mathcal{Q}, \mathcal{A})$ such that $x \notin (\mathcal{Q}, \mathcal{A})$, then $\exists$ τ_1-closed set $(\mathcal{P}_1, \mathcal{A})$ and τ_2- closed set $(\mathcal{P}_2, \mathcal{A})$ such that $x \in (\mathcal{P}_1, \mathcal{A}), (\mathcal{Q}, \mathcal{A}) \subseteq (\mathcal{P}_1, \mathcal{A})$ int-$(\mathcal{P}_2, \mathcal{A})$ and $(\mathcal{P}_1, \mathcal{A}) \sqcap (\mathcal{P}_2, \mathcal{A}) = \emptyset$.

Definition 8.22. A space $\mathcal{X}$ is known as bi- Alexo Normal space if $\exists$ τ_1- closed set $(\mathcal{P}_1, \mathcal{A})$ and τ_2-closed set $(\mathcal{P}_2, \mathcal{A})$ such that $(\mathcal{P}, \mathcal{A}) \sqsubseteq (\mathcal{P}_1, \mathcal{A})$, $(\mathcal{Q}, \mathcal{A}) \sqsubseteq (\mathcal{P}_2, \mathcal{A})$ and $(\mathcal{P}_1, \mathcal{A}) \sqcap (\mathcal{P}_2, \mathcal{A}) = \emptyset$ for $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{Q}, \mathcal{A})$ open sets over $\mathcal{X}$ and $(\mathcal{P}, \mathcal{A}) \sqcap (\mathcal{Q}, \mathcal{A}) = \emptyset$.

Definition 8.23. [90] A space $\mathcal{X}$ is known as bi- Alexo $\mathcal{C}$ (completely)-Normal space if $(\mathcal{P}, \mathcal{A})$, $(\mathcal{G}, \mathcal{A}) \sqsubseteq \mathcal{X}$ with $(\mathcal{P}, \mathcal{A}) \cap$ cl-$(\mathcal{G}, \mathcal{A})_{\tau_1}$ and $(\mathcal{P}, \mathcal{A}) \cap (\mathcal{Q}, \mathcal{A}) = \emptyset$, $\exists$ τ_1 closed set $\mathcal{U}$ and τ_2 closed set $\mathcal{V}$ such that $(\mathcal{P}, \mathcal{A}) \sqsubseteq \mathcal{U}$ & $(\mathcal{G}, \mathcal{A}) \sqsubseteq \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Definition 8.24. [51] A space $\mathcal{X}$ is known as bi- Alexo strongly pairwise Normal space if for each τ_1-open set $(\mathcal{P}, \mathcal{A})$ and τ_2-open set $(\mathcal{G}, \mathcal{A})$ with $(\mathcal{P}, \mathcal{A}) \cap (\mathcal{G}, \mathcal{A}) = \emptyset$, $\exists$ τ_1 closed set $\mathcal{U}$ and τ_2 closed set $\mathcal{V}$ such that $(\mathcal{P}, \mathcal{A}) \sqsubseteq$ int- $\mathcal{U}_{\tau_1}$ & $(\mathcal{G}, \mathcal{A}) \sqsubseteq$ int- $\mathcal{V}_{\tau_2}$ and $\mathcal{U}$ $\cap$ $\mathcal{V} = \emptyset$.

Remark 8.25. 1) Every closed subspace of regular space is regular.
2) Every closed subspace of normal is normal.

8.0.2 Alexandroff Soft Bitopology

Definition 8.26. An Alexandroff Soft Bitopology is a non-empty set endowed with two soft topologies having arbitrary intersection of open sets is open. Thus, a triplet $(\mathcal{X}, \tau_1^s, \tau_2^s)$ is known as Alexandroff Soft Bitopological Spaces(ASBS). Elements of τ_1^s and τ_2^s are closed sets and their complements are open sets in $\mathcal{X}$.

Definition 8.27. Let us consider $(\mathcal{X}, \tau_1^s, \tau_2^s)$ be an Alexandroff Soft Bitopological spaces and $\mathcal{A} \sqsubseteq \mathcal{X}$. Then, $(\mathcal{X}, \tau_1\mathcal{A}^s, \tau_2\mathcal{A}^s)$ is a subspace in $\mathcal{X}$ where $\tau_1\mathcal{A}^s = \{\mathcal{U} \cap \mathcal{A}^s : \mathcal{U} \in \tau_1^s\}$ and $\tau_2\mathcal{A}^s = \{\mathcal{V} \cap \mathcal{A}^s : \mathcal{V} \in \tau_2^s\}$.

Definition 8.28. A soft subset $\mathcal{A}$ in ASBS is said to be a $(1,2)^s$-soft dense in $\mathcal{X}$ if and only if τ_1^s-$\overline{(\tau_2^s\mathcal{A})} = \mathcal{X}$. The set of $(1,2)^s$-soft dense sets in $\mathcal{X}$ is denoted by $(1,2)^s$-$SD(\mathcal{X})$.

Definition 8.29. An Alexo soft bi- $\mathcal{T}_0$ space is defined as -
For points $p, q \in \mathcal{X}$ and $p \neq q$, $\exists$ τ_1^s- soft closed set $(\mathcal{R}, \mathcal{A}^s)$ and τ_2^s-soft closed set $(\mathcal{S}, \mathcal{A}^s)$ such that $p \in (\mathcal{R}, \mathcal{A}^s)$ and $q \notin (\mathcal{R}, \mathcal{A}^s)$ or $q \in (\mathcal{S}, \mathcal{A}^s)$ and $p \notin (\mathcal{S}, \mathcal{A}^s)$.

Definition 8.30. A space $(\mathcal{X}, \tau_1^s\tau_2^s)$ is called Alexo soft bi- $\mathcal{T}_1$- space if $\exists$ τ_1^s-soft closed set $(\mathcal{R}, \mathcal{A}^s)$ and τ_2^s- soft closed set $(\mathcal{S}, \mathcal{A}^s)$ such that $p \in (\mathcal{R}, \mathcal{A}^s)$ and $q \notin (\mathcal{R}, \mathcal{A}^s)$ and $q \in (\mathcal{S}, \mathcal{A}^s)$ and $p \notin (\mathcal{S}, \mathcal{A}^s)$, for points $p, q \in \mathcal{X}$ and $p \neq q$.

Definition 8.31. A space $(\mathcal{X}, \tau_1^s, \tau_2^s)$ is known as Alexo soft bi- $\mathcal{T}_2$- space if $\exists$ τ_1^s soft closed set $(\mathcal{R}, \mathcal{A}^s)$ and τ_2^s soft closed set $(\mathcal{S}, \mathcal{A}^s)$ such that $p \in (\mathcal{R}, \mathcal{A}^s)$, $q \in (\mathcal{S}, \mathcal{A}^s)$ and $(\mathcal{R}, \mathcal{A}^s) \sqcap (\mathcal{S}, \mathcal{A}) = \emptyset$, for $p, q \in \mathcal{X}$ such that $p \neq q$

Definition 8.32. Alexo Soft bi-Regular space - for each $x \in \mathcal{X}$ and a τ_2^s soft open set $(\mathcal{S}, \mathcal{A}^s)$ such that $p \notin (\mathcal{S}, \mathcal{A}^s)$, then $\exists$ τ_1^s-soft closed set $(\mathcal{R}_1, \mathcal{A}^s)$ and τ_2^s- soft closed set $(\mathcal{R}_2, \mathcal{A}^s)$ such that $p \in (\mathcal{R}_2, \mathcal{A}^s), (\mathcal{S}, \mathcal{A}^s) \subseteq (\mathcal{R}_1, \mathcal{A}^s)$ and $(\mathcal{R}_1, \mathcal{A}^s) \sqcap (\mathcal{R}_2, \mathcal{A}^s) = \emptyset$.

Definition 8.33. A space $\mathcal{X}$ is called as Alexo soft bi- Normal space if $\exists$ τ_1^s- soft closed set $(\mathcal{R}_1, \mathcal{A}^s)$ and τ_2^s-closed set $(\mathcal{R}_2, \mathcal{A}^s)$ such that $(\mathcal{R}, \mathcal{A}^s) \sqsubseteq (\mathcal{R}_1, \mathcal{A}^s)$, $(\mathcal{S}, \mathcal{A}^s) \sqsubseteq (\mathcal{R}_2, \mathcal{A}^s)$ and $(\mathcal{R}_1, \mathcal{A}^s) \sqcap (\mathcal{R}_2, \mathcal{A}^s) = \emptyset$ for $(\mathcal{R}, \mathcal{A}^s)$ and $(\mathcal{S}, \mathcal{A}^s)$ soft open sets over $\mathcal{X}$ and $(\mathcal{R}, \mathcal{A}^s)$ $\sqcap$ $(\mathcal{S}, \mathcal{A}^s) = \emptyset$.

8.0.3 Generalized closed sets in Alexandroff Soft Bitopological Spaces

This section begins with basic introductions before defining $(1,2) - \breve{g}$-soft closed sets in ASBS and presenting various findings.

Definition 8.34. [37] Let us consider $(\mathcal{X}, \tau_1^s, \tau_2^s)$ be an Alexandroff Soft bitopological spaces and $\mathcal{A}^s \sqsubseteq \mathcal{X}$. Thus, $\mathcal{A}^s$ is called $\tau_1^s\tau_2^s$-soft open if $\mathcal{A}^s = \mathcal{B}^s \cup \mathcal{C}^s$ where $\mathcal{B}^s \in \tau_1^s$ and $\mathcal{C}^s \in \tau_2^s$.

Definition 8.35. [37] Let $\mathcal{P} \subseteq \mathcal{Q}$. Then,
1) $\tau_1^s\tau_2^s$-soft closure of $\mathcal{Q}$ is denoted by $\tau_1^s\tau_2^s - cl(\mathcal{Q})$ and defined as $\tau_1^s\tau_2^s - cl(\mathcal{Q}) = \cap\{\mathcal{R} : \mathcal{Q} \subseteq \mathcal{R}, \mathcal{R}$ is $\tau_1^s\tau_2^s$-soft closed$\}$
2) $\tau_1^s\tau_2^s$-soft interior of $\mathcal{Q}$ is denoted by $\tau_1^s\tau_2^s - int(\mathcal{Q})$ and defined as $\tau_1^s\tau_2^s - int(\mathcal{Q}) = \cup\{\mathcal{R} : \mathcal{Q} \subseteq \mathcal{R}, \mathcal{R}$ is $\tau_1^s\tau_2^s$-soft open$\}$.

Definition 8.36. A subset $\mathcal{F}^s$ of $\mathcal{X}$ is known as $(1,2) - \breve{g}$ soft closed set if $\tau_1^s\tau_2^s$-cl- $\mathcal{F}^s$ $\sqsubseteq \mathcal{U}^s$, whenever $\mathcal{F}^s \sqsubseteq \mathcal{U}^s$ and $\mathcal{U}^s$ is $\tau_1\tau_2$-soft open in $\mathcal{X}$.

Example 8.1. *Let $\mathcal{X} = \{x_1, x_2, x_3\}$, $\mathcal{E} = \{a_1, a_2\}$ be a set of parameters and $\mathcal{A} \sqsubseteq \mathcal{E}$. Define $\mathcal{F}_A^s = \{(a_1, \{x_1, x_2\}), (a_2, \{x_2, x_3\})\}$. Then*
$\mathcal{F}_{A_1}^s = \{(a_1, \{x_1\})\}$
$\mathcal{F}_{A_2}^s = \{(a_1, \{x_2\}$
$\mathcal{F}_{A_3}^s = \{(a_1, \{x_1, x_2\})$
$\mathcal{F}_{A_4}^s = \{(a_2, \{x_2)\}$

$\mathcal{F}^s_{A_5} = \{(a_2, \{x_3)\}$
$\mathcal{F}^s_{A_6} = \{(a_2, \{x_2, x_3\})$
$\mathcal{F}^s_{A_7} = \{(a_1, \{x_1\}), (a_2, \{x_2\})\}$
$\mathcal{F}^s_{A_8} = \{(a_1, \{x_1\}), (a_2, \{x_3\})\}$
$\mathcal{F}^s_{A_9} = \{(a_1, \{x_1\}), (a_2, \{x_2, x_3\})\}$
$\mathcal{F}^s_{A_{10}} = \{(a_1, \{x_2\}), (a_2, \{x_2\})\}$
$\mathcal{F}^s_{A_{11}} = \{(a_1, \{x_2\}), (a_2, \{x_3\})\}$
$\mathcal{F}^s_{A_{12}} = \{(a_1, \{x_2\}), (a_2, \{x_2, x_3\})\}$
$\mathcal{F}^s_{A_{13}} = \{(a_1, \{x_1, x_2\}), (a_2, \{x_2\})\}$
$\mathcal{F}^s_{A_{14}} = \{(a_1, \{x_1, x_2\}), (a_2, \{x_3\})\}$
$\mathcal{F}^s_{A_{15}} = \mathcal{F}^s_A$
$\mathcal{F}^s_{A_{16}} = \mathcal{F}^s_\emptyset$
are all soft subsets of $\mathcal{F}^s_A$*.*
Now, consider $\mathcal{F}^s_A = \{\mathcal{F}_{A_2}, \mathcal{F}_{A_3}, \mathcal{F}_{A_5}, (a_1, \{x_1, x_3\}), (a_1, \{x_2, x_3\})\}$*. Let* τ^s_1 *and* τ^s_2 *be two topologies given by* $\tau^s_1 = \{\mathcal{F}^s_\emptyset, \mathcal{F}^s_A, \mathcal{F}^s_{A_2}\}$ *and* $\tau_2 = \{\mathcal{F}^s_\emptyset, \mathcal{F}^s_A\}$*. Then,* $\tau^s_1\tau^s_2 = \{\mathcal{F}^s_\emptyset, \mathcal{F}^s_A, \mathcal{F}^s_{A_2}\}$ *is* $\tau^s_1\tau^s_2$ *soft open. Clearly,* $\mathcal{F}^s_{A_3}, \{(a_1, \{x_2, x_3\})\}$ *are* $(1,2) - \breve{g}$*-soft closed sets.*

Theorem 8.37. *The union of two* $(1,2) - \breve{g}$*-soft closed sets is also* $(1,2) - \breve{g}$*-soft closed set.*

Proof. Let $\mathcal{A}^s$ and $\mathcal{B}^s$ be two $(1,2) - \breve{g}$-soft closed sets. Then, $\tau^s_1\tau^s_2$-cl- $\mathcal{A}^s \sqsubseteq \mathcal{U}^s$, whenever $\mathcal{A}^s \sqsubseteq \mathcal{U}^s$ and $\mathcal{U}^s$ is $\tau^s_1\tau^s_2$-soft open in $\mathcal{X}$ and $\tau^s_1\tau^s_2$-cl- $\mathcal{B}^s \sqsubseteq \mathcal{V}^s$, whenever $\mathcal{B}^s \sqsubseteq \mathcal{V}^s$ and $\mathcal{V}^s$ is $\tau^s_1\tau^s_2$-soft open in $\mathcal{X}$.
Thus, $\tau^s_1\tau^s_2$-cl- $(\mathcal{A}^s \cup \mathcal{B}^s) \sqsubseteq (\mathcal{U}^s \cup \mathcal{V}^s)$, whenever $(\mathcal{A}^s \cup \mathcal{B}^s) \sqsubseteq (\mathcal{U}^s \cup \mathcal{V}^s)$ and $(\mathcal{U}^s \cup \mathcal{V}^s)$ is $\tau_1\tau_2$-soft open in $\mathcal{X}$. Hence $(\mathcal{A}^s \cup \mathcal{B}^s)$ is also $(1,2) - \breve{g}$-soft closed set. □

Remark 8.38. The intersection of two $(1,2) - \breve{g}$-soft closed sets may not be a $(1,2) - \breve{g}$-soft closed set.

Proof. This follows from example 8.1 since $\mathcal{F}_{A_3} \cap \{(a_1, \{x_2, x_3\})\} = \mathcal{F}_{A_2}$ which is not $(1,2) - \breve{g}$-soft closed set. □

Theorem 8.39. *Every* $\tau^s_1\tau^s_2$*- soft closed set implies* $(1,2) - \breve{g}$*-soft closed set.*

Proof. We suppose $\mathcal{F}^s_A$ be $\tau^s_1\tau^s_2$- soft closed set in $\mathcal{F}^s_B$. Therefore, $\tau^s_1\tau^s_2$- cl$(\mathcal{F}^s_A) = \mathcal{F}^s_A \sqsubseteq \mathcal{F}^s_B$ whenever soft closed set $\mathcal{F}^s_A \sqsubseteq \mathcal{F}^s_B$ and $\mathcal{F}^s_B$ is $\tau^s_1\tau^s_2$ soft open. This implies that $\mathcal{F}^s_A$

is $(1,2)-\breve{g}$ soft closed set. But the converse of this may not be true which follows from example 8.1. □

Definition 8.40. [97] Let $\mathcal{F}_A^s$ and $\mathcal{F}_A^s$ be two soft sets and $\mathcal{F}_A^s \sqsubseteq \mathcal{F}_B^s$. Then, $\mathcal{F}_A^s$ is called $(1,2)-gsg$-soft closed set if $\tau_1^s\tau_2^s$ cl$(\mathcal{F}_A^s) \sqsubseteq \mathcal{F}_C^s$ Whenever $\mathcal{F}_A^s \sqsubseteq \mathcal{F}_C^s$ and $\mathcal{F}_C^s$ is semi-generalized soft open set in $\tau_1^s\tau_2^s$.

Theorem 8.41. *A $(1,2)-gsg$-soft closed set is $(1,2)-\breve{g}$-soft closed set.*

Proof. Let us consider $\mathcal{F}_A^s$ be a $(1,2)-gsg$-soft closed set. Then, $\tau_1^s\tau_2^s$ cl$(\mathcal{F}_A^s) \sqsubseteq \mathcal{F}_C^s$ Whenever $\mathcal{F}_A^s \sqsubseteq \mathcal{F}_C^s$ and $\mathcal{F}_C^s$ is semi-generalized soft open set in $\tau_1^s\tau_2^s$. Since any soft open set is semi-generalized soft open set, thus every $(1,2)-gsg$-soft closed set is $(1,2)-\breve{g}$-soft closed set. □

Remark 8.42. 1) $\tau_1^s\tau_2^s$-soft closed set implies $(1,2)-\breve{g}$-soft closed, to the contrary, it is not true.
2) $(1,2)-gsg$-soft closed implies $(1,2)-\breve{g}$ soft closed, to the contrary, it is false.

Theorem 8.43. *If a subset $\mathcal{F}_A^s$ of $\mathcal{X}$ is $(1,2)-\breve{g}$-soft closed. Then, $\tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s) \setminus \mathcal{F}_A^s$ does n't contain any non-void $\tau_1^s\tau_2^s$-soft closed set.*

Proof. Let $\mathcal{F}_B^s$ be $\tau_1^s\tau_2^s$-soft closed subset of $\tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s) \setminus \mathcal{F}_A^s$. Then, $\mathcal{F}_B^s \sqsubseteq \tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s)$ and $\mathcal{F}_B^s \cap \mathcal{F}_A^s = \emptyset$.
Therefore, $\mathcal{X} \setminus \mathcal{F}_B^s$ is $\tau_1^s\tau_2^s$-soft open set.
Since $\mathcal{F}_B^s \cap \mathcal{F}_A^s = \emptyset$, $\mathcal{F}_A^s \sqsubset \mathcal{X} \setminus \mathcal{F}_B^s$. But $\mathcal{F}_A^s$ is $(1,2)-\breve{g}$-soft closed, then, $\tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s) \sqsubseteq \mathcal{X} \setminus \mathcal{F}_B^s$ and consequently, $\mathcal{F}_B^s \sqsubseteq \mathcal{X}\setminus$ $\tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s)$.
Therefore, $\mathcal{F}_B^s \sqsubseteq \tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s)$ $\cap\mathcal{X}\setminus$ $\tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s)$.
Hence, $\mathcal{F}_B^s$ is empty. □

Theorem 8.44. *Let $\mathcal{F}_A^s$ be a $(1,2)-\breve{g}$-soft closed subset of $\mathcal{X}$. If $\mathcal{F}_A^s \sqsubseteq \mathcal{F}_B^s \sqsubset \tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s)$. Then, $\mathcal{F}_B^s$ is also $(1,2)-\breve{g}$-soft closed.*

Proof. We suppose $\mathcal{U}^s$ be a $\tau_1^s\tau_2^s$-soft open set with $\mathcal{F}_B^s \sqsubseteq \mathcal{U}^s$. Then, $\mathcal{F}_A^s \sqsubseteq \mathcal{U}^s$.
Since $\mathcal{F}_A^s$ is $(1,2)-\breve{g}$-soft closed, $\tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s) \sqsubseteq \mathcal{U}^s$.
Since $\mathcal{F}_B^s \sqsubseteq \tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s) \sqsubseteq \mathcal{U}^s$, this implies $\tau_1^s\tau_2^s$-cl$(\mathcal{F}_B^s) \sqsubseteq \tau_1^s\tau_2^s$-cl$(\mathcal{F}_A^s) \sqsubseteq \mathcal{U}^s$.
Thus, $\mathcal{F}_B^s$ is also $(1,2)-\breve{g}$-soft closed. □

8.0.4 Application of Alexandroff Soft Bitopological Approximations using $(1,2)-\breve{g}$-soft closed set in data reduction problems

In this section, we used Pawlak's rough approximation theory to produce new set approximations in bitopology using $(1,2)-\breve{g}$-soft closed sets. We studied various properties of rough sets on Alexandroff Soft bitopological approximation spaces. Finally, we give an application of these approximations to data reduction in multi-valued information systems.

8.0.4.1 Brief Description

Pawlak is deemed as the originator of Rough set theory [98]. This theory is the result of applying pure mathematics to the study of data systems that are characterised by ambiguity and uncertainty. This is often the new mathematical tool to traumatize soft computing still as uncertainty of information besides fuzzy pure mathematics. Attribute reduction is usually being a haul in data systems. Recently, this problem has become a major concern among more and more researchers. Rough approximations are used to deal with this problem. It has been successfully applied in various fields like artificial intelligence, machine learning, pattern recognition, decision analysis, cognitive sciences, intelligent decision making and process control etc. Zhi Pei *et al.* [99] in 2007 explored the relationship between topology and generalized rough sets. In [100] Abu-Donia mentioned three varieties of lower and higher approximations of any set with relevancy any relation supported right neighborhood and generalized these three varieties of approximations into two ways that employing a finite variety of any binary relations. This paper studies a number of basic ideas of rough pure mathematics by employing a finite family of any (reflexive,tolerance, dominance, equivalence) relations. A.S Salama [101] worked on reduction of data sets in information system by using topology with rough sets. Abu-Donia with Salama have generalized the classical rough approximation spaces using topological near open sets called $\delta\beta$-open sets [102, 103]. Salama [51] initiated new idea of lower and upper approximations by using two topological structures and used these approximations as an application. She has used bitopological approximations to resolve the matter of attribute reduction in multi-valued information systems in her paper [92].

Likewise, we used this conception of approximations in Alexandroff Bitopological spaces with the usage of $(1,2)-\breve{g}$-soft closed set in data reduction in multi-valued information system.

8.0.4.2 Alexandroff Soft Bitopological Approximation Spaces

The essential rough set approximations lower and upper coincide with the topological interior and the closure operation respectively. The topological interior and closure of a subset $\mathcal{G} \sqsubset \mathcal{U}$ are defined as follows:

1) $\mathcal{G}^{\circ} = \sqcup\{\mathcal{H} \sqsubseteq \mathcal{U} : \mathcal{H} \in \tau^{s}, \mathcal{H} \subseteq \mathcal{G}\}$.

2) $\bar{\mathcal{G}} = \sqcap\{\mathcal{I} \sqsubseteq \mathcal{U} : \mathcal{H} \in (\tau^{s})^{c}, \mathcal{G} \subseteq \mathcal{I}\}$.

The partition characterizes a topological space, called Approximation space $\mathcal{G} = (\mathcal{U}, \mathcal{R})$ where $\mathcal{U}$ is called the universe and $\mathcal{R}$ is an equivalence relation [101, 104]. The equivalence class $[x]_{\mathcal{R}}$ is the key apparatus for defining rough approximations. The lower and upper approximations are defined as follows:

1) $\underline{\mathcal{R}}(\mathcal{A}) = \{x \in \mathcal{U} : [x]_{\mathcal{R}} \sqsubseteq \mathcal{A}\}$.

2) $\overline{\mathcal{R}}(\mathcal{A}) = \{x \in \mathcal{U} : [x]_{\mathcal{R}} \sqcap \mathcal{A} = \emptyset\}$.

Also, the positive region and negative region are $POS_{\mathcal{R}}(\mathcal{A}) = \underline{\mathcal{R}}(\mathcal{A})$ and $NEG_{\mathcal{R}}(\mathcal{A}) = \mathcal{U} \setminus \overline{\mathcal{R}}(\mathcal{A})$ respt. Thus, borderline region is $BN_{\mathcal{R}}(\mathcal{A}) = \underline{\mathcal{R}}(\mathcal{A}) \setminus \overline{\mathcal{R}}(\mathcal{A})$ of $\mathcal{U}$. The accuracy measures are used to discover the degree of completeness of knowledge. Pawlak defined the accuracy measure as below:

$\alpha_{\mathcal{R}}(\mathcal{A}) = \frac{|\underline{\mathcal{R}}(\mathcal{A})|}{|\overline{\mathcal{R}}(\mathcal{A})|}$ where $\mathcal{A} \neq \emptyset$ and $|\mathcal{U}|$ is the cardinality of $\mathcal{U}$.

A topological base generated by binary relations can produce a lower and upper approximations. The family of all right blocks $\mathcal{S}_{\mathcal{R}} = \{\mathcal{R}_{i-\mathcal{R}}(x); x \in \mathcal{U}\}$ is a subbase for topology $\tau^{s}_{\mathcal{R}}$ on the universal set. Similarly, the family of all left blocks $\mathcal{S}_{\mathcal{L}} = \{\mathcal{R}_{i-\mathcal{L}}(x); x \in \mathcal{U}\}$ is a subbase for another topology $\tau^{s}_{\mathcal{L}}$. Thus, the approximation space $BIT^{s} = (\mathcal{U}, \mathcal{R}, \tau^{s}_{\mathcal{R}}, \tau^{s}_{\mathcal{L}})$ is known as Alexandroff Soft Bitopological Approximation Space(ASBAS).

A subset $\mathcal{A}^{s}$ of $\mathcal{U}$ is known as $(1,2)-\breve{g}$-soft closed set if $\tau^{s}_{\mathcal{R}}\tau^{s}_{\mathcal{L}}$-cl- $\mathcal{A}^{s} \sqsubseteq \mathcal{X}^{s}$, whenever $\mathcal{A}^{s} \sqsubseteq \mathcal{X}^{s}$ and $\mathcal{X}^{s}$ is $\tau^{s}_{\mathcal{R}}\tau^{s}_{\mathcal{L}}$-soft open in $\mathcal{U}$. The complement of $(\mathcal{R},\mathcal{L})-\breve{g}$- soft closed set is $(\mathcal{R},\mathcal{L})-\breve{g}$-soft open in Alexandroff Soft bitopological approximation space $BIT^{s} = (\mathcal{U}, \mathcal{R}, \tau^{s}_{\mathcal{R}}, \tau^{s}_{\mathcal{L}})$. The family of all $(\mathcal{R},\mathcal{L})-\breve{g}$-open is denoted by $(\breve{g})\mathcal{O}(\mathcal{U})$ and the family of all closed $(\mathcal{R},\mathcal{L})-\breve{g}$ sets is denoted by $(\breve{g})\mathcal{C}(\mathcal{U})$

Now, let $BIT^{s} = (\mathcal{U}, \mathcal{R}, \tau^{s}_{\mathcal{R}}, \tau^{s}_{\mathcal{L}})$ be an Alexandroff Soft Bitopological Space. $\mathcal{R}$ lower approximation and $\mathcal{R}$ upper approximation of subset $\mathcal{A} \sqsubseteq \mathcal{U}$ is defined as-

1) $\underline{BIT}^s_{\mathcal{R}}(\mathcal{A}) = \sqcup\{\mathcal{B} \in \mathcal{RO}(\mathcal{U}) : \mathcal{B} \sqsubseteq \mathcal{A}\}$

2) $\overline{BIT}^s_{\mathcal{R}}(\mathcal{A}) = \sqcap\{\mathcal{G} \in \mathcal{RC}(\mathcal{U}) : \mathcal{A} \sqsubseteq \mathcal{G}\}$.

Let $BIT^s = (\mathcal{U}, \mathcal{R}, \tau^s_{\mathcal{R}}, \tau^s_{\mathcal{L}})$ be an Alexandroff Soft Bitopological space. $\mathcal{L}$ lower approximation and $\mathcal{L}$ upper approximation of subset $\mathcal{A} \sqsubseteq \mathcal{U}$ is defined as-

1) $\underline{BIT}^s_{\mathcal{L}}(\mathcal{A}) = \sqcup\{\mathcal{B} \in \mathcal{LO}(\mathcal{U}) : \mathcal{B} \sqsubseteq \mathcal{A}\}$

2) $\overline{BIT}^s_{\mathcal{L}}(\mathcal{A}) = \sqcap\{\mathcal{G} \in \mathcal{LC}(\mathcal{U}) : \mathcal{A} \sqsubseteq \mathcal{G}\}$.

Let $BIT^s = (\mathcal{U}, \mathcal{R}, \tau^s_{\mathcal{R}}, \tau^s_{\mathcal{L}})$ be a Alexandroff Soft Bitopological space. $\breve{g}$ lower approximation and $\breve{g}$ upper approximation of subset $\mathcal{A} \sqsubseteq \mathcal{U}$ is defined as-

1) $\underline{BIT}^s_{\breve{g}}(\mathcal{A}) = \sqcup\{\mathcal{B} \in \breve{g}\mathcal{O}(\mathcal{U}) : \mathcal{B} \sqsubseteq \mathcal{A}\}$

2) $\overline{BIT}^s_{\breve{g}}(\mathcal{A}) = \sqcap\{\mathcal{G} \in \breve{g}\mathcal{C}(\mathcal{U}) : \mathcal{A} \sqsubseteq \mathcal{G}\}$.

The accuracy measures for the above defined approximations are specified as below:
$\alpha_{\mathcal{R}}(\mathcal{A}) = \frac{|\mathcal{A} \sqcap \underline{BIT}^s_{\mathcal{R}}(\mathcal{A})|}{|\mathcal{A} \sqcup \overline{BIT}^s_{\mathcal{R}}(\mathcal{A})|}$, $\alpha_{\mathcal{L}}(\mathcal{A}) = \frac{|\mathcal{A} \sqcap \underline{BIT}^s_{\mathcal{L}}(\mathcal{A})|}{|\mathcal{A} \sqcup \overline{BIT}^s_{\mathcal{L}}(\mathcal{A})|}$, $\alpha_{\breve{g}}(\mathcal{A}) = \frac{|\mathcal{A} \sqcap \underline{BIT}^s_{\breve{g}}(\mathcal{A})|}{|\mathcal{A} \sqcup \overline{BIT}^s_{\breve{g}}(\mathcal{A})|}$.

Example 8.2. *Let $\mathcal{U} = \{1, 2, 3, 4, 5\}$ be a universal set and three relations $\mathcal{R}$ on $\mathcal{U}$ are defined as:*

$\mathcal{R}_1 = \{(1, 1), (1, 5), (2, 3), (2, 4), (3, 5), (4, 1), (4, 5), (5, 5)\}$.

$\mathcal{R}_2 = \{(1, 1), (1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (3, 5), (4, 1), (4, 5), (5, 5)\}$.

$\mathcal{R}_3 = \{(1, 1), (1, 5), (2, 3), (2, 4), (3, 3), (3, 5), (4, 1), (4, 5), (4, 4), (5, 5)\}$.

Then, $\mathcal{S}_{\mathcal{R}} = \{\{1, 5\}, \{3, 4\}, \{5\}\}$.

$\mathcal{S}_{\mathcal{L}} = \{\{1, 4\}, \{1, 3, 4, 5\}, \{2\}\}$.

Thus, the topologies generated with these relations are-

1) $\tau^s_{\mathcal{R}} = \{\mathcal{U}, \emptyset, \{1, 5\}, \{3, 4\}, \{5\}, \{3, 4, 5\}, \{1, 3, 4, 5\}\}$

2) $\tau^s_{\mathcal{L}} = \{\mathcal{U}, \emptyset, \{1, 4\}, \{2\}, \{1, 2, 4\}, \{1, 3, 4, 5\}\}$.

Three types of accuracies $\alpha_{\mathcal{R}}(\mathcal{A}), \alpha_{\mathcal{L}}(\mathcal{A}), \alpha_{\breve{g}}(\mathcal{A})$ for some subsets of $\mathcal{U}$ are calculated as below in the table 8.1.

$\mathcal{A} \sqsubseteq \mathcal{U}$	$\alpha_{\mathcal{R}}(\mathcal{A})$	$\alpha_{\mathcal{L}}(\mathcal{A})$	$\alpha_{\breve{g}}(\mathcal{A})$
$\{2, 5\}$	1/3	1/3	1
$\{1, 2, 5\}$	2/3	1	1
$\{1, 3, 4\}$	1/2	2/3	1
$\{3, 4, 5\}$	3/5	3/4	1
$\{1, 3, 4, 5\}$	4/5	4/5	1
$\{2, 3, 4, 5\}$	3/5	4/5	1

TABLE 8.1: Three types of accuracy measures

We see that by using $\alpha_{\breve{g}}$ accuracy measure the degree of exactness equals to 1, which consequently means $\breve{g}$ accuracy measure is the best accuracy measure among all.

For the above approximations, any subset $\mathcal{A} \sqsubseteq \mathcal{U}$ has the following regions:

1) The $\mathcal{R}$internal edge, $\underline{Edg}_{\mathcal{R}}(\mathcal{A}) = \mathcal{A} - \underline{BIT}^s_{\mathcal{R}}(\mathcal{A})$.

2) The $\mathcal{L}$internal edge, $\underline{Edg}_{\mathcal{L}}(\mathcal{A}) = \mathcal{A} - \underline{BIT}^s_{\mathcal{L}}(\mathcal{A})$.

3) The $\breve{g}$internal edge, $\underline{Edg}_{\breve{g}}(\mathcal{A}) = \mathcal{A} - \underline{BIT}^s_{\breve{g}}(\mathcal{A})$.

4) The $\mathcal{R}$external edge, $\overline{Edg}_{\mathcal{R}}(\mathcal{A}) = \mathcal{A} - \overline{BIT}^s_{\mathcal{R}}(\mathcal{A})$.

5) The $\mathcal{L}$external edge, $\overline{Edg}_{\mathcal{L}}(\mathcal{A}) = \mathcal{A} - \overline{BIT}^s_{\mathcal{L}}(\mathcal{A})$.

6) The $\breve{g}$external edge, $\overline{Edg}_{\breve{g}}(\mathcal{A}) = \mathcal{A} - \overline{BIT}^s_{\breve{g}}(\mathcal{A})$.

7) The $\mathcal{R}$boundary, $BON_{\mathcal{R}}(\mathcal{A}) = \overline{BIT}^s_{\mathcal{R}}(\mathcal{A}) - \underline{BIT}^s_{\mathcal{R}}(\mathcal{A})$.

8) The $\mathcal{L}$boundary, $BON_{\mathcal{L}}(\mathcal{A}) = \overline{BIT}^s_{\mathcal{L}}(\mathcal{A}) - \underline{BIT}^s_{\mathcal{L}}(\mathcal{A})$.

9) The $\breve{g}$boundary, $BON_{\breve{g}}(\mathcal{A}) = \overline{BIT}^s_{\breve{g}}(\mathcal{A})$ - $\underline{BIT}^s_{\breve{g}}(\mathcal{A})$.

Proposition 8.45. *For $BIT^s = (\mathcal{U}, \mathcal{R}, \tau^s_{\mathcal{R}}, \tau^s_{\mathcal{L}})$ and a subset $\mathcal{A} \sqsubseteq \mathcal{U}$, we have:*

1) The $\mathcal{R}$boundary, $BON_{\mathcal{R}}(\mathcal{A}) = \underline{Edg}_{\mathcal{R}}(\mathcal{A}) \sqcup \overline{Edg}_{\mathcal{R}}(\mathcal{A})$.

2) The $\mathcal{L}$boundary, $BON_{\mathcal{L}}(\mathcal{A}) = \underline{Edg}_{\mathcal{L}}(\mathcal{A}) \sqcup \overline{Edg}_{\mathcal{L}}(\mathcal{A})$.

3) The $\breve{g}$boundary, $BON_{\breve{g}}(\mathcal{A}) = \underline{Edg}_{\breve{g}}(\mathcal{A}) \sqcup \overline{Edg}_{\breve{g}}(\mathcal{A})$.

Proof. 1) $BON_{\mathcal{R}}(\mathcal{A}) = \overline{BIT}^s_{\mathcal{R}}(\mathcal{A})$ - $\underline{BIT}^s_{\mathcal{R}}(\mathcal{A}) = (\overline{BIT}^s_{\mathcal{R}}(\mathcal{A})$ - $\mathcal{A}) \sqcup (\mathcal{A}$ - $\underline{BIT}^s_{\mathcal{R}}(\mathcal{A}))$ but $\overline{Edg}_{\mathcal{R}}(\mathcal{A}) = \mathcal{A} - \overline{BIT}^s_{\mathcal{R}}(\mathcal{A})$ and $\underline{Edg}_{\mathcal{R}}(\mathcal{A}) = \mathcal{A} - \underline{BIT}^s_{\mathcal{R}}(\mathcal{A})$, thus we have $BON_{\mathcal{R}}(\mathcal{A}) = \underline{Edg}_{\mathcal{R}}(\mathcal{A}) \sqcup \overline{Edg}_{\mathcal{R}}(\mathcal{A})$.

Similarly, using the above approximations, second and third part can be proved. □

Also, there is a connection between classical lower and upper approximations and that of our approach, the following proposition give the connection between them, where $\underline{\mathcal{R}}(\mathcal{A})$ and $\overline{\mathcal{R}}(\mathcal{A})$ are the classical lower and upper approximations of rough sets.

Proposition 8.46. *For any Alexandroff Soft bitopological approximation space, $BIT^s = (\mathcal{U}, \mathcal{R}, \tau^s_{\mathcal{R}}, \tau^s_{\mathcal{L}})$ and a subset $\mathcal{S} \sqsubseteq \mathcal{U}$, we have:*

1) $\overline{\mathcal{R}}(\mathcal{S})$ - $\underline{BIT}^s_{\mathcal{R}}(\mathcal{S}) = \overline{Edg}(\mathcal{S}) \sqcup \underline{Edg}_{\mathcal{R}}(\mathcal{S})$.

2) $\overline{\mathcal{R}}(\mathcal{S})$ - $\underline{BIT}^s_{\mathcal{L}}(\mathcal{S}) = \overline{Edg}(\mathcal{S}) \sqcup \underline{Edg}_{\mathcal{L}}(\mathcal{S})$.

3) $\overline{\mathcal{R}}(\mathcal{S})$ - $\underline{BIT}^s_{\breve{g}}(\mathcal{S}) = \overline{Edg}(\mathcal{S}) \sqcup \underline{Edg}_{\breve{g}}(\mathcal{S})$.

4) $\overline{BIT}^s_{\mathcal{R}}(\mathcal{S})$ - $\underline{\mathcal{R}}(\mathcal{S}) = \overline{Edg}_{\mathcal{R}}(\mathcal{S}) \sqcup \underline{Edg}(\mathcal{S})$.

5) $\overline{BIT}^s_{\mathcal{L}}(\mathcal{S})$ - $\underline{\mathcal{R}}(\mathcal{S}) = \overline{Edg}_{\mathcal{L}}(\mathcal{S}) \sqcup \underline{Edg}(\mathcal{S})$.

6) $\overline{BIT}^s_{\breve{g}}(\mathcal{S})$ - $\underline{\mathcal{R}}(\mathcal{S}) = \overline{Edg}_{\breve{g}}(\mathcal{S}) \sqcup \underline{Edg}(\mathcal{S})$.

For any Alexandroff Soft Bitopological approximation space $BIT^s = (\mathcal{U}, \mathcal{R}, \tau^s_{\mathcal{R}}, \tau^s_{\mathcal{L}})$, a subset $\mathcal{A} \sqsubseteq \mathcal{U}$ is called-

1) $\mathcal{R}$ definable set if $\overline{BIT}^s_{\mathcal{R}}(\mathcal{A}) = \underline{BIT}^s_{\mathcal{R}}(\mathcal{A})$ or $BON_{\mathcal{R}}(\mathcal{A}) = \emptyset$.

2) $\mathcal{L}$ definable set if $\overline{BIT}^s_{\mathcal{L}}(\mathcal{A}) = \underline{BIT}^s_{\mathcal{L}}(\mathcal{A})$ or $BON_{\mathcal{L}}(\mathcal{A}) = \emptyset$.

3) $\breve{g}$ definable set if $\overline{BIT}^s_{\breve{g}}(\mathcal{A}) = \underline{BIT}^s_{\breve{g}}(\mathcal{A})$ or $BON_{\breve{g}}(\mathcal{A}) = \emptyset$.

4) $\mathcal{R}$ rough if $\overline{BIT}^s_{\mathcal{R}}(\mathcal{A}) \neq \underline{BIT}^s_{\mathcal{R}}(\mathcal{A})$ or $BON_{\mathcal{R}}(\mathcal{A}) \neq \emptyset$.

5) $\mathcal{L}$ rough if $\overline{BIT}^s_{\mathcal{L}}(\mathcal{A}) \neq \underline{BIT}^s_{\mathcal{L}}(\mathcal{A})$ or $BON_{\mathcal{L}}(\mathcal{A}) \neq \emptyset$.

6) $\breve{g}$ rough if $\overline{BIT}^s_{\breve{g}}(\mathcal{A}) \neq \underline{BIT}^s_{\breve{g}}(\mathcal{A})$ or $BON_{\breve{g}}(\mathcal{A}) \neq \emptyset$.

7) Roughly $\mathcal{R}$ definable, if $\underline{BIT}^s_{\mathcal{R}}(\mathcal{A}) \neq \emptyset and \overline{BIT}^s_{\mathcal{R}}(\mathcal{A}) \neq \mathcal{U}$.

8) Roughly $\mathcal{L}$ definable, if $\underline{BIT}^s_{\mathcal{L}}(\mathcal{A}) \neq \emptyset and \overline{BIT}^s_{\mathcal{L}}(\mathcal{A}) \neq \mathcal{U}$.

9) Roughly $\breve{g}$ definable, if $\underline{BIT}^s_{\breve{g}}(\mathcal{A}) \neq \emptyset$ and $\overline{BIT}^s_{\breve{g}}(\mathcal{A}) \neq \mathcal{U}$.

10) Internally $\mathcal{R}$ undefinable, if $\underline{BIT}^s_{\mathcal{R}}(\mathcal{A}) = \emptyset$ and $\overline{BIT}^s_{\mathcal{R}}(\mathcal{A}) \neq \mathcal{U}$.

11) Internally $\mathcal{L}$ undefinable, if $\underline{BIT}^s_{\mathcal{L}}(\mathcal{A}) = \emptyset$ and $\overline{BIT}^s_{\mathcal{L}}(\mathcal{A}) \neq \mathcal{U}$.

12) Internally $\breve{g}$ undefinable, if $\underline{BIT}^s_{\breve{g}}(\mathcal{A}) = \emptyset$ and $\overline{BIT}^s_{\breve{g}}(\mathcal{A}) \neq \mathcal{U}$.

13) Externally $\mathcal{R}$ undefinable, if $\underline{BIT}^s_{\mathcal{R}}(\mathcal{A}) \neq \emptyset$ and $\overline{BIT}^s_{\mathcal{R}}(\mathcal{A}) = \mathcal{U}$.

14) Externally $\mathcal{L}$ undefinable, if $\underline{BIT}^s_{\mathcal{L}}(\mathcal{A}) \neq \emptyset$ and $\overline{BIT}^s_{\mathcal{L}}(\mathcal{A}) = \mathcal{U}$.

15) Externally $\breve{g}$ undefinable, if $\underline{BIT}^s_{\breve{g}}(\mathcal{A}) \neq \emptyset$ and $\overline{BIT}^s_{\breve{g}}(\mathcal{A}) = \mathcal{U}$.

16) Totally $\mathcal{R}$ undefinable, if $\underline{BIT}^s_{\mathcal{R}}(\mathcal{A}) = \emptyset$ and $\overline{BIT}^s_{\mathcal{R}}(\mathcal{A}) = \mathcal{U}$.

17) Totally $\mathcal{L}$ undefinable, if $\underline{BIT}^s_{\mathcal{L}}(\mathcal{A}) = \emptyset$ and $\overline{BIT}^s_{\mathcal{L}}(\mathcal{A}) = \mathcal{U}$.

18) Totally $\breve{g}$ undefinable, if $\underline{BIT}^s_{\breve{g}}(\mathcal{A}) = \emptyset$ and $\overline{BIT}^s_{\breve{g}}(\mathcal{A}) = \mathcal{U}$.

Proposition 8.47. *[92] For any space $BIT^s = (\mathcal{U}, \mathcal{R}, \tau^s_{\mathcal{R}}, \tau^s_{\mathcal{L}})$ and $\forall x, y \in \mathcal{U}$, we have:*
1) $\overline{BIT}_{\mathcal{R}}(x) = \overline{BIT}_{\mathcal{R}}(y)$, if $x \in \overline{BIT}_{\mathcal{R}}(y)$ & $y \in \overline{BIT}_{\mathcal{R}}(x)$.
2) $\overline{BIT}_{\mathcal{L}}(x) = \overline{BIT}_{\mathcal{L}}(y)$, if $x \in \overline{BIT}_{\mathcal{L}}(y)$ & $y \in \overline{BIT}_{\mathcal{L}}(x)$.
3) $\overline{BIT}_{\breve{g}}(x) = \overline{BIT}_{\breve{g}}(y)$, if $x \in \overline{BIT}_{\breve{g}}(y)$ & $y \in \overline{BIT}_{\breve{g}}(x)$.

Proof. Since $\overline{BIT}_{\mathcal{R}}(x)$ is the smallest $\mathcal{R}$ closed set containing x and $cl_{\tau_{\mathcal{R}}}(y)$ is $\mathcal{R}$ closed set containing x (by the definition of $\mathcal{R}$ upper approximation), thus $\overline{BIT}_{\mathcal{R}}(x) \sqsubseteq \overline{BIT}_{\mathcal{R}}(y)$ and by symmetry, $\overline{BIT}_{\mathcal{R}}(y) \sqsubseteq \overline{BIT}_{\mathcal{R}}(x)$. Similarly, second and third part can be solved. □

8.0.5 Data devaluation in Multi-esteemed information frameworks using Alexandroff Soft Bitopological Approximation Spaces

In this section, we reduce the attributes in information system with the use of Alexandroff Soft Bitopological Approximation Spaces.

A decision table consists of independent attributes named condition attributes and dependent attributes are called decision attributes.

By devaluation of the information table, we mean a smaller subset $\mathcal{R} \sqsubseteq \mathcal{S}$ of attributes that safeguard the nature of approximations.
The subset $\mathcal{R}' \sqsubset \mathcal{R} \sqsubseteq \mathcal{S}$ is a reduct as for $BIT^s = (\mathcal{U}, \mathcal{C}, \tau^s_{\mathcal{R}}, \tau^s_{\mathcal{L}})$ if it is a minimal subset of $\mathcal{B}$ which keeps the nature of order unchanged.
Dependency of attributes plays a crucial role in information system. The set of all attributes $\mathcal{A} \sqsubseteq \mathcal{C}$ depends totally on the set of attributes $\mathcal{B} \sqsubseteq \mathcal{C}$, denoted by $\mathcal{B} \to \mathcal{A}$ if the set of all values of attributes from $\mathcal{A}$ are contained in the values of attributes from $\mathcal{B}$.
Let us consider $\mathcal{R}$ and $\mathcal{S}$ be subsets of $\mathcal{S}$. Then, dependency can be defined as-
$\mathcal{R}$ depends on $\mathcal{S}$ with reference to $\tau^s_{\mathcal{R}}$ if $\gamma_{\mathcal{R}}(\mathcal{R}, \mathcal{S}) = \frac{|POS_{\mathcal{R}}(\mathcal{R})|}{|\mathcal{U}|}$, and concerning $\tau^s_{\mathcal{L}}$ if $\gamma_{\mathcal{L}}(\mathcal{R}, \mathcal{S}) = \frac{|POS_{\mathcal{L}}(\mathcal{R})|}{|\mathcal{U}|}$.

Example 8.3. *Consider the multi-valued information system given in the table below: Let the set* $\mathcal{X} = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7\}$ *be the set of objects and* $\mathcal{C} = \{D_1, D_2, D_3\}$ *be condition attributes and Decision* $= \mathcal{D}$ *is the decision attribute. In this, we need to determine the condition attribute that support the decision attribute.*

$\mathcal{D}$	$\mathcal{C}_3$	$\mathcal{C}_2$	$\mathcal{C}_1$	$\mathcal{X}$
Yes	$\{c\}$	$\{a,b,c\}$	$\{a_0\}$	$\mathfrak{x}_1\mathfrak{x}_1$
No	$\{c,d\}$	$\{a,b\}$	$\{a_0,a\}$	$\mathfrak{x}_2$
Yes	$\{c\}$	$\{a,c\}$	$\{b\}$	$\mathfrak{x}_3$
No	$\{d\}$	$\{a,c,d\}$	$\{a\}$	$\mathfrak{x}_4$
$Maybe$	$\{c,d\}$	$\{d\}$	$\{a\}$	$\mathfrak{x}_5$
Yes	$\{c\}$	$\{a,b\}$	$\{a,b\}$	$\mathfrak{x}_6$
$Maybe$	$\{c,d\}$	$\{a,b,c\}$	$\{a_0,b\}$	$\mathfrak{x}_7$

TABLE 8.2: Multi-valued Information System

Now, we define a binary relations on $\mathcal{X}$ *as follows:*
$\mathcal{R}_{\mathcal{P}\sqsubseteq\mathcal{Q}} = \{(p,q) : f_{\mathcal{P}\sqsubseteq\mathcal{Q}}(p) \sqsubseteq f_{\mathcal{P}\sqsubseteq\mathcal{Q}}(q), \forall \mathcal{P} \sqsubseteq \mathcal{Q}, \mathcal{P} \neq \emptyset, \forall\ p,q \in \mathcal{X}\}$.

Using this relation, we construct relations on condition attributes and then generate two topologies we need in the reduction process which are given by:

$\tau^s_{\mathcal{R}} = \{\mathcal{X}, \emptyset, \{x_2\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}, \{x_3, x_7\},$
$\{x_2, x_4\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_2, x_5\}, \{x_2, x_6\}, \{x_2, x_7\},$
$\{x_4, x_5\}, \{x_5, x_6\}, \{x_5, x_7\}, \{x_6, x_7\}, \{x_3, x_4, x_7\},$
$\{x_1, x_4, x_7\}, \{x_2, x_3, x_7\}, \{x_1, x_2, x_7\}, \{x_3, x_5, x_7\},$
$\{x_1, x_5, x_7\}, \{x_3, x_6, x_7\}, \{x_1, x_6, x_7\}, \{x_1, x_3, x_7\},$
$\{x_2, x_4, x_5\}, \{x_2, x_4, x_6\}, \{x_2, x_4, x_7\},$
$\{x_4, x_5, x_6\}, \{x_4, x_5, x_7\}, \{x_2, x_3, x_4, x_7\},$
$\{x_1, x_2, x_4, x_7\}, \{x_3, x_4, x_5, x_7\},$
$\{x_1, x_4, x_5, x_7\}, \{x_2, x_4, x_5, x_6, x_7\},$
$\{x_2, x_3, x_4, x_5, x_6, x_7\}, \{x_1, x_2, x_4, x_5, x_6, x_7\}\}$
$\tau^s_{\mathcal{L}} = \{\mathcal{X}, \emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_1, x_3, x_7\},$
$\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_1, x_6\}, \{x_2, x_3\}$
$\{x_2, x_4\}, \{x_2, x_5\}, \{x_1, x_6\}, \{x_3, x_4\}, \{x_3, x_5\},$
$\{x_3, x_6\}, \{x_1, x_2, x_3, x_7\}, \{x_4, x_5\}, \{x_4, x_6\},$
$\{x_1, x_3, x_4, x_7\}, \{x_5, x_6\}, \{x_1, x_3, x_5, x_7\},$
$\{x_1, x_3, x_6, x_7\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\},$
$\{x_1, x_2, x_6\}, \{x_1, x_3, x_4\},$
$\{x_1, x_3, x_5\}, \{x_1, x_3, x_6\}, \{x_1, x_4, x_5\},$
$\{x_1, x_4, x_6\}, \{x_1, x_5, x_6\}, \{x_2, x_3, x_4\}, \{x_2, x_3, x_5\},$
$\{x_2, x_3, x_6\}, \{x_2, x_4, x_5\}, \{x_2, x_4, x_6\},$
$\{x_1, x_2, x_3, x_5, x_7\}, \{x_1, x_2, x_3, x_6, x_7\}, \{x_3, x_4, x_6\}, \{x_4, x_5, x_6\},$
$\{x_1, x_3, x_4, x_5, x_7\}, \{x_1, x_3, x_5, x_6, x_7\}, \{x_1, x_2, x_3, x_4\},$
$\{x_1, x_2, x_3, x_5\}, \{x_1, x_2, x_3, x_6\},$
$\{x_2, x_3, x_4, x_5\}, \{x_2, x_3, x_4, x_6\},$
$\{x_1, x_2, x_3, x_4, x_7\}, \{x_3, x_4, x_5, x_6\},$
$\{x_1, x_2, x_3, x_4, x_5\}, \{x_2, x_3, x_4, x_5, x_6\},$
$\{x_1, x_2, x_3, x_4, x_6\}, \{x_1, x_2, x_3, x_4, x_5, x_6\}$

These topologies can be considered as the basic knowledge for our sysytem which can generate decision rules.

The discernible subsets of the decision attribute are:

$\mathcal{D}1$ = *Decision(Yes)* = $\{x_1, x_3, x_6\}$

$\mathcal{D}2 = Decision(No) = \{\mathfrak{x}_2,,\mathfrak{x}_4\}$

$\mathcal{D}3 = DecisionMaybe = \{\mathfrak{x}_5, \mathfrak{x}_7\}$

After many calculations using the topologies right and left, the $\mathcal{R}$lowerapproximations and $\mathcal{R}$upper approximation of the subset $\mathcal{D}_i$ are given by:

$\underline{BIT}^s_{\mathcal{R}}(\mathcal{D}_1) = \{\mathfrak{x}_1, \mathfrak{x}_6\}$, $\underline{BIT}^s_{\mathcal{R}}(\mathcal{D}_1) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_6, \mathfrak{x}_7\}$, *then* $BON^s_{\mathcal{R}}(\mathcal{D}_1) = \{\mathfrak{x}_3, \mathfrak{x}_7\}$.

$\underline{BIT}^s_{\mathcal{R}}(\mathcal{D}_2) = \{\mathfrak{x}_2, \mathfrak{x}_4\}$, $\underline{BIT}^s_{\mathcal{R}}(\mathcal{D}_2) = \{\mathfrak{x}_2, \mathfrak{x}_4\}$, *then* $BON^s_{\mathcal{R}}(\mathcal{D}_2) = \emptyset$.

$\underline{BIT}^s_{\mathcal{R}}(\mathcal{D}_3) = \{\mathfrak{x}_5, \mathfrak{x}_7\}$, $\underline{BIT}^s_{\mathcal{R}}(\mathcal{D}_3) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7\}$, *then* $BON^s_{\mathcal{R}}(\mathcal{D}_3) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_6\}$.

Also, the $\mathcal{L}$lower approximation and $\mathcal{L}$upper approximation are given by:

$\underline{BIT}^s_{\mathcal{L}}(\mathcal{D}_1) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_6\}$, $\underline{BIT}^s_{\mathcal{L}}(\mathcal{D}_1) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_6, \mathfrak{x}_7\}$, *then* $BON^s_{\mathcal{L}}(\mathcal{D}_1) = \{\mathfrak{x}_7\}$.

$\underline{BIT}^s_{\mathcal{L}}(\mathcal{D}_2) = \{\mathfrak{x}_2, \mathfrak{x}_4\}$, $\underline{BIT}^s_{\mathcal{L}}(\mathcal{D}_2) = \{\mathfrak{x}_2, \mathfrak{x}_4\}$, *then* $BON^s_{\mathcal{L}}(\mathcal{D}_2) = \emptyset$.

$\underline{BIT}^s_{\mathcal{L}}(\mathcal{D}_3) = \{\mathfrak{x}_5, \mathfrak{x}_7\}$, $\underline{BIT}^s_{\mathcal{L}}(\mathcal{D}_3) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_7\}$, *then* $BON^s_{\mathcal{L}}(\mathcal{D}_3) = \{\mathfrak{x}_1, \mathfrak{x}_3\}$.

Also, the $\breve{g}$lower approximation and $\breve{g}$upper approximation are given below:

$\underline{BIT}^s_{\breve{g}}(\mathcal{D}_1) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_6\}$, $\underline{BIT}^s_{\breve{g}}(\mathcal{D}_1) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_6, \mathfrak{x}_7\}$, *then* $BON^s_{\breve{g}}(\mathcal{D}_1) = \{\mathfrak{x}_7\}$.

$\underline{BIT}^s_{\breve{g}}(\mathcal{D}_2) = \{\mathfrak{x}_2, \mathfrak{x}_4\}$, $\underline{BIT}^s_{\breve{g}}(\mathcal{D}_2) = \{\mathfrak{x}_2, \mathfrak{x}_4\}$, *then* $BON^s_{\breve{g}}(\mathcal{D}_2) = \emptyset$.

$\underline{BIT}^s_{\breve{g}}(\mathcal{D}_3) = \{\mathfrak{x}_5, \mathfrak{x}_7\}$, $\underline{BIT}^s_{\breve{g}}(\mathcal{D}_3) = \{\mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_7\}$, *then* $BON^s_{\breve{g}}(\mathcal{D}_3) = \{\mathfrak{x}_3\}$.

The conclusions about the accurate approach using $(1,2) - \breve{g}$ sets with respect to information given in the table 8.2 are:

1) The decision value "Yes" is not exactly such that $\mathfrak{x}_7$ is in the boundary region.

2) The decision value "No" is exactly 1 such that its boundary region is empty.

3) The decision value "May be" is not exactly such that $\mathfrak{x}_3$ is in the boundary region.

In this chapter, we firstly defined Alexandroff Soft Bitopological Spaces and investigated its properties along with the generalization of closed sets in it. Further, we analyzed the concept of bitopological approximations by giving the definition of Alexandroff Soft Bitopological Approximation Spaces and gave an application to data reduction in information systems. The comparison of three accuracy measures were given which showed that the measure using $(1,2) - \breve{g}$-soft closed sets is the best accuracy measure than other measures for information systems.

Chapter 9

Conclusion and Future Scope

This whole work structured new type of topological spaces and examined their properties rigorously. New Separation axioms along with their properties were generated. Thereafter, we studied the concept of generalization of closed sets in newly developed spaces, introducing new generalized closed sets and investigated their properties. The new type of topological spaces namely Alexandroff Soft Topological Spaces, Alexandroff Soft Bitopological Spaces and Fuzzy Alexandroff Soft Topological Spaces have their own importance and can be used in different fields of decision-making problems, data devaluation, image recognition etc. The new notions like separation axioms, generalized closed sets in these spaces were introduced.

Our work provides an introductory platform for newly developed spaces but potentially useful research in theoretical as well as applicable directions can be made. One possible inspirational thought is that we can make use of programming language in topological sense to make our manual work easier. Also, we can define new algorithms to deal with real life problems of decision-making.

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List of Publications

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4) International Webinar Series on "Advances in Mathematics" organized by Tata Institute of Social Sciences, Tuljapur, 22-27 June, 2020.
5) One week faculty development programme on "Mathematics: A Practical Approach in Science and Technology" organized by Deogiri institute of Engineering and management studies from 28th June- 3rd July, 2020.
6) International webinar series on "Applications of Mathematics" organized by Gurucharan College, Silchar, Assam, India from 20th Aug-22nd Aug, 2020.
7) International online workshop on " New trends in Fuzzy and Rough set theory and its applications" organized by Manipal University, Jaipur from 25th Sept to 29th Sept, 2020.

8) 5-day workshop on "Data Sciences and advanced computing" organized by VIT-AP University, from 13th-14th March and 19th-21st March 2021.

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An Advanced Uncertainty Measure Using Fuzzy Soft Sets: Application to Decision-Making Problems

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Abstract: In this paper, uncertainty has been measured in the form of fuzziness which arises due to imprecise boundaries of fuzzy sets. Uncertainty caused due to human's cognition can be decreased by the use of fuzzy soft sets. There are different approaches to deal with the measurement of uncertainty. The method we proposed uses fuzzified evidence theory to calculate total degree of fuzziness of the parameters. It consists of mainly four parts. The first part is to measure uncertainties of parameters using fuzzy soft sets and then to modulate the uncertainties calculated. Afterward, the appropriate basic probability assignments with respect to each parameter are produced. In the last, we use Dempster's rule of combination to fuse independent parameters into integrated one. To validate the proposed method, we perform an experiment and compare our outputs with grey relational analysis method. Also, a medical diagnosis application in reference to COVID-19 has been given to show the effectiveness of advanced method by comparing with other method.

Key words: fuzzy soft sets; Dempster–Shafer theory; grey relational analysis; entropy; belief measures and medical diagnosis

1 Introduction

The fuzzy logics have emerged as a very important and useful topic in past recent years. It has aroused as an important mathematical tool to deal with uncertainties and vagueness of data. Zadeh[1] presented the concept of fuzzy set theory in 1965 as a transformation of classical set theory.

It can solve the problems of decision-making and deal with the problem of vagueness, uncertainty, and imprecision of data. Various theories like classical set theory[2], fuzzy set theory[1], probability theory, possibility theory[3], and Dempster–Shafer evidence theory[4,5] have been given to deal with certain types of uncertainties. Each theory has its own merits and demerits. Soft set theory is one of the theories initiated by Molodstov[6] in 1999 which can give exact solutions to various engineering and computer science problems. Fuzzy soft theory was given by Maji et al.[7] This theory has wider applications which can be easily found in Refs. [8–13]. Fuzzy soft sets can solve the problems of decision-making in real life. It deals with uncertainties and vagueness of data. Uncertainty refers to epistemic situations involving imperfect or unknown information. There are different forms of uncertainty, namely, fuzziness which arises due to imprecise boundaries, non-specificity (imprecision), discord and strife, etc. Measuring uncertainty is an open issue. Many belief entropies like Deng entropy[14], W-entropy[15], Hohel uncertainty measure[16], Dubois and Prade measure[17], Pan and Deng[18] uncertainty measure, etc., are introduced to deal with this open issue. They measure the uncertainty of parameters in different forms. Also, there are different approaches to solve decision making problems using fuzzy soft sets. Hou[19] made use of grey relational analysis to take care of the issues of problems in making

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Alexandroff soft topological spaces

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ABSTRACT

In this paper, we characterized another kind of topological space known as Alexandroff Soft Topological space. It is characterized over a general set $\mathcal{X}$ alongside an arbitrary set of parameters. This space satisfies a more grounded condition that an arbitrary intersection of open sets are open. We have likewise contemplated different ideas like basis of a topology, sub base, subspace, closure of a space and so on. Further, different separation axioms known as Alexo T_i-spaces have been presented alongside their properties. This space is also the parametrized type of general topology.

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1. Introduction & preliminaries

Nowadays, researchers are trying to develop novel approaches for solving the fuzzy complex problems [18–30]. There are disparate types of topological spaces namely discrete topology, indiscrete topology, bitopology [7], soft topology [12], fuzzy topology [15], intuitionistic fuzzy topology [3], Alexandroff spaces [2], and nano topology [14] etc. Each one of them is unique in relation to each other in some specific circumstances.

Molodtsov [9] gave a new peculiar theory named as soft set theory in 1999. He implemented this theory effectively in numerous ways such as functions smoothness, theory of games etc. Shabir and Naz [12] proposed notion of soft topology as a parametrized family of topological spaces by giving various definitions. They also defined a topology corresponding to each parameter in a space and explained results related to them. They have also introduced various separation axioms. After that H. Hazra [6] had introduced notions of topological structures in soft set settings. D.N Georgiou [4] studied soft topological space and gave different properties and results related to it. Maji [8] solved various decision making problems. Aktaş and Cagman [5] explained the algebraic nature of soft set theory.

Alexandroff spaces were first introduced by P. Alexandrov, after his name in 1937, with the name of diskrete Raume (1937) [11], where he produced the characterization in context of sets and neighbourhood. These spaces had not been studied properly and systematically. So, F.G Arenas in 1999 took initiative to study these spaces and studied all the properties of topology in it as they played an interesting role in place of finite spaces in digital topology and also it follows from the fact that these spaces have all the properties of finite spaces which are relevant to such theory. These spaces have a great property which differentiates it from general topology like every intersection of an open set is open.

In this paper, we have started with some results of general topology and soft set theory as pre requisites and then obtain a generalisation of them in Alexandrov soft topology presenting new interpretations, classifications, and many concepts related to it. This paper is divided into four sections. First section is the introductory part containing preliminaries as well. Second part defines Alexandroff Soft topological space. This part describes various properties like basis, subbase, subspace, closure of the space etc with various results. Third section explaine separation axioms and their related results along with the examples. Few notations which have been used in the paper are as below-.

$(\mathcal{X}, \tau_{A_s})$ denotes an Alexandroff Soft Topological Space, $\mathcal{A}$ denotes the arbitrary set of parameters and Alexo T_i denotes various separation axioms.

The basic definitions and results which are required for further work:

Definition 1.1. [1] "A set X along with a system $\mathcal{F}$ of subsets is said to be an Alexandroff space (or σ- space), if the following points fulfilled:

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Fuzzy Alexandroff soft topological spaces

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ABSTRACT

The main purpose of this paper is to establish a new type of topological space with the use of Fuzzy soft sets and Alexandroff spaces. We defined Fuzzy Alexandroff Topological Spaces and studied their topological properties. Further, we investigated two major properties of topology namely connectedness and compactness by giving the definitions of c_{f_A}-connectedness, c_{f_i}-connectness and c_{f_A}-compactness. A few examples have additionally been given which can show the utilization of this spaces in the field of Physics. We likewise detailed different outcomes identified with them.

1. Introduction

Topology can be generalized in many ways. Different types of topology has their own importance. Zadeh [6] in 1965 proposed the novel theory of fuzzy sets which has proved to be useful in almost every sphere of sciences. Many mathematicians [21–35] studied this concept for many years and gave important results on them. C.L Chang [7] defined the topology on fuzzy sets in the context of gradeness of open sets. After that, R.Lowen [8] redefined fuzzy topological spaces and gave different important results related to it. He also studied the property of compactness in fuzzy topological spaces. Similarly, Molodstov [4] gave the peculiar concept of soft sets which can eradicate the problems caused by the use of classical methods in solving various engineering problems. Maji [3] elucidated the theory of fuzzy soft sets and used this in solving decision-making problems. In 2012, Sanjay [20] constructed topology on fuzzy soft sets and studied various topological properties like fuzzy soft base, fuzzy soft subbase etc in fuzzy soft topological spaces. Connectedness and compactness are two important properties of topology which have been studied for so many years. Ajmal [9] gave the concept of c_i-connectedness in fuzzy topology which is the strongest form of connectedness among c_M-connectedness [11], c_5 -connectedness [10], O_q-connectedness [13] etc. Ruth and Selvam [14] gave a new approach of connectedness in fuzzy soft topology. In the same way, many authors explored the concept of compactness in fuzzy topology as well as fuzzy soft topology [8,15–17]. Alexandroff spaces [2] possesses a great property known as arbitrary intersection of open sets is open which differentiates it from other kind of topologies.It has been named after Russian Topologist Pavel Alexandroff in 1937. After that, F.G Arenas studied these spaces and found that it has the properties of finite spaces which can be used in the field of digital topology [2]. Timothy [18] gave a note on Alexandroff spaces and studied various properties of it.

Now, in this paper, our main purpose was to introduce a new kind of topology using the concept of fuzzy soft sets and Alexandroff spaces. This kind of topology is known as Fuzzy Alexandroff Soft Topological Spaces. We have also studied various topological properties of it and gave the notion of fuzzy alexandroff soft base and fuzzy alexandroff soft subbase. We further explored the concept of connectedness and compactness and gave the definition of c_{f_A}- connectedness, c_{f_i}-connectedness and c_{f_A}-compactness in Fuzzy Alexandroff Soft Topological Spaces along with their results and examples. This paper is divided into five sections. First two sections contains the introduction and preliminaries which are required for our main work. Third section explained the main work of paper along with the important results. Fourth and fifth section is the elaboration of notion of connectedness and compactness in this newly developed topological space. Throughout the paper, $\left(X, \tau_f, \mu_{f_\varepsilon}\right)$ denotes the Fuzzy Alexandroff Soft Topological Spaces and ε is the arbitrary set of parameters.

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rw^*-closed sets in Alexandroff Spaces

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Abstract. This paper explained and defined the notion of regular weakly-star closed (briefly known as rw^*-closed) sets in alexandroff spaces in which every point has a minimal neighbourhood. We discuss the characterizations and study their properties based on set theory along with the notion of rw^*-open sets.

1. Introduction

Alexandroff spaces is a topology which supports the property that an arbitrary intersection of family of open sets is open. It has been named after Russian topologist Pavel Alexandrov and known by the name "Discrete Raume". This kind of topology also states that every point has minimal neighbourhood. This space has all the properties of finite spaces which make relevance to digital topology. Indeed, we can say that Alexandrov discrete spaces is a generalization of finite topologial spaces. These spaces are mostly determined by specialization preorders. Francisco [8] studied Alexandroff spaces properly and produced various relevant results related to their topological properties. He produced the characterization in context of sets and neighbourhood in Alexandroff spaces. Alexandroff spaces have applications in Reimannian geometry [15], digital topology [6], path topologies of spacetime and linear orders [14] etc. Generalisation of closed sets constantly assumed a significant role in topological spaces and contributed to the hypothesis of separation axioms,covering lemmas etc. Many mathematicians contributed to the theory of generalisation of closed sets by giving different classes of generalised closed sets. Regular open, w-closed, g-closed, rw-closed sets were introduced by Tong [5], Sundaram [9], Levine [3], Benchalli [12] respt. Pratulanda and Mamun [11] introduced g^*-closed sets in Alexandroff spaces and investigate some of its characteristics and showed that g^*-closed sets did not have the same kind of results as that of generalised closed sets and obtained a new separtion axiom, namely, T_w- axiom. Amar kumar *et al* [13] studied the notion of g^*-closed sets, $g\lambda$-closed and λ^* closed sets in Alexandroff spaces and introduced various separation axioms namely $T_{5w/8}$, $T_{3w/8}$ and T_w and showed that $T_{5w/8}$ which can be placed between $T_{3w/8}$ and T_w.

This paper developed the notion of rw^*-closed sets in Alexandroff spaces and explored various properties of these sets. In the whole paper, a space $(\mathcal{X}, \tau_{\mathcal{A}})$ or simply $\mathcal{X}$ represents Alexandroff spaces and $\mathcal{R}$ and $\mathcal{Q}$ denotes the set of real numbers and rational numbers respectively.