A MATHEMATICAL STUDY OF STABILITY AND BIFURCATION ANALYSIS OF MUTUAL COMPETITION BETWEEN PLANT POPULATIONS UNDER ALLELOPATHIC EFFECT USING DELAY DIFFERENTIAL EQUATION

Thesis Submitted For the Award of the Degree of

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In

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By

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DECLARATION BY THE CANDIDATE

I declare that the thesis entitles "***A Mathematical Study of Stability and Bifurcation Analysis of Mutual Competition between Plant Populations under Allelopathic Effect using Delay Differential Equation***" submitted for award of degree of Doctor of Philosophy in Mathematics, Lovely Professional University, Phagwara, is my own work conducted under the supervision of Dr. Pankaj Kumar (Supervisor), Associate Professor in Department of Mathematics at School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab, India. I further declare that to the best of my knowledge the thesis does not contain any part of any work which has been submitted for the award of any degree in this university or in any other University/Deemed University without proper citation.

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CERTIFICATE

This is to certify that the work entitled "***A Mathematical Study of Stability and Bifurcation Analysis of Mutual Competition between Plant Populations under Allelopathic Effect using Delay Differential Equation***" is a piece of research work done by Mr. Dipesh under my guidance and supervision for the degree of Doctor of Philosophy of Mathematics, Lovely Professional University, Phagwara (Punjab) India. To the best of my knowledge and belief the thesis:

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Dedicated

To God

And

My Family

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ABSTRACT

In the proposed work, plant growth dynamics are studied considering delay in nutrient uptake rate, utilization rate and nutrient use efficiency rate under the effect of allelochemicals realized by other plant populations. Plant growth depends largely on the availability of allelochemicals present near about it or presence in soil. Presence of allelochemicals affects the plant growth and plant developments. This impedance in nutrients supply and utilization are taken as the delay parameter, which plays a key role in this entire study. In the proposed work, the models for single plant growth, tree growth and plant population are formulated considering the delay in growth dynamics under the effect of allelochemicals. The proposed models have been analyzed mathematically and the results have been verified numerically and further validated with already existing statistical data of plant population under the effect of the allelochemicals.

Mathematically, the positivity and boundedness of all analytical solutions is established using comparison theorem. All possible interior and feasible interior and exterior equilibrium are calculated. Local stability analysis of interior equilibrium is established. Stability analysis of interior equilibrium considering delay parameter resulted into Hopf-bifuraction showing the complex dynamical behavior. Nature of the roots has been studied in detail using Rouche's theorem. Sensitivity analysis and directional analysis of state variable with respect to model parameters is done for almost all the models using 'Direct Method' and 'Centre Manifold Theory' respectively. Numerical simulation is done using MATLAB software using dde23 command, where all the model parameters have been assigned different numerical value. This helped to find the critical value of the delay parameter below which the system exhibited stability and above this critical value, the system lost stability and Hopf-bifuraction occurred.

In chapter-1. The general introduction about the development of plant dynamics under the influence of allelochemicals is given. The remarkable work done by the researcher is cited and the gaps have been identified through extensive literature review. All the important concepts of plant physiology and the necessary mathematical concept required for their study have also been described. It also includes the proposed objectives of the study and summery of all the chapters.

In chapter-2. The analysis of mutual competition of the growth of plant with particular emphasis on time reliant variations in their densities has been done. We are proposing a mathematical model of plant population growth with competitive and allelopathic impacts that are stimulating as well as inhibitory on each other. By using "Routh-Hurwitz criteria", we have calculated the equilibrium point and analysis of stability has been carried out by taking non-zero equilibrium point. Delay parameter destabilizes the system when allelopathic effect is supposed to be of stimulatory in nature. It is observed that when there is no realizes of allelochemical our system is stable. At $\tau < 2.77,$ the system is asymptotically stable and $\tau \geq 2.77$ hopf bifurcation is observed. The numerical results are substantiated using MATLAB.

In chapter-3. We have proposed a mathematical model in order to study the allelopathy interference between plants that has traditionally been debated with respect to the development of allelochemicals. A competition mathematical model is used to analyze the coexistence of two competing plant species with special focus on time-dependent difference in their density and allelopathic impacts. Equilibrium points are calculated and using the "Routh-Hurwitz theorem", certain explicit formulations defining the stability and path of the Hopf-bifurcation. The system is stable below the critical value of delay parameter $\tau = 2.22$ and losses stability and show Hopf-bifuraction above the critical value. Numerical simulation and graphical support are provided for analytical findings using dde23 command in MATLAB software.

In chapter-4. A mathematical model is present for the study of the impact of allelochemistry on plants population involving delay. The proposed model has been constructed availing a system of non-linear delay differential equations. By incorporating the fact that impact of one or more toxicants on the ecosystem is beneficial for the development and wellbeing of a single or interactive biological organism. Delay parameter destabilizes the process when allelopathic effect is predicted to be in a stimulating way in nature. At $\tau < 1.47$ system shows asymptotic stability and $\tau \geq 1.47$ hopf bifurcation is observed as the delay parameter value reaches the critical point. Furthermore, the stability and length of these bifurcating periodic solutions are also analyzed. The sensitivity analysis of the system variables w.r.t. the system parameter is done, to know how the uncertainty in the output of a mathematical model can be divided and allocated to different source of uncertainty in its inputs using “Direct Method”. MATLAB is used to provide numerical analysis and graphical help for theoretical results.

In chapter-5. A mathematical model is framed that depicts the ecological phenomenon that the effect of density of allelochemicals on plant populations involving delays. The model is constructed using a system of non-linear delay differential equations. In the model, there are three state variables: plant populations (P_1 & P_2) and density of allelochemical (T). It is assumed that in the presence of density of allelochemicals, plant populations are affected. If further delays the conversion of resource into the density of allelochemicals and hence, affect the plant population adversely. This effect is observed by using delay in the competition model. Equilibrium points are calculated and using the Routh-Hurwitz theorem, certain explicit formulations defining the stability. For the critical value of delay parameter $\tau = 1.73$, Hopf-bifuraction is observed where by system fluctuates its behaviors from being stable to unstable. Further, the directions, of these bifurcating periodic solutions are also studied. Numerical simulation and graphical support are provided for analytical findings using MATLAB.

In chapter-6. We have proposed a mathematical model for competing organisms in the presence of allelochemicals with differing time-dependent densities**.** A competition mathematical model is proposed to understand the effect of allelochemicals on each other’s. It is shown that when one plant produces an allelochemical, it gives stimulatory effect to itself and inhibitory effect to the other plant. By incorporating the time lag the effect of allelochemicals has been studied. We have calculated the equilibrium point and analysis of stability has been carried out by taking non-zero equilibrium point. The system changes its behaviors from stable to unstable while passing through the critical value of delay parameter $\tau = 0.47$ and Hopf-bifuraction occurred. The model has been verified with literature data for the effect of black walnut plant on T. astivum plant growth by Zahid et.al (2016). The numerical results are substantiated using MATLAB.

In chapter-7. Competition may have an impact on plant population dynamics; there is no indication that this biological phenomenon has a detrimental impact on its long-term survival. Mutual competition in dynamical phenomenon’s which is affecting the population dynamics and have significant implications for population stability. In this paper, we have studied the effect of

allelochemical on plant population using competing mathematical model with the help of delay differential equation. Equilibrium point is calculated. The effect of allelochemicals is studied by introducing the delay. By the addition of delay in the dynamical system, the stability was measured. We observe that when the value of delay parameter $\tau < 1.266$, system shows asymptotically stability and when the value of delay parameter $\tau \geq 1.266$, systems show Hopf-bifurcation. The local and global stability analysis about non-zero equilibrium point was calculated with the help of Routh-Hurwitz criteria and Benedixon-Dulac criterion respectively. With the help of purposed mathematical model, the effect of allelochemicals can be control, and further plant growth enhanced. MATLAB software is used to provide numerical analysis and sgraphical help for theoretical results.

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General Introduction

1.1 Introduction

“Allelopathy” The term emerges from the Greek words, allelon, which interpret 'mutual' or 'for each other' and pathos, interpret 'suffering' or 'feeling.' The word literally meant that influence of one plant on another’s. When coining allelopathy, the word 'allelopathy' has caused considerable confusion, since the interactions involved are rarely reciprocal, as allelon may suggest, and are not necessarily harmful, as the word end-pathy usually infers. Allelopathy in soil can adversely affect forest, agricultural crops and vegetation by destroying their characteristics and productivity. It is widely recognized that some trees and weeds produce allelochemicals that can interfere with the germination, development, and robustness of nearby crops. Allelochemicals affects an particular plant development, crop, there by infect the plant natives dynamics which in turn effect the plant and crop production. It studies the growth of individual plant and plant population under varying environment conditions. Plants are the big producers in the environment. Plant development depends on the environment factors includes allelochemicals, water, temperature and availability of nutrient in soil. These factors in turn depend on variable like net assimilation rate, runoff, organic matter, microbial mineralization and immobilization of nutrients in the soil. Initially, allelochemicals was thought to be a facet in the interconnection of plants, possibly due to similarities in ecological system, and certain plants are harmful for the health of the animals. Expanding crops and trees simultaneously is not a new concept; farmers in both poor and advanced countries use it. Allelopathy has gained interest through the production of anti-herbivorous chemistry coat-tails through plants. Now days, it is widely agreed that the diversified range of allelochemicals in plants is mainly due to herbivores, especially insects and contagious organisms. In literature it has been proposed in the past, but with limited evidence to support it, that allelopathy may play a role in the conservation of species diversity.

The term "Allelopathy" was originated in German by the renowned Austrian plant physiologist Hans Molisch in his latest book, Der Ein flusse iner Pflanze auf die andere – Allelopathie1, published shortly before his death in 1937 written in the book history of allelopathy by wills. Most scholars have concluded that Molisch meant the former sense of emotion, but this is incorrect. When formulation of allelopathy was rising, Molisch desired that the word literally meant the influence of one plant on another. One might vainly argue that 'allelopathy' is phonemically best suited to the opinion of plant competition. While Molisch is rarely considered initiator of the science of allelopathy, this is a misunderstanding since the massive text of Molisch's is actually concerned about the effect of allelopathy of ethylene, now commonly used more as a food hormone than a plant interaction agent.

In ecological interactions, Molisch foreshadowed the significance of plant component, but in less detail. It's widely known that practically any substance that inhibits plant activity at high concentrations is likely to be stimulatory at low concentrations, and vice versa. As a plant physiologist, Molisch was fully aware of this, claiming that he used the word allelopathy to include both inhibitory and stimulating chemical interactions. In the sixteenth century, Paracelsus observed this duality of substances, saying, "All things are poison and not poison; only the dose does not produce a poison." The bulk of allelopathic research has been on inhibitory effects, although stimulation effects are likely to be so natural and perhaps modest that they are often neglected. Allelopathy has also been defined as "chemical warfare amongst plants," which takes our attention; perhaps releasing allelochemicals aren't always harmless.

While many trees, weeds, and crops are planted together in multi-story crop production, tree-crop-weed interactions are apparent. In the analyses of tree-crop-weed interactions, experts have widely ignored allelopathy as a likely cause. Allelopathy is a condition in which one plant causes harm to another by producing allelochemical that inhibits germination and development. Planting trees on damaged and unused land can substantially enhance the diversity of natural forest species. These plantings appear to stimulate forest area evolution under the right conditions by altering understory temperature conditions and soil quality, resulting in a more suitable ecosystem for the creation of biological diversity. Many allelochemicals have been detected in different parts of the test plant by different scholars; some of which may be inhibitory. Terpineol, cinnam aldehyde, geranial essential oil, limonene, phenol, and tartaric acid are all found in Tamarindus indica.

In 1984, E.L. Rice, who has long been considered the dean of American allelopathy investigators, has re-defined 'allelopathy' in taking into account the substances' harmful and beneficial effects as any explicitly or implicitly influence of one plant on another through the production of chemical compounds that escape into the environment. In recent decades, there has been a movement among allelopathy scientists to return to the previous, simpler definition of allelopathy, i.e., the inhibitory impact of one plant on another caused by chemical compounds released.

In the 1960s and 1970s, E.L. Rice and his research mates have shown that allelochemicals released from plants can harmful nitrifying bacteria and reduce the equilibrium of ammo-nium to nitrate in the soil. "Allelopathy" occurs in 1993 OED Add-ons sequence and is characterized by the "harmful process by which one organism influences others in the vicinity by leaking or releasing toxic or inhibitory substances into the environment: generally restricted to certain interactions between higher plants."

Until the Fifth Edition in 2002, the finest version, the Shorter Oxford English Dictionary, removed the phrase as well the definition itself was abbreviated to: "the process by which one organism harms or affects others nearby by releasing allelochemistry," which sounds more like a description of antibiotics.

Another misunderstanding has arisen in latest decades, because the zoologists have borrowed the word "allelopathy" to describe chemically dependent interactions between sessile animals, particularly invertebrates such as sponges and corals. Allelopathy is an interesting research. It is well established that plants can be abundant in both the variety and amount of so-called secondary metabolites.

Varahamihara and Brhatsamhita (about 500 AD) is the first known Indian text to mention allelopathy. While cultivating a certain crop, it was suggested that seed be cultivated, cut, and transformed into soil. Maybe the goal was to encourage the growth of plants. Floral and agricultural science in India extends back thousands of years, with the earliest literature dating back to the early Vedic period (1500 B.C.). The Vedas, or Books of Knowledge, are ancient books from the Vedic era.

India is basically an agricultural country and about seventy percent of the population is engaged in agriculture practices. To feed such a large populations farmer are using extensive amount of pesticides, insecticides, herbicides and chemical fertilizer to increase the yield. Due to excessive use of all these chemicals the fertility of the soil is decreasing, consequently diminishing the yield of agricultural crops which are also containing allelochemicals are even harmful to human health. Agricultural having traditionally feared that such crops would "contaminate" soil for different crops. Nevertheless, the releasing of nitrogen-rich organic chemicals from the formation of nuts and seeds is undeniably advantageous to succeeding crops. Among the most researched organisms, having compounds that have been shown to affect other plants. On farms, integrate trees give wind barriers, organic material sources, protection, and soil stabilizer to avoid desertification while also generating additional revenue. In temperate zones, the assimilation of trees into the agriculture land base using tree-crop intercropping systems clearly shows real potential. Many fields of "applied allelopathy" include the analysis of some forms of what is commonly referred to as "soil disease" or "soil exhaustion" or "replanting issues."

Modeling is widely used in research environment for purposes such as integrating information or quantitatively analyzing hypotheses, and for modeling, the system of concern must be defined. In horticulture and agricultural science, the system of interest is usually a plant and very often a complex of interacting plants.

Many researchers have used time delays of various types into biological models. The use of differential equations was prominent in the modeling of the various components of the system involved in the absorption of nutrients by roots under the influence of allelopathy. In the real life, an individual plant population's growth rates usually do not quickly respond to changes in its own population, but instead with a time lag. Allelochemicals in the soil, absorption frequency, and nutrient availability during plant growth development might all be factors that contributed to the time delay. The use of dde in all such time lag models has been highly useful. In generally, the behaviors of delay differential equations are significantly more complex than the dynamics of ordinary differential equations. Due to the time lag, a stable equilibrium loses its stability and becomes unstable, resulting in fluctuation. The role of allelochemicals in plant growth delay is a relatively recent topic of research. It is important to apply this principle of delay differential equations in all possible variations of time component, in order to provide a better understanding of the mechanism of plant-soil interaction under the influence of allelopathy.

Mathematical models are accurate representations of the resulting collection of environmental and ecological data. It is not feasible to incorporate all infinite variables and components in a mathematical model of physical phenomena, As a result, just the relevant data is analyses in order to gain a logical understanding of the dynamics' nature.. The proposed research will utilize mathematical models to evaluate quantitatively the harm process of plants and decrease in agricultural productivity related to decreasing variables such as the presence of allelochemicals in the soil.

1.2 Literature Review

Studies of Allelochemicals in plant growth are likely to begin in ancient times. The word allelopathic got fame after Australian Professor Hans Molisch defines it for a very first time. In 19th century allelopathic became the center of attraction for many researchers and large number of researches have been carried out. Wills [1] further bind up the work of Molisch by publishing a book “Der Ein flusse iner Pflanze auf die andere – Allelopathie1”. In the middle Ages, the modeling of the plant-soil relationship began with Hiltner [2]. Models on a single-root scale have provided insight into the complex dynamic phenomenon that occurs around the roots and the models that deal with it are called rhizosphere models that are essentially the soil area directly surrounding the root. Hornung [3] has studied a wealth of serious mathematical literature on the mathematical modeling of fluid flow in porous media that can be used effectively to understand the mechanism of plant soil interaction. A. Mukhopadhyay et al. [4] proposed an updated delay differential equation model for the growth of two species of plankton having competitive and allelopathic effects on each other. The model system showed a stable limit cycle oscillation when the allelopathic effect is of a stimulating nature. If the allelopathic model being studied by us is of a harmful form (AS system), periodic fluctuations in the plankton population can be successfully observed. On the other side, if the pattern is explore is of a harmful form (AI system), periodic fluctuations are not possible.

Thornley [5] was the first to use mathematical modeling to a number of topics in plant physiology to predict the effects of environmental factors like as humidity, temperature, radiation exposure and carbon dioxide concentration on photosynthesis cycle, fluid transport, respiration, transpiration and stomach reactions. Plant development models come in a variety of shapes and sizes, with varied degrees of complexity based on their intended utilization. Nonlinear mathematical models are used to analyzed the impact of one allelochemicals on another biological organisms interacting with each other. Shukla and Agrawal [6] conclude in their study of allelochemicals on biotic species that a single toxicant or a combined effect more allelochemicals, which depends on population-dependent or constant and the subset of dynamic’s system affected by allelochemicals and with extreme sign such as decreased height, deformity, lesion, etc. has been suggested. Allelochemistry is thought to possess unique biotic properties and the reactions of the organism are external manifestations of these properties. To view the characteristic reactions of the organism to allelochemistry plant development models come in a variety of shapes and sizes, with varied degrees of complexity based on their intended utilization. Johnson and Loveet [7] observed that Characteristic responses of the organism to the allelochemicals, i.e., Beneficial or attraction at low allelochemical concentrations and harmful repulsion as the concentration increases. Allelopathic effect on two competing species was investigated in order to investigate the coexistence of two competing plant species in bounded surroundings, supposing that each species produces allelochemical that affects the other. Dubey and Hussain [8] find that when there is no diffusion, it demonstrates that the two competing native species reach optimum concentrations that are smaller in magnitude than their respective initial density independent carrying capabilities. The word ‘process-based model’ was proposed to show the models that identified relationship between plant functional processes and abiotic factors. These models within an individual plant were given by Marcelis et al. [9]; Battagluia and Sands [10]. There are two forms of allelochemistry. Plant outputs are governed by age and plant stress, and they mirror the atmosphere's related dynamics. A general instance may be the decrease in allelochemicals content in live plants as the plants become older, but a particular case could be cyclical growth. Liu et al. [11] found that Volatilization, leaching, root exudation, and degradation of plant wastes is all ways that plants discharge allelochemicals into the ecosystem. Studying the cause-and-effect link in allelopathy requires knowledge of how chemicals are created in live plants and their long-term destiny in the ecosystem. However, in the literature, complex situations are mentioned. Zha et al. [12] studied the ratio-dependent predator-prey model with stage structure and time delay. And found in their research that the present level of the predator has an instant impact on the growth of the maturation prey, so that the growth of the predator is affected by the amount of the maturation prey. They have proper conditions for the propagation and extinction of the system. The dynamics of two-species allelopathic competition with optimal harvesting model under certain parametric conditions, the model has a special positive equilibrium point that is asymptotically stable globally whenever it is locally stable. The criteria for the nature of bionomic equilibrium are discussed by Gupta et al. [13] and the optimal harvesting strategy is based on the Pontryagin Maximum Principle and observed that the points of bionomical equilibrium depend on the coefficients of toxicity.

In such models, Plant morphological (i.e., the geometrical and structural connectedness between organs) is rarely treated explicitly. As a result, models relating development activities to plant morphogenesis have been developed. Temperature, soil properties, and water quality were all used to build the 3-D development of plant structure through time, as well as the interplay of these activities with environmental variables. This class of models is known as functional-structural (FS) plant models and has been developed by Godin et al. [14]. Models that take into account interactions between plant architecture, organ functions (e.g., root absorption, leaf photosynthesis, sap flow conduction) And growth mechanisms have been extensively developed over the last decade by Perttunen et al. [15], Reffye et al. [16], Prusinkiewicz [17]. Wolf et al. [18] studied the modeling of long-term response of crop to fertilizer and nitrogen. The presence of two organic forms of nitrogen i.e., ammonium and nitrate demonstrate the complex behavior of nitrogen available in soil.

Mathematical modeling in plant biology: consequences of physiological methods for resource management written by Louis, Gross [19] in 1990 applied mathematical models to a broad variety of physiology topics, but all of them modeled different processes such as photosynthesis, respiration, fluid transfer, transpiration, stomatological processes, etc. Our key objective is to examine the effect of environmental factors, such as radiation intake, humidity, wind, carbon dioxide concentration and temperature, on process levels. Misra and Saxena [20] have performed theoretical studies using a mathematical model for two interconnected and dispersing structures of organisms under the influence of pollutants.

Mishra and Kalra [21-22] have provided a very sound two-bay mathematical model for modeling the impact of toxic metal on separate plant development, and also gave another mathematical model for the study of the influence of toxic metal on the structural dry weight of the plant. The analysis of local and global stability examined. The numerical simulation is performed using MATLAB. Kalra and Kumar [23] proposed a two-compartment mathematical model for modeling the A two-compartment model of the influence of hazardous metals on plant development dynamics with delay. The toxicity symptoms in the model restricted root absorption, and when the toxic metal intake rate increased, the nutrient content and structural dry weight of the root and shoot compartments decreased. To investigate the harmful impact of toxic metals on plant biomass, researchers combined toxic metals and soil chemistry studied by Kalra and Kumar [24] gave another worth full two compartments mathematical modeling under the influence of environmental pollutants on plant biomass with a delay. The numerical simulation is done using MATLAB.

Ecological, phenomenon that’s nutrient pool and plant population density are adversely affected by the presence of excessive toxic metal. Kalra and Kumar [25] studied the role of delays in growth of plant population growth due to toxic metal. The local and global stability analysis has been done. Mishra et al. [26] provided a very sound two-bay mathematical model for modeling on conservation of forestry biomass introducing variable taxation for harvesting, age-structured forest biomass (premature, semi-mature, mature trees) to be industrialized, and considered that taxation is entirely based on the degree of harvesting of semi-mature and mature forest trees. The two Prey and predator species conform to the Logistic act cap in the absence of prey and predator species. Srinivas et al. [27] studied the influence of delay in predation of a two species allelopathic system having unspecific growth rates has a delay in predation and dynamics model is explored when both species have allelopathic effects on each other and is the natural rate of growth of both species can differ from time to time. Favini et al. [28] studies the modeling of fractional model in allelopathic stimulatory phytoplankton species. Allelochemicals have been uniformly present in plants and that one of their most essential physio-biochemical functions is the defense against their enemies. Hooda et al. [29] studied the allelopathic effects of vigna mungo on germination and growth. The effect of leaf leaching of Parthenium, Hyptis and Tridax on the germination of black grams has been observed. Seeds were put in petridises containing 0.5, 1.0 and 5.0 per cent leaf leaching of each weed extract. Preety and Kumar [30] studied the two compartments mathematical model to analyze the state variables in structural dry weight and nutrient concentration. Lima et al. [31] worked on the allelochemicals (inhibitory and stimulatory) and examined the factor affecting the plant development and crop production as shown in figure1.1

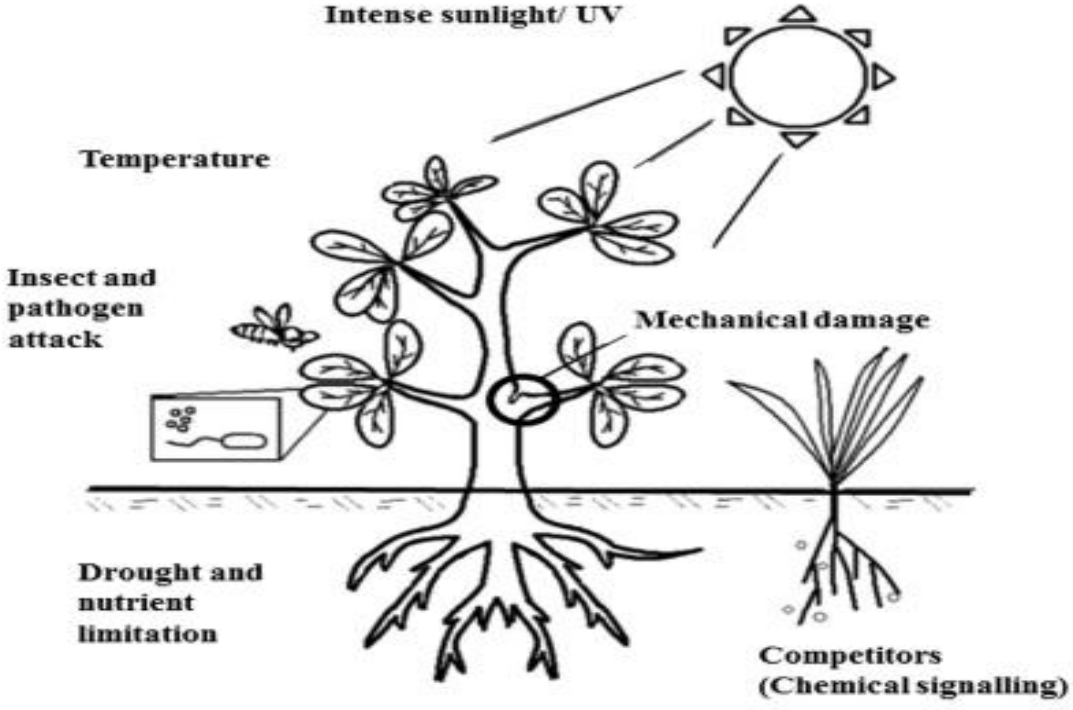

Figure1.1 Different parameter's effecting the plant development.

Which shows how the root, shoot, leaf compart of a plant is affected by the different parameters (Temperature, sunlight/UV, Competitors, Rain, Drought etc.). Kubiaczyk and Saker [32] studied the oscillatory and stability of dynamic equations. They used the equation proposed by Mackey and Glass for a "dynamic illness" involving respiratory disorders and a models proposed by Nazarenko to studied the regulation of a single cell population. Lenbury and Giang [33] studied non-linear dynamic system for plant development using time lag and utilized on Nicholson's model in ecosystem. Roose and Szalai [34] have studied prolongation and bifurcation analysis of delay differential equations. Chaudhary and Gejji [35] studied the conditions for existence of uniqueness theorems for multi-term fractional delay differential equations. Tahir [36] studied the oscillatory behavior of the first order solution delay differential equations. This analysis sets out ample conditions for the oscillation of the first order delay differential equation with positive and negative coefficients. Huang et al. [37] studied the dynamics of delay differential equations with its applications. Janka et al. [38] worked on dynamic of root-shoot interactivity and barrier in plant development, balancing between root-shoot interactions during development of plant development. Yanar et al. [39] do worked on allelopathic impact on plant extracts against seed germination of various plants and it was discovered that using plant extracts as herbicides to control weeds will be quite effective beneficial for crop production. Huang et al. [40] studied stability of non-linear dynamics system with the help of direct Lyapunova method. Berezansky and Braverman [41] studied boundedness and determination of dde with mixed nonlinearity. Kubiaczyk and Saker [42] studied the oscillatory behavior and stability of dynamic system using time lag. Li and Wei [43] examine the nature of roots of 4^{th} degree exponential polynomial. Ruan [44] integrated a time delay of some type in biological models by numerous researchers working with the dynamics of single species populations. With the use of normal-form theory and Centre multiple theorems Wang et al. [45] studied the direction and stability of hopf bifuraction in a phytoplankton-zooplankton model. Rihan [46-47] used the adjoint equation and direct method to create a complete concept for sensitivity analysis of mathematical models with delays and computational tools, parameter estimates, and sensitivity analysis. To further identify the basic biological processes, Wu [48] designed a framework for sensitivity analysis of functional structural plant models in his research and examine the direction of continuous evolutionary change. Ingalls et al. [49] developed a novel approach for analyzing the sensitivity of delay differential equations with respect to parameters. Dubey and Hussain [50] examined the negative effects of pollution on forestry biomass in a diffusive system with periodic delays. Naresh et al. [51] studied the impact of heavy metal on plant growth using delay differential equations. Danny et al. [52] worked on allelopathic and intraspecific growth competition effects establishment of direct sown miscanthus and reduce the inhibitory effects of higher seed. Kunz et al. [53] examined the effect of allelochemicals on crops and also tell us about the biochemicals impact play a measure role in weed repressive on cover crops. Ling Yuan et al. [54] examined the role of allelochemicals in alien plants. Allelopathy may cause coexistence of closely related species or particular species dominant, based on the negative connection between phylogenetic distance and allelopathy. Upadhyay et al. [55] analyzed the effect of industrialization and pollution on forestry resources. Hussain [56] examined how allelochemicals' destiny in the soil is determined by their interactions with the soil complex, soil microbes, and environmental substances and allelochemicals production is influenced by different parameters as shown in figure1.2

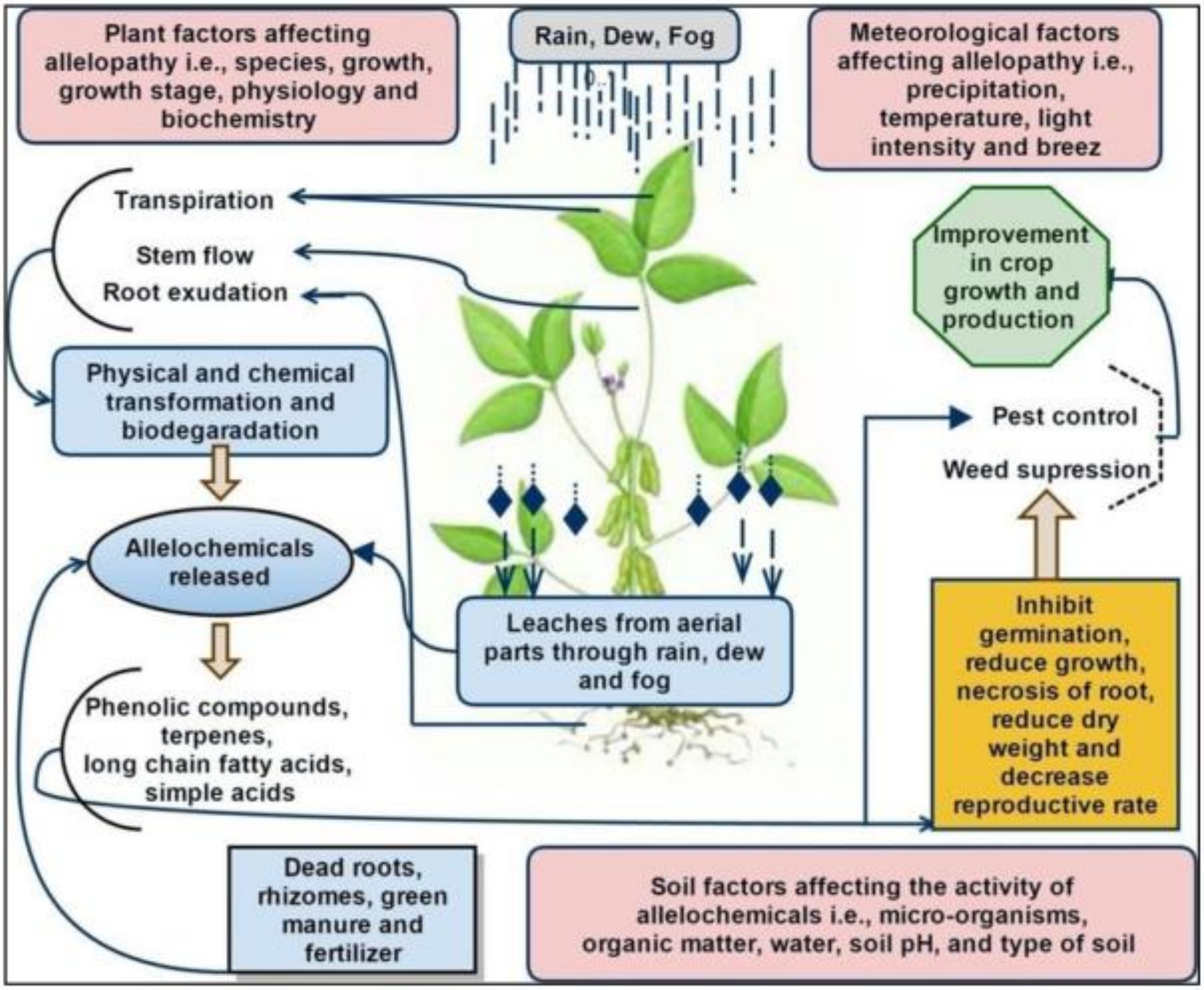

Figure1.2 Allelochemicals' destiny in the soil

Frey and Patil [57] identified the different method for calculating the sensitivity analysis and also tell us about that each method has its own key assumptions and limitations. Eigentler [58] observed the inclusion of local intraspecific competition in dynamics for dry land vegetation patterns has a weighty role on dynamic system. Meiners et al. [59] studied interspecific competition due to allelopathy in rice with paddy weeds. Bocharova and Rihan [60] worked on numerical modelling in the biosciences utilizing delay differential equations, as well as identifying appropriate computing approaches for the numerical solution in the biology, and comparing them to those used by bio-modellers. Kuznetsov [61] worked on nonlinear dynamics system and their bifuraction by different values of parameters. Lenbury and Giang [62] worked on nonlinear delay differential equations involving population increase. The term Hopf bifurcation is characterized by Hopf [63] mean's the oscillation behavior of a dynamic system. When a complex conjugate pair of eigenvalues becomes entirely imaginary, this is referred to as Hopf bifurcation. This means that hops can only be bifurcated in systems of two or greater dimensions. Marsden and McCracken [64] work on Hopf bifuraction and also tell us about its application. Kuang [65] shown how the famous MATLAB-based dde23 solved by Shampine and Thompson for delay differential equations can be utilized to numerically solve most delay differential equations and do stability analysis. Huang et al. [41] examined the global stability for the dde with nonlinear system. The delay differential equation has a history that spans more than 200 years. The first implementation of the delay differential equation is found in geometry and number theory. However, the subject gained popularity after 1940 due to its application in engineering systems and control. In chapter 2 of the book "Delay-Differential Equation" [66], author explains the general theory of delay differential equation and solution of existence and uniqueness. Engelborghs et al. [67] developed numerical techniques and software package for delay differential equation bifurcation analysis. Dickinson and Galinas [68] proposed the "Direct approach" for sensitivity analysis of ordinary differential equations with uncertain parameters. The sensitivity of the solution is measured by partial derivatives with respect to the model parameter. Arino et al. [69] worked on delay differential equations and utilization in mathematics. Ruan et al. or Paret and Nussbaum [70-71] studied on stability and bifurcation in delay differential equations with two delays. Shampine and Thompson [72] generate a MATLAB code: DDE23 in order to solve delay differential equation with fixed delays. Engelborghs et al. [73] develop a MATLAB code: DDE-BIFTOOL, for numerical bifuraction analysis systems of delay differential equations with multiple fixed, discrete delays.

1.3Motivation and Background (Research Gap)

Many scientists, professors, researchers have made several contributions to plant biology and modeling in the field of allelopathy. Most of the work performed on allelopathy in plant biology and modeling is based on statistical data and techniques. During review literature, it is found that very less work involves mathematical models in the field of plant competition involving allelochemical interactions. The even more motivating factor for this proposed research work is the rare use of delay differential equations in these interactions. This actually gives the required gap and scope of proposed work.

1.4 Scope of study (Hypothesis)

It is a well-known fact that allelopathy play a significance role in plant populations dynamics. The problem related to plant growth dynamics and crop production is the main problem facing by many scientists, professors and researchers. There is dire need to propose such mathematical model that can overcome the existing problems that related to plant population development and crop production. With the help of ordinary differential equation in competing mathematical model we can study the allelochemicals effects in plant development and crop production. Further we introduced the delay in model to understand the allelochemicals effect in plants and crops. Using Competiting mathematical model the overall growth of the plant and crop production enhanced.

1.5 Research objectives of the study

In the view of the above in this thesis, following problems have been studied using mathematical model.

1. To study the stimulatory and inhibitory allelopathic effect on Competiting plant populations using delay differential equation.
2. To study the stability and bifuraction analysis of proposed plant competition model.
3. To do the directional analysis and sensitivity analysis of the state variable with respect to model parameter.
4. To verify the proposed model by comparison with existing experimental data.

1.6 Basic concept of general plant populations used in the thesis

1.6.1 Structure and Storage

Plant models are mostly based on a gross over simplification of the real system. Plant material is divided into two categories: structure and storage, a simplicity that appears to have physiological importance [75].

Let P denotes the plant being considered. P_G and P_S be the structural and storage components of the plant respectively.

$$P = P_G + P_S \tag{1.1}$$

Defining the word “growth” simply in term of plant development, and differentiating Eq. (1.1) w.r.t. time (t), it follows that there are two contributes to the total growth rate of plant:

$$\frac{dP}{dt} = \frac{dP_G}{dt} + \frac{dP_S}{dt} \tag{1.2}$$

The structural growth rate $\frac{P_G}{dt}$, and the storage growth rate $\frac{P_S}{dt}$, together make up the total growth rate $\frac{dP}{dt}$. Thus, it is possible for a plant to have a negative total growth rate, made up of a positive structural growth rate and a negative storage rate.

1.6.2 Translocation

Plants would be unable to develop if they were unable to carry materials to areas where they could be used. Although many mechanisms are innately non-polar, the arguments of sources and sinks exert polarity on the systems; transport is primarily a polar phenomenon. In such instances, changing the sign of sources and sinks might affect the direction of transit.

1.6.3 Light Interception by Plants and Crops

The interaction of the system with the light climate is a significant factor of many plants and crop growth models. The mechanism in which this occurs dictates the light flux density falling on each area or crop surface, which in turn affects the amount and pattern of growth through photosynthesis.

Let I_0 be the light flus density (assumed uniform) that is incident on a field of aera A_f. If J_0 is the maximum light flux available for absorption, then

$$J_0 = A_F \, I_0 \tag{1.3}$$

Let J_c be the actual light flux absorbed by a crop covering a field. The overall efficiency of light absorption by the crop f_c, is given by

$$f_c = \frac{J_c}{J_0} \tag{1.4}$$

Let the crop consists of $n_c \, J_p$ be the light flux absorbed by the crop, so that J_p is the light flux absorbed by an average plant. If f_p is the efficiency of light absorption of the plants in the crop, f_p is given by

$$f_p = \frac{J_c}{n_c J_p} \tag{1.5}$$

1.6.4 Photosynthesis

An equation which is frequently used to describe steady-state photosynthesis response to light and carbon dioxide is the rectangular hyperbola

$$P_S = \frac{\alpha I \beta C}{\alpha I + \beta C} \tag{1.6}$$

where P_S is the steady-state photosynthesis rate (no distinction is made here between net and gross photosynthesis), I is the flux density, C is the carbon dioxide density, α and β are constant. This equation is popular because it often gives an acceptable description of actual response and is manageable.

1.6.5 Transport of Substrate

The flux of substrate, B (referred to as the mass transfer rate) from partition i to partition $(i-1)$ is given by

$$B = \frac{s_i - s_{i-1}}{r_i} \tag{1.7}$$

where r_i is the transport resistance between ith and $(\mathrm{i}-1)\mathrm{th}$ compartment and s_i is the substrate concentration in the ith compartment.

1.6.6 Plant Growth Curve

Plant growth is sluggish in the beginning (lag phase). After that, the plant's activity compared to the additional significantly (exponential phase). As a result of the obstruction of nutrients, the growth rate decreases slightly (stationary phase). The plant growth curve is a classic parabolic or S-shaped curve derived by plotting growth and time (Figure 1.3)

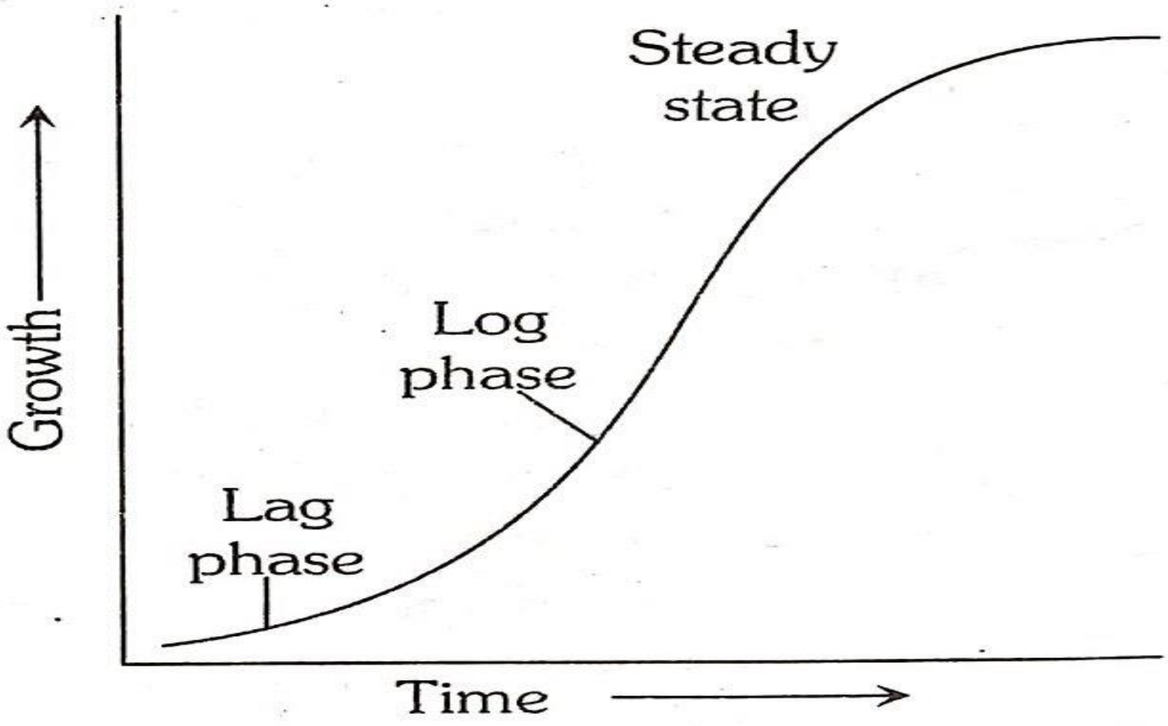

Figure1.3 Plant Growth Curve

1.6.7 Plant Growth Rate

At the individual, population, and ecological levels of organization, the plant-resource interaction is the influence of a limiting resource on plants. At the individual level, a growth-rate dependency on nutrient availability can be written as [76]

$$r(R) = \eta \mu_m W_r \frac{R}{k_r + R} \tag{1.8}$$

where R the availability of nutrient is η is the nutrient use efficiency. W_r is the proportion of total biomass allocated to root mass, μ_m is the supplies-saturated rate of supplies uptake per unit of root mass and k_r is a half-saturation constant for nutrient uptake.

1.7 Mathematical Portion

1.7.1 Existence of Unique, Bounded and Positive Solution of Delay differential equation.

A delay differential equation is a differential equation in which the present time derivative depends on the solution and derivatives of earlier times. Here an initial history function, rather than an initial condition, needs to be defined. A delayed state variable can be used to demonstrate the past dependence of a differential equation. The derivative of the state variables not required in this case. The corresponding delay differential equation with a single delay $\tau > 0$ is given by [77]

$$x(t) = f\big(x, x(t), x(t-\tau)\big) \tag{1.9}$$

Assume that $f(t,x,y)$ and $f_x(t,x,y)$ are continuous on R^3. Let $s \in R$ and $\phi: [s-\tau, s] \to R$ be continuous. We seek a solution $x(t)$ of equation (1.9) satisfying

$$x(t) = \phi(t), t\, \epsilon[s-\tau, s], x(0) = x_0 \tag{1.10}$$

And satisfying equation (1.9) on $t\epsilon[s-\tau, s]$ for some $\sigma > 0$.

Theorem 1.7.1 (a) (Existence of unique solution).

Let $f(t,x,y)$ and $f_x(t,x,y)$ are continuous on R^3. Let $s \in R$ and $\phi: [s-\tau, s] \to R$ be constant. Then $\exists\ \sigma > s$ and a unique solution of the initial-value problem (1.9) - (1.10) on $[s-\tau, s]$.

Theorem 1.7.1 (b) Boundedness of solution.

Let f satisfy the hypothesis of theorem1.1 and let $x: [s-\tau, s] \to R$ be the non-continue able solution of the initial value problem (1.9) - (1.10). If exists $\sigma > \infty$ then $\lim_{t\to\sigma^-} |x(t)| = \infty$

Remark 1.7.1 (c) Theorems 1.7.1 (a) and 1.7.1 (b) enlarge immediately to the case that $x \in R^n$ and $f: R \times R^n \times R^n \to R^n$, it also enlarges to multiple discrete delays $\tau_0 < \tau_1 < \cdots \tau_m$ where $f = f\big(t, y(t), y(t-\tau_0), \dots \dots \dots \dots .., y(t-\tau_m)\big)$.

Theorem 1.7.1 (d) Positivity of solution

Suppose that $f: R \times R^n_+ \times R^n_+ \to R^n$ satisfies the hypothesis of theorem 1.7.1(a) and remark 1.7.1 (c) and for all i, t and for all $x, y \in R^n$:

$$x_i = 0 \Rightarrow f_i(t,x,y) \geq 0$$

If the initial data ϕ in equation (2) satisfy $\phi \geq 0$, then the corresponding solution $x(t)$ of equation (1.10) satisfy $x(t) \geq 0$ for all $t \geq s$ where it is defined.

Theorem 1.7.2 Stability by Variational Matrix Method

Let an autonomous system of equations be

$$\frac{dy}{dt} = f(y) \tag{1.11}$$

where y is an n-tuple vector i.e., $y = (y_1, y_2, \dots \dots \dots .. y_n)$. Let $\phi(t)$ be the solution of system (1.11). The linear part of the expansion of the system (1.11) about $\phi(t)$ is given by the variational equation of the system (1.11) with respect to (t), written as

$$\frac{dx}{dt} = f_y\big(\phi(t)\big)x \tag{1.12}$$

where $f_y\big(\phi(t)\big) = \frac{df_i}{(dy_j)_{n\times n}}$ at $\phi(t)$. Since the stability of the variational system depicts the stability of any solution of a non- linear system governed by it, so stability of $x = 0$ of (1.12) determines the stability of $y = \phi(t)$ of (1.11). Particularly, when $\phi(t) = \phi_0$, a constant, the system (1.11) becomes

$$\frac{dx}{dt} = Ax$$

where $A = f_y(\phi_0)$. Since a small perturbation of the system (1.11) is represented by system (1.12), so the stability of $y = \phi_0$ of (5) actually gives the stability of the solution of $x = 0$ of (4). The description of stability of every solution of $x' = Ax$ is given by following theorems [78].

Theorem 1.7.2 (a) Every solution of the system $x' = Ax$ (where $A = (a_{ij})$ is a constant matrix) is asymptotically stable, if A have all the characteristic root that consist of negative real part.

Theorem 1.7.2 (b) All the solutions of the system $x' = Ax$ are bounded and stable when all of A's characteristic roots with multiplicities larger than one have negative real parts, but all of its roots with multiplicities less than one have non-negative real parts., then. Following theorem [78] to determine the sign of real parts of the roots of characteristic equation

Theorem 1.7.2 (c) Hurwitz's Theorem

A required and adequate condition for the negativity of the real parts of all the roots of the polynomial

$L(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots \dots \dots \dots \dots \dots \dots + a_n$ with real coefficients is the positivity of all the principal diagonals of the minors of the Hurwitz matrix

$$H_n=\begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & : & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & : & 0 \\ : & : & : & : & : & : & : & : \\ 0 & 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

& With the help of Routh- Hurwitz we can calculate the stability of the system. Whether the system are stable or not stable. For stability we see the first Colum of Hurwitz must be positive & $a_1>1$. If any value of first Colum is negative & $a_1<1$, then the system is unstable. When all the row element in any one row of the Hurwitz is zero. The polynomial whole coefficient is the element of the row just above the row of zero. The Hurwitz is called auxiliary polynomial. The order of auxiliary polynomial is always even.

1.7.3 Hopf-Bifurcation

Hopf's crucial contribution was the extension from two dimensions to higher dimensions. Sometimes Hopf bifurcation is also called as “Poincaré-Andronov-Hopf bifurcation” [79]. The Hopf-bifurcation theorem illustrates how topological characteristics of a flow change when one or more variable change. The essential finding of flows is that if the static point is hyperbolic, i.e., the generalized linear flow's eigen values at the stationary point all have non-zero real pars, then the linearized flow totally determines the flow's local behaviour. Hence, bifurcations of stationary points can only occur at parameter values for which a stationary point is non-hyperbolic. More, precisely, a bifurcation value of a parameter is a value at which the qualitative nature of the flow changes.

The Hopf bifuraction is several orders of magnitude harder to analyze since it involves a non-hyperbolic stationary point with linearized eigenvalue $\mp i\omega$, and thus a two-dimensional center manifold and bifurcating solution are periodic rather than stationary.

Theorem 1.7.3 (a) Hopf-Bifuraction Theorem

Let consider one parameter family of delay equation

$$x(t) = F(x_t, \mu) \tag{1.13}$$

where $F: C \times R \to R^n$ is a twice continuously differential in its arguments and $x = 0$ is a steady state for all values of $\mu: F(0, \mu) \equiv 0$.

We may linearize F about $\phi = 0$ as follows

$$F(\phi, \mu) = L(\mu)\phi + f(\phi, \mu)$$

where $L(\mu): C \to R^n$ is a bounded linear operator and f is higher order

$$\lim_{\phi \to 0} \frac{|\mathrm{f}(\phi, \mu)|}{||\phi||} = 0$$

Following is the characteristic equation about L:

$$|\lambda I - A(\mu, \lambda)| = 0, A_{ij}(\mu) = L(\mu)_i(e_\lambda e_j)$$

The roots of this equation constitute the main assumption.

(H) the characteristic equation will be having a pair of simple roots $\mp i\omega$ with$\omega_0 \neq 0$ and no other root that is an integer multiple of iw_0 for $\mu = 0$.

Here a root of order one means [81] a simple root. If the characteristic equation is written as $h(\mu, \lambda) = 0$, then **(H)** implies $h_\lambda(0, i\omega_0) \neq 0$. So, by the implicit function theorem, there exists a continuously differentiable family of roots $\lambda = \lambda(\mu) + i\omega(\mu)$ for small μ satisfying $\lambda(0) = i\omega_0$. In particular, $\alpha(0) = 0$ and $\omega(0) = \omega_0$. Next assumption is that as μ increase through zero, the line of imaginary axis is crossed transversally by these roots. Actually, the assumption is:

$$\alpha' > 0 \tag{1.14}$$

In case $\alpha'(0) < 0$, we always ensure that equation (1.14) holds by changing the sign of the parameter i.e., we take parameter $\nu = -\mu$. Thus, the positive sign is basically a normalization which that if $\mu < 0$, then the pair of roots a negative real part and if $\mu > 0$, then it has positive real part.

Theorem 1.7.3 (b) let (**H**) and equation (1.14) hold. Then there exist $\varepsilon_0 > 0$, real valued even function $\mu(\varepsilon)$ and $\mathrm{T}(\varepsilon) > 0$ satisfying $\mu(0) = 0$ and $\mathrm{T}(\varepsilon) = \frac{2\pi}{\omega_0}$, and a non-constant $\mathrm{T}(\varepsilon)$ periodic function $p(t, \varepsilon)$ with all functions being continuously differentiable in ε for $|\varepsilon| < \varepsilon_0$, such that $p(t, \varepsilon)$ is a solution of equation (1.13) and $p(t, \varepsilon) = \varepsilon q(t, \varepsilon)$ where $q(t, 0)$ is a $\frac{2\pi}{\omega_0}$ periodic solution of $q = L(0)q$.

Moreover, there exist $\mu_0, \beta_0, \delta > 0$ such that if equation (1.13) has a non-constant periodic solution $x(t)$ of period P for some μ satisfying $|\mu| < \mu_0$ with $\max(x_t) < \beta_0$ and $\left|p - \frac{2\pi}{\omega_0}\right| < \delta$, then $\mu = \mu(\varepsilon)$ and $x(t) = p(t + \theta, \varepsilon)$ for some $|\varepsilon| < \varepsilon_0$ and some θ.

If F is five times continuously differentiable then:

$$\mu(\varepsilon) = \mu_1 \varepsilon^2 + O(\varepsilon^4) \tag{1.15}$$

$$T(\varepsilon) = \frac{2\pi}{\omega_0}[1 + \tau_1\varepsilon^2 + O(\varepsilon^4)] \quad (1.16)$$

If all other characteristics roots for $\mu = 0$ have strictly negative real parts expect for $\mp i\omega$ then $p(t, \varepsilon)$ is asymptotically stable if $\mu_1 > 0 <$ and unstable if $\mu_1 < 0$.

1.7.4 Sensitivity Analysis of state Variable with respect to model Parameter

Study how various source of uncertainty in mathematical model contribute to the model overall uncertainty. There are number of methods to do sensitivity analysis of system without delay, but there are only a few methods for sensitivity analysis of systems involving delay. The knowledge of how a small change in model parameter can bring change in the state variable can be a great help in modelling process. It helps in elimination of ineffective and irrelevant parameters. It gives a complete insight into the overall behavior of the proposed model.

If all the parameters in the given system (1.9) - (1.10) are considered to be constants, then sensitivity analysis includes just the calculation of partial derivative of solution with respect to each parameter [55]. The matrix of sensitivity functions is of the form:

$$S(t) = S(t, \alpha) = \left|\frac{\partial}{\eth\alpha}\right|^T x(t, \alpha) \quad (1.17)$$

Its jth column is: $S_j(t, \alpha) = \left[\frac{\partial x_j(t,\alpha)}{\partial \alpha_1}, \frac{\partial x_j(t,\alpha)}{\partial \alpha_2}, \dots, \frac{\partial x_j(t,\alpha)}{\partial \alpha_n}\right]^T$

This column vector gives sensitivity of the solution $x_j(t, \alpha)$ for small change in parameter $\alpha_1, i = 1,2, \dots \dots n$

Theorem 1.7.4 (a) $S(t)$ satisfies the delay differential equation:

$$S'(t) = j(t)S(t) + j_\tau(t)S(t - \tau) + B(t), t \geq 0 \quad (1.18)$$

where $j(t) = \frac{\partial}{\eth x} f(t, x, x_\tau), j_\tau(t) = \frac{\partial}{\eth x_\tau}(t, x, x_\tau), B(t) = \frac{\partial}{\eth\alpha} f(t, x, x_\tau)$

1.8 Summary

This thesis consists of seven chapters whose detail is as follows:

In chapter-1, the general introduction of the subject matter is given. Some important concept of plant population is mentioned. All the necessary mathematical concepts have been described, on the basis which the analytical and numerical analysis of all the proposed mathematical model have become possible. It also includes the detailed study of the work done in the field of plant development in the form of literature review.

In chapter-2 To understand the allelopathic effect, mathematical modeling is used in competing plant population. The analysis of mutual competition of the growth of plant with particular emphasis on time reliant variations in their densities has been done in our present work. We are proposing a mathematical model of plant population growth with competitive and allelopathic impacts that are stimulating as well as inhibitory on each other. Equilibrium points are calculated and stability analysis is performed about non-zero equilibrium point. Delay parameter destabilizes the system when allelopathic effect is supposed to be of stimulatory in nature. The system shows asymptotic stability when delay parameter having value below the critical point. Hopf bifurcation is observed when the value of delay parameter crosses the critical point. The numerical results are substantiated using MATLAB

In chapter-3 A mathematical model is proposed to study the allelopathy interference between plants has traditionally been debated with respect to the development of allelochemicals. A competition mathematical model is used to analyze the coexistence of two competing plant species with special focus on time-dependent difference in their density and allelopathic impacts. Equilibrium points are calculated and using the Routh-Hurwitz theorem, certain explicit formulations defining the stability and path of the Hopf-bifurcation periodic solutions bifurcating from Hopf-bifurcation at the positive equilibrium are obtained. Numerical simulation and graphical support are provided for analytical findings using dde23 command in MATLAB software.

In chapter-4 A mathematical model is present for the study of the impact of allelochemistry on plants population involving delay. The model is constructed using a system of non-linear delay differential equations incorporating the fact that impact of one or more toxicants on the ecosystem is beneficial for the development and wellbeing of a single or interactive biological organism. Delay parameter destabilizes the process when allelopathic effect is predicted to be in a stimulating way in nature. The system demonstrates asymptotic stability whenever the delay parameter is below the critical point. Hopf bifurcation is observed as the delay parameter value reaches the critical point. Furthermore, the stability and length of these bifurcating periodic solutions are also analyzed. The sensitivity analysis of the system variables w.r.t. the system parameter is done. MATLAB is used to provide numerical analysis and graphical help for theoretical results.

In chapter-5 A mathematical model is applied to investigate the effect of density of allelochemicals on plant population involving delay. The model is constructed using a system of non-linear delay differential equations. In the model, there are three state variables: plant populations (P_1 & P_2) and density of allelochemical (T). It is assumed that in the presence of density of allelochemicals, plant populations are affected. If further delays the conversion of resource into the density of allelochemicals and hence, affect the plant population adversely. This effect is observed by using delay in the competition model. Equilibrium points are calculated. Using the Routh-Hurwitz theorem, certain explicit formulations defining the stability and path of the Hopf-bifurcation periodic solutions at the positive equilibrium are obtained. Further, the directions, of these bifurcating periodic solutions are also studied. Numerical simulation and graphical support are provided for analytical findings using MATLAB.

In chapter-6 A mathematical model is developed for the competing organisms in the presence of allelochemicals with differing time-dependent densities**.** When plant produced allelochemicals, it affects their surrounding plants. A competition mathematical model is proposed in this chapter. It is shown that when one plant produces an allelochemical, it gives stimulatory effect to itself and inhibitory effect to the other plant. Equilibrium points are calculated and stability analysis is performed about non-zero equilibrium point. Hopf-bifuraction is observed with the help of delay parameter introducing in term of allelochemicals. Model is verified with already existing data for the effect of allelochemical on plant growth by Zahid et.al (2016). MATLAB code dde23 is used to support the analytical results numerically.

In chapter-7 Mutual Competition has an impact on plant population dynamics. There is no indication that this biological phenomenon has a detrimental impact on its long-term survival. This dynamical phenomenon’s which is affecting the population dynamics has significant implications for population stability. In this paper, the effect of allelochemical on plant population using competing mathematical model with the help of delay differential equation is studied. Equilibrium point is calculated. The effect of allelochemicals is studied by introducing the delay and stability was measured. We observe that when the value of delay parameter $\tau<1.266$, system shows asymptotically stability and value of delay parameter $\tau\geq1.266$, systems show Hopf-bifuraction. The local and global stability analysis about non-zero equilibrium point was calculated using Routh-Hurwitz criteria and Benedixon-Dulac criterion respectively. Further we calculate the directional analysis of competing plant populations. With this purposed mathematical model, the effect of allelochemicals can be control, and further plant growth enhanced.

Chapter-2

The Study of Stimulatory and Inhibitory Allelopathy Effect on plant Growth Competition

2.1 INTRODUCTION

Allelopathy is generally recognized as the adverse impact of one plant on another plant due to chemical released in atmosphere. ‘Allelopathy’, in the light of the simultaneous inhibitory and stimulating effect of substance, as ‘anyone either direct or indirect favourable or unfavourable effect by a plant (including microorganism) on second by development of chemical compound that run off into the atmosphere’ [1]. The effect of allelopathy is more likely to be sutble, and allelopathy may affect species and/or their life stage differentially. Fluctuations in densities of plant populations under mutual competition are an important component of study of plant physiology. Physical environment, changes in required nutrients and combination of these are important factors which lead to the change in size and densities of plant populations. It has been observed that excessive increase in the population of a plant may influence adversely in growth of second plant or on number of plants by releasing allelochemicals which may prove inhibitory or stimulatory to each other. Allelopathy is the process of releasing phytochemicals by plants into the neighboring environment. It may inhibit the growth and germination of neighboring plants. A plant can excrete a phytochemical toxin such as hydrogen sulphide or volatile fatty acids which can be harmful to another single plant population, more than one plant population or even to itself [80]. In an investigation in old-field in Illinois, eight out of nine bioassays showed inhibition and phytotoxicity on each other [81]. The effect of toxic inhibition phytoplankton caused by some other phytoplankton has been seen by many researchers [82]. The allelochemical influence on herbaceous plants and soil microbial by four species of trees was tested [83]. For sustainable crop production, it is essential to take benefit of both, stimulatory and inhibitory effect of allelopathic plants for regulating plant growth [84]. Root were more sensitive than shoots to the inhibitors produced, and wheat shoots were inhibited more than oat shoots, but oat shoots were inhibited more than wheat roots [85]. The allelopathic impact of leaf residues of Eucalyptus plantation on Soybean crop was studied. The results showed increase in DNA damage frequency of soybean nuclei proportional to Eucalyptus ground leaves used [86]. Many physiological and biochemical process of plants such as photosynthetic rates, respiration, roots reduction etc. are affected by allelochemicals [87]. A meta-analysis of about four hundred studies was undertaken to measure the allelopathic effects of one plant species over the other plant species including itself. The results showed overall 25% reduction in plant performance under allelopathic effects [88]. It has been observed that Centaurea maculosa is an invasive species found in the western part of US, which override indigenous species of plant by discharging the phytotoxin via its roots [89].Experimentally, it has been proved that allelopathic bacteria may grow and mature in that areas in which soil crop is grown repeatedly, and reduced product results cannot be altered back by using any minerals [90]. A review was conducted by taking into consideration, a variety of weed and crop species which manifest inhibitory and stimulatory allelopathic interference in all kinds of possible clinical and field [91].The role of intervention in the process of germination of seeds and the growth of seedlings of crop species should be important, as this weed is perennial and will therefore be maintained. An adequate grant of these compound's incorporation to other phenolic acids. They are also inhibitory to the growth and establishment of seedlings [92]. In numerous plants and several organs including fruits and leaves the allelopathy were found (Inderjit1996, Sazada et al., 2009) [91]. Lepidium sativum was used as a sample to check the poisonous effects of chromium and cadmium. As a result it has been reported that the toxic elements badly affects the development of root and dry biomass [93]. Plant population species require time to attain full maturity and then they release a phytochemical substance which becomes toxic or stimulatory to other plant species. Thus, delay term comes into play in the dynamics. The effect of time lag in plant development affected by toxicant has been studied using system of non-linear delay differential equations [30, 24, 23, 94]. A mathematical model was formulated using delay differential equations to study the growth of two planktonic species under mutual competition and allelopathic effects [4]. Some of these studies specific that allelochemicals, specifically phenolic acid flavonoids and tannins, from alive plants or their remaining parts or exudates hinder it nitrification [96,97]. A detailed analysis was given for several types of stabilities like conditional stability, bifurcation in predator-prey process along with discrete delays and absolute stability [98]. The nature of zeros of exponential characteristic equation was analyzed in detail [99]. The stability analysis of a non-linear system of delay differential equations about equilibrium point is carried out [100]. In the field of agricultural production and plant population allelopathy has appreciable application potential.

The effect of allelochemicals in not immediate, but takes some time to mature, which leads to the scope of introduction of delay parameter. The application of delay parameter in plant competition under allelopathic effect has not been a part of many studies, but has a lot of potential to be explored. In view of this idea, therefore, in this paper the problem related role of delay in plant population competition under the allelopathy effect is studied using the following mathematical model.

2.2 MATHEMATICAL MODEL

2.2.1 Assumptions of the model:

1. It is assumed that the one of the two plant populations release allelochemicals which exhibit both stimulatory and inhibitory effects on the other population.
2. There is inter specific and intra specific competition between plant populations.
3. There is a delay in growth of one plant population due to release of allelochemical released by other, which is represented by parameter τ.

2.2.2 Model Formulation:

Let $P_1$and P_2 be two competing plant populations. It is assumed that the release of allopathic substance is not instantaneous, but some discrete time delay is needed for plant population to mature. We introduce this time delay τ in release of stimulatory allelochemical by first plant population. The competition mechanism is governed by the following system:

$$\frac{dP_1}{dt} = P_1C_1 - \alpha_1 {P_1}^2 - \beta_1 P_1 P_2 + \gamma_1 {P_1}^2 P_2 \tag{1}$$

$$\frac{dP_2}{dt} = P_2C_2 - \alpha_2 {P_2}^2 - \beta_2 P_1 P_2 + \gamma_2 P_1(t-\tau) {P_2}^2 \tag{2}$$

where: $P_1(0) > 0, P_2(0) > 0$ for all t and $P_1(t-\tau) =$ constant for $t \in [0, \tau]$.

The parameters considered in this system are: $C_1, C_2$the rates of cell proliferation per hour; α_1, α_2 the rateof intra specific competition of 1st and 2ndplant population respectively; β_1, β_2 therate of inter specific competition of 1st and 2ndplant population respectively; $\gamma_1, \gamma_2$are the rates of stimulatory allelochemical release of 1st and 2nd plant population species respectively. The units of, α_1, α_2,$\beta_1, \beta_2$and $\gamma_1, \gamma_2$are per hour per cell and the unit of time is hours.

2.3 Equilibrium Points

At the steady state $P_1(t-\tau) \cong P_1$. The steady state equilibrium points are:

$E_1(0,0), E_2\left(\frac{C_1}{\alpha_1}, 0\right)$,$E_3\left(0, \frac{C_2}{\alpha_2}\right)$ and $E_4({P_1}^*, {P_2}^*)$

The non-zero equilibrium $E_4({P_1}^*, {P_2}^*)$ is calculated as follow:

$$\frac{d{P_1}^*}{dt} = 0 \Rightarrow {P_2}^* = \frac{\alpha_1 {P_1}^* - C_1}{\gamma_1 {P_1}^* - 1}$$

$$\frac{d{P_2}^*}{dt} = 0 \Rightarrow (\alpha_1 - \gamma_1\beta_2){P_1}^{*2} + (C_2\gamma_1 - \alpha_1\alpha_2 + \beta_2 - C_1){P_1}^* + (C_1\alpha_2 - C_2) = 0$$

Which is a quadratic equation in ${P_1}^*$, giving two roots as:

$${P_1}^* = \frac{-(C_2\gamma_1 - \alpha_1\alpha_2 + \beta_2 - C_1) \pm \sqrt{(C_2\gamma_1 - \alpha_1\alpha_2 + \beta_2 - C_1)^2 - 4(\alpha_1 - \gamma_1\beta_2)(C_1\alpha_2 - C_2)}}{2(\alpha_1 - \gamma_1\beta_2)}$$

2.4 Stability of Equilibrium $\mathbf{E_4({P_1}^*, {P_2}^*)}$under Stimulatory Allelopathic Effect and Hopf-bifurcation

The system of equations governing the plant population competition mechanism at the equilibrium $E_4({P_1}^*, {P_2}^*)$ is given by:

$$\frac{d{P_1}^*}{dt} = {P_1}^* C_1 - \alpha_1 {P_1}^{*2} - \beta_1 {P_1}^* {P_2}^* + \gamma_1 {P_1}^{*2} {P_2}^* \tag{3}$$

$$\frac{d{P_2}^*}{dt} = {P_2}^* C_2 - \alpha_2 {P_2}^{*2} - \beta_2 {P_1}^* {P_2}^* + \gamma_2 {P_1}^*(t-\tau) {P_2}^{*2} \tag{4}$$

The characteristic equation associated with the system of equations (3)-(4) is given by:

$$\lambda^2 + a\lambda + b + c\mathrm{e}^{-\lambda\tau} = 0 \tag{5}$$

Where $a = (C_1 - 2\alpha_1{P_1}^* - \beta_1{P_2}^* + 2\gamma_1{P_1}^*{P_2}^* + C_2 - 2C_2{P_2}^* - \beta_2{P_1}^*)$,

$$b = (C_1 - 2\alpha_1{P_1}^* - \beta_1{P_2}^* + 2\gamma_1{P_1}^*{P_2}^*)(C_2 - 2C_2{P_2}^* - \beta_2{P_1}^*) + \beta_1{P_2}^*\left(\gamma_1{P_1}^{*2} - \beta_1{P_1}^*\right),$$

$$c = \gamma_2{P_2}^{*2}\left(\gamma_1{P_1}^{*2} - \beta_1{P_1}^*\right)$$

When $\tau = 0$, the equation (5) becomes:

$$\lambda^2 + a\lambda + b + c = 0 \tag{6}$$

By Routh-Hurwitz's criteria, coefficient of equation (6) will have negative real part i.e., the system is balanced if

$$(\mathcal{H}_1): a > 0;$$

$$(\mathcal{H}_2): (\mathscr{b} + c) > 0$$

Which obviously is true.

Now, we would like to check the shifting of negative real part of the roots to positive real parts with variations in the values of τ.

Let $\lambda = i\omega$ be a root of equation (5), then equation (5) becomes:

$$(\mathrm{i}\omega)^2 + a(\mathrm{i}\omega) + \mathscr{b} + c\mathrm{e}^{-(\mathrm{i}\omega)\tau} = 0$$

$$\Rightarrow -\omega^2 + a(\mathrm{i}\omega) + \mathscr{b} + \ c(\cos\omega\tau - \mathrm{i}\sin\omega\tau\ ) = 0$$

Separating real and imaginary parts:

$$\omega^2 - \mathscr{b} = c\cos\omega\tau \tag{7}$$

$$a\omega = c\sin\omega\tau \tag{8}$$

Squaring and adding:

$$\omega^4 + (a^2 - 2\mathscr{b})\omega^2 + (a^2 - c^2) = 0 \tag{9}$$

The two roots of equation (9) are:

$$\omega_{1,2}^2 = \frac{(2\mathscr{b} - a^2) \pm \sqrt{(a^2 - 2\mathscr{b})^2 - 4(a^2 - c^2)}}{2} \tag{10}$$

None of the two roots $\omega_{1,2}^2$ is positive if:

$(\mathcal{H}_3)$: $(2\mathscr{b} - a^2) < 0\ and\ (a^2 - c^2) > 0\ or\ (a^2 - 2\mathscr{b}) < 4(a^2 - c^2)$

That means equation (10) does not have positive roots if condition $(\mathcal{H}_3)$ holds.

We have the following lemma [17].

Lemma 2.4.1 If $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold, then all the roots of equation (5) have negative real parts for all $\tau \geq 0$.

On the other hand, if:

$(\mathcal{H}_4)$: $(a^2 - c^2) < 0\ or\ \ (2\mathscr{b} - a^2) > 0\ and\ (a^2 - 2\mathscr{b})^2 = 4(a^2 - c^2)$

Then, +ve root of equation (7) is ω_1^2.

On the same basis, if:

$(\mathcal{H}_5)$: $(a^2 - c^2) > 0\ or\ \ (2\mathscr{b} - a^2) > 0\ and\ (a^2 - 2\mathscr{b})^2 > 4(a^2 - c^2)$

Then, two +ve roots of equation (7) are $\omega_{1,2}^2$.

In both- $(\mathcal{H}_4)$and $(\mathcal{H}_5)$, the equation (5) has purely imaginary roots when τ takes certain values. The critical values $\tau_j^\pm$ of τ can be calculated from the system of equations (7)-(8), given by:

$$\tau_j^\pm = \frac{1}{\omega_{1,2}} cos^{-1}\left(\frac{\omega_{1,2}^2 - \mathscr{b}}{c}\right) + \frac{2j\pi}{\omega_{1,2}}, j = 0,1,2, \ldots \tag{11}$$

The above discussion can be condensed in succeeding lemma [17].

Lemma 2.4.2 (I) If $(\mathcal{H}_1)-(\mathcal{H}_2)\ and\ (\mathcal{H}_4)$ hold and $\tau=\tau_j^+$, then there a pair of purely imaginary roots in equation (5) is $\pm i\omega_1$.

(II) If $(\mathcal{H}_1)-(\mathcal{H}_2)\ and\ (\mathcal{H}_5)$ hold and $\tau=\tau_j^+\left(\tau=\tau_j^-\ respectively\right)$, then there a pair of purely imaginary roots in equation (5) is $\pm i\omega_1(\pm i\omega_2\ respectively)$.

Our expectation is the shifting of negative real part of some roots of equation (5) to positive real part when $\tau>\tau_j^+$ and $\tau<\tau_j^-$. To look into this possibility, let us denote:

$$\tau_j^{\pm}=\mu_j^{\pm}(\tau)+i\omega_j^{\pm}(\tau); j=0,1,2,3,\ldots$$

The roots of equation (5) satisfy: $\mu_j^{\pm}\left(\tau_j^{\pm}\right)=0, \omega_j^{\pm}\left(\tau_j^{\pm}\right)=\omega_{1,2}$

The preceding transversality can be confirmed and condition holds:

$$\frac{d}{d\tau}\left(Re\ \lambda_j^+\left(\tau_j^+\right)\right)>0\ and\ \frac{d}{d\tau}\left(Re\ \lambda_j^-\left(\tau_j^-\right)\right)<0$$

It concludes that $\tau_j^{\pm}$ are bifurcating values. The succeeding postulate gives the scattering of the zeros of the equation (5) [17].

Theorem 2.1 Let $\tau_j^+(j=0,1,2,3,\ldots)$ be defined by equation (11).

(I) If $(\mathcal{H}_1),(\mathcal{H}_2)$ hold, then all the roots of equation (5) have -ve real parts $\forall\ \tau\geq 0$.

(II) If $(\mathcal{H}_1),(\mathcal{H}_2)\ and\ (\mathcal{H}_4)$ hold and when $\tau\in[0,\tau_0^+)$, then all the roots of equation (5) have -ve real components. When $\tau=\tau_0^+$, then equation (5) has a pair of purely imaginary roots $\pm i\omega_1$. When $\tau>\tau_0^+$, There is always at least a component with a positive real component of the equation (5).

(III) If $(\mathcal{H}_1),(\mathcal{H}_2)\ and\ (\mathcal{H}_5)$ hold and there is an integer m that is positive.

$0<\tau_0^+<\tau_0^-<\tau_1^+<\tau_1^- ---<\tau_{m-1}^-<\tau_m^+$ and There are m transitions between equilibrium and uncertainty. This means, when $\tau\in[0,\tau_0^+),(\tau_0^-,\tau_1^+),---,(\tau_{m-1}^-,\tau_m^+)$, all the roots of equation (5) have negative real parts. When $\tau\in(\tau_0^+,\tau_0^-),(\tau_1^+,\tau_1^-),---,(\tau_{m-1}^+,\tau_{m-1}^-)$ and $\tau>\tau_m^+$, At least one root in equation (5) has a positive real component.

2.5 Stability of Equilibrium $E_4(P_1^*, P_2^*)$ under inhibitory effect

$$p_{ij}P_i^2+q_{ij}P_i+c_{ij}=0\ i,j=1,2\ and\ i\neq j \tag{12}$$

$$And\quad p_{i,j}=b_{ji}\gamma_i-\alpha_i\gamma_i$$

$$q_{ij} = R_i\gamma_j - R_j\gamma_i - a_ia_j - b_{ij}b_{ji}$$

$$c_{ij} = R_ia_j - R_jb_{ij}$$

$$P_1^* = \frac{1}{2p_{ij}}\left(-q_{ij} \pm \sqrt{q_{ij}^2 - 4p_{ij}r_{ij}}\right) \tag{13}$$

With the condition $p_{ij} \neq 0, q_{ij}^2 - 4p_{ij}r_{ij} \geq 0, P_1^* > 0$ (14)

The perturbed system (1) and (2) around equilibrium (P_1^*, P_2^*) can be written as

$$\frac{dx}{dt} = Sx + Ty + \delta_1xy + \delta_2x^2 + \delta_2x^2y$$

$$\frac{dy}{dt} = Ux + Vy + Wx(t-\tau) + \varepsilon_1xy + \varepsilon_2(t-\tau)y^2 + \varepsilon_3y^2 + \varepsilon_1x(t-\tau)y^2 \tag{15}$$

Where $x = P_1 - P_1^*, y = P_2 - P_2^*$

$$S = R_1 - 2a_1P_1^* - b_{12}P_2^* - 2\gamma_1P_1^*P_2^*, T = -P_1^*(b_{12} + \gamma_1P_1^*)$$

$$U = -b_{21}P_2^*, V = R_2 - 2a_2P_2^* - b_{21}P_1^* - 2\gamma_2P_1^*P_2^*$$

$$W = -P_2^*\gamma_2$$

$$\delta_1 = -(b_{12} + 2\gamma_1P_1^*), \delta_2 = -(a_1 + \gamma_1P_2^*), \delta_3 = -\gamma_1$$

$$\varepsilon_1 = -b_{21}, \varepsilon_2 = -\gamma_2, \varepsilon_3 = -(a_2 + \gamma_2P_1^*), \varepsilon_1 = -2\gamma_2P_2^* \tag{16}$$

Retaining only the linear term in equation (15) the liberalized system become

$$\frac{dx}{dt} = Sx + Ty$$

$$\frac{dy}{dt} = Ux + Vy + Wx(t-\tau) \tag{17}$$

The system (17) is same as

$$\frac{d^2x}{dt^2} - (S+V)\frac{dx}{dt} + (SV - TU)x - TWx(t-\tau) \tag{18}$$

We'll use the form for our solution $x(t) = be^{\lambda t}, y(t) = e^{\lambda t}$

$\therefore b = \frac{T}{(\lambda - S)}$ and we have such a characteristic equation that applies to it.

$$\Delta(\lambda,\tau) = \lambda^2 - (S+V)\lambda + (SV - TU) - TWe^{-\lambda t} = 0 \tag{19}$$

Now we're looking at the unusual positive equation's local and global superior properties.

$E_4 = (P_1^*, P_2^*)$ and put $\lambda = \alpha + i\omega$ in equation (19) and separating real and imaginary part, we get:

$$\alpha^2 - \omega^2 - (S+U)\alpha + (SV - TU) + TWe^{-\alpha t}sin\omega\tau = 0$$

$$2\alpha\omega - (S+U)\omega + TWe^{-\alpha t}cos\omega\tau = 0 \tag{20}$$

With the help of Routh-Hurwitz criteria. We can calculate the stability equation (19) and if λ is true or the sign of α satisfy (20), if λ is complex.

Theorem 2.2 A set of adequate and reliable condition for E_4 to be asymptotically stable for $\tau \geq 0$ is following

2.2.1 The real part of all root $\Delta(\lambda, 0) = 0$ are negative.

2.2.2 $\forall$ real part $m \; and \; \tau \geq 0$, $\Delta(im, \tau) \neq 0$, where $i = \sqrt{-1}$.

Proof 2.2.1 From (19) if $\tau = 0$

$$\Delta(\lambda, 0) = \lambda^2 - (S + v)\lambda + (SV - TU - TW) = 0$$

$$\lambda = \frac{1}{2}[(S + V) \pm \sqrt{(S + V)^2 - 4(SV - TV - TV)}]$$

$(S + V)$ is an always negative & $(S + V)^2 - 4(SV - TV - TW) = (S - V)^2 + (TV + TW) \geq 0$

Therefore $\Delta(\lambda, 0)$ has no imaginary root. Then the cases are:

1.$(SV - TV - TW) < 0, \Delta(\lambda, 0) = 0$, has one positive & one negative root.

2. $(SV - TV - TW) = 0, \Delta\,(\lambda, 0) = 0$, has one zero & one negative root.

3. When $(SV - TV - TW) > 0$, Then both the root of $\Delta(\lambda, 0) = 0$ is negative,

We can prove $(SV - TV - TW) > 0$, Hence 2.1 proved.

Proof 2.2.2 if $\Delta(i\omega_0, \tau_0) = 0$ for real ω_0, firstly when $\omega_0 = 0$

$\Delta(0, \tau_0) = (SV - TV - TW) \neq 0,$ Secondly $\omega_0 \neq 0,$

Let $\Delta(i\omega_0, \tau_0) = -\omega_0^2 - i(S + V)\omega_0 + (SV - TU) - TV(cos\omega_0\tau_0 - isin\omega_0\tau_0),$

Separating real and imaginary part:

$$\omega_0^2 + (SV - TU) - TWcos\omega_0\tau_0 = 0$$

$$-(S + W)\omega_0 + TWsin\omega_0\tau_0 = 0 \tag{21}$$

Squaring and adding (21), we get

$$\omega_0^4 + (S^2 + V^2 + 2TU)\omega_0^2 + (SV - TU + TW)(SV - TU - TW) = 0 \tag{22}$$

$(S^2 + V^2 + 2TU)$ remains positive. It's enough to think about the symbol of $(SV - TU + TW)(SV - TU - TW)$ for existence of real root of (22).

Since $TW > 0 \; and \; (SV - TU - TW) > 0,$ we have $(SV - TU + TW) > 0$, hence $\Delta(i\omega_0, \tau_0) \neq 0$, for some original number ω_0. This fulfils the requirement of 2.2

Theorem 2.3 The unique interior E_4 of the system (1) & (2) with allelopathy inhibitory is locally asymptotically stable for all $\tau \geq 0$.

Proof. With the help of theorem (2), the unique equilibrium E_4 of inhibitory allelopathy system always asymptotically stable if $\tau \geq 0$, hence delay is harmless in inhibitory.

2.6 NUMERICAL SOLUTION

The following set of parametric values is taken to represent graphically the dynamics depicted by the system of equations (1)- (2).

$C_1 = 2, C_2 = 1, \alpha_1 = 0.07, \alpha_2 = 0.08, \beta_1 = 0.05, \beta_2 = 0.015, \gamma_1 = 0.0008, \gamma_2 = 0.003$

With $P_1(0) = 10$ and $P_2(0) = 10$

The change of behaviour of the system of equations (1)-(2) under the stimulatory allelopathic effect from being stable to complex dynamics about the equilibrium $E_4({P_1}^*, {P_2}^*)$ for different values of delay parameter τ is shown below:

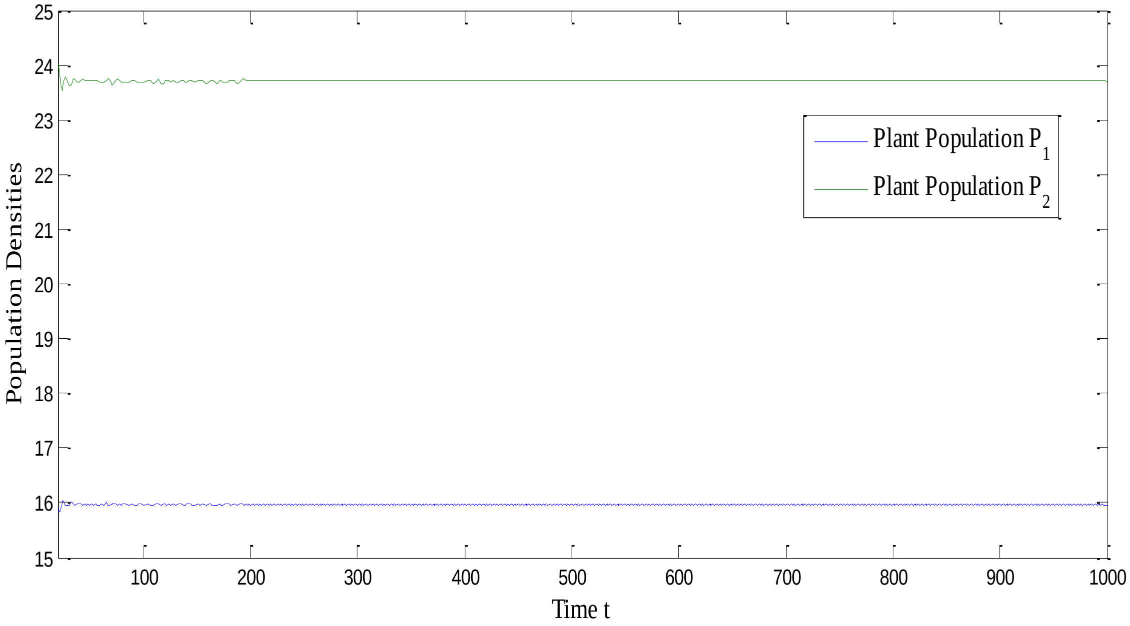

Figure 2.1 At $\tau = 0$, the equilibrium $E_4(15.99, 23.67)$ is stable under stimulatory allelopathic effect.

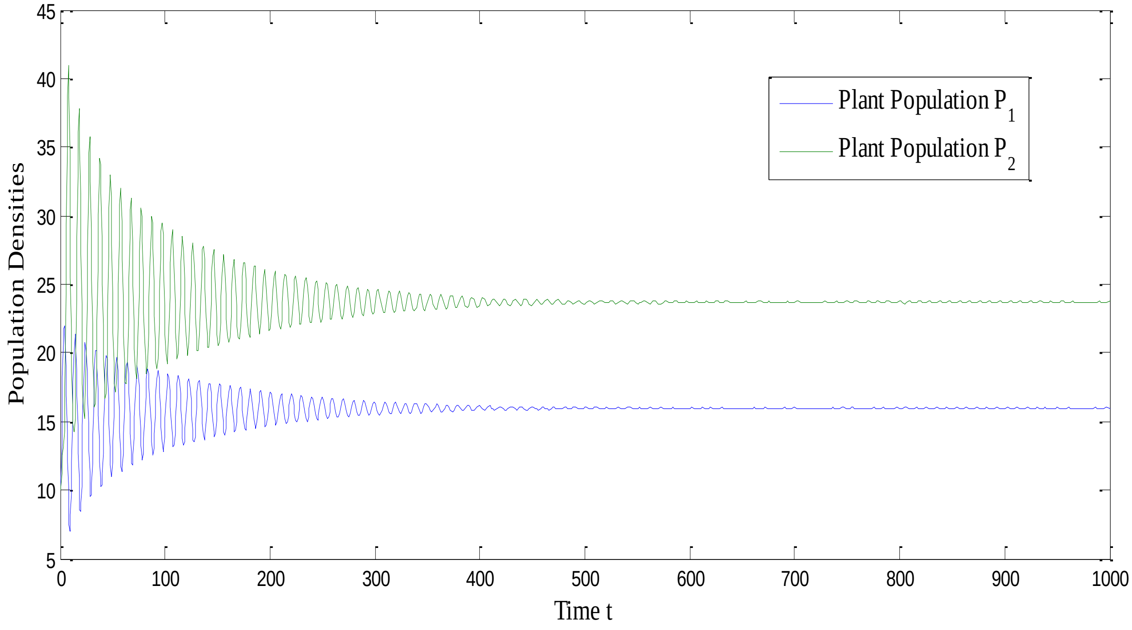

Figure 2.2 The equilibrium $E_4(15.99,23.67)$ is asymptotically stable under stimulatory allelopathic effect when delay is less than critical vale i.e., when $\tau < 2.77$

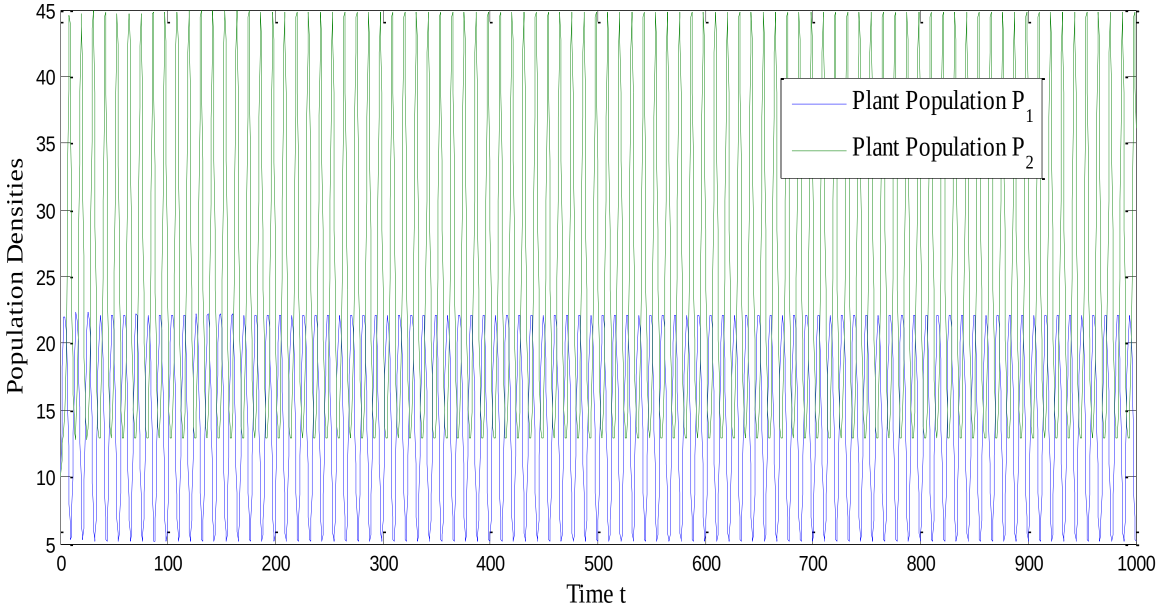

Figure 2.3 The equilibrium $E_4(15.99,23.67)$ losses stability and shows Hopf-bifurcation under stimulatory allelopathic effect when delay crosses the critical vale i.e., when $\tau \geq 2.77$.

The change of behaviour of the system of equations (1)-(2) under the inhibitory allelopathic effect about the equilibrium $E_4({P_1}^*, {P_2}^*)$under the same set of parameters but taking $\gamma_1 = -0.0008, \gamma_2 = -0.003$ is shown below:

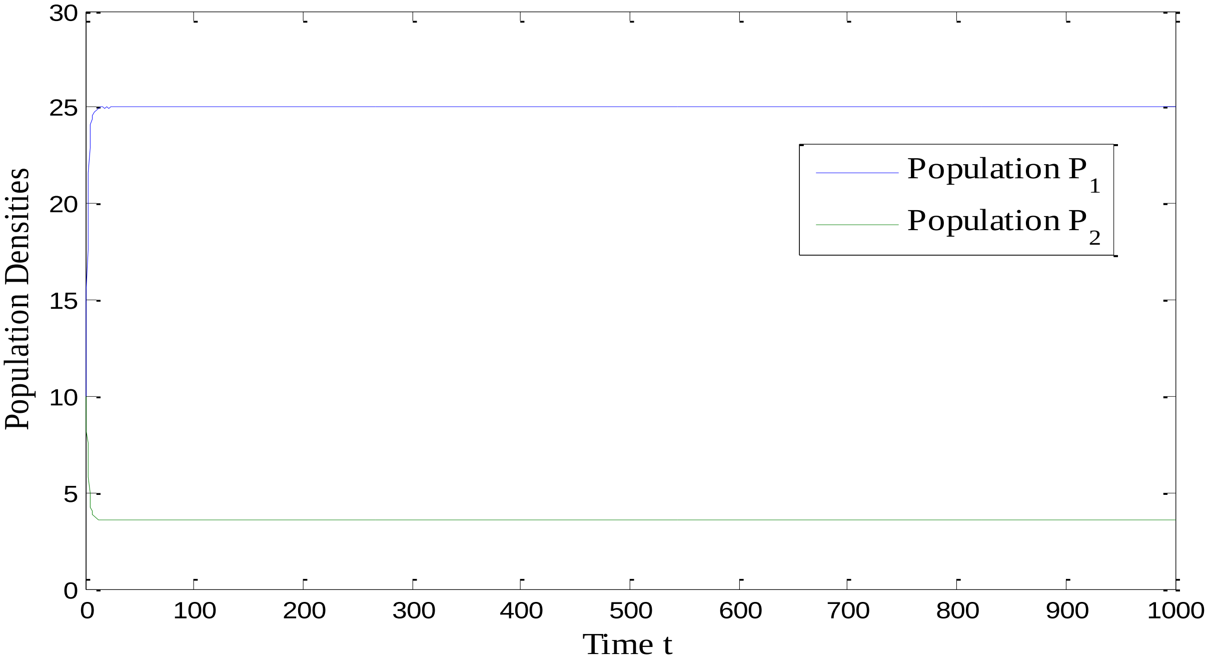

Figure 2.4 The equilibrium $E_4(25.00,3.57)$ showing inhibitory allelopathic effect.

2.7 CONCLUSION

The importance of delay in plant population competition under the stimulatory and inhibitory allelopathic effect has been studied by using proposed model. Delay disrupts the system and initiates complex behaviour with limit cycles and stable periodic solutions through Hopf-bifurcation, when the allelopathic effect is assumed to be stimulatory in nature. The stability examination of the equilibrium E_4 has been carried out. In the absence of delay, the equilibrium E_4 is absolutely stable as shown in figure 2.1. The same fact is also supplemented by $(\mathcal{H}_1) - (\mathcal{H}_2)$ as in lemma 1. When the value of delay parameter τ is below the critical point i.e., $\tau < 2.77$, the equilibriumE_4 starts losing stability and leads to asymptotical stability as shown in figure 2.2. The moment, the delay parameter τ crosses the critical value i.e., $\tau \geq 2.77$, the equilibriumE_4 exhibits the complex dynamics in the form of Hopf bifurcation. The stable periodic solutions having large amplitude and limit cycle trajectory are demonstrated in figure 2.3. This observation of complex behaviour shown by the system (1)-(2) is in agreement with

$(\mathcal{H}_4) - (\mathcal{H}_5)$ as in lemma 2. Figure 2.4 shows the inhibitory allelopathic effect where the value of population density P_2 goes down from 10 to 3.57 as shown in equilibrium E_4.

Chapter-3

Analysis of the Two Mutually Competing Plant Population under Allelopathy

3.1 INTRODUCTION

Allelopathy is generally understood as the adverse impact that one plant has on another plant due to chemicals released into the atmosphere. Unfortunately, however, there has been considerable variation and confusion in the definition and use of allelochemistry. On the surface of the planet, living creatures choose only those behaviours that increase their chances of surviving and passing their genetic on to the next generation. Fluctuations in plant population densities under reciprocal rivalry are an important component of the plan physiology analysis. Changes in the physical environment of the required nutrients and the combination of these are important factors that lead to changes in the size and density of the plant population [1]. Xueyong Zhou and Jingan Cui describe the stability and Hopf-bifuraction analysis of an eco-epidemiological model with delay [101]. The geometric stability switches criteria in a delay differential system with delay dependency parameters [102]. To construct and model of single-species growth in which single parts of the species go through two stages of development: adolescent and robust [103]. Soot-root interaction and resource partitioning playing role plant growth and development can be defined using mathematical modeling in dynamics [38]. Lgwe have shown that modeling of the rising of differential effects of the first inter-competition on the ecodiversity value; competition between two phytoplankton species [104]. Thornely introduced work on mathematical modelling of specific plant development processes, and his models have been used to a wide range of plant biology problems [74]. The soil-nutrient-plant interaction relationship for plant growth can be classified as single community and specified ecosystem. Various plant developments are influenced by nutrients that have an impact on population growth dynamics, which influences standing crop production. Models which were created to explain specific plant development in a crop categorized by Benamin and Hardwick based on the premise that supplies are allocated in a right way [105]. Chattopadhyay also suggested a model for the study of the influence of the toxic agent on a competitive two-species environment withdrawing the clear account of the amount of the toxicant in the model [106]. Many plants physico-chemical processes such as photosynthetic rates, respiration, root reduction, etc. is impaired by allelochemicals [107]. Huang et al. examined a study of the global equilibrium of the non-linear delay differential equation method concerning development of plants [41]. The existence of the zeros of the empirical given polynomial was studied in detail [99]. Kalra & Kumar have studied the effect of toxic metal on plant population with time lag and analyzing that the plant population density decreases with increasing in quantity of toxic metal [95]. Ruan [44] integrated a time delay of some type in biological models by numerous researchers working with the dynamics of single species populations. Plant population dynamics are given a temporal delay because, according to the population growth phenomena, the biomass change is impacted by both the previous and current populations. Kalra and Kumar investigated plant population species that need time to mature and then produce a physicochemical substance that is inhibitory or stimulatory to other plant species. As a result, the delay concept comes into play in the complexity. A non-linear delay differential equation system was used to investigate the effect of the delay parameter on toxicant-affected plant growth dynamics [30,23,24,108]. The mechanism of the exponential characteristic equation's zeros was studied in depth [98]. In this environment, it's valuable noticing by May that the time delay investigations, in which he found that if a physiological or other process has a potentially stable negativity process, but it's applied with a time lag that's longer than the system's organic frequency magnitude the outcome is instability, not stability [109]. Arino at al. [69] worked on delay differential equations and applications in mathematics. Even so, for the most part aspect in nature known as allelopathy, where a single plant species can create a pollutant in the atmosphere affecting a plant species, received relatively little interest in its research. Phytoplankton research is a use full concern with an environment view. All of those are creatures that drift easily and move slowly in an aquatic system. The organism, which are single cells and tiny in size, are extremely useful in sea species. Abbas et al. [28] developed a fractional model for distinct phytoplankton species in which one species generated allelochemicals that stimulated the growth of another. According to Roy et al. [110] allelopathy is expected to decrease competitive exclusion and promote phytoplankton diversity in aquatic ecosystems where lots of species fights for a limited number of resources, due to which competition more occur in between species. Grzyb et al. [111] worked on the environmental factors which affect the crop residue and talk about the application of crop residue and tells us how the various parameter like temperature, soil qualities, the chemical makeup of crop wastes, rain fall, the physical state of microorganism populations, and the carbon-nitrogen balancing in land are influence the biological phenomena. We can control the growth rate and path of processing of agricultural wastes in soil by adjusting qualities of these parameter. Tanveer et al. [112] studied on Waste-land weeds in the fields have an allelopathic effect on crops via their leaf leachates and rhizospheric soils. Weed aquatic plant extracts had a substantially stronger inhibitory impact than their mycorrhizae soils. Furthermore, plants' root development was suppressed more by plant phenolic content than their shoot production was suppressed by their mycorrhizae environments.

We claim that all methods used in the modelling process are moderate and low such that responses to the IVBP are identical and constant at all significant periods. Till date, unusual use of delay differential equation in the domain of allelopathy using competitive plant populations. In the light of the above, we analysis the role of delay in the rivalry of the community of plants under the allelopathic effect when one population releases allelochemicals and harm other, while the other organism doesn’t develop any allelochemicals and discovers that the affecting population may be pushed to death.

3.2 MATHEMATICAL MODEL

The competing plant population is constrained by three state variables: P_1 and P_2 are two competing plant populations respectively, and T is the concentration of allelochemical generated by the species P_2. The competition mechanism is governed by the following system:

$$\frac{dP_1}{dt} = S - \beta_1 P_1 P_2 - a_1 P_1 \tag{1}$$

$$\frac{dP_2}{dt} = \beta_1 P_1(t-\tau) P_2 - b_1 P_2 - a_2 P_2 \tag{2}$$

$$\frac{dT}{dt} = b_2 P_2 - \mathrm{c}_1 T \tag{3}$$

Where $P_1(0) > 0, P_2(0) > 0, T(0) > 0 \forall\ t\ \&\ P_1(t-\tau) = constant\ for\ t\ \&\ t \in [0,\tau]$.

S is the availability of environment factor which have plant population P_1(Constant) , β_1 is the rate of mutual competition between plant population P_1 & P_2 respectively. a_1 & a_2 is the natural decay rate of plant population P_1 & P_2 resp., $b_1$is the environment factor which is stimulating for P_2 (water, nutrient, co₂ gas, etc.) b_2 & c_1 growth rate and natural decay of coefficient of T respectively. The units of a_1, a_2, b_1, b_2 and c_1 are per hour per cell and the unit of time is hours.

These properties would make evaluating the model's suitability for physiological processes much easier. Although it appears to be self-evident that delays exist in nature, they may play a significant influence in system dynamics.

3.3 BOUNDEDNESS

Solutions boundedness of the model is given as:

$$A = P_1 + P_2$$

$$\frac{dA}{dt} = \frac{d(P_1 + P_2)}{dt} = \frac{dP_1}{dt} + \frac{dP_2}{dt}$$

Let $\Omega = min(a_1, a_2\ , \beta_1, b_2, b_1,)\ and\ P_1(t - \tau) \sim P_1$

$$\frac{dA(t)}{dt} \leq \mathrm{S} - \Omega$$

As $t \to \infty$ and us applying the comparison theorem

$$0 \leq (P_1 + P_2) \leq \frac{S}{\Omega}$$

S0, $0 \leq P_1 + P_2 \leq \frac{S}{\Omega}$ and

$$\frac{dT}{dt} \geq -c_1 T$$

Using comparison theorem

When t→ ∞, $\frac{dT(t)}{dt} \to 0$

$$T \leq \frac{1}{c_1}\text{, So}$$

$$0 \leq T \leq \frac{1}{c_1}$$

Therefore, the mention equations of the dynamics system (1) - (3) lie in 3D plane.

$D = [(P_1, P_2, T)\ \varepsilon\ R_+^3 \colon 0 \leq P_1 + P_2 \leq \frac{S}{\Omega}, 0 \leq T \leq \frac{1}{c_1}]$ as t→ ∞, for all positive initial value

Where $P_1(0) > 0, P_2(0) > 0, T(0) > 0 \forall\ t\ \&\ P_1(t - \tau) = constant\ for\ t\ \&\ t \in [-\tau, 0]\} \in D \subset R_+^3$

Where $\Omega = min(a_1, a_2, \beta_1, b_2, b_1,)$

3.4 POSITIVITY OF SOLUTION

Positivity means that the system is sustainable. One has to prove that all the solution has been represent by equation (1)-(3), Where preliminary constrain is $P_1(0) > 0, P_2(0) > 0, T(0) > 0 \; \forall \; t \; and \;\; P_1(t-\tau)T = \text{constant}$ for t ∈ $[0, \tau]$. The solution (P_1(t), P_2(t), T(t)) of the dynamic system remains +ve for every $t > 0$. By using equation (3),

$$\frac{dT}{dt} \geq -c_1 T$$

$$T \geq e^{-c_1 t}$$

Similarly, we can calculate P_1, P_2.

3.5 EQUILIBRIUM POINT

$$\frac{dP_1^*}{dt} = 0$$

$$\frac{dP_1^*}{dt} = S - \beta_1 P_1^* P_2^* - a_1 P_1^*$$

$$S - \beta_1 P_1^* P_2^* - a_1 P_1^* = 0$$

$$P_1^* = \frac{S}{a_1 + \beta_1 P_2^*}$$

& Similarly, we can calculate P_2^*, T^*

$$P_2^* = \frac{\beta_1 S - a_1 b_1 - a_2 a_3}{\beta_1 b_1 + \beta_1 a_2}$$

$$T^* = \frac{b_2 P_2^*}{c_1}$$

3.6 STABILITY OF EQUILIBRIUM $\mathbf{E^*(P_1^*, P_2^*, T^*)}$ AND HOPF- BIFURCATION

The balancing competition process of $E^*(P_1^*, P_2^*, T^*)$ the plant is specified by the equation (1)-(3) scheme.

$$\frac{dP_1^*}{dt} = S - \beta_1 P_1^* P_2^* - a_1 P_1^*$$

$$\frac{dP_2^*}{dt} = \beta_1 P_1^*(t-\tau) P_2^* - b_1 P_2^* - a_2 P_2^*$$

$$\frac{dT^*}{dt} = b_2 P_2^* - c_1 T^*$$

The system's characteristic equation is:

$$\lambda^3 + x_1\lambda^2 + x_2\lambda + x_3 + e^{-\lambda\tau}(y_1\lambda^2 + y_2\lambda + y_3) = 0 \qquad (4)$$

Where $x_1 = b_1 + a_2 + \beta_1 P_2 + a_1 + c_1$

$x_2 = \beta_1 b_1 P_2 + \beta_1 a_2 P_2 + a_1 b_1 + a_1 a_2 + c_1 b_1 + c_1 a_2 + a_1 c_1 + \beta_1 c_1 P_2$

$x_3 = \beta_1 b_1 c_1 P_2 + \beta_1 a_2 c_1 P_2 + a_1 b_1 c_1 + a_1 a_2 c_1$

$y_1 = \beta_1$

$y_2 = \beta_1 c_1 - \beta_1 a_1 - \beta_1^2 P_1 P_2 - \beta_1^2 P_2$

$y_3 = \beta_1^2 (P_1 - 1) P_2 c_1 - \beta_1 a_1 c_1$

Clearly $x_1, x_2, x_3, y_1, y_2, y_3$ all are " + ve".

Clearly $\lambda = \mathrm{i}\phi$ is a solution of Eq. (4) iff

$$(\mathrm{i}\phi)^3 + x_1(\mathrm{i}\phi)^2 + x_2(\mathrm{i}\phi) + x_3 + e^{-\mathrm{i}\phi\tau}(y_1(\mathrm{i}\phi)^2 + y_2(\mathrm{i}\phi) + y_3) = 0 \qquad (5)$$

Separating real and imaginary part:

$$x_3 - x_1\phi^2 + (y_3 - y_1\phi^2)\cos(\phi\tau) + y_2\phi\sin(\phi\tau) = 0 \qquad (6)$$

$$x_2 - \phi^3 + y_2\phi\cos(\phi\tau) - (y_3 - y_1\phi^2)\sin(\phi\tau) = 0 \qquad (7)$$

Squaring and adding (6) and (7)

$$\phi^6 + ({x_1}^2 - {y_1}^2 - 2x_2)\phi^4 + ({x_2}^2 - {y_2}^2 + 2\,y_1 y_3 - 2\,x_1 x_3)\phi^2 + ({x_3}^2 - {y_3}^2) = 0 \qquad (8)$$

Put

${x_1}^2 - {y_1}^2 - 2x_2 = j$

${x_2}^2 - {y_2}^2 + 2\,y_1 y_3 - 2\,x_1 x_3 = k$

${x_3}^2 - {y_3}^2 = l$

And put $\phi^2 = z$, we get

$$z^3 + jz^2 + kz + l = 0 \qquad (9)$$

If $l < 0$, Eq. (8) will give minimum one positive real root

Let $h(z) = z^3 + jz^2 + kz + l$

$h(0) = l < 0, \quad \lim_{y\to\infty} \mathrm{h(z)} = \infty, \ \exists\, z_0 \in (0, \infty)$

$h(z_0) = 0$ if

$l \geq 0$ then essential constrain for Eq. (8) to have +ve real root is

$D = j^2 - 3k \geq 0$, where D is the discriminant

$h(z) = z^3 + jz^2 + kz + l$

$h'(z) = 3z^2 + 2jz + k$

$$h'(z) = 0 \ \rightarrow 3z^2 + 2jz + k = 0 \qquad (10)$$

$$y_{1,2} = \frac{-2j \mp \sqrt{4j^2 - 12k}}{6} = \frac{-j \mp \sqrt{D}}{3} \qquad (11)$$

Eq. (10) will not contain any real roots if $D < 0$. So, that $h(z)$ is raising monotonic function in z, $h(0) = l \geq 0$ Eq. (8) will not contain any positive real roots.

Clearly if $D \geq 0$, then $z_1 = \frac{-j+\sqrt{D}}{3}$ is local minima of h(z)

If $l \geq 0$ then Eq. (9) will contain positive root if $z_1 > 0$ and $h(z_1) > 0$

Let us suppose that either $z_1 \leq 0$ or $z_1 > 0$ and $h(z_1) > 0$

If $z_1 \leq 0, \sin h(z)$ is increasing for $z \geq z_1$ and $h(0) = l \geq 0$

It follows that $h(z)$ do not contain positive real zero.

If $z_1 > 0$ and $h(z_1) > 0, \sin z_2 = \frac{-j+\sqrt{D}}{3}$ is local maxima value,

It follows that $h(z) \leq h(z_2)$

$h(0) = l \geq 0$, As $h(z)$ do not give any positive real root.

Lemma 3.1 Suppose $z_1$is defined by Eq. (11)

(1.1) If $l < 0$, Eq. (9) will give at the minimum one positive real root.

(1.2) If $l \geq 0$ and $D = j^2 - 3k < 0$, then equation (9) doesn't contain any positive root.

(1.3) If $l \geq 0$, then Eq. (9) has positive root if $z_1 > 0$ and $h(z_1) \leq 0$.

Let Eq. (9) contains a positive root. W L O G, let us assume that it contains three positive roots z_1, z_2, z_3. Then Eq. (8) will contain three positive roots

$\phi_1 = \sqrt{z_1}, \phi_2 = \sqrt{z_2}, \phi_3 = \sqrt{z_3}$

From (7)

$$\sin \phi\tau = \frac{x_2\phi - \phi^3}{d}$$

$$\tau = \frac{1}{\phi}\left[\sin^{-1}\frac{(x_2\phi - \phi^3)}{d} + 2(i-1)\pi\right] : i = 1,2,3$$

$$\tau_p^{(i)} = \frac{1}{\phi p}\left[\sin^{-1}\frac{(x_2\phi - \phi^3 p)}{d} + 2(i-1)\pi\right] : p = 1,2,3 \text{ i} = 0,1,2 \ldots\ldots \tag{12}$$

Above equation is the set of pure imaginary roots of Eq. (8)

If

$$\tau = \tau_p^{(i)}, p = 1,2,3; i = 0,1,2 \ldots..$$

$$\lim_{j\to\infty} \tau_p^{(i)} = \infty, \qquad \text{p} = 1,2,3$$

$$\tau_0 = \tau_{p_0}^{(i_0)} = \min\nolimits_{1\leq p\leq 3, i\geq 1}[\tau_p^{(i)}], \phi_0 = \phi_{p_0}, \; z_0 = z_{p_0} \tag{13}$$

Lemma 3.2 Suppose that $x_1 \geq 0, (x_3 + d), x_1, x_2(x_3 - d) > 0$

(2.1) If $l \geq 0$ and $D = j^2 - 3k < 0$ then the obtained roots of Eq. 4 will be negative real when $\tau \geq 0$

(2.2) If $l < 0 \; or \; l \geq 0, z_1 > 0$ and $h(z_1) \leq 0$ then the obtained roots of Eq. 4 will be negative real when $\forall \tau \, \epsilon (0, \tau_0)$,

Proof: When $\tau = 0$, then equation (4) become

$$\lambda^3 + (x_1 + y)\lambda^2 + (x_2 + y_2)\lambda + (x_3 + y_3) = 0 \qquad (14)$$

By using Routh- Hurwitz's Criteria

the obtained roots of Eq. 5 will be negative iff

$(x_3 + y_3) \geq 0, (x_1 + y_1)(x_2 + y_2) - (x_3 + y_3) > 0$

If $l \geq 0$ and $D = j^2 - 3k < 0$,

Lemma 1(1.2) show that equation (4) has no real roots $\forall \; \tau \geq 0$

If $l < 0 \; or \; l \geq 0, z_1 > 0$ & $h(z_1) \leq 0$,

Lemma 1(1.1) & (1.2) implies when $\tau \neq \tau_p^{(i)}, p = 1,2,3, i \geq 1$, eq. (4) doesn't have a root with the real component of zero, and even the value of τ is minimal, thus it is completely imaginary.

Lemma 3.3 Suppose$z_0 = \phi_0^2$, if $\tau = \tau_0$. Then $\text{Sign}[\varphi'(\tau_0)] = \text{Sign}[f'(z_0)]$.

Proof: Let us assume that

$\lambda(\tau) = \varphi(\tau) + i\phi(\tau)$

Be the root of (4) and put $\varphi(\tau_0) = 0, \phi(\tau_0) = \phi_0$ be sure that $\mp\phi_0$ are simply imaginary root of equation (4) with $\tau = \tau_0$ and $\lambda(\tau)$ satisfying transversely constrains, suppose that $f'(z_0) \neq 0$. By substituting $\lambda(\tau)$ in equation (4) and differentiating w.r.t. λ, we get

$$\frac{d\lambda}{d\tau}\big[3\lambda^2 + 2x_1\lambda + x_2 + \big\{(y_1\lambda^2 + y_2\lambda + y_3)(-\tau) + (2y_1\lambda + y_2)e^{-\lambda\tau}\big]$$

$$= \lambda(y_1\lambda^2 + y_2\lambda + y_3)e^{-\lambda\tau}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2x_1\lambda + x_2)e^{-\lambda\tau}}{\lambda(y_1\lambda^2 + y_2\lambda + y_3)} + \frac{(2y_1\lambda + y_2)}{\lambda(y_1\lambda^2 + y_2\lambda + y_3)} - \frac{\tau}{\lambda}$$

From (5), (6) and (7):

$$\varphi'(\tau_0) = \text{Re}\left[\frac{(3\lambda^2 + 2x_1\lambda + x_2)e^{-\lambda\tau}}{\lambda(y_1\lambda^2 + y_2\lambda + y_3)}\right] + \text{Re}\left[\frac{(2y_1\lambda + y_2)}{\lambda(y_1\lambda^2 + y_2\lambda + y_3)}\right] = \frac{[3\phi_0^6 + 2j\phi_0^4 + k\phi_0^2]}{[(y_3 - y_1\phi^2)^2 + (y_2\phi)^2]}$$

Here $[(y_3 - y_1\phi^2)^2 + (y_2\phi)^2] > 0$ & $\phi_0 > 0$

It is shows that $\text{Sign}[\varphi'(\tau_0)] = \text{Sign}[f'(z_0)]$.

Theorem 3.1: Let τ_0 and ϕ_0 be defined by equation (12) and $z_0 = \phi_0^2$. Let $(x_1 + y_1) > 0, (x_3 + y_3) > 0, (x_1 + y_1)(x_2 + y_2) > 0$.

(1) If $l \geq 0$ and $D = j^2 - 3k < 0$, then the equilibrium $E^*(P_1^*, P_2^*, T^*)$ of the system (1)-(3) is absolutely stable.

(ii) If $l < 0$ and $l \geq 0, z_1 > 0$ and $\mathrm{f}(z_1) \leq 0$, then the equilibrium $E^*(P_1^*, P_2^*, T^*)$ of the equation (1)-(3) is asymptotically stable $\forall\ \tau \in [0, \tau_0)$.

(iii) If the criteria (ii) are true for $\tau = \tau_0$ & $f'(z_0) \neq 0$, then the equation exhibits Hopf-bifuraction at $E^*(P_1^*, P_2^*, T^*)$.

3.7 NUMERICAL EXAMPLE

The dynamics described by the numerical solution are graphically using dde23 command in MATLAB software and described by the following set of parameters:

$$S = 0.4, \beta_1 = 1.98, a_1 = 0.59, b_1 = 0.5, a_2 = 0.6, b_2 = 0.9, c_1 = 0.5.$$

The dynamics behaviour of (1)-(3) shows by different value of delay as shown below:

$$(E^*(P_1^* = 0.2156, P_2^* = 0.4511, T^* = 1.7289)$$

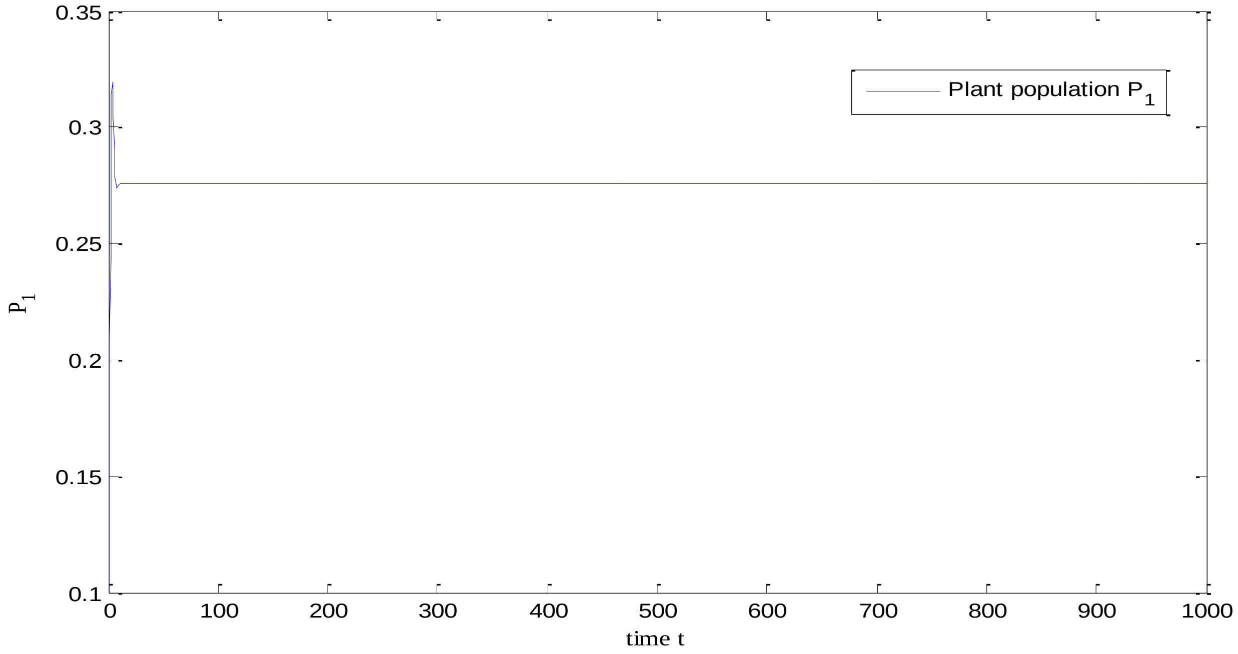

Figure 3.1 When there is no delay the plant population P_1 is stable, at $\tau = 0$.

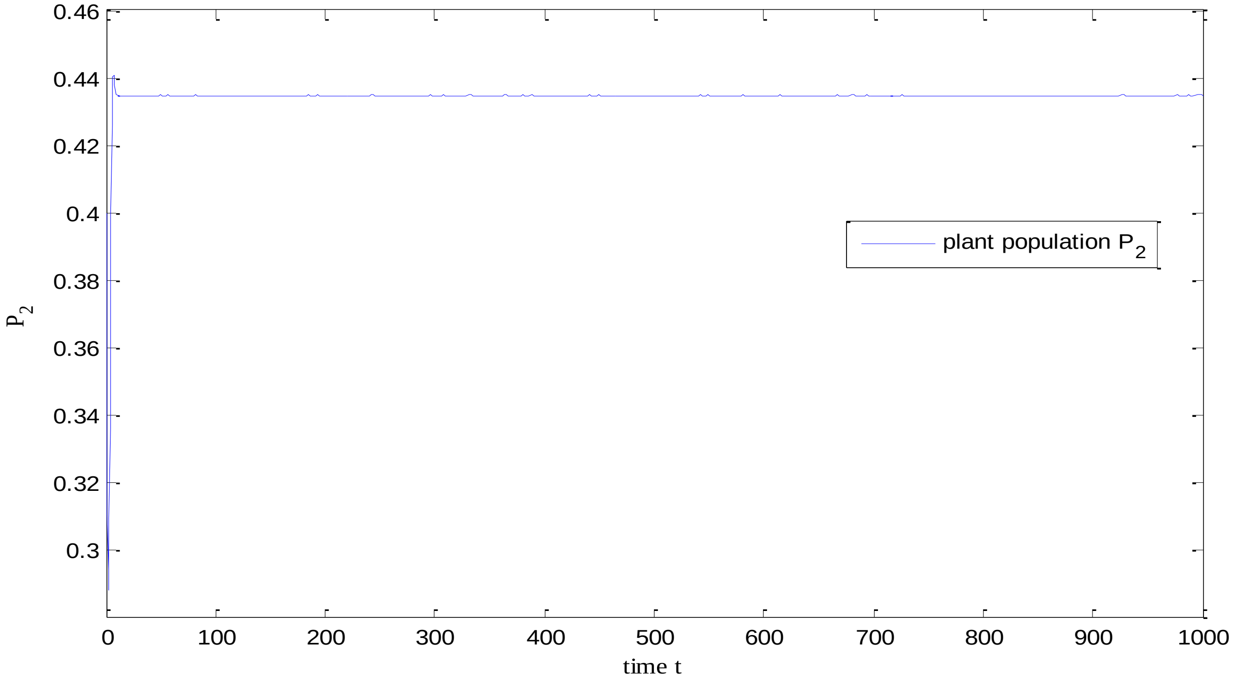

Figure 3.2 When there is no delay the plant population P_2 is stable, at $\tau = 0$.

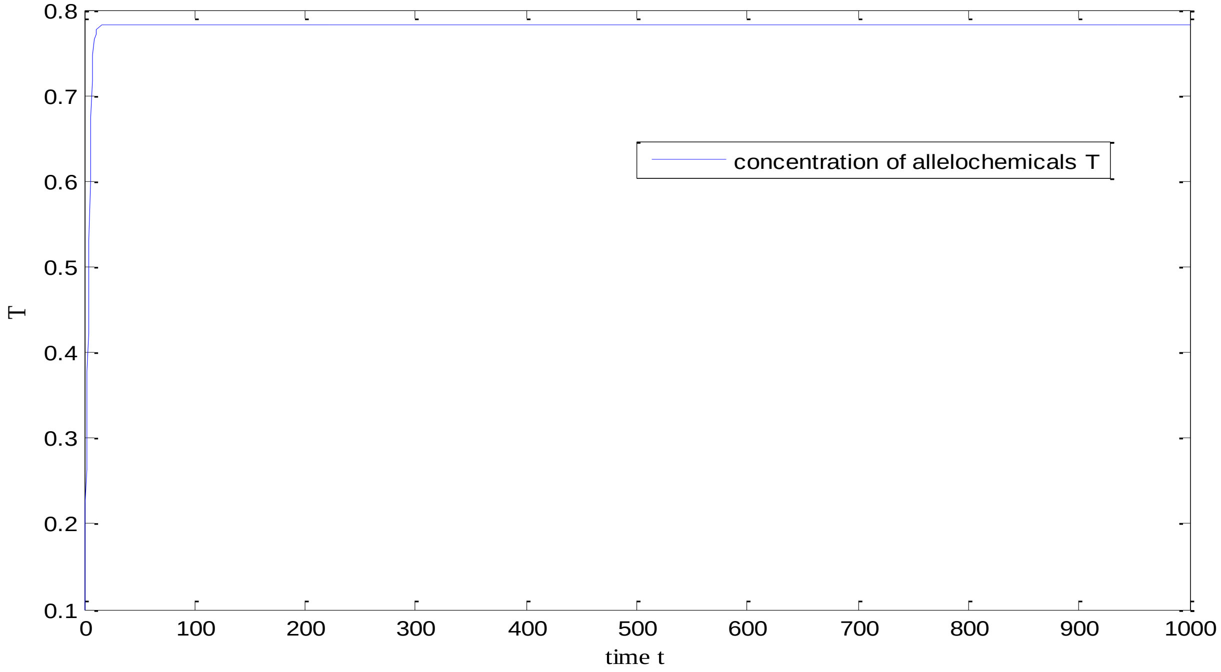

Figure 3.3 When there is no delay, the concentration of allelochemicals T is stable at $\tau = 0$.

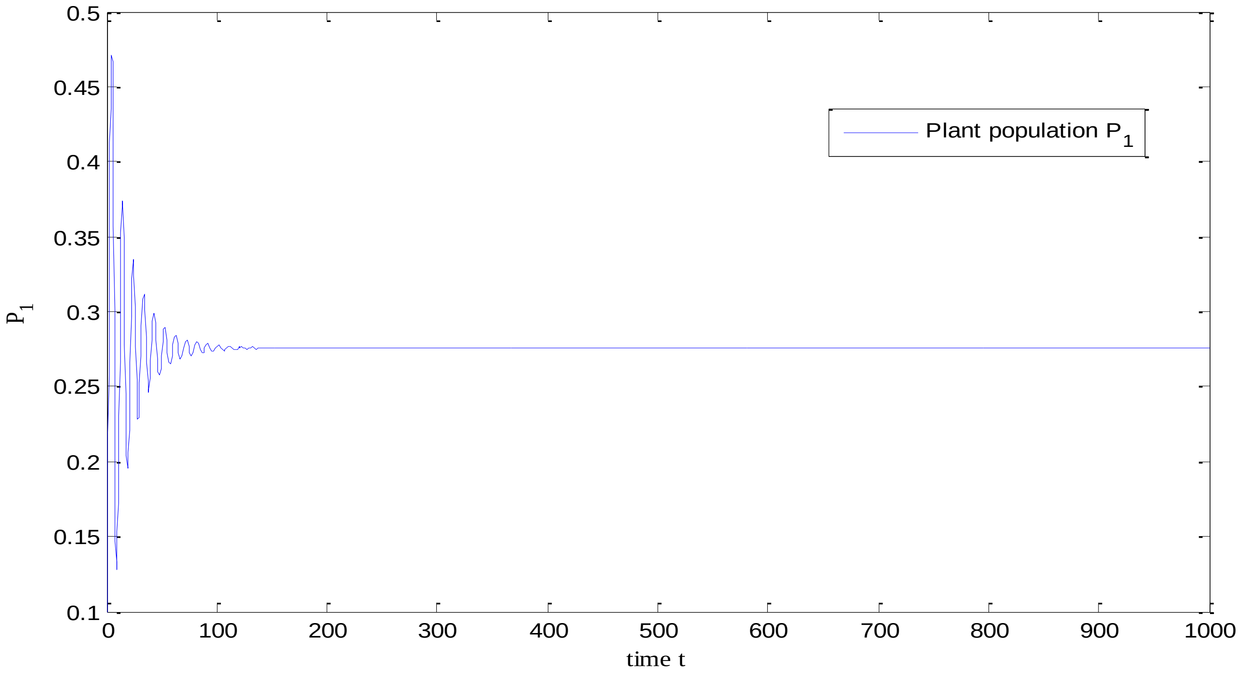

Figure 3.4 The plant population P_1 shows asymptotically stable at $\tau < 2.222$.

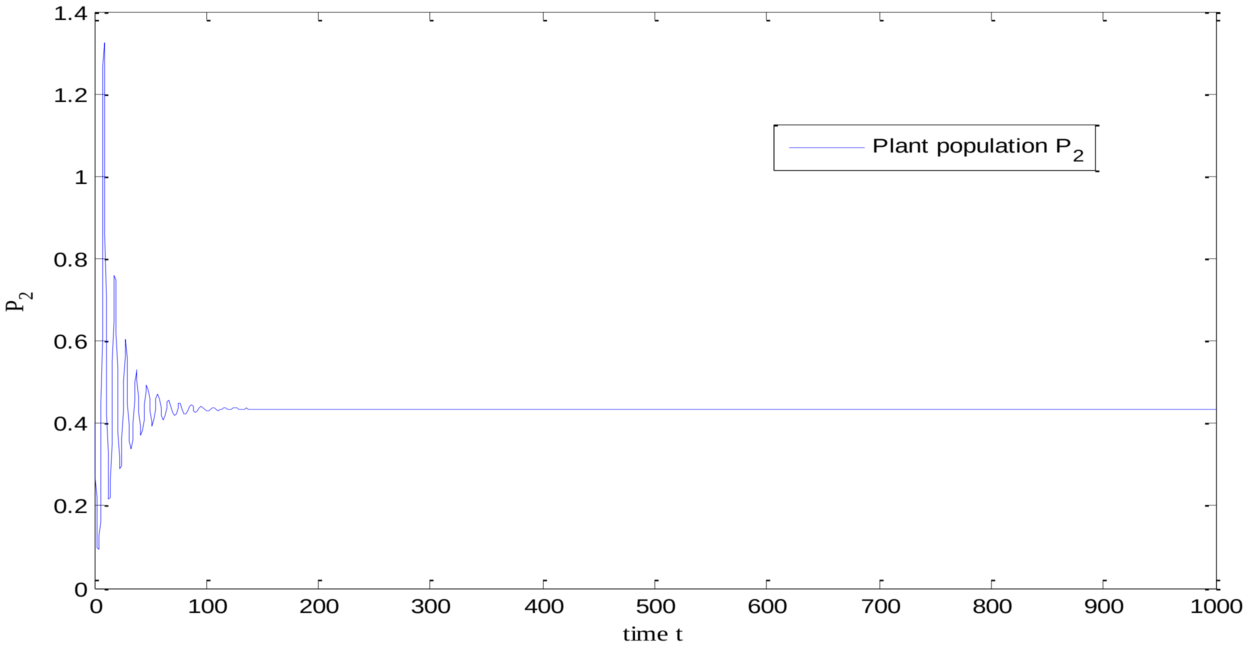

Figure 3.5 The plant population P_2 shows asymptotically stable at $\tau < 2.222$.

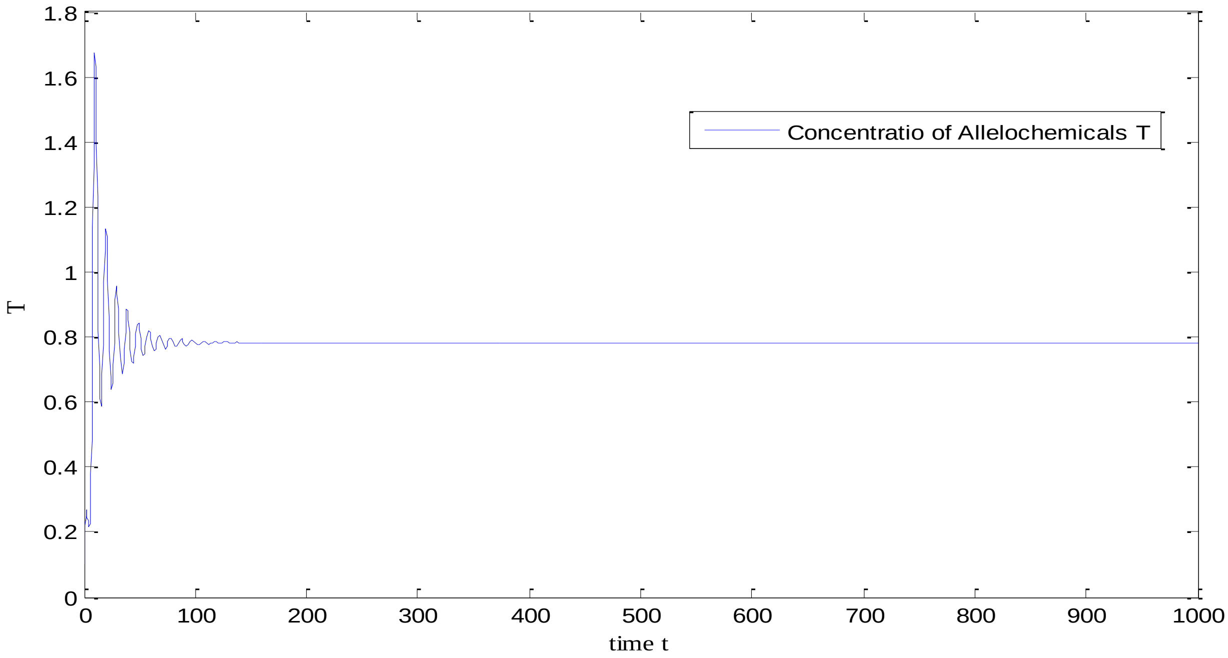

Figure 3.6 The concentration of allelochemicals T shows asymptotically stable at $\tau < 2.222$.

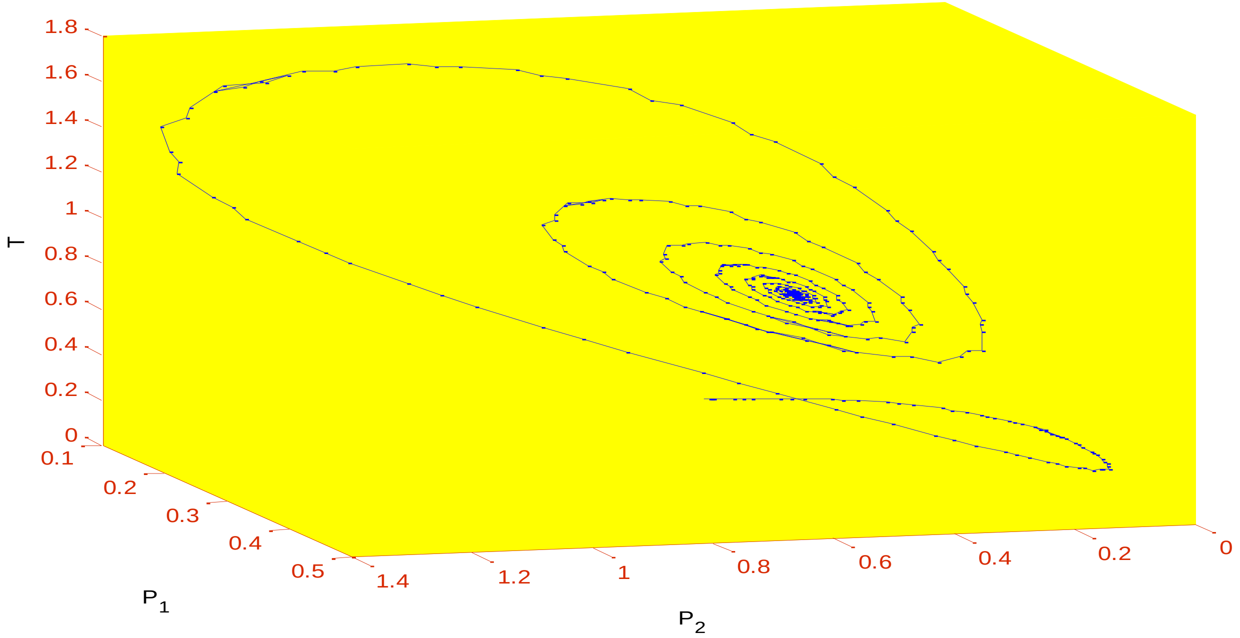

Figure 3.7 Phase space of Plant population P_1, P_2 and Conc. of Allelochemicals T when delay is $\tau < 2.222$.

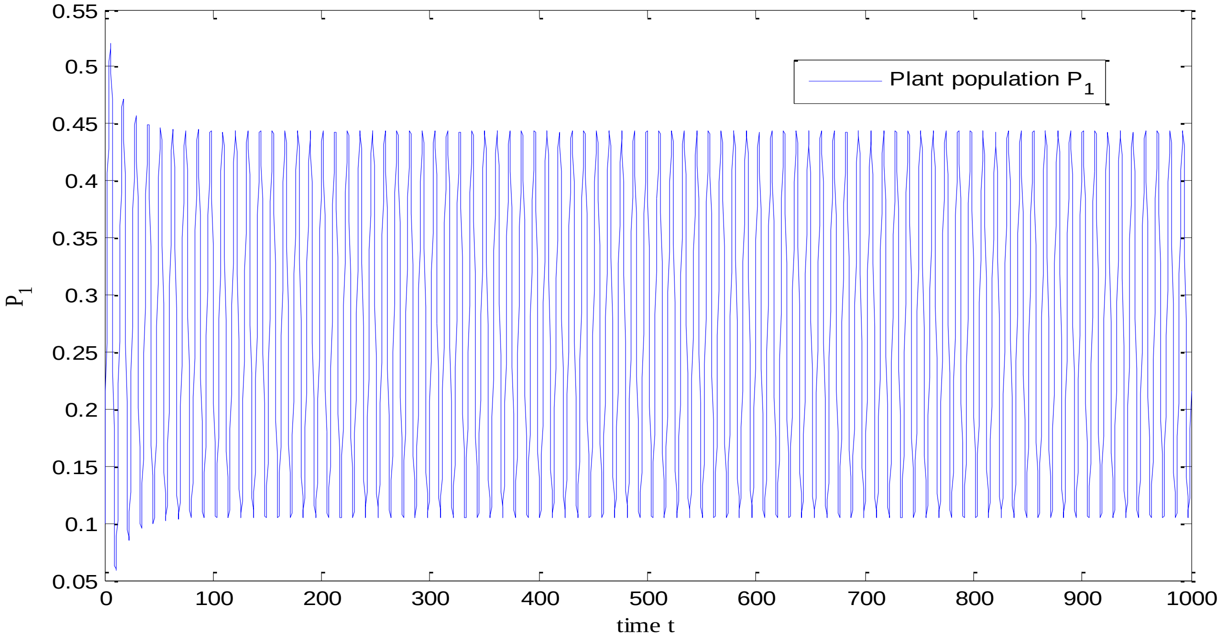

Figure 3.8 The plant population P_1 losses its asymptotically and Hopf-bifuraction obtained at $\tau \geq 2.222$

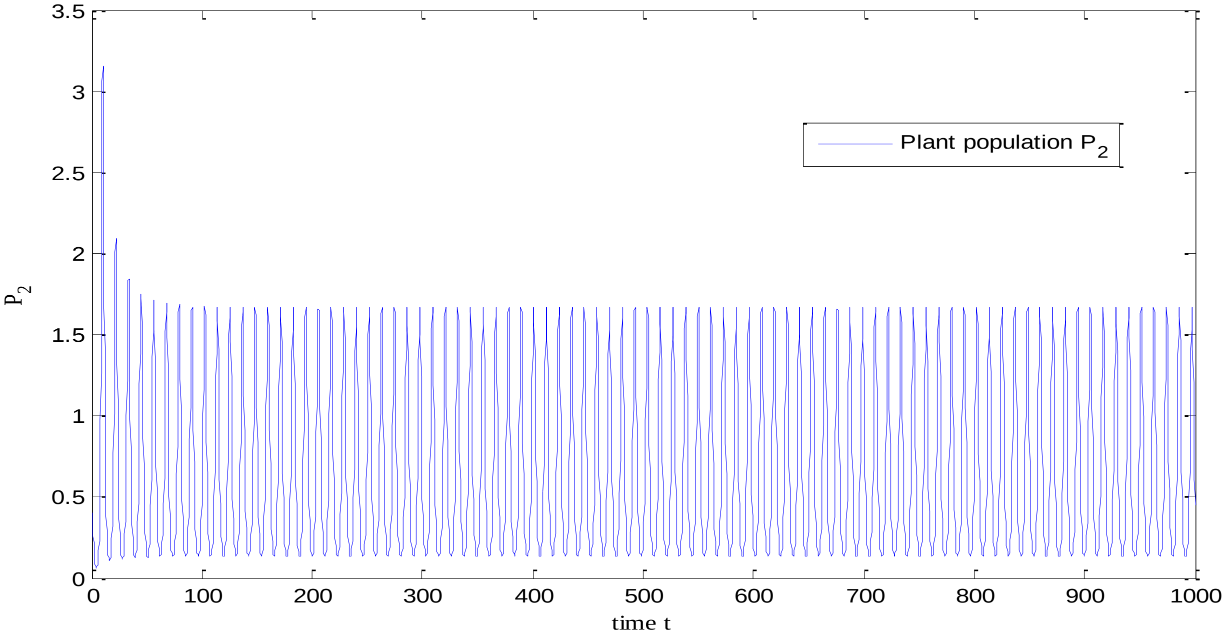

Figure 3.9 The plant population P_2 losses its asymptotically and Hopf-bifuraction obtained at $\tau \geq 2.222$

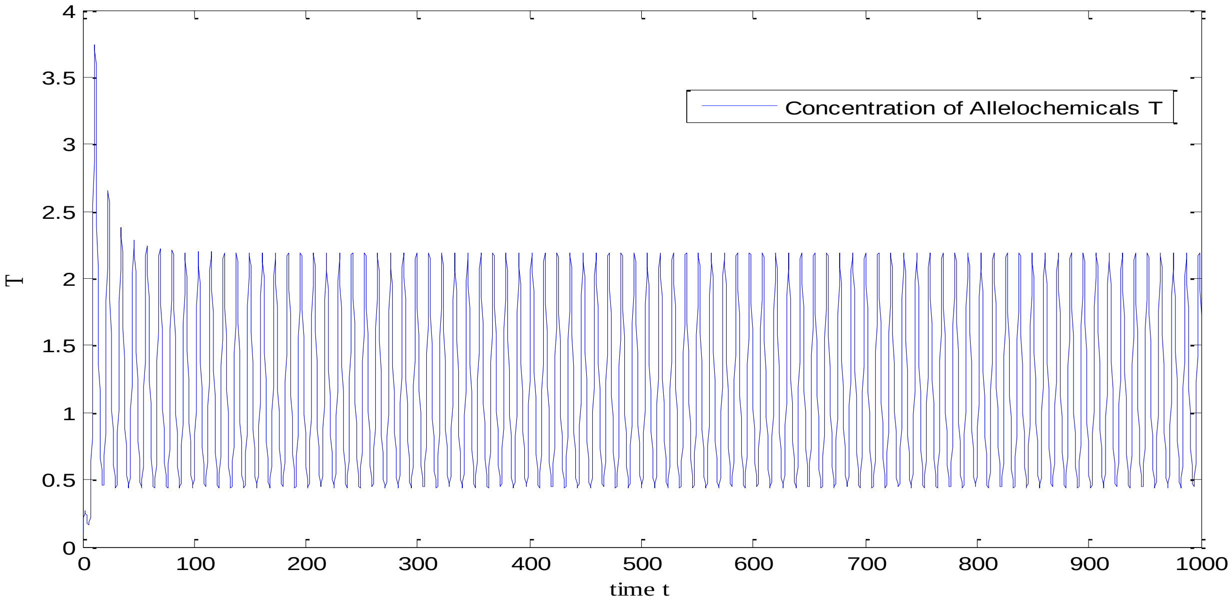

Figure 3.10 The concentration of allelochemicals T losses its asymptotically and Hopf-bifuraction obtained $\tau \geq 2.222$.

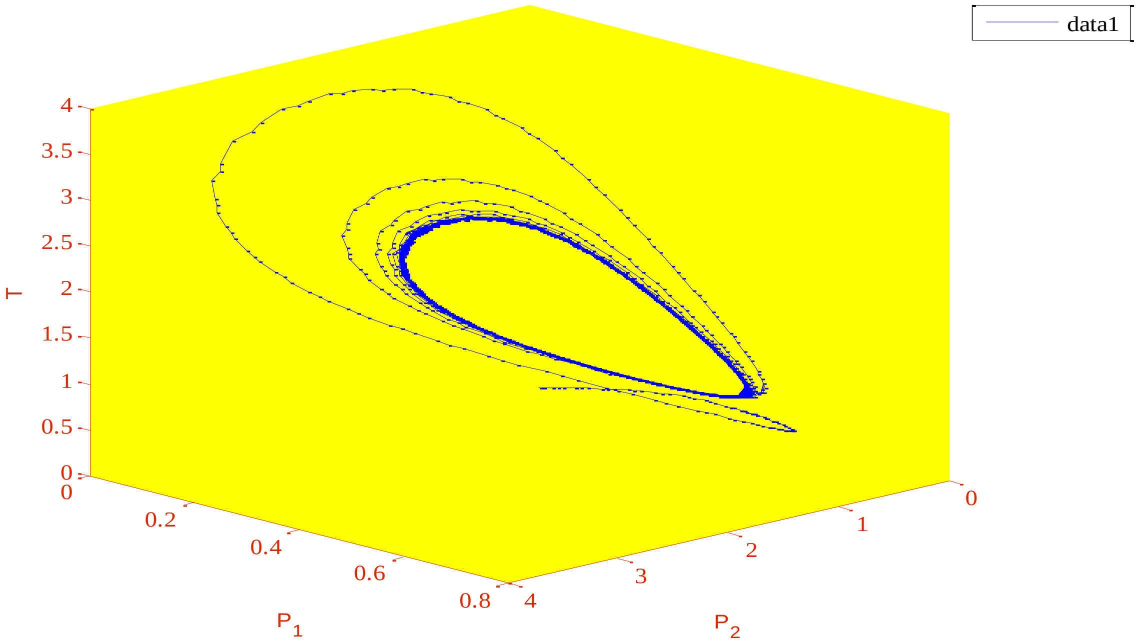

Figure 3.11 Phase space of Plant population P_1, P_2 and Conc. of Allelochemicals T when delay is $\tau \geq 2.222$.

3.8 CONCLUSION: An mathematical model is proposed in this paper for investigating influence of time lag (delay) on two competing plant population underneath influence of allelochemicals. We assume from the literature review that the plant population is influenced by allelochemicals. Delay interrupts the mechanism and initiates dynamic behaviour by Hopf-bifurcation with minimal intervals. The stability analysis of equilibrium $E^{*}(P_1^{*}, P_2^{*}, T^{*})$ is done. We observe that in the absence of delay, plant population is stable with the help of Routh-Hurwitz's method. But if the value of time lag (delay) is less than the critical value, plant population losses its stability and move for the asymptotically stability of plant population. When the value of time lag (delay) is greater than the critical value, system loses the asymptotically stability and it starts oscillation and periodic solutions occur leading to hopf-bifuraction.

The future scope of this model includes the practical use of delay differential equation for verification of the statistical data obtained from the agriculture field. It will help the farmers to predict in advanced. The rate of growth as well as the factor's that adversely affectively, the growth under the interval of time-legs and delay.

Chapter-4

Effect of Allelochemical on Two mutually Competiting Plant Population

4.1 INTRODUCTION

Allelopathy is generally understood as the adverse impact that one plant has on another plant due to chemicals released into the atmosphere. Unfortunately, however, there has been considerable variation and confusion in the definition and use of allelochemistry. Fluctuations in plant population densities under reciprocal rivalry are an important component of the plan physiology analysis. Changes in the physical environment of the required nutrients and the combination of these are important factors that lead to changes in the size and density of the plant population [1]. Biochemical interactions between plants of all sorts, including the microorganism usually placed in the plant genus. Of instance, these microorganisms comprise several microscopic algae, fungi and bacteria [113]. The biological species of all terrestrial and marine environments is impacted by the effect of toxicants or toxins released into the atmosphere on their habitats [114]. In particular, these toxicants or toxins are released from development of food processes and from pollution or may be generated by the species itself [115,116,117]. The effects of environmental pollution on the growth of plant communities have been studied [118]. Sulphur dioxide has had an impact on crop yield, growth, photosynthetic activity and secretion [119]. The cause of the release of toxicants to the atmosphere is natural. Even so, there will be circumstances in the natural world, as with the situation of allelopathy, where pollutants created by a higher plant can affect other competitive plant species in the habitat. It's already been observed that, in this case, plant leaves and flowers perform an important role in the quality of impurities [120]. Even though amount of pollutant released by the production facility is not very high, it is adequate to have a strong effect on the soil micro - organisms and to greatly influence the development of the origin as well as the surrounding plants. The influence of a toxin released to the atmosphere on the inhabitants, considering that perhaps the population size rate of growth depending on the absorption of the toxic substance by the environmental load power [121]. A non-linear system is used to analyses the concurrent the effectiveness on various pollutants, each becoming more toxic than another, also on behavioral environment using the ordinary differential equilibrium method [122]. A situational of approximately four hundred studies was conducted to assess the allelopathic impact of one species of plants on the other plant species, including itself. Overall, the results indicate a 25% decrease in plant production under allelopathic effects [30]. It has been noted that centaurea maculosis is an exotic species located in the Western United States, which overwhelms native plant species by dispersing the plant toxin from its root [24]. Experimentally, allelopathic microbes have been found to be able to grow and develop in environments where as such is an cultivated regularly and the decreased yield could be changed by the use of certain chemicals [23]. The study of the global equilibrium in non-linear delay differential equation is used to investigation the population growth [41]. The accumulation of heavy metals could detriment influence behaviour using a two-bay mathematical model [21]. Analyze the role of toxic effects on plant biomass over time [123]. Analysed the development of two planktonic organisms in the sense of reciprocal rivalry and allelopathic effects [4]. A detailed evaluation was carried for different forms of stability, such as total stability, conditional stability and bifurcation in predator-prey systems with discrete delays [99]. The existence of the zeros of the empirical given polynomial was studied in detail [100]. The study of time lag on plant growth under the effect of toxic metal was studied [95]. Even so, one of the most important aspects in nature known as allelopathy, where a single plant species can create a pollutant in the atmosphere affecting a plant species, received relatively little interest in its research [124].

From the above-mentioned related research work, it is noted that the toxicants released into the atmosphere are detrimental to the populations living in the ecosystem and that an analysis of these is thus completely important. Throughout the 1990s, mathematical simulations have become valuable methods for studying and forecasting the behaviour of biogeochemical cycles, yet modelling the impact of environmental toxins on population genetics in each biodiversity is a growing interest area of study in the field of environmental science. It should also be remembered that many of these experiments have been observational in the old days and little measures have been taken to explain these processes using mathematical models.

4.2 MATHEMATICAL MODEL:

The following system of non-linear delay differential equations with three system parameters determines the dynamics of competing plant populations: P_1 & P_2 is two competing plant populations and T is the concentration of allelochemical produced by the species P_2. The competition mechanism is governed by the following system:

$$\frac{dP_1}{dt} = c_1P_1 - \alpha_1 {P_1}^2 - \beta_1 P_1 P_2 + r_1 {P_1}^2 P_2 \tag{1}$$

$$\frac{dP_2}{dt} = c_2P_2 - \alpha_2 {P_2}^2 - \beta_2 P_1 P_2 + r_2 P_1(t-\tau) {P_2}^2 \tag{2}$$

$$\frac{dT}{dt} = r_3P_2 - r_4T - r_5P_1T \tag{3}$$

Where $P_1(0) > 0$, $P_2(0) > 0$, $T(0) > 0$ $\forall\, t$ and $P_1(t-\tau)$ = constant for $t \in [0, \tau]$.

Where P_1, P_2 the competing plant populations of 1st and 2nd species, T are is the concentration of allelochemical produced by the species P_2. c_1, c_2 are the rate of cell proliferation of 1st species and 2nd species per hour. α_1, α_2 are the change of intra specific competition of 1st and 2nd plant population respectively. β_1, β_2 are the change of inter specific competition of 1st and 2nd plant population resp. r_1, r_2 are the rate of stimulatory allelochemical effect on 1st and 2nd plant population. r_3 is the Inflow of the allelopathic effect due to P_2. r_4 is the natural decay rate of T. r_5 is the decay of T due to intersection with P_1. τ is the delay parameters. The units of α_1, α_2, β_1, β_2 and r_1, r_2, r_3, r_4, r_5 are per hour per cell and the unit of time is hours.

4.3 POSITIVITY OF SOLUTION

Positivity means that the system is sustainable. One has to prove that all the solution given by equation one to three are positive, where initial constraints are $P_1 > 0, P_2 > 0, T > 0$ $\forall\, t$ and $P_1(t-\tau)$= constant for t for $t \in [0, \tau]$, the resultants ($P_1(t)$, $P_2(t)$, $T(t)$) of system remain positive $\forall\, t > 0$.

And from the system, we take equation (3)

$$\frac{dT}{dt} \geq -(r_4 + r_5P_1)T$$

$$\frac{dT}{dt} \geq -\theta T - \theta P_1 T$$

$$\frac{dT}{dt} \geq -\theta T(1 + P_1)$$

$$\frac{dT}{dt} \geq -\theta T[1 + \frac{(c_1 + c_2)}{\eta}]$$

$$\frac{dT}{T} \geq -\theta[1 + \frac{(c_1 + c_2)}{\eta}]dt$$

$$T \geq e^{-\theta[1+\frac{(c_1+c_2)}{\eta}]t}$$

$$T \geq 0 \text{ for all t}$$

And similarly, we can calculate for P_1andP_2.

4.4 BOUNDEDNESS OF SOLUTIONS

The boundedness of solutions of the model is given by:

$$W = P_1 + P_2$$

$$\frac{dW}{dt} = \frac{d(P_1 + P_2)}{dt} = \frac{dP_1}{dt} + \frac{dP_2}{dt}$$

$$= c_1P_1 - \alpha_1 \, {P_1}^2 - \beta_1P_1P_2 + r_1{P_1}^2P_2 + c_2P_2 - \ \alpha_2{P_2}^2 - \beta_2P_1P_2 + r_2P_1(t-\tau){P_2}^2$$

Let $\eta = min(\alpha_1, \alpha_2, \ \beta_1, \beta_2, r_1, r_2) \ and \ P_1(t-\tau) \sim P_1$

$$\frac{dW(t)}{dt} \leq \eta(P_1 + P_2) - (c_1 + c_2)$$

$$0 \leq \eta(P_1 + P_2) - (c_1 + c_2)$$

Applying the comparison theorem

S0, $0 \leq P_1 + P_2 \leq \frac{(c_1+c_2)}{\eta}$ and

$$\frac{dT}{dt} = r_3P_2 - r_4T - r_5P_1T$$

Let $\theta = min(r_3, r_4, r_5)$

$$\frac{dT(t)}{dt} \leq \theta P_2 - \theta T$$

$$\frac{dT(t)}{dt} \leq \theta\frac{(c_1 + c_2)}{\eta} - \theta T$$

When t→ ∞, $\frac{dT(t)}{dt} \to 0$

$$0 \leq \ \ \theta\frac{(c_1 + c_2)}{\eta} - \theta T$$

$$\theta T \leq \ \ \theta\frac{(c_1 + c_2)}{\eta}$$

$$0 \leq T \leq \frac{(c_1 + c_2)}{\eta}$$

Hence, all the equation of the system of equation (1) - (3) lie in three – dimensional region

S= $[(P_1, P_2, T)\ \varepsilon\ R_+^3 : 0 \le P_1 + P_2 \le \frac{(c_1+c_2)}{\eta}, 0 \le T \le \frac{(c_1+c_2)}{\eta}]$ as t→ ∞, for all positive initial value {P_1(0) > 0, P_2(0) > 0, T(0), $P_1(t-\tau) \sim P_1$= Constant for all $t\ \varepsilon[-\tau, 0]\}\ \varepsilon\ S \subset R_+^3$, Where $\eta = min(\alpha_1, \alpha_2, \beta_1, \beta_2, r_1, r_2)$ and $\theta = min(r_3, r_4, r_5)$.

4.5 EQUILIBRIUM POINT OF THE MODEL

The interior equilibrium point $E^*(P_1^*, P_2^*, T^*)$ of system (1)-(3) is calculates as:

$$\frac{dP_1^*}{dt} = 0,$$

$$\frac{dP_1^*}{dt} = c_1P_1^* - \alpha_1\ P_1{}^{*2} - \beta_1 P_1^* P_2^* + r_1 P_1{}^{*2} P_2^*$$

$$c_1P_1^* - \alpha_1\ P_1{}^{*2} - \beta_1 P_1^* P_2^* + r_1 P_1{}^{*2} P_2^* = 0$$

$$P_1^*[c_1 - \alpha_1\ P_1^* - \beta_1 P_2^* + r_1 P_1^* P_2^*] = 0$$

$$c_1 - \alpha_1\ P_1^* - \beta_1 P_2^* + r_1 P_1^* P_2^* = 0$$

$$-\alpha_1\ P_1^* + r_1 P_1^* P_2^* + c_1 - \beta_1 P_2^* = 0$$

$$P_1^*(r_1 P_2^* - \alpha_1) + (c_1 - \beta_1 P_2^*) = 0$$

$$P_1^* = \frac{\beta_1 P_2^* - c_1}{r_1 P_2^* - \alpha_1}$$

And similarly, for $P_2^* and\ T^*$

$P_2^* = \frac{\beta_2 P_1^* - c_2}{r_2 P_1^* - \alpha_2}$, but the value of P_1^*,

$$T^* = \frac{r_3 P_2^*}{r_4 - r_5 P_1^*}$$

$$\frac{dP_2^*}{dt} = 0$$

Which is quadratic equation in P_2^* , giving two roots, $(\beta_1 r_2 - \alpha_2 r_1)$ are not zero.

$$(\beta_1 r_2 - \alpha_2 r_1) P_2^{*2} - (\alpha_1\alpha_2 - c_1 r_2 - \beta_1\beta_2 + r_1 c_2) P_2^* + (c_1\beta_2 - \alpha_1 c_2) = 0$$

$$P_2^* = \frac{(\alpha_1\alpha_2 - c_1 r_2 - \beta_1\beta_2 + r_1 c_2) \pm \sqrt{(\alpha_1\alpha_2 - c_1 r_2 - \beta_1\beta_2 + r_1 c_2)^2 - 4(\beta_1 r_2 - \alpha_2 r_1)(c_1\beta_2 - \alpha_1 c_2)}}{2(\beta_1 r_2 - \alpha_2 r_1)}$$

4.6 STABILITY OF EQUILIBRIUM $E^*(P_1^*, P_2^*, T^*)$ AND HOPF-BIFURACTION.

Here the dynamic behaviour for equilibrium points $E^*(P_1^*, P_2^*, T^*)$ of system given by equation one to three is analyzed. The E^* equilibrium empirical characteristics equation is expressed as shown:

$$\psi^3 + a_1\psi^2 + a_2\psi + a_3 + (b_1\psi^2 + b_2\psi + b_3)e^{-\psi\tau} = 0 \qquad (4)$$

Where

$$a_1 = 2\alpha_1 P_1 - c_1 + r_4 - c_2 + 2\alpha_2 P_2$$

$$a_2 = r_2$$

$$a_3 = -2\alpha_2 P_2 c_1 r_4 - 4\alpha_1\alpha_2 P_1 P_2 r_4 + c_1 c_2 r_4 - 2\alpha_1 P_1 c_2 r_4$$

$$b_1 = 2\alpha_1 P_1 r_4 - c_1 r_4 - c_1 c_2 - 2c_2\alpha_1 P_1 - r_4 c_2 - 2\alpha_2 P_2 c_1 + 4\alpha_1\alpha_2 P_1 P_2 - 2\alpha_2 P_2 r_4$$

$$b_2 = 2\alpha_1 P_1$$

$$b_3 = r_4 c_1 - 2\alpha_1 P_1 r_4$$

Clearly a_1, a_2, a_3, b_1, b_2, b_3 are all positive.

If and only if $\psi = i\vartheta$ is a response of (4)

$$(i\vartheta)^3 + a_1(i\vartheta)^2 + a_2(i\vartheta) + a_3 + (\, b_1(i\vartheta)^2 + \ b_2(i\vartheta) + \ b_3)e^{-i\vartheta t} = 0 \qquad (5)$$

Distinguishing between actual and imagined parts:

$$a_3 - \ a_1\vartheta^2 + (\, b_3 - \ b_1\vartheta^2)\cos(\vartheta\tau) + \ b_2\vartheta\sin(\vartheta\tau) = 0 \qquad (6)$$

$$a_2 - \vartheta^3 + \ b_2\vartheta\cos(\vartheta\tau) - (\, b_3 - \ b_1\vartheta^2)\sin(\vartheta\tau) = 0 \qquad (7)$$

Squaring and adding parts:

$$\vartheta^6 + \left({a_1}^2 - {b_1}^2 - 2a_2\right)\vartheta^4 + \left({a_2}^2 - {b_2}^2 + 2\, b_1\, b_3 - 2\, a_1\, a_3\right)\vartheta^2 + \left({a_3}^2 - {b_3}^2\right) = 0 \qquad (8)$$

Put

$$\mathrm{a_1}^2 - \mathrm{b_1}^2 - 2\mathrm{a}_2 = \mathrm{v}$$

$$\mathrm{a_2}^2 - \mathrm{b_2}^2 + 2\,\mathrm{b}_1\,\mathrm{b}_3 - 2\,\mathrm{a}_1\,\mathrm{a}_3 = \mathrm{b}$$

$$\mathrm{a_3}^2 - \mathrm{b_3}^2 = \mathrm{c}$$

And $\vartheta^2 = z$, We get

$$z^3 + vz^2 + bz + c = 0 \qquad (9)$$

If $c < 0$ then (9) does have at minimum a real root that is positive.

Let $\mathrm{j(z)} = z^3 + vz^2 + bz + c$

$$j(0) = c < 0. \lim_{y\to\infty} j(z) = \infty, \exists z_0 \,\epsilon(0,\infty)$$

$$j(z_0) = 0$$

if $c \geq 0$, then necessary condition for equation to have positive root is

$S = v^2 - 3b \geq 0$, S is the discriminant & condition for positive root.

$$\mathrm{j(z)} = z^3 + vz^2 + bz + c$$

$$j'(\mathrm{z}) = 3z^2 + 2vz + b$$

$$j'(\mathrm{z}) = 0 \rightarrow 3z^2 + 2vz + b = 0 \quad (10)$$

$$z_{1,2} = \frac{-2v \pm \sqrt{4v^2 - 12b}}{6} = \frac{-v \pm \sqrt{S}}{3} \quad (11)$$

Let $z_1 = \frac{-v - \sqrt{S}}{3}$ and $z_2 = \frac{-v + \sqrt{S}}{3}$

Clearly $z_1 < z_2$

So $j(z_1) < j(z_2)$

Therefore $j(z)$ is monotonically increasing function.

If $S < 0$ is valid for (10) have no root. As a consequence, j(z) monotone increasing function in z. $j(0) = c \geq 0$, (8) seems to have no positives origins.

Clearly if $S \geq 0$ then $z_1 = \frac{-v \pm \sqrt{S}}{3}$ is j(z) of z

If $c \geq 0$ then (9) is real, if $z_1 > 0$ and $\mathrm{j(z_1)} > 0$

Assume that either $z_1 \leq 0 \; or \; z_1 > 0 \; and \; j(z_1) > 0$

If $z_1 \leq 0, \mathrm{sinj}(z)$ is growing rapidly for $\mathrm{z} \geq \mathrm{z_1}, \mathrm{j}(0) = \mathrm{c} \geq 0$

As a result, j(z) does not have a positive real zero.

If $z_1 > 0$ and $j(z_1) > 0, sinz_2 = \frac{-v \pm \sqrt{S}}{3}$ is the value of the local maxima.

It follows that $j(z_1) \leq j(z_2)$. So $j(0) = c \geq 0$, As $j(z)$ have no positive root.

Lemma 4.1 Assume y_1 is given by (11)

4.1.1 $c < 0$, then in (9) exist at least one positive root.

4.1.2 $c \geq 0$ and $S = v^2 - 3b < 0,$ have no positive root in (9).

4.1.3 $c \geq 0$, then (9) have positive root if $z_1 > 0$ and $j(z_1) \leq 0$

Proof Suppose equation (9) has positive root. WLOG, confirm three positive roots are z_1, z_2, z_3, (8) one on either hand, is detrimental roots.

$$\vartheta_1 = \sqrt{z_1}, \, \vartheta_2 = \sqrt{z_2}, \vartheta_3 = \sqrt{z_3}$$

Derived (7)

$$\sin \vartheta\tau = \frac{a_2\vartheta - \vartheta^3}{d}$$

$$\tau = \frac{1}{\vartheta}\left[\sin^{-1}\frac{(a_2\vartheta - \vartheta^3)}{d} + 2(j-1)\pi\right] : j = 1,2,3$$

$$\tau_k^{(j)} = \frac{1}{\vartheta k}\left[\sin^{-1}\frac{(a_2\vartheta - \vartheta^3 k)}{d} + 2(j-1)\pi\right] : k = 1,2,3 \;\; \text{j=0,1,2……}$$

So, there's a set of strictly complexed valued function where real value is zero and equation roots (8). When

$$\tau = \tau_k^{(j)}, k = 1,2,3; j = 0,1,2 \ldots..$$

$$\lim_{j\to\infty} \tau_k^{(j)} = \infty, \qquad \mathrm{k} = 1,2,3,4$$

$$\tau_0 = \tau_{k_0}^{(j_0)} = \min_{1\le k\le 3, j\ge 1} [\tau_k^{(j)}], \vartheta_0 = \vartheta_{k_0}, \qquad z_0 = z_{k_0} \tag{12}$$

Lemma 4.2 Suppose that $a_1 \ge 0, (a_3 + d),\ a_1,\ a_2(\ a_3 - d) > 0$

4.2.1 If $c \ge 0$ and $S = v^2 - 3b < 0, \forall\tau \ge 0$ equation (4) has negative component.

4.2.2 If $c < 0\ or\ c \ge 0, z_1 > 0$ and $j(z_1) \le 0, \quad \forall\tau\, \epsilon(0, \tau_0)$ then all component of (4) have negative roots.

A show: if $\tau = 0$, then (4) become

$$\psi^3 + (a_1 + b_1)\psi^2 + (a_2 + b_2)\psi + (a_3 + b_3) = 0 \tag{13}$$

According to Routh- Hurwitz’s Criteria

(5) have negative roots iff

$$(a_3 + b_3) \ge 0, (a_1 + b_1)(a_2 + b_2) - (a_3 + b_3) > 0$$

If $c \ge 0$ and $S = v^2 - 3b < 0,$

Lemma 4.1.2 Show $\forall\ \tau \ge 0$, there are no roots in equation (4) because the actual element is null. When $c < 0\ or\ c \ge 0, z_1 > 0$ and $j(z_1) \le 0,$

Lemma 4.1.1 & 4.1.2 implies when $\tau \ne \tau_k^{(j)}, k = 1,2,3, j \ge 1$, since equation (4) has no origin with null linear system and is the expected value of, these have non - integer roots, and conclusion of the lemma is obtained.

4.7 SENSITIVITY ANALYSIS OF STATE VARIABLE W.R.T. MODEL PARAMETER:

The model contains constant parameters in this article. The 'Direct Method' is used for estimating the coefficients of generalized sensitivity. The direct approach is based on assuming all parameters as fixed then somehow, by utilizing the sensitivity equations with initial solution of equations, the sensitivity coefficients are calculated. When all the parameters (α_1, α_2, β_1, β_2, r_1, r_2, r_3, r_4, r_5) in the proposed system (1) - (3) are assumed to be constant, then sensitivity analysis may simply involve finding the solution's partial derivatives with regard to each parameter in this case. As an example, if parameter β_1 considered then partial derivatives of the solution $(P_{1,}, P_2, T)$ with respect to β_1 gives rise to following set of sensitivity equation:

$$\frac{dS_1}{dt} = (C_1 - 2\alpha_1 P_1 + \beta_1 P_2 + 2P_1 P_2 r_1)S_1 + (\beta_1 P_2 + r_1 P_1^2)S_2 - P_1 P_2 \qquad (14)$$

$$\frac{dS_2}{dt} = (C_2 - 2\alpha_2 P_2 + \beta_2 P_1 + 2r_2 P_2 P_1(t-\tau))S_2 + (\beta_2 P_2 + r_2 P_1^2(t-\tau))S_1 \qquad (15)$$

$$\frac{dS_3}{dt} = -Tr_5 S_1 + r_3 S_2 - (r_4 + r_5 P_1)S_3 \qquad (16)$$

$$Where\ S_1 = \frac{\partial P_1}{\partial \beta_1}, S_2 = \frac{\partial P_2}{\partial \beta_1}, S_3 = \frac{\partial T}{\partial \beta_1}$$

This sensitivity equation system (14) - (16) is then solved together with the original equation system (1)-(3) to approximate the sensitivity of the variable $(P_{1,}, P_2, T)$ to the parameter β_1. The system variables' sensitivity to the function, and the related approach and statement apply.

4.7.1 SENSITIVITY OF VARIABLE TO PARAMETER β_1

In Figure1 and Figure2, the variable consumption coefficient β_1 causes no difference and adjustment in the quality of the state vector allelochemical P_1 and concentration of allelochemicals T which ultimately remain stable and tends to zero, as we decrease the value of $\alpha = 0.05\ to\ \alpha = 0.09$. It predicts the lesser sensitivity of state variable P_1 and T to the variable β_1. However, in the case of the equal rang of value of β_1, the state variable amount of allelochemicals P_2 undergoes under the considerable change as shown by (α_1, α_2, β_1, β_2, r_1, r_2, $r_{3,}$ $r_{4,}$ r_5). It shows increase in the rate of allelochemical with decrease in the delayed value of coefficient. It remains stable as well.

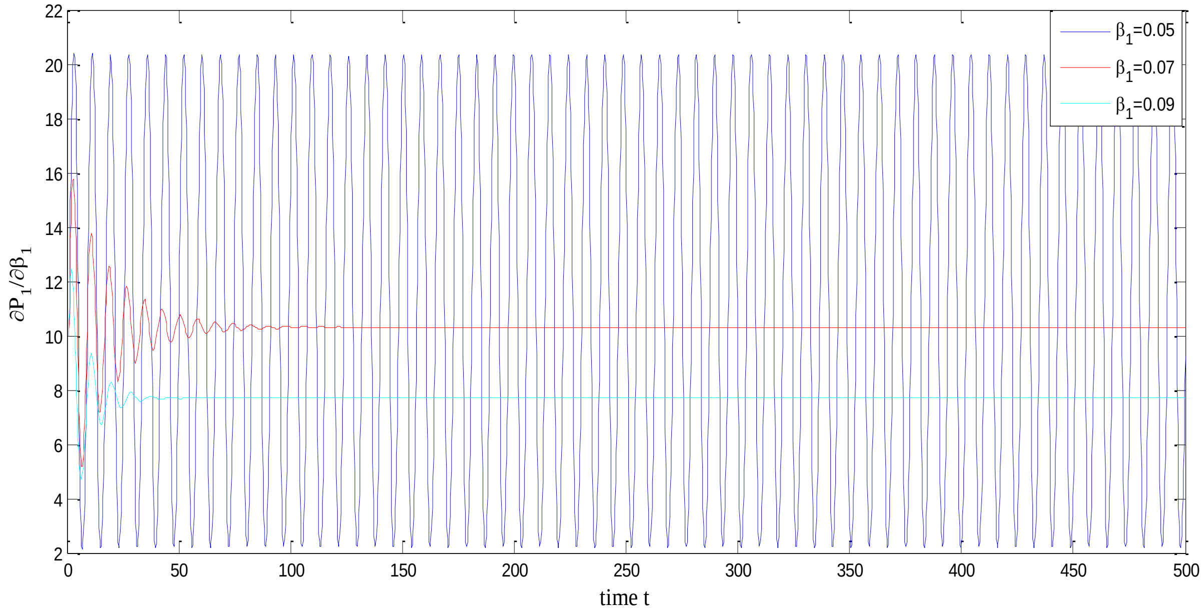

Figure 4.1 Time series graph between partial change in allelochemicals P_1 for different value of coefficient β_1.

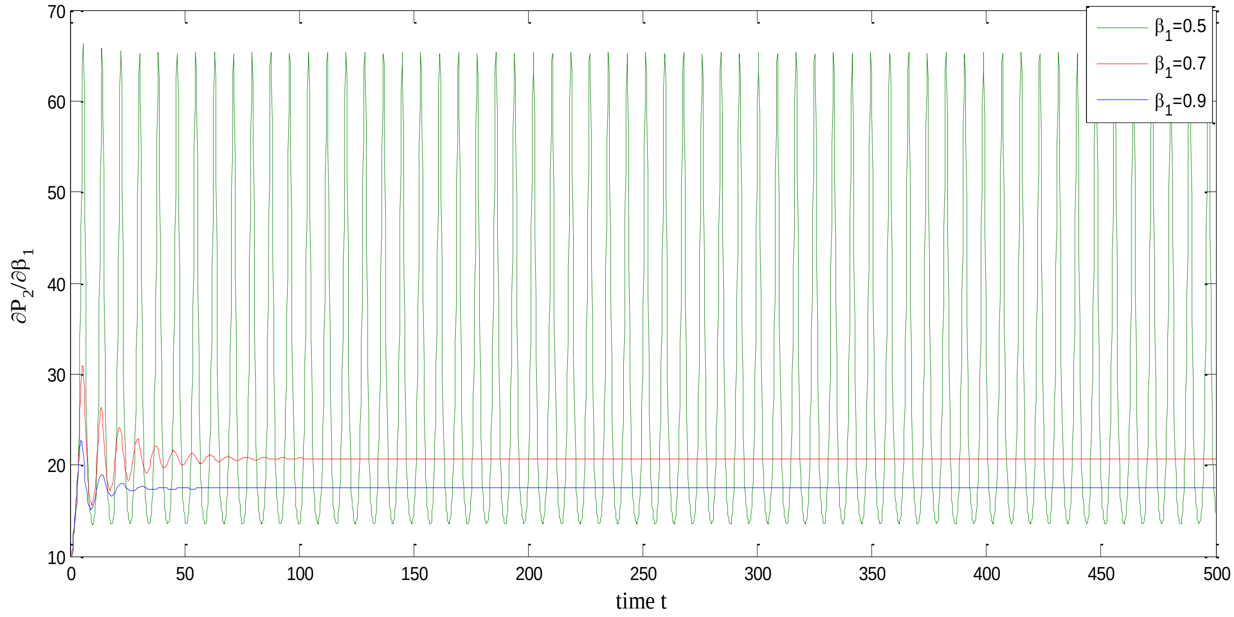

Figure 4.2 Time series graph between partial change in allelochemicals P_2 for different value of coefficient β_1.

4.7.2 SENSITIVITY OF VARIABLE TO PARAMETER α_1

In Figure3 and Figure4, the variable consumption coefficient α_1 may not induce a lot of variance and shift in the state's quality variable allelochemical P_1 and concentration of allelochemicals T which ultimately remain stable and tends to zero, as we decrease the value of $\alpha_1 = 0.07\ to\ \alpha_3 = 0.11$.It assumes that the system will be less responsive. variable P_1 and T to the variable β_1. On the other hand, equal rang of value of α_1, the state variable amount of allelochemicals P_2 undergoes under the considerable change as shown by (α_1, α_2, β_1, β_2, r_1, r_2, r_3, r_4, r_5). It shows increase in the rate of allelochemical with decrease in the delayed value of coefficient. It remains stable as well.

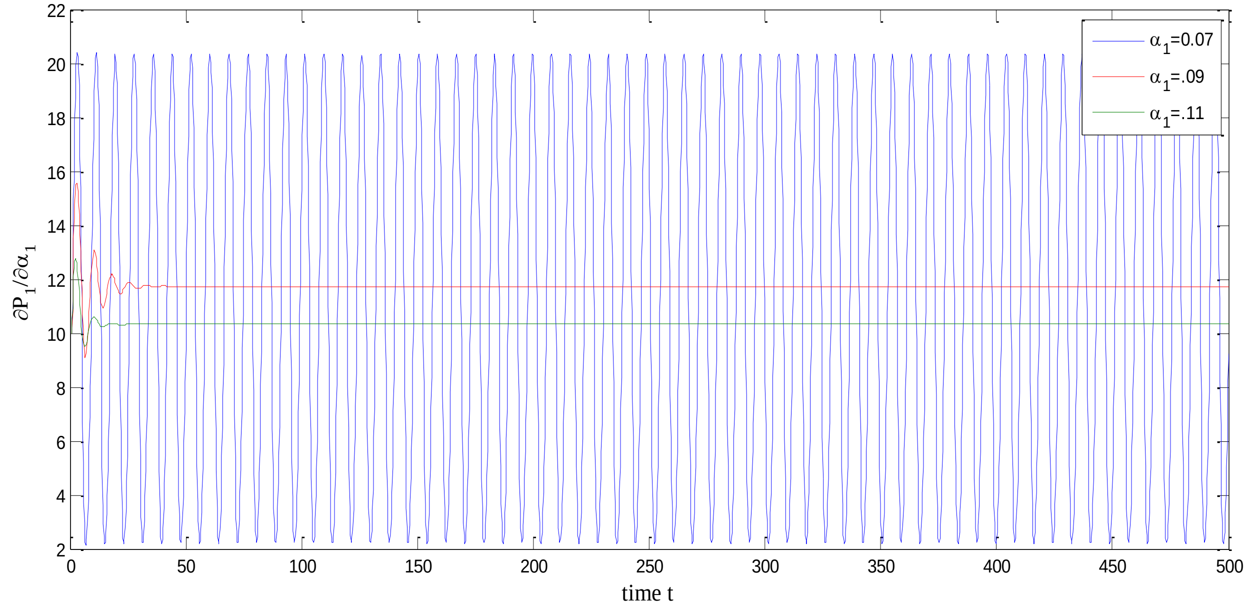

Figure 4.3 Time series graph between partial change in allelochemicals P_1 for different value of coefficient α_1.

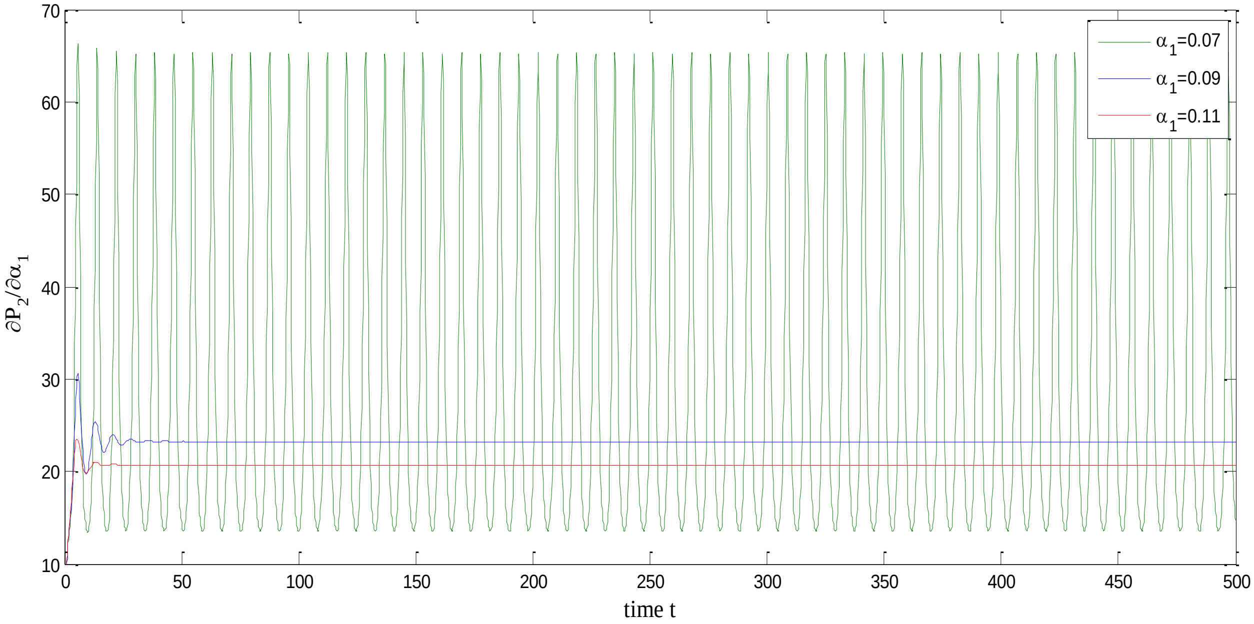

Figure 4.4 Time series graph between partial changes in allelochemicals P_2 for different value of coefficient α_1.

4.8 NUMERICAL EXAMPLE

The calculation is conducted with MATLAB to coordinate the analytical result with the aid of a numerical.

$$C_1 = 2, C_2 = 1, \alpha_1 = 0.07, \alpha_2 = 0.08, \beta_1 = 0.05, \beta_2 = 0.015, r_1 = 0.0008, r_2 = 0.0038,$$
$$r_3 = 0.5, r_4 = 0.1, r_5 = 0.5$$

The behaviour of the system of equations (1) -(3) for various delay values is displayed as:

$$E^*(P_1^* = 9.2387, P_2^* = 14.6050, T^* = 2.1893)$$

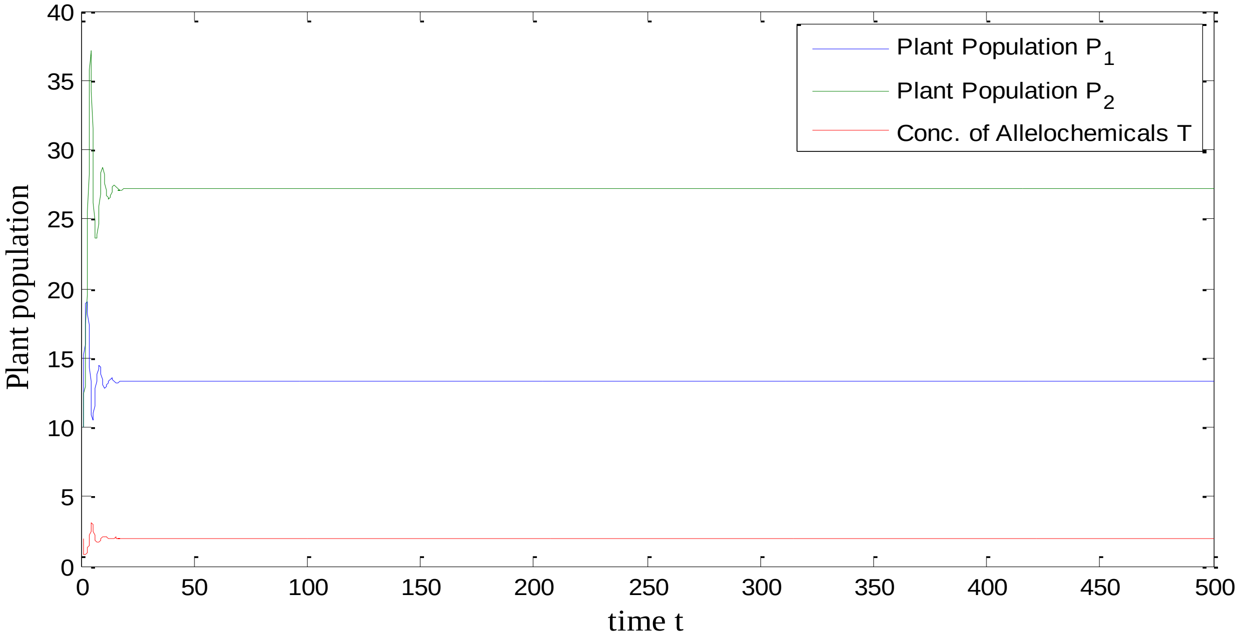

Figure 4.5 The interior equilibrium points $E(P_1^* = 9.2387, P_2^* = 14.6050, T^* = 2.1893)$ are stable when there is no delay, i.e., $\tau = 0$.

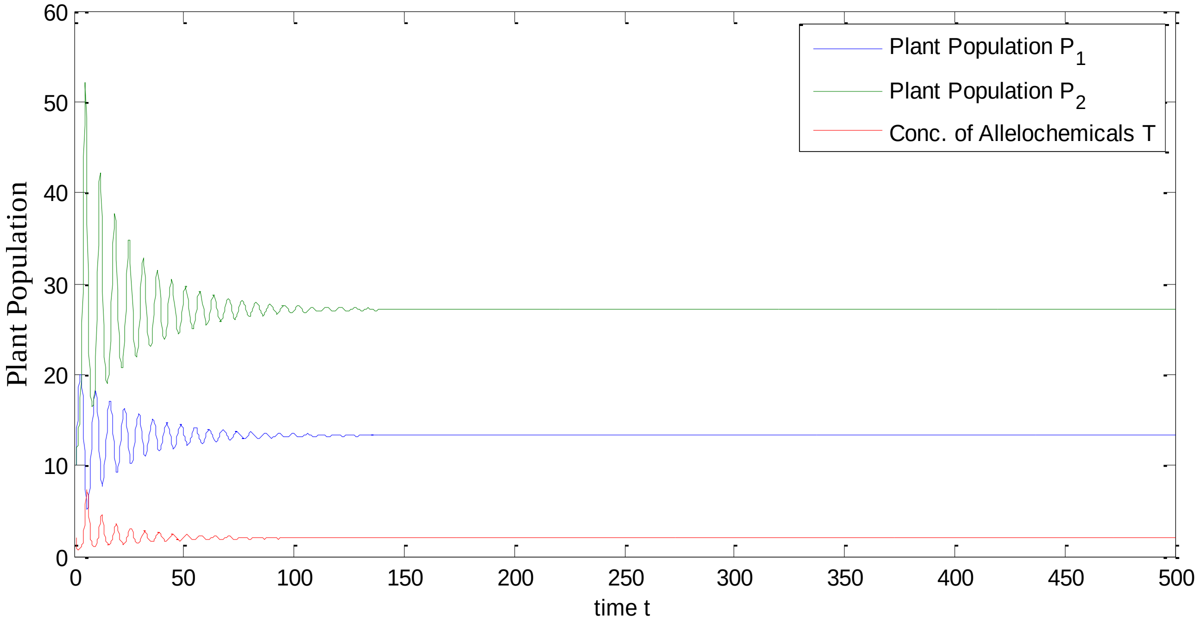

Figure 4.6 When delay $\tau \leq 1.47$,the interior equilibrium points $E(P_1^* = 9.2387, P_2^* = 14.6050, T^* = 2.1893)$ is asymptotically stable.

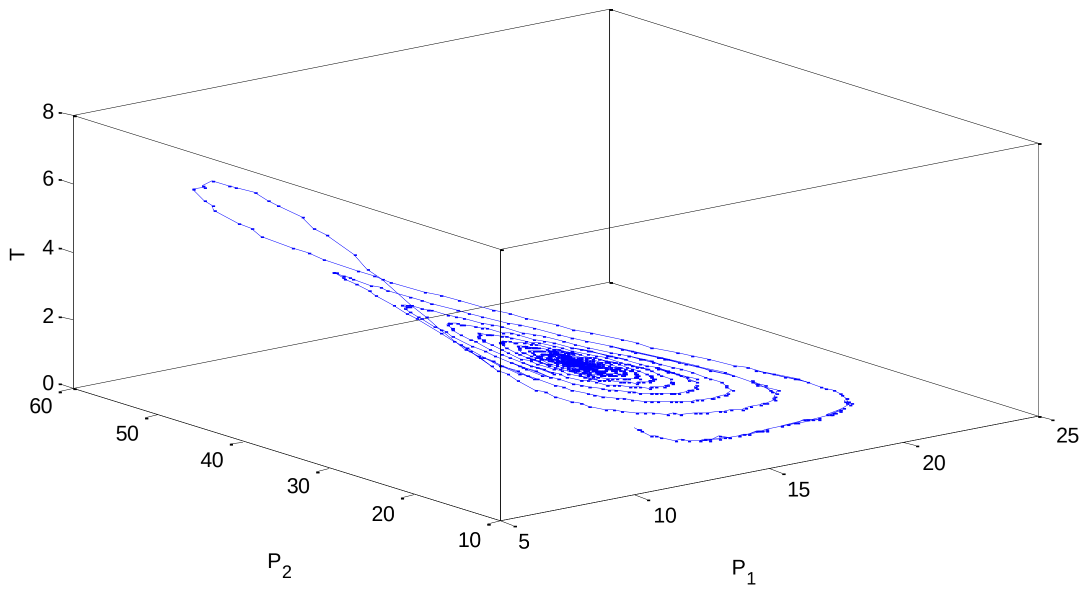

Figure 4.7 Phase space of Plant population P_1, P_2 and Conc. of Allelochemicals T when delay is $\tau \leq 1.47$.

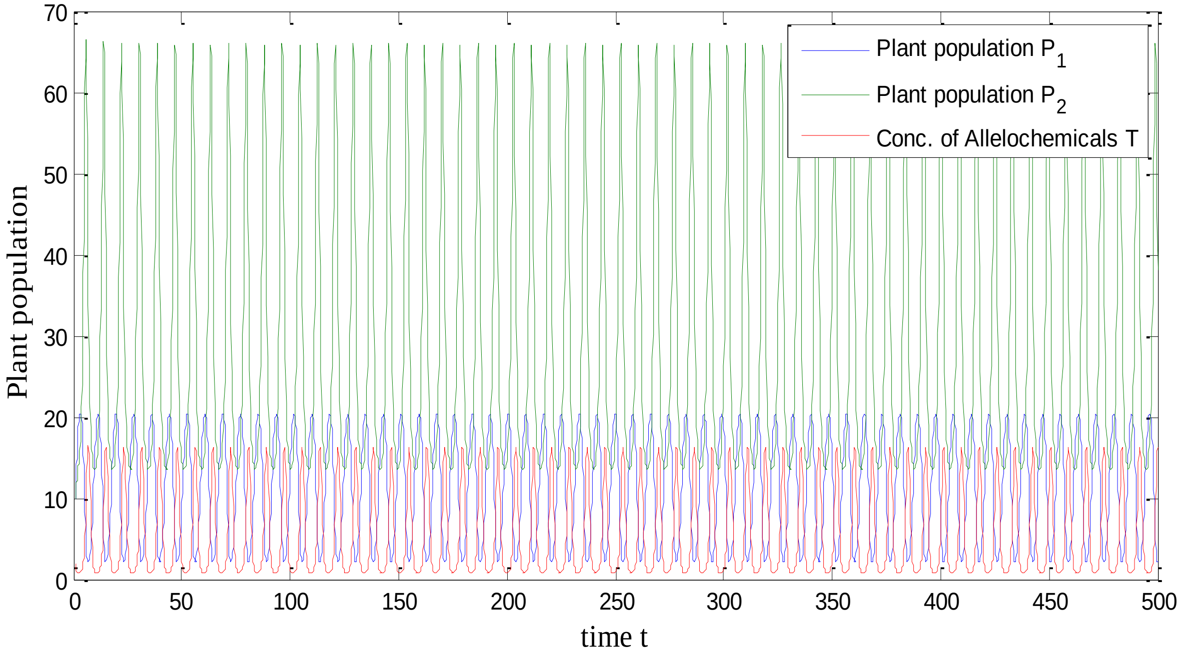

Figure 4.8 The interior equilibrium point $E(P_1^* = 9.2387, P_2^* = 14.6050, T^* = 2.1893)$ losses its stability and Hopf-bifuraction occurs when delay $\tau > 1.47$.

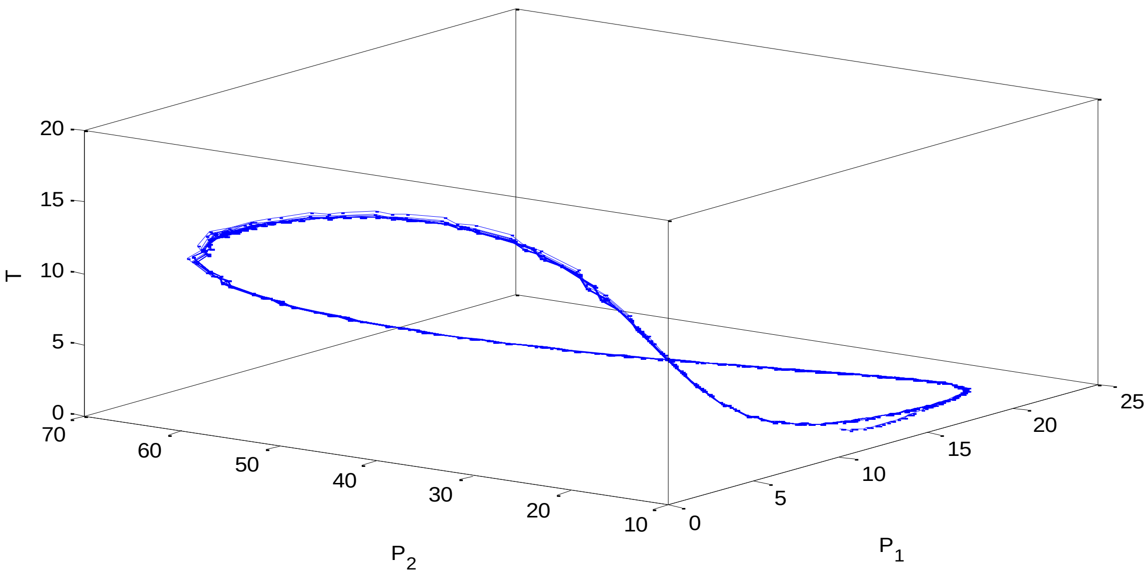

Figure 4.9 Phase space of Plant population P_1, P_2 and Conc. of Allelochemicals T when delay is $\tau > 1.47$. The bifurcation periodic solution is orbitally, as asymptotically stable.

4.9 CONCLUSION

A mathematical model is proposed to investigate the effect of delay on two competing plant population under the allelopathic effect. We assume from the literature review that the plant population is influenced by allelochemicals. Delay interrupts the mechanism and initiates dynamic behaviour by Hopf-bifurcation with minimal intervals and predictable intermittent remedies, while the allelopathic effect is believed to be stimulatory in nature. The stability analysis of interior equilibrium $E^*(P_1^*, P_2^*, T^*)$ is done. It is concluded that when there is no delay. Equilibrium $E^*(P_1^*, P_2^*, T^*)$ is completely stable (Figure 4.5) as proved by lemma2 using Routh-Hurwitz's criteria. But the critical value of the parameter delay is produced under the same set of parameters, below which the system is asymptotically stable (Figure 4.6 and 4.7), and above that critical value of parameter (Figure 4.8 and 4.9) proved by lemma1. The process displays oscillation, which is Hopf-bifuraction, when going through the critical value.

In this article, the sensitivity of system variables with respect to model parameters is done using the "Direct approach". It demonstrates how the sensitivity functions allow one to recognize specific parameters and enhance the view of the importance of the delay played by different parameters of the model (Figure 4.1 to Figure 4.4).

Farmers, ecologists and scientists will profit from the proposed mathematical model coping with delays in plant populations under the control of allelochemicals. The numerical simulation is descriptive in nature and would prove to be economical in terms of time and resources expended on huge experiments.

Chapter-5

Plant Competition Involving Release of Allelochemicals by both the Plant Population

5.1 INTRODUCTION

Allelopathy is a sub-discipline of biochemical ecology which is dealing with the influence of chemicals generated by plant or phytoplankton on the development, production and distribution of several other plants and phytoplankton in natural ecosystems or agricultural system. Allelopathy is a significant factor affecting the growth of phytoplankton. In this article, our research work is based on allelochemicals. In 1937, after Rice [1] had utilized to phytoplankton groups, Molisch [1] first used the word allelopathy. Allelopathy is the influence of one plant on another plant by the release into the atmosphere of secondary metabolites. These secondary metabolites release called as "allelochemicals".

In the 1980s, a new concept was created in the mathematical model that was produced of a toxicant with an impact on a population. The issue of using a statistical method to qualitatively estimate toxic compounds have an effect on the organisms is a really efficient method of doing so. The influence of the linear model allelochemicals on different phases are examined by Hallam et al. [121,125], Hallam and Luna [126], Luna and Hallam [127], Freedman and Shukla [128], Ghosh et al. [129], He and Wang [130], Das et al. [131] and others. Maynard-Simth [132] first worked on mathematical model to understand the allelopathy interaction between two competing species. Maynard-Simth found competition of two Lotka-Volterra species with an additional concept identifying the adverse impact on other organisms of allelochemicals produced by one inhibition effect of one organism. Largely based on the Maynard-Simth model, other researchers [133,134,4,135] studied various forms of mathematical model. Another assumption made produced by the observer is that throughout the production of allelochemicals (stimulatory/inhibitory), the increase in population of one species could affect the growth and development of other species. Delays exist in all biochemical phenomena in nature [136]. A time delay is added in plant population dynamics, as the population growth phenomenon, the biomass change is influenced by the former population as well as the present population. Kalra and

Kumar studied on plant population species take time to achieve full maturity and then release a phytochemical material to other plant species that becomes poisonous or stimulatory. So, in the complexities, the delay concept comes into effect. The influence of the delay parameter on toxicant-effected plant growth dynamics was analyzed using a non-linear delay differential equation system [30,29,95,23,108]. Mukhopadhyay et al. [4] alteration the Maynard-Simth [132] worked on a delay differential equation model. They found the distinct delay necessary for the development of the harmful substance to be generated by the organisms. The author observed that because of discrete delay, there are no stabilizing impacts on the complexity of the model organism. The nature of zeros of exponential characteristic equation was analyzed in detail [98].

We further improved the most fundamental mathematical model for allelopathy in the appearance of populations of plants in this paper. In order to take part in the relationship between species and also generate allelochemicals, a discrete delay is inserted into the model development of organisms.

5.2 MATHEMATICAL MODEL:

5.2.1 Assumptions of the model:

1. In this model P_1 and P_2 are two competiting plant population and T is the density of allelochemicals.

2. It is assumed that both the plant populations release allelochemicals which exhibit inhibitory and stimulatory effects on the other population.

3. There is inter specific and intra specific competition between plant populations.

4. There is a delay in growth of both of the plant population due to release of allelochemical released by them which is represented by parameter τ.

5.2.2 Mathematical Formulation:

Three input variables are included in the following nonlinear delay differential equations method defines the complexities of the competing plant population: P_1 & P_2 are two competing plant populations and T is the density of allelochemicals. The mechanism of competition is regulated by the following system:

$$\frac{dP_1}{dt} = a_1 P_1 - a_1 \, {P_1}^2 - P_1 T + P_1 P_2 \, T \tag{1}$$

$$\frac{dP_2}{dt} = b_1P_2 - \ b_1{P_2}^2 - P_2T + \ P_1P_2\,T \qquad (2)$$

$$\frac{dT}{dt} = -c_1T^2 + d_1P_1(t-\tau)T + d_2P_2(t-\tau)T \qquad (3)$$

The variables and parameter considered in this model are interpreted in Table 1:

Table 1: Description of the variable and parameter of the system (1) – (3)

P_1	The competing plant population of 1st species.
P_2	The competing plant population of 2nd species.
T	The density of allelochemicals.
P_1T	Change in the density of Inter specific competition of P_1 plant population.
P_2T	Change of Inter specific competition of P_2 plant population.
$P_1P_2\,T$	It produced the stimulatory and inhibitory effect for plant populations.
a_1	The rate of cell proliferation of 1st species per hours.
b_1	The rate of cell proliferation of 2nd species per hours.
c_1	The rate of decay of T.
d_1	The rate of interaction of plant population P_1.
d_2	The rate of interaction of plant population P_2.
τ	Delay parameter.

All the parameter takes positive value, $a_1, b_1, c_1, d_1, d_2 > 0$. Where $P_1(0) > 0$, $P_2(0) > 0$, $T(0) > 0$ for all t and $P_1(t-\tau)$ & $P_2(t-\tau)$ = constant for $t \in [0, \tau]$.

5.3 EQUILIBRIUM POINT OF SOLUTION

We calculated an intrinsic equilibrium E^* of model. There is only one possible equilibrium $E^*(P_1^*, P_2^*, T^*)$ in the system of equations (1)-(3).

where $P_1^* = \frac{b_1c_1+d_2(1-\sqrt{b_1/a_1}}{d_2+d_1\sqrt{b_1/a_1}}$, $P_2^* = \frac{c_1\sqrt{a_1b_1}-d_1(1-\sqrt{b_1/a_1}}{d_2+d_1\sqrt{b_1/a_1}}$,

$T^* = \sqrt{a_1b_1}$ and E^* exist under the following conditions $c_1\sqrt{a_1b_1} \le d_1 + d_2, c_1a_1 + d_1 > d_1\sqrt{b_1/a_1}, c_1b_1 + d_2 > d_2\sqrt{a_1/b_1}$.

Theorem 5.1 Consider the exponential polynomial:

$f(\lambda. e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$

$$= \lambda^n + P_1^0\lambda^{n-1} + \cdots + P_{n-1}^0\lambda^n + P_n^0 + [P_1^1\lambda^{n-1} + \cdots + P_{n-1}^1\lambda^n + P_n^1]e^{-\lambda\tau_1} + \cdots$$
$$+ [P_1^m\lambda^{n-1} + \cdots + P_{n-1}^m\lambda^n + P_n^m]e^{-\lambda\tau_m},$$

Where $\tau_i \geq 0 (i = 0,1, \dots . m)$ and P_j^i $(i = 0,1,2 \dots m\colon j = 1,2..n)$ are constant. As $\tau_1, \tau_2 \dots \tau_m)$ vary, the number of the zeros in the sequence of $f(\lambda. e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$ and if a zero occurs on or passes the virtual axes will the open right half layout alter.

This conclusion has been proven up with Ruan and Wei by using Rouche's theorem.

5.4 STABILITY OF EQUILIBRIUM $E^*(P_1^*, P_2^*, T^*)$ AND LOCAL HOPF-BIFURACTION.

The dynamic behavior for equilibrium points $E^*(P_1^*, P_2^*, T^*)$ of the model given by (1) - (3) is investigated. The exponential characteristic equation for equilibrium E^* is given by:

$$\lambda^3 + \mathcal{A}\lambda^2 + \beta_1\lambda + (\beta_2\lambda + c_2)e^{-\lambda\tau} = 0 \qquad (4)$$

Where $\mathcal{A} = a_1P_1^* + b_1P_2^* + c_1T^*$

$\beta_1 = b_1c_1P_2^*T^* + a_1P_1^*(b_1P_2^* + c_1T^*) - P_1^*P_2^*T^{*2}$

$\beta_2 = d_2P_2^*T^*(1 - P_1^*) + d_1P_1^*T^*(1 - P_2^*)$

$c_2 = a_1d_2P_1^*P_2^*T^*(1 - P_1^*) + d_1P_1^*P_2^*T^*(1 - P_1^*) + d_2P_1^*P_2^*T^{*2}(1 - P_2^*) + b_1d_1P_1^*P_2^*(1 - P_2^*)$

& $\mathcal{A}, \beta_1, \beta_2, c_2$ all are +ve.

Clearly $\boldsymbol{\lambda} = \mathbf{i\omega}$ is a solution of equation (4) if and only if

$$(i\omega)^3 + \mathcal{A}(i\omega)^2 + \beta_1(i\omega) + (\beta_2(i\omega) + 2)e^{-i\omega t} = 0 \qquad (5)$$

Distinguishing between actual and imagined parts:

$$-\mathcal{A}\omega^2 + c_2\cos(\omega\tau) + \beta_2\omega\sin(\omega\tau) = 0 \qquad (6)$$

$$-\omega^3 + \beta_1\omega + \beta_2\omega\cos(\omega\tau) - c_2\sin(\omega\tau) = 0 \qquad (7)$$

Squaring and adding parts:

$$\omega^6 + p\omega^4 + q\omega^2 + r = 0 \qquad (8)$$

$p = \mathcal{A}^2 - 2\beta_1, q = \beta_1^2 - \beta_2^2, r = -c_2^2,$

Put $x = \omega^2$ then (8) become

$$x^3 + px^2 + qx + r = 0 \qquad (9)$$

If $r < 0$ then equation (9) has at least one positive real root

Let $h(x) = x^3 + px^2 + qx + c$

$h(0) = r < 0. \lim_{y\to\infty} h(x) = \infty, \exists y_0 \epsilon (0, \infty)$

$h(x_0) = 0$ if $r \geq 0$, And there is a condition that must be satisfied in order for the equation to have a positive origin.

$D = p^2 - 3q \geq 0$

$\text{h(x)} = x^3 + px^2 + qx + c$

$h'(\text{x}) = 3x^2 + 2px + q$

$$h'(\text{x}) = 0 \rightarrow 3x^2 + 2px + q = 0 \tag{10}$$

$$x_{1,2} = \frac{-2p \pm \sqrt{4p^2 - 12q}}{6} = \frac{-p \pm \sqrt{D}}{3} \tag{11}$$

If $D < 0$ Then there are no roots in equation (10). So, h(x) monotone increasing functions in x.

$h(0) = r \geq 0$ equation (8) seems to have no positive real roots.

Obviously, if $D \geq 0$ then $x_1 = \frac{-p \pm \sqrt{D}}{3}$ are local maxima of h(x)

If $r \geq 0$ then equation (9) has positive root if $x_1 > 0\ and\ h(x_1) > 0$

Assume that either $x_1 \leq 0\ or\ x_1 > 0\ and\ h(x_1) > 0$

If $x_1 \leq 0, \sinh(x)\ is\ increasing\ for\ x \geq x_1\ and\ h(0) = r \geq 0$

As a result, h(x) does not have a definite absolute zero.

If $x_1 > 0$ and $h(x_1) > 0, sinx_2 = \frac{-p \pm \sqrt{D}}{3}$ is local maxima value.

It follows that $h(x_1) \leq h(x_2)$

$h(0) = r \geq 0$, As $h(x)$ have no positive real root.

Lemma 5.1 Assume that $y_1$is given by equation (11)

1.$r < 0$, equation (9) contains at least one positive real root.

2. $r \geq 0$ and $D = p^2 - 3q < 0$, then equation (9) has no positive root.

3.$r \geq 0$, then equation (9) has positive root if $x_1 > 0\ and\ h(x) \leq 0$

Suppose equation (9) has positive root. WLOG assume there are three substantial roots of x_1, x_2, x_3,

Equation (8), on the other hand, has positive roots.

$\omega_1 = \sqrt{x_1}, \omega_2 = \sqrt{x_2}, \omega_3 = \sqrt{x_3}$

From (7)

$$\sin \omega\tau = \frac{\beta_2 \omega - \omega^3}{d}$$

$$\tau = \frac{1}{\omega}\left[\sin^{-1}\frac{(\beta_2\omega - \omega^3)}{d} + 2(j-1)\pi\right] : j = 1,2,3 \qquad (12)$$

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left[\sin^{-1}\frac{(\beta_2\omega - \omega^3 k)}{d} + 2(j-1)\pi\right] : k = 1,2,3 \text{ j=0,1,2……}$$

Then it is a pair of purely imaginary roots of equation (8)

When

$$\tau = \tau_k^{(j)}, k = 1,2,3; j = 0,1,2 \ldots..$$

$$\lim_{j\to\infty} \tau_k^{(j)} = \infty, \qquad \text{k} = 1,2,3,4$$

$$\tau_0 = \tau_{k_0}^{(j_0)} = \min_{1\le k\le 3, j\ge 1}[\tau_k^{(j)}], \omega_0 = \omega_{k_0}, x_0 = x_{k_0} \qquad (13)$$

Lemma 5.2 Suppose that $\mathcal{A} \ge 0, d, \mathcal{A}, \beta_2, d > 0$

1. If $r \ge 0$ and $D = p^2 - 3q < 0$ then all roots of equation (4) have negative real part $\forall\, \tau \ge 0$

2. If $r < 0\ or\ r \ge 0, x_1 > 0$ and $h(x_1) \le 0$ then all the roots of equation (4) have negative part $\forall\, \tau\, \epsilon (0, \tau_0)$,

Proof When $\tau = 0$, then equation (4) become

$$\lambda^3 + \mathcal{A}\lambda^2 + (\beta_1 + b_2)\lambda + c_3 = 0 \qquad (14)$$

Acc. to Routh- Hurwitz's Criteria

All roots of equation (5) have negative part iff

$$c_2 \ge 0, \mathcal{A}(\beta_1 + b_2) - c_3 > 0$$

If $r \ge 0$ and $D = p^2 - 3q < 0$,

Lemma 5.1(2) show there are no roots in equation (4) since the real element is zero $\forall\, \tau \ge 0$

When $r < 0\ or\ r \ge 0, x_1 > 0$ and $h(x_1) \le 0$,

Lemma 5.1(1) & 5.1(2) implies when $\tau \ne \tau_k^{(j)}, k = 1,2,3, j \ge 1$, Equation (4) has completely imaginary roots but it has no root with zero real elements and is the minimal value of and conclusion of the lemma is obtained.

5.5 DIRECTION AND STABILITY OF HOPF-BIFURCATING SOLUTION

A group of continuous functions that bifurcate at the critical value of the positive steady state is produced. The path, stability and the bifurcating periodic solutions' periodicity is also significant. A specific formula to calculate the characteristics of the Hopf-bifuraction at the complex level, using the standard principle and several reductions due to Hassard, Kazarion and Wan, will be obtained in this section (1981).

$Let\ u_1 = P_1 - P_1^*, u_2 = P_2 - P_2^*, u_3 = T - T^*$ and normalizing the delay τ by time scaling $t \rightarrow \frac{t}{\tau}$, system is transformed into

$$\frac{du_1}{dt} = -a_1P_1^*u_1 - P_1^*u_3 + P_1^*P_2^*u_3 + P_1^*T^*u_2 - a_1u_1^2 - (1 - P_2^*)u_1u_3 + P_1^*u_2u_3 + T^*u_1u_2 + u_1u_2u_3$$

$$\frac{du_2}{dt} = -b_1P_2^*u_2 - P_2^*u_3 + P_1^*P_2^*u_3 + P_2^*T^*u_1 - b_1u_1^2 - (1 - P_1^*)u_2u_3 + P_2^*u_1u_3 + T^*u_1u_2 + u_1u_2u_3 \qquad (15)$$

$$\frac{du_3}{dt} = -c_1T^* + d_1T^*u_1(t-1) + d_2T^*u_2(t-1) - c_1u_3^2 + d_1u_1(t-1)u_3 + d_2u_2(t-1)u_3$$

Thus, in that phase, we can deal $C = C\big((-1,0), R_+^3\big)$.WLOG, denote the critical value $\tau_j\ by\ \tau_0$. Let $\tau = \tau_0 + \mu$, then $\mu = 0$ is the value of Hopf-bifurcation of the system (10). For the simplification of notations, we rewrite (15) as

$$u'(t) = L_\mu(u_t) + F(\mu, u_t) \qquad (16)$$

Where $u(t) = (u_1(t), u_2(t), u_3(t)\ )^T \in R^3, u_t(\theta) \in C$ is defined by $u_t(\theta) = u_t(t + \theta)$, and $L_\mu: C \rightarrow R,\ \ F: R \times C \rightarrow R$ are given, respectively by

$$L_\mu\emptyset = (\tau_0 + \mu)\begin{bmatrix} -a_1P_1^* & P_1^*T^* & -P_1^* + P_1^*P_2^* \\ P_2^*T^* & -b_1P_2^* & -P_2^* + P_1^*P_2^* \\ 0 & 0 & -c_1T^* \end{bmatrix}\begin{bmatrix} \emptyset_1(0) \\ \emptyset_2(0) \\ \emptyset_3(0) \end{bmatrix} + (\tau_0 + \mu)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_1T^* & d_2T^* & 0 \end{bmatrix}\begin{bmatrix} \emptyset_1(-1) \\ \emptyset_2(-1) \\ \emptyset_3(-1) \end{bmatrix}$$

And

And $F(\mu, \emptyset) = (\tau_0 + \mu)\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$

Respectively, where

$$F_1 = -a_1\emptyset_1^2(0) - (1 - P_2^*)\emptyset_1(0)\emptyset_3(0) + P_1^*\emptyset_2(0)\emptyset_3(0) + T^*\emptyset_1(0)\emptyset_2(0) + \emptyset_1(0)\emptyset_2(0)\emptyset_3(0)$$

$$F_2 = -b_1\emptyset_2^2(0) - (1 - P_1^*)\emptyset_2(0)\emptyset_3(0) + P_2^*\emptyset_1(0)\emptyset_3(0) + T^*\emptyset_1(0)\emptyset_2(0) + \emptyset_1(0)\emptyset_2(0)\emptyset_3(0)$$

$$F_3 = -c_1\emptyset_3^2(0) + d_1\emptyset_1(-1)\emptyset_3(0) + d_2\emptyset_2(-1)\emptyset_3(0),$$

$\emptyset(\theta) = \left(\emptyset_1(\theta), \emptyset_2(\theta), \emptyset_3(\theta)\right)^T \in C\left((-1,0), R\right)$.

By the Riesz representation theorem, there exist a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1,0]$, such that such that $L_\mu \emptyset = \int_{-1}^0 d\,\eta(\theta,0)\emptyset(\theta)$ for $\emptyset \in C$.

In fact, we can choose

$$\eta(\theta,\mu) = (\tau_0+\mu)\begin{bmatrix} -a_1P_1^* & P_1^*T^* & -P_1^*+P_1^*P_2^* \\ P_2^*T^* & -b_1P_2^* & -P_2^*+P_1^*P_2^* \\ 0 & 0 & -c_1T^* \end{bmatrix}\delta(\theta)$$

$$+(\tau_0+\mu)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_1T^* & d_2T^* & 0 \end{bmatrix}\delta(\theta+1)$$

Here δ is the Direct delta function. For $\emptyset \in C\left([-1,0], {R_+}^3\right)$, define

$$\mathcal{A}(\mu)\emptyset = \begin{cases} \dfrac{d\emptyset(\theta)}{d\theta}, & \theta \in [-1,0) \\ \int_{-1}^0 d\,\eta(\theta,0)\emptyset(\theta), & \theta = 0. \end{cases} \quad \text{And } R(\mu)\emptyset = \begin{cases} 0, & \theta \in [-1,0) \\ F(\mu,\emptyset) & \theta = 0. \end{cases}$$

Then the system (16) is equivalent to

$$u'(t) = \mathcal{A}(\mu)\emptyset + R(\mu)u_t \tag{17}$$

For $\psi \in C^1\left([-1,0], {R_+}^3\right)$, define

$$\mathcal{A}^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in [-1,0) \\ \int_{-1}^0 d\,\eta^T(-t,0)\psi(-t), & s = 0. \end{cases} \quad \text{And bi-linear inner product}$$

$$< \psi(s), \emptyset(\theta) > = \overline{\psi(0)}\emptyset(0) - \int_{-1}^{0}\int_{\xi=\theta}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)\,d\xi \tag{18}$$

$\mathcal{A}^*$ and $\mathcal{A} = \mathcal{A}(0)$ are algebraic operators and $i\omega_0$ are eigen values of $\mathcal{A}(0)$. As a result, they are coefficients of $\mathcal{A}^*$. Suppose that $q(\theta) = q(0)e^{i\omega_0\theta}$ is an eigen vector of $\mathcal{A}(0)$ corresponding to the eigen value $i\omega_0$. Then $\mathcal{A}(0) = i\omega_0\, q(\theta)$. When $\theta = 0$, we obtain $\left[i\omega_0 I - \int_{-1}^0 d\eta(\theta)e^{i\omega_0\theta}\right]q(0) = 0$, which yields $q(0) = (1,\sigma_1,\rho_1)^T$ where

$$\sigma_1 = \frac{(P_1^*-P_1^*P_2^*)P_2^*T^* + (P_2^*-P_1^*P_2^*)(i\omega_0+a_1P_1^*)}{P_1^*T^*(P_2^*-P_1^*P_2^*) - (P_1^*-P_1^*P_2^*)(i\omega_0+b_1P_2^*)}$$

$$\rho_1 = \frac{P_1^*P_2^*T^{*2} - (i\omega_0+a_1P_1^*)(i\omega_0+b_1P_2^*)}{P_1^*T^*(P_2^*-P_1^*P_2^*) - (P_1^*-P_1^*P_2^*)(i\omega_0+b_1P_2^*)}$$

Similarly, it can be verified that $q^*(s) = D(1,\sigma_2,\rho_2)e^{i\omega_0\tau_0 s}$ is the eigen value of $\mathcal{A}^*$ corresponding to $-i\omega_0$, where

$$\sigma_2 = \frac{(P_1^* - P_1^*P_2^*)P_2^*T^* + (P_2^* - P_1^*P_2^*)(a_1P_1^* - i\omega_0)}{P_1^*T^*(P_2^* - P_1^*P_2^*) - (P_1^* - P_1^*P_2^*)(b_1P_2^* - i\omega_0)}$$

$$\rho_2 = \frac{P_1^*P_2^*T^{*2} - (i\omega_0 + a_1P_1^*)(b_1P_2^* - i\omega_0)}{P_1^*T^*(P_2^* - P_1^*P_2^*) - (P_1^* - P_1^*P_2^*)(b_1P_2^* - i\omega_0)}$$

In order to assure $< q^*(s), q(\theta) > = 1$, and we calculate the value of D.

With the help of (13), $< q^*(s), q(\theta) >$

$$= \overline{D}(1, \overline{\sigma_2}, \overline{\rho_2})(1, \sigma_1, \rho_1)^T - \int_{-1}^{0}\int_{\xi=\theta}^{\theta} \overline{D}(1, \overline{\sigma_2}, \overline{\rho_2}) e^{-i\omega_0\tau_0(\xi-\theta)} d\eta(\theta)(1, \sigma_1, \rho_1)^T e^{i\omega_0\tau_0}\, d\xi$$

$$= \overline{D}\left\{1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} - \int_{-1}^{0}(1, \overline{\sigma_2}, \overline{\rho_2})\, \theta e^{i\omega_0\tau_0\theta}(1, \sigma_1, \rho_1)^T\right\}$$

$$= \overline{D}\left\{1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} + \tau_0\overline{\sigma_2}T^*(\beta\rho_1 - \alpha\sigma_1)e^{i\omega_0\tau_0}\right\}$$

Hence, we can choose

$$\overline{D} = \frac{1}{(1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} + \tau_0\overline{\sigma_2}P_3^*(\beta\rho_1 - \alpha\sigma_1)e^{i\omega_0\tau_0})}$$

Such that $< q^*(s), q(\theta) > = 1, < q^*(s), \overline{q(\theta)} > = 0$.

Following the algorithm given by Hassard, B.D., Kazarinoff, N.D., Wan, Y.H and to calculate the parameters, representing the central surface, have used the same notations like there. C_0 at $\mu = 0$. Let $u_t$be the output of equation (11) with $\mu = 0$. Define

$z(t) =< q^*(s), u_t(\theta) >$,

$$W(t, \theta) = u_t(\theta) - 2Re\big(z(t)q(\theta)\big) \qquad (19)$$

On the center manifold C_0,

$W(t, \theta) = W\left(z(t), \overline{z(t)}, \theta\right)$ Where

$W(z, \overline{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots,$

z and $\overline{z}$ are local coordinates for centre manifold C_0 in the direction of q^* and $\overline{q^*}$ .And W is +ve if u_t is +ve. Only real solutions should be considered. For solution $u_t \in C_0$ of equation (11), since $\mu = 0$,

$z'(t) = i\omega_0\tau_0 z + < \overline{q^*}(\theta), F\big(0, W(z, \overline{z}, \theta) + 2Re\big(z(t)q(\theta)\big)\big) >$

$= i\omega_0\tau_0 z + \overline{q^*}(0)\, F\big(0, W(z, \overline{z}, 0) + 2Re\big(z(t)q(\theta)\big)\big)$

$\equiv i\omega_0\tau_0 z + \overline{q^*}(0)F_0(z, \overline{z})$

We rewrite this equation as

$$z'(t) = i\omega_0\tau_0 z(t) + g(z, \overline{z}) \qquad (20)$$

Where

$g(z,\overline{z}) = \overline{q^*}(0)F_0(z,\overline{z})$

$$g(z,\overline{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\frac{\overline{z}^2}{2} + g_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots \qquad (21)$$

Noticing

As $u_t(\theta) = (u_{1t}, u_{2t}, u_{3t}\ ) = W(t,\theta) + z\,q(\theta) + \overline{z}\overline{q(\theta)}$ and $q(0) = (1, \sigma_1, \rho_1)^T e^{i\omega_0\tau_0\theta}$, we have

$$u_{1t}(0) = z + \overline{z} + W_{20}{}^{(1)}(0)\frac{z^2}{2} + W_{11}{}^{(1)}(0)\, z\overline{z} + W_{02}{}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots,$$

$$u_{2t}(0) = \sigma_1 z + \overline{\sigma_1}\,\overline{z} + W_{20}{}^{(2)}(0)\frac{z^2}{2} + W_{11}{}^{(2)}(0)\, z\overline{z} + W_{02}{}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots,$$

$$u_{3t}(0) = \rho_{1_1} z + \overline{\rho_{1_1}}\,\overline{z} + W_{20}{}^{(3)}(0)\frac{z^2}{2} + W_{11}{}^{(3)}(0)\, z\overline{z} + W_{02}{}^{(3)}(0)\frac{\overline{z}^2}{2} + \cdots,$$

$$u_{1t}(-1) = ze^{-i\omega_0\tau_0} + \overline{z}e^{i\omega_0\tau_0} + W_{20}{}^{(1)}(-1)\frac{z^2}{2} + W_{11}{}^{(1)}(-1)\, z\overline{z} + W_{02}{}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots,$$

$$u_{2t}(-1) = \sigma_1 e^{-i\omega_0\tau_0} z + \overline{\sigma_1}\, e^{i\omega_0\tau_0}\,\overline{z} + W_{20}{}^{(2)}(-1)\frac{z^2}{2} + W_{11}{}^{(2)}(-1)\, z\overline{z} + W_{02}{}^{(2)}(-1)\frac{\overline{z}^2}{2} + \cdots,$$

Thus, comparison of coefficients with equation (21) gives:

$$\begin{aligned} g_{20} = -2\tau_0\overline{D}[&a + (1-P_2^*)\rho_1 - \sigma_1(P_1^*\rho_1 + T^*) + \overline{\sigma_2}(b_1\sigma_1^2 + (1-P_1^*)\sigma_1\rho_1) - \sigma_1 T^* - \rho_1 P_2^*) \\ &+ \overline{\sigma_2}\sigma_1(c\rho_1 - d_1 e^{-i\omega_0\tau_0} - e\sigma_1 e^{-i\omega_0\tau_0})]; \end{aligned}$$

$$\begin{aligned} g_{11} = -2\tau_0\overline{D}[&a + (1-P_2^*)Re\{\rho_1\} - P_1^* Re\{\overline{\rho_1}\sigma_1\} - T^* Re\{\rho_1\} + \overline{\sigma_2}(\sigma_1\overline{\rho_1}b_1 \\ &+ (1-P_1^*)Re\{\sigma_1\overline{\rho_1}\} - P_2^* Re\{\overline{\rho_1}\} - T^* Re\{\sigma_1\}) + \overline{\rho_2}(c_1\rho_1\overline{\rho_1} - d_1 Re\{\rho_1 e^{i\omega_0\tau_0}\} \\ &- d_2 Re\{\rho_1\overline{\sigma_1}e^{i\omega_0\tau_0}\})]; \end{aligned}$$

$$\begin{aligned} g_{02} = -2\tau_0\overline{D}\big[&a + (1-P_2^*)\overline{\rho_1} - \overline{\sigma_1}(P_1^*\overline{\rho_1} + T^*) + \overline{\sigma_2}\big(b_1\overline{\sigma_1^2} + (1-P_1^*)\overline{\sigma_1}\,\overline{\rho_1} - \overline{\sigma_1}T^* - \overline{\rho_1}P_2^*\big) \\ &+ \overline{\rho_2}\,\overline{\rho_1}\big(c_1\overline{\rho_1} - d_1 e^{i\omega_0\tau_0} - e\overline{\sigma_1}e^{i\omega_0\tau_0}\big)\big]; \end{aligned}$$

$$\begin{aligned} &g_{21} = -2\tau_0\overline{D}[a_1\Big(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\Big) + (1-P_2^*)(\tfrac{1}{2}W_{20}^{(1)}(0)\overline{\rho_1} + W_{11}^{(1)}(0)\rho_1 + \tfrac{1}{2}W_{20}^{(3)}(0) + \\ &W_{11}^{(3)}(0)) - (2Re\{\sigma_1\rho_1) - P_1^*\Big(\tfrac{1}{2}W_{20}^{(2)}(0)\overline{\rho_1} + \tfrac{1}{2}W_{20}^{(3)}(0)\overline{\rho_1} + W_{11}^{(1)}(0)\rho_1 + W_{11}^{(3)}(0)\sigma_1\Big) - \\ &T^*\Big(\tfrac{1}{2}W_{20}^{(2)}(0) + \tfrac{1}{2}W_{20}^{(1)}(0)\overline{\sigma_1} + W_{11}^{(1)}(0)\sigma_1\Big) + \overline{\sigma_2}\Bigg(b_1\Big(W_{20}^{(2)}(0)\overline{\sigma_1} + 2W_{11}^{(2)}(0)\sigma_1\Big) + \\ &(1-P_1^*)\Big(\tfrac{1}{2}W_{20}^{(2)}(0)\overline{\sigma_1} + W_{11}^{(2)}(0)\sigma_1 + W_{11}^{(3)}(0)\overline{\sigma_1}\Big) - (2Re\{\sigma_1\overline{\rho_1}\} + \sigma_1\rho_1) - \\ &P_2^*\Big(\tfrac{1}{2}W_{20}^{(1)}(0)\overline{\rho_1} + W_{20}^{(3)}(0) + W_{11}^{(1)}(0)\rho_1 + W_{11}^{(3)}(0)\Big) - T^*\Big(\tfrac{1}{2}W_{20}^{(2)}(0) + W_{20}^{(1)}(0)\overline{\sigma_1} + \end{aligned}$$

$W_{11}^{(2)}(0) + W_{11}^{(1)}(0)\sigma_1\Big)\Big) + \overline{\rho_2}(c_1\Big(W_{20}^{(3)}(0)\overline{\rho_1} + 2W_{11}^{(1)}(0)\rho_1\Big) - d_1\Big(W_{20}^{(1)}(-1)\overline{\rho_1} +$

$W_{11}^{(1)}(-1)\rho_1 + \frac{1}{2}W_{20}^{(3)}(0)e^{i\omega_0\tau_0} + W_{11}^{(3)}(0)e^{-i\omega_0\tau_0}\Big) - d_2(\frac{1}{2}W_{20}^{(1)}(-1)\overline{\rho_1} + W_{11}^{(2)}(-1)\rho_1 +$

$\frac{1}{2}W_{20}^{(3)}(0)\overline{\sigma_1}e^{i\omega_0\tau_0} + W_{11}^{(3)}(0)\sigma_1 e^{-i\omega_0\tau_0}))].$

For calculating g_{21}, The calculation of must be prioritized of $W_{20}(\theta)$ and $W_{11}(\theta)$. From equations (17) and (19):

$$W' = {u_t}' - z'q - \overline{z}'q = \begin{cases} \mathcal{A}W - 2Re\left[\overline{q^*}(0)F_0 q(\theta)\right], & \theta \in [-1,0) \\ \mathcal{A}W - 2Re\left[\overline{q^*}(0)F_0 q(0)\right] + F_0, & \theta = 0 \end{cases}$$

Let $W' = \mathcal{A}W + H(z,\overline{z},\theta)$ (22)

Where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + H_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots, \qquad (23)$$

On the other hand, on C_0 near the origin $W' = W_z z' + W_z\overline{z}'$

Expanding the above series and computing the coefficients, we get

$[\mathcal{A} - 2i\omega_0 I]W_{20}(\theta) = -H_{20}(\theta),$

$$\mathcal{A}W_{11}(\theta) = -H_{11}(\theta) \qquad (24)$$

By (16), we know that for $\theta \in [-1,0)$,

$H(z,\overline{z},\theta) = -\overline{q^*}(0)\overline{F_0}q(\theta) - \overline{q^*}(0)\overline{F_0}\,\overline{q}(\theta) = -gq(\theta) - \overline{g}\,\overline{q}(\theta)$

Comparing the coefficient with (18) we get for $\theta \in [-1,0)$, that

$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\,\overline{q}(\theta),$

$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\,\overline{q}(\theta).$

From (22), (24) and definition of $\mathcal{A}$ we obtain

$W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g_{02}}\,\overline{q}(\theta)$

Solving for $W_{20}(\theta)$:

$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{02}}}{3\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_1 e^{2i\omega_0\tau_0\theta},$

And similarly

$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{11}}}{\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_2$

Where E_1 and E_2 are both three dimensional vectors, and calculating by putting the $\theta = 0$ in H. In fact since

$H(z, \overline{z}, \theta) = -2Re\left[\overline{q^*}(0)F_0 q(0)\right] + F_0$, we have

$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\,\overline{q}(\theta) + F_{z^2}$,

$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\,\overline{q}(\theta) + F_{z\overline{z}}$

Where $F_0 = F_{z^2}\frac{z^2}{2} + F_{z\overline{z}}z\overline{z} + F_{\overline{z}^2}\frac{\overline{z}^2}{2} + \cdots$

Hence combining the definition of $\mathcal{A}$,

$\int_{-1}^{0} d\,\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(0) + g_{20}q(0) + \overline{g_{02}}\,\overline{q}(0) - F_{z^2}$ and

$\int_{-1}^{0} d\,\eta(\theta)W_{11}(\theta) = g_{11}q(0) - \overline{g_{11}}\,\overline{q}(0) - F_{z\overline{z}}$

Notice that

$\left[i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta}d\eta(\theta)\right]q(0) = 0$ and $\left[-i\omega_0\tau_0 I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}d\eta(\theta)\right]\overline{q}(0) = 0$

Implies

$\left[2i\omega_0\tau_0 I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta}d\eta(\theta)\right]E_1 = F_{z^2}$ and $-\left[\int_{-1}^{0} d\eta(\theta)\right]E_2 = F_{z\overline{z}}$

Hence,

$$\begin{bmatrix} (2i\omega_0 + a_1P_1^*) & -P_1^*P_3^* & P_1^* - P_1^*P_2^* \\ -P_2^*T^* & (2i\omega_0 + b_1P_2^*) & P_2^* - P_1^*P_2^* \\ -d_1T^*e^{-2i\omega_0\tau_0} & -d_2T^*e^{-2i\omega_0\tau_0} & (2i\omega_0 + c_1T^*) \end{bmatrix} E_1$$

$$= -2\begin{bmatrix} a_1 + (1 - P_2^*)\rho_1 - \sigma_1(P_1^*\rho_1 + T^*) \\ (b_1\sigma_1^* + (1 - P_1^*)\sigma_1\rho_1 - \sigma_1T^* - \rho_1P_2^*) \\ \rho_1(c_1\rho_1 - d_1e^{-i\omega_0\tau_0} - d_2\sigma_1e^{-i\omega_0\tau_0}) \end{bmatrix}$$

And

$$\begin{bmatrix} a_1P_1^* & -P_1^*P_3^* & P_1^* - P_1^*P_2^* \\ -P_2^*T^* & b_1P_2^* & P_2^* - P_1^*P_2^* \\ -d_1T^* & -d_2T^* & d_2T^* \end{bmatrix} E_2$$

$$= -2\begin{bmatrix} a_1 + (1 - P_2^*)Re\{\rho_1\} - P_1^*Re\{\overline{\rho_1}\sigma_1\} - T^*Re\{\sigma_1\} \\ \sigma_1\overline{\sigma_1}b_1 + (1 - P_1^*)Re\{\sigma_1\overline{\rho_1}\} - P_2^*Re\{\overline{\rho_1}\} - T^*Re\{\sigma_1\} \\ c_1\rho_1\overline{\rho_1} - d_1Re\{\rho_1\} - d_2Re\{\rho_1\overline{\sigma_1}\}e^{i\omega_0\tau_0} \end{bmatrix}$$

Thus g_{21} shows by the parameters.

Based on the above analysis, each g_{ij} can be used to evaluate by the parameters. As a consequence, the following values are calculated:

$C_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}$,

$\mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_0)\}}$, $\beta_2 = 2Re\{C_1(0)\}$,

$$T_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\,\{\lambda'(\tau_0)\}}{\omega_0\tau_0} \tag{25}$$

Theorem 5.2 The value of μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0$ $(\tau < \tau_0)$. The value of β_2 determines the stability of bifurcating solutions: the bifurcating periodic solutions are orbitally asymptotically stable (unstable) if $\beta_2 < 0$ $(\beta_2 > 0\,)$. The value of T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ $(T_2 < 0\,)$.

5.6 NUMERICAL SOLUTION

MATLAB is used to simulate the empirical outcome with the use of a numerical. For the different values of set, the system's display of actions as fellow:

$$a_1 = 1, b_1 = 1.4, c_1 = 1, d_1 = 1, d_2 = 2$$

The behavior of the system of equation from being stable to complex dynamic about the equilibrium $E^*(P_1^*, P_2^*, T^*)$ for different values of delay parameter τ is shown below:

And $E^*(P_1^* = 0.8687, P_2^* = 0.8969, T^* = 0.9824)$

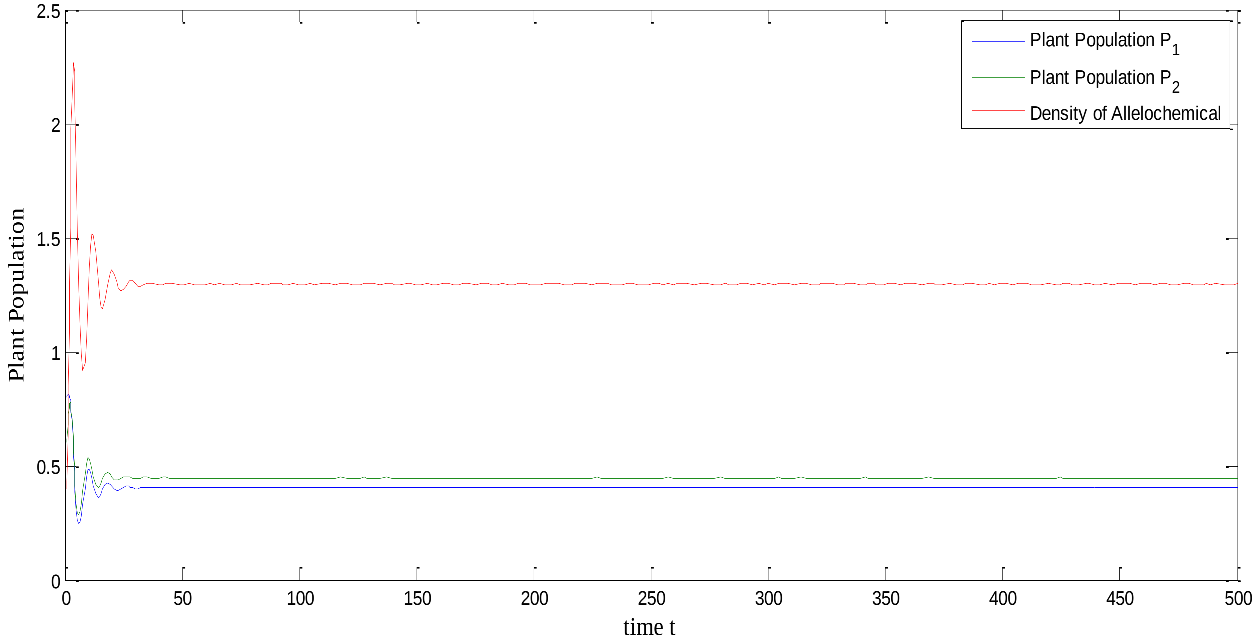

Figure 5.1 The interior equilibrium points $E^{*}(0.8687,0.8960,0.9824)$ of system stable when there is no delay $(\tau = 0)$.

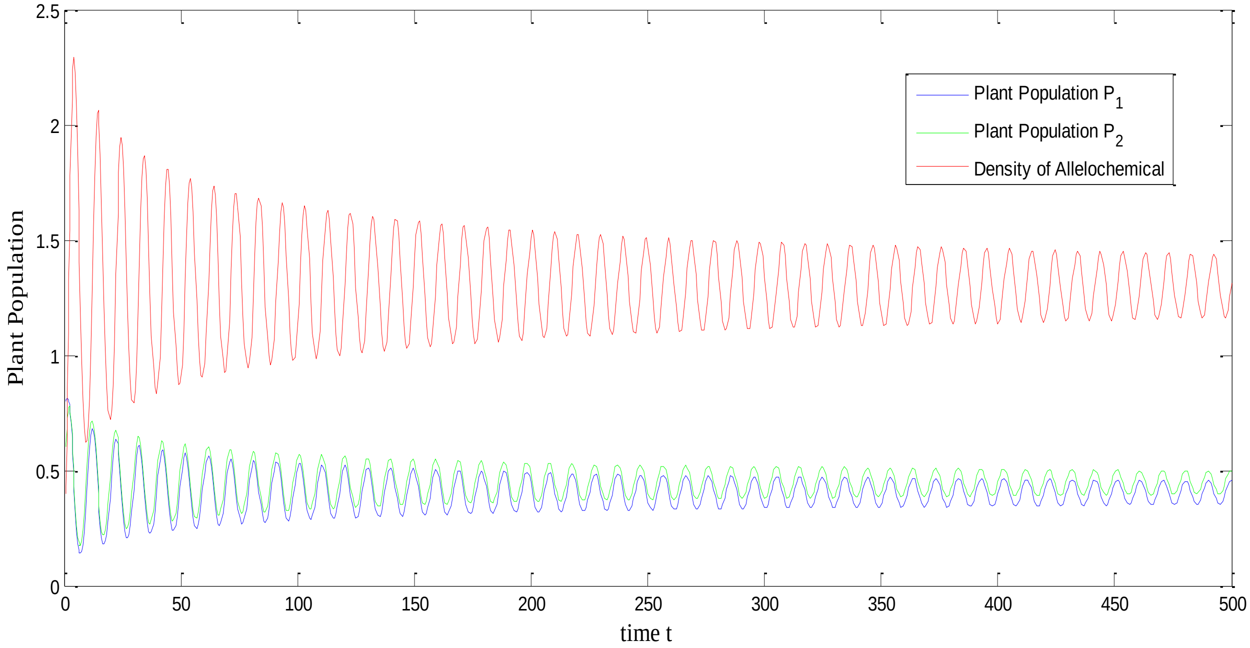

Figure 5.2 When delay $\tau < 1.7387$, the interior equilibrium point $E^{*}(0.8687,0.8960,0.9824)$ is asymptotically stable.

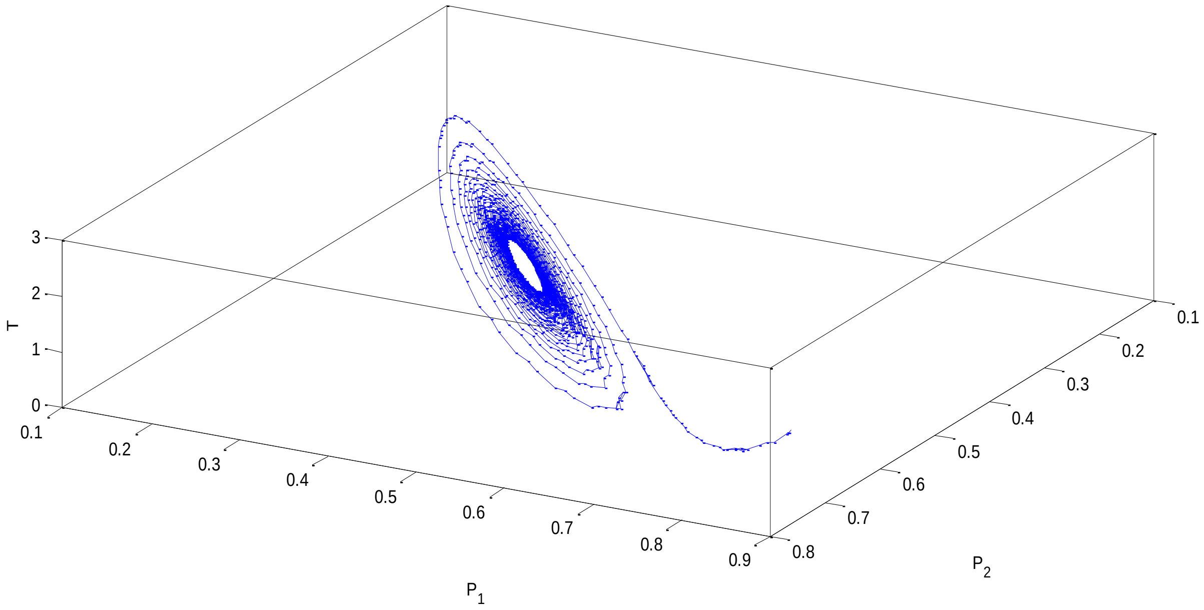

Figure 5.3 Phase diagram of Plant Populations P_1 & P_2, and Density of Allelochemical T when $\tau < 1.7387$,

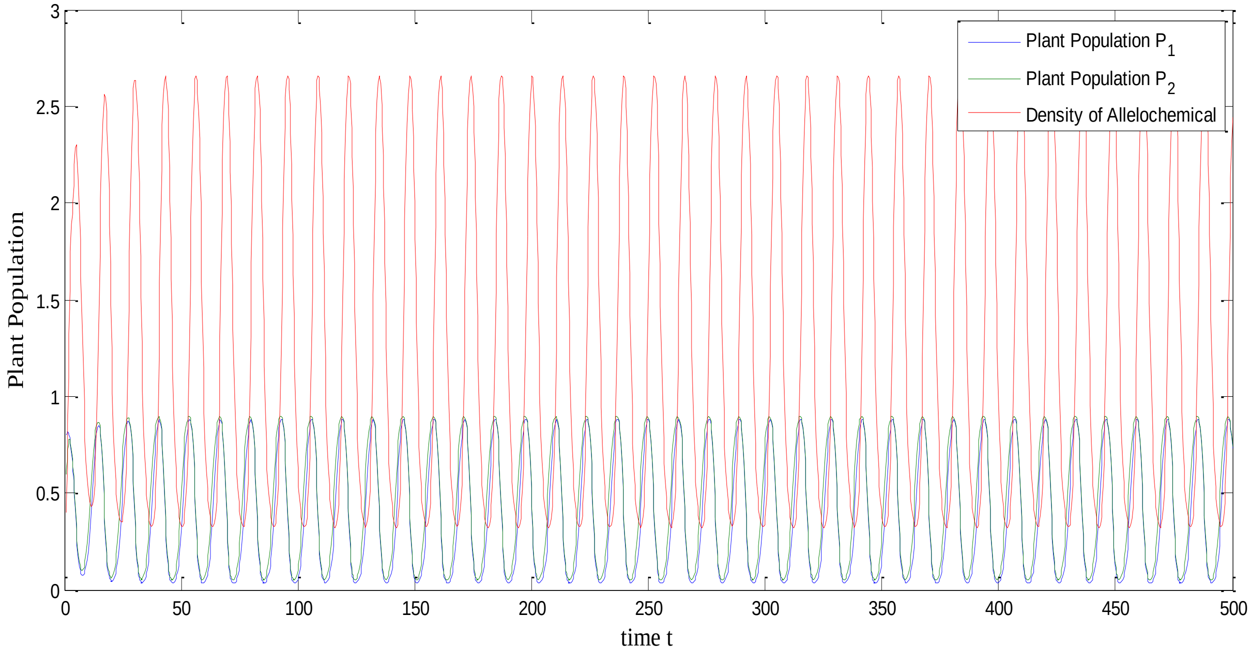

Figure 5.4 The interior equilibrium points $E^*(0.8687,0.8960,0.9824)$ losses its stability and Hopf- bifurcation occurs when $\tau \geq 1.7387$.

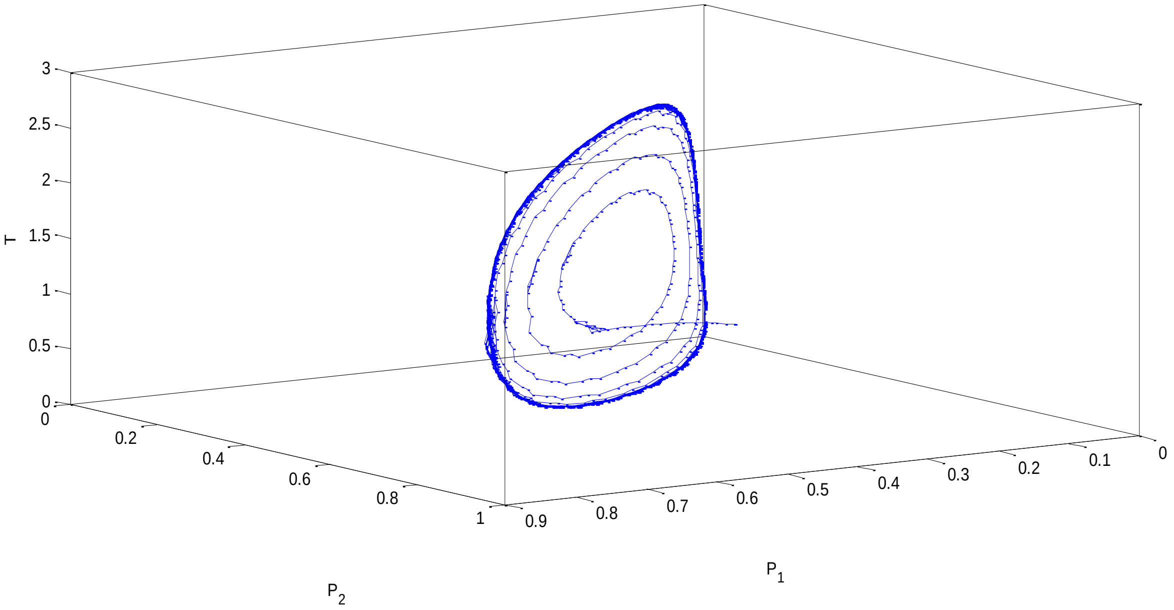

Figure 5.5 Phase diagram of Plant Populations P_1 & P_2, and Density of Allelochemical T when $\tau \geq 1.7387$. The bifurcation periodic solution is orbitally, asymptotically stable.

5.7 CONCLUSION:

A mathematical study assuming that plant population support one another when they are exposed to allelochemical concentration are the result of delay on an inter competing model composed of two plant population and one density of allelochemicals. The application of delay is said to destabilize the system's organ of stability using Routh-Hurwitz criteria. The model's internal equilibrium has been examined for system stability. When there is no delay, equilibrium point is stable $(\tau = 0)$ and when the value of delay parameter τ is below the critical point i.e., $\tau < 1.7387$, the equilibrium starts losing stability and leads to asymptotical stability. When the value of delay parameter τ is greater than or equal to the critical value $\tau \geq 1.7387$, the equilibrium exhibits the complex dynamics in the form of Hopf-bifuraction. The stable periodic solutions having large amplitude and limit cycle trajectory by the system. In addition, the specific procedure that defines the direction and stability of the Hopf-bifurcating solution is obtained.

Chapter-6

Mutual Competition of Two Plant Population under Allelochemicals and its Verification with Existing Data

6.1 INTRODUCTION

Allelopathic interactions arise when one plant produces a chemical that prevents the germination, development, or occurrence of another plant. Allelochemicals released by plants are also known as secondary metabolism. The analysis of biochemical mechanisms between living organisms is known as environmental ecology. Allelochemical interactions, for example, include compounds that allow organisms of one species to influence the development, health, or behavior of organisms of another species [137]. Friedman et al. conclude that actinomycetes that are toxic to plants or antifungal can inhibit plant growth or productivity of crops, either implicitly or explicitly, by inhibiting plant growth or mycorrhizal fungi [138]. Cherrington and Elliott studied that inhibition pseudomonads were discovered on wheat crop, sweet potato, lentil, and grain, as well as on downy brome. Soil management and area had little impact on conquering numbers. Both inhibition and stimulation isolates are discovered, with the inhibitory excludes outnumbering the stimulatory isolates [139]. Anwar et al. examined that the three methanol extract components of the combiflash from Lantana camara were bioassayed at 50%, 75%, and 100% concentrations utilising implantation rate criteria, panicle inhibition, and maximum root size [140]. Niggli et al. conclude that Average daily quantities of Boskoop apples were lowered by rape straw and oak bark, and both bark mulches increased fruit production [141]. Numerous vast and sometimes ecologically destructive artificial herbicides may be reduced by the effective use of allelopathic plant & microbes, or organic ingredients created by bacteria and fungi. Rice determines the amount of phenolic and other secondary metabolites in plants is increased by a variety of impacts [80]. Kurniadie and kato-noguchi examined that Lantana camara's allelopathic properties may encourage its invasion ability and the establishment of intense monospecies populations [142]. Qu et al. studied that root extracts have a substantial allelopathic influence on plant development and soil microorganism’s production by imitating root exudate, revealing novel clues into the function of plant-soil symbioses in determining invasion efficacy [143].

Todhunter and Beineke studied the change in black walnut development in the Indiana- black walnut plant growth system in the area with tall fescue or without. Growth of black walnut affected by fescue due to allelopathy, and compared it when naturally occurring in ground [144]. Weidenhamer et al. developed a strategy for discriminating between resources competition and allelopathic activity in trees that they suggested had some potential [145]. Misra & Kalra researched how the accumulation of heavy metals could detriment influence behaviors using a two-bay mathematical model [21]. A detailed evaluation was carried for different forms of stability, such as total stability, conditional stability along with discrete delays and bifurcation in predator-prey structure [4]. Kumar and Kalra analyses the influence of time gap in growth process of plant by adopting a two-compartment mathematical model [30]. Allelopathy is used in pest management indirectly by selecting allelochemicals from a variety of materials, such as plants or microorganisms, and using them as fertilizers instead of toxic substances. Since most organic fertilizers are broken down easily by specific microorganisms, they are not permanent compounds in the atmosphere like certain synthetic herbicides. Kumar studied the impact of heavy metals on plant population growth a mathematical model [25]. Ruan and Wei studied the existence and utilization of exponential polynomial (of third degree) roots, and used the outcomes to construct a delay model for testosterone secretion regulation [146]. Sorina et al. has shown in his research that when comparison to different extracts from the plants, the found that different from the leaf surface of black walnut had a greater allelopathy inhibitory effect [147]. Shibu and Gillespie studied on allelopathy in black walnut alley cropping in the Midwest: spatially difference in soil juglone in a black walnut–corn alley cropping method, and observed that polyethylene root inhibitors were found to be effective in prohibiting juglone from entering the alley where related crop populations are typically planted. This means that root thinning, nutrient injection, and root discing will all be used to reduce juglone's negative effects [148]. Chaubey et al. worked in Uttarakhand Himalaya on allelopathy possessions of Festering walnut leaf of radish seed germination and growth of sprout and found that Walnut leaf extracts have a concentration-dependent inhibitory effect on radish germinating seeds [149]. Allelochemicals found in the soils tend to have a detrimental impact on crop germination percentage and plant duration in reaction to rhizobial concentrations of weeds. Bertin et al. studied that allelochemicals are produced by almost all plants, but they play no direct role in their growth or development [150]. Khan et al. and Safdar et al. studied that the water extracts of parthenium were found to have phytotoxic impact on the developing and growing conditions of mungbean, soybean, and maize [151,152]. Until a huge breeding population can be developed to develop commercial production of essential agricultural crops that are allelopathic to identified weeds, several more screening programmers' must be performed.

6.2 MATHEMATICAL MODEL

The competing plant population is led by the system of non-linear delay differential equations having three variables which are stated as: P_1 & P_2 be two competing plant populations and T is the concentration of allelochemical produced by the species P_2. The competition mechanism is governed by the following system:

$$\frac{dP_1}{dt} = S - dP_1 - \beta_1 P_1(t-\tau)P_2 \tag{1}$$

$$\frac{dP_2}{dt} = hT - cP_2{}^2 \tag{2}$$

$$\frac{dT}{dt} = \beta_1 P_1 P_2 - \delta T \tag{3}$$

Where $P_1(0) > 0, P_2(0) > 0, T(0) > 0 \ \forall\ t$ and $P_1(t-\tau)T = \text{constant for } t$ for $t \in [0,\tau]$.

Table 1 The description of the parameters used in the model (1) -(3) is given in the Table 1.

Parameters	Description
S	Constant inflow of nutrients
d	Nature decay rate of plant population P_1
β_1	Rate of mutual competition between P_1 and P_2
h	Rate of stimulatory effect of allelochemicals for plant population P_2
c	Rate of intraspecific competition for plant population P_2
δ	Natural decay rate of allelochemicals
τ	Delay parameter

6.3 BOUNDEDNESS OF SOLUTIONS

The boundedness of solutions of the model (1) - (3) is given by:

$w = P_1 + P_2$ using (1) and (2)

$$\frac{\mathrm{dw}}{\mathrm{dt}} = \frac{\mathrm{dP_1}}{\mathrm{dt}} + \frac{\mathrm{dP_2}}{\mathrm{dt}}$$

And $\Phi = \min(d, h, \beta_1)$

$$\frac{dw}{dt} \leq S - \Phi$$

As $t \to \infty$ $w \leq \frac{S}{\Phi}$ Using comparison theorem

$$\text{So, } 0 \leq P_1 + P_2 \leq \frac{\mathrm{S}}{\Phi}$$

And from (3)

$$\frac{dT}{dt} \geq -\delta T$$

Using comparison theorem $t \to \infty$ $T \leq \frac{1}{\delta}$

$$\text{So } 0 \leq T \leq \frac{1}{\delta}$$

All solution lying in the Region

$Z = [(P_1, P_2, T) \in R^3 +: 0 \leq P_1 + P_2 \leq \frac{\mathrm{S}}{\Phi}, 0 \leq T < \frac{1}{\delta}]$

As t→ ∞ for all positively initial value.

6.4 POSITIVITY OF SOLUTION

Positivity means that the system is sustainable. One has to prove that all the solution has been given by equation (1)-(3), Where initial condition Where $P_1(0) > 0, P_2(0) > 0, T(0) > 0 \; \forall \; t \; and \; P_1(t - \tau)T = \text{constant} \; for \; t \; for \; t \in [0, \tau]$. the solution ($P_1$(t), P_2(t), T(t)) of the model remains +ve for all time $t > 0$.

From equation (3)

$$\frac{dT}{dt} \geq -\delta T$$

$$T \geq e^{-\delta t}$$

Similarly, we can calculate P_1, P_2.

6.5 EQUILIBRIUM POINT

$$\frac{dP_1^*}{dt} = S - dP_1^* - \beta_1 P_1^*(t-\tau)P_2^* = 0$$

$$S - P_1^*(d + \beta_1 P_2^*) = 0$$

$$P_1^* = \frac{S}{(d + \beta_1 P_2^*)} \tag{4}$$

$$P_2^* = \frac{h\beta_1 P_1^*}{c}$$

$$T^* = \frac{\beta_1 P_1^* P_2^*}{\delta}$$

Put the value of P_2^* in (4) and we get a quadric equation in P_1^*

$$\beta_1^2 h P_1^{*2} + cdP_1^* - Sc = 0$$

$$P_1^* = \frac{-(cd) \pm \sqrt{(cd)^2 + 4\beta_1^2 hSc}}{2\beta_1^2 h}$$

6.6 STABILITY OF EQUILIBRIUM $\mathbf{E^*(P_1^*, P_2^*, T^*)}$ AND HOPF- BIFURCATION

The equation scheme of the $E^*(P_1^*, P_2^*, T^*)$ balance competition process of the plant is defined by

$$\frac{\mathrm{d}P_1^*}{\mathrm{d}t} = S - dP_1^* - \beta_1 P_1^*(t-\tau)P_2^*$$

$$\frac{\mathrm{d}P_2^*}{\mathrm{d}t} = hT^* - cP_2{}^{*2}$$

$$\frac{\mathrm{d}T^*}{\mathrm{d}t} = \beta_1 P_1^* P_2^* - \delta T^*$$

Characteristic equation of the system is:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + (b_1\lambda^2 + b_2\lambda + b_3)e^{-\lambda\tau} = 0 \tag{5}$$

$$\text{Where } a_1 = \delta + 2cP_2^* + d$$

$$a_2 = 2c\delta P_2^* - h\beta_1 P_1^* + \delta d + 2cP_2^* d$$

$$a_3 = 2c\delta P_2^* d - h\beta_1 P_1^* d$$

$$b_1 = \beta_1 P_2^*$$

$$b_2 = (\beta_1 \delta P_2^* + 2c\beta_1 P_2{}^{*2})$$

$$b_3 = 2c\delta\beta_1 P_2{}^{*2} - h\beta_1^2 P_1^* P_2^* + h\beta_1^2 P_2^*$$

Clearly a_1, a_2, a_3, b_1, b_2, b_3 are all positive.

Clearly $\boldsymbol{\lambda} = \mathbf{i\omega}$ is a solution of equation (5) if and only if

$$(i\omega)^3 + a_1(i\omega)^2 + a_2(i\omega) + a_3 + (\, b_1(i\omega)^2 + \, b_2(i\omega) + \, b_3)e^{-i\omega t} = 0 \qquad (6)$$

Separating real and imaginary parts:

$$a_3 - \, a_1\omega^2 + (\, b_3 - \, b_1\omega^2)\cos(\omega\tau) + \, b_2\omega\sin(\omega\tau) = 0 \qquad (7)$$

$$a_2 - \omega^3 + \, b_2\omega\cos(\omega\tau) - (\, b_3 - \, b_1\omega^2)\sin(\omega\tau) = 0 \qquad (8)$$

Squaring and adding parts:

$$\omega^6 + \left({a_1}^2 - {b_1}^2 - 2a_2\right)\omega^4 + \left({a_2}^2 - {b_2}^2 + 2\, b_1\, b_3 - 2\, a_1\, a_3\right)\omega^2 + \left({a_3}^2 - {b_3}^2\right) = 0 \qquad (9)$$

Put

$$ {a_1}^2 - {b_1}^2 - 2a_2 = a$$

$$ {a_2}^2 - {b_2}^2 + 2\, b_1\, b_3 - 2\, a_1\, a_3 = b$$

$$ {a_3}^2 - {b_3}^2 = c$$

$$\text{And } \omega^2 = y$$

We get

$$y^3 + ay^2 + by + c = 0 \qquad (10)$$

If $c < 0$ then equation (9) has at least one positive real root

$$\text{Let h(y)} = y^3 + ay^2 + by + c$$

$$h(0) = c < 0, \quad \lim_{y\to\infty} \text{h(y)} = \infty, \;\; \exists\, y_0 \in (0,\infty)$$

$$h(y_0) = 0$$

if $c \geq 0$ then necessary condition for equation (9) to have positive real root is

$$D = a^2 - 3b \geq 0$$

$$h(y) = y^3 + ay^2 + by + c$$

$$h'(y) = 3y^2 + 2ay + b$$

$$h'(y) = 0 \; \to 3y^2 + 2ay + b = 0 \qquad (11)$$

$$y_{1,2} = \frac{-2a \mp \sqrt{4a^2 - 12b}}{6} = \frac{-a \mp \sqrt{D}}{3} \qquad (12)$$

If $D < 0$ then equation (11) does not have any real roots. So, the function $h(y)$ monotone increasing function in y,

$$h(0) = c \geq 0$$

equation (9) has no positive real roots.

Clearly if $D \geq 0$, then $y_1 = \frac{-a+\sqrt{D}}{3}$ is local minima of h(y)

If $c \geq 0$ then equation (10) has positive root if $y_1 > 0$ and $h(y_1) > 0$

Assume that either $y_1 \leq 0$ or $y_1 > 0$ and $h(y_1) > 0$

If $y_1 \leq 0, \sin h(y)$ is increasing for $y \geq y_1$ and $h(0) = c \geq 0$

It follows that $h(y)$ has no positive real zero.

If $y_1 > 0$ and $h(y_1) > 0, \sin y_2 = \frac{-a+\sqrt{D}}{3}$ is local maxima value,

It follows that $h(y_1) \leq h(y_2)$

$h(0) = c \geq 0$, as $h(y)$does not have positive real root.

Lemma 6.l Suppose $y_1$is defined by equation (12)

1.1. If$c < 0$, equation (10) contains at least one positive real root.

1.2. If $c \geq 0$ and $D = a^2 - 3b < 0$, then equation (10) has no positive root

1.3. If$c \geq 0$, then equation (9) has positive root if $y_1 > 0$ and $h(y_1) \leq 0$

Suppose equation (10) has positive root. W L O G, assume that it has three positive roots y_1, y_2, y_3. Then equation (9) has three positive roots

$$\omega_1 = \sqrt{y_1}, \omega_2 = \sqrt{y_2}, \omega_3 = \sqrt{y_3}$$

From (8)

$$\sin \omega\tau = \frac{a_2\omega - \omega^3}{d}$$

$$\tau = \frac{1}{\omega}\left[\sin^{-1}\frac{(a_2\omega - \omega^3)}{d} + 2(j-1)\pi\right] : j = 1,2,3 \tag{13}$$

$$\text{j=0,1,2} \quad \tau_k^{(j)} = \frac{1}{\omega k}\left[\sin^{-1}\frac{(a_2\omega - \omega^3 k)}{d} + 2(j-1)\pi\right] : k = 1,2,3\ldots\ldots$$

Then it is a pair of purely imaginary roots of equation (9)

When

$$\tau = \tau_k^{(j)}, k = 1,2,3; j = 0,1,2 \ldots..$$

$$\lim_{j\to\infty} \tau_k^{(j)} = \infty, \qquad \text{k = 1,2,3,4}$$

$$\tau_0 = \tau_{k_0}^{(j_0)} = \min_{1\leq k\leq 3, j\geq 1}[\tau_k^{(j)}], \omega_0 = \omega_{k_0}, \; y_0 = y_{k_0} \tag{14}$$

Lemma 6.2 Suppose that $a_1 \geq 0, (a_3 + d), \; a_1, \; a_2(\, a_3 - d) > 0$

2.1. If $c \geq 0$ and $D = a^2 - 3b < 0$ then all roots of equation (5) have negative real part for all $\tau \geq 0$

2.2. If $c < 0 \; or \; c \geq 0, y_1 > 0$ and $h(y_1) \leq 0$ then all the roots of equation (5) have negative part for all $\tau \, \epsilon (0, \tau_0)$,

Proof When $\tau = 0$, then equation (4) become

$$\lambda^3 + (a_1 + b_1)\lambda^2 + (a_2 + b_2)\lambda + (a_3 + b_3) = 0 \tag{15}$$

According to Routh- Hurwitz's Criteria

All roots of equation (6) have negative part iff

$$(a_3 + b_3) \geq 0, (a_1 + b_1)(a_2 + b_2) - (a_3 + b_3) > 0$$

$$\text{If } c \geq 0 \text{ and } D = a^2 - 3b < 0,$$

Lemma 1(1.2) show that equation (5) has no roots with zero real part $\forall \; \tau \geq 0$

When $c < 0 \; or \; c \geq 0, y_1 > 0$ and $h(y_1) \leq 0$,

Lemma (1.1) & (1.2) implies when $\tau \neq \tau_k^{(j)}, k = 1,2,3, j \geq 1$, equation (5) has no root with zero real part & is the minimum value of τ so that equation (5) has purely imaginary roots.

6.7 Verify the model with Existing Exp. Data

In this part, we attempt to validate the proposed model using established experimental results on Lantana camara and Wheat plant densities in the presence of allelochemicals.

The data related to this is given by Zahid et.al (2016).

Table 2 Plant densities of Wheat (P_1) and Lantana camara (P_2) under the effect of allelopathy.

	h	Decrease in (%)		h	Increase in (%)
P₁	2	73.1	P₂	2	68.42
	2.4	67.1		2.4	161.13
	2.7	63.5		2.7	171.69

6.8 NUMERICAL SOLUTION

The calculation is conducted with MATLAB to coordinate the analytical result with the aid of a numerical. The percentage decrease and increase of plant population densities are taken in a finite rang of allelochemicals taken in above mentioned experimental data Zahid et.al (2016).

$S = 10, d = 1, \beta_1 = 1, h = 2, c = 1, \delta = 1.$

The behavior of the system of equations (1) -(3) for various delay values is displayed as:

$E^*(P_1^* = 2.0000, P_2^* = 3.9999, T^* = 7.9994)$

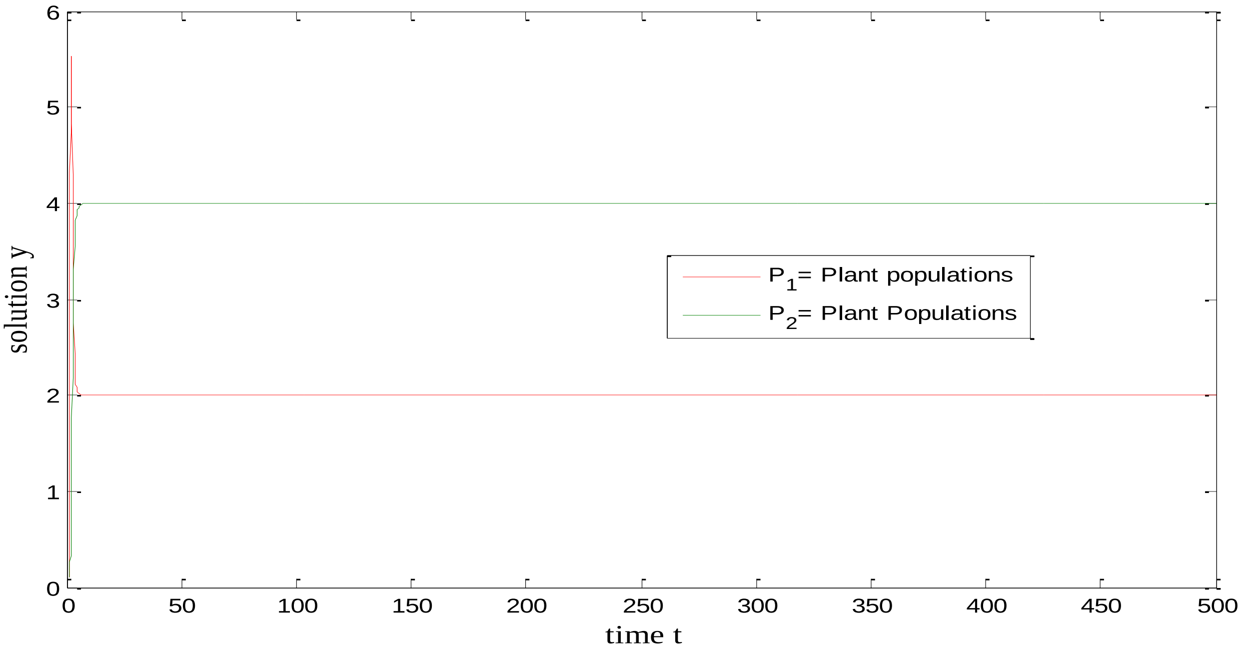

Figure 6.1 The uniform Equilibrium $E^*(P_1^*, P_2^*, 0)$ is absolute stable in the absence of conc. of allelochemicals.

The following table 3 shows that the growth of plant population gets increase with increasing the conc. of allelochemicals.

Table 3 Effect of increasing rate of plant population P$_2$

	H	Plant Population	Increase in (%)
P$_2$	2	5.845	68.42
	2.4	6.445	161.13
	2.7	6.867	171.69

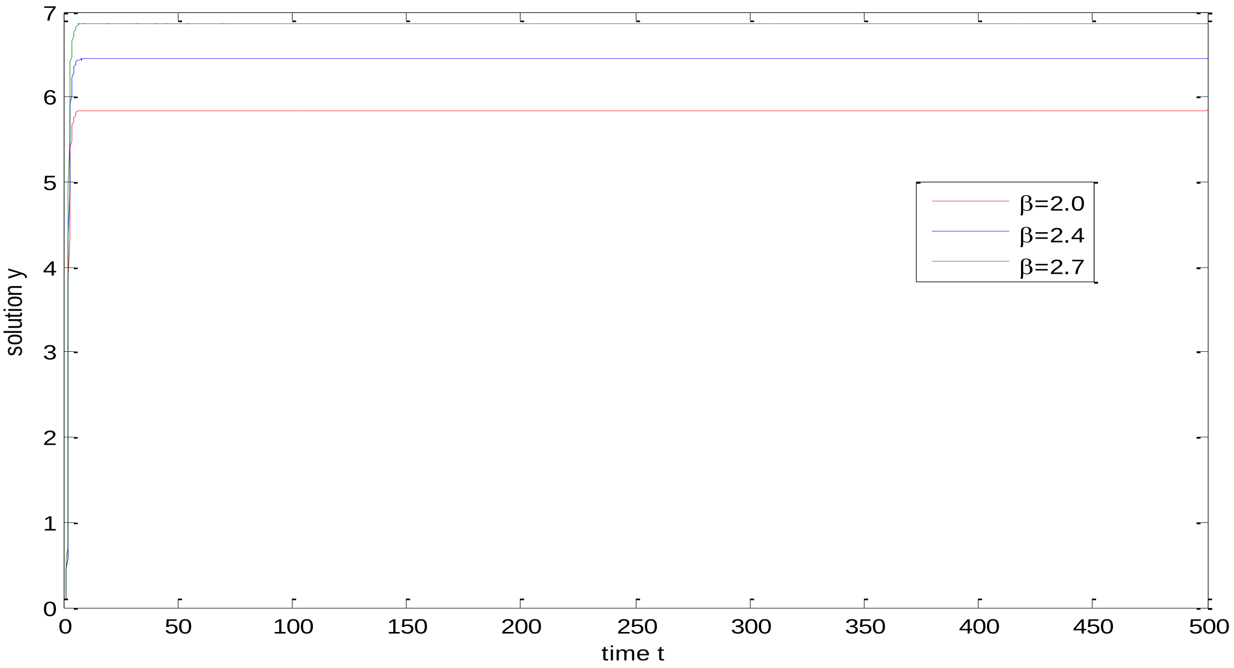

Figure 6.2 Course of plant population showing increase in their value with increasing β.

The following table 4 shows that the growth of plant population gets decrease with increasing the conc. of allelochemicals

Table 4 Effect of decreasing rate of plant population P$_1$

	H	Plant Population	Decrease in (%)
P$_1$	2	1.4612	73.1
	2.4	1.3429	67.1
	2.7	1.2715	63.5

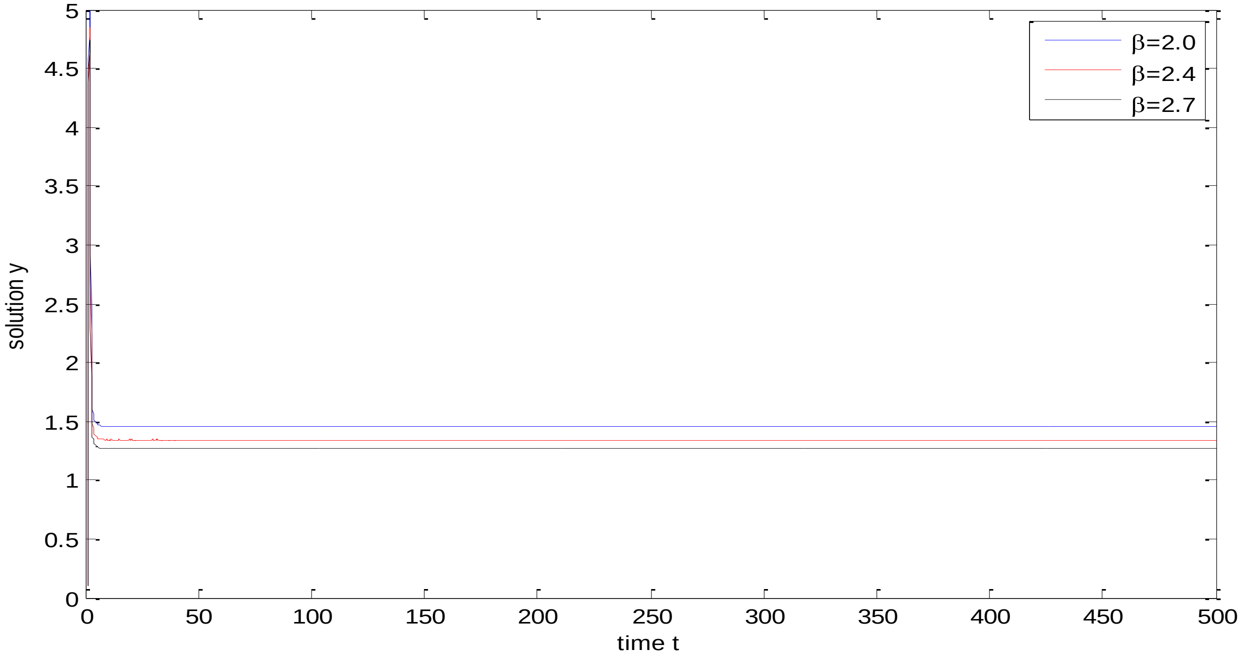

Figure 6.3 Course of plant population showing decrease in their value with increasing β.

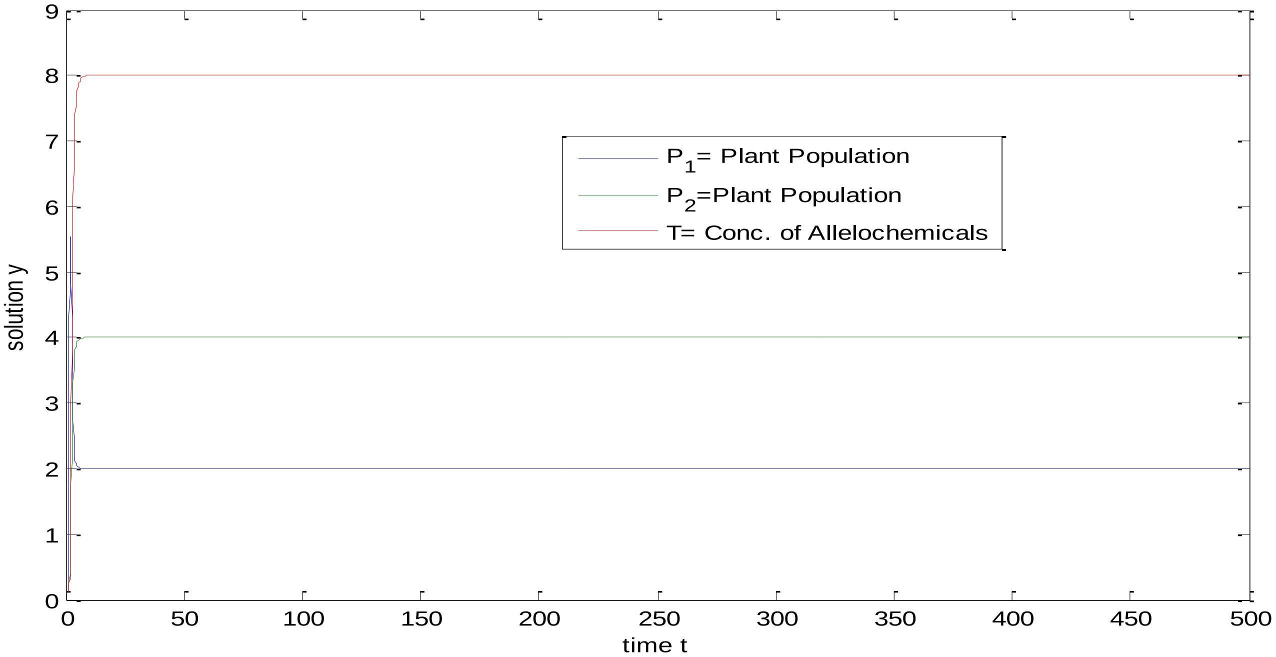

Figure 6.4 The Equilibrium $E^*(P_1^*, P_2^*, T^*)$ is absolutely stable in the absence of delay i.e., $\tau = 0$.

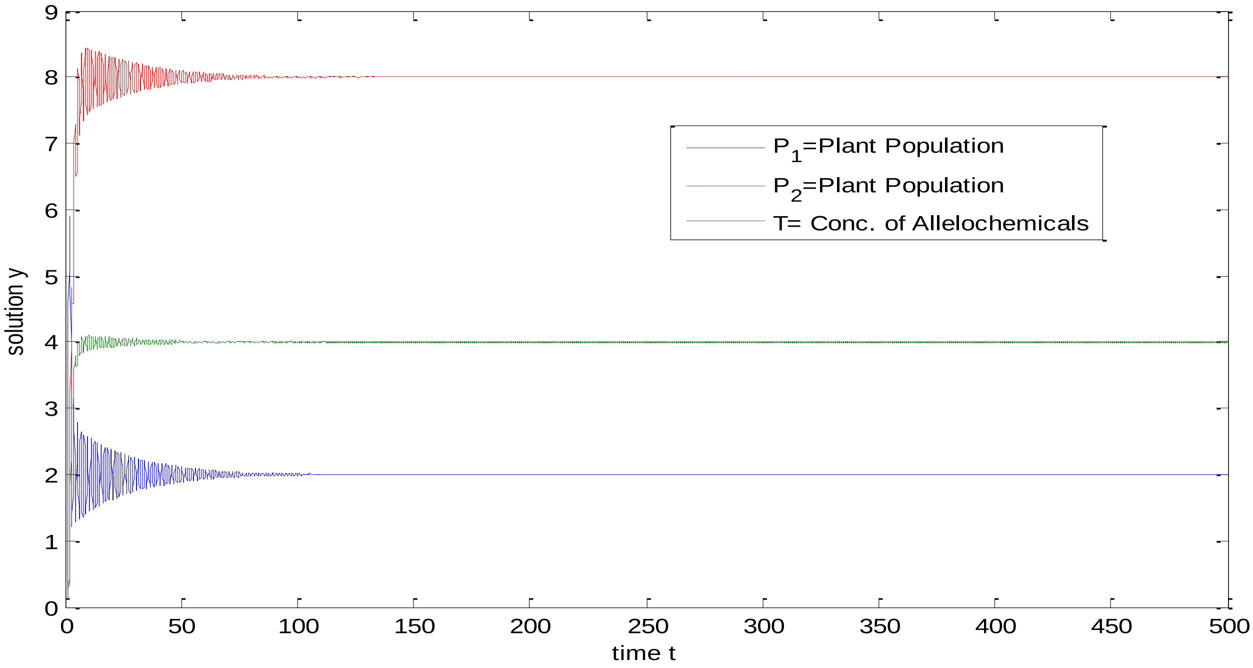

Figure 6.5 The equilibrium $E^*(P_1^*, P_2^*, T^*)$ is asymptotically stable when delay is less than the critical value. i.e., $\tau < 0.47$.

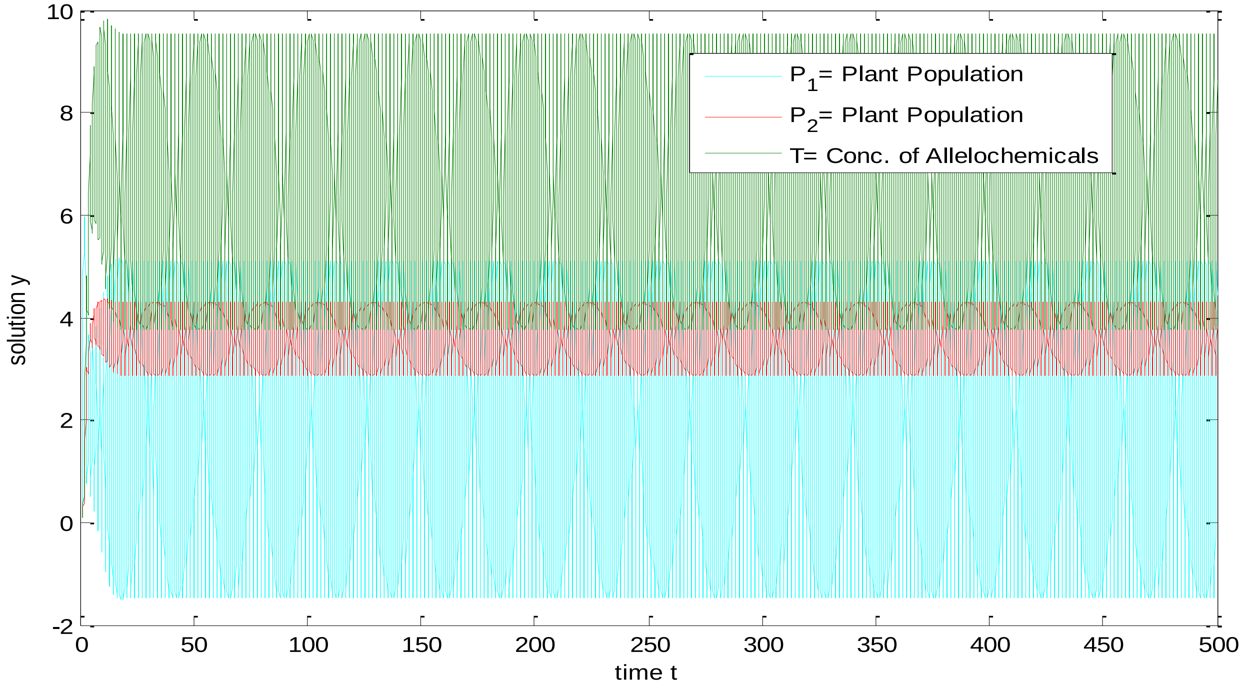

Figure 6.6 The equilibrium $E^*(P_1^*, P_2^*, T^*)$ losses its stability and Hopf-bifuraction takes place when delay i.e., $\tau \geq 0.47$.

6.9 CONCLUSION

With the use of the proposed model, the impact of delay on population of plant under the influence of allelochemicals is analyzed. Lantana camara plant affects the Wheat plant by producing allelochemicals. Equilibrium (E^*) is absolutely stable in the absence of allelochemicals. The experimental data considered in this paper in table3 and table4 shows how the Lantana camara(P_2) shows stimulating effect for itself and when it affects the Wheat plant(P_1), it shows inhibitory affect for Wheat plant. With the help of comparison theorem, we calculate positivity and boundedness of the model. Stability of equilibrium E^* is studied. Using Routh Hurwitz criteria, equilibrium (E^*) is absolutely stable in the non-appearance of allelochemicals as represented in Figure 6.4. The equilibrium (E^*) is asymptotically stable when delay is less than the critical value i.e. $\tau < 0.47$. The equilibrium E^* losses its stability and Hopf-bifuraction take place for with delay values that are more than critical value. i.e. $\tau \geq 0.47$.

CHAPTER-7

Effect of Time Delay on Directional and Stability analysis of Plant Competition for Allelochemicals Study

7.1 INTRODUCTION

Allelopathy shows all directly or indirectly effect from living substance from one plant population to another plant populations. The term coined fairly recently in 1937 by Austrian professor Hans Molisch, cited by Rice [80]. Allelopathy occurs when one plant population species inhibits the development of other plant population species through releasing the allelochemicals, is known as “allelochemicals” [137]. All physical and biological components are interconnected, and their balance is critical for the world's long-term sustainable growth and strength. Allelopathy is useful in plant, virus, and insect management, and in the research of optimal agricultural methods. Allelochemicals have been utilized as environmentally benign herbicides for generations, yet there are extremely many biological plants developed from allelochemicals on the industry. Maynard-Smith proposed a mathematical model for representing the allelochemicals relationship between two competing plant populations and worked on Lotka-Volterra competition model with adding extra term to determine the allelochemicals effect produced by one plant population on another plant population [153]. Chattopadhyay worked on the interaction between two plant populations under the allelopathic effect [106]. Allelopathy is form of physiology connection that takes place among different organism of the same species. Estrada et al. worked on Lotka-Volterra model on the basis of group cultivation and exclude the interspecific competition term because growing curve of the harmful species is same for crops and all bacteria varying the concentration of two species [154]. Allelochemical concentrations in plants start to decline with maturity, although they can vary as a defensive response to stress over the period of a plant's life. The components in the ecosystem are time-dependent, unexpected, and variable, and it should be treated as stochastic. The stochastic variation of each of the many process variables has a major impact on the dynamics of biological community. Since unpredictability is an intrinsic characteristic of any ecosystem [155,156]. For that we study the local and global stability about the non-zero equilibrium point. Saito give the necessary and sufficient condition for balanced system with the help of Lyapunov function to determine the solution in the complexed plane [157]. Huang et al. focused on research of the non-linear delay differential equation method's global equilibrium in terms of growth dynamics [41]. The presence of the empirical provided polynomial's zeros was investigated in depth by Raun and Wei [98,99]. Maynard-Simth [132] was changed by Mukhopadhyay et al. [4] to a delay differential dynamics system. They identified the specific delay required for the formation of the harmful allelochemicals that the microorganisms would produce. Fathalla A. Rihan studied delay differential equations in bioinformatics, including based assessment and sensitivity analysis; also create an application that verifies delay differential equation models' dependability with cell development dynamics [47]. Allelopathy also plays a significant role to eliminating the natural antagonists and competitor, because allelochemicals from plant may affect the living organism and product [158]. Many methods for allelopathic studies were presented in the literature, each meeting specific characteristics in various regions. The optimal method setup is important for explaining allelopathic relationships, which includes well-designed linked molecular biology techniques and field experiments [159]. From a sustainability and environmental approach, the impact of harmful substances on biological communities seems to be of major importance. With the help of mathematical models, Das et al. [160], Ghosh et al. [161], Hallam and Clark [162], Hallam et al. [121,125] and others examined the impact of allelochemicals on different ecosystems. Huang et.al examined a study of the global equilibrium of the non-linear delay differential equation method concerning population growth [41]. Abiotic and biotic variables influence the quality of allelochemicals in natural plants. Naresh et al. investigated the impact of toxic impact on plant growth across time [123]. Dipesh and kumar studied the stimulatory and inhibitory allelopathic effect of two competing plant populations [163]. Yafia worked on competition tumor and immune systems have inhibitory effect and explicit the direction analysis of the system [164].

Time lags play a significant role in the living organism, and it can have complicated influence on dynamic system such as generated oscillation, stability and periodic solutions. With the help of Lyapunov function, Zhen and Ma give the sufficient condition for local and global stability with delay for dynamics systems [165]. Zhang et al. study on the occurrence of hopf-bifuraction in the dynamical system and existing stability of limit cycle bifuraction from Hopf-bifuraction [166]. Kumar and Dipesh studied the effect of time delay on dynamic of plant competition under allelopathy [167.168]. Zhang identify the condition of dynamical system in which periodical solution bifurcate from the stable steady state [169]. Shi et al. observed that allelochemicals affect the several physiological and biochemicals process in plant populations, photosynthetic efficiency and metabolism and so on [87]. Allelopathic organisms have been shown to proliferate and bloom in environments where a unique crop is planted regularly, and the lower product consequences can't be reversed with the use of minerals [90]. Xiang et al. studied on the boundedness, continuity, global attractor and extinction of differential equations [170,171].

Allelopathic potential may affect plant's intrinsic quality, and it isn't always the truth that one component of a competing plant community is suppressed while the other improves. It's possible that both sides are influenced, but not to the same amount, so equilibrium is formed among them throughout the fight for a common resource. We suggest that all methodologies utilized in the simulation analysis are moderate and low, resulting in equal and consistent responses to the initial boundary value problems across all relevant periods. Because there is no confusion, the fundamental denotes a variable that is relative to its cases. In the domain of allelopathy in competitive plant populations, delayed differential equations have never been used before. In the light of the above, we analysis the role of delay in the competition of the community of plants under the allelopathic effect when one population releases allelochemicals and harm other, while the other organism doesn't develop any allelochemicals and discovers that the affecting population may be pushed to death.

7.2 MATHEMATICAL MODEL

A set of nonlinear ordinary differential equations lead the classic two-species Lotka-Volterra-type competition model.

$$\frac{dP_1}{dt} = P_1(a_1 - \alpha_1 P_1 - \beta_{12} P_2) \tag{1}$$

$$\frac{dP_2}{dt} = P_2(a_2 - \alpha_2 P_2 - \beta_{21} P_1) \tag{2}$$

Where $P_1 and\ P_2$are the two Competiting plant populations species respectively, $a_1\ and\ a_2$ are the cell proliferation rate of 1st and 2nd plant populations respectively, $\alpha_1 and\ \alpha_2$ are the intraspecific competition of 1st and 2nd plant population species. $\beta_{12}\ and\ \beta_{21}$ are the interspecific competition of 1st and 2nd plant population species. The units of $a_1, a_2, \alpha_1, \alpha_2$ number of days per cell and days is the unit of time. The development of each species accompanies the logistic equation of growth with intrinsic growth rate a_1, a_2 and carrying capacity $\frac{a_1}{\alpha_1}$ and $\frac{a_2}{\alpha_2}$ respectivily.

We take an additional term into the Lotka-Volterra integral equation of second plant population species to the model for the allelopathic interactions.

$$\frac{dP_1}{dt} = P_1(a_1 - \alpha_1 P_1 - \beta_{12} P_2) \tag{3}$$

$$\frac{dP_2}{dt} = P_2(a_2 - \alpha_2 P_2 - \beta_{21} P_2 + \gamma P_1 P_2) \tag{4}$$

γ denote the role of allelopathic substance realized by second plant population species and affects the first plant population with having all parameters have same biotic explanation above. Release of allelochemicals is not suddenly, but some discrete time delay is required for plant population to mature. This time delay is introduced in the release of allelochemicals by the 1st plant population. The competition model is defined for the system as:

$$\frac{dP_1}{dt} = P_1(a_1 - \alpha_1 P_1 - \beta_{12} P_2) \tag{5}$$

$$\frac{dP_2}{dt} = P_2(a_2 - \alpha_2 P_2 - \beta_{21} P_2 + \gamma P_1(t-\tau) P_2) \tag{6}$$

Where $P_1 > 0, P_1 > 0$ $for\ all\ t$ & $P_1(t-\tau) =$ constant for $t \in [0, \tau]$

For a well posed model we check basis properties like positivity, boundedness, equilibrium and stability because these properties would make it is easier to determine the appropriateness of the biological system. Although it seems, delay occur in the nature and play a significance role to understand the dynamic of the system.

7.3 BOUNDEDNESS OF SOLUTION

Let $B = P_1 + P_2$

$$\frac{dB}{dt} = \frac{dP_1}{dt} + \frac{dP_2}{dt}$$

$$\text{Let } \phi = \min(\alpha_1, \alpha_2, \beta_{12}, \beta_{21}, \gamma)\ and\ P_1(t-\tau) = P_1)$$

$$\frac{dB(t)}{dt} \le \phi(P_1 + P_2) - (a_1 + a_2)$$

$$\Rightarrow 0 \le \phi B - (a_1 + a_2)$$

$$\Rightarrow 0 \le B \le \frac{(a_1 + a_2)}{\phi}$$

Thus, all solution of (5)-(6) lying in the region $D = [(P_1, P_2) \in R_+^2 : 0 \le B \le \frac{(a_1+a_2)}{\phi}]$ as $t \to \infty$ for all positively initial value $P_1 > 0, P_1 > 0\ for\ all\ t\ \&\ P_1(t-\tau) =$ constant for t $\epsilon[0, \tau] \in \mathrm{D} \subset R_+^2, where\ \phi = \min(\alpha_1, \alpha_2, \beta_{12}, \beta_{21}, \gamma)$.

7.4 POSITIVITY OF THE SOLUTION

Positivity means that the system is feasible. One has to prove that all the give solution of equation (5)-(6) are positive with initial condition $P_1 > 0, P_1 > 0\ for\ all\ t\ \&\ P_1(t-\tau) =$ constant for t $\epsilon[0, \tau]$. The consequence (P_1, P_2) remain positive $\forall\ t > 0$.

From equation (5)

$$\frac{dP_1}{dt} \ge -\alpha_1 P_1$$

$$\Rightarrow \frac{dP_1}{P_1} \ge -\alpha_1 dt$$

$$\Rightarrow P_1 \ge e^{-\alpha_1 t}$$

And from equation (6)

$$\frac{dP_2}{dt} \ge -(\alpha_2 - \beta_{21} - \gamma P_1)P_2$$

$$\frac{dP_2}{P_2} \ge -(\alpha_2 - \beta_{21} - \gamma P_1)dt$$

$$\Rightarrow P_2 \ge e^{-\left(\alpha_2 - \beta_{21} - \gamma \frac{(a_1+a_2)}{\phi}\right)t}$$

For all t $P_1 \ge 0, P_2 \ge 0$,

7.5 EQUILIBRIUM POINTS

Various equilibrium points for the model (5)-(6) are $\mathrm{E}_{00}, \mathrm{E}_{a0}, \mathrm{E}_{0a}, \mathrm{E}^*$ exists with no restrictions on the variables that make the systems

$$\mathrm{E}_{00}: (0,0) \quad \text{(zero equilibrium point, unstable)}$$

$$\mathrm{E}_{a0}: \left(\frac{a_1}{\alpha_1}, 0\right) \quad \text{(axial equilibrum point, unstable)}$$

$$\mathrm{E}_{0a}: \left(0, \frac{a_2}{\alpha_2}\right) \quad \text{(axial equilibrum point, unstable)}$$

$$\mathrm{E}^*: (\mathrm{P}_1^*, \mathrm{P}_2^*)\ (\text{Non} - \text{zero equilibrium point, Stable})$$

Further we study and calculate the non-zero equilibrium point $\mathrm{E}^*(\mathrm{P}_1^*, \mathrm{P}_2^*)$

From equation (5)

$$\frac{dP_1^*}{dt} = 0 \Rightarrow P_1^*(a_1 - \alpha_1 P_1^* - \beta_{12} P_2^*)$$

$$P_1^* \neq 0, So\ (a_1 - \alpha_1 P_1^* - \beta_{12} P_2^*) = 0$$

$$P_1^* = \frac{a_1 - \beta_{12} P_2^*}{\alpha_1} \tag{7}$$

And similarly

$$P_2^* = \frac{a_2}{\alpha_2 + \beta_{21} - \gamma P_1^*} \tag{8}$$

Put the value of P_2 in equation (7), we get a quadratic equation in P_1 which is

$$\alpha_1 \gamma P_1^2 - (\gamma a_1 + \alpha_1 \alpha_2 + \alpha_1 \beta_{21}) P_1 + a_1 \alpha_2 + a_1 \beta_{21} - \beta_{12} a_1 = 0$$

Which have two roots

$$P_{1,2}^* = \frac{(\gamma a_1 + \alpha_1 \alpha_2 + \alpha_1 \beta_{21}) \pm \sqrt{(\gamma a_1 + \alpha_1 \alpha_2 + \alpha_1 \beta_{21})^2 - 4\alpha_1 \gamma (a_1 \alpha_2 + a_1 \beta_{21} - \beta_{12} a_1)}}{2\alpha_1 \gamma} \tag{9}$$

7.6 LOCAL STABILITY OF THE EQUILIBRIUM $\boldsymbol{E^*(P_1^*, P_2^*)}$

The equation scheme of the $E^*(P_1^*, P_2^*)$ balance competition of the plants is defined as

$$\frac{dP_1^*}{dt} = P_1^*(a_1 - \alpha_1 P_1^* - \beta_{12} P_2^*)$$

$$\frac{dP_2^*}{dt} = P_2^*(a_2 - \alpha_2 P_2^* - \beta_{21} P_2^* + \gamma P_1^*(t - \tau) P_2^*)$$

The character equation of the model is

$$\lambda^2 + a\lambda + b + ce^{-\lambda t} = 0 \tag{10}$$

Where $a = 2\alpha_2 P_2^* + 2\beta_{21} P_2^* + 2\gamma P_1^* P_2^* - a_2 - a_1 + 2\alpha_1 P_1^* + \beta_{12} P_2^*$

$$b = a_1 a_2 - 2a_1 \alpha_2 P_2^* - 2a_1 \beta_{21} P_2^* - 2a_1 \gamma P_1^* P_2^* - 2\alpha_1 P_1^*(b_2 - 2\alpha_2 P_2^* - 2\beta_{21} P_2^* - 2\gamma P_1^* P_2^*)$$
$$+ \beta_{21} P_2^*(2\alpha_2 P_2^* - a_2 + 2\beta_{21} P_2^* + 2\gamma P_1^* P_2^*)$$

$$c = \beta_{21} \gamma P_1^* P_2^{*2}$$

Clearly a, b, c is all positive.

When we put $\tau = 0$ in equation (10), we get

$$\lambda^2 + a\lambda + b + c = 0 \tag{11}$$

With the help of Routh-Hurwitz criteria, root of equ. (11) will be negative real part, i.e., the system is stable iff:

(H_1): $a > 0$, (H_2): $(b + c) > 0$ Which is true.

Now examine, how the negative real component of the root moves for the positive real component root when the value of τ differ.

Let $\lambda = i\omega$ be the root of equ. (10), then equ. (10) become

$$(i\omega)^2 + a(i\omega) + b + ce^{-i\omega t} = 0$$

$$\Rightarrow\ -\omega^2 + a(i\omega) + b + c(\cos\omega\tau - i\sin\omega\tau) = 0$$

Separate real and imaginary part, we get

$$\omega^2 - b = c\cos\omega\tau \tag{12}$$

$$a\omega = c\sin\omega\tau \tag{13}$$

Squaring and adding (12) and (13), we get

$$\omega^4 + (a^2 - 2b)\omega^2 + (a^2 - c^2) = 0 \tag{14}$$

Equations (14) have two roots

$$\omega_{1,2}^2 = \frac{(2b - a^2) \pm \sqrt{(a^2 - 2b)^2 - 4(a^2 - c^2)}}{2} \tag{15}$$

None of the two roots $\omega_{1,2}^2$ is positive if:

$$(H_3): (2b - a^2) < 0\ and\ (a^2 - c^2) > 0\ or\ (a^2 - 2b) < 4(a^2 - c^2)$$

So, equation (15) doesn't have positive root if condition (H_3) hold.

Lemma 7.1 If $(X_1) - (X_2)$ hold, then all the root of equation of (10) have $-ve$ real part $\forall\ \tau \geq 0$.

On the other hand, if

$$(H_4): (a^2 - c^2) < 0\ or\ (2b - a^2) > 0\ and\ (a^2 - 2b)^2 = 4(a^2 - c^2)$$

Then, positive root of equation (12) is ω_1^2. On the other hand, if

$$(H_5): (a^2 - c^2) > 0\ or\ (2b - a^2) > 0\ and\ (a^2 - 2b)^2 > 4(a^2 - c^2)$$

Then, equation (12) has two $+ve$ roots are $\omega_{1,2}^2$.

In both (H_4) and (H_5), the Equ. (10) has purely imaginary root when τ takes different values. The critical value $\tau_j^{\pm}$ of τ can be calculated from the equation (12) – (13), given by

$$\tau_l^{\pm} = \frac{1}{\omega_{1,2}} \cos^{-1}\frac{(\omega_{1,2}^2 - b)}{c} + \frac{2l\pi}{\omega_{1,2}}, l = 0,1,2, \ldots \tag{16}$$

The above knowledge can be described S. Ruan [13].

Lemma 7.2 (i) If $(H_1) - (H_2)$ and (H_4) hold and $\tau = \tau_l^{+}$, then equation (10) has a pair of imaginary roots $\pm i\omega_1$.

(ii) If $(H_1) - (H_2)$ and (H_5) hold and $\tau = \tau_l^- (\tau = \tau_l^+ \text{ resp.})$, then Equ. (10) has a pair of imaginary roots $\pm i\omega_2(\pm i\omega_1)$ respe.

Our hypothesis is that the negative real component of some equation roots will move to the positive real component when $\tau > \tau_l^+$ & $\tau < \tau_l^+$. Let us have a look at this possibility:

$$\tau_l^{\pm} = \mu_l^{\pm}(\tau) + i\omega_l^{\pm}(\tau); l = 0,1,2,3 \ldots.$$

The roots of equation (10) fulfill. $\mu_l^{\pm}\left(\tau_l^{\pm}\right) = 0,\ \omega_l^{\pm}\left(\tau_l^{\pm}\right) = \omega_{1,2}$

The preceding initial boundary criterion can be checked.

$$\frac{d}{d\tau}\left(Re\lambda_l^+(\tau_l^+)\right) > 0 \text{ and } \frac{d}{d\tau}\left(Re\lambda_l^-(\tau_l^-)\right) < 0$$

It concludes that τ_l^+ are the bifurcating values. The distribution of the equation's (10) zeros is determined by the Raun, S.

Theorem 7.1 Let $\tau_l^+(l = 0,1,2,3 \ldots)$ be defined by equation (16).

(1) If $(H_1), (H_2)$ hold, then all of root (10) have negative real part $\forall \tau \geq 0$.

(2) If $(H_1), (H_2)$ and (H_4) hold and when $\tau \epsilon [0, \tau_0^+)$, then all of root (10) have negative real parts. When $\tau = \tau_0^+$, then (10) has a pair of purely imaginary roots $\pm i\theta_1$. When $\tau > \tau_0^+$, (10) has at least one $+ve$ real part root.

(3) If $(H_1), (H_2)$ and (H_5) hold, then positive integer n such that $0 < \tau_0^+ < \tau_0^- < \tau_1^+ < \tau_1^+ \ldots\ldots\ldots < \tau_{n-1}^- < \tau_n^+$ and there are n switches from stability to instability. This means, when $\tau \epsilon [0, \tau_0^+), (\tau_0^-, \tau_1^+) \ldots\ldots\ldots (\tau_{n-1}^+, \tau_n^+)$ all the roots of Equ. (10) have $-ve$ real parts. When $\tau \epsilon [0, \tau_0^+), (\tau_0^-, \tau_1^+) \ldots\ldots\ldots (\tau_{n-1}^+, \tau_{n-1}^-)$ and $\tau > \tau_n^+$, equation (10) has at least one root with real parts.

7.7 GLOBAL STABILITY OF THE EQUILIBRIUM $E^*(P_1^*, P_2^*)$

In this part, we'll look at the conditions that make E^* a global attraction. According to the preceding section's formula for (a), it's evident that it's always positive and doesn't change its sign for any permissible parametric value, excluding out the occurrence of a Hopf-bifuraction periodic solution in the aera of E^*. The Benedixon-Dulac criterion is used to prove that a periodic orbit around E^* does not change.

$$h(P_1, P_2) = \frac{1}{P_1, P_2},\ h(P_1, P_2) > 0\ \forall\ P_1, P_2 > 0. \text{ We find}$$

$$d_1(P_1, P_2) = (a_1 P_1 - \alpha_1 P_1^2 - \beta_{12} P_1 P_2)$$

$$d_2(P_1, P_2) = (a_2P_2 - \alpha_2P_2^2 - \beta_{21}P_2^2 + \gamma P_1P_2^2)$$

$$\Delta(P_1, P_2) = \frac{\partial}{\partial P_1}(d_1, h) + \frac{\partial}{\partial P_2}(d_2, h)$$

And calculate that

$$\Delta(P_1, P_2) = -\frac{\alpha_1}{P_2} - -\frac{\alpha_2}{P_1} - \gamma P_2 \tag{17}$$

$$\Delta(P_1, P_2) < 0 \ \forall \ P_1, P_2 > 0;$$

and all parameters are $+ve$. With the help of Benedixon-Dulac criterion, we observe that there is no periodic path in the first quadrant near E^*. Now, we state the global stability condition E^* under different condition satisfies by the parameter complex with the dynamics system.

Theorem 7.2 Assume E^* is local asymptotically stable and let

$$4\alpha_1(\alpha_2 - \gamma\left(e^{-\lambda t}\right)^2) \geq (\beta_{21} + \beta_{12})^2 \tag{18}$$

Then E^* is global asymptotically stable with positive region of P_1, P_2 plane.

Proof: The Lyapunov function

$$B(P_1, P_2) = \sum_{i=12}\left[P_i - P_i^* - P_i^* \log\left(\frac{P_i}{P_i^*}\right)\right] \tag{19}$$

Which is positive $\forall \ (P_1, P_2) \in D$ and $(P_1, P_2) \neq (P_1^*, P_2^*)$, $B(P_1^*, P_2^*) = 0$

We do the derivate of (19) with respect to time along the solution path of model of (5)-(6) is

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + \sum_{i=1,2}\frac{\partial B}{\partial P_i}\frac{dP_i}{dt} = \sum_{i=1,2}\frac{P_i - P_i^*}{P_i}\frac{dP_i}{dt}$$

$$= (P_1 - P_1^*)(a_1 - \alpha_1P_1 - \beta_{12}P_2) + (P_2 - P_2^*)\left(a_2 - \alpha_2P_2 - \beta_{21}P_2 + \gamma e^{-\lambda t}P_2^2\right) \tag{20}$$

Using the result $a_2 = \alpha_2P_2^* + \beta_{21}P_1^* + \gamma e^{-\lambda t}(P_2^*)^2$, $a_1 = \alpha_1P_1^* + \beta_{12}P_2^*$ and after solving the mathematical computation, we get

$$\frac{dB}{dt} = -X^TJX$$

Where $X = \begin{bmatrix} P_1 - P_1^* \\ P_2 - P_2^* \end{bmatrix}$, and

$$J = \begin{bmatrix} \alpha_1 & \frac{1}{2}(\beta_{12} + \beta_{21} - \gamma e^{-\lambda t}P_2^*) \\ \frac{1}{2}(\beta_{12} + \beta_{21} - \gamma e^{-\lambda t}P_2^*) & \alpha_2 - \gamma\left(e^{-\lambda t}\right)^2 \end{bmatrix}$$

The matrix J is absolute $+ve$, if

$$4\alpha_1\left(\alpha_2-\gamma\left(e^{-\lambda t}\right)^2\right) \geq \left(\beta_{12}+\beta_{21}-\gamma e^{-\lambda t}P_2^*\right)^2 \geq (\beta_{21}+\beta_{12})^2 \tag{21}$$

Which hold the condition

$$4\alpha_1(\alpha_2-\gamma\left(e^{-\lambda t}\right)^2) \geq (\beta_{21}+\beta_{12})^2$$

$$\Rightarrow B(P_1,P_2) < 0 \forall\ (P_1,P_2) \in D\ and\ (P_1,P_2) \neq (P_1^*,P_2^*).$$

So, local asymptotically stable for E^*, inexistence of regular solution with the condition (20), which shows global stability of co-existence of equilibrium point, under these condition E^* is global stable.

7.8 DIRECTION AND STABILITY OF HOPF-BIFURCATING SOLUTION

A group of continuous functions that bifurcate at the critical value of the positive steady state is produced. The path, stability and the bifurcating periodic solution's periodicity is also significant. A specific formula to calculate the characteristics of the Hopf-bifuraction at the complex level, using the standard principle and several reductions due to Hassard, Kazarion and Wan, will be obtained in this section (1981).

$Let\ x_1 = P_1 - P_1^*, x_2 = P_2 - P_2^*,$ and normalizing the delay τ by time scaling, $t \to \frac{t}{\tau}$, system is transformed into

$$\frac{dx_1}{dt} = a_1x_1 + a_1P_1^* - \alpha_1x_1^2 - \alpha_1P_1^{*2} - 2\alpha_1x_1P_1^* - \beta_{12}x_1P_2^* - \beta_{12}x_2P_1^* - \beta_{12}x_1x_2$$

$$\frac{dx_2}{dt} = a_2x_2 + a_2P_2^* - \alpha_2x_2^2 - \alpha_2P_2^{*2} - 2\alpha_2x_2P_2^* - \beta_{21}x_1P_2^* - \beta_{21}x_2P_1^* - \beta_{21}x_1x_2$$

$$+ 2\gamma P_1(t-1)P_2^*x_2 + \gamma x_1(t-1)u_2^2 + \gamma x_1(t-1)P_2^{*2} \tag{22}$$

Thus, in that phase, we can deal $C = C\left((-1,0), R_+^2\right)$.WLOG, denote the critical value $\tau_j\ by\ \tau_0$. Let $\tau = \tau_0 + \mu,$ then $\mu = 0$ is the value of Hopf-bifurcation of the system (22-24).

For the simplification of notations, we rewrite (22) as

$$x'(t) = k_\mu(x_t) + m(\mu, x_t) \tag{24}$$

Where $x(t) = (x_1(t), x_2(t),\ )^T \in R^2, x_t(\theta) \in C$ is defined by $x_t(\theta) = x_t(t+\theta)$, and $k_\mu: C \to R,\ \ m: R \times C \to R$ are given, respectively by

$$k_\mu\emptyset = (\tau_0+\mu)\begin{bmatrix} a_1 - 2\alpha_1P_1^* - \beta_1P_2^* & -\beta_1P_1^* \\ 0 & a_2 \end{bmatrix}\begin{bmatrix}\phi_1(0) \\ \phi_2(0)\end{bmatrix} + (\tau_0+\mu)\begin{bmatrix} 0 & 0 \\ \gamma P_2^{*2} & 0\end{bmatrix}\begin{bmatrix}\phi_1(-1) \\ \phi_2(-1)\end{bmatrix}$$

And $m(\mu,\emptyset) = (\tau_0+\mu)\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$

Where $m_1 = b_1\phi_1(0)\phi_2(0), m_2 = \gamma\phi_1(0)\phi_2(0)$

And

$$\emptyset(\theta) = \left(\emptyset_1(\theta), \emptyset_2(\theta)\right)^T \in C\left((-1,0), R\right).$$

According to the Riesz representation theorem, ∃ a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1,0]$, s.t.

$$k_\mu\emptyset = \int_{-1}^{0} d\,\eta(\theta, 0)\emptyset(\theta) \text{ for } \emptyset \in C.$$

$$\eta(\theta, \mu) = (\tau_0 + \mu)\begin{bmatrix} a_1 - 2\alpha_1 P_1^* - \beta_1 P_2^* & -\beta_1 P_1^* \\ 0 & 2_2 \end{bmatrix}\delta(\theta) + (\tau_0 + \mu)\begin{bmatrix} 0 & 0 \\ \gamma P_2^{*2} & 0 \end{bmatrix}\delta(\theta + 1)$$

Here, δ is the Direct delta function, for $\phi \epsilon C([-1,0], R_+^2)$, define as

$$\mathcal{B}(\mu)\emptyset = \begin{cases} \frac{d\emptyset(\theta)}{d\theta}, & \theta \in [-1,0) \\ \int_{-1}^{0} d\,\eta(\theta, 0)\emptyset(\theta), & \theta = 0. \end{cases} \text{and } R(\mu)\emptyset = \begin{cases} 0, & \theta \in [-1,0) \\ m(\mu, \emptyset) & \theta = 0. \end{cases}$$

Then the system (23) is equivalent to

$$x'(t) = \mathcal{B}(\mu)\emptyset + R(\mu)x_t \text{ For} \tag{25}$$

$\psi \in C^1\left([-1,0], {R_+}^2\right)$, define

$$\mathcal{B}^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in [-1,0) \\ \int_{-1}^{0} d\,\eta^T(-t, 0)\psi(-t), & s = 0. \end{cases} \text{ And bi-linear inner product}$$

$$< \psi(s), \emptyset(\theta) > = \overline{\psi(0)}\emptyset(0) - \int_{-1}^{0}\int_{\xi=\theta}^{\theta} \overline{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)\,d\xi \tag{26}$$

$\mathcal{B}^*$ and $\mathcal{B} = \mathcal{B}(0)$ are algebraic operators and $i\omega_0$ are eigen values of $\mathcal{B}(0)$. As a result, they are coefficients of $\mathcal{A}^*$. Suppose that $q(\theta) = q(0)e^{i\omega_0\theta}$ is an eigen vector of $\mathcal{B}(0)$ corresponding to the eigen value $i\omega_0$. Then $\mathcal{B}(0) = i\omega_0\, q(\theta)$. When $\theta = 0$, we obtain

$\left[i\omega_0 I - \int_{-1}^{0} d\eta(\theta)e^{i\omega_0\theta}\right]q(0) = 0$, Which option $q(0) = (1, \sigma_1,)^T$

Where $\sigma_1 = \frac{a_1 - 2\alpha_2 P_1^* - \beta_1 P_2^* + i\omega_0}{-\beta_1 P_1^*}$

Similarly, we can verify that $q^*(s) = D(1, \sigma_2)e^{i\omega_0\tau_0 s}$ is the eigen value of $\mathcal{B}^*$ corresponding to $-i\omega_0$,

Where $\sigma_2 = \frac{a_1 - 2\alpha_2 P_1^* - \beta_1 P_2^* - i\omega_0}{-\beta_1 P_1^*}$

In order to assure$< q^*(s), q(\theta) > = 1$, and we calculate the value of D.

With the help of (11), $< q^*(s), q(\theta) >$

$$= \overline{D}(1,\overline{\sigma_2})(1,\sigma_1)^T - \int_{-1}^{0}\int_{\xi=\theta}^{\theta} \overline{D}(1,\overline{\sigma_2})e^{-i\omega_0\tau_0(\xi-\theta)}d\eta(\theta)(1,\sigma_1)^T e^{i\omega_0\tau_0}\, d\xi$$

$$= \overline{D}\left\{1+\sigma_1\overline{\sigma_2} - \int_{-1}^{0}(1,\overline{\sigma_2})\,\theta e^{i\omega_0\tau_0\theta}(1,\sigma_1)^T\right\} \overline{D}\{1+\sigma_1\overline{\sigma_2}+\tau_0\overline{\sigma_2}P_2^*(\gamma_2-\beta_2)e^{i\omega_0\tau_0}\}$$

Hence

$$\overline{D} = \frac{1}{(1+\sigma_1\overline{\sigma_2}+\rho_1\overline{\rho_2}+\tau_0\overline{\sigma_2}P_2^*\gamma e^{i\omega_0\tau_0})}$$

Such that $< q^*(s), q(\theta) > = 1, < q^*(s), \overline{q(\theta)} > = 0$.

Following the algorithm given by Hassard, B.D., Kazarinoff, N.D., Wan, Y.H. and to calculate the parameters, representing the central surface, have used the same notations like there. C_0 at $\mu = 0$. Let $u_t$be the output of equation (15) with $\mu = 0$. Define

$$z(t) =< q^*(s), x_t(\theta) >, \qquad W(t,\theta) = x_t(\theta) - 2Re\big(z(t)q(\theta)\big) \tag{27}$$

On the Centre manifoldC_0, $W(t,\theta) = W\left(z(t),\overline{z(t)},\theta\right)$

Where $W(z,\overline{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots,$

z and $\overline{z}$ are local coordinates for centre manifold C_0 in the direction of q^* and $\overline{q^*}$. And W is +ve if u_t is+ve. Only real solutions should be considered. For solution $x_t \in C_0$ of equation (15), since $\mu = 0$,

$z'(t) = i\omega_0\tau_0 z + < \overline{q^*}(\theta), F\big(0, W(z,\overline{z},\theta) + 2Re\big(z(t)q(\theta)\big)\big) >$

$= i\omega_0\tau_0 z + \overline{q^*}(0)\, F\big(0, W(z,\overline{z},0) + 2Re\big(z(t)q(\theta)\big)\big)$

$\equiv i\omega_0\tau_0 z + \overline{q^*}(0)F_0(z,\overline{z})$

We rewrite this equation as

$$z'(t) = i\omega_0\tau_0 z(t) + g(z,\overline{z}) \tag{28}$$

Where $g(z,\overline{z}) = \overline{q^*}(0)F_0(z,\overline{z})$

$$g(z,\overline{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\frac{\overline{z}^2}{2} + g_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots \tag{29}$$

Noticing

As $x_t(\theta) = (x_{1t}, x_{2t}\ ) =\ W(t,\theta) + z\,q(\theta) + \overline{z}\overline{q(\theta)}$ and $q(0) = (1,\sigma_1)^T e^{i\omega_0\tau_0\theta}$, we have

$x_{1t}(0) = z + \overline{z} + {W_{20}}^{(1)}(0)\frac{z^2}{2} + {W_{11}}^{(1)}(0)\, z\overline{z} + {W_{02}}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots,$

$x_{2t}(0) = \sigma_1 z + \overline{\sigma_1}\,\overline{z} + {W_{20}}^{(2)}(0)\frac{z^2}{2} + {W_{11}}^{(2)}(0)\, z\overline{z} + {W_{02}}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots,$

$$x_{1t}(-1) = ze^{-i\omega_0\tau_0} + \bar{z}e^{i\omega_0\tau_0} + W_{20}{}^{(1)}(-1)\frac{z^2}{2} + W_{11}{}^{(1)}(-1)\, z\bar{z} + W_{02}{}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots,$$

$$x_{2t}(-1) = \sigma_1 e^{-i\omega_0\tau_0}z + \overline{\sigma_1}\, e^{i\omega_0\tau_0}\,\bar{z} + W_{20}{}^{(2)}(-1)\frac{z^2}{2} + W_{11}{}^{(2)}(-1)\, z\bar{z} + W_{02}{}^{(2)}(-1)\frac{\bar{z}^2}{2} + \cdots,$$

Comparison of coefficients with equation

$$g_{20} = \bar{D}(1,\sigma_1)f_{z^2}, \qquad g_{02} = \bar{D}(1,\sigma_1)f_{\bar{z}^2}$$

$$g_{11} = \bar{D}(1,\sigma_1)f_{z\bar{z}}, \quad g_{21} = \bar{D}(1,\sigma_1)f_{z^2\bar{z}}$$

For calculating g_{21}, The calculation of must be prioritized of $W_{20}(\theta)$ and $W_{11}(\theta)$. From equations (24) and (26):

$$W' = x_t{}' - z'q - \bar{z}'q = \begin{cases} \mathcal{B}W - 2Re\left[\overline{q^*}(0)F_0q(\theta)\right], & \theta \in [-1,0) \\ \mathcal{B}W - 2Re\left[\overline{q^*}(0)F_0q(0)\right] + F_0, & \theta = 0 \end{cases}$$

Let $$W' = \mathcal{B}W + H(z,\bar{z},\theta) \tag{29}$$

Where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + H_{21}(\theta)\frac{z^2\bar{z}}{2} + \cdots \tag{30}$$

But at the other side, on C_0 close the origin $W' = W_z z' + W_z\bar{z}'$

We obtain by multiplying the above series by the variables.

$[\mathcal{B} - 2i\omega_0 I]W_{20}(\theta) = -H_{20}(\theta)$,

$$\mathcal{B}W_{11}(\theta) = -H_{11}(\theta) \tag{31}$$

By (23), we know that for $\theta \in [-1,0)$,

$H(z,\bar{z},\theta) = -\overline{q^*}(0)\overline{F_0}q(\theta) - \overline{q^*}(0)\overline{F_0}\,\bar{q}(\theta) = -gq(\theta) - \bar{g}\,\bar{q}(\theta)$

Comparing the coefficient with (26) we get for $\theta \in [-1,0)$, that

$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\,\bar{q}(\theta)$, $H_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\,\bar{q}(\theta)$.

From (29), (31) and definition of $\mathcal{A}$ we obtain

$W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g_{02}}\,\bar{q}(\theta)$

Solving for $W_{20}(\theta)$:

$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{02}}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_1e^{2i\omega_0\tau_0\theta}$,

And similarly

$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g_{11}}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_2$

Where E_1 and E_2 are both three dimensional vectors, and can be calculating by putting the $\theta = 0$ in H. In fact since

$H(z,\bar{z},\theta) = -2Re\left[\overline{q^*}(0)F_0 q(0)\right] + F_0$, we have $H_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\,\bar{q}(\theta) + F_{z^2}$,

$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\,\bar{q}(\theta) + F_{z\bar{z}}$, Where $F_0 = F_{z^2}\frac{z^2}{2} + F_{z\bar{z}}z\bar{z} + F_{\bar{z}^2}\frac{\bar{z}^2}{2} + \cdots$

Hence combining the definition of $\mathcal{A}$,

$\int_{-1}^{0} d\,\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(0) + g_{20}q(0) + \overline{g_{02}}\,\bar{q}(0) - F_{z^2}$ and

$\int_{-1}^{0} d\,\eta(\theta)W_{11}(\theta) = g_{11}q(0) - \overline{g_{11}}\,\bar{q}(0) - F_{z\bar{z}}$ Notice that

$\left[i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta}d\eta(\theta)\right]q(0) = 0$ and $\left[-i\omega_0\tau_0 I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}d\eta(\theta)\right]\bar{q}(0) = 0 \Rightarrow$

$\left[2i\omega_0\tau_0 I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta}d\eta(\theta)\right]E_1 = F_{z^2}$ and $-\left[\int_{-1}^{0} d\eta(\theta)\right]E_2 = F_{z\bar{z}}$

Hence $\begin{bmatrix} 2i\omega_0 + a_1 - 2\alpha_1 P_1^* - \beta_1 P_2^* & -\beta_1 P_1^* \\ (-\beta_{21}P_2^* + \gamma P_2^{*2})e^{-2i\omega_0\tau_0\theta} & a_2 + 2i\omega_0 \end{bmatrix} E_1 = -2\begin{bmatrix} 0 \\ -P_2^*\gamma\sigma_1 e^{-i\omega_0\tau_0\theta} \end{bmatrix}$

$$\begin{bmatrix} a_1 - 2\alpha_1 P_1^* - \beta_1 P_2^* & -\beta_1 P_1^* \\ -\beta_{21}P_2^* + \gamma P_2^{*2} & a_2 \end{bmatrix} E_1 = -2\begin{bmatrix} 0 \\ -P_2^*\gamma\sigma_1 e^{-i\omega_0\tau_0\theta} \end{bmatrix}$$

And g_{21} can be shows by the parameters

On the basis of above calculations, every g_{ii} can be calculated with the help of parameters. And these quantities can be calculated:

$$C_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2} \;\; and\; \mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_0)\}},$$

$$\beta_2 = Re\{C_1(0)\}$$

$$T_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\omega_0\tau_0} \tag{32}$$

Theorem 7.3 The direction of Hopf-bifuraction is calculated by the value of μ_2: if $\mu_2 > 0 (\mu_2 < 0)$, then the hopf-bifuraction is saturated and there is a regular bifuraction exist for $\tau > \tau_0$ $(\tau < \tau_0)$. With the help of β_2, we can calculate the stability of bifurcating solutions: the bifuraction cyclic solutions are asymptotically stable if $\beta_2 > 0$ $(\beta_2 < 0)$. The value of T_2 calculate the cycle of bifurcating cyclic solutions, whether the cyclic increase or decease $T_2 > 0$ $(T_2 < 0)$.

7.8 NUMERICAL EXAMPLE

In order to give the assist to the local stability outcomes for the dynamics and calculated the numerically solution of the system (5)-(6) with the help of MATLAB software. Different values for different parameters are as follow

$$a_1 = 3, \alpha_1 = 0.07, \beta_{12} = 0.04, a_2 = 1, \alpha_2 = 0.08, \beta_{21} = 0.0015, \gamma = 0.4$$

The different value of the delay parameter, the dynamics behaviour of the system is complexed around the equilibrium points E^* as shown in graphs:

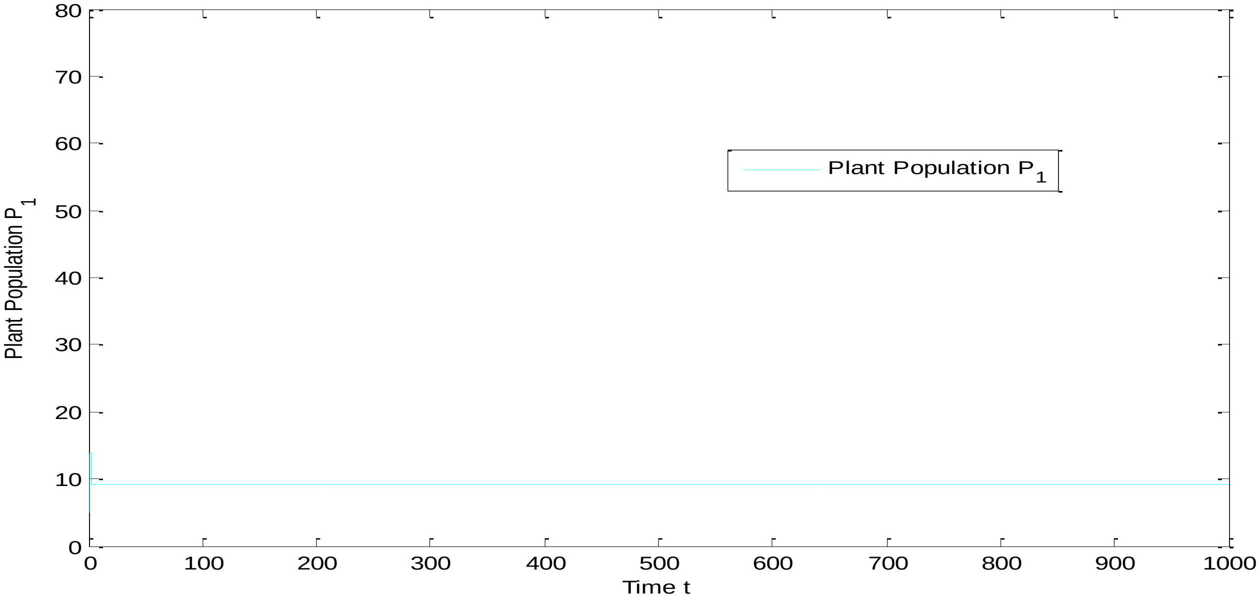

Figure 7.1 When there is no delay the plant population P_1 is stable, at $\tau = 0$.

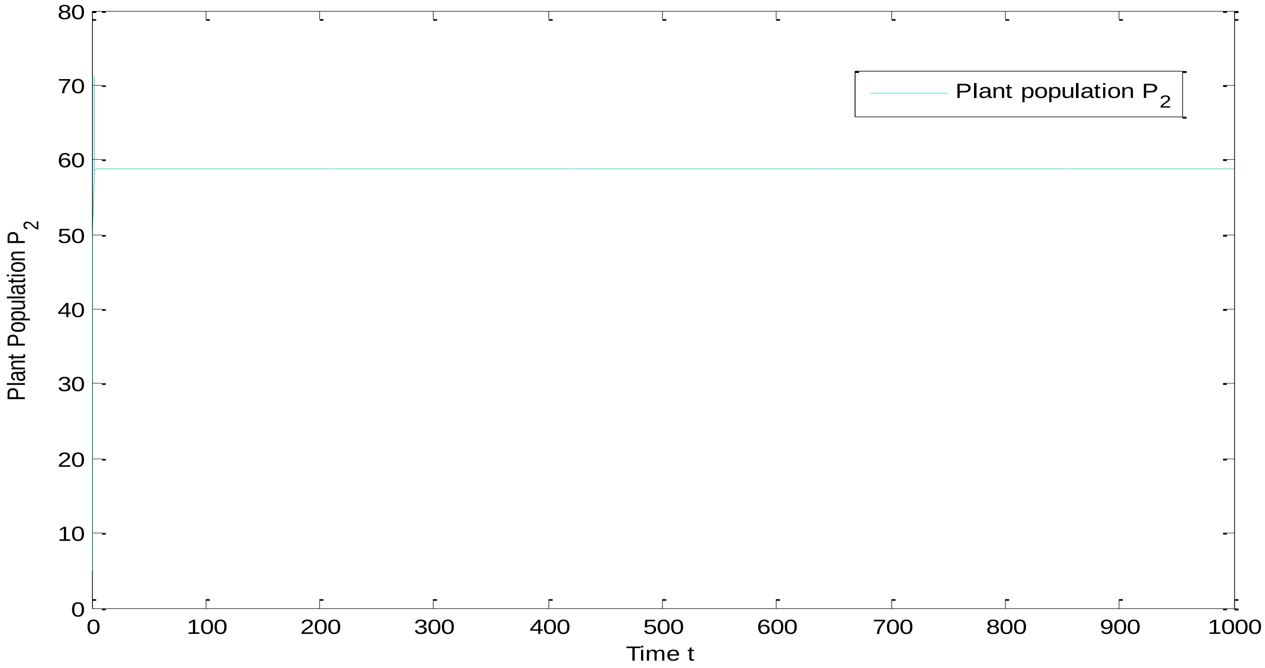

Figure 7.2 When there is no delay the plant population P_2 is stable, at $\tau = 0$.

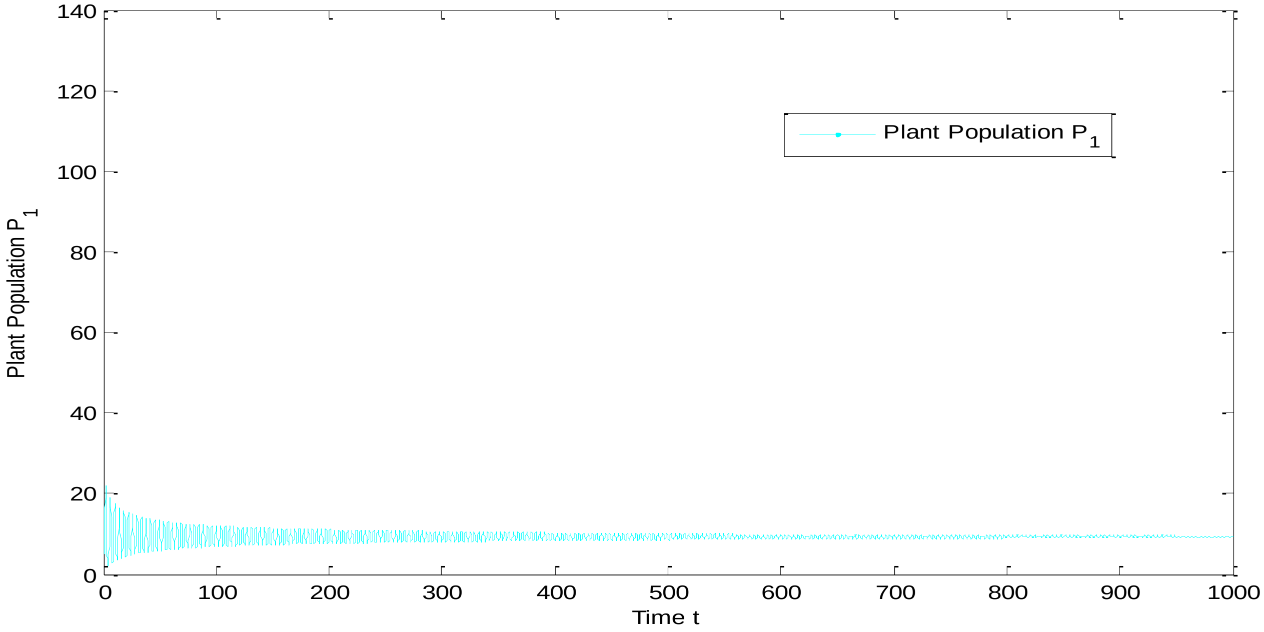

Figure 7.3 The plant population P_1 shows asymptotically stable at $\tau < 1.26$

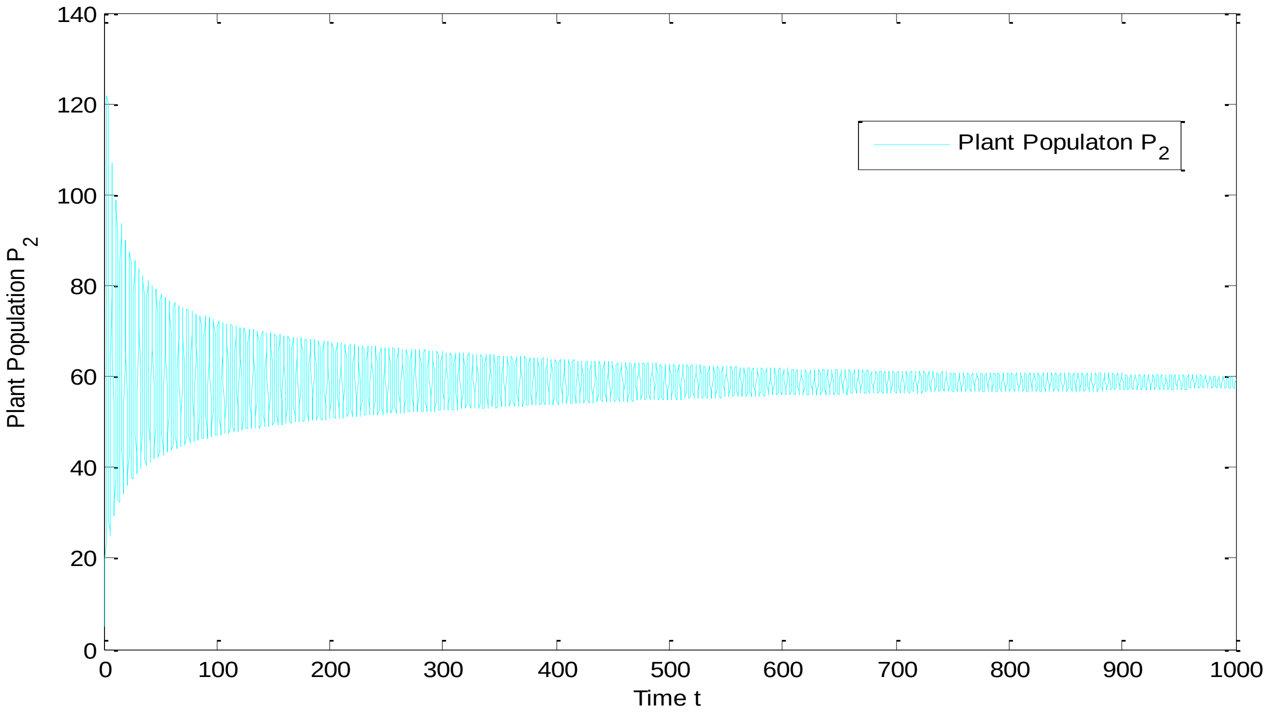

Figure 7.4 The plant population P_2 shows asymptotically stable at $\tau < 1.26$

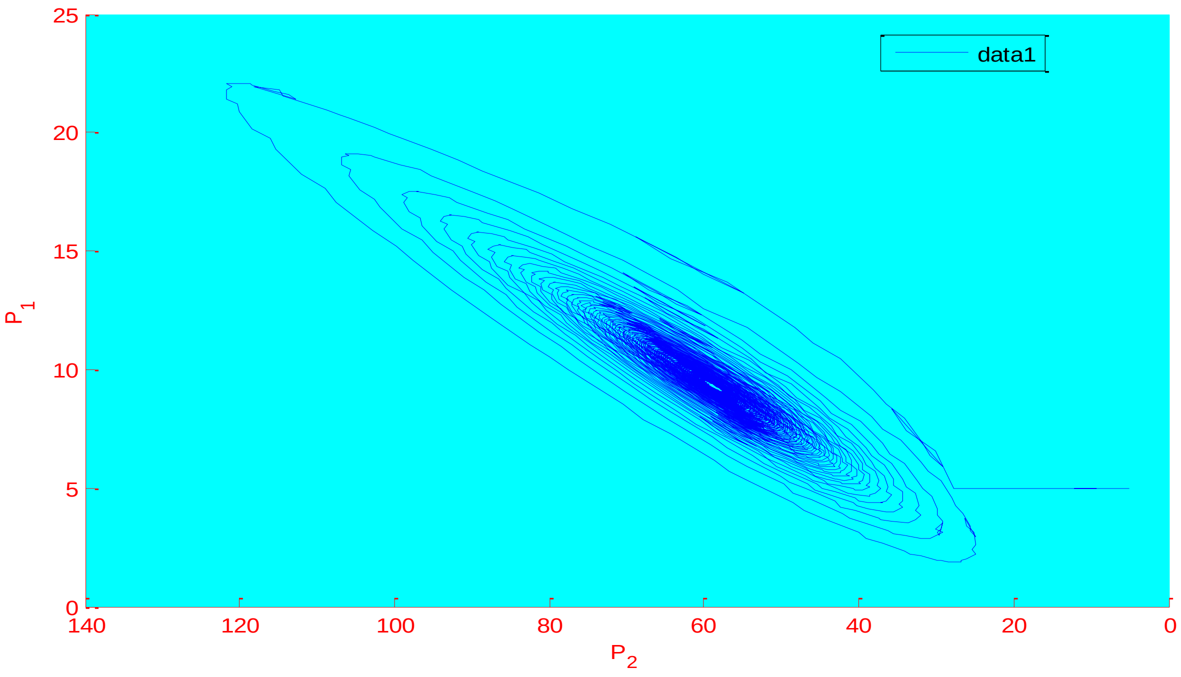

Figure 7.5 Phase diagram of plant populations when the value of delay is less than the critical value. i.e., $\tau < 1.26$

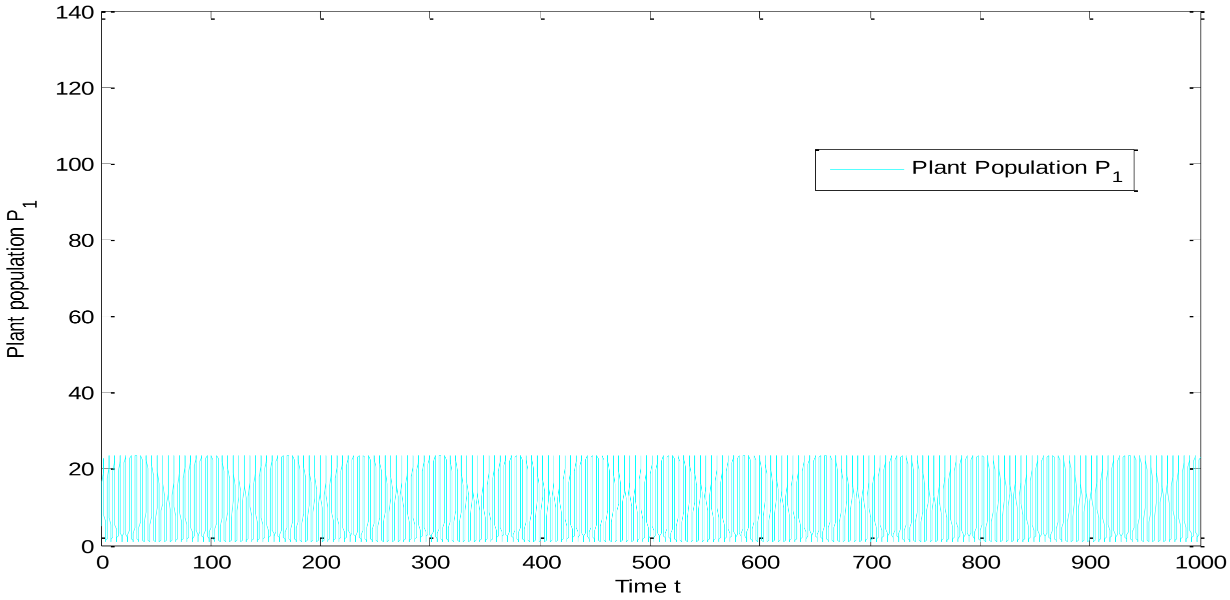

Figure 7.6 The plant population P_1 losses its asymptotically and Hopf-bifuraction obtained at $\tau \geq 1.26$.

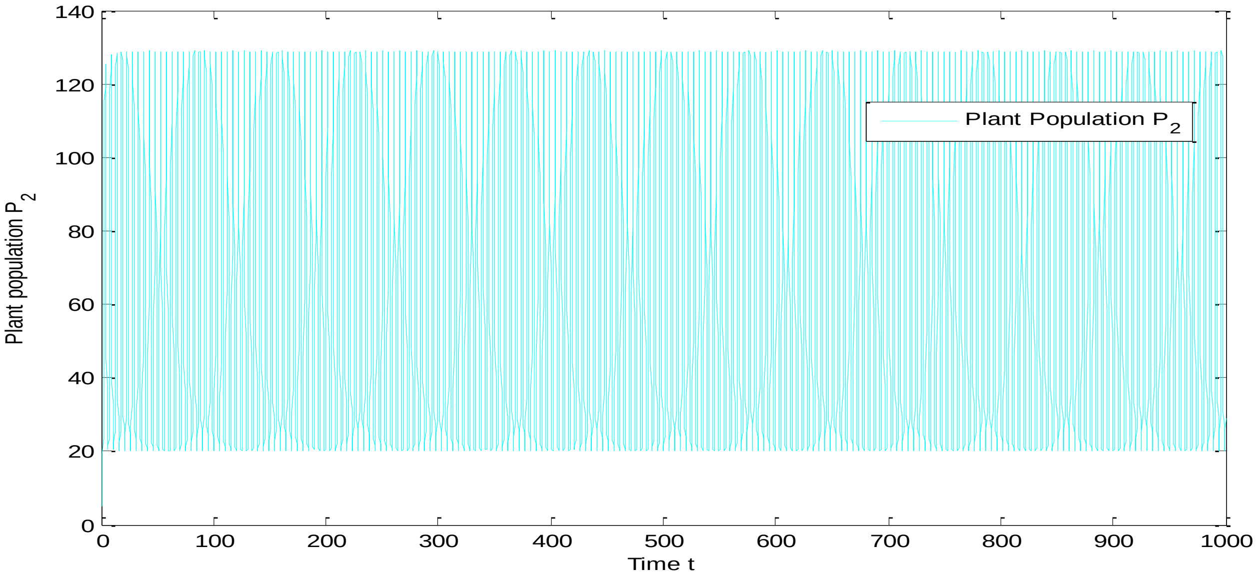

Figure 7.7 The plant population P_2 losses its asymptotically and Hopf-bifuraction obtained at $\tau \geq 1.26$.

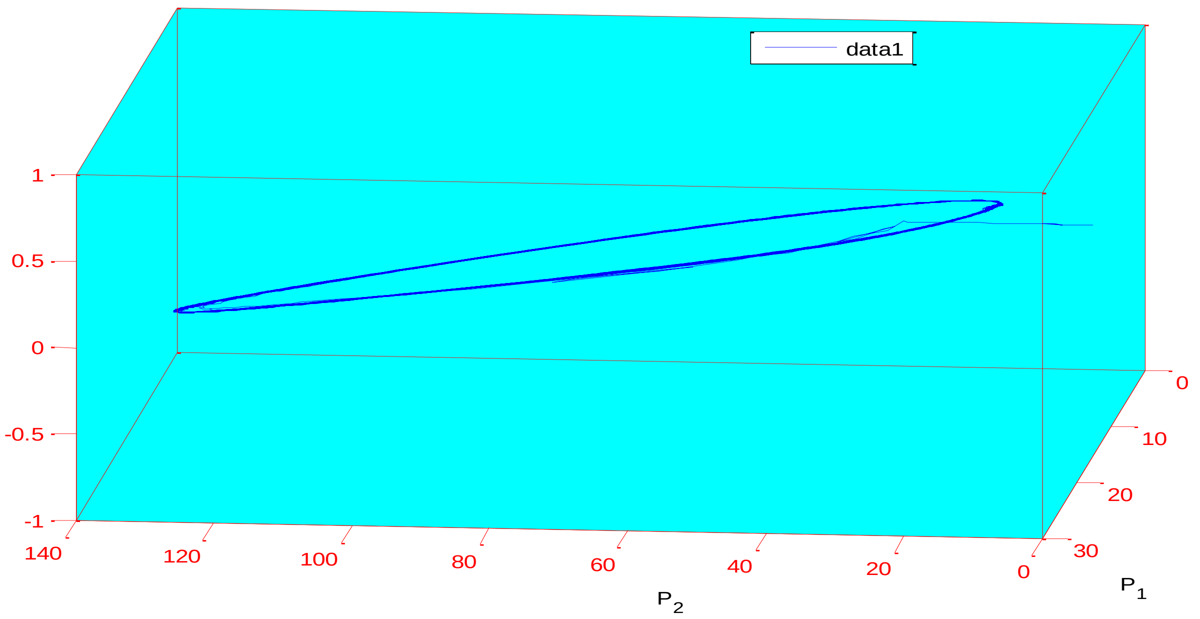

Figure 7.8 Phase diagram of the plant population when the value of delay is greater than the critical value. i.e., $\tau \geq 1.26$.

After that we choose $\alpha_2 = 1.49$ and others parameters are same, we observe the stability of the plant population, E^* became global stable. Figure 9 shows that plant population density of P_1

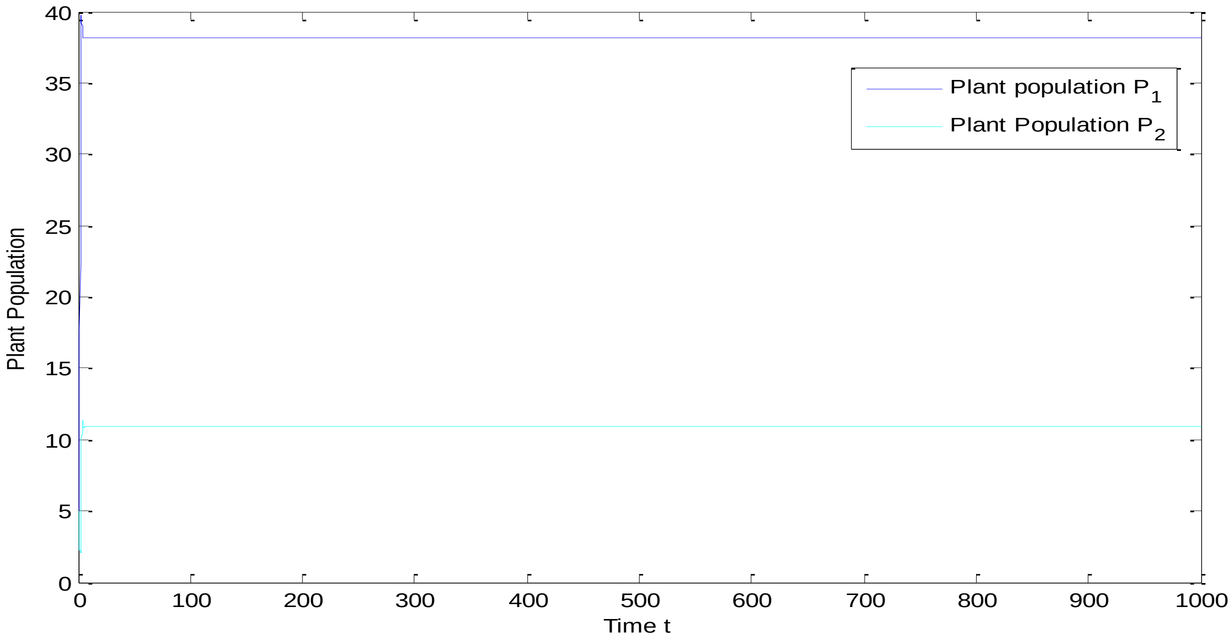

Figure 7.9 The plant population density of the dynamics system (5)-(6) is stable with parametric value.

plant population increases faster as compare the plant population P_2. The plant population P_1, P_2 attain their equilibrium state and remains all future time. It shows the global stability E^* and determination of the plant population P_1 and P_2.

Other possible parameter variables will provide the same type of dynamics as those described above.

7.9 CONCLUSION:

In this paper, we study the stability of two dynamics competitive plant population under the influence of allelopathy. The ordinary differential equation is used to investigate the dynamics of the system. Delay plays a significant role to understand the complex behaviour of the dynamics system. With the help of comparison theorem, we calculated the positivity and boundedness of the system. Using Routh-Hurwitz criteria, the stability carried about non-zero equilibrium point E^* and observe that when there is no delay, equilibrium point $E^*$is absolutely stable. But when delay is less than the critical value, system losses its stability and shows asymptotically stability. The equilibrium point E^* losses it's asymptotically stability when the value of delay is greater than the critical value and Hopf-bifuraction take place. Also, we calculate the global stability of the dynamics system with the help of Benedixon-Dulac criterion, and observe that there is no periodic path near equilibrium point. At lower concentration of plant population's different biotic structure of allelochemicals interaction is responsible for increased convergence between simulation and experimental values (158). In addition the specific procedure that defines the direction analysis is obtained. Finally, a simple model is presented and has the potential to utilize the dynamical behaviors of two plant population in ecosystem, allelopathic and competitive.

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Future Work

Mathematical Modelling of Turf Grass species and their application in Cancer treatment

Objective

- To identifies the better plant quality using mathematical modelling for medicinal applications.
- To identifies the different compounds in the plants and their comparison for cancer treatment.