# STUDY OF STABILITY AND BIFURCATION ANALYSIS OF PREY- PREDATOR DYNAMICS USING DELAY DIFFERENTIAL EQUATION

Thesis Submitted for the Award of the Degree of

## DOCTOR OF PHILOSOPHY

**in**
**(Mathematics)**

**By**

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**Supervised By:**

**Dr. Pankaj Kumar**

**LOVELY PROFESSIONAL UNIVERSITY**
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## ***DECLARATION BY THE CANDIDATE***

I, hereby declared that the presented work in the thesis entitled "**Study of Stability and Bifurcation Analysis of Prey-Predator Dynamics using Delay Differential Equation**" in fulfilment of degree of **Doctor of Philosophy** in **Mathematics** is outcome of research work carried out by me under the supervision **Dr. Pankaj Kumar**, working as Associate Professor,in the Department of Mathematics at School of Chemical Engineering and Physical Sciences of Lovely Professional University, Phagwara, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of another investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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# *CERTIFICATE*

This is to certify that the work reported in the Ph. D. thesis entitled "**Study of Stability and Bifurcation Analysis of Prey-Predator Dynamics using Delay Differential Equation**" submitted in fulfillment of the requirement for the reward of degree of **Doctor of Philosophy** in the Department of Mathematics at School of Chemical Engineering and Physical Sciences, Lovely Professional University, Punjab, India, is a research work carried out by Shiv Raj, 41900721, is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

Pankaj Kumar

**(Signature of Supervisor)**

Name of supervisor: **Dr. Pankaj Kumar**

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**Dedicated**

**To God**

**And**

**My Family**

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# ABSTARCT

In the proposed work, prey-predator dynamics are studied using the delay involving predation of mature prey, the effect of toxin-producing prey on predator populations, multi-team prey-predator scenario, interaction of commensal species with delay and a single prey and a pair of predators using a delay differential equation and a functional response with a square root.

A method that has worked well for integrating and synthesising environmental and ecological data is mathematical modelling. Natural objects are not directly addressed in mathematical ecology. It works with mathematical constructs and procedures that serve as analogues to natural phenomena and processes. The mathematical models do not include all of the knowledge we may have about nature; rather, they just include the knowledge that is most relevant to the task at hand. By understanding the logic behind our theories about nature, mathematical modelling enables us to avoid making convincing arguments that may not be true or are true only in particular circumstances. The suggested models have undergone mathematical analysis, and the outcomes have undergone numerical verification.

Mathematically, the positivity and bounds of all analytical solutions are established using the comparison theorem. All possible and feasible interior and exterior equilibrium conditions are calculated. Calculation of stability carried out about interior equilibrium values and Hopf bifurcation takes place when delay crosses specific value. The nature of the roots has been studied in detail using Roache’s theorem. Sensitivity analysis and directional analysis of state variables with respect to model parameters are done for almost all the models using the "Direct Method" and "Centre Manifold Theory," respectively. Numerical simulation is done using MATLAB software using the dde23 command, where all the model parameters have been assigned different numerical values. This made it easier to identify the delay parameter's critical value, over which the system lost stability and

experienced a Hopf bifurcation. Below this specific value, the structure displayed stability.

# List of Publications

1. Kumar, P., & Raj, S. (2021). Modelling and analysis of prey-predator model involving predation of mature prey using delay differential equations. *Numerical Algebra, Control and Optimization*. 2022, Volume12, Issue 4: 783-791. Doi: 10.3934/naco.2021035

2. Kumar, P., & Raj, S. (2022, May). Modelling the Effect of Toxin Producing Prey on Predator Population using Delay Differential Equations. In *Journal of Physics: Conference Series* (Vol. 2267, No. 1, p. 012077). IOP Publishing. DOI 10.1088/1742-6596/2267/1/012077

3. Raj, S.., & Kumar, P. (2023). MODELLING THE MULTITEAM PREY–PREDATOR DYNAMICS USING THE DELAY DIFFERENTIAL EQUATION. *Malaysian Journal of Science*, *42*(1), 42–61. https://doi.org/10.22452/mjs.vol42no1.5

4. Modelling the Prey-Predator Dynamics Involving Commensal Species Under the Effect of Time Lag. https://www.neuroquantology.com/article.php?id=8391

5. Study of the stability of a single prey and a pair of predators using a delay differential equation and a functional response with a square root. (Paper communicated)

## LIST OF FIGURES

Throughout the thesis where in graphs $solutiony$ written stands for population densities or population dynamics. In graphs y axis represents population densities and x axis represent time.

# LIST OF TABLES

# TABLE OF CONTENTS

# Chapter 1

# General Introduction

## 1.1 Introduction

Word Ecology is combination of two words Eco + Logos in Greek it means Household +Learning about. Thus, ecology mean to understand the life of species. The word “ECOLOGY” firstly used by German Zoologist, Ernst Haeckel, in 1866 to explain the “economics” of living organism. Eugene Odum is the inventor of modern ecology. In his honour University of Georgia established Odum, Eugene P. The School of Ecology, the only independent college of ecology in the world, celebrated its 13th anniversary on January 9, 2021.

One of the fundamental concepts of ecology is the analysis of the interactions between predators and prey. One of the notable issues with concept of predator-prey relations is “paradox of enrichment” first used by Michael Rosen Zweig in 1971. He explained sequel in six- predator-prey models were increasing the food available to prey caused the predators population to destabilize.

According to Arditi and Ginzburg (1989), field findings consistently contradict these assumptions. In 1992, Ginzburg and Akcakaya and Luck, Arditi and Berryman investigated in biological control paradox (1920), which states that it is impossible to simultaneously achieve a very low and constant prey equilibrium density using a classical model. Although there are countless incidents of the introduction of predators and parasites in Classical biological management that led to the maintenance of exotic pests at sparser and more stable populations, as noted by different authors namely Turnbull and Chant in 1961, DeBach in 1974, Hagen and Franz in 1973.

Ratio-dependent predator-prey theory has recently attracted attention by Arditi and Ginzburg 1989; Berryman 1990. Although not new, this model is not well-known in ecology theory. Despite the fact that they resolved issues with conventional models as well as enhancement and biocontrol agents' paradoxes.

Thomas Malthus in 1798 gave information on the Principle of Population provided the first pure mathematical calculations on population. According to Malthus, when population grows logarithmically, the things that affect it either remain the same or rise arithmetically. Malthus' Concept of Population was transformed into a mathematical equation by Verhulst forty years later, in 1838. The Logistic equation is

$$\frac{dM}{dt} = nM\left(1 - \frac{M}{K}\right)$$

M = density of population

n = rate of change

K = equilibrium density

This equation is constantly criticized for being too straightforward, yet it is always useful in the creation of single-species population structure, especially when it is extended to discontinuous increases in population that are time-lag and volume dependent. This theory describes the behaviour of numerous single organisms in both lab and real-world settings studied by many researchers in different decades namely Gause in 1934, Allee et al. in 1949, Thomas et al. in 1980, Berryman and Millstein in 1990.

In Lotka theory of Physical Biology firstly we observed Population structure theory. Lotka presented the first framework of predator-prey relationship in addition to handling the logistic equation, which he referred to as the Rule of Population Increase from First Principle. Not only did he create the prey-predator model that included two species instead of just one, postulated by Volterra in 1928, as interpreted by Chapmans in 1931. He contends that population growth is inversely correlated with biomass production. Some few months later, in 1935, Nicholson and Bailey develop a discrete-time model to describe how insect parasitic organisms and their hosts engage. The Lotka-Volterra model and this one includes concepts that are similar Royama 1971. The stabilization of the Nicholson-Bailey formulas involved numerous complex computations.

It is fact that Volterra 1928 and Gause 1934 both used the logistic equation for their two species competition model, but not succeed to make it perfect for modelling predator-prey interactions. Lwslie (1948) was first to drive Logistic predator equation. The incorporation of a predator model equation was the next significant advancement in the idea of predator-prey relationships. The prey mortality rate must be a nonlinear system of prey population, according to Solomon 1949 and Holling 1959–66, who noted that a predator can just deal a fixed supply of preys in a short period of time.

In 1977, Holling derived the Michaelis-Menten equation of enzyme kinetics, from which his disc equation was constructed. The relationship between individual organisms, as noted by Berryman and Getz in 1981 and 1984, is the source of population structure; hence, the expression must be calculated as the per-capita rate of variation. The half-saturation point, a physiological response that specifies the amount of alternative food accessible to predators, has an ecological value based on population. The product, not the predator/prey ratio, as shown by Leslie's equation.

An effective tool for integrating and synthesising environmental and ecological data is mathematical modelling. Natural objects are not directly addressed in mathematical ecology. It focuses with mathematical constructs and procedures that serve as analogues to natural phenomena and processes. The mathematical structure does not include all of the knowledge we may have about nature; rather, they just include the knowledge that is most relevant to the task at hand.

Now a new equation derived by Michaelis-Menten-Holling so that behaviour of predator from one species to other can be observed. Whenever functional response taken in prey equation, we get parabolic prey isoclines which is behaviour of ratio-dependent prey equation. By understanding the logic behind our theories about nature, mathematical modelling enables us to avoid unnecessary convincing arguments that might not be real or are only valid in particular circumstances. The number of people inside a population fluctuates, occasionally dramatically, over the life of an organism.

Numerous creatures, such as algae, insects, fish, frogs, and others underwent similar abrupt transformations, as seen by ecologists. Ecologists were intrigued to comprehend such significant shift, and early studies came to the conclusion that resource availability was always crucial. Researchers found that populations always exhibit a drop trend for availability to the restricted resources when sources for example food, nesting locations, or refuges are few. Such bottom-up management in environments refers to habitats where the availability of nutrients, productivity, and primary producers' characteristics keep the environment in good condition.

In reality, these two methods of population reduction cooperate to track down population fluctuations over period. Another extremely significant point is that diseases and insects also can affect population structure. Species with massive, cycle-like population fluctuations provide some astounding instances of population variations. These cycles frequently co-occur in the same region with other species' population cycles. Red foxes, for instance, feed on hares, grouse, and voles in northern Sweden. Studies on these species show that each of the prey animals has connected population cycles, with population maxima every three to four years.

Vegetation is the main food source for grouse, voles, and hares, and their populations are reliant on it. With more food, population size is influenced by bottom-up control. When there are many voles and there is intense competition for food, the voles start fighting over little amounts of food, which changes their hunting behaviour and causes the population to drop. So, in communities of these animals, food availability is crucial. When there aren't any voles around, foxes prefer to eat other small creatures like grouse and hares. Foxes should grow when there is plenty of food available. Since this system is unmanageable for experimentation, we are unable to anticipate how far any of these influences the population loop will go. However, theories suggest that food and hunting cooperate to restrict population levels. Prey must have access to enough of food in order to survive and reproduce while also avoiding becoming prey for predators.

The snowshoe hare's learning demonstrates the impact of food availability and predator escape on population levels. Ecologists have used mathematical models to study the nature of prey-predator dynamics since choosing between eating food and predator escape is difficult in the field, just like businessmen and meteorological researchers do. This new relation gives primitive rectilinear Lotka-Volterra predator isocline and a new paradox enrichment and biological control. This isocline structure help in development of modern prey-predator theory & provide a mathematical way to population ecologists understand the facts that affect population dynamics.

All of these time lag models have benefited greatly from the inclusion of delay differential equations. Delay differential equations have not yet been used to the study of prey-predator systems. When delay is applied to any concept that received negative feedback, it results in the system & oscillatory behaviour. There are concepts that provide negative feedback in mathematical models dealing with dynamics. It also offers a very good opportunity to delay this term in order to look at the dynamic behaviour shown by the overall growth program. It is important to apply this principle of delay differential equations in time and circumstance. If we take the initial Lotka Volterra equation, the prey population increased at a very greater incidence to add a limit to the prey equation to allow this meaningful. This change renders the configuration of the corresponding point and makes the device stable.

For two species interaction models, both Volterra and Gause use logistic equation but do not end up making this appropriate for interaction between prey-predator. Even if the population formula where it is introduced always operates on slow population trends, including functional response in the prey-predator model is crucial since it describes the behaviour of seeking predators on a quick time scale.

Ecology is defined as the study of interactions between different organisms and their surroundings. It concerns the underlying principles that govern all animals as well plants. Ecology is therefore described as a tool to understand the coexistence of organisms. A.G. Tansely was the first to invent the word ecosystem. Ecosystem is made up of two terms:

$$Ecosystem = Eco\ (Environment) + System\ (\ Interaction)$$

The ecology is the basic fundamental unit of species and their surroundings interacting with one other also with their individual components, claims American ecologist E.P. Odum. Food webs include, but are not limited to, detrital food webs, soil food webs, marine food webs, ocean food webs, arctic food webs, grassland food webs, and microbial food webs. Abiotic, or non-living, environment and a live biological community come together to form an ecosystem. Green plants classified as producers (autotrophs), consumers (heterotrophs), and decomposers (saprotrophs) make up the living community. For a species to survive, it must receive energy. For instance, plants need the sun's energy to grow; some animals consume plants; and other species consume other species.

A food chain is the order in which members of a biological community are fed by other members. A food chain highlights the dependence of various living creatures in a given ecosystem in terms of their feeding relationships and the transfer of energy from the sun to producers, consumers, and decomposers. Even a food web forms when several food chains in an ecosystem intersect each other.

$$Producers \rightarrow Primary\ Consumers \rightarrow Secondary\ Consumers \rightarrow Ternary\ Consumers$$

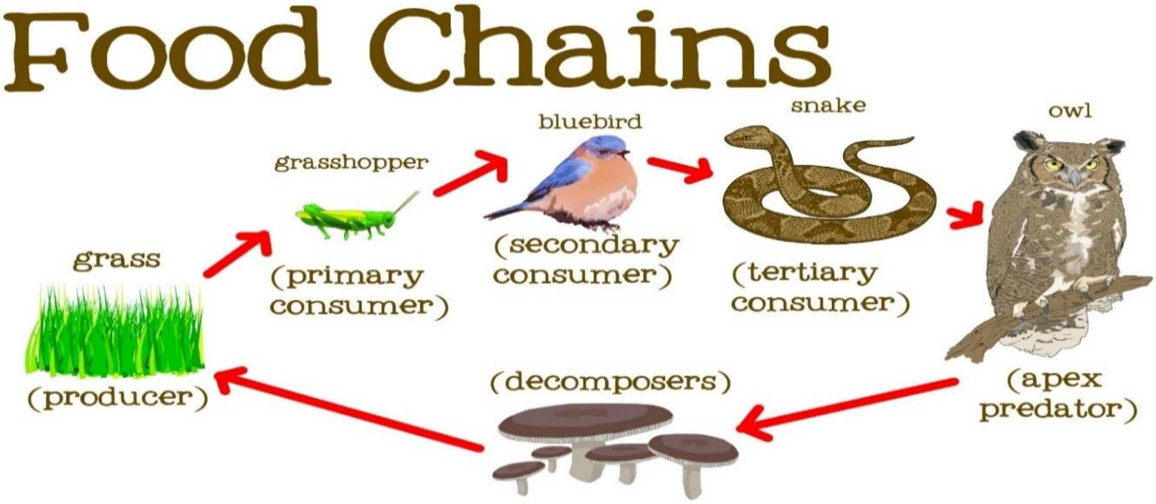

**Figure 1.1** Grazing and Detritus food chain

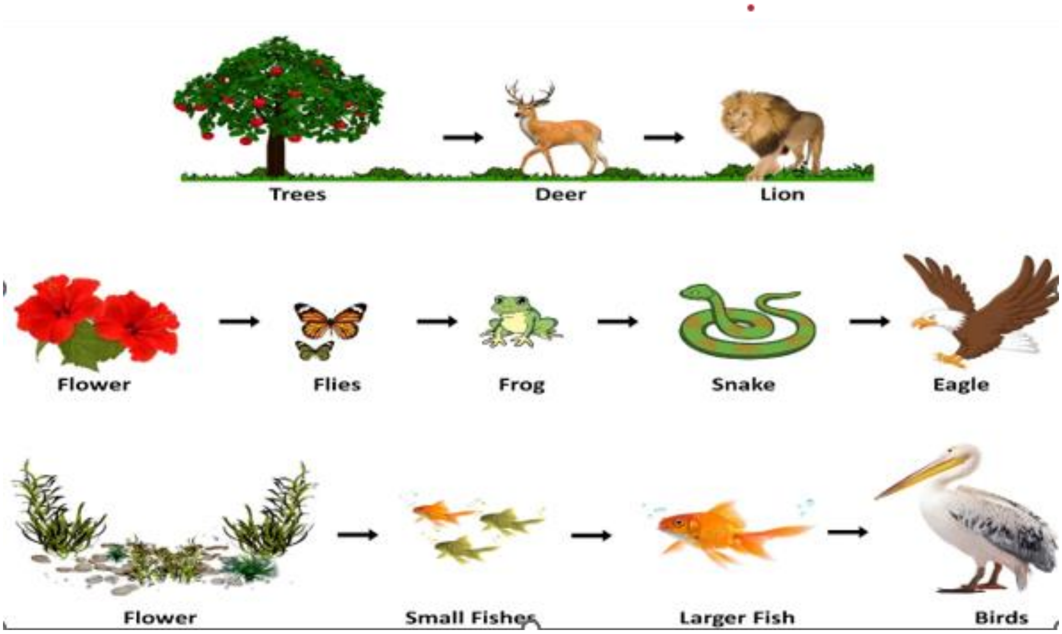

**Figure 1.2** Grazing food chains

An ecological community's species feeding relationships are presented graphically in a food web. Even the energy connection and transmission between them are specified. In these food webs, several food chains are interconnected. The ecology is more stable the more chains it has.

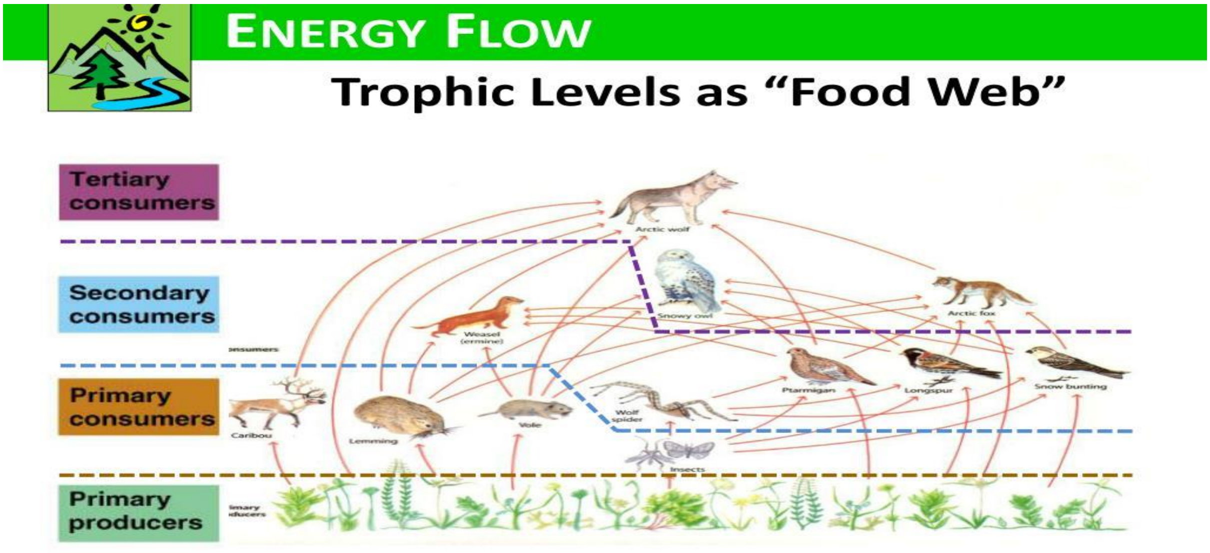

**Figure 1.3** Food Web Diagram

The trophic level of an organism refers to its place within a population or a food chain. The producers (plants, algae, and bacteria) occupy the lowest tropic level of an ecosystem. They absorb sunlight and transform it into food energy. It generates carbohydrates that are high in energy using solar energy. All living things ultimately derive their vitality from the sun.

An animal that has been physically and physiologically modified to consume plant matter is an herbivore. Herbivores often have mouthparts suited to rasping or grinding due to their diet of plants. For e.g., Elephant, Deer, Grasshopper, etc. A species that gets all of its nutrition and energy needs from a diet that predominantly or exclusively consists of animal flesh, whether through hunting or rummaging, is known as a carnivore, which is characterized as a meat eater. Tiger, Jaguar, Egret, Falcon, etc. are a few examples. An omnivore is a type of animal that may receive its calories and nutrients from a meal that includes a variety of food sources, including plants, animals, algae, fungi, and bacteria. such as Beer, Turtle, etc. Bacteria and fungus are decomposer species that feed on and decompose decaying protoplasm to restore organic material components to ecological cycles.

Herbivore Producers Carnivore Omnivore Decomposer

**Figure 1.4** Examples of organisms at different tropic levels

The term "food web" refers to the way in which different food chains interact. In nature, some organisms operate in the ecosystem not just at a single trophic level but more than one trophic level. It means that it may derive its food from more than one source. The organism may get eaten itself by another organism always present top of trophic level. It may also survive on different organisms of lower trophic levels. In this way, various food chains are linked together in an ecosystem and the intersection of these food chains forms a complex network called the food web. Hundreds of species are often connected by their eating patterns in real food webs. Essential parts of ecosystem ecology are food webs.

Figure 1.1 shows a food web in a unique habitat with many interconnected food chains. The stability of an ecosystem in nature is greatly influenced by the food webs. For instance, a decline in the number of rabbits would inevitably lead to a decline in the population of carnivorous consumers who love to eat rabbit. Consequently, it would be dependent on two key factors:

a) Length of food- chain: - The length of a food chain would depend on the diversity of the species and how they ate. The length of the food chain would increase with the variety of species' eating patterns.

b) Choices at different stages for consumers: - The more choices there are, the more interlocking the pattern will be. Food webs are substantially more complicated in deep oceans, seas, and other environments with a diverse range of species.

In the actual world, a species' population growth rate frequently takes time to adjust to changes in either its own population or the population of an interacting species. A few components that introduce time delay into the model include the age structure of the population which affects birth and death rates, puberty period, gestation period which varies from 20 days for mice to 645 days for elephants, mealtime, response time, storage of food, wealth restoration durations, and malnutrition coefficient in carnivores' interactions.

According to Haines and Crouch “Mathematical modelling is a process in which real-life situations and relations are expressed by using mathematics. One of the pillars of mathematics education is mathematical modelling. Through the technique of mathematical modelling, a real-world issue is translated into a mathematical structure. Modelling entails creating circumstances from real life or changing difficulties from mathematical explanations into situations that are acceptable or realistic. As Sir Arthur Eddington once said, "Do not believe an experiment unless it is confirmed by theory”. The problem may or may not be fully resolved as a consequence of the modelling process, but it will be made clearer.

Mathematics has played a significant part in the development of a framework that makes it easier to understand the enormously complicated ecological processes in nature. Mathematical ecology has a long and illustrious history. Ecologists are frequently interested in how people, communities, and ecosystems change through time and place; luckily, a field of mathematics known as dynamical systems theory was created expressly to address dynamics. Today, math is still important and may perhaps be becoming more so in the advancement of ecology.

From above discussion in this thesis, following problems have been studied using mathematical models:

1. Modelling and analysis of prey- predator model involving predation of mature prey using delay differential equation.

2. Modelling the effect of toxin producing prey on predator population using delay differential equations.
3. Modelling the multi-team prey-predator dynamics using the delay differential equation.
4. Modelling the prey- predator dynamics involving commensal species under the effect of time lag.
5. Study of the stability of a single prey and a pair of predators using a delay differential equation and a functional response with a square root.

## 1.2 Literature Review

Over the years, numerous mathematical models have been used in the comprehensive study of mathematical biology and population dynamics. The Lotka-Volterra model, also called a Prey-Predator model, and [1] made the most significant contributions to theoretical population biology. It is a coupled non-linear ODE system that is used to describe the interaction of two species, one of which is referred to as the prey species and the other as the predator species. The integro-differential equation model was first suggested by [2], who quickly understood that time delay must be considered for accurate modelling. But he was unable to determine whether an equilibrium point was stable. When solving the updated model using [2], [3] provided the precise approach, employing the delay function's most basic form. Limit cycle prey-predator interaction models considered the limited prey population and the restricted hunger of the predators [4-5]. Mass action predation, in contrast to Holling type-II, has a non-zero handling time, as demonstrated by [6]. It has been demonstrated that transmissible diseases can afflict the predator under a variety of physical situations and circumstances [7].

If we suppose that the functional predation responses are of Holling type II, the Hopf-bifurcation is necessarily supercritical [8]. It has also been demonstrated that many diseases, such as rabies and hepatitis, require an incubation period before becoming infection vectors [9-10]. It has been demonstrated that the susceptible predator consumes the prey according to mass action law, whereas the infected predator consumes according to Holling type-II response [11].

The ratio dependent model presented in [12] evolved as an alternative to the models presented in [1] and [2]. Many animals have two stages in their lives: immature and mature. Each level has a unique set of behavioral characteristics. Many authors [13-15] have also provided stage-structured prey-predator models. A mathematical model in which the predator category consumes the immature prey extensively was proposed [16]. Positive equilibrium stability qualities for continual input of refuge preys in Lotka-Volterra prey-predator systems have been explored in detail [17].

In a predator-prey scenario with discrete delays, a full investigation was presented for a number of stabilities such as absolute, conditional, and bifurcation [18]. The zeros of the exponential characteristic equation were thoroughly examined [19]. [20] investigated the stability of a non-linear DDE about non-zero equilibrium locations. A plankton model is an evolutionary model, and many have been produced and studied in the past. Several articles have attempted to evaluate the significance of various hydrological parameters in the generation of plankton blooms, as well as appropriate functional responses to explain the drop-in predator population due to toxin-producing prey [21].

These data indicate that toxic compounds, as well as poisonous prey, play an important role in predator population development and have an impact on prey-predator confrontations. Peter A. Abrams talked on theories about the dynamics and equilibrium of species and traits, as well as how predatory and anti-predator qualities can adapt to environmental changes. Stability appears to be the least likely where there is coevolution and a bi-directional axis of prey vulnerability. An increase in prey protection ability is less likely than an increase in predator assault capacity to result in evolutionary responses in its companion [22]. Alan A. Berryman discovered that prey-predator ratios, rather than prey numbers alone, determine functional responses. The most serious criticism leveled at logistic prey-predator models is that they do not precisely follow environmental rules. After all, predator reproduction does not necessitate prey death [23].

Buike Arthur L. Ma Jr. and Ernest F. B Enfield discovered a plethora of toxicities in predator—prey interactions. Any toxicity data is thought to make an individual more vulnerable to predation. Prediction could be better if species behaved oddly. This is a critical concern for the vast majority of invertebrate organisms. Sherberger et al. investigated the effect of heat shock on floating insects. They were investigating if thermally stressed nymphs are more vulnerable to banded sculpture predation [24]. M. J. Naxadua and B. A. Kroft concluded that the fungicide benomyl's direct toxicity and prey poisoning caused a predatory mite's lifelong immature depression[25]. Ja Ksh et al. discover two types of adaptations to prey by carnivorous vertebrates and illustrate their link in an economic model [26].

According to the Toxin Ology Notes, poisonous animals can be researched as a substance, and toxicity must be investigated from any viewpoint that is necessary for life, including parasites and population control.B. A. Perret and T. A. Freyvo Gel Fascinating links between toxinology and ecology may be formed, which will be critical for the survival of biodiversity. As a result, toxinology can only be assumed as a limit in its early stages[27]. Lincoln Pierson Brower and Linda Susan Fink investigated whether wild birds could be trained to avoid naturally harmful insects by sight or taste. However, authors have proven that unconditioned taste rejection of hazardous substances by wild birds exists. In actuality, such unconditional responses to aposematic visual and taste markers of many insects appear to be as necessary or crucial as conditioned reactions[28].

Single species tests, according to John Cairns, Jr., can reliably predict reactions at later stages. Various levels of environmental realism can be reached in the laboratory or under laboratory conditions at various phases of biological organization, according to the author. The authors concluded that the toxicity test balancing at various phases of biological organization is required to produce a credible risk evaluation [29]. Maria R. Servedio discovered the classical conditioning model for predator remembering and forgetting tests, as well as the intensive one-trial and no forgetting over time, which can occur with very toxic prey. The author discovered that the extremely poisonous prey conditions accelerate the development of coloration by stabilizing very brilliant mutations, with each succeeding mutation creating a somewhat higher conspicuousness of the prey[30].

Prey species with comparable shape and behavior that are initially killed and consumed may differ greatly in their usefulness as food for generalist arthropod predators, according to Soren Woft and Darid H. Wise [31]. Jefferson T. Turner and Patricio A. Tester discovered that interactions between hazardous phytoplanktones and their zooplancton graders are intricate. Some zooplankters ingest toxic phytoplankton without apparent harm, whereas others are harmful. Their mechanisms of action, toxicity, and solubility differ from phycotoxins and impact grazers in different ways [32].

Akira Mori and Gordon M. Burghardt proposed that the anti-predator nature of animals was shaped by a variety of environmental factors. When it comes to ectotherms, one of the most important variables is the ambient temperature, as most physiological functions rely on the body's temperature. The authors investigated the anti-predator behavior of a Japanese grass serpent's erect temperature [33]. According to research, R. tigvinuz shifts from diverse passive responses to active flight responses when the temperature rises in order to combat predators.Thomas N. Sherratt came to the conclusion that the average likelihood of attacking defenseless models and their defenseless imitations should decrease in a sigmoidal manner with an increase in the availability of defenseless alternative prey and that the choice to drill predators generally should be fairly indifferent to the probability of the possible defense of prey items[34].

Michael P. Speed observed that the elusive presence of an appetizing prey might inhibit predator learning and accelerate predator forgetting, to the prey's benefit. In terms of learning and memory, cryptic may be an anti-signal[35]. Ruxton et al.

observed that the toxicity and coloration of prey species are frequently varied. The authors also discovered a substantial relationship between signal intensity and the average toxicity of each signal level. When equilibrium is reached, predators change their behavior and appear to ignore the signal[36].

Christina G. Halpin and Candy Rowe concur that the rejection risk is influenced not only by an individual's investment in safeguards, but also by the investment of others in the same group. As a result, a predator's taste rejection may destabilize defense expenditure and stimulate prey defensive heterogeneity[37]. The trained predators, John Skelhorn and Candy Rowe, observed that there are important selective factors in aposematism and that they continue to make strategic decisions to consume guarded prey based on their acquired knowledge of the prey's poisonous and nutritional quality[38].

Stephanie O'Donnell et al. discovered that invasive species cause many extinctions while also being difficult to eradicate. Because indigenous prey is critical to invasive predator problems, CTA could be employed to alter invasive predator feeding behavior. The authors concluded that CTA may be a viable strategy for reducing the ecological consequences of invasive species [39]. According to David W.M. Nelson et al., generalist predators use natural and acquired behaviors to reject unfavorable foods and effectively choose ideal preserved forging, in order to choose the presence of prey variable in diseases, diet, defenses, and availability. Expert predators do not exhibit such behavioral variety, maybe because their food is of predictable consistency [40].

istein Haugsten Holen discovered that if the protective toxin is sufficiently unpleasant, a mimicric complex may be less helpful to the predatory than it should be if the models are slightly toxic and better resistant against predation. Taste imitation may also reduce the profitability of the imitation complex while increasing predation security [41]. Richard Shine discovered that most dunnarts ate only a portion of a toad before discarding it, and there was no sign of toad poisoning in a single animal. After only one or two encounters, both dunnarts learnt to avoid toads as prey. Predators denied toads over the duration of the study (22 days), indicating that toads are hazardous in terms of long-term retention [42].

According to Christina G. Halpin, European starlings (Sturnus vulgaris) increase their consumption by artificially increasing their nutritional level and lower their intake by discontinuing protein enrichment. This demonstrates that birds may detect and utilise this information in their food foraging decisions to detect the nutrient content of deadly prey, therefore improving new opportunities for evolving prey defenses. This is obviously visible. Nutritional disparities between people could result in similarly hazardous prey, which could explain why some species undergo ontogenetic protection measures [43].

According to D. Peacock et al., innovative strategies are crucial for the survival of endangered species against feral cats. According to new research, if a cat has effectively killed a dangerous species, a single feral cat would actively destroy endangered beast populations. Toxic collars and toxic implants fitted or implanted during surveillance or reintroduction programming may empower the offending animals if a target animal is killed multiple times [44].

Flubendiamide, according to R.S.Ramos et al., tends to linger on the surface longer. The authors believe that the insecticides chlorfenapyr, chlorantraniliprole, flubendiamide, spinosad, and indoxacarb are the most interesting to use with C. Furthermore, these chemicals protect B communities. Pallescent and more environmentally friendly pest management strategies [45].

Coral gobies were mentioned as a possible prey by Barbara Gratzer et al., however E. Fasciatus' non-toxic monitoring fish clearly preferred the Gobiodon. When faced with a goby, the predator did not choose one animal over the other. Poisonous gobies were frequently caught, but they were evacuated quickly, repeatedly, and alive, contrary to our expectations. This unusual avoidance after capture indicates the substantial risk of these gobies living on the ground due to their skin toxin [46].

M. Banerjee and E. Venturino have explored a prey-toxic prey-predator interaction device without accounting for the time lag of the gestation period. However, the time lag associated with the gestation period was not considered by the authors of this paper. The predator's replication after predating the prey is instantaneous in the technique proposed by M. Banerjee and E. Venturino, and the practical form of the predator exhibiting avoidance behavior in the presence of enormous amounts of toxic prey is considered. However, in the natural world, it is more practical to tolerate the temporal lag caused by the gestation period. M. Banerjee and E. Venturino's analysis is broadened by incorporating time delay into the predator's.

The types of behavioral features demonstrated by animals to recognize predators cannot be determined by examining the relationship between prey population and strike rate [51]. One prey is dangerous to the predator, whereas the other is not. The hunting processes of both prey groups are tracked individually by Monod-Haldane and Holling type II functional outputs [52].

Using an ordinary differential equation, the authors assessed the predator-prey relationship in three dimensions in three species. The computation was based on the subjective population numbers of two prey animals and one predator creature that share their habitats [53]. In research, a mathematical model of hunts for two competing prey species was developed and tested. The rate of growth and functional responses could be general nonlinear functions. The findings point to the presence of an intraspecific interference component, which is a critical characteristic determining the system's dynamics [54].

Three prey-predator populations could be asymptotically stabilized over time by using nonlinear feedback control inputs. The functional parameter limit, beneath which variables converge to limit cycles, is established [55]. Many animals like to live in groups called herds. Different groups may cooperate, compete, or create a predator-prey relationship since they share the same habitat. A prior study [56] reported a new paradigm for predator-prey teams.

We looked at two prey populations and two predator populations in this study. When two prey species exist in different environments and can defend themselves in groups, only one of the two predators can move between them. A predator influences both group trends and pesticide effectiveness [57]. A study investigated the long-term unexpected behavior of the at-risk group using a stochastic logistic differential equation that calculates ecosystem function [59]. To maintain a stable food web, the predator alternates between prey with varying relative densities [60]. A study discovered the conditions under which species become extinct in the system and identified the essential characteristics for all species to persist indefinitely [61]. By completing an initial examination of the model's normalised version, researchers demonstrated the existence of dynamics in genuine predator-prey systems that are well-represented by fundamental conditions [62]. Few studies have employed mathematical modeling to investigate how prey maturation affects a predator-prey model. To analyze predatory behavior on mature prey, delay differential equations can be used [63].

A mathematical model that accounts for time lag is required to analyze the dynamic behavior of these types of biological systems [64]. In the absence of climate fluctuation, the criteria for local asymptotic stability were met. The probabilistic technique was created by the authors by incorporating Gaussian white noise notions into all regular equations [65]. Under some conditions, the system has a stationary distribution that is ergodic. Under certain conditions, the system's solution stays globally asymptotically stable [66]. A three-species prey-predator system was used in one study, with the predator being a layman by nature because it feeds on two prey creatures [67].

In many environments, prey cooperation benefits both prey populations. If one prey is dangerous to a predator and the other is weak, the predator may pursue the weak prey [68]. Another study found that nonconstant solutions require significant diffusions, interspecific contests, slower prey intrinsic growth rates, and higher predator intrinsic reduction rates [69].

The interplay of multiple prey and a single predator is frequently unstable. It is extremely unlikely that all three species will dwell in the same location. In practice, the predator always triumphs [70]. When two prey species compete directly, a predator model with temporal delays and a weak Allee effect in the prey's development function is used. Despite its simplicity, the system exhibits a wide spectrum of dynamic behavior, such as biostability at the equilibrium point [71]. To study delay models in population dynamics, we used the model proposed by [72]. Using stochastic Lyapunov functionals, one study proposed numerous conditions for extinction and survival in the mean of the three species [73]. A high level of fear of a prey animal and a higher quality of life for second prey may boost that species' chances of survival [74]. If the impulsive duration exceeds a certain threshold, the structure is usually stable [75]. A dynamic system modelled by multiple teams consists of two preys and one predator. Individuals from both prey groups would support one another during an attack, and the rate of predation for each group would differ [76].

Researchers can see the impact of prey collaboration by determining the types of interactions in a multiteam system. Such a study's findings would be comparable to those reported by Poole on Leslie-Gower computations [77]. The amount of easily digested food influences a predator's behavior toward a specific prey species. A biosynthetic suppression strategy regulates these genes, and therefore the predator population's lifestyle [78]. Survival and density variations in prey species increase protection evolution based on parameters. Incorporating a predator's optimal meal selection into the model improved cohabitation and reduced overall density variations [79].

A fourth-order nonlinear differential equation can be utilized to represent the system when there is just one predator [80]. The authors employed the press perturbation approach to assess the intensity of overall impacts of connections between phytoplankton and bacteria in the stable state under these coexistence conditions. This implies that, despite the fact that plankton and bacteria compete for a common resource, inorganic phosphorus, a mutualistic scenario may emerge as a result of the mechanism by which element carbon moves from plankton to bacteria [81]. Because of its generality and significance, the predator-prey relationship has drawn the attention of a growing number of applied mathematicians and ecologists. While developing a variety of sophisticated models for two or more interacting animal systems [82], the effects of population, age distribution, time delay, functional response, switching, and other aspects were all considered.

According to the author, Hydra plays an important function in prey population management. According to the author, hydra population density climbed from June to a high in late July or early August, then stayed stable or slightly fell until late August [85]. The essential idea that the parameters are cyclic functions of time modifies the total set of differential equations that reflect predator-prey dynamics. The author also determined that the system has a periodic solution [86].

Commensal and mutualistic interactions are common among terrestrial vertebrate species and have significant but mostly unmeasured effects on individual fitness. Because all terrestrial vertebrate commensal and mutualistic relationships occur spontaneously [87]. Dingoes had an impact on the past human economy by competing for large prey because of their severe impact on kangaroo population reduction in current ecology and the human economy. The authors acknowledge that identifying all independent situations and individuals who interact to generate an example of this type in our hypothesis archaeological enquiry is difficult [88].

The author arrived to extremely generic nonlinear mutualism models and some simple tests for determining if a nonlinear mutualism model is globally stable or stable in a finite region. The author also discovered that mutualistic systems are more mathematically tractable than competition and prey-predator dynamics [91]. The researcher came to the conclusion that how to analyze a scenario when its key findings appear to be empirically incorrect. Because the theory is analytically valid and, in some ways, tautological, the author attempted to figure out what went wrong to produce its empirical falsification [93]. In multispecies ecosystems, the author identified a relationship between complexity and sustainability [97].

The authors used a multi-framework to integrate two delays and the Holling II impulse response. The normal form approach and the Centre manifold theorem are also utilized to produce explicit equations for establishing the trajectory of the Hopf bifurcation and the existence of bifurcating solutions [98]. According to the authors' findings [101], the strength of commensal mediation eventually decreases as a result of a steady pace of resource depletion.

The importance of the system's species interactions, such as qualitative stability, time dependence, predation, mutualism, commensalism, and so on, is an intriguing and important problem to investigate. In terms of various sorts of functional responses, several prey-predator models have been examined [102]. When analyzing the dynamics of the predator-prey relationship, the authors discovered two main paths. The authors also noted that, while this technique is clearly useful for gaining a thorough understanding of each particular predator-prey relationship, it is inefficient for drawing broad conclusions about two significant ecological issues [103].

Researchers used a chaotic ecological model to increase ecological intuition, and it was determined that simplified mathematical models have served and will continue to play an important role in understanding exactly what sorts of behaviors may occur in natural ecosystems [104]. When surrounding variations are taken into account, researchers discovered that providing more food is insufficient to influence the behavior of a predator-prey ecosystem [105].

## 1.3 Motivation and Background (Research Gap)

Many scientists, professors, researchers have made several contributions in modelling of prey-predator dynamics. From my Literature Review, I have found that majority of work on prey-predator dynamics involves ordinary differential equations. There is a huge scope for inclusion of delay differential equations in scenarios such as predation of mature prey, team up of the different species of prey to delay the predation rate, effect of toxin producing prey etc.

## 1.4 Scope of the Study (Hypothesis)

Majority of work on prey-predator dynamics involves ordinary differential equations. There is a huge scope for inclusion of delay differential equations in scenarios such as predation of mature prey, team up of the different species of prey to delay the predation rate, effect of toxin producing prey etc.

## 1.5 Research Objectives of the Study

In the view of the above in this thesis, following problems have been studied using mathematical model.

1. To model the prey-predator dynamics involving mature prey using the delay differential equation and study the stability and bifurcation analysis.
2. To model the effect of toxin producing prey on predator population using the delay differential equation and study the stability, direction and bifurcation analysis.
3. To model the multi-team prey-predator dynamics using the delay differential equation and study the stability, direction and bifurcation analysis.

## 1.6 Basic Terms used in the Thesis

**1.6.1 Predation: -** An ecological interaction called predation occurs when one species kills and consumes another. Based on the behaviour of a predator that catches and kills a prey before consuming it, predatory behaviour refers to the ecological process by which energy is transferred from living animal to live animal. Predation is a well-known species interaction that has been studied extensively over the years. Predation is conceptualised as a relationship in which the predator benefits and the prey suffers. Of fact, in cruel reality, prey is frequently not just "harmed," but also killed. Depending on the kind of predator.

**1.6.2 Gestation Period: -** The gestation period is the span of time during which a mammal foetus develops from conception to birth. The length of this time varies depending on the species. For instance, the gestation period of a cat typically lasts 58–65 days whereas that of an elephant lasts 645 days.

**1.6.3 Prey and Predator: -** The term "predator" refers to a creature that hunts and eats specific other organisms. The animals that predators consume are referred to as prey. Predators are often either carnivorous or omnivorous. Some examples of Predators are lions, tigers, sharks, and snakes etc. Depending on their position in the food chain, predators may potentially become prey to other creatures. For instance, a snake may be both an eagle's prey and a mouse's predator.

**1.6.4 Delay Differential Equation: -** A special type of differential equation where the values of the function's values from previous periods are used to define the derivative of the unknown function at a specific time.

**1.6.5 Commensal Species: -** Commensalism is a type of relationship between two living things where one of them gains from the other without hurting it. Commensalism can take the form of brief encounters between organisms or a permanent connection.

**1.6.6 Stability: -** Here, we'll keep a watch out for the delay parameter's partial value, below that my system is stable and above which complicated behaviour begins to manifest. As we talk about the real quantity in model. None of which can even be negative, some prove the criteria mathematically by proving the positivity of model.

**1.6.7 Positivity: -** As we talk about the real quantity in model. None of which can even be negative, some prove the criteria mathematically by proving the positivity of model.

**1.6.8 Boundedness: -** Again as we are taking something in real. The sum total and combination of which always lies between nothing to something finite. We prove this assumption mathematically by showing that system is bounded.

**1.6.9 Equilibrium Point: -** In the absence of time variation that is when there is no disturbance in the system. We put the derivative of the system equal to zero.

**1.6.10 Food Webs and Chains: -** The activity of feeding connects the lives of various animals in a wildlife habitat. Energy and food ingredients locked up in plant and animal tissues are passed through a chain as they develop and are consumed by others. Food chains interconnect there, and the resulting network is known as a food web. A community's animal and plant populations are naturally balanced. If the equilibrium is thrown off, the entire web might suffer.

## 1.7 Mathematical Preliminaries

### 1.7.1 Existence of Unique, Bounded and Positive Solution of Delay Differential Equation

A special type of differential equation where the values of the function's values from previous periods are used to define the derivative of the unknown function at a specific time. Here an initial history function, rather than an initial condition, needs to be defined. A delayed state variable can be used to demonstrate the past dependence of a differential equation. The derivative of the state variables not required in this case. The corresponding DDE with a single delay $\tau > 0$ is given by

$$\dot{x}(t) = f(x, x(t), x(t-\tau)) \qquad (1.1)$$

Assume that $f(x, y, t)$ and $f_x(x, y, t)$ are continuous on $R^3$. Let $s \in R$ and $\emptyset: [s-r, s] \to R$ be continuous. We seek a solution $x(t)$ of equation (1.1) satisfying, in function $f(x, y, t), y$ stands for $\frac{dx}{dt}$.

$$x(t) = \mathcal{F}(t), t \in [s-r, s], x(0) = x_0 \qquad (1.2)$$

And satisfying equation (1.1) on $t \in [s,\ s + \sigma]$ for some $\sigma > 0$.

**Theorem 1 (Existence of Unique Solution)** Let $f(x,y,t)$ and $f_x(x,y,t)$ are continuous on $R^3$. Let $s \in R$ and $\emptyset: [s-r,s] \to R$ be continuous. Then there exists $\sigma > s$ *and a unique solution of the IVP* $(1.1)-(1.2)$*on* $[s-r,\sigma]$.

**Theorem 2 (Boundedness of Solution).** Let $f$ satisfy the hypothesis of theorem 1.7.1 and let $x$: $[s-r, \sigma) \longrightarrow R$ be the noncontinuable solution of the IVP (1.1)- (1.2). If $\sigma < \infty$ then limit $_{t\to\sigma-}|x(t)| = \infty$

**Remark** Theorems 1 and 2 can be elaborated for the value when $x \in R^n$ and $f$: $R \times R^n \times R^n \longrightarrow R^n$, is the function then we can elaborate this to multiple times of discrete values of delay $r_0 < r_1 < \cdots < r_m$ where $f = f\,(t, \mathrm{w}(t), \mathrm{w}\,(t-r_0), \mathrm{w}(\,t-r_1), \ldots, \mathrm{w}\,(t-r_m))$.

**Theorem 3 (Positivity of Solution).** Suppose that $f$: $R\times\ R_+^n \times R_+^n \to R^n$satisfies the hypothesis of theorem 1.7.2 and remark 1.7.3 and for all $i$, $t$ and for all $x, y \in\ R_+^n, x_i = 0\ \Rightarrow\ f_i(x,y,t) \geq 0$ If the initial data $\emptyset$ in equation (1.2) satisfy $\emptyset \geq 0$, then the corresponding solution $x(t)$ of equation (1.1) satisfy $x(t) \geq 0$ for all $t \geq s$ where it is defined.

## 1.7.2 Stability by Variational Matrix Method

Let an autonomous system of equations be

$$\frac{dy}{dt} = \mathrm{f(y)} \tag{1.3}$$

Where y is an n-tuple vector i.e., $y = (y_1, y_2, ---y_n)$. Let$\phi(t)$be the solution of system (1.3). The linear part of the expansion of the system (1.3) about $\phi(t)$ is given by the variational equation of the system (1.3) with respect to $\phi(t)$, written as

$$\frac{dx}{dt} = f_y\ (\phi(\mathrm{t}))\mathrm{x} \tag{1.4}$$

Where $f_y\big(\phi(\mathrm{t})\big) = \frac{df_i}{(dy_j)_{n\times n}}$ at $\phi(t)$. Given that every solution to a non-linear system regulated by a variational system would also be stable, so stability of $x = 0$ of (1.4) determines the stability of $y = \phi(t)$ of (1.3). Particularly, when$\phi(t) = \phi_0$, a constant, the system (1.3) becomes

$$\frac{dx}{dt} = Ax$$

Where $A = f_y\ (\phi_0)$. Since a small perturbation of the system (1.3) is represented by system (1.4), so the stability of $y = \phi_0$ of (1.3) actually gives the stability of the solution of$x = 0$ of (1.4). The description of stability of every solution of $x' = Ax$ is given by following theorems.

**Theorem 4** Whenever all latent values of $A$ have no positive real parts, then every answer of the system $x' = Ax \ where\ A = a_{ij}$ is a constant matrix, is stable asymptotically.

**Theorem 5** If all the eigen values of $A$ with multiplicity greater than one has negative real parts and all its roots with multiplicity one has non-positive real parts, then all the solutions of the system $x' = Ax$ are bounded and hence stable.

Following theorem determine the sign of real parts of the latent values of characteristic polynomial.

## Hurwitz Theorem

$$P(\lambda) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + - - - + b_n$$

A required and sufficient criterion for all of the polynomial roots of the polynomial $P(\lambda)$ having real portions to be negative is that the value of minors for all principal diagonals of Hurwitz matrix must have positive value

$$H_n = \begin{bmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \dots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

**Theorem 6** Let $\varsigma_1, \varsigma_2, \varsigma_3, \dots \varsigma_m$ are all non-negative $\varsigma_i^j (j = 0,1,2, \dots m; i = 1,2, \dots, n)$ are constants. As $\varsigma_1, \varsigma_2, \varsigma_3, \dots \varsigma_m$ changes, the sum of the orders of the zeros of exponential polynomial

$$P(\chi, e^{-\chi\varsigma_1}, \dots, e^{-\chi\varsigma_m}) = \chi^n + \zeta_1^0\chi^{n-1} + \cdots + \zeta_{n-1}^0\chi^n + \zeta_n^0 +$$
$$[\zeta_1^1\chi^{n-1} + \cdots + \zeta_{n-1}^1\chi^n + \zeta_n^1]e^{-\chi\varsigma_1} + \cdots$$
$$+ [\zeta_1^m\chi^{n-1} + \cdots + \zeta_{n-1}^m\chi^n + \zeta_n^m]e^{-\chi\varsigma_m}$$

Ruan and Wei proved this theorem using Rouches theorem.

### 1.7.3 Hopf-Bifurcation

The extension from two dimensions to higher dimensions was a key contribution made by Hopf. Sometimes Hopf bifurcation is also called as "Poincare- Andronov -Hopf bifurcation". The Hopf-bifurcation Theorem explains how a flow's topological characteristics change when one or more parameters change. The essential finding about flows is that the linearized flow entirely determines the local behaviour of the flow if the stationary point is hyperbolic, or if all of the latent values of the linearized flow at the stationary point have non-zero real parts. Therefore, stationary point bifurcations can only take place at parameter values when the stationary point is not hyperbolic. More, precisely, a bifurcation value of a parameter is a value at which the qualitative nature of the flow changes. It is far more challenging to analyse the Hopf bifurcation since it calls for a non-hyperbolic stationary point with linearized latent values in a two-dimensional centre manifold, as well as periodic rather than stationary bifurcating solutions.

### Theorem 7 Hopf-Bifurcation Theorem

Consider one family of delay polynomials with a single parameter

$$x'(t) = F(x_t, \mu) \tag{1.5}$$

Where $F: C \times R \to R^n$ is a continuous differentiable function in its domain and $x$=0 is a steady state for arbitrary values of $\mu: F(0, \mu) \equiv 0$

We have to linearize function $F$ about $\phi = 0$ as follows $F(\phi, \mu) = L(\mu)\phi + f(\phi, \mu)$

Where $L(\mu): c \to R^n$ is a bounded linear function and $f$ has large order:

$$\lim_{\phi \to 0} \frac{|f(\phi, \mu)|}{\|\phi\|} = 0$$

Following is the characteristic polynomial about $L$:

$$|\lambda I - A(\mu, \lambda)| = 0, A_{ij}(\mu) = L(\mu)_i(e_\lambda e_j)$$

The roots of this equation constitute the main assumption.

**(H)**The characteristic equation will be having a pair of simple conjugates eigen values $\mp i\omega \; with \; \omega_0 \neq 0$ and also no eigen value is scalar multiple of $i\omega_0 \; for \; \mu = 0$. Here a root of order one means a simple root. If we write characteristic polynomial in the form $h(\mu, \lambda) = 0$ then **(H)** implies $h_\lambda(0, i\omega_0) \neq 0$. So, by using theorem for implicit function, there exists a continuously differentiable set of solution $\lambda = \lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ for small $\mu$ satisfying $\lambda(0) = i\omega_0$. In particular $\alpha(0) = 0$ and $\omega(0) = \omega_0$. Next assumption is that as $\mu$ increases through zero, the line of imaginary axis is crossed transversally by these roots. Actually, the assumption is:

$$\alpha'(0) > 0 \tag{1.6}$$

In case $\alpha'(0) < 0$, we always ensure that equation (1.6) holds by changing the sign of the parameter i.e., we take parameter $\nu = -\mu$. The positive sign is essentially a normalization, ensuring that the pair of latent values has a positive real portion if $\mu > 0$ and a negative real part otherwise.

**Theorem 8** Let **(H)** and equation (1.6) both satisfied. Then $\exists \; \varepsilon_0 > 0$, a real valued well defined even function $\mu(\varepsilon)$ and $T(\varepsilon) > 0 \; satisfying \; \mu(0) > 0 \; and \; T(\varepsilon) = {}^{2\pi}\!/_{\omega_0}$, and a non-constant $T(\varepsilon) - periodic \; function \; p(t, \varepsilon)$

having continuous differentiability for all functions in $\varepsilon$ for $|\varepsilon| < \varepsilon_0$, such that $p(t, \varepsilon)$ is a result of equation (1.5) and

$p(t, \varepsilon) = \varepsilon q(t, \varepsilon)\ where\ q(t, 0)\ is\ a\ {}^{2\pi}/_{\omega_0}\ periodic\ solution\ of\ q' = L(0)q.$

Moreover, there exist $\mu_0, \beta_0, \delta > 0$, such that if equation (1.5) has a non-constant periodic solution $x(t)$ of period $P$ for some $\mu$ satisfying $\mu < \mu_0$ with $max_t|x_t| < \beta_0$ and $\left|P - {}^{2\pi}/_{\omega_0}\right| < \delta, then\ \mu = \mu(\varepsilon)\ and\ x(t) = p(t + \theta, \varepsilon)\ for\ some\ |\varepsilon| < \varepsilon_0\ and\ some\ \theta.$

If F is continuously differentiable five times, then:

$$\mu(\varepsilon) = \mu_1 \varepsilon^2 + O(\varepsilon^4) \tag{1.7}$$

$$T(\varepsilon) = \frac{2\pi}{\omega_0}[1 + \tau_1 \varepsilon^2 + O(\varepsilon^4)] \tag{1.8}$$

If all other eigen values for $\mu = 0$ have only negative real components, with the exception of $\mp i\omega\ then\ p(t, \varepsilon)$ is asymptotically stable if $\mu_1 > 0\ and\ unstable\ if\ \mu_1 < 0.$

### 1.7.4 Calculation of Sensitivity with Respect to State Variables of the Model Parameters

Systematic evaluation of the effects of model parameters on system solutions is called sensitivity analysis. There are number of methods to do sensitivity analysis of systems without delay, but there are only a few methods for sensitivity analysis of systems involving delays. The knowledge of how a small change in model parameter can bring change in the state variable, can be a great help in modelling process. It helps in elimination of ineffective and irrelevant parameters. It gives a complete insight into the overall behaviour of the proposed model. The computation of partial derivatives of the solution with respect to each parameter is all that the sensitivity analysis involves if all the parametric values in the given system are assumed to be nonvarying. The sensitivity function matrix has the following format:

$$\mathrm{S(t)} \equiv \mathrm{S(t,\alpha)} = \left[\frac{\partial}{\partial \alpha}\right]^T x(t,\alpha)$$

Its $j^{th}$ $coloumn\ is:$ $S_j(t,\alpha) = \left[\frac{\partial \alpha_j(t,\alpha)}{\partial \alpha_1}, \frac{\partial \alpha_j(t,\alpha)}{\partial \alpha_2}, \dots, \frac{\partial \alpha_j(t,\alpha)}{\partial \alpha_n}\right]^T$

This column vector gives sensitivity of the solution $x_j(t,\alpha)$ for small change in parameter

$\alpha_i, i = 1,2,3, \dots, n.$

**Theorem 9** $S(t)$ satisfies the delay differential equation:

$$S'(t) = J(t)S(t) + J_\tau S(t-\tau) + B(t), t \geq 0$$

$$Where\ J(t) = \frac{\partial}{\partial x} f(t,x,x_\tau), J_\tau(t) = \frac{\partial}{\partial \alpha_\tau} f(t,x,x_\tau), B(t) = \frac{\partial}{\partial \alpha} f(t,x,x_\tau)$$

## 1.8 Summary

This thesis consists of six chapters whose details is as follows:

In chapter-1, a broad overview of the subject is provided. Some key topics in prey-predator dynamics are discussed. All of the essential mathematical ideas have also been provided, allowing for the analytical and numerical examination of all of the suggested mathematical models. It also provides a comprehensive examination of previous work in the subject of prey-predator dynamics in the form of a literature review.

In chapter-2, In this chapter an interesting scenario assumed. While making this mathematical model we assumed that predator consume only mature prey due to this delay occur in system which we have studied using the DDE. A stability analysis is conducted about such a non-zero equilibrium point after calculating all probable equilibrium points. As the delay parameter's value falls below the critical point, the system destabilises and asymptotically stabilises. Once the value of the lag parameter reaches the critical point, Hopf bifurcation is shown. Moreover, sensitivity analysis has been carried out to examine the impact of additional factors on the model parameters. Very few researchers have investigated the impact of prey maturation in a predator-prey model with the help of mathematical modelling. As time delay has a highly intricate effect on the operational behaviour of the system, such as stability calculation, attractivity, periodic vibration, bifurcation analysis, and so on. Many intriguing results have been achieved using the suggested model. Using MATLAB, the numerical findings are supported.

In chapter-3, a mathematical model is proposed to study the effect of toxin producing prey on predator population using delay differential equations. The associated state variables are Prey populations and predator populations. The assumption is that the toxicity released by prey population adversely affects the predator population. The feasible interior equilibrium is calculated. Hopf bifurcation is observed about the critical value of delay parameter. Analytical findings are supported using MATLAB simulation.

In chapter-4, the delay differential equation is used to explore the multi-team prey-predator dynamics in a mathematical model. Many creatures in nature travel in herds and form teams to get from one area to another. This helps them in reducing the risk of predation. A lag in time brought on by the age structure, maturation period, and feeding time is a major factor in real-time prey–predator dynamics that result in periodic solutions and the bifurcation phenomenon. This study analysed the behaviour of teamed-up prey populations against predation by using a mathematical model. The following variables were considered: the prey population $P_{r_1}$, the prey population $P_{r_2}$, and the predator population $P_{r_3}$. The interior equilibrium point was calculated. A local satiability analysis was performed to ensure a feasible interior equilibrium. The effect of the delay parameter on the dynamics was examined. A Hopf bifurcation was noted when the delay parameter crossed the critical value. Direction analysis was performed using the centre manifold theorem. The graphs of analytical results were plotted using MATLAB.

In chapter-5, The study of prey-predator dynamics including commensal species under the influence of time lag is suggested using a mathematical model. In this work, a multi-species system is developed with a delay in which alternating prey are connected to two competing animals, a predator species, and a commensal species. The internal equilibrium point is calculated with and as the state variables under consideration. The practicable interior equilibrium is subjected to an investigation of local satiability. Study is done on how the delay parameter affects the behaviour. Hopf bifurcation happens when the delay parameter reaches a specific threshold. The direction is also determined using the centre manifold theorem. Analytical findings were supported visually using MATLAB.

In chapter-6, the stability of a solitary prey and a pair of predators is investigated mathematically using a delay differential equation and a functional response with square root. In this paper we consider a prey-predator model with a logistic growth delay for the prey population and a square root functional response. A new competitor predator introduced into the system with a density-dependent death rate. Calculating the interior equilibrium point and doing a stability check about non-zero equilibrium point. The system's behaviour changes from stable to unstable, whenever the value of delay is greater than critical value, Hopf-Bifurcation is seen. Numerical simulation done using MATLAB to support analytical finding.

# Chapter-2

# Modelling and Analysis of Prey-Predator Model Involving Predation of Mature Prey using Delay Differential Equations

## 2.1 Introduction

Mathematical biology and population dynamics has been extensively studied using various mathematical models over the years. The most important contributions toward theoretical population biology were given by [1] and [2] which is also known as Lotka-Volterra model or a Prey-Predator model. The dynamics of two populations, one of which is referred to as the prey population and the other as the predator population, are described by a linked non-linear ODE system. However, [2] soon realized that time delay needs to be included for realistic modelling and proposed an integro-differential equation model. But he could not analyse the stability of equilibrium point.

The exact method to solve the modified model by [2] was given by [3] who used the simplest form of delay function. The limited prey population and limited predator appetite were considered in limit cycle prey-predator interaction models [4-5]. Nevertheless, Holling type-II has non-zero handling time, as shown by [6], whereas mass action predation offers zero handling time. It has been shown that the predator can be affected by transmissible diseases under various physical conditions and circumstances [7].

If the functional predation responses are assumed to be of Holling type-II, the Hopf-bifurcation is invariably of a supercritical character [8]. It has also been proved that there are many diseases such as rabies, hepatitis etc. which need incubation period before becoming infection vectors [9-10]. It has been proved that susceptible predator consumes the prey as per mass action law, but the infected predator consumes as per Holling type-II response [11]. The ratio dependent model given by [12] evolved as an alternate to models given by [1] and [2].

Many animals' life histories are divided into two phases immature and mature. There are various behavioural traits for each level. Many writers have also presented stage-structure prey-predator models [13–15]. A mathematical scenario was put up in which the predator species heavily devoured the young prey [16]. In-depth research on positive equilibrium has been done on the stability characteristics for continual input of refuge preys in Lotka-Volterra prey-predator models [17].

A complete investigation was given for a number of stabilities, such as absolute stability, conditioned stability, and bifurcated in a predator-prey scenario with discrete delays [18]. In-depth analysis was done on the exponential characteristic equation's zeros [19]. The stability of equilibrium points is examined using a non-linear set of DDE [20].

In the view of above, therefore, this work aims to analyse the prey-predator relationship with delay differential equations involving predation of mature prey is performed. In the beginning section, after formulation of the basic model, the interior non-zero equilibrium is calculated. The stability study of this non-zero equilibrium is conducted in the next section in both the absence and presence of delay. Next, sensitivity analysis was done to determine how different factors affected the state variables. At last, conclusion part has been written involving the major findings of the analysis and biological applications of the proposed model.

## 2.2 Mathematical Model

### 2.2.1 Assumptions of the Model:

1. Prey population has maturity time $\tau$.
2. Predators consume only mature prey.
3. The mechanism that controls this prey-predator dynamic is as follows: Where, $P_r(0) > 0, P_d(0) > 0$ and $P_r(t-\tau) = c(constant)$ for $t \in [0, \tau]$
4. Positive constants are used for all the parameters.

## 2.2.2 Model Formulation:

Let $P_r$ and $P_d$ stand for the relative masses of prey and predator. Prey populations are thought to mature with time $\tau$. Further, it is considered that only fully grown prey is consumed by predators. The mechanism that controls this prey-predator dynamic is as follows:

$$\frac{dP_r}{dt} = bP_r - \alpha {P_r}^2 - \beta P_r P_d \qquad (1)$$

$$\frac{dP_d}{dt} = -dP_d - \delta {P_d}^2 + \gamma P_r(t-\tau)P_d \qquad (2)$$

Where: $P_r(0) > 0, P_d(0) > 0$ and $P_r(t-\tau) = c(constant)$ for $t \in [0, \tau]$.

These variables are taken into account by this system:

| Parameter | Description |
|---|---|
| $b$ | Birth rate of the prey population intrinsically |
| $\alpha$ | Prey intraspecific competition rate |
| $\beta$ | Rate of inter-specific competing |
| $d$ | Mortality rates among predatory species |
| $\gamma$ | Rate of inter-specific competing |
| $\delta$ | Predator intraspecific competition frequency |
| $\tau$ | Delay variable |

It is reasonable to take all of the parameters to be constant positive values.

## 2.3 Non-Zero Equilibrium

At the steady state $P_r(t-\tau) \cong P_r(t)$. Calculations for the non-zero equilibrium $E^*({P_r}^*, {P_d}^*)$ are as follows:

$\frac{dP_r{}^*}{dt} = 0 \Rightarrow P_d{}^* = \frac{b-\alpha P_r{}^*}{\beta}$

$\frac{dP_d{}^*}{dt} = 0 \Rightarrow P_d{}^* = \frac{\gamma P_r{}^* - d}{\delta}$

On comparing these values of $P_d{}^*$, we get: $P_r{}^* = \frac{b\delta - \alpha\delta - d\beta}{(\gamma\beta+\delta)}$

$\Rightarrow P_d{}^* = \frac{\gamma\beta b - \alpha\beta\gamma + d\beta}{(\gamma\beta+\delta)\beta}$

As a result, we have a non-zero equilibrium point:

$E^*(P_r{}^*, P_d{}^*) = E^*\left(\frac{b\delta-\alpha\delta-d\beta}{(\gamma\beta+\delta)}, \frac{\gamma\beta b-\alpha\beta\gamma+d\beta}{(\gamma\beta+\delta)\beta}\right)$

## 2.4 Investigation of Stability and Hopf-Bifurcation for Non-Zero of Equilibrium $\boldsymbol{E^*(P_r{}^*, P_d{}^*)}$

The equations that regulate the process of population competition at steady point $E^*(P_r{}^*, P_d{}^*)$ are as follows:

$\frac{dP_r{}^*}{dt} = bP_r{}^* - \alpha P_r{}^{*2} - \beta P_r{}^* P_d{}^*$ (3)

$\frac{dP_d{}^*}{dt} = -dP_d{}^* - \delta P_d{}^{*2} + \gamma P_r{}^*(t-\tau) P_d{}^*$ (4)

With respect to the set of equations (3)– (4), the characteristic equation is provided by:

$\lambda^2 + a_1\lambda + a_2 + a_3 e^{-\lambda\tau} = 0$ (5)

Where $a_1 = (2\alpha P_r{}^* + \beta P_d{}^* + d + 2\delta P_d{}^* - b)$,

$a_2 = (d + 2\delta P_d{}^*)(2\alpha P_r{}^* + \beta P_d{}^* - b),\ a_3 = \beta\gamma P_r{}^* P_d{}^*$

When $\tau = 0$, the equation (5) changes to:

$\lambda^2 + a_1\lambda + a_2 + a_3 = 0$ (6)

Hurwitz's roots of equation (6) must have a negative real component in order for the system to meet the Routh-criterion; hence, the system is stable if:

$(L_1)$: $a_1 > 0$;

$(L_2)$: $(a_2 + a_3) > 0$

which definitely holds true.

Now, with changes in the values of $\tau$, we want to examine the shifting of the roots' negative real portions to positive real parts.

If equation (5) has a root of $\lambda = i\omega$ then equation (5) transformed to:

$$(i\omega)^2 + a_1(i\omega) + a_2 + a_3 e^{-(i\omega)\tau} = 0$$

$$\Rightarrow -\omega^2 + a_1(i\omega) + a_2 + \ a_3(\cos\omega\tau - i sin\ \omega\tau\ ) = 0$$

Equating real and imaginary parts:

$$\omega^2 - a_2 = a_3 \cos\omega\tau \qquad (7)$$

$$a_1\omega = a_3 sin\ \omega\tau \qquad (8)$$

Square and add equation (7) and (8):

$$\omega^4 + (a_1^2 - 2a_2)\omega^2 + (a_1^2 - a_3^2) = 0 \qquad (9)$$

The two solutions (roots)of equation (9) are:

$$\omega_{1,2}^2 = \frac{(2a_2 - a_1^2) \pm \sqrt{(a_1^2 - 2a_2)^2 - 4(a_1^2 - a_3^2)}}{2} \qquad (10)$$

None of the two solutions (roots) $\omega_{1,2}^2$ is positive if:

$(L_3)$: $(2a_2 - a_1^2) < 0\ and\ (a_1^2 - a_3^2) > 0\ or\ (a_1^2 - 2a_2) < 4(a_1^2 - a_3^2)$

Therefore, if condition $(L_3)$ is true, equation (10) does not possess positive roots. The following lemma is available [18].

**Lemma 2.4.1** If $(L_1) - (L_2)$ are true, all the roots (solution) of equation (5) have negative real parts for all$\tau \geq 0$.

Besides, if:

$(L_4)$: $(a_1^2 - a_3^2) < 0 \ or \ (2a_2 - a_1^2) > 0 \ and \ (a_1^2 - 2a_2)^2 = 4(a_1^2 - a_3^2)$

Then, positive solution(root) of equation (7) is $\omega_{1,2}^2$.

It follows that, if:

$(L_5)$: $(a_1^2 - a_3^2) > 0 \ or \ (2a_2 - a_1^2) > 0 \ and \ (a_1^2 - 2a_2)^2 > 4(a_1^2 - a_3^2)$

Then, two positive solutions(roots) of equation (7) are $\omega_{1,2}^2$.

When takes particular values, equation (5) in both cases has roots that are entirely imaginary. The system of equations (7)-(8), which is provided may be used to compute the critical values $\tau_j^{\pm}$ of $\tau$.

When takes certain values, the equation (5) in both $(L_4)$and $(L_5)$, has totally imaginary roots. The system of equations (7)-(8), which is provided by: may be used to compute the critical values $\tau_j^{\pm}$ of $\tau$,

$$\tau_j^{\pm} = \frac{1}{\omega_{1,2}} cos^{-1}\left(\frac{\omega_{1,2}^2 - a_2}{a_3}\right) + \frac{2j\pi}{\omega_{1,2}}, j = 0,1,2, \dots \qquad (11)$$

The following lemma [18] can be used to summarise the ideas from above.

**Lemma 2.4.2** (I) If $(L_1) - (L_2) \ and \ (L_4)$ hold and $\tau = \tau_j^{+}$, so equation (5) has two roots that are totally imaginary, $\pm i\omega_1$.

(II) If $(L_1) - (L_2) \ and \ (L_5)$ hold and $\tau = \tau_j^{+}\left(\tau = \tau_j^{-} \ respectively\right)$, in that case equation (5) has two roots that are totally imaginary,$\pm i\omega_1(\pm i\omega_2 \ respectively)$.

Our anticipation is that when $\tau > \tau_j^+$ and $\tau < \tau_j^-$, some solutions of equation (5) will transition from having negative real parts to having positive real parts. Let's notice the following to investigate its possibilities:

$\tau_j^{\pm} = \mu_j^{\pm}(\tau) + i\omega_j^{\pm}(\tau); j = 0,1,2,3, \ldots$

The eigen values of equation (5) satisfy: $\mu_j^{\pm}\left(\tau_j^{\pm}\right) = 0, \omega_j^{\pm}\left(\tau_j^{\pm}\right)$ $=\omega_{1,2}$

$We\ can\ confirm\ that\ the\ transversality\ criterion\ stated\ below\ is\ true$:

$\frac{d}{d\tau}\left(Re\ \lambda_j^+\left(\tau_j^+\right)\right) > 0\ and\ \frac{d}{d\tau}\left(Re\ \lambda_j^-\left(\tau_j^-\right)\right) < 0$

This means that the values $\tau_j^{\pm}$ are bifurcating values. The dispersion of the zeros in equation (5) is provided by the following hypothesis [18].

**Theorem:2.5** Let $\tau_j^+ (j = 0,1,2,3, \ldots)$ perceive equation (11).

(I) All the latent values of equation (5) have negative real portions for any $\tau \geq 0$ if $(L_1), (L_2)$ are true.

(II) If $(L_1), (L_2)\ and\ (L_4)$ are true and $\tau \in [0, \tau_0^+)$, is true, then all latent values of equation (5) have negative real portions. When value of $\tau = \tau_0^+$, then equation (5) supposed to have a pair of totally imaginary roots i.e., $\pm i\omega_1$. When $\tau > \tau_0^+$ in this case equation (5) has at least one latent value with a positive real component

(III) If $(L_1), (L_2)\ and\ (L_5)$ are true, then there must be a positive integer m such that the transition from stability to instability of $0 < \tau_0^+ < \tau_0^- < \tau_1^+ < \tau_1^- \ldots < \tau_{m-1}^- < \tau_m^+$ and there are m occurs. This indicates that all the latent values of equation (5) have negative real portions when $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), \ldots, (\tau_{m-1}^-, \tau_m^+)$. When $\tau \in (\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), \ldots, (\tau_{m-1}^+, \tau_{m-1}^-)$ and $\tau > \tau_m^+$ in this case equation (5) has at least one latent value with a positive part.

## 2.6 Numerical Simulation

The collection of parametric values shown below is thought to visually depict the behaviour revealed by the mathematical equations (1)– (2):

$\gamma = 1.2, \alpha = 1, b = 1.7, \ \beta = 1, d = 0.9, \delta = 1$

The behaviour of the set of mathematical equations (1)– (2) changes from stable to complex dynamics close to the equilibrium $E^*({P_r}^*, {P_d}^*)$ for different values of the delay parameter $\tau$, as shown in the accompanying diagram:

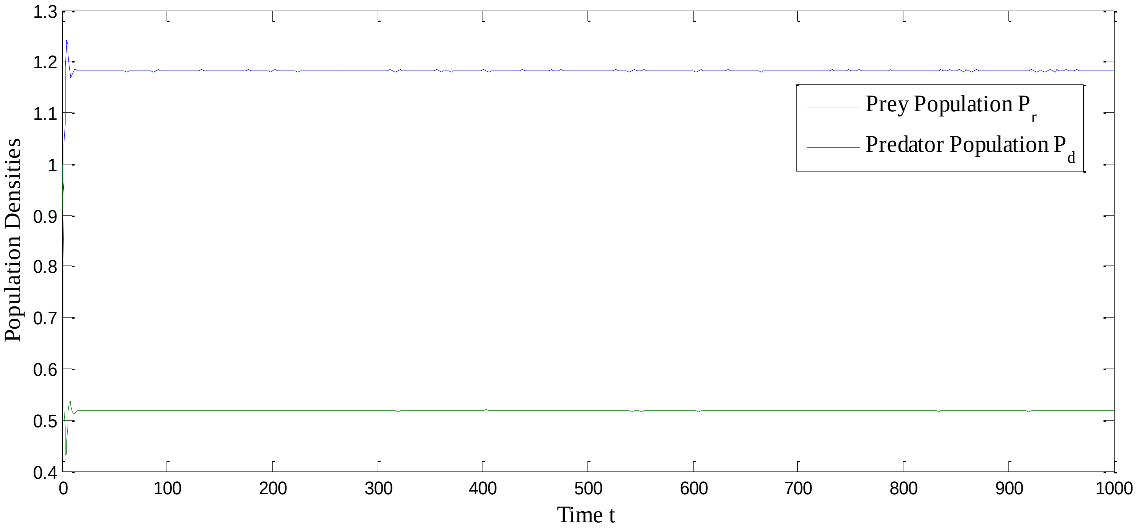

**Figure 2.1.** When there is no delay i.e., when $\tau = 0$ the equilibrium $E^*({P_r}^*, {P_d}^*)$ is stable

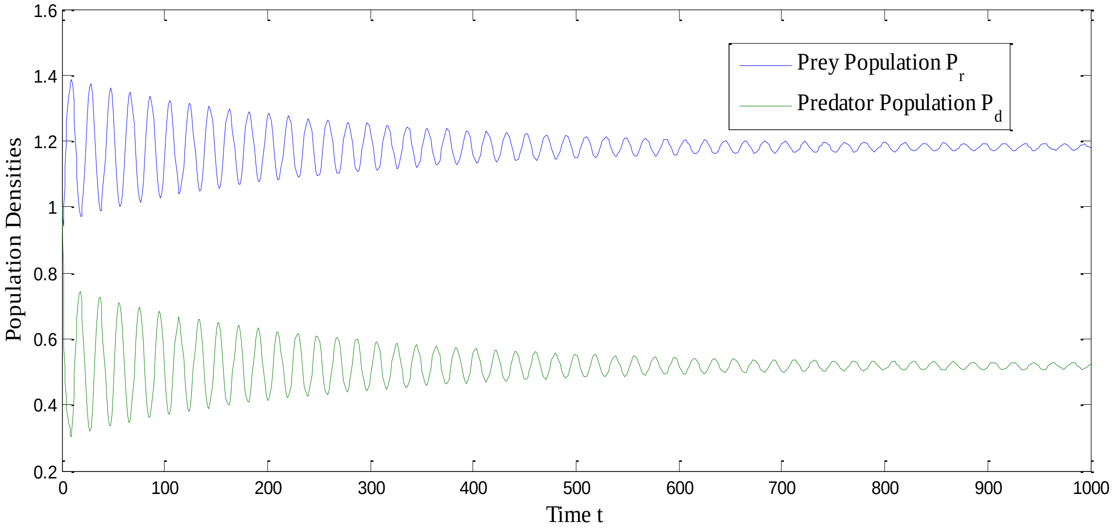

**Figure 2.2** Whenever the delay is lower than the threshold value, i.e., $\tau < 8.23$, the equilibrium $E^*({P_r}^*, {P_d}^*)$ is asymptotically stable.

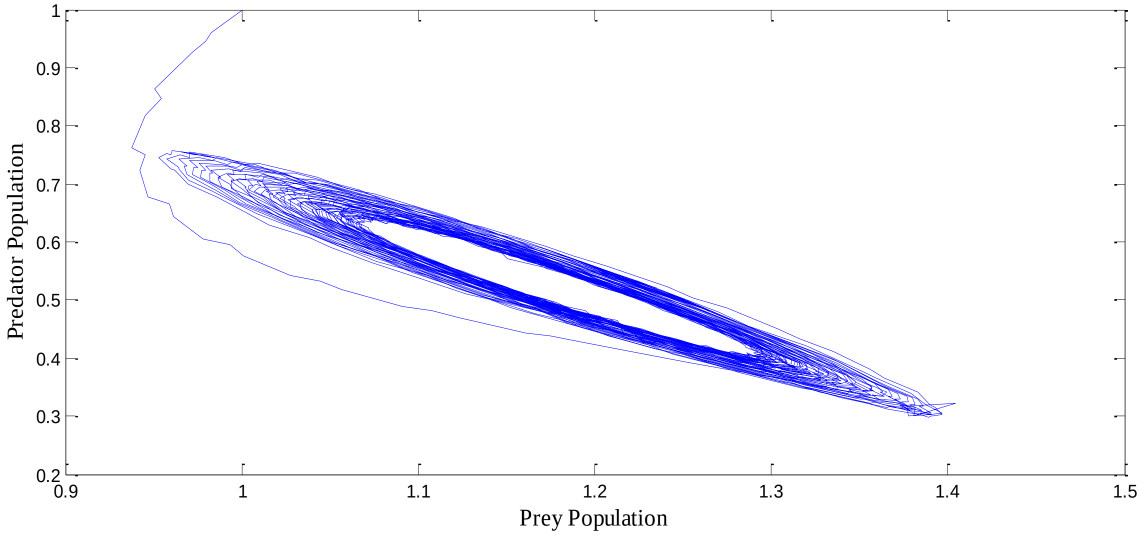

**Figure 2.3** Whenever the delay is lower than the threshold value, $\tau < 8.23$, the phase plane of equilibrium $E^*({P_r}^*, {P_d}^*)$ is visible.

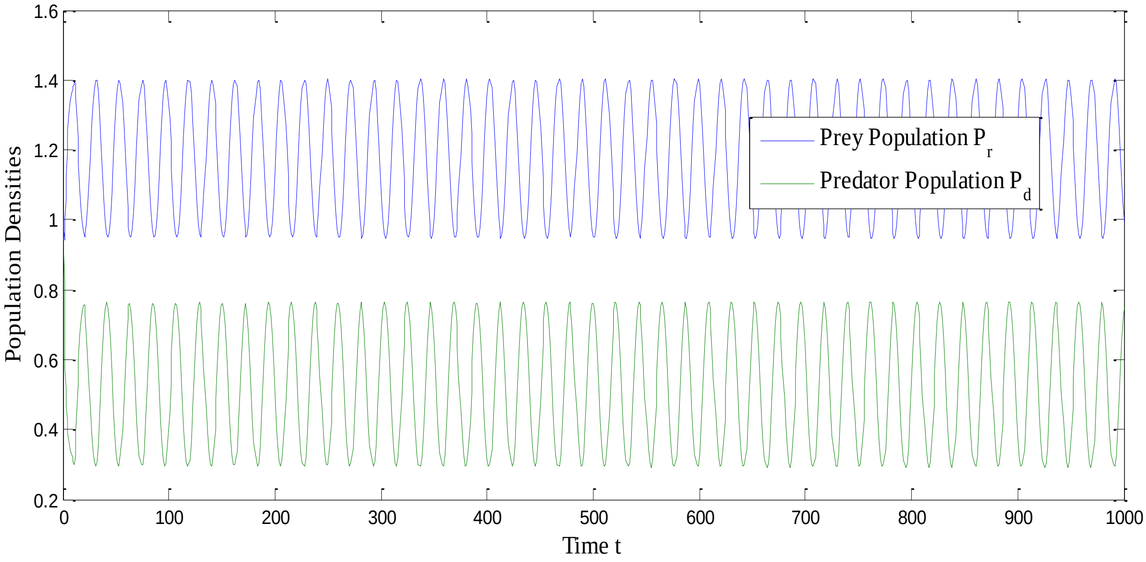

**Figure 2.4** As the value of delay passes the threshold value, i.e., when $\tau \geq 8.23$, the equilibrium $E^*({P_r}^*, {P_d}^*)$ loses stability and exhibits Hopf bifurcation**.**

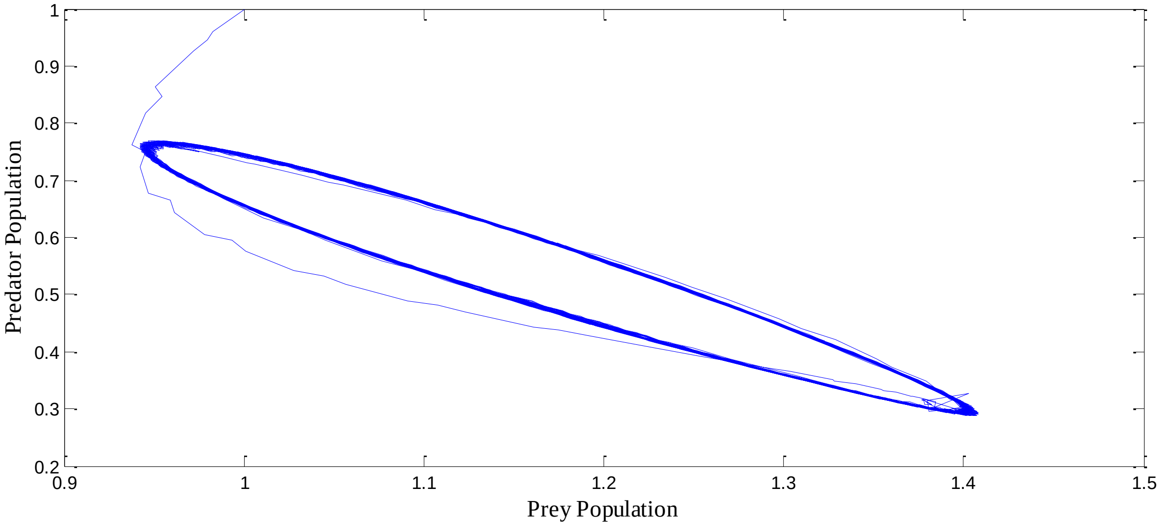

**Figure 2.5** Whenever the value of delay exceeds the threshold value, i.e., $\tau \geq 8.23$, the phase plane of equilibrium $E^*({P_r}^*, {P_d}^*)$ changes.

## 2.7 Analysing (Sensitivity) How State Variables are Affected by Model Parameters

Understanding the significance and impact of other system components on the stability of the dynamic system is made possible by the determination of sensitivity. Here the effect of Intra specific competition rate among prey $\alpha$ on predator population $P_d$ and the effect of Intra specific competition rate among predator $\delta$ on prey population $P_r$ has been depicted with the help of Figure 2.6 and Figure 2.7.

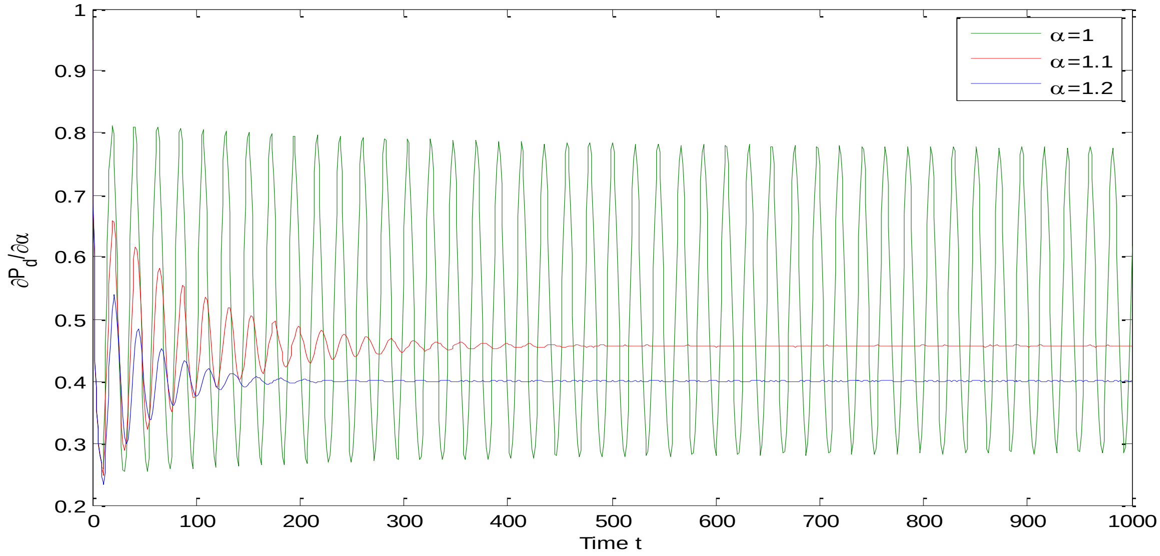

**Figure 2.6** Time series graph shows partial variations in $P_d$ (predator counts) for various parameter values of $\alpha$.

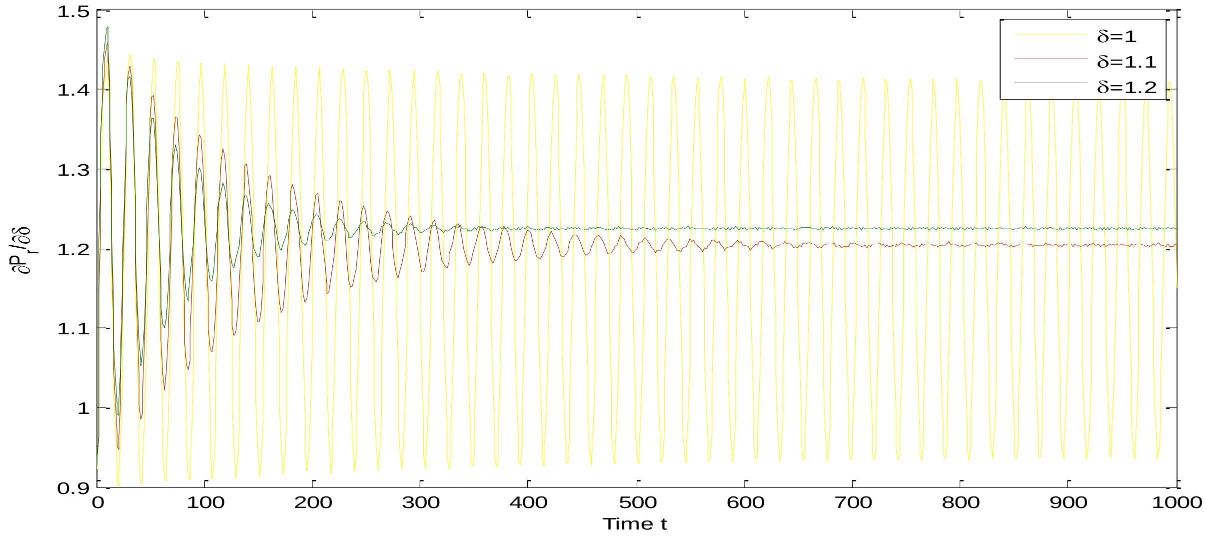

**Figure 2.7** Time series graph shows partial variations in $P_r$ (predator counts) for various parameter values of $\delta$.

## 2.8 Conclusion

With the support of the suggested model, the study of the prey-predator population scenario including the predation of only mature prey is carried out utilising DDE. During Hopf-bifurcation, delay causes the system to break down and sets off extraordinary nature with limit cycles and stable periodic solutions.

Calculation of stability is done about the non-zero equilibrium point $E^*$. The equilibrium point $E^*$show stability whenever no delay in the system, as seen in figure 2.1. The identical information is supplied by $(L_1) - (L_2)$, just like in lemma 1. Figures 2.2 and 2.3 illustrate how the equilibrium $E^*$ begins to lose stability and transition to asymptotical stability as the delay parameter's value falls below the certain threshold point, i.e., $\tau < 8.23$. The non-zero equilibrium value $E^*$ displays complicated nature in the form of a Hopf bifurcation as soon as the delay parameter reaches the critical threshold, which is i.e., $\tau \geq 8.2$,. Figures 2.3 and 2.4 show examples of stable periodic solutions with large amplitudes and limit cycle trajectories. This observation of the system's complicated behaviour (1)-(2) agrees with $(L_4) - (L_5)$ as stated in lemma 2.

Sensitivity analysis of the proposed model has been done. The existing competition within a species (Intra specific competition) rate among prey and predator has been chosen as the parameters whose effect is seen on prey and predator populations. Figure 2.6 shows that as the Intra specific competition rate among prey increases from 1.0 to 1.2, the predator population tends to move toward stability. Figure 2.7 shows that as the Intra specific competition rate among predators increases from 1.0 to 1.2, the prey population tends to move toward stability.

# Chapter-3

# Modelling the Effect of Toxin Producing Prey on Predator Population Using Delay Differential Equations

## 3.1 Introduction

Theoretical studies of predator-prey environments have a long tradition of mathematical ecology, dating back to the popular Lotka-Volterra equations. This technique has been used in a variety of biological systems. The global rise in toxic prey blooms over the last three decades has piqued interest in studying the dynamics of toxin-producing prey and their regulation. A plankton model is an evolutionary replica, and various plankton replica have been developed and researched in the past. Numerous publications have attempted to evaluate the relevance of different aspects that govern the features of the earth's water, particularly its mobility in relation to land, as well as suitable sorts of functional responses to explain the drop in predator population owing to toxin-producing prey [21].

These findings suggest that toxic substances, as well as toxic prey, play a crucial role in predator population dynamics and have a substantial impact on prey-predator confrontations. Peter A. Abrams talked on concepts well about evolution and equilibrium of species and characteristics, in addition to how predatory and anti-predator qualities might adapt to environmental changes. There is coevolution and a bi-directional axis of prey susceptibility, stability seems to be the least possible. An increase in prey defense ability is less likely to result in evolutionary countermeasures in its companion than an increase in predator attack ability [22]. Alan A. Berryman found that functional responses are influenced by prey-predator ratios rather than prey abundance singly. The most serious criticism levelled with logistic prey-predator scenarios is that they do not precisely follow environmental laws. After all, predator reproduction does not require the death of prey [23].

Buike Arthur L. Ma Jr. and Ernest F. B Enfield found a large number of toxicities for predators—prey encounters. Any toxicity data is considered to increase the vulnerability of an individual to predation. Prediction may be improved if species were behaving abnormally. This is an important consideration in the majority of invertebrate organisms. The effect of thermal shock on floating insects has been studied by Sherberger et al. One variable they were testing is whether thermally stressed nymphs are more sensitive to banded sculpture predation [24].

M. J. Naxad ua and B. A. Kroft concluded that the direct toxicity and prey poisoning of the fungicide benomyl induced a predatory mite's lifelong immature depression[25]. Ja Ksh et al.finds the two forms of adaptations to their prey by carnivorous vertebrates, and they demonstrate their connection in an economic model [26]. The Toxin Ology Notes found that poisonous animals can be studied as a substance and toxicity must be examined from any angle that is critical for survival, including parasitizes and population management.T. A. Freyvo Gel and B. A. Perret Fascinating ties between toxinology and ecology may be developed and in turn essential for the survival of biodiversity. So toxinology can only be presumed as a boundary in its budding region[27].

Lincoln Pierson Brower and Linda Susan Fink tested that wild birds may become programmed to avoid naturally poisonous insects either visually or through taste. Authors have confirmed, however, that unconditioned taste rejection of noxious chemicals by wild birds also exists. In reality, such unconditional reactions to the aposematic visual and taste indicators of many insects always seem as essential or important as conditioned reactions[28]. John Cairns, Jr. noted that single species experiments can be used to accurately predict responses at other stages. The author has observed that at various stages of biological organisation varying levels of environmental reality can be achieved in the laboratory or in laboratory conditions. In order to provide a reliable risk assessment, the authors have concluded that the toxicity test balance at various stages of biological organisation [29].

The classical conditioning paradigm for the tests of predator remembering and forgetting and then the intense one-trial and no forgetting over time, which can happen with very toxic prey, was found by Maria R. Servedio. Author has observed that conditions required of the extremely toxic prey facilitate the evolution of colouring by fixing very bright mutations, with successive mutations each causing a slightly higher conspicuousness of the prey[30]. Soren Woft and Darid H. Wise showed that prey families with comparable structural and behavioural associations that are initially killed and devoured differ greatly in their usefulness as meal for generalist arthropod predators [31].

The interactions between toxic phytoplanktone and its zooplancton graders were found to be complicated by Jefferson T. Turner and Patricio A. Tester. Some zooplankters consume some toxic phytoplankters without obvious harm, while others are damaging. Their modes of action, toxicity and solubility vary with the phycotoxins and affect the grazers in various ways [32]. Akira Mori & Gordon M. Burghardt suggested that the anti-predator nature of animals was built by several contextual influences. The environmental temperature is one of the most significant variables when ectotherms in which most physiological processes rely on the body's temperature. The authors looked at the erect temperature of the Japanese grass serpent anti-predator behaviour of an ectotherm [33].

Authors have shown that R. tigvinuz is changing its anti-predatory action with rising temperature from various passive responses to active flying responses.Thomas N. Sherratt concluded that the average likelihood of attacking defenseless models and their defenseless imitations should decrease in sigmoidal manner, with an increasing availability of defenseless alternative prey and that the decision to drill predators in general should be reasonably indifferent to the probability of the possible defense of prey items[34].

Michael P. Speed discovered that the elusive presence of an edible prey can slow predator learning and speed up predator forgetting, to the benefit of the prey. Cryptic may be an anti-signal in terms of learning and memory[35]. Ruxton et al. noted that both toxicity and colouring of prey species are often variable. Authors also observed that there is a strong association between signal intensity and the average toxicity of each signal level. When equilibrium is achieved, predators are changing their behaviour, and now appear to ignore the signal visibly[36].

Christina G. Halpin and Candy Rowe agree that the rejection risk is affected not only by an individual's investment in protections, but also by the investment of other individuals in the same population. Consequently, a predators taste rejection may lead to destabilization in defense expenditure and encourage heterogeneity in prey defense[37]. The trained predators, John Skelhorn and Candy Rowe, noted, there are an important selective forces in aposematism and continue to adopt strategic decisions to eat guarded prey on the basis of their gained knowledge of the prey's toxic and nutritional quality[38]. Stephanie O'Donnell et al. discovered that invasive species are responsible for many extinctions but are also difficult to eliminate. Since indigenous prey is key to invasive predator’s issues, CTA could be used to change invasive predators' feeding behavior. Authors concluded that CTA may offer an effective method to minimize invasive species' ecological effects [39].

David W.M. Nelson et al. pointed out that generalist’s predators use natural and acquired behaviours to eliminate unfavourable foods and choose suitable preserves forging effectively, in order to choose the presence of prey variable in pathogens, diet, defences and availability. Expert predators also do not show such versatility in their behaviour, perhaps because their food is of predictable consistency [40].

Øistein Haugsten Holen observed that a mimicric complex can be less beneficial to the predatory if the protective toxin is sufficiently distasteful than it should be if the models are mildly toxic, and better defended against predation. Also taste imitation may decrease profitability of the imitation complex and increase predation security [41]. Richard Shine found that most dunnarts ate partial toads until they were discarded, and there were no evidence of toad poisoning in a single specimen. After one or two meetings, both dunnarts quickly learned to avoid toads as a prey. For the length of the sample (22 days), Predators refused toads, meaning that toads are harmful in the awareness of the long-term retention [42].

Christina G. Halpin  believe that European starlings (Sturnus vulgaris) raise their intake by artificially raising their nutrient level and decrease their intake by ceasing protein enrichment. This shows that birds can sense and use this information in their decisions on food forage to detect the nutrient content of poisonous prey, thereby improving new prospects for evolving prey defenses. This can be seen clearly. Nutritional differences among people might lead to equally toxic prey that are unprecedented and could explain why some species experience ontogenetic defense strategies [43] .

D. Peacock et al. indicated that new techniques are critically necessary for the sustainability of endangered species against feral cats. There is emerging research that if a cat has destroyed a daunting species effectively, a single feral cat will actively destroy endangered beast populations. In the event that a target animal is killed many times, toxic collars and toxic implants fitted or implanted during surveillance or reintroduction programmers may empower the offending pet [44]. R.S.Ramos et al. found that flubendiamide was tending to stay on the surface longer. The authors proposed that the most promising compounds for use with C are pro-insecticides. In- cludens these pro-insecticides also maintain B communities. pallescens and allow advanced pest control systems to be more sustainable [45].

Barbara Gratzer et al. noted that coral gobies were potentially a prey, but E. The non-toxic monitoring fish of Fasciatus obviously favoured the Gobiodon. The predator did not prefer one animal to the other when targeted against a goby. Contrary to our hopes, poisonous gobies were often caught, but they were expelled instantly, repeatedly, and alive. This rare post-capture avoidance confirms the high risk of these gobies living on the ground due to their skin toxin [46].

M. Banerjee and E. Venturino recently investigated a prey-toxic prey-predator interacting device without taking into account the gestation period's time lag. However, the writers of this report did not take into account the gestation period's time lag. In the method proposed by M. Banerjee and E. Venturino, the predator's replication after predating the prey is instantaneous, and the practical form of the predator exhibiting avoidance behaviour in the presence of vast amounts of toxic prey is taken into account. However, in the natural world, it is more practical to accept the gestation period's time lag. Taking this into account, M. Banerjee and E. Venturino's study is expanded by integrating time delay into the predator's dynamical equation [47].

## 3.2 Mathematical Model

### 3.2.1 Assumptions of The Model

1. One out of two prey is toxic.
2. The predator can distinguish between the two prey species.
3. When the poisonous prey consumes too much and hence kills too many predators, the toxic prey's intake decreases.
4. Predator takes time $\tau$ for the gestation period.

## 3.3 Mathematical Formulation

Assume a delayed prey-predator structure in which the predator consumes food from two separate prey populations $N_1\ and\ N_3$. We suppose that either of the two is poisonous. Let $N_1, N_2\ and\ N_3$ represent the prey, predator, and poisonous prey populations, respectively. The latter, if consumed, will cause serious harm to the predator. We further suppose that the predator can distinguish between the two prey populations, and that if the poisonous prey is consumed in excess, killing too many predators, the latter would reduce its intake. This is represented by a Monod-Haldane type model equation that rises to a maximum and then falls for higher values of the poisonous prey population. This sort of functional response accounts for the fact that predation rates drop with increasing poisonous prey density [48]. The predator is expected to take time for the reproductive cycle in this scenario [49,50]. Accordingly, the model can be expressed as

$$\frac{dN_1}{dt} = a_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - a_2 N_1 N_2 - a_3 N_1 N_3 \tag{1}$$

$$\frac{dN_2}{dt} = b_1 N_2 \left(1 - \frac{N_2}{K_2}\right) - b_2 N_1 N_2 - b_3 N_2 N_3 \tag{2}$$

$$\frac{dN_3}{dt} = c_1 N_1 (t-\tau) N_3 - c_2 N_2 N_3 - c_3 N_3 \tag{3}$$

The above system is associated with the initial functions mentioned below

$(N_1(\theta), N_2(\theta), N_3(\theta)) \epsilon c_+ = c(c-\tau, 0), R_+^3), N_1(0), N_2(0), N_3(0) > 0$

Specification of the variables and parameters

| Parameters | Description |
|---|---|
| $a_1$ | The growth rate of the harmful prey $N_1$ |
| $a_2$ | The inhibitory effects of the two competing prey populations |
| $a_3$ | The rates of predation of both prey $N_1$by predator $N_3$ |

| | |
|---|---|
| $b_1$ | The growth rate of the prey $N_2$ |
| $b_2$ | The inhibitory effects of the two competing prey populations |
| $b_3$ | The rates of predation of both prey $N_2$by predator $N_3$ |
| $c_1$ | The rates of toxin liberation by the harmful prey $N_1$ reducing the growth of predator $N_3$ |
| $c_2$ | The conversion rates of predator $N_3$ |
| $c_3$ | Natural death rate of predator population $N_3$ |
| $K_1$ | Carrying capacity of prey population $N_1$ |
| $K_2$ | Carrying capacity of prey population $N_2$ |
| $\tau$ | Delay parameter |

## 3.4 Equilibrium Point

$$\frac{dN_3}{dt} = 0$$

$$\Rrightarrow \;\; c_1N_1N_3 - c_2N_2N_3 - c_3N_3 = 0$$

$$N_3(c_1N_1 - c_2N_2 - c_3) = 0$$

$$N_3 = 0, \quad c_1N_1 - c_2N_2 - C_3 = 0$$

$$\Rrightarrow N_1 = \frac{c_2N_2 + c_3}{c_1}$$

$$\frac{dN_1}{dt} = 0$$

$$a_1N_1\left(1 - \frac{N_1}{K_1}\right) - a_2N_1N_2 - a_3N_1N_2 = 0$$

$$N_1\left\{a_1\left(1-\frac{N_1}{K_1}\right)-a_2N_2-a_3N_2\right\}=0$$

$$a_1\left(1-\frac{N_1}{K_1}\right)-a_2N_2-a_3N_2=0$$

Put the value of $N_1$

$$a_1\left(1-\frac{c_2N_2+c_3}{c_1K_1}\right)-a_2N_2-a_3N_2=0$$

$$a_1(c_1K_1-\ c_2N_2+\ c_3)-\ a_2c_1K_1N_2-\ a_3c_1K_1N_2=0$$

$$a_1c_1K_1-\ a_1c_2N_2+\ a_1c_3-\ a_2c_1K_1N_2-\ a_3c_1K_1N_2=0$$

$$N_2(-a_1c_2-a_2c_1K_1-a_3c_1K_1)=\ -a_1c_1K_1-a_1c_3$$

$$\mathrm{N_2}=\ \frac{\mathrm{a_1c_1K_1}+\ \mathrm{a_1c_3}}{\mathrm{a_1c_2}+\mathrm{a_2c_1K_1}+\mathrm{a_3c_1K_1}}$$

## 3.5 Local Stability

In this part, we will just look at the interior equilibrium's stability and local Hopf-bifurcation. The system of equations (1), (2), and (3) clearly has a statistically significant and positive equilibrium.

$$\begin{vmatrix}\eta-z_1 & z_2 & z_3\\ z_4 & \eta-z_5 & z_6\\ z_7 & z_8 & \eta-z_9\end{vmatrix}=0$$

$$\begin{vmatrix}\eta-\left(a_1-\dfrac{2N_1}{K_1}-a_2N_2-a_3N_3\right) & -b_2N_2 & c_1N_3\\ -a_2N_1 & \eta-\left(b_1-\dfrac{2N_2}{K_2}-b_2N_1-b_3N_3\right) & -c_2N_3\\ -a_3N_1 & -b_3N_2 & \eta-(c_1N_1-c_2N_2-c_3)\end{vmatrix}$$
$$=0$$

After simplification, we get characteristic equation

$$\eta^3+A_1\eta^2+A_2\eta+A_3+e^{-\eta\tau}(B_1\eta+B_2)=0 \tag{4}$$

Where $A_1 = -(a_{11} + a_{22})$, $A_2 = a_{23}a_{32} + a_{11}a_{22} - a_{12}a_{21}, B_1 = -a_{31}a_{13}$, $A_3 = a_{11}a_{23}a_{32} - a_{13}a_{21}a_{32}, B_2 = a_{13}a_{31}a_{22} - a_{12}a_{31}a_{23}$

For the equilibrium point to be stable, all of the latent roots of the characteristic equation (4) must have real part with a negative sign. It is tough to determine under what circumstances all latent values of the equation (4) would have real part with negative sign. Assume the value of delay $\tau = 0$ equation (4) is obtained. At the interior equilibrium $E^*(N_1^*, N_2^*, N_3^*)$, the characteristic equation of the above system of equations (1), (2), and (3) is

$$\eta^3 + A_1\eta^2 + (A_2 + B_1)\eta + (A_3 + B_2) = 0 \quad (5)$$

According to the Routh-Hurwitz criterion

If $A_1 > 0$, $(A_3 + B_2) > 0$,$A_1(A_2 + B_1) > (A_3 + B_2)$ (6)

So the real parts of all the roots of eq. (5) would be negative.

$A_3 + B_2 = 0$ if we assume $\eta = 0$ to be a solution of (4). As a result, this statement violates the previous condition (6). Therefore, corresponding result obtained for $\eta = 0$ is not acceptable for (4). Assume that for any $\tau \geq 0$ with $\mu > 0$ is a (4) solution, then

$$-i\mu - A_1\mu + iA_2\mu + A_3 + (cos\mu\tau - i\sin\mu\tau)(iB_1\mu + B_2) = 0 \quad (7)$$

Distinguishing between the real and the imaginary values,

$$A_3 - A_1\mu^2 + B_2\cos\mu\tau + B_1\sin\mu\tau = 0 \quad (8)$$

$$A_2\mu - \mu^3 + B_1\mu\cos\mu\tau - B_2\sin\mu\tau = 0 \quad (9)$$

which results in

$$\mu^6 + x\mu^4 + y\mu^2 + z = 0 \quad (10)$$

of which

$$x = A^2 - 2A_2, y = A_2^2 - B_1^2 - 2A_1A_3, z = A_3^2 - B_2^2$$

If we set $\alpha = \mu^2$, equation (10) becomes,

$$\alpha^3 + x\alpha^2 + y\alpha + z = 0 \tag{11}$$

and $h(\alpha) = (\alpha^3 + p\alpha^2 + q\alpha + ɤ)$

## 3.6 Lemma 1

The polynomial equation (11) yields the following results.

1) Condition for the equation (11) has at least one positive root is

   $z < 0$

2) Condition for the equation (11) do not possess positive root is $z \geq 0$ and $(x^2 - 3y) \leq 0$

3) Condition for the equation (11) must have a positive solution is $z \geq 0$ and $(x^2 - 3y) > 0$ and also satisfies the following condition

$$\upsilon = \frac{-x \pm (x^2 - 3y)}{3} > 0 \; and \; h(\upsilon) \leq 0$$

Proof. For equation (11) has at least one positive real solution let us take

$$\mu_0 = \sqrt{\alpha_0}$$

From equation (8) and equation (9), we obtain,

$$\cos \mu_0 \tau = \frac{-\left(B_1\mu_0^2(A_2 - \mu_0^2) + (A_3 - A_1\mu_0^2)(B_2)\right)}{(B_2)^2 + (B_1\mu_0)^2} \tag{12}$$

$$\tau_j = \frac{1}{\mu_0} \, aɤc \cos\left(\frac{-\left(B\mu_0^2(A - \mu_0^2) + (A_2 - A\mu_0^2)(B_1)\right)}{(B_2)^2 + (B_1\mu_0)^2} + 2j\pi\right) \tag{13}$$

where j = 0, 1, 2,3...

## 3.7 Lemma 2

Assume $h(\alpha_0) = (3\alpha^2 + 2x\alpha_0 + y_0)$ and the conditions in (6) are satisfied. For (j=0,1,2...), denote $\mu\eta(\tau) = \alpha(\tau) + i\mu(\tau)$ be the root of equation (4) satisfying $\alpha(\tau_j) = 0, \mu(\tau_j) = \mu_0$, where

$$\tau_j = \frac{1}{\mu}\, arc \cos\left(\frac{-\left(B_1\mu^2(A_2 - \mu^2) + (A_3 - A_1\mu^2)(B_2)\right)}{(B_2)^2 + (B_1\mu_0)^2} + 2j\pi\right)$$

then $\pm i\mu_0$ are simple roots. If the transversality condition

$$\alpha^j(\tau_j) = \frac{R_e\eta(\tau)}{d\tau}\bigg|_{\eta = i\,\mu_0} \quad f = 0$$

At any equilibrium point where $\tau = \tau_j$ and, a Hopf bifurcation exists for the system of equations (1), (2), and (3).

Proof. Let the root of the equation (4) be $\eta = \eta(\tau)$. As $\eta(\tau)$ is substituted into eq. (4) and both sides of equation (4) are differentiated with respect to τ, it follows that

$$\left[(3\eta^2 + 2A_1\eta + A_2) + \left((\eta^2B_3 + \eta B_1 + B_2)(-\tau) + (2\lambda B_3 + B_1)\right)e^{-\eta\tau}\right]\frac{d\mu}{d\tau} = \eta(\eta^2B_3 + \eta B_1 + B_2)e^{-\eta\tau}$$

Thus

$$\frac{d\eta}{d\tau} = \frac{(3\eta^2 + 2A_1 + A_2)}{\eta(\eta B_1 + B_2)} + \frac{(B_1)}{\eta(\eta B_1 + B_2)}$$

From (8)-(10), we have

$$a^j(\tau_0) = R_e\left[\frac{(3\eta^2 + 2A_1\eta + A_2)e^{\eta\tau}}{\eta(\eta B_1 + B_2)}\right] + R_e\left[\frac{(B_1)}{\eta(\eta B_1 + B_2)}\right]$$

$$= \overset{\triangle}{1}\,[3\mu^6 + 2(A^2 - 2A_2)\mu^4 + (A_2^2 - B_1^2 - 2A_1A_3)\mu^2]$$

$$= \overset{\Delta}{1}(3\mu^6 + 2p\mu^4 + q\mu^2)$$

$$= \mu_0^2 \ \overset{\Delta}{h^j}(\alpha)$$

where $\Delta = B_1^2\mu_0^2 + B_2^2$.Notice that $\eta' > 0$ and $\mu_0 > 0$, we conclude that $\text{sign}[\alpha^j(\tau_0)] = \text{sign}[h^j(\alpha_0)]$

This demonstrates the lemma.

Using Lemma 2, we can easily establish the following findings on the stability and bifurcation of the mathematical equations (1), (2), and (3).

**3.8 Theorem 1.** Suppose that (6) is satisfied

1) The non-zero equilibrium point $E^*$ of the system of mathematical system (1), (2) and (3) is stable asymptotically for all $\tau \geq 0$ and the required condition for which all latent values of equation (4) have real part with negative sign is that $z \geq 0$ and $(x^2 - 3y) \leq 0$.

2) The non-zero equilibrium point $E^*$ of the mathematical frame (1), (2) and (3) is stable asymptotically for all the values of delay $\tau \epsilon [0, \tau_0]$ is that either $z < 0$ or $z \geq 0$ and $(x^2 - 3y) > 0$ hold.

3) The non-zero equilibrium point $E^*$ of the mathematical frame (1), (2) and (3) show Hopf bifurcation the value of delay when $\tau = \tau_0$ and all conditions as stated in (2) and $h^j(\alpha_0)\ f = 0$ hold.

4) All parameters are expected to be nonnegative and to be time constant.

## 3.9 Numerical Example

The following parametric values are used to graphically represent the behavior indicated by the following mathematical equations (1) -(3)

$a_1 = 0.68$, $a_2 = 1.22$, $a_3 = 1.35$

$b_1 = 0.66$, $b_2 = 0.2$, $b_3 = 0.9$

$c_1 = 1.63$, $c_2 = 1$, $c_3 = 0.58$

The response of the mathematical equations (1)-(3) varies with the delay parameter, shifting from stable to complicated behavior around the equilibrium $E^*(N_1^*, N_2^*, N_3^*)$.

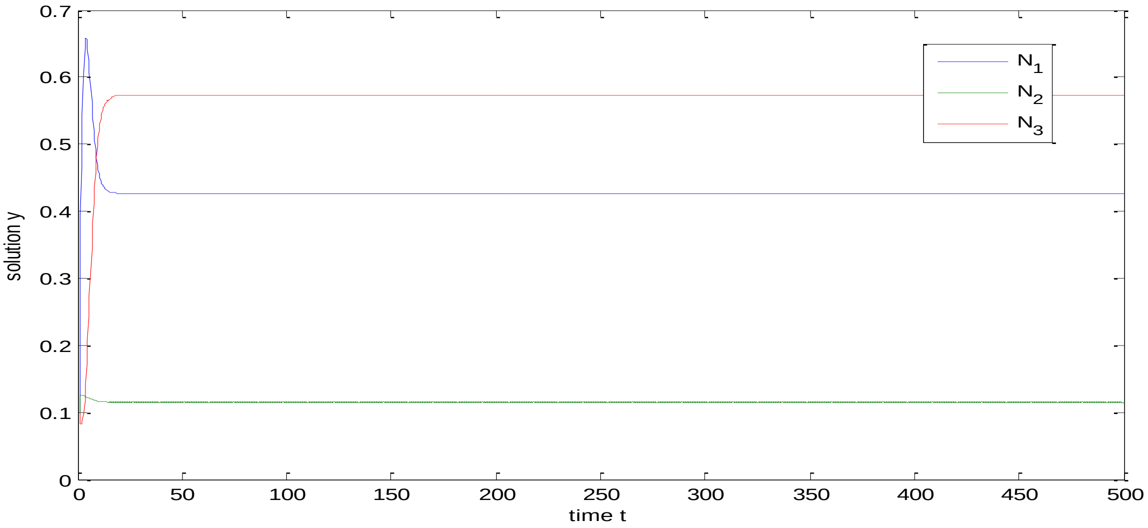

**Figure 3.1** When the value of delay parameter is zero i.e., when $\tau = 0$ the non-zero equilibrium point $E^*(N_1^*, N_2^*, N_3^*)$ is stable

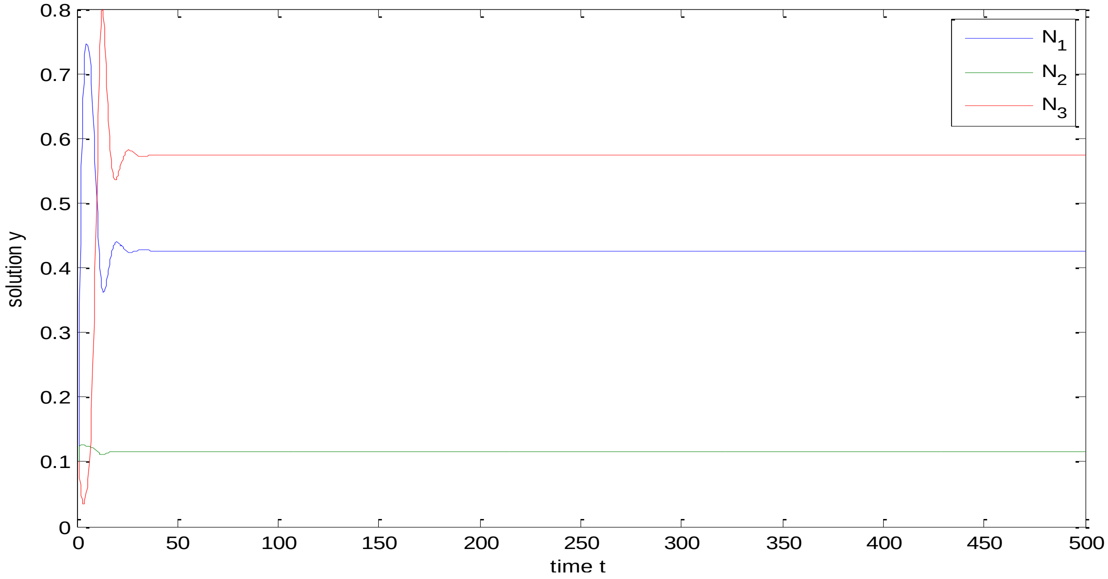

**Figure 3.2** When the value of delay parameter is smaller than the thershold value, i.e., when $\tau < 4.48$ the equilibrium $E^*(N_1^*, N_2^*, N_3^*)$ is asymptotically stable.

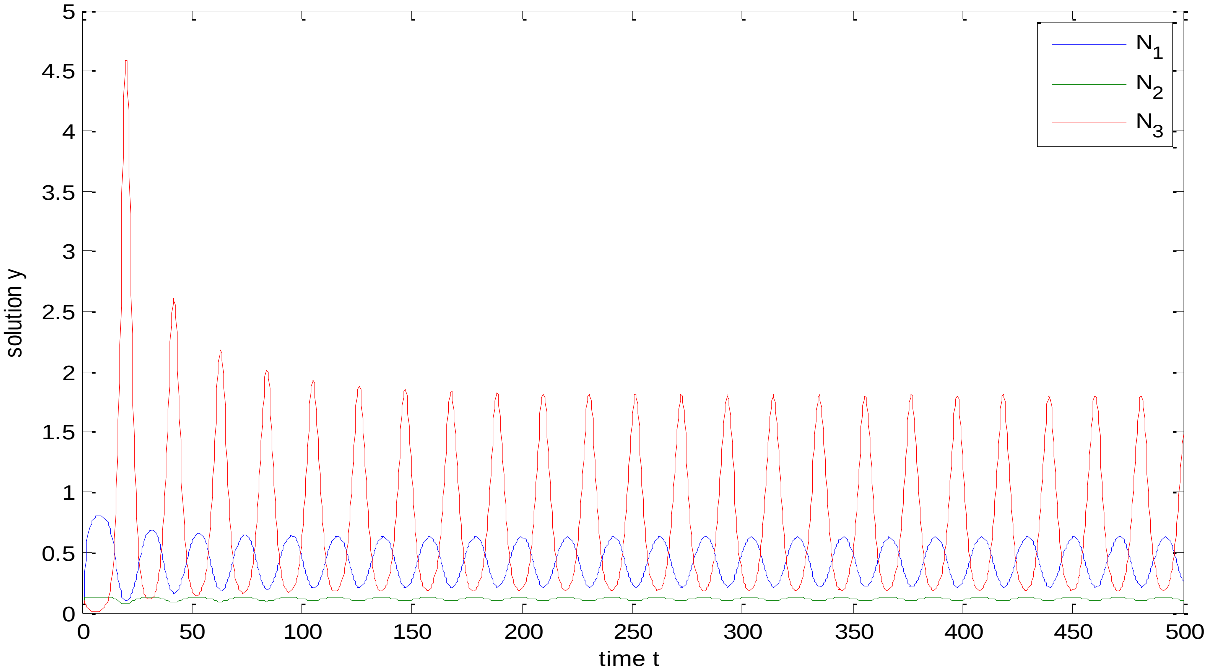

**Figure 3.3** As the delay exceeds the critical value, that is, when $\tau \geq 4.48$, the equilibrium $E^*(N_1^*, N_2^*, N_3^*)$ loses stability and exhibits Hopf bifurcation.

## 3.10 Conclusion:

The analysis of prey-predator model to study the effect of toxin producing prey on predator population using delay differential equations. The calculation of stability is carried about non-zero equilibrium point $E^*(N_1^*, N_2^*, N_3^*)$. When the value of delay parameter is zero i.e., when $\tau = 0$ the non-zero equilibrium point $E^*(N_1^*, N_2^*, N_3^*)$ is stable shown in **Figure 3.1.** The toxicity released by prey population adversely affects the predator population. When the value of delay parameter is smaller than the thershold value, i.e., when $\tau < 4.48$ the equilibrium $E^*(N_1^*, N_2^*, N_3^*)$ is asymptotically stable as shown in **Figure 3.2**. As the delay exceeds the critical value, that is, when $\tau \geq 4.48$, the equilibrium $E^*(N_1^*, N_2^*, N_3^*)$ loses stability and exhibits Hopf bifurcation shown in the **Figure 3.3**. Hence, it is concluded that time lag is a key factor which should be included in the mathematical model in order to study the dynamical behaviour of these types of biological systems.

# Chapter4

## Modelling the Multiteam Prey-Predator Dynamics Using the Delay Differential Equation

### 4.1 Introduction

Numerous applied mathematicians and ecologists have increasingly focused on the predator–prey relationship because of its generality and significance. The impacts of population, age distribution, time delay, functional response, switching, and other characteristics have all been taken into account while developing a number of complex systems for at least two interacting class of breed in an ecosystem [61, 64, 65, 75, 78].

Every species inside this natural world exists in the wild. Some population lives alone, while others live in herds, flocks, colonies, or packs. Living with or close to other animals can help certain species survive and guarantee that each member's needs are met through collaboration. Furthermore, building a team and interacting with others is a fundamental tool through which a team member can consistently achieve positive outcomes and readily meet their needs. This study focused on a system in which herds of prey coexist and are attacked by the same type of predator. The problem of multiteam games is relatively new. For certain animals, forming a team more considerably improves their efficiency of food search as a group compared with when this activity is performed alone and reduces the danger of predation.

Two prey and one predator sometimes interact in an unstable way. The likelihood of all three species coexisting in a particular area is quite low. Practically, the predator always wins [70]. Determining the types of interactions in a multiteam eco-system can enable researchers to recognise the importance of prey teamwork. The finding of such a study would be similar to those reported by Poole on Leslie–Gower computations [77]. The sort of behavioural traits displayed by animals to recognise predators cannot be determined by looking at the link between the population of prey and attack rate [51]. By performing an initial assessment of the model's normalised form, a study demonstrated the presence of dynamics in practical systems of predator-prey that are substantially analogous to basic circumstances [62].

A study identified the necessary criteria for all species to survive indefinitely and determined the condition when species becomes extinct in the system [61]. By utilising a stochastic logistic differential equation that calculates ecosystem function, a study examined the long-term unexpected behaviour of the at-risk group [59]. Another study reported that strong diffusions or interspecific competitions or slower prey intrinsic growth rates and faster predator intrinsic reduction rates are required to obtain a nonconstant solution [69]. To maintain a stable food web, the predator spends its time between preys with different relative densities [60]. The amount of food that can be quickly digested affects a predator's behaviour toward a particular kind of prey. These genes and therefore the predator population's lifestyle is regulated by a biosynthetic repression approach [78]. Protection evolution in prey species is improved by survival and decreased density fluctuations based on parameters. The inclusion of a predator's optimal meal choice into the model enhanced cohabitation and reduced overall density variations [79]. If the impulsive duration surpasses a threshold value, the structure remains typically stable [75].

Many types of animals prefer to be together in a herd. Because different groups share the same habitat, they may cooperate, compete, or form a predator–prey relationship. New paradigm for predator–prey teams were reported in a previous study [56]. By applying nonlinear feedback control inputs, three prey–predator populations could be stabilised over time asymptotically. The functional parameter limit is set under which variables converge to limit cycles [55].

The system exhibits a stationary distribution that is ergodic under certain conditions. The system's solution remains globally asymptotically stable under certain conditions [66]. Two preys and one predator comprise a dynamic system modelling multiple teams. During an attack, the individuals of both prey groups would support one another and the pace of predation for both groups would differ [76]. A mathematical simulation of two competing prey species was developed in research and evaluated. The pace of growth and functional responses might be nonlinear functions that are general in nature. The findings suggest the presence of a crucial characteristic governing the system's dynamics that is termed as an intraspecific interference factor [54]. The criteria for local asymptotic stability were achieved in the lack of climate fluctuation. The authors defined the probabilistic approach by including Gaussian white noise notions into all regular equations [65]. The authors evaluated the predator–prey relationship in three species in three dimensions by using an ordinary differential equation. The estimate took into account the subjective population numbers of two prey species and one predator species that coexist in the same environment [53].

Prey cooperation benefits both prey populations in many ecosystems. A predator may keep pursuing the weaker prey if one prey is dangerous and the other is weak [68]. In the presence of only one predator in the system, a fourth-order nonlinear differential system of equation can be used to represent the given mathematical system [80]. A study used a three-species prey–predator system considering that a predator is layman by nature because it survives on two prey animals [67]. In this study, we considered two prey populations and two predator populations. When two prey species live in two types of habitats and can defend themselves in groups, only one of the two predators can move between the two types of habitats [58]. Trends of a group and the effectiveness of herbicide are both affected by a predator [57].

In the case of direct competition between two prey populations, a predator model is used that incorporates temporal delays and a weak Allee effect in the prey's growth function. Despite its simplicity, the system exhibits a wide range of dynamic behaviour, such as the equilibrium point's biostability [71]. A high degree of fear of a prey animal and a higher quality of living for second prey may improve the chances of living of that species [74]. One prey poses a threat to the predator, while the other is uninjured. Monod-Haldane and Holling type II functional responses separately track the predation processes of both prey groups [52].

Research that employed stochastic Lyapunov functionals put forward several prerequisites for extinction and persistence in the median of the three species [73]. To examine delay models in population dynamics, we used the model adopted by [72]. There haven't been many research that use mathematical modelling to examine how prey maturity affects a predator-prey model. Predation of mature prey may be investigated through delay differential equations [63]. Time lag is a crucial factor that should be included in the mathematical model to examine the dynamic behaviour of these types of biological systems [64].

The delay model has not been employed in any research on the dynamics of prey-predator systems. This work used delay differential equations to analyse the dynamics of multiteam prey-predator systems.

## 4.2 Mathematical Frame

A two-prey, one-predator delay differential model's dynamics were investigated in this work where both preys support each other to prevent predation. The predator is expected to take time $\tau$ during the gestation phase in this scenario. Thus, the model can be represented as follows:

$$\frac{dP_{r1}(t)}{dt} = a_1 P_{r1}(t)\big(1 - P_{r1}(\mathrm{t})\big) - P_{r1}(t)P_{r3}(t) + P_{r3}(t)P_{r2}(t)P_{r1}(t) \tag{1}$$

$$\frac{dP_{r2}(t)}{dt} = b_1 P_{r2}(t)\big(1 - P_{r2}(\mathrm{t})\big) - P_{r2}(t)P_{r3}(t) + P_{r3}(t)P_{r2}(t)P_{r1}(t) \tag{2}$$

$$\frac{dP_{r3}(t)}{dt} = -c_1 P_{r3}^2(t) + d_1 P_{r1}(t-\tau)P_{r3}(t) + e_1 P_{r2}(t-\tau)P_{r3}(t) \tag{3}$$

Where delay $\tau > 0$ is the lag time necessary for the predator's gestation period; $P_{r1}$ (t) and $P_{r2}$ (t) are the populations of the two teams of preys, respectively; and $P_{r3}$ (t) is the population of predators. All the parameters have positive values, that is, the values of $a_1$, $b_1$, $c_1$, $d_1$, and $e_1$ are more than zero. The system (Poole, 1974; Vance, 1978; Abrams et al., 1993) is related to the following starting functions:

$$(P_{r1}(\theta), P_{r2}(\theta), P_{r3}(\theta)) \in \ C_{+} = C\big((-\tau, 0), R_{+}^{3}\big), P_{r3},\ P_{r2}, P_{r1}) > 0$$

Explanation of the variables and the parameters:

| **Variables/Parameters** | **Elucidation** |
|---|---|
| $P_{r1}$ | First prey population |
| $P_{r2}$ | Second prey population |
| $P_{r3}$ | Predator population |
| $a_1$ | Natural growth rate of $P_{r1}$ |
| $b_1$ | Natural growth rate of $P_{r2}$ |
| $c_1$ | Death rate of predator population due to mutual competition. |
| $d_1$ | Rate of predation of $P_{r1}$ |
| $e_1$ | Rate of predation of $P_{r2}$ |
| $\tau$ | Delay parameter |

## 4.3 Equilibrium Point

The systems (Poole, 1974; Vance, 1978; Abrams et al., 1993) have eight equilibria with specific nonnegativity requirements. Because delay does not play any role in the stability of the results in the other seven equilibria, we only concentrate on the stability and local Hopf bifurcation of the inner equilibrium in this section. Equivalently, set equation (1) to zero to determine the equilibrium point,

$$\frac{dP_{r1}}{dt} = 0$$

$$a_1 P_{r1} - a_1 P_{r1}^2 - P_{r1}P_{r3} + P_{r1}P_{r2}P_{r3} = 0$$

$$P_{r1}(a_1 - a_1 P_{r1} - P_{r3} + P_{r2}P_{r3}) = 0$$

Either $P_{r1} = 0 \ or \ a_1 - a_1 P_{r1} - P_{r3} + P_{r2}P_{r3} = 0$

$$\frac{dP_{r2}}{dt} = 0$$

$$b_1 P_{r2} - b_1 P_{r2}^2 - P_{r2}P_{r3} + P_{r1}P_{r2}P_{r3} = 0$$

$$P_{r2}(b_1 - b_1 P_{r2} - P_{r3} + P_{r1}P_{r3} = 0$$

$$\frac{dP_{r3}}{dt} = 0$$

$$- c_1 P_{r3}^2 + d_1 P_{r1}P_{r3} + e_1 P_{r2}P_{r3} = 0$$

$$P_{r3}(- c_1 P_{r3} + d_1 P_{r1} + e_1 P_{r2}) = 0$$

As $P_{r1} \neq 0, P_{r2} \neq 0, P_{r3} \neq 0$ so we have three equations in $P_{r1}, P_{r2}, P_{r3}$

$$a_1 - a_1 P_{r1} - P_{r3} + P_{r2}P_{r3} = 0$$

$$a_1(1 - P_{r1}) + (-1 + P_{r2})P_{r3} = 0 \tag{4}$$

$$b_1 - b_1 P_{r2} - P_{r3} + P_{r1}P_{r3} = 0$$

$$b_1(1 - P_{r2}) + (-1 + P_{r1})P_{r3} = 0 \tag{5}$$

$$- c_1 P_{r3} + d_1 P_{r1} + e_1 P_{r2} = 0 \tag{6}$$

Multiply (4) by $-(-1 + P_{r1})$ and (5) by $(-1 + P_{r1})$ then add

$$a_1(1 - P_{r1})(-)(-1 + P_{r1}) - (-1 + P_{r1})(-1 + P_{r2})P_{r3} = 0$$

$$b_1(1 - P_{r1})(-1 + P_{r2}) + (-1 + P_{r1})(-1 + P_{r2})P_{r3} = 0$$

Add these equations

$$a_1(1 - P_{r1})(1 - P_{r1}) + b_1(1 - P_{r1})(-1 + P_{r2}) = 0$$

$$a_1(1 - P_{r1})^2 - b_1(1 - P_{r2})^2 = 0$$

$$a_1(1 - P_{r1})^2 = b_1(1 - P_{r2})^2$$

$$\frac{a_1}{b_1}(1 - P_{r1})^2 = b_1(1 - P_{r2})^2$$

Taking square root

$$\sqrt{\frac{a_1}{b_1}}(1 - P_{r1}) = (1 - P_{r2})$$

$$P_{r2} = 1 - \sqrt{\frac{a_1}{b_1}}\,(1 - P_{r1})$$

Put this value in (5)

$$\left[1 - \left\{1 - \sqrt{\frac{a_1}{b_1}}\,(1 - P_{r1})\right\}\right] b_1 + (-1 + P_{r1})P_{r3} = 0$$

$$\left[1 - 1 + \sqrt{\frac{a_1}{b_1}}(1 - P_{r1})\right] b_1 + (-1 + P_{r1})P_{r3} = 0$$

$$b_1\left[\sqrt{\frac{a_1}{b_1}}(1 - P_{r1})\right] + (-1 + P_{r1})P_{r3} = 0$$

$$\sqrt{a_1 b_1}(1 - P_{r1}) + (-1 + P_{r1})P_{r3} = 0$$

$$(-1 + P_{r1})P_{r3} = -\sqrt{a_1 b_1}\,(1 - P_{r1})$$

$$(-1 + P_{r1})P_{r3} = \sqrt{a_1 b_1}\,(-1 + P_{r1})$$

$$P_{r3} = \sqrt{a_1 b_1}$$

Let $E^*(P_{r1}^*,\ P_{r2}^*,\ P_{r3}^*)$ denote the interior equilibrium

where

$$P_{r3}^* = \sqrt{a_1 b_1}, \qquad P_{r2}^* = \frac{c_{1\,\sqrt{a_1 b_1}} - d_1\left(1 - \sqrt{b_1/a_1}\right)}{e_1 + d_1\sqrt{b_1 a_1}}, P_{r1}^* = \frac{b_1 c_1 + e_1\left(1 - \sqrt{\frac{b_1}{a_1}}\right)}{e_1 + d_1\sqrt{\frac{b_1}{a_1}}}$$

$$c_1\sqrt{a_1 b_1} \leq d_1 + e, \qquad c_1 a_1 + d_1 > d_1\sqrt{b_1/a_1}, \qquad c_1 b_1 + e_1 > e_1\sqrt{a_1/b_1}$$

## 4.4 Stability

Now, we calculate stability of the aforementioned system of equation

$$\frac{dP_{r1}}{dt} = a_1 P_{r1} - a_1 P_{r1}^2 - P_{r1}P_{r3} + P_{r1}P_{r2}P_{r3}$$

$$\frac{dP_{r2}}{dt} = b_1 P_{r2} - b_1 P_{r2}^2 - P_{r2}P_{r3} + P_{r1}P_{r2}P_{r3}$$

$$\frac{dP_{r3}}{dt} = -c_1 P_{r3}^2 + d_1 P_{r1}(t-\tau)P_{r3} + e_1 P_{r2}(t-\tau)P_{r3}$$

Differentiation w.r.t. $P_{r1}$

$$m_1 = a_1 - 2a_1P_{r1} - P_{r3} + P_{r2}P_{r3}, \qquad m_2 = P_{r2}P_{r3}, \qquad m_3 = d_1P_{r3}e^{-\lambda\tau}$$

Differentiation w.r.t. $P_{r2}$

$$m_4 = P_{r1}P_{r3}, \qquad m_5 = b_1 - 2b_1P_{r2} - P_{r3} + P_{r1}P_{r3,} \qquad m_6 = e_1P_{r3}e^{-\lambda\tau}$$

Differentiation w.r.t. $P_{r3}$

$$m_7 = -P_{r1} + P_{r1}P_{r2}, \qquad m_8 = -P_{r2} + P_{r1}P_{r2}, \qquad m_9 = -2c_1P_{r3}$$

Let $v_1 = P_{r1} - P_{r1}^*, v_2 = P_{r2}P_{r2}^*$ and $v_3 = P_{r3}P_{r3}^*$then equations (1), (2) and (3) can be expressed in this form

$$\frac{dv_1}{dt} = -a_1P_{r1}^* - P_{r1}^*v_3 + P_{r1}^*P_{r2}^*v_3 + P_{r1}^*P_{r3}^*v_2 - a_1v_1^2 - (1 - P_{r2}^*)v_1v_3 + P_{r1}^*v_2v_3 + P_{r3}^*v_1v_2 + v_1v_2v_3 \qquad (7)$$

$$\frac{dv_2}{dt} = -b_1P_{r2}^*v_2 - P_{r2}^*v_3 + P_{r1}^*P_{r2}^*v_3 + P_{r2}^*P_{r3}^*v_1 - b_1v_2^2 - (1 - P_{r1}^*)v_2v_3 + P_{r2}^*v_1v_2 + P_{r3}^*v_1v_2 + v_1v_2v_3 \qquad (8)$$

$$\frac{dv_3}{dt} = -c_1P_{r3}^*v_3 - d_1P_{r3}^*v_1(t-\tau) + e_1P_{r3}^*v_2(t-\tau) - c_1v_3^2 - d_1v_1(t-\tau)v_3 + e_1v_2(t-\tau)v_3 \qquad (9)$$

The stability of the equilibrium $E^*(P_{r1}^*,\ P_{r2}^*,\ P_{r3}^*)$ can be examined by investigating the stability of the origin for equations (7), (8), and (9). Now, we compute the linearised system's characteristics equations (7), (8), and (9) at (0, 0, 0)

$$\begin{vmatrix} \lambda - m_1 & -m_2 & -m_3 \\ -m_4 & \lambda - m_5 & -m_6 \\ -m_7 & -m_8 & \lambda - m_9 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda - (a_1 - 2a_1P_{r1} - P_{r3} + P_{r2}P_{r3}) & -P_{r2}P_{r3} & -d_1P_{r3}e^{-\lambda\tau} \\ -P_{r1}P_{r3} & \lambda - (b_1 - 2b_1P_{r2} - P_{r3} + P_{r1}P_{r3}) & -e_1P_{r3}e^{-\lambda\tau} \\ P_{r1} - P_{r1}P_{r2} & P_{r2} - P_{r1}P_{r2} & \lambda + 2c_1P_{r3} \end{vmatrix} = 0$$

After simplification, we obtain the characteristic equation

$$\lambda^3 + X\lambda^2 + Y_1\lambda + e^{-\lambda\tau}(Y_2\lambda + Z_2) = 0 \tag{10}$$

When τ = 0, equation (10) becomes

$$\lambda^3 + X\lambda^2 + (Y_1 + Y_2)\lambda + Z_2 = 0 \tag{11}$$

The Routh–Hurwitz criteria implies that with $\tau$ = 0, the equilibrium point $E^*$ is locally asymptotically stable if

$$(\boldsymbol{H_1}) X > 0, \qquad (Y_1 + Y_2) > 0, \qquad Z_2 > 0, \qquad X(\,Y_1 + Y_2) > Z_2\ hold$$

Let us consider $(\boldsymbol{H_1})$ condition is fulfilled. Then, equation (11) with $\tau = \tau_j(j = 0, 1, \dots\dots)$ has only complex solution with conjugation $\pm i\omega_0$

$$\tau_j = \frac{1}{\omega_0}\left[arccos\frac{\omega_0^2(\omega_0^2Y_2 + Z_2X - Y_1Y_2)}{Y_2^2\omega_0^2 + Z_2^2} + 2j\pi\right]$$

We have the following conditions:

1. For all solution of equation (11) have real part with negative sign is that $\tau \epsilon\, [0, \tau_0)$.
2. For all solution of equation (11) have only pair of conjugated complex solution $\pm i\omega_0$ and real part with negative sign only if $\tau = \tau_0$.

If $\lambda = 0$ is a solution of (11) if $Z_2$ = 0. This condition found contrary to the third prerequisite in $(\boldsymbol{H_1})$, implying that $\lambda = 0$ is not corresponds to a solution of (11). So, assume that the condition when delay is greater and equal to zero i.e., $\tau \geq 0, and\ i\omega$ with $\omega > 0$ is a valid solution of (11).

$$-i\,\omega^3 - X\omega^2 + iY_1\omega + e^{-\omega\tau}(iY_2\omega + Z_2) = 0$$

$$-i\,\omega^3 - X\omega^2 + iY_1\omega + (cos\,\omega\tau - isin\,\omega\tau)\,(iY_2\omega + Z_2) = 0$$

$$i\,(-\omega^3 + Y_1\omega + Y_2\omega cos\,\omega\tau - Z_2 sin\,\omega\tau) + (-X\omega^2 + Z_2 cos\,\omega\tau + Y_2\omega sin\,\omega\tau) = 0$$

Separating real and imaginary parts

$$-\omega^3 + Y_1\omega + Y_2\omega cos\,\omega\tau - Z_2 sin\,\omega\tau = 0 \qquad (12)$$

$$-X\omega^2 + Z_2 cos\,\omega\tau + Y_2\omega sin\,\omega\tau = 0 \qquad (13)$$

which give $\omega^6 + \alpha\omega^4 + \beta\omega^2 + \gamma = 0$ (14)

where $\alpha = X^2 - 2Y_1, \beta = Y_1^2 - Y_2^2, \gamma = -Z_2^2$

Let $l = \omega^2$, then equation (14) becomes

$$l^3 + \alpha l^2 + \beta l + \gamma = 0 \qquad (15)$$

Supposed $l_1, l_2\ and\ l_3$ are the roots of equation (15) and connected by

Sum of the roots $l_3 + l_2 + l_1 = -\alpha$ (16)

Product of the roots $l_3\, l_2\, l_1 = -\gamma$ (17)

Thus, equation (17) has either only one or all three positive real roots.

Depending on the value of determinant $\Delta_1$of the equation (15)

where $\Delta_1 = \left(\frac{S}{2}\right)^2 + \left(\frac{T}{3}\right)^3$ and $T = \beta - \frac{1}{3}\alpha^2,\ S = \frac{2}{27}\alpha^3 - \frac{1}{3}\alpha\beta + \gamma$

Three situations are possible for the solution of (15):

a) If $\Delta_1 > 0$, then one real root and a pair of imaginary roots can be obtained for equation (15). When the real root is positive, it can be written as follows:

$$l_1 = \sqrt[3]{\frac{-S}{2} + \sqrt{\Delta_1}} + \sqrt[3]{\frac{-S}{2} - \sqrt{\Delta_1}} - \frac{1}{3}\alpha$$

b) If $\Delta_1 = 0$, all three real roots and two repeated roots are obtained for equation (15). If $\alpha > 0$, we obtain only one positive root, $l_1 = 2\sqrt[3]{\frac{-S}{2}} - \frac{1}{3}\alpha. If\ \alpha < 0$, we obtain only one positive root, $l_1 = 2\sqrt[3]{\frac{-S}{2}} - \frac{1}{3}\alpha$ for ,

$\sqrt[3]{\frac{-S}{2}} > -\frac{1}{3}\alpha$ and three positive real roots for $\frac{\alpha}{6} < \sqrt[3]{\frac{-S}{2}} < -\frac{1}{3}\alpha$, $l_1 = 2\sqrt[3]{\frac{-S}{2}} - \frac{1}{3}\alpha$, $l_2 = l_3 = -\sqrt[3]{\frac{-S}{2}} - \frac{1}{3}\alpha$

c) If $\Delta_1 < 0$, we obtain all the three roots are real and distinct, $l_1 = 2\sqrt{\frac{|T|}{3}}\cos\left(\frac{\xi}{3}\right) - \frac{\alpha}{3}$

$$l_2 = \sqrt{\frac{|T|}{3}}cos\left(\frac{\xi}{3} + \frac{2\pi}{3}\right) - \frac{\alpha}{3}, \qquad l_3 = 2\sqrt{\frac{|T|}{3}}\cos\left(\frac{\xi}{3} + \frac{4\pi}{3}\right) - \frac{\alpha}{3}$$

Where $\cos\xi = \left(-\frac{S}{2\sqrt{\left(\frac{|T|}{3}\right)^3}}\right)$, $0 < \xi < \pi$. Moreover, if $\alpha > 0$, one and only one positive solution exists. Otherwise, if $\alpha < 0$, we obtain all three real positive roots or only one positive real solution. It is equivalent to max $(l_1, l_2, l_3)$ only when we obtain one positive real root. The number of positive real roots depends on the sign of $\alpha$. Equation (15) has only one positive real root when $\alpha \geq 0$ is present. Otherwise, we obtained three positive real roots. When $\alpha = X^2 - 2Y_1 > 0$, one positive real root is obtained for (15). Let the obtained positive real root be denoted by symbol $l_0$. Then, equation (14) would have only one positive real root $\omega_0 = \sqrt{l_0}$. From equation (13), we have

$$cos\omega_0\tau = \frac{\omega_0^2(\omega_0^2 Y_2 + Z_2 X - Y_1 Y_2)}{Y_2^2\omega_0^2 + Z_2^2}$$

Express

$$\tau_j = \frac{1}{\omega_0}\left[arccos\frac{\omega_0^2(\omega_0^2 Y_2 + Z_2 X - Y_1 Y_2)}{Y_2^2\omega_0^2 + Z_2^2} + 2j\pi\right] \tag{18}$$

where $j = 0, 1, 2, 3, 4, 5, \ldots..$ $\pm i\omega_0$ obtained a root of equation (10) when $\tau = \tau_j$. Furthermore, if $(H_1)$ standards are fulfilled, all the roots of equation (10) with $\tau = 0$ have negative real values. We determine the outcomes of lemma1 by summarising the preceding discussion and using the lemma provided. The proof is completed with the following outcomes from theorem and lemma.

**4.5 Theorem1** Assume the condition in (H1) is fulfilled. If $\tau \epsilon [0, \tau_0)$, then the zero solution of equations (7), (8), and (9) is asymptotically stable.

For delayed functional differential equations, the following components may be found by applying the conventional Hopf bifurcation theorem.

**4.6 Lemma 1** Let $n(l_0) = (3l_0^2 + 2\alpha l_0 + \beta) \neq 0$ and condition in $(H_1)$ are satisfied. For $(j = 0, 1, \dots)$, $\lambda(\tau) = \delta(\tau) + i\omega(\tau)$ is denoted as the solution of equation (10) that fulfils the condition $\delta(\tau_j) = 0, \omega(\tau_j) = \omega_0$ were

$$\tau_j = \frac{1}{\omega_0}\left[arccos\frac{\omega_0^2(\omega_0^2 Y_2 + Z_2 X - Y_1 Y_2)}{Y_2^2\omega_0^2 + Z_2^2} + 2j\pi\right]$$

Then, $\pm i\omega_0$ are pair of simple roots. If the transversality condition $(H_1)$ $\delta'(\tau_J) = \frac{Re\,\lambda(\tau)}{d\tau}\Big|_{\lambda = i\omega_0} \neq 0$ holds good, we obtain a Hopf bifurcation for (7), (8) and (9) at $v = 0 \; and \; \tau = \tau_j$.

Proof. Assume $\lambda = \lambda\,(\tau)$ is a solution of equation (10). Put $\lambda\,(\tau)$ in (10) and differentiating with respect to $\tau$ on both sides, we get

$$\left[(3\lambda^2 + 2X\lambda + Y_1) + \big((\lambda Y_2 + Z_2)(-\tau) + Y_2\big)e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = \lambda(\lambda Y_2 + Z_2)e^{-\lambda\tau}$$

Thus

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2X\lambda + Y_1)e^{\lambda\tau}}{\lambda(\lambda Y_2 + Z_2)} + \frac{Y_2}{\lambda(\lambda Y_2 + Z_2)} - \frac{\tau}{\lambda}$$

From (12)– (15), we have

$$\begin{aligned}
-\alpha'(\tau_J) &= Re\left[\frac{(3\lambda^2 + 2X\lambda + Y_1)e^{\lambda\tau}}{\lambda(\lambda Y_2 + Z_2)}\right] + Re\left[\frac{Y_2}{\lambda(\lambda Y_2 + Z_2)}\right] \\
&= \frac{1}{\Omega}[3\omega_0^6 + 2(X^2 - 2Y_1)\omega_0^4 + (Y_1^2 - 2XZ_2 - Y_2^2)\omega_0^2] \\
&= \frac{1}{\Omega}(3\omega_0^6 + 2\alpha\omega_0^4 + \beta\omega_0^2) \\
&= \frac{\omega_0^2}{\Omega}(3\omega_0^4 + 2\alpha\omega_0^2 + \beta)
\end{aligned}$$

$$= \frac{\omega_0^2}{\Omega}(3l_0^2 + 2\alpha l_0 + \beta)$$

$$= \frac{\omega_0^2}{\Omega} n(l_0)$$

where $\Omega = Y_2^2 \omega_0^2 + Z_2^2$

$n(l_0) = 3l_0^2 + 2\alpha l_0 + \beta$. Observed, when $\Omega > 0$ and $\omega_0 > 0$

We find that

Sign $[\delta'(\tau_J)] = sign[n(l_0)]$ proves the theorem.

## 4.7 Direction Analysis and Stability of The Hopf Bifurcation Solution

As we showed in the last section, a set of solutions may be determined as bifurcates from the favourable steady state $E^*$ at a critical level of $\tau$. These bifurcating periodic solutions' direction, stability, and period need to be identified. In this part, we use normal form theory and the centre manifold theorem at the critical point $\tau_j$ to construct accurate equations characterising the characteristics of the Hopf bifurcation at the specific critical value.

Normalizing delay value $\tau$ by the time scaling $t \longrightarrow \frac{t}{\tau}$ system (7), (8), and (9) is transformed into

$$\frac{dv_1}{dt} = -a_1 P_{r1}^* v_1 - P_{r1}^* v_3 + P_{r1}^* P_{r2}^* v_3 + P_{r1}^* P_{r3}^* v_2 - a v_1^2 - (1 - P_{r2}^*) v_1 v_3 + P_{r1}^* v_2 v_3 + P_{r3}^* v_1 v_2 + v_1 v_2 v_3 \qquad (19)$$

$$\frac{dv_2}{dt} = -b_1 P_{r2}^* v_2 - P_{r2}^* v_3 + P_{r1}^* P_{r2}^* v_3 + P_{r2}^* P_{r3}^* v_1 - b v_2^2 - (1 - P_{r1}^*) v_2 v_3 + P_{r2}^* v_1 v_3 + P_{r3}^* v_1 v_2 + v_1 v_2 v_3 \qquad (20)$$

$$\frac{dv_3}{dt} = -c_1P_{r3}^*v_3 + d_1P_{r3}^*v_1(t-1) + e_1P_{r3}^*v_2(t-1)$$
$$- c_1v_3^2 + d_1v_1(t-1)v_3 + e_1v_2(t-1)v_3 \qquad (21)$$

Take phase plane $C = C((-1,0), R_+^3)$. WLOG, denote the critical value $\tau_j$ by $\tau_0$. If $\tau = \tau_0 + \sigma$, then $\sigma = 0$ is a value for Hopf bifurcation for equations (19)– 21). To facilitate notational ease, we construct (19)– (21) in this manner.

$$v'(t) = L_\sigma(v_t) + G(\sigma, v_t) \qquad (22)$$

where $v(t) = v_1(t), v_2(t), v_3(t) \, \epsilon \, R^3, v_t(\theta) = v(t+\theta)$ and

$$L_\sigma\varphi = (\tau_0+\sigma)\begin{bmatrix} -a_1P_{r1}^* & P_{r1}^*P_{r3}^* & -P_{r1}^* + P_{r1}^*P_{r2}^* \\ P_{r2}^*P_{r3}^* & -b_1P_{r2}^* & -P_{r2}^* + P_{r1}^*P_{r2}^* \\ 0 & 0 & -c_1P_{r3}^* \end{bmatrix}\begin{bmatrix} \varphi_1\,(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{bmatrix} +$$

$$(\tau_0+\sigma)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_1P_{r3}^* & e_1P_{r3}^* & 0 \end{bmatrix}\begin{bmatrix} \varphi_1(-1) \\ \varphi_2(-1) \\ \varphi_3(-1) \end{bmatrix} \text{ and}$$

$$G(\sigma,\varphi) = (\tau_0+\sigma)\begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \text{respectively, were}$$

$$G_1 = -a_1\varphi_1^2(0) - (1-P_{r2}^*)\varphi_1(0)\varphi_3(0) + P_{r1}^*\varphi_2(0)\varphi_3(0)$$
$$+ P_{r3}^*\varphi_2(0)\varphi_1(0) + \varphi_3(0)\varphi_2(0)\varphi_1(0),$$
$$G_2 = -b_1\varphi_1^2(0) - (1-P_{r1}^*)\varphi_3(0)\varphi_2(0) + P_{r2}^*\varphi_3(0)\varphi_1(0)$$
$$+ P_{r3}^*\varphi_2(0)\varphi_1(0) + \varphi_3(0)\varphi_2(0)\varphi_1(0),$$
$$G_3 = -c_1\varphi_3^2(0) + d_1\varphi_1(-1)\varphi_3(0) + c_1\varphi_2(-1)\varphi_3(0),$$
$$\varphi(0) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta))^T \epsilon \, C(C-1,0), R).$$

Reisz representation theorem utilisation, we can find a function $\sphericalangle(\theta,\sigma)$ of bounced variation for $\theta \, \epsilon \, [-1, 0)$ as

$$L_\sigma\varphi = \int_{-1}^{0} d\sphericalangle(\theta, 0)\,\varphi(0) for\ \varphi \, \epsilon \, C.$$

We choose

$$\sphericalangle(\theta,\sigma) = (\tau_0+\sigma)\begin{bmatrix} -a_1P_{r1}^* & P_{r1}^*P_{r3}^* & -P_{r1}^* + P_{r2}^*P_{r1}^* \\ P_{r2}^*P_{r3}^* & -b_1P_{r2}^* & -P_{r2}^* + P_{r2}^*P_{r1}^* \\ 0 & 0 & -c_1P_{r3}^* \end{bmatrix}\chi(\theta) +$$

$$(\tau_0+\sigma)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_1P_{r3}^{*} & e_1P_{r3}^{*} & 0 \end{bmatrix}\chi(\theta+1)$$

where $\chi$ denotes the Delta Dirac rule for any $\varphi\,\epsilon\,C([-1,0],R_{+}^{3})$,

Let us define a function

$$A(\sigma)\varphi=\begin{cases} d\varphi(0) & \theta\,\epsilon\,[-1,0) \\ \int_{-1}^{0} d\sphericalangle(\theta,\varphi) & \theta=0 \end{cases}$$

$$\mathrm{H}\,(\sigma)\,\varphi=\begin{cases} 0 & \theta\,\epsilon[-1,0) \\ G(\sigma,\varphi) & \theta=0 \end{cases}$$

Then, the system (22) is equivalent to

$$v_t'=X(\sigma)v_t+H(\sigma)v_t \tag{23}$$

For $\varepsilon\,\epsilon\,C'([-1,0],R_{+}^{3})$, define

$$X^{*}\epsilon(s)=\begin{cases} \dfrac{-d\epsilon(s)}{d\theta} & s\,\epsilon\,[-1,0, \\ \int_{-1}^{0} d\sphericalangle^{T}\,(-1,0)\varepsilon(-t) & s=0 \end{cases}$$

The bilinear inner product is as follows:

$$<\varepsilon(s),\varphi(\theta)>=\in(0)\,\varphi(0)-\int_{-1}^{0}\int_{v=0}^{0}\epsilon(v-\theta)d\sphericalangle(\theta)\varphi(v)dv \tag{24}$$

$X^{*}\ and\ X(0)$are adjoint operators; thus, $i\omega_0$ are the eigen values of $X(0)$. They are the eigen values of $X^{*}$. Suppose that $\beta(\theta)=\beta(0)e^{i\omega_{0}\theta}$is an eigen vector of $X(0)$ corresponding to the eigenvalue $i\omega_{0.}$ Then, $X(0)=i\omega_0\beta(\theta)$. When $\theta=0$, we obtain

$$\left[i\omega_0 I-\int_{-1}^{0} d\sphericalangle(\theta)e^{i\omega_0\theta}\right]\beta(0)=0$$

which yields $\beta(0)-(1,x_1y_1)^{T}$,

where

$$x_1=\frac{(P_{r1}^{*}-P_{r2}^{*}P_{r1}^{*})P_{r3}^{*}P_{r2}^{*}+(P_{r2}^{*}-P_{r2}^{*}P_{r1}^{*})(i\omega_0+a_1P_{r1}^{*})}{P_{r1}^{*}P_{r3}^{*}(P_{r2}^{*}-P_{r2}^{*}P_{r1}^{*})-(P_{r1}^{*}-P_{r2}^{*}P_{r1}^{*})(i\omega_0+P_{r2}^{*}b_1)}$$

$$y_1 = \frac{P_{r2}^{*}P_{r3}^{*2}P_{r1}^{*} - (i\omega_0 + a_1P_{r1}^{*})(i\omega_0 + b_1P_{r2}^{*})}{P_{r1}^{*}P_{r3}^{*}(P_{r2}^{*} - P_{r2}^{*}P_{r1}^{*}) - (P_{r1}^{*} - P_{r2}^{*}P_{r1}^{*})(i\omega_0 + P_{r2}^{*}b_1)}$$

Similarly, it can be verified that $\beta^{*}(s) = D(1, x_2y_2)e^{i\omega_0\tau_0 s}$is the eigen vector of $X^{*}$corresponding to $-i\omega_{0,}$

where

$$x_2 = \frac{P_{r2}^{*}P_{r3}^{*}(P_{r1}^{*} - P_{r2}^{*}P_{r1}^{*}) + (P_{r2}^{*} - P_{r2}^{*}P_{r1}^{*})(a_1P_{r1}^{*} - i\omega_0)}{P_{r1}^{*}P_{r3}^{*}(P_{r2}^{*} - P_{r1}^{*}P_{r2}^{*}) - (P_{r1}^{*} - P_{r1}^{*}P_{r2}^{*})(b_1P_{r2}^{*} - i\omega_0)}$$

$$y_2 = \frac{P_{r1}^{*}P_{r2}^{*}P_{r3}^{*2} - (a_1P_{r1}^{*} - i\omega_0)(b_1P_{r2}^{*} - i\omega_0)}{P_{r1}^{*}P_{r3}^{*}(P_{r2}^{*} - P_{r2}^{*}P_{r1}^{*}) - (P_{r1}^{*} - P_{r2}^{*}P_{r1}^{*})(b_1P_{r2}^{*} - i\omega_0)}$$

To assume $< \beta^{*}(s), \beta(\theta) \geq 1$, we have to calculate the value of D. From [24], we obtain $< \beta^{*}(s), \beta(\theta) >$

$$= \bar{D}(1, \bar{x}_2, \bar{y}_2)(1, x_1, y_1)^T - \int_{-1}^{0}\int_{v=0}^{\theta} \bar{D}(1, \bar{x}_2, \bar{y}_2)\, e^{-i\omega_0\tau_0(v-\theta)} d$$

$$\sphericalangle\, (\theta)(1, x_1, y_1)^T e^{-\omega_0\tau_0} dv$$

$$= \bar{D}\{1 + x_1\bar{x}_2 + y_1\bar{y}_2 - \int_{-1}^{0} (1, \bar{x}_2, \bar{y}_2)\, \theta e^{i\omega_0\tau_0\theta} d \sphericalangle (\theta)(1, x_1, y_1)^T\}$$

$$= \bar{D}\left\{1 + x_1\bar{x}_2 + y_1\bar{y}_2 - \tau_0\, \bar{y}_2 P_{r3}^{*}(d_1x_1 + e_1y_2)e^{i\omega_0\tau_0}\right\}$$

Hence, we can choose

$$\bar{D} = \frac{1}{1 + x_1\bar{x}_2 + y_1\bar{y}_2 + \tau_0\bar{y}_2P_{r3}^{*}(d_1\sigma_{1+}e_1y)e^{i\omega_0\tau_0}}$$

Such that

$$< \beta^{*}(s), \beta(\theta) \geq 1, < \beta^{*}(s), \beta(\theta) = 0$$

Continue the coordinates defining the vector by following the algorithm and using the same notations as their manifold $c_0$ at $\sigma = 0$. Let $v_t$ be a solution of equation (23) with $\sigma = 0$. Define

$$m(t) = < \beta^{*}(s), v_t(\theta) > \tag{25}$$

$$V(t, \theta) = v_t(\theta) - 2Re\, m(t)\beta(\theta) \tag{26}$$

According to manifold, we obtain centre $C_0$. Accordingly,

$$V(t,\theta) = V(m(t)\,\bar{m}(t),\theta),$$

where

$$V(m,\bar{m},\theta) = V_{20}(\theta)\frac{m^2}{2} + V_{11}(\theta)m\bar{m} + V_{02}(\theta)\frac{\bar{m}^2}{2} + \cdots$$

$m$ and $\bar{m}$ are local values for the manifold centre $C_0$ in the direction of $\beta^*$ and $\bar{\beta}^*$. When $V$ is real, $v_t$ is real. We assume only the real solution. For solution $v_t \in C_0$ of (23), since $\sigma = 0$,

$$m'(t) = i\omega_0\tau_0 m + < \bar{\beta}^*(\theta), G(0, V(m,\bar{m},\theta) + 2Re\{m(t)\beta(\theta)\}) >$$

$$= i\omega_0\tau_0 m + \bar{\beta}^*(0)G(0, V(m,\bar{m},0) + 2Re\{m(t)\beta(\theta)\})$$

$$= i\omega_0\tau_0 m + \bar{\beta}^*(0)\, G_0(m,\bar{m}) \tag{27}$$

where $s(m,\bar{m}) = \bar{\beta}^*(0)\ \ G_0(m,\bar{m})$

$$= s_{20}(\theta)\frac{m^2}{2} + s_{11}(\theta)m\bar{m} + s_{02}(\theta)\frac{\bar{m}^2}{2} + s_{21}\frac{m^2\bar{m}}{2} + \cdots \tag{28}$$

Noticing

$$v_t(\theta) = (v_{it}, v_{2t}, v_{3t}) = V(t,\theta) + m\beta(\theta) + \bar{m}\bar{\beta}(\theta)$$

and $\beta(0) = (1, x_1, y_1)^T e^{i\omega_0\tau_0\theta}$, we have

$$v_{1t}(0) = m + \bar{m} + V_{20}^{(1)}\frac{m^2}{2} + V_{11}^{(1)}(0)m\bar{m} + V_{02}^{(1)}(0)\frac{\bar{m}^2}{2} + \cdots,$$

$$v_{2t}(0) = x_1 m + \bar{x}_1\bar{m} + V_{20}^{(2)}\frac{m^2}{2} + V_{11}^{(2)}(0)m\bar{m} + V_{02}^{(2)}(0)\frac{\bar{m}^2}{2} + \cdots,$$

$$v_{3t}(0) = y_1 m + \bar{y}_1\bar{m} + V_{20}^{(3)}(0)\frac{m^2}{2} + V_{11}^{(3)}(0)m\bar{m} + V_{02}^{(3)}(0)\frac{\bar{m}^2}{2} + \cdots,$$

$$v_{1t}(-1) = me^{-i\omega_0\tau_0} + \bar{m}e^{i\omega_0\tau_0} + V_{20}^{(1)}(-1)\frac{m^2}{2} + V_{11}^{(1)}(-1)m\bar{m}$$

$$+ V_{02}^{(1)}(-1)\frac{\bar{m}^2}{2} + \cdots,$$

$$v_{2t}(-1) = x_1 e^{-i\omega_0\tau_0} + \bar{x}e^{i\omega_0\tau_0}\bar{m} + V_{20}^{(2)}(-1)\frac{m^2}{2} + V_{11}^{(2)}(-1)m\bar{m}$$

$$+ V_{02}^{(2)}(-1)\frac{\bar{m}^2}{2} + \cdots,$$

Comparing coefficients with (28), we have

$$
\begin{aligned}
&s_{20} = -2\tau_0\overline{D}[a_1 + (1-P^*_{r2})y_1 - x_1(P^*_{r1}y_1 + P^*_{r3}) + \bar{x}_2(b_1x_1^2 + \\
&(1-P^*_{r1})x_1y_1) - x_1P^*_{r3} - y_1P^*_{r2} + \bar{y}_2y_1\big(c_1y_1 - d_1e^{-i\omega_0\tau_0} - e_1x_1e^{-i\omega_0\tau_0}\big)],
\end{aligned}
$$

$$
\begin{aligned}
&s_{11} = -2\tau_0\overline{D}[a_1 + (1-P^*_{r2})Re\{y_1\} - P^*_{r1}Re\{\bar{y}_1x_1\} - P^*_{r3}Re\{x_1\} + \\
&\bar{x}_2(x_1\bar{x}_1b_1 + (1-P^*_{r1})Re\{x_1\bar{y}_1\} - P^*_{r2}Re\{\bar{y}_1\} - P^*_{r3}\{x_1\} + \bar{y}_2(c_1y_1\bar{y}_1 - \\
&d_1Re\big\{y_1e^{i\omega_0\tau_0}\big\} - e_1Re\big\{y_1\bar{x}_1e^{i\omega_0\tau_0}\big\})],
\end{aligned}
$$

$$
\begin{aligned}
s_{02} = -2\tau_0\overline{D}\big[&a_1 + (1-P^*_{r2})\bar{y}_1 - \bar{x}_1(P^*_{r1}\bar{y}_1 + P^*_{r3}) \\
&+ \bar{x}_2(b_1\bar{x}_1^2 + (1-P^*_{r1})\bar{x}_1\bar{y}_1) - \bar{x}_1P^*_{r3} - \bar{y}_1P^*_{r2} \\
&+ \bar{y}_2\bar{y}_1\big(c_1\bar{y}_1 - d_1e^{-i\omega_0\tau_0}\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
&s_{21} = -2\tau_0\overline{D}[a_1\Big(V_{20}^{(1)}(0) + 2V_{11}^{(1)}(0)\Big) + (1-P^*_{r2})(\tfrac{1}{2}\,V_{20}^{(1)}(0)\bar{y}_1 + \\
&V_{11}^{(1)}(0)y_1 + \tfrac{1}{2}\,V_{20}^{(3)}(0) + V_{11}^{(3)}(0) - (2Re\{x_1\bar{y}_1\} + x_1y_1) - \\
&P^*_{r1}\Big(\tfrac{1}{2}\,W_{20}^{(2)}(0)\bar{y}_1 + \tfrac{1}{2}\,V_{20}^{(3)}(0)\bar{x}_1 + V_{11}^{(2)}(0)y_1 + V_{11}^{(3)}(0)x_1\Big) - \\
&P^*_{r3}\Big(\tfrac{1}{2}\,V_{20}^{(2)}(0) + \tfrac{1}{2}V_{20}^{(1)}(0)\bar{x}_1 + V_{11}^{(2)}(0) + V_{11}^{(1)}(0)x_1\Big) + \bar{x}_2\Big(b_1V_{20}^{(2)}(0)\bar{x}_1 + \\
&2V_{11}^{(2)}(0)x_1\Big) + (1-P^*_{r1})(\tfrac{1}{2}V_{20}^{(2)}(0)\bar{y}_1 + V_{11}^{(3)}(0)x_1 - (2Re\{x_1\bar{y}_1\} + x_1y_1) - \\
&P^*_{r2}\Big(\tfrac{1}{2}V_{20}^{(1)}(0)\bar{y}_1 + \tfrac{1}{2}\,V_{20}^{(3)}(0) + V_{11}^{(1)}(0)y_1 + V_{11}^{(1)}(0)y_1 + V_{11}^{(3)}(0) - \\
&P^*_{r3}\Big(\tfrac{1}{2}\,V_{20}^{2}(0) + \tfrac{1}{2}\,V_{20}^{(1)}(0)\bar{x}_1 + V_{11}^{(2)}(0) + V_{11}^{(1)}(0)x_1\Big)\Big) + \bar{y}_2(c_1\Big(V_{20}^{(3)}(0)\bar{y}_1 + \\
&2V_{11}^{(3)}(0)y_1\Big) - d_1\Big(\tfrac{1}{2}V_{20}^{(1)}(-1)\bar{y}_1 + V_{11}^{(1)}(-1)y_1 + \tfrac{1}{2}V_{20}^{(3)}(0)e^{i\omega_0\tau_0} + \\
&V_{11}^{(3)}(0)e^{-i\omega_0\tau_0}\Big) - e_1(\tfrac{1}{2}V_{20}^{(1)}(-1)\bar{y}_1 + V_{11}^{(2)}(-1)y_1 + \tfrac{1}{2}V_{20}^{(3)}(0)\bar{x}_1e^{i\omega_0\tau_0} + \\
&V_{11}^{(3)}x_1e^{-i\omega_0\tau_0}))]
\end{aligned}
$$

Because of the presence of $V_{20}(\theta)$ and $V_{11}(\theta)$ in $s_{21}$, we need to further compute them. From (23) and (26), we have

$$
\begin{aligned}
V' &= v_t' - m'\beta - \overline{m}'\beta \\
&= \begin{cases} XV - 2Re\big\{\bar{\beta}^*(0)G_0\beta(\theta)\big\}, & \theta \in [-1,0), \\ XV - 2Re\big\{\bar{\beta}^*(0)G_0\beta(0)\big\} + G_0 & \theta = 0 \end{cases} \\
&\triangleq XV + N(m,\overline{m},\theta),
\end{aligned}
$$

where

$$N(m, \overline{m}, \theta) = N_{20}(\theta)\frac{m^2}{2} + N_{11}(\theta)m\overline{m} + N_{02}(\theta)\frac{\overline{m}^2}{2} + N_{21}\frac{m^2\overline{m}}{2}$$

$$+ \cdots \qquad (29)$$

On the other hand, on $C_0$near the origin

$$V' = V_m m' + V_{\overline{m}}\overline{m}'$$

Expanding the aforementioned series and comparing the coefficient, we obtain

$$[X - 2i\omega_0 I]V_{20}(\theta) = -N_{20}(\theta) \qquad (30)$$

$$XV_{11}(\theta) = -N_{11}(\theta) \qquad (31)$$

From (27), we know that for $\theta \epsilon [-1, 0)$,

$$N(m, \overline{m}, \theta) = -\beta^{\bar{y}}(0)\bar{G}_0\beta(\theta) - \beta^*(0)\bar{G}_0\bar{\beta}(\theta) = -s\beta(\theta) - \bar{s}\bar{\beta}(\theta).$$

Comparing the coefficient with (30), we obtain $\theta \epsilon [-1, 0]$ that

$$N_{20}(\theta) = -s_{20}\beta(\theta) - \bar{s}_{02}\bar{\beta}(\theta)$$

$$N_{11}(\theta) = -s_{11}\beta(\theta) - \bar{s}_{11}\bar{\beta}(\theta)$$

From (29), (30), and (31) and the definition of $X$, we obtain

$$V_{20}(\theta) = 2i\omega_0\tau_0 V_{20}(\theta) + s_{20}\beta(\theta) + \bar{s}_{02}\bar{\beta}(\theta)$$

Solving for $V_{20}(\theta)$, we obtain

$$V_{20}(\theta) = \frac{is_{20}}{\omega_0\tau_0}\beta(0)e^{i\omega_0\tau_{0\theta}} + \frac{i\bar{s}_{02}\bar{\beta}(0)}{3\omega_0\tau_0}e^{-i\omega_0\tau_0\theta} + P_1 e^{2i\omega_0\tau_0\theta}$$

Similarly,

$$V_{11}(\theta) = \frac{-is_{11}}{\omega_0\tau_0}\beta(0)e^{i\omega_0\tau_{0\theta}} + \frac{i\bar{s}_{11}\bar{\beta}(0)}{\omega_0\tau_0}e^{-i\omega_0\tau_0\theta} + P_2$$

Where, by putting $\theta = 0$ in N, it is possible to calculate the three-dimensional vectors $P_1$and $P_z$. Accordingly, we obtain

$$N(m, \overline{m}, \theta) = -2Re\{\bar{\beta}^*(0)G_0\beta(0)\} + G_0$$

when

$$N_{20}(\theta) = -s_{20}\beta(\theta) - \bar{s}_{02}\bar{\beta}(\theta) + G_{m^2}$$

$$N_{11}(\theta) = -s_{11}\beta(\theta) - \bar{s}_{11}\bar{\beta}(\theta) + F_{m\overline{m}}$$

where

$$G_0 = G_{m^2}\frac{m^2}{2} + G_{m\bar{m}}m\bar{m} + G_{\bar{m}^2}\frac{\bar{m}^2}{2} + \cdots$$

Combining the definition of $X$, we obtain

$$\int_{-1}^{0} d \sphericalangle (\theta) V_{20}(\theta) = 2i\omega_0\tau_0 V_{20}(0) + s_{20}\beta(0) + \bar{s}_{02}\bar{\beta}(0) - G_{m^2}$$

and

$$\int\limits_{-1}^{0} d \sphericalangle (\theta) V_{11}(\theta) = s_{11}\beta(0) - \bar{s}_{11}\bar{\beta}(0) - G_{m\bar{m}}$$

Notice that

$$i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta} d \sphericalangle (\theta))\beta(0) = 0$$

and

$$-i\omega_0\tau_0 I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}\ d \sphericalangle (\theta))\bar{\beta}(0) = 0$$

We have

$$2i\omega_0\tau_0 I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta} d \sphericalangle (\theta))P_1 = G_{m^2}$$

Similarly, we have

$$-\left(\int_{-1}^{0} d \sphericalangle (\theta)\right) P_2 = G_{m\bar{m}}$$

Hence, we obtain

$$\begin{bmatrix} 2i\omega_0 + a_1 P_{r1}^* & -P_{r1}^* P_{r3}^* & P_{r1}^* - P_{r2}^* P_{r1}^* \\ -P_{r3} P_{r2}^* & 2i\omega_0 + b_1 P_{r2}^* & P_{r2}^* - P_{r2}^* P_{r1}^* \\ -d_1 P_{r3}^* e^{-2i\omega_0\tau_0} & -e_1 P_{r3}^* e^{-2i\omega_0\tau_0} & 2i\omega_0 + c_1 P_{r3}^* \end{bmatrix} P_1 =$$
$$-2\begin{bmatrix} a_1 + (1 - P_{r2}^*)y_1 - x_1(P_{r1}^* y_1 + P_{r3}^*) \\ b_1 x_1^2 + (1 - P_{r1}^*)x_1 y_1) - x_1 P_{r3}^* - y_1 P_{r2}^* \\ y_1(c_1 y_1 - d_1 e^{-i\omega_0\tau_0} - e_1 x_1 e^{-i\omega_0\tau_0}) \end{bmatrix}$$

and

$$\begin{bmatrix} a_1 P_{r1}^* & -P_{r1}^* P_{r3}^* & -P_{r2}^* P_{r1}^* + P_{r1}^* \\ -P_{r2}^* P_{r3} & b_1 P_{r2}^* & -P_{r2}^* P_{r1}^* + P_{r2}^* \\ -d_1 P_{r3}^* & -e_1 P_{r3}^* & c_1 P_{r3}^* \end{bmatrix} P_2$$

$$= -2\begin{bmatrix} a_1 + (1 - P_{r2}^*)Re\{y_1\} - P_{r1}^* Re\{\bar{y}_1 x_1\} - P_{r3}^* Re\{x_1\} \\ x_1\bar{x}_1 b_1 + (1 - P_{r1}^*)Re\{x_1\bar{y}_1\} - P_{r2}^* Re\{\bar{y}_1\} - P_{r3}^* Re\{x_1\}) \\ c_1 y_1\bar{y}_1 - d_1 Re\{y_1\} - e_1 Re\{y_1\bar{x}_1\}e^{i\omega_0\tau_0}) \end{bmatrix}$$

Then, $s_{21}$can be denoted by the variables.

We determined that $s_{ij}$can be calculated using the variables. Thus, we computed these quantities as follows:

$$Z_2(0) = \frac{i}{2\omega_0\tau_0}\left(s_{11}s_{20} - 2|s_{11}|^2 - \frac{|s_{02}|^2}{3}\right) + \frac{s_{21}}{2} \tag{32}$$

$$\sigma_2 = -\frac{Re\{Z_2(0)\}}{Re\{\lambda'(\tau_0)\}} \tag{33}$$

$$\beta_2 = 2Re\{Z_2(0)\}$$

$$T_2 = -\frac{I_m\{Z_2(0)\} + \sigma_2 I_m\{\lambda'(Z_0)\}}{\tau_0\omega_0} \tag{34}$$

Theorem. $\sigma_2$ calculates the direction of the Hopf bifurcation: if $\sigma_2$ <0$(\sigma_2 > 0)$, we obtain the supercritical Hopf bifurcation. When $\tau > \tau_0$ $(\tau < \tau_0)$, we observed the bifurcating period solutions. $P_2$ indicates that the bifurcating periodic solution is stable. If $\beta_2 < 0$ $(\beta_2 > 0)$, we observe that bifurcating periodic solutions are arbitrary and asymptotically stable (unstable). The bifurcating periodic solution is determined by $T_2$. When $T_2$> 0 ($T_2$ < 0), the period increases (decreases), respectively.

## 4.8 Numerical Example

In this part, we used MATLAB to perform a numerical simulation of the system (Poole, 1974; Abrams et al., 1993). We use these parametric values:

**Set 1**

($a_1$ = 1.2; $b_1$ = 1.4; $c_1$ = 1; $d_1$= 1; $e_1$ = 2)

We can observe the positive interior equilibrium point when the initial value is 0.2, 0.4, or 0.6.

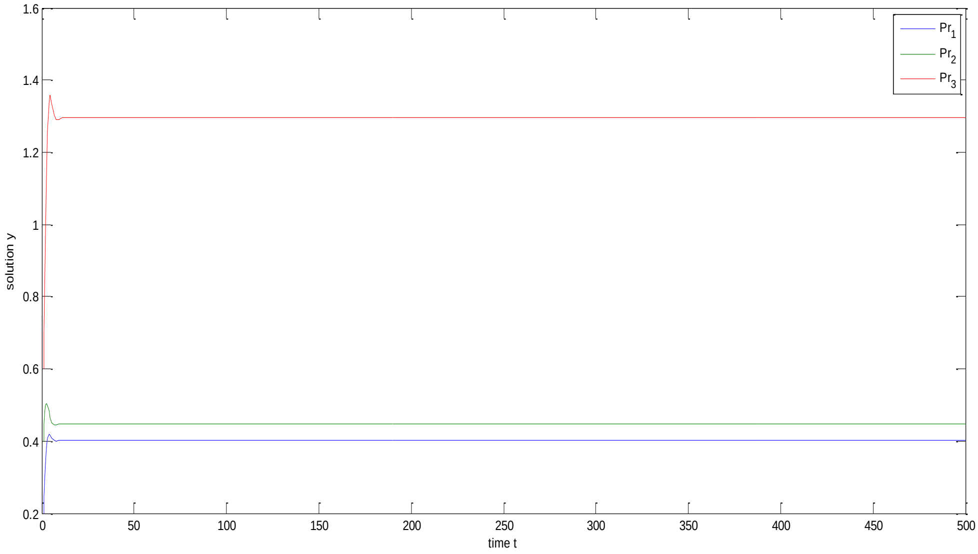

**Figure 4.1** When the value of delay is zero i.e., $\tau = 0$, the system is stable

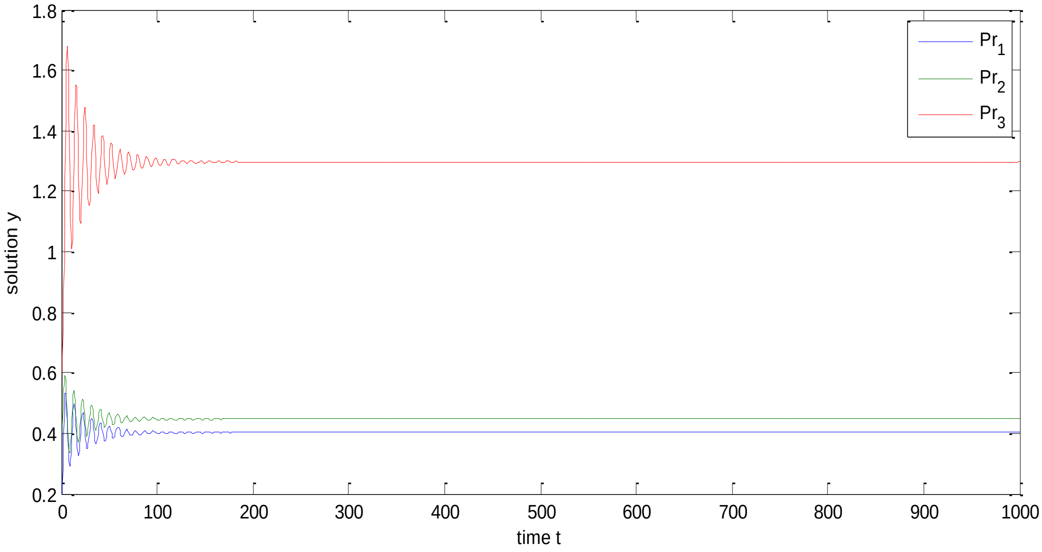

**Figure 4.2** System show Asymptotically stable behaviour when value of delay is less then threshold point$\tau = 1.5 < \tau_0 = 1.7387$

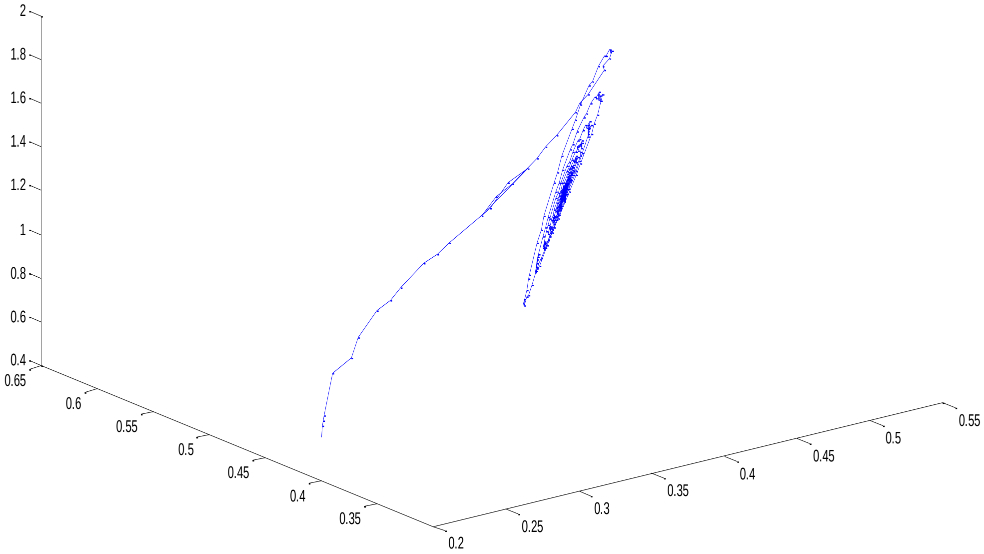

**Figure 4.3** Phase plane graph for asymptotically stable when $\tau = 1.5 < \tau_0 = 1.7387$

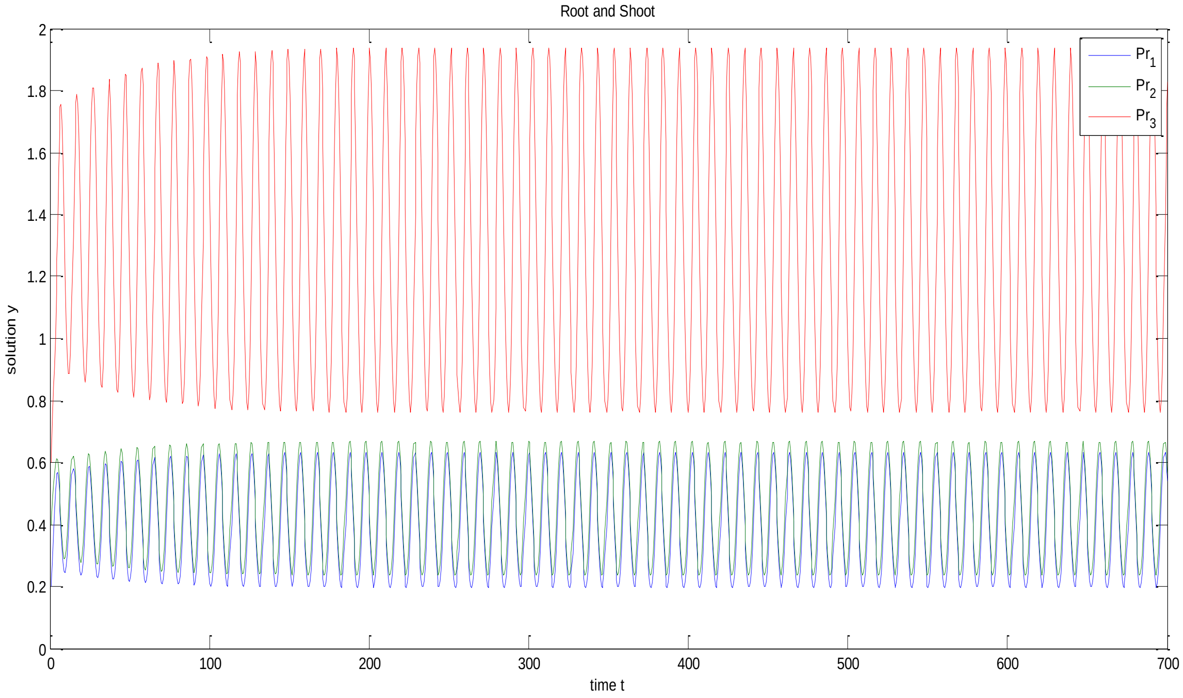

**Figure 4.4** Hopf Bifurcation when $\tau = 1.85 > \tau_0 = 1.7387$

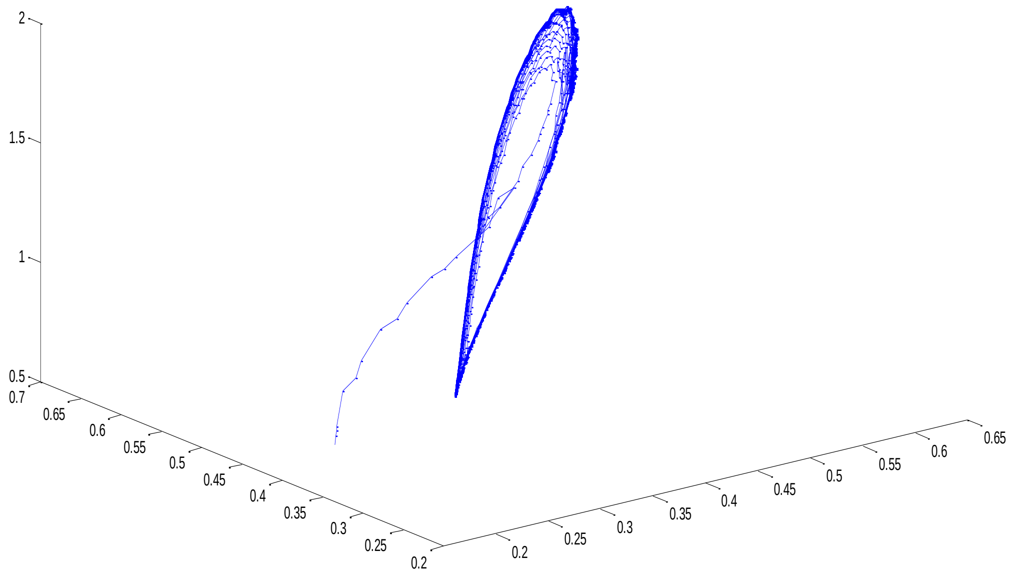

**Figure 4.5** Phase plane graph for Hopf Bifurcation when $\tau = 1.85 > \tau_0 = 1.7387$

**Set 2 (**$a_1 = 1$; $b_1 = 1.44$; $c_1 = 1$; $d_1 = 1$; $e_1 = 1.2$)

The positive interior equilibrium point is obtained when the initial value is 0.2, 0.4, and 0.6.

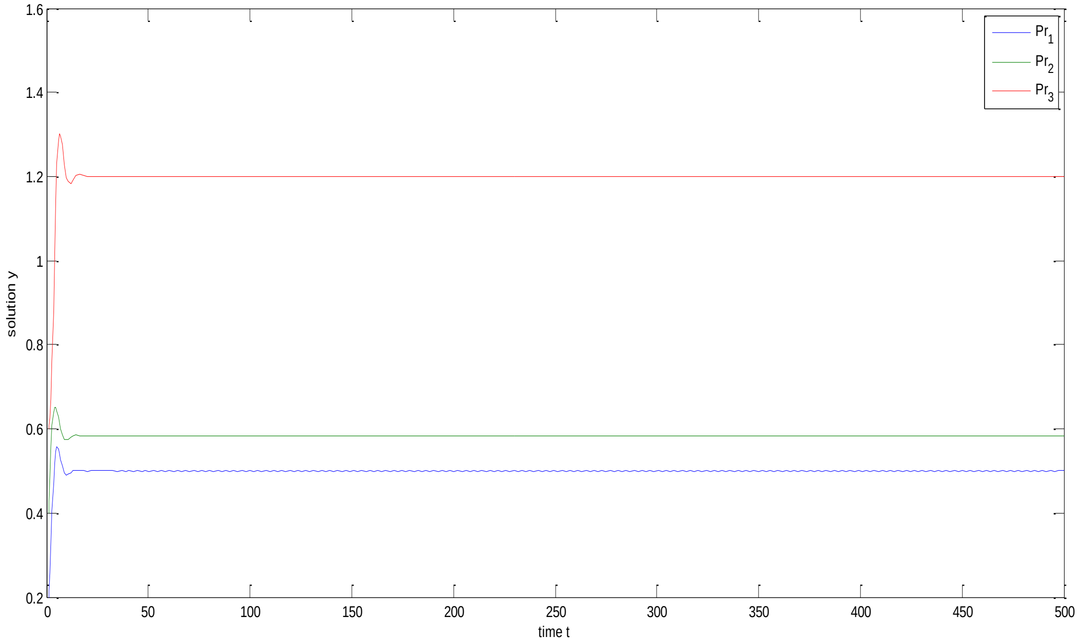

**Figure 4.6** In the absence of delay, the system is stable

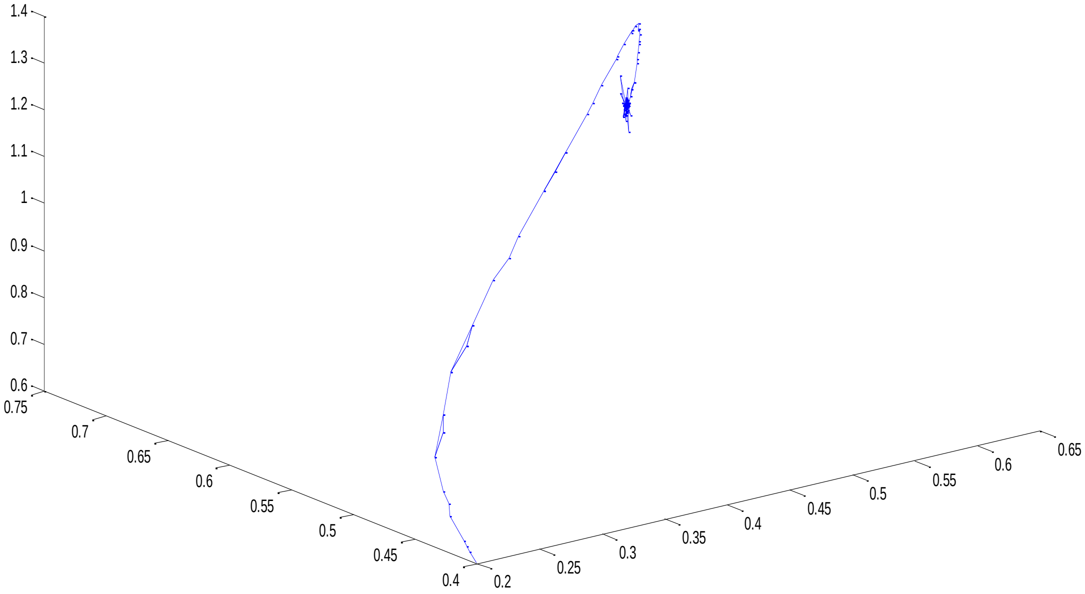

**Figure 4.7** Phase plane graph in the absence of delay in the system

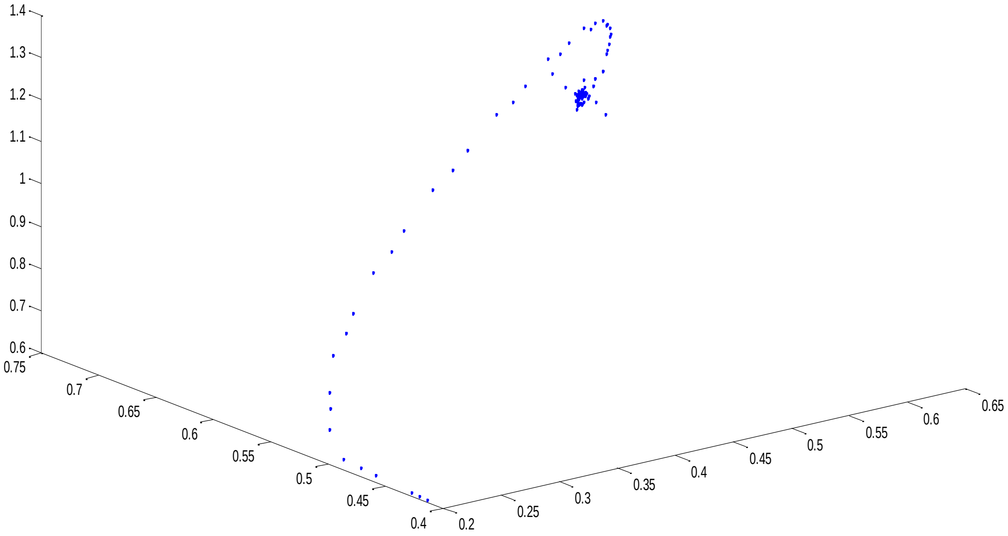

**Figure 4.8** Phase plane graph in the absence of delay in the system

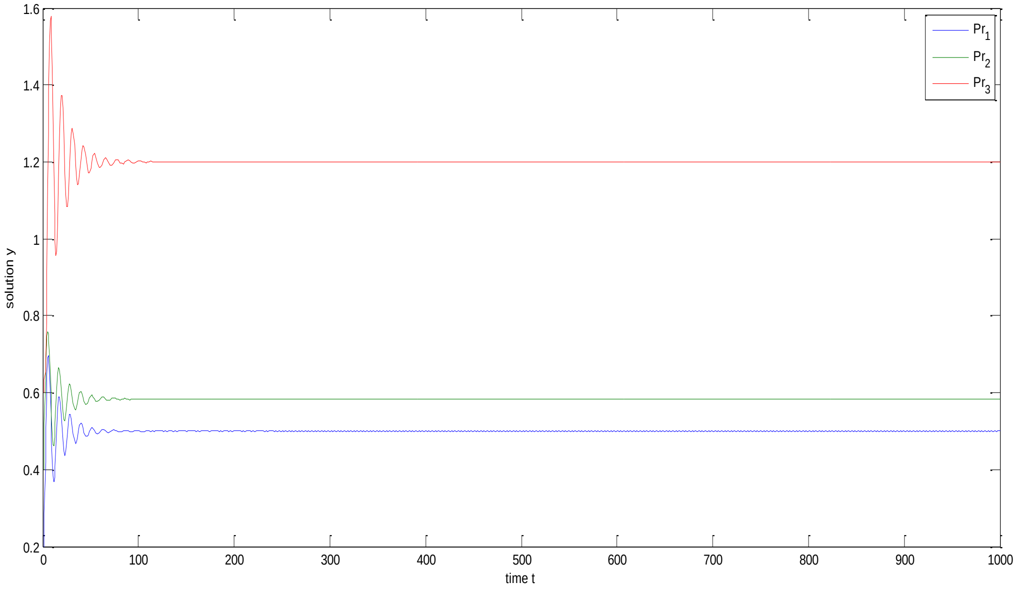

**Figure 4.9** Asymptotically stable when $\tau < \tau_0 = 1.7387$

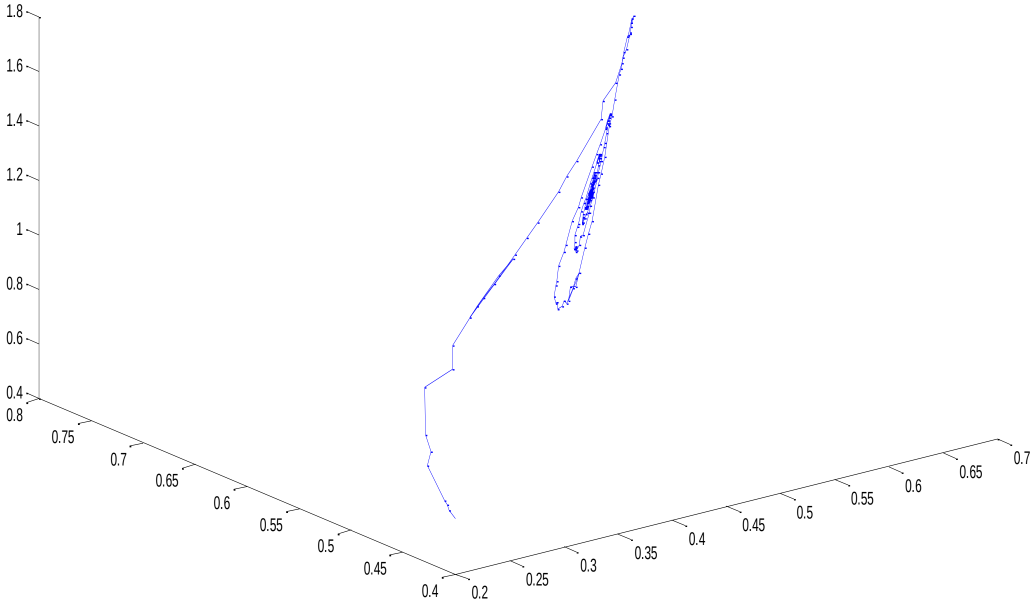

**Figure 4.10** Phase plane graph for asymptotically stable when

$\tau < \tau_0 = 1.7387$

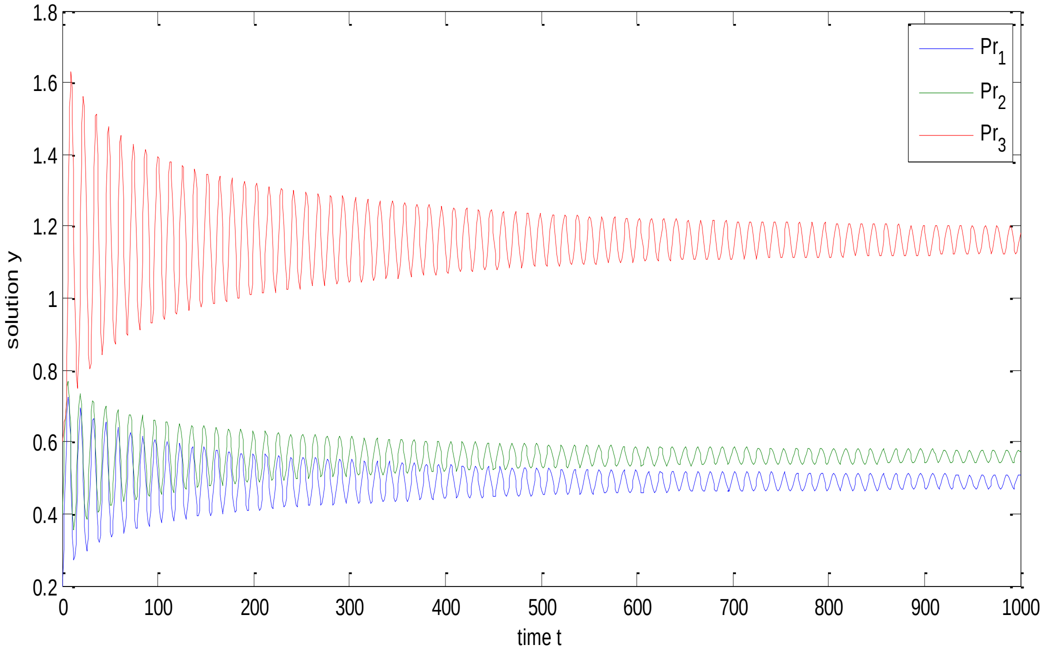

**Figure 4.11** Hopf Bifurcation when $\tau = 2.5 > \tau_0 = 1.7387$

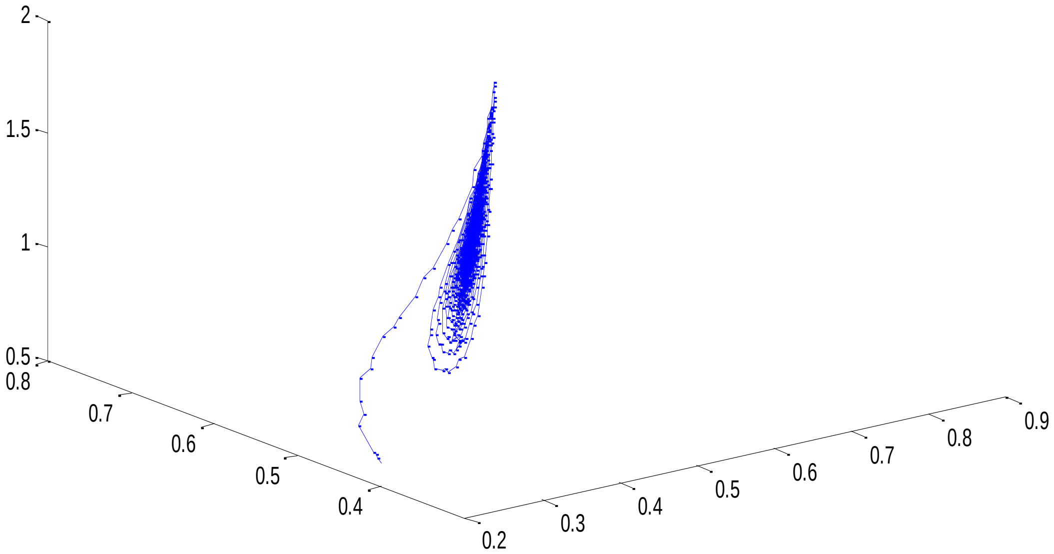

**Figure 4.12** Phase plane graph for Hopf bifurcation when $\tau = 2.5 > \tau_0 = 1.7387$

## 4.9 Conclusion

Certain species, such as zebras and gazelles, form teams because it reduces the predation risk. Time delay caused by the age structure, maturation period, and feeding time is a major factor in real-time prey–predator dynamic that results in periodic solutions and the bifurcation phenomenon. This study investigated the impact of lag time on a multiteam prey–predator dynamic by examining two prey and one predator and considering that the two prey populations support each other when they are susceptible to predation. The insertion of a time delay destabilizes the system's stable equilibrium point. For set 1, the system is absolutely stable in the absence of delay (i.e., $\tau = 0$; Figure 4.1). The same finding is analytically supported by Ruth–Hurwitz's criteria. The system is asymptotically stable when the value of delay is less than the critical value (i.e., $\tau < 1.7387$; Figures 4.2). The Hopf bifurcation is observed when the delay parameter passes a critical value (i.e., $\tau \geq 1.7387$; Figures 4.3). For set 2, the system is absolutely stable in the absence of delay (i.e., $\tau = 0$; Figures 4.6).

The same finding is analytically supported by Ruth–Hurwitz's criteria. The system is asymptotically stable when the value of delay is less than the critical value (i.e., $\tau < 1.7387$; Figures 4.9). The Hopf bifurcation is observed when the delay parameter passes a critical value (i.e., $\tau \geq 1.7387$; Figures 4.11). These graphs have their basics covered in lemmas. Furthermore, the technique used to determine the direction and stability of a Hopf bifurcation solution is constructed using normal form theory and the centre manifold reduction hypothesis. Numerical results are substantiated using the dde23 code of MATLAB.

## Chapter-5

# Modelling the Prey-Predator Dynamics Involving Commensal Species Under the Effect of Time Lag

## 5.1 Introduction

The importance of the system's species interactions, such as qualitatively stability, time dependent, predation, mutualism, commensalism etc., are interesting essential issue to research. Several prey-predator models have been investigated in terms of various types of functional responses [97, 86, 91, 87, 101, 83, 64, 92] etc.

The researcher concluded that how to investigate a scenario when its major results appear to be empirically wrong. Because the theory is analytically correct and, in a way, tautological, the author attempted to determine what happened to cause its empirical falsification [93]. The writers found that when investigating the dynamics of the predator-prey relationship, they used two key paths. The authors also remarked that while this technique is undoubtedly important for a comprehensive knowledge of every individual predator-prey relationship, it is ineffective for making broad conclusions about two major ecological issues [103]. Author discovered a link between complexity and sustainability in multispecies ecosystems [97].

The basic hypothesis that the coefficients are periodic functions of time modifies the overall system of differential equations representing predator-prey dynamics. Author also calculated that system has periodic solution [86]. The author concluded extremely generic nonlinear mutualism models and some easy tests to evaluate a nonlinear mutualism model if it is globally stable or stable in a finite region. Author also discovered that mutualistic systems are more accessible to mathematical study than competition and prey-predator dynamics [91]. The author presented that Hydra plays a significant role in prey population control. The author observed that hydra population density increased from June to a peak in late July or early August, then remained stable or slightly decreased through late August [85]. Commensal and mutualistic interactions occur often across terrestrial vertebrate species and have significant but mostly unmeasured consequences on individual fitness. Because all commensal and mutualistic relationships among terrestrial vertebrates occurs naturally [87].

Researchers used a chaotic ecological model to improve ecological intuition, and it was concluded that simplified arithmetic models have served and will continue to play an important role in comprehending exactly what kinds of behaviors may occur in natural habitats [104]. Authors used the press perturbation approach to evaluate the intensity of overall impacts of connections among phytoplankton and bacteria at the stable state under these coexistence circumstances. This implies that a mutualistic condition might arise as a result of the reason when element carbon pass from phytoplankton to bacteria, even though phytoplankton and bacteria fight for the same resource, inorganic phosphorus [81].

Dingoes affected the previous human economy through competition for big prey because they have such a severe impact on kangaroo population reduction in modern ecology and the human economy. Authors acknowledge that there is a problem in isolating all independent circumstances and individuals who collaborate to generate an example of this kind in our hypothesis archaeological query [88].

Researchers observed that, when surrounding fluctuations are considered, the provision of more food is insufficient to govern the behaviour of a predator-prey ecosystem [105]. Authors used a multi-framework with Holling II impulse response and two lags. Explicit equations used for estimating the path of the Hopf bifurcation as well as the stability of bifurcating periodic results are also derived using the normal form technique and the Centre manifold theorem [98]. Authors demonstrated that a continuous pace of resource harvesting results in a decline in commensal density, which eventually diminishes the intensity of commensal mediation [101].

Authors focused to analysis a Multiple Species Sync an environment having Commensal Predator-Prey Couple and Host Predator-Prey Pair (Normal Steady state). The system includes of a Predator and a prey that depends only on prey to survived, and two Carriers for whom above prey predator are commensal. The Regular Constant Condition has been formed. If all of eigen values are negative, if values are real, and have negative real character, if the values are not real (complex), the solution is sustainable [83].

The functional mechanisms that happen when invasive commensal species intrude on native forest observed [82]. The authors demonstrated a numerical investigation on an ecological model that includes a commensal and a host with restricted resources. This model takes into account a wide range of values for the parameters and numerous restrictions are depicted. The interactions of the organisms have been found [102]. Authors used multiple sync-ecosystem which consists of a commensal predator & two hosts with assumption species have a plenty of food. Using the model equations, all viable equilibrium states are discovered in two phases, and conditions for its stability are addressed [99]. Researchers investigates the viability of a multiple sync eco-system with a commensal death rate. The system is made up of a commensal and multiple hosts. The global stability is established using an appropriately designed Lyapunov's function-pair, and the organisms' increases are statistically determined using the R-K Forth order technique [100].

A multiple system with two competing species that are logistically developing in a vicinity studied. Using numerical modelling, the reason of coexistence was determined to be commensalism [89]. Authors studied a multiple ecosystem wherein two organisms associate directly and the other is a predator organism that predates on both mutual organisms, all while dealing with limited resources. Researchers examined the effect of increasing values of the inhibition coefficient of the second mutual organisms and solve this model by R-K forth order [95].

The current study assessed the behaviour of land use changes between 2007 and 2016, identifying the impact of these changes on the vegetation of the Atlantic Forest. In the Land Change Modeler, the evaluation of land use changes and ecological losses was modelled, and the benefits and losses for each category, as well as their future scenarios, were determined. Landscape metrics were calculated using the ArcGIS V-LATE plugin [96]. Author proposed and investigated a multiple commensal relationship model with Allee effect and one party that cannot exist alone. Sufficient criteria are discovered to ensure the border equilibrium's and positive equilibrium's local and global stability, respectively. The Allee effect causes instability in the system, although it is controllable [84].

Authors used AB fractional derivative to analyse specific computational properties of a three-species prey–predator model in mathematical biology [90]. Time lag is a key factor which should be included in the mathematical model in order to study the dynamical behaviour of these types of biological systems [64].

## 5.2 Mathematical Model

We presented a three-species paradigm with time lag that contains two competing populations $P_1$, $P_2$ and a predator organism $P_d$ that is partly linked to alternative prey and is likely to benefit commensal species. The mathematical model is represented by following non-linear ordinary differential equations:

$$\frac{dP_1}{dt} = p_1P_1\left(1-\frac{P_1}{K_1}\right) - x_1P_1P_2 + b_1P_1P_d \tag{1}$$

$$\frac{dP_2}{dt} = p_2P_2\left(1-\frac{P_2}{K_2}\right) - y_1P_1P_2 - A_1c_1P_2P_d \tag{2}$$

$$\frac{dP_d}{dt} = A_1c_1e_1P_2(t-\tau)P_d + (1-A_1)P_d - z_1P_d - z_2P_d^2 \tag{3}$$

Where $\tau > 0$ is the time lag necessary for predator's gestation period. $P_1$, $P_2$ and $P_d$ represent the populations of commensal, focus prey, and predator species at any given moment (t). All of the variables are greater than zero, i.e., $p_1, p_2, x_1, y_1, b_1, A_1, c_1, e_1, z_1$ *and* $z_2$. $A_1$is a time-independent constant with an alternative resource as its origin. If $A_1 = 1$, the predator is solely dependent on prey species. At $A_1 = 0$, both the predator and prey populations expand without connection. Predation of prey is nil under this state, and predator populations depending on other food sources. Here we are not considered such type of model. A predator which depends on two sources for food is given by $0 < A_1 < 1$.

Table summarizes the parameters and other variables used in the model.

Elucidation of the variables and the parameters:

| Variables/Parameters | Description |
|---|---|
| $P_1$ | Density of commensal species |
| $P_2$ | Prey population |
| $P_d$ | Predator population |
| $p_1$ | Natural growth rate of $P_1$ |
| $p_2$ | Natural growth rate of $P_2$ |
| $K_1$ | Environmental carrying capacity for $P_1$ |
| $K_2$ | Environmental carrying capacity for $P_2$ |
| $x_1$ | Interspecific competition coefficient between $P_1$ and $P_2$ |
| $y_1$ | Interspecific competition coefficient between $P_1$ and $P_2$ |
| $b_1$ | The presence of a predator increases the interspecific commensalism coefficient |
| $z_1$ | Death rate of predator species |
| $z_2$ | Intraspecific competition rate of predator species |
| $\tau$ | Gestation period of Predator |
| $c_1$ | Predation rate of predator |
| $e_1$ | Prey conversion of biomass rate to predator biomass |
| $A_1$ | Time independent constant |

---

## 5.3 Equilibrium Point

$$\frac{dP_d}{dt} = 0$$

$$A_1c_1e_1P_2P_d + (1 - A_1)P_d - z_1P_d - z_2P_d^2 = 0$$

$$P_d[A_1c_1e_1P_2 + (1 - A_1) - z_1 - z_2P_d] = 0$$

Either $P_d = 0$ or $A_1c_1e_1P_2 + (1 - A_1) - z_1 - z_2P_d = 0$

$$A_1c_1e_1P_2 = -(1 - A_1) + z_1 + z_2P_d$$

$$P_2 = \frac{-(1 - A_1) + z_1 + z_2P_d}{A_1c_1e_1} \tag{4}$$

$$\frac{dP_2}{dt} = 0$$

$$p_2P_2\left(1 - \frac{P_2}{K_2}\right) - y_1P_1P_2 - A_1c_1P_2P_d = 0$$

$$P_2\ \ [\,p_2\left(1 - \frac{P_2}{K_2}\right) - y_1P_1 - A_1c_1P_d] = 0$$

Either $P_2 = 0$ or $[\,p_2\left(1 - \frac{P_2}{K_2}\right) - y_1P_1 - A_1c_1P_d] = 0$

$$[\,p_2\left(1 - \frac{P_2}{K_2}\right) - y_1P_1 - A_1c_1P_d] = 0$$

Substitute the value of $P_2$ from equation (4)

$$[\,p_2\left(1 - \frac{\frac{-(1 - A_1) + z_1 + z_2P_d}{A_1c_1e_1}}{K_2}\right) - y_1P_1 - A_1c_1P_d] = 0$$

$$[\,p_2\left(1 - \{\frac{-(1 - A_1) + z_1 + z_2P_d}{A_1c_1e_1K_2}\}\right) - y_1P_1 - A_1c_1P_d] = 0$$

$$p_2[A_1c_1e_1K_2 + (1 - A_1) - z_1 - z_2P_d] - A_1c_1e_1K_2y_1P_1 - A_1^2\,c_1^2e_1K_2\,P_d = 0$$

$$(-A_1^2\,c_1^2e_1K_2\,P_d - p_2z_2)P_d - A_1c_1e_1K_2y_1P_1 + p_2A_1c_1e_1K_2 + p_2(1 - A_1) - p_2z_1 = 0 \tag{5}$$

$$\frac{dP_1}{dt} = 0$$

$$p_1P_1\left(1-\frac{P_1}{K_1}\right)-x_1P_1P_2+b_1P_1P_d=0$$

$$P_1[p_1\left(1-\frac{P_1}{K_1}\right)-x_1P_2+b_1P_d\ ]=0$$

Either$P_1=0\ or\ [p_1\left(1-\frac{P_1}{K_1}\right)-x_1P_2+b_1P_d\ ]=0$

$$[p_1\left(1-\frac{P_1}{K_1}\right)-x_1P_2+b_1P_d\ ]=0$$

$$[p_1\left(1-\frac{P_1}{K_1}\right)-x_1P_2+b_1P_d\ ]=0$$

$$P_1[p_1\left(1-\frac{P_1}{K_1}\right)-x_1P_2+b_1P_d\ ]=0$$

Substitute the value of $P_2$ from from equation (4)

$$P_1[p_1\left(1-\frac{P_1}{K_1}\right)-x_1\{\frac{-(1-A_1)+z_1+z_2P_d}{A_1c_1e_1}\}+b_1P_d\ ]=0$$

$$(A_1c_1e_1K_1b_1-x_1K_1z_2)P_d-A_1p_1c_1e_1P_1+A_1e_1p_1c_1K_1+x_1K_1(1-A_1)-x_1K_1z_1$$
$$=0 \qquad (6)$$

Multiply (5) by $-p_1$ and (6) by $K_2y_1$ then add we get

$$P_d=\frac{(1-A_1)(p_1p_2-K_2K_1x_1y_1)+A_1K_2p_1p_2e_1c_1-A_1K_2p_1p_2e_1c_1-A_1K_1K_2p_1y_1e_1c_1+K_2K_1x_1y_1z_1}{p_1p_2z_2+K_2A_1^2\ c_1^2p_1e_1+A_1K_1K_2b_1y_1e_1c_1-K_1K_2x_1y_1z_2}$$

(7)

## 5.4 The Model's Equilibrium Points

The system of equation (1)-(3) have eight feasible equilibrium point

## 5.5 $E_1$ Is a Point of Trivial Equilibrium (0,0,0)

## 5.6 Point of Axial Equilibrium

Put $P_2=0$ and $P_d=0$ in equation (1)

$$p_1P_1\left(1-\frac{P_1}{K_1}\right)=0$$

$$\left(1-\frac{P_1}{K_1}\right)=0$$

$$1=\frac{P_1}{K_1}$$

$$P_1=K_1$$

$$E_2 \equiv (K_1, 0,0)$$

Put $P_1 = 0$ and $P_d = 0$ in equation (2)

$$p_2P_2\left(1 - \frac{P_2}{K_2}\right) = 0$$

$$\left(1 - \frac{P_2}{K_2}\right) = 0$$

$$P_2 = K_2$$

$$E_3 \equiv (0, K_2, 0)$$

Put $P_1 = 0$ and $P_2 = 0$ in equation (3)

$$(1 - A_1)P_d - z_1P_d - z_2P_d^2 = 0$$

$$[(1 - A_1) - z_1 - z_2P_d]P_d = 0$$

$$(1 - A_1) - z_1 - z_2P_d = 0 \quad \text{as } P_d \neq 0$$

$$(1 - A_1) - z_1 = z_2P_d$$

$$P_d = \frac{(1 - A_1) - z_1}{z_2} \quad \text{let } m_1 = (1 - A_1) - z_1$$

$$E_4 \equiv \left(0,0, \frac{m_1}{z_2}\right) \quad \text{exists if } A_1 + z_1 < 1 \tag{8}$$

## 5.7 Boundary Equilibrium Point:

When $P_2 = 0$ from equation (3)

we get $P_d = \frac{m_1}{z_2}$

From equation (1) we get

$$p_1P_1\left(1 - \frac{P_1}{K_1}\right) + b_1P_1P_d = 0$$

$$\left[p_1\left(1 - \frac{P_1}{K_1}\right) + b_1P_d\right] P_d = 0$$

$$p_1\left(1 - \frac{P_1}{K_1}\right) + b_1\frac{m_1}{z_2} = 0$$

$$p_1\left(1 - \frac{P_1}{K_1}\right) = -b_1\frac{m_1}{z_2}$$

$$P_1 = \frac{b_1m_1K_1 + p_1K_1z_2}{p_1z_2}$$

$$E_5 \equiv \left(\frac{b_1m_1K_1 + p_1K_1z_2}{p_1z_2}, 0, \frac{m_1}{z_2}\right) = (X_1, 0, Z_1) \tag{9}$$

Thus $E_5$ exists if (8) is satisfied.

Similarly, when $P_d = 0$ we obtain

$$E_6 \equiv \left(\frac{-p_2 x_1 K_2 K_1 + p_1 K_1 p_2}{p_1 p_2 - y_1 x_1 K_2 K_1}, \frac{-p_1 y_1 K_2 K_1 + p_1 K_2 p_2}{p_1 p_2 - y_1 x_1 K_2 K_1}, 0\right) = \left( X_2, Y_{2,} 0\right) \quad (10)$$

Thus $E_6$ exists only if

$$p_2 > y_1 K_1 \qquad (11)$$

$$p_1 > x_1 K_2 \qquad (12)$$

And when $P_1 = 0$ we obtain

$$E_7 \equiv \left(0, \frac{-c_1 m_1 K_2 A_1 + z_2 K_2 p_2}{z_2 p_2 + A_1^2\, c_1^2 e_1 K_2}, \frac{c_1 e_1 p_2 K_2 A_1 + m_1 p_2}{z_2 p_2 + A_1^2\, c_1^2 e_1 K_2}, 0\right) = \left( 0, Y_{3,} Z_3\right)$$

## 5.8 Interior Equilibrium Point: $E_8 \equiv (P_1^*, P_2^*, P_d^*) \equiv \left( X_4, Y_{4,} Z_4\right)$.

Substitute the value of $P_d$ in (6) to find the value of $P_1$ we get

$P_1^* = \frac{B_1}{B_2}$ , $P_2^* = \frac{B_3}{B_2}$, $P_d^* = \frac{B_5}{B_6}$ where

$$\begin{aligned} B_1 = {} & (A_1 K_1 b_1 e_1 c_1 - x_2 K_1 z_2)\{A_1 K_2 p_1 p_2 e_1 c_1 + p_1 p_2 (1 - A_1) - A_1 K_1 K_2 p_1 e_1 c_1 y_1 \\ & - K_1 K_2 x_2 y_1 (1 - A_1) + K_1 K_2 x_2 z_1 y_1\} + (p_1 p_2 z_2 + A_1^2\, c_1^2 p_1 e_1 K_2 \\ & + K_1 K_2 A_1 b_1 y_1 e_1 c_1 - K_1 K_2 x_2 y_1 z_2)(A_1 K_1 p_1 e_1 c_1 + K_1 x_2 (1 - A_1) \\ & - x_2 K_1 z_1) \end{aligned}$$

$$B_2 = (p_1 p_2 z_2 + A_1^2\, c_1^2 p_1 e_1 K_2 + A_1 K_1 K_2 b_1 e_1 c_1 y_1 - K_1 K_2 x_2 y_1 z_2)(A_1 p_1 e_1 c_1)$$

$$\begin{aligned} B_3 = {} & \{-(1 - A_1) + z_1\}(p_1 p_2 z_2 + A_1^2\, c_1^2 p_1 e_1 K_2 + A_1 K_1 K_2 b_1 e_1 c_1 y_1 - K_1 K_2 x_2 z_2 y_1) \\ & + z_2\{A_1 K_2 p_1 e_1 c_1 p_2 + p_1 p_2 (1 - A_1) - p_1 p_2 z_1 - A_1 K_1 K_2 p_1 e_1 c_1 y_1 \\ & - K_1 K_2 x_2 y_1 (1 - A_1) + K_1 K_2 y_1 x_2 z_1 \end{aligned}$$

$$\begin{aligned} B_5 = {} & (1 - A_1)(p_1 p_2 - K_2 K_1 x_1 y_1) + A_1 K_2 p_1 p_2 e_1 c_1 - A_1 K_2 p_1 p_2 e_1 c_1 \\ & - A_1 K_1 K_2 p_1 y_1 e_1 c_1 + K_2 K_1 x_1 y_1 z_1 \end{aligned}$$

$$B_6 = p_1 p_2 z_2 + K_2 A_1^2\, c_1^2 p_1 e_1 + A_1 K_1 K_2 b_1 y_1 e_1 c_1 - K_1 K_2 x_1 y_1 z_2$$

## 5.9 Stability

Now stability of above system of equation is calculated

$$n_1 = p_1 - \frac{2p_1P_1}{K_1} - x_1P_2 + b_1P_d$$

$$n_2 = -y_1P_2$$

$$n_3 = 0$$

$$n_4 = -x_1P_1$$

$$n_5 = p_2 - \frac{2p_2P_2}{K_2} - y_1P_1 - A_1c_1P_d$$

$$n_6 = A_1c_1e_1P_de^{-\lambda\tau}$$

$$n_7 = b_1P_1$$

$$n_8 = -A_1c_1P_2$$

$$\mathrm{n_9 = (1 - A_1) - z_1 - 2z_2P_d}$$

## 5.10 Dynamical Behaviour when $\tau = 0$

variational matrix for system (1)-(3) is

W=

$$\begin{vmatrix} p_1 - \frac{2p_1P_1}{K_1} - x_1P_2 + b_1P_d & -y_1P_2 & 0 \\ -x_1P_1 & p_2 - \frac{2p_2P_2}{K_2} - y_1P_1 - A_1c_1P_d & A_1c_1e_1P_de^{-\lambda\tau} \\ b_1P_1 & -A_1c_1P_2 & (1 - A_1) - z_1 - 2z_2P_d \end{vmatrix} = 0$$

At $E_1 \equiv (0,0,0)$ variational matrix reduced to

$$W_1 = \begin{vmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & (1 - A_1) - z_1 \end{vmatrix} = 0$$

The latent values of the characteristic matrix $W_1$at equilibrium point $E_1$are $p_1$, $p_2$ and $(1 - A_1) - z_1$, so the critical value $E_1$ of the system (1)-(3) is not stable.

At $E_2 \equiv (K_1, 0,0)$ the characteristic matrix transformed to

$$W_2=\begin{vmatrix} p_1-\frac{2p_1K_1}{K_1} & -x_1K_1 & b_1K_1 \\ 0 & p_2-y_1K_1 & 0 \\ 0 & 0 & (1-A_1)-z_1 \end{vmatrix}=0$$

The latent root of the system is $-p_1$, $p_2-y_1K_1$ and $(1-A_1)-z_1$ and the system (1)-(3) is stable if $p_2-y_1K_1<0$ that is $p_2<y_1K_1$ and $(1-A_1)-z_1<0$.

At $E_3\equiv(0,K_2,0)$ the characteristic matrix is transformed to

$$W_3=\begin{vmatrix} p_1-x_1K_2 & 0 & 0 \\ -y_1K_2 & -p_2 & -A_1c_1K_2 \\ 0 & 0 & (1-A_1)-z_1 \end{vmatrix}=0$$

The latent root of the system is $p_1-x_1K_2$, $-p_2$and $(1-A_1)-z_1$ and system (1)-(3) is stable if

$p_1-x_1K_2<0$ and $(1-A_1)-z_1<0$.

At $E_4\equiv\left(0,0,\frac{m_1}{z_2}\right)$ the characteristic matrix transformed to

$$W_4=\begin{vmatrix} p_1+\frac{b_1m_1}{z_2} & 0 & 0 \\ 0 & p_2-c_1A_1\frac{m_1}{z_2} & 0 \\ 0 & c_1e_1A_1\frac{m_1}{z_2} & (1-A_1)-z_1-2m_1 \end{vmatrix}=0$$

The latent roots of the characteristic matrix $W_4$ at equilibrium point $E_4$ are $p_1+\frac{b_1m_1}{z_2}$ , $p_2-c_1A_1\frac{m_1}{z_2}$ and $(1-A_1)-z_1-2m_1$, so equilibrium point of the system (1)-(3) is unstable.

At $E_5$ the latent roots are $-p_1X_1$, $p_2-y_1X_1-c_1Z_1A_1$and $-z_2Z_1$ so the critical values of the system (1)-(3) stable only if $p_2<y_1X_1+c_1Z_1A_1$.

At equilibrium point $E_6$, the characteristic equation of W is:

$$(c_1e_1A_1Y_2+m_1-\lambda)(\lambda^2+\left(p_1\frac{X_2}{K_1}+p_2\frac{Y_2}{K_2}\right)\lambda+p_1p_2\frac{X_2Y_2}{K_1K_2}-x_1y_1X_2Y_2)=0$$

(11)

$E_6$ is asymptotically stable locally if and only if $c_1e_1A_1Y_2+1<A_1+z_1$.

The characteristic equation of W is found at equilibrium point $E_7$:

$$(p_1-x_1Y_3+b_1Z_3-\lambda)\left(\lambda^2+\left(p_2\frac{Y_3}{K_2}+z_2Z_3\right)\lambda+p_2\frac{Y_3}{K_2}z_2Z_3+A_1^2\,c_1^2e_1Y_3Z_3\right)=0$$

(12)

For the point $E_7$ is to be a locally asymptotically stable if this condition is satisfied $p_1 + b_1 Z_3 < x_1 Y_3$.

Find out if the latent values have a real value with negative sign by applying the Gerschgorin circle of the variational matrix W at equilibrium point $E_8$

$$\frac{p_1}{K_1} > x_1 + b_1 \tag{13}$$

$$\frac{p_2}{K_2} > y_1 + c_1 A_1 \tag{14}$$

$$z_2 > e_1 c_1 A_1 \tag{15}$$

Following are some observations based on the preceding discussion:

a. The ratio of $P_1's$ intrinsic growth rate to carrying capacity is smaller than the sum of its commensalism rate ($b_1$) and interspecific competing rate ($x_1$).

b. The ratio of its intrinsic growth rate to carrying capacity is smaller than the total of interspecific competition rate ( $y_1$) and the product of predation rate ($c_1$) of $P_2$ with other feed ingredient constant $A_1$.

c. Predator intraspecific competition rate is smaller than the product of predation rate ($c_1$) predator conversion efficiency, and substitute food supply constant $A_1$.

## 5.11 The Behaviour of System When Delay $\tau > 0$

The system's characteristic equations (1)-(3) at the equilibrium point $E_8$ is

$$\begin{vmatrix} -\frac{p_1 X_4}{K_1} - \lambda & -x_1 X_4 & b_1 X_4 \\ -y_1 Y_4 & -\frac{p_2 Y_4}{K_2} - \lambda & -c_1 A_1 Y_4 \\ 0 & A_1 c_1 e_1 Z_4 e^{-\lambda\tau} & -z_2 Z_4 - \lambda \end{vmatrix} = 0$$

The characteristic equation is

$$\lambda^3 + T_1\lambda^2 + T_2\lambda + T_3 + e^{-\lambda\tau}(L_1\lambda + L_2) = 0 \tag{16}$$

Where $T_1 = \frac{p_1 X_4}{K_1} + \frac{p_2 Y_4}{K_2} + z_2 Z_4$

$$T_2 = -x_1 y_1 X_4 Y_4 + \frac{p_1 p_2 X_4 Y_4}{K_1 K_2} + \frac{p_1 z_2 X_4 Z_4}{K_1} + \frac{p_2 z_2 Y_4 Z_4}{K_2}$$

$$T_2 = -x_1 y_1 z_2 X_4 Y_4 Z_4 + \frac{z_2 p_1 p_2 X_4 Y_4 Z_4}{K_1 K_2}$$

$$L_1 = A_1^2\, c_1^2 e_1 Y_4 Z_4$$

$$L_2 = A_1 e_1 b_1 y_1 c_1 X_4 Y_4 Z_4 + \frac{A_1^2\, c_1^2 e_1 p_1 X_4 Y_4 Z_4}{K_1}$$

Equilibrium point is stable if all latent values of (16) must have negative real part. It is not easy to find the condition when all eigen values of equation (16) have negative real part. When delay $\tau = 0$ then equation (16) reduces to

$$\lambda^3 + T_1\lambda^2 + (T_2 + L_1)\lambda + (T_3 + L_2) = 0 \tag{17}$$

By Routh-Hurwitz criterion

$(\boldsymbol{H_1})\ if\ T_1 > 0,\ (T_3 + L_2) > 0,\ T_1(\,T_2 + L_1) > (T_3 + L_2)\ hold$ if and only if every latent value of equation (17) has real value with negative sign.

Let us assume $\lambda = 0$ is solution of (16) then $(T_3 + L_2) = 0$. Thus, this result opposes to the second assumption given in $(\boldsymbol{H_1})$. So, $\lambda = 0$ does not satisfy equation (16). Let us consider that for some $\tau \geq 0$, $i\omega$ with $\omega > 0$ is a solution of (16), then

$$-i\,\omega^3 - T_1\omega^2 + iT_2\omega + T_3 + \ e^{-\omega\tau}(iL_1\omega + L_2) = 0$$

$$-i\,\omega^3 - T_1\omega^2 + iT_2\omega + T_3 + \ (cos\ \omega\tau - isin\ \omega\tau)(iL_1\omega + L_2) = 0 \tag{18}$$

Separating imaginary and real parts

$$-T_1\omega^2 + T_3 + \ L_2cos\ \omega\tau + L_1isin\ \omega\tau = 0 \tag{19}$$

$$-\omega^3 + T_2\omega + L_1 \cos\omega\tau - L_2isin\ \omega = 0 \tag{20}$$

Which gives

$$\omega^6 + f\omega^4 + g\omega^2 + h = 0 \tag{21}$$

Where $f = T_1^2 - 2\mathrm{T}_2$, $g = T_2^2 - L_1^2 - 2\mathrm{T}_1\mathrm{T}_3$, $h = T_3^2 - L_2^2$.

Let $u = \omega^2$, equation (21) becomes,

$$u^3 + fu^2 + gu + h = 0 \tag{22}$$

$$k(u) = u^3 + fu^2 + gu + h$$

**5.12 Lemma 1** we have the following results, for the polynomial equation (22)

1) Condition for the equation (22) has at least one positive solution is $z < 0$.
2) Condition for the equation (22) only roots with no positive sign is $z \geq 0$ and $(x^2 - 3y) \leq 0$.
3) Condition for the equation (22) has only the solutions with positive sign is $z \geq 0$ and $(x^2 - 3y) > 0$

$$\upsilon = \frac{-x \pm (x^2 - 3y)}{3} > 0\ and\ h(\upsilon) \leq 0$$

Proof. Let us assume that at least one positive root for equation (22)

$$\mu_0 = \sqrt{\alpha_0}$$

From equation (19) and equation (20), we obtain,

$$\cos \mu_0 \tau = \frac{-\left(L_1 \mu_0^2 (T_2 - \mu_0^2) + (T_3 - T_1 \mu_0^2)(L_2)\right)}{(L_2)^2 + (L_1 \mu_0)^2} \tag{23}$$

$$\tau_j = \frac{1}{\mu_0}\, arc\cos\left(\frac{-\left(L_1 \mu_0^2 (T_2 - \mu_0^2) + (T_3 - T_1 \mu_0^2)(L_2)\right)}{(L_2)^2 + (L_1 \mu_0)^2} + 2j\pi\right) \tag{24}$$

where j takes values 0, 1, 2,3...

**5.13 Lemma 2** Consider $k(u_0) = \left(3{u_0}^2 + 2fu_0 + g\right) \neq 0$ and the assumption in $(\boldsymbol{H_1})$ is satisfied. For the values of (j takes values 0,1,2...), denote $\mu\eta(\tau) = \alpha(\tau) + i\mu(\tau)$ be the solution of equation (4) satisfying $\alpha(\tau_j) = 0, \mu(\tau_j) = \mu_0$, where

$$\tau_j = \frac{1}{\mu}\, arc\cos\left(\frac{-\left(L_1 \mu^2 (T_2 - \mu^2) + (T_3 - T_1 \mu^2)(L_2)\right)}{(L_2)^2 + (L_1 \mu_0)^2} + 2j\pi\right)$$

then $\pm i\mu_0$ are simple roots. If the transversality condition

$(\boldsymbol{H_2})$ $\alpha^j(\tau_j) = \left.\frac{R_e \eta(\tau)}{d\tau}\right|_{\eta = i\,\mu_0} \neq 0$

At any equilibrium point where $\tau = \tau_j$ and, a Hopf bifurcation exists for the system of equations (1), (2), and (3).

Proof. Let the root of the equation (17) be $\eta = \eta(\tau)$. As $\eta(\tau)$ is substituted into eq. (17) and differentiating with respect to τ, it results into

$$\left[(3\eta^2 + 2T_1\eta + T_2) + \left((\eta^2 L_1 + \eta L_2)(-\tau) + L_1\right)e^{-\eta\tau}\right]\frac{d\mu}{d\tau} = \eta(\eta L_1 + L_2)e^{-\eta\tau}$$

Thus

$$\left(\frac{d\eta}{d\tau}\right)^{-1} == \frac{(3\eta^2 + 2T_1 + T_2)}{\eta(\eta L_1 + L_2)} + \frac{(L_1)}{\eta(\eta L_1 + L_2)} - \frac{\tau}{\eta}$$

From (19)-(21), we have

$$a^j(\tau_0) = R_e\left[\frac{(3\eta^2 + 2T_1\eta + T_2)e^{\eta\tau}}{\eta(\eta L_1 + L_2)}\right] + R_e\left[\frac{(L_1)}{\eta(\eta L_1 + L_2)}\right]$$

$$= \frac{1}{\Delta}\left[3\mu^6 + 2\left({T_1}^2 - 2T_2\right)\mu^4 + (T_2^2 - L_1^2 - 2T_1T_3)\mu^2\right]$$

$$= \frac{1}{\Delta}(3\mu^6 + 2\mu^4 + g\mu^2)$$

$= \frac{\mu_0^2}{\Delta}\, k'(u)$

where $\Delta = L_1^2\mu_0^2 + L_2^2$.Notice that $\Delta > 0$ and $\mu_0 > 0$, we observed that $\text{sign}\left[\alpha^j(\tau_0)\right] = \text{sign}\left[k^j(u_0)\right]$

This demonstrates the lemma.

We can easily calculate the results on the stability and bifurcation of the system of equations (1), (2), and (3) using Lemma number 2.

## 5.14 Assessment of the Hopf-Bifurcation Solution's Stability and Direction

We shift the non-zero interior equilibrium $E_8 \equiv (P_1^*, P_2^*, P_d^*)$ to the origin by the translation $v_1(t) = P_1(t) - P_1^*,\ v_2(t) = P_2(t) - P_2^*,\ v_3(t) = P_d(t) - P_d^*$ and using the time scaling $t \longrightarrow \frac{t}{\tau}$ for normalizing the delay $\tau$ then system of mathematical equation (1)-(3) is converted into

$$\frac{dv_1}{dt} = c_{11}v_1 + c_{12}v_2 + c_{13}v_3 + c_{14}v_1^2 + c_{15}v_1v_2 + c_{16}v_1v_3 \tag{25}$$

$$\frac{dv_2}{dt} = c_{21}v_1 + c_{22}v_2 + c_{23}v_3 + c_{24}v_2^2 + c_{25}v_1v_2 + c_{26}v_2v_3 \tag{26}$$

$$\frac{dv_3}{dt} = c_{31}v_2(t-1) + c_{32}v_3 + c_{33}v_3^2 + c_{34}v_2(t-1)\ v_3 \tag{27}$$

Where $c_{11} = -\frac{p_1P_1^*}{K_1}$ , $c_{12} = -x_1P_1^*$ , $c_{13} = b_1P_1^*$ , $c_{14} = -\frac{p_1}{K_1}$ , $c_{15} = -x_1$, $c_{16} = b_1$, $c_{21} = -y_1P_2^*$, $c_{22} = -\frac{p_2P_2^*}{K_2}$, $c_{23} = -c_1A_1P_2^*$ , $c_{24} = -\frac{p_2}{K_2}$, , $c_{25} = -y_1$, $c_{26} = -c_1A_1$, $c_{31} = c_1A_1e_1P_d^*$ ,$c_{32} = -z_2P_d^*$ , $c_{33} = -z_2$, $c_{34} = -c_1A_1e_1$.

Thus, we can proceed our calculation in the phase $C = C((-1,0), R_+^3)$. WLOG, denote the critical value $\tau_j$by $\tau_0$. Let $\tau = \tau_0 + \delta$, then $\delta = 0$ is value of Hopf bifurcation to the system (25)-(27). For the simplicity of representation, we write (25)-(27) as

$$v'(t) = H_\delta(v_t) + G(\delta, v_t) \tag{28}$$

Where $v(t) = (v_1(t), v_2(t), v_3(t))^T \epsilon R^3, v_t(\theta)\epsilon C$ is defined as $v_t(\theta) = v(t+\theta)$, and $H_\delta: C \to R$, $G: C \times R \to R$, given by respectively.

$H_\delta\xi = (\tau_0 + \delta)E_1\xi(0) + (\tau_0 + \delta)E_2\xi(-1)$ where $E_1$ *and* $E_2$ are defined as

$$E_1 = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & c_{32} \end{vmatrix}, E_2 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{31} & 0 \end{vmatrix} \text{ and } G(\delta, \xi) = (\tau_0 + \delta) \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$

$$G_1 = \ c_{14}\xi_1^2(0) + c_{15}\xi_1(0)\xi_2(0) + c_{16}\xi_1(0)\xi_3(0),$$

$$G_2 = \ c_{24}\xi_2^2(0) + c_{25}\xi_1(0)\xi_2(0) + c_{26}\xi_2(0)\xi_3(0),$$

$$G_3 = \ c_{33}\xi_3^2(0) + c_{34}\xi_2(-1)\xi_3(0),$$

where$\xi(\theta) = (\xi_1(\theta), \xi_2(\theta), \xi_3(\theta))^T \epsilon \ C((-1, 0), R)$. Use of Riesz representation theorem, help us to find a function ε($\theta$,δ) of the bounded variation for $\theta \ belong \ in \ the \ interval \ [-1, 0]$, such that

$$H_\delta \xi = \int_{-1}^{0} d\varepsilon(\theta, 0)\, \xi(\theta) for \ \xi \ \epsilon \ C.$$

We can choose $\varepsilon(\theta, 0) = \begin{cases} \frac{d\xi(\theta)}{d\theta} & \theta \ \epsilon \ [-1,0) \\ \int_{-1}^{0} d\varepsilon(\theta, 0)\, \xi(\theta) & \theta = 0 \end{cases}$

and J ($\delta$) $\xi = \begin{cases} 0 & \theta \ \epsilon [-1, 0) \\ G(\delta, \xi) & \theta = 0 \end{cases}$

the system (28) is equivalent to

$$v_t' = I(\delta) v_t + J(\delta) v_t.$$

For $\varphi \ \epsilon \ C'([-1,0], R_+^3)$, define

$$I^* \varphi(s) = \begin{cases} \dfrac{-d\varphi(s)}{d\theta} & s \ \epsilon \ [-1,0), \\ \displaystyle\int_{-1}^{0} d\varepsilon^T(-t, 0)\varphi(-t) & s = 0 \end{cases}$$

And bilinear inner product.

$< \varphi(s), \xi(\theta) > = \ \bar{\varphi}(0)\, \xi(0) - \int_{-1}^{0} \int_{v=0}^{0} \bar{\varphi}(\xi - \theta) d\varepsilon(\theta) \xi(\zeta) d\zeta.$

Since $I^*$ and $I = I(0)$ are adjoint operators. Then by the results obtained earlier, we observed that $i\omega_0$ is an eigen values of $I(0)$. Thus, we conclude that this is an eigen values of $I^*$. Let us assume that

$g(\theta) = g(0)e^{i\omega_0}$ is an eigen vector of $I(0)$ corresponding to eigen value $-i\omega_{0,}$ Then $I(0) = i\omega_0 \, g(\theta)$. When $\theta = 0$, we obtain $\left[ i\omega_0 I - \int_{-1}^{0} d\varepsilon(\theta) e^{i\omega_0} \right] g(0) = 0$, which gives $g(0) = (1, \sigma_1, \rho_1)^T$ where

$$\sigma_1 = \frac{c_{31}c_{21} + c_{23}(i\omega_0 - c_{11})}{c_{12}c_{23} + c_{13}(i\omega_0 - c_{22})} \tag{29}$$

$$\rho_1 = \frac{(i\omega_0 - c_{11})(i\omega_0 - c_{22}) - c_{12}c_{21}}{c_{12}c_{23} + c_{13}(i\omega_0 - c_{22})} \tag{30}$$

Same, it can be calculated that

$g^*(s) = D((1, \sigma_2, \rho_2)e^{i\omega_0\tau_0 s}$ is the latent vector of $I^*$corresponding to $-i\omega_0\tau_0$ where

$$\sigma_2 = \frac{c_{31}c_{21} - c_{23}(i\omega_0 + c_{11})}{c_{12}c_{23} - c_{13}(i\omega_0 + c_{22})} \tag{31}$$

$$\rho_2 = \frac{(i\omega_0 + c_{11})(i\omega_0 + c_{22}) - c_{12}c_{21}}{c_{12}c_{23} - c_{13}(i\omega_0 + c_{22})} \tag{32}$$

Where

$$\bar{E} = \frac{1}{(1 + \sigma_1\bar{\sigma}_2 + \rho_1\bar{\rho}_2 + \tau_0\bar{\rho}_2\rho_1 c_{31}e^{-i\omega_0\tau_0})}$$

$$< g^*(s), g(\theta) >= 1, < g^*(s), \bar{g}\,(\theta) >= 0$$

Now we calculate the coefficients, specifying important quantities of the periodic solution. We use the same calculation process as done by [6] in Hssard et al.

$$l_{20} = 2\bar{E}(N_{11} + \bar{\sigma}_2 N_{21} + \bar{\rho}_2 N_{31})$$

$$l_{11} = \bar{E}(N_{12} + \bar{\sigma}_2 N_{22} + \bar{\rho}_2 N_{32})$$

$$l_{02} = 2\bar{E}(N_{13} + \bar{\sigma}_2 N_{23} + \bar{\rho}_2 N_{33})$$

$$l_{21} = 2\bar{E}(N_{14} + \bar{\sigma}_2 N_{24} + \bar{\rho}_2 N_{34})\ Where$$

$$N_{11} = c_{14} + c_{16}\rho_1 + c_{15}\sigma_1$$

$$N_{13} = c_{14} + c_{16}\bar{\rho}_1 + c_{15}\bar{\sigma}_1$$

$$N_{12} = 2c_{14} + c_{16}\rho_1 + c_{16}\bar{\rho}_1 + c_{15}\sigma_1 + c_{15}\bar{\sigma}_1$$

$$\begin{aligned} N_{14} = {} & 2c_{14}M_{11}^{(1)}(0) + c_{15}M_{11}^{(2)}(0) + c_{16}M_{11}^{(3)}(0) + c_{14}M_{20}^{(1)}(0) + c_{15}\frac{M_{20}^{(2)}(0)}{2} \\ & + c_{16}\frac{M_{20}^{(3)}(0)}{2} + c_{16}M_{11}^{(1)}(0)\rho_1 + c_{15}\frac{M_{20}^{(1)}(0)}{2}\bar{\rho}_1 + c_{15}M_{11}^{(1)}(0)\sigma_1 \\ & + c_{15}\frac{M_{20}^{(1)}(0)}{2}\bar{\sigma}_1 \end{aligned}$$

$$N_{21} = c_{25}\sigma_1 + c_{26}\rho_1\sigma_1 + c_{24}\sigma_1^2$$

$$N_{22} = c_{25}\sigma_1 + c_{26}\bar{\rho}_1\sigma_1 + c_{25}\bar{\sigma}_1 + c_{26}\rho_1\bar{\sigma}_1 + 2c_{24}\sigma_1\bar{\sigma}_1$$

$$N_{23} = c_{25}\bar{\sigma}_1 + c_{26}\bar{\rho}_1 + c_{24}\bar{\sigma}_1^2$$

$$N_{24} = c_{25}M_{11}^{(2)}(0) + c_{25}\frac{M_{20}^{(2)}(0)}{2} + c_{26}M_{11}^{(2)}(0)\rho_1 + c_{26}\frac{M_{20}^{(2)}(0)}{2}\bar{\rho}_1 + c_{25}M_{11}^{(1)}(0)\sigma_1 + 2c_{24}M_{11}^{(2)}(0)\sigma_1 + c_{26}M_{11}^{(3)}(0)\sigma_1 + c_{25}\frac{M_{20}^{(1)}(0)}{2}\bar{\sigma}_1 + c_{24}M_{20}^{(2)}(0)\bar{\sigma}_1 + c_{26}\frac{M_{20}^{(3)}(0)}{2}\bar{\sigma}_1$$

$$N_{31} = c_{33}\rho_1^2 + c_{34}e^{-i\omega_0\tau_0}\rho_1\sigma_1$$

$$N_{32} = 2c_{33}\rho_1\bar{\rho}_1 + c_{34}e^{-i\omega_0\tau_0}\bar{\rho}_1\sigma_1 + c_{34}e^{-i\omega_0\tau_0}\rho_1\bar{\sigma}_1$$

$$N_{33} = c_{33}\bar{\rho}_1{}^2 + c_{34}e^{-i\omega_0\tau_0}\bar{\rho}_1\bar{\sigma}_1$$

$$N_{34} = c_{34}M_{11}^{(2)}(-1)\rho_1 + 2c_{33}M_{11}^{(3)}(0)\rho_1 + c_{34}\frac{M_{20}^{(2)}(-1)}{2}\bar{\rho}_1 + c_{33}M_{20}^{(3)}(0)\bar{\rho}_1 + c_{34}e^{-i\omega_0\tau_0}M_{11}^{(3)}(0)\sigma_1 + \frac{1}{2}c_{34}e^{-i\omega_0\tau_0}M_{20}^{(3)}(0)\bar{\sigma}_1$$

However

$$M_{20}(\theta) = \frac{il_{20}}{\omega_0\tau_0}t(0)\,e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{02}\bar{q}(0)}{3\omega_0\tau_0}\,e^{-i\omega_0\tau_0\theta} + F_1e^{2i\omega_0\tau_0\theta}$$

And

$$M_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_0}q(0)\,e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{\omega_0\tau_0}\,e^{-i\omega_0\tau_0\theta} + F_2,$$

Where $F_1$ *and* $F_2$ are both three-dimensional vector quantity, and can be calculated by

$$\begin{vmatrix} 2i\omega_0 - c_{11} & -c_{12} & -c_{13} \\ -c_{21} & 2i\omega_0 - c_{22} & -c_{23} \\ 0 & -c_{31}e^{-2i\omega_0\tau_0\theta} & 2i\omega_0 - c_{32} \end{vmatrix} F_1 = 2\begin{bmatrix} N_{11} \\ N_{21} \\ N_{31} \end{bmatrix}$$

And

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & c_{31} & c_{32} \end{vmatrix} F_2 = -2\begin{bmatrix} N_{12} \\ N_{22} \\ N_{32} \end{bmatrix}$$

Then $l_{21}$ can be written by the use of parameters.

From the above discussion, we can observe that every $l_{ij}$ determined by parameters used. Thus, we obtain the result:

$$D_1(0) = \frac{i}{2\omega_0\tau_0}\left(l_{11}\,l_{20} - 2l_{11}^2 - \frac{l_{02}^2}{3}\right) + \frac{l_{21}}{2},$$

$$n_1 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_0)\}}$$

$$n_2 = 2Re\{C_1(0)\}.$$

$$T_2 = -\frac{Im\{C_1(0)\} + n_1 Im\{\lambda'(\tau_0)\}}{\omega_0 \tau_0} \tag{33}$$

Which determines the quantities of bifurcation periodic solutions in the Centre manifold at critical values at $\tau_0$.By the result of Hassard et al. we have the following

Theorem. $n_1$determines propagation of the Hopf-bifurcation: if $n_1 > 0$ ($n_1$ <0), then the Hopf-bifurcation is supercritical (subcritical) and the bifurcating period solutions exits if $\tau > \tau_0$ $(\tau < \tau_0)$; $n_2$ determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions are arbitically asymptotically stable when $n_2 < 0$ and periodic solutions are arbitically unstable if $n_1 > 0; and\, T_2$ finds the bifurcating periodic solution. The period increases when $T_2 > 0$ and the period decreases when $T_2 < 0$

## 5.15 Numerical Example

In this part, we used MATLAB to do a numerical simulation of system [1]-[3]. We use these parametric values:

$$p_1 = 3\ p_2 = 3\ K_1 = 5\ b_1 = 0.9$$

$$e_1 = 0.5\ y_1 = 0.6\ z_1 = 0.2\ A_1 = 0.91$$

$$z_2 = 0.5\ K_2 = 4\ x_1 = 0.14\ c_1 = 1.8$$

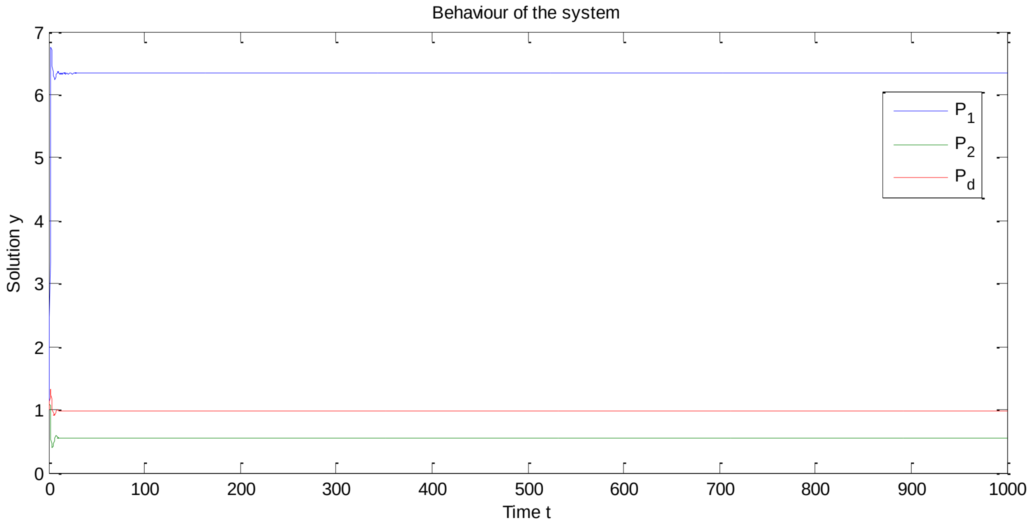

**Figure 5.1** In absence of delay the system is stable

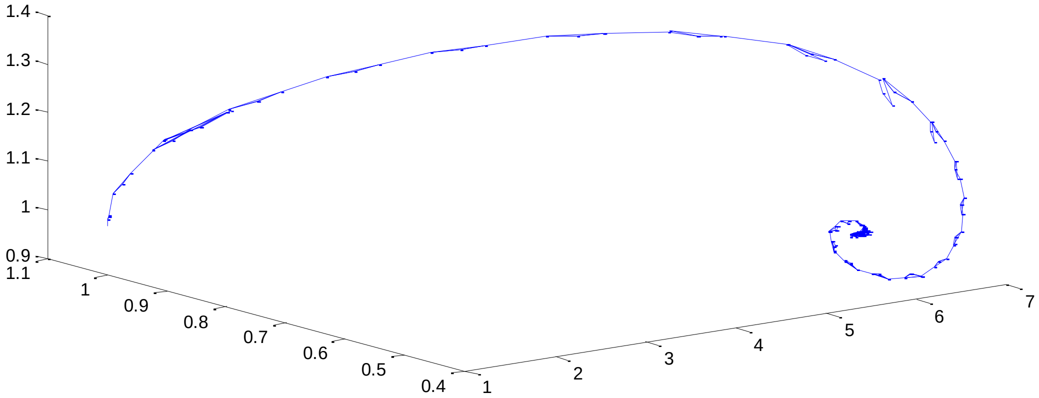

**Figure 5.2** Phase plane graph in absence of delay

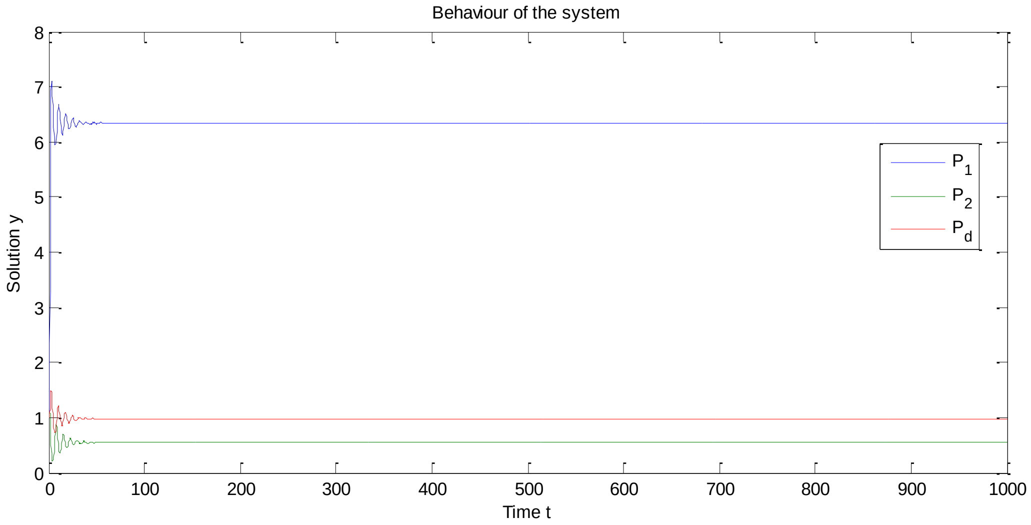

**Figure 5.3** Asymptotically stable when $\tau = 0.8 < \tau_0 = 1.30997$

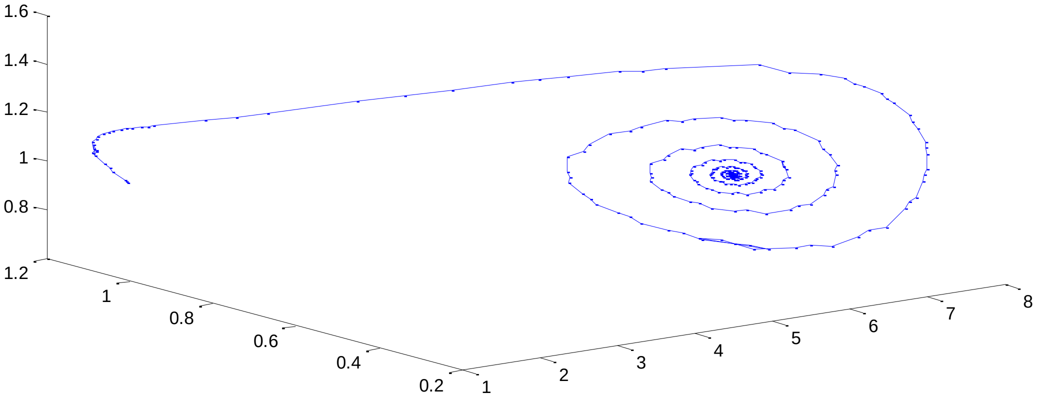

**Figure 5.4** Phase plane graph for Asymptotically stable when $\tau = 0.8 < \tau_0 = 1.30997$

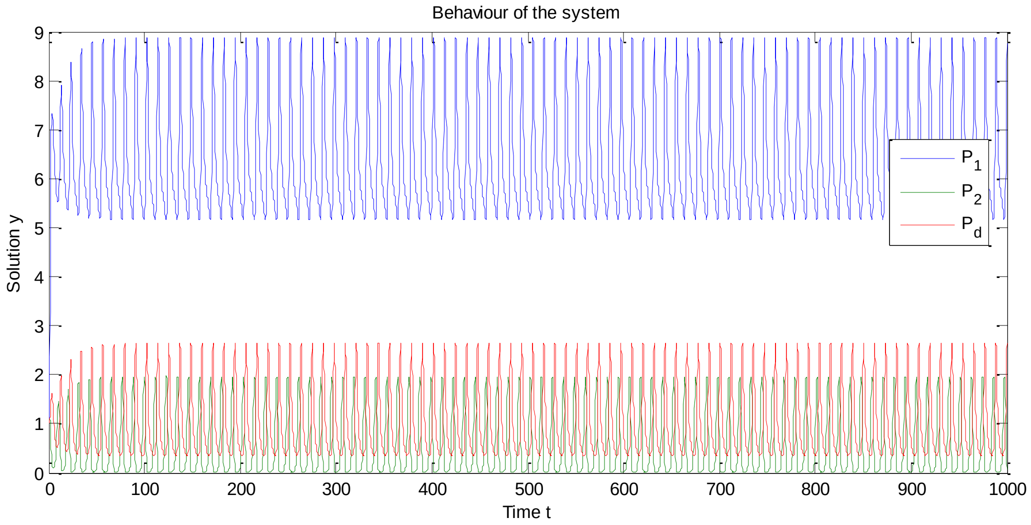

**Figure 5.5** Hopf Bifurcation when $\tau = 1.7 > \tau_0 = 1.30997$

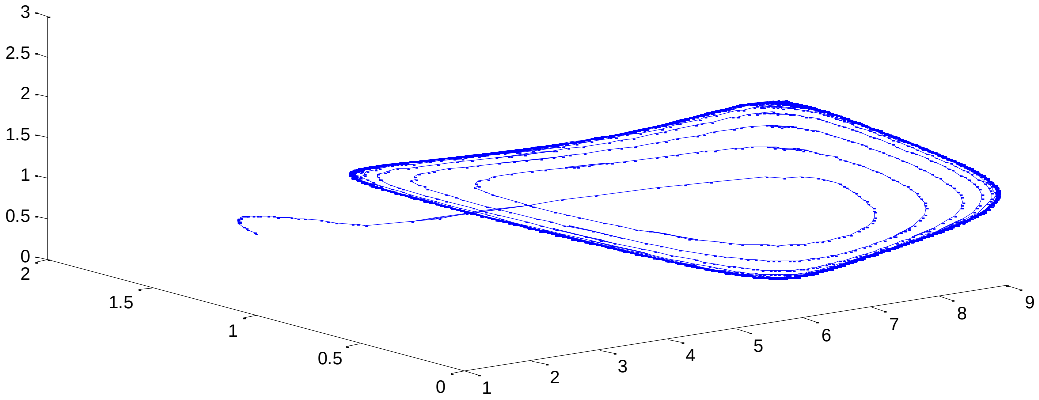

**Figure 5.6** Phase plane graph for Hopf Bifurcation when $\tau = 1.7 > \tau_0 = 1.30997$

## 5.16 Conclusion

In this study, we proposed a three-species time-lag model that includes two species that are in competition with one another and a predatory creature that is partly connected to alternate prey and is probably beneficial to commensal species. Initially, the system shows stable coexisting, but when alternate food is introduced, the system oscillates. This research, on the other hand, shows that supplying a variety of diets does not lead the system to oscillate. The extra foodstuffs may increase or decrease the possibility of coexisting equilibrium level, according to analytic and statistical analysis. In the absence of delay the system is stable (figure 5.1). Phase plane graph in absence of delay (figure 5.2). System is asymptotically stable (figure 5.3). Phase plane graph for asymptotically stable system (figure 5.4). Hopf Bifurcation diagram shown in (figure 5.5). Phase Plane graph for Hopf Bifurcation shown in (figure 5.6). Limit cycles for the non-zero equilibrium point when the time delay surpasses a critical value arise whenever the time delay is applied. According to observation, the crucial level of the delay parameter rises as $A_1's$ numerical value does.

# Chapter 6

# Study of the Stability of a Single Prey and a Pair of Predators using a Delay Differential Equation and A Functional Response with a Square Root

## 6.1 Introduction

In a predator-prey context, the fundamental and secondary components of predation were presented by C. S. Holling. Because of its significance, the functional response has been studied further in an attempt to understand its characteristics [106]. N. Macdonald studied it is normal to add a temporal delay in the component denoting the dependency of the predator on the prey in deterministic models of prey-predator interaction [107]. When studying the general differential equations system that characterizes predator-prey dynamics, J. M. Cushing made the assumption that the parameters are cyclic expressions of time [86]. Len Nunney observed persistent predator-prey interactions. The author discovered that discrete-generation predator-prey models are stabilized by sigmoid functional responses [108].

Andrew Sih found an interesting fact any tactic that lowers the danger of predation can be included in the broad definition of refuse usage [109]. The prevailing theories hold that prey is encountered by predators at random and that the tropic function is solely dependent on prey availability. This strategy is not always acceptable, according to Roger Arditi et al. The tropic function must be considered on the long-time scale of population dynamics, rather than the short behavioral time scale on which the models operate [12]. Yang Kuang et al. studied ratio-dependent prey-predator delayed systems [110].

Chris Cosner and team proposed this unified scientific method for the generation of many forms of functional responses. The deduction is based on the idea of mass action, but with the important modification that the nature of predator spatial distribution and predation opportunities are taken into account implicitly [111]. Jill McGrady-Steed et al. discovered that species richness influences population size and stability. Many fundamental theories suggest that complex speciose societies are much less stable than simple ones. Functional groups did not indicate individual population stability. Inter - specific relationships and statistical averaging may both aid in reducing the temporal changes of functional groups generated by aggregating a large number of species [112]. In 20th century several types of prey-predator dynamics studied by different researchers. These dynamics related to density dependent predation by [113, 114, 116, 118, 115].

Peter A. Abrams et al. examine the characteristics of the dynamics as well as the responses of the mean frequencies of each predator to death rates imposed on it or its competition. The dynamics study for this model exposes a variety of previously unknown behaviors, like chaotic oscillations and long-term transients that differ greatly from the final patterns of oscillations [117]. Meng Fan et al. meticulously analyze the behavior of a nonautonomous predator-prey system using the Beddington-DeAngelis functional response. For the cyclic (almost periodic) situation, a positive (just about) periodic answer and a border (just about) periodic solution are examined, together with persistence, mortality, and global asymptotic stable (generic nonautonomous situation) [119].

When presented with a crowd of prey, many predators become confused and less efficient in their attacks. J. M. Jeschke et al. investigated the impact of predator perplexity on predator dynamics in order to gain knowledge on the environmental, ethological, and evolutionary consequences of this phenomena. The authors developed the very first functional responder model that takes perplexity into consideration and subjectively and statistically compares it to actual data from multi predator/prey systems [120]. Dongmei Xiao [121] proved physiologically the elimination of the singularity of the origin, which was deemed "pathological behaviour" for a ratio-dependent prey - predator system, and the prevention of mutual extinction as a probable outcome of the predator prey interaction.

Researchers establish a functional reaction that accounts for both prey densities. As a result, depending on how the two prey densities interact, the functional response exhibits types II (manipulating prey densities) and III (switching predator behaviour) studied by E. Vanleeuwen [19]. The authors looked at the consequences of linear changes to intrinsically unstable continuous-time prey-predator models in a (small) region of the origin. P. E. Kloeden et al., in particular, proved that based on the parameters chosen, either one universal attractor or a repulsive state arises [122].

Valerio Ajraldi et al. demonstrate certain basic assumptions for traditional two-population system [123]. S. N. Matia et al. discovered approach for herd behaviour as a self-defence mechanism outperforms the strategy of toxic production. The model also exhibits ecologically important dynamics around the origin due to prey herd behaviour [124]. Yun Kang et al. came to the conclusion that species are vulnerable to extinction in the presence of Allee effects and that starting state has a significant impact on both the survival of prey and its matching predator. Researchers also noted that illness could be able to prevent prey from going extinct due to predation and promote cohabitation, while the disease-driven extinction cannot be stopped by a predator [125].

Hongxia Liu et al. investigated a prey-predator dynamics model with a square root response function using a stable-dependent impulse. The existence, uniqueness, and attraction for first order cyclic result first investigated using differential equation geometry theory and the successor function technique. The stability first order cycle is next examined using the Poincaré criterion for spontaneous differential equations [126]. Sourav Kumar Sasmal et al. investigate the behavior of a prey-predator model theory. This model incorporates collective prey defense against predation via the Monod-Haldane functional response, in addition to a fear effect that slows prey growth rate when a predator is around [127].

Jai Prakash Tripathi et al. described universal attractivity and permanence as basic functions using the continuation theory from coincidence degree theory, which improves on the standard conditions derived by employing limits of the important parameters and the presence of periodic solutions. This method outperforms the Brouwer fixed point theory in the Crowley-Martin basis function [128]. Yang et. al. explored the partial differential equation system's characteristic roots, along with the Turing destabilization and Hopf bifurcation. Analysing the normal form of the Turing-Hopf bifurcation [129] yielded the phase diagram. Jawdat Alebraheem suggests a predator-dependent operational and numerical response-based autonomous predator-prey model. The model's coexistence and extinction conditions were validated and identified using Kolmogorov analysis [130].

A multi-species model with delay that has two competing prey populations and a predator organism that is somewhat connected to other prey and helps commensal species. When alternate meal is provided, the system oscillates after initially exhibiting steady coexistence. This research, on the other hand, shows that supplying a variety of diets does not lead the system to oscillate [131].

## 6.2 Mathematical Model

$$\frac{dR}{dT} = sR\left(1 - \frac{R(T-\tau)}{K_1}\right) - b_1\sqrt{R}D_1 - b_2\sqrt{R}D_2 \tag{1}$$

$$\frac{dD_1}{dT} = b_1c_1\sqrt{R}D_1 - f_1D_1 \tag{2}$$

$$\frac{dD_2}{dT} = b_2c_2\sqrt{R}D_2 - f_2D_2 - gD^2 \tag{3}$$

With the initial conditions $R(0) > 0, D_1(0) > 0, D_2(0) > 0$ above system (1)-(3) can be non- depersonalised by introducing new relations:

$$\hat{p} = \frac{R}{K_1}, z_1 = \frac{D_1b_1}{s\sqrt{K_1}}, z_2 = \frac{D_1b_2}{s\sqrt{K_1}}, t = \frac{sT}{2}$$

and new parameters are used

$$s_1 = \frac{2c_1b_1\sqrt{K_1}}{s}, s_2 = \frac{2c_2b_2\sqrt{K_1}}{s}, h_1 = \frac{f_1}{c_1b_1\sqrt{K_1}}, h_2 = \frac{f_2}{c_2b_2\sqrt{K_1}}, h_3 = \frac{gs}{c_2b_2^2}$$

After conversion non-dimensional system become:

$$\frac{d\hat{p}}{2dt} = \hat{p}\big(1 - \hat{p}(t-\tau)\big) - \sqrt{\hat{p}}Z_1 - \sqrt{\hat{p}}Z_2 \tag{4}$$

$$\frac{dz_1}{dt} = s_1z_1\big(\sqrt{\hat{p}} - h_1\big) \tag{5}$$

$$\frac{dz_2}{dt} = s_2z_2\big(\sqrt{\hat{p}} - h_2 - h_3z_2\big) \tag{6}$$

The system (4)-(6) has singularity when Jacobian calculated, because in (4)-(6) terms are not free from square root. To understand the system of equations (4)-(6) we assume $\hat{p} = p^2$,

$$\frac{dp}{dT} = p\big(1 - p^2(t - \tau)\big) - z_1 - z_2 \tag{7}$$

$$\frac{dz_1}{dT} = s_1 z_1 (p - h_1) \tag{8}$$

$$\frac{dz_2}{dT} = s_2 z_2 (p - h_2 - h_3 z_2) \tag{9}$$

With initial conditions $p(0) > 0, z_1 > 0, z_2 > 0$

**Table:** Meaning of parameters used in equation (1)-(3)

| **Symbols** | **Meaning** |
|---|---|
| R | Prey population at time t |
| $D_1$ | Predator population at time t |
| $D_2$ | Population of second predator at time t |
| $h_j$ | Death rate |
| $c_j$ | Predation rate |
| g | Intra-specific competition rate |
| $K_1$ | Carrying capacity |
| $b_1$ | Conversion efficiency |
| $s$ | Intrinsic growth rate of prey |

The subscript $j = 1 \; or \; 2$

## 6.3 Equilibrium Points of the System:

We calculate equilibrium points of (7)-(9)

$$p(1 - p^2) - z_1 - z_2 = 0 \tag{10}$$

$$s_1 z_1 (p - h_1) = 0 \tag{11}$$

$$s_2 z_2 (p - h_2 - h_3 z_2) = 0 \tag{12}$$

After simplification (10)-(12) system of equation. We calculated five points of equilibrium for the system (7)-(9).

1) First, we calculate Trivial equilibrium points i.e., origin $G_0 = (0,0,0)$
2) Second, we calculate equilibrium point is axial $G_A = (1,0,0)$
3) Third, we calculate equilibrium point is boundary $G_{B_1} = \big(h_{1,}h_1(1 - h_1^2), 0\big)\ and\ G_{B_2} = (\tilde{p}, 0, \tilde{z_2})$

Where $\tilde{z_2} = \tilde{p}(1 - \tilde{p}^2) and\ \tilde{p}$ is the first positive eigen root of the cubic equation.

$$h_3\tilde{p}^3 + (1 - h_3)\tilde{p} - h_2 = 0 \qquad (13)$$

if $h_1 < 1$ then $G_{B_1}$the first boundary equilibrium point exists and $G_{B_2}$ exists only if $\tilde{p}\ < 1$

4) Forth, the interior equilibrium points $G^* = (p^*, z_1^*, z_2^*)$ where $p^* = h_1, z_1^* = h_1(1 - h_1^2) - \frac{d_1 - d_2}{d_3}, and\ \ z_2^* = \frac{d_1 - d_2}{d_3}$.

## 6.4 Dynamical Behaviour when $\tau = 0$

To study the nature of the system of equation (7)-(9) about the five feasible equilibrium points the variational matrix is

$$W = \begin{bmatrix} 1 - 3p^2 & -1 & -1 \\ s_1 z_1 & s_1(p - h_1) & 0 \\ s_2 z_2 & 0 & s_2(p - h_2 - 2h_3 z_2) \end{bmatrix}$$

Char. Polynomial of W corresponding to eigen value $G_0$ is given by

$$(1 - \lambda)(\ s_1 h_1 + \lambda)(s_2 h_2 + \lambda) = 0 \qquad (14)$$

Because Equation (14) contains one positive root definitely, so $G_0$ is unstable.

At the equilibrium point $G_A$, the characteristic polynomial of W is

$$(2 +\ \lambda)(\ s_1(1 - h_1) -\ \lambda)(\ s_2(1 - h_2) -\ \lambda) = 0$$

Eigen values of $W$ at $G_A$ are $s_1(1 - h_1), -2\ and\ s_2(1 - h_2)$, so, the equilibrium points $G_A$ of the system (7)-(9) is stable if $h_1 > 1 and\ h_2 > 1$.

One interesting result also obtained, if the equilibrium point $G_A$ is stable in that case the remaining equilibrium points $G_{B_1}, G_{B_2} and\ G^*$ does not exists, and if equilibrium point $G_{B_1}, G_{B_2} and\ G^*$ exists then the equilibrium point $G_A$ is not stable. The characteristic equation of W at $G_{B_1}$is

$$(s_2(h_1 - h_2) - \lambda)(\lambda^3 + (3h_1^2 - 1)\lambda + s_1 h_1) = 0 \qquad (15)$$

Only if $h_2 > h_1$and $h_1 > \frac{1}{\sqrt{3}}$ then equilibrium points $G_{B_1}$of the system (7)-(9) locally asymptotically stable. During calculation we notice that when $G_{B_1}$(equilibrium point )is stable then $G^*$(equilibrium point) does not exists and vice-versa.

At $G_{B_2}$, the characteristic polynomial of W is

$$(s_1(\tilde{p} - h_1) - \lambda)(\lambda^2 + (3\tilde{p}^2 + s_2 h_3 \tilde{z}_2 - 1)\lambda + s_2 \tilde{z}_2) = 0 \qquad (16)$$

if $h_1 > \tilde{p}\ and\ \tilde{p} > \frac{1}{\sqrt{3}}$ then the point $G_{B_2}$ is local asymptotic stable. At point $G^*$ the characteristic polynomial of the system (7)-(9) is given by

$$\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 = 0 \qquad (17)$$

Where $J_1 = s_2 h_3 z_2^* + 3h_1^2 - 1, J_2 = (3h_1^2 - 1)s_2 h_3 z_2^* + s_2 z_2^* + s_1 z_1^*\ and\ J_3 = s_1 h_3 z_1^* s_2 z_2^*$.

By Routh- Hurwitz criteria the equilibrium points $G^*$in system (7)-(9) is asymptotically locally stable if $h_1 > \frac{1}{\sqrt{3}}$.

## 6.5 Study Behaviour when $\tau > 0$

The characteristic equation of (7)-(9) around any equilibrium point is

$$\begin{vmatrix} 1 - p^2 - 2p^2 e^{-\lambda\tau} - \lambda & -1 & -1 \\ s_1 z_1 & s_1(p - h_1) - \lambda & 0 \\ s_2 z_2 & 0 & s_2(p - h_2 - 2h_3 z_2) - \lambda \end{vmatrix} = 0$$

After simplification

$$\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3 + e^{-\lambda\tau}(Q_1\lambda^2 + Q_2\lambda + Q_3) = 0 \qquad (18)$$

Where $L_1 = -1 + s_1 h_1 + s_2 h_2 - s_1 p - s_2 p + p^2 + 2s_2 h_3 z_2$

$$Q_1 = 2p^2$$

$$\begin{aligned} L_2 = {} & -s_1 h_1 - s_2 h_2 + s_1 h_1 s_2 h_2 + s_1 p + s_2 p - s_1 s_2 h_1 p - s_1 s_2 h_2 p + s_1 s_2 p^2 \\ & + s_1 h_1 p^2 + s_2 h_2 p^2 - s_1 p^3 - s_2 p^3 + s_1 z_1 + s_2 z_2 - 2s_2 z_2 h_3 \\ & + 2s_1 s_2 h_1 z_2 h_3 - 2s_1 s_2 p z_2 h_3 + 2s_2 z_2 h_3 p^2 \end{aligned}$$

$$Q_2 = 2s_1 h_1 p^2 + 2s_2 h_2 p^2 - 2s_1 p^3 - 2s_2 p^3 + 4s_2 z_2 h_3 p^2$$

$$L_3 = -s_1h_1s_2h_2 + s_1h_1s_2p + s_1ps_2h_2 - s_1s_2p^2 + s_1h_1s_2h_2p^2 - s_1h_1s_2p^3 - s_1h_2s_2p^3 + s_1s_2p^4 + s_1z_1s_2h_2 - s_1z_1s_2p + s_1h_1s_2z_2 - 2s_1h_1s_2h_3z_2 - s_1ps_2z_2 + 2s_1h_3ps_2z_2 + 2s_1h_1s_2h_3z_2p^2 - 2s_1p^3s_2h_3z_2 + 2s_1z_1s_2h_3z_2,$$

$$Q_3 = 2s_1h_1s_2h_2p^2 - 2s_1h_1s_2p^3 - 2s_1h_2s_2p^3 + 4s_1s_2p^2h_1h_3z_2 - 4s_1s_2p^3h_3z_2$$

If the eigen values of the characteristic polynomial (18) do not have positive real part, then equilibrium point is stable. It is not an easy task when all latent roots of (18) have no positive real part. When $\tau = 0$ equation (18) transforms to

$$\lambda^3 + (L_1 + Q_1)\lambda^2 + (L_2 + Q_2)\lambda + L_3 + Q_3 = 0 \qquad (19)$$

By Routh-Hurwitz criterion

**(H)** If $(L_1 + Q_1) > 0$, $(L_3 + Q_3) > 0$, $(L_1 + Q_1)\,(L_2 + Q_2) > (L_3 + Q_3)$, then equation (19) has all the roots negative real parts.

Let us suppose that if $\lambda = 0$ is a solution of (18) then $L_3 + Q_3 = 0$, a contradiction to fact in **(H).** Hence, $\lambda = 0$ is not eigen value of equation (18). Suppose that for some arbitrary $\tau \geq 0, i\omega$ with a condition $\omega > 0$ is a solution of (18), so

$$-i\omega^3 - L_1\omega^2 + iL_2\omega + L_3 + (cos\omega\tau - isin\omega\tau)(-Q_1\omega + iQ_2\omega + Q_3) = 0 \qquad (20)$$

Separate real and imaginary parts,

$$L_3 - L_1\omega^2 + (Q_3 - Q_1\omega^2)\cos\omega\tau + Q_2\omega\sin\omega\tau = 0 \qquad (21)$$

$$L_2\omega - \omega^3 + Q_2\omega cos\omega\tau - (Q_3 - Q_1\omega^2)sin\omega\tau = 0 \qquad (22)$$

Which gives $\omega^6 + l\omega^4 + m\omega^2 + n = 0$ (23)

Where $l = L_1^2 - Q_1^2 - 2L_2, m = L_2^2 - Q_2^2 + 2Q_1Q_3 - 2L_1L_3, n = L_3^2 - Q_3^2$

Let $\alpha = \omega^2$, then equation (23) becomes

$$\alpha^3 + l\alpha^2 + m\alpha + n = 0 \qquad (24)$$

Where $o(\alpha) = \alpha^3 + l\alpha^2 + m\alpha + n)$

**6.6 Lemma1:** For equation (24), we have results

a) If $n < 0$, the equation (24) has one positive root definitely.

b) If $n \geq 0$, and $(l^2 - 3m) \leq 0$, then equation (24) has all root negative.

c) If $n \geq 0$, and $(l^2 - 3m) > 0$, then positive real root for equation (24) iff.

$$\alpha_1^* = \frac{-l \pm \sqrt{l^2 - 3\mathrm{m}}}{3} > 0 \ and \ o(\alpha_1^* \leq 0.)$$

Assume equation (24) has a minimum one positive real root $\omega_0 = \sqrt{\alpha_0}$.

From equation (21)-(22), we obtain,

$$cos\omega_0\tau = \frac{-\left(Q_2\omega_0^2(L_2 - \omega_0^2) + (L_3 - L_1\omega_0^2)(Q_3 - Q_1\omega_0^2)\right)}{(Q_3 - Q_1\omega_0^2)^2 + (Q_2\omega_0)^2} \qquad (25)$$

$$\tau_i = \frac{1}{\omega_0}\arccos\left(\frac{-\left(Q_2\omega_0^2(L_2 - \omega_0^2) + (L_3 - L_1\omega_0^2)(Q_3 - Q_1\omega_0^2)\right)}{(Q_3 - Q_1\omega_0^2)^2 + (Q_2\omega_0)^2} + 2j\pi\right) \qquad (26)$$

Where i =0,1,2,3…

**6.7 Lemma2**: Suppose $o(\alpha_0) = (3\alpha_0^2 + 3l\alpha_0 + m)$ and condition in (**H**) are satisfied. For (i=0,1,2,3,4…), let $\lambda(\tau) = \beta(\tau) + i\omega(\tau)$ be the latent value of equation (18) satisfying $\beta(\tau_i) = 0, \omega(\tau_i) = \omega_0$, where

$$\tau_i = \frac{1}{\omega_0}\arccos\left(\frac{-\left(Q_2\omega_0^2(L_2 - \omega_0^2) + (L_3 - L_1\omega_0^2)(Q_3 - Q_1\omega_0^2)\right)}{(Q_3 - Q_1\omega_0^2)^2 + (Q_2\omega_0)^2} + 2i\pi\right)$$

Then $\pm i\omega_0$ are latent values. If condition of transversality condition

$$(H_0) \qquad \beta'(\tau_I) = \frac{Re\lambda(\tau)}{d\tau}|_{\lambda=i\omega_0} \neq 0 \text{exists,}$$

then Hopf-Bifurcation observed in system (7)-(9) at any arbitrary equilibrium point and $\tau = \tau_i$.

Proof. Suppose $\lambda = \lambda(\tau)$ be the solution of equation (18) put $\lambda(\tau)$ in equation (18) and differentiating equation (18) w.r.t $\tau$, we observed that

$$\left[(3\lambda^2 + 2L_1\lambda + L_2) + ((\lambda^2 Q_1 + \lambda Q_2 + Q_3)(-\tau) + (2\lambda Q_1 + Q_2))e^{-\lambda t}\right]\frac{d\lambda}{d\tau}$$

$$= \lambda(\lambda^2 Q_1 + \lambda Q_2 + Q_3)e^{-\lambda t}$$

Thus,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2\lambda_1 + L_2)e^{\lambda\tau}}{\lambda(\lambda^2 Q_1 + \lambda Q_2 + Q_3)} + \frac{(2\lambda Q_1 + Q_2)}{\lambda(\lambda^2 Q_1 + \lambda Q_2 + Q_3)} - \frac{\tau}{\lambda}$$

From equation (21)-(23), we have

$$\beta'(\tau_0) = Re\left[\frac{(3\lambda^2 + 2\lambda_1 + L_2)e^{\lambda\tau}}{\lambda(\lambda^2 Q_1 + \lambda Q_2 + Q_3)}\right] + Re\left[\frac{(2\lambda Q_1 + Q_2)}{\lambda(\lambda^2 Q_1 + \lambda Q_2 + Q_3)}\right]$$

$$= \frac{1}{\Lambda}[3\omega_0^2 + 2(L_1^2 - Q_1^2 - 2L_2)\omega_0^4 + (L_2^2 - Q_2^2 + 2Q_1Q_2 - 2L_1L_3)\omega_0^2]$$

$$= \frac{1}{\Lambda}(3\omega_0^6 + 2l\omega_0^4 + m\omega_0^2)$$

$$= \frac{\omega_0^2}{\Lambda}o'(\alpha_0),$$

Where $\Lambda = (Q_3 - Q_1\omega_0^2)^2 + (Q_2\omega_0)^2$. Note that when $\Lambda > 0$ and $\omega_0 > 0$, we concluded that $sign[\beta'(\tau_0)] = sign[o'(\alpha_0)]$

This proves the lemma.

At equilibrium point $G_A$, the equation (18) becomes

$$\lambda^3 + R_1\lambda^2 + R_2\lambda + e^{-\lambda\tau}(R_3\lambda^2 + R_4\lambda + R_5) = 0 \qquad (27)$$

$$and\ l, m, n\ becomes\ R_1^2 - R_3^2 - 2R_2, R_2^2 - R_4^2 + 2R_3R_5\ and\ - R_5^2\ respectively, and\ \tau_j$$

Becomes

$$\tau_A = \frac{1}{\omega_0}\arccos\left(\frac{-\left(R_4\omega_0^2(R_2 - \omega_0^2) - R_1\omega_0^2(R_5 - R_3\omega_0^2)\right)}{(R_5 - R_3\omega_0^2)^2 + (R_4\omega_0)^2}\right) \qquad (28)$$

Where

$$R_1 = -s_1 - s_2 + s_1h_1 + s_2h_2,$$

$$R_2 = s_1s_2 - s_1s_2h_1 - s_1s_2h_2 + s_1s_2h_1h_2, R_3 = 2,$$

$$R_4 = -2s_1 - 2s_2 + 2s_1h_1 + 2s_2h_2$$

$$R_5 = -2s_1s_2(-1 + h_1)(1 - h_2)$$

**6.8 Theorem1:** Suppose that axioms in **(H)** and $(\boldsymbol{H_0})$ are satisfied. Then

a. The system (18) latent points $G_A$ are asymptotically stable for every $\tau\epsilon[0, \tau_A)$.

b. At the point of equilibrium $G_A$ the system of equation (18) show Hopf-bifurcation, when $\tau = \tau_A$ in the same way equilibrium point $G_{B1}$,the equation (15) changes to

$$\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3 + e^{-\lambda\tau}(S_4\lambda^2 + S_5\lambda) = 0 \qquad (29)$$

$$l, m, n\ becomes,$$

$$l = S_1^2 - S_4^2 - 2S_2, m = S_2^2 - S_5^2 - 2S_1S_3, n = S_3^2\ and\ \tau_i becomes$$

$$\tau_{B1} = \frac{1}{\omega_0} arccos\left(\frac{-(S_5\omega_0^2(S_2 - \omega_0^2) - S_4\omega_0^2(S_3 - S_1\omega_0^2))}{(S_4\omega_0^2)^2 + (S_5\omega_0)^2}\right) \qquad (30)$$

Where, $S_1 = -1 - s_2h_1 + h_1^2 + s_2h_2$ ,$S_2 = h_1s_1 + h_1s_2 - s_1h_1^3 - s_2h_1^3 - s_2h_2 + s_2h_1^2h_2$, $S_3 = -s_1s_2h_1^2 + s_1s_2h_1^4 + s_1s_2h_1h_2 - s_1s_2h_1^2h_2$, $S_4 = 2h_1^2$, $S_5 = -2s_2h_1^3 + 2s_2h_1^3 + 2s_2h_1^2h_2$.

### 6.9 Theorem2. Consider that **(H)** and **($H_0$)** are fulfilled and

a. If $(l^2 - 3m) \leq 0$, all eigen values of (29) cannot have positive real parts for every $\tau \geq 0$, and so the latent values $G_{B1}$of the system (7)-(9) is stable asymptotic for every $\tau \geq 0$.

b. If $(l^2 - 3m) > 0, \alpha_1^* > 0\ and\ o(\alpha_1^*) \leq 0$, holds then the equilibrium points $G_{B1}$of the system (7)-(9) is stable asymptotic for every $\tau \in [0, \tau_{B1})$.

c. If every statement in (b) and $o'(\alpha_0) \neq 0$ satisfied, then the system (7)-(9) show a Hopf-bifurcation at point of equilibrium $G_{B1}$, when $\tau = \tau_{B1}$.

At equilibrium point $G_{B2}$, the equation (15) becomes

$$\lambda^3 + W_1\lambda^2 + W_2\lambda + W_3 + e^{-\lambda\tau}(W_4\lambda^2 + W_5\lambda + W_6) = 0 \qquad (31)$$

$l, m, n\ changes\ to\ l = W_1^2 - W_4^2 - 2W_2, m = W_2^2 - W_5^2 + 2W_4W_6 - 2W_1W_3, n = W_3^2 - W_6^2$ respectively, and $\tau_i$ becomes

$$\tau_{B2} = \frac{1}{\omega_0} arccos\left(\frac{-(W_5\omega_0^2(W_2 - \omega_0^2) + (W_3 - W_1\omega_0^2)(W_6 - W_4\omega_0^2)}{(W_6 - W_4\omega_0^2)^2 + (W_5\omega_0)^2}\right) \qquad (32)$$

Where

$W_1 = -1 + s_1h_1 - s_1\hat{p} + \hat{p}^2 + s_2h_3$

$W_2 = -s_1h_1 + s_1\hat{p} + s_1h_1p^2 - s_1p^3 + s_2\hat{z}_2 - s_2h_3\hat{z}_2 + s_1h_1s_2h_3\hat{z}_2 - s_1s_2ph_3\hat{z}_2 + s_2h_3p^2\widehat{z_2}$

$$W_3 = s_1s_2h_1\hat{z}_2 - s_1s_2h_1h_3\hat{z}_2 - s_1s_2\hat{p}\hat{z}_2 + s_1s_2ph_3\hat{z}_2 + s_1s_2h_1h_3\widehat{p^2}z_2 - s_1s_2p^3h_3\hat{z}_2$$

$W_4 = 2\hat{p}^2, W_5 = 2s_1h_1\hat{p}^2 - 2s_1\hat{p}^2 - 2s_1\hat{p}^3 + 2s_2h_3\hat{p}^2\hat{z}_2, W_6 = 2s_1s_2h_1h_3\hat{p}^2\hat{z}_2 - 2s_1s_2\hat{p}^3\hat{z}_2$

**6.10 Theorem3.** Let (**H**) and ($H_0$) are satisfied and

a) If $n \geq 0\ and\ (l^2 - 3m) \leq 0$, all solution of equation (31) has no real part with positive sign for all $\tau \geq 0$, and so $G_{B_2}$ the point of equilibrium of mathematical equation (7)-(9) is stable asymptotic for every $\tau \geq 0$.

b) If either $n < 0\ or\ n \geq 0\ and\ (l^2 - 3m) > 0, \alpha_1^* > 0\ and\ o(\alpha_1^*) \leq 0 satisfy$ then $G_{B_2}$ the equilibrium points of the system (7)-(9) is stable asymptotic for every $\tau \epsilon [0, \tau_{B_2})$.

c) If all axioms in (b) and $o'(\alpha_0) \neq 0$ hold, then the system (7)-(9) at $G_{B_2}$ occurred a Hopf-Bifurcation, when $\tau = \tau_{B_2}$. Similarly at the equilibrium point $G^*$, the equation (18) becomes,

$$\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 + e^{-\lambda\tau}(P_4\lambda^2 + P_5\lambda) = 0 \tag{33}$$

$l, m, n$ becomes $l = P_1^2 - P_4^2 - 2P_2, m = P_2^2 - P_5^2 - 2P_1P_3, n = P_3^2$ respectively and $\tau_i$ becomes

$$\tau_0 = \frac{1}{\omega_0} arccos\left(\frac{-(P_5\omega_0^2(P_2 - \omega_0^2) - P_4\omega_0^2(P_3 - P_1\omega_0^2))}{(P_4\omega_0^2)^2 + (P_5\omega_0)^2}\right) \tag{34}$$

$$P_1 = -1 + h_1^2 + s_2h_3z_2^*, P_2 = s_1z_1^* + s_2z_2^* - s_2h_3z_2^* + s_2h_3h_1^2z_2^*, P_3 = s_1s_2h_3z_1^*z_2^*$$

$$P_4 = 2h_1^2, P_5 = 2s_2h_3h_1^2z_2^*.$$

**6.11 Theorem 4** Let (**H**) and ($H_0$) are satisfied and if

a) If $(l^2 - 3m) \leq 0$, for every solution of equation (33) have no positive real parts for every $\tau \geq 0$, and hence $G^*$ point of equilibrium for (7)-(9) is stable asymptotic for every $\tau \geq 0$.

b) If $(l^2 - 3m) > 0$, $\alpha_1^* > 0$ and $o(\alpha_1^*) \leq 0$ satisfied then $G^*$ the point of equilibrium of the system (7)-(9) is stable asymptotic for every $\tau \epsilon [0, \tau_0)$.

c) If all axioms stated in (b) and $o'(\alpha_0) \neq 0$ satisfied, then the system (7)-(9) at $E^*$ equilibrium point shows a Hopf-Bifurcation when $\tau = \tau_0$.

## 6.12 Numerical Simulation

Numerical simulation performed with the help of MATLAB software. We choose the parametric values for the parameters

$$s_1 = 1, s_2 = 1 \; and \;\; h_3 = 0.58$$

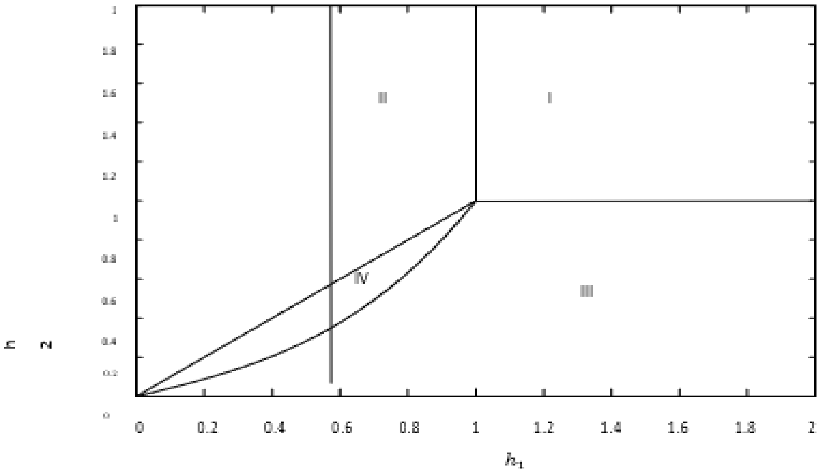

Figure 6.1

For each equilibrium point of the system, the stability areas with respect to variables $h_1 and\ h_2$ are displayed (see figure 6.1).

For all these graphs Initial value is fixed (0.41, 0.11,0.11)

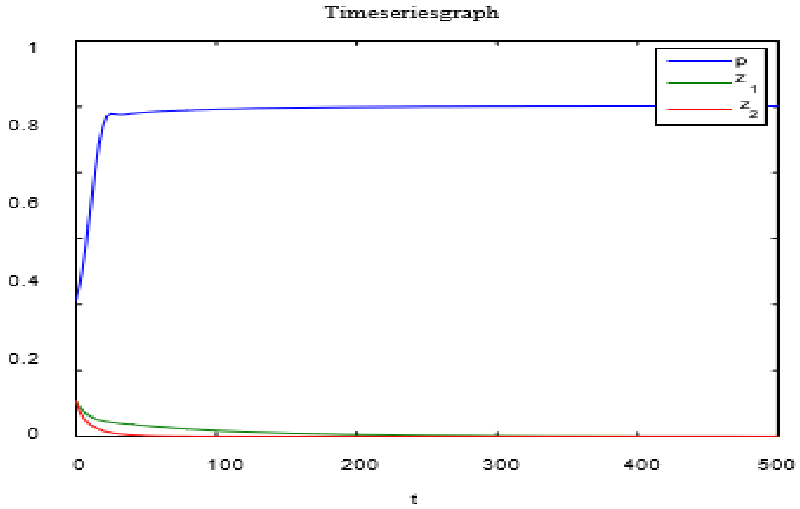

Figure 6.2. The system (7)-(9) has stable asymptotic axial equilibrium points $G_A$

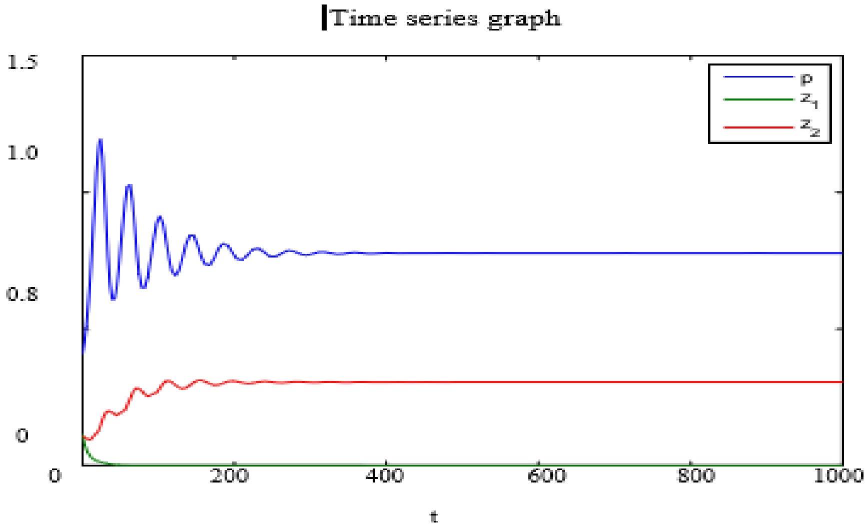

Figure 6.3. The axial equilibrium points $G_{B1}$ of systems (7)-(9) are asymptotically stable.

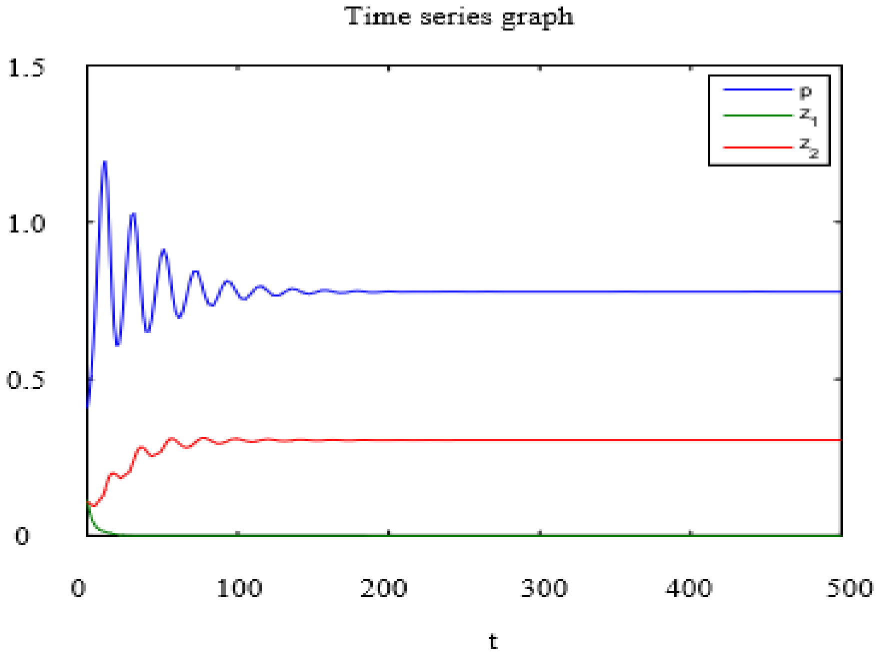

Figure 6.4. The system (7)-(9)'s axial equilibrium points $G_{B1}$ are stable asymptotic points.

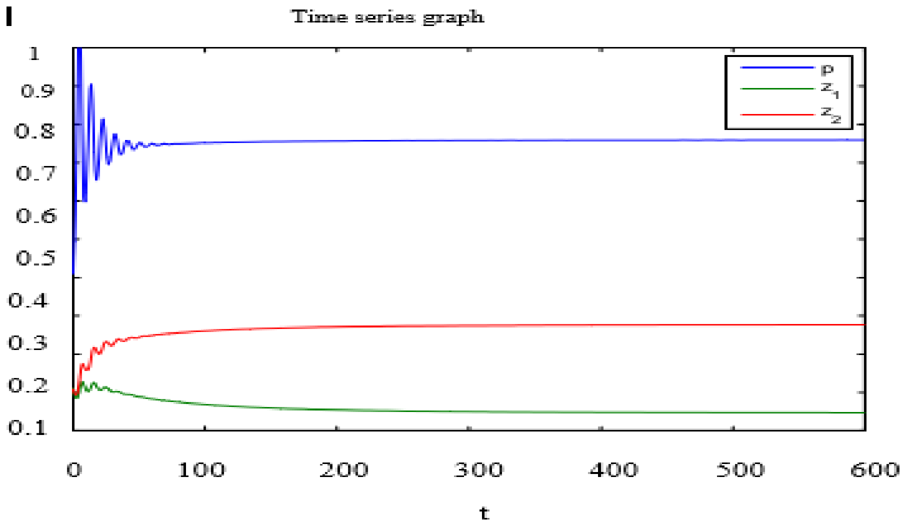

Figure 6.5. The system (7)-(9) has stable asymptotic axial equilibrium points $G^*$.

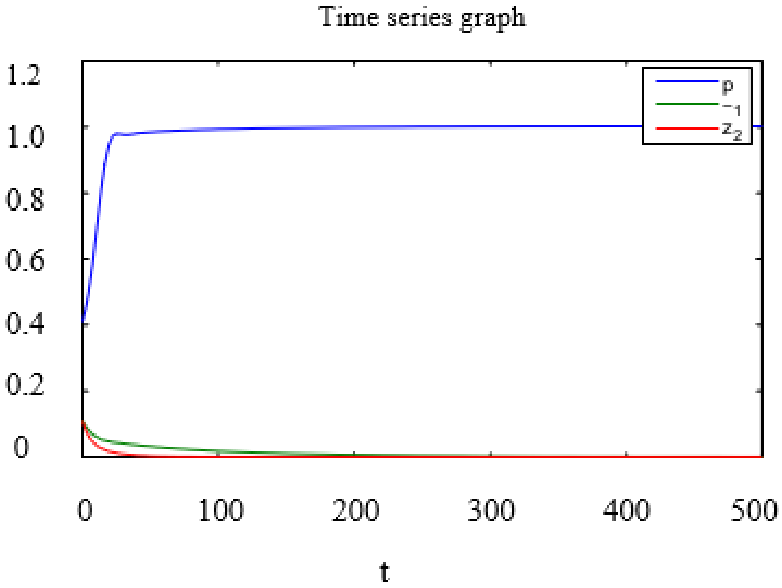

Figure 6.6. When $\tau = 0.3 < \tau_A = 0.757968$, the axial equilibrium points $G_A$of the system (7)-(9) are stable asymptotically.

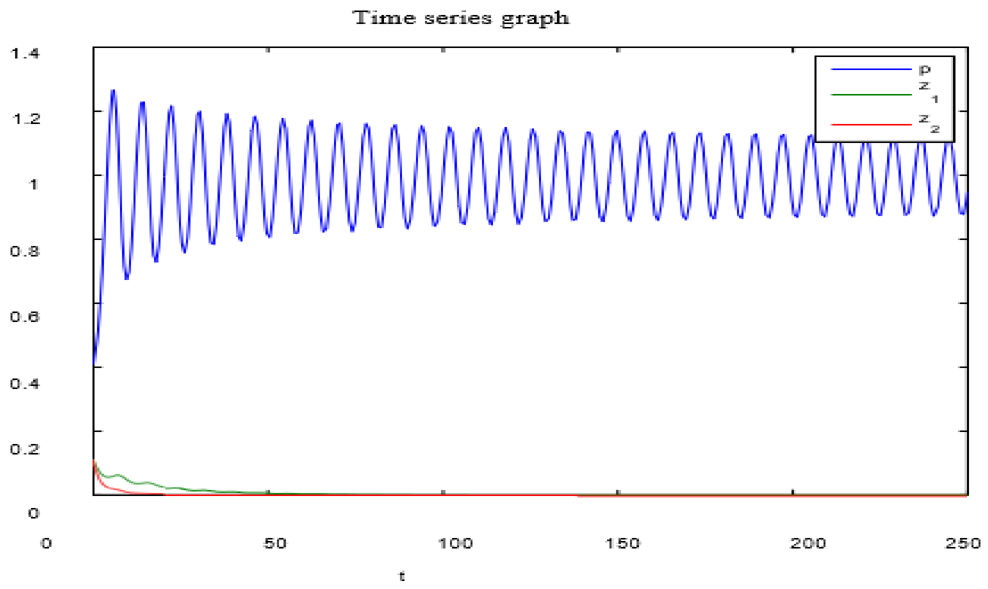

Figure 6.7. The axial equilibrium points $G_A$ of the system (7)–(9) lose stability and undergo a Hopf-Bifurcation at $\tau = 0.79 > \tau_A = 0.75$

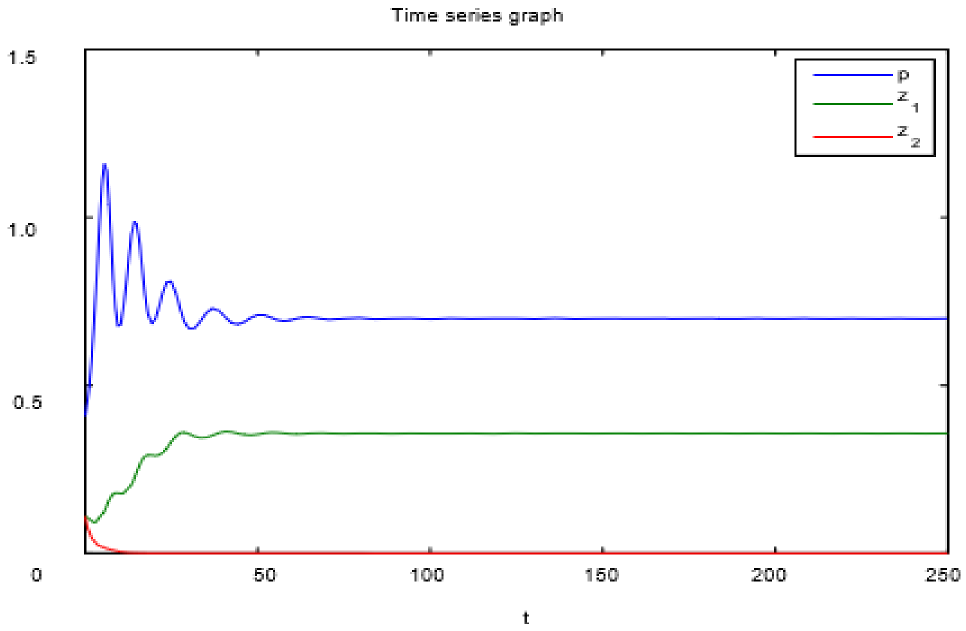

Figure 6.8. When $\tau = 0.75 < \tau_{B1} = 0.891635$ the axial equilibrium points $G_{B1}$of the system (7)-(9) are asymptotically stable.

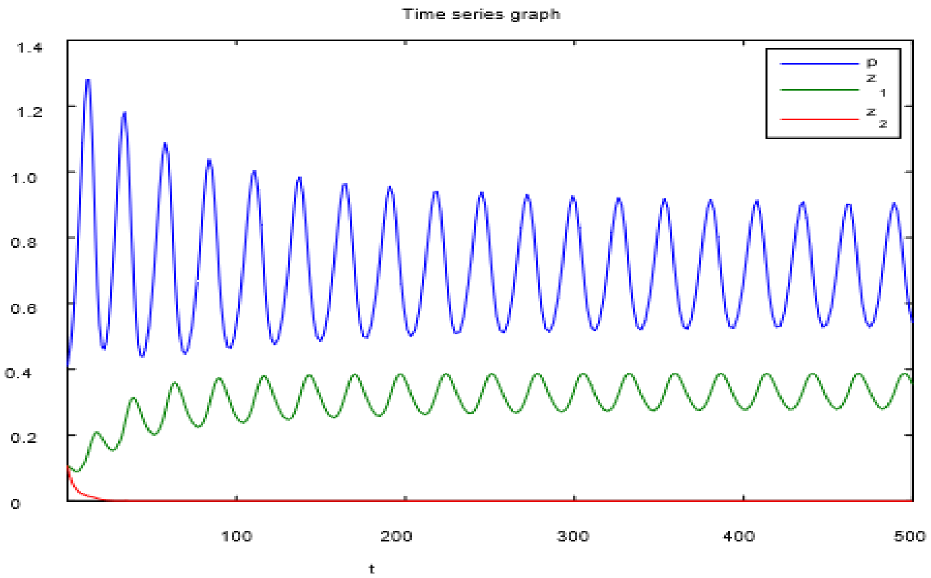

Figure 6.9. The axial equilibrium points $E_{B1}$ of the system (7)-(9) are unstable at $\tau = 0.9 > \tau_{B1} = 0.891635$ and Hopf bifurcation takes place.

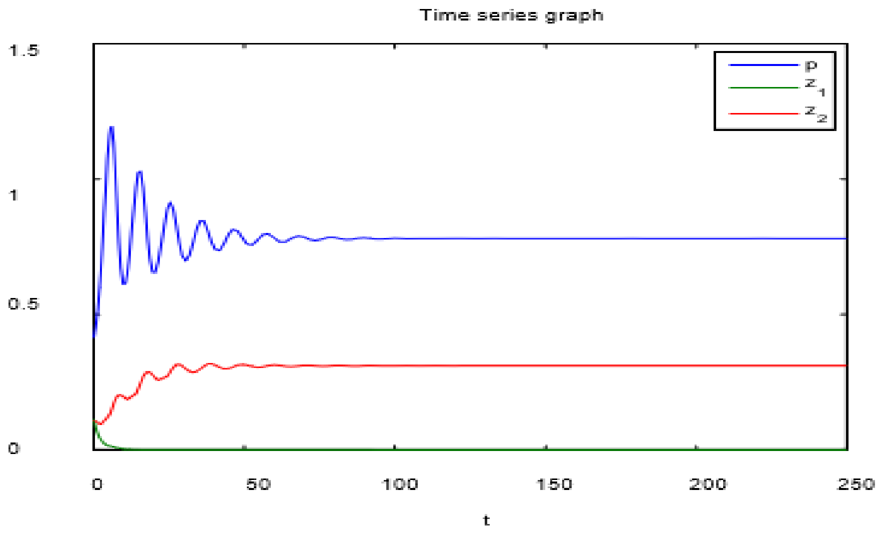

Figure 6.10. The system's axial equilibrium points E B2 are stable asymptotically at $\tau = 0.75 < \tau_{B2} = O.919727$.

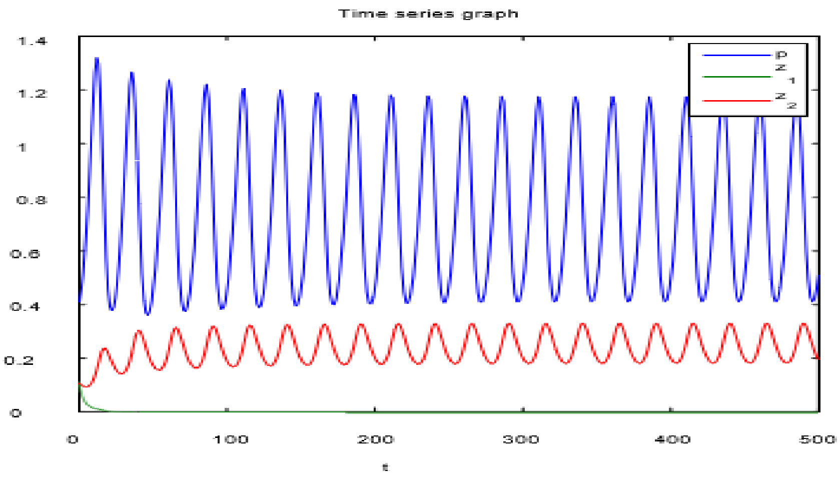

Figure 6.11. The axial equilibrium points $E_{B2}$ of the system (7)-(9) lose stability and undergo Hopf-Bifurcation at $\tau = 0.95 > \tau_{B2} = 0.919727$

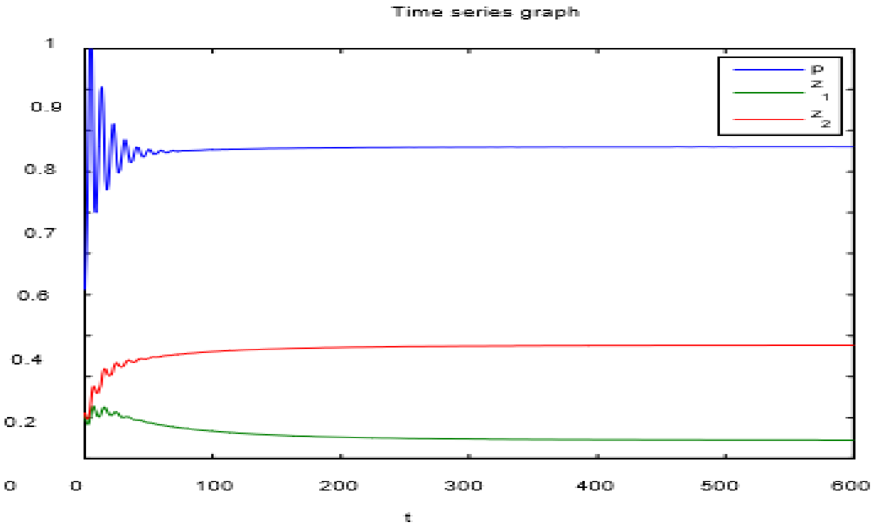

Figure 6.12. When $\tau = 0.79 < \tau_0 = 0.920973$, the system (7)-(9)'s positive interior equilibrium points G* are asymptotically stable.

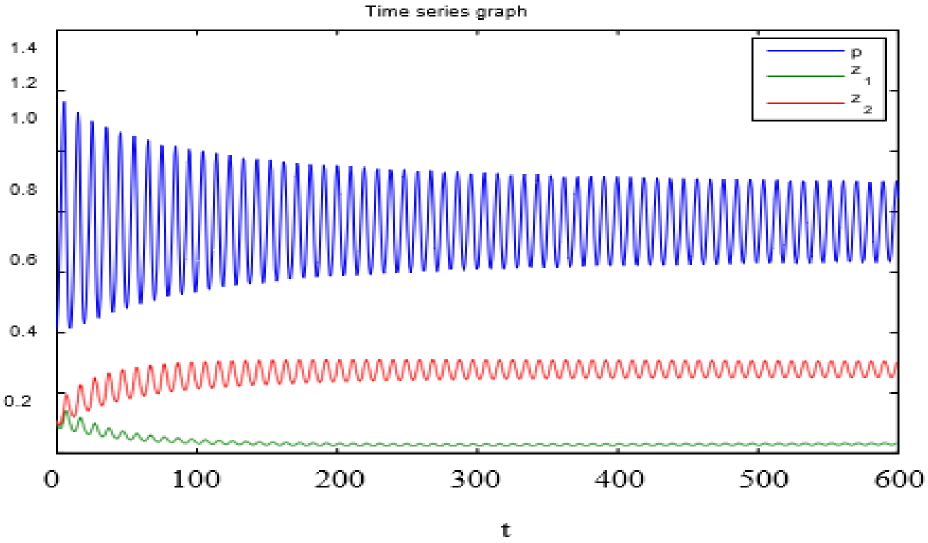

Figure 6.13 When $\tau = 0.924 > \tau_0 = 0.920973$, the positive internal equilibrium points $E^*$ of the system (7)-(9) lose their stability, resulting in a Hopf-Bifurcation.

For the above set of parameters and $h_1 = 1.1$ and $h_2 = 1.6$, $G_A(1,0,0)$ axial equilibrium point is stable asymptotic (see figure 6.2). Choose $d_1 = .7$ and $d_2 = 1.6$ and other parameters remain same points $G_{B_2}(0.7781, 0.0000, 0.3070)$ the boundary equilibrium point is stable asymptotic locally (see figure 6.3). If we set $h_1 = 1.7$ and $h_2 = 0.6$, $G_{B_2}(0.7781, 0.0000, 0.3070)$ the boundary equilibrium point, is stable local asymptotic (see figure 6.4). If we set $h_1 = 0.76$ and $h_2 = 0.6$ we observed that $G^*(0.7600, 0.0452, 0.2760)$ interior equilibrium point is local asymptotic stable (see figure 6.5). If we set $h_1 = 1.1$ and $h_2 = 1.6$ and delay $\tau = 0.3 < \tau_A = 0.75$, then $G_A(1,1,0)$ the axial equilibrium point is stable locally asymptotic (see figure 6.6) and system (7)-(9) at $\tau_A = 0.75$ showed Hopf-Bifurcation (see figure 6.7). If we set $h_1 = 1.7$ and $h_2 = 0.6$ $G_{B_2}(0.7781, 0.0000, 0.3070)$ the boundary equilibrium points when $\tau = 0.8 < \tau_{B_2} = 0.919727(Figure\ 6.10)$ and if we take $\tau_{B_2} = 0.919727$ the system (7)-(9) showed Hopf-Bifurcation (see figure 6.11). If we take $h_1 = 0.76$ and $h_2 = 0.6$ then "$G^*(0.7600, 0.0452, 0.2760)$ is stable asymptotic when delay $\tau = 0.79 < \tau_0 = 0.920973$ $(Figure\ 6.12)$ and at $\tau_0 = 0.920973$ in the system (7)-(9) Hopf-bifurcation is observed (see figure 6.13). According to the numerical simulation above, the duration of the delay is the smallest when just the system's prey population exists and longest when every of the system's species survive i.e., $\tau_A < \tau_{B_1} < \tau_{B_2} < \tau_0$ .

## 6.13 Conclusion

Limit cycles arise spontaneously under suitable situations that involve model parameters in Valerio Ajrajdi et al [123] prey-predator model. In this paper, we explored a one-prey, multiple predator system with a delay in the prey's logistic growth rate and a square root functional response. The system becomes even more stable than it was because of the addition of another rival predator [123], and limit cycles don't appear in the system by itself.

(1) The prey population will survive and the predator population will die out if the ratio of their mortality rates to the square root of the prey population's carrying capacity is larger than the product of their particular conversion efficiency and predation rates. The prey population (R) and one of the predator populations ($D_1$) survive, and the predator population ($D_2$) that exhibits a density dependent mortality rate goes extinct, if the ratio of the predator's ($D_1$) death rate to the product of its conversion efficiency and predation rate is less than the ratio of the predator's ($D_2$) death rate to the product of its conversion efficiency and predation rate.

(2) If the predator's mortality rate ($D_1$) to the total amount of its conversion efficiency and predation rate falls short of the predator's mortality rate ($D_2$) to the total value of its conversion efficiency and predation frequency, both the predator and prey populations ($D_2$) will survive, but the predator population ($D_1$) will disappear.

(3) When the predator's conversion efficiency, predation percentage, and density are all combined together, the predator's mortality rate ($D_1$) falls short of the prey population's equilibrium point. When the ratio of the predator's mortality rate ($D_1$) to the sum of its conversion efficiency and predation rate is greater than the square root of one-third of the prey population's available capacity, coexistence occurs in the system.

(4) When the effect of time lag on the system is considered, limit cycles appear for all steady state when the time delay exceeds a given limits.

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