FIXED POINT THEOREMS FOR VARIOUS MAPPINGS IN ABSTRACT SPACES

Thesis submitted for the Award of the Degree of

DOCTOR OF PHILOSOPHY
in
MATHEMATICS

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Transforming Education Transforming India

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2023

CERTIFICATE

This is to certify that the work reported in the Ph.D. thesis entitled "**Fixed Point Theorems for Various Mappings in Abstract Spaces**" submitted in fulfillment of the requirement for the award of degree of Doctor of Philosophy (Ph.D.) in the Department of Mathematics, is a research work carried out by Astha Malhotra, 41900779, is bonafide record of her original work carried out under my supervision and that no part of the thesis has been submitted for any other degree, diploma or equivalent course.

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DECLARATION

I, hereby declare that the presented work in the thesis entitled "**Fixed Point Theorems for Various Mappings in Abstract Spaces**" in fulfilment of degree of Doctor of Philosophy (Ph.D.) is outcome of the research work carried out by me under the supervision Dr. Deepak Kumar, working as Associate Professor and Assistant Dean, in the Department of Mathematics of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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Abstract

Fixed point theory has always been an inevitable part of mathematical analysis, being a combination of analytical, topological, and geometrical aspects of mathematics. With vast applications in the fields of mathematics like approximation theory, game theory, optimization theory, mathematical modeling, graph theory, and interdisciplinary fields like simulation functions in physics and Nash equilibrium in economics, to name a few. Apart from numerous applications, a constructive proof of a fixed point theorem renders an algorithm in the form of an iterative scheme to find a fixed point of a map. Not only fixed points but also coincidence points and common fixed points are significant because they are an extended part of fixed points for a pair of maps. The theory of fixed points in multivalued maps is as essential as the theory of single-valued maps. Even after much-acclaimed work in the literature on the existence and uniqueness of fixed points, many questions remain unanswered since there are numerous examples known that possess fixed points but do not satisfy some or all hypotheses of the results in the literature.

The objective of the research work in this thesis is to find a generalized approach for establishing the existence and uniqueness of fixed points for different contraction maps as well as non-contractive maps in abstract spaces and to introduce more generalized metric spaces. Each chapter exhibits fixed point results in various abstract metric spaces. Some of these spaces are well known in the literature while others have been introduced as a result of the research work.

Chapter 1 gives a brief introduction to the research work along with some notations and definitions used throughout the thesis. The chapter also presents a short summary giving an overview of each chapter.

Chapter 2 is devoted to fixed point results in orthogonal metric space that are extended with some generalized contraction maps like orthogonal α-η-$G\mathcal{F}$-contraction, orthogonal α-type $\mathcal{F}$-contraction, orthogonal TAC-type S-contraction, orthogonal TAC-contraction, orthogonal *Suzuki-Berinde* type F-contraction, and orthogonal $\mathcal{F}$-weak contraction. As an application, the existence and uniqueness of the solution for a first order differential equation are discussed.

Chapter 3 generalizes and unifies the fixed point results in relation theoretic metric space, briefly written as $\mathcal{R}$-metric space. Using $\mathcal{F}$-weak expansive map, multivalued counter part of $\mathcal{F}$-contraction, $\mathcal{F}$-weak contraction, almost $\mathcal{F}$-contraction,

and α-type $\mathcal{F}$-contraction an attempt is made to extend the literature of fixed point results in $\mathcal{R}$-metric space. Many examples as well as a potential application of determining the existence of a solution for a non-homogeneous, non-linear Volterra integral equation endowed with a binary relation $\mathcal{R}$ are included in this chapter, along with stability results.

Chapter 4 presents a novel class of metric space termed as C^*-algebra valued $\mathcal{R}$-metric space, which generalizes C^*-algebra valued metric space. We also introduce the idea of C^*-algebra valued $\mathcal{R}$-contractive map and corresponding fixed point results, as well as the existence and uniqueness of coincidence and common fixed points using the Picard-Jungck iteration process. The results are generalized enough to derive fixed point, coincidence point, and common fixed point results in C^*-algebra valued ordered metric space, C^*-algebra valued metric space, and metric space. As an application, the results obtained are applied to C^*-algebra valued metric space endowed with a directed graph.

Chapter 5 emphasizes to introduce the idea of bipolar $\mathcal{R}$-metric space that exists by associating an arbitrary binary relation $\mathcal{R}$ with bipolar metric space and to illustrate some fixed point results. The notions of $\mathcal{F}_{\mathcal{R}}$-contractive map and $\mathcal{F}_{\mathcal{R}}$-expansive map are presented, and fixed point results are discussed in these settings. Furthermore, for certain specified conditions, the results reduce to a novel fixed point result in bipolar metric space and extends some well known results in the literature.

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List of Abbreviations

- $\mathbb{N}$ — set of natural numbers
- $\mathbb{N}'$ — $\mathbb{N} \cup \{0\}$
- $\mathbb{Z}$ — set of integers
- $\mathbb{Q}$ — set of rational numbers
- $\mathbb{Q}^c$ — set of irrational numbers
- $\mathbb{R}$ — set of real numbers
- $\mathbb{R}^+$ — set of positive real numbers
- $\mathbb{C}$ — set of complex numbers
- $M_2(\mathbb{R})$ — set of all 2×2 matrices over $\mathbb{R}$
- $M_2(\mathbb{C})$ — set of all 2×2 matrices over $\mathbb{C}$
- $\mho$, Λ — non-empty sets
- $\mathcal{R}$ — amorphous binary relation on $\mho$
- w.r.t — with respect to
- s.t — such that
- c.t.b — claimed to be
- iff — if and only if
- a.e — almost everywhere
- $\forall$ — for all
- $\exists$ — there exists
- A^H — conjugate transpose of matrix A
- C^*_{AV} — C^*-algebra valued
- $\mathcal{N}(\mho)$ — set of all non-empty subsets of $\mho$
- $\mathcal{K}(\mho)$ — set of all compact subsets of $\mho$
- $\mathcal{CB}(\mho)$ — set of all closed and bounded subsets of $\mho$

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Chapter 1

General Introduction

1.1 Introduction

The theory of fixed point analysis has always been an integral part of mathematical analysis. This field of research has grown in prominence over the last century and a half being an excellent combination of algebraic, topological, and geometrical aspects of mathematics. For a metric space $(\mho, d)$ and a self-map $\omega : \mho \to \mho$, a point $\varrho \in \mho$ is c.t.b a fixed point of ω if $\omega\varrho = \varrho$. For example, if $\omega\varrho = \varrho^3$ where $\omega : [-1, 1] \to [-1, 1]$, then ω has 3 fixed points, which are $-1, 0$ and 1. The following graph shows the fixed points obtained:

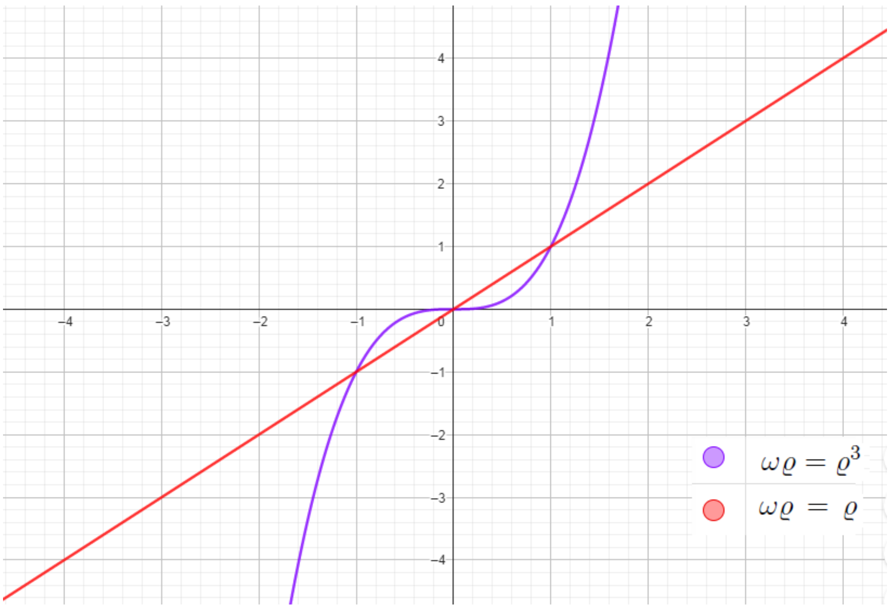

Figure 1.1: Fixed points of function $\omega\varrho = \varrho^3$.

The result that establishes the existence of at least one fixed point, subject to certain conditions, is known as a fixed point theorem. Picard (1890) introduced an iterative scheme under which a sequence $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ in (ρ,σ) defined by $\varrho_{\varpi+1} = \omega\varrho_{\varpi}\ \forall\ \varpi \in \mathbb{N}$, where $\omega : [\rho,\sigma] \to (-\infty,+\infty)$ is continuous and differentiable on (ρ,σ) and $|\omega^{'}\varrho| \leqslant L$ for some $L < 1$, converges to a solution of an equation $\omega\varrho = \varrho$. The manner in which this sequence was described constituted one of the turning points in the history of fixed point analysis and it is frequently used to demonstrate the existence and uniqueness of a fixed point. Brouwer (1911) explored the topological aspect of fixed point theory with his result which states that *"Every continuous map from a unit ball of* $\mathbb{R}^n$ *into itself has a fixed point."* Banach (1922) came up with a classical and the most celebrated principle called *"Banach Contraction Principle"* for the existence and uniqueness of the fixed point of a self-map on a complete metric space along with a contractive condition. Thereafter, numerous generalizations of the *Banach Contraction Principle* have been presented by the researchers (for reference, see Kannan (1968), Ćirić (1974), Czerwik (1993), Rhoades (2001), Ran & Reurings (2004), Long-Guang & Xian (2007), Berinde (2008), Wardowski (2012), Ma et al. (2014), Wardowski & Dung (2014), Alam & Imdad (2015), Sintunavarat (2016*a*), Gordji et al. (2017), Karapınar et al. (2019), Khalehoghli et al. (2020), Nazam et al. (2021)). Another class of map was given by Wang (1984) which initiated the idea of expansive map and established some fixed point results in this setting and henceforth, many researchers came up with the generalized expansive condition and proved certain fixed point results in various spaces (see Khan et al. (1986), Daffer & Kaneko (1992), Imdad & Khan (2004), Mustafa et al. (2010), Shahi et al. (2012), Górnicki (2016), Imdad & Alfaqih (2018), Gubran et al. (2019), Yeşilkaya & Aydın (2020), Rossafi et al. (2021), Gupta et al. (2022)). All fixed point results discussed till now are for single-valued self-map. However, the research in the framework of a multivalued map, also addressed as a set-valued map, was initiated by Kakutani (1941) and Wallace (1941) wherein the former extended the fixed point results of Brouwer (1911) for a set-valued map and the later studied the fixed points for trees, which in a major sense is related to finding the fixed point of a multivalued map. Strother (1953) after extensively studying continuity in Strother (1951) answered an open question concerning the fixed point results for a multivalued map. The research on the multivalued maps and their fixed point results was continued by Markin (1968) and by the most renowned paper of Nadler Jr (1969). Extensive literature in this area has thrived since then (see Plunkett (1956), Covitz & Nadler (1970), Lim (1985), Czerwik (1998), Rus et al. (2003), Feng & Liu (2006), Klim

& Wardowski (2007), Tahat et al. (2012), Ali & Abbas (2017), Chifu & Petruşel (2017), Ghanifard et al. (2020), Nazam et al. (2021), Abbas et al. (2021), Debnath (2022) and references cited therein).
The immediate application of some of these fixed point theorems is on linear and non-linear systems of equations, integral equations as well as on differential equations. Many non-linear problems can be converted to an equivalent fixed point form and eventually, the existence of a solution can be ascertained using a suitable fixed point tool. Thus, one can conclude that fixed point analysis has a wide scope of research in non-linear analysis (see Joshi & Bose (1985), Zeidler & Wadsack (1993)). Furthermore, the majority of developments in fixed point theory have been published in various monographs (see Goebel & Kirk (1990), Kirk & Sims (2001), Agarwal et al. (2001), Granas & Dugundji (2003), Agarwal et al. (2009), Almezel et al. (2014), Subrahmanyam (2018)). The profoundness of this theory is also due to its vast application in many other fields of mathematics like approximation theory, game theory, optimization theory, mathematical engineering, etc., besides various other interdisciplinary applications in the fields of economics, electronics, physics, and biology.

1.2 Notations and Definitions

This section gives a brief introduction to the research work along with some notations and definitions used throughout the thesis. To begin with, we define metric space.

Definition 1.2.1. *(Fréchet (1906)) On a set* $\mho$*, we say a map* $d : \mho \times \mho \to [0, +\infty)$ *is a metric if* $\forall\ \varrho, \varsigma, \sigma \in \mho$*, the following are satisfied:*

(i) $d(\varrho, \varsigma) = 0$ *iff* $\varrho = \varsigma$*;*

(ii) $d(\varrho, \varsigma) = d(\varsigma, \varrho)$*;*

(iii) $d(\varrho, \varsigma) \leqslant d(\varrho, \sigma) + d(\sigma, \varsigma)$*.*

Then, $(\mho, d)$ *is c.t.b a* ***metric space****.*

Definition 1.2.2. *(see Rudin (1991)) For a vector space* Θ*, over the field* $\mathbb{F}$*,* ***norm*** *is a function* $\|.\| : \Theta \to \mathbb{R}$ *s.t:*

(i) $\|\varrho + \varsigma\| \leqslant \|\varrho\| + \|\varsigma\|$*;*

(ii) $\|k\varrho\| = |k|\|\varrho\|$;

(iii) $\|\varrho\| > 0$ *if* $\varrho \neq 0$;

$\forall\ \varrho, \varsigma \in \Theta$ *and* $k \in \mathbb{F}$.

Definition 1.2.3. *(see Rosen (1991)) A binary relation '$\preceq$' is c.t.b a* ***partially ordered relation*** *on set* $\mho$ *if it satisfies the following:*

(i) reflexive, that is, $\varrho \preceq \varrho\ \forall\ \varrho \in \mho$;

(ii) antisymmetric, that is, if $\varrho \preceq \varsigma$ *and* $\varsigma \preceq \varrho$ *then* $\varrho = \varsigma\ \forall\ \varrho, \varsigma \in \mho$;

(iii) transitive, that is, if $\varrho \preceq \sigma$ *and* $\sigma \preceq \varsigma$ *then* $\varrho \preceq \varsigma\ \forall\ \varrho, \varsigma, \sigma \in \mho$.

Throughout the thesis, let $\mathbb{B}$ denote a unital C^*-algebra with the unit $I_{\mathbb{B}}$ and zero element $\theta_{\mathbb{B}}$. Let $\mathbb{B}_+ = \{\nu \in \mathbb{B} : \theta_{\mathbb{B}} \preceq \nu\}$ and $\mathbb{B}' = \{\nu \in \mathbb{B} : \nu\nu' = \nu'\nu\ \forall\ \nu' \in \mathbb{B}\}$.

Definition 1.2.4. *(Ma et al. (2014)) On a set* $\mho$, *let* $d : \mho \times \mho \to \mathbb{B}_+$ *be a map s.t* $\forall\ \varrho, \varsigma, \sigma \in \mho$, *the following holds:*

(i) $\theta_{\mathbb{B}} \preceq d(\varrho, \varsigma)$;

(ii) $d(\varrho, \varsigma) = \theta_{\mathbb{B}}$ *iff* $\varrho = \varsigma$;

(iii) $d(\varrho, \varsigma) = d(\varsigma, \varrho)$;

(iv) $d(\varrho, \varsigma) \preceq d(\varrho, \sigma) + d(\sigma, \varsigma)$.

Then, $(\mho_{\mathbb{B}}, d)$ *is c.t.b a* C^*_{AV}***-metric space*** *whereas* d *is a* C^*_{AV}*-metric.*

Lemma 1.2.1. *(Douglas (2012), Murphy (2014)) In* $\mathbb{B}$, *the following holds:*

(1) for $\nu \in \mathbb{B}_+$ *where* $\|\nu\| < \frac{1}{2}$, *then* $I_{\mathbb{B}} - \nu$ *is invertible and* $\|\nu(I_{\mathbb{B}} - \nu)^{-1}\| < 1$;

(2) for $\nu, \nu' \in \mathbb{B}$ *with* $\nu, \nu' \succeq \theta_{\mathbb{B}}$ *and* $\nu'\nu = \nu\nu'$, *then* $\nu'\nu \succeq \theta_{\mathbb{B}}$;

(3) If $\nu \in \mathbb{B}'$ *and* $\nu^*, \nu^{**} \in \mathbb{B}$ *where* $\nu^* \succeq \nu^{**} \succeq \theta_{\mathbb{B}}$ *and* $I_{\mathbb{B}} - \nu \in \mathbb{B}'_+$ *is invertible operator, then*
$$(I_{\mathbb{B}} - \nu)^{-1}\nu^* \succeq (I_{\mathbb{B}} - \nu)^{-1}\nu^{**}.$$

Definition 1.2.5. *(Gordji et al. (2017)) A set* $\mho$ *with a binary relation '$\perp$' is c.t.b an* ***orthogonal set*** *(denoted by $\perp$-set) when* $\exists\ \varrho_0 \in \mho$ *implies either* $[\varrho \perp \varrho_0\ \forall\ \varrho \in \mho]$ *or* $[\varrho_0 \perp \varrho\ \forall\ \varrho \in \mho]$. *The element* ϱ_0 *is called an orthogonal element.*

Definition 1.2.6. *(Gordji et al. (2017)) For an orthogonal set* $(\mho, \perp)$*, a sequence* $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}} \subset \mho$ *is c.t.b an* ***orthogonal sequence*** *(denoted by* $\perp$*-sequence) when either* $[\varrho_\varpi \perp \varrho_{\varpi+1} \ \forall \ \varpi \in \mathbb{N}]$ *or* $[\varrho_{\varpi+1} \perp \varrho_\varpi \ \forall \ \varpi \in \mathbb{N}]$.

Definition 1.2.7. *(Gordji et al. (2017)) A set* $\mho$ *along with metric* d *and a binary relation '*$\perp$*' is c.t.b an* ***orthogonal metric space*** *(written as* $(\mho, d_\perp)$*), if:*

(i) $(\mho, d)$ *is a metric space;*

(ii) $(\mho, \perp)$ *is an orthogonal set.*

Definition 1.2.8. *(Gordji et al. (2017)) On an orthogonal metric space* $(\mho, d_\perp)$*, let* $\Omega : \mho \to \mho$ *be a self-map. Then,*

(i) Ω *is c.t.b* ***orthogonally continuous*** *(denoted by* $\perp$*-continuous) if for every* $\perp$*-sequence* $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ *with* $\varrho_\varpi \to \varrho$ *implies* $\Omega\varrho_\varpi \to \Omega\varrho$ *as* $\varpi \to +\infty$*. In addition,* Ω *is* $\perp$*-continuous on entire space* $\mho$ *if* Ω *is orthogonally continuous at every point* $\varrho \in \mho$.

(ii) orthogonal metric space $(\mho, d_\perp)$ *is c.t.b a* ***complete orthogonal metric space*** *(denoted by* $\perp$*-complete), if each* $\perp$*-Cauchy sequence in* $\mho$ *is convergent in* $\mho$.

(iii) Ω *is c.t.b* ***orthogonal preserving*** *(written as* $\perp$*-preserving) if* $\varrho \perp \varsigma$ *implies* $\Omega\varrho \perp \Omega\varsigma$ *and* Ω *is weakly* $\perp$*-preserving if* $\varrho \perp \varsigma$ *implies* $\Omega\varrho \perp \Omega\varsigma$ *or* $\Omega\varsigma \perp \Omega\varrho$.

Definition 1.2.9. *(Mutlu & Gürdal (2016)) For two non-empty sets* $\mho$ *and* Λ*, a map* $d : \mho \times \Lambda \to [0, +\infty)$ *is c.t.b a* ***bipolar metric*** *if the following are satisfied:*

(i) $d(\varrho, \varsigma) = 0$ *iff* $\varrho = \varsigma \ \forall \ (\varrho, \varsigma) \in \mho \times \Lambda$;

(ii) $d(\varrho, \varsigma) = d(\varsigma, \varrho) \ \forall \ \varrho, \varsigma \in \mho \cap \Lambda$;

(iii) $d(\varrho_1, \varsigma_2) \leqslant d(\varrho_1, \varsigma_1) + d(\varrho_2, \varsigma_1) + d(\varrho_2, \varsigma_2) \ \forall \ \varrho_1, \varrho_2 \in \mho$ *and* $\varsigma_1, \varsigma_2 \in \Lambda$.

The triplet $(\mho, \Lambda, d)$ *is c.t.b a* ***bipolar metric space***.

Definition 1.2.10. *(Mutlu & Gürdal (2016)) In a bipolar metric space* $(\mho, \Lambda, d)$

(i) a point is c.t.b ***left, right or central point*** *depending if it belongs to* $\mho$*,* Λ *or* $\mho \cap \Lambda$ *respectively.*

(ii) a sequence $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ *on the set* $\mho \times \Lambda$ *is c.t.b a* ***bisequence*** *on* $(\mho, \Lambda, d)$*.*

(iii) a bisequence $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ *on the set* $\mho \times \Lambda$ *is c.t.b* ***convergent*** *if both the sequences* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *and* $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}}$ *are convergent to respective right and left point. In addition, if both* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *and* $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}}$ *converge to the same centre point, then the bisequence* $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ *is c.t.b* ***biconvergent****.*

Definition 1.2.11. *(Lipschutz (1964)) On* $\mho$*, a relation* $\mathcal{R}$ *is s.t* $\mathcal{R} \subseteq \mho \times \mho$*.*

Definition 1.2.12. *(Kolman et al. (1996)) For a subset* Z *of* $\mho$*, we say a relation* $\mathcal{R}$ *is restricted to* Z *(denoted by* $\mathcal{R}|_Z$*) when* $\mathcal{R} = \mathcal{R} \cap Z^2$*.*

Definition 1.2.13. *(Alam & Imdad (2015)) For a relation* $\mathcal{R}$*, we say* $\varrho, \varsigma \in \mho$ *are* $\mathcal{R}$***-comparative*** *(denoted by* $[\varrho, \varsigma] \in \mathcal{R}$*) if either* $(\varrho, \varsigma) \in \mathcal{R}$ *or* $(\varsigma, \varrho) \in \mathcal{R}$*.*

Definition 1.2.14. *(Khalehoghli et al. (2020)) For a relation* $\mathcal{R}$*, a sequence* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}} \subset \mho$ *is c.t.b an* $\mathcal{R}$***-sequence*** *if* $(\varrho_\varpi, \varrho_{\varpi+1}) \in \mathcal{R} \ \forall \ \varpi \in \mathbb{N}$*.*

Definition 1.2.15. *(Khalehoghli et al. (2020)) A metric space* $(\mho, d)$ *together with a relation* $\mathcal{R}$ *is c.t.b an* $\mathcal{R}$***-metric space****. It is usually written as* $(\mho, d_\mathcal{R})$*.*

Definition 1.2.16. *Let* $(\mho, d_\mathcal{R})$ *be an* $\mathcal{R}$*-metric space and let* $\phi : \mho \to \mho$*, then*

(i) (Alam & Imdad (2015)) a relation $\mathcal{R}$ *is c.t.b* $d_\mathcal{R}$***-self-closed*** *on* $\mho$ *if for an arbitrary* $\mathcal{R}$*-sequence* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}} \subset \mho$ *s.t* $\lim\limits_{\varpi\to+\infty} \varrho_\varpi = \varrho$ *implies existence of a sub-sequence* $\{\varrho_{\varpi_k}\}_{k\in\mathbb{N}} \subseteq \{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *s.t* $[\varrho_{\varpi_k}, \varrho] \in \mathcal{R} \ \forall \ k \in \mathbb{N}$*.*

(ii) (Khalehoghli et al. (2020)) $(\mho, d_\mathcal{R})$ *is c.t.b* $\mathcal{R}$***-complete*** *if each* $\mathcal{R}$*-Cauchy sequence is convergent.*

(iii) (Khalehoghli et al. (2020)) ϕ *is c.t.b* $\mathcal{R}$***-continuous*** *at* $\varrho \in \mho$ *if for an arbitrary* $\mathcal{R}$*-sequence* $\{\varrho_n\}_{n\in\mathbb{N}} \subset \mho$ *with* $\lim\limits_{n\to+\infty} \varrho_n = \varrho$ *implies* $\lim\limits_{n\to+\infty} \phi\varrho_n = \phi\varrho$*. Also,* ϕ *is c.t.b* $\mathcal{R}$*-continuous on* $\mho$ *if* $\forall \ \varrho \in \mho$*,* ϕ *is* $\mathcal{R}$*-continuous at* ϱ*.*

(iv) (Khalehoghli et al. (2020)) ϕ *is c.t.b* $\mathcal{R}$***-preserving*** *if for every* $(\varrho, \varsigma) \in \mathcal{R}$ *implies* $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$*.*

Remark 1.2.2. *(Khalehoghli et al. (2020)) Every continuous map is* $\mathcal{R}$*-continuous but not conversely.*

Example 1.2.3. *Consider* $\mho = (-\infty, 0]$ *with usual metric* d*. Let the relation* $\mathcal{R}$ *on* $\mho$ *be defined as* $(\varrho, \varsigma) \in \mathcal{R}$ *iff* $\varrho^2 = \varsigma^2$*. Then,* $(\mho, d_{\mathcal{R}})$ *is an* $\mathcal{R}$*-metric space. Define a self-map* ϕ *on* $\mho$ *s.t*

$$\phi(\varrho) = \begin{cases} 0 & for\ \varrho \in \mho \cap \mathbb{Z}; \\ -\varrho^2 & otherwise. \end{cases}$$

Then, ϕ *is an* $\mathcal{R}$*-continuous map on* $\mho$ *but it is discontinuous at every non-integer points of* $\mho$*.*

Definition 1.2.17. *(Wardowski (2012)) Denote* $\mathfrak{F}$ *as a class of all functions* $\mathcal{F} : \mathbb{R}^+ \to (-\infty, +\infty)$ *s.t the following holds:*

$(\mathcal{F}_1)$ *for* $\varrho, \varsigma \in \mathbb{R}^+$*, where* $\varrho < \varsigma$ *implies* $\mathcal{F}(\varrho) < \mathcal{F}(\varsigma)$*;*

$(\mathcal{F}_2)$ *for each sequence* $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$*, where* $\varrho_\varpi \in \mathbb{R}^+$ *s.t*

$$\lim_{\varpi \to +\infty} \varrho_\varpi = 0 \text{ iff } \lim_{\varpi \to +\infty} \mathcal{F}(\varrho_\varpi) = -\infty;$$

$(\mathcal{F}_3)$ *for some* $\gamma \in (0, 1)$*, we have* $\varsigma \in \mathbb{R}^+$ *s.t* $\lim_{\varsigma \to 0^+} \varsigma^\gamma \mathcal{F}(\varsigma) = 0$.

In addition to the above, we denote $\mathfrak{F}' = \{\mathcal{F} : \mathbb{R}^+ \to (-\infty, +\infty)$ s.t $\mathcal{F}$ satisfies $(\mathcal{F}_1), (\mathcal{F}_2), (\mathcal{F}_3), (\mathcal{F}_4)\}$, where $(\mathcal{F}_4)$: $\mathcal{F}(\inf U) = \inf\big(\mathcal{F}(U)\big)\ \forall\ U \subset \mathbb{R}^+$ with $\inf U > 0$.

Definition 1.2.18. *(Piri & Kumam (2014)) Denote* Δ_F*, the family of all maps* $F : \mathbb{R}^+ \to (-\infty, +\infty)$ *s.t:*

(F_1) *for* $\varrho, \varsigma \in \mathbb{R}^+$ *if* $\varrho \leqslant \varsigma$ *implies* $F(\varrho) \leqslant F(\varsigma)$*;*

(F_2) $\inf F = -\infty$*;*

(F_3) F *is continuous in* $\mathbb{R}^+$*.*

Lemma 1.2.4. *(Secelean (2013)) Define* $F : \mathbb{R}^+ \to (-\infty, +\infty)$ *as an increasing map and let* $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ *be a sequence s.t* $\varrho_\varpi \in \mathbb{R}^+\ \forall\ \varpi \in \mathbb{N}$*. Then, the following holds:*

(i) If $F(\varrho_\varpi) \to -\infty$ *implies* $\varrho_\varpi \to 0$*;*

(ii) If $\inf F = -\infty$ *and* $\varrho_\varpi \to 0$ *implies* $F(\varrho_\varpi) \to -\infty$*.*

Lemma 1.2.5. *(Górnicki (2016)) If* $\omega : \mho \to \mho$ *is a surjective map on metric space* $(\mho, d)$*, then* ω *has a right inverse.*

Definition 1.2.19. *For a metric space* $(\mho, d)$*, a self-map* $\omega : \mho \to \mho$ *is c.t.b*

(i) (Samet et al. (2012)) an α***-admissible map****, where* $\alpha : \mho^2 \to [0, +\infty)$*, if for each* $\varrho, \varsigma \in \mho$ *with* $1 \leqslant \alpha(\varrho, \varsigma)$ *implies* $1 \leqslant \alpha(\omega\varrho, \omega\varsigma)$*.*

(ii) (Salimi et al. (2013)) an α***-admissible map w.r.t*** η*, with* $\alpha, \eta : \mho^2 \to \mathbb{R}^+$*, if for* $\varrho, \varsigma \in \mho$ *with* $\eta(\varrho, \varsigma) \leqslant \alpha(\varrho, \varsigma)$ *implies* $\eta(\omega\varrho, \omega\varsigma) \leqslant \alpha(\omega\varrho, \omega\varsigma)$*.*

(iii) (Alizadeh et al. (2014)) a ***cyclic*** $(\hat{\alpha}, \beta)$***-admissible map****, with* $\hat{\alpha}, \beta : \mho \to \mathbb{R}^+$*, if:*

(a) For any $\varrho \in \mho$*,* $1 \leqslant \hat{\alpha}(\varrho)$ *implies* $1 \leqslant \beta(\omega\varrho)$*;*

(b) For any $\varrho \in \mho$*,* $1 \leqslant \beta(\varrho)$ *implies* $1 \leqslant \hat{\alpha}(\omega\varrho)$*.*

(iv) (Sintunavarat (2015)) a ***weak*** α***-admissible map****, where* $\alpha : \mho^2 \to \mathbb{R}^+$*, if for each* $\varrho \in \mho$ *with* $1 \leqslant \alpha(\varrho, \omega\varrho)$ *implies* $1 \leqslant \alpha(\omega\varrho, \omega\omega\varrho)$*.*

*(v) (Sintunavarat (2016*b*)) an* α***-admissible map type S****, where* $\alpha : \mho^2 \to \mathbb{R}^+$ *and real number* s *with* $s \geqslant 1$*, if for* $\varrho, \varsigma \in \mho$ *we have* $s \leqslant \alpha(\varrho, \varsigma)$ *implies* $s \leqslant \alpha(\omega\varrho, \omega\varsigma)$*.*

*(vi) (Sintunavarat (2016*b*)) a* ***weak*** α***-admissible map type S****, where* $\alpha : \mho^2 \to \mathbb{R}^+$ *and real number* s *with* $s \geqslant 1$*, if for* $\varrho \in \mho$ *we have* $s \leqslant \alpha(\varrho, \omega\varrho)$ *implies* $s \leqslant \alpha(\omega\varrho, \omega\omega\varrho)$*.*

(vii) (Mongkolkeha & Sintunavarat (2018)) a ***cyclic*** $(\hat{\alpha}, \beta)$***-admissible map type S****, with* $\hat{\alpha}, \beta : \mho \to \mathbb{R}^+$ *and real number* s *where* $s \geqslant 1$*, if:*

(a) For any $\varrho \in \mho$*,* $\hat{\alpha}(\varrho) \geqslant s$ *implies* $\beta(\omega\varrho) \geqslant s$*;*

(b) For any $\varrho \in \mho$*,*$\beta(\varrho) \geqslant s$ *implies* $\hat{\alpha}(\omega\varrho) \geqslant s$*.*

Remark 1.2.6. *Following are few observations from (Sintunavarat (2016*b*)):*

(i) Each α*-admissible map is weak* α*-admissible map.*

(ii) Each α*-admissible map type S is weak* α*-admissible map type S.*

(iii) The class of α*-admissible map is different from* α*-admissible map type S.*

Remark 1.2.7. *The family of cyclic $(\hat{\alpha}, \beta)$-admissible maps is different from the family of cyclic $(\hat{\alpha}, \beta)$-admissible maps type S (see Mongkolkeha & Sintunavarat (2018)).*

Definition 1.2.20. *(Hussain & Salimi (2014)) Denote $\mathfrak{G}$, the set of all maps $G : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ s.t $\forall\ \hbar_1, \hbar_2, \hbar_3, \hbar_4 \in [0, +\infty)$ with $\hbar_1.\hbar_2.\hbar_3.\hbar_4 = 0$ we have $\wp > 0$, s.t*

$$G(\hbar_1, \hbar_2, \hbar_3, \hbar_4) = \wp.$$

Example 1.2.8. *Let $G(\hbar_1, \hbar_2, \hbar_3, \hbar_4) = \wp.e^{K.(\hbar_1.\hbar_2.\hbar_3.\hbar_4)}$ where $\wp > 0$ and K is a non-negative real constant, then $G \in \mathfrak{G}$.*

Definition 1.2.21. *(Ansari (2014)) A function $\mathcal{C} : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is c.t.b a C-**class function** if $\mathcal{C}$ is continuous map s.t:*

(i) $\mathcal{C}(\varrho, \varsigma) \leqslant \varrho$;

(ii) $\mathcal{C}(\varrho, \varsigma) = \varrho$ *implies either* $\varrho = 0$ *or* $\varsigma = 0$;

$\forall\ (\varrho, \varsigma) \in [0, +\infty) \times [0, +\infty)$.
Throughout the thesis, the family of C-class function is denoted by $\mathfrak{C}$.

Define a functional $\mathcal{D} : \mathcal{N}(\mho) \times \mathcal{N}(\mho) \to [0, +\infty) : \mathcal{D}(U, V) = \inf\{d(\rho, \sigma) \text{ s.t } \rho \in U,\ \sigma \in V\}$ and **Pompeiu-Hausdorff functional** (see Chifu & Petruşel (2017)) $\mathcal{H} : \mathcal{N}(\mho) \times \mathcal{N}(\mho) \to [0, +\infty) \cup \{+\infty\}$ s.t $\mathcal{H}(U, V) = \max\{\sup_{\rho \in U} d(\rho, V), \sup_{\sigma \in V} d(\sigma, U)\}$.
The upcoming lemma can be obtained from the result given in (Czerwik (1998)) for b-metric space.

Lemma 1.2.9. *For a metric space $(\mho, d)$, $\mathcal{D}(\rho, V) \leqslant d(\rho, \sigma) + \mathcal{D}(\sigma, V)\ \forall\ \rho, \sigma \in \mho$ and $V \subseteq \mho$.*

1.3 Chapterwise Summary

This section of the chapter provides an overview of the work done in each chapter of the thesis.

In Chapter 2, we have introduced generalized contraction maps like orthogonal α-η-$G\mathcal{F}$-contraction, orthogonal α-type $\mathcal{F}$-contraction, orthogonal TAC-type

S-contraction, orthogonal TAC-contraction, orthogonal *Suzuki-Berinde* type F-contraction, and orthogonal $\mathcal{F}$-weak contraction together with some of their weaker versions in orthogonal metric space. Further, the results are applied to establish the existence and uniqueness of the solution for first order differential equation. The fixed point results discussed are proper extensions of some of the results present in the literature.

In Chapter 3, we extend the fixed point results in the relation theoretic metric space introduced by Alam & Imdad (2015), briefly written as $\mathcal{R}$-metric space, by putting forward the fixed point results using F-weak expansive map followed by the fixed point results that are subjected to contraction conditions corresponding to the multivalued counterpart of $\mathcal{F}$-contraction, $\mathcal{F}$-weak contraction, almost $\mathcal{F}$-contraction and α-type $\mathcal{F}$-contraction in $\mathcal{R}$-metric space. Next, we discuss the existence of the solution for a non-homogeneous, non-linear Volterra integral and its stability using the idea of Hyers-Ulam stability.

Chapter 4 introduces the notion of $C^*_{AV}\mathcal{R}$-metric space which generalizes the class of C^*_{AV}-metric space. The first section introduces the idea of $C^*_{AV}\mathcal{R}$-contractive map and $C^*_{AV}\mathcal{R}$-metric space along with some fixed point results which, in turn, generalizes and integrates some well-known outcomes in the literature. The second section discusses the existence and uniqueness of coincidence points and common fixed points in $C^*_{AV}\mathcal{R}$-metric space using the technique of Picard-Jungck iteration. Here, the results proved are for a pair of self-maps in the $C^*_{AV}\mathcal{R}$-metric space which is generalized enough to deduce coincidence and common fixed point results in C^*_{AV}-ordered metric space, C^*_{AV}-metric space, and metric space. As an application, the coincidence and common fixed point results are applied on C^*_{AV}-metric space endowed with a directed graph.

In Chapter 5, on associating an amorphous binary relation $\mathcal{R}$ with the bipolar metric space, we introduce the notion of bipolar $\mathcal{R}$-metric space together with the fixed point results. Further, we introduce the notions of $\mathcal{F}_{\mathcal{R}}$-contractive map and $\mathcal{F}_{\mathcal{R}}$-expansive map. The results provides a fixed point result in the setting of $\mathcal{F}_{\mathcal{R}}$-contractive map followed by fixed point deductions with $\mathcal{F}_{\mathcal{R}}$-expansive map in bipolar $\mathcal{R}$-metric space. Under a specific condition, the results are reduced to a novel fixed point result in bipolar metric space with respect to an expansive map and to some result of literature, thus substantiating their utility.

The thesis ends with the bibliography followed by the list of publications, paper presented in conferences, and workshop attended.

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Chapter 2

Fixed Point Results in Orthogonal Metric Space

2.1 Introduction

Gordji et al. (2017) gave the notion of orthogonal sets and subsequently, orthogonal metric space and proved the Banach fixed point result in this space. Over the period of time, number of authors have deduced fixed point results in an orthogonal metric space (for reference, see Ramezani & Baghani (2017a,b), Ahmadi et al. (2018), Senapati et al. (2018), Kanwal et al. (2020), Sawangsup et al. (2020), Yang & Bai (2020), Beg et al. (2021), Chandok & Radenović (2022), Gnanaprakasam et al. (2022)).

In this chapter, we generalize the contraction maps in an orthogonal metric space and associated fixed point results, which are inspired by the previous work. The contraction maps like orthogonal α-η-$G\mathcal{F}$-contraction, orthogonal α-type $\mathcal{F}$-contraction, orthogonal TAC-type S-contraction, orthogonal TAC-contraction, orthogonal *Suzuki-Berinde* type F-contraction, and orthogonal $\mathcal{F}$-weak contraction together with some of their weaker versions are discussed. Also, various fixed point results owing to these generalized contraction conditions are proved, which indeed extends the results given in Hussain & Salimi (2014), Gopal et al. (2016), Chandok et al. (2016), Baghani et al. (2016), Hussain & Ahmad (2017) and Sawangsup et al. (2020). The results are used to show that the solution of a first order ordinary differential equation exists and is unique. The fixed point results demonstrated in this chapter are a proper extension of several results in the literature. The results proved in this chapter are part of [1,2].

2.2 Generalized Contraction Maps and Fixed Point Results

This section is divided into five subsections, each of which introduces generalized contraction maps in an orthogonal metric space and consequently, explores various fixed point results owing to these weaker contraction conditions.

2.2.1 Orthogonal α-η-$G\mathcal{F}$-Contraction

Hussain & Salimi (2014) presented the notion of α-η-$G\mathcal{F}$-contraction map. In this subsection, we first introduce some of the basic definitions including orthogonal α-η-$G\mathcal{F}$-contraction and orthogonal α-η-$G\mathcal{F}$-weak contraction and later, prove certain fixed point results under these settings. Further, the definitions and results are supported by the examples.

Definition 2.2.1. *For an orthogonal metric space $(\mho, d_\perp)$ with two maps $\alpha, \eta : \mho^2 \to \mathbb{R}^+$, a self-map $\Omega : \mho \to \mho$ is c.t.b an **orthogonal α-η-continuous map** (denoted by $\perp$-α-η-continuous) if for some $\varrho \in \mho$ and an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ where $\eta(\varrho_\varpi, \varrho_{\varpi+1}) \leqslant \alpha(\varrho_\varpi, \varrho_{\varpi+1})\ \forall\ \varpi \in \mathbb{N}$ and $\lim_{\varpi\to+\infty} \varrho_\varpi = \varrho$ implies*

$$\lim_{\varpi\to+\infty} \Omega\varrho_\varpi = \Omega\varrho.$$

Example 2.2.1. *Let $\mho = [0, +\infty)$ equipped with usual metric. Let $\varrho \perp \varsigma$ iff $\varrho.\varsigma \in \{\varrho, \varsigma\}$. Then, $(\mho, d_\perp)$ is an orthogonal metric space. Let $\Omega : \mho \to \mho$ be defined as*

$$\Omega(\varrho) = \begin{cases} 0 & \varrho \in [0, 1); \\ 1/2 & \textit{otherwise}. \end{cases}$$

Define $\alpha, \eta : \mho^2 \to \mathbb{R}^+$ where

$$\alpha(\varrho, \varsigma) = \begin{cases} 4 & \varrho, \varsigma \in [0, 1); \\ 1/4 & \textit{otherwise}, \end{cases}$$

[1] Malhotra, A., and Kumar, D. (2022). Generalized Contraction Mappings and Fixed Point Results in Orthogonal Metric Space. Applied Mathematics E-Notes, 22, 393-426.

[2] Kumar, D. and Malhotra, A. (2022). Orthogonal F-weak Contraction Mapping in Orthogonal Metric Space, Fixed Points and Applications. Filomat. (Accepted)

and, $\eta(\varrho,\varsigma)=1\ \forall\ \varrho,\varsigma\in\mho$. *Thus, for* $\alpha(\varrho,\varsigma)\geqslant\eta(\varrho,\varsigma)$, *we must have* $\varrho,\varsigma\in[0,1)$. *The sequence* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$, *defined as*

$$\varrho_\varpi=\begin{cases}0 & \varpi=2m-1,\ \forall\quad m\in\mathbb{N};\\ 1/2^m & \varpi=2m,\ \forall\ m\in\mathbb{N},\end{cases}$$

is an $\perp$*-sequence in* $\mho$ *and* $\alpha(\varrho_\varpi,\varrho_{\varpi+1})\geqslant\eta(\varrho_\varpi,\varrho_{\varpi+1})\ \forall\ \varpi\in\mathbb{N}$. *Also, since* $\varrho_\varpi\to 0$ *as* $\varpi\to+\infty$ *then* $\lim\limits_{\varpi\to+\infty}\Omega\varrho_\varpi=0=\Omega 0$. *Hence,* Ω *is* $\perp$-α-η*-continuous however, it is not a continuous map.*

Definition 2.2.2. *For an orthogonal metric space* $(\mho,d_\perp)$ *with two maps* $\alpha,\eta:\mho^2\to\mathbb{R}^+$, *a self-map* $\Omega:\mho\to\mho$ *is c.t.b an* ***orthogonal*** α-η-$G\mathcal{F}$***-contraction*** *(denoted by* $\perp$-α-η-$G\mathcal{F}$*-contraction) if* $\forall\ \varrho,\varsigma\in\mho$ *with* $\varrho\perp\varsigma$, $d_\perp(\Omega\varrho,\Omega\varsigma)>0$ *and* $\eta(\varrho,\Omega\varrho)\leqslant\alpha(\varrho,\varsigma)$, *we have*

$$G\Big(d_\perp(\varrho,\Omega\varrho),d_\perp(\varsigma,\Omega\varsigma),d_\perp(\varrho,\Omega\varsigma),d_\perp(\varsigma,\Omega\varrho)\Big)+\mathcal{F}\Big(d_\perp(\Omega\varrho,\Omega\varsigma)\Big)\leqslant\mathcal{F}\Big(d_\perp(\varrho,\varsigma)\Big),$$

where $G\in\mathfrak{G}$ *and* $\mathcal{F}\in\mathfrak{F}$.

Example 2.2.2. *Consider* $\mho=\{0,2,4,\ldots,2^k,\ldots\}$ *along with usual metric space. Let* $\varrho\perp\varsigma$ *iff* $\varrho.\varsigma\in\{0\}$. *Then,* $(\mho,d_\perp)$ *is an orthogonal metric space. Let* $\Omega:\mho\to\mho$ *be defined as*

$$\Omega(\varrho)=\begin{cases}2^{m-1} & for\ \varrho=2^m\ where\ m\in\mathbb{N}-\{1\};\\ 0 & \varrho\in\{0,2\}.\end{cases}$$

Define $\alpha,\eta:\mho^2\to\mathbb{R}^+$ *as,*

$$\alpha(\varrho,\varsigma)=\begin{cases}1 & \varrho\in\{0,2\};\\ 5/2 & otherwise,\end{cases}$$

and,

$$\eta(\varrho,\varsigma)=\begin{cases}1/2 & \varrho\in\{0,2\};\\ 1 & otherwise.\end{cases}$$

Now by above, we have

(i) for $d_\perp(\Omega\varrho,\Omega\varsigma)>0$, *we must have either* $\varrho\in\{0,2\}$ *and* $\varsigma=2^m$ *where* $m\in\mathbb{N}-\{1\}$ or $\varrho=2^m$ *where* $m\in\mathbb{N}-\{1\}$ *and* $\varsigma\in\{0,2\}$.

(ii) for $\varrho\perp\varsigma$, *either* $\varrho=0$ *or* $\varsigma=0$.

Thus, for (i) and (ii) to hold together, we have either $\varrho = 0$ *and* $\varsigma = 2^m$ *where* $m \in \mathbb{N} - \{1\}$ or $\varrho = 2^m$ *where* $m \in \mathbb{N} - \{1\}$ *and* $\varsigma = 0$.
Consider $\varrho = 0$ *and* $\varsigma = 2^m$ *where* $m \in \mathbb{N} - \{1\}$. *Then for such choice of* ϱ *and* ς, *we have* $\eta(\varrho, \Omega\varrho) < \alpha(\varrho, \varsigma)$. *So for* $\mathcal{F}(\mu) = \ln(\mu)$ *and* $\wp = 0.5$, *we have*

$$G\Big(d_\perp(\varrho, \Omega\varrho), d_\perp(\varsigma, \Omega\varsigma), d_\perp(\varrho, \Omega\varsigma), d_\perp(\varsigma, \Omega\varrho)\Big) + \mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) = \wp + \ln 2^{m-1} \quad (2.1)$$

and,

$$\mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big) = \ln(2^m). \quad (2.2)$$

Thus from (2.1) and (2.2), we obtain

$$G\Big(d_\perp(\varrho, \Omega\varrho), d_\perp(\varsigma, \Omega\varsigma), d_\perp(\varrho, \Omega\varsigma), d_\perp(\varsigma, \Omega\varrho)\Big) + \mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \leqslant \mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big).$$

Hence, Ω *is* $\perp$-α-η-$G\mathcal{F}$*-contraction on* $\mho$.

Definition 2.2.3. *For an orthogonal metric space* $(\mho, d_\perp)$ *with two maps* $\alpha, \eta : \mho^2 \to \mathbb{R}^+$, *a self-map* $\Omega : \mho \to \mho$ *is c.t.b an* ***orthogonal*** α-η-$G\mathcal{F}$***-weak contraction*** *(denoted by* $\perp$-α-η-$G\mathcal{F}$*-weak contraction) if* $\forall\ \varrho, \varsigma \in \mho$ *with* $\varrho \perp \varsigma$, $d_\perp(\Omega\varrho, \Omega\varsigma) > 0$ *and* $\eta(\varrho, \Omega\varrho) \leqslant \alpha(\varrho, \varsigma)$, *we have*

$$\begin{aligned} &G\Big(d_\perp(\varrho, \Omega\varrho), d_\perp(\varsigma, \Omega\varsigma), d_\perp(\varrho, \Omega\varsigma), d_\perp(\varsigma, \Omega\varrho)\Big) + \mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \\ &\leqslant \mathcal{F}\left(\max\left\{d_\perp(\varrho, \varsigma), d_\perp(\varrho, \Omega\varrho), d_\perp(\varsigma, \Omega\varsigma), \frac{d_\perp(\varrho, \Omega\varsigma) + d_\perp(\varsigma, \Omega\varrho)}{2}\right\}\right), \end{aligned}$$

where $G \in \mathfrak{G}$ *and* $\mathcal{F} \in \mathfrak{F}$.

Remark 2.2.3. *From the above definitions, we can conclude that every* $\perp$-α-η-$G\mathcal{F}$*-contraction is an* $\perp$-α-η-$G\mathcal{F}$*-weak contraction.*

Theorem 2.2.4. *For* $(\mho, d_\perp)$ *an* $\perp$*-complete metric space with* ϱ_0 *as an orthogonal element, suppose* $G \in \mathfrak{G}$ *and* $\mathcal{F} \in \mathfrak{F}$. *Let* $\alpha, \eta : \mho^2 \to \mathbb{R}^+$ *and* $\Omega : \mho \to \mho$ *be a self-map s.t:*

(I) Ω *is* $\perp$*-preserving;*

(II) Ω *is* α*-admissible map w.r.t* η*;*

(III) $\exists\ \varrho_0 \in \mho$ *s.t* $\eta(\varrho_0, \Omega\varrho_0) \leqslant \alpha(\varrho_0, \Omega\varrho_0)$*;*

(IV) Ω *is* $\perp$-α-η*-continuous;*

(V) Ω is $\perp$-α-η-$G\mathcal{F}$-contraction.

Then, Ω possesses a fixed point. In addition, if $\forall\ \varrho, \varsigma \in \mho$ s.t $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ implies $\eta(\varrho, \varrho) \leqslant \alpha(\varrho, \varsigma)$, then Ω possesses a unique fixed point.

Proof. Consider $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ be a sequence in $\mho$ where $\varrho_{\varpi+1} = \Omega\varrho_\varpi = \Omega^{\varpi+1}\varrho_0$ for each $\varpi \in \mathbb{N}$. Since $\eta(\varrho_0, \Omega\varrho_0) \leqslant \alpha(\varrho_0, \Omega\varrho_0)$, so using α-admissibility of Ω w.r.t η, we get

$$\eta(\varrho_1, \varrho_2) = \eta(\Omega\varrho_0, \Omega^2\varrho_0) \leqslant \alpha(\Omega\varrho_0, \Omega^2\varrho_0) = \alpha(\varrho_1, \varrho_2),$$

repetitive use of α-admissibility of Ω w.r.t η, we obtain

$$\eta(\varrho_{\varpi-1}, \varrho_\varpi) \leqslant \alpha(\varrho_{\varpi-1}, \varrho_\varpi) \quad \forall\ \varpi \in \mathbb{N}.$$

Also, as $\varrho_0, \Omega\varrho_0 \in \mho$ where $(\mho, \perp)$ is an $\perp$-set then the repeated use of $\perp$-preserving property of Ω, gives

$$[\varrho_{\varpi-1} \perp \varrho_\varpi\ \forall\ \varpi \in \mathbb{N}] \quad \text{or} \quad [\varrho_\varpi \perp \varrho_{\varpi-1}\ \forall\ \varpi \in \mathbb{N}].$$

Using contractive property of Ω, we get

$$\begin{aligned} &G\Big(d_\perp(\varrho_{\varpi-1}, \Omega\varrho_{\varpi-1}), d_\perp(\varrho_\varpi, \Omega\varrho_\varpi), d_\perp(\varrho_{\varpi-1}, \Omega\varrho_\varpi), d_\perp(\varrho_\varpi, \Omega\varrho_{\varpi-1})\Big) \\ &\qquad +\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big) \leqslant \mathcal{F}\Big(d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)\Big). \end{aligned} \tag{2.3}$$

Since, we have

$$d_\perp(\varrho_\varpi, \varrho_{\varpi+1}).d_\perp(\varrho_{\varpi-1}, \varrho_\varpi).d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi+1}).d_\perp(\varrho_\varpi, \varrho_\varpi) = 0,$$

so $\exists\ \wp > 0$, s.t

$$G(d_\perp(\varrho_\varpi, \varrho_{\varpi+1}), d_\perp(\varrho_{\varpi-1}, \varrho_\varpi), d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi+1}), d_\perp(\varrho_\varpi, \varrho_\varpi)) = \wp. \tag{2.4}$$

On using (2.4) in (2.3), we obtain

$$\begin{aligned} \wp + \mathcal{F}(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)) &\leqslant \mathcal{F}(d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)), \\ \text{that is,} \quad \mathcal{F}(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})) &\leqslant \mathcal{F}(d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)) - \wp \\ &\leqslant \mathcal{F}(d_\perp(\varrho_{\varpi-2}, \varrho_{\varpi-1})) - 2\wp \\ &\leqslant \cdots \leqslant \mathcal{F}(d_\perp(\varrho_0, \varrho_1)) - \varpi\wp. \end{aligned} \tag{2.5}$$

Taking limit as $\varpi \to +\infty$ in (2.5) and by using $(\mathcal{F}_2)$ property of $\mathcal{F}$, we have

$$\lim_{\varpi \to +\infty} d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) = 0. \tag{2.6}$$

Further, by $(\mathcal{F}_3)$ property of $\mathcal{F}$, $\exists$ some $0 < \gamma < 1$, s.t

$$\lim_{\varpi \to +\infty} \Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma \mathcal{F}(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})) = 0. \tag{2.7}$$

Using (2.6) and (2.7) in (2.5), we get

$$\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma \Big(\mathcal{F}(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})) - \mathcal{F}(d_\perp(\varrho_0, \varrho_1))\Big) \leqslant -\varpi\wp\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma \leqslant 0.$$

On letting $\varpi \to +\infty$ in above, we have $\lim_{\varpi \to +\infty} \varpi\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma = 0$. So, $\exists\ \varpi_0 \in \mathbb{N}$, s.t

$$\begin{aligned} \varpi\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma &\leqslant 1 \quad \forall\, \varpi \geqslant \varpi_0, \\ \text{implies} \quad d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) &\leqslant \frac{1}{\varpi^{1/\gamma}} \quad \forall\, \varpi \geqslant \varpi_0. \end{aligned}$$

Now, for $\varpi^* > \varpi > \varpi_0$ and using triangle inequality, we obtain

$$d_\perp(\varrho_\varpi, \varrho^*_\varpi) \leqslant \sum_{i=\varpi}^{\varpi^*-1} d_\perp(\varrho_i, \varrho_{i+1}) \leqslant \sum_{i=1}^{+\infty} d_\perp(\varrho_i, \varrho_{i+1}) \leqslant \sum_{i=1}^{+\infty} \frac{1}{\varpi^{1/\gamma}}.$$

As $0 < \gamma < 1$, so convergence of $\sum_{i=1}^{+\infty} \frac{1}{\varpi^{1/\gamma}}$ implies $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ is an $\perp$-Cauchy sequence and since $\mho$ is $\perp$-complete, we have $\lim_{\varpi \to +\infty} \varrho_\varpi = \varrho$. Therefore, by $\perp$-α-η-continuity of Ω, we get

$$\begin{aligned} \lim_{\varpi \to +\infty} \Omega\varrho_\varpi = \lim_{\varpi \to +\infty} \varrho_{\varpi+1} &= \Omega\varrho, \\ \text{that is,} \quad \varrho &= \Omega\varrho. \end{aligned}$$

Thus, Ω possesses a fixed point. Next, let ς be s.t $\Omega\varsigma = \varsigma$ and $\varrho \perp \varsigma$ then by given condition $\eta(\varrho, \varrho) \leqslant \alpha(\varrho, \varsigma)$. On using $\perp$-α-η-$G\mathcal{F}$-contraction of Ω over ϱ and ς, we obtain

$$G\Big(d_\perp(\varrho, \Omega\varrho), d_\perp(\varsigma, \Omega\varsigma), d_\perp(\varrho, \Omega\varsigma), d_\perp(\varsigma, \Omega\varrho)\Big) + \mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \leqslant \mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big).$$

Since,

$$d_\perp(\varrho, \Omega\varrho).d_\perp(\varsigma, \Omega\varsigma).d_\perp(\varrho, \Omega\varsigma).d_\perp(\varsigma, \Omega\varrho) = 0,$$

so $\exists \wp > 0$, s.t

$$G\Big(d_\perp(\varrho, \Omega\varrho), d_\perp(\varsigma, \Omega\varsigma), d_\perp(\varrho, \Omega\varsigma), d_\perp(\varsigma, \Omega\varrho)\Big) = \wp.$$

Therefore,

$$\begin{aligned} \wp + \mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) &\leqslant \mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big), \\ \text{implies} \quad \wp + \mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big) &\leqslant \mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big), \end{aligned}$$

that holds only if $\varrho = \varsigma$. Hence, Ω possesses a unique fixed point. □

Example 2.2.5. *Consider the orthogonal metric space and $\perp$-α-η-$G\mathcal{F}$-contraction map Ω defined in Example 2.2.2, then*

(i) $(\mho, d_\perp)$ *is $\perp$-complete: Suppose $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ be any $\perp$-Cauchy sequence in $\mho$. Then, we have a sub-sequence $\{\varrho_{\varpi_k}\}$ of $\{\varrho_\varpi\}$ s.t $\varrho_{\varpi_k} = 0\ \forall\ k \geqslant 1$, that is, $\varrho_{\varpi_k} \to 0$ as $\varpi \to +\infty$. Since this happens with any $\perp$-Cauchy sequence in $\mho$, so we have $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ convergent in $\mho$.*

(ii) Ω is $\perp$-preserving: Since $0 \perp \varsigma\ \forall\ \varsigma \in \mho$, then $\Omega 0 = 0 \perp \Omega\varsigma\ \forall\ \varsigma \in \mho$.

(iii) Ω is α-admissible w.r.t η: From the definition of α, η and Ω, we can concluded that Ω is α-admissible map w.r.t η.

(iv) Ω is $\perp$-continuous: For any convergent $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$, we have $\varrho_\varpi \to 0$ as $\varpi \to +\infty$. Then, $\Omega\varrho_\varpi \to \Omega 0 = 0$ as $\varpi \to +\infty$.

Since all the hypotheses of Theorem 2.2.4 hold, so Ω possesses a fixed point viz. $\varrho = 0$.

Theorem 2.2.6. *For $(\mho, d_\perp)$ an $\perp$-complete metric space with ϱ_0 as an orthogonal element, let $G \in \mathfrak{G}$ and $\mathcal{F} \in \mathfrak{F}$. Let $\alpha, \eta : \mho^2 \to \mathbb{R}^+$ and $\Omega : \mho \to \mho$ be a self-map s.t:*

(I) Ω is $\perp$-preserving;

(II) Ω is α-admissible map w.r.t η;

(III) $\exists\ \varrho_0 \in \mho$ s.t $\eta(\varrho_0, \Omega\varrho_0) \leqslant \alpha(\varrho_0, \Omega\varrho_0)$;

(IV) Ω is $\perp$-α-η-continuous;

(V) Ω is $\perp$-α-η-$G\mathcal{F}$-weak contraction.

Then, Ω possesses a fixed point. In addition, if $\forall\ \varrho, \varsigma \in \mho$ s.t $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ implies $\eta(\varrho, \varrho) \leqslant \alpha(\varrho, \varsigma)$, then Ω possesses a unique fixed point.

Proof. Working on the lines of Theorem 2.2.4, we obtain an $\perp$-sequence $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ in $\mho$, s.t

$$\begin{aligned} &G\Big(d_\perp(\varrho_{\varpi-1}, \Omega\varrho_{\varpi-1}), d_\perp(\varrho_{\varpi}, \Omega\varrho_{\varpi}), d_\perp(\varrho_{\varpi-1}, \Omega\varrho_{\varpi}), d_\perp(\varrho_{\varpi}, \Omega\varrho_{\varpi-1})\Big) \\ &+\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_{\varpi})\Big) \leqslant \mathcal{F}\Bigg(\max\Bigg\{d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi}), d_\perp(\varrho_{\varpi-1}, \Omega\varrho_{\varpi-1}), \\ &d_\perp(\varrho_{\varpi}, \Omega\varrho_{\varpi}), \frac{d_\perp(\varrho_{\varpi-1}, \Omega\varrho_{\varpi}) + d_\perp(\varrho_{\varpi}, \Omega\varrho_{\varpi-1})}{2}\Bigg\}\Bigg). \end{aligned} \tag{2.8}$$

Since, we have

$$d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1}).d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi}).d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi+1}).d_\perp(\varrho_{\varpi}, \varrho_{\varpi}) = 0,$$

so $\exists\ \wp > 0$, that gives

$$G(d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1}), d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi}), d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi+1}), d_\perp(\varrho_{\varpi}, \varrho_{\varpi})) = \wp. \tag{2.9}$$

On using (2.9) in (2.8), we get

$$\begin{aligned} &\wp + \mathcal{F}(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_{\varpi})) \\ &\leqslant \mathcal{F}\Bigg(\max\Bigg\{d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi}), d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1}), \frac{d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi+1})}{2}\Bigg\}\Bigg) \\ &\leqslant \mathcal{F}\Bigg(\max\Bigg\{d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi}), d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1}), \frac{d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi}) + d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1})}{2}\Bigg\}\Bigg) \\ &= \mathcal{F}\Bigg(\max\Bigg\{d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi}), d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1})\Bigg\}\Bigg). \end{aligned}$$

Case (i): Let $\max\Bigg\{d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi}), d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1})\Bigg\} = d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1})$, then

$$\wp + \mathcal{F}\Big(d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1})\Big) \leqslant \mathcal{F}(d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1})),$$

which is not true for any $\wp > 0$.

Case (ii): Let $\max\Bigg\{d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi}), d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1})\Bigg\} = d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi})$, then

$$\wp + \mathcal{F}\Big(d_\perp(\varrho_{\varpi}, \varrho_{\varpi+1})\Big) \ \leqslant\ \mathcal{F}(d_\perp(\varrho_{\varpi-1}, \varrho_{\varpi})),$$

$$\text{thus,}\quad \mathcal{F}\Big(d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\Big) \leqslant \mathcal{F}(d_\perp(\varrho_{\varpi-1},\varrho_\varpi)) - \wp$$
$$= \mathcal{F}(d_\perp(\varrho_{\varpi-1},\varrho_\varpi)) - 2\wp \leqslant \cdots \leqslant \mathcal{F}(d_\perp(\varrho_0,\varrho_1)) - \varpi\wp.$$

The result now follows on the lines of Theorem 2.2.4. □

Remark 2.2.7. *In the upcoming result, we exclude the condition of $\perp$-α-η-continuity of Ω and instead consider a weaker condition.*

Theorem 2.2.8. *For $(\mho, d_\perp)$ an $\perp$-complete metric space with ϱ_0 as an orthogonal element, let $G \in \mathfrak{G}$ and $\mathcal{F} \in \mathfrak{F}$. Let $\alpha, \eta : \mho^2 \to \mathbb{R}^+$ and $\Omega : \mho \to \mho$ be a self-map s.t:*

(I) *Ω is $\perp$-preserving;*

(II) *Ω is α-admissible map w.r.t η;*

(III) *$\exists\, \varrho_0 \in \mho$ s.t $\eta(\varrho_0, \Omega\varrho_0) \leqslant \alpha(\varrho_0, \Omega\varrho_0)$;*

(IV) *If $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ is an $\perp$-sequence in $\mho$ s.t $\eta(\varrho_\varpi, \varrho_{\varpi+1}) \leqslant \alpha(\varrho_\varpi, \varrho_{\varpi+1})$ and $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$, then*

$$[\varrho_\varpi \perp \varrho\ \forall\, \varpi] \quad \textit{or} \quad [\varrho \perp \varrho_\varpi\ \forall\, \varpi]$$

and,

$$[\eta(\Omega\varrho_\varpi, \Omega^2\varrho_\varpi) \leqslant \alpha(\Omega\varrho_\varpi, \varrho)] \quad \textit{or} \quad [\eta(\Omega^2\varrho_\varpi, \Omega^3\varrho_\varpi) \leqslant \alpha(\Omega^2\varrho_\varpi, \varrho)]\ \forall\, \varpi \in \mathbb{N};$$

(V) *Ω is $\perp$-α-η-$G\mathcal{F}$-contraction.*

Then, Ω possesses a fixed point. In addition, if for each $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ implies $\eta(\varrho, \varrho) \leqslant \alpha(\varrho, \varsigma)$, then Ω possesses a unique fixed point.

Proof. Working on the lines of Theorem 2.2.4, we obtain an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ s.t

$$\eta(\varrho_\varpi, \varrho_{\varpi+1}) \leqslant \alpha(\varrho_\varpi, \varrho_{\varpi+1}) \quad \text{and} \quad \lim_{\varpi\to+\infty} \varrho_\varpi = \varrho.$$

Here, we say that ϱ is a fixed point of Ω in $\mho$. By the given condition, we have

$$[\varrho_\varpi \perp \varrho\ \forall\, \varpi \in \mathbb{N}] \quad \textit{or} \quad [\varrho \perp \varrho_\varpi\ \forall\, \varpi \in \mathbb{N}],$$

and,

$$[\eta(\varrho_{\varpi+1}, \varrho_{\varpi+2}) \leqslant \alpha(\varrho_{\varpi+1}, \varrho)] \quad or \quad [\eta(\varrho_{\varpi+2}, \varrho_{\varpi+3}) \leqslant \alpha(\varrho_{\varpi+2}, \varrho)] \; \forall \; \varpi \in \mathbb{N}.$$

Thus, $\exists$ a sub-sequence $\{\varrho_{\varpi_s}\}$ of $\{\varrho_\varpi\}$, s.t

$$\eta(\varrho_{\varpi_s}, \Omega\varrho_{\varpi_s}) \leqslant \alpha(\varrho_{\varpi_s}, \varrho).$$

Since, Ω is $\perp$-α-η-$G\mathcal{F}$-contraction, we obtain

$$\begin{aligned} \mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi_s}, \Omega\varrho)\Big) &< G\Big(d_\perp(\varrho_{\varpi_s}, \Omega\varrho_{\varpi_s}), d_\perp(\varrho, \Omega\varrho), d_\perp(\varrho_{\varpi_s}, \Omega\varrho), \\ &\quad d_\perp(\varrho, \Omega\varrho_{\varpi_s})\Big) + \mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi_s}, \Omega\varrho)\Big) \\ &\leqslant \mathcal{F}\Big(d_\perp(\varrho_{\varpi_s}, \varrho)\Big), \\ \text{that is,} \quad \mathcal{F}(d_\perp(\Omega\varrho_{\varpi_s}, \Omega\varrho)) &\leqslant \mathcal{F}(d_\perp(\varrho_{\varpi_s}, \varrho)). \end{aligned}$$

From $(\mathcal{F}_1)$ property of $\mathcal{F}$, we have

$$d_\perp(\Omega\varrho_{\varpi_s}, \Omega\varrho) < d_\perp(\varrho_{\varpi_s}, \varrho). \tag{2.10}$$

Letting $s \to +\infty$ in (2.10), gives $d_\perp(\varrho, \Omega\varrho) = 0$. Thus, Ω possesses a fixed point. Further, the uniqueness of fixed point follows on the lines of Theorem 2.2.4. □

Theorem 2.2.9. *For $(\mho, d_\perp)$ an $\perp$-complete metric space with ϱ_0 as an orthogonal element, suppose $G \in \mathfrak{G}$ and $\mathcal{F} \in \mathfrak{F}$. Let $\alpha, \eta : \mho^2 \to \mathbb{R}^+$ and $\Omega : \mho \to \mho$ be a self-map s.t:*

(I) Ω is $\perp$-preserving;

(II) Ω is α-admissible map w.r.t η;

(III) $\exists\ \varrho_0 \in \mho$ s.t $\eta(\varrho_0, \Omega\varrho_0) \leqslant \alpha(\varrho_0, \Omega\varrho_0)$;

(IV) If $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ is an $\perp$-sequence in $\mho$ s.t $\eta(\varrho_\varpi, \varrho_{\varpi+1}) \leqslant \alpha(\varrho_\varpi, \varrho_{\varpi+1})$ and $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$, then

$$[\varrho_\varpi \perp \varrho \; \forall \; \varpi] \quad or \quad [\varrho \perp \varrho_\varpi \; \forall \; \varpi]$$

and,

$$[\eta(\Omega\varrho_\varpi, \Omega^2\varrho_\varpi) \leqslant \alpha(\Omega\varrho_\varpi, \varrho)] \quad or \quad [\eta(\Omega^2\varrho_\varpi, \Omega^3\varrho_\varpi) \leqslant \alpha(\Omega^2\varrho_\varpi, \varrho)] \; \forall \; \varpi \in \mathbb{N};$$

(V) Ω is $\perp$-α-η-$G\mathcal{F}$-weak contraction.

Then, Ω possesses a fixed point. In addition, if for each $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ implies $\eta(\varrho, \varrho) \leqslant \alpha(\varrho, \varsigma)$, then Ω possesses a unique fixed point.

Proof. Working on the lines of Theorem 2.2.8, we obtain a sub-sequence $\{\varrho_{\varpi_s}\}$ of an $\perp$-sequence $\{\varrho_{\varpi}\}$ with $\eta(\varrho_{\varpi_s}, \Omega\varrho_{\varpi_s}) \leqslant \alpha(\varrho_{\varpi_s}, \varrho)$, s.t

$$\begin{aligned} \mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi_s}, \Omega\varrho)\Big) &< G\Big(d_\perp(\varrho_{\varpi_s}, \Omega\varrho_{\varpi_s}), d_\perp(\varrho, \Omega\varrho), d_\perp(\varrho_{\varpi_s}, \Omega\varrho), d_\perp(\varrho, \Omega\varrho_{\varpi_s})\Big) \\ &+ \mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi_s}, \Omega\varrho)\Big) \\ &\leqslant \mathcal{F}\Bigg(\max\bigg\{d_\perp(\varrho_{\varpi_s}, \varrho), d_\perp(\varrho_{\varpi_s}, \Omega\varrho_{\varpi_s}), d_\perp(\varrho, \Omega\varrho), \\ &\quad \frac{d_\perp(\varrho_{\varpi_s}, \Omega\varrho) + d_\perp(\varrho, \Omega\varrho_{\varpi_s})}{2}\bigg\}\Bigg). \end{aligned} \tag{2.11}$$

From $(\mathcal{F}_1)$ property of $\mathcal{F}$ in (2.11), we have

$$\begin{aligned} d_\perp(\Omega\varrho_{\varpi_s}, \Omega\varrho) &< \max\bigg\{d_\perp(\varrho_{\varpi_s}, \varrho), d_\perp(\varrho_{\varpi_s}, \Omega\varrho_{\varpi_s}), d_\perp(\varrho, \Omega\varrho), \\ &\quad \frac{d_\perp(\varrho_{\varpi_s}, \Omega\varrho) + d_\perp(\varrho, \Omega\varrho_{\varpi_s})}{2}\bigg\}. \end{aligned} \tag{2.12}$$

Letting $s \to +\infty$ in (2.12), gives

$$d_\perp(\varrho, \Omega\varrho) = 0.$$

Thus, Ω possesses a fixed point. Further, the uniqueness of fixed point follows on the lines of Theorem 2.2.4. □

2.2.2 Orthogonal α-type $\mathcal{F}$-Contraction

The idea of α-type $\mathcal{F}$-contraction was discussed by Gopal et al. (2016), and the results proved were generalization of the contraction results in Ćirić (1974), Wardowski (2012), Wardowski & Dung (2014).
In this subsection, we first discuss some basic definitions and prove fixed point results related to orthogonal α-type $\mathcal{F}$-contraction and some of its weaker contraction conditions.

Definition 2.2.4. *For an orthogonal metric space $(\mho, d_\perp)$ and for $\alpha : \mho^2 \to \mathbb{R}^+$, a self-map $\Omega : \mho \to \mho$ is c.t.b an* ***orthogonal*** α***-type*** $\mathcal{F}$***-contraction*** *(denoted by $\perp$-α type $\mathcal{F}$-contraction) if $\exists\ \wp > 0$, $\mathcal{F} \in \mathfrak{F}$ with $\varrho \perp \varsigma$ and $d_\perp(\Omega\varrho, \Omega\varsigma) > 0$ s.t $\forall\ \varrho, \varsigma \in \mho$, we have*

$$\wp + \alpha(\varrho,\varsigma)\mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \leqslant \mathcal{F}\Big(d_\perp(\varrho,\varsigma)\Big).$$

Example 2.2.10. *Let $\mho = \mathbb{R}^+$, $d_\perp(\varrho,\varsigma) = |\varrho - \varsigma|$ and $\varrho \perp \varsigma$ iff either $\varrho = 0$ or $\varsigma = 0$. Then, $(\mho, d_\perp)$ is an orthogonal metric space. Let $\Omega : \mho \to \mho$ be defined as*

$$\Omega(\varrho) = \begin{cases} 3/2 & \varrho \in [10, 20); \\ 0 & \textit{otherwise}. \end{cases}$$

Let $\alpha : \mho^2 \to [0, +\infty)$ be defined as $\alpha(\varrho,\varsigma) = 3/2 \quad \forall\ \varrho,\varsigma \in \mho$. Define $\mathcal{F}(\mu) = \ln(\mu)$. For $d_\perp(\Omega\varrho, \Omega\varsigma) > 0$ and $\varrho \perp \varsigma$ to hold simultaneously, we have either $\varrho = 0$ and $\varsigma \in [10, 20)$ or $\varrho \in [10, 20)$ and $\varsigma = 0$.

Let $\varrho = 0$ and $\varsigma \in [10, 20)$. Then,

$$\wp + \alpha(0,\varsigma)\ln(d_\perp(\Omega 0, \Omega\varsigma)) = \wp + \frac{3}{2}\ln(d_\perp(0, 3/2)) = \wp + \frac{3}{2}\ln(3/2) \tag{2.13}$$

and,

$$\ln(d_\perp(0,\varsigma)) = \ln(\varsigma). \tag{2.14}$$

From (2.13), (2.14) and for $\wp = 1$, we can conclude that Ω is $\perp$-α type $\mathcal{F}$-contraction. The case for $\varrho \in [10, 20)$ and $\varsigma = 0$ holds on the similar lines.

Definition 2.2.5. *For an orthogonal metric space $(\mho, d_\perp)$ and for $\alpha : \mho^2 \to \mathbb{R}^+$, a self-map $\Omega : \mho \to \mho$ is c.t.b an* ***orthogonal*** α***-type*** $\mathcal{F}$***-weak contraction*** *(denoted by $\perp$-α type $\mathcal{F}$-weak contraction) if $\exists\ \wp > 0$ and $\mathcal{F} \in \mathfrak{F}$ s.t $\forall\ \varrho,\varsigma \in \mho$ with $\varrho \perp \varsigma$ and $d_\perp(\Omega\varrho, \Omega\varsigma) > 0$, we have*

$$\wp + \alpha(\varrho,\varsigma)\mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \leqslant \mathcal{F}\Bigg(\max\Bigg\{d_\perp(\varrho,\varsigma), d_\perp(\varrho, \Omega\varrho), d_\perp(\varsigma, \Omega\varsigma), \frac{d_\perp(\varrho, \Omega\varsigma) + d_\perp(\varsigma, \Omega\varrho)}{2}\Bigg\}\Bigg).$$

Remark 2.2.11. *From the above definitions, we can concluded that each $\perp$-α type*

$\mathcal{F}$-contraction is an $\perp$-α type $\mathcal{F}$-weak contraction.

Theorem 2.2.12. *For $(\mho, d_\perp)$ an $\perp$-complete metric space with $s \geqslant 1$ and ϱ_0 as an orthogonal element, suppose $\mathcal{F} \in \mathfrak{F}$. Let $\alpha : \mho^2 \to \mathbb{R}^+$ and $\Omega : \mho \to \mho$ be a self-map s.t:*

(I) Ω is $\perp$-preserving;

(II) Ω is weak α-admissible map type S;

(III) $\exists$ some $\varrho_0 \in \mho$ with $s \leqslant \alpha(\varrho_0, \Omega\varrho_0)$;

(IV) Ω is $\perp$-continuous;

(V) Ω is $\perp$-α type $\mathcal{F}$-contraction.

Then, Ω possesses a fixed point. In addition, if for each $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ implies $s \leqslant \alpha(\varrho, \varsigma)$, then Ω possesses a unique fixed point.

Proof. On defining a sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ where $\varrho_{\varpi+1} = \Omega\varrho_\varpi = \Omega^{\varpi+1}\varrho_0$ for each $\varpi \in \mathbb{N}$ and since $\varrho_0, \Omega\varrho_0 \in \mho$ where $(\mho, \perp)$ is an $\perp$-set then the repeated use of $\perp$-preserving property of Ω, gives

$$[\varrho_{\varpi-1} \perp \varrho_\varpi \ \forall\, \varpi \in \mathbb{N}] \quad \text{or} \quad [\varrho_\varpi \perp \varrho_{\varpi-1} \ \forall\, \varpi \in \mathbb{N}],$$

thus, $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ is an $\perp$-sequence in $\mho$.
Now, by the given condition $\alpha(\varrho_0, \varrho_1) = \alpha(\varrho_0, \Omega\varrho_0) \geqslant s$ and as Ω is weak α-admissible map type S, we have $\alpha(\varrho_1, \varrho_2) \geqslant s$ continuing, we get $\alpha(\varrho_{\varpi-1}, \varrho_\varpi) \geqslant s$, and

$$\begin{aligned}\mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) = \mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big) &\leqslant s\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big)\\ &\leqslant \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big).\end{aligned}$$

Using $\perp$-α type $\mathcal{F}$-contraction condition of Ω and for $\wp > 0$, we get

$$\begin{aligned}\wp + \mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) &\leqslant \wp + s\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big)\\ &\leqslant \wp + \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big)\\ &\leqslant \mathcal{F}\Big(d_\perp(\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big),\\ \text{that is,} \quad \mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) &\leqslant \mathcal{F}\Big(d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)\Big) - \wp\\ &\leqslant \mathcal{F}\Big(d_\perp(\varrho_{\varpi-2}, \varrho_{\varpi-1})\Big) - 2\wp\\ &\leqslant \cdots \leqslant \mathcal{F}\Big(d_\perp(\varrho_0, \varrho_1)\Big) - \varpi\wp. \qquad (2.15)\end{aligned}$$

Taking limit as $\varpi \to +\infty$ in (2.15) and by $(\mathcal{F}_2)$ property of $\mathcal{F}$, we have

$$\lim_{\varpi \to +\infty} d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) = 0. \tag{2.16}$$

And further, by $(\mathcal{F}_3)$ property of $\mathcal{F}$, $\exists\, \gamma \in (0,1)$, s.t

$$\lim_{\varpi \to +\infty} \Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma \mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) = 0. \tag{2.17}$$

From (2.15), we conclude that

$$\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma \Big(\mathcal{F}(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})) - \mathcal{F}(d_\perp(\varrho_0, \varrho_1))\Big) \leqslant -\varpi \Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma \wp. \tag{2.18}$$

On letting $\varpi \to +\infty$ in (2.18) and using (2.16) and (2.17), we obtain

$$\lim_{\varpi \to +\infty} \varpi \Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma = 0.$$

So, $\exists$ some $\varpi_1 \in \mathbb{N}$, s.t

$$d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) < \frac{1}{\varpi^{1/\gamma}} \quad \forall\, \varpi \geqslant \varpi_1.$$

Consider $\varpi^* > \varpi > \varpi_1$, then by triangle inequality, we obtain

$$\begin{aligned} d_\perp(\varrho_\varpi, \varrho_{\varpi^*}) &\leqslant d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) + d_\perp(\varrho_{\varpi+1}, \varrho_{\varpi+2}) + \cdots + d_\perp(\varrho_{\varpi^*-1}, \varrho_{\varpi^*}) \\ &\leqslant \sum_{i=1}^{+\infty} d_\perp(\varrho_i, \varrho_{i+1}) = \sum_{i=1}^{+\infty} \frac{1}{i^{1/\gamma}}. \end{aligned}$$

As series $\sum_{i=1}^{+\infty} \frac{1}{i^{1/\gamma}}$ is convergent, we get $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ is an $\perp$-Cauchy sequence and since $\mho$ is $\perp$-complete, so $\exists\ \varrho \in \mho$ s.t $\lim_{\varpi \to +\infty} \varrho_\varpi = \varrho$. Further, since Ω is $\perp$-continuous, we have

$$\begin{aligned} \lim_{\varpi \to +\infty} \Omega\varrho_\varpi = \lim_{\varpi \to +\infty} \varrho_{\varpi+1} &= \Omega\varrho, \\ \text{that is,} \quad \varrho &= \Omega\varrho. \end{aligned}$$

Thus, Ω possesses a fixed point. Next, let ς be s.t $\Omega\varsigma = \varsigma$ and $\varrho \perp \varsigma$. Then by given condition, we obtain $\alpha(\varrho, \varsigma) \geqslant s$. Using $\perp$-α type $\mathcal{F}$-contraction property of Ω, we have

$$\wp + \mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \leqslant \wp + s\mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \leqslant \wp + \alpha(\varrho, \varsigma)\mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big)$$

$$\leqslant \mathcal{F}\Big(d_\perp(\varrho,\varsigma)\Big),$$

$$\text{that is,}\quad \wp+\mathcal{F}\Big(d_\perp(\varrho,\varsigma)\Big) \leqslant \mathcal{F}\Big(d_\perp(\varrho,\varsigma)\Big). \tag{2.19}$$

Now, (2.19) holds only if $\varrho=\varsigma$. Hence, Ω possesses a unique fixed point. □

Example 2.2.13. *The self-map Ω defined in Example 2.2.10 satisfies all hypotheses of above theorem and thus possesses a fixed point $\varrho=0$.*

Theorem 2.2.14. *For $(\mho, d_\perp)$ an $\perp$-complete metric space with $s\geqslant 1$ and ϱ_0 as an orthogonal element, suppose $\mathcal{F}\in\mathfrak{F}$. Let $\alpha:\mho^2\to\mathbb{R}^+$ and $\Omega:\mho\to\mho$ be a self-map s.t:*

(I) Ω is $\perp$-preserving;

(II) Ω is weak α-admissible map type S;

(III) $\exists\ \varrho_0\in\mho$ with $s\leqslant\alpha(\varrho_0,\Omega\varrho_0)$;

(IV) Ω is $\perp$-continuous;

(V) Ω is $\perp$-α type $\mathcal{F}$-weak contraction.

Then, Ω possesses a fixed point. In addition, if for each $\varrho,\varsigma\in\mho$ with $\varrho\perp\varsigma$, $\Omega\varrho=\varrho$ and $\Omega\varsigma=\varsigma$ implies $s\leqslant\alpha(\varrho,\varsigma)$, then Ω possesses a unique fixed point.

Proof. Working on the lines of Theorem 2.2.12, we obtain an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ with $\alpha(\varrho_\varpi,\varrho_{\varpi+1})\geqslant s\quad\forall\ \varpi\in\mathbb{N}$.

$$\begin{aligned}
\mathcal{F}\Big(d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\Big) &= \mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1},\Omega\varrho_\varpi)\Big)\\
&\leqslant s\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1},\Omega\varrho_\varpi)\Big)\\
&\leqslant \alpha(\varrho_{\varpi-1},\varrho_\varpi)\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1},\Omega\varrho_\varpi)\Big).
\end{aligned}$$

Since Ω is $\perp$-α type $\mathcal{F}$-weak contraction, so

$$\begin{aligned}
\wp+\mathcal{F}\Big(d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\Big) \leqslant&\wp+\alpha(\varrho_{\varpi-1},\varrho_\varpi)\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1},\Omega\varrho_\varpi)\Big)\\
\leqslant&\mathcal{F}\Bigg(\max\Bigg\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi),d_\perp(\varrho_{\varpi-1},\Omega\varrho_{\varpi-1}),\\
&d_\perp(\varrho_\varpi,\Omega\varrho_\varpi),\frac{d_\perp(\varrho_{\varpi-1},\Omega\varrho_\varpi)+d_\perp(\varrho_\varpi,\Omega\varrho_{\varpi-1})}{2}\Bigg\}\Bigg)\\
=&\mathcal{F}\Bigg(\max\Bigg\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi),d_\perp(\varrho_\varpi,\varrho_{\varpi+1}),
\end{aligned}$$

$$\left.\left.\frac{d_\perp(\varrho_{\varpi-1},\varrho_{\varpi+1})}{2}\right\}\right)$$

$$\leqslant \mathcal{F}\left(\max\left\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_\varpi,\varrho_{\varpi+1}), \frac{d_\perp(\varrho_{\varpi-1},\varrho_\varpi)+d_\perp(\varrho_\varpi,\varrho_{\varpi+1})}{2}\right\}\right),$$

$$\text{implies} \quad \wp + \mathcal{F}\Big(d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\Big) \leqslant \mathcal{F}\left(\max\left\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\right\}\right). \tag{2.20}$$

Case (i): Let $\max\left\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\right\} = d_\perp(\varrho_\varpi,\varrho_{\varpi+1})$, then by (2.20)

$$\wp + \mathcal{F}\Big(d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\Big) \leqslant \mathcal{F}(d_\perp(\varrho_\varpi,\varrho_{\varpi+1})),$$

which does not hold true for any $\wp > 0$.

Case (ii): Let $\max\left\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\right\} = d_\perp(\varrho_{\varpi-1},\varrho_\varpi)$, then by (2.20)

$$\begin{aligned}
\wp + \mathcal{F}\Big(d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\Big) &\leqslant \mathcal{F}(d_\perp(\varrho_{\varpi-1},\varrho_\varpi)), \\
\text{that is,} \quad \mathcal{F}\Big(d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\Big) &\leqslant \mathcal{F}(d_\perp(\varrho_{\varpi-1},\varrho_\varpi)) - \wp \\
&= \mathcal{F}(d_\perp(\varrho_{\varpi-1},\varrho_\varpi)) - 2\wp \\
&\leq \cdots \leqslant \mathcal{F}(d_\perp(\varrho_0,\varrho_1)) - \varpi\wp.
\end{aligned}$$

The result now follows on the lines of Theorem 2.2.12. □

Remark 2.2.15. *In the upcoming result, we weaken the condition of $\perp$-continuity of Ω.*

Theorem 2.2.16. *For $(\mho, d_\perp)$ an $\perp$-complete metric space with $s \geqslant 1$ and ϱ_0 as an orthogonal element, suppose $\mathcal{F} \in \mathfrak{F}$. Let $\alpha : \mho^2 \to \mathbb{R}^+$ and $\Omega : \mho \to \mho$ be a self-map s.t:*

(I) Ω *is* $\perp$*-preserving;*

(II) Ω *is weak* α*-admissible map type S;*

(III) $\exists$ *some* $\varrho_0 \in \mho$ *with* $s \leqslant \alpha(\varrho_0, \Omega\varrho_0)$*;*

(IV) If $\exists$ *an* $\perp$*-sequence* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *with* $\alpha(\varrho_\varpi,\varrho_{\varpi+1}) \geqslant s$ *and* $\varrho_\varpi \to \varrho$ *as* $\varpi \to +\infty$*, then* $\alpha(\varrho_\varpi,\varrho) \geqslant s$ *and* either $[\varrho_\varpi \perp \varrho \ \forall\, \varpi \in \mathbb{N}]$ or $[\varrho \perp \varrho_\varpi \ \forall\, \varpi \in \mathbb{N}]$*;*

(V) Ω is $\perp$-α type $\mathcal{F}$-contraction.

Then, Ω possesses a fixed point. In addition, if for each $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ implies $s \leqslant \alpha(\varrho, \varsigma)$, then Ω possesses a unique fixed point.

Proof. Proceeding on the lines of Theorem 2.2.12, one can obtain $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ an $\perp$-sequence where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ and $\alpha(\varrho_\varpi, \varrho_{\varpi+1}) \geqslant s$. Then, by given condition we have $\alpha(\varrho_\varpi, \varrho) \geqslant s$ and either

$$[\varrho_\varpi \perp \varrho \,\forall\, \varpi \in \mathbb{N}] \quad \text{or} \quad [\varrho \perp \varrho_\varpi \,\forall\, \varpi \in \mathbb{N}].$$

Using $\perp$-preserving property of Ω, we get

$$[\Omega\varrho_\varpi \perp \Omega\varrho \,\forall\, \varpi \in \mathbb{N}] \quad \text{or} \quad [\Omega\varrho \perp \Omega\varrho_\varpi \,\forall\, \varpi \in \mathbb{N}].$$

Since Ω is an $\perp$-α type $\mathcal{F}$-contraction, so we have

$$\begin{aligned} \mathcal{F}\Big(d_\perp(\varrho_{\varpi+1}, \Omega\varrho)\Big) \leqslant \wp + \mathcal{F}\Big(d_\perp(\varrho_{\varpi+1}, \Omega\varrho)\Big) &= \wp + \mathcal{F}\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) \\ &\leqslant \wp + s\mathcal{F}\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) \\ &\leqslant \wp + \alpha(\varrho_\varpi, \varrho)\mathcal{F}\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) \\ &\leqslant \mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho)\Big). \end{aligned} \tag{2.21}$$

Using $(\mathcal{F}_1)$ property of $\mathcal{F}$ in (2.21), we obtain

$$d_\perp(\varrho_{\varpi+1}, \Omega\varrho) < d_\perp(\varrho_\varpi, \varrho).$$

Let $\varpi \to +\infty$, we get $d_\perp(\varrho, \Omega\varrho) = 0$. Thus, Ω possesses a fixed point. Further, the uniqueness of the fixed point of Ω follows on the lines of Theorem 2.2.12. □

Theorem 2.2.17. *For $(\mho, d_\perp)$ an $\perp$-complete metric space with $s \geqslant 1$ and ϱ_0 as an orthogonal element, suppose $\mathcal{F} \in \mathfrak{F}$. Let $\alpha : \mho^2 \to \mathbb{R}^+$ and $\Omega : \mho \to \mho$ be a self-map s.t:*

(I) Ω is $\perp$-preserving;

(II) Ω is weak α-admissible map type S;

(III) $\exists$ some $\varrho_0 \in \mho$ with $s \leqslant \alpha(\varrho_0, \Omega\varrho_0)$;

(IV) If $\exists$ an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ with $\alpha(\varrho_\varpi, \varrho_{\varpi+1}) \geqslant s$ and $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$, then $\alpha(\varrho_\varpi, \varrho) \geqslant s$ and either $[\varrho_\varpi \perp \varrho \,\forall\, \varpi \in \mathbb{N}]$ or $[\varrho \perp \varrho_\varpi \,\forall\, \varpi \in \mathbb{N}]$*;*

(V) Ω is $\perp$-α type $\mathcal{F}$-weak contraction.

Then, Ω possesses a fixed point. In addition, if each $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ implies $s \leqslant \alpha(\varrho, \varsigma)$, then Ω possesses a unique fixed point.

Proof. By the working of Theorem 2.2.16, we obtain

$$\begin{aligned}
\mathcal{F}\Big(d_\perp(\varrho_{\varpi+1}, \Omega\varrho)\Big) &\leqslant \wp + \mathcal{F}\Big(d_\perp(\varrho_{\varpi+1}, \Omega\varrho)\Big) \\
&= \wp + \mathcal{F}\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) \\
&\leqslant \wp + s\mathcal{F}\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) \\
&\leqslant \wp + \alpha(\varrho_\varpi, \varrho)\mathcal{F}\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) \\
&\leqslant \mathcal{F}\Bigg(\max\Bigg\{d_\perp(\varrho_\varpi, \varrho), d_\perp(\varrho_\varpi, \Omega\varrho_\varpi), d_\perp(\varrho, \Omega\varrho), \\
&\quad \frac{d_\perp(\varrho_\varpi, \Omega\varrho) + d_\perp(\varrho, \Omega\varrho_\varpi)}{2}\Bigg\}\Bigg). \qquad (2.22)
\end{aligned}$$

Using ($\mathcal{F}_1$) property of $\mathcal{F}$ in (2.22), we obtain

$$d_\perp(\varrho_{\varpi+1}, \Omega\varrho) < \max\Bigg\{d_\perp(\varrho_\varpi, \varrho), d_\perp(\varrho_\varpi, \Omega\varrho_\varpi), d_\perp(\varrho, \Omega\varrho), \frac{d_\perp(\varrho_\varpi, \Omega\varrho) + d_\perp(\varrho, \Omega\varrho_\varpi)}{2}\Bigg\}.$$

Let $\varpi \to +\infty$, we get

$$d_\perp(\varrho, \Omega\varrho) = 0.$$

Thus, Ω possesses a fixed point. Further, the uniqueness of the fixed point of Ω follows on the lines of Theorem 2.2.16. □

Remark 2.2.18. *It should be noted that Theorem 2.2.12, Theorem 2.2.14, Theorem 2.2.16 and Theorem 2.2.17 proved above are valid even if Ω is considered as an α-admissible map type S.*

Theorem 2.2.19. *For $(\mho, d_\perp)$ an $\perp$-complete metric space with ϱ_0 as an orthogonal element, suppose $\mathcal{F} \in \mathfrak{F}$. Let $\alpha : \mho^2 \to \mathbb{R}^+$ and $\Omega : \mho \to \mho$ be a self-map s.t:*

(I) Ω is $\perp$-preserving;

(II) Ω is weak α-admissible map;

(III) $\exists$ some $\varrho_0 \in \mho$ with $1 \leqslant \alpha(\varrho_0, \Omega\varrho_0)$;

(IV) Ω *is* $\perp$*-continuous;*

(V) Ω *is* $\perp$-α *type* $\mathcal{F}$*-contraction.*

Then, Ω *possesses a fixed point. In addition, if for each* $\varrho, \varsigma \in \mho$ *with* $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ *and* $\Omega\varsigma = \varsigma$ *implies* $1 \leqslant \alpha(\varrho, \varsigma)$*, then* Ω *possesses a unique fixed point.*

Proof. On defining a sequence $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ in $\mho$, where $\varrho_{\varpi+1} = \Omega\varrho_\varpi = \Omega^{\varpi+1}\varrho_0$ for each $\varpi \in \mathbb{N}$ and since $\varrho_0, \Omega\varrho_0 \in \mho$, where $(\mho, \perp)$ is an $\perp$-set then the repeated use of $\perp$-preserving property of Ω, gives

$$[\varrho_{\varpi-1} \perp \varrho_\varpi \ \forall\, \varpi \in \mathbb{N}] \quad or \quad [\varrho_\varpi \perp \varrho_{\varpi-1} \ \forall\, \varpi \in \mathbb{N}],$$

thus, $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ is an $\perp$-sequence in $\mho$. Now, by given condition $\alpha(\varrho_0, \varrho_1) = \alpha(\varrho_0, \Omega\varrho_0) \geqslant 1$ then by the weak α-admissibility of Ω, we have $\alpha(\Omega\varrho_0, \Omega\Omega\varrho_0) = \alpha(\varrho_1, \varrho_2) \geqslant 1$ continuing, we get $\alpha(\varrho_{\varpi-1}, \varrho_\varpi) \geqslant 1$, and

$$\mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) = \mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big) \leqslant \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big).$$

Using $\perp$-α type $\mathcal{F}$-contraction condition of Ω and for $\wp > 0$, we have

$$\begin{aligned} \wp + \mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) &\leqslant \wp + \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big) \\ &\leqslant \mathcal{F}\Big(d_\perp(\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big), \\ \text{that is,} \quad \mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) &\leqslant \mathcal{F}\Big(d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)\Big) - \wp \\ &\leqslant \mathcal{F}\Big(d_\perp(\varrho_{\varpi-2}, \varrho_{\varpi-1})\Big) - 2\wp \\ &\leqslant \cdots \leqslant \mathcal{F}\Big(d_\perp(\varrho_0, \varrho_1)\Big) - \varpi\wp. \end{aligned} \tag{2.23}$$

Taking limit as $\varpi \to +\infty$ in (2.23) and by $(\mathcal{F}_2)$ property of $\mathcal{F}$, we have

$$\lim_{\varpi \to +\infty} d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) = 0. \tag{2.24}$$

And further, by $(\mathcal{F}_3)$ property of $\mathcal{F}$, $\exists\, \gamma \in (0, 1)$, implies

$$\lim_{\varpi \to +\infty} \Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma \mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) = 0. \tag{2.25}$$

From (2.23), we conclude that

$$\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma \Big(\mathcal{F}(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})) - \mathcal{F}(d_\perp(\varrho_0, \varrho_1))\Big) \leqslant -\varpi\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big)^\gamma \wp. \tag{2.26}$$

On letting $\varpi \to +\infty$ in (2.26) and by (2.24), (2.25), we get

$$\lim_{\varpi \to +\infty} \varpi \Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) \Big)^\gamma = 0.$$

Thus, $\exists$ some $\varpi_1 \in \mathbb{N}$, s.t

$$d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) < \frac{1}{\varpi^{1/\gamma}} \quad \forall\, \varpi \geqslant \varpi_1.$$

Consider $\varpi^* > \varpi > \varpi_1$, then by triangle inequality, we obtain

$$\begin{aligned} d_\perp(\varrho_\varpi, \varrho_{\varpi^*}) &\leqslant d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) + d_\perp(\varrho_{\varpi+1}, \varrho_{\varpi+2}) + \cdots + d_\perp(\varrho_{\varpi^*-1}, \varrho_{\varpi^*}) \\ &\leqslant \sum_{i=1}^{+\infty} d_\perp(\varrho_i, \varrho_{i+1}) = \sum_{i=1}^{+\infty} \frac{1}{i^{1/\gamma}}. \end{aligned}$$

As series $\sum_{i=1}^{+\infty} \frac{1}{i^{1/\gamma}}$ is convergent, we get $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ is an $\perp$-Cauchy sequence and since $\mho$ is $\perp$-complete, so $\exists\ \varrho \in \mho$ for which

$$\lim_{\varpi \to +\infty} \varrho_\varpi = \varrho.$$

Further, since Ω is $\perp$-continuous, so we have

$$\begin{aligned} \lim_{\varpi \to +\infty} \Omega \varrho_\varpi = \lim_{\varpi \to +\infty} \varrho_{\varpi+1} &= \Omega \varrho, \\ \text{that is,} \quad \varrho &= \Omega \varrho. \end{aligned}$$

Thus, Ω possesses a fixed point. Next, let ς be s.t $\Omega\varsigma = \varsigma$ and $\varrho \perp \varsigma$. Then by given condition, we get $\alpha(\varrho, \varsigma) \geqslant 1$. Using $\perp$-α type $\mathcal{F}$-contraction property of Ω, we have

$$\begin{aligned} \wp + \mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) &\leqslant \wp + \alpha(\varrho, \varsigma)\mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \leqslant \mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big), \\ \text{that is,} \quad \wp + \mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big) &\leqslant \mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big). \end{aligned} \tag{2.27}$$

Now, (2.27) holds only if $\varrho = \varsigma$. Hence, Ω possesses a unique fixed point. □

Example 2.2.20. *Let $\mho = (-\infty, +\infty)$ along with usual metric space and define $\varrho \perp \varsigma$ iff $\varrho = k\varsigma\ \forall\, \varsigma \in \mho$ and for some fixed $k \in \mathbb{Z}$. Then, $(\mho, d_\perp)$ is an orthogonal metric space. Let $\Omega : \mho \to \mho$ be defined as*

$$\Omega(\varrho) = \begin{cases} 22/25 & for\ \varrho \in \mho - [-1, 1]; \\ 0 & otherwise. \end{cases}$$

Let $\alpha : \mho^2 \to \mathbb{R}^+$ *as* $\alpha(\varrho, \varsigma) = 1 \ \forall \ \varrho, \varsigma \in \mho$. *For* $\varrho \perp \varsigma$ *and* $d_\perp(\Omega\varrho, \Omega\varsigma) > 0$ *to hold together, we must have either* $\varrho = 0$ *and* $\varsigma \in \mho - [-1, 1]$ or $\varrho \in \mho - [-1, 1]$ *and* $\varsigma = 0$. *Consider* $\varrho = 0$ *and* $\varsigma \in \mho - [-1, 1]$ *along with* $\mathcal{F}(\mu) = \ln(\mu)$ *and* $\wp = -\ln(22/25) > 0$, *we have*

$$\wp + \alpha(\varrho, \varsigma)\mathcal{F}\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) = \wp + \ln(22/25) = 0, \tag{2.28}$$

$$\text{and,} \quad \mathcal{F}\Big(d_\perp(\varrho, \varsigma)\Big) = \ln(|\varsigma|), \quad \text{where } \varsigma \in \mho - [-1, 1]. \tag{2.29}$$

So, from (2.28) and (2.29), we can conclude that Ω *is* $\perp$-α *type* $\mathcal{F}$*-contraction although,* Ω *is not continuous. Also, the space* $(\mho, d_\perp)$ *is* $\perp$*-complete (because of completeness of metric space* $(\mho, d_\perp)$*) and the self-map* Ω *is weak* α*-admissible and* $\perp$*-preserving. Next, to check* $\perp$*-continuity of* Ω*, let* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *be an* $\perp$*-sequence in* $\mho$ *which is convergent. Then, we have* $\varrho_\varpi \to 0$ *as* $\varpi \to +\infty$*, that is,* $\lim_{\varpi\to+\infty} \Omega\varrho_\varpi = 0 = \Omega 0$. *Thus,* Ω *is* $\perp$*-continuous. Since, each hypothesis of Theorem 2.2.19 is satisfied, so* Ω *possesses a fixed point viz.* $\varrho = 0$.

Theorem 2.2.21. *For* $(\mho, d_\perp)$ *an* $\perp$*-complete metric space with* ϱ_0 *as an orthogonal element, suppose* $\mathcal{F} \in \mathfrak{F}$. *Let* $\alpha : \mho^2 \to \mathbb{R}^+$ *and* $\Omega : \mho \to \mho$ *be a self-map s.t:*

(I) Ω *is* $\perp$*-preserving;*

(II) Ω *is weak* α*-admissible map;*

(III) $\exists$ *some* $\varrho_0 \in \mho$ *with* $1 \leqslant \alpha(\varrho_0, \Omega\varrho_0)$*;*

(IV) Ω *is* $\perp$*-continuous;*

(V) Ω *is* $\perp$-α *type* $\mathcal{F}$*-weak contraction.*

Then, Ω *possesses a fixed point. In addition, if for each* $\varrho, \varsigma \in \mho$ *with* $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ *and* $\Omega\varsigma = \varsigma$ *implies* $1 \leqslant \alpha(\varrho, \varsigma)$*, then* Ω *possesses a unique fixed point.*

Proof. By the working done in Theorem 2.2.19, we obtain an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ with $\alpha(\varrho_\varpi, \varrho_{\varpi+1}) \geqslant 1 \quad \forall \, \varpi \in \mathbb{N}$.

$$\mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) = \mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big) \leqslant \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big).$$

The result now follows on the lines of Theorem 2.2.14. □

Remark 2.2.22. *In the upcoming result, we weaken the condition of $\perp$-continuity of Ω.*

Theorem 2.2.23. *For $(\mho, d_\perp)$ an $\perp$-complete metric space with ϱ_0 as an orthogonal element, suppose $\mathcal{F} \in \mathfrak{F}$. Let $\alpha : \mho^2 \to \mathbb{R}^+$ and $\Omega : \mho \to \mho$ be a self-map s.t:*

(I) Ω *is $\perp$-preserving;*

(II) Ω *is weak α-admissible map;*

(III) $\exists$ *some* $\varrho_0 \in \mho$ *with* $1 \leqslant \alpha(\varrho_0, \Omega\varrho_0)$*;*

(IV) If $\exists$ *an $\perp$-sequence* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *with* $\alpha(\varrho_\varpi, \varrho_{\varpi+1}) \geqslant 1$ *and* $\varrho_\varpi \to \varrho$ *as* $\varpi \to +\infty$*, then* $\alpha(\varrho_\varpi, \varrho) \geqslant 1$ *and* either $[\varrho_\varpi \perp \varrho \; \forall \, \varpi \in \mathbb{N}]$ or $[\varrho \perp \varrho_\varpi \; \forall \, \varpi \in \mathbb{N}]$*;*

(V) Ω *is $\perp$-α type $\mathcal{F}$-contraction.*

Then, Ω possesses a fixed point. In addition, if for each $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ implies $1 \leqslant \alpha(\varrho, \varsigma)$, then Ω possesses a unique fixed point.

Proof. By the working done in Theorem 2.2.19, one can obtain $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ an $\perp$-sequence, where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ and $\alpha(\varrho_\varpi, \varrho_{\varpi+1}) \geqslant 1$. Then by given condition, we have $\alpha(\varrho_\varpi, \varrho) \geqslant 1$ and either

$$[\varrho_\varpi \perp \varrho \; \forall \, \varpi \in \mathbb{N}] \quad \text{or} \quad [\varrho \perp \varrho_\varpi \; \forall \, \varpi \in \mathbb{N}].$$

Using $\perp$-preserving property of Ω, we get

$$[\Omega\varrho_\varpi \perp \Omega\varrho \; \forall \, \varpi \in \mathbb{N}] \quad \text{or} \quad [\Omega\varrho \perp \Omega\varrho_\varpi \; \forall \, \varpi \in \mathbb{N}].$$

As, Ω is an $\perp$-α type $\mathcal{F}$-contraction, so

$$\begin{aligned}\mathcal{F}\Big(d_\perp(\varrho_{\varpi+1}, \Omega\varrho)\Big) \leqslant \wp + \mathcal{F}\Big(d_\perp(\varrho_{\varpi+1}, \Omega\varrho)\Big) &= \wp + \mathcal{F}\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) \\ &\leqslant \wp + \alpha(\varrho_\varpi, \varrho)\mathcal{F}\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) \\ &\leqslant \mathcal{F}\Big(d_\perp(\varrho_\varpi, \varrho)\Big). \qquad (2.30)\end{aligned}$$

Using $(\mathcal{F}_1)$ property of $\mathcal{F}$ in (2.30), we obtain

$$d_\perp(\varrho_{\varpi+1}, gp) \leqslant d_\perp(\varrho_\varpi, \varrho).$$

On letting $\varpi \to +\infty$, we get

$$d_{\perp}(\varrho, \Omega\varrho) = 0.$$

Thus, Ω possesses a fixed point. Further, the uniqueness follows on the lines of Theorem 2.2.19. ☐

Theorem 2.2.24. *For $(\mho, d_{\perp})$ an $\perp$-complete metric space with ϱ_0 as an orthogonal element, suppose $\mathcal{F} \in \mathfrak{F}$. Let $\alpha : \mho^2 \to \mathbb{R}^+$ and $\Omega : \mho \to \mho$ be a self-map s.t:*

(I) Ω is $\perp$-preserving;

(II) Ω is weak α-admissible map;

(III) $\exists$ some $\varrho_0 \in \mho$ with $1 \leqslant \alpha(\varrho_0, \Omega\varrho_0)$;

(IV) If $\exists$ an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ with $\alpha(\varrho_\varpi, \varrho_{\varpi+1}) \geqslant 1$ and $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$, then $\alpha(\varrho_\varpi, \varrho) \geqslant 1$ and either $[\varrho_\varpi \perp \varrho \; \forall \; \varpi \in \mathbb{N}]$ or $[\varrho \perp \varrho_\varpi \; \forall \; \varpi \in \mathbb{N}]$*;*

(V) Ω is $\perp$-α type $\mathcal{F}$-weak contraction.

Then, Ω possesses a fixed point. In addition, if for each $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$, $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ implies $1 \leqslant \alpha(\varrho, \varsigma)$, then Ω possesses a unique fixed point.

Proof. The proof follows from the working of Theorem 2.2.23 followed by working done in Theorem 2.2.17. ☐

Remark 2.2.25. *It should be noted that Theorem 2.2.19, Theorem 2.2.21, Theorem 2.2.23 and Theorem 2.2.24 proved above are valid even if Ω is considered as an α-admissible map.*

2.2.3 Orthogonal *TAC*-Contraction

TAC-type contractive map was introduced by Chandok et al. (2016). Inspired by work done, in this subsection we put forward the notion of orthogonal *TAC*-type S-contraction map, orthogonal weak *TAC*-type S-rational contraction, orthogonal *TAC*-contraction map and orthogonal weak *TAC*-rational contraction that further extends our approach towards contraction principles and fixed point results in orthogonal metric space.

Let Ψ denotes the set of maps $\psi_1 : [0, +\infty) \to [0, +\infty)$, which are continuous and monotonically increasing with $\psi_1^{-1}(\{0\}) = 0$ and let Φ denotes the set of maps $\phi_1 : [0, +\infty) \to [0, +\infty)$, which are continuous where $\lim_{\varpi \to +\infty} \phi_1(\varrho_\varpi) = 0$ implies $\lim_{\varpi \to +\infty} \varrho_\varpi = 0$.

Definition 2.2.6. *For an orthogonal metric space $(\mho, d_\perp)$, a self-map $\Omega : \mho \to \mho$ is c.t.b an **orthogonal* TAC*-type S-contraction*** *(denoted by $\perp$-TAC-type S-contraction) if for $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$, $s \geqslant 1$ and $\hat{\alpha}(\varrho).\beta(\varsigma) \geqslant s$ implies*

$$\psi_1\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \leqslant \mathcal{C}\Big(\psi_1(d_\perp(\varrho, \varsigma)), \phi_1(d_\perp(\varrho, \varsigma))\Big),$$

where $\hat{\alpha}, \beta : \mho \to [0, +\infty)$, $\mathcal{C} \in \mathfrak{C}$, $\psi_1 \in \Psi$ and $\phi_1 \in \Phi$.

Example 2.2.26. *Let $\mho = \mathbb{R}$, $d_\perp(\varrho, \varsigma) = |\varrho - \varsigma|$ and $\varrho \perp \varsigma$ iff $\varrho.\varsigma = 0$. Then, $(\mho, d_\perp)$ is an orthogonal metric space. Let $\Omega : \mho \to \mho$ be defined as*

$$\Omega(\varrho) = \begin{cases} -\varrho/7 & \varrho \in [0, +\infty); \\ 0 & \textit{otherwise}. \end{cases}$$

Let $\hat{\alpha}, \beta : \mho \to [0, +\infty)$ be defined as

$$\hat{\alpha}(\varrho) = \begin{cases} 2 & \varrho \in [0, +\infty); \\ 0 & \textit{otherwise}, \end{cases}$$

and,

$$\beta(\varrho) = \begin{cases} 2 & \varrho \in (-\infty, 0]; \\ 0 & \textit{otherwise}. \end{cases}$$

Also, $\mathcal{C} : [0, +\infty)^2 \to \mathbb{R}$ be defined as $\mathcal{C}(\varrho, \varsigma) = \varrho - \varsigma$ and $\psi_1, \phi_1 : [0, +\infty) \to [0, +\infty)$ as $\psi_1(\varrho) = \dfrac{3\varrho}{2}$ and $\phi_1(\varrho) = \dfrac{3\varrho}{4}$. Now, for $\varrho \perp \varsigma$ and $\hat{\alpha}(\varrho)\beta(\varsigma) \geqslant s = 2$ to hold simultaneously, we must have either $\varrho = 0$ and $\varsigma \in (-\infty, 0]$ or *$\varrho \in [0, +\infty)$ and $\varsigma = 0$.*

Case (i)*: For $\varrho = 0$ and $\varsigma \in (-\infty, 0]$, we have*

$$\psi_1\Big(d_\perp(\Omega 0, \Omega\varsigma)\Big) = 0, \tag{2.31}$$

$$\textit{and,} \quad \mathcal{C}\Big(\psi_1(d_\perp(0, \varsigma)), \phi_1(d_\perp(0, \varsigma))\Big) = \mathcal{C}\Big(\psi_1(|\varsigma|), \phi_1(|\varsigma|)\Big) = \mathcal{C}\left(\frac{3|\varsigma|}{2}, \frac{3|\varsigma|}{4}\right) = \frac{3|\varsigma|}{4}. \tag{2.32}$$

Case (ii): *For $\varrho \in [0, +\infty)$ and $\varsigma = 0$, we have*

$$\psi_1\Big(d_\perp(\Omega\varrho, \Omega 0)\Big) = \psi_1\Big(d_\perp(\Omega\varrho, 0)\Big) = \psi_1(|\varrho|/7) = \frac{3|\varrho|}{14}, \tag{2.33}$$

$$\text{and,} \quad \mathcal{C}\Big(\psi_1(d_\perp(\varrho, 0)), \phi_1(d_\perp(\varrho, 0))\Big) = \mathcal{C}\Big(\psi_1(|\varrho|), \phi_1(|\varrho|)\Big) = \mathcal{C}\bigg(\frac{3|\varrho|}{2}, \frac{3|\varrho|}{4}\bigg) = \frac{3|\varrho|}{4}. \tag{2.34}$$

From (2.31), (2.32), (2.33) and (2.34), we have Ω as $\perp$-TAC-type S-contraction.

Definition 2.2.7. *For an orthogonal metric space $(\mho, d_\perp)$, a self-map $\Omega : \mho \to \mho$ is c.t.b an **orthogonal weak** **TAC-type S-rational contraction** (denoted by $\perp$-weak TAC-type S-rational contraction) if for $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$, $s \geqslant 1$ and $\hat{\alpha}(\varrho).\beta(\varsigma) \geqslant s$, implies*

$$d_\perp(\Omega\varrho, \Omega\varsigma) \leqslant \mathcal{C}\Big(M^*(\varrho, \varsigma), \phi_1(M^*(\varrho, \varsigma))\Big),$$

where $\hat{\alpha}, \beta : \mho \to [0, +\infty)$, $\mathcal{C} \in \mathfrak{C}$, $\phi_1 \in \Phi$ and,
$M^*(\varrho, \varsigma) = \max\left\{d_\perp(\varrho, \varsigma), \dfrac{\Big(1 + d_\perp(\varrho, \Omega\varrho)\Big)d_\perp(\varsigma, \Omega\varsigma)}{1 + d_\perp(\varrho, \varsigma)}\right\}.$

Definition 2.2.8. *For an orthogonal metric space $(\mho, d_\perp)$, a self-map $\Omega : \mho \to \mho$ is c.t.b an **orthogonal** **TAC-contraction** (denoted by $\perp$-TAC-contraction) if for $\varrho, \varsigma \in \mho$ with $\varrho \perp \varsigma$ and $\hat{\alpha}(\varrho).\beta(\varsigma) \geqslant 1$, implies*

$$\psi_1\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) \leqslant \mathcal{C}\Big(\psi_1(d_\perp(\varrho, \varsigma)), \phi_1(d_\perp(\varrho, \varsigma))\Big),$$

where $\hat{\alpha}, \beta : \mho \to [0, +\infty)$, $\mathcal{C} \in \mathfrak{C}$, $\psi_1 \in \Psi$ and $\phi_1 \in \Phi$.

Example 2.2.27. *Let $\mho = [0, +\infty)$, $d_\perp(\varrho, \varsigma) = |\varrho - \varsigma|$ and $\varrho \perp \varsigma$ iff $\varrho.\varsigma \in \{\frac{\varrho}{2}, \frac{\varsigma}{2}\}$. Then, $(\mho, d_\perp)$ is an orthogonal metric space. Let $\Omega : \mho \to \mho$ be defined as*

$$\Omega(\varrho) = \begin{cases} \varrho/3 & \varrho \in [0, 2]; \\ 5/7 & \text{otherwise.} \end{cases}$$

Let $\hat{\alpha}, \beta : \mho \to [0, +\infty)$ be defined as

$$\hat{\alpha}(\varrho) = \begin{cases} 1 & \varrho \in [0, 2]; \\ 0 & \text{otherwise,} \end{cases}$$

and,

$$\beta(\varrho)=\begin{cases}2 & \varrho\in[0,2];\\ 0 & otherwise.\end{cases}$$

Also, $\mathcal{C}:[0,+\infty)^2\rightarrow\mathbb{R}$ *be defined as* $\mathcal{C}(\varrho,\varsigma)=\varrho-\varsigma$ *and* $\psi_1,\phi_1:[0,+\infty)\rightarrow[0,+\infty)$ *as* $\psi_1(\varrho)=\varrho$ *and* $\phi_1(\varrho)=\varrho/3$. *Now, for* $\varrho\perp\varsigma$ *and* $\hat{\alpha}(\varrho)\beta(\varsigma)\geqslant 1$ *to hold simultaneously, we must have either* $\varrho=0$ *and* $\varsigma\in[0,2]$ or $\varrho\in[0,2]$ *and* $\varsigma=0$. *Considering* $\varrho\in[0,2]$ *and* $\varsigma=0$, *we have*

$$\psi_1\Big(d_\perp(\Omega\varrho,\Omega 0)\Big)=d_\perp(\Omega\varrho,0)=\varrho/3, \tag{2.35}$$

$$\begin{aligned}\textit{and,}\quad \mathcal{C}\Big(\psi_1(d_\perp(\varrho,0)),\phi_1(d_\perp(\varrho,0))\Big)=\mathcal{C}\Big(\psi_1(\varrho),\phi_1(\varrho)\Big)&=\psi_1(\varrho)-\phi_1(\varrho)\\ &=\varrho-\varrho/3=\frac{2}{3}\varrho.\end{aligned} \tag{2.36}$$

From (2.35) and (2.36), we have Ω *as* $\perp$*-TAC-contraction which is not a continuous map.*

Definition 2.2.9. *For an orthogonal metric space* $(\mho,d_\perp)$, *a self-map* $\Omega:\mho\rightarrow\mho$ *is c.t.b an* ***orthogonal weak*** **TAC*****-rational contraction*** *(denoted by* $\perp$-weak TAC*-rational contraction) if for* $\varrho,\varsigma\in\mho$ *with* $\varrho\perp\varsigma$ *and* $\hat{\alpha}(\varrho).\beta(\varsigma)\geqslant 1$, *implies*

$$d_\perp(\Omega\varrho,\Omega\varsigma)\leqslant\mathcal{C}\Big(M^*(\varrho,\varsigma),\phi_1(M^*(\varrho,\varsigma))\Big),$$

where $\hat{\alpha},\beta:\mho\rightarrow[0,+\infty)$, $\mathcal{C}\in\mathfrak{C}$, $\phi_1\in\Phi$ *and,*
$M^*(\varrho,\varsigma)=\max\Bigg\{d_\perp(\varrho,\varsigma),\dfrac{\Big(1+d_\perp(\varrho,\Omega\varrho)\Big)d_\perp(\varsigma,\Omega\varsigma)}{1+d_\perp(\varrho,\varsigma)}\Bigg\}.$

Theorem 2.2.28. *For* $(\mho,d_\perp)$ *an* $\perp$*-complete metric space with* $s\geqslant 1$ *and* ϱ_0 *as an orthogonal element, let* $\hat{\alpha},\beta:\mho\rightarrow[0,+\infty)$ *and* $\Omega:\mho\rightarrow\mho$ *be a cyclic* $(\hat{\alpha},\beta)$*-admissible map type S s.t:*

(I) Ω *is* $\perp$*-preserving;*

(II) If $\exists$ *some* ϱ_0 *in* $\mho$ *with* $\hat{\alpha}(\varrho_0)\geqslant s$ *and* $\beta(\varrho_0)\geqslant s$;

(III) Ω *is* $\perp$*-continuous;*

(IV) Ω *is* $\perp$*-TAC-type S-contraction.*

Then, Ω possesses a fixed point. Moreover, if $\hat{\alpha}(\varrho) \geqslant s$ and $\beta(\varsigma) \geqslant s \ \forall \ \varrho, \varsigma \in \mho$ where $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ with $\varrho \perp \varsigma$, then Ω possesses a unique fixed point.

Proof. As $(\mho, d_\perp)$ is an orthogonal set, then for $\varrho_0 \Omega \varrho_0 \in \mho$, we have

$$[\varrho_0 \perp \Omega\varrho_0] \quad or \quad [\Omega\varrho_0 \perp \varrho_0]. \tag{2.37}$$

Define a sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$, where $\varrho_{\varpi+1} = \Omega\varrho_\varpi = \Omega^{\varpi+1}(\varrho_0) \ \forall \ \varpi \in \mathbb{N}$. Using $\perp$-preserving property of Ω in (2.37), we obtain that $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ is an $\perp$-sequence in $\mho$. Next, by given condition, $\hat{\alpha}(\varrho_0) \geqslant s$ and as Ω is cyclic $(\hat{\alpha}, \beta)$-admissible map type S, we have $\beta(\varrho_1) = \beta(\Omega\varrho_0) \geqslant s$. Continuing in similar way, we get $\hat{\alpha}(\varrho_{\varpi-1}) \geqslant s$ and $\beta(\varrho_\varpi) \geqslant s$ for each $\varpi \in \mathbb{N}'$. Then, $\hat{\alpha}(\varrho_{\varpi-1})\beta(\varrho_\varpi) \geqslant s$. Let us denote $\zeta_\varpi = d_\perp(\varrho_\varpi, \varrho_{\varpi+1})$. Since, Ω is $\perp$-TAC-type S-contraction, we have

$$\begin{aligned} \psi_1\Big(\zeta_\varpi\Big) = \psi_1\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) &= \psi_1\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big) \\ &\leqslant \mathcal{C}\Big(\psi_1(\zeta_{\varpi-1}), \phi_1(\zeta_{\varpi-1})\Big) \leqslant \psi_1(\zeta_{\varpi-1}). \end{aligned} \tag{2.38}$$

Also, ψ_1 is monotonically increasing map, so $\zeta_\varpi \leqslant \zeta_{\varpi-1} \quad \forall \ \varpi \in \mathbb{N}$, thus, $\{\zeta_\varpi\}_{\varpi\in\mathbb{N}}$ is a decreasing and as each $\zeta_\varpi \in \mathbb{R}^+$, so $\exists$ some $\zeta \in [0, +\infty)$, s.t $\lim\limits_{\varpi\to+\infty} \zeta_\varpi = \zeta$. On taking limit as $\varpi \to +\infty$ in (2.38), we obtain

$$\begin{aligned} &\psi_1(\zeta) \leqslant \mathcal{C}\Big(\psi_1(\zeta), \phi_1(\zeta)\Big) \leqslant \psi_1(\zeta), \\ \text{that is,} \qquad &\mathcal{C}(\psi_1(\zeta), \phi_1(\zeta)) = \psi_1(\zeta). \end{aligned}$$

By using definition of C-class function, we obtain either $\psi_1(\zeta) = 0$ or $\phi_1(\zeta) = 0$. From either of the cases we have $\zeta = 0$, that is,

$$\lim_{\varpi\to+\infty} \zeta_\varpi = \lim_{\varpi\to+\infty} d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) = 0.$$

So, for some $l = \epsilon/\varpi^* > 0 \ \exists$ some $\varpi_l \in \mathbb{N}$ s.t

$$d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) < l \quad \forall \ \varpi > \varpi_l. \tag{2.39}$$

Let $\varpi, \varpi^* \in \mathbb{N}$ where $\varpi > \varpi_l$. Using triangle inequality and (2.39), we get

$$\begin{aligned} d_\perp(\varrho_\varpi, \varrho_{\varpi+\varpi^*}) &\leqslant d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) + d_\perp(\varrho_{\varpi+1}, \varrho_{\varpi+2}) + \cdots + d_\perp(\varrho_{\varpi+\varpi^*-1}, \varrho_{\varpi+\varpi^*}) \\ &< \varpi^* l = \epsilon. \end{aligned}$$

Thus, we have, $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ as an $\perp$-Cauchy sequence in $\mho$. Since $\mho$ is $\perp$-complete,

$\exists\ \varrho \in \mho$ s.t $\lim\limits_{\varpi \to +\infty} \varrho_\varpi = \varrho$. In addition, as Ω is an $\perp$-continuous map, so

$$\begin{aligned} \lim_{\varpi \to +\infty} \Omega\varrho_\varpi = \lim_{\varpi \to +\infty} \varrho_{\varpi+1} &= \Omega\varrho, \\ \text{that is,} \quad \varrho &= \Omega\varrho. \end{aligned}$$

Hence, Ω possesses a fixed point. Next, let $\varsigma \in \mho$ be s.t $\Omega\varsigma = \varsigma$ and $\varrho \perp \varsigma$ then, $\hat{\alpha}(\varrho)\beta(\varsigma) \geqslant s$. Using $\perp$-TAC-type S-contraction condition of Ω, we obtain

$$\begin{aligned} \psi_1\Big(d_\perp(\varrho,\varsigma)\Big) = \psi_1\Big(d_\perp(\Omega\varrho,\Omega\varsigma)\Big) &\leqslant \mathcal{C}\Big(\psi_1(d_\perp(\varrho,\varsigma)), \phi_1(d_\perp(\varrho,\varsigma))\Big) \\ &\leqslant \psi_1\Big(d_\perp(\varrho,\varsigma)\Big), \\ \text{that is,} \quad \mathcal{C}\Big(\psi_1(d_\perp(\varrho,\varsigma)), \phi_1(d_\perp(\varrho,\varsigma))\Big) &= \psi_1(d_\perp(\varrho,\varsigma)). \end{aligned}$$

On using definition of C-class function we obtain, either $\psi_1(d_\perp(\varrho,\varsigma)) = 0$ or $\phi_1(d_\perp(\varrho,\varsigma)) = 0$. From both the cases, we get $d_\perp(\varrho,\varsigma) = 0$. Hence, Ω possesses a unique fixed point. □

Example 2.2.29. *Example 2.2.26, satisfies all hypotheses of Theorem 2.2.28 and thus possesses a fixed point viz.* $\varrho = 0$.

Corollary 2.2.30. *For $\perp$-complete metric space $(\mho, d_\perp)$ with $s \geqslant 1$ and ϱ_0 as an orthogonal element, let $\hat{\alpha}, \beta : \mho \to [0,+\infty)$ and $\Omega : \mho \to \mho$ be a cyclic $(\hat{\alpha},\beta)$-admissible map type S s.t:*

(I) Ω is $\perp$-preserving;

(II) If $\exists$ some ϱ_0 in $\mho$ with $\hat{\alpha}(\varrho_0) \geqslant s$ and $\beta(\varrho_0) \geqslant s$;

(III) If $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ is an $\perp$-sequence where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ along with $\beta(\varrho_\varpi) \geqslant s$ for each $\varpi \in \mathbb{N}$, implies $\beta(\varrho) \geqslant s$ and either $[\varrho_\varpi \perp \varrho\ \forall\ \varpi]$ or $[\varrho \perp \varrho_\varpi\ \forall\ \varpi]$;

*(IV) Ω is $\perp$-*TAC*-type S contraction.*

Then, Ω possesses a fixed point. Moreover, if $\hat{\alpha}(\varrho) \geqslant s$ and $\beta(\varsigma) \geqslant s\ \forall\ \varrho, \varsigma \in \mho$ where $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ with $\varrho \perp \varsigma$, then Ω possesses a unique fixed point.

Proof. Working on the lines of Theorem 2.2.28, we obtain $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ an $\perp$-sequence in $\mho$ s.t $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ and also, $\beta(\varrho_\varpi) \geqslant s$ for each $\varpi \in \mathbb{N}$. Then, by given condition, we obtain $\beta(\varrho) \geqslant s$ and *either* $[\varrho_\varpi \perp \varrho\ \forall\ \varpi]$ *or* $[\varrho \perp \varrho_\varpi\ \forall\ \varpi]$. Thus $\hat{\alpha}(\varrho_\varpi)\beta(\varrho) \geqslant s$, implies

$$\begin{aligned} \psi_1\Big(d_\perp(\varrho_{\varpi+1},\Omega\varrho)\Big) = \psi_1\Big(d_\perp(\Omega\varrho_\varpi,\Omega\varrho)\Big) &\leqslant \mathcal{C}\Big(\psi_1(d_\perp(\varrho_\varpi,\varrho)), \phi_1(d_\perp(\varrho_\varpi,\varrho))\Big) \\ &\leqslant \psi_1\Big(d_\perp(\varrho_\varpi,\varrho)\Big). \end{aligned}$$

Taking limit as $\varpi \to +\infty$ and using continuity of $\mathcal{C}$, ψ_1 and ϕ_1, we have $d_\perp(\varrho, \Omega\varrho) = 0$. Thus, Ω possesses a fixed point in $\mho$. Also, by Theorem 2.2.28 we obtain uniqueness of fixed point. □

Theorem 2.2.31. *For $\perp$-complete metric space $(\mho, d_\perp)$ with $s \geqslant 1$ and ϱ_0 as an orthogonal element, let $\hat{\alpha}, \beta : \mho \to [0, +\infty)$ and $\Omega : \mho \to \mho$ be a cyclic $(\hat{\alpha}, \beta)$-admissible map of type S s.t:*

(I) Ω is $\perp$-preserving;

(II) If $\exists$ some ϱ_0 in $\mho$ with $\hat{\alpha}(\varrho_0) \geqslant s$ and $\beta(\varrho_0) \geqslant s$;

(III) Ω is $\perp$-continuous;

*(IV) Ω is $\perp$-*weak TAC*-type S-rational contraction.*

Then, Ω possesses a fixed point. In addition, if $\hat{\alpha}(\varrho) \geqslant s$ and $\beta(\varsigma) \geqslant s$ $\forall$ $\varrho, \varsigma \in \mho$ where $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ with $\varrho \perp \varsigma$, then Ω possesses a unique fixed point.

Proof. By the working done in Theorem 2.2.28, we can obtain an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ with $\hat{\alpha}(\varrho_{\varpi-1})\beta(\varrho_\varpi) \geqslant s$ for every $\varpi \in \mathbb{N}$. By using $\perp$-*weak TAC*-type S-rational contraction of Ω, we get

$$\begin{aligned} \zeta_\varpi = d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) &= d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi) \\ &\leqslant \mathcal{C}\Big(M^*(\varrho_{\varpi-1}, \varrho_\varpi), \phi_1(M^*(\varrho_{\varpi-1}, \varrho_\varpi))\Big) \\ &\leqslant M^*(\varrho_{\varpi-1}, \varrho_\varpi), \end{aligned} \tag{2.40}$$

$$\begin{aligned} \text{where,}\quad M^*(\varrho_{\varpi-1}, \varrho_\varpi) &= \max\left\{ d_\perp(\varrho_{\varpi-1}, \varrho_\varpi), \frac{\Big(1 + d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)\Big) d_\perp(\varrho_\varpi, \varrho_{\varpi+1})}{1 + d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)} \right\} \\ &= \max\{\zeta_{\varpi-1}, \zeta_\varpi\}. \end{aligned}$$

Suppose for some $\varpi_0 \in \mathbb{N}$, we have $M^*(\varrho_{\varpi_0-1}, \varrho_{\varpi_0}) = \zeta_{\varpi_0}$, that is,

$$\zeta_{\varpi_0} > \zeta_{\varpi_0-1}. \tag{2.41}$$

Then, by (2.40), we have

$$\begin{aligned} &\zeta_{\varpi_0} \leqslant \mathcal{C}\Big(\zeta_{\varpi_0}, \phi_1(\zeta_{\varpi_0})\Big) \leqslant \zeta_{\varpi_0}, \\ \text{that is,}\quad &\mathcal{C}\Big(\zeta_{\varpi_0}, \phi_1(\zeta_{\varpi_0})\Big) = \zeta_{\varpi_0}. \end{aligned}$$

By using definition of C-class function, we have either $\zeta_{\varpi_0} = 0$ or $\phi_1(\zeta_{\varpi_0}) = 0$. From either of the cases, we get $\zeta_{\varpi_0} = 0$ which is a contradiction to (2.41). Thus for each $\varpi \in \mathbb{N}$, we have $\zeta_{\varpi} \leqslant \zeta_{\varpi-1}$, where $\{\zeta_{\varpi}\}_{\varpi\in\mathbb{N}}$ is a decreasing and as each $\zeta_{\varpi} \in \mathbb{R}^+$, so $\exists$ some $\zeta \in [0, +\infty)$, s.t $\lim_{\varpi\to+\infty} \zeta_{\varpi} = \zeta$. On taking limit as $\varpi \to +\infty$ in (2.40), we get $\mathcal{C}\Big(\zeta, \phi_1(\zeta)\Big) = \zeta$, which implies either $\zeta = 0$ or $\phi_1(\zeta) = 0$ that is, $\zeta = 0$, and thus, $\lim_{\varpi\to+\infty} \zeta_{\varpi} = \lim_{\varpi\to+\infty} d_{\perp}(\varrho_{\varpi}, \varrho_{\varpi+1}) = 0$. Now, for some $l = \epsilon/\varpi^* > 0$, $\exists$ some $\varpi_l \in \mathbb{N}$ with,

$$d_{\perp}(\varrho_{\varpi}, \varrho_{\varpi+1}) < l \quad \forall\, \varpi > \varpi_l. \tag{2.42}$$

Let $\varpi, \varpi^* \in \mathbb{N}$ where $\varpi > \varpi_l$. Using triangle inequality and (2.42), we get

$$\begin{aligned} d_{\perp}(\varrho_{\varpi}, \varrho_{\varpi+\varpi^*}) &\leqslant d_{\perp}(\varrho_{\varpi}, \varrho_{\varpi+1}) + d_{\perp}(\varrho_{\varpi+1}, \varrho_{\varpi+2}) + \cdots + d_{\perp}(\varrho_{\varpi+\varpi^*-1}, \varrho_{\varpi+\varpi^*}) \\ &< \varpi^* l = \epsilon. \end{aligned}$$

Thus, we have, $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ as an $\perp$-Cauchy sequence in $\mho$. Since $\mho$ is $\perp$-complete, $\exists\, \varrho \in \mho$, s.t $\lim_{\varpi\to+\infty} \varrho_{\varpi} = \varrho$. As Ω is an $\perp$-continuous map, so

$$\begin{aligned} \lim_{\varpi\to+\infty} \Omega\varrho_{\varpi} = \lim_{\varpi\to+\infty} \varrho_{\varpi+1} &= \Omega\varrho, \\ \text{that is,} \quad \varrho &= \Omega\varrho. \end{aligned}$$

Hence, Ω possesses a fixed point. For uniqueness, let ς be s.t $\Omega\varsigma = \varsigma$ and $\varrho \perp \varsigma$ then by given condition $\hat{\alpha}(\varrho)\beta(\varsigma) \geqslant s$. Using $\perp$-*weak TAC*-type S-rational contraction of Ω, we obtain

$$d_{\perp}(\varrho, \varsigma) = d_{\perp}(\Omega\varrho, \Omega\varsigma) \leqslant \mathcal{C}\Big(M^*(\varrho, \varsigma), \phi_1(M^*(\varrho, \varsigma))\Big), \tag{2.43}$$

$$\text{where,} \quad M^*(\varrho, \varsigma) = \max\left\{d_{\perp}(\varrho, \varsigma), \frac{\Big(1 + d_{\perp}(\varrho, \Omega\varrho)\Big) d_{\perp}(\varsigma, \Omega\varsigma)}{1 + d_{\perp}(\varrho, \varsigma)}\right\} = d_{\perp}(\varrho, \varsigma).$$

Thus from (2.43), we get

$$d_{\perp}(\varrho, \varsigma) \leqslant \mathcal{C}\Big(d_{\perp}(\varrho, \varsigma), \phi_1(d_{\perp}(\varrho, \varsigma))\Big) \leqslant d_{\perp}(\varrho, \varsigma),$$

which implies $d_{\perp}(\varrho, \varsigma) = 0$. Hence, Ω possesses a unique fixed point. □

Corollary 2.2.32. *For $\perp$-complete metric space $(\mho, d_{\perp})$ with $s \geqslant 1$ and ϱ_0 as an orthogonal element, let $\hat{\alpha}, \beta : \mho \to [0, +\infty)$ and $\Omega : \mho \to \mho$ be a cyclic $(\hat{\alpha}, \beta)$-admissible map of type S s.t:*

(I) Ω *is* $\perp$*-preserving;*

(II) If $\exists$ *some* ϱ_0 *in* $\mho$ *with* $\hat{\alpha}(\varrho_0) \geqslant s$ *and* $\beta(\varrho_0) \geqslant s$*;*

(III) If $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *is an* $\perp$*-sequence where* $\varrho_\varpi \to \varrho$ *as* $\varpi \to +\infty$ *along with* $\beta(\varrho_\varpi) \geqslant s$ *for each* $\varpi \in \mathbb{N}$*, implies* $\beta(\varrho) \geqslant s$ *and* either $[\varrho_\varpi \perp \varrho \ \forall \ \varpi]$ or $[\varrho \perp \varrho_\varpi \ \forall \ \varpi]$;

(IV) Ω *is* $\perp$-weak TAC-*type S-rational contraction.*

Then, Ω *possesses a fixed point. In addition, if* $\hat{\alpha}(\varrho) \geqslant s$ *and* $\beta(\varsigma) \geqslant s \ \forall \ \varrho, \varsigma \in \mho$ *where* $\Omega\varrho = \varrho$ *and* $\Omega\varsigma = \varsigma$ *with* $\varrho \perp \varsigma$*, then* Ω *possesses a unique fixed point.*

Proof. With reference to the working of Theorem 2.2.31, we can obtain an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ and $\beta(\varrho_\varpi) \geqslant s$ for each $\varpi \in \mathbb{N}$. By given condition, $\beta(\varrho) \geqslant s$ and *either* $[\varrho_\varpi \perp \varrho \ \forall \ \varpi]$ *or* $[\varrho \perp \varrho_\varpi \ \forall \ \varpi]$ which implies $\hat{\alpha}(\varrho_\varpi)\beta(\varrho) \geqslant s$. On using $\perp$-*weak TAC*-type S-rational contraction of Ω, we get

$$d_\perp(\varrho_{\varpi+1}, \Omega\varrho) = d_\perp(\Omega\varrho_\varpi, \Omega\varrho) \leqslant \mathcal{C}\Big(M^*(\varrho_\varpi, \varrho), \phi_1(M^*(\varrho_\varpi, \varrho))\Big), \tag{2.44}$$

$$\text{where, } M^*(\varrho_\varpi, \varrho) = \max\left\{d_\perp(\varrho_\varpi, \varrho), \frac{\Big(1 + d_\perp(\varrho_\varpi, \Omega(\varrho_\varpi))\Big)d_\perp(\varrho, \Omega(\varrho))}{1 + d_\perp(\varrho_\varpi, \varrho)}\right\}.$$

Taking limit as $\varpi \to +\infty$ in (2.44), we obtain $d_\perp(\varrho, \Omega\varrho) = 0$. Thus, ϱ is a fixed point of Ω and the uniqueness of the fixed point follows on the lines of Theorem 2.2.31. □

Remark 2.2.33. *In the upcoming result, we consider* Ω *to be a cyclic* $(\hat{\alpha}, \beta)$*-admissible map.*

Theorem 2.2.34. *For* $\perp$*-complete* $(\mho, d_\perp)$ *with* ϱ_0 *as an orthogonal element, let* $\hat{\alpha}, \beta : \mho \to [0, +\infty)$ *are defined on* $\mho$ *and* $\Omega : \mho \to \mho$ *be a cyclic* $(\hat{\alpha}, \beta)$*-admissible map on* $\mho$ *s.t:*

(I) Ω *is* $\perp$*-preserving;*

(II) If $\exists$ *some* ϱ_0 *in* $\mho$ *with* $\alpha(\varrho_0) \geqslant 1$ *and* $\beta(\varrho_0) \geqslant 1$*;*

(III) Ω *is* $\perp$*-continuous;*

(IV) Ω *is* $\perp$*-TAC-contraction.*

Then, Ω *possesses a fixed point. In addition, if* $\hat{\alpha}(\varrho) \geqslant 1$ *and* $\beta(\varsigma) \geqslant 1 \ \forall \ \varrho, \varsigma \in \mho$ *where* $\Omega\varrho = \varrho$ *and* $\Omega\varsigma = \varsigma$ *with* $\varrho \perp \varsigma$*, then* Ω *possesses a unique fixed point.*

Proof. As $(\mho, d_\perp)$ is an $\perp$-set, then for $\varrho_0, \Omega\varrho_0 \in \mho$, we have

$$[\varrho_0 \perp \Omega\varrho_0] \quad or \quad [\Omega\varrho_0 \perp \varrho_0]. \tag{2.45}$$

Define a sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ where $\varrho_{\varpi+1} = \Omega\varrho_\varpi = \Omega^{\varpi+1}(\varrho_0)\ \forall\ \varpi \in \mathbb{N}$. Using $\perp$-preserving property of Ω in (2.45), we obtain that $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ is an $\perp$-sequence in $\mho$. Next, by given condition, $\hat{\alpha}(\varrho_0) \geqslant 1$ and by cyclic $(\hat{\alpha}, \beta)$-admissibility of Ω, we have $\beta(\varrho_1) = \beta(\Omega\varrho_0) \geqslant 1$. Repetitive use of cyclic $(\hat{\alpha}, \beta)$-admissibility of Ω, we get $\hat{\alpha}(\varrho_{\varpi-1}) \geqslant 1$ and $\beta(\varrho_\varpi) \geqslant 1\ \forall\ \varpi \in \mathbb{N}'$. Then, $\hat{\alpha}(\varrho_{\varpi-1})\beta(\varrho_\varpi) \geqslant 1$. Let us denote $\zeta_\varpi = d_\perp(\varrho_\varpi, \varrho_{\varpi+1})$. Using $\perp$-TAC-contraction of Ω, we have

$$\begin{aligned}\psi_1\Big(\zeta_\varpi\Big) = \psi_1\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) &= \psi_1\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big)\\ &\leqslant \mathcal{C}\Big(\psi_1(\zeta_{\varpi-1}), \phi_1(\zeta_{\varpi-1})\Big) \leqslant \psi_1(\zeta_{\varpi-1}).\end{aligned} \tag{2.46}$$

Since, ψ_1 is monotonically increasing function, so $\zeta_\varpi \leqslant \zeta_{\varpi-1}\ \forall\ \varpi \in \mathbb{N}$, thus, $\{\zeta_\varpi\}_{\varpi\in\mathbb{N}}$ is a decreasing and as each $\zeta_\varpi \in \mathbb{R}^+$, so $\exists$ some $\zeta \in [0, +\infty)$, s.t $\lim_{\varpi\to+\infty} \zeta_\varpi = \zeta$. On taking limit as $\varpi \to +\infty$ in (2.46), we obtain

$$\begin{aligned}\psi_1(\zeta) \leqslant \mathcal{C}\Big(\psi_1(\zeta), \phi_1(\zeta)\Big) &\leqslant \psi_1(\zeta),\\ \text{that is,}\quad \mathcal{C}(\psi_1(\zeta), \phi_1(\zeta)) &= \psi_1(\zeta).\end{aligned}$$

By using definition of C-class function, we obtain either $\psi_1(\zeta) = 0$ or $\phi_1(\zeta) = 0$. From either of the cases we have $\zeta = 0$, that is,

$$\lim_{\varpi\to+\infty} \zeta_\varpi = \lim_{\varpi\to+\infty} d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) = 0.$$

So, for some $l = \epsilon/\varpi^* > 0$, $\exists$ some $\varpi_l \in \mathbb{N}$ s.t

$$d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) < l \quad \forall\ \varpi > \varpi_l. \tag{2.47}$$

Let $\varpi, \varpi^* \in \mathbb{N}$ where $\varpi > \varpi_l$. Using triangle inequality and (2.47), we get

$$\begin{aligned}d_\perp(\varrho_\varpi, \varrho_{\varpi+\varpi^*}) &\leqslant d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) + d_\perp(\varrho_{\varpi+1}, \varrho_{\varpi+2}) + \cdots + d_\perp(\varrho_{\varpi+\varpi^*-1}, \varrho_{\varpi+\varpi^*})\\ &< \varpi^* l = \epsilon.\end{aligned}$$

Thus, we have $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ as an $\perp$-Cauchy sequence in $\mho$. Since $\mho$ is $\perp$-complete, $\exists\ \varrho \in \mho$ with $\lim_{\varpi\to+\infty} \varrho_\varpi = \varrho$. As Ω is an $\perp$-continuous map, so

$$\lim_{\varpi\to+\infty} \Omega\varrho_\varpi = \lim_{\varpi\to+\infty} \varrho_{\varpi+1} = \Omega\varrho,$$

that is, $\varrho = \Omega\varrho$.

Hence, Ω possesses a fixed point. Next, let ς be s.t $\Omega\varsigma = \varsigma$ and $\varrho \perp \varsigma$ then by given condition $\hat{\alpha}(\varrho)\beta(\varsigma) \geqslant 1$. Using $\perp$-TAC-contraction of Ω, we obtain

$$\begin{aligned} \psi_1\Big(d_\perp(\varrho,\varsigma)\Big) = \psi_1\Big(d_\perp(\Omega\varrho,\Omega\varsigma)\Big) &\leqslant \mathcal{C}\Big(\psi_1(d_\perp(\varrho,\varsigma)),\phi_1(d_\perp(\varrho,\varsigma))\Big) \\ &\leqslant \psi_1\Big(d_\perp(\varrho,\varsigma)\Big), \\ \text{implies}\quad \mathcal{C}\Big(\psi_1(d_\perp(\varrho,\varsigma)),\phi_1(d_\perp(\varrho,\varsigma))\Big) &= \psi_1(d_\perp(\varrho,\varsigma)). \end{aligned}$$

On using definition of C-class function, we obtain either $\psi_1(d_\perp(\varrho,\varsigma)) = 0$ or $\phi_1(d_\perp(\varrho,\varsigma)) = 0$. From both the cases, we get $d_\perp(\varrho,\varsigma) = 0$. Hence, Ω possesses a unique fixed point. □

Example 2.2.35. *Consider the space defined in Example 2.2.27. Then, $\mho$ is $\perp$-complete and also such Ω is $\perp$-preserving. Next, we have*

(i) Cyclic $(\hat{\alpha},\beta)$-admissibility of Ω: Since for $\varrho \in [0,2]$ we get $\hat{\alpha}(\varrho) \geqslant 1$ implies $\beta(\Omega\varrho) = \beta(\varrho/3) \geqslant 1$, similarly, for $\varrho \in [0,2]$ we get $\beta(\varrho) \geqslant 1$ implies $\hat{\alpha}(\Omega\varrho) = \hat{\alpha}(\varrho/3) \geqslant 1$.

(ii) $\perp$-continuity of Ω: Since for $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ an $\perp$-sequence in $\mho$, then $\varrho_\varpi \to 0$. So we have, $\{\Omega\varrho_\varpi\} \to 0 = \Omega 0$.

Since all hypotheses of Theorem 2.2.34 hold, so Ω possesses a fixed point which is $\varrho = 0$.

Remark 2.2.36. *The above theorem holds even if instead of taking Ω as an $\perp$-continuous map we consider a weaker condition as discussed in the following result.*

Corollary 2.2.37. *For $\perp$-complete metric space $(\mho, d_\perp)$ with ϱ_0 as an orthogonal element, suppose $\hat{\alpha},\beta : \mho \to [0,+\infty)$ are defined on $\mho$ and $\Omega : \mho \to \mho$ be a cyclic $(\hat{\alpha},\beta)$-admissible map on $\mho$ s.t:*

(I) Ω is $\perp$-preserving;

(II) If $\exists$ some ϱ_0 in $\mho$ with $\hat{\alpha}(\varrho_0) \geqslant 1$ and $\beta(\varrho_0) \geqslant 1$;

(III) If $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ is an $\perp$-sequence where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ along with $\beta(\varrho_\varpi) \geqslant 1$ for each $\varpi \in \mathbb{N}$, implies $\beta(\varrho) \geqslant 1$ and either $[\varrho_\varpi \perp \varrho \;\forall\, \varpi]$ or $[\varrho \perp \varrho_\varpi \;\forall\, \varpi]$*;*

*(IV) Ω is $\perp$-*TAC*-contraction.*

Then, Ω possesses a fixed point. In addition, if $\hat{\alpha}(\varrho) \geqslant 1$ and $\beta(\varsigma) \geqslant 1 \ \forall \ \varrho, \varsigma \in \mho$ where $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ with $\varrho \perp \varsigma$, then Ω possesses a unique fixed point.

Proof. Working on the footprints of Theorem 2.2.34, we obtain $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ an $\perp$-sequence in $\mho$ with $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ and also, $\beta(\varrho_\varpi) \geqslant 1 \ \forall \ \varpi \in \mathbb{N}$. Then, by given condition, we obtain $\beta(\varrho) \geqslant 1$ and *either* $[\varrho_\varpi \perp \varrho \ \forall \ \varpi]$ *or* $[\varrho \perp \varrho_\varpi \ \forall \ \varpi]$. Thus $\hat{\alpha}(\varrho_\varpi)\beta(\varrho) \geqslant 1$, implies

$$\psi_1\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) \leqslant \mathcal{C}\Big(\psi_1(d_\perp(\varrho_\varpi, \varrho)), \phi_1(d_\perp(\varrho_\varpi, \varrho))\Big) \leqslant \psi_1\Big(d_\perp(\varrho_\varpi, \varrho)\Big).$$

Taking limit as $\varpi \to +\infty$ and using continuity of $\mathcal{C}$, ψ_1 and ϕ_1, we have $d_\perp(\varrho, \Omega\varrho) = 0$. Thus, Ω possesses a fixed point in $\mho$. Also, the uniqueness of fixed point can be proved on the lines of Theorem 2.2.34. □

Theorem 2.2.38. *For $\perp$-complete metric space $(\mho, d_\perp)$ with ϱ_0 as an orthogonal element, let $\hat{\alpha}, \beta : \mho \to [0, +\infty)$ and $\Omega : \mho \to \mho$ be a cyclic $(\hat{\alpha}, \beta)$-admissible map on $\mho$ s.t:*

(I) Ω is $\perp$-preserving;

(II) If $\exists$ some ϱ_0 in $\mho$ with $\hat{\alpha}(\varrho_0) \geqslant 1$ and $\beta(\varrho_0) \geqslant 1$;

(III) Ω is $\perp$-continuous;

*(IV) Ω is $\perp$-*weak TAC*-rational contraction.*

Then, Ω possesses a fixed point. In addition, if $\hat{\alpha}(\varrho) \geqslant 1$ and $\beta(\varsigma) \geqslant 1 \ \forall \ \varrho, \varsigma \in \mho$ where $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ with $\varrho \perp \varsigma$, then Ω possesses a unique fixed point.

Proof. By the working done in Theorem 2.2.34, we obtain an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ with $\hat{\alpha}(\varrho_{\varpi-1})\beta(\varrho_\varpi) \geqslant 1$ for every $\varpi \in \mathbb{N}$. By using $\perp$-*weak TAC*-rational contraction of Ω, we get

$$\begin{aligned} \zeta_\varpi = d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) = d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi) &\leqslant \mathcal{C}\Big(M^*(\varrho_{\varpi-1}, \varrho_\varpi), \phi_1(M^*(\varrho_{\varpi-1}, \varrho_\varpi))\Big) \\ &\leqslant M^*(\varrho_{\varpi-1}, \varrho_\varpi), \end{aligned} \tag{2.48}$$

$$\begin{aligned} \text{where,} \quad M^*(\varrho_{\varpi-1}, \varrho_\varpi) &= \max\left\{ d_\perp(\varrho_{\varpi-1}, \varrho_\varpi), \frac{\Big(1 + d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)\Big) d_\perp(\varrho_\varpi, \varrho_{\varpi+1})}{1 + d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)} \right\} \\ &= \max\{\zeta_{\varpi-1}, \zeta_\varpi\}. \end{aligned}$$

Suppose for some $\varpi_0 \in \mathbb{N}$, we have $M^*(\varrho_{\varpi_0-1}, \varrho_{\varpi_0}) = \zeta_{\varpi_0}$, that is,

$$\zeta_{\varpi_0} > \zeta_{\varpi_0-1}. \tag{2.49}$$

Then by (2.48), we have

$$\begin{aligned} \zeta_{\varpi_0} \leqslant \mathcal{C}\Big(\zeta_{\varpi_0}, \phi_1(\zeta_{\varpi_0})\Big) &\leqslant \zeta_{\varpi_0}, \\ \text{that is,} \quad \mathcal{C}\Big(\zeta_{\varpi_0}, \phi_1(\zeta_{\varpi_0})\Big) &= \zeta_{\varpi_0}. \end{aligned}$$

Using definition of C-class function, we have either $\zeta_{\varpi_0} = 0$ or $\phi_1(\zeta_{\varpi_0}) = 0$. From either of the cases, we get $\zeta_{\varpi_0} = 0$ which is a contradiction to (2.49). Hence for each $\varpi \in \mathbb{N}$, we have $\zeta_{\varpi} \leqslant \zeta_{\varpi-1}$, thus, $\{\zeta_{\varpi}\}_{\varpi \in \mathbb{N}}$ is a decreasing and as each $\zeta_{\varpi} \in \mathbb{R}^+$, so $\exists$ some $\zeta \in [0, +\infty)$, with $\lim_{\varpi \to +\infty} \zeta_{\varpi} = \zeta$. On taking limit as $\varpi \to +\infty$ in (2.48), we get $\mathcal{C}\Big(\zeta, \phi_1(\zeta)\Big) = \zeta$, which implies either $\zeta = 0$ or $\phi_1(\zeta) = 0$ that is, $\zeta = 0$, thus we obtain,

$$\lim_{\varpi \to +\infty} \zeta_{\varpi} = \lim_{\varpi \to +\infty} d_{\perp}(\varrho_{\varpi}, \varrho_{\varpi+1}) = 0.$$

Now, for some $l = \epsilon/\varpi^* > 0$, $\exists$ some $\varpi_l \in \mathbb{N}$ s.t,

$$d_{\perp}(\varrho_{\varpi}, \varrho_{\varpi+1}) < l \quad \forall\, \varpi > \varpi_l. \tag{2.50}$$

Let $\varpi, \varpi^* \in \mathbb{N}$ where $\varpi > \varpi_l$. Using triangle inequality and (2.50), we get

$$\begin{aligned} d_{\perp}(\varrho_{\varpi}, \varrho_{\varpi+\varpi^*}) &\leqslant d_{\perp}(\varrho_{\varpi}, \varrho_{\varpi+1}) + d_{\perp}(\varrho_{\varpi+1}, \varrho_{\varpi+2}) + \cdots + d_{\perp}(\varrho_{\varpi+\varpi^*-1}, \varrho_{\varpi+\varpi^*}) \\ &< \varpi^* l = \epsilon. \end{aligned}$$

Thus, we have $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ as an $\perp$-Cauchy sequence in $\mho$. Since $\mho$ is $\perp$-complete, $\exists\ \varrho \in \mho$, with $\lim_{\varpi \to +\infty} \varrho_{\varpi} = \varrho$. As Ω is an $\perp$-continuous map, so

$$\begin{aligned} \lim_{\varpi \to +\infty} \Omega\varrho_{\varpi} = \lim_{\varpi \to +\infty} \varrho_{\varpi+1} &= \Omega\varrho, \\ \text{that is,} \quad \varrho &= \Omega\varrho. \end{aligned}$$

Hence, Ω possesses a fixed point. Next, let ς be s.t $\Omega\varsigma = \varsigma$ and $\varrho \perp \varsigma$ then, by given condition $\hat{\alpha}(\varrho)\beta(\varsigma) \geqslant 1$. Using $\perp$-*weak TAC*-rational contraction of Ω, we obtain

$$d_{\perp}(\varrho, \varsigma) = d_{\perp}(\Omega\varrho, \Omega\varsigma) \leqslant \mathcal{C}\Big(M^*(\varrho, \varsigma), \phi_1(M^*(\varrho, \varsigma))\Big), \tag{2.51}$$

$$\text{where, } M^*(\varrho,\varsigma) = \max\left\{ d_\perp(\varrho,\varsigma), \frac{\left(1+d_\perp(\varrho,\Omega\varrho)\right)d_\perp(\varsigma,\Omega\varsigma)}{1+d_\perp(\varrho,\varsigma)} \right\} = d_\perp(\varrho,\varsigma).$$

Thus from (2.51), we get

$$d_\perp(\varrho,\varsigma) \leqslant \mathcal{C}\Big(d_\perp(\varrho,\varsigma), \phi_1(d_\perp(\varrho,\varsigma))\Big) \leqslant d_\perp(\varrho,\varsigma),$$

which implies $d_\perp(\varrho,\varsigma) = 0$. Hence, Ω possesses a unique fixed point. □

Remark 2.2.39. *The above theorem also holds if we drop $\perp$-continuity of Ω and instead consider a weaker condition, as discussed in the following corollary.*

Corollary 2.2.40. *For $\perp$-complete $(\mho, d_\perp)$ with ϱ_0 as an orthogonal element, let $\hat{\alpha}, \beta : \mho \to [0,+\infty)$ and $\Omega : \mho \to \mho$ be a cyclic $(\hat{\alpha},\beta)$-admissible map on $\mho$ s.t:*

(I) *Ω is $\perp$-preserving;*

(II) *If $\exists$ some ϱ_0 in $\mho$ with $\hat{\alpha}(\varrho_0) \geqslant 1$ and $\beta(\varrho_0) \geqslant 1$;*

(III) *If $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ is an $\perp$-sequence where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ along with $\beta(\varrho_\varpi) \geqslant 1$ for each $\varpi \in \mathbb{N}$, implies $\beta(\varrho) \geqslant 1$ and* either $[\varrho_\varpi \perp \varrho \; \forall \; \varpi]$ or $[\varrho \perp \varrho_\varpi \; \forall \; \varpi]$*;*

(IV) *Ω is $\perp$-*weak TAC*-rational contraction.*

Then, Ω possesses a fixed point. In addition, if $\hat{\alpha}(\varrho) \geqslant 1$ and $\beta(\varsigma) \geqslant 1 \; \forall \; \varrho,\varsigma \in \mho$ where $\Omega\varrho = \varrho$ and $\Omega\varsigma = \varsigma$ with $\varrho \perp \varsigma$, then Ω possesses a unique fixed point.

Proof. With reference to working of Theorem 2.2.38, one can obtain an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ and $\beta(\varrho_\varpi) \geqslant 1$ for each $\varpi \in \mathbb{N}$. By given condition, $\beta(\varrho) \geqslant 1$ and *either* $[\varrho_\varpi \perp \varrho \; \forall \; \varpi]$ *or* $[\varrho \perp \varrho_\varpi \; \forall \; \varpi]$ which implies $\hat{\alpha}(\varrho_\varpi)\beta(\varrho) \geqslant 1$. On using $\perp$-*weak TAC*-rational contraction of Ω, we get

$$d_\perp(\varrho_{\varpi+1},\Omega\varrho) = d_\perp(\Omega\varrho_\varpi,\Omega\varrho) \leqslant \mathcal{C}\Big(M^*(\varrho_\varpi,\varrho), \phi_1(M^*(\varrho_\varpi,\varrho))\Big), \tag{2.52}$$

$$\text{where, } M^*(\varrho_\varpi,\varrho) = \max\left\{ d_\perp(\varrho_\varpi,\varrho), \frac{\left(1+d_\perp(\varrho_\varpi,\Omega(\varrho_\varpi))\right)d_\perp(\varrho,\Omega(\varrho))}{1+d_\perp(\varrho_\varpi,\varrho)} \right\}.$$

Taking limit as $\varpi \to +\infty$ in (2.52), we get $d_\perp(\varrho,\Omega\varrho) = 0$. Thus, ϱ is a fixed point of Ω and the uniqueness follows on the lines of Theorem 2.2.38. □

Example 2.2.41. *Consider $\mho$ be the interval $[0,+\infty)$ with usual metric space and let $\varrho \perp \varsigma$ iff $\varrho \leqslant \varsigma \; \forall \; \varsigma \in \mho$. Then, $(\mho, d_\perp)$ is $\perp$-complete. Let $\Omega : \mho \to \mho$ be* *defined as*

$$\Omega(\varrho) = \begin{cases} \dfrac{17\varrho}{19} & \varrho \in [0, 1/2); \\ 15\varrho^2 & otherwise. \end{cases}$$

Clearly, here Ω is $\perp$-preserving and $\perp$-continuous map but not a continuous map. Define $\hat{\alpha}, \beta : \mho \to \mathbb{R}^+$ as

$$\hat{\alpha}(\varrho) = \begin{cases} 3/2 & \varrho \in [0, 1/2); \\ 0 & otherwise, \end{cases}$$

and,

$$\beta(\varrho) = \begin{cases} 5/4 & \varrho \in [0, 1/2); \\ 0 & otherwise. \end{cases}$$

Define $\mathcal{C} : [0, +\infty)^2 \to \mathbb{R}$ as $\mathcal{C}(\varrho, \varsigma) = \varrho - \varsigma$ and $\phi_1 : [0, +\infty) \to [0, +\infty)$ as $\phi_1(\varrho) = \varrho/2$. For $\varrho \in [0, 1/2)$, $\hat{\alpha}(\varrho) \geqslant 1$ implies $\beta(\Omega\varrho) = \beta(\frac{17\varrho}{19}) \geqslant 1$ and vice-versa, thus Ω is a cyclic $(\hat{\alpha}, \beta)$-admissible map. Next, for $\hat{\alpha}(\varrho)\beta(\varsigma) \geqslant 1$ and $\varrho \perp \varsigma$ to hold simultaneously, we must have either

$$[\varrho = 0 \text{ and } \varsigma \in [0, 1/2)] \quad or \quad [\varsigma = 0 \text{ and } \varrho \in [0, 1/2)].$$

Considering $\varrho = 0$ and $\varsigma \in [0, 1/2)$, we get

$$d_\perp(\Omega 0, \Omega\varsigma) = d_\perp\left(0, \frac{17\varsigma}{19}\right) = \frac{17\varsigma}{19}, \tag{2.53}$$

$$\text{and, } \mathcal{C}\Big(M^*(0, \varsigma), \phi_1(M^*(0, \varsigma))\Big) = \mathcal{C}(\varsigma, \varsigma/2) = \varsigma/2. \tag{2.54}$$

*From (2.53) and (2.54), we conclude that ς is an $\perp$-*weak TAC*-rational contraction. Thus, by Theorem 2.2.38 we conclude that Ω possesses a fixed point viz. $\varrho = 0$.*

2.2.4 Orthogonal Suzuki-Berinde type F-Contraction

Recently, Hussain & Ahmad (2017) introduced the idea of *Suzuki-Berinde* type F-contraction and established certain fixed point result, which is a generalization of Piri & Kumam (2014). In this subsection, we put forward the notation of orthogonal *Suzuki-Berinde* type F-contraction and explore the fixed point results.

Definition 2.2.10. *For an orthogonal metric space* $(\mho, d_\perp)$ *and for* $F \in \Delta_F$, *a self-map* $\Omega : \mho \to \mho$ *is c.t.b an* ***orthogonal*** **Suzuki-Berinde** ***type*** F***-contraction map*** *(denoted by* $\perp$*-S-B type* F*-contraction) if* $\exists\ \wp > 0$ *with* $L \geqslant 0$ *s.t for each* $\varrho, \varsigma \in \mho$, *where* $d_\perp(\Omega\varrho, \Omega\varsigma) > 0$ *and* $\varrho \perp \varsigma$, *we have*

$$\begin{aligned} \frac{1}{2} d_\perp(\varrho, \Omega\varrho) &< d_\perp(\varrho, \varsigma) \text{ implies} \\ \wp + F\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) &\leqslant F\Big(d_\perp(\varrho, \varsigma)\Big) + L.\min\Big\{d_\perp(\varrho, \Omega\varrho), d_\perp(\varrho, \Omega\varsigma), d_\perp(\varsigma, \Omega\varrho)\Big\}. \end{aligned}$$

Example 2.2.42. *Let* $\mho = [0, 7/2]$ *with* $d_\perp(\varrho, \varsigma) = |\varrho - \varsigma|$ *and* $\varrho \perp \varsigma$ *iff* $\varrho.\varsigma = \varsigma\ \forall\ \varsigma \in \mho$. *Then,* $(\mho, d_\perp)$ *is an orthogonal metric space (with* $\varrho = 1$ *as an orthogonal element). Let* $\Omega : \mho \to \mho$ *be defined as*

$$\Omega(\varrho) = \begin{cases} 1 & \varrho \in [0, 7/2); \\ 2/7 & \text{otherwise.} \end{cases}$$

For $\varrho \perp \varsigma$ *and* $d_\perp(\Omega\varrho, \Omega\varsigma) > 0$, *we must have either* $\varrho = 1$ *and* $\varsigma = 7/2$ *or* $\varrho = 7/2$ *and* $\varsigma = 1$. *Consider* $\varrho = 1$ *and* $\varsigma = 7/2$. *Then for* $F(\mu) = \ln(\mu)$ *and* $0 < \wp < 1$, *we have*

$$\wp + F\Big(d_\perp(\Omega\varrho, \Omega\varsigma)\Big) = \wp + \ln(2/7), \tag{2.55}$$

$$\text{and,} \quad F\Big(d_\perp(\varrho, \varsigma)\Big) + L.\min\Big\{d_\perp(\varrho, \Omega\varrho), d_\perp(\varrho, \Omega\varsigma), d_\perp(\varsigma, \Omega\varrho)\Big\} = \ln(5/2). \tag{2.56}$$

From (2.55) and (2.56), we can conclude that Ω *is* $\perp$*-S-B type* F*-contraction.*

Theorem 2.2.43. *For* $\perp$*-complete metric space* $(\mho, d_\perp)$ *with* ϱ_0 *as an orthogonal element, let* $F \in \Delta_F$ *and* $\Omega : \mho \to \mho$ *be* $\perp$*-preserving,* $\perp$*-continuous and* $\perp$*-S-B type* F*-contraction. Then,* Ω *possesses a fixed point. Moreover, if* $\varrho \perp \varsigma\ \forall\ \varrho, \varsigma \in \mho$ *where* $\Omega\varrho = \varrho$ *and* $\Omega\varsigma = \varsigma$, *then* Ω *possesses a unique fixed point.*

Proof. Let $\{\varrho\}_{\varpi\in\mathbb{N}}$ be a sequence in $\mho$, where $\varrho_{\varpi+1} = \Omega\varrho_\varpi = \Omega^{\varpi+1}\varrho_0\ \forall\ \varpi \in \mathbb{N}$. Since ϱ_0 is an orthogonal element, so we have $[\varrho_0 \perp \Omega\varrho_0]$ or $[\Omega\varrho_0 \perp \varrho_0]$. Repetitive use of $\perp$- preserving property of Ω, we obtain $\{\varrho\}_{\varpi\in\mathbb{N}}$ as an $\perp$-sequence in $\mho$. If for some $\varpi_0 \in \mathbb{N}$, we have $\varrho_{\varpi_0} = \varrho_{\varpi_0+1} = \Omega\varrho_{\varpi_0}$ then we are done. Suppose $\varrho_\varpi \neq \varrho_{\varpi+1}\ \forall\ \varpi \in \mathbb{N}$, that is, $d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) > 0$. As, $\frac{1}{2}d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) = \frac{1}{2}d_\perp(\varrho_\varpi, \Omega\varrho_\varpi)$ $< d_\perp(\varrho_\varpi, \Omega\varrho_{\varpi+1})$, and since Ω is an $\perp$-S-B type F-contraction map, therefore

$$\begin{aligned} \wp + F\Big(d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big) &= \wp + F\Big(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi)\Big) \\ &\leqslant F\Big(d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)\Big) + \end{aligned}$$

$$
\begin{aligned}
& & L.\min\Big\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_{\varpi-1},\varrho_{\varpi+1}), d_\perp(\varrho_\varpi,\varrho_\varpi)\Big\}, \\
\text{that is,}\quad F\Big(d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\Big) &\leqslant& F\Big(d_\perp(\varrho_{\varpi-1},\varrho_\varpi)\Big)-\wp \\
&\leqslant& F\Big(d_\perp(\varrho_{\varpi-2},\varrho_{\varpi-1})\Big)-2\wp \\
&\leqslant& \cdots \leqslant F\Big(d_\perp(\varrho_0,\varrho_1)\Big)-\varpi\wp. \qquad (2.57)
\end{aligned}
$$

Taking limit as $\varpi \to +\infty$ in (2.57) and using (F_2) and Lemma 1.2.4, gives $\lim_{\varpi\to+\infty} d_\perp(\varrho_\varpi,\varrho_{\varpi+1})=0$. Thus, for some $\epsilon/\varpi^*=l>0\ \exists\ \varpi_l\in\mathbb{N}$, with

$$
d_\perp(\varrho_\varpi,\varrho_{\varpi+1})<l \quad \forall\ \varpi>\varpi_l. \qquad (2.58)
$$

Let $\varpi,\varpi^*\in\mathbb{N}$ where $\varpi>\varpi_l$. Using triangle inequality and (2.58), we have

$$
\begin{aligned}
d_\perp(\varrho_\varpi,\varrho_{\varpi+\varpi^*}) &\leqslant d_\perp(\varrho_\varpi,\varrho_{\varpi+1})+d_\perp(\varrho_{\varpi+1},\varrho_{\varpi+2})+\cdots+d_\perp(\varrho_{\varpi+\varpi^*-1},\varrho_{\varpi+\varpi^*}) \\
&< \varpi^* l=\epsilon.
\end{aligned}
$$

Therefore, $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ is an $\perp$-Cauchy sequence in $\mho$. Since $\mho$ is $\perp$-complete, so $\exists\ \varrho\in\mho$ where $\lim_{\varpi\to+\infty}\varrho_\varpi=\varrho$. Since, Ω is an $\perp$-continuous map, so

$$
\begin{aligned}
\lim_{\varpi\to+\infty}\varrho_{\varpi+1}=\lim_{\varpi\to+\infty}\Omega\varrho_\varpi &= \Omega\varrho, \\
\text{that is,}\quad \varrho &= \Omega\varrho.
\end{aligned}
$$

Thus, Ω possesses a fixed point. Let ς be s.t $\Omega\varsigma=\varsigma$ so by given condition $\varrho\perp\varsigma$. Suppose $\varrho\neq\varsigma$, that is, $d_\perp(\varrho,\varsigma)>0$. Also, $\frac{1}{2}d_\perp(\varrho,\varrho)=0=\frac{1}{2}d_\perp(\varrho,\Omega\varrho)<d_\perp(\varrho,\varsigma)$ and, using $\perp$-S-B type F-contraction of Ω, we get

$$
\begin{aligned}
F\Big(d_\perp(\varrho,\varsigma)\Big)=F\Big(d_\perp(\Omega\varrho,\Omega\varsigma)\Big) &<\wp+F\Big(d_\perp(\Omega\varrho,\Omega\varsigma)\Big) \\
&\leqslant F\Big(d_\perp(\varrho,\varsigma)\Big) \\
&\quad + L.\min\Big\{d_\perp(\varrho,\Omega\varrho), d_\perp(\varrho,\Omega\varsigma), d_\perp(\varsigma,\Omega\varrho)\Big\}, \\
\text{that is, } F\Big(d_\perp(\varrho,\varsigma)\Big) &< F\Big(d_\perp(\varrho,\varsigma)\Big),
\end{aligned}
$$

which does not hold. Hence, Ω possesses a unique fixed point. □

Example 2.2.44. *Consider the orthogonal metric space and a self-map discussed in Example 2.2.42. Then, we have*

(i) $(\mho,d_\perp)$ *is* $\perp$*-complete: For any* $\perp$*-Cauchy sequence* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *in* $\mho$, $\exists$ *a subsequence* $\{\varrho_{\varpi_k}\}$ *where* $\varrho_{\varpi_k}=1\ \forall\ k\geqslant 1$*, that is,* $\{\varrho_{\varpi_k}\}$ *is convergent. Thus, we have* $(\mho,d_\perp)$ *as* $\perp$*-complete.*

(ii) Ω *is* $\perp$*-preserving: Since* $1 \perp \varsigma \;\forall\, \varsigma \in \mho$*, then* $\Omega(1) = 1 \perp \Omega\varsigma \;\forall\, \varsigma \in \mho$*.*

(iii) Ω *is* $\perp$*-continuous: For any* $\perp$*-sequence* $\{\varrho_\varpi\} \to \varrho$ *as* $\varpi \to +\infty$*, then* $\varrho = 1$*. Thus,* $\{\Omega\varrho_\varpi\} \to \Omega(1) = 1$ *as* $\varpi \to +\infty$*.*

Since all hypotheses of Theorem 2.2.43 hold, so Ω *possesses a fixed point viz.* $\varrho = 1$*.*

Corollary 2.2.45. *For* $\perp$*-complete* $(\mho, d_\perp)$ *with* ϱ_0 *as an orthogonal element. Let* $\Omega : \mho \to \mho$ *be* $\perp$*-preserving,* $\perp$*-S-B type* F*-contraction and if* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *is an* $\perp$*-sequence in* $\mho$ *with* $\varrho_\varpi \to \varrho$ *as* $\varpi \to +\infty$ *implies*

$$[\varrho_\varpi \perp \varrho \;\forall\, \varpi] \quad \textit{or} \quad [\varrho \perp \varrho_\varpi \;\forall\, \varpi].$$

Then, Ω *possesses a fixed point. Moreover, if* $\varrho \perp \varsigma \;\forall\, \varrho, \varsigma \in \mho$ *where* $\Omega\varrho = \varrho$ *and* $\Omega\varsigma = \varsigma$*, then* Ω *possesses a unique fixed point.*

Proof. On the lines of Theorem 2.2.43, we obtain an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ with $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$. Thus, by given hypothesis, either $[\varrho_\varpi \perp \varrho \;\forall\, \varpi]$ or $[\varrho \perp \varrho_\varpi \;\forall\, \varpi]$. Suppose for some $\varpi_0 \in \mathbb{N}$,

$$\frac{1}{2} d_\perp(\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0}) \geqslant d_\perp(\varrho_{\varpi_0}, \varrho) \quad \text{or} \quad \frac{1}{2} d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0}) \geqslant d_\perp(\Omega\varrho_{\varpi_0}, \varrho), \tag{2.59}$$

$$\begin{aligned} \text{implies} \quad 2d_\perp(\varrho_{\varpi_0}, \varrho) &\leqslant d_\perp(\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0}) \leqslant d_\perp(\varrho_{\varpi_0}, \varrho) + d_\perp(\varrho, \Omega\varrho_{\varpi_0}), \\ \text{that is,} \quad d_\perp(\varrho_{\varpi_0}, \varrho) &\leqslant d_\perp(\varrho, \Omega\varrho_{\varpi_0}). \end{aligned} \tag{2.60}$$

From (2.59) and (2.60), we get

$$d_\perp(\varrho_{\varpi_0}, \varrho) \leqslant d_\perp(\varrho, \Omega\varrho_{\varpi_0}) \leqslant \frac{1}{2} d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0}). \tag{2.61}$$

Since, $\frac{1}{2} d_\perp(\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0}) < d_\perp(\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0})$. So by contraction condition of Ω, we obtain

$$\begin{aligned} F\Big(d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0})\Big) &< \wp + F\Big(d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0})\Big) \\ &\leqslant F\Big(d_\perp(\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0})\Big) + L.\min\Big\{d_\perp(\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0}), \\ &\qquad d_\perp(\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0}), d_\perp(\Omega\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0})\Big\}, \\ \text{that is,} \quad F\Big(d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0})\Big) &< \wp + F\Big(d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0})\Big) \leqslant F\Big(d_\perp(\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0})\Big). \end{aligned}$$

By (F_1), we obtain

$$d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0}) < d_\perp(\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0}). \tag{2.62}$$

Using triangle inequality and (2.61) in (2.62), we get

$$\begin{aligned} d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0}) &< d_\perp(\varrho_{\varpi_0}, \varrho) + d_\perp(\varrho, \Omega\varrho_{\varpi_0}) \\ &\leqslant \frac{1}{2}d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0}) + \frac{1}{2}d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0}) \\ &= d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0}), \end{aligned}$$

which is a contradiction. Thus, we have

$$\frac{1}{2}d_\perp(\varrho_{\varpi_0}, \Omega\varrho_{\varpi_0}) < d_\perp(\varrho_{\varpi_0}, \varrho) \quad \text{or} \quad \frac{1}{2}d_\perp(\Omega\varrho_{\varpi_0}, \Omega^2\varrho_{\varpi_0}) < d_\perp(\Omega\varrho_{\varpi_0}, \varrho) \quad \forall\, \varpi \in \mathbb{N}.$$

Since Ω is an $\perp$-S-B type F-contraction map, so we obtain

$$\begin{aligned} \wp + F\Big(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)\Big) &\leqslant F\Big(d_\perp(\varrho_\varpi, \varrho)\Big) + \\ & L.\min\Big\{d_\perp(\varrho_\varpi, \Omega\varrho_\varpi), d_\perp(\varrho_\varpi, \Omega\varrho), d_\perp(\varrho, \Omega\varrho_\varpi)\Big\}. \end{aligned} \tag{2.63}$$

On taking limit as $\varpi \to +\infty$ in (2.63) and using (F_2) along with Lemma 1.2.4, we get

$$\begin{aligned} \lim_{\varpi\to+\infty} d_\perp(\Omega\varrho_\varpi, \Omega\varrho) &= 0, \\ \text{that is,} \quad d_\perp(\varrho, \Omega\varrho) &= 0. \end{aligned}$$

Thus, Ω possesses a fixed point. The uniqueness of fixed point follows on the lines of Theorem 2.2.43. □

2.2.5 Orthogonal $\mathcal{F}$-weak Contraction

Inspired by the work done in Baghani et al. (2016), Sawangsup et al. (2020), in this section, we put forward the notation of orthogonal $\mathcal{F}$-weak contraction and establish some fixed point theorems with orthogonal $\mathcal{F}$-weak contraction in complete orthogonal metric space.

Definition 2.2.11. *A self-map Ω on $\mho$, where $(\mho, d_\perp)$ is an orthogonal metric space and $\mathcal{F} \in \mathfrak{F}$, is c.t.b an **orthogonal $\mathcal{F}$-weak contraction** (denoted by $\perp_{\mathcal{F}}$-*

weak contraction) if $\exists\ \wp > 0$ *s.t for each* $\varrho, \varsigma \in \mho$ *with* $\varrho \perp \varsigma$ *and* $d_\perp(\Omega\varrho, \Omega\varsigma) > 0$ *implies*

$$\wp + \mathcal{F}(d_\perp(\Omega\varrho, \Omega\varsigma)) \leqslant \mathcal{F}\left(\max\left\{d_\perp(\varrho,\varsigma), d_\perp(\varrho,\Omega\varrho), d_\perp(\varsigma,\Omega\varsigma), \frac{d_\perp(\varrho,\Omega\varsigma)+d_\perp(\varsigma,\Omega\varrho)}{2}\right\}\right). \tag{2.64}$$

Remark 2.2.46. *From (2.64), we can infer that every* $\perp_{\mathcal{F}}$*-contraction is* $\perp_{\mathcal{F}}$*-weak contraction. However, following example substantiates that the converse need not hold true.*

Example 2.2.47. *Let* $\mho = \{0, 1, 2, 3, 4\}$ *with usual metric* $d_\perp$*.*
Let $R = \{(0,0), (0,1), (0,2), (0,3), (0,4), (4,0), (4,1), (4,2), (4,3), (4,4)\}$*. Define* $\varrho \perp \varsigma$ *iff* $(\varrho, \varsigma) \in R$*. Clearly,* $(\mho, \perp)$ *is an orthogonal set (with* 0 *and* 4 *as orthogonal element). Let* $\Omega : \mho \to \mho$ *be defined as* $\Omega(0) = 0 = \Omega(1) = \Omega(4)$*,* $\Omega(2) = 1$*,* $\Omega(3) = 2$*. Let* $\mathcal{F}(\mu) = \ln(\mu)$*. It can be verified that* Ω *is* $\perp_{\mathcal{F}}$*-weak contraction however,* Ω *is not* $\perp_{\mathcal{F}}$*-contraction since for* $\varrho = 4$ *and* $\varsigma = 3$*,* $\wp + \mathcal{F}(d_\perp(\Omega\varrho, \Omega\varsigma)) \leqslant \mathcal{F}(d_\perp(\varrho, \varsigma))$ *does not hold for any* $\wp > 0$*.*

Theorem 2.2.48. *For an* $\perp$*-complete metric space* $(\mho, d_\perp)$ *and* $\mathcal{F} \in \mathfrak{F}$*, if a self-map* Ω *on* $\mho$ *is* $\perp$*-continuous,* $\perp_{\mathcal{F}}$*-weak contraction and* $\perp$*-preserving. Then,* Ω *possesses a unique fixed point in* $\mho$*.*

Proof. As $(\mho, \perp)$ is an orthogonal set, therefore, let there be an orthogonal element $\varrho_0 \in \mho$ where

$$[\varrho \perp \varrho_0 \quad \forall\ \varrho \in \mho] \quad \text{or} \quad [\varrho_0 \perp \varrho \quad \forall\ \varrho \in \mho]. \tag{2.65}$$

As ϱ_0, $\Omega(\varrho_0) \in \mho$ then by (2.65), we have $[\Omega(\varrho_0) \perp \varrho_0]$ or $[\varrho_0 \perp \Omega(\varrho_0)]$. Define a sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$, where $\varrho_{\varpi+1} = \Omega(\varrho_\varpi)\ \forall\ \varpi \in \mathbb{N}$. Since Ω is $\perp$-preserving. Therefore, $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ is an $\perp$-sequence. Let us consider $\eta_\varpi = d_\perp(\varrho_\varpi, \varrho_{\varpi+1})$ for $\varpi = 0, 1, 2, \ldots$. If for some $\varpi_0 \in \mathbb{N}$,

$$\begin{aligned} \eta_{\varpi_0} &= d_\perp(\varrho_{\varpi_0}, \varrho_{\varpi_{0+1}}) = 0, \\ \text{that is,}\quad \varrho_{\varpi_0} &= \varrho_{\varpi_{0+1}} = \Omega(\varrho_{\varpi_0}), \end{aligned}$$

which gives that Ω possesses a fixed point. Instead, let $\eta_\varpi \neq 0\ \forall\ \varpi \in \mathbb{N}$. As Ω is $\perp_{\mathcal{F}}$-weak contraction, so $\forall\ \varpi \in \mathbb{N}$, we have

$$\mathcal{F}(\eta_\varpi) = \mathcal{F}(d_\perp(\varrho_\varpi, \varrho_{\varpi+1}) = \mathcal{F}(d_\perp(\Omega\varrho_{\varpi-1}, \Omega\varrho_\varpi))$$

$$
\begin{aligned}
\leqslant &\mathcal{F}\Bigg(\max\Bigg\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_{\varpi-1},\Omega\varrho_{\varpi-1}), d_\perp(\varrho_\varpi,\Omega\varrho_\varpi),\\
&\frac{d_\perp(\varrho_{\varpi-1},\Omega\varrho_\varpi)+d_\perp(\varrho_\varpi,\Omega\varrho_{\varpi-1})}{2}\Bigg\}\Bigg)-\wp\\
=&\mathcal{F}\left(\max\left\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_\varpi,\varrho_{\varpi+1}), \frac{d_\perp(\varrho_{\varpi-1},\varrho_{\varpi+1})+d_\perp(\varrho_\varpi,\varrho_\varpi)}{2}\right\}\right)-\wp\\
\leqslant &\mathcal{F}\left(\max\left\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_\varpi,\varrho_{\varpi+1}), \frac{d_\perp(\varrho_{\varpi-1},\varrho_\varpi)+d_\perp(\varrho_\varpi,\varrho_{\varpi+1})}{2}\right\}\right)-\wp\\
=&\mathcal{F}\left(\max\left\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\right\}\right)-\wp.
\end{aligned}
$$

If $\max\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_\varpi,\varrho_{\varpi+1})\} = d_\perp(\varrho_\varpi,\varrho_{\varpi+1})$ then from above, we have

$$\mathcal{F}(\eta_\varpi) \leqslant \mathcal{F}(\eta_\varpi) - \wp,$$

which is a contradiction (for $\wp > 0$). Thus, we obtain $\max\Big\{d_\perp(\varrho_{\varpi-1},\varrho_\varpi), d_\perp(\varrho_\varpi, \varrho_{\varpi+1})\Big\} = d_\perp(\varrho_{\varpi-1},\varrho_\varpi)\ \forall\ \varpi \in \mathbb{N}$ and,

$$\mathcal{F}(\eta_\varpi) \leqslant \mathcal{F}(\eta_{\varpi-1}) - \wp \leqslant \mathcal{F}(\eta_{\varpi-2}) - 2\wp \leqslant \cdots \leqslant \mathcal{F}(\eta_0) - \varpi\wp. \tag{2.66}$$

Taking limit as $\varpi \to +\infty$ in (2.66), we obtain $\lim_{\varpi\to+\infty} \mathcal{F}(\eta_\varpi) = -\infty$. Using $(\mathcal{F}_2)$, we obtain

$$\lim_{\varpi\to+\infty} \eta_\varpi = 0. \tag{2.67}$$

By $(\mathcal{F}_3)$ property $\exists\ \gamma \in (0,1)$ s.t

$$\lim_{\varpi\to+\infty} \eta_\varpi^\gamma \mathcal{F}(\eta_\varpi) = 0. \tag{2.68}$$

From (2.66), we have $\eta_\varpi^\gamma\mathcal{F}(\eta_\varpi) - \eta_\varpi^\gamma\mathcal{F}(\eta_0) \leqslant -\eta_\varpi^\gamma\varpi\wp$. Taking $\varpi \to +\infty$ and using (2.67) and (2.68), we get

$$\lim_{\varpi\to+\infty} \varpi\eta_\varpi^\gamma = 0. \tag{2.69}$$

On observing (2.69), we get that $\exists\ \varpi_1 \in \mathbb{N}$ where $\forall\ \varpi > \varpi_1$, we have

$$\eta_\varpi \leqslant \frac{1}{\varpi^{\frac{1}{\gamma}}} \quad \forall\ \varpi > \varpi_1. \tag{2.70}$$

Consider $\varpi, \varpi^* \in \mathbb{N}$ with $\varpi^* > \varpi > \varpi_1$, using (2.70) and triangle inequality of metric space $d_\perp$, we get

$$\begin{aligned} d_{\perp}(\varrho_{\varpi^*}, \varrho_{\varpi}) &\leqslant d_{\perp}(\varrho_{\varpi^*}, \varrho_{\varpi^*-1}) + \cdots + d_{\perp}(\varrho_{\varpi+2}, \varrho_{\varpi+1}) + d_{\perp}(\varrho_{\varpi+1}, \varrho_{\varpi}) \\ &= \eta_{\varpi^*-1} + \cdots + \eta_{\varpi+1} + \eta_{\varpi} < \sum_{i=1}^{+\infty} \eta_i \leqslant \sum_{i=1}^{+\infty} 1/i^{1/\gamma}. \end{aligned}$$

As $\sum_{i=1}^{+\infty} 1/i^{1/\gamma}$ is convergent (for $\gamma \in (0,1)$), we get $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ is an $\perp$-Cauchy sequence and since $\mho$ is $\perp$-complete, we have $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ is convergent, that is, $\exists\ \varrho \in \mho$ with $\lim_{\varpi\to+\infty} \varrho_{\varpi} = \varrho$. Using $\perp$-continuity of Ω, we get

$$\begin{aligned} \lim_{\varpi\to+\infty} \Omega(\varrho_{\varpi}) &= \Omega(\varrho), \\ \text{implies} \quad \lim_{\varpi\to+\infty} \varrho_{\varpi+1} &= \Omega(\varrho). \end{aligned}$$

Thus, $\varrho = \Omega(\varrho)$. Hence, ϱ is a fixed point of Ω. Let ϱ^* be s.t $\Omega\varrho^* = \varrho^*$ which implies

$$\Omega^{\varpi}(\varrho^*) = \varrho^* \quad \forall\ \varpi \in \mathbb{N}.$$

By (2.65), we have

$$[\varrho_0 \perp \varrho^*] \quad \text{or} \quad [\varrho^* \perp \varrho_0].$$

Since Ω is $\perp$-preserving, therefore

$$[\Omega^{\varpi}(\varrho_0) \perp \Omega(\varrho^*)] \quad \text{or} \quad [\Omega(\varrho^*) \perp \Omega^{\varpi}(\varrho_0)].$$

Also, Ω is $\perp_{\mathcal{F}}$-weak contraction, thus

$$\begin{aligned} \mathcal{F}(d_{\perp}(\varrho_{\varpi}, \varrho^*)) =& \mathcal{F}(d_{\perp}(\Omega^{\varpi}\varrho_0, \varrho^*)) \\ =& \mathcal{F}(d_{\perp}(\Omega\varrho_{\varpi-1}, \Omega\varrho^*)) \\ \leqslant& \mathcal{F}\Bigg(\max\Big\{d_{\perp}(\varrho_{\varpi-1}, \varrho^*), d_{\perp}(\varrho_{\varpi-1}, \Omega\varrho_{\varpi-1}), d_{\perp}(\varrho^*, \Omega\varrho^*), \\ & \frac{d_{\perp}(\varrho_{\varpi-1}, \Omega\varrho^*) + d_{\perp}(\varrho^*, \Omega\varrho_{\varpi-1})}{2}\Big\}\Bigg) - \wp \\ =& \mathcal{F}\left(\max\left\{d_{\perp}(\varrho_{\varpi-1}, \varrho^*), d_{\perp}(\varrho_{\varpi-1}, \varrho_{\varpi}), d_{\perp}(\varrho_{\varpi}, \varrho^*)\right\}\right) - \wp. \end{aligned}$$

Next, we have following cases:

Case (i): Let $\max\{d_{\perp}(\varrho_{\varpi-1}, \varrho^*), d_{\perp}(\varrho_{\varpi-1}, \varrho_{\varpi}), d_{\perp}(\varrho_{\varpi}, \varrho^*)\} = d_{\perp}(\varrho_{\varpi}, \varrho^*)$ then

$\forall\ \varpi \in \mathbb{N}$, we obtain $\mathcal{F}(d_\perp(\varrho_\varpi, \varrho^*)) \leqslant \mathcal{F}(d_\perp(\varrho_\varpi, \varrho^*)) - \wp$, which does not hold for any $\wp > 0$.

Case (ii): Let $\max\{d_\perp(\varrho_{\varpi-1}, \varrho^*), d_\perp(\varrho_{\varpi-1}, \varrho_\varpi), d_\perp(\varrho_\varpi, \varrho^*))\} = d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)$ then $\forall\ \varpi \in \mathbb{N}$, we get

$$\mathcal{F}(d_\perp(\varrho_\varpi, \varrho^*)) \leqslant \mathcal{F}(d_\perp(\varrho_{\varpi-1}, \varrho_\varpi)) - \wp = \mathcal{F}(\eta_{\varpi-1}) - \wp.$$

Using (2.66), we get

$$\mathcal{F}(d_\perp(\varrho_\varpi, \varrho^*)) \leqslant \mathcal{F}(\eta_{\varpi-1}) - \wp \leqslant \cdots \leqslant \mathcal{F}(\eta_0) - \varpi\wp.$$

Taking $\varpi \to +\infty$, we get $\lim\limits_{\varpi\to+\infty} \mathcal{F}(d_\perp(\varrho_\varpi, \varrho^*)) = -\infty$. By $(\mathcal{F}_2)$ property,

$$\lim_{\varpi\to+\infty} d_\perp(\varrho_\varpi, \varrho^*) = 0,$$

implies $\varrho = \varrho^*$.

Case (iii): Let $\max\{d_\perp(\varrho_{\varpi-1}, \varrho^*), d_\perp(\varrho_{\varpi-1}, \varrho_\varpi), d_\perp(\varrho_\varpi, \varrho^*))\} = d_\perp(\varrho_{\varpi-1}, \varrho^*)$ then $\forall\ \varpi \in \mathbb{N}$, we get

$$\begin{aligned} \mathcal{F}(d_\perp(\varrho_\varpi, \varrho^*)) &\leqslant \mathcal{F}(d_\perp(\varrho_{\varpi-1}, \varrho^*)) - \wp \\ &\leqslant \mathcal{F}(d_\perp(\varrho_{\varpi-2}, \varrho^*)) - 2\wp \leqslant \cdots \leqslant \mathcal{F}(d_\perp(\varrho_0, \varrho^*)) - \varpi\wp. \end{aligned}$$

Taking $\varpi \to +\infty$ and using $(\mathcal{F}_2)$ property, we obtain $\varrho = \varrho^*$. Thus, we conclude that Ω has a unique fixed point in $\mho$. □

Remark 2.2.49. *Theorem 2.2.48 proved above provides a proper extension of Theorem 3.10 and Theorem 3.3 of Baghani et al. (2016) and Sawangsup et al. (2020) respectively. The example discussed below further substantiates the outcome.*

Example 2.2.50. *Consider the orthogonal metric space discussed in Example 2.2.47. Then, the map defined in it can be verified for $\perp$-continuous and $\perp$-preserving. Also, $\mho$ is $\perp$-complete since for any arbitrary $\perp$-Cauchy sequence $\{\varrho_\varpi\}$ in $\mho$, $\exists$ a sub-sequence $\{\varrho_{\varpi_k}\}$ of $\{\varrho_\varpi\}$ s.t $\varrho_{\varpi_k} = 0\ \forall\ k \geqslant k_1$ or $\varrho_{\varpi_k} = 4$ $\forall\ k \geqslant k_2$ for some $k_1, k_2 \in \mathbb{N}$. Thus, $\{\varrho_{\varpi_k}\}$ converges to 0 or 4. Therefore, $\{\varrho_\varpi\}$ is convergent. Since all hypotheses of Theorem 2.2.48 hold so Ω has a unique fixed point which is $\varrho = 0$, even though Ω is not $\perp_{\mathcal{F}}$-contraction.*

Theorem 2.2.51. *For an $\perp$-complete metric space $(\mho, d_\perp)$ and $\mathcal{F} \in \mathfrak{F}$, if a self-map Ω on $\mho$ is $\perp_{\mathcal{F}}$-weak contraction and $\perp$-preserving s.t*

(I) $\mathcal{F}$ is continuous;

(II) If $\exists$ an $\perp$-sequence $\{\varrho_\varpi\}$ in $\mho$ is s.t for $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$, we have $\varrho_\varpi \perp \varrho \quad \forall\, \varpi \in \mathbb{N}$ or $\varrho \perp \varrho_\varpi \quad \forall\, \varpi \in \mathbb{N}$.

Then, Ω possesses a unique fixed point in $\mho$.

Proof. By the working done in Theorem 2.2.48, it can be shown that there is an $\perp$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$, where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$. We show that ϱ is the desired fixed point. However, once the existence of fixed point is established then the uniqueness follows similar to Theorem 2.2.48. Suppose on the contrary that $d_\perp(\varrho, \Omega\varrho) \geq 0$.

Case (i): If $\{\varpi \in \mathbb{N} : \Omega\varrho_\varpi = \Omega\varrho\}$ is infinite. Then, $\exists$ sub-sequence $\{\varrho_{\varpi_i}\}$ of $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ s.t $\Omega\varrho_{\varpi_i} = \Omega\varrho$ implies $\varrho_{\varpi_i+1} = \Omega\varrho$. Taking limit $\varpi \to +\infty$, we get $\varrho = \Omega\varrho$, which is a contradiction.

Case (ii): If $\{\varpi \in \mathbb{N} : \Omega\varrho_\varpi = \Omega\varrho\}$ is finite, that is, for some $\varpi_0 \in \mathbb{N}$, $d_\perp(\Omega\varrho_\varpi, \Omega\varrho) > 0 \quad \forall\, \varpi > \varpi_0$. By given condition, we have $[\varrho_\varpi \perp \varrho \quad \forall\, \varpi \in \mathbb{N}]$ or $[\varrho \perp \varrho_\varpi \quad \forall\, \varpi \in \mathbb{N}]$. Since Ω is $\perp$-preserving, therefore

$$[\Omega\varrho_\varpi \perp \Omega\varrho \quad \forall\, \varpi \in \mathbb{N}] \quad \text{or} \quad [\Omega\varrho \perp \Omega\varrho_\varpi \quad \forall\, \varpi \in \mathbb{N}].$$

As Ω is $\perp_{\mathcal{F}}$-weak contraction, we have

$$\begin{aligned}
&\wp + \mathcal{F}(d_\perp(\Omega\varrho_\varpi, \Omega\varrho)) \\
&\leqslant \mathcal{F}\Bigg(\max\Big\{d_\perp(\varrho_\varpi, \varrho), d_\perp(\varrho_\varpi, \Omega\varrho_\varpi), d_\perp(\varrho, \Omega\varrho), \frac{d_\perp(\varrho_\varpi, \Omega\varrho) + d_\perp(\varrho, \Omega\varrho_\varpi)}{2}\Big\}\Bigg) \\
&\leqslant \mathcal{F}\Bigg(\max\Big\{d_\perp(\varrho_\varpi, \varrho), d_\perp(\varrho_\varpi, \varrho_{\varpi+1}), d_\perp(\varrho, \Omega\varrho), \\
&\frac{d_\perp(\varrho_\varpi, \varrho) + d_\perp(\varrho, \Omega\varrho) + d_\perp(\varrho, \varrho_{\varpi+1})}{2}\Big\}\Bigg).
\end{aligned} \tag{2.71}$$

Since, $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$. Therefore, $\exists\, \varpi_1 \in \mathbb{N}$, s.t $d_\perp(\varrho_\varpi, \varrho) = 0 \quad \forall\, \varpi > \varpi_1$. Hence, for each $\varpi > \max\{\varpi_0, \varpi_1\}$, we obtain

$$\max\Bigg\{d_\perp(\varrho_\varpi, \varrho), d_\perp(\varrho_\varpi, \varrho_{\varpi+1}), d_\perp(\varrho, \Omega\varrho), \frac{d_\perp(\varrho_\varpi, \varrho) + d_\perp(\varrho, \Omega\varrho) + d_\perp(\varrho, \varrho_{\varpi+1})}{2}\Bigg\}$$

$$= d_\perp(\varrho, \Omega\varrho).$$

As $\mathcal{F}$ is continuous, on letting limit $\varpi \to +\infty$ in (2.71), we get

$$\wp + \mathcal{F}(d_\perp(\Omega\varrho, \Omega\varrho)) \leqslant \mathcal{F}(d_\perp(\Omega\varrho, \Omega\varrho)),$$

which does not hold. Thus, Ω has a fixed point ϱ in $\mho$. □

Corollary 2.2.52. *For $(\mho, d_\perp)$, an $\perp$-complete metric space and $\mathcal{F} \in \mathfrak{F}$, if a map $\Omega : \mho \to \mho$ is $\perp_\mathcal{F}$-weak contraction and $\perp$-preserving s.t*

(I) $\mathcal{F}$ is continuous;

(II) If $\exists$ an $\perp$-sequence $\{\varrho_\varpi\}$ in $\mho$ is s.t for $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$, we have $\varrho_\varpi \perp \varrho \quad \forall \quad \varpi \in \mathbb{N}$ or $\varrho \perp \varrho_\varpi \quad \forall\, \varpi \in \mathbb{N}$.

Then, Ω in $\mho$ possesses a unique fixed point. Further, for each $\varrho^* \in \mho$ the Picard sequence $\{\Omega^\varpi(\varrho^*)\}_{\varpi\in\mathbb{N}}$ converges to fixed point ϱ of Ω.

Proof. By the working done in Theorem 2.2.51, Ω possesses a unique fixed point. We show that Picard sequence $\{\Omega^\varpi(\varrho^*)\}_{\varpi\in\mathbb{N}}$ converges to fixed point ϱ, that is $\lim_{\varpi\to+\infty} \Omega^\varpi(\varrho^*) = \varrho$. Since $\varrho^* \in \mho$ is any arbitrary point and $\mho$ is an orthogonal set, therefore $[\varrho^* \perp \varrho_0]$ or $[\varrho_0 \perp \varrho^*]$, and as Ω is $\perp$-preserving, thus

$$[\Omega^\varpi(\varrho^*) \perp \Omega^\varpi(\varrho_0) \quad \forall\, \varpi \in \mathbb{N}] \quad \text{or} \quad [\Omega^\varpi(\varrho_0) \perp \Omega^\varpi(\varrho^*) \quad \forall\, \varpi \in \mathbb{N}].$$

Using $\perp_\mathcal{F}$-weak contraction of Ω, we obtain

$$\begin{aligned}
&\wp + \mathcal{F}(d_\perp(\Omega^\varpi(\varrho^*), \varrho_\varpi)) \\
&= \wp + \mathcal{F}(d_\perp(\Omega^\varpi(\varrho^*), \Omega^\varpi(\varrho_0)) \\
&= \wp + \mathcal{F}(d_\perp(\Omega(\Omega^{\varpi-1}(\varrho^*)), \Omega(\Omega^{\varpi-1}(\varrho_0))) \\
&\leqslant \mathcal{F}\Bigg(\max\Bigg\{d_\perp(\Omega^{\varpi-1}(\varrho^*), \varrho_{\varpi-1}), d_\perp(\Omega^{\varpi-1}(\varrho^*), \Omega^\varpi(\varrho^*)), d_\perp(\varrho_{\varpi-1}, \varrho_\varpi), \\
&\frac{d_\perp(\Omega^{\varpi-1}(\varrho^*), \varrho_\varpi) + d_\perp(\varrho_{\varpi-1}, \Omega^\varpi(\varrho^*))}{2}\Bigg\}\Bigg).
\end{aligned} \tag{2.72}$$

Taking limit as $\varpi \to +\infty$ in (2.72) and since $\mathcal{F}$ is continuous, we get

$$\wp + \mathcal{F}(d_\perp(\lim_{\varpi\to+\infty} \Omega^\varpi\varrho^*, \varrho)) \leqslant \mathcal{F}(d_\perp(\lim_{\varpi\to+\infty} \Omega^\varpi\varrho^*, \varrho)),$$

which holds iff $\lim_{\varpi\to+\infty} \Omega^\varpi(\varrho^*) = \varrho$. Hence, Ω is a Picard operator. □

2.3 Application

We now apply the outcome of Theorem 2.2.48 to show the existence and uniqueness of the solution of the ordinary differential equation given by:

$$\begin{aligned} \theta'(\varrho) - \Omega(\varrho, \theta(\varrho)) = 0 \quad \text{a.e} \quad \varrho \in \hat{\mathrm{I}} = [0, T]; \\ \theta(0) = a \quad \text{for} \quad a \geqslant 1, \end{aligned} \tag{2.73}$$

where, $\Omega : \hat{\mathrm{I}} \times (-\infty, +\infty) \to (-\infty, +\infty)$ is an integrable function which satisfies the following:

(I) $\Omega(\varrho, \varsigma) \geqslant 0 \ \forall \ \varrho \in \hat{\mathrm{I}}$ and $\varsigma \geqslant 0$;

(II) $\exists \ \alpha(\varrho) \in \mathcal{L}^1(\hat{\mathrm{I}})$ and $\wp > 0$ s.t

$$\Big|\Omega(\varrho, \mathfrak{X}(\varrho)) - \Omega(\varrho, \mathfrak{Y}(\varrho))\Big| \leqslant \frac{\alpha(\varrho)}{e^{\wp}}\Big|\mathfrak{X}(\varrho) - \mathfrak{Y}(\varrho)\Big|$$

for each $\mathfrak{X}, \mathfrak{Y} \in \mathcal{L}^1(\hat{\mathrm{I}})$ s.t $\mathfrak{X}(\varrho)\mathfrak{Y}(\varrho) \geqslant \mathfrak{X}(\varrho)$ or $\mathfrak{X}(\varrho)\mathfrak{Y}(\varrho) \geqslant \mathfrak{Y}(\varrho)$.

Theorem 2.3.1. *The differential equation given in (2.73) along with condition (I) and (II) has a unique solution.*

Proof. Let $\mho = \{\mathfrak{X} \in C(\hat{\mathrm{I}}, (-\infty, +\infty)) : \mathfrak{X}(\varrho) > 0 \ \forall \ \varrho \in \hat{\mathrm{I}}\}$ and define a relation on $\mho$ as

$$\mathfrak{X} \perp \mathfrak{Y} \quad \text{iff} \quad \mathfrak{X}(\varrho)\mathfrak{Y}(\varrho) \geqslant \mathfrak{X}(\varrho) \quad \text{or} \quad \mathfrak{X}(\varrho)\mathfrak{Y}(\varrho) \geqslant \mathfrak{Y}(\varrho) \quad \forall \ \varrho \in \hat{\mathrm{I}}.$$

Then, $(\mho, \perp)$ is an orthogonal set. Let $A(\varrho) = \int_0^{\varrho} |\alpha(\varrho)| d\varrho$. So that $A' = |\alpha(\varrho)|$ a.e $\varrho \in \hat{\mathrm{I}}$. Define a map $d_\perp : \mho \times \mho \to [0, +\infty)$ by

$$d_\perp(\mathfrak{X}, \mathfrak{Y}) = \|\mathfrak{X} - \mathfrak{Y}\| = \sup_{\varrho \in \hat{\mathrm{I}}} e^{-A(\varrho)} | \ \mathfrak{X}(\varrho) - \mathfrak{Y}(\varrho) \ | \quad \forall \ \mathfrak{X}, \mathfrak{Y} \in \mho.$$

Now, since $(\mho, d_\perp)$ is $\perp$-complete. Let $\{\mathfrak{X}_\varpi\}_{\varpi \in \mathbb{N}}$ be a $\perp$-Cauchy sequence in $\mho$ then we can conclude that $\{\mathfrak{X}_\varpi\}$ converges to a point $\mathfrak{X}$ in $C(\hat{\mathrm{I}})$. It is enough if we show that $\mathfrak{X} \in \mho$. Let $\varrho \in \hat{\mathrm{I}}$ fixed then

$$\mathfrak{X}_\varpi(\varrho)\mathfrak{X}_{\varpi+1}(\varrho) \geqslant \mathfrak{X}_\varpi(\varrho) \quad \text{or} \quad \mathfrak{X}_\varpi(\varrho)\mathfrak{X}_{\varpi+1}(\varrho) \geqslant \mathfrak{X}_{\varpi+1}(\varrho).$$

As $\mathfrak{X}_\varpi(\varrho) > 0 \quad \forall \ \varpi \in \mathbb{N}$, then $\exists$ a sub-sequence $\{\mathfrak{X}_{\varpi_k}\}$ of $\{\mathfrak{X}_\varpi\}$ for which $\mathfrak{X}_{\varpi_k} \geqslant 1$

and since $\mathfrak{X}_{\varpi} \to \mathfrak{X}$ as $\varpi \to +\infty$ so $\mathfrak{X}_{\varpi_k} \to \mathfrak{X}$ as $\varpi \to +\infty$ implies $\mathfrak{X}(\varrho) \geqslant 1$. Thus, $\mathfrak{X} \in \mho$. Define a map $\mathcal{Y} : \mho \to \mho$ as:

$$(\mathcal{Y}\mathfrak{X})(\varrho) = \beta + \int_0^{\varrho} \Omega(t, \mathfrak{X}(t))dt.$$

Then:

(1) $\mathcal{Y}$ is $\perp$-preserving. Let $\mathfrak{X} \perp \mathfrak{Y}$, then

$$(\mathcal{Y}\mathfrak{X})(\varrho) = \beta + \int_0^{\varrho} \Omega(t, \mathfrak{X}(t))dt \geqslant 1,$$

which shows that $(\mathcal{Y}\mathfrak{X})(\varrho)(\mathcal{Y}\mathfrak{Y})(\varrho) \geqslant (\mathcal{Y}\mathfrak{Y})(\varrho)$ or $(\mathcal{Y}\mathfrak{X})(\varrho)(\mathcal{Y}\mathfrak{Y})(\varrho) \geqslant (\mathcal{Y}\mathfrak{X})(\varrho)$. Therefore, $\mathcal{Y}\mathfrak{X} \perp \mathcal{Y}\mathfrak{Y}$.

(2) $\mathcal{Y}$ is $\perp$-continuous. Let $\{\mathfrak{X}_{\varpi}\}$ be an $\perp$-sequence in $\mho$ which converges to $\mathfrak{X} \in \mho$. Then, it is well evident from previous working that $\mathfrak{X}(\varrho) \geqslant 1$ implies $\mathfrak{X}_{\varpi}(\varrho) \perp \mathfrak{X}(\varrho)$ for each $\varpi \in \mathbb{N}$ and $\varrho \in \hat{\mathrm{I}}$. So, we have

$$\begin{aligned} e^{-A(\varrho)}|(\mathcal{Y}\mathfrak{X}_{\varpi})(\varrho) - (\mathcal{Y}\mathfrak{X})(\varrho)| &\leqslant e^{-A(\varrho)} \int_0^{\varrho} |\Omega(t, \mathfrak{X}_{\varpi}(t)) - \Omega(t, \mathfrak{X}(t))|dt \\ &\leqslant e^{-A(\varrho)} \int_0^{\varrho} |\mathfrak{X}_{\varpi}(t)) - \mathfrak{X}(t))| \frac{|\alpha(t)|}{e^{\wp}} e^{-A(t)} e^{A(t)} dt \\ &\leqslant e^{-A(\varrho)} e^{-\wp} d_{\perp}(\mathfrak{X}_{\varpi}, \mathfrak{X}) \int_0^{\varrho} |\alpha(t)| e^{A(t)} dt \\ &\leqslant e^{-A(\varrho)} e^{-\wp} d_{\perp}(\mathfrak{X}_{\varpi}, \mathfrak{X})(e^{A(\varrho)-1}). \end{aligned}$$

Since above inequality holds for any arbitrary $\varrho \in \hat{\mathrm{I}}$ and $\varpi \in \mathbb{N}$. So, we have

$$d_{\perp}(\mathcal{Y}\mathfrak{X}_{\varpi}, \mathcal{Y}\mathfrak{X}) \leqslant e^{-\wp}(1 - e^{-\|\alpha\|_1}) d_{\perp}(\mathfrak{X}_{\varpi}, \mathfrak{X}) \quad \forall \quad \varpi \in \mathbb{N}.$$

Thus, $\mathcal{Y}\mathfrak{X}_{\varpi} \to \mathcal{Y}\mathfrak{X}$.

(3) $\mathcal{Y}$ is $\perp_{\mathcal{F}}$-weak contraction.
Let $\mathfrak{X}, \mathfrak{Y} \in \mho$ s.t $\mathfrak{X} \perp \mathfrak{Y}$ and $d_{\perp}(\mathcal{Y}\mathfrak{X}, \mathcal{Y}\mathfrak{Y}) > 0$, then for each $\varrho \in \hat{\mathrm{I}}$, we obtain

$$\begin{aligned} |(\mathcal{Y}\mathfrak{X})(\varrho) - (\mathcal{Y}\mathfrak{Y})(\varrho)| &\leqslant \int_0^{\varrho} |\Omega(t, \mathfrak{X}(t)), \Omega(t, \mathfrak{Y}(t))|dt \\ &\leqslant \int_0^{\varrho} e^{-\wp} |\alpha(t)||\mathfrak{X}(t) - \mathfrak{Y}(t)| e^{-A(t)} e^{A(t)} dt \\ &\leqslant e^{-\wp} d_{\perp}(\mathfrak{X}, \mathfrak{Y}) \int_0^{\varrho} |\alpha(t)| e^{A(t)} \\ &\leqslant e^{-\wp} d_{\perp}(\mathfrak{X}, \mathfrak{Y})(e^{A(\varrho)} - 1), \\ \text{that is,} \quad e^{-A(\varrho)}|(\mathcal{Y}\mathfrak{X})(\varrho) - (\mathcal{Y}\mathfrak{Y})(\varrho)| &\leqslant e^{-A(\varrho)}(e^{A(\varrho)} - 1) e^{-\wp} d_{\perp}(\mathfrak{X}, \mathfrak{Y}) \end{aligned}$$

$$\begin{aligned} &\leqslant (1-e^{-A(\varrho)})e^{-\wp}d_\perp(\mathfrak{X},\mathfrak{Y}) \\ &\leqslant (1-e^{-\|\alpha\|_1})e^{-\wp}d_\perp(\mathfrak{X},\mathfrak{Y}). \end{aligned}$$

It follows that $d_\perp(\mathcal{Y}\mathfrak{X},\mathcal{Y}\mathfrak{Y}) \leqslant e^{-\wp}d_\perp(\mathfrak{X},\mathfrak{Y})$. Taking logarithm, we get

$$\begin{aligned} \wp + \ln(d_\perp(\mathcal{Y}\mathfrak{X},\mathcal{Y}\mathfrak{Y})) \leqslant \ln\Bigg(\max\Bigg\{ & d_\perp(\mathfrak{X},\mathfrak{Y}), d_\perp(\mathfrak{X},\mathcal{Y}\mathfrak{X}), d_\perp(\mathfrak{Y},\mathcal{Y}\mathfrak{Y}), \\ & \frac{d_\perp(\mathfrak{X},\mathcal{Y}\mathfrak{Y}) + d_\perp(\mathfrak{Y},\mathcal{Y}\mathfrak{X})}{2}\Bigg\}\Bigg). \end{aligned}$$

On defining $\mathcal{F}:\mathbb{R}^+ \to (-\infty,+\infty)$ as $\mathcal{F}(\mu) = \ln(\mu)$, we obtain that $\mathcal{Y}$ is $\perp_{\mathcal{F}}$-weak contraction. Therefore, using Theorem 2.2.48, $\mathcal{Y}$ has a unique fixed point and hence differential equation has a unique solution. □

2.4 Conclusion

Under some specific conditions, the results proved in this chapter are reduced to many well-known fixed point results in the literature. Consider the binary relation $\varrho \perp \varsigma$ iff $\varrho,\varsigma \in \mho \quad \forall\, \varrho,\varsigma \in \mho$ then $(\mho, d_\perp)$ is an orthogonal metric space (for any metric $d_\perp$ on $\mho$) with every element in $\mho$ as an orthogonal element. Infact, in such a case the orthogonal metric space $(\mho, d_\perp)$ reduces to metric space $(\mho, d)$ then:

(I) With the above condition, Theorem 2.2.4 and Theorem 2.2.8 reduces to Theorem 2.1 and Theorem 2.2 respectively of Hussain & Salimi (2014).

(II) Theorem 3.8 of Gopal et al. (2016) can be deduced from Theorem 2.2.21 under specific condition as mentioned above along with Ω as an α-admissible map.

(III) Theorem 8, Theorem 12 of Chandok et al. (2016) are particular cases of Theorem 2.2.28 and Corollary 2.2.37, Theorem 2.2.31 and Corollary 2.2.40 respectively with respect to the above orthogonal metric space.

(IV) Theorem 2.1 of Hussain & Ahmad (2017) can be deduced from Corollary 2.2.45 along with specific condition as mentioned above.

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Chapter 3

Fixed Point Results in Relation Theoretic Metric Space

3.1 Introduction

Alam & Imdad (2015) introduced the idea of relation theoretic metric space, briefly written as $\mathcal{R}$-metric space (notation introduced by Khalehoghli et al. (2020)), wherein the given metric space is combined with an amorphous binary relation, $\mathcal{R}$. Since then, fixed point results for various maps in the relation theoretic metric have been studied (see Imdad et al. (2018), Prasad et al. (2020), Alam et al. (2021), Prasad (2021), Khan et al. (2022)).

Motivated by the work done in the literature on $\mathcal{R}$-metric space, in this chapter we first put forward the fixed point results using $\mathcal{F}$-weak expansive map followed by the fixed point results that are subjected to contraction conditions corresponding to the multivalued counterpart of $\mathcal{F}$-contraction, $\mathcal{F}$-weak contraction, almost $\mathcal{F}$-contraction and α-type $\mathcal{F}$-contraction in $\mathcal{R}$-metric space. Next, we discuss the solution of a non-homogeneous, non-linear Volterra integral along with its stability using the idea of Hyers-Ulam stability (Hyers (1941), Ulam (1960)). The results of this chapter have been presented in [3,4,5].

[3]Malhotra, A., and Kumar, D. (2022). Some fixed point results using F-weak expansive mapping in relation theoretic metric space. Journal of Physics: Conference Series, IOP Publishing, 2267(1), 012040.

[4]Malhotra, A., and Kumar, D. (2023). Fixed Point Results for Multivalued Mapping in R-Metric Space. Sahand Communications in Mathematical Analysis, 20(2), 109-121.

[5]Malhotra, A., and Kumar, D. (2023). Existence and Stability of Solution for a Nonlinear Volterra Integral Equation with binary relation via Fixed Point Results. (Communicated).

3.2 Generalized Expansive Maps and Fixed Point Results

Herein, we first introduce the notions of relational type $\mathcal{F}$-expansive map and relational type $\mathcal{F}$-weak expansive map in the setting of $\mathcal{R}$-metric space.

Definition 3.2.1. *For an $\mathcal{R}$-metric space $(\mho, d_{\mathcal{R}})$, $\phi : \mho \to \mho$ is c.t.b a* ***relational type $\mathcal{F}$-expansive map*** *for $\mathcal{F} \in \mathfrak{F}$, if $\exists\ \wp > 0$ s.t for each $\varrho, \varsigma \in \mho$ with $(\varrho, \varsigma) \in \mathcal{R}$, we have*

$$\mathcal{F}(d_{\mathcal{R}}(\phi\varrho, \phi\varsigma)) \geqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma)) + \wp.$$

Definition 3.2.2. *For an $\mathcal{R}$-metric space $(\mho, d_{\mathcal{R}})$, $\phi : \mho \to \mho$ is c.t.b a* ***relational type $\mathcal{F}$-weak expansive map*** *for $\mathcal{F} \in \mathfrak{F}$, if $\exists\ \wp > 0$ s.t for each $\varrho, \varsigma \in \mho$ with $(\varrho, \varsigma) \in \mathcal{R}$, we have*

$$\mathcal{F}(d_{\mathcal{R}}(\phi\varrho, \phi\varsigma)) \geqslant \mathcal{F}\Big(\mathcal{M}(\varrho, \varsigma)\Big) + \wp,$$

where $\mathcal{M}(\varrho, \varsigma) = \max\Bigg\{d_{\mathcal{R}}(\varrho, \varsigma), d_{\mathcal{R}}(\varrho, \phi\varrho), d_{\mathcal{R}}(\varsigma, \phi\varsigma), \dfrac{d_{\mathcal{R}}(\varrho, \phi\varsigma) + d_{\mathcal{R}}(\varsigma, \phi\varrho)}{2}\Bigg\}.$

Example 3.2.1. *Let $\mho = \{0, 1, 2, 3, 4, 5, 6\}$ equipped with metric $d_{\mathcal{R}}(\varrho, \varsigma) = |\varrho - \varsigma|$. Let relation $\mathcal{R} = \{(\varrho, \varsigma) : \varrho \neq \varsigma,\ \varrho.\varsigma = 0\ \varrho, \varsigma < 5\}$. Then, $(\mho, d_{\mathcal{R}})$ is an $\mathcal{R}$-metric space. Define $\phi : \mho \to \mho$ as $\phi(0) = 0$, $\phi(1) = 3$, $\phi(2) = 4$, $\phi(3) = 6$, $\phi(4) = 5$, $\phi(5) = 1$ and $\phi(6) = 2$. Then, ϕ is a relational type $\mathcal{F}$-weak expansive map but clearly ϕ is not $\mathcal{F}$-weak expansive map.*

Remark 3.2.2. *Every relational type $\mathcal{F}$-weak expansive map is a relational type $\mathcal{F}$-expansive map. Since, if ϕ is a relational type $\mathcal{F}$-weak expansive map, then for $\mathcal{F} \in \mathfrak{F}$, $\exists\ \wp > 0$ s.t for each ϱ, ς with $(\varrho, \varsigma) \in \mathcal{R}$, we have*

$$\begin{aligned}\mathcal{F}(d_{\mathcal{R}}(\phi\varrho, \phi\varsigma)) &\geqslant \max\Bigg\{d_{\mathcal{R}}(\varrho, \varsigma), d_{\mathcal{R}}(\varrho, \phi\varrho), d_{\mathcal{R}}(\varsigma, \phi\varsigma), \frac{d_{\mathcal{R}}(\varrho, \phi\varsigma) + d_{\mathcal{R}}(\varsigma, \phi\varrho)}{2}\Bigg\} + \wp \\ &\geqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma)) + \wp.\end{aligned}$$

Hence, ϕ is relational type $\mathcal{F}$-expansive map. Given Example 3.2.1 further verifies a relational type $\mathcal{F}$-weak expansive map which is also relational type $\mathcal{F}$-expansive map.

Theorem 3.2.3. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\phi : \mho \to \mho$ be a surjective map s.t:*

(I) ϕ is relational type $\mathcal{F}$-weak expansive map;

(II) ϕ is $\mathcal{R}$-preserving;

(III) $\exists$ some $\varrho_0 \in \mho$ s.t $(\varrho_0, \varsigma) \in \mathcal{R}\ \forall\ \varsigma \in \phi(\mho)$;

(IV) ϕ is $\mathcal{R}$-continuous.

Then, ϕ possesses a unique fixed point.

Proof. Define $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}'}$, a sequence in $\mho$ where $\varrho_1 = \phi\varrho_0,\ \varrho_2 = \phi\varrho_1,\ \ldots,\ \varrho_{\varpi+1} = \phi\varrho_\varpi$. On using condition *(III)*, we have $(\varrho_0, \phi\varrho_0) \in \mathcal{R}$, that is, $(\varrho_0, \varrho_1) \in \mathcal{R}$ and since ϕ is $\mathcal{R}$-preserving, so $(\phi\varrho_0, \phi\varrho_1) = (\varrho_1, \varrho_2) \in \mathcal{R}$. Proceeding in similar way, we obtain that $(\varrho_\varpi, \varrho_{\varpi+1}) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}'$. Next, for $\varrho_n, \varrho_{n+1} \in \mho$ and since ϕ is a surjective map so by Lemma 1.2.5, there is a right inverse ϕ^* of ϕ s.t $\phi^*\varrho_{\varpi+1} = \varrho_\varpi$. Since, ϕ is relational type $\mathcal{F}$-weak expansive map, therefore $\exists\ \wp > 0$ s.t

$$\begin{aligned}
\mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi+2}, \varrho_{\varpi+1})) &= \mathcal{F}(d_{\mathcal{R}}(\phi\varrho_{\varpi+1}, \phi\varrho_\varpi)) \geqslant \mathcal{F}(\mathcal{M}(\varrho_{\varpi+1}, \varrho_\varpi)) + \wp \\
&\geqslant \mathcal{F}(\mathcal{M}(\phi^*\varrho_{\varpi+2}, \phi^*\varrho_{\varpi+1})) + \wp \\
&= \mathcal{F}(\max\Big\{ d_{\mathcal{R}}(\phi^*\varrho_{\varpi+2}, \phi^*\varrho_{\varpi+1}), d_{\mathcal{R}}(\phi^*\varrho_{\varpi+2}, \varrho_{\varpi+2}), \\
&\quad d_{\mathcal{R}}(\phi^*\varrho_{\varpi+1}, \varrho_{\varpi+1}), \frac{d_{\mathcal{R}}(\phi^*\varrho_{\varpi+2}, \varrho_{\varpi+1}) + d_{\mathcal{R}}(\phi^*\varrho_{\varpi+1}, \varrho_{\varpi+2})}{2} \Big\} \\
&\quad + \wp \\
&\geqslant \mathcal{F}(d_{\mathcal{R}}(\phi^*\varrho_{\varpi+2}, \phi^*\varrho_{\varpi+1})) + \wp.
\end{aligned}$$

Thus, ϕ^* is a relational $\mathcal{F}$-contraction map on $\mho$, therefore, by Theorem 3.2 of Sawangsup et al. (2017), ϕ^* possesses a unique fixed point, that is, $\exists\ \zeta \in \mho$ s.t $\phi^*\zeta = \zeta$. Now, $\zeta = \phi\phi^*\zeta = \phi\zeta$, hence, ϕ possesses a unique fixed point. □

Corollary 3.2.4. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\phi : \mho \to \mho$ be a surjective map s.t:*

(I) ϕ is relational type $\mathcal{F}$-expansive map;

(II) ϕ is $\mathcal{R}$-preserving;

(III) $\exists$ some $\varrho_0 \in \mho$ s.t $(\varrho_0, \varsigma) \in \mathcal{R}\ \forall\ \varsigma \in \phi(\mho)$;

(IV) ϕ is $\mathcal{R}$-continuous.

Then, ϕ possesses a unique fixed point.

Proof. By Remark 3.2.2, every relational type $\mathcal{F}$-weak expansive map is relational type $\mathcal{F}$-expansive map so proof now follows from Theorem 3.2.3. $\square$

3.3 Generalized Multivalued Contraction Maps and Fixed Point Results

We now prove fixed point results for multivalued maps in an $\mathcal{R}$-metric space subject to generalized contractions. But before proceeding to the results, we first define relation between two subsets of an $\mathcal{R}$-metric space and $\mathcal{R}$-continuity for multivalued maps.

Definition 3.3.1. *For an $\mathcal{R}$-metric space $(\mho, d_{\mathcal{R}})$, two non-empty subsets $\hat{U}, \hat{V}$ of $\mho$ we say, $(\hat{U}, \hat{V}) \in \mathcal{R}$ if $(\rho, \sigma) \in \mathcal{R}$ for each $\rho \in \hat{U}$ and $\sigma \in \hat{V}$.*

Example 3.3.1. *Let $\mho = \mathbb{R}$ with usual metric and define $\mathcal{R} = \{(\varrho, \varsigma) \in \mho^2$ iff $\varrho.\varsigma < 0\}$. Then, for subsets $\hat{U} = (-\infty, 0)$ and $\hat{V} = (0, +\infty)$ of $\mho$, we have $(\hat{U}, \hat{V}) \in \mathcal{R}$.*

Definition 3.3.2. *For an $\mathcal{R}$-metric space $(\mho, d_{\mathcal{R}})$, a multivalued map $\phi : \mho \to \mathcal{K}(\mho)$ is c.t.b an $\mathcal{R}_{\mathcal{H}}$-**continuous** at $\varrho \in \mho$ if for any $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho$ with $d_{\mathcal{R}}(\varrho_\varpi, \varrho) \to 0$ as $\varpi \to +\infty$, we have $\mathcal{H}(\phi\varrho_\varpi, \phi\varrho) \to 0$ as $\varpi \to +\infty$. Also, ϕ is c.t.b $\mathcal{R}_{\mathcal{H}}$-**continuous on** $\mho$ if $\forall\ \varrho \in \mho$, $\mathcal{R}_{\mathcal{H}}$-continuous at ϱ.*

It should be noted that the above definition holds true if we consider a multivalued map $\phi : \mho \to \mathcal{CB}(\mho)$.

Theorem 3.3.2. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\phi : \mho \to \mathcal{K}(\mho)$ be a multivalued map s.t:*

(I) $\exists\ \varrho_0 \in \mho$ s.t $(\varrho_0, \varsigma) \in \mathcal{R}\ \forall\ \varsigma \in \phi\varrho_0$;

(II) For each $(\varrho, \varsigma) \in \mathcal{R}$, we have $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$;

(III) Either ϕ is $\mathcal{R}_{\mathcal{H}}$-continuous or for any $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ s.t $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$, we have $(\varrho_\varpi, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$;

(IV) If for some $\mathcal{F} \in \mathfrak{F}$, $\exists\ \wp > 0$ with $\mathcal{H}(\phi\varrho, \phi\varsigma) > 0$ s.t for every $(\varrho, \varsigma) \in \mathcal{R}$, we have

$$\wp + \mathcal{F}(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma)).$$

Then, ϕ possesses a fixed point.

Proof. Let $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ be a sequence in $\mho$ where $\varrho_{\varpi+1} \in \phi\varrho_\varpi \ \forall \ \varpi \in \mathbb{N}'$. By condition *(I)*, we obtain $(\varrho_0, \varrho_1) \in \mathcal{R}$. On using condition *(II)*, we have

$$(\phi\varrho_0, \phi\varrho_1) = (\varrho_1, \varrho_2) \in \mathcal{R}. \tag{3.1}$$

On repetitive use of condition *(II)* in (3.1), we get $(\varrho_\varpi, \varrho_{\varpi+1}) \in \mathcal{R}, \quad \forall \ \varpi \in \mathbb{N}'$. Thus, $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ is an $\mathcal{R}$-sequence in $\mho$. If $\varrho_1 \in \phi\varrho_1$ then we are done. Suppose $\varrho_1 \notin \phi\varrho_1$ and since $\phi\varrho_1$ is compact subset of $\mho$ then $d_\mathcal{R}(\varrho_1, \phi\varrho_1) > 0$. Now, $d_\mathcal{R}(\varrho_1, \phi\varrho_1) \leqslant \mathcal{H}(\phi\varrho_0, \phi\varrho_1)$ thus by $(\mathcal{F}_1)$, we have

$$\begin{aligned} \mathcal{F}(d_\mathcal{R}(\varrho_1, \phi\varrho_1)) &\leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \\ &\leqslant \mathcal{F}(d_\mathcal{R}(\varrho_0, \varrho_1)) - \wp. \end{aligned}$$

If $\varrho_k \in \phi\varrho_k$ for some $k \in \mathbb{N}'$ then we are done. Suppose $\varrho_\varpi \notin \phi\varrho_\varpi \ \forall \ \varpi \in \mathbb{N}'$ and since $\phi\varrho_\varpi$ is compact subset of $\mho$ then, $d_\mathcal{R}(\varrho_\varpi, \phi\varrho_\varpi) > 0 \ \forall \ \varpi \in \mathbb{N}'$.

$$\begin{aligned} \text{As,} \quad d_\mathcal{R}(\varrho_\varpi, \phi\varrho_\varpi) \leqslant d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1}) &\leqslant \mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi), \\ \text{that is,} \quad \mathcal{F}(d_\mathcal{R}(\varrho_\varpi, \phi\varrho_\varpi)) \leqslant \mathcal{F}(d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1})) &\leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \\ &\leqslant \mathcal{F}(d_\mathcal{R}(\varrho_{\varpi-1}, \varrho_\varpi)) - \wp \\ &< \mathcal{F}(d_\mathcal{R}(\varrho_{\varpi-1}, \varrho_\varpi)). \quad (3.2) \end{aligned}$$

Thus, $\{\zeta_\varpi = d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1})\}_{\varpi\in\mathbb{N}'}$ is a decreasing sequence of non-negative real number. Let $\lim_{\varpi\to+\infty} \zeta_\varpi = \zeta \geqslant 0$. Next, by (3.2) we have

$$\mathcal{F}(\zeta_\varpi) \leqslant \mathcal{F}(\zeta_{\varpi-1}) - \wp \leqslant \mathcal{F}(\zeta_{\varpi-2}) - 2\wp \leqslant \cdots \leqslant \mathcal{F}(\zeta_0) - \varpi\wp. \tag{3.3}$$

On letting $\varpi \to +\infty$ in (3.3) we get, $\lim_{\varpi\to+\infty} \zeta_\varpi = 0$. By $(\mathcal{F}_3)$, $\exists \ \gamma \in (0,1)$ s.t $\lim_{\varpi\to+\infty} \zeta_\varpi^\gamma \mathcal{F}(\zeta_\varpi) = 0$. Using (3.3), we have

$$\zeta_\varpi^\gamma \mathcal{F}(\zeta_\varpi) - \zeta_\varpi^\gamma \mathcal{F}(\zeta_0) \leqslant -\varpi\zeta_\varpi^\gamma \wp. \tag{3.4}$$

Taking limit as $\varpi \to +\infty$ in (3.4), we get $\lim_{\varpi\to+\infty} \varpi\zeta_\varpi^\gamma = 0$. Thus, $\exists \ \varpi_0 \in \mathbb{N}$ with $\varpi \geqslant \varpi_0$, $\zeta_\varpi \leqslant \dfrac{1}{\varpi^{1/\gamma}}$. Consider $\varpi^*, \varpi \in \mathbb{N}$, where $\varpi^* > \varpi > \varpi_0$ and

$$\begin{aligned} d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi^*}) &\leqslant d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1}) + d_\mathcal{R}(\varrho_{\varpi+1}, \varrho_{\varpi+2}) + \cdots + d_\mathcal{R}(\varrho_{\varpi^*-1}, \varrho_{\varpi^*}) \\ &= \zeta_\varpi + \zeta_{\varpi+1} + \cdots + \zeta_{\varpi^*-1} \end{aligned}$$

$$= \sum_{j=\varpi}^{\varpi^*-1} \zeta_j \leqslant \sum_{j=\varpi}^{+\infty} \zeta_j \leqslant \sum_{j=\varpi}^{+\infty} \frac{1}{j^{1/\gamma}}. \tag{3.5}$$

Since the series $\sum_{j=\varpi}^{+\infty} \frac{1}{j^{1/\gamma}}$ is convergent and on letting $\varpi \to +\infty$ in (3.5), we obtain that $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ is an $\mathcal{R}$-Cauchy sequence and using $\mathcal{R}$-completeness of $\mho$, $\exists\ \varrho \in \mho$ s.t $\lim_{\varpi\to+\infty} \varrho_\varpi = \varrho$. We now claim that $\varrho \in \phi\varrho$.

Case (i): Let ϕ be a $\mathcal{R}_{\mathcal{H}}$-continuous map. Since $\varrho_{\varpi+1} \in \phi\varrho_\varpi$, we have

$$d_{\mathcal{R}}(\varrho_{\varpi+1}, \phi\varrho) \leqslant \mathcal{H}(\phi\varrho_\varpi, \phi\varrho). \tag{3.6}$$

Taking limit as $\varpi \to +\infty$ in (3.6) and using $\mathcal{R}$-continuity of ϕ, we get

$$d_{\mathcal{R}}(\varrho, \phi\varrho) = \lim_{\varpi\to+\infty} d_{\mathcal{R}}(\varrho_{\varpi+1}, \phi\varrho) \leqslant \lim_{\varpi\to+\infty} \mathcal{H}(\phi\varrho_\varpi, \phi\varrho) = 0.$$

So, we conclude that $\varrho \in \phi\varrho$.

Case (ii): We have an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ s.t $\varrho_{\varpi+1} \in \phi\varrho_\varpi\ \forall\ \varpi \in \mathbb{N}'$ and $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ then, $(\varrho_\varpi, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$. On the contrary, suppose that $\varrho \notin \phi\varrho$ then $\exists\ \varpi' \in \mathbb{N}$ with $\varrho \notin \{\varrho_\varpi\}$ for every $\varpi > \varpi'$ which further gives $\mathcal{H}(\phi\varrho, \phi\varrho_\varpi) > 0$ and also by given condition, we have $(\varrho_\varpi, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}'$.

$$\mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi+1}, \phi\varrho)) \leqslant \wp + \mathcal{F}(\mathcal{H}(\phi\varrho_\varpi, \phi\varrho)) < \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho)). \tag{3.7}$$

On letting $\varpi \to +\infty$ in (3.7), we obtain $d_{\mathcal{R}}(\varrho, \phi\varrho) = 0$ which is not true. Hence, $\varrho \in \phi\varrho$. □

Theorem 3.3.3. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\phi : \mho \to \mathcal{CB}(\mho)$ be a multivalued map s.t:*

(I) $\exists\ \varrho_0 \in \mho$ *s.t* $(\varrho_0, \varsigma) \in \mathcal{R}\ \forall\ \varsigma \in \phi\varrho_0$*;*

(II) For each $(\varrho, \varsigma) \in \mathcal{R}$*, we have* $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$*;*

(III) Either ϕ *is* $\mathcal{R}_{\mathcal{H}}$*-continuous or for any* $\mathcal{R}$*-sequence* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *s.t* $\varrho_\varpi \to \varrho$ *as* $\varpi \to +\infty$*, we have* $(\varrho_\varpi, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$*;*

(IV) If for some $\mathcal{F} \in \mathfrak{F}'$*,* $\exists\ \wp > 0$ *with* $\mathcal{H}(\phi\varrho, \phi\varsigma) > 0$ *s.t for every* $(\varrho, \varsigma) \in \mathcal{R}$*, we have*

$$\wp + \mathcal{F}(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma)).$$

Then, ϕ *possesses a fixed point.*

Proof. By the working done in Theorem 3.3.2, we obtain an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ in $\mho$, where $\varrho_{\varpi+1} \in \phi\varrho_\varpi \ \forall \ \varpi \in \mathbb{N}'$. If $\varrho_1 \in \phi\varrho_1$ then, we are done. Suppose $\varrho_1 \notin \phi\varrho_1$ and since $\phi\varrho_1$ is closed subset of $\mho$ then, $d_{\mathcal{R}}(\varrho_1, \phi\varrho_1) > 0$. Now, $d_{\mathcal{R}}(\varrho_1, \phi\varrho_1) \leqslant \mathcal{H}(\phi\varrho_0, \phi\varrho_1)$ thus by $(\mathcal{F}_1)$, we have

$$\mathcal{F}(d_{\mathcal{R}}(\varrho_1, \phi\varrho_1)) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp. \tag{3.8}$$

Thereby using $(\mathcal{F}_4)$ property of $\mathcal{F}$, we get

$$\mathcal{F}(d_{\mathcal{R}}(\varrho_1, \phi\varrho_1)) = \mathcal{F}(\inf_{\varsigma\in\phi\varrho_1} d_{\mathcal{R}}(\varrho_1, \varsigma)) = \inf_{\varsigma\in\phi\varrho_1} \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varsigma)), \tag{3.9}$$

on using (3.8) in (3.9) and the fact that $\varrho_2 \in \phi\varrho_1$, we observe that

$$\inf_{\varsigma\in\phi\varrho_1} \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp.$$

So from above equation, we have $\mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp$. If $\varrho_2 \in \phi\varrho_2$ then we are done, else for $\varrho_2 \notin \phi\varrho_2$ we have $\varrho_3 \in \phi\varrho_2$ so that

$$\mathcal{F}(d_{\mathcal{R}}(\varrho_2, \varrho_3)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) - \wp.$$

Continuing in a similar manner, we obtain $\mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)) - \wp$ and thus $d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) < d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)$, that is, $\{d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})\}_{\varpi\in\mathbb{N}'}$ is a decreasing sequence of non-negative real numbers. Now, by the working of Theorem 3.3.2 we conclude that ϕ possesses a fixed point. □

Theorem 3.3.4. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\phi : \mho \to \mathcal{K}(\mho)$ be a multivalued map s.t:*

(I) $\exists \ \varrho_0 \in \mho$ *with* $(\varrho_0, \varsigma) \in \mathcal{R} \ \forall \ \varsigma \in \phi\varrho_0$*;*

(II) *For each* $(\varrho, \varsigma) \in \mathcal{R}$*, we have* $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$*;*

(III) *Either* ϕ *is* $\mathcal{R}_{\mathcal{H}}$*-continuous or for any* $\mathcal{R}$*-sequence* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$*, where* $\varrho_\varpi \to \varrho$ *as* $\varpi \to +\infty$*, we have* $(\varrho_\varpi, \varrho) \in \mathcal{R} \ \forall \ \varpi \in \mathbb{N}$*;*

(IV) *If for some* $\mathcal{F} \in \mathfrak{F}$*,* $\exists \ \wp > 0$ *with* $\mathcal{H}(\phi\varrho, \phi\varsigma) > 0$ *so that for every* $(\varrho, \varsigma) \in \mathcal{R}$*, we have*

$$\wp + \mathcal{F}(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}\left(\max\left\{d_{\mathcal{R}}(\varrho, \varsigma), D(\varrho, \phi\varrho), D(\varsigma, \phi\varsigma), \frac{D(\varrho, \phi\varsigma) + D(\varsigma, \phi\varrho)}{2}\right\}\right).$$

Then, ϕ *possesses a fixed point.*

Proof. By the working done in Theorem 3.3.2, an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ is obtained, where $\varrho_{\varpi+1} \in \phi\varrho_\varpi \ \forall\ \varpi \in \mathbb{N}'$. Suppose $\varrho_1 \notin \phi\varrho_1$, then we have

$$0 < d_{\mathcal{R}}(\varrho_1, \phi\varrho_1) \ \leqslant\ \mathcal{H}(\phi\varrho_0, \phi\varrho_1).$$

Further, using condition *(IV)*, we have

$$\begin{aligned}\wp + \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \ &\leqslant\ \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_0, \varrho_1), D(\varrho_0, \phi\varrho_0), D(\varrho_1, \phi\varrho_1), \\ &\qquad \frac{D(\varrho_0, \phi\varrho_1) + D(\varrho_1, \phi\varrho_0)}{2}\Bigg\}\Bigg). \end{aligned} \tag{3.10}$$

Next, the following observations can be easily made for a multivalued map and in addition using Lemma 1.2.9, we have

$$\begin{aligned} D(\varrho_0, \phi\varrho_0) \ &\leqslant\ d_{\mathcal{R}}(\varrho_0, \varrho_1), \\ D(\varrho_1, \phi\varrho_1) \ &\leqslant\ \mathcal{H}(\phi\varrho_0, \phi\varrho_1), \\ D(\varrho_1, \phi\varrho_0) \ &=\ 0, \\ \text{and,}\quad D(\varrho_0, \phi\varrho_1) \ &\leqslant\ d_{\mathcal{R}}(\varrho_0, \varrho_1) + \mathcal{H}(\phi\varrho_0, \phi\varrho_1). \end{aligned}$$

Using above in (3.10), we get

$$\begin{aligned}\wp + \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \ &\leqslant\ \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_0, \varrho_1), \mathcal{H}(\phi\varrho_0, \phi\varrho_1), \frac{D(\varrho_0, \phi\varrho_1)}{2}\Bigg\}\Bigg) \\ &\leqslant\ \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_0, \varrho_1), \mathcal{H}(\phi\varrho_0, \phi\varrho_1), \\ &\qquad \frac{d_{\mathcal{R}}(\varrho_0, \varrho_1) + \mathcal{H}(\phi\varrho_0, \phi\varrho_1)}{2}\Bigg\}\Bigg). \end{aligned} \tag{3.11}$$

If $d_{\mathcal{R}}(\varrho_0, \varrho_1) < \mathcal{H}(\phi\varrho_0, \phi\varrho_1)$, then by (3.11) we obtain

$$\wp + \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)),$$

which is a contradiction. Thus, we have $\mathcal{H}(\phi\varrho_0, \phi\varrho_1) < d_{\mathcal{R}}(\varrho_0, \varrho_1)$ and by (3.11), we get $\wp + \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1))$. Further, suppose $\varrho_\varpi \notin \phi\varrho_\varpi \ \forall\ \varpi \in \mathbb{N}'$,

$$\begin{aligned}\wp + \mathcal{F}(\mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1})) \ &\leqslant\ \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}), D(\varrho_\varpi, \phi\varrho_\varpi), D(\varrho_{\varpi+1}, \phi\varrho_{\varpi+1}), \\ &\qquad \frac{D(\varrho_\varpi, \phi\varrho_{\varpi+1}) + D(\varrho_{\varpi+1}, \phi\varrho_\varpi)}{2}\Bigg\}\Bigg). \end{aligned} \tag{3.12}$$

Again, we have the following observations:

$$\begin{aligned} D(\varrho_\varpi, \phi\varrho_\varpi) &\leqslant d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1}), \\ D(\varrho_{\varpi+1}, \phi\varrho_{\varpi+1}) &\leqslant \mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1}), \\ D(\varrho_{\varpi+1}, \phi\varrho_\varpi) &= 0, \\ \text{and,} \quad D(\varrho_\varpi, \phi\varrho_{\varpi+1}) &\leqslant d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1}) + \mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1}). \end{aligned}$$

Using above in (3.12), we get

$$\begin{aligned} \wp + \mathcal{F}(\mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1})) \leqslant \ & \mathcal{F}\Bigg(\max\Bigg\{d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1}), \mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1}), \\ & \frac{d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1}) + \mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1})}{2}\Bigg\}\Bigg). \end{aligned} \tag{3.13}$$

If $d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1}) \leqslant \mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1})$, then by (3.13) we have

$$\wp + \mathcal{F}(\mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1})) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1})),$$

which is a contradiction. Thus, we have $\mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1}) < d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1})$ and by (3.13), we get

$$\wp + \mathcal{F}(\mathcal{H}(\phi\varrho_\varpi, \phi\varrho_{\varpi+1})) \leqslant \mathcal{F}(d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi+1})).$$

Now, by the working of Theorem 3.3.2 we conclude that ϕ possesses a fixed point.

□

Theorem 3.3.5. *For an $\mathcal{R}$-complete metric space $(\mho, d_\mathcal{R})$, let $\phi : \mho \to \mathcal{CB}(\mho)$ be a multivalued map s.t:*

(I) $\exists\ \varrho_0 \in \mho$ *s.t* $(\varrho_0, \varsigma) \in \mathcal{R}\ \forall\ \varsigma \in \phi\varrho_0$*;*

(II) For each $(\varrho, \varsigma) \in \mathcal{R}$*, we have* $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$*;*

(III) Either ϕ is $\mathcal{R}_\mathcal{H}$-continuous or for any $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ s.t $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$, we have $(\varrho_\varpi, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$;

(IV) If for some $\mathcal{F} \in \mathfrak{F}'$, $\exists\ \wp > 0$ with $\mathcal{H}(\phi\varrho, \phi\varsigma) > 0$ s.t for every $(\varrho, \varsigma) \in \mathcal{R}$, we have

$$\wp + \mathcal{F}(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}\Bigg(\max\Bigg\{d_\mathcal{R}(\varrho, \varsigma), D(\varrho, \phi\varrho), D(\varsigma, \phi\varsigma), \frac{D(\varrho, \phi\varsigma) + D(\varsigma, \phi\varrho)}{2}\Bigg\}\Bigg).$$

Then, ϕ possesses a fixed point.

Proof. Using the working done in Theorem 3.3.4, we obtain an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ where $\varrho_{\varpi+1} \in \phi\varrho_\varpi\ \forall\ \varpi \in \mathbb{N}'$. Suppose $\varrho_1 \notin \phi\varrho_1$, then we have

$$\begin{aligned}\wp + \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) &\leqslant \mathcal{F}\Big(\max\Big\{d_{\mathcal{R}}(\varrho_0, \varrho_1), D(\varrho_0, \phi\varrho_0), D(\varrho_1, \phi\varrho_1), \\ &\qquad \frac{D(\varrho_0, \phi\varrho_1) + D(\varrho_1, \phi\varrho_0)}{2}\Big\}\Big) \\ &\leqslant d_{\mathcal{R}}(\varrho_0, \varrho_1)).\end{aligned}$$

Since, $\phi\varrho_1$ is closed and $\varrho_2 \in \phi\varrho_1$, then by working done in Theorem 3.3.3, we obtain

$$\begin{aligned}\inf_{\varsigma\in\phi\varrho_1} \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) &\leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp \\ \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp.\end{aligned}$$

The existence of fixed point now follows on the lines of Theorem 3.3.3. □

Example 3.3.6. *Consider* $\mho = \{0, 1, 2, 3, 4\}$ *and* $d(\varrho, \varsigma) = |\varrho - \varsigma|$. *Define* $\phi : \mho \to \mathcal{CB}(\mho)$ *as:*

$$\phi\varrho = \begin{cases} \{0, 1\} & for\ \varrho = 3; \\ \{0\} & otherwise. \end{cases}$$

Let $\mathcal{R} = \{(0, 0), (0, 1), (0, 3)\}$ *and* $\mathcal{F}(\mu) = \ln(\mu)$. *We have* $(\mho, d_{\mathcal{R}})$ *is an* $\mathcal{R}$-*complete metric space. Also,* ϕ *is* $\mathcal{R}_{\mathcal{H}}$-*continuous and satisfies condition* (II) *of Theorem 3.3.5. Next, for* $(\varrho, \varsigma) \in \mathcal{R}$ *with* $\mathcal{H}(\phi\varrho, \phi\varsigma)) > 0$, *we have* $\varrho = 0$, $\varsigma = 3$ *and*

$$\wp + \mathcal{F}(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}\Big(\max\Big\{d_{\mathcal{R}}(\varrho, \varsigma), D(\varrho, \phi\varrho), D(\varsigma, \phi\varsigma), \frac{D(\varrho, \phi\varsigma) + D(\varsigma, \phi\varrho)}{2}\Big\}\Big).$$

Thus, the given $\mathcal{R}$-*metric space satisfies all the condition of Theorem 3.3.5 and hence,* ϕ *has a fixed point which is* $\varrho = 0$.

Remark 3.3.7. *It should be noted that the given example in the absence of relation* $\mathcal{R}$ *does not satisfy the multivalued contraction condition given in Altun et al. (2015) and Acar et al. (2014). Also noted that the given relation* $\mathcal{R}$ *is not orthogonal thus, the results of Sharma & Chandok (2020) cannot be applied.*

Example 3.3.8. *Let* $\mho = \mho_1 \cup \mho_2$, *where* $\mho_1 = [0, 1/2]$ *and* $\mho_2 = (1/2, 1)$. *Define* *a metric* $d : \mho \times \mho \to [0, +\infty)$ *as:*

$$d(\varrho, \varsigma) = \begin{cases} 0.01 + \mid \varrho - \varsigma \mid & if \ \varrho \neq \varsigma; \\ 0 & if \ \varrho = \varsigma. \end{cases}$$

Also, define $\mathcal{R} = \{(\varrho, \varsigma) \in \mho^2 \ s.t \ \varrho.\varsigma \in \{\varrho, \varsigma\}\}$. *Then,* $(\mho, d_{\mathcal{R}})$ *is an* $\mathcal{R}$*-metric space. Let a multivalued map* $\phi : \mho \to \mathcal{CB}(\mho)$ *be defined as:*

$$\phi\varrho = \begin{cases} \{0\} & if \ \varrho \in \mho_1; \\ \{0, 1/3\} & if \ \varrho \in \mho_2. \end{cases}$$

Let $\mathcal{F}(\mu) = \ln(\mu)$*, then we have* $(\mho, d_{\mathcal{R}})$ *is an* $\mathcal{R}$*-complete metric space and in addition, the following conditions hold:*

(i) *For* $0 \in \mho$, $(0, \phi 0) \in \mathcal{R}$.

(ii) *For any* $(\varrho, \varsigma) \in \mathcal{R}$, $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$.

(iii) *For any* $\mathcal{R}$*-sequence* $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ *with* $\varrho_\varpi \to \varrho$ *as* $\varpi \to +\infty$ *then* $\varrho = 0$ *and thus* $(\varrho_\varpi, \varrho) \in \mathcal{R} \ \forall \ \varpi \in \mathbb{N}$.

(iv) *Let* $(\varrho, \varsigma) \in \mathcal{R}$ *then* $\varrho = 0$ *or/and* $\varsigma = 0$. *Consider* $\varrho = 0$ *(the case for* $\varsigma = 0$ *follows similarly), then we have the following cases:*

Case (i): *Let* $\varsigma \in \mho_1$ *then*

$$\begin{aligned} &\wp + \ln(\mathcal{H}(\phi\varrho, \phi\varsigma)) = \wp + \ln(\mathcal{H}(0, 0)) \to -\infty, \\ &and, \quad \ln(d_{\mathcal{R}}(\varrho, \varsigma)) = \ln(d_{\mathcal{R}}(0, \varsigma)) \geqslant -\infty. \end{aligned}$$

Thus, contraction condition

$$\wp + \ln(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \ln(d_{\mathcal{R}}(\varrho, \varsigma)),$$

holds for any finite $\wp > 0$.

Case (ii): *Let* $\varsigma \in \mho_2$ *then,*

$$\begin{aligned} &\wp + \ln(\mathcal{H}(\phi\varrho, \phi\varsigma)) = \wp + \ln(\mathcal{H}(\{0\}, \{0, 1/3\})) = \wp + \ln(0.3433), \\ &and, \ \ln(d_{\mathcal{R}}(\varrho, \varsigma)) = \ln(d_{\mathcal{R}}(0, \varsigma)) = \ln(0.01 + \varsigma) > \ln(0.51). \end{aligned}$$

Thus, contraction condition

$$\wp + \ln(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \ln(d_{\mathcal{R}}(\varrho, \varsigma)),$$

holds for $0 < \wp < 0.396$.

Hence, from the above example, we conclude that a multivalued map ϕ satisfies all condition of Theorem 3.3.3 and thus possesses a fixed point which is $\varrho = 0$.

Remark 3.3.9. *However, it should be noted here that since the space $(\mho, d_{\mathcal{R}})$ is an incomplete metric space thus results of Wardowski (2012), Wardowski & Dung (2014), Acar et al. (2014), Altun et al. (2015) are not applicable.*

Theorem 3.3.10. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\phi : \mho \to \mathcal{CB}(\mho)$ be a multivalued map s.t:*

(I) $\exists\ \varrho_0 \in \mho$ *with* $(\varrho_0, \phi\varrho_0) \in \mathcal{R}$;

(II) *For each* $(\varrho, \varsigma) \in \mathcal{R}$ *implies* $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$;

(III) *Either ϕ is $\mathcal{R}_{\mathcal{H}}$-continuous on $\mho$ or there is an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$, where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ implies $(\varrho_\varpi, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$;*

(IV) *If for some $\mathcal{F} \in \mathfrak{F}'$, there are two constants $\wp > 0$ and $\kappa \geqslant 0$ s.t for every $(\varrho, \varsigma) \in \mathcal{R}$ with $\mathcal{H}(\phi\varrho, \phi\varsigma) > 0$, we have*

$$\wp + F(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma) + \kappa\mathcal{D}(\varsigma, \phi\varrho)).$$

Then, ϕ possesses a fixed point.

Proof. On defining a sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ where $\varrho_{\varpi+1} \in \phi\varrho_\varpi\ \forall\ \varpi \in \mathbb{N}'$, we have $(\varrho_0, \varrho_1) \in \mathcal{R}$ by using condition *(I)*. Further, on using condition *(II)*, we obtain $(\phi\varrho_0, \phi\varrho_1) \in \mathcal{R}$, that is, $(\varrho_1, \varrho_2) \in \mathcal{R}$. The repetitive use of condition *(II)* yields that, $(\varrho_\varpi, \varrho_{\varpi+1}) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}'$. Thus, $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ is an $\mathcal{R}$-sequence in $\mho$. Next, if $\varrho_1 \in \phi\varrho_1$ then we are done. Suppose $\varrho_1 \notin \phi\varrho_1$ and since $\phi\varrho_1$ is closed subset of $\mho$ implies $\mathcal{D}(\varrho_1, \phi\varrho_1) > 0$. Now, $\mathcal{D}(\varrho_1, \phi\varrho_1) \leqslant \mathcal{H}(\phi\varrho_0, \phi\varrho_1)$ thus by $(\mathcal{F}_1)$, we have

$$\begin{aligned} \mathcal{F}(\mathcal{D}(\varrho_1, \phi\varrho_1)) &\leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1) + \kappa\mathcal{D}(\varrho_1, \phi\varrho_0)) - \wp \\ &= \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp. \end{aligned} \tag{3.14}$$

Thereby using $(\mathcal{F}_4)$ property of $\mathcal{F}$, we obtain

$$\mathcal{F}(\mathcal{D}(\varrho_1, \phi\varrho_1)) = \mathcal{F}\Big(\inf_{\varsigma\in\phi\varrho_1} d_{\mathcal{R}}(\varrho_1, \varsigma)\Big) = \inf_{\varsigma\in\phi\varrho_1} \mathcal{F}\Big(d_{\mathcal{R}}(\varrho_1, \varsigma)\Big). \tag{3.15}$$

On using (3.14) in (3.15) and as $\varrho_2 \in \phi\varrho_1$, we obtain that

$$\begin{aligned}\mathcal{F}(\mathcal{D}(\varrho_1, \phi\varrho_1)) = \inf_{\varsigma\in\phi\varrho_1} \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varsigma)) &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) \\ &\leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \\ &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp \\ &< \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)).\end{aligned} \tag{3.16}$$

So from (3.16), we have $d_{\mathcal{R}}(\varrho_1, \varrho_2) \leqslant d_{\mathcal{R}}(\varrho_0, \varrho_1)$. Again, if $\varrho_2 \in \phi\varrho_2$ then we are done, else suppose $\varrho_2 \notin \phi\varrho_2$, we obtain $d_{\mathcal{R}}(\varrho_2, \varrho_3) \leqslant d_{\mathcal{R}}(\varrho_1, \varrho_2) - \wp$. Continuing in a similar manner suppose $\varrho_\varpi \notin \phi\varrho_\varpi\ \forall\ \varpi \in \mathbb{N}_0$, we obtain $d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) < d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)$, that is, $\{d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})\}_{\varpi\in\mathbb{N}_0}$ is a decreasing sequence of non-negative real numbers and as

$$\begin{aligned}\mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)) - \wp \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-2}, \varrho_{\varpi-1})) - 2\wp \\ &\leqslant \cdots \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \varpi\wp.\end{aligned} \tag{3.17}$$

On letting $\varpi \to +\infty$ in (3.17), we get $\lim_{\varpi\to+\infty} d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) = 0$. By $(\mathcal{F}_3)$, $\exists\ 0 < \gamma < 1$ s.t

$$\lim_{\varpi\to+\infty}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}))^\gamma \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) = 0. \tag{3.18}$$

Then, from (3.17), we have

$$\begin{aligned}&(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}))^\gamma \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) - (d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}))^\gamma \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) \\ &\leqslant -\varpi\wp(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}))^\gamma.\end{aligned} \tag{3.19}$$

Taking limit as $\varpi \to +\infty$ in (3.19) and using (3.18), we have

$$\lim_{\varpi\to+\infty} \varpi(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}))^\gamma = 0.$$

Thus, $\exists\ \varpi_1 \in \mathbb{N}_0$ s.t for $\varpi \in \mathbb{N}_0$ with $\varpi \geqslant \varpi_1$, we obtain

$$d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) \leqslant \frac{1}{\varpi^{1/\gamma}}. \tag{3.20}$$

Next, we show that $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}_0}$ is an $\mathcal{R}$-Cauchy sequence. Consider $\varpi, \varpi^* \in \mathbb{N}_0$, where $\varpi^* > \varpi \geqslant \varpi_1$. By (3.20) and triangle inequality, we obtain

$$\begin{aligned} d_{\mathcal{R}}(\varrho_{\varpi^*}, \varrho_{\varpi}) &\leqslant d_{\mathcal{R}}(\varrho_{\varpi}, \varrho_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_{\varpi+2}) + \cdots + d_{\mathcal{R}}(\varrho_{\varpi^*-1}, \varrho_{\varpi^*}) \\ &\leqslant \sum_{i=\varpi}^{+\infty} d_{\mathcal{R}}(\varrho_i, \varrho_{i+1}) \leqslant \sum_{i=\varpi}^{+\infty} \frac{1}{i^{1/\gamma}}. \end{aligned} \tag{3.21}$$

Using the convergence of the series $\sum_{i=\varpi}^{+\infty} \frac{1}{i^{1/\gamma}}$ in (3.21), we get that $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}_0}$ is an $\mathcal{R}$-Cauchy sequence and using $\mathcal{R}$-completeness of $\mho$, $\exists\ \varrho^* \in \mho$ s.t $\lim_{\varpi \to +\infty} \varrho_{\varpi} = \varrho^*$. We now claim that $\varrho^* \in \phi\varrho^*$.

Case (i): Let ϕ be an $\mathcal{R}_{\mathcal{H}}$-continuous map. Since $\varrho_{\varpi+1} \in \phi\varrho_{\varpi}$, we have

$$\mathcal{D}(\varrho_{\varpi+1}, \phi\varrho^*) \leqslant \mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*). \tag{3.22}$$

Taking limit as $\varpi \to +\infty$ in (3.22) and by $\mathcal{R}$-continuity of ϕ, we get

$$\mathcal{D}(\varrho^*, \phi\varrho^*) = \lim_{\varpi \to +\infty} \mathcal{D}(\varrho_{\varpi+1}, \phi\varrho^*) \ \leqslant \ \lim_{\varpi \to +\infty} \mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*) = 0.$$

Thus, $\varrho^* \in \phi\varrho^*$.

Case (ii): Let there be an $\mathcal{R}$-sequence $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}'}$ where $\varrho_{\varpi} \to \varrho^*$ as $\varpi \to +\infty$ implies $(\varrho_{\varpi}, \varrho^*) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}'$. Instead, let $\varrho^* \notin \phi\varrho^*$ then $\exists\ \varpi' \in \mathbb{N}'$ s.t $\varrho^* \notin \{\varrho_{\varpi}\}\ \forall\ \varpi > \varpi'$ implies $\mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*) > 0$ and also by given condition, we have $(\varrho_{\varpi}, \varrho^*) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}'$.

$$\mathcal{F}(\mathcal{D}(\varrho_{\varpi+1}, \phi\varrho^*)) \leqslant \wp + \mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*)) < \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi}, \varrho^*)). \tag{3.23}$$

On letting $\varpi \to +\infty$ in (3.23), we obtain $\mathcal{D}(\varrho^*, \phi\varrho^*) = 0$ which is not true. Hence, $\varrho^* \in \phi\varrho^*$. □

Corollary 3.3.11. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, define a multivalued map $\phi : \mho \to \mathcal{K}(\mho)$ s.t:*

(I) $\exists\ \varrho_0 \in \mho$, where $(\varrho_0, \phi\varrho_0) \in \mathcal{R}$;

(II) For every $(\varrho, \varsigma) \in \mathcal{R}$ implies $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$;

(III) Either ϕ is $\mathcal{R}_{\mathcal{H}}$-continuous on $\mho$ or there is an $\mathcal{R}$-sequence $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$, where $\varrho_{\varpi} \to \varrho$ as $\varpi \to +\infty$ implies $(\varrho_{\varpi}, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$;

(IV) If for some $\mathcal{F} \in \mathfrak{F}$, there are two constants $\wp > 0$ and $\kappa \geqslant 0$ with $\mathcal{H}(\phi\varrho, \phi\varsigma) >$

0 *s.t for each* $(\varrho, \varsigma) \in \mathcal{R}$*, we have*

$$\wp + \mathcal{F}(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma) + \kappa\mathcal{D}(\varsigma, \phi\varrho)).$$

Then, ϕ *possesses a fixed point.*

Proof. Working on the lines of Theorem 3.3.10, we obtain an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ in $\mho$ where $\varrho_{\varpi+1} \in \phi\varrho_\varpi \ \forall \ \varpi \in \mathbb{N}'$. Suppose $\varrho_1 \notin \phi\varrho_1$ and as $\phi\varrho_1$ is compact, so $\exists$ some $\varrho_2 \in \phi\varrho_1$, with

$$\begin{aligned} \mathcal{D}(\varrho_1, \phi\varrho_1) = d_{\mathcal{R}}(\varrho_1, \varrho_2) &\leqslant \mathcal{H}(\phi\varrho_0, \phi\varrho_1), \\ \text{that is,} \quad \mathcal{F}(\mathcal{D}(\varrho_1, \phi\varrho_1)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) &\leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \\ &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1) + \kappa\mathcal{D}(\varrho_1, \phi\varrho_0)) - \wp \\ &= \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp. \end{aligned} \tag{3.24}$$

So from (3.24), we have $d_{\mathcal{R}}(\varrho_1, \varrho_2) \leqslant d_{\mathcal{R}}(\varrho_0, \varrho_1)$. Again, if $\varrho_2 \in \phi\varrho_2$ then we are done, else for $\varrho_2 \notin \phi\varrho_2$, we have $d_{\mathcal{R}}(\varrho_2, \varrho_3) \leqslant d_{\mathcal{R}}(\varrho_1, \varrho_2)$. Continuing in a similar manner, we obtain $d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) < d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)$, that is, $\{d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})\}_{\varpi\in\mathbb{N}_0}$ is a decreasing sequence of non-negative real numbers. The proof now follows from Theorem 3.3.10. □

Theorem 3.3.12. *For an* $\mathcal{R}$*-complete metric space* $(\mho, d_{\mathcal{R}})$*, let* $\alpha : \mho \times \mho \to [0, +\infty)$ *and* $\phi : \mho \to \mathcal{CB}(\mho)$ *be a multivalued* α*-admissible map s.t:*

(I) $\exists \ \varrho_0 \in \mho$*, where* $(\varrho_0, \phi\varrho_0) \in \mathcal{R}$ *and* $\alpha(\varrho_0, \phi\varrho_0) \geqslant 1$*;*

(II) For every $(\varrho, \varsigma) \in \mathcal{R}$ *implies* $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$*;*

(III) Either ϕ *is* $\mathcal{R}_{\mathcal{H}}$*-continuous on* $\mho$ *or there is an* $\mathcal{R}$*-sequence* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$*, where* $\varrho_\varpi \to \varrho$ *as* $\varpi \to +\infty$ *implies* $\alpha(\varrho_\varpi, \varrho) \geqslant 1$ *and* $(\varrho_\varpi, \varrho) \in \mathcal{R} \ \forall \ \varpi \in \mathbb{N}$*;*

(IV) If for some $\mathcal{F} \in \mathfrak{F}'$*,* $\exists \ \wp > 0$ *with* $\mathcal{H}(\phi\varrho, \phi\varsigma) > 0$ *s.t for every* $(\varrho, \varsigma) \in \mathcal{R}$*, we have*

$$\wp + \alpha(\varrho, \varsigma)\mathcal{F}(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma)).$$

Then, ϕ *possesses a fixed point.*

Proof. By the working done in Theorem 3.3.10, we obtain an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ where $\varrho_{\varpi+1} \in \phi\varrho_\varpi \ \forall \ \varpi \in \mathbb{N}'$. By condition *(I)*, we get $\alpha(\varrho_0, \varrho_1) \geqslant 1$. Using multivalued α-admissibility of ϕ, we have $\alpha(\phi\varrho_0, \phi\varrho_1) \geqslant 1$ implies $\alpha(\varrho_1, \varrho_2) \geqslant 1$.

Continuing in a similar way, we get $\alpha(\varrho_\varpi, \varrho_{\varpi+1}) \geqslant 1 \ \forall \ \varpi \in \mathbb{N}'$. Next, if $\varrho_1 \in \phi\varrho_1$, then we are done. Suppose $\varrho_1 \notin \phi\varrho_1$ and as $\phi\varrho_1$ is closed, so we have $\mathcal{D}(\varrho_1, \phi\varrho_1) > 0$ and consequently $\mathcal{D}(\varrho_1, \phi\varrho_1) \leqslant H(\phi\varrho_0, \phi\varrho_1)$, and thereby using $(\mathcal{F}_4)$ property of $\mathcal{F}$, we obtain

$$\begin{aligned}
\mathcal{F}(\mathcal{D}(\varrho_1, \phi\varrho_1)) &= \mathcal{F}\Big(\inf_{\varsigma\in\phi\varrho_1} d_{\mathcal{R}}(\varrho_1, \varsigma)\Big) = \inf_{\varsigma\in\phi\varrho_1} \mathcal{F}\Big(d_{\mathcal{R}}(\varrho_1, \varsigma)\Big) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) \\
&\leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \alpha(\varrho_0, \varrho_1)\mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \\
&\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp < \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)). \qquad (3.25)
\end{aligned}$$

So from (3.25), we have $d_{\mathcal{R}}(\varrho_1, \varrho_2) \leqslant d_{\mathcal{R}}(\varrho_0, \varrho_1)$. Again, if $\varrho_2 \in \phi\varrho_2$ then we are done, else suppose $\varrho_2 \notin \phi\varrho_2$, we have $\mathcal{D}(\varrho_2, \phi\varrho_2) > 0$, $\mathcal{D}(\varrho_2, \phi\varrho_2) \leqslant H(\phi\varrho_1, \phi\varrho_2)$ and,

$$d_{\mathcal{R}}(\varrho_2, \varrho_3) \leqslant d_{\mathcal{R}}(\varrho_1, \varrho_2).$$

Further, if $\varrho_\gamma \in \phi\varrho_\gamma$ for some $\gamma \in \mathbb{N}'$, then the fixed point is obtained. Suppose $\varrho_\varpi \notin \phi\varrho_\varpi \ \forall \ \varpi \in \mathbb{N}'$ and as $\phi\varrho_\varpi$ is closed so $\mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi) > 0$ thus, we have

$$\begin{aligned}
\mathcal{F}(\mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi)) &= \inf_{\varsigma\in\phi\varrho_\varpi} \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varsigma)) \\
&\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \\
&\leqslant \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \\
&\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)) - \wp < \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)). \qquad (3.26)
\end{aligned}$$

Hence, $\{d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})\}_{\varpi\in\mathbb{N}'}$ is a decreasing sequence of non-negative real numbers. From (3.26), we get

$$\mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)) - \wp. \qquad (3.27)$$

On letting $\varpi \to +\infty$ in (3.27), then $\lim_{\varpi\to+\infty} d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) = 0$. By $(\mathcal{F}_3)$, $\exists\ 0 < \gamma < 1$ s.t

$$\lim_{\varpi\to+\infty} (d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}))^\gamma \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) = 0. \qquad (3.28)$$

Then from (3.27), we have

$$(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}))^\gamma \Bigg(\mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) - \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1))\Bigg) \leqslant -\varpi\wp(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}))^\gamma. \qquad (3.29)$$

Taking limit as $\varpi \to +\infty$ in (3.29) and using (3.28), we have

$$\lim_{\varpi \to +\infty} \varpi(d_{\mathcal{R}}(\varrho_{\varpi}, \varrho_{\varpi+1}))^{\gamma} = 0.$$

Thus, $\exists\ \varpi_1 \in \mathbb{N}_0$ so that for $\varpi \in \mathbb{N}_0$ with $\varpi \geqslant \varpi_1$, we obtain

$$d_{\mathcal{R}}(\varrho_{\varpi}, \varrho_{\varpi+1}) \leqslant \frac{1}{\varpi^{1/\gamma}}. \tag{3.30}$$

We now show that $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}_0}$ is an $\mathcal{R}$-Cauchy sequence. Consider $\varpi, \varpi^* \in \mathbb{N}_0$ where $\varpi^* > \varpi \geqslant \varpi_1$. By (3.30) and triangle inequality, we get

$$\begin{aligned} d_{\mathcal{R}}(\varrho_{\varpi^*}, \varrho_{\varpi}) &\leqslant d_{\mathcal{R}}(\varrho_{\varpi}, \varrho_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_{\varpi+2}) + \cdots + d_{\mathcal{R}}(\varrho_{\varpi^*-1}, \varrho_{\varpi^*}) \\ &\leqslant \sum_{i=\varpi}^{+\infty} d_{\mathcal{R}}(\varrho_i, \varrho_{i+1}) \leqslant \sum_{i=\varpi}^{+\infty} \frac{1}{i^{1/\gamma}}. \end{aligned} \tag{3.31}$$

Using the convergence of the series $\sum_{i=\varpi}^{+\infty} \frac{1}{i^{1/\gamma}}$ in (3.31), we get that $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}_0}$ is an $\mathcal{R}$-Cauchy sequence and since $\mho$ is an $\mathcal{R}$-complete metric space, thus $\exists\ \varrho^* \in \mho$ s.t $\lim_{\varpi \to +\infty} \varrho_{\varpi} = \varrho^*$. We now claim that $\varrho^* \in \phi\varrho^*$.

Case (i): Let ϕ be an $\mathcal{R}_{\mathcal{H}}$-continuous map. Since $\varrho_{\varpi+1} \in \phi\varrho_{\varpi}$, we have

$$\mathcal{D}(\varrho_{\varpi+1}, \phi\varrho^*) \leqslant \mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*). \tag{3.32}$$

Taking limit as $\varpi \to +\infty$ in (3.32) and using $\mathcal{R}$-continuity of ϕ, we obtain

$$\mathcal{D}(\varrho^*, \phi\varrho^*) = \lim_{\varpi \to +\infty} \mathcal{D}(\varrho_{\varpi+1}, \phi\varrho^*) \leqslant \lim_{\varpi \to +\infty} \mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*) = 0.$$

So, $\varrho^* \in \phi\varrho^*$.

Case (ii): Let there be an $\mathcal{R}$-sequence $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}'}$ s.t $\varrho_{\varpi} \to \varrho^*$ as $\varpi \to +\infty$ implies $(\varrho_{\varpi}, \varrho^*) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}'$. Let $\varrho^* \notin \phi\varrho^*$, then $\exists\ \varpi' \in \mathbb{N}'$ s.t $\varrho^* \notin \{\varrho_{\varpi}\}$ for every $\varpi > \varpi'$ implies $\mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*) > 0$ and also by given condition, we have $(\varrho_{\varpi}, \varrho^*) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}'$. Now,

$$\begin{aligned} \mathcal{F}(\mathcal{D}(\varrho_{\varpi+1}, \phi\varrho^*)) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*)) &\leqslant \alpha(\varrho_{\varpi}, \varrho^*)\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*)) \\ &< \wp + \alpha(\varrho_{\varpi}, \varrho^*)\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi}, \phi\varrho^*)) \\ &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi}, \varrho^*)). \end{aligned} \tag{3.33}$$

On letting $\varpi \to +\infty$ in (3.33), we get $\mathcal{D}(\varrho^*, \phi\varrho^*) = 0$ which is a contradiction. Hence, $\varrho^* \in \phi\varrho^*$. □

Corollary 3.3.13. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\alpha : \mho \times \mho \to [0, +\infty)$ and $\phi : \mho \to \mathcal{K}(\mho)$ be a multivalued α-admissible map s.t:*

(I) $\exists\ \varrho_0 \in \mho$, where $(\varrho_0, \phi\varrho_0) \in \mathcal{R}$ and $\alpha(\varrho_0, \phi\varrho_0) \geqslant 1$;

(II) For each $(\varrho, \varsigma) \in \mathcal{R}$ implies $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$;

(III) Either ϕ is $\mathcal{R}_{\mathcal{H}}$-continuous on $\mho$ or there is an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$, where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$ implies $\alpha(\varrho_\varpi, \varrho) \geqslant 1$ and $(\varrho_\varpi, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$;

(IV) If for some $\mathcal{F} \in \mathfrak{F}$, $\exists\ \wp > 0$ with $\mathcal{H}(\phi\varrho, \phi\varsigma) > 0$ s.t for every $(\varrho, \varsigma) \in \mathcal{R}$, we get

$$\wp + \alpha(\varrho, \varsigma)\mathcal{F}(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma)).$$

Then, ϕ possesses a fixed point.

Proof. Working on the footprints of Theorem 3.3.10, we obtain an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ where $\varrho_{\varpi+1} \in \phi\varrho_\varpi\ \forall\ \varpi \in \mathbb{N}'$. Also, from the working in Theorem 3.3.12, we get $\alpha(\varrho_\varpi, \varrho_{\varpi+1}) \geqslant 1\ \forall\ \varpi \in \mathbb{N}'$. Next, suppose $\varrho_1 \notin \phi\varrho_1$ and as $\phi\varrho_1$ is compact, so $\exists$ some $\varrho_2 \in \phi\varrho_1$, s.t

$$\begin{aligned} \mathcal{D}(\varrho_1, \phi\varrho_1) = d_{\mathcal{R}}(\varrho_1, \varrho_2) &\leqslant H(\phi\varrho_0, \phi\varrho_1), \\ \text{that is,} \quad \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \phi\varrho_1)) &\leqslant \mathcal{F}(H(\phi\varrho_0, \phi\varrho_1)) \\ &\leqslant \alpha(\varrho_0, \varrho_1)\mathcal{F}(H(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp. \end{aligned}$$

Next, if $\varrho_2 \in \phi\varrho_2$, then the fixed point is obtained. Suppose $\varrho_2 \notin \phi\varrho_2$ and as $\phi\varrho_2$ is compact so $\exists$ some $\varrho_3 \in \phi\varrho_2$ s.t

$$\begin{aligned} \mathcal{D}(\varrho_2, \phi\varrho_2) = d_{\mathcal{R}}(\varrho_2, \varrho_3) &\leqslant H(\phi\varrho_1, \phi\varrho_2), \\ \text{that is,} \quad \mathcal{F}(d_{\mathcal{R}}(\varrho_2, \phi\varrho_3)) &\leqslant \mathcal{F}(H(\phi\varrho_1, \phi\varrho_2)) \\ &\leqslant \alpha(\varrho_1, \varrho_2)\mathcal{F}(H(\phi\varrho_1, \phi\varrho_2)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) - \wp. \end{aligned}$$

If $\varrho_k \in \phi\varrho_k$ for some $k \in \mathbb{N}'$, then we are done. Suppose $\varrho_\varpi \notin \phi\varrho_\varpi\ \forall\ \varpi \in \mathbb{N}'$ and as $\phi\varrho_\varpi$ is compact, so $\exists\ \varrho_{\varpi+1} \in \phi\varrho_\varpi$, s.t

$$\begin{aligned} \mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi) = d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) &\leqslant H(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi), \\ \text{that is,} \quad \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) &\leqslant \mathcal{F}(H(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \\ &\leqslant \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}(H(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \\ &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)) - \wp. \end{aligned}$$

Hence, $\{d_{\mathcal{R}}(\varrho_{\varpi}, \varrho_{\varpi+1})\}_{\varpi \in \mathbb{N}'}$ is a decreasing sequence of non-negative real numbers. The proof now follows on the line of Theorem 3.3.12. □

Corollary 3.3.14. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\phi : \mho \to \mho$ be a self-map s.t:*

(I) $\exists\ \varrho_0 \in \mho$, *where* $(\varrho_0, \phi\varrho_0) \in \mathcal{R}$;

(II) ϕ *is $\mathcal{R}$-preserving;*

(III) Either ϕ is $\mathcal{R}$-continuous on $\mho$ or there is an $\mathcal{R}$-sequence $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$, where $\varrho_{\varpi} \to \varrho$ as $\varpi \to +\infty$ implies $(\varrho_{\varpi}, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$;

(IV) $\exists$ some $\wp > 0$ s.t for every $(\varrho, \varsigma) \in \mathcal{R}$ with $d_{\mathcal{R}}(\phi\varrho, \phi\varsigma) > 0$, we have

$$d_{\mathcal{R}}(\phi\varrho, \phi\varsigma) \leqslant e^{-\wp} d_{\mathcal{R}}(\varrho, \varsigma).$$

Then, ϕ possesses a fixed point.

Proof. Considering ϕ as a self-map on $\mho$ with $\mathcal{F}(\mu) = \ln(\mu)$ in Corollary 3.3.13, we obtain the result. □

Theorem 3.3.15. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\phi : \mho \to \mathcal{CB}(\mho)$ be a multivalued α-admissible map s.t:*

(I) $\exists\ \varrho_0 \in \mho$, *where* $(\varrho_0, \phi\varrho_0) \in \mathcal{R}$;

(II) For each $(\varrho, \varsigma) \in \mathcal{R}$ implies $(\phi\varrho, \phi\varsigma) \in \mathcal{R}$;

(III) Either ϕ is $\mathcal{R}_{\mathcal{H}}$-continuous on $\mho$ or there is an $\mathcal{R}$-sequence $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$, where $\varrho_{\varpi} \to \varrho$ as $\varpi \to +\infty$ implies $\alpha(\varrho_{\varpi}, \varrho) \geqslant 1$ and $(\varrho_{\varpi}, \varrho) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$;

(IV) If for some $\mathcal{F} \in \mathfrak{F}'$, $\exists\ \wp > 0$ s.t for every $(\varrho, \varsigma) \in \mathcal{R}$ with $\mathcal{H}(\phi\varrho, \phi\varsigma) > 0$, we have

$$\wp + \alpha(\varrho, \varsigma)\mathcal{F}(\mathcal{H}(\phi\varrho, \phi\varsigma)) \leqslant \mathcal{F}\Big(\max\Big\{d_{\mathcal{R}}(\varrho, \varsigma), \mathcal{D}(\varrho, \phi\varrho), \mathcal{D}(\varsigma, \phi\varsigma), \frac{\mathcal{D}(\varrho, \phi\varsigma) + \mathcal{D}(\varsigma, \phi\varrho)}{2}\Big\}\Big).$$

Then, ϕ possesses a fixed point.

Proof. Working on the lines of Theorem 3.3.10, we get an $\mathcal{R}$-sequence $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}'}$ with $\varrho_{\varpi+1} \in \phi\varrho_{\varpi}\ \forall\ \varpi \in \mathbb{N}'$. Also, from the working in Theorem 3.3.12, we obtain $\alpha(\varrho_{\varpi}, \varrho_{\varpi+1}) \geqslant 1\ \forall\ \varpi \in \mathbb{N}'$. Next, if $\varrho_1 \in \phi\varrho_1$, then we are done. Suppose

$\varrho_1 \notin \phi\varrho_1$ and as $\phi\varrho_1$ is closed, so we have $\mathcal{D}(\varrho_1, \phi\varrho_1) > 0$ and consequently $\mathcal{D}(\varrho_1, \phi\varrho_1) \leqslant H(\phi\varrho_0, \phi\varrho_1)$. Thereby using $(\mathcal{F}_4)$ property of $\mathcal{F}$, we get

$$\begin{aligned}\mathcal{F}(\mathcal{D}(\varrho_1, \phi\varrho_1)) &= \inf_{\varsigma\in\phi\varrho_1} \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) \\ &\leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \alpha(\varrho_0, \varrho_1)\mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)). \quad (3.34)\end{aligned}$$

Next, for some $\wp > 0$, using condition (IV) in (3.34), we have

$$\begin{aligned}\wp + \alpha(\varrho_0, \varrho_1)\mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant &\mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_0, \varrho_1), \mathcal{D}(\varrho_0, \phi\varrho_0), \mathcal{D}(\varrho_1, \phi\varrho_1), \\ &\frac{\mathcal{D}(\varrho_0, \phi\varrho_1) + \mathcal{D}(\varrho_1, \phi\varrho_0)}{2}\Bigg\}\Bigg) \\ \leqslant &\mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_0, \varrho_1), \mathcal{H}(\phi\varrho_0, \phi\varrho_1), \frac{\mathcal{D}(\varrho_0, \phi\varrho_1)}{2}\Bigg\}\Bigg) \\ \leqslant &\mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_0, \varrho_1), \mathcal{H}(\phi\varrho_0, \phi\varrho_1), \\ &\frac{d_{\mathcal{R}}(\varrho_0, \varrho_1) + \mathcal{H}(\phi\varrho_0, \phi\varrho_1)}{2}\Bigg\}\Bigg). \quad (3.35)\end{aligned}$$

If $d_{\mathcal{R}}(\varrho_0, \varrho_1) < \mathcal{H}(\phi\varrho_0, \phi\varrho_1)$, then by (3.35) we have

$$\wp + \alpha(\varrho_0, \varrho_1)\mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)),$$

which is a contradiction for $\wp > 0$, where $\alpha(\varrho_0, \varrho_1) \geqslant 1$. Thus, we have $\mathcal{H}(\phi\varrho_0, \phi\varrho_1) < d_{\mathcal{R}}(\varrho_0, \varrho_1)$ and by (3.34) and (3.35), we get

$$\mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) \leqslant \alpha(\varrho_0, \varrho_1)\mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp.$$

Further, suppose that $\varrho_\varpi \notin \phi\varrho_\varpi \ \forall \ \varpi \in \mathbb{N}'$. As $\phi\varrho_\varpi$ is closed, so we have $\mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi) > 0$ and consequently $\mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi) \leqslant H(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)$. On using $(\mathcal{F}_4)$ property of $\mathcal{F}$, we get

$$\begin{aligned}\mathcal{F}(\mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi)) = \inf_{\varsigma\in\phi\varrho_\varpi} \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varsigma)) &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \\ &\leqslant \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)). \quad (3.36)\end{aligned}$$

Again, for some $\wp > 0$, using condition (IV), we have

$$\wp + \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \leqslant \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi), \mathcal{D}(\varrho_{\varpi-1}, \phi\varrho_{\varpi-1}),$$

$$\mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi), \frac{\mathcal{D}(\varrho_{\varpi-1}, \phi\varrho_\varpi) + \mathcal{D}(\varrho_\varpi, \phi\varrho_{\varpi-1})}{2}\Bigg\}\Bigg)$$
$$\leqslant \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi), \mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi), \frac{\mathcal{D}(\varrho_{\varpi-1}, \phi\varrho_\varpi)}{2}\Bigg\}\Bigg)$$
$$\leqslant \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi), \mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi), \frac{d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi) + \mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)}{2}\Bigg\}\Bigg). \tag{3.37}$$

If $d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi) \leqslant \mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)$, then by (3.37) we have

$$\wp + \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)),$$

which is a contradiction for $\wp > 0$, where $\alpha(\varrho_{\varpi-1}, \varrho_\varpi) \geqslant 1$. Thus, we have $\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi) < d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)$ and by (3.36) and (3.37), we get

$$\mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) \leqslant \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi)) - \wp.$$

Now, by the working done in Theorem 3.3.12, we obtain that $\exists\ \varrho^* \in \mho$ s.t $\lim_{\varpi\to+\infty} \varrho_\varpi = \varrho^*$. We now claim that $\varrho^* \in \phi\varrho^*$.

Case (i): Let ϕ be an $\mathcal{R}_{\mathcal{H}}$-continuous map. Since $\varrho_{\varpi+1} \in \phi\varrho_\varpi$, we have

$$\mathcal{D}(\varrho_{\varpi+1}, \phi\varrho^*) \leqslant \mathcal{H}(\phi\varrho_\varpi, \phi\varrho^*). \tag{3.38}$$

Taking limit as $\varpi \to +\infty$ in (3.38) and using $\mathcal{R}$-continuity of ϕ, we obtain

$$\mathcal{D}(\varrho^*, \phi\varrho^*) = \lim_{\varpi\to+\infty} \mathcal{D}(\varrho_{\varpi+1}, \phi\varrho^*) \leqslant \lim_{\varpi\to+\infty} \mathcal{H}(\phi\varrho_\varpi, \phi\varrho^*) = 0.$$

So, $\varrho^* \in \phi\varrho^*$.

Case (ii): Let there be an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$, where $\varrho_\varpi \to \varrho^*$ as $\varpi \to +\infty$ implies $(\varrho_\varpi, \varrho^*) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}'$. Let $\varrho^* \notin \phi\varrho^*$, then $\exists\ \varpi' \in \mathbb{N}'$ s.t $\varrho^* \notin \{\varrho_\varpi\}$ for every $\varpi > \varpi'$ implies $\mathcal{H}(\phi\varrho_\varpi, \phi\varrho^*) > 0$ and also by given condition, we have $(\varrho_\varpi, \varrho^*) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}'$. Now,

$$\begin{aligned}\mathcal{F}(\mathcal{D}(\varrho_{\varpi+1}, \phi\varrho^*)) &\leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_\varpi, \phi\varrho^*)) \\ &\leqslant \alpha(\varrho_\varpi, \varrho^*)\mathcal{F}(\mathcal{H}(\phi\varrho_\varpi, \phi\varrho^*)) \\ &< \wp + \alpha(\varrho_\varpi, \varrho^*)\mathcal{F}(\mathcal{H}(\phi\varrho_\varpi, \phi\varrho^*)) \\ &\leqslant \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_\varpi, \varrho), \mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi), \mathcal{D}(\varrho, \phi\varrho),\end{aligned}$$

$$\left.\left.\frac{\mathcal{D}(\varrho_\varpi,\phi\varrho)+\mathcal{D}(\varrho,\phi\varrho_\varpi)}{2}\right\}\right),$$

$$\text{implies, } \mathcal{D}(\varrho_{\varpi+1},\phi\varrho^*) \leqslant \max\Big\{d_{\mathcal{R}}(\varrho_\varpi,\varrho),\mathcal{D}(\varrho_\varpi,\phi\varrho_\varpi),\mathcal{D}(\varrho,\phi\varrho),$$

$$\left.\frac{\mathcal{D}(\varrho_\varpi,\phi\varrho)+\mathcal{D}(\varrho,\phi\varrho_\varpi)}{2}\right\} \tag{3.39}$$

On letting $\varpi \to +\infty$ in (3.39), we obtain $\mathcal{D}(\varrho^*,\phi\varrho^*)=0$ which is not true. Hence, $\varrho^* \in \phi\varrho^*$. □

Corollary 3.3.16. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$, let $\phi:\mho\to\mathcal{K}(\mho)$ be a multivalued α-admissible map s.t:*

(I) $\exists\ \varrho_0 \in \mho$, where $(\varrho_0,\phi\varrho_0)\in\mathcal{R}$;

(II) For each $(\varrho,\varsigma)\in\mathcal{R}$ implies $(\phi\varrho,\phi\varsigma)\in\mathcal{R}$;

(III) Either ϕ is $\mathcal{R}_{\mathcal{H}}$-continuous on $\mho$ or there is an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$, where $\varrho_\varpi\to\varrho$ as $\varpi\to+\infty$ implies $\alpha(\varrho_\varpi,\varrho)\geqslant 1$ and $(\varrho_\varpi,\varrho)\in\mathcal{R}\ \forall\ \varpi\in\mathbb{N}$;

(IV) If for some $\mathcal{F}\in\mathfrak{F}$, $\exists\ \wp>0$ s.t for each $(\varrho,\varsigma)\in\mathcal{R}$ with $\mathcal{H}(\phi\varrho,\phi\varsigma)>0$, we have

$$\wp+\alpha(\varrho,\varsigma)\mathcal{F}(\mathcal{H}(\phi\varrho,\phi\varsigma))\leqslant\mathcal{F}\left(\max\left\{d_{\mathcal{R}}(\varrho,\varsigma),\mathcal{D}(\varrho,\phi\varrho),\mathcal{D}(\varsigma,\phi\varsigma),\frac{\mathcal{D}(\varrho,\phi\varsigma)+\mathcal{D}(\varsigma,\phi\varrho)}{2}\right\}\right).$$

Then, ϕ possesses a fixed point.

Proof. Working on the lines of Theorem 3.3.10, we get an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ with $\varrho_{\varpi+1}\in\phi\varrho_\varpi\ \forall\ \varpi\in\mathbb{N}'$. Also, from the working in Theorem 3.3.12, we obtain $\alpha(\varrho_\varpi,\varrho_{\varpi+1})\geqslant 1\ \forall\ \varpi\in\mathbb{N}'$. Suppose $\varrho_1\notin\phi\varrho_1$ and as $\phi\varrho_1$ is compact, so $\exists$ some $\varrho_2\in\phi\varrho_1$, s.t

$$\begin{aligned}
\mathcal{D}(\varrho_1,\phi\varrho_1)=d_{\mathcal{R}}(\varrho_1,\varrho_2) &\leqslant H(\phi\varrho_0,\phi\varrho_1),\\
\text{that is,}\quad \mathcal{F}(\mathcal{D}(\varrho_1,\phi\varrho_1))=\mathcal{F}(d_{\mathcal{R}}(\varrho_1,\varrho_2)) &\leqslant \mathcal{F}(H(\phi\varrho_0,\phi\varrho_1))\\
&\leqslant \alpha(\varrho_0,\varrho_1)\mathcal{F}(H(\phi\varrho_0,\phi\varrho_1))\\
\wp+\mathcal{F}(d_{\mathcal{R}}(\varrho_1,\phi\varrho_1)) &\leqslant \wp+\alpha(\varrho_0,\varrho_1)\mathcal{F}(H(\phi\varrho_0,\phi\varrho_1)),
\end{aligned} \tag{3.40}$$

for some $\wp>0$. Next, using condition (IV) in (3.40), we have

$$
\begin{aligned}
\wp + \alpha(\varrho_0, \varrho_1)\mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant & \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_0, \varrho_1), \mathcal{D}(\varrho_0, \phi\varrho_0), \mathcal{D}(\varrho_1, \phi\varrho_1), \\
& \frac{\mathcal{D}(\varrho_0, \phi\varrho_1) + \mathcal{D}(\varrho_1, \phi\varrho_0)}{2}\Bigg\}\Bigg) \\
\leqslant & \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_0, \varrho_1), \mathcal{H}(\phi\varrho_0, \phi\varrho_1), \frac{\mathcal{D}(\varrho_0, \phi\varrho_1)}{2}\Bigg\}\Bigg) \\
\leqslant & \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_0, \varrho_1), \mathcal{H}(\phi\varrho_0, \phi\varrho_1), \\
& \frac{d_{\mathcal{R}}(\varrho_0, \varrho_1) + \mathcal{H}(\phi\varrho_0, \phi\varrho_1)}{2}\Bigg\}\Bigg). \qquad (3.41)
\end{aligned}
$$

If $d_{\mathcal{R}}(\varrho_0, \varrho_1) < \mathcal{H}(\phi\varrho_0, \phi\varrho_1)$, then by (3.41) we have

$$
\wp + \alpha(\varrho_0, \varrho_1)\mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)),
$$

which is a contradiction for $\wp > 0$ and $\alpha(\varrho_0, \varrho_1) \geqslant 1$. Thus, we have $\mathcal{H}(\phi\varrho_0, \phi\varrho_1) < d_{\mathcal{R}}(\varrho_0, \varrho_1)$. By (3.40) and (3.41), we get

$$
\mathcal{F}(d_{\mathcal{R}}(\varrho_1, \varrho_2)) \leqslant \alpha(\varrho_0, \varrho_1)\mathcal{F}(\mathcal{H}(\phi\varrho_0, \phi\varrho_1)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varrho_1)) - \wp.
$$

Further, suppose $\varrho_\varpi \notin \phi\varrho_\varpi \ \forall\ \varpi \in \mathbb{N}'$. As $\phi\varrho_\varpi$ is compact, so $\exists$ some $\varrho_{\varpi+1} \in \phi\varrho_\varpi$, s.t

$$
\begin{aligned}
\mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi) = d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) &\leqslant H(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi), \\
\text{thus, } \mathcal{F}(\mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi)) = \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1})) &\leqslant \mathcal{F}(H(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \\
&\leqslant \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}(H(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)), \\
\text{that is, } \wp + \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \phi\varrho_\varpi)) &\leqslant \wp + \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}(H(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)),
\end{aligned}
\qquad (3.42)
$$

for some $\wp > 0$. Again, on using condition (IV) in (3.42), we have

$$
\begin{aligned}
& \wp + \alpha(\varrho_{\varpi-1}, \varrho_\varpi)\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1}, \phi\varrho_\varpi)) \\
& \leqslant \mathcal{F}\Bigg(\max\Bigg\{d_{\mathcal{R}}(\varrho_{\varpi-1}, \varrho_\varpi), \mathcal{D}(\varrho_{\varpi-1}, \phi\varrho_\varpi), \mathcal{D}(\varrho_\varpi, \phi\varrho_\varpi), \\
& \frac{\mathcal{D}(\varrho_{\varpi-1}, \phi\varrho_\varpi) + \mathcal{D}(\varrho_\varpi, \phi\varrho_{\varpi-1})}{2}\Bigg\}\Bigg)
\end{aligned}
$$

$$\leqslant \mathcal{F}\bigg(\max\bigg\{d_{\mathcal{R}}(\varrho_{\varpi-1},\varrho_{\varpi}),\mathcal{H}(\phi\varrho_{\varpi-1},\phi\varrho_{\varpi}),\frac{\mathcal{D}(\varrho_{\varpi-1},\phi\varrho_{\varpi})}{2}\bigg\}\bigg)$$
$$\leqslant \mathcal{F}\bigg(\max\bigg\{d_{\mathcal{R}}(\varrho_{\varpi-1},\varrho_{\varpi}),\mathcal{H}(\phi\varrho_{\varpi-1},\phi\varrho_{\varpi}),\frac{d_{\mathcal{R}}(\varrho_{\varpi-1},\varrho_{\varpi})+\mathcal{H}(\phi\varrho_{\varpi-1},\phi\varrho_{\varpi})}{2}\bigg\}\bigg). \tag{3.43}$$

If $d_{\mathcal{R}}(\varrho_{\varpi},\varrho_{\varpi+1}) \leqslant \mathcal{H}(\phi\varrho_{\varpi},\phi\varrho_{\varpi+1})$, then by (3.43) we have

$$\wp + \alpha(\varrho_{\varpi},\varrho_{\varpi+1})\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi},\phi\varrho_{\varpi+1})) \leqslant \mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi},\phi\varrho_{\varpi+1})),$$

which is a contradiction for $\wp > 0$ and $\alpha(\varrho_{\varpi},\varrho_{\varpi+1}) \geqslant 1$. Thus, we have $\mathcal{H}(\phi\varrho_{\varpi},\phi\varrho_{\varpi+1}) < d_{\mathcal{R}}(\varrho_{\varpi},\varrho_{\varpi+1})$. By (3.42) and (3.43), we get

$$d_{\mathcal{R}}(\varrho_{\varpi},\varrho_{\varpi+1}) \leqslant \alpha(\varrho_{\varpi-1},\varrho_{\varpi})\mathcal{F}(\mathcal{H}(\phi\varrho_{\varpi-1},\phi\varrho_{\varpi})) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1},\varrho_{\varpi})).$$

Now, by the working done in Theorem 3.3.15 we conclude that ϕ has a fixed point. □

3.4 Existence of Solution to Non-linear Volterra Integral Equation with Binary Relation

The results obtained in the preceding section will now be implemented on a non-homogeneous, non-linear Volterra integral equation equipped with a binary relation in order to substantiate the existence of its solution. Consider $\mho = C([0,1],\mathbb{R}^+)$, that is, set of all continuous functions from $[0,1]$ to $\mathbb{R}^+$ and define $d:\mho\times\mho\to[0,+\infty)$ as

$$d(\Gamma,\Upsilon) = \|\Gamma(\varrho)-\Upsilon(\varrho)\|_{\infty,[0,1]}.$$

Let relation $\mathcal{R}$ be defined as $\mathcal{R} = \{(\Gamma,\Upsilon)\in\mho\times\mho : \Gamma(\varrho).\Upsilon(\varrho)\geqslant 0\ \forall\ \varrho\in[0,1]\}$. Define $\phi:\mho\to\mho$ as

$$\phi\Gamma(\varrho) = \xi(\varrho) + \Omega\bigg(\int_0^{\varrho}\mathcal{K}(\varrho,\varsigma,\Gamma(\varsigma))\mathrm{d}\varsigma\bigg), \tag{3.44}$$

where $\xi(\varrho)$ is a continuous non-negative real valued function on $[0,1]$, kernel $\mathcal{K}(\varrho,\varsigma,\Gamma(\varsigma))$ is a continuous and Ω is a linear operator on $\mho$ so $\exists\ \wp > 0$ s.t

$\|\Omega(\Gamma)\| \leqslant \wp\|\Gamma\|$ and

$$\|\Omega\| = \sup\left\{\frac{\|\Omega(\Gamma)\|}{\|\Gamma\|} : \Gamma \neq 0 \text{ and } \Gamma \in \mho\right\}.$$

Clearly, here ϕ is a non-homogeneous, non-linear Volterra integral equation along with $\mathcal{R}$ as defined above.

Theorem 3.4.1. *Let $(\mho, d_{\mathcal{R}})$ be defined as above. Let ϕ be a self-map on $\mho$ given by (3.44) where kernel $\mathcal{K}(\varrho, \varsigma, \Gamma(\varsigma))$ s.t for $(\Gamma, \Upsilon) \in \mathcal{R}$,*

$$|\mathcal{K}(\varrho, \varsigma, \Gamma(\varsigma)) - \mathcal{K}(\varrho, \varsigma, \Upsilon(\varsigma))| \leqslant e^{-\wp}|\Gamma(\varsigma) - \Gamma(\varsigma)|,$$

where $\varsigma \in [0, 1]$ and for some $\wp > 0$. Then the non-homogeneous, non-linear Volterra equation (3.44) possesses a solution.

Proof. For $\mho = \mathrm{C}([0, 1], \mathbb{R}^+)$ with $d_{\mathcal{R}}(\Gamma, \Upsilon) = \|\Gamma(\varrho) - \Upsilon(\varrho)\|_{\infty,[0,1]}$ and relation $\mathcal{R}$ on $\mho$ given by $\mathcal{R} = \{(\Gamma, \Upsilon) \in \mho \times \mho : \Gamma(\varrho).\Upsilon(\varrho) \geqslant 0 \ \forall \ \varrho \in [0, 1]\}$, then $(\mho, d_{\mathcal{R}})$ an $\mathcal{R}$-complete metric space. We now show that ϕ given by (3.44) satisfies all hypotheses of Corollary 3.3.14.

(i) $\exists \ \hat{O}$ (zero function) in $\mho$ with $(\hat{O}, \phi\hat{O}) \in \mathcal{R}$.

(ii) For $(\Gamma, \Upsilon) \in \mathcal{R}$, we have $\Gamma(\varrho).\Upsilon(\varrho) \geqslant 0$ implies $\phi\Gamma(\varrho).\phi\Upsilon(\varrho) \geqslant 0$.

(iii) By the definition of ϕ in equation (3.44), we have ϕ is $\mathcal{R}$-continuous.

(iv) Let $(\Gamma, \Upsilon) \in \mathcal{R}$ with $d_{\mathcal{R}}(\phi\Gamma, \phi\Upsilon) > 0$, then

$$\begin{aligned} |\phi(\Gamma(\varrho)) - \phi(\Upsilon(\varrho))| &= \left|\Omega\left(\int_0^{\varrho} \mathcal{K}(\varrho, \varsigma, \Gamma(\varsigma))\mathrm{d}\varsigma\right) - \right. \\ &\quad \left. \Omega\left(\int_0^{\varrho} \mathcal{K}(\varrho, \varsigma, \Upsilon(\varsigma))\mathrm{d}\varsigma\right)\right| \\ &\leqslant \|\Omega\| \int_0^{\varrho} \Big|\mathcal{K}(\varrho, \varsigma, \Gamma(\varsigma)) - \mathcal{K}(\varrho, \varsigma, \Upsilon(\varsigma))\Big|\mathrm{d}\varsigma, \\ \text{that is, } d_{\mathcal{R}}(\phi(\Gamma(\varrho)), \phi(\Upsilon(\varrho))) &\leqslant \|\Omega\| e^{-\wp} d_{\mathcal{R}}(\Gamma(\varrho), \Upsilon(\varrho)) \\ &= e^{(\wp - \ln\|\Omega\|)} d_{\mathcal{R}}(\Gamma(\varrho), \Upsilon(\varrho)). \end{aligned}$$

Thus, the non-homogeneous non-linear Volterra integral equation defined in (3.44) satisfies hypotheses of Corollary 3.3.14 therefore, the integral equation possesses a solution. □

3.5 Hyers-Ulam Stability of Solution of Non-linear Volterra Integral Equation with Binary Relation

Now, we present the Hyers-Ulam stability due to Hyers (1941) and Ulam (1960) in the frame of an $\mathcal{R}$-metric space $(\mho, d_{\mathcal{R}})$. Let $\phi : \mho \to \mho$ be a self-map with the fixed point ϱ, that is,

$$\phi\varrho = \varrho, \tag{3.45}$$

and let $\varsigma \in \mho$ satisfies the inequality

$$d_{\mathcal{R}}(\phi\varsigma, \varsigma) \leqslant \epsilon, \tag{3.46}$$

with $\epsilon > 0$, then the fixed point equation given in (3.45) is c.t.b Hyers-Ulam stable if $\exists\ \delta > 0$ s.t for each ϵ-solution ς^* of (3.46), where $\epsilon > 0$, $\exists$ solution ϱ^* of (3.45) with

$$d_{\mathcal{R}}(\varrho^*, \varsigma^*) \leqslant \delta\epsilon.$$

Next, we show that the fixed point equation $\phi\Gamma = \Gamma$ where ϕ is defined by (3.44) is Hyers-Ulam stable. But before that consider the iterative scheme given as:

$$\phi\Gamma_{\varpi}(\varrho) = \xi(\varrho) + \Omega\bigg(\int_0^{\varrho} \mathcal{K}(\varrho, \varsigma, \Gamma_{\varpi}(\varsigma))\mathrm{d}\varsigma\bigg) = \Gamma_{\varpi+1}(\varrho),$$

for $\varpi \in \mathbb{N}$. Then,

$$\begin{aligned}
|\Gamma_{\varpi+1}(\varrho) - \Gamma_{\varpi}(\varrho)| \leqslant & \bigg|\Omega\bigg(\int_0^{\varrho} \mathcal{K}(\varrho, \varsigma_1, \Gamma_{\varpi}(\varsigma_1))\mathrm{d}\varsigma_1 - \int_0^{\varrho} \mathcal{K}(\varrho, \varsigma_1, \Gamma_{\varpi-1}(\varsigma_1))\mathrm{d}\varsigma_1\bigg)\bigg| \\
\leqslant & \|\Omega\| e^{-\wp} \int_0^{\varrho} |\Gamma_{\varpi}(\varsigma_1) - \Gamma_{\varpi-1}(\varsigma_1)|\mathrm{d}\varsigma_1 \\
\leqslant & \Big(\|\Omega\| e^{-\wp}\Big)^2 \int_0^{\varrho}\int_0^{\varsigma_1} |\Gamma_{\varpi-1}(\varsigma_2) - \Gamma_{\varpi-2}(\varsigma_2)|\mathrm{d}\varsigma_2\mathrm{d}\varsigma_1 \\
\leqslant & \Big(\|\Omega\| e^{-\wp}\Big)^{\varpi-1} \int_0^{\varrho}\int_0^{\varsigma_1}\cdots\int_0^{\varsigma_{\varpi-2}} |\Gamma_2(\varsigma_{\varpi-1}) - \\
& \Gamma_1(\varsigma_{\varpi-1})|\mathrm{d}\varsigma_{\varpi-1}\ldots\mathrm{d}\varsigma_2\ \mathrm{d}\varsigma_1 \\
\leqslant & \Big(\|\Omega\| e^{-\wp}\Big)^{\varpi-1} d(\Gamma_1, \phi\Gamma_1) \int_0^{\varrho}\int_0^{\varsigma_1}\cdots\int_0^{\varsigma_{\varpi-2}} \mathrm{d}\varsigma_{\varpi-1}\ldots\mathrm{d}\varsigma_2\ \mathrm{d}\varsigma_1,
\end{aligned}$$

thus, $|\Gamma_{\varpi+1}(\varrho) - \Gamma_{\varpi}(\varrho)| \leqslant (\|\Omega\| e^{-\wp})^{\varpi-1} \dfrac{\varrho^{\varpi-1}}{(\varpi-1)!} d_{\mathcal{R}}(\Gamma_1, \phi\Gamma_1)$.

Theorem 3.5.1. *For an $\mathcal{R}$-complete metric space $(\mho, d_{\mathcal{R}})$ as defined in Theorem 3.4.1 along with the condition $(\Gamma, \Upsilon) \in \mathcal{R}$ for every ϵ-solution Γ^* and Υ^*, then the fixed point equation $\phi\Gamma = \Gamma$ is Hyer-Ulam stable.*

Proof. By the working done in Theorem 3.4.1, $\exists\, \Gamma^*$ in $\mho$, where $\phi\Gamma^* = \Gamma^*$. Let Υ^* be an ϵ-solution of fixed point equation, then $(\Gamma^*, \Upsilon^*) \in \mathcal{R}$, and by the working done above, we have $d_{\mathcal{R}}(\phi^{\varpi}\Upsilon^*, \Gamma^*) \leqslant \epsilon$, since $\phi^{\varpi}\Upsilon^*$ converges to Γ^* as $\varpi \to +\infty$. Also,

$$\begin{aligned}
d_{\mathcal{R}}(\Gamma^*, \Upsilon^*) &\leqslant d_{\mathcal{R}}(\Gamma^*, \phi^{\varpi}\Upsilon^*) + d_{\mathcal{R}}(\phi^{\varpi}\Upsilon^*, \Upsilon^*) \\
&\leqslant d_{\mathcal{R}}(\Gamma^*, \phi^{\varpi}\Upsilon^*) + d_{\mathcal{R}}(\Upsilon^*, \phi\Upsilon^*) + d_{\mathcal{R}}(\phi\Upsilon^*, \phi^2\Upsilon^*) + \cdots + \\
&\quad d_{\mathcal{R}}(\phi^{\varpi-1}\Upsilon^*, \phi^{\varpi}\Upsilon^*) \\
&\leqslant d_{\mathcal{R}}(\Gamma^*, \phi^{\varpi}\Upsilon^*) + d_{\mathcal{R}}(\Upsilon^*, \phi\Upsilon^*) + a\, d_{\mathcal{R}}(\Upsilon^*, \phi\Upsilon^*) + \frac{a^2}{2!} d_{\mathcal{R}}(\Upsilon^*, \phi\Upsilon^*) \\
&\quad + \cdots + \frac{a^{\varpi-1}}{(\varpi-1)!} d_{\mathcal{R}}(\Upsilon^*, \phi\Upsilon^*) \\
&\leqslant (1 + e^a)\epsilon, \qquad (3.47)
\end{aligned}$$

where $a = \|\Omega\| e^{-\wp}$. Thus, from (3.47) we obtain $d_{\mathcal{R}}(\Gamma^*, \Upsilon^*) \leqslant \delta\epsilon$. So, the fixed point equation $\phi\Gamma = \Gamma$, where ϕ is defined by (3.44) is Hyers-Ulam stable. □

3.6 Conclusion

In this chapter, we have introduced a novel approach to prove the fixed point results for certain types of expansive maps and multivalued maps on a $\mathcal{R}$-metric space that extends, unifies and generalizes the results on multivalued and single valued maps in the literature. However, under certain conditions the results proved in this chapter are reduced to some well known results of the literature.

(I) If in Theorem 3.2.3 or Corollary 3.2.4 we consider relation $\mathcal{R}$ as universal relation then we obtain the equivalent counterpart of Theorem 2.1 of Górnicki (2016).

(II) If the binary relation $\mathcal{R}$ in Theorem 3.3.2 and Theorem 3.3.3 is considered to be a universal relation on $\mho$, then the main results of Altun et al. (2015) are obtained. Under a similar condition in Theorem 3.3.5, a result equivalent to one given by Acar et al. (2014) is deduced.

(III) If the binary relation $\mathcal{R}$ in Theorem 3.3.2 and Theorem 3.3.3 is considered to be an orthogonal relation, that is, there is some $\varrho_0 \in \mho$ where $(\varrho_0, \varsigma) \in \mathcal{R}$ $\forall\, \varsigma \in \mho$ or $(\varsigma, \varrho_0) \in \mathcal{R}\ \forall\, \varsigma \in \mho$, then the main results of Sharma & Chandok (2020) are deduced.

(IV) On considering $\mathcal{R}$ as a universal relation and ϕ as a single valued self-map on $\mho$ in Theorem 3.3.2 and Theorem 3.3.5, the corresponding results of Wardowski (2012) and Wardowski & Dung (2014) are obtained.

(V) On taking into account the amorphous binary relation $\mathcal{R}$ as a universal relation in Theorem 3.3.10 and Corollary 3.3.11, we obtain Theorem 2.2 and Remark 3.3 of Altun et al. (2016).

(VI) If in Theorem 3.3.15 and Corollary 3.3.16 of the present chapter, we consider $\alpha(\varrho, \varsigma) = 1$, ϕ as a single valued self-map on $\mho$ and $\mathcal{R} = \mho \times \mho$ then we obtain Theorem 2.2 and Theorem 2.4 of Mınak et al. (2014) and Wardowski & Dung (2014), respectively.

(VII) On considering ϕ as a single valued self-map on $\mho$, an amorphous binary relation $\mathcal{R}$ as a universal relation and $\kappa = 0$ in Theorem 3.3.10 and Corollary 3.3.11, we obtain Theorem 2.1 of Wardowski (2012).

(VIII) If we consider ϕ as a single valued self-map on $\mho$, an amorphous binary relation $\mathcal{R}$ to be orthogonal (that is, there is some $\varsigma_0 \in \mho$ s.t $(\varsigma_0, \varrho) \in \mathcal{R}$ $\forall\, \varrho \in \mho$ or $(\varrho, \varsigma_0) \in \mathcal{R}\ \forall\, \varrho \in \mho$) and $\kappa = 0$ in Theorem 3.3.10 and Corollary 3.3.11, we obtain Theorem 3.10, Theorem 3.3 and Theorem 3.3 of Baghani et al. (2016), Mani et al. (2021) and Sawangsup et al. (2020), respectively.

(IX) If in Theorem 3.3.15 and Corollary 3.3.16 we consider $\alpha(\varrho, \varsigma) = 1$, ϕ as a single valued self-map on $\mho$ and $\mathcal{R} = \mho \times \mho$ then we obtain Theorem 2.2 of Mınak et al. (2014).

(X) On considering ϕ as a single valued self-map on $\mho$, an amorphous binary relation $\mathcal{R}$ as a universal relation and $\alpha(\varrho, \varsigma) = 1$ in Theorem 3.3.12 and Corollary 3.3.13, we obtain Theorem 2.1 of Wardowski (2012).

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Chapter 4

Fixed Point Results in C^*-Algebra Valued $\mathcal{R}$-Metric Space

4.1 Introduction

Ma et al. (2014) put forward the idea of a novel metric space named C^*_{AV}-metric spaces and established some of the fixed point results subjected to fairly new contractions as well as expansion maps that, over a period of time, has been generalized by many (see Ma & Jiang (2015), Shen et al. (2018), Chandok et al. (2019), Ghanifard et al. (2020), Mustafa et al. (2021), Shagari et al. (2023) and references cited therein).

Inspired by the work done by Ma et al. (2014) and Alam & Imdad (2015), in this chapter we put forward the idea of $C^*_{AV}\mathcal{R}$-metric space which generalizes the class of C^*_{AV}-metric space. Also, we introduce the idea of $C^*_{AV}\mathcal{R}$-contractive map and related fixed point results along with the existence and uniqueness of coincidence points and common fixed points using Picard-Jungck iteration process in $C^*_{AV}\mathcal{R}$-metric space. As an application, the results obtained are applied on C^*_{AV}-metric space together with a directed graph. The results proved in this chapter have been discussed in [6,7].

[6]Malhotra, A., Kumar, D., and Park, C. (2022). C^*-algebra valued $\mathcal{R}$-metric space and fixed point theorems. AIMS Mathematics, 7(4), 6550-6554.

[7]Malhotra, A., and Kumar, D. (2023). Coincidence Point and Common Fixed Point in C*-algebra Valued $\mathcal{R}$-metric Space using Picard-Jungck Iteration Process with Application in Graph Theory. (Communicated).

4.2 Generalized Contraction Maps and Fixed Point Results

Definition 4.2.1. *For a set* $\mho$ *together with* $\mathbb{B}$ *and relation* $\mathcal{R}$*, define* $d : \mho \times \mho \to \mathbb{B}_+$*. Then,* $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ *is c.t.b a* $C^*_{AV}\mathcal{R}$***-metric space*** *if:*

(i) $(\mho, \mathbb{B}, d)$ *is a* C^*_{AV}*-metric space;*

(ii) $\mathcal{R}$ *is a binary relation on* $\mho$.

Definition 4.2.2. *For a* $C^*_{AV}\mathcal{R}$*-metric space* $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$*,* $\varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ *is c.t.b a* $C^*_{AV}\mathcal{R}$***-contractive map*** *if* $\forall\ \varrho, \varsigma \in \mho_{\mathbb{B}}$ *with* $(\varrho, \varsigma) \in \mathcal{R}$*,* $\exists\ \nu \in \mathbb{B}$ *where* $\|\nu\| < 1$ *s.t*

$$d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) \preceq \nu^* d_{\mathcal{R}}(\varrho, \varsigma)\nu.$$

Example 4.2.1. *Let* $\mho = \mathbb{R}$*,* $\mathbb{B} = M_2(\mathbb{R})$ *with involution on* $\mathbb{B}$ *defined as* $A^* = A^t$ $\forall\ A \in \mathbb{B}$*, where* A^t *denotes the transpose of matrix* A *and zero element* $\theta_{\mathbb{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \hat{0}$*. For* $A = [a_{ij}]$*, let*

$$\|A\| = \max_{1 \leqslant i,j \leqslant 2} |a_{ij}|.$$

Define $d : \mho \times \mho \to \mathbb{B}_+$ *as*

$$d(\varrho, \varsigma) = \begin{bmatrix} |\varrho - \varsigma| & 0 \\ 0 & |\varrho - \varsigma| \end{bmatrix}.$$

In such case, for $A = [a_{ij}], B = [b_{ij}] \in \mathbb{B}$*, we say* $A \preceq B$ *iff* $a_{ij} \leqslant b_{ij}\ \forall\ i, j = 1, 2$*. Define* $\mathcal{R}$ *as* $(\varrho, \varsigma) \in \mathcal{R}$ *iff* $\varrho.\varsigma = 0$ *s.t* $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ *is a* $C^*_{AV}\mathcal{R}$*-metric space. Let* $\varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ *be defined as*

$$\varphi(\varrho) = \begin{cases} 3/25 & for\ \varrho \in \mathbb{N}; \\ 0 & otherwise. \end{cases}$$

Now, for $(\varrho, \varsigma) \in \mathcal{R}$*, we must have either* ϱ *or* ς *or both to be zero. Consider* $\varsigma = 0$*, then we have the following cases:*

Case (i)*: If* $\varrho \in \mho_{\mathbb{B}} - \mathbb{N}$*. Then, we have* $\varphi\varrho = 0$ *and eventually* $d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) = d_{\mathcal{R}}(0, 0) = \hat{0}$*. For any* $A \in \mathbb{B}$ *with* $\|A\| < 1$*, we have* $A^* d_{\mathcal{R}}(\varrho, 0)A \succeq \hat{0}$ *and thus*

$$d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) \preceq A^* d_{\mathcal{R}}(\varrho, \varsigma)A.$$

Case (ii): *If $\varrho \in \mathbb{N}$. Then*

$$d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) = d_{\mathcal{R}}(3/25, 0) = \begin{bmatrix} 3/25 & 0 \\ 0 & 3/25 \end{bmatrix}, \tag{4.1}$$

and, for $A = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix}$, we have

$$A^* d_{\mathcal{R}}(\varrho, \varsigma) A = A^t d_{\mathcal{R}}(\varrho, \varsigma) A = \begin{bmatrix} \varrho/5 & 0 \\ 0 & \varrho/5 \end{bmatrix}. \tag{4.2}$$

Thus, from (4.1) and (4.2), we obtain $d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) \preceq A^* d_{\mathcal{R}}(\varrho, \varsigma) A$ for any $\varrho \in \mathbb{N}$. The case when $\varrho = 0$ can be proved in a similar manner as above. Hence, φ is a $C^*_{AV}\mathcal{R}$-contractive map.

Example 4.2.2. *Consider $\mho = [0, 1]$, $\mathbb{B} = M_2(\mathbb{C})$ with involution on $\mathbb{B}$ defined as $A^* = A^H \ \forall\ A \in \mathbb{B}$, zero element $\theta_{\mathbb{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \hat{0}$. For $A = [a_{ij}]$, let $\|A\| = \max_{1 \leqslant i,j \leqslant 2} |a_{ij}|$. Define $d : \mho \times \mho \to \mathbb{B}_+$ as*

$$d(\varrho, \varsigma) = \begin{bmatrix} |\varrho - \varsigma|^{\zeta} & 0 \\ 0 & |\varrho - \varsigma|^{\zeta} \end{bmatrix}, \text{where } \zeta \geqslant 1.$$

In such a case, for $A = [a_{ij}], B = [b_{ij}] \in \mathbb{B}$, we say $A \preceq B$ iff $|a_{ij}| \leqslant |b_{ij}| \ \forall\ i, j = 1, 2$. Let $\mathcal{R}$ be defined as $(\varrho, \varsigma) \in \mathcal{R}$ iff $\varrho.\varsigma = 0$ then, $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ is a $C^*_{AV}\mathcal{R}$-metric space. Let $\varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ as

$$\varphi(\varrho) = \begin{cases} \varrho/4 & for\ \varrho \in \mho_{\mathbb{B}} \cap \mathbb{Q}; \\ 0 & otherwise. \end{cases}$$

Now, for $(\varrho, \varsigma) \in \mathcal{R}$, we must have either ϱ or ς or both to be zero. Consider $\varsigma = 0$, then

Case (i): *Let $\varrho \in \mho_{\mathbb{B}} - \mathbb{Q}$. Then, we have $d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) = d_{\mathcal{R}}(0, 0) = \hat{0}$. For each $A \in \mathbb{B}$ with $\|A\| < 1$, we have $\hat{0} \preceq A^* d_{\mathcal{R}}(\varrho, 0) A$ and thus $d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) \preceq A^* d_{\mathcal{R}}(\varrho, \varsigma) A$.*

Case (ii): *If $\varrho \in \mho_{\mathbb{B}} \cap \mathbb{Q}$. Then*

$$d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) = d_{\mathcal{R}}(\varrho/4, 0) = \begin{bmatrix} (\varrho/4)^{\zeta} & 0 \\ 0 & (\varrho/4)^{\zeta} \end{bmatrix}, \tag{4.3}$$

and, for $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$, *we have*

$$A^* d_{\mathcal{R}}(\varrho, \varsigma) A = A^H d_{\mathcal{R}}(\varrho, \varsigma) A = \begin{bmatrix} \varrho^{\zeta}/2 & 0 \\ 0 & \varrho^{\zeta}/2 \end{bmatrix}. \tag{4.4}$$

Thus, from (4.3) and (4.4), we obtain $d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) \preceq A^* d_{\mathcal{R}}(\varrho, \varsigma) A$ *for any* $\varrho \in \mho_{\mathbb{B}} \cap \mathbb{Q}$. *The case when* $\varrho = 0$ *can be proved in a similar manner as above. Hence,* φ *is a* $C^*_{AV}\mathcal{R}$*-contractive map.*

Definition 4.2.3. *Let* $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ *be a* $C^*_{AV}\mathcal{R}$*-metric space and* $\varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ *be a self-map, then*

(i) an $\mathcal{R}$***-sequence*** $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}} \subset \mho_{\mathbb{B}}$ *is c.t.b* ***convergent to*** $\varrho \in \mho_{\mathbb{B}}$ *if for any* $\epsilon > 0$, $\exists\ \varpi_0 \in \mathbb{N}$ *s.t* $\|d_{\mathcal{R}}(\varrho_{\varpi}, \varrho)\| \leqslant \epsilon\ \forall\ \varpi \geqslant \varpi_0$.

(ii) φ *is c.t.b* $\mathcal{R}$***-continuous*** *at* $\varrho \in \mho_{\mathbb{B}}$ *if for any* $\mathcal{R}$*-sequence* $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}} \subset \mho_{\mathbb{B}}$ *with* $\lim\limits_{\varpi \to +\infty} \|d_{\mathcal{R}}(\varrho_{\varpi}, \varrho)\| = \theta_{\mathbb{B}}$ *implies* $\lim\limits_{\varpi \to +\infty} \|d_{\mathcal{R}}(\varphi\varrho_{\varpi}, \varphi\varrho)\| = \theta_{\mathbb{B}}$. *Also,* φ *is* $\mathcal{R}$***-continuous on*** $\mho_{\mathbb{B}}$ *if* $\forall\ \varrho \in \mho_{\mathbb{B}}$, φ *is* $\mathcal{R}$*-continuous at* ϱ.

(iii) an $\mathcal{R}$***-sequence*** $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}} \subset \mho_{\mathbb{B}}$ *is c.t.b* $\mathcal{R}$***-Cauchy*** *if for any* $\epsilon > 0$, $\exists\ \varpi_0 \in \mathbb{N}$ *s.t* $\|d_{\mathcal{R}}(\varrho_{\varpi}, \varrho_{\varpi^*})\| \leqslant \epsilon\ \forall\ \varpi, \varpi^* \geqslant \varpi_0$.

(iv) $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ *is c.t.b a* ***complete*** $C^*_{AV}\mathcal{R}$***-metric space*** *if each* $\mathcal{R}$*-Cauchy sequence is convergent in* $\mho$.

(v) a subset $Z_{\mathbb{B}}$ *of* $\mho_{\mathbb{B}}$ *is c.t.b a complete* $C^*_{AV}\mathcal{R}$*-subspace if* $(Z_{\mathbb{B}}, d_{\mathcal{R}})$ *is a complete* $C^*_{AV}\mathcal{R}$*-metric space.*

Example 4.2.3. *Consider* $\mho = [0, 3]$ *and* $\mathbb{B}$ *be the set of all* 2×2 *diagonal matrices on* $\mathbb{C}$. *Let involution on* $A \in \mathbb{B}$ *be defined as* $A^* = A^H$, *where* A^H *is the conjugate transpose of matrix* $A = [a_{ij}]$ *and* $\|A\| = \max\limits_{1 \leqslant i,j \leqslant 2} |\ a_{ij}\ |$. *Define* $d : \mho \times \mho \to \mathbb{B}_+$ *as*

$$d(\varrho, \varsigma) = \begin{bmatrix} |\ \varrho - \varsigma\ |^{\lambda} & 0 \\ 0 & |\ \varrho - \varsigma\ |^{\lambda} \end{bmatrix}, \text{ where } \lambda \geqslant 1.$$

Let $\mathcal{R}$ *be defined as* $(\varrho, \varsigma) \in \mathcal{R}$ *iff* $\varrho.\varsigma = 0$ *then,* $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ *is a* $C^*_{AV}\mathcal{R}$*-metric space. Let* $\varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ *be defined as*

$$\varphi(\varrho) = \begin{cases} \varrho/4 & for\ \rho \in \mho_{\mathbb{B}} \cap \mathbb{Q}; \\ 0 & otherwise. \end{cases}$$

For any convergent $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$, we must have $\lim_{\varpi\to+\infty}\varrho_\varpi = 0$ and so, $\lim_{\varpi\to+\infty}\varphi\varrho_\varpi = 0 = \varphi 0$. Thus, φ is an $\mathcal{R}$-continuous map.

Theorem 4.2.4. *Let $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ be a $C^*_{AV}\mathcal{R}$-metric space and let $Z_{\mathbb{B}}$ be a complete $C^*_{AV}\mathcal{R}$-subspace of $\mho_{\mathbb{B}}$. If $\varphi:\mho_{\mathbb{B}}\to\mho_{\mathbb{B}}$ is a self-map on $\mho_{\mathbb{B}}$ s.t:*

(I) $\varphi(\mho_{\mathbb{B}})\subseteq Z_{\mathbb{B}}$;

(II) φ is $\mathcal{R}$-preserving;

(III) $\exists$ some $\varrho_0\in\mho_{\mathbb{B}}$ s.t $(\varrho_0,\varsigma)\in\mathcal{R}\ \forall\ \varsigma\in\varphi(\mho_{\mathbb{B}})$;

(IV) φ is $C^*_{AV}\mathcal{R}$-contractive map;

(V) Either φ is $\mathcal{R}$-continuous or $\mathcal{R}$ is $d_{\mathcal{R}}$-self closed on $Z_{\mathbb{B}}$.

Then, φ possesses a unique fixed point.

Proof. Define a sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ in $\mho_{\mathbb{B}}$ where $\varrho_1=\varphi\varrho_0$, $\varrho_{\varpi+1}=\varphi^{\varpi}\varrho_0=\varphi\varrho_\varpi\ \forall\ \varpi\in\mathbb{N}$. By condition *(III)* for some $\varrho_0\in\mho_{\mathbb{B}}$, $(\varrho_0,\varphi\varrho_0)\in\mathcal{R}$, that is $(\varrho_0,\varrho_1)\in\mathcal{R}$. Since, φ is $\mathcal{R}$-preserving, so we have $(\varphi\varrho_0,\varphi\varrho_1)=(\varrho_1,\varrho_2)\in\mathcal{R}$. On continuous use of $\mathcal{R}$-preserving property of φ, we get $(\varrho_\varpi,\varrho_{\varpi+1})\in\mathcal{R}\ \forall\ \varpi\in\mathbb{N}'$. Thus, $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ is a $\mathcal{R}$-sequence in $\mho_{\mathbb{B}}$. Next, by using condition *(IV)*, we obtain

$$\begin{aligned} d_{\mathcal{R}}(\varrho_{\varpi+1},\varrho_\varpi) &= d_{\mathcal{R}}(\varphi\varrho_\varpi,\varphi\varrho_{\varpi-1})\preceq \nu^* d_{\mathcal{R}}(\varrho_\varpi,\varrho_{\varpi-1})\nu \\ &= \nu^* d_{\mathcal{R}}(\varphi\varrho_{\varpi-1},\varphi\varrho_{\varpi-2})\nu \\ &\preceq (\nu^*)^2 d_{\mathcal{R}}(\varrho_{\varpi-1},\varrho_{\varpi-2})\nu^2\preceq\cdots\preceq(\nu^*)^{\varpi}\vartheta\nu^{\varpi}, \end{aligned} \tag{4.5}$$

where $\vartheta=d_{\mathcal{R}}(\varrho_1,\varrho_0)$ and $\nu\in\mathbb{B}$ with $\|\nu\|<1$. Let $\varpi>\varpi^*$, for $\varpi,\varpi^*\in\mathbb{N}'$, and using triangle inequality along with (4.5), we get

$$\begin{aligned} d_{\mathcal{R}}(\varrho_{\varpi+1},\varrho_{\varpi^*}) &\preceq d_{\mathcal{R}}(\varrho_{\varpi+1},\varrho_\varpi)+\cdots+d_{\mathcal{R}}(\varrho_{\varpi^*+2},\varrho_{\varpi^*+1})+d_{\mathcal{R}}(\varrho_{\varpi^*+1},\varrho_{\varpi^*}) \\ &\preceq \sum_{\xi=\varpi^*}^{\varpi}(\nu^*)^{\xi}\vartheta\nu^{\xi}=\sum_{\xi=\varpi^*}^{\varpi}(\nu^{\xi})^*\vartheta^{1/2}\vartheta^{1/2}\nu^{\xi} \\ &= \sum_{\xi=\varpi^*}^{\varpi}(\vartheta^{1/2}\nu^{\xi})^*(\vartheta^{1/2}\nu^{\xi})=\sum_{\xi=\varpi^*}^{\varpi}|\vartheta^{1/2}\nu^{\xi}|^2\preceq\left\|\sum_{\xi=\varpi^*}^{\varpi}|\vartheta^{1/2}\nu^{\xi}|^2\right\|I_{\mathbb{B}} \\ &\preceq \sum_{\xi=\varpi^*}^{\varpi}\|\vartheta^{1/2}\|^2\|\nu^{\xi}\|^2 I_{\mathbb{B}}\preceq\|\vartheta^{1/2}\|^2\sum_{\xi=\varpi^*}^{+\infty}\|\nu\|^{2\xi}I_{\mathbb{B}} \\ &= \|\vartheta^{1/2}\|^2\frac{\|\nu\|^{2\varpi^*}}{(1-\|\nu\|)}I_{\mathbb{B}}\to\theta_{\mathbb{B}}\quad\text{as}\quad\varpi^*\to+\infty. \end{aligned}$$

Thus, $\{\varrho_\varpi = \varphi\varrho_{\varpi-1}\}_{\varpi\in\mathbb{N}}$ is an $\mathcal{R}$-Cauchy sequence in $Z_{\mathbb{B}}$ so $\exists\ \varrho \in Z_{\mathbb{B}} \subset \mho_{\mathbb{B}}$ with $\lim\limits_{\varpi\to+\infty} \varrho_\varpi = \varrho$.

Case (i): Consider φ be a $\mathcal{R}$-continuous map. Since $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ is $\mathcal{R}$-sequence with $\lim\limits_{\varpi\to+\infty} \varrho_\varpi = \varrho$. Then,

$$\varrho = \lim_{\varpi\to+\infty} \varrho_{\varpi+1} = \lim_{\varpi\to+\infty} \varphi\varrho_\varpi = \varphi\varrho.$$

Thus, φ possesses a fixed point.

Case (ii): Consider $\mathcal{R}$ be $d_{\mathcal{R}}$-self closed on $Z_{\mathbb{B}}$. Since $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ is an $\mathcal{R}$-sequence where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$. Then, $\exists$ a sub-sequence $\{\varrho_{\varpi_k}\}_{k\in\mathbb{N}}$ of $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ s.t $[\varrho_{\varpi_k}, \varrho] \in \mathcal{R}|_{Z_{\mathbb{B}}}$. Now,

$$d_{\mathcal{R}}(\varrho_{\varpi_k+1}, \varphi\varrho) = d_{\mathcal{R}}(\varphi\varrho_{\varpi_k}, \varphi\varrho) \preceq \nu^* d_{\mathcal{R}}(\varrho_{\varpi_k}, \varrho)\nu \to \theta_{\mathbb{B}} \quad \text{as} \quad k \to +\infty.$$

Therefore, $\varrho_{\varpi_k} \to \varphi\varrho$ as $k \to +\infty$ and by uniqueness of limit, we have $\varrho = \varphi\varrho$. Thus, φ possesses a fixed point. Next, let ς be s.t $\varphi\varsigma = \varsigma$ infact $\varphi^\varpi\varsigma = \varsigma$. By condition *(III)*, $\exists\ \varrho_0 \in \mho_{\mathbb{B}}$ s.t $(\varrho_0, \varsigma) = (\varrho_0, \varphi\varsigma) \in \mathcal{R}$. Since φ is $\mathcal{R}$-preserving, so $(\varphi\varrho_0, \varphi\varsigma) \in \mathcal{R}$ implies $(\varphi^\varpi\varrho_0, \varphi^\varpi\varsigma) \in \mathcal{R}$. On using contractive condition of φ, we have

$$\begin{aligned} d_{\mathcal{R}}(\varrho_\varpi, \varsigma) = d_{\mathcal{R}}(\varphi^\varpi\varrho_0, \varphi^\varpi\varsigma) &\preceq \nu^* d_{\mathcal{R}}(\varphi^{\varpi-1}\varrho_0, \varphi^{\varpi-1}\varsigma)\nu \\ &\preceq (\nu^*)^2 d_{\mathcal{R}}(\varphi^{\varpi-2}\varrho_0, \varphi^{\varpi-2}\varsigma)(\nu)^2 \\ &\preceq \cdots \preceq (\nu^*)^\varpi d_{\mathcal{R}}(\varrho_0, \varsigma)(\nu)^\varpi. \end{aligned} \tag{4.6}$$

Taking limit as $\varpi \to +\infty$ in (4.6), we get $d_{\mathcal{R}}(\varrho, \varsigma) = \theta_{\mathbb{B}}$. Hence, φ possesses a unique fixed point. □

Example 4.2.5. *Consider the $C^*_{AV}\mathcal{R}$-metric space as discussed in Example 4.2.2, where the defined self-map φ on $\mho_{\mathbb{B}}$ is a $C^*_{AV}\mathcal{R}$-contractive map and $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ is a complete $C^*_{AV}\mathcal{R}$-metric space. Also, $\exists\ \varrho_0 = 0 \in \mho_{\mathbb{B}}$ s.t $(\varrho_0, \varphi\varrho_0) \in \mathcal{R}$. Further, φ is $\mathcal{R}$-preserving (since for any $(\varrho, \varsigma) \in \mathcal{R}$ implies $\varrho = 0$ or/and $\varsigma = 0$ implies $(\varphi\varrho, \varphi\varsigma) \in \mathcal{R}$) and for any convergent $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ we must have $\lim\limits_{\varpi\to+\infty} \varrho_\varpi = 0$ and so is $\lim\limits_{\varpi\to+\infty} \varphi\varrho_\varpi = 0 = \varphi 0$. Thus, φ is a $\mathcal{R}$-continuous map. Therefore, by Theorem 4.2.4 (case when $Z_{\mathbb{B}} = \mho_{\mathbb{B}}$), φ possesses a fixed point viz. $\varrho = 0$.*

The upcoming theorem proves an analogues result of Kannan (1968) endowed with a binary relation $\mathcal{R}$ under similar setting.

Theorem 4.2.6. *Let $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ be a $C^*_{AV}\mathcal{R}$-metric space and let $Z_{\mathbb{B}}$ be a complete $C^*_{AV}\mathcal{R}$-subspace of $\mho_{\mathbb{B}}$. If $\varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ is a self-map on $\mho_{\mathbb{B}}$ s.t:*

(I) $\varphi(\mho_{\mathbb{B}}) \subseteq Z_{\mathbb{B}}$;

(II) φ *is* $\mathcal{R}$*-preserving;*

(III) There exists some $\varrho_0 \in \mho_{\mathbb{B}}$ *where* $(\varrho_0, \varsigma) \in \mathcal{R}\ \forall\ \varsigma \in \varphi(\mho_{\mathbb{B}})$;

(IV) For all $\varrho, \varsigma \in \mho_{\mathbb{B}}$ *with* $(\varrho, \varsigma) \in \mathcal{R}$, $\exists\ \nu \in \mathbb{B}'_+$, *where* $\|\nu\| < 1/2$ *s.t*

$$d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) \preceq \nu(d_{\mathcal{R}}(\varphi\varrho, \varrho) + d_{\mathcal{R}}(\varphi\varsigma, \varsigma));$$

(V) Either φ *is* $\mathcal{R}$*-continuous or* $\mathcal{R}$ *is* $d_{\mathcal{R}}$*-self closed on* $Z_{\mathbb{B}}$.

Then, φ *possesses a unique fixed point.*

Proof. By working done in Theorem 4.2.4, we get an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ in $\mho_{\mathbb{B}}$ where $(\varrho_\varpi, \varrho_{\varpi+1}) \in \mathcal{R}$ for $\varpi \in \mathbb{N}'$. Using condition *(IV)*, we get

$$\begin{aligned} d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_\varpi) = d_{\mathcal{R}}(\varphi\varrho_\varpi, \varphi\varrho_{\varpi-1}) &\preceq \nu(d_{\mathcal{R}}(\varphi\varrho_\varpi, \varrho_\varpi) + d_{\mathcal{R}}(\varphi\varrho_{\varpi-1}, \varrho_{\varpi-1})) \\ &= \nu(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_\varpi) + d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi-1})), \\ \text{therefore,} \quad (I_{\mathbb{B}} - \nu)d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_\varpi) &\preceq \nu d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi-1}). \end{aligned}$$

Now, $\nu \in \mathbb{B}'_+$ and $\|\nu\| < 1/2$. Thus, by Lemma 1.2.1, $(I_{\mathbb{B}} - \nu)$ and $\nu(I_{\mathbb{B}} - \nu)^{-1} \in \mathbb{B}'_+$ with $\|\nu(I_{\mathbb{B}} - \nu)^{-1}\| < 1$, so we have

$$\begin{aligned} d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_\varpi) &\preceq \nu(I_{\mathbb{B}} - \nu)^{-1} d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi-1}), \\ &= b d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi-1}) \\ &\preceq \cdots \preceq b^\varpi d_{\mathcal{R}}(\varrho_1, \varrho_0) = b^\varpi \vartheta, \end{aligned} \tag{4.7}$$

where $b = \nu(I_{\mathbb{B}} - \nu)^{-1}$ and $\vartheta = d_{\mathcal{R}}(\varrho_1, \varrho_0)$. Let $\varpi > \varpi^*$, for $\varpi, \varpi^* \in \mathbb{N}$, and using triangle inequality along with (4.7), we get

$$\begin{aligned} d_{\mathcal{R}}(\varrho_{\varpi^*}, \varrho_{\varpi+1}) &\preceq d_{\mathcal{R}}(\varrho_{\varpi^*}, \varrho_{\varpi^*+1}) + d_{\mathcal{R}}(\varrho_{\varpi^*+1}, \varrho_{\varpi^*+2}) + \cdots + d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi+1}) \\ &= \sum_{\xi=\varpi^*}^{\varpi} b^\xi \vartheta = \sum_{\xi=\varpi^*}^{\varpi} b^{\xi/2} b^{\xi/2} \vartheta^{1/2} \vartheta^{1/2} = \sum_{\xi=\varpi^*}^{\varpi} (b^{\xi/2}\vartheta^{1/2})^*(b^{\xi/2}\vartheta^{1/2}) \\ &= \sum_{\xi=\varpi^*}^{\varpi} |b^{\xi/2}\vartheta^{1/2}|^2 \preceq \left\| \sum_{\xi=\varpi^*}^{\varpi} |b^{\xi/2}\vartheta^{1/2}|^2 \right\| I_{\mathbb{B}} \\ &\preceq \|\vartheta^{1/2}\|^2 \sum_{\xi=\varpi^*}^{+\infty} \|b\|^\xi I_{\mathbb{B}} = \|\vartheta^{1/2}\|^2 \frac{\|b\|^{\varpi^*}}{1 - \|b\|} I_{\mathbb{B}} \to \theta_{\mathbb{B}} \text{ as } \varpi^* \to +\infty. \end{aligned}$$

Thus, $\{\varrho_\varpi = \varphi\varrho_{\varpi-1}\}_{\varpi\in\mathbb{N}'}$ is an $\mathcal{R}$-Cauchy sequence in $Z_\mathbb{B}$ and as $Z_\mathbb{B}$ is complete $C^*_{AV}\mathcal{R}$-subspace of $\mho_\mathbb{B}$ so, $\exists\ \varrho \in Z_\mathbb{B} \subset \mho_\mathbb{B}$ s.t $\lim_{\varpi\to+\infty} \varrho_\varpi = \varrho$.

Case (i): Consider φ be an $\mathcal{R}$-continuous map. Then

$$\varrho = \lim_{\varpi\to+\infty} \varrho_{\varpi+1} = \lim_{\varpi\to+\infty} \varphi\varrho_\varpi = \varphi\varrho.$$

Thus, φ possesses a fixed point.

Case (ii): Consider $\mathcal{R}$ be $d_\mathcal{R}$-self closed on $Z_\mathbb{B}$. Since $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ is a $\mathcal{R}$-sequence where $\varrho_\varpi \to \varrho$ as $\varpi \to +\infty$. Then, $\exists$ a sub-sequence $\{\varrho_{\varpi_k}\}_{k\in\mathbb{N}}$ of $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ s.t $[\varrho_{\varpi_k}, \varrho] \in \mathcal{R}|_{Z_\mathbb{B}}$. Now,

$$d_\mathcal{R}(\varrho_{\varpi_k+1}, \varphi\varrho) = d_\mathcal{R}(\varphi\varrho_{\varpi_k}, \varphi\varrho) \preceq \nu(d_\mathcal{R}(\varphi\varrho_{\varpi_k}, \varrho_{\varpi_k}) + d_\mathcal{R}(\varphi\varrho, \varrho)).$$

Taking norm on both sides, we get

$$\begin{aligned} \|d_\mathcal{R}(\varrho_{\varpi_k+1}, \varphi\varrho)\| &\leqslant \|\nu\|(\|d_\mathcal{R}(\varphi\varrho_{\varpi_k}, \varrho_{\varpi_k})\| + \|d_\mathcal{R}(\varphi\varrho, \varrho))\|) \\ &= \|\nu\|(\|d_\mathcal{R}(\varrho_{\varpi_{k+1}}, \varrho_{\varpi_k})\| + \|d_\mathcal{R}(\varphi\varrho, \varrho)\|). \end{aligned}$$

Taking limit as $k \to +\infty$ on both sides, we get

$$\|d_\mathcal{R}(\varrho, \varphi\varrho)\| \leqslant \|\nu\|\|d_\mathcal{R}(\varphi\varrho, \varrho)\|.$$

For $\|\nu\| < 1/2$, above holds only when $\|d_\mathcal{R}(\varphi\varrho, \varrho)\| = 0$. Thus, φ possesses a fixed point. Next, let ς be s.t $\varphi\varsigma = \varsigma$ infact $\varphi^\varpi\varsigma = \varsigma$. By condition *(III)*, $\exists\ \varrho_0 \in \mho_\mathbb{B}$ where $(\varrho_0, \varsigma) = (\varrho_0, \varphi\varsigma) \in \mathcal{R}$. Since φ is $\mathcal{R}$-preserving, so $(\varphi\varrho_0, \varphi\varsigma) \in \mathcal{R}$ implies $(\varphi^\varpi\varrho_0, \varphi^\varpi\varsigma) \in \mathcal{R}$ for $\varpi \in \mathbb{N}$. On using condition *(IV)*, we have

$$\begin{aligned} d_\mathcal{R}(\varrho_\varpi, \varsigma) = d_\mathcal{R}(\varphi^\varpi\varrho_0, \varphi^\varpi\varsigma) &\preceq \nu(d_\mathcal{R}(\varphi^\varpi\varrho_0, \varphi^{\varpi-1}\varrho_0) + d_\mathcal{R}(\varphi^\varpi\varsigma, \varphi^{\varpi-1}\varsigma)) \\ &= \nu d_\mathcal{R}(\varrho_\varpi, \varrho_{\varpi-1}) \preceq \cdots \preceq \nu^\varpi d_\mathcal{R}(\varrho_1, \varrho_0). \quad (4.8) \end{aligned}$$

On taking limit as $\varpi \to +\infty$ in (4.8), we get $d_\mathcal{R}(\varrho, \varsigma) = \theta_\mathbb{B}$. Hence, φ possesses a unique fixed point. □

In the next theorem, we establish the $\mathcal{R}$ analog of the Chatterjea (1972) contractive condition for a $C^*_{AV}\mathcal{R}$-metric space.

Theorem 4.2.7. *Let $(\mho_\mathbb{B}, d_\mathcal{R})$ be a $C^*_{AV}\mathcal{R}$-metric space and let $Z_\mathbb{B}$ be a complete $C^*_{AV}\mathcal{R}$-subspace of $\mho_\mathbb{B}$. If $\varphi : \mho_\mathbb{B} \to \mho_\mathbb{B}$ is a self-map on $\mho_\mathbb{B}$ s.t:*

(I) $\varphi(\mho_{\mathbb{B}}) \subseteq Z_{\mathbb{B}}$*;*

(II) φ *is* $\mathcal{R}$*-preserving;*

(III) There exists some $\varrho_0 \in \mho_{\mathbb{B}}$ *with* $(\varrho_0, \varsigma) \in \mathcal{R}\ \forall\ \varsigma \in \varphi(\mho_{\mathbb{B}})$*;*

(IV) For all $\varrho, \varsigma \in \mho_{\mathbb{B}}$ *with* $(\varrho, \varsigma) \in \mathcal{R}$*,* $\exists\ \nu \in \mathbb{B}'_+$*, where* $\|\nu\| < 1/2$ *s.t*

$$d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) \preceq \nu(d_{\mathcal{R}}(\varphi\varrho, \varsigma) + d_{\mathcal{R}}(\varphi\varsigma, \varrho));$$

(V) Either φ *is* $\mathcal{R}$*-continuous or* $\mathcal{R}$ *is* $d_{\mathcal{R}}$*-self closed on* $Z_{\mathbb{B}}$*.*

Then, φ *possesses a unique fixed point.*

Proof. Working on the lines of Theorem 4.2.4, we obtain an $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ in $\mho_{\mathbb{B}}$ where $(\varrho_\varpi, \varrho_{\varpi+1}) \in \mathcal{R}$ for $\varpi \in \mathbb{N}'$. Using condition *(IV)*, we get

$$\begin{aligned}
d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_\varpi) &= d_{\mathcal{R}}(\varphi\varrho_\varpi, \varphi\varrho_{\varpi-1}) \\
&\preceq \nu(d_{\mathcal{R}}(\varphi\varrho_\varpi, \varrho_{\varpi-1}) + d_{\mathcal{R}}(\varphi\varrho_{\varpi-1}, \varrho_\varpi)) \\
&= \nu d_{\mathcal{R}}(\varphi\varrho_\varpi, \varrho_{\varpi-1}) \\
&= \nu d_{\mathcal{R}}(\varphi\varrho_\varpi, \varphi\varrho_{\varpi-2}) \\
&\preceq \nu(d_{\mathcal{R}}(\varphi\varrho_\varpi, \varphi\varrho_{\varpi-1}) + d_{\mathcal{R}}(\varphi\varrho_{\varpi-1}, \varphi\varrho_{\varpi-2})) \\
&\preceq \nu(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_\varpi) + d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi-1})), \\
\text{therefore,} \quad (I_{\mathbb{B}} - \nu)d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_\varpi) &\preceq \nu d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi-1}).
\end{aligned}$$

Now, $\nu \in \mathbb{B}'_+$ and $\|\nu\| < 1/2$. Thus, by Lemma 1.2.1, $(I_{\mathbb{B}} - \nu)$ and $\nu(I_{\mathbb{B}} - \nu)^{-1} \in \mathbb{B}'_+$ with $\|\nu(I_{\mathbb{B}} - \nu)^{-1}\| < 1$, so we have

$$\begin{aligned}
d_{\mathcal{R}}(\varrho_{\varpi+1}, \varrho_\varpi) &\preceq \nu(I_{\mathbb{B}} - \nu)^{-1} d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi-1}) \\
&= b d_{\mathcal{R}}(\varrho_\varpi, \varrho_{\varpi-1}) \preceq \cdots \preceq b^\varpi d_{\mathcal{R}}(\varrho_1, \varrho_0) = b^\varpi \vartheta,
\end{aligned}$$

where $b = \nu(I_{\mathbb{B}} - \nu)^{-1}$ and $\vartheta = d_{\mathcal{R}}(\varrho_1, \varrho_0)$. By the working done in Theorem 4.2.6, we obtain that $\{\varrho_\varpi = \varphi\varrho_{\varpi-1}\}_{\varpi\in\mathbb{N}'}$ is an $\mathcal{R}$-Cauchy sequence in $Z_{\mathbb{B}}$, and since $Z_{\mathbb{B}}$ is complete $C^*_{AV}\mathcal{R}$-subspace of $\mho_{\mathbb{B}}$, so $\exists\ \varrho \in Z_{\mathbb{B}} \subset \mho_{\mathbb{B}}$ s.t $\lim_{\varpi\to+\infty} \varrho_\varpi = \varrho$.

Case (i): Consider φ be a $\mathcal{R}$-continuous map. Then

$$\varrho = \lim_{\varpi\to+\infty} \varrho_{\varpi+1} = \lim_{\varpi\to+\infty} \varphi\varrho_\varpi = \varphi\varrho.$$

Thus, φ possesses a fixed point.

Case (ii): Consider $\mathcal{R}$ be $d_{\mathcal{R}}$-self closed on $Z_{\mathbb{B}}$. Since $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}'}$ is $\mathcal{R}$-sequence where $\varrho_{\varpi} \to \varrho$ as $\varpi \to +\infty$. Then, $\exists$ a sub-sequence $\{\varrho_{\varpi_k}\}_{k\in\mathbb{N}}$ of $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ s.t $[\varrho_{\varpi_k}, \varrho] \in \mathcal{R}|_{Z_{\mathbb{B}}}$. Now,

$$d_{\mathcal{R}}(\varrho_{\varpi_k+1}, \varphi\varrho) = d_{\mathcal{R}}(\varphi\varrho_{\varpi_k}, \varphi\varrho) \preceq \nu(d_{\mathcal{R}}(\varphi\varrho_{\varpi_k}, \varrho) + d_{\mathcal{R}}(\varphi\varrho, \varrho_{\varpi_k})).$$

Taking norm on both sides, we get

$$\begin{aligned} \|d_{\mathcal{R}}(\varrho_{\varpi_k+1}, \varphi\varrho)\| &\leqslant \|\nu\|(\|d_{\mathcal{R}}(\varphi\varrho_{\varpi_k}, \varrho)\| + \|d_{\mathcal{R}}(\varphi\varrho, \varrho_{\varpi_k})\|) \\ &= \|\nu\|(\|d_{\mathcal{R}}(\varrho_{\varpi_{k+1}}, \varrho)\| + \|d_{\mathcal{R}}(\varphi\varrho, \varrho_{\varpi_k})\|). \end{aligned}$$

Taking limit as $k \to +\infty$ on both sides, we get

$$\|d_{\mathcal{R}}(\varrho, \varphi\varrho)\| \leqslant \|\nu\|\|d_{\mathcal{R}}(\varphi\varrho, \varrho)\|.$$

For $\|\nu\| < 1/2$, above holds only when $\|d_{\mathcal{R}}(\varphi\varrho, \varrho)\| = 0$. Thus, φ possesses a fixed point. Next, if ς is another fixed point of φ in $\mho_{\mathbb{B}}$, that is, $\varphi\varsigma = \varsigma$ infact $\varphi^{\varpi}\varsigma = \varsigma$. By condition *(III)*, $\exists$ $\varrho_0 \in \mho_{\mathbb{B}}$ s.t $(\varrho_0, \varsigma) = (\varrho_0, \varphi\varsigma) \in \mathcal{R}$. Since φ is $\mathcal{R}$-preserving, so $(\varphi\varrho_0, \varphi\varsigma) \in \mathcal{R}$ implies $(\varphi^{\varpi}\varrho_0, \varphi^{\varpi}\varsigma) \in \mathcal{R}$ for $\varpi \in \mathbb{N}$. On using condition *(IV)*, we have

$$\begin{aligned} d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma) = d_{\mathcal{R}}(\varphi^{\varpi}\varrho_0, \varphi^{\varpi}\varsigma) &\preceq \nu(d_{\mathcal{R}}(\varphi^{\varpi}\varrho_0, \varphi^{\varpi-1}\varsigma) + d_{\mathcal{R}}(\varphi^{\varpi}\varsigma, \varphi^{\varpi-1}\varrho_0)) \\ &= \nu(d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma) + d_{\mathcal{R}}(\varsigma, \varrho_{\varpi-1})), \\ \text{therefore,} \quad (I_{\mathbb{B}} - \nu)d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma) &\preceq \nu d_{\mathcal{R}}(\varsigma, \varrho_{\varpi-1}) \\ d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma) &\preceq \frac{\nu}{(I_{\mathbb{B}} - \nu)} d_{\mathcal{R}}(\varsigma, \varrho_{\varpi-1}) \\ &\preceq \cdots \preceq \frac{\nu^{\varpi}}{(I_{\mathbb{B}} - \nu)^{\varpi}} d_{\mathcal{R}}(\varsigma, \varrho_0). \end{aligned} \tag{4.9}$$

Using Lemma 1.2.1 and taking limit as $\varpi \to +\infty$ in (4.9), we get $d_{\mathcal{R}}(\varrho, \varsigma) = \theta_{\mathbb{B}}$. Hence, φ possesses a unique fixed point. □

Remark 4.2.8. *The results proved in Theorem 4.2.4, 4.2.6 and 4.2.7 holds true if $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ is considered complete $C^*_{AV}\mathcal{R}$-metric space.*

Example 4.2.9. *Consider $\mho = [0, 1)$ together with usual metric and let the unital C^*_{AV}-metric space $\mathbb{B} = (-\infty, +\infty)$ together with $\|\nu\| = |\nu|$, for $\nu, \gamma \in \mathbb{B}$ we have $\nu \preceq \gamma$ iff $\nu \leqslant \gamma$ and involution given by $\nu^* = \nu$. Define a relation $\mathcal{R}$ on $\mho_{\mathbb{B}}$ as*

$(\varrho,\varsigma)\in\mathcal{R}$ *iff* $\varrho.\varsigma\in\{\varrho,\varsigma\}$ *and let* $\varphi:\mho_{\mathbb{B}}\to\mho_{\mathbb{B}}$ *be defined as*

$$\varphi(\varrho)=\begin{cases}0 & for\ \varrho\in[0,2/9];\\ 1/9 & for\ \varrho\in(2/9,1),\end{cases}$$

then, $\mho_{\mathbb{B}}$ *is a complete* $C^*_{AV}\mathcal{R}$*-metric space. Next, for* $(\varrho,\varsigma)\in\mathcal{R}$ *we have either* $\varrho=0$ *or/and* $\varsigma=0$*. Let us consider* $\varsigma=0$*, so the following cases arise:*

Case (i)*: If* $\varrho\in[0,2/9]$*, then*

$$d_{\mathcal{R}}(\varphi\varrho,\varphi\varsigma)=d_{\mathcal{R}}(0,0)=0 \tag{4.10}$$

$$and,\quad d_{\mathcal{R}}(\varrho,\varsigma)=d_{\mathcal{R}}(\varrho,0)=\varrho. \tag{4.11}$$

So for any $\nu\in\mathbb{B}$ *with* $\|\nu\|<1$*, from (4.10) and (4.11), we obtain*

$$d_{\mathcal{R}}(\varphi\varrho,\varphi\varsigma)\preceq\nu^*d_{\mathcal{R}}(\varrho,\varsigma)\nu.$$

Case (ii)*: If* $\varrho\in(2/9,1)$*, then*

$$d_{\mathcal{R}}(\varphi\varrho,\varphi\varsigma)=d_{\mathcal{R}}(1/9,0)=\frac{1}{9} \tag{4.12}$$

$$and,\quad d_{\mathcal{R}}(\varrho,\varsigma)=d_{\mathcal{R}}(\varrho,0)=\varrho. \tag{4.13}$$

So for $\nu=\frac{1}{\sqrt{2}}$ *and from (4.12) and (4.13), we obtain*

$$d_{\mathcal{R}}(\varphi\varrho,\varphi\varsigma)\preceq\nu^*d_{\mathcal{R}}(\varrho,\varsigma)\nu.$$

Thus, φ *is* $C^*_{AV}\mathcal{R}$*-contractive map. Also,* φ *is* $\mathcal{R}$*-preserving and* $\mathcal{R}$*-continuous. Thus, by Theorem 4.2.4 (case when* $Z_{\mathbb{B}}=\mho_{\mathbb{B}}$*), we obtain that* φ *possesses a unique fixed point which in this case is* $\varrho=0$*.*

Remark 4.2.10. *The metric space* $(\mho,d)$ *considered in the above example is an incomplete metric space and thus violates the applicability of fixed point results proved in Banach (1922) and Ma et al. (2014).*

Example 4.2.11. *Let* $\mho=\{0,1,2,3,4\}$ *and* $\mathbb{B}=M_2(\mathbb{R})$ *with* $A^*=A$ *for each* $A\in\mathbb{B}$ *and* $\|A\|=\max_{1\leqslant i,j\leqslant 2}|a_{i,j}|$*. Let* $\theta_{\mathbb{B}}=\hat{0}=\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$*. Define* $d:\mho\times\mho\to\mathbb{B}_+$ *as*

$$d(\varrho,\varsigma)=\begin{cases}\hat{0} & for\ \varrho=\varsigma;\\ I_{\mathbb{B}} & otherwise,\end{cases}$$

where $I_{\mathbb{B}}$ denotes an identity matrix of order 2.
Define $\mathcal{R} = \{(0,0),(0,1),(1,1),(2,2),(2,4),(3,2),(3,4),(4,4)\}$, then $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ is a complete $C^*_{AV}\mathcal{R}$-metric space. Define a self-map $\varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ as $\varphi(0) = 0 = \varphi(1)$, $\varphi(2) = 1 = \varphi(3) = \varphi(4)$. Then, φ is an $\mathcal{R}$-preserving map (since for any $(\varrho,\varsigma) \in \mathcal{R}$, $(\varphi\varrho, \varphi\varsigma) = (0,0)$ or $(1,1)$). Also, as $0 \in \mho_{\mathbb{B}}$ where $(0,\varphi\varsigma) \in \mathcal{R}\ \forall\ \varsigma \in \mho_{\mathbb{B}}$ and φ is $\mathcal{R}$-continuous as for any $\mathcal{R}$-sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ with $\varrho_\varpi \to \varrho$, we have $\varphi\varrho_\varpi$ converge to either 0 or 1. Next, for a given $A \in \mathbb{B}$ with $0 < \|A\| < 1/2$, we have

$$d_{\mathcal{R}}(\varphi 0, \varphi 4) = d_{\mathcal{R}}(0,1) = I_{\mathbb{B}} > A(d_{\mathcal{R}}(\varphi 0, 0) + d_{\mathcal{R}}(\varphi 4, 4)) = A(\hat{0} + I_{\mathbb{B}}) = AI_{\mathbb{B}}.$$

Here, φ satisfies all the hypotheses of Theorem 4.2.6 (case when $Z_{\mathbb{B}} = \mho_{\mathbb{B}}$) and hence, φ possesses a unique fixed point viz. $\varrho = 0$.

Remark 4.2.12. *The self-map discussed in the above example does not satisfy Kannan (1968) contraction condition.*

Example 4.2.13. *Let $\mho = [0,2)$ equipped with usual metric and let C^*_{AV}-metric space $\mathbb{B} = (-\infty, +\infty)$ together with $\|\nu\| = |\nu|$, for $\nu, \gamma \in \mathbb{B}$, $\nu \preceq \gamma$ iff $\nu \leqslant \gamma$ and an involution given by $\nu^* = \nu$. Define a relation $\mathcal{R}$ on $\mho$ as $(\varrho,\varsigma) \in \mathcal{R}$ iff $\varrho.\varsigma \in \{0\}$ and let $\varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ be defined as*

$$\varphi(\varrho) = \begin{cases} \dfrac{\varrho^2}{6} & for\ \varrho \in [0,1); \\ 0 & otherwise. \end{cases}$$

Next, for $(\varrho,\varsigma) \in \mathcal{R}$ we have $\varrho = 0$ or/and $\varsigma = 0$. Let $\varsigma = 0$ so we have the following cases:

***Case (i)**: If $\varrho \in [0,1)$, then*

$$d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) = d_{\mathcal{R}}\Big(\frac{\varrho^2}{6}, 0\Big) = \frac{\varrho^2}{6} \tag{4.14}$$

$$\text{and, } \nu(d_{\mathcal{R}}(\varphi\varrho, \varsigma) + d_{\mathcal{R}}(\varphi\varsigma, \varrho)) = \nu\left(d_{\mathcal{R}}\Big(\frac{\varrho^2}{6}, 0\Big) + d_{\mathcal{R}}(0, \varrho)\right) = \nu\left(\frac{\varrho^2}{6} + \varrho\right). \tag{4.15}$$

Then for $\nu = 1/3$ and from (4.14) and (4.15), we obtain

$$d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) \preceq \nu(d_{\mathcal{R}}(\varphi\varrho, \varsigma) + d_{\mathcal{R}}(\varphi\varsigma, \varrho)).$$

***Case (ii)**: If $\varrho \in [1,2)$, then*

$$d_{\mathcal{R}}(\varphi\varrho, \varphi\varsigma) = d_{\mathcal{R}}(0,0) = 0 \tag{4.16}$$

$$\text{and,}\quad \nu(d_{\mathcal{R}}(\varphi\varrho,\varsigma)+d_{\mathcal{R}}(\varphi\varsigma,\varrho))=\nu(d_{\mathcal{R}}(0,0)+d_{\mathcal{R}}(0,\varrho))=\nu\varrho. \tag{4.17}$$

Then, from (4.16) and (4.17) and for any value of $\nu\in(0,1/2)$, we conclude that φ satisfies the contraction hypothesis of Theorem 4.2.7 (case when $Z_{\mathbb{B}}=\mho_{\mathbb{B}}$). Moreover, φ is $\mathcal{R}$-continuous and $\mathcal{R}$-preserving. Thus, by Theorem 4.2.7 (case when $Z_{\mathbb{B}}=\mho_{\mathbb{B}}$), φ possesses a unique fixed point which is $\varrho=0$.

Remark 4.2.14. *The metric space $(\mho,d)$ considered in the above example is an incomplete metric space and thus violates the applicability of Theorem 2.3 of Ma et al. (2014) and Chatterjea (1972) contraction theorem.*

4.3 Coincidence and Common Fixed Point Results

This section deals with coincidence point and common fixed point results in $C^*_{AV}\mathcal{R}$-metric space $(\mho_{\mathbb{B}},d_{\mathcal{R}})$. Firstly, we introduce the notions of $\mathcal{R}_{\varphi}$-preserving, $\mathcal{R}$-compatible, $\mathcal{R}$-precomplete, $\mathcal{R}$-continuous, $\mathcal{R}$-contractive and weak $\mathcal{R}$-contractive in the framework of $C^*_{AV}\mathcal{R}$-metric space.

Definition 4.3.1. *Let $(\mho_{\mathbb{B}},d_{\mathcal{R}})$ be a $C^*_{AV}\mathcal{R}$-metric space and ψ,φ be two self-maps on $\mho_{\mathbb{B}}$, then*

*(i) ψ is c.t.b $\mathcal{R}_{\varphi}$-**preserving** if $(\varphi\varrho,\varphi\varsigma)\in\mathcal{R}$ implies $(\psi\varrho,\psi\varsigma)\in\mathcal{R}$.*

*(ii) ψ,φ are c.t.b $\mathcal{R}$-**compatible** if for any $\mathcal{R}$-sequence $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ in $\mho_{\mathbb{B}}$ s.t $\{\psi\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ and $\{\varphi\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ are two $\mathcal{R}$-sequences and $\lim\limits_{\varpi\to+\infty}\psi\varrho_{\varpi}=\lim\limits_{\varpi\to+\infty}\varphi\varrho_{\varpi}$ we have,*

$$\|d_{\mathcal{R}}(\varphi(\psi\varrho_{\varpi}),\psi(\varphi\varrho_{\varpi}))\|\to\theta_{\mathbb{B}}\quad \text{as } \varpi\to+\infty.$$

*(iii) a subset $Z_{\mathbb{B}}$ of $\mho_{\mathbb{B}}$ is c.t.b an $\mathcal{R}$-**precomplete subspace** of $\mho_{\mathbb{B}}$, if for every $\mathcal{R}$-Cauchy sequence $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ in $Z_{\mathbb{B}}$, we have $\lim\limits_{\varpi\to+\infty}\|d_{\mathcal{R}}(\varrho_{\varpi},\varrho)\|=\theta_{\mathbb{B}}$ where $\varrho\in\mho_{\mathbb{B}}$.*

*(iv) maps ψ,φ are c.t.b $\mathcal{R}$-**contractive** if for any $\varrho,\varsigma\in\mho_{\mathbb{B}}$ with $(\varphi\varrho,\varphi\varsigma)\in\mathcal{R}$, we have*

$$d_{\mathcal{R}}(\psi\varrho,\psi\varsigma)\leqslant\delta^* d_{\mathcal{R}}(\varphi\varrho,\varphi\varsigma)\delta,$$

where $\delta\in\mathbb{B}$ and $\|\delta\|<1$.

(v) maps ψ,φ are c.t.b ***weak $\mathcal{R}$-contractive*** *if for any $\varrho,\varsigma\in\mho_{\mathbb{B}}$ with $(\varphi\varrho,\varphi\varsigma)\in\mathcal{R}$, we have*

$$d_{\mathcal{R}}(\psi\varrho,\psi\varsigma)\leqslant\delta^{*}\zeta^{*}(\varrho_{\varpi},\varrho_{\varpi+1})\delta,$$

where $\delta\in\mathbb{B}$ with $\|\delta\|<1$, where

$$\zeta^{*}(\varrho_{\varpi},\varrho_{\varpi+1})=\max\Big\{d_{\mathcal{R}}(\varphi\varrho_{\varpi},\varphi\varrho_{\varpi+1}),d_{\mathcal{R}}(\psi\varrho_{\varpi},\varphi\varrho_{\varpi}),d_{\mathcal{R}}(\psi\varrho_{\varpi+1},\varphi\varrho_{\varpi+1}),\frac{d_{\mathcal{R}}(\psi\varrho_{\varpi},\varphi\varrho_{\varpi+1})+d_{\mathcal{R}}(\psi\varrho_{\varpi+1},\varphi\varrho_{\varpi})}{2}\Big\}.$$

Example 4.3.1. *Let $\mho=\left(-1+\dfrac{1}{\varpi},1-\dfrac{1}{\varpi}\right)$ with $d(\varrho,\varsigma)=\mid\varrho-\varsigma\mid$, where $\varpi\in\mathbb{N}$ with $\mathbb{B}=\mathbb{R}$ and define $\mathcal{R}=\{(\varrho,\varsigma)\in\mho^{2}:\varrho.\varsigma>0\}$. Let $\psi,\varphi:\mho_{\mathbb{B}}\to\mho_{\mathbb{B}}$ be defined as*

$$\psi(\varrho)=\begin{cases}\frac{1}{2} & for\ \varrho\in(-1+\frac{1}{\varpi},0];\\ \frac{1}{9} & for\ \varrho\in(0,1-\frac{1}{\varpi}),\end{cases}\quad and,\ \varphi(\varrho)=\begin{cases}\frac{-1}{2} & for\ \varrho\in(-1+\frac{1}{\varpi},0];\\ \frac{1}{9} & for\ \varrho\in(0,1-\frac{1}{\varpi}).\end{cases}$$

Then for any $\varrho,\varsigma\in\mho_{\mathbb{B}}$ with $(\varphi\varrho,\varphi\varsigma)\in\mathcal{R}$, we have $(\psi\varrho,\psi\varsigma)\in\mathcal{R}$. Thus, ψ is $\mathcal{R}_{\varphi}$-preserving.

Remark 4.3.2. *It can be observed from Definition 4.3.1 that on considering the self-map φ on $\mho_{\mathbb{B}}$ as identity map, we obtain that self-map ψ is $\mathcal{R}$-preserving.*

Example 4.3.3. *Consider $\mho=[0,3]$ and $\mathbb{B}$ be the set of all 2×2 diagonal matrices on $\mathbb{C}$. Let involution on element A of $\mathbb{B}$ be defined as $A^{*}=A^{H}$, that is, conjugate transpose of matrix $A=[a_{ij}]$ and $\|A\|=\max\limits_{1\leqslant i,j\leqslant 2}\mid a_{ij}\mid$. Define metric $d:\mho\times\mho\to\mathbb{B}_{+}$ as*

$$d(\varrho,\varsigma)=\begin{bmatrix}\mid\varrho-\varsigma\mid & 0\\ 0 & \lambda\mid\varrho-\varsigma\mid\end{bmatrix},\ where\ \lambda\geqslant 0.$$

Let $\mathcal{R}=\{\varrho,\varsigma\in\mho:$ either $\varrho=0$ or $\varsigma=0\}$, then $(\mho_{\mathbb{B}},d_{\mathcal{R}})$ is a $C^{*}_{AV}\mathcal{R}$-metric space. Define self-maps ψ,φ on $\mho_{\mathbb{B}}$ as:

$$\psi(\varrho)=\begin{cases}\dfrac{\varrho}{3} & for\ \varrho\in[0,2];\\ 0 & otherwise,\end{cases}\quad and,\ \varphi(\varrho)=\begin{cases}\dfrac{\varrho^{2}}{11} & for\ \varrho\in[0,5/2];\\ 0 & otherwise.\end{cases}$$

Then, for any $\mathcal{R}$-sequence $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}'}$ in $\mho_{\mathbb{B}}$, $\psi\varrho_{\varpi}$ and $\varphi\varrho_{\varpi}$ are $\mathcal{R}$-sequences with $\lim\limits_{\varpi\to+\infty}\psi\varrho_{\varpi}=\lim\limits_{\varpi\to+\infty}\varphi\varrho_{\varpi}$ s.t $d_{\mathcal{R}}(\psi(\varphi\varrho_{\varpi}),\varphi(\psi\varrho_{\varpi}))\to 0$ as $\varpi\to+\infty$. Thus ψ,φ are $\mathcal{R}$-compatible.

Example 4.3.4. *Let* $\mho = [0, 1)$ *and* $Z = (0, 1)$ *be a subset of* $\mho$ *with* $\mathbb{B} = (-\infty, +\infty)$ *and metric* $d : \mho \times \mho \to \mathbb{B}_+$ *defined as*

$$d(\varrho, \varsigma) = \begin{cases} 0 & for\ \varrho = \varsigma; \\ 1 & for\ \varrho \neq \varsigma. \end{cases}$$

Let $\mathcal{R} = \{\varrho, \varsigma \in \mho : d_{\mathcal{R}}(\varrho, \varsigma) < 1\}$*. Then, for any* $\mathcal{R}$*-Cauchy sequence* $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}'}$ *in* $Z_{\mathbb{B}}$ *is convergent in* $\mho_{\mathbb{B}}$*. Thus,* $Z_{\mathbb{B}}$ *is an* $\mathcal{R}$*-precomplete subspace of* $\mho_{\mathbb{B}}$*.*

Remark 4.3.5. *Every precomplete subspace is an* $\mathcal{R}$*-precomplete subspace, however converse does not hold true. In Example 4.3.4,* $Z_{\mathbb{B}}$ *is an* $\mathcal{R}$*-precomplete subspace of* $\mho_{\mathbb{B}}$ *but not a precomplete subspace of* $\mho_{\mathbb{B}}$ *since the sequence* $\{1 - \frac{1}{\varpi}\}_{\varpi \in \mathbb{N}}$ *is a Cauchy sequence in* $Z_{\mathbb{B}}$ *which is not convergent in* $\mho_{\mathbb{B}}$*.*

Theorem 4.3.6. *Let* $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ *be a* $C^*_{AV}\mathcal{R}$*-metric space with* $Z_{\mathbb{B}}$ *as an* $\mathcal{R}$*-precomplete subspace of* $\mho_{\mathbb{B}}$ *and let* ψ, φ *be two self-maps which are* $\mathcal{R}$*-contractive,* $\mathcal{R}$*-continuous and* $\mathcal{R}$*-compatible. Also, let* ψ *be* $\mathcal{R}_\varphi$*-preserving where* $\psi(\mho_{\mathbb{B}}) \subseteq \varphi(\mho_{\mathbb{B}}) \cap Z_{\mathbb{B}}$ *and* $\exists\ \varrho_0 \in \mho_{\mathbb{B}}$ *with* $(\psi\varrho_0, \varphi\varrho_0) \in \mathcal{R}$*. Then,* ψ, φ *have a coincidence point. In addition, if for any two coincidence points* γ_1 *and* γ_2 *of* ψ, φ*, that is, there exist some* $\varsigma_1, \varsigma_2 \in \mho_{\mathbb{B}}$ *with* $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ *and* $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$ *we have* $(\varsigma_1, \varsigma_2) \in \mathcal{R}$*, then* ψ, φ *possess a unique coincidence point. Moreover, if* ψ, φ *are two weakly compatible maps then they possess a unique common fixed point.*

Proof. Define Picard-Jungck sequences $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}'}$ and $\{\varsigma_\varpi\}_{\varpi \in \mathbb{N}'}$, where $\varsigma_\varpi = \varphi\varrho_{\varpi+1} = \psi\varrho_\varpi\ \forall\ \varpi \in \mathbb{N}'$. By given condition, $\exists$ some $\varrho_0 \in \mho_{\mathbb{B}}$ s.t $(\psi\varrho_0, \varphi\varrho_0) \in \mathcal{R}$, that is, $(\varphi\varrho_1, \varphi\varrho_0) \in \mathcal{R}$. Since ψ is $\mathcal{R}_\varphi$-preserving, we obtain

$$(\psi\varrho_1, \psi\varrho_0) \in \mathcal{R}, \quad \text{that is,} \quad (\varphi\varrho_2, \varphi\varrho_1) \in \mathcal{R}.$$

On repetitive use of $\mathcal{R}_\varphi$-preserving property of ψ, we get $\{\varsigma_\varpi\}_{\varpi \in \mathbb{N}'}$, $\{\psi\varrho_\varpi\}_{\varpi \in \mathbb{N}'}$ and $\{\varphi\varrho_\varpi\}_{\varpi \in \mathbb{N}'}$ are $\mathcal{R}$-sequences. Now,

$$\begin{aligned} d_{\mathcal{R}}(\varsigma_\varpi, \varsigma_{\varpi+1}) =& d_{\mathcal{R}}(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}) \\ =& d_{\mathcal{R}}(\psi\varrho_\varpi, \psi\varrho_{\varpi+1}) \\ \leqslant& \delta^* d_{\mathcal{R}}(\varphi\varrho_\varpi, \varphi\varrho_{\varpi+1})\delta \\ \leqslant& (\delta^*)^2 d_{\mathcal{R}}(\varphi\varrho_{\varpi-1}, \varphi\varrho_\varpi)(\delta)^2 = (\delta^*)^2 d_{\mathcal{R}}(\varsigma_{\varpi-2}, \varsigma_{\varpi-1})(\delta)^2 \\ \leqslant& \cdots \leqslant (\delta^*)^\varpi d_{\mathcal{R}}(\varphi\varrho_1, \varphi\varrho_2)(\delta)^\varpi = (\delta^*)^\varpi d_{\mathcal{R}}(\varsigma_0, \varsigma_1)(\delta)^\varpi. \end{aligned} \tag{4.18}$$

We next show that $\{\varsigma_\varpi\}_{\varpi \in \mathbb{N}'}$ is an $\mathcal{R}$-Cauchy sequence. Let $p, q \in \mathbb{N}'$ with $p < q$

then, we have

$$\begin{aligned}
d_{\mathcal{R}}(\varsigma_p,\varsigma_q) &\leqslant d_{\mathcal{R}}(\varsigma_p,\varsigma_{p+1})+d_{\mathcal{R}}(\varsigma_{p+1},\varsigma_{p+2})+\cdots+d_{\mathcal{R}}(\varsigma_{q-1},\varsigma_q)\\
&\leqslant \sum_{\gamma=p}^{q}(\delta^*)^{\gamma}d_{\mathcal{R}}(\varsigma_0,\varsigma_1)(\delta)^{\gamma}\\
&= \sum_{\gamma=p}^{q}\left(\Big[d_{\mathcal{R}}(\varsigma_0,\varsigma_1)\Big]^{1/2}\delta^{\gamma}\right)^*\left(\Big[d_{\mathcal{R}}(\varsigma_0,\varsigma_1)\Big]^{1/2}\delta^{\gamma}\right)\\
&\leqslant \left\|\sum_{\gamma=p}^{+\infty}\left|\Big[d_{\mathcal{R}}(\varsigma_0,\varsigma_1)\Big]^{1/2}\delta^{\gamma}\right|^2\right\|I_{\mathbb{B}}\\
&= \left\|\Big[d_{\mathcal{R}}(\varsigma_0,\varsigma_1)\Big]^{1/2}\right\|^2\frac{\|\delta\|^{2p}}{1-\|\delta\|}I_{\mathbb{B}}\to\theta_{\mathbb{B}}\quad\text{as}\quad p\to+\infty. \qquad (4.19)
\end{aligned}$$

From (4.19), we observe that $\{\varsigma_{\varpi}\}_{\varpi\in\mathbb{N}'}$ is an $\mathcal{R}$-Cauchy sequence in $Z_{\mathbb{B}}$ and since $Z_{\mathbb{B}}$ is $\mathcal{R}$-precomplete subspace of $\mho_{\mathbb{B}}$ so, $\exists$ some $\varsigma\in\mho_{\mathbb{B}}$ s.t

$$\begin{aligned}
\lim_{\varpi\to+\infty}\varsigma_{\varpi} &= \varsigma,\\
\text{that is,}\quad \lim_{\varpi\to+\infty}\varphi\varrho_{\varpi} &= \lim_{\varpi\to+\infty}\psi\varrho_{\varpi}=\varsigma. \qquad (4.20)
\end{aligned}$$

On using $\mathcal{R}$-compatibility of ψ and φ, we have

$$\lim_{\varpi\to+\infty}d_{\mathcal{R}}(\varphi(\psi\varrho_{\varpi}),\psi(\varphi\varrho_{\varpi}))=\theta_{\mathbb{B}}.$$

On using $\mathcal{R}$-continuity of ψ in (4.20), we obtain

$$\lim_{\varpi\to+\infty}\psi(\varphi\varrho_{\varpi})=\lim_{\varpi\to+\infty}\psi(\psi\varrho_{\varpi})=\psi\varsigma.$$

Again, by using $\mathcal{R}$-continuity of φ in (4.20), we obtain

$$\lim_{\varpi\to+\infty}\varphi(\varphi\varrho_{\varpi})=\lim_{\varpi\to+\infty}\varphi(\psi\varrho_{\varpi})=\varphi\varsigma.$$

Further,

$$d_{\mathcal{R}}(\varphi\varsigma,\psi\varsigma)=\lim_{\varpi\to+\infty}d_{\mathcal{R}}(\varphi(\psi\varrho_{\varpi}),\psi(\varphi\varrho_{\varpi}))=\theta_{\mathbb{B}}.$$

Therefore, ς is the coincidence point of ψ,φ. Next, let γ_1,γ_2 be two distinct coincidence point of ψ,φ, that is, there exist $\varsigma_1,\varsigma_2\in\mho_{\mathbb{B}}$ with $\varsigma_1\neq\varsigma_2$ with $(\varsigma_1,\varsigma_2)\in\mathcal{R}$. Now, $\psi\varsigma_1=\varphi\varsigma_1=\gamma_1$ and $\psi\varsigma_2=\varphi\varsigma_2=\gamma_2$, thus

$$d_{\mathcal{R}}(\psi\varsigma_1,\psi\varsigma_2)\leqslant\delta^*d_{\mathcal{R}}(\varphi\varsigma_1,\varphi\varsigma_2)\delta,\ \text{that is,}\ d_{\mathcal{R}}(\gamma_1,\gamma_2)\leqslant\delta^*d_{\mathcal{R}}(\gamma_1,\gamma_2)\delta,$$

which does not hold since $\|\delta\| < 1$. Hence, ψ and φ have a unique coincidence point. We now consider ψ, φ to be weakly compatible, that is, $\psi\varphi\varsigma = \varphi\psi\varsigma$ where ς is unique coincidence point of ψ, φ. Let $\upsilon \in \mho_{\mathbb{B}}$ where $\psi\varsigma = \varphi\varsigma = \upsilon$, then we have

$$\psi\upsilon = \psi\varphi\varsigma = \varphi\psi\varsigma = \varphi\upsilon.$$

Thus, υ is another coincidence point of ψ, φ which implies $\varsigma = \upsilon$, so we obtain ς as a unique common fixed point of ψ, φ. □

Theorem 4.3.7. *Let $(\mho_{\mathbb{B}}, d_{\mathcal{R}})$ be a $C^*_{AV}\mathcal{R}$-metric space with $Z_{\mathbb{B}}$ as an $\mathcal{R}$-precomplete subspace of $\mho_{\mathbb{B}}$ and suppose ψ, φ are two self-maps which are weak $\mathcal{R}$-contractive, $\mathcal{R}$-continuous and $\mathcal{R}$-compatible. Also, let ψ be $\mathcal{R}_\varphi$-preserving with $\psi(\mho_{\mathbb{B}}) \subseteq \varphi(\mho_{\mathbb{B}}) \cap Z_{\mathbb{B}}$ and $\exists\ \varrho_0 \in \mho_{\mathbb{B}}$ where $(\psi\varrho_0, \varphi\varrho_0) \in \mathcal{R}$. Then, ψ, φ have a coincidence point. In addition, if for any two coincidence points γ_1 and γ_2 of ψ, φ, that is, there exist some $\varsigma_1, \varsigma_2 \in \mho_{\mathbb{B}}$ with $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ and $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$ we have $(\varsigma_1, \varsigma_2) \in \mathcal{R}$, then ψ, φ possess a unique coincidence point. Furthermore, if ψ, φ are two weakly compatible maps then they possess a unique common fixed point.*

Proof. On defining a Picard-Jungck sequences $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ and $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}'}$ in $\mho_{\mathbb{B}}$ as defined in Theorem 4.3.6, we conclude that $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}'}$, $\{\psi\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ and $\{\varphi\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ are $\mathcal{R}$-sequences. Now,

$$\begin{aligned} d_{\mathcal{R}}(\varsigma_\varpi, \varsigma_{\varpi+1}) &= d_{\mathcal{R}}(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}) \\ &= d_{\mathcal{R}}(\psi\varrho_\varpi, \psi\varrho_{\varpi+1}) \leqslant \delta^*\zeta^*(\varrho_\varpi, \varrho_{\varpi+1})\delta, \qquad (4.21) \\ \text{where,} \quad \zeta^*(\varrho_\varpi, \varrho_{\varpi+1}) &= \max\Big\{ d_{\mathcal{R}}(\varphi\varrho_\varpi, \varphi\varrho_{\varpi+1}), d_{\mathcal{R}}(\psi\varrho_\varpi, \varphi\varrho_\varpi), \\ &\quad d_{\mathcal{R}}(\psi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+1}), \\ &\quad \frac{d_{\mathcal{R}}(\psi\varrho_\varpi, \varphi\varrho_{\varpi+1}) + d_{\mathcal{R}}(\psi\varrho_{\varpi+1}, \varphi\varrho_\varpi)}{2} \Big\} \\ &= \max\Big\{ d_{\mathcal{R}}(\varphi\varrho_\varpi, \varphi\varrho_{\varpi+1}), d_{\mathcal{R}}(\varphi\varrho_{\varpi+1}, \varphi\varrho_\varpi), \\ &\quad d_{\mathcal{R}}(\varphi\varrho_{\varpi+2}, \varphi\varrho_{\varpi+1}), \\ &\quad \frac{d_{\mathcal{R}}(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+1}) + d_{\mathcal{R}}(\varphi\varrho_{\varpi+2}, \varphi\varrho_\varpi)}{2} \Big\} \\ &\leqslant \max\Big\{ d_{\mathcal{R}}(\varphi\varrho_\varpi, \varphi\varrho_{\varpi+1}), d_{\mathcal{R}}(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}), \end{aligned}$$

$$\left.\frac{d_{\mathcal{R}}(\varphi\varrho_{\varpi+1},\varphi\varrho_{\varpi+2})+d_{\mathcal{R}}(\varphi\varrho_{\varpi},\varphi\varrho_{\varpi+1})}{2}\right\}$$

$$\leqslant \max\left\{d_{\mathcal{R}}(\varphi\varrho_{\varpi},\varphi\varrho_{\varpi+1}),d_{\mathcal{R}}(\varphi\varrho_{\varpi+1},\varphi\varrho_{\varpi+2})\right\}.$$

Let $\max\left\{d_{\mathcal{R}}(\varphi\varrho_{\varpi},\varphi\varrho_{\varpi+1}),d_{\mathcal{R}}(\varphi\varrho_{\varpi+1},\varphi\varrho_{\varpi+2})\right\} = d_{\mathcal{R}}(\varphi\varrho_{\varpi+1},\varphi\varrho_{\varpi+2})$, then from (4.21), we have

$$d_{\mathcal{R}}(\varphi\varrho_{\varpi+1},\varphi\varrho_{\varpi+2}) \leqslant \delta^* d_{\mathcal{R}}(\varphi\varrho_{\varpi+1},\varphi\varrho_{\varpi+2})\delta,$$

which is a contradiction since $\|\delta\| < 1$.
If, $\max\left\{d_{\mathcal{R}}(\varphi\varrho_{\varpi},\varphi\varrho_{\varpi+1}),d_{\mathcal{R}}(\varphi\varrho_{\varpi+1},\varphi\varrho_{\varpi+2})\right\} = d_{\mathcal{R}}(\varphi\varrho_{\varpi},\varphi\varrho_{\varpi+1})$ and from (4.21), we get

$$d_{\mathcal{R}}(\varphi\varrho_{\varpi+1},\varphi\varrho_{\varpi+2}) \leqslant \delta^* d_{\mathcal{R}}(\varphi\varrho_{\varpi},\varphi\varrho_{\varpi+1})\delta.$$

By Theorem 4.3.6, we obtain that φ,ψ possess a unique coincidence point and common fixed point. □

4.4 Application in Graph Theory

In this section, we establish an association of the results proved in the previous section with a directed graph. Let $\Xi = (V(\Xi), E(\Xi))$ be a directed graph, where $V(\Xi)$ be the set of vertices which coincides with $\mho$ and $E(\Xi)$ be the set of directed edges along with all loops, that is, $E(\Xi)$ is a subset of $V(\Xi) \times V(\Xi)$. We also assume that Ξ has no parallel edges in it.

Theorem 4.4.1. *For $(\mho_{\mathbb{B}}, d)$ a C^*_{AV}-metric space endowed with a directed graph Ξ and let $Z_{\mathbb{B}}$ be a subspace of $\mho_{\mathbb{B}}$ where every Cauchy sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $Z_{\mathbb{B}}$ with $(\varrho_\varpi, \varrho_{\varpi+1}) \in E(\Xi)$ is convergent in $\mho_{\mathbb{B}}$. Let $\psi, \varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ be self-maps s.t:*

(I) For any $\varrho, \varsigma \in \mho_{\mathbb{B}}$ with $(\varphi\varrho, \varphi\varsigma) \in E(\Xi)$, we have

$$d(\psi\varrho, \psi\varsigma) \leqslant \delta^* d(\varphi\varrho, \varphi\varsigma)\delta,$$

where $\delta \in \mathbb{B}$ and $\|\delta\| < 1$;

(II) *For each* $\varrho \in \mho_{\mathbb{B}}$ $\exists$ *a sequence* $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *in* $\mho_{\mathbb{B}}$ *where* $(\varrho_{\varpi}, \varrho_{\varpi+1}) \in E(\Xi)$ *with* $\lim\limits_{\varpi \to +\infty} \|d(\varrho_{\varpi}, \varrho)\| = \theta_{\mathbb{B}}$ *implies*

$$\lim_{\varpi \to +\infty} \|d(\psi\varrho_{\varpi}, \psi\varrho)\| = \theta_{\mathbb{B}} \text{ and, } \lim_{\varpi \to +\infty} \|d(\varphi\varrho_{\varpi}, \varphi\varrho)\| = \theta_{\mathbb{B}};$$

(III) *For any sequence* $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *in* $\mho_{\mathbb{B}}$ *where* $(\varrho_{\varpi}, \varrho_{\varpi+1}) \in E(\Xi)$ *s.t* $\{\psi\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *and* $\{\varphi\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *are two sequences with* $(\psi\varrho_{\varpi}, \psi\varrho_{\varpi+1}) \in E(\Xi)$, $(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1}) \in E(\Xi)$ *and* $\lim\limits_{\varpi \to +\infty} \psi\varrho_{\varpi} = \lim\limits_{\varpi \to +\infty} \varphi\varrho_{\varpi}$, *we have*

$$\|d(\varphi(\psi\varrho_{\varpi}), \psi(\varphi\varrho_{\varpi}))\| \to \theta_{\mathbb{B}} \quad \text{as } \varpi \to +\infty;$$

(IV) *If* $(\varphi\varrho, \varphi\varsigma) \in E(\Xi)$ *implies* $(\psi\varrho, \psi\varsigma) \in E(\Xi)$*;*

(V) $\psi(\mho_{\mathbb{B}}) \subseteq \varphi(\mho_{\mathbb{B}}) \cap Z_{\mathbb{B}}$*;*

(VI) $\exists$ *some* $\varrho_0 \in \mho_{\mathbb{B}}$ *s.t* $(\psi\varrho_0, \varphi\varrho_0) \in E(\Xi)$*.*

Then, ψ, φ *have a coincidence point. In addition, if for any two coincidence points* γ_1 *and* γ_2 *of* ψ, φ*, that is, there exist some* $\varsigma_1, \varsigma_2 \in \mho_{\mathbb{B}}$ *with* $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ *and* $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$ *we have* $(\varsigma_1, \varsigma_2) \in E(\Xi)$*, then* ψ, φ *possess a unique coincidence point. Moreover, if* ψ, φ *are two weakly compatible maps then they possess a unique common fixed point.*

Proof. Define a Picard-Jungck sequences $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}'}$ and $\{\varsigma_{\varpi}\}_{\varpi \in \mathbb{N}'}$ with $\varsigma_{\varpi} = \varphi\varrho_{\varpi+1} = \psi\varrho_{\varpi}$ $\forall$ $\varpi \in \mathbb{N}'$. From condition *(V)*, $\exists$ some $\varrho_0 \in \mho_{\mathbb{B}}$ s.t $(\psi\varrho_0, \varphi\varrho_0)$ make an edge, that is, $(\varphi\varrho_1, \varphi\varrho_0)$ make an edge. On using condition *(IV)*, we obtain

$$(\psi\varrho_1, \psi\varrho_0) \in E(\Xi), \quad \text{that is,} \quad (\varphi\varrho_2, \varphi\varrho_1) \in E(\Xi).$$

On repetitive use of condition *(IV)*, we get that $(\varsigma_{\varpi}, \varsigma_{\varpi+1}) \in E(\Xi)$, $(\psi\varrho_{\varpi}, \psi\varrho_{\varpi+1}) \in E(\Xi)$ and $(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1}) \in E(\Xi)$ $\forall$ $\varpi \in \mathbb{N}'$. Now,

$$\begin{aligned} d(\varsigma_{\varpi}, \varsigma_{\varpi+1}) = d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}) =& d(\psi\varrho_{\varpi}, \psi\varrho_{\varpi+1}) \\ \leqslant& \delta^* d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1})\delta \\ \leqslant& \cdots \leqslant (\delta^*)^{\varpi} d(\varphi\varrho_1, \varphi\varrho_2)(\delta)^{\varpi} \\ =& (\delta^*)^{\varpi} d(\varsigma_0, \varsigma_1)(\delta)^{\varpi}. \end{aligned} \tag{4.22}$$

We next show that $\{\varsigma_{\varpi}\}_{\varpi \in \mathbb{N}'}$ is a Cauchy sequence, where $(\varsigma_{\varpi}, \varsigma_{\varpi+1})$ form an edge

$\forall\, \varpi \in \mathbb{N}'$. Let $p, q \in \mathbb{N}'$ with $p < q$ then, we have

$$\begin{aligned}
d(\varsigma_p, \varsigma_q) \leqslant & d(\varsigma_p, \varsigma_{p+1}) + d(\varsigma_{p+1}, \varsigma_{p+2}) + \cdots + d(\varsigma_{q-1}, \varsigma_q) \\
& \leqslant \sum_{\gamma=p}^{q} (\delta^*)^\gamma d(\varsigma_0, \varsigma_1)(\delta)^\gamma \\
& \leqslant \sum_{\gamma=p}^{+\infty} (\Big[d(\varsigma_0, \varsigma_1)\Big]^{1/2} \delta^\gamma)^* (\Big[d(\varsigma_0, \varsigma_1)\Big]^{1/2} \delta^\gamma) = \left\| \sum_{\gamma=p}^{+\infty} \left| \Big[d(\varsigma_0, \varsigma_1)\Big]^{1/2} \delta^\gamma \right|^2 \right\| I_{\mathbb{B}} \\
& = \left\| \Big[d(\varsigma_0, \varsigma_1)\Big]^{1/2} \right\|^2 \frac{\|\delta\|^{2p}}{1 - \|\delta\|} I_{\mathbb{B}} \to \theta_{\mathbb{B}} \quad \text{as} \quad p \to +\infty. \qquad (4.23)
\end{aligned}$$

Thus, $\{\varsigma_\varpi\}_{\varpi \in \mathbb{N}'}$ is a Cauchy sequence in $Z_{\mathbb{B}}$ and since every Cauchy sequence in $Z_{\mathbb{B}}$ is convergent in $\mho_{\mathbb{B}}$, so $\exists$ some $\varsigma \in \mho_{\mathbb{B}}$ s.t $\lim_{\varpi \to +\infty} \varsigma_\varpi = \varsigma$, that is, $\lim_{\varpi \to +\infty} \varphi \varrho_\varpi = \lim_{\varpi \to +\infty} \psi \varrho_\varpi = \varsigma$.
On using condition *(II)*, we obtain

$$\begin{aligned}
\lim_{\varpi \to +\infty} \varphi(\varphi \varrho_\varpi) &= \lim_{\varpi \to +\infty} \varphi(\psi \varrho_\varpi) = \varphi \varsigma, \\
\text{and,} \quad \lim_{\varpi \to +\infty} \psi(\varphi \varrho_\varpi) &= \lim_{\varpi \to +\infty} \psi(\psi \varrho_\varpi) = \psi \varsigma.
\end{aligned}$$

On using condition *(III)*, we have

$$\lim_{\varpi \to +\infty} d(\varphi(\psi \varrho_\varpi), \psi(\varphi \varrho_\varpi)) = \theta_{\mathbb{B}}.$$

Further,

$$d(\varphi \varsigma, \psi \varsigma) = \lim_{\varpi \to +\infty} d(\varphi(\psi \varrho_\varpi), \psi(\varphi \varrho_\varpi)) = \theta_{\mathbb{B}}$$

Therefore, ς is the coincidence point of ψ, φ. Next, let γ_1, γ_2 be s.t there exist $\varsigma_1, \varsigma_2 \in \mho_{\mathbb{B}}$ with $\varsigma_1 \neq \varsigma_2$ with $(\varsigma_1, \varsigma_2) \in E(\Xi)$. Now, $\psi \varsigma_1 = \varphi \varsigma_1 = \gamma_1$ and $\psi \varsigma_2 = \varphi \varsigma_2 = \gamma_2$, thus

$$d(\psi \varsigma_1, \psi \varsigma_2) \leqslant \delta^* d(\varphi \varsigma_1, \varphi \varsigma_2) \delta \text{ that is, } d(\gamma_1, \gamma_2) \leqslant \delta^* d(\gamma_1, \gamma_2) \delta,$$

which does not hold since $\|\delta\| < 1$. Hence, ψ, φ have a unique coincidence point. We now consider ψ, φ to be weakly compatible, that is, $\psi \varphi \varsigma = \varphi \psi \varsigma$ where ς is a unique coincidence point of ψ, φ. Let $\upsilon \in \mho_{\mathbb{B}}$ be s.t $\psi \varsigma = \varphi \varsigma = \upsilon$, then we have

$$\psi \upsilon = \psi \varphi \varsigma = \varphi \psi \varsigma = \varphi \upsilon.$$

Thus, υ is another coincidence point of ψ, φ which implies $\varsigma = \upsilon$, so we obtain ς

as a unique common fixed point of ψ, φ. $\square$

Theorem 4.4.2. *For $(\mho_{\mathbb{B}}, d)$ be a C^*_{AV}-metric space together with a directed graph Ξ and let $Z_{\mathbb{B}}$ be a subspace of $\mho_{\mathbb{B}}$ where every Cauchy sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $Z_{\mathbb{B}}$ where $(\varrho_\varpi, \varrho_{\varpi+1}) \in E(\Xi)$ is convergent in $\mho_{\mathbb{B}}$. Let $\psi, \varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ be self-maps s.t:*

(I) For any $\varrho, \varsigma \in \mho_{\mathbb{B}}$ with $(\varphi\varrho, \varphi\varsigma) \in E(\Xi)$, we have

$$d(\psi\varrho, \psi\varsigma) \leqslant \delta^* \zeta^*(\varrho, \varsigma)\delta,$$

where $\delta \in \mathbb{B}$ with $\|\delta\| < 1$ and,

$$\zeta^*(\varrho, \varsigma) = \max\left\{d(\varphi\varrho, \varphi\varsigma), d(\psi\varrho, \varphi\varrho), d(\psi\varsigma, \varphi\varsigma), \frac{d(\psi\varrho, \varphi\varsigma) + d(\psi\varsigma, \varphi\varrho)}{2}\right\};$$

(II) For each $\varrho \in \mho_{\mathbb{B}}$ $\exists$ a sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho_{\mathbb{B}}$ where $(\varrho_\varpi, \varrho_{\varpi+1}) \in E(\Xi)$ with $\lim_{\varpi\to+\infty} \|d(\varrho_\varpi, \varrho)\| = \theta_{\mathbb{B}}$ implies

$$\lim_{\varpi\to+\infty} d(\psi\varrho_\varpi, \psi\varrho) = \theta_{\mathbb{B}} \text{ and, } \lim_{\varpi\to+\infty} d(\varphi\varrho_\varpi, \varphi\varrho) = \theta_{\mathbb{B}};$$

(III) For any sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho_{\mathbb{B}}$ where $(\varrho_\varpi, \varrho_{\varpi+1}) \in E(\Xi)$ s.t $\{\psi\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ and $\{\varphi\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ are two sequences with $(\psi\varrho_\varpi, \psi\varrho_{\varpi+1}) \in E(\Xi)$, $(\varphi\varrho_\varpi, \varphi\varrho_{\varpi+1}) \in E(\Xi)$ and $\lim_{\varpi\to+\infty} \psi\varrho_\varpi = \lim_{\varpi\to+\infty} \varphi\varrho_\varpi$, we have

$$\|d(\varphi(\psi\varrho_\varpi), \psi(\varphi\varrho_\varpi))\| \to \theta_{\mathbb{B}} \quad \text{as } \varpi \to +\infty;$$

(IV) If $(\varphi\varrho, \varphi\varsigma) \in E(\Xi)$ implies $(\psi\varrho, \psi\varsigma) \in E(\Xi)$;

(V) $\psi(\mho_{\mathbb{B}}) \subseteq \varphi(\mho_{\mathbb{B}}) \cap Z_{\mathbb{B}}$;

(VI) $\exists$ some $\varrho_0 \in \mho_{\mathbb{B}}$ where $(\psi\varrho_0, \varphi\varrho_0) \in E(\Xi)$.

Then, ψ, φ have a coincidence point. In addition, if for any two coincidence points γ_1 and γ_2 of ψ, φ, that is, there exist some $\varsigma_1, \varsigma_2 \in \mho_{\mathbb{B}}$ with $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ and $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$ we have $(\varsigma_1, \varsigma_2) \in E(\Xi)$, then ψ, φ possess a unique coincidence point. Furthermore, if ψ, φ are two weakly compatible maps then they possess a unique common fixed point.

Proof. On defining a Picard-Jungck sequences $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}'}$ and $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}'}$ in $\mho_{\mathbb{B}}$ as defined in Theorem 4.4.1, we conclude that $(\varsigma_\varpi, \varsigma_{\varpi+1}), (\psi\varrho_\varpi, \psi\varrho_{\varpi+1}), (\varphi\varrho_\varpi, \varphi\varrho_{\varpi+1}) \in$

$E(\Xi)\ \forall\ \varpi \in \mathbb{N}'$. Now,

$$d(\varsigma_{\varpi}, \varsigma_{\varpi+1}) = d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}) = d(\psi\varrho_{\varpi}, \psi\varrho_{\varpi+1}) \leqslant \delta^* \varpi^*(\varrho_{\varpi}, \varrho_{\varpi+1})\delta, \tag{4.24}$$

$$\begin{aligned}\text{where } \varpi^*(\varrho_{\varpi}, \varrho_{\varpi+1}) &= \max\Big\{ d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1}), d(\psi\varrho_{\varpi}, \varphi\varrho_{\varpi}), d(\psi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+1}), \\ &\quad \frac{d(\psi\varrho_{\varpi}, \varphi\varrho_{\varpi+1}) + d(\psi\varrho_{\varpi+1}, \varphi\varrho_{\varpi})}{2}\Big\} \\ &= \max\Big\{ d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1}), d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1}), d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}), \\ &\quad \frac{d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+1}) + d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+2})}{2}\Big\} \\ &\leqslant \max\Big\{ d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1}), d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}), \\ &\quad \frac{d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}) + d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1})}{2}\Big\}.\end{aligned}$$

Let $\max\Big\{ d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1}), d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}), \frac{d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}) + d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1})}{2}\Big\} = d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2})$, then from (4.24), we obtain

$$d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}) \leqslant \delta^* d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2})\delta,$$

which is a contradiction since $\|\delta\| < 1$. Thus, $\max\Big\{ d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1}), d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}), \frac{d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}) + d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1})}{2}\Big\} = d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1})$ and from (4.24), we get

$$d(\varphi\varrho_{\varpi+1}, \varphi\varrho_{\varpi+2}) \leqslant \delta^* d(\varphi\varrho_{\varpi}, \varphi\varrho_{\varpi+1})\delta.$$

Now, proceeding on the lines of Theorem 4.4.1 we obtain that φ, ψ have a unique coincidence point and common fixed point. □

4.5 Consequences

The present section substantiates that the results proved in this chapter are proper extension of several well-known results found in the literature. Deductions of fixed point, coincidence point and common fixed point results proved in this chapter can be done in different spaces.

4.5.1 Results in C^*-algebra Valued Ordered Metric Space

Let $\mathcal{R}$ be considered as a partially ordered relation, that is, $\mathcal{R} :=\preceq$, then following results are obtained.

Corollary 4.5.1. *Let $(\mho_{\mathbb{B}}, d_{\preceq})$ be a C^*_{AV}-ordered metric space and let $Z_{\mathbb{B}}$ be a subspace of $\mho_{\mathbb{B}}$ where every Cauchy sequence $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ in $Z_{\mathbb{B}}$ where $\varrho_{\varpi} \preceq \varrho_{\varpi+1}$ is convergent in $\mho_{\mathbb{B}}$. Let $\psi, \varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ be self-maps s.t:*

(I) For any $\varrho, \varsigma \in \mho_{\mathbb{B}}$ with $\varphi\varrho \preceq \varphi\varsigma$, we have

$$d_{\preceq}(\psi\varrho, \psi\varsigma) \leqslant \delta^* d_{\preceq}(\varphi\varrho, \varphi\varsigma)\delta,$$

where $\delta \in \mathbb{B}$ and $\|\delta\| < 1$;

(II) For each $\varrho \in \mho_{\mathbb{B}}$ $\exists$ a sequence $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ in $\mho_{\mathbb{B}}$, where $\varrho_{\varpi} \preceq \varrho_{\varpi+1}$ with $\lim\limits_{\varpi\to+\infty} \|d_{\preceq}(\varrho_{\varpi}, \varrho)\| = \theta_{\mathbb{B}}$ implies

$$\lim_{\varpi\to+\infty} d_{\preceq}(\psi\varrho_{\varpi}, \psi\varrho) = \theta_{\mathbb{B}} \text{ and, } \lim_{\varpi\to+\infty} d_{\preceq}(\varphi\varrho_{\varpi}, \varphi\varrho) = \theta_{\mathbb{B}};$$

(III) For any sequence $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ in $\mho_{\mathbb{B}}$, where $\varrho_{\varpi} \preceq \varrho_{\varpi+1}$ s.t $\{\psi\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ and $\{\varphi\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ are two sequences with $\psi\varrho_{\varpi} \preceq \psi\varrho_{\varpi+1}$, $\varphi\varrho_{\varpi} \preceq \varphi\varrho_{\varpi+1}$ and $\lim\limits_{\varpi\to+\infty} \psi\varrho_{\varpi} = \lim\limits_{\varpi\to+\infty} \varphi\varrho_{\varpi}$, we have

$$\|d_{\preceq}(\varphi(\psi\varrho_{\varpi}), \psi(\varphi\varrho_{\varpi}))\| \to \theta_{\mathbb{B}} \quad \text{as } \varpi \to +\infty;$$

(IV) If $\varphi\varrho \preceq \varphi\varsigma$ implies $\psi\varrho \preceq \psi\varsigma$;

(V) $\psi(\mho_{\mathbb{B}}) \subseteq \varphi(\mho_{\mathbb{B}}) \cap Z_{\mathbb{B}}$;

(VI) $\exists$ some $\varrho_0 \in \mho_{\mathbb{B}}$ with $\psi\varrho_0 \preceq \varphi\varrho_0$.

Then, ψ, φ have a coincidence point. In addition, if for any two coincidence points γ_1 and γ_2 of ψ, φ, that is, there exist some $\varsigma_1, \varsigma_2 \in \mho_{\mathbb{B}}$ with $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ and $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$ we have $\varsigma_1 \preceq \varsigma_2$, then ψ, φ possess a unique coincidence point. Moreover, if ψ, φ are two weakly compatible maps then they possess a unique common fixed point.

Corollary 4.5.2. *For* $(\mho_{\mathbb{B}}, d_{\preceq})$ *a* C^*_{AV}*-ordered metric space and let* $Z_{\mathbb{B}}$ *be a subspace of* $\mho_{\mathbb{B}}$ *where every Cauchy sequence* $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ *in* $Z_{\mathbb{B}}$ *where* $\varrho_{\varpi} \preceq \varrho_{\varpi+1}$ *is convergent in* $\mho_{\mathbb{B}}$*. Let* $\psi, \varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ *be self-maps s.t:*

(I) For any $\varrho, \varsigma \in \mho_{\mathbb{B}}$ *with* $\varphi\varrho \preceq \varphi\varsigma$*, we have*

$$d_{\preceq}(\psi\varrho, \psi\varsigma) \leqslant \delta^*\zeta^*(\varrho, \varsigma)\delta,$$

where $\delta \in \mathbb{B}$ *with* $\|\delta\| < 1$ *and,*

$$\zeta^*(\varrho, \varsigma) = \max\left\{d_{\preceq}(\varphi\varrho, \varphi\varsigma), d_{\preceq}(\psi\varrho, \varphi\varrho), d_{\preceq}(\psi\varsigma, \varphi\varsigma), \frac{d_{\preceq}(\psi\varrho, \varphi\varsigma) + d_{\preceq}(\psi\varsigma, \varphi\varrho)}{2}\right\};$$

(II) For each $\varrho \in \mho_{\mathbb{B}}$ $\exists$ *a sequence* $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ *in* $\mho_{\mathbb{B}}$ *where* $\varrho_{\varpi} \preceq \varrho_{\varpi+1}$ *with* $\lim_{\varpi\to+\infty} \|d_{\preceq}(\varrho_{\varpi}, \varrho)\| = \theta_{\mathbb{B}}$ *implies*

$$\lim_{\varpi\to+\infty} \|d_{\preceq}(\psi\varrho_{\varpi}, \psi\varrho)\| = \theta_{\mathbb{B}} \text{ and, } \lim_{\varpi\to+\infty} \|d_{\preceq}(\varphi\varrho_{\varpi}, \varphi\varrho)\| = \theta_{\mathbb{B}};$$

(III) For any sequence $\{\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ *in* $\mho_{\mathbb{B}}$*, where* $\varrho_{\varpi} \preceq \varrho_{\varpi+1}$*, s.t* $\{\psi\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ *and* $\{\varphi\varrho_{\varpi}\}_{\varpi\in\mathbb{N}}$ *are two sequences with* $\psi\varrho_{\varpi} \preceq \psi\varrho_{\varpi+1}$*,* $\varphi\varrho_{\varpi} \preceq \varphi\varrho_{\varpi+1}$ *and* $\lim_{\varpi\to+\infty} \psi\varrho_{\varpi} = \lim_{\varpi\to+\infty} \varphi\varrho_{\varpi}$*, we have*

$$\|d_{\preceq}(\varphi(\psi\varrho_{\varpi}), \psi(\varphi\varrho_{\varpi}))\| \to \theta_{\mathbb{B}} \quad \text{as } \varpi \to +\infty;$$

(IV) If $\varphi\varrho \preceq \varphi\varsigma$ *implies* $\psi\varrho \preceq \psi\varsigma$*;*

(V) $\psi(\mho_{\mathbb{B}}) \subseteq \varphi(\mho_{\mathbb{B}}) \cap Z_{\mathbb{B}}$*;*

(VI) $\exists$ *some* $\varrho_0 \in \mho_{\mathbb{B}}$ *with* $\psi\varrho_0 \preceq \varphi\varrho_0$*.*

Then, ψ, φ *have a coincidence point. In addition, if for any two coincidence points* γ_1 *and* γ_2 *of* ψ, φ*, that is, there exist some* $\varsigma_1, \varsigma_2 \in \mho_{\mathbb{B}}$ *with* $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ *and* $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$ *we have* $\varsigma_1 \preceq \varsigma_2$*, then* ψ, φ *possess a unique coincidence point. Moreover, if* ψ, φ *are two weakly compatible maps then they possess a unique common fixed point.*

4.5.2 Results in C^*-algebra Valued Metric Space

Let $\mathcal{R}$ be the universal relation, then the following results are obtained.

Corollary 4.5.3. *For $(\mho_{\mathbb{B}}, d)$ be a C^*_{AV}-metric space and let $Z_{\mathbb{B}}$ be a subspace of $\mho_{\mathbb{B}}$ where every Cauchy sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $Z_{\mathbb{B}}$ is convergent in $\mho_{\mathbb{B}}$. Let $\psi, \varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ be self-maps s.t:*

(I) For any $\varrho, \varsigma \in \mho_{\mathbb{B}}$, we have

$$d(\psi\varrho, \psi\varsigma) \leqslant \delta^* d(\varphi\varrho, \varphi\varsigma)\delta,$$

where $\delta \in \mathbb{B}$ and $\|\delta\| < 1$;

(II) For each $\varrho \in \mho_{\mathbb{B}}$ $\exists$ a sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho_{\mathbb{B}}$ with $\lim\limits_{\varpi\to+\infty} \|d(\varrho_\varpi, \varrho)\| = \theta_{\mathbb{B}}$ implies

$$\lim_{\varpi\to+\infty} \|d(\psi\varrho_\varpi, \psi\varrho)\| = \theta_{\mathbb{B}} \quad \textit{and,} \quad \lim_{\varpi\to+\infty} \|d(\varphi\varrho_\varpi, \varphi\varrho)\| = \theta_{\mathbb{B}};$$

(III) For any sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $\mho_{\mathbb{B}}$ s.t $\{\psi\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ and $\{\varphi\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ are two sequences with $\lim\limits_{\varpi\to+\infty} \psi\varrho_\varpi = \lim\limits_{\varpi\to+\infty} \varphi\varrho_\varpi$, we have

$$\|d(\varphi(\psi\varrho_\varpi), \psi(\varphi\varrho_\varpi))\| \to \theta_{\mathbb{B}} \quad \textit{as } \varpi \to +\infty;$$

(IV) $\psi(\mho_{\mathbb{B}}) \subseteq \varphi(\mho_{\mathbb{B}}) \cap Z_{\mathbb{B}}$.

Then, ψ, φ have a coincidence point. In addition, if for any two coincidence points γ_1 and γ_2 of ψ, φ, that is, there exist some $\varsigma_1, \varsigma_2 \in \mho_{\mathbb{B}}$ with $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ and $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$, then ψ, φ possess a unique coincidence point. Moreover, if ψ, φ are two weakly compatible maps then they possess a unique common fixed point.

Corollary 4.5.4. *For $(\mho_{\mathbb{B}}, d)$ be a C^*_{AV}-metric space and let $Z_{\mathbb{B}}$ be a subspace of $\mho_{\mathbb{B}}$ where every Cauchy sequence $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ in $Z_{\mathbb{B}}$ is convergent in $\mho_{\mathbb{B}}$. Let $\psi, \varphi : \mho_{\mathbb{B}} \to \mho_{\mathbb{B}}$ be self-maps s.t:*

(I) For any $\varrho, \varsigma \in \mho_{\mathbb{B}}$, we have

$$d(\psi\varrho, \psi\varsigma) \leqslant \delta^* \zeta^*(\varrho, \varsigma)\delta,$$

where $\delta \in \mathbb{B}$ with $\|\delta\| < 1$ and,

$$\zeta^*(\varrho, \varsigma) = \max\left\{ d(\varphi\varrho, \varphi\varsigma), d(\psi\varrho, \varphi\varrho), d(\psi\varsigma, \varphi\varsigma), \frac{d(\psi\varrho, \varphi\varsigma) + d(\psi\varsigma, \varphi\varrho)}{2} \right\};$$

(II) *For each* $\varrho \in \mho_{\mathbb{B}}$ $\exists$ *a sequence* $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *in* $\mho_{\mathbb{B}}$ *with* $\lim_{\varpi \to +\infty} \|d(\varrho_{\varpi}, \varrho)\| = \theta_{\mathbb{B}}$ *implies*

$$\lim_{\varpi \to +\infty} \|d(\psi\varrho_{\varpi}, \psi\varrho)\| = \theta_{\mathbb{B}} \quad \textit{and,} \quad \lim_{\varpi \to +\infty} \|d(\varphi\varrho_{\varpi}, \varphi\varrho)\| = \theta_{\mathbb{B}};$$

(III) *For any sequence* $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *in* $\mho_{\mathbb{B}}$ *s.t* $\{\psi\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *and* $\{\varphi\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *are two sequences with* $\lim_{\varpi \to +\infty} \psi\varrho_{\varpi} = \lim_{\varpi \to +\infty} \varphi\varrho_{\varpi}$*, we have*

$$\|d(\varphi(\psi\varrho_{\varpi}), \psi(\varphi\varrho_{\varpi}))\| \to \theta_{\mathbb{B}} \quad \textit{as } \varpi \to +\infty;$$

(IV) $\psi(\mho_{\mathbb{B}}) \subseteq \varphi(\mho_{\mathbb{B}}) \cap Z_{\mathbb{B}}$.

Then, ψ, φ *have a coincidence point. In addition, if for any two coincidence points* γ_1 *and* γ_2 *of* ψ, φ*, that is, there exist some* $\varsigma_1, \varsigma_2 \in \mho_{\mathbb{B}}$ *with* $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ *and* $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$*, then* ψ, φ *possess a unique coincidence point. Moreover, if* ψ, φ *are two weakly compatible maps then they possess a unique common fixed point.*

4.5.3 Results in Metric Space

Let $\mathcal{R}$ be universal relation on $\mho$ along with $\mathbb{B} = \mathbb{R}$, then the following results are obtained.

Corollary 4.5.5. *For* $(\mho, d)$ *a metric space, let* Z *be a subspace of* $\mho$ *where every Cauchy sequence* $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *in* Z *is convergent in* $\mho$*. Let* $\psi, \varphi : \mho \to \mho$ *be self-maps s.t:*

(I) *For any* $\varrho, \varsigma \in \mho$*, we have*

$$d(\psi\varrho, \psi\varsigma) \leqslant \nu d(\varphi\varrho, \varphi\varsigma),$$

where $0 < \nu < 1$*;*

(II) *For each* $\varrho \in \mho$ $\exists$ *a sequence* $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *in* $\mho$ *where* $\lim_{\varpi \to +\infty} d(\varrho_{\varpi}, \varrho) = 0$ *implies* $\lim_{\varpi \to +\infty} d(\psi\varrho_{\varpi}, \psi\varrho) = 0$ *and,* $\lim_{\varpi \to +\infty} d(\varphi\varrho_{\varpi}, \varphi\varrho) = 0$*;*

(III) *For any sequence* $\{\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *in* $\mho$ *s.t* $\{\psi\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$*,* $\{\varphi\varrho_{\varpi}\}_{\varpi \in \mathbb{N}}$ *are two sequences with* $\lim_{\varpi \to +\infty} \psi\varrho_{\varpi} = \lim_{\varpi \to +\infty} \varphi\varrho_{\varpi}$*, we have*

$$\lim_{\varpi \to +\infty} d(\varphi(\psi\varrho_{\varpi}), \psi(\varphi\varrho_{\varpi})) = 0;$$

(IV) $\psi(\mho) \subseteq \varphi(\mho) \cap Z$.

Then, ψ, φ have a coincidence point. In addition, if for any two coincidence points γ_1 and γ_2 of ψ, φ, that is, there exist some $\varsigma_1, \varsigma_2 \in \mho$ with $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ and $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$, then ψ, φ possess a unique coincidence point. Moreover, if ψ, φ are two weakly compatible maps then they possess a unique common fixed point.

Corollary 4.5.6. *For $(\mho, d)$ a metric space, let Z be a subspace of $\mho$ where every Cauchy sequence $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ in Z is convergent in $\mho$. Let $\psi, \varphi : \mho \to \mho$ be self-maps s.t:*

(I) *For any $\varrho, \varsigma \in \mho$, we have*

$$d(\psi\varrho, \psi\varsigma) \leqslant \nu\zeta^*(\varrho, \varsigma),$$

where $0 < \nu < 1$ and,

$$\zeta^*(\varrho, \varsigma) = \max\left\{d(\varphi\varrho, \varphi\varsigma), d(\psi\varrho, \varphi\varrho), d(\psi\varsigma, \varphi\varsigma), \frac{d(\psi\varrho, \varphi\varsigma) + d(\psi\varsigma, \varphi\varrho)}{2}\right\};$$

(II) *For each $\varrho \in \mho$, $\exists$ a sequence $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ in $\mho$, where $\lim_{\varpi \to +\infty} d(\varrho_\varpi, \varrho) = 0$ implies $\lim_{\varpi \to +\infty} d(\psi\varrho_\varpi, \psi\varrho) = 0$ and, $\lim_{\varpi \to +\infty} d(\varphi\varrho_\varpi, \varphi\varrho) = 0$;*

(III) *For any sequence $\{\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ in $\mho$ s.t $\{\psi\varrho_\varpi\}_{\varpi \in \mathbb{N}}$, $\{\varphi\varrho_\varpi\}_{\varpi \in \mathbb{N}}$ are two sequences with $\lim_{\varpi \to +\infty} \psi\varrho_\varpi = \lim_{\varpi \to +\infty} \varphi\varrho_\varpi$, we have*

$$\lim_{\varpi \to +\infty} d(\varphi(\psi\varrho_\varpi), \psi(\varphi\varrho_\varpi)) = 0;$$

(IV) $\psi(\mho) \subseteq \varphi(\mho) \cap Z$.

Then, ψ, φ have a coincidence point. In addition, if for any two coincidence points γ_1 and γ_2 of ψ, φ, that is, there exist some $\varsigma_1, \varsigma_2 \in \mho$ with $\psi\varsigma_1 = \varphi\varsigma_1 = \gamma_1$ and $\psi\varsigma_2 = \varphi\varsigma_2 = \gamma_2$, then ψ, φ possess a unique coincidence point. Furthermore, if ψ, φ are two weakly compatible maps then they possess a unique common fixed point.

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Chapter 5

Fixed Point Results in Bipolar $\mathcal{R}$-Metric Space

5.1 Introduction

Mutlu & Gürdal (2016) coined the notion of the bipolar metric space which deals with two abstract spaces by defining metric d on the cartesian product of these spaces. This theory indeed generalized the metric space where only one space is involved. Many authors have presented fixed point results in bipolar metric space together with contractive maps (see Kishore et al. (2018), Gürdal et al. (2020), Mutlu et al. (2020), Gaba et al. (2021), Roy et al. (2022) and references cited therein).

With the intend to further generalize the idea of bipolar metric space, by this chapter, we first introduce the notion of bipolar $\mathcal{R}$-metric space wherein by associating an arbitrary binary relation $\mathcal{R}$ with bipolar metric space, fixed point result is obtained. Next, we move a step forward and introduce the notions of $\mathcal{F}_{\mathcal{R}}$-contractive map and $\mathcal{F}_{\mathcal{R}}$-expansive map along with some fixed point results in a bipolar $\mathcal{R}$-metric space . Under a certain specific condition, the results reduces to novel fixed point result in bipolar metric space with respect to an expansive map. The results of this chapter are part of the research papers presented in [8,9].

[8]Malhotra, A., and Kumar, D. (2022). Bipolar $\mathcal{R}$-metric space and fixed point result. International Journal of Nonlinear Analysis and Applications, 13(2), 709-712.

[9]Malhotra, A., and Kumar, D. (2023). Fixed Point Results using $F_{\mathcal{R}}$-contractive map and $F_{\mathcal{R}}$-expansive map in Bipolar $\mathcal{R}$-metric space. (Communicated).

5.2 Bipolar $\mathcal{R}$-metric space and Generalized Contraction Maps

To begin with, we first put forward some of the terminologies used along with the supportive lemma. At the end of the section, an example is discussed that helps to validate the result proved.

Definition 5.2.1. *Two non-empty sets* $\mho$ *and* Λ *together with metric* $d : \mho \times \Lambda \to [0, +\infty)$ *and binary relation* $\mathcal{R} \subseteq \mho \times \Lambda$ *is c.t.b a **bipolar** $\mathcal{R}$**-metric space** (denoted by* $(\mho, \Lambda, d_{\mathcal{R}})$*) if:*

(i) $(\mho, \Lambda, d)$ *is a bipolar metric space;*

(ii) $\mathcal{R}$ *is a binary relation on* $\mho \times \Lambda$.

Definition 5.2.2. *In a bipolar* $\mathcal{R}$*-metric space* $(\mho, \Lambda, d_{\mathcal{R}})$*:*

(i) a bisequence $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ *in* $\mho\times\Lambda$ *is c.t.b an* $\mathcal{R}$***-bisequence*** *if* $(\varrho_\varpi, \varsigma_{\varpi+1}) \in \mathcal{R}$ *or* $(\varrho_{\varpi+1}, \varsigma_\varpi) \in \mathcal{R}\ \forall\ \varpi \in \mathbb{N}$.

(ii) an $\mathcal{R}$*-bisequence* $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ *is c.t.b a **convergent*** $\mathcal{R}$***-bisequence*** *if both* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *and* $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}}$ *are convergent to respective right and left point.*

(iii) an $\mathcal{R}$*-bisequence* $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ *is c.t.b a **biconvergent*** $\mathcal{R}$***-bisequence*** *if both* $\{\varrho_\varpi\}_{\varpi\in\mathbb{N}}$ *and* $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}}$ *are convergent to the same central point.*

(iv) a map $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ *is c.t.b a **bipolar*** $\mathcal{R}$***-continuous*** *if* $\forall$ *convergent* $\mathcal{R}$*-bisequence* $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ *in* $\mho \times \Lambda$ *s.t*

$$\varrho_\varpi \to \varsigma \quad \textit{and} \quad \varsigma_\varpi \to \varrho \quad \textit{as} \quad \varpi \to +\infty,$$
$$\textit{implies,} \quad \psi\varrho_\varpi \to \psi\varsigma \quad \textit{and} \quad \psi\varsigma_\varpi \to \psi\varrho \quad \textit{as} \quad \varpi \to +\infty.$$

(v) $(\mho, \Lambda, d_{\mathcal{R}})$ *is c.t.b a **complete bipolar*** $\mathcal{R}$***-metric space*** *if every Cauchy* $\mathcal{R}$*-bisequence is convergent* $\mathcal{R}$*-bisequence.*

(vi) a map $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ *is c.t.b a* $\mathcal{F}_{\mathcal{R}}$***-contractive map*** *for some* $\mathcal{F} \in \mathfrak{F}$ *if* $\exists$ *some* $\wp > 0$ *s.t for* $(\varrho, \varsigma) \in \mathcal{R}$*, we have*

$$\wp + \mathcal{F}(d_{\mathcal{R}}(\psi\varrho, \psi\varsigma)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma)).$$

Lemma 5.2.1. *In a bipolar $\mathcal{R}$-metric space, each convergent Cauchy $\mathcal{R}$-bisequence implies it is biconvergent $\mathcal{R}$-bisequence.*

Proof. Let $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ be a convergent Cauchy $\mathcal{R}$-bisequence, that is, there is some $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}} \subset \mho \times \Lambda$ with $\varrho_\varpi \to \varsigma$ (in Λ) and $\varsigma_\varpi \to \varrho$ (in $\mho$) as $\varpi \to +\infty$. Let $\varpi, \varpi_0, \varpi^* \in \mathbb{N}$ where $\varpi, \varpi^* > \varpi_0$, then

$$d_{\mathcal{R}}(\varrho, \varsigma) \leqslant d_{\mathcal{R}}(\varrho, \varsigma_{\varpi^*}) + d_{\mathcal{R}}(\varrho_\varpi, \varsigma) + d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi^*}).$$

Taking limit as $\varpi, \varpi^* \to +\infty$, we get $\varrho = \varsigma$. Hence, $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ is biconvergent $\mathcal{R}$-bisequence. □

Theorem 5.2.2. *For a complete bipolar $\mathcal{R}$-metric space $(\mho, \Lambda, d_{\mathcal{R}})$, let $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ a map s.t:*

(I) $\psi(\mho) \subseteq \mho$ *and* $\psi(\Lambda) \subseteq \Lambda$;

(II) $\exists$ *some* $0 < \lambda < 1$ *with* $d_{\mathcal{R}}(\psi\varrho, \psi\varsigma) \leqslant \lambda d_{\mathcal{R}}(\varrho, \varsigma)$ *for each* $(\varrho, \varsigma) \in \mathcal{R}$;

(III) $\exists$ *some* $(\varrho_0, \varsigma_0) \in \mho \times \Lambda$ *with* $(\varrho_0, \varsigma_0) \in \mathcal{R}$ *and* $(\varrho_0, \psi\varsigma_0) \in \mathcal{R}$;

(IV) ψ *is bipolar $\mathcal{R}$-continuous;*

(V) For each $(\varrho, \varsigma) \in \mathcal{R}$, *we have* $(\psi\varrho, \psi\varsigma) \in \mathcal{R}$.

Then, ψ possesses at least one fixed point.

Proof. Let the bisequence $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}'}$ in $\mho \times \Lambda$, where $\psi\varrho_{\varpi-1} = \varrho_\varpi$ and $\psi\varsigma_{\varpi-1} = \varsigma_\varpi \quad \forall\, \varpi \in \mathbb{N}'$. By condition *(III)*, we obtain that $\exists$ some $(\varrho_0, \varsigma_0) \in \mho \times \Lambda$ where $(\varrho_0, \varsigma_0) \in \mathcal{R}$ and $(\varrho_0, \varsigma_1) = (\varrho_0, \psi\varsigma_0) \in \mathcal{R}$. On using condition *(V)*, we have

$$(\psi\varrho_0, \psi\varsigma_0) = (\varrho_1, \varsigma_1) \in \mathcal{R} \quad \text{and} \quad (\psi\varrho_0, \psi\varsigma_1) = (\varrho_1, \varsigma_2) \in \mathcal{R},$$

continuing this process, we get $(\varrho_\varpi, \varsigma_\varpi) \in \mathcal{R}$ and $(\varrho_\varpi, \varsigma_{\varpi+1}) \in \mathcal{R}$, $\forall\, \varpi \in \mathbb{N}'$. Thus $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}'}$ is an $\mathcal{R}$-bisequence. Now from condition *(II)*, we obtain

$$d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1}) = d_{\mathcal{R}}(\psi\varrho_\varpi, \psi\varsigma_\varpi) \leqslant \lambda d_{\mathcal{R}}(\varrho_\varpi, \varsigma_\varpi) \leqslant \cdots \leqslant \lambda^{\varpi+1} d_{\mathcal{R}}(\varrho_0, \varsigma_0).$$

Furthermore,

$$d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1}) = d_{\mathcal{R}}(\psi\varrho_{\varpi-1}, \psi\varsigma_\varpi) \leqslant \lambda d_{\mathcal{R}}(\varrho_{\varpi-1}, \varsigma_\varpi) \leqslant \cdots \leqslant \lambda^{\varpi} d_{\mathcal{R}}(\varrho_0, \varsigma_1).$$

Next, for some $\varpi, \varpi^* \in \mathbb{N}'$ with $\varpi^* > \varpi$, we obtain

$$\begin{aligned}
d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi^*}) &\leqslant d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi^*}) \\
&\leqslant d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+2}) + d_{\mathcal{R}}(\varrho_{\varpi+2}, \varsigma_{\varpi+2}) \\
&\quad + d_{\mathcal{R}}(\varrho_{\varpi+2}, \varsigma_{\varpi^*}) \\
&\leqslant (d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+2}, \varsigma_{\varpi+2}) + \cdots + d_{\mathcal{R}}(\varrho_{\varpi^*-1}, \varsigma_{\varpi^*-1})) \\
&\quad + (d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+2}) + \cdots + d_{\mathcal{R}}(\varrho_{\varpi^*-1}, \varsigma_{\varpi^*})) \\
&\leqslant \sum_{k=\varpi}^{\varpi^*-2} d_{\mathcal{R}}(\varrho_{k+1}, \varsigma_{k+1}) + \sum_{k=\varpi}^{\varpi^*-1} d_{\mathcal{R}}(\varrho_k, \varsigma_{k+1}) \\
&\leqslant \sum_{k=\varpi}^{+\infty} \lambda^{k+1} d_{\mathcal{R}}(\varrho_0, \varsigma_0) + \sum_{k=\varpi}^{+\infty} \lambda^k d_{\mathcal{R}}(\varrho_0, \varsigma_1), \\
&= \frac{\lambda^{\varpi+2} d_{\mathcal{R}}(\varrho_0, \varsigma_0)}{1-\lambda} + \frac{\lambda^{\varpi+1} d_{\mathcal{R}}(\varrho_0, \varsigma_1)}{1-\lambda} \to 0 \quad \text{as} \quad \varpi \to +\infty.
\end{aligned}$$

Thus $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi \in \mathbb{N}'}$ is a Cauchy $\mathcal{R}$-bisequence and since $(\mho, \Lambda, d_{\mathcal{R}})$ is a complete bipolar $\mathcal{R}$-metric space, so $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi \in \mathbb{N}'}$ is convergent $\mathcal{R}$-bisequence. By Lemma 5.2.1, $\exists\ \eta \in \mho \cap \Lambda$ s.t

$\varrho_\varpi \to \eta$ and $\varsigma_\varpi \to \eta$ as $\varpi \to +\infty$.

As ψ is bipolar $\mathcal{R}$-continuous, so we have

$$\begin{aligned}
&\lim_{\varpi \to +\infty} \psi \varrho_\varpi = \psi \eta \quad \text{and} \quad \lim_{\varpi \to +\infty} \psi \varsigma_\varpi = \psi \eta, \\
\text{that is,} \quad &\lim_{\varpi \to +\infty} \varrho_{\varpi+1} = \psi \eta \quad \text{and} \quad \lim_{\varpi \to +\infty} \varsigma_{\varpi+1} = \psi \eta, \\
&\text{then,} \quad \eta = \psi \eta.
\end{aligned}$$

Thus, ψ possesses at least one fixed point. □

Example 5.2.3. *Let $\mho = [0, 1/2]$ and $\Lambda = [-1/2, 0]$ where for $(\varrho, \varsigma) \in \mho \times \Lambda$ we define $d(\varrho, \varsigma) = |\varrho - \varsigma|$. Define $\mathcal{R}$ on $\mho \times \Lambda$ as $(\varrho, \varsigma) \in \mathcal{R}$ iff $\varrho.\varsigma = 0$. Define $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ as:*

$$\psi(\varrho) = \begin{cases} \dfrac{29\varrho}{73} & for\ \varrho \in [0, 1/2]; \\ \dfrac{-\varrho^2}{6} & for\ \varrho \in [-1/2, 0). \end{cases}$$

Clearly, $\psi(\mho) \subseteq \mho$ and $\psi(\Lambda) \subseteq \Lambda$. For $(\varrho, \varsigma) \in \mathcal{R}$ we have $(\psi\varrho, \psi\varsigma) \in \mathcal{R}$. Also, ψ is a bipolar $\mathcal{R}$-continuous, since for any convergent $\mathcal{R}$-bisequence $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi \in \mathbb{N}} \in \mho \times \Lambda$, we have $\varrho_\varpi \to 0$ and $\varsigma_\varpi \to 0$ as $\varpi \to +\infty$ then $\psi\varrho_\varpi \to \psi 0 = 0$ and

$\psi\varsigma_\varpi \to \psi 0 = 0$ as $\varpi \to +\infty$. Next, we verify condition (II) *of Theorem 5.2.2. For $(\varrho, \varsigma) \in \mathcal{R}$ either $\varrho = 0$ and/or $\varsigma = 0$, therefore we have the following cases:*

***Case (i)**: If $\varrho = 0$ and $\varsigma \in [-1/2, 0)$ and for $\lambda = 1/11$, we have*

$$d_{\mathcal{R}}(\psi\varrho, \psi\varsigma) \leqslant \lambda d_{\mathcal{R}}(\varrho, \varsigma).$$

***Case (ii)**: If $\varrho \in (0, 1/2]$ and $\varsigma = 0$ and for $\lambda \in (29/73, 1)$, we have*

$$d_{\mathcal{R}}(\psi\varrho, \psi\varsigma) \leqslant \lambda d_{\mathcal{R}}(\varrho, \varsigma).$$

Since all hypotheses of Theorem 5.2.2 hold, so ψ possesses a fixed point viz. 0.

Theorem 5.2.4. *For a complete bipolar $\mathcal{R}$-metric space $(\mho, \Lambda, d_{\mathcal{R}})$, let $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ be a map s.t for some $\mathcal{F} \in \mathfrak{F}$, the following holds:*

(I) $\psi(\mho) \subseteq \mho$ and $\psi(\Lambda) \subseteq \Lambda$;

(II) ψ is $\mathcal{F}_{\mathcal{R}}$-contractive map;

(III) $\exists$ some $(\varrho_0, \varsigma_0) \in \mho \times \Lambda$ where $(\varrho_0, \varsigma_0) \in \mathcal{R}$ and $(\varrho_0, \psi\varsigma_0) \in \mathcal{R}$;

(IV) ψ is bipolar $\mathcal{R}$-continuous;

(V) For each $(\varrho, \varsigma) \in \mathcal{R}$, we have $(\psi\varrho, \psi\varsigma) \in \mathcal{R}$.

Then, ψ possesses a fixed point. Furthermore, if there are two fixed point ϱ, ϱ^* then $(\varrho, \varrho^*) \in \mathcal{R}$ and in such case ψ possesses a unique fixed point.

Proof. Define a bisequence $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi \in \mathbb{N}'}$ in $\mho \times \Lambda$ with $\psi\varrho_{\varpi-1} = \varrho_\varpi$ and $\psi\varsigma_{\varpi-1} = \varsigma_\varpi$. Since $(\varrho_0, \varsigma_0) \in \mho \times \Lambda$, by condition *(III)*, we obtain

$$(\varrho_0, \varsigma_0) \in \mathcal{R} \quad \text{and} \quad (\varrho_0, \psi\varsigma_0) \in \mathcal{R}.$$

On using $\mathcal{R}$-preserving property of ψ, we obtain

$$(\psi\varrho_0, \psi\varsigma_0) \in \mathcal{R} \quad \text{and} \quad (\psi\varrho_0, \psi\varsigma_1) \in \mathcal{R}.$$

Repetitive use of $\mathcal{R}$-preserving property of ψ, we get

$$(\varrho_\varpi, \varsigma_\varpi) \in \mathcal{R} \quad \text{and} \quad (\varrho_\varpi, \varsigma_{\varpi+1}) \in \mathcal{R},$$

$\forall\ \varpi \in \mathbb{N}'$. Thus, $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi\in\mathbb{N}}$ is an $\mathcal{R}$-bisequence. Now,

$$\begin{aligned}\mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1})) &= \mathcal{F}(d_{\mathcal{R}}(\psi\varrho_\varpi, \psi\varsigma_\varpi)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varsigma_\varpi)) - \wp \\ &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varsigma_{\varpi-1})) - 2\wp \\ &\leqslant \cdots \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varsigma_0)) - (\varpi+1)\wp. \end{aligned} \tag{5.1}$$

Letting $\varpi \to +\infty$ in (5.1) and using $(\mathcal{F}_2)$ property of $\mathcal{F}$, we have $\lim_{\varpi\to+\infty} d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1}) = 0$. Further using $(\mathcal{F}_3)$ property of $\mathcal{F}$, we obtain that $\exists\ \varpi \in (0,1)$ s.t

$$\lim_{\varpi\to+\infty} \Big(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1})\Big)^{\varpi} \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1})) = 0. \tag{5.2}$$

Using (5.1) in (5.2), we obtain

$$\begin{aligned}\Big(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1})\Big)^{\varpi} \Big(\mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1})) &- \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varsigma_0))\Big) \\ &\leqslant -(\varpi+1)\Big(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1})\Big)^{\varpi}\wp. \end{aligned} \tag{5.3}$$

Taking limit as $\varpi \to +\infty$ in (5.3), we get $\lim_{\varpi\to+\infty} (\varpi+1)\Big(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1})\Big)^{\varpi} = 0$. Thus, $\exists$ some $\varpi^* \in \mathbb{N}'$ s.t for each $\varpi \geqslant \varpi^*$, we get

$$d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1}) \leqslant \frac{1}{(\varpi+1)^{1/\varpi}}\ . \tag{5.4}$$

Since,

$$\begin{aligned}\mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1})) &= \mathcal{F}(d_{\mathcal{R}}(\psi\varrho_{\varpi-1}, \psi\varsigma_\varpi)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varsigma_\varpi)) - \wp \\ &\leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-2}, \varsigma_{\varpi-1})) - 2\wp \\ &\leqslant \cdots \leqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varsigma_1)) - \varpi\wp. \end{aligned} \tag{5.5}$$

Taking limit as $\varpi \to +\infty$ in (5.5), we get $\lim_{\varpi\to+\infty} d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1}) = 0$. By using $(\mathcal{F}_3)$ property of $\mathcal{F}$, we obtain that $\exists\ \varpi^* \in (0,1)$ so that

$$\lim_{\varpi\to+\infty} \Big(d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1})\Big)^{\varpi^*} \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1})) = 0. \tag{5.6}$$

Using (5.6) in (5.5), we have

$$\bigg(d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1})\bigg)^{\varpi^*} \bigg(\mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1})) - \mathcal{F}(d_{\mathcal{R}}(\varrho_0, \varsigma_1))\bigg) \leqslant -\varpi\Big(d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1})\Big)^{\varpi^*}\wp. \tag{5.7}$$

Taking limit as $\varpi \to +\infty$ in (5.7), we get $\lim\limits_{\varpi\to+\infty} \varpi\Big(d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma_{\varpi+1})\Big)^{\varpi^*} = 0$. Thus, $\exists$ some $\varpi^{**} \in \mathbb{N}'$ s.t for each $\varpi \geqslant \varpi^{**}$, we obtain

$$d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma_{\varpi+1}) \leqslant \frac{1}{\varpi^{1/\varpi^*}}. \tag{5.8}$$

Consider $\varpi, \varpi', \varpi^* \in \mathbb{N}'$ where $\varpi' = \max\{\varpi^*, \varpi^{**}\}$ and $\varpi^* > \varpi > \varpi'$, we have

$$\begin{aligned} d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma_{\varpi^*}) &\leqslant d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+2}) + \cdots + \\ &\quad d_{\mathcal{R}}(\varrho_{\varpi^*-1}, \varsigma_{\varpi^*}) \\ &= \Big(d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+2}, \varsigma_{\varpi+2}) + \cdots + d_{\mathcal{R}}(\varrho_{\varpi^*-1}, \varsigma_{\varpi^*-1})\Big) \\ &\quad + \Big(d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma_{\varpi+1}) + d_{\mathcal{R}}(\varrho_{\varpi+1}, \varsigma_{\varpi+2}) + \cdots + d_{\mathcal{R}}(\varrho_{\varpi^*-1}, \varsigma_{\varpi^*})\Big) \\ &\leqslant \sum_{\gamma=1}^{+\infty} d_{\mathcal{R}}(\varrho_{\gamma}, \varsigma_{\gamma}) + \sum_{\delta=1}^{+\infty} d_{\mathcal{R}}(\varrho_{\delta}, \varsigma_{\delta+1}). \end{aligned} \tag{5.9}$$

Using (5.4) and (5.8) in (5.9), we obtain

$$d_{\mathcal{R}}(\varrho_{\varpi}, \varsigma_{\varpi}) \leqslant \sum_{\gamma=1}^{+\infty} \frac{1}{(\gamma+1)^{\frac{1}{\varpi}}} + \sum_{\delta=1}^{+\infty} \frac{1}{\Big(\delta\Big)^{\frac{1}{\varpi^*}}}. \tag{5.10}$$

Since, (5.10) is a convergent series, so we have $(\{\varrho_{\varpi}\}, \{\varsigma_{\varpi}\})_{\varpi\in\mathbb{N}'}$ is a Cauchy $\mathcal{R}$-bisequence on complete bipolar $\mathcal{R}$-metric space. Using Lemma 5.2.1, we obtain that $\exists\ \sigma \in \mho \cap \Lambda$ with

$$\lim_{\varpi\to+\infty} \varrho_{\varpi} = \sigma \quad \text{and} \quad \lim_{\varpi\to+\infty} \varsigma_{\varpi} = \sigma.$$

Using $\mathcal{R}$-continuity of ψ, we get

$$\lim_{\varpi\to+\infty} \psi\varrho_{\varpi} = \lim_{\varpi\to+\infty} \varrho_{\varpi+1} = \psi\sigma \quad \text{and} \quad \lim_{\varpi\to+\infty} \psi\varsigma_{\varpi} = \lim_{\varpi\to+\infty} \varsigma_{\varpi+1} = \psi\sigma,$$

that is, $\sigma = \psi\sigma$.

Thus, ψ possesses a fixed point. Next, let $\sigma^* \in \mho \cap \Lambda$ be s.t $\psi\sigma^* = \sigma^*$, then $(\sigma, \sigma^*) \in \mathcal{R}$. Now,

$$\mathcal{F}(d_{\mathcal{R}}(\sigma, \sigma^*)) \leqslant \wp + \mathcal{F}(d_{\mathcal{R}}(\psi\sigma, \psi\sigma^*)) \leqslant \mathcal{F}(d_{\mathcal{R}}(\sigma, \sigma^*)),$$

which holds only if $\sigma = \sigma^*$. Hence, ψ possesses a unique fixed point. □

Example 5.2.5. *Let* $\mho = \{1, 2, 3\}$, $\Lambda = \{3, 4, 5\}$ *together with metric* $d : \mho \times \Lambda \to$

$[0, +\infty)$ *defined as* $d(\varrho, \varsigma) = |\varrho - \varsigma|$ *and a binary relation* $\mathcal{R} \subset \mho \times \Lambda$ *defined as*

$$\mathcal{R} = \{(1,3), (2,3), (3,3)\}.$$

Define a map $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ *as* $\psi(1) = 2$, $\psi(2) = 3$, $\psi(3) = 3$, $\psi(4) = 4$, $\psi(5) = 4$. *Clearly* $\psi(\mho) \subseteq \mho$, $\psi(\Lambda) \subseteq \Lambda$ *and for any* $(\varrho, \varsigma) \in \mathcal{R}$ *implies* $(\psi(\varrho), \psi(\varsigma)) \in \mathcal{R}$. *Also, for any convergent* $\mathcal{R}$*-bisequence* $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi \in \mathbb{N}} \in \mho \times \Lambda$, *we have* $\lim\limits_{\varpi \to +\infty} \varrho_\varpi = 3$ *and* $\lim\limits_{\varpi \to +\infty} \varsigma_\varpi = 3$ *thus* $\lim\limits_{\varpi \to +\infty} \psi \varrho_\varpi = \psi 3 = 3$ *and* $\lim\limits_{\varpi \to +\infty} \psi \varsigma_\varpi = \psi 3 = 3$. *Next, to show that* ψ *is* $\mathcal{F}_{\mathcal{R}}$*-contractive map, where* $\mathcal{F}(\mu) = \ln(\mu) + \mu$, *let us consider the following cases:*

Case (i)*: Let* $(\varrho, \varsigma) = (1,3)$. *Then,*

$$\wp + \mathcal{F}(d(\psi\varrho, \psi\varsigma)) = \wp + \ln(d(2,3)) + d(2,3) = \wp + 1,$$

$$\textit{and,} \quad \mathcal{F}(d(\varrho, \varsigma)) = \ln(d(1,3)) + d(1,3) = \ln(2) + 2.$$

So, the $\mathcal{F}_{\mathcal{R}}$*-contractive condition holds in this case for any* $\wp \in (0, \ln(2) + 1)$.

Case (ii)*: Let* $(\varrho, \varsigma) = (2,3)$. *Then,*

$$\wp + \mathcal{F}(d(\psi\varrho, \psi\varsigma)) \to -\infty, \quad \textit{and,} \quad \mathcal{F}(d(\varrho, \varsigma)) = \ln(d(2,3)) + d(2,3) = 1.$$

So, the $\mathcal{F}_{\mathcal{R}}$*-contractive condition holds in this case for any* $\wp > 0$.

Case (iii)*: Let* $(\varrho, \varsigma) = (3,3)$. *Then,*

$$\wp + \mathcal{F}(d(\psi\varrho, \psi\varsigma)) \to -\infty, \quad \textit{and,} \quad \mathcal{F}(d(\varrho, \varsigma)) \to -\infty.$$

So, the $\mathcal{F}_{\mathcal{R}}$*-contractive condition holds in this case for any* $\wp > 0$.
Since the conditions (I)-(V) of Theorem 5.2.4 hold, so ψ *possesses fixed points which are* $\varrho = 3$ *and* $\varrho = 4$.

5.3 Generalized Expansive Map and Fixed Point Result

Definition 5.3.1. *For a bipolar* $\mathcal{R}$*-metric space* $(\mho, \Lambda, d_{\mathcal{R}})$, $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ *is c.t.b a* $\mathcal{F}_{\mathcal{R}}$***-expansive map*** *for* $\mathcal{F} \in \mathfrak{F}$ *if* $\exists$ *some* $\wp > 0$ *s.t for* $(\varrho, \varsigma) \in \mathcal{R}$, *we* *have*

$$\mathcal{F}(d_{\mathcal{R}}(\psi\varrho, \psi\varsigma)) \geqslant \mathcal{F}(d_{\mathcal{R}}(\varrho, \varsigma)) + \wp.$$

Theorem 5.3.1. *For a complete bipolar $\mathcal{R}$-metric space $(\mho, \Lambda, d_{\mathcal{R}})$, let $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ be a surjective map with ψ^* as right inverse of ψ s.t for some $\mathcal{F} \in \mathfrak{F}$, the following holds:*

(I) $\psi(\mho), \psi^*(\mho) \subseteq \mho$ *and* $\psi(\Lambda), \psi^*(\Lambda) \subseteq \Lambda$*;*

(II) ψ *is* $\mathcal{F}_{\mathcal{R}}$*-expansive map;*

(III) $\exists$ *some* $(\varrho_0, \varsigma_0) \in \mho \times \Lambda$ *s.t* $(\varrho_0, \varsigma_0) \in \mathcal{R}$, $(\varrho_0, \psi\varsigma_0) \in \mathcal{R}$ *and* $(\varrho_0, \psi^*\varsigma_0) \in \mathcal{R}$*;*

(IV) ψ *and* ψ^* *are both bipolar* $\mathcal{R}$*-continuous;*

(V) For each $(\varrho, \varsigma) \in \mathcal{R}$, *we have* $(\psi\varrho, \psi\varsigma) \in \mathcal{R}$ *and* $(\psi^*\varrho, \psi^*\varsigma) \in \mathcal{R}$.

Then, ψ possesses a fixed point. Furthermore, if there are two fixed point ϱ, ϱ^* then $(\varrho, \varrho^*) \in \mathcal{R}$ and in such case ψ possesses a unique fixed point.

Proof. Define $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi \in \mathbb{N}'}$ be a bisequence in $\mho \times \Lambda$, where

$$\psi\varrho_{\varpi-1} = \varrho_\varpi \quad \text{and} \quad \psi\varsigma_{\varpi-1} = \varsigma_\varpi, \quad \forall\, \varpi \in \mathbb{N}$$

Now, proceeding on the lines of Theorem 5.2.4, we obtain that $(\{\varrho_\varpi\}, \{\varsigma_\varpi\})_{\varpi \in \mathbb{N}'}$ is an $\mathcal{R}$-bisequence. Since, $(\varrho_\varpi, \varsigma_\varpi) \in \mathcal{R}$ for $\varpi \in \mathbb{N}$ and ψ is surjective so we have $\psi^* : \mho \cup \Lambda \to \mho \cup \Lambda$ s.t

$$\psi^*\varrho_\varpi = \varrho_{\varpi-1} \quad \text{and} \quad \psi^*\varsigma_{\varpi+1} = \varsigma_\varpi \qquad \forall\, \varpi \in \mathbb{N}'.$$

Next,

$$\begin{aligned} \mathcal{F}(d_{\mathcal{R}}(\varrho_\varpi, \varsigma_{\varpi+1})) &= \mathcal{F}(d_{\mathcal{R}}(\psi\varrho_{\varpi-1}, \psi\varsigma_\varpi)) \\ &\geqslant \mathcal{F}(d_{\mathcal{R}}(\varrho_{\varpi-1}, \varsigma_\varpi)) + \wp \\ &= \mathcal{F}(d_{\mathcal{R}}(\psi^*\varrho_\varpi, \psi^*\varsigma_{\varpi+1})) + \wp. \end{aligned} \tag{5.11}$$

By (5.11) and Theorem 5.2.4, we obtain that $\exists$ unique $\sigma \in \mho \cap \Lambda$ s.t,

$$\begin{aligned} \psi^*\sigma &= \sigma, \\ \text{that is, } \psi\sigma &= \sigma. \end{aligned}$$

Thus, ψ possesses a unique fixed point. □

5.4 Consequences

In this section, we reduce the results proved in previous section for fixed point result in bipolar metric space $(\mho, \Lambda, d)$ under an expansive map and some fixed point result in the literature.

Theorem 5.4.1. *For a complete bipolar metric space* $(\mho, \Lambda, d)$*, let* $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ *be a surjective map with* ψ^* *as right inverse of* ψ *where* ψ *and* ψ^* *are continuous map with* $\psi(\mho), \psi^*(\mho) \subseteq \mho$*,* $\psi(\Lambda), \psi^*(\Lambda) \subseteq \Lambda$ *and for some* $\mathcal{F} \in \mathfrak{F}$*,* $\exists\ \wp > 0$ *s.t:*

$$\mathcal{F}(d(\psi\varrho, \psi\varsigma)) \geqslant \mathcal{F}(d(\varrho, \varsigma)) + \wp.$$

Then, ψ *possesses a unique fixed point.*

Proof. If in Theorem 5.3.1, we consider $\mathcal{R} = \mho \times \Lambda$ then the above result is obtained. □

Theorem 5.4.2. *(Mani et al. (2023)) For a complete bipolar metric space* $(\mho, \Lambda, d)$*, let* $\psi : \mho \cup \Lambda \to \mho \cup \Lambda$ *be a map where* ψ *is a continuous map with* $\psi(\mho) \subseteq \mho$*,* $\psi(\Lambda) \subseteq \Lambda$ *and for some* $\mathcal{F} \in \mathfrak{F}$*,* $\exists\ \wp > 0$ *s.t:*

$$\wp + \mathcal{F}(d(\psi\varrho, \psi\varsigma)) \leqslant \mathcal{F}(d(\varrho, \varsigma)).$$

Then, ψ *possesses a unique fixed point.*

Proof. If in Theorem 5.2.4, we consider $\mathcal{R} = \mho \times \Lambda$ then the above result is obtained. □

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List of Publications

1. **Malhotra, A.**, Kumar, D., and Park, C. (2022). C^*-algebra valued $\mathcal{R}$-metric space and fixed point theorems. AIMS Mathematics, 7(4), 6550-6554. (SCI, Impact Factor 2.739)

2. **Malhotra, A.**, and Kumar, D. (2022). Some fixed point results using F-weak expansive mapping in relation theoretic metric space. Journal of Physics: Conference Series, IOP Publishing, 2267(1), 012040. (ESCI & SCOPUS, SJR 0.21)

3. **Malhotra, A.**, and Kumar, D. (2022). Generalized Contraction Mappings and Fixed Point Results in Orthogonal Metric Space. Applied Mathematics E-Notes, 22, 393-426. (SCOPUS, SJR 0.27)

4. **Malhotra, A.**, and Kumar, D. (2022). Bipolar $\mathcal{R}$-metric space and fixed point result. International Journal of Nonlinear Analysis and Applications, 13(2), 709-712. (ESCI)

5. **Malhotra, A.**, and Kumar, D. (2023). Fixed Point Results for Multivalued Mapping in R-Metric Space. Sahand Communications in Mathematical Analysis, 20(2), 109-121. (ESCI & SCOPUS, SJR 0.19)

6. Kumar, D. and **Malhotra, A.** (2022). Orthogonal F-weak Contraction Mapping in Orthogonal Metric Space, Fixed Points and Applications. Filomat. (Accepted, SCI, Impact Factor 0.8)

7. **Malhotra, A.**, and Kumar, D. (2023). Existence and Stability of Solution for a Nonlinear Volterra Integral Equation with binary relation via Fixed Point Results. (Communicated)

8. **Malhotra, A.**, and Kumar, D. (2023). Coincidence Point and Common Fixed Point in C*-algebra Valued $\mathcal{R}$-metric Space using Picard-Jungck Iteration Process with Application in Graph Theory. (Communicated)

9. **Malhotra, A.**, and Kumar, D. (2023). Fixed Point Results using $F_{\mathcal{R}}$-contractive map and $F_{\mathcal{R}}$-expansive map in Bipolar $\mathcal{R}$-metric space. (Communicated)

Papers Presented in Conferences

1. **Malhotra, A.**, and Kumar, D. (2021). Some fixed point results using F-weak expansive mapping in relation theoretic metric space, "RAFAS-2021" held at Lovely Professional University, Phagwara, India on June 25-26, 2021.

2. **Malhotra, A.**, and Kumar, D. (2022). Some Fixed Point Results for Multivalued Mapping in R-Metric Space, "ICRANFAA-2022" held at Andhra University, Visakhapatnam, India on January 29-30, 2022.

Workshop and Conferences Attended

1. **International e-Conference on Fixed Point Theory and its Applications to Real World Problems** organized by Department of Mathematics, Government Post Graduate College Maldevta, on June 27, 2020.

2. **International e-Conference on Nonlinear Analysis and its Application** organized by Department of Mathematics, Dayanand Science College, Latur, from July 27-29, 2020.

3. **One Week International e-Faculty Development Programme on Fixed Point Theory and its Applications**, organized by School of Mathematics and Statistics, School of Basic Science, Manipal University Jaipur, from September 15-19, 2020.

4. **One Week Short Term Training Program on Computational Software (MATLAB & MATHEMATICA)** organized by Applied Mathematics and Humanities Department, Sardar Vallabhbhai National Institute of Technology, Surat, from October 05–09, 2020.

5. **Workshop titled Various Applications of Fixed Point Theory** organized by King Fahd University of Petroleum and Minerals Department of Mathematics & Statistics Fixed Point Theory & Applications Research Group, from December 14-15, 2020.