A STUDY ON THE METHODS FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

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By

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Abstract

Fractional calculus is the generalized version of traditional calculus. Mathematical models are more meaningful if they involve the fractional differential equations (FDEs), and to solve those equations, effective techniques are needed. There are many analytical, semi analytical and numerical techniques that have been used to solve the ordinary and partial differential equations.

This work is about exploring analytical and numerical methods to solve the time-fractional differential equations. Differential transform method (DTM) is a semi analytical method that has evolved over years to solve FDEs. The differential quadrature method (DQM) is a numerical technique that has also been modified to cater to the FDEs.

These methods are well established for the ordinary and partial differential equations and they are used here to solve fractional differential equations.

In this work we solve

- the one dimensional fractional Bagley Trovik equation and fractional relaxation oscillation equation using DTM.
- the one dimensional fractional Burgers' equation and the inverse problem on fractional Fisher's equation using modified DQM.
- the two dimensional fractional diffusion equation using a hybrid method D(TQ)M.

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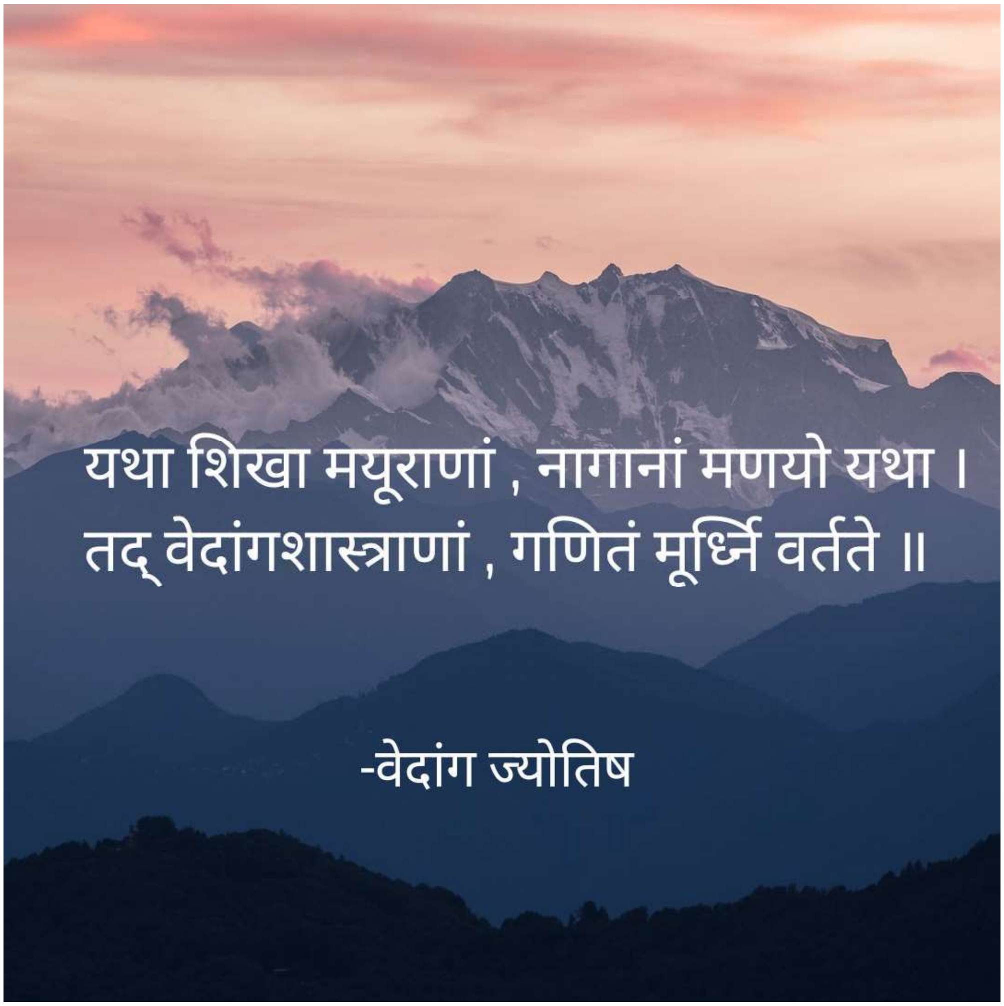
Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

Certification by the Supervisor: It is hereby certified that the information, and details presented above are true to the best of my knowledge.

April 2022



I would dedicate this thesis to my mother Smt. Vimla Devshali and my father Sh. Vishweshwar Dutt Devshali.

List of tables

Table of contents

List of figures

Chapter 1

Introduction

1.1 Fractional Calculus

In the creation, the only constant is change. And what can be a better way to study change, than calculus! Issac Newton and Gottfried Leibniz are equally credited for discovering calculus in the mid 17th century.

The notation propounded by Leibniz for the n^{th} order derivative of a function $f(x)$ was

Fig. 1.1 Issac Newton

Fig. 1.2 Gottfried Leibniz

given by $\frac{d^n f(x)}{dx^n}$, where n is a positive integer. The idea of derivative can be better understood by considering the distance as a function of time. Its first derivative w.r.t. time gives speed, second derivative is understood as acceleration. The third derivative of distance function w.r.t. time can be felt as a jerk and fourth order derivative is termed as jounce. Can one somehow visualize the three fourth derivative of distance function w.r.t. time or a derivative of the order π !!

In 1695, L'hopital wrote a letter to Leibniz asking what would happen for $'n = 1/2'$ in his notation. And Leibniz prophecized " This is an apparent paradox from which, one day, useful consequences will be drawn." And thus began the journey of fractional calculus around 326 years ago.

The term fractional calculus is a misnomer [1] as it not only caters to the fractional values of n but to any real or complex valued n. So if the integer order calculus is known as calculus, the

arbitrary order calculus can be termed as 'super-calculus'. Yet its popular name 'fractional calculus' will be used in this thesis.
Fractional calculus [2] is one of the best tools to characterize long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviors, power laws, allometric scaling laws, and so on. So the corresponding mathematical models are fractional differential equations. Their evolutions behave in a much more complicated way. So to study the corresponding dynamics is much more difficult. Although the existence theorems [3, 4] for the fractional differential equations can be similarly obtained, not all the classical theory of differential equation can be directly applied to the fractional differential equations.
To get a deeper understanding of this, some special functions need some explanation first.

1.1.1 Gamma function

Euler's gamma function [5] is one of the most fundamental functions of the fractional calculus. It is a generalization of factorial function $\eta!$ and allows η to take non integer and even complex values. It is defined as

$$\Gamma(\eta) = \int_0^\infty e^{-t} t^{\eta-1} dt.$$

For $\eta = x + iy$,

$$\Gamma(x+iy) = \int_0^\infty e^{-t}\ t^{x-1} [cos(y\ logt) + i\ sin(y\ logt)] dt.$$

The expression $cos(y\ logt) + i\ sin(y\ logt)$ is bounded for all t; e^{-t} provides convergence at infinity and for the convergence at zero, $x = Re(\eta)$ must be greater than one. Therefore , the gamma function converges in the right half of the complex plane $Re(\eta) > 0$. Moreover the gamma function has simple poles at $\eta = -k,\ (k = 0, 1, 2, ...)$.
The gamma function satisfies $\Gamma(\eta+1) = \eta\Gamma(\eta)$ and $\Gamma(\eta+1) = \eta!$. The limit representation of $\Gamma(\eta)$ is given as

$$\Gamma(\eta) = \lim_{\eta\to\infty} \frac{k^{\eta} k!}{\eta(\eta+1)...(\eta+k)}$$

where $-m < Re(\eta) \leq -m+1$ and m is a positive integer.

1.1.2 Mittag-Leffler function

Mittag-Leffler function is the generalized exponential function.

Fig. 1.3 Leonhard Euler

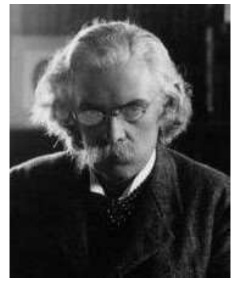


Fig. 1.4 Mittag-Leffler

Its one parameter form is represented as

$$E_{\aleph}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\aleph k+1)}, \aleph > 0 \tag{1.1}$$

and two parameter form is defined as

$$E_{\aleph,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\aleph k+\beta)}, \ \aleph > 0, \beta > 0. \tag{1.2}$$

Clearly $E_1(z) = E_{1,1}(z) = e^z$.

1.2 Development of the definition of fractional derivative

The definition of integer order derivative is unique, which is not the case with the arbitrary order deriative. This 326 years old topic took long time to grow and get popular. Euler came up with Gamma function in 1729.
Lacroix's in 1819 extended the n^{th} integer order derivative $D^n x^p = \frac{p!}{(p-n)!} x^{p-n}$ to the derivative of arbitrary order $\aleph$ by replacing the factorials by gamma function yielding:

$$D^{\aleph} x^p = \frac{\Gamma(p+1)}{\Gamma(p-\aleph+1)} x^{p-\aleph}. \tag{1.3}$$

Fourier in 1822 extended the definition for sine and cosine function for derivative of arbitrary order $\aleph$:

$$D^{\aleph} sin(x) = sin(x + \frac{\aleph \pi}{2}).$$

Abel provided the first application of fractional calculus in 1823 with his tautochrone problem. Liouville was the first mathematician to study fractional calculus in depth. He published three mammoth memoirs in 1832. He began with the arbitrary derivative for the exponential

Fig. 1.5 Fourier

Fig. 1.6 Lacroix

function. The n^{th} order derivative of e^{ax} is $a^n e^{ax}$, where n is an integer. Liouville extended this naturally to the derivative of arbitrary order $\aleph$ as:

$$D^{\aleph} e^{ax} = a^{\aleph} e^{ax}.$$

He framed some definitions of fractional derivative but those were not suitable for a wide class of functions.

G. F. B. Riemann developed his theory on fractional integration, but his work was published in 1892 when he was no more. He came up with a definition for the fractional integral which is similar to Riemann-Liouville fractional integral. In 1867 Grunwald used the definition of the derivative as a limit of a difference quotient to define the fractional derivative. Letnikov completed the proof for Grunwald-Letnikov fractional derivative in 1868.

$$D^{\aleph} f(x) = \lim_{h \to 0} \frac{1}{h^{\aleph}} \sum_{k=0}^{\frac{x-a}{h}} (-1)^k \frac{\Gamma(\aleph + 1)}{k!\Gamma(\aleph - k + 1)} f(x - kh), \ \aleph > 0, \tag{1.4}$$

where a is the lower limit of differentiation. In 1890 Oliver Heaviside's operational calculus was an important step in the development of applications of fractional derivatives. In 1899 onwards, Mittag-Leffler gave Mittag-Leffler function in one parameter form which added to the field of fractional calculus. In the twentieth century G. H. Hardy and John Littlewood studied the fractional integrals. In 1967, M. Caputo developed his definition of fractional derivative. He basically framed his definition so that integer order initial conditions could be used in a fractional differential equation. It is the most widely used definition till date. The modern notion of fractional differentiation originates from the formula for n^{th} integral:

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_n) dx_1 dx_2 \dots dx_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt. \tag{1.5}$$

If D is the differential operator and $J(= D^{-1})$ is the integral operator, then Riemann-Liouville fractional integral of order $\aleph$ is defined as

$$J^{\aleph} f(x) = \frac{1}{\Gamma(\aleph)} \int_0^x (x-s)^{\aleph-1} f(s)ds. \tag{1.6}$$

From this, the Riemann-Liouville fractional derivative (RLFD) is defined so that

$$D^{\aleph} J^{\aleph} f(x) = f(x) \tag{1.7}$$

and is given as

$$D^{\aleph} f(x) = \frac{1}{\Gamma(n-\aleph)} \frac{d^n}{dx^n} \int_0^x (x-s)^{n-\aleph-1} f(s)ds, \; n-1 < \aleph < n \tag{1.8}$$

where n is a positive integer. The Caputo's definition of fractional derivative is obtained by the reversal of order of operators in (1.7) and for positive integer n it is given as

$$D^{\aleph} f(x) = \frac{1}{\Gamma(n-\aleph)} \int_0^x (x-s)^{n-\aleph-1} f^n(s)ds, \; n-1 < \aleph < n. \tag{1.9}$$

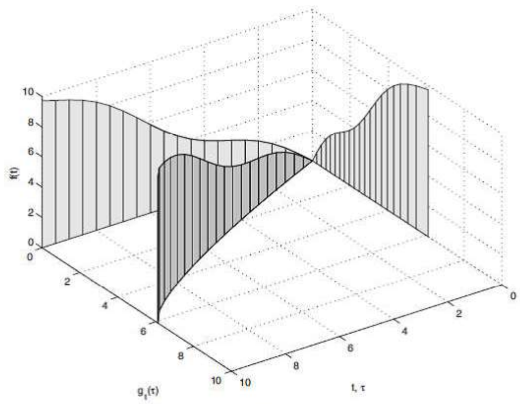
Thus, the 2.5^{th} derivative of a function $f(x)$, $D^{2.5} f(x) = D^3 J^{0.5} f(x)$ in the Riemann-Liouville's way and $D^{2.5} f(x) = J^{0.5} D^3 f(x)$ in the Caputo's way to obtain the fractional derivative of a function. From this point onwards, the Riemann-Liouville and Caputo derivatives would be represented as $\mathbb{D}$ and D respectively. The relation in these derivatives is given as

$$\mathbb{D}_a^{\aleph} y(x) = D_a^{\aleph} y(x) + \sum_{p=0}^{n-1} \frac{y^{(p)}(a)}{\Gamma(p-\aleph+1)} (x-a)^{p-\aleph}. \tag{1.10}$$

Both these definitions coincide for the initial conditions.

Having defined the most popular fractional derivatives, the definition of a fractional differential equation comes naturally as any equation containing the unknown function and its fractional derivatives. Podlubny [2] has discussed the existence and uniqueness theorems for linear fractional differential equations (FDE) with continuous coefficients and then for a general FDE. To start with, in standard calculus, differentiation is symbolized by the operator D and $D^n f(t), \; n = 1, 2, ...$ represents n^{th} derivative of $f(t)$. The particular $n-$fold integral of $f(t)$

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \ldots \int_0^{t_{n-1}} d(\xi) d\xi$$



"Live fence" and its shadows: ${}_0I_t^1 f(t)$ a ${}_0I_t^\alpha f(t)$,
for $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, $0 \le t \le 10$.

Fig. 1.7 'Live fence' and its shadows

is represented as $D^{-n}f(t),\ n = 1,2,\ldots$ The study of the operator $D^{\aleph}$, where $\aleph$ (not necessarily an integer), constitutes the soul of fractional calculus. The integer order derivatives and integrals have evident physical interpretations. For example, the first derivative of distance function w.r.t. time represents the speed; the first integral of a function of one variable w.r.t. that variable represents the area under curve. The fractional calculus being the superset of the standard calculus is expected to have such physical interpretations, but in literature not many illustrations exist on this.

Some authors [6] consider the fractional operators as linear filters and also seek their geometrical interpretation in the fractal geometry. Podlubny [3] provides a physical interpretation of the fractional integration in terms of two different time scales, viz. the homogeneous, equably flowing scale and the inhomogeneous time scale and introduced ' shadows on the walls' as shown in the Figure 1.7. In last ten years, it was proved that the fractional calculus is suitable for describing the properties of real materials as polymers [2]. The new fractional order models are more suitable as compared to the integer order models.

The first application of a half derivative was done by Abel [1] in 1823 to solve the integral equation for the tautochrone problem. That problem deals with the determination of shape of the curve such that the time descent of a frictionless point mass sliding down along the curve under the action of gravity is independent of the starting point [5].

Now the Tautochrone problem [2] can be understood and role of Caputo derivative can be seen in its solution clearly. Tautochrone problem is to find a curve in the (x,y) plane such that the time required for a particle to slide down the curve to its lowest point is independent of its initial placement on the curve; assuming the homogeneous gravity field and no friction.

Let us s fix the lowest point of a curve at the origin and the position of a curve in the positive quadrant of the plane, denoted by (x, y) the initial point and (x^*, y^*) any point intermediate between $(0,0)$ and (x, y).

According to the energy conservation law it may be written that

$$\frac{m}{2}(\frac{d\sigma}{dt})^2 = mg(y - y*)$$

where σ is the length along the curve measured from the origin, m is the mass of the particle and g is the gravitational acceleration. Considering $\frac{d\sigma}{dt} < 0$ and $\sigma = \sigma(y*(t))$ the formula can be re-written in the form of the integral equation:

$$\sigma' \frac{dy*}{dt} = -\sqrt{2g(y - y*)}$$

which is integrated from $y^* = y$ to $y^* = 0$ and from $t = 0$ to $t = T$. After some calculations the obtained integral equation is

$$\int_0^y \frac{\sigma'(y*)}{\sqrt{(y - y*)}} dy* = \sqrt{2g}T.$$

Here one can easily recognize the Caputo integral and write

$$CD_0^{1/2}\sigma(y) = \frac{\sqrt{2g}}{\Gamma(1/2)}T.$$

Note that T is the time of descent, so it is a constant. By applying the half-integral to both sides of the equation and by using the formulas for the composition of the Caputo derivatives and for the fractional integral of the constant , the relation between the length along the curve and the initial position in y direction is obtained,

$$\sigma(y) = \frac{\Gamma(1)\sqrt{2g}T}{\Gamma(1/2)\Gamma(3/2)} y^{1/2} = \frac{2\sqrt{2g}T}{\pi} y^{1/2}.$$

The formula describing coordinates of points generating the curve can be written by the help of the relation:

$$\frac{d\sigma}{dy} = \sqrt{1 + (\frac{dx}{dy})^2},$$

which after the substitution of $\sigma(y)$ gives

$$\frac{dx}{dy} = \sqrt{\frac{2gT^2}{\pi^2 y} - 1}.$$

It can be shown that the solution of this equation is so-called tautochrone , i.e. one arch of the cycloid which arises by rolling of the circle, whose parametric equations are given as,

$$x = \frac{A}{2}[u + sin\ u],$$
$$y = \frac{A}{2}[1 - cos\ u]$$

where

$$A = \frac{2gT^2}{\pi^2}.$$

Thus the knowledge of the rules of fractional calculus is very useful for solving this type of integral equations.

The fractional ordinary differential equation may be written as,

$$D^{\aleph} y(t) = f(t,y), \quad \aleph \in (n-1,n], t \in (0,1], n \geq 1 \text{ is an integer.}$$

Here $D^{\aleph}$ represents either Riemann-Liouville derivative or the Caputo derivative.

All the work done in this thesis is for solving the fractional differential equation of the type

$$_0D_t^{\aleph} y(t) = f(t, y(t)), 0 < t < T \tag{1.11}$$

$$y^{(k)}(0) = y_0^{(k)}, k = 0,1,2,\ldots \lceil \aleph \rceil - 1 \tag{1.12}$$

where $y_0^{(k)}$ may be arbitrary real numbers and $\aleph > 0$. Here $_0D_t^{\aleph} y(t)$ is Caputo derivative of $y(t)$ defined as

$$_0D_t^{\aleph} y(t) = \frac{1}{\Gamma(n-\aleph)} \int_0^t (t-u)^{n-\aleph-1} y^{(n)}(u) du, \tag{1.13}$$

where $n = \lceil \aleph \rceil$ is the smallest integer $\geq \aleph$

The following section mainly looks into the various methods adopted to solve the fractional differential equations over the time. The exact solutions of fractional differential equations have been found by using Laplace transforms, Fourier transforms and Differential transform method etc. and numerical solutions have been found in various ways as briefed in the review of literature.

1.3 Literature Review

The existence and uniqueness of the solutions for equations (1.11) and (1.12) was studied by Podlubny [2] and Diethelm [10]. Numerical methods were used to find the solutions, some of which are explained ahead.
Fractional differential equation was written as Abel-Volterra integral equation by Lubich. He used the convolution quadrature method to approximate the fractional integral and obtained approximate solutions of FDEs [11].
The fractional Riemann-Liouville derivative was written by using Hadamard finite part integral by Diethelm. He used a quadrature formula to approximate the integral [12].
Diethelm and Luchko expressed the FDE as a Mittag-Leffler type function and used convolution quadrature again along with discretised operational calculus to get the approximate the Mittag-Leffler type function [13].
Collocation method was used by Blank to approximate the FDE [14] while Podlubny [2] used the Grunwald and Letnikov method to approximate the fractional derivative. They also defined an implicit finite difference method for solving 1.11 and 1.12 with order of convergence as $O(h)$, h being the step size. The above mentioned FDE was solved with restrictive conditions with order of convergence as $O(h^2)$ by Gorenflo [15].
Diethelm et al. in [16] approximated the integral in the Volterra integral equation equivalent to 1.11 and 1.12 using piecewise linear interpolation polynomial and introduced a fractional Adam's type predictor corrector method. They proved the order of convergence to be $min(2, 1+\aleph)$ for $\aleph \in (0,2]$ if ${}_0^C D_t^{\aleph} y(t) \in C^2[0,T]$. This method was modified by Deng [17] with the order of convergence as $min(2, 1+2\aleph)$ for $\aleph \in (0,1]$. Deng and Zao [18] introduced Jacobi predictor corrector approach based on the polynomial interpolation, with which any desired convergence order can be obtained. Ford et al. [19] used non polynomial collocation method and achieved good convergence without considering any smoothness of the solution.
Recently, the authors in [20] used two approaches to solve 1.11 and 1.12. Direct discretization of the integral form of the fractional differential operator with order $\aleph \in (0,1)$ led to the order of convergence of $O(h^{3-\aleph})$. With the discretization of the integral form of the FDE, fractional Adams type method for a non linear FDE of any order $\aleph > 0$. The order of convergence of the numerical method was proved to be $O(h^3)$ for $\aleph \geq 1$ and $O(h^{1+2\aleph})$ for $0 < \aleph < 1$ for sufficiently smooth solutions.
To begin with, Wyss [21] considered the time fractional diffusion equation and the solution is given in closed form in terms of Fox functions, and Schneider and Wyss [22] considered the time fractional diffusion and wave equations. The corresponding Green functions are obtained in closed form for arbitrary space dimensions in terms of Fox functions and their

properties are exhibited. Using the similarity method and the method of Laplace transform, Gorenflo et al. [23] proved that the scale-invariant solutions of the mixed problem of signaling type for the time-fractional diffusion-wave equation are given in terms of the Wright function . The reduced equation for the scale-invariant solutions is given in terms of the Caputo-type modification of the Erdelyi-Kober fractional differential operator.

Agrawal [24] considered a time fractional diffusion-wave equation in a bounded space domain. The fractional time derivative is described in the Caputo sense. Using the finite sine transform technique and the Laplace transform, the solution is expressed in terms of the Mittag-Leffler functions. Results showed that for fractional time derivatives of order $1/2$ and $3/2$, the system exhibits, respectively, slow diffusion and mixed diffusion-wave behaviours.

Mainardi et al. [25] considered the basic models for anomalous transport provided by the integral equation for a continuous time random walk (CTRW) and by the time fractional diffusion equation. They compared the corresponding fundamental solutions of these equations in order to investigate numerically the increasing quality of the approximation with advancing time.

Chen et al. [26] derived the analytical solution for the time-fractional telegraph equation by the method of separation of variables. Luchko and Gorenflo [27] developed an operational method for solving fractional differential equations with Caputo derivatives and the obtained solutions are expressed through Mittag-Leffler type functions. However, the analytic solutions of most fractional differential equations are not usually expressed explicitly. As a consequence, many authors have discussed approximate solutions of the FDE, which are reviewed in the next section.

Most popular numerical solution techniques for equations involving fractional differential operators are based on random walk models [36], the finite difference method [29], the finite element method [30], numerical quadrature [**?**], the method of Adomian decomposition [32, 33], Monte Carlo simulation [34] or the matrix transform method [35]. Examples of random walk model methods include the work done by Gorenflo and Mainardi, who constructed random walk models for the space fractional diffusion processes [36] and the L´evy-Feller diffusion processes [37], based on the Gr¨unwald-Letnikov discretisation of the fractional derivatives occurring in the spatial pseudo-differential operator.

Liu, Anh and Turner [38] proposed a computationally effective method of lines technique for solving space fractional partial differential equations. They transformed the space FDE into a system of ordinary differential equations that was then solved using backward differentiation formulas.

Tadjeran et al. [39] examined a practical numerical method that is second-order accurate in time and in space to solve a class of initial-boundary value fractional diffusion equations with variable coefficients on a finite domain. An approach based on the classical Crank-Nicolson method combined with spatial extrapolation was used to obtain temporally and spatially second-order accurate numerical estimates. Stability, consistency, and (therefore) convergence of the method were also examined. Finite difference methods are also applicable in the numerical solution of time FDEs.

Zhuang and Liu [40] analysed an implicit difference approximation for the time fractional diffusion equation, and discussed the stability and convergence of the method. Yuste and Acedo [41] proposed an explicit finite difference method and a new von Neumann type stability analysis for the fractional sub-diffusion equation. However, they did not give the convergence analysis and pointed out that it is not such an easy task when implicit methods are considered. Yuste [42] proposed a weighted average finite difference method for fractional diffusion equations. Its stability is analyzed by means of a recently proposed procedure akin to the standard von Neumann stability analysis.

Diethelm, Ford and Freed [43] presented an Adams-type predictor-corrector method for the time FDE. They also discussed several modifications of the basic algorithm designed to improve the performance of the method. In their later work [44], the authors presented a detailed error analysis, including error bounds under various assumptions on the equation.

Authors who have applied finite difference methods to solve time-space FDEs include Liu et al. [45] who investigated a fractional order implicit finite difference approximation for the space-time fractional diffusion equation with initial and boundary values. Liu et al. [46] also investigated the stability and convergence of the difference methods for the space-time fractional advection-diffusion equation.

Gorenflo and Abdel-Rehim [47] discussed the convergence of the Gr¨unwald-Letnikov scheme for a time-fractional diffusion equation in one spatial dimension. These difference schemes can also be interpreted as discrete random walks. Lin and Xu [48] constructed a stable and high order scheme to efficiently solve the time-fractional diffusion equation. The proposed method is based on a finite difference scheme in time and Legendre spectral methods in space. Stability and convergence of the method were rigorously established. Zhuang et al. [49] presented an implicit numerical method for the time-space fractional Fokker-Planck equation and discussed its stability and convergence.

Podlubny et al. [50] presented a matrix approach for the solution of time- and space-fractional partial differential equations. The method is based on the idea of a net of discretisation nodes, where solutions at every desired point in time and space are found simultaneously by the solution of an appropriate linear system. The structure of the linear system, involving triangular strip matrices, is exploited in the numerical solution algorithm. Aside from discrete random walk approaches and the finite difference method, other numerical methods for solving the

space or time, or space-time fractional partial differential equations have been proposed.
Fix and Roop [51] proved existence and uniqueness of the least squares finite-element solution of a fractional order two-point boundary value problem. Optimal error estimates were proved for piecewise linear trial elements. Ervin and Roop [52] presented a theoretical framework for the Galerkin finite element approximation to the steady state fractional advection dispersion equation. Appropriate fractional derivative spaces were defined and shown to be equivalent to the usual fractional dimension Sobolev spaces. Existence and uniqueness results were proven, and error estimates for the Galerkin approximation derived.
Kumar and Agrawal [**?**] presented a numerical method for the solution of a class of FDEs for which there is a link between the FDE and a Volterra type integral equation. The FDEs are expressed as initial value problems involving the Caputo fractional derivative, and this allows the reduction to the Volterra type integral equation. The authors proposed using quadratic interpolation functions over three successive grid points, which allows the integrals to be computed, thereby yielding a system of algebraic equations to be solved. The scheme handles both linear and nonlinear problems.
Agrawal [53] revisited earlier work by Yuan and Agrawal [54] on a memory-free formulation (MFF) for the numerical solution of an FDE. The original MFF used Gauss-Laguerre quadrature to approximate the integral that appears in the definition of the Caputo fractional derivative. The new, modified MMF introduced a change of variable in the integration that overcome some of the limitations of the original formulation, by correctly computing initial compliance and more accurately simulating creep response over a much greater duration.
Jafari and Daftardar-Gejji [55] proposed an Adomian decomposition method for solving the linear/nonlinear fractional diffusion and wave equations. Momani and Odibat [56] developed two reliable algorithms, the Adomian decomposition method and variational iteration method, to construct numerical solutions of the space-time FADE in the form of a rapidly convergent series with easily computable components. However, they did not give its theoretical analysis.
Al-Khaled and Momani [57] gave an approximate solution for a fractional diffusion-wave equation using the decomposition method. Ili´c et al. [58] considered a fractional-in-space diffusion equation with homogeneous and nonhomogeneous boundary conditions in one dimension, respectively. They derived the analytic solutions by a spectral representation, and constructed numerical approximations by the matrix transform method (MTM), which is an exciting new method with great promise for higher dimensions.
Research on numerical methods for higher dimensional FDEs has been limited to date. Roop [59] investigated the computational aspects of the Galerkin approximation using continuous piecewise polynomial basis functions on a regular triangulation of the bounded domain in R^2.
Meerschaert et. al. [60] derived practical numerical methods to solve two-dimensional

fractional dispersion equations with variable coefficients on a finite domain and obtained first order accuracy in space and time. Zhuang and Liu [61] proposed a finite difference approximation for the two-dimensional time fractional diffusion equation and discussed the convergence and stability of the numerical method. Tadjeran and Meerschaert [62] presented a second-order accurate numerical method for the two-dimensional fractional superdiffusive differential equation. This numerical method combined the alternating directions implicit (ADI) approach with a Crank-Nicolson discretisation and a Richardson extrapolation to obtain an unconditionally stable second-order accurate finite difference method. The stability and the consistency of the method were established.

An interesting and not so common method called the differential transform method (DTM) was first applied in the engineering domain by Zhou [63]. The DTM is numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure. Arikoglu and Ozkol implemented this analytical technique for the field of fractional calculus, for solving fractional type differential equations, named as Fractional Differential Transform Method (FDTM) [64].

The majority of the previous research in this field has focused on one-dimensional problems, and research on the numerical methods of the higher dimensional fractional differential equations are limited to date.

1.4 A word on analytic and numerical methods

The difference between analytic and numerical approach to a problem is synonymous to the difference in pure and applied mathematics. Practically both involve each other, there is no fine demarcation between the two. As per the literature, however, mathematicians have used this classification immensely.

Analytic approach gives deep insight of the problem leading to the exact solution or an approximate solution while numerical approach is all about getting the best approximate solution. Analytic approach uses the methods of analysis and numerical approach uses the methods of numerical analysis. First approach gives a comprehensive general solution while the second approach gives mostly a problem based single solution.

For example solving a differential equation using any transformation or special functions is considered analytical and using approximate derivatives/ initial solutions etc. to solve the differential equation comes in the numerical methods category.

In this thesis, the fractional differential equations have been solved using both ideologies.

Out of many analytic and numerical methods, differential transform method and differential quadrature method are used to solve fractional differential equations. These two methods have a huge literature on the solution of ordinary and partial differential equations. They have been used to solve various fractional differential equations in this work.

1.5 Differential Transform Method

Many problems involving higher order derivatives are dealt by using the transformations. This method has a history of usage to solve linear as well as non linear ordinary differential equations. The popular Laplace transform can solve the linear ODE nicely but for non linear ODEs there are issues. Differential transform method fills the gap there.
The differential transform method was conceptualized by Zhou [63] in the context of electric circuit analysis. It is an iterative procedure for obtaining analytic Taylor series solution. The differential transform of the k^{th} order of a function $\psi(x)$ is given by

$$\Psi(k) = \frac{1}{k!}\left(\frac{d^k\psi(x)}{dx^k}\right)_{x=x_0}$$

and inverse differential transform of $\Psi(k)$ is

$$\psi(x) = \sum_{k=0}^{\infty}\Psi(k)(x-x_0)^k.$$

Differential transform of the different types of functions of x are mentioned in the following table 1.1

Table 1.1 Differential transforms

Relation in $\phi(x)$, $\psi(x)$ and $\omega(x)$	Relation in $\Phi(k)$, $\Psi(k)$ and $\Omega(k)$
$\phi(x) = \psi(x) \pm \omega(x)$	$\Phi(k) = \Psi(k) \pm \Omega(k)$
$\phi(x) = c\psi(x)$	$\Phi(k) = c\Psi(k)$
$\phi(x) = \psi(x)\omega(x)$	$\Phi(k) = \sum_{i=0}^{k}\Psi(i)\Omega(k-i)$
$\phi(x) = D^m\psi(x)$	$\Phi(k) = \frac{(k+m)!}{k!}\Psi(k+m)$
$\phi(x) = x^m$	$\Phi(k) = \underline{\delta}(k-m)$
$\phi(x) = e^{\eta x}$	$\Phi(k) = \frac{\eta^k}{k!}$

where $\underline{\delta}(k-m)$ is the Kronecker delta function. It takes the value 1 when $k=m$ and takes 0 when $k \neq m$.

Consider a linear ordinary differential equation $\frac{dy}{dx}=x^2+x$ with initial condition $y(0)=0$. Using calculus, its exact solution is obtained as $y=\frac{x^3}{3}+\frac{x^2}{2}$. Applying differential transform on the differential equation, one gets, $Y(k+1)=\frac{1}{k+1}[\underline{\delta}(k-2)+\underline{\delta}(k-1)]$ and on taking the inverse differential transform, the solution is obtained as

$$y(x)=0+0.x+\frac{1}{2}.x^2+\frac{1}{3}.x^3+0.x^4+\text{all zero terms}$$

which is same as the exact solution.

Now consider a non linear ODE $\frac{dy}{dx}+4xy=2x$, with conditions $y(0)=1, y'(0)=0$. The exact solution of the problem can be verified to be $\frac{1}{2}(1+e^{-2x^2})$. Solving the problem by DTM , the following recurrence relation is obtained:

$$(k+1)Y(k+1)+4\sum_{r=0}^{k}\underline{\delta}(r-1)Y(k-r)=2\underline{\delta}(k-1)$$

and on taking the inverse differential transform, the solution is obtained as,

$$y(x)=1-x^2+x^4-2x^6/3+x^8/3+\ldots.$$

On re arrangement of terms in the above series, one gets

$$y(x)=\frac{1}{2}+\frac{1}{2}\left(1+(-2x^2)+\frac{(-2x^2)^2}{2!}+\frac{(-2x^2)^3}{3!}+\ldots\right)$$

which is same as the exact solution.

From these examples, it is clear that DTM can efficiently be used to solve linear as well as non linear ODEs. For non linear ODEs, the only challenge lies in checking the nature of the series obtained.

One of the objectives of this research work is to investigate on the usage of differential transform method for solving the fractional differential equations. For this, the corresponding definitions would modify as follows:

The fractional differential transform of the k^{th} order of function $\psi(x)$ would inherently involve the fractional derivative of order $\aleph$. Considering Caputo derivative the differential transform of $\psi(x)$ is given as

$$\Psi_{\aleph}(k)=\frac{1}{\Gamma(\aleph k+1)}\left[\left(D_{x_0}^{\aleph}\right)^(k)\psi(x)\right]_{x=x_0} \tag{1.14}$$

and inverse differential transform is given as

$$\psi(x) = \sum_{k=0}^{\infty} \Psi_{\aleph}(k)(x - x_0)^{k\aleph}. \tag{1.15}$$

Fractional differential transforms of the initial conditions are given as

$$\Psi(k) = \begin{cases} \frac{1}{(k\aleph)!}\left[\frac{d^{k\aleph}\psi(x)}{dx^{k\aleph}}\right]_{x=x_0}, & \text{if } k\aleph \in \mathbb{Z}^+, k = 0,1,...,\left(\frac{n}{\aleph}\right) - 1, \text{n is the order of the FDE} \\ 0, & \text{otherwise.} \end{cases} \tag{1.16}$$

Let us assume that $\Phi(k)$, $\Psi(k)$ and $\Omega(k)$ are the fractional differential transforms of $\phi(x)$, $\psi(x)$ and $\omega(x)$ respectively. The results for the stated functions and their differential transforms [64, 65] are mentioned in the Table 2.1. The proofs of some of the results are mentioned ahead.

Table 1.2 Fractional differential transforms

Relation in $\phi(x)$, $\psi(x)$ and $\omega(x)$	Relation in $\Phi(k)$, $\Psi(k)$ and $\Omega(k)$
$\phi(x) = \psi(x) \pm \omega(x)$	$\Phi(k) = \Psi(k) \pm \Omega(k)$
$\phi(x) = c\psi(x)$	$\Phi(k) = c\Psi(k)$
$\phi(x) = \psi(x)\omega(x)$	$\Phi(k) = \sum_{i=0}^{k} \Psi(i)\Omega(k-i)$
$\phi(x) = D_{x_0}^{\aleph}\psi(x)$	$\Phi(k) = \frac{\Gamma(\aleph k+\aleph+1)}{\Gamma(\aleph k+1)}\Psi(k+1)$
$\phi(x) = (x - x_0)^q;\ q = n\aleph;\ n \in \mathbb{Z}$	$\Phi(k) = \underline{\delta}\left(k - \frac{q}{\aleph}\right)$
$\phi(x) = D_{x_0}^{\xi}\psi(x), m-1 < \xi \leq m$	$\Phi(k) = \frac{\Gamma(\aleph k+\xi+1)}{\Gamma(\aleph k+1)}\Psi\left(k+\frac{\xi}{\aleph}\right)$

Theorem 1. *If* $\phi(x) = \psi(x) \pm \omega(x)$ *then* $\Phi(k) = \Psi(k) \pm \Omega(k)$.

Proof. By definition, the right hand side can be written as,

$$\begin{aligned} \phi(x) &= \sum_{k=0}^{\infty} \Psi(k)(x - x_0)^{k\aleph} \pm \sum_{k=0}^{\infty} \Omega(k)(x - x_0)^{k\aleph} \\ &= \sum_{k=0}^{\infty} [\Psi(k) \pm \Omega(k)](x - x_0)^{k\aleph} \end{aligned}$$

Thus, $\Phi(k) = \Psi(k) \pm \Omega(k)$. □

Theorem 2. *If* $\phi(x) = c\psi(x)$ *then* $\Phi(k) = c\Psi(k)$.

Proof. By definition of inverse differential transform,

$$\begin{aligned}\phi(x) &= c\sum_{k=0}^{\infty}\Psi(k)(x-x_0)^{k\aleph}\\ &= \sum_{k=0}^{\infty}c\Psi(k)(x-x_0)^{k\aleph}\end{aligned}$$

Therefore, $\Phi(k) = c\Psi(k)$. □

Theorem 3. *If* $\phi(x) = \psi(x)\omega(x)$ *then* $\Phi(k) = \Sigma_{r=0}^{k}\Psi(r)\Omega(k-r)$.

Proof. By definition,

$$\begin{aligned}\phi(x) &= \psi(x)\omega(x)\\ &= \sum_{k=0}^{\infty}\Psi(k)(x-x_0)^{k\aleph}.\sum_{k=0}^{\infty}\Omega(k)(x-x_0)^{k\aleph} \quad = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\Psi(r)\Omega(k-r)(x-x_0)^{k\aleph}\end{aligned}$$

Thus, $U(k) = \Sigma_{r=0}^{k}\Psi(r)\Omega(k-r)$. □

Theorem 4. *If* $\phi(x) = D_{x_0}^{\aleph}\psi(x)$ *then* $\Phi(k) = \frac{\Gamma(\aleph k+\aleph+1)}{\Gamma(\aleph k+1)}\Psi(k+1)$.

Proof. Applying the definition,

$$\begin{aligned}\Phi(k) &= \frac{1}{\Gamma(\aleph k+1)}\left(D_{x_0}^{\aleph}\right)^{k}\left[D_{x_0}^{\aleph}\psi(x)\right]_{x=x_0}\\ &= \frac{1}{\Gamma(\aleph k+1)}\left[\left(D_{x_0}^{\aleph}\right)^{k+1}\psi(x)\right]_{x=x_0}\\ &= \frac{\Gamma(\aleph k+\aleph+1)}{\Gamma(\aleph k+1)\Gamma(\aleph k+\aleph+1)}\left[\left(D_{x_0}^{\aleph}\right)^{k+1}\psi(x)\right]_{x=x_0}\\ &= \frac{\Gamma(\aleph k+\aleph+1)}{\Gamma(\aleph k+1)}\Psi(k+1)\end{aligned}$$

Therefore, $\Phi(k) = \frac{\Gamma(\aleph k+\aleph+1)}{\Gamma(\aleph k+1)}\Psi(k+1)$. □

Theorem 5. *If* $\phi(x) = (x-x_0)^q; \quad q = n\aleph; \quad n\in\mathbb{Z}$ *then* $\Phi(k) = \underline{\delta}\left(k-\frac{q}{\aleph}\right)$.

Proof. Here $\underline{\delta}(k-n)$ is Kronecker delta function defined as

$$\underline{\delta}(k-n) = \begin{cases}1, & \text{if } k=n.\\ 0, & \text{otherwise.}\end{cases} \tag{1.17}$$

By definition $\phi(x)=\sum_{k=0}^{\infty}(x-x_0)^{k\aleph}\underline{\delta}\left(k-\frac{q}{\aleph}\right)$.
Therefore, $\Phi(k)=\underline{\delta}\left(k-\frac{q}{\aleph}\right)$. □

Theorem 6. *If* $\phi(x)=D_{x_0}^{\xi}\psi(x), m-1<\xi\leq m$ *then* $\Phi(k)=\frac{\Gamma(\aleph k+\xi+1)}{\Gamma(\aleph k+1)}\Psi\left(k+\frac{\xi}{\aleph}\right)$.

Proof. By definition of inverse differential transform,

$$\begin{aligned}\Phi(k) &= \frac{1}{\Gamma(\aleph k+1)}\left(D_{x_0}^{\aleph}\right)^k\left[D_{x_0}^{\xi}\psi(x)\right]_{x=x_0}\\ &= \frac{1}{\Gamma(\aleph k+1)}\left[(D_{x_0})^{\aleph k+\xi}\,\psi(x)\right]_{x=x_0}\\ &= \frac{\Gamma(\aleph k+\xi+1)}{\Gamma(\aleph k+1)\Gamma(\aleph k+\xi+1)}\left[(D_{x_0})^{\aleph k+\xi}\,\psi(x)\right]_{x=x_0}\\ &= \frac{\Gamma(\aleph k+\xi+1)}{\Gamma(\aleph k+1)}\Psi\left(k+\frac{\xi}{\aleph}\right)\end{aligned}$$

Therefore, $\Phi(k)=\frac{\Gamma(\aleph k+\xi+1)}{\Gamma(\aleph k+1)}\Psi\left(k+\frac{\xi}{\aleph}\right)$. □

In 2009, Turkish mathematician Keskin [66] had devised method of reduced differential transform (RDTM) involving reduction of a variable . Consider a function $\phi(x,t)$ representable as a product of a function of x only and a function of t only, i.e. $\phi(x,t)=\delta(x)\theta(t)$. Using the properties mentioned above, the function $\phi(x,t)$ can be written as follows:

$$\phi(x,t)=\sum_{i=0}^{\infty}\Delta(i)x^i\sum_{j=0}^{\infty}\Theta(j)t^j=\sum_{k=0}^{\infty}\Phi_k(x)(t-t_0)^k$$

where $\Phi_k(x)$ is called spectrum function of 't' dimension of $\phi(x,t)$. If function $\phi(x,t)$ is analytic with respect to both independent variables in the domain of interest, then

$$\Phi_k(x)=\frac{1}{k!}\left[\frac{\partial^k\phi(x,t)}{\partial x^k}\right]_{t=t_0} \tag{1.18}$$

where the t-dimensional spectrum function $\Phi_k(x)$ is the transformed function.
The inverse differential transform of the function $\Phi_k(x)$ is given as

$$\phi(x,t)=\sum_{k=0}^{\infty}\Phi_k(x)(t-t_0)^k. \tag{1.19}$$

The fractional counterpart [67] of RDTM is modified fractional differential transform method (MFDTM). Suppose that the variable of interest is t, then the Taylor series of the function

$\phi(x,t)$ can be defined as follows:

$$\phi(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\aleph k+1)} \left[\frac{\partial^{\aleph k}}{\partial t^{\aleph k}} \phi(x,t) \right]_{t=t_0} (t-t_0)^{\aleph k}. \tag{1.20}$$

The differential transform of the function $\phi(x,t)$ is thus given as

$$\Phi(x) = \frac{1}{\Gamma(\aleph k+1)} \left[\frac{\partial^{\aleph k}}{\partial t^{\aleph k}} \phi(x,t) \right]_{t=t_0}. \tag{1.21}$$

The inverse transform is then given as

$$\phi(x,t) = \sum_{k=0}^{\infty} \Phi(x)(t-t_0)^{\aleph k}. \tag{1.22}$$

The properties related to the MFDTM are in the Table 1.3. Extending the method for three

Table 1.3 Modified fractional differential transforms for one dimensional space variables

Relation in $\phi(x,t)$, $\psi(x,t)$ and $\omega(x,t)$	Relation in $\Phi(x)$, $\Psi(x)$ and $\Omega(x)$
$\phi(x,t) = \psi(x,t) \pm \omega(x,t)$	$\Phi_k(x) = \Psi_k(x) \pm \Omega_k(x)$
$\phi(x,t) = c\psi(x,t)$	$\Phi_k(x) = c\Psi_k(x)$
$\phi(x,t) = f(x)\omega(x,t)$	$\Phi_k(x) = f(x)\Omega_k(x)$
$\phi(x,t) = (x-x_0)^m (t-t_0)^{n\aleph}$	$\Phi_k(x) = (x-x_0)^m \underline{\delta}(k-n)$
$\phi(x,t) = D_{x_0}^{M\aleph} \psi(x,t)$	$\Phi_k(x) = \frac{\Gamma((k+M)\aleph+1)}{\Gamma(\aleph k+1)} \Psi_{k+1}(x)$

dimensions (two dimensional space variables) i.e. for a function $w(x,y,t)$, continuously differentiable and analytic in the domain of interest, the MFDT is given as,

$$\Phi_k(x,y) = \frac{1}{\Gamma(\aleph k+1)} \left[D_t^{\aleph k} \phi(x,y,t) \right]_{t=t_0}, k = 0,1,2,\ldots. \tag{1.23}$$

The inverse MFDT of $\Phi_k(x,y)$ is given as

$$\phi(x,y,t) = \sum_{k=0}^{\infty} \Phi_k(x,y)(t-t_0)^{\aleph k}. \tag{1.24}$$

Combining equations (1.23) and (1.24),

$$\phi(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\aleph k+1)} \left[D_t^{\aleph k} \phi(x,y,t) \right]_{t=t_0} (t-t_0)^{\aleph k}. \tag{1.25}$$

The properties related to the MFDTM are in the Table 1.4. The property $\phi(x,y,t) = (x-x_0)^m (y-y_0)^{m_1} (t-t_0)^{n\aleph}$ has a differential transform $\Phi_k(x,y) = (x-x_0)^m (y-y_0)^{m_1} \underline{\delta}(k-n)$, is proved here as the result shows a variation [67–69] in the literature.

Table 1.4 Modified fractional differential transforms for two dimensional space variables

Relation in $\phi(x,y,t)$, $\psi(x,y,t)$ and $\omega(x,y,t)$	Relation in $\Phi(x,y)$, $\Psi(x,y)$ and $\Omega(x,y)$
$\phi(x,y,t) = \psi(x,y,t) \pm \omega(x,y,t)$	$\Phi_k(x,y) = \Psi_k(x,y) \pm \Omega_k(x,y)$
$\phi(x,y,t) = c\psi(x,y,t)$	$\Phi_k(x,y) = c\Psi_k(x,y)$
$\phi(x,y,t) = f(x,y)\omega(x,y,t)$	$\Phi_k(x,y) = f(x,y)\Omega_k(x,y)$
$\phi(x,y,t) = (x-x_0)^m (y-y_0)^{m_1} (t-t_0)^{n\aleph}$	$\Phi_k(x,y) = (x-x_0)^m (y-y_0)^{m_1} \underline{\delta}(k-n)$
$\phi(x,y,t) = D_{x_0}^{M\aleph} \psi(x,y,t)$	$\Phi_k(x,y) = \frac{\Gamma((k+M)\aleph+1)}{\Gamma(\aleph k+1)} \Psi_{k+1}(x,y)$

For a function
$\phi(x,y,t) = (x-x_0)^m (y-y_0)^{m_1} (t-t_0)^{n\aleph}$, its differential transform is written as,
$\phi(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\aleph+1)} \frac{\partial^{k\aleph}\phi}{\partial t^{k\aleph}}$
$= \sum_{k=0}^{\infty} (x-x_0)^m (y-y_0)^{m_1} \frac{1}{\Gamma(k\aleph+1)} \frac{\partial^{k\aleph}(t-t_0)^{n\aleph}}{\partial t^{k\aleph}} (t-t_0)^{k\aleph}$
$= \sum_{k=0}^{\infty} \Phi(x,y,k)(t-t_0)^{k\aleph}$
And

$$\Phi(x,y,k) = (x-x_0)^m (y-y_0)^{m_1} \frac{1}{\Gamma(k\aleph+1)} \frac{\partial^{k\aleph}(t-t_0)^{n\aleph}}{\partial t^{k\aleph}}$$

Or

$$\Phi(x,y,k) = (x-x_0)^m (y-y_0)^{m_1} \begin{cases} 0, & \text{if } k\aleph \neq n\aleph \\ 1, & \text{if } k\aleph = n\aleph. \end{cases}$$

Or

$$\Phi(x,y,k) = (x-x_0)^m (y-y_0)^{m_1} \begin{cases} 0, & \text{if } k \neq n \\ 1, & \text{if } k = n \end{cases}$$

Or

$$\Phi(x,y,k) = (x-x_0)^m (y-y_0)^{m_1} \underline{\delta}(k-n)$$

where $\underline{\delta}$ is the Kronecker delta.

1.6 What is B-spline?

First of all spline, literally is a device used by the draftsmen, aircraft and shipbuilding industries to draw a curve through the pre plotted points such that the curve, the slope of the curve and the curvature of the curve are continuous.
The idea of spline can also be established by the problem of fitting a polynomial passing through points whose functional values are there. For two such points, a linear polynomial can be fitted, for five such points, a biquadratic polynomial can be fitted and so on. More the number of points, higher will be the degree of the polynomial. And higher degree polynomials are cumbersome to deal with. To resolve this issue, piecewise polynomials came to the rescue; i.e. instead of using a single polynomial over the whole domain, the function can be approximated by many polynomials defined over the sub domains. These piecewise polynomials are continuous as they are polynomials but may not be continuously differentiable on the interval of approximation. The concept of spline addresses this issue. The spline constructs a piecewise polynomial approximation that interpolates the given functional values ensuring that it is also smooth upto a certain degree.
Consider a uniform partition $x_0 < x_1 < \ldots < x_{n-1} < x_n$ of the domain $[a,b]$ with $x_0 = a, x_n = b$. The x_i are called knots. A function $S(x)$ is called a spline of degree k if it is a k^{th} degree polynomial $P(x)$ in each of the interval $[x_i, x_{i+1}], i = 0,1,2,...n-1$ with the property that $P(x)$ and its first $(k-1)$ derivatives are also continuous everywhere in the domain $[x_0, x_n]$. Thus a spline $S(x)$ on domain $[x_0, x_n]$ can be defined as $S(x) = \sum_{i=1}^{n} P_i(x)$ Here, $P(x)$ is a k^{th} degree polynomial in each partition.
Schoenberg in 1946 coined the term B-spline which is an abbreviation for basis spline. A B-spline is defined as a spline function that has minimum support w.r.t. a given degree, smoothness and domain partition. The core aspect of the basis function is the knot sequence x_i. Let X be a set of $N+1$ real numbers called knots such that $x_0 \leq x_1 \leq x_2 \leq \ldots x_N$. The set X is called the knot vector. If the knots are equi-spaced, the knot vector is said to be uniform. Each B-spline function of degree k covers k intervals or $k+1$ knots.
The B-spline basis function happens to have various degrees and forms. For a zero degree B-spline i.e. $k = 0$, the basis function is just a step function given as

$$B_{m,0} = \begin{cases} 1 & x \in [x_m, x_{m+1}) \\ 0 & otherwise. \end{cases}$$

Thus, a zero degree B-spline is equal to zero at all points except on the clopen interval $[x_m, x_{m+1})$. B-spline of first degree is like a tent function which is non-zero for two knot spans $[x_m, x_{m+1})$ and $[x_{m+1}, x_{m+2})$ and is given as

$$B_{m,1} = \begin{cases} \frac{x - x_m}{x_{m+1} - x_m} & x \in [x_m, x_{m+1}) \\ \frac{x_{m+2} - x}{x_{m+2} - x_{m+1}} & x \in [x_{m+1}, x_{m+2}) \\ 0 & otherwise. \end{cases}$$

The quadratic B-spline basis function looks like Gaussian curve. It is non-zero between three knot spans $[x_m, x_{m+1}), [x_{m+1}, x_{m+2})$ and $[x_{m+2}, x_{m+3})$ and is given as

$$B_{m,2} = \begin{cases} \frac{(x-x_m)^2}{(x_{m+2}-x_m)(x_{m+1}-x_m)} & x \in [x_m, x_{m+1}) \\ \frac{(x-x_m)(x_{m+2}-x)}{(x_{m+2}-x_m)(x_{m+2}-x_{m+1})} + \frac{(x_{m+3}-x)(x-x_{m+1})}{(x_{m+3}-x_{m+1})(x_{m+2}-x_{m+1})} & x \in [x_{m+1}, x_{m+2}) \\ \frac{(x_{m+3}-x)^2}{(x_{m+3}-x_{m+1})(x_{m+3}-x_{m+2})} & x \in [x_{m+2}, x_{m+3}) \\ 0 & otherwise. \end{cases}$$

The next in line is the cubic B-spline (CBS) and it too takes the bell shape. The associated basis function is given by the formula

$$B_{m,3} = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3 & x \in [x_{m-2}, x_{m-1}) \\ (x - x_{m-2})^3 - 4(x - x_{m-1})^3 & x \in [x_{m-1}, x_m) \\ (x_{m+2} - x)^3 - 4(x_{m+1} - x)^3 & x \in [x_m, x_{m+1}) \\ (x_{m+2} - x)^3 & x \in [x_{m+1}, x_{m+2}) \\ 0 & otherwise \end{cases}$$

where $h = x_{m+1} - x_m$ is the pre defined step size.
And in the similar manner standard quartic, quintic etc. forms of B-spline can be defined. Let $T_m(x)$ be the trigonometric B-spline (TBS) with knots at the points x_i where the N knots $a = x_1 < x_2 < ... < x_N = b$ are uniformly distributed with a width of h. Then the TBS, $T_1, T_2, ..., T_N$ form a basis for the functions on [a, b].

The m^{th} order TBS $T_{j,m}(x)$ for $j = 1,2,\ldots,N$ is given as

$$T_{j,m}(x) = \frac{sin\frac{x-x_j}{2}T_{j,m-1}(x) + sin\frac{x_{j+m}-x}{2}T_{j+1,m-1}(x)}{sin\frac{x_{j+m}-x_j}{2}}. \tag{1.30}$$

$T_{j,m}$ is a piecewise trigonometric function with properties like C^∞ continuity, non negativity and partition of unity. These can be classified according to their degree.
The zero degree TBS basis function for $m = 1,2,3,...$ is defined as

$$TB_{m,0} = \begin{cases} 1 & x \in [x_m, x_{m+1}) \\ 0 & otherwise. \end{cases}$$

The first degree TBS basis function can be written as

$$TB_{m,1} = \frac{1}{W_1}\begin{cases} P(x_m) & x \in [x_m, x_{m+1}) \\ Q(x_{m+2}) & x \in [x_{m+1}, x_{m+2}) \\ 0 & otherwise. \end{cases}$$

The second degree TBS basis function can be written as

$$TB_{m,2} = \frac{1}{W_2}\begin{cases} P^2(x_m) & x \in [x_m, x_{m+1}) \\ P(x_m)Q(x_{m+2}) + Q(x_{m+3})P(x_{m+1}) & x \in [x_{m+1}, x_{m+2}) \\ Q^2(x_{m+3}) & x \in [x_{m+2}, x_{m+3}) \\ 0 & otherwise. \end{cases}$$

The formula for the cubic TBS basis function at the knot points is specified by

$$TB_{m,3} = \frac{1}{W_3}\begin{cases} P^3(x_m) & x \in [x_m, x_{m+1}) \\ P(x_m)(P(x_m)Q(x_{m+2}) + Q(x_{m+3})P(x_{m+1}) + Q(x_{m+4})P^2(x_{m+1})) & x \in [x_{m+1}, x_{m+2}) \\ Q(x_{m+4})(P(x_{m+1})Q(x_{m+3}) + Q(x_{m+4})P(x_{m+2}) + P(x_m)Q^2(x_{m+3}) & x \in [x_{m+2}, x_{m+3}) \\ Q^3(x_{m+4}) & x \in [x_{m+3}, x_{m+4}) \\ 0 & otherwise \end{cases}$$

Table 1.5 Value of $B_m(x)$ for the CBS and its derivatives at the knots

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
$B_m(x)$	0	1	4	1	0
$B_m'(x)$	0	3/h	0	-3/h	0
$B_m''(x)$	0	6/h^2	-12/h^2	-6/h^2	0

where $W_1 = \frac{1}{sin\frac{h}{2}}$,
$W_2 = \frac{1}{sin\frac{h}{2}sinh}$,
$W_3 = \frac{1}{sin(\frac{h}{2})sin(h)sin(\frac{3h}{2})}$,
$P(x_m) = sin(\frac{x-x_m}{2})$and
$Q(x_m) = sin(\frac{x_m-x}{2})$.
Throughout this work, the modified cubic B-spline [107] basis function $\mathbb{B}_1, \mathbb{B}_2, ..., \mathbb{B}_N$ is given as:

$$\begin{aligned}
\mathbb{B}_1(x) &= B_1(x) + 2B_0(x), \\
\mathbb{B}_2(x) &= B_2(x) - B_0(x), \\
\mathbb{B}_m(x) &= B_m(x) \quad \text{for} \quad m = 3,4,...,N-2, \\
\mathbb{B}_{N-1}(x) &= B_{N-1}(x) - B_{N+1}(x), \\
\mathbb{B}_N(x) &= B_N(x) + 2B_{N+1}(x).
\end{aligned}$$

And the modified trigonometric cubic form is given as:

$$\begin{aligned}
\mathbb{TB}_1(x) &= T_1(x) + 2T_0(x), \\
\mathbb{TB}_2(x) &= T_2(x) - T_0(x), \\
\mathbb{TB}_m(x) &= T_m(x) \quad \text{for} \quad m = 3,4,...,N-2, \\
\mathbb{TB}_{N-1}(x) &= T_{N-1}(x) - T_{N+1}(x), \\
\mathbb{TB}_N(x) &= T_N(x) + 2T_{N+1}(x).
\end{aligned}$$

Table 1.6 Value of $TB_m(x)$ for the cubic TBS and its derivatives at the knots

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
$TB_m(x)$	0	D_1	D_2	D_1	0
$TB_m^{'}(x)$	0	D_3	0	D_4	0
$TB_m^{''}(x)$	0	D_5	D_6	D_5	0

The values of $D_i's$ in the Table 1.6 are as follows:

$$
\begin{aligned}
D_1 &= \frac{sin^2(h/2)}{sin(h)sin(3h/2)},\\
D_2 &= \frac{2}{1+2cos(h)},\\
D_3 &= \frac{-3}{4sin(3h/2)},\\
D_4 &= \frac{3}{4sin(3h/2)},\\
D_5 &= \frac{3(1+3cos(h)}{16sin^2(h/2)(2cos(h/2)+cos(3h/2))},\\
D_6 &= \frac{-3cos^2(h/2)}{sin^2(h/2)(2+4cos(h))}.
\end{aligned}
$$

1.7 Differential Quadrature Method

The differential quadrature method is a numerical solution technique for solving IVPs and BVPs. The method was conceived by Bellman and Casti in early seventies. It has been modified since then and has been used to solve problems of physical sciences and engineering. For instance, fluid mechanics, structural mechanics, lubrication mechanics, transport processes, biosciences etc. This method was supposed to be an alternate choice for finite difference method and finite element method.

This method was brought up as an analogy to integral quadrature where a definite integral is approximated as a weighted sum of the integrand on a set of points. This concept was extended by Bellman and Casti in 1971 for the derivatives under the name of differential quadrature.

This method reduces the differential equation into a system of algebraic equations or ordinary differential equations. The space derivatives are written as the weighted sum of the functional values at the pre defined grid points of the given domain. The choice of weights is of the utmost importance. There are various test functions that are used to calculate the weighting coefficients or just weights such as the Lagrange interpolation polynomial, cubic B-Spline, trigonometric B-spline, modified B-spline etc.

It is a well-known fact that a function $u(x,t)$ can be estimated by interpolation at N discrete grid points as

$$u(x,t) = \sum_{j=1}^{N} c_j u(x_j,t), \quad 1 \le i \le N$$

where c_j is the basis function and $u(x_j,t)$ is the value of the function $u(x,t)$ at arbitrary knot point x_j in the domain $[a,b]$. The first derivative of the same function, by virtue of DQM, can then be written as,

$$\frac{\partial u(x_i,t)}{\partial x} \approx \sum_{j=1}^{N} a_{ij} u(x_j,t), \quad 1 \le i \le N \tag{1.35}$$

Similarly, the second derivative can be approximated as

$$\frac{\partial^2 u(x_i,t)}{\partial x^2} \approx \sum_{j=1}^{N} b_{ij} u(x_j,t), \quad 1 \le i \le N \tag{1.36}$$

where a_{ij} and b_{ij} are the weight coefficients to be determined by a suitable method. The literature speaks of many ordinary and partial differential equations that have been solved by DQM and the weight coefficients have been calculated using cubic B-splines [88], modified cubic B-spline [89, 90], quintic B-spline [91], Lagrange polynomial [126], trigonometric B-splines [93] etc. In this work a hybrid B-spline is used to get a basis function and hence the weight coefficients.

1.7.1 Hybrid B-spline

The linear combination of the CBS and the cubic TBS basis functions is defined as

$$H_m(x) = \beta B_m(x) + \gamma TB_m(x) \tag{1.37}$$

where $\gamma = 1-\beta$ and $TB_m(x)$ depicts modified cubic TBS and $B_m(x)$ depicts modified CBS function. When $\gamma = 0, \beta = 1$, the basis function becomes standard cubic B-spline and in case $\gamma = 1, \beta = 0$, the basis function turns into a trigonometric cubic B-spline.

The first order derivative approximation at the knots $x_i, i = 1,2,...N$ can be written as

$$H_m^{'}(x_i) = \sum_{j=1}^{N} a_{ij} H_m(x_j) \text{for} \quad m = 1,2,...N.$$

For the first knot point x_1 the approximation is given as

$$H_m'(x_1) = \sum_{j=1}^{N} a_{1j} H_m(x_j) \text{for} \quad m = 1,2,...N.$$

which gives a tri-diagonal system of equations of the form $AX = B$ where

$$A = \begin{bmatrix} \gamma(2D_1+D_2)+6\beta & \gamma D_1+\beta & & & & & \\ 0 & \gamma D_2+4\beta & \gamma D_1+\beta & & & & \\ & \gamma D_1+\beta & \gamma D_2+4\beta & \gamma D_1+\beta & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma D_1+\beta & \gamma D_2+4\beta & \gamma D_1+\beta & \\ & & & & \gamma D_1+\beta & \gamma D_2+4\beta & 0 \\ & & & & & \gamma D_1+\beta & \gamma(2D_1+D_2)+6\beta \end{bmatrix}$$

$$X = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1N-1} \\ a_{1N} \end{bmatrix},$$

$$B = \begin{bmatrix} -2D_4\gamma - \frac{6}{h}\beta \\ \frac{(D_3-D_4)}{2}\gamma + \frac{6}{h}\beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This tri-diagonal system of equations can be solved for the weight coefficients $a_{11}, a_{12}, ..., a_{1N}$ using the Thomas algorithm.

Now for the first derivative at the second knot x_2 can be approximated as follows

$$H_m'(x_2) = \sum_{j=1}^{N} a_{2j} H_m(x_j) \text{ for} \quad m = 1,2,...N.$$

The tri-diagonal system for x_2 is given as

$$A = \begin{bmatrix} \gamma(2D_1+D_2)+6\beta & \gamma D_1+\beta & & & & & \\ 0 & \gamma D_2+4\beta & \gamma D_1+\beta & & & & \\ & \gamma D_1+\beta & \gamma D_2+4\beta & \gamma D_1+\beta & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma D_1+\beta & \gamma D_2+4\beta & \gamma D_1+\beta & \\ & & & & \gamma D_1+\beta & \gamma D_2+4\beta & 0 \\ & & & & & \gamma D_1+\beta & \gamma(2D_1+D_2)+6\beta \end{bmatrix},$$

$$X = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2N-1} \\ a_{2N} \end{bmatrix},$$

$$B = \begin{bmatrix} D_4\gamma - \frac{3}{h}\beta \\ 0 \\ D_3\gamma + \frac{3}{h}\beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Solution of the above system gives the weight coefficients $a_{21}, a_{22}, ..., a_{2N}$. In the similar fashion, the tri-diagonal system for the first order derivative approximation at the point x_{N-1} is given as,

$$A = \begin{bmatrix} \gamma(2D_1+D_2)+6\beta & \gamma d_1+\beta & & & & & \\ 0 & \gamma D_2+4\beta & \gamma D_1+\beta & & & & \\ & \gamma D_1+\beta & \gamma D_2+4\beta & \gamma D_1+\beta & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma D_1+\beta & \gamma D_2+4\beta & \gamma D_1+\beta & \\ & & & & \gamma D_1+\beta & \gamma D_2+4\beta & 0 \\ & & & & & \gamma D_1+\beta & \gamma(2D_1+D_2)+6\beta \end{bmatrix},$$

$$X = \begin{bmatrix} a_{N-11} \\ a_{N-12} \\ \vdots \\ a_{N-1N-1} \\ a_{N-1N} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ D_4\gamma - \frac{3}{h}\beta \\ 0 \\ D_3\gamma + \frac{3}{h}\beta \end{bmatrix}.$$

Solution of the above system gives the weight coefficients $a_{N-11}, a_{N-12}, ..., a_{N-1N}$. And on the last knot x_N, the respective tri-diagonal system becomes

$$A = \begin{bmatrix} \gamma(2D_1+D_2)+6\beta & \gamma D_1+\beta & & & & & \\ 0 & \gamma D_2+4\beta & \gamma D_1+\beta & & & & \\ & \gamma D_1+\beta & \gamma D_2+4\beta & \gamma D_1+\beta & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma D_1+\beta & \gamma D_2+4\beta & \gamma D_1+\beta & \\ & & & & \gamma D_1+\beta & \gamma D_2+4\beta & 0 \\ & & & & & \gamma D_1+\beta & \gamma(2D_1+D_2)+6\beta \end{bmatrix},$$

$$X = \begin{bmatrix} a_{N1} \\ a_{N2} \\ \vdots \\ a_{NN-1} \\ a_{NN} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ (D_4 - D_3)\gamma - \frac{6}{h}\beta \\ 2D_3\gamma + \frac{6}{h}\beta \end{bmatrix}.$$

This tri-diagonal system gives the weight coefficients $a_{N1}, a_{N2}, ..., a_{NN}$. The weight coefficients for the approximation of second order derivatives [108] can be calculated by the following formula:

$$b_{ij} = \begin{cases} 2a_{ij}\left(a_{ii} - \frac{1}{x_i - x_j}\right) & \text{for} \quad i \neq j \\ -\sum_{i=1, i\neq j}^{N} a_{ij} & \text{for} \quad i = j. \end{cases}$$

1.7.2 Two dimensional differential quadrature method

For any function $\Psi = \Psi(x, y, t)$ having its field on rectangle $0 \leq x \leq a, 0 \leq y \leq b$, let the rectangle be further divided into a grid by taking M points on x-axis and N points on the y-axis. Then, the p^{th} derivative [105] of the said function at a point $x = x_i$ along any line $y = y_j$ is

$$\frac{\partial^p \Psi}{\partial x^p}(x_i, y_j, t) = \sum_{k=1}^{M} A_{ik}^{(p)} \Psi(x_k, y_j, t); \;\; i = 1, 2, \ldots, M \tag{1.38}$$

and the q^{th} derivative at a point $y = y_j$ along the line $x = x_i$ is given as

$$\frac{\partial^q \Psi}{\partial y^q}(x_i, y_j, t) = \sum_{k=1}^{N} B_{jk}^{(q)} \Psi(x_i, y_k, t); \;\; j = 1, 2, \ldots, N \tag{1.39}$$

where $p \in \mathbb{Z}^+, 1 \leq i \leq M, 1 \leq j \leq N$ and $A_{ik}^{(p)}$ and $B_{jl}^{(q)}$ are respective weight coefficients (WCs) that give the approximate p^{th} and q^{th} derivative at the knots. The WCs $A_{ik}^{(1)}$ and $B_{jk}^{(1)}$ are calculated by the same process as mentioned in the section 1.7.1. Then to approximate the second order derivative, the following formula is to be used.

$$A_{ik}^{(2)} = \begin{cases} 2A_{ik}^{(1)}\left(A_{ii}^{(1)} - \frac{1}{x_i - x_k}\right) & \text{for} \quad i \neq k \\ -\sum_{i=1, i\neq k}^{M} A_{ik}^{(2)} & \text{for} \quad i = k \end{cases}$$

and

$$B_{jk}^{(2)} = \begin{cases} 2B_{jk}^{(1)}\left(B_{jj}^{(1)} - \frac{1}{y_j - y_k}\right) & \text{for} \quad j \neq k \\ -\sum_{k=1, j\neq k}^{N} B_{jk}^{(2)} & \text{for} \quad j = k. \end{cases}$$

Having obtained the weight coefficients corresponding to both space variables x and y, the space-derivatives of first order and second order can be approximated as in equations (1.38) and (1.39). And once the space derivatives get approximated, the time derivative can be treated using some transformation or by using discretization, leading to the solution of fractional differential equations.

1.8 Objectives of the research work

The objectives as approved in the state of the art (SOTA) seminar are as follows:

- To apply analytic methods such as differential transform method to solve fractional differential equations.
- To find an efficient technique to find the solution of fractional differential equation.
- To solve the fractional differential equations in higher dimensions.

1.8.1 Organisation of the thesis

This thesis aims to find an efficient technique to solve the fractional differential equations. The methods of differential transform and differential quadrature are well established for the ordinary and partial differential equations already. In this work, these methods have been applied with some variations on the different fractional differential equations.

In the second chapter, the fractional Bagley Torvik equation and the fractional relaxation oscillation equation has been solved, using the differential transform method.

In the third chapter, Fractional Burgers' equation has been solved using the differential quadrature method. One great application of any method is to solve the inverse problems. The inverse problem related to the fractional Fisher equation has also been solved by the same methodology.

In the fourth chapter, a hybrid method D(TQ)M is proposed to solve a two dimensional fractional diffusion equation. This method consists of the treatment of the equation with DTM and then DQM and then obtaining the final solution by using inverse DTM.

The fifth chapter is about the conclusions and the future scope of this work.

Chapter 2

Solution of one dimensional fractional differential equation using Differential Transform Method

In this chapter differential transform method has been used to solve one dimensional fractional differential equations namely the fractional Bagley Torvik equation (FBTE) and the fractional relaxation oscillation equation (FROE).
In day to day life one comes across structures containing elastic and viscoelastic components. Mathematical model for such kind of material is well described by Bagley Torvik equation.

$$Lz''(x) + Mz^{3/2}(x) + Nz(x) = \psi(x); \quad L \neq 0, \quad N \in \mathbb{R}. \tag{2.1}$$

This fractional differential equation was given by P. J. Torvik and R. L. Bagley [72] in 1984 in their work on the manifestation of the non integer order derivative in the behavior of real materials. They concluded with the finest of statements, "The fractional derivative appears naturally in the behavior of real materials. Thus, there is some basis for suspecting that the utility of constitutive relationships involving fractional derivatives for describing the behavior of real materials may not be just a happy coincidence."
This equation has been since solved for its analytic and numerical solution by many scholars. Igor Podlubny [2] in 1999 proposed a numerical method for inhomogeneous BT equation and also solved the constant coefficient Bagley Torvik (BT) equation analytically using fractional Green's $G_3(t)$ function. Afterwards many authors worked upon the numerical solution of BT equation such as Leszczynski et. al. [73], Erturk et. al. [74], El-Sayed et. al.[75], Edwards et. al.[76], Diethelm et. al. [77] etc. There were authors [78] who worked upon the analytic solutions rather than numerical ones. Ray and Bera [79] had found the analytic solution of

BT equation by Adomian method verified by Podlubny's results. The same has been used as a benchmark for our result in the second example. Arikoglu and Ozkol [64] applied DTM to BT equation in brief for specified initial conditions and a specific function. Yücel Çenesiz et al [71] solved the BT equation with generalized Taylor collocation method. T. Mekkaoui, Z. Hammouch [80] used fractional iteration method to get an approximate solution of BT equation.

In this chapter fractional DTM is implemented on general and particular forms of BT equation and comparison has been made with the analytic solution and the solution by generalized Taylor collocation method.

On the other hand the relaxation oscillation equation is a significant equation of relaxation and oscillation processes, based on the behavior of physical system's return to equilibrium after being disturbed. The process of relaxation oscillation is significant in the phenomenon where the physical system tends to return to equilibrium after being disturbed. The process of relaxation oscillation is omnipresent as in a pneumatic hammer, the scratching noise of a knife on a plate, the waving of a flag in the wind, the periodic re-occurrence of epidemics and of economical crises [81], the beating of the heart [82] etc. Such process is depicted as an ordinary linear differential equation of order 1 or 2. For instance $D^1y(t)+Ay(t)=f(t)$ is a relaxation equation, where time is the independent variable and A is a positive constant and $D^2y(t)+Ay(t)=f(t)$ is an oscillation equation.

Francesko Mainardi [83] in 1995 came up with its fractional analogy. The process of relaxation oscillation in many branches of Physics and Biology [81] can better be explained by fractional relaxation oscillation (FRO) equation.

It is one of the simplest fractional order differential equations given as:

$$D^{\aleph}y(t)+Ay(t)=f(t) \quad t>0; \quad y^{(k)}(0)=0, \quad k=0,1,2,...,n-1 \tag{2.2}$$

where $n-1<\aleph\leqq n$.

For $0<\aleph\leqq 1$, this equation is called fractional relaxation equation and for $1<\aleph\leqq 2$, it is fractional oscillation equation specifically. The first kind of system was tagged as ultraslow processes and latter as intermediate processes by [83] Mainardi.

The analytic solution of this equation in terms of Green's function [2] is given as

$$y(x)=\int_0^x G_2(x-\tau)f(\tau)d\tau \tag{2.3}$$

where

$$G_2(x)=x^{\aleph-1}E_{\aleph,\aleph}(-Ax^{\aleph}).$$

Many methods have been employed to find the solution of this equation. In this chapter fractional DTM is implemented on general FRO equation and its particular forms employing Caputo's definition of fractional derivative. A comparison has been made with the analytic solution and the solution by cubic B-spline wavelet collocation method.The series solution obtained using DTM is compared with the existing results. Two examples are given to demonstrate the validity and applicability of the method.

2.1 Implementation of method on fractional Bagley Torvik Equation (FBTE)

Consider equation (2.1) equipped with the initial conditions $z(0) = 0, \quad z'(0) = 0$.
The differential transform (Section:1.5) of equation (2.1) can be written as,

$$Z(k+4) = \frac{\Gamma(\frac{k}{2}+1)\,[\Psi(k) - N \quad Z(k)] - M \quad \Gamma(\frac{k}{2}+2.5)Z(k+3)}{L \quad \Gamma(\frac{k}{2}+3)}. \tag{2.4}$$

Using inverse differential transform, the series solution can be obtained as:

$$z(x) = \Sigma_{k=0}^{\infty} Z(k)(x-x_0)^{k\aleph}. \tag{2.5}$$

It is to be noted that the initial conditions will give $Z(k)$ for $k = 0, 1, 2, 3$ and the rest of them are to be obtained from the recurrence relation (2.4).

2.2 Numerical examples on FBTE

2.2.1 Example

Consider the BT equation

$$z''(x) + z^{3/2}(x) + z(x) = 1 + x$$

with initial conditions $z(0) = 1$ and $z'(0) = 1$ i.e. $L = 1, M = 1, N = 1$ and $\psi(x) = 1 + x$ in generalized BT equation (2.1). Using the results of section 1.5, the fractional differential transform of the equation can be written as

$$Z(k+4) = \frac{\Gamma(\frac{k}{2}+1)\,[\delta(k) + \delta(k-2) - Z(k)] - \Gamma(\frac{k}{2}+2.5)Z(k+3)}{.}\Gamma(\frac{k}{2}+3) \tag{2.6}$$

The initial conditions imply

$$Z(0) = 1, \quad Z(1) = 0, \quad Z(2) = 1, \quad Z(3) = 0.$$

From equation (2.4) we obtain

$$Z(k) = 0 \quad \forall \quad k > 3.$$

Thus the solution by inverse fractional differential transform is

$$z(x) = 1 + x.$$

The solution obtained is exactly same as the analytic solution.

2.2.2 Example

Consider the BT equation

$$z''(x) + 0.5z^{3/2}(x) + 0.5z(x) = 8$$

with initial conditions $z(0) = 0$ and $z'(0) = 0$ i.e. $L = 1, M = 0.5, N = 0.5$ and $\psi(x) = 8$ in generalized BT equation (2.1). The fractional differential transform 1.5 of the equation can be written as

$$Z(k+4) = \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k}{2}+3)} \left[\frac{-0.5\Gamma(\frac{k}{2}+2.5)Z(k+3)}{\Gamma(\frac{k}{2}+1)} - 0.5Z(k) + 8\delta(k) \right]. \tag{2.7}$$

The initial conditions imply

$$Z(k) = 0 \quad for \quad k = 0, 1, 2, 3.$$

From equation 2.7 we obtain

Table 2.1 Comparison of fractional DTM solution of FBTE with GTCM and analytic solution

height x	GTCM	Analytic solution	Fractional DTM
0	0	0	0
0.1	0.036485547	0.036487479	0.036487479
0.2	0.140634716	0.140639621	0.140639621
0.3	0.307476229	0.307484627	0.307484626
0.4	0.533271294	0.533284110	0.533284108
0.5	0.814735609	0.814756950	0.814756938
0.6	1.148805808	1.148837428	1.148837373
0.7	1.532521264	1.532565443	1.532565254
0.8	1.962974991	1.963029298	1.963028738
0.9	2.437455982	2.437334072	2.437332615
1.0	2.954070000	2.952584099	2.952580672

Thus the approximate solution can be given as

$$\begin{aligned} z(x) = 4x^2 &- 1.203604445x^{2.5} + 0.333333333x^3 - 0.085971745x^{3.5} - 0.145833333x^4 \\ &+ 0.0716431216x^{4.5} - 0.023958333x^5 + 0.0067301113x^{5.5} - 0.000934500x^6 \\ &+ 0.0002564803x^{6.5} - 0.000740702x^7 + 0.0095069366x^{7.5} - 0.427803803x^8. \end{aligned}$$

The same problem has been solved in [71] by using Generalized Taylor Collocation Method (GTCM). The comparison is mentioned in Table 2.1 and it leaves no doubt in the capability of the fractional DTM to solve a fractional differential equation. The results of our calculations are in agreement with the analytical solution [79] by Adomian decomposition method.

2.3 Implementation of method on fractional relaxation oscillation equation (FROE)

Consider equation (2.2) equipped with the general initial conditions. Applying fractional differential transform 1.5 to equation (2.2), it can be written as,

$$Y(k+1) = \frac{\Gamma(\aleph k+1)\,(F(k) - AY(k))}{\Gamma(\aleph k+\aleph+1)}. \tag{2.8}$$

Using inverse differential transform, the series solution can be obtained as:

$$y(t) = \Sigma_{k=0}^{\infty} Y(k)(t-t_0)^{k\aleph}. \tag{2.9}$$

It is to be noted that the initial conditions will give $Y(k)$ for $k = 0, 1, 2, 3, ..., n-1$ and the rest of them are to be obtained from the obtained recursive relation.

2.4 Numerical examples on FROE

To show the effectiveness of the method, two examples are discussed. The solution obtained by the fractional differential transform method has been compared with the analytic solution in both the examples.

2.4.1 Example

Consider the fractional relaxation oscillation equation

$$D^{1/2}y(x) + y(x) = 0$$

with initial condition $y(0) = 1$ i.e. $A = 1, f(x) = 0$ in generalized FRO equation. Using theorems in section 3, the fractional differential transform of the equation can be written as

$$Y(k+1) = -\frac{\Gamma(0.5k+1)}{\Gamma(0.5k+1.5)}Y(k). \tag{2.10}$$

The initial condition imply

$$Y(0) = 1.$$

And the approximate solution by inverse fractional differential transform is

$$\begin{aligned}y(x) = 1 - 1.12837916714x^{0.5} + 1x - 0.7522527780x^{1.5} + 0.4999999999x^2 - \\ 0.3009011111x^{2.5} + 0.1666666665x^3 - 0.0859717459x^{3.5} + 0.0416666665x^4 - \\ 0.0191048323x^{4.5} + 0.0083333332x^5 - 0.0034736058x^{5.5} + 0.0013888888x^6 - \\ 0.0005344008x^{6.5} + 0.0001984126x^7 - 0.0000712534x^{7.5} + 0.0000248015x^8 - \\ 0.0000083827x^{8.5} + 0.0000027357x^9 - 0.00000088238549x^{9.5} + \\ 0.00000028254315x^{10} - 0.000000086163217x^{10.5}.\end{aligned}$$

The table 2.2 shows the results from wavelet collocation method [85] for J=5 , analytic method , fractional DTM and the last column depicts the absolute error in the solution

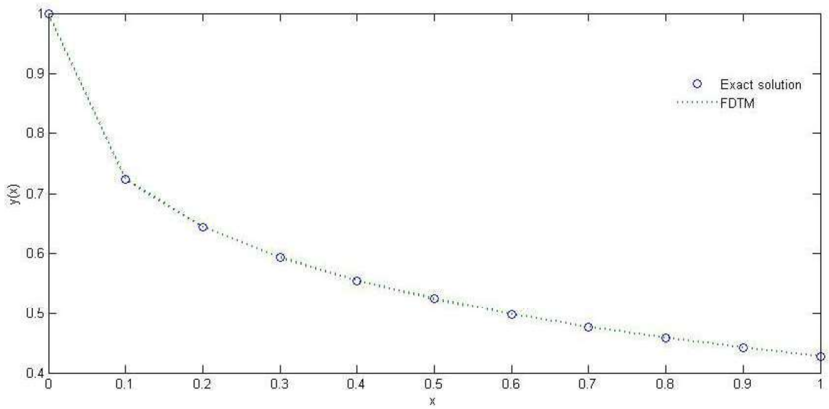


Fig. 2.1 Comparison of solutions obtained in example 2.4.1

Table 2.2 Comparison of fractional DTM solution of FRO equation with existing solutions

x	Wavelet collocation method [85]	Analytic solution(A)	Fractional DTM(F)	Error(|A-F|)
0.0	1.0000000	1.0000000	1.0000000	0.0000e-000
0.1	0.7235784	0.7235784	0.7235784	1.3611e-013
0.2	0.6437882	0.6437882	0.6437882	9.9298e-013
0.3	0.5920184	0.5920184	0.5920184	1.8488e-012
0.4	0.5536062	0.5536062	0.5536062	2.2161e-012
0.5	0.5231565	0.5231565	0.5231565	3.9851e-011
0.6	0.4980245	0.4980245	0.4980245	2.3950e-010
0.7	0.4767027	0.4767027	0.4767027	1.0558e-009
0.8	0.4582460	0.4582460	0.4582460	3.8471e-009
0.9	0.4420214	0.4420214	0.4420213	1.2184e-008
1.0	0.4275835	0.4275835	0.4275835	3.4561e-008

obtained by analytic method and fractional DTM. It's explicit that with fractional DTM the results are fairly near to the analytic solution.

2.4.2 Example

Consider another FRO equation

$$D^{3/2}y(x) + y(x) = 0$$

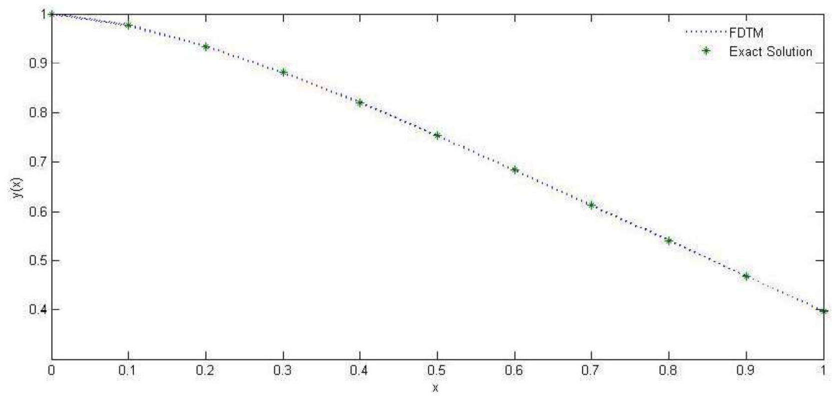


Fig. 2.2 Comparison of solutions obtained in example 2.4.2

with initial conditions $y(0) = 1$ and $y'(0) = 0$ i.e. $A = 1, f(x) = 0$ in generalized FRO equation.
Using theorems in section (1.5), the fractional differential transform of the equation can be written as

$$Y(k+1) = -\frac{\Gamma(1.5k+1)}{\Gamma(1.5k+2.5)}Y(k). \tag{2.11}$$

The initial conditions imply

$$Y(0) = 1.$$

The approximate solution is

$$\begin{aligned} y(x) = 1 - 0.75225277806367508x^{1.5} + 0.16666666666666669x^3 - \\ 0.019104832458760004x^{4.5} + 0.0013888888888888892x^6 - \\ 7.1253454391645706e-5x^{7.5} + 2.7557319223985897e-6x^9 - \\ 8.4037687620988591e-8x^{10.5} + 2.08767569878681e-9x^{12} - \\ 4.3304444506789613e-11x^{13.5} + 7.6471637318198184e-13x^{15} - \\ 1.1677471805518486e-14x^{16.5} + 1.561920696858623e-16x^{18} - \\ 1.8497133837075119e-18x^{19.5} + 1.9572941063391266e-20x^{21}. \end{aligned}$$

Table 2.3 is self explanatory and strengthens the efficiency and applicability of fractional DTM. Besides many applications of FBTE in viscoelasticity problems and of FROE in many realms of nature, these serve more like miniature fractional differential equations. Their solution by any method would lead to innovation in the bigger family of fractional differential

Table 2.3 Comparison of fractional DTM solution of FRO equation with existing solutions

x	Wavelet collocation method [85]	Analytic solution(A)	Fractional DTM(F)	Error(|A-F|)
0.0	1.0000000	1.0000000	1.0000000	0.0000e-000
0.1	0.9763777	0.9763777	0.9763777	0.0000e-000
0.2	0.9340362	0.9340362	0.9340362	0.0000e-000
0.3	0.8808084	0.8808084	0.8808084	1.1102e-016
0.4	0.8200563	0.8200563	0.8200563	0.0000e-000
0.5	0.7540488	0.7540488	0.7540488	0.0000e-000
0.6	0.6845298	0.6845298	0.6845298	0.0000e-000
0.7	0.6129215	0.6129215	0.6129215	0.0000e-000
0.8	0.5404169	0.5404169	0.5404169	1.1102e-016
0.9	0.4680306	0.4680306	0.4680306	0.0000e-000
1.0	0.3966293	0.3966293	0.3966293	0.0000e-000

equations. In this chapter, we have solved the two equations using fractional differential transform method and established it with two examples. The results are as good as the available analytic solutions.

Chapter 3

Solution of one dimensional fractional differential equation using Differential Quadrature Method

This chapter is about the application of the modified differential quadrature method (DQM) on the fractional differential equations. In first part of the chapter the fractional Burgers' equation has been solved using DQM with a hybrid cubic modified B-spline (section:1.7) and in the second part an inverse problem has been solved for fractional Fisher's equation by the same methodology.

3.1 The fractional Burgers' equation (FBE)

In 1915, a British mathematician Harry Bateman gave a partial differential equation relating the velocity and its space and time derivatives along with the initial and boundary conditions. The same equation, in 1948, was used by a Dutch physicist Johannes Martinus Burgers to model the turbulence. To regard his contribution in fluid mechanics, this equation is addressed as Burgers' equation [94, 95] and is given as:

$$u_t + uu_x = \nu u_{xx}, \quad 0 < x < L, \quad 0 < t < \tau \tag{3.1}$$

with conditions,

$$u(x,0) = \psi(x), \quad 0 < x < L$$

and

$$u(0,t) = \xi_1(t), \quad u(L,t) = \xi_2(t), \quad 0 < t < \tau$$

where u is the velocity, x is the space coordinate, t is time coordinate, ν is the kinematic viscosity and $\psi(x), \xi_1(t), \xi_2(t)$ are the fuctions depending upon specific conditions of the problem. Burgers' equation arises as a result of combining the phenomenon of nonlinear advection with linear diffusion of non-Newtonian fluid (e.g. glycerin, paints) flow. These fluids hold the characteristics of elasticity and retention of memory. Burgers' equation (3.1) has the same importance in the theory of viscoelasticity as the energy-mass equation has in the theory of relativity. With the advent and exploration of fractional calculus, it was understood that the fractional Burgers' equation is a better model for analyzing fluid motion. The equation was modified to fractional form by writing the derivative with respect to time by the Caputo differential operator and thus the fractional Burgers' equation is given as:

$$\frac{\partial^{\aleph} u(x,t)}{\partial t^{\aleph}} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \nu \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad 0 < \aleph < 1 \tag{3.2}$$

with conditions,

$$u(x,0) = \psi(x), \quad a \leq x \leq b$$

and

$$u(a,t) = \xi_1(t), \quad u(b,t) = \xi_2(t), \quad t \geq 0$$

where $\aleph$ is the order of the fractional derivative and rest of the variables have the same meaning as for equation (3.1).

The fractional Burgers' equation (3.2) has been solved for approximate solutions by analytical methods such as variational iteration method [96, 97], Adomian decomposition method [98] as well as by numerical methods like finite element method [99], finite difference method [100] etc.

In this chapter, the solution of time fractional Burgers' equation has been introduced with differential quadrature method (DQM). It is a way of approximating the derivative of a function by linear addition of the values of the function at nodal points of the domain. This method is well known among all numerical methods for its computational efficiency and accuracy.

The fractional time derivative has been discretized using modified Lubich's approximation [124]. We have used a convex combination of modified CBS and cubic TBS functions to calculate the weight coefficients in order to approximate the space derivatives. This hybrid B-spline function enables one to get the solutions with standard spline only as well as with trigonometric spline only. Moreover it holds the properties of both standard and trigonometric splines simultaneously.

3.1.1 Discretization of time derivative

In 1986, Lubich [124] devised the ν^{th} order ($\nu \leq 6$) derivative approximation of the $\aleph^{th}$ derivative by corresponding coefficients of the generating functions $\delta^{\aleph}(\zeta)$ given as

$$\delta^{\aleph}(\zeta) = \left(\sum_{i=1}^{\nu} \frac{1}{i}(1-\zeta)^i \right)^{\aleph}. \tag{3.3}$$

For $\nu = 3$ and for all $|\zeta| \leq 1$ equation (3.3) takes the form

$$\delta^{\aleph}(\zeta) = \sum_{k=0}^{\infty} l_k^{3,\aleph} \zeta^k \tag{3.4}$$

where

$$l_k^{3,\aleph} = \left(\frac{11}{6}\right)^{\aleph} \sum_{j=0}^{k} \sum_{m=0}^{j} \left(\frac{4}{7+\sqrt{(-39)}}\right)^{m} \left(\frac{4}{7-\sqrt{(-39)}}\right)^{j-m} l_m^{1,\aleph} l_{j-m}^{1,\aleph} l_{k-j}^{1,\aleph}. \tag{3.5}$$

The symbols are well described in reference [124].
The relation in Riemann-Liouville and Caputo fractional derivative

$$\mathbb{D}_a^{\aleph} f(x) = D_a^{\aleph} f(x) + \sum_{p=0}^{n-1} \frac{f^{(p)}(a)}{\Gamma(p-\aleph+1)}(x-a)^{p-\aleph} \tag{3.6}$$

can be modified as

$$D^{\aleph} f(x,t) = \mathbb{D}^{\aleph} f(x,t) - \sum_{p=0}^{m-1} \frac{f^{(p)}(x,0)}{\Gamma(p+1)} \cdot \frac{\Gamma(p+1)t^{p-\aleph}}{\Gamma(p+1-\aleph)}$$

or

$$D^{\aleph} f(x,t) = \mathbb{D}^{\aleph} f(x,t) - \sum_{p=0}^{m-1} \frac{f^{(p)}(x,0)}{\Gamma(p+1)} [\mathbb{D}^{\aleph} t^p]. \tag{3.7}$$

Using equation (3.4) in (3.7), the Caputo time derivative takes the form

$$D^{\aleph} f(x,t) \approx \frac{1}{\kappa^{\aleph}} \sum_{k=0}^{n} l_k^{\aleph} f(x,t_{n-k}) - \frac{1}{\kappa^{\aleph}} \sum_{k=0}^{n} l_k^{\aleph} f(x,0) + O(\kappa^3). \tag{3.8}$$

The above expression is the required discretized fractional time derivative involved in fractional Burgers' equation.

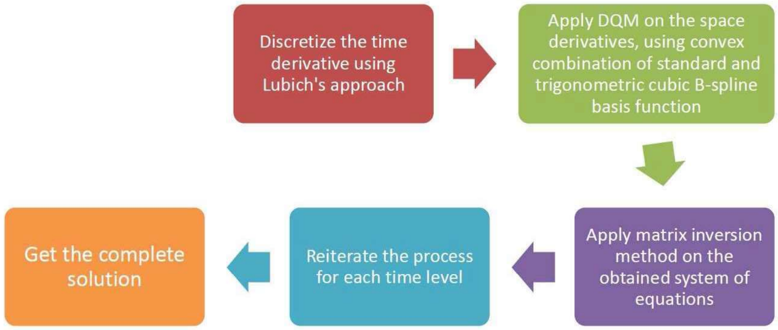


Fig. 3.1 Brief methodology

3.2 Implementation of method

On putting together the above-defined methodology with discretized time derivative and space derivatives, the numerical scheme for equation (3.2) :

$$\kappa^{\aleph} u_i^{n-1} \sum_{j=1}^{N} a_{ij} u_j^n - \kappa^{\aleph} \nu \sum_{j=1}^{N} b_{ij} u_j^n = \sum_{k=1}^{n} l_k^{\aleph} u_i^1 - \sum_{k=1}^{n} l_k^{\aleph} u_i^{n-k+1} + \kappa^{\aleph} f_i^n; \quad i = 2, 3, \ldots, N-1 \tag{3.9}$$

which on simplification takes the form of linear equations given as $CY = F$ where

$$C = \begin{cases} \kappa^{\aleph}(u_i^n a_{ij} - \nu b_{ij}) & \text{if } i \neq j \\ l_1^{\aleph} + \kappa^{\aleph}(u_i^n a_{ij} - \nu b_{ij}) & \text{if } i = j, \end{cases} \tag{3.10}$$

$Y = [u_2^n \quad u_3^n \quad \ldots u_{N-1}^n]'$ and F is a column matrix with elements $\kappa^{\aleph}(f_\varepsilon^n - u_\varepsilon^{n-1} a_{\varepsilon 1} u_1^n - u_\varepsilon^{n-1} a_{\varepsilon N} u_N^n + \nu b_{\varepsilon 1} u_1^n + \nu b_{\varepsilon N} u_N^n) + \sum_{k=1}^{n} l_k^{\aleph} u_\varepsilon^1 - \sum_{k=2}^{n} l_k^{\aleph} u_\varepsilon^{n-k+1}$ where ε varies from 2 to $N-1$.

On solving this matrix system, we obtain the solution at one time level and reiteration of this leads to the solution at next time levels.

The algorithm is briefly described in Figure4.1.

3.3 Stability of the scheme

In this work, we have approximated the derivatives by *specific* sums, so the obtained numerical scheme has to be checked for its stability [109]. In layman language, stability of a system ensures that the errors do not grow, as one moves ahead with the recursions.

As discussed in previous sections, Burgers' fractional differential equation can be written as a numerical scheme which can be translated in matrix form [110] as

$$Gu^{j+1} = Hu^{j} + d^{j}$$

and can eventually be reframed as

$$u^{j+1} = Mu^{j} + f^{j}. \tag{3.11}$$

For stability, norm [110] of matrix M from equation (3.11) compatible with norm of u must satisfy $\|M\| \leq 1$. Moreover the system is stable if the largest of the moduli of the eigenvalues (spectral radius) of the matrix M satisfies $\rho(M) \leq 1$.
The system is considered to be convergent if

$$\rho(M) \leq \|M\| \leq 1 \quad \text{as} \quad N \to \infty. \tag{3.12}$$

Stability and convergence have been discussed example-wise, for a particular number of partitions and time level in the section on numerical examples.

3.3.1 Error analysis

The properties of cubic B-spline interpolation can be used to get the error bounds of approximations of equations (1.35) and (1.36) as described in [111, 112]. We have considered the hybrid B-spline basis function as a convex combination of modified standard cubic and trigonometric cubic B-splines. Here we assume that the linear combination of any two cubic splines is a cubic spline.

Theorem 7. *Let $u(x) \in C^6[a,b]$. Then, the approximations*

$$u'(\tau_i) \approx U'(\tau_i) = \sum_{j=1}^{N} a_{ij} U(\tau_j), 1 \leq i \leq N$$

and

$$u''(\tau_i) \approx U''(\tau_i) = \sum_{j=1}^{N} b_{ij} U(\tau_j), 1 \leq i \leq N$$

have the error bounds for $1 \leq i \leq N$ as follows:

$$|u'(\tau_i) - U'(\tau_i)| = O(h^4),$$

$$|u''(\tau_i) - U''(\tau_i)| = O(h^2)$$

Proof: *Let $H(x)$ be a cubic spline interpolant of u(x) defined by*

$$H(x) = \sum_{m=1}^{N} c_m H_m(x) = \sum_{m=1}^{N} c_m[\eta T_m(x) + (1-\eta)B_m(x)]$$

where c_m are the constant coefficients obtained by using the interpolation conditions as described in section 2 of [112]. Now using the triangular inequality,

$$|u'(\tau_i) - U'(\tau_i)| \leq |u'(\tau_i) - H'(\tau_i)| + |H'(\tau_i) - U'(\tau_i)|. \tag{3.13}$$

Referring theorem 2.1 and 2.2 of [111], first expression in the inequality 3.13 is of order 4 and the second expression can be analyzed as follows:

$$\begin{aligned}
H'(\tau_i) - U'(\tau_i) &= \sum_{m=1}^{N} c_m H'_m(\tau_i) - \sum_{j=1}^{N} a_{ij} U(\tau_j) \\
&= \sum_{m=1}^{N} c_m[\eta T'_m(\tau_i) + (1-\eta)B'_m(\tau_i)] - \sum_{j=1}^{N} a_{ij} U(\tau_j) \\
&= \sum_{m=1}^{N} c_m \left[\eta \sum_{j=1}^{N} a_{ij} T_m(\tau_j) + (1-\eta) \sum_{j=1}^{N} a_{ij} B_m(\tau_j)\right] - \sum_{j=1}^{N} a_{ij} U(\tau_j) \\
&= \sum_{j=1}^{N} a_{ij} \left[H(\tau_j) - U(\tau_j)\right] \\
&= a_{i1}[H(\tau_1) - U(\tau_1)] + \sum_{j=2}^{N-1} a_{ij} \left[H(\tau_j) - U(\tau_j)\right] + a_{in}[H(\tau_n) - U(\tau_n)] \\
&= O(h^4) + 0 + O(h^4) \\
&= O(h^4)
\end{aligned}$$

In a similar manner, for the following inequality

$$|u''(\tau_i) - U''(\tau_i)| \leq |u''(\tau_i) - H''(\tau_i)| + |H''(\tau_i) - U''(\tau_i)|$$

the value of

$$\begin{aligned}
H''(\tau_i) - U''(\tau_i) &= \sum_{j=1}^{N} b_{ij} \left[H(\tau_j) - U(\tau_j) \right] \\
&= b_{i1}[H(\tau_1) - U(\tau_1)] + \sum_{j=2}^{N-1} b_{ij} \left[H(\tau_j) - U(\tau_j) \right] + b_{in}[H(\tau_n) - U(\tau_n)] \\
&= O(h^4).
\end{aligned}$$

And using the interpolation conditions as described in section 2 of [112], $u''(\tau_i) - H''(\tau_i) = \frac{h^2}{12} u''''(\tau_i) + O(h^4)$. *Thereby proving that* $|u''(\tau_i) - U''(\tau_i)|$ *is of* $O(h^2)$. *Thus, the error bounds are established for the approximations.*

3.4 Numerical examples on FBE

In this segment three examples have been discussed to validate the presented method. Other than exact solutions (u_{exact} in tables) our results ($u_{dqm_{bspline}}$ in tables) have been compared with DQM where the weight coefficients have been calculated using Lagrange polynomial [113, 114] with CGL grid points ($u_{dqm_{lpcgl}}$ in tables). The accuracy of the proposed method has been mentioned by presenting L_2 and L_∞ errors defined as

$$L_2 \quad \text{error} = \sqrt{h \sum_{j=1}^{N} |u_{exact} - u_{numerical}|^2} \tag{3.14}$$

and

$$L_\infty \quad \text{error} = \max |u_{exact} - u_{numerical}|. \tag{3.15}$$

All the calculations are done with MATLAB programming. For the sake of simplicity, we have considered the viscosity parameter ν to be unity, throughout this chapter.

3.4.1 Example

Examine equation (3.2) with $\nu = 1$ under the conditions

$$u(x,0) = 0, \quad u(0,t) = t^2, \quad u(1,t) = et^2, \quad 0 \le x \le 1, \quad t \ge 0 \tag{3.16}$$

and

Table 3.1 Pointwise error in exact and numerical solutions in example 3.4.1 with $\aleph = 0.5, N = 181, \eta = 0.5$

x	$A(u_{exact})$	$B(u_{dqm_{bspline}})$	$C(u_{dqm_{lpcgl}})$	a.e. in A and B	a.e. in A and C
0.0	1	1	1	0	0
0.1	1.105170918	1.106402844	1.106396405	0.001231926	0.001225487
0.2	1.221402758	1.223784549	1.223778603	0.002381791	0.002375844
0.3	1.349858808	1.353292948	1.353287917	0.003434141	0.003429109
0.4	1.491824698	1.496181607	1.496177658	0.004356909	0.00435296
0.5	1.648721271	1.653816094	1.653813339	0.005094823	0.005092069
0.6	1.8221188	1.827675917	1.827674465	0.005557116	0.005555665
0.7	2.013752707	2.019350154	2.01935012	0.005597446	0.005597413
0.8	2.225540928	2.230521345	2.230522817	0.004980416	0.004981889
0.9	2.459603111	2.462927637	2.462930439	0.003324525	0.003327328
1.0	2.718281828	2.718281828	2.718281828	0	0

$$f(x,t) = \frac{2t^{2-\aleph}e^x}{\Gamma(3-\aleph)} + t^4e^{2x} - t^2e^x. \tag{3.17}$$

The exact solution of the problem is given as $u(x,t) = t^2e^x$. The equation has been solved by using the methodology as reported in section 1.7. The results are recorded in Table 1 and Figure 3.2. Here a.e. stands for absolute error.

As discussed in section 3.3, stability and convergence have been verified for this example. The spectral radius (0.4757) and norm of the matrix (0.4842) fulfil the desired stability criterion given by equation (3.12).

The hybrid spline i.e. the convex combination shows an unexpected trend as it must be affected by variation in η but the performance of the scheme does not change with the values of η as is recorded in Table 3.1. Therefore, it can be said that the research work which is just replicating the standard cubic spline work into trigonometric spline, is not different, in case of fractional Burgers' equation.

Since the time step is inversely proportional to the execution time in any algorithm, it plays a major role in the numerical scheme. In our discretization scheme, the function at a time, depends upon all the previous time levels i.e. there is *global dependence* of time so we used the time step of 0.01 and obtained decent results within 16 seconds.

The same example is solved by Esen et. al. [115] using the Galerkin method with time step 0.00025, $\aleph = 0.5, \nu = 1, T = 1$ and different values of N. They reported L_∞ error of the order 10^{-3}. In our example we took time step of 0.01, and it resulted in an error of order 10^{-2}.

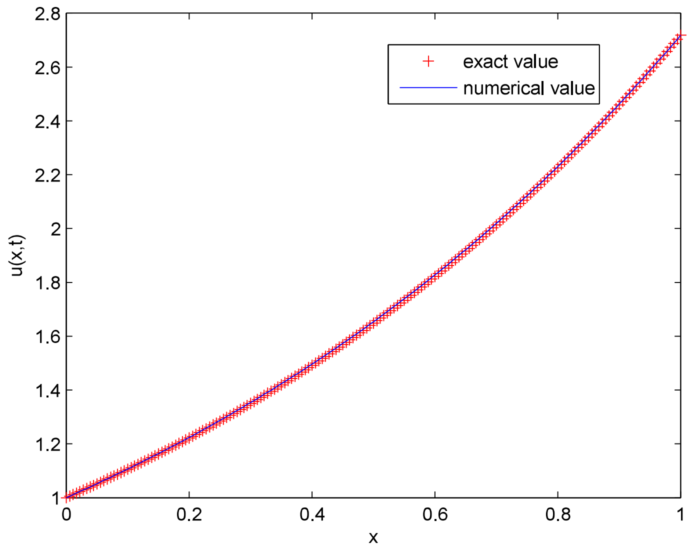


Fig. 3.2 Comparison of numerical and exact solution for example 3.4.1 with N=181, ν=1, η=0.5 and $\aleph$=0.5

Thus the time step of 0.01 is resulting in error of order 10^{-2}. A smaller time step leads to even reduced error.
Moreover we compared our results with DQM with Lagrange polynomial with CGL points and observed from Table 1 that the results are comparable.

3.4.2 Example

Examine equation (3.2) with $\nu = 1$ under the conditions

$$u(x,0) = 0, \quad u(0,t) = t, \quad u(1,t) = -t, \quad 0 \le x \le 1, \quad t \ge 0 \tag{3.18}$$

and

$$f(x,t) = \frac{cos\pi x}{\Gamma(2-\aleph)} t^{1-\aleph} - 0.5\pi t^2 sin2\pi x + \pi^2 t cos\pi x. \tag{3.19}$$

The exact solution of the problem is given as $u(x,t) = tcos\pi x$.

This equation is solved using the procedure as described in section (1.7) and the obtained results are presented in Table 2 and Figure 3.3. The condition regarding stability and

Table 3.2 Pointwise error in exact and numerical solutions for example 3.4.2 with $\aleph = 0.5, N = 181, \eta = 0.5$

x	$A(u_{exact})$	$B(u_{dqm_{bspline}})$	$C(u_{dqm_{lpcgl}})$	a.e. in A and B	a.e. in A and C
0.0	1	1	1	0	0
0.1	0.951056516	0.950826031	0.950752904	0.000230485	0.000303612
0.2	0.809016994	0.808637779	0.808573079	0.000379215	0.000443915
0.3	0.587785252	0.587397487	0.587350518	0.000387765	0.000434735
0.4	0.309016994	0.308773601	0.308749004	0.000243393	0.00026799
0.5	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.6	-0.309016994	-0.308773601	-0.308749004	0.000243393	0.00026799
0.7	-0.587785252	-0.587397487	-0.587350518	0.000387765	0.000434735
0.8	-0.809016994	-0.808637779	-0.808573079	0.000379215	0.000443915
0.9	-0.951056516	-0.950826031	-0.950752904	0.000230485	0.000303612
1.0	-1	-1	-1	0	0

Table 3.3 Error norms for different values of N in example 3.4.2 for fixed parameters

N	L_∞ error	L_2 error
21	9.7918e-04	0.0261
51	4.7483e-04	0.0180
101	4.1647e-04	0.0164
121	4.1068e-04	0.0163
151	4.0614e-04	0.0161
181	4.0366e-04	0.0160
1001	3.9818e-04	0.0159

convergence given by equation (3.12) is getting satisfied for the obtained values of spectral radius (0.4469) and norm (0.4473).

For fractional order $\aleph = 0.5$, time level $\kappa = 0.01$ and $\eta = 0.5$ in the convex combination term, the results are presented for various values of N in Table 3. It is inferred that L_∞ and L_2 errors reduce as N increases.

3.4.3 Example

Examine fractional Burgers' equation (3.2) with a unit viscosity constant under the conditions

$$u(x,0) = 0, \quad u(0,t) = t^2, \quad u(1,t) = -t^2, \quad 0 \le x \le 1, \quad t \ge 0 \tag{3.20}$$

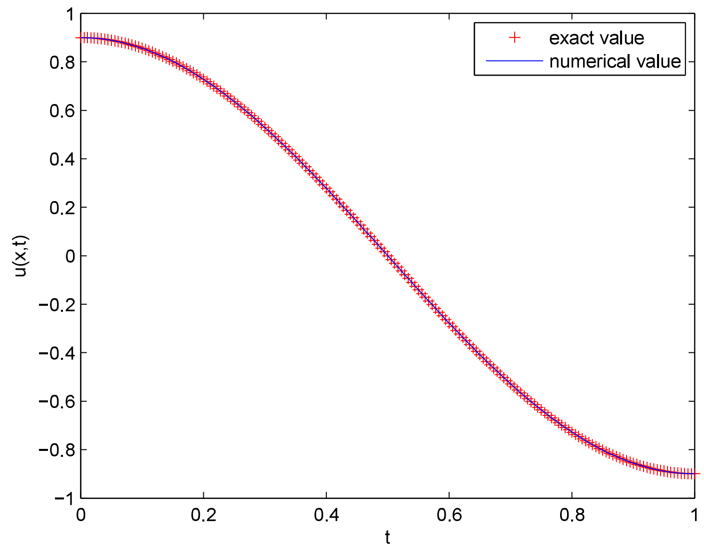


Fig. 3.3 Comparison of numerical and exact solution for example 3.4.2 with N=181, ν=1, η=0.5 and $\aleph$=0.5

and

$$f(x,t) = \frac{2t^{2-\aleph}}{\Gamma(3-\aleph)}(sin\pi x + cos\pi x) + \pi t^4 cos2\pi x + t^2\pi^2(sin\pi x + cos\pi x). \tag{3.21}$$

The exact solution of the problem is given as $u(x,t) = t^2(sin\pi x + cos\pi x)$. This equation is solved using the proposed methodology and the results are presented in Table 4 and Figure 3.4.

The spectral radius for this problem is 0.4441 and norm of the matrix as discussed in section 3.3 is 0.4466 which establishes the stable nature of the scheme.

The Figures 3.5, 3.6 and 3.7 represent the solution surface for the three examples respectively for time in the range [0:0.1:1].

We observe from all three numericals that the standard and trigonometric B-splines are behaving in the same way for fractional Burgers' equation. Also the reference [102] suggests that the Lagrange polynomial with CGL points based DQM is better than the B-Spline based DQM, but that was for ordinary and partial diffrential equations. For fractional Burgers' equation, our examples suggest that DQM with both approaches to calculate the weight coefficients are comparable.

A numerical scheme is executed on the fractional Burgers' equation by hybridizing cubic and trigonometric B-splines for the differential quadrature method. The time fractional derivative

Table 3.4 Pointwise error in exact and numerical solutions in example 3.4.3 with $\aleph = 0.5, N = 181, \eta = 0.5$

x	$A(u_{exact})$	$B(u_{dqm_{bspline}})$	$C(u_{dqm_{lpcgl}})$	a.e. in A and B	a.e. in A and C
0.0	1	1	1	0	0
0.1	1.260073511	1.260006533	1.259927293	6.69776E-05	0.000146217
0.2	1.396802247	1.396124783	1.396058361	0.000677463	0.000743886
0.3	1.396802247	1.395204587	1.39515773	0.00159766	0.001644516
0.4	1.260073511	1.257600653	1.257577727	0.002472857	0.002495783
0.5	1.000000000	0.997058789	0.997063467	0.002941211	0.002936533
0.6	0.642039522	0.639249341	0.639283924	0.002790181	0.002755598
0.7	0.221231742	0.219158367	0.219222568	0.002073375	0.002009174
0.8	-0.221231742	-0.222337667	-0.222248088	0.001105925	0.001016346
0.9	-0.642039522	-0.642357727	-0.642254071	0.000318205	0.000214549
1.0	-1	-1	-1	0	0

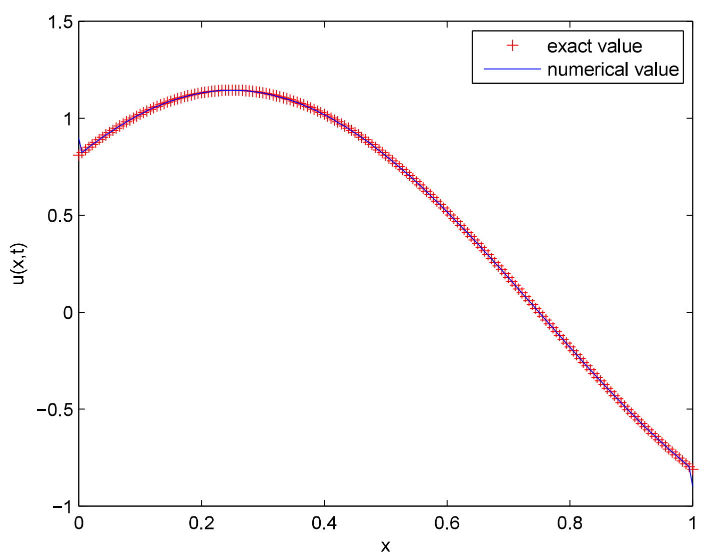


Fig. 3.4 Comparison of numerical and exact solution for example 3.4.3 with N=181, ν=1, η=0.5 and $\aleph$=0.5

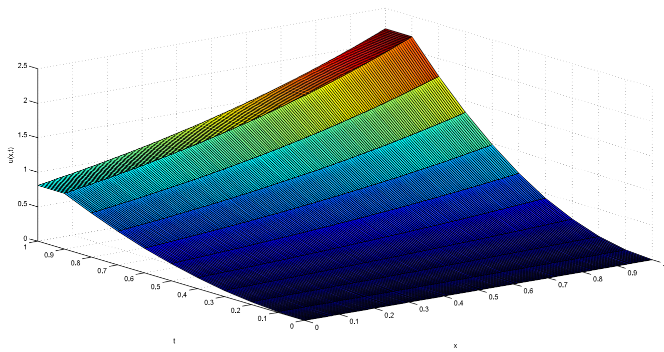


Fig. 3.5 Solution surface of example 3.4.1

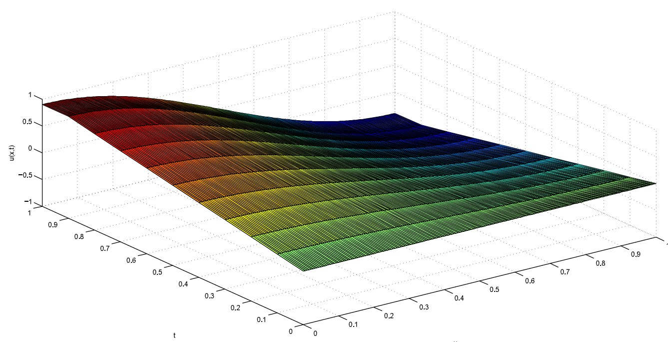


Fig. 3.6 Solution surface of example 3.4.2

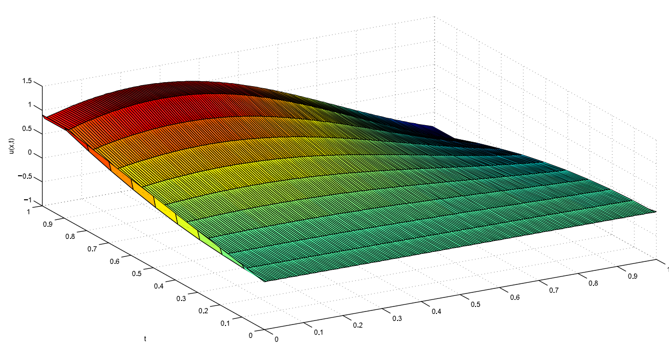


Fig. 3.7 Solution surface of example 3.4.3

Table 3.5 Error norms for different values of η for fixed parameters

η	Example 3.4.1		Example 3.4.2		Example 3.4.3	
	L_∞ error	L_2 error	L_∞ error	L_2 error	L_∞ error	L_2 error
0	0.0058	0.0613	0.0031	0.0385	0.000404	0.160
0.2	0.0058	0.0613	0.0031	0.0385	0.000404	0.160
0.5	0.0058	0.0613	0.0031	0.0385	0.000404	0.160
0.7	0.0058	0.0613	0.0031	0.0385	0.000404	0.160
1	0.0058	0.0613	0.0031	0.0385	0.000404	0.160

was discretized by Lubich's approach and space derivatives by differential quadrature. The scheme is verified on three examples where the numerical results were available in literature for the first example. The presented numerical scheme has a global dependence on time, so the best feasible time step was chosen to be 0.01. It is observed that if the time step is reduced further, the errors would reduce but execution time of the program would increase. From all the presented examples, it can be depicted that the L_∞ as well as L_2 errors remain unaffected with the variation of value of η for a fixed value of N. Thus, it can be concluded that the results due to standard and trigonometric B-splines are same for fractional Burgers' equation and so DQM is independent of choice of spline (standard or trigonometric). The above discussed concept can be verified for other fractional differential equations for generalized results.

The acquired numerical outcomes agreed with the exact solutions, along with the stability conditions. Hence it can be concluded that the presented scheme can further be implemented to find the solution of space and time-space fractional differential equations.

3.5 The fractional Fisher's equation (FFE)

The Fisher's equation is a reaction diffusion equation which represents the problem of biological invasion. It finds its place in ecology, physiology, and in general phase transition problems etc. and is considered a population growth model. Its equation is given as

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + \aleph u(x,t)(1 - \frac{u(x,t)}{u_\infty}) \tag{3.22}$$

where u is the population, x is the space coordinate, t is time coordinate and D, $\aleph$, u_∞ are positive constant parameters that represent diffusivity, growth rate and number of individuals respectively.

Using the following transformations:

$$\tau = \aleph t,$$

$$z = x\sqrt{\frac{\aleph}{D}}$$

and

$$v = \frac{u(x,t)}{u_\infty}.$$

Its much popular dimensionless form is given as:

$$\frac{\partial v(z,\tau)}{\partial \tau} = \frac{\partial^2 v(z,\tau)}{\partial z^2} + v(1-v).$$

Writing the time derivative as Caputo's fractional derivative, one easily gets the fractional form of Fisher's equation:

$$\frac{\partial^\aleph v(z,\tau)}{\partial \tau^\aleph} = \frac{\partial^2 v(z,\tau)}{\partial z^2} + v(1-v).$$

In the recent past the fractional differential equations have proved to be better models [2]. Fractional partial differential equations (FPDEs) have generated quite interest in mathematics fraternity. Besides modeling the physical, biological and chemical processes [1, 2], they have applications in sampling, signal processing etc. There are various analytic and numerical methods [117–120] for solving nonlinear FPDEs. In this chapter, the focus is to solve the inverse problem of fractional Fisher's equation using additional information.

3.6 Inverse problem

For a completely known physical system, the mathematical description of its uniqueness, stability, existence of a solution etc. and all the parameters must be known. But if one of the unknown parameters describing this system is to be found from some extra information, then that kind of problem is called an inverse problem.

In literature, there are mainly three categories of inverse problem viz. coefficient inverse problems, boundary value inverse problems and evolutionary inverse problems. In this work we intend to discuss the boundary value inverse problem on fractional Fisher equation.

Inverse problems came into sight around twentieth century. It is a mathematical problem in which some unknown fact can be found out with the help of information in hand. This unknown element can be a coefficient, a boundary condition or an initial condition primarily.

In this work, the inverse problem is to find the unknown boundary condition. The problem is solved in two parts, wherein the first part is a direct problem defined on one part say P of the domain and the inverse problem is then defined on the second say $\bar{P}$ as mentioned ahead.
In this work, we consider time fractional inverse Fisher's equation of order $\aleph, \quad 0 \leq \aleph \leq 1$ for the function $u(x,t)$ as

$$\frac{\partial^{\aleph} u(x,t)}{\partial t^{\aleph}} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)[1-u(x,t)] + f(x,t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T \tag{3.23}$$

with initial condition

$$u(x,0) = a(x), \quad 0 \leq x \leq 1, \tag{3.24}$$

and boundary conditions

$$u(x^{@},t) = b(t), \quad u(1,t) = c(t), \quad 0 < x^{@} \leq 1, \quad 0 \leq t \leq T \tag{3.25}$$

where $a(x)$, $b(t)$, $c(t)$ are known functions and $a(x)$ is continuous while $b(t)$ and $c(t)$ are infinitely differentiable and T is the known time level. The function $u(x,t)$ and boundary value $u(0,t)$ is to be determined. This problem can be bifurcated on two sub-parts of the domain i.e. $P = \{x : 0 \leq x \leq x^{@}\}$ and on $\bar{P} = \{x : x^{@} \leq x \leq 1\}$.
The direct problem is stated as

$$\frac{\partial^{\aleph} u(x,t)}{\partial t^{\aleph}} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)[1-u(x,t)] + f(x,t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T \tag{3.26}$$

with initial condition

$$u(x,0) = a(x), \quad x^{@} \leq x \leq 1,$$

and boundary conditions

$$u(x^{@},t) = b(t), \quad u(1,t) = c(t), \quad 0 < x^{@} \leq 1, \quad 0 \leq t \leq T.$$

The inverse problem is then defined as

$$\frac{\partial^{\aleph} u(x,t)}{\partial t^{\aleph}} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)[1-u(x,t)] + f(x,t), \quad 0 \leq x \leq x^{@}, \quad 0 \leq t \leq T \tag{3.27}$$

with initial condition

$$u(x,0) = a(x), \quad 0 \leq x \leq x^{@},$$

Table 3.6 Discretization of derivatives

Direct/Inverse problem	*Discretization*
Direct	$\frac{\partial^{\aleph} u}{\partial t^{\aleph}} \approxeq \frac{1}{\tau^{\aleph}} \sum_{k=1}^{n} l_k^{\aleph} u(x, t_{n-k+1}) - \frac{1}{\tau^{\aleph}} \sum_{k=1}^{n} l_k^{\aleph} u(x, 1)$
Direct	$\frac{\partial^2 u}{\partial x^2} \approxeq \sum_{j=1}^{n} b_{ij} u_j(x,t)$
Inverse	$\frac{\partial^{\aleph} u}{\partial t^{\aleph}} \approxeq \frac{1}{\tau^{\aleph}} \sum_{k=1}^{n} l_k^{\aleph} u(x, t_{n-k+1}) - \frac{1}{\tau^{\aleph}} \sum_{k=1}^{n} l_k^{\aleph} u(x, 1)$
Inverse	$\frac{\partial^2 u}{\partial x^2} \approxeq \frac{6}{5} \frac{u(x_{i+1},\, t_{k+1}) - 2u(x_i,\, t_{k+1}) + u(x_{i-1},\, t_{k+1})}{h^2}$

and boundary conditions

$$u(x^{@}, t) = b(t), \quad 0 \leq t \leq T.$$

In this chapter, the direct problem is solved with the help of differential quadrature method (DQM) using B-spline [130] and comparison has been made with the solutions obtained by DQM using Chebyshev-Gauss-Lobatto (CGL) points [114] and the inverse problem is solved using fourth order accurate compact approximation finite difference method.

3.6.1 The discretizations

Consider a uniform mesh of size h and τ on x and t axes respectively such that $h = \frac{1}{N}$ and $\tau = \frac{T}{N}$. The grid points are defined by

$$x_i = ih, \quad i = 1, 2, ..., N,$$
$$t_k = k\tau, \quad k = 1, 2, ..., N.$$

Let $u_i^k \approxeq u(x_i, t_k)$ be the numerical approximation and $x^{@}$ be arbitrarily any point in the grid, according to which we bifurcate the x-axis into two parts. As $x^{@}$ is an interior point, let $x^{@} = mh = x_m$ for some integer $2 \leq m \leq N-1$. The direct problem and then the inverse problem are solved on the suitable parts. The solution of inverse problem [128, 129] depends upon the solution of direct problem. For the direct problem, the second order derivative is discretized using the DQM [130] and for inverse problem, fourth order accurate compact approximation finite difference method [131] is used to discretize the second order derivative. In both cases, Caputo's definition of fractional derivative has been used and is discretized using Lubich's approach as in [105, 124, 130].

The symbols have usual meanings as described in the reference [130].

3.7 Solution of the direct problem

Using the discretizations as in Table 3.6, the equation (3.26) can be written in the following form:

$$\frac{1}{\tau^{\aleph}}\sum_{k=1}^{N} l_k^{\aleph} u_i^{N-k+1} - \frac{1}{\tau^{\aleph}}\sum_{k=1}^{N} l_k^{\aleph} u_i^{1} = \sum_{j=1}^{N} b_{ij}u_j^N + u_i^N(1-u_i^N) + f_i^N. \tag{3.28}$$

Taking $i = m+1$ to $N-1$ equation (3.28) results into the matrix system $AU = B$ where

$$A = \begin{bmatrix} l_1^{\aleph} - (1+b_{(m+1)2} - u_{m+1}^{N-1})\tau^{\aleph} & -b_{(m+1)3}\tau^{\aleph} & \dots & b_{(m+1)(N-1)}\tau^{\aleph} \\ -b_{(m+2)2}\tau^{\aleph} & l_1^{\aleph} - (1+b_{(m+2)3} - u_{m+2}^{N-1})\tau^{\aleph} & \dots & b_{(m+2)(N-1)}\tau^{\aleph} \\ \vdots & \ddots & & \\ -b_{(N-1)2}\tau^{\aleph} & -b_{(N-1)3}\tau^{\aleph} & & l_1^{\aleph} - (1+b_{(N-1)(N-1)} - u_{N-1}^{N-1})\tau^{\aleph} \end{bmatrix},$$

$$U = \begin{bmatrix} u_{m+1}^N & u_{m+2}^N & \dots & u_{N-1}^N \end{bmatrix}'$$

and

$$B = \begin{bmatrix} \tau^{\aleph}(f_{m+1}^N + b_{(m+1)1}u_1^N + b_{(m+1)N}U_N^N) + \sum_{k=1}^{N} l_k^{\aleph} u_{m+1}^1 - \sum_{k=2}^{N-1} l_k^{\aleph} u_{m+1}^{N-k+1} \\ \tau^{\aleph}(f_{m+2}^N + b_{(m+2)1}u_1^N + b_{(m+2)N}U_N^N) + \sum_{k=1}^{N} l_k^{\aleph} u_{m+2}^1 - \sum_{k=2}^{N-1} l_k^{\aleph} u_{m+2}^{N-k+1} \\ \vdots \\ \tau^{\aleph}(f_{N-1}^N + b_{(N-1)1}u_1^N + b_{(N-1)N}U_N^N) + \sum_{k=1}^{N} l_k^{\aleph} u_{N-1}^1 - \sum_{k=2}^{N-1} l_k^{\aleph} u_{N-1}^{N-k+1} \end{bmatrix}.$$

The matrix system can be solved for $u(x,t)$ in part $\bar{P}$ of the domain.

3.8 Solution of inverse problem

The inverse problem of the fractional Fisher's equation is defined in equation (3.27) and on applying the discretizations as mentioned in Table 3.6, the following formulation is obtained:

$$\frac{1}{\tau^{\aleph}}\sum_{k=1}^{N} l_k^{\aleph} u_i^{N-k+1} - \frac{1}{\tau^{\aleph}}\sum_{k=1}^{N} l_k^{\aleph} u_i^{1} = \frac{6}{5}\frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{h^2} + u_i^k(1-u_i^k) + f_i^{k+1}. \tag{3.29}$$

Varying i from 2 to $m-1$ equation (3.27) results in the matrix system $C\tilde{U} = D$ where

$$C = \begin{bmatrix} 1 & -2 & 1 & 0\ldots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1\ldots & 0 & 0 & 0 \\ \vdots & \ddots & & & & & \\ 0 & 0 & 0 & 0\ldots & 0 & 1 & -2 \\ 0 & 0 & 0 & 0\ldots & 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{U} = \begin{bmatrix} u_1^{k+1} & u_2^{k+1} & \ldots & u_{m-1}^{k+1} & u_m^{k+1} \end{bmatrix}',$$

$$D = -h^2 \begin{bmatrix} u_2^k(1-u_2^k) + f_2^{k+1} - \frac{1}{k^{\aleph}}(\sum_{k=1}^N l_k^{\aleph} u_2^{N-k+1} + \sum_{k=1}^N l_k^{\aleph} u_2^1) \\ u_3^k(1-u_3^k) + f_3^{k+1} - \frac{1}{k^{\aleph}}(\sum_{k=1}^N l_k^{\aleph} u_3^{N-k+1} + \sum_{k=1}^N l_k^{\aleph} u_3^1) \\ \vdots \\ u_{m-1}^k(1-u_{m-1}^k) + f_{m-1}^{k+1} - \frac{1}{k^{\aleph}}(\sum_{k=1}^N l_k^{\aleph} u_{m-1}^{N-k+1} + \sum_{k=1}^N l_k^{\aleph} u_{m-1}^1) + u_m^{k+1} \\ u_m^k(1-u_m^k) + f_m^{k+1} - \frac{1}{k^{\aleph}}(\sum_{k=1}^N l_k^{\aleph} u_m^{N-k+1} + \sum_{k=1}^N l_k^{\aleph} u_m^1) + u_{m+1}^{k+1} - 2u_m^{k+1} \end{bmatrix}.$$

Using the solution of direct problem, inverse problem can be solved and required boundary condition can be obtained.

3.9 Numerical examples on FFE

The method presented above can be used to solve inverse problems and some examples are given to establish the same. We have used the DQM with hybrid B-spline in solving the direct problem and compared the results with the DQM with CGL points.

3.9.1 Example

Consider equation (3.23) - (3.25) with the following conditions:
$u(x,0) = 0, \quad 0 \le x \le 1,$
$u(x^{@},t) = e^{x^{@}} t^2, \quad 0 \le t \le T,$
$u(1,t) = e^1 t^2, \quad 0 \le t \le T$ where $x^{@} = 0.5$. The exact solution of the problem is $u(x,t) = e^x\, t^2$ and

$$f(x,t) = \frac{2e^x t^{2-\aleph}}{\Gamma(3-\aleph)} - 2e^x t^2 + e^{2x} t^4.$$

For fixed value of $k = 0.01, T = 1$, $\aleph = 0.2$ the absolute errors were found for various number of mesh points and recorded in the Table 3.7.

Table 3.7 Table of errors in DQM with B-spline and DQM with CGL points for different values of N and $\aleph = 0.2$ for example 3.9.1

N	Direct/Inverse problem	DQM with B-spline	DQM with CGL points
7	Direct	8.1778e-06	3.7370e-06
7	Inverse	0.0070	0.0515
9	Direct	5.6622e-06	3.2963e-06
9	Inverse	0.0062	0.0374
11	Direct	4.5481e-06	3.0955e-06
11	Inverse	0.0058	0.0290
21	Direct	3.1091e-06	2.7561e-06
21	Inverse	0.0049	0.0125
41	Direct	2.7580e-06	2.6711e-06
41	Inverse	0.0045	0.0045
45	Direct	2.7162e-06	2.6665e-06
45	Inverse	0.0044	0.0037
51	Direct	2.7162e-06	2.6620e-06
51	Inverse	0.0044	0.0029

From the Table 3.7 it can be concluded that $N = 41$ is the best choice for the number of grid points. For $N = 41,\ k = 0.01,\ T = 0.1$, the change in the behaviour of the solution can be observed with different values of $\aleph$.

3.9.2 Example

Consider equation (3.23) - (3.25) with the following conditions:
$u(x,0) = 0, \quad 0 \leq x \leq 1,$
$u(x^@,t) = 0, \quad 0 \leq t \leq T,$
$u(1,t) = -t, \quad 0 \leq t \leq T$ where $x^@ = 0.5$. The exact solution of the problem is $u(x,t) = t\ cos\pi x$ and

$$f(x,t) = \frac{t^{1-\aleph}\ cos\ \pi x}{\Gamma(2-\aleph)} + t\ cos\pi x\ (\pi^2 - 1 + tcos\pi x).$$

3.9.3 Example

Consider equation (3.23) - (3.25) with the following conditions:
$u(x,0) = 0, \quad 0 \leq x \leq 1,$

Table 3.8 Table of errors in DQM with B-spline and DQM with CGL points for different values of $\aleph$

$\aleph$	Problem	DQM with B-spline	DQM with CGL points
0.1	Direct	2.8166e-06	2.7282e-06
0.1	Inverse	0.0027	0.0027
0.25	Direct	2.7173e-06	2.6313e-06
0.25	Inverse	0.0057	0.0057
0.5	Direct	2.3503e-06	2.2748e-06
0.5	Inverse	0.0177	0.0177
0.75	Direct	1.5337e-06	1.4804e-06
0.75	Inverse	0.0317	0.0317
1.0	Direct	1.1091e-06	1.1862e-06
1.0	Inverse	0.0471	0.0471

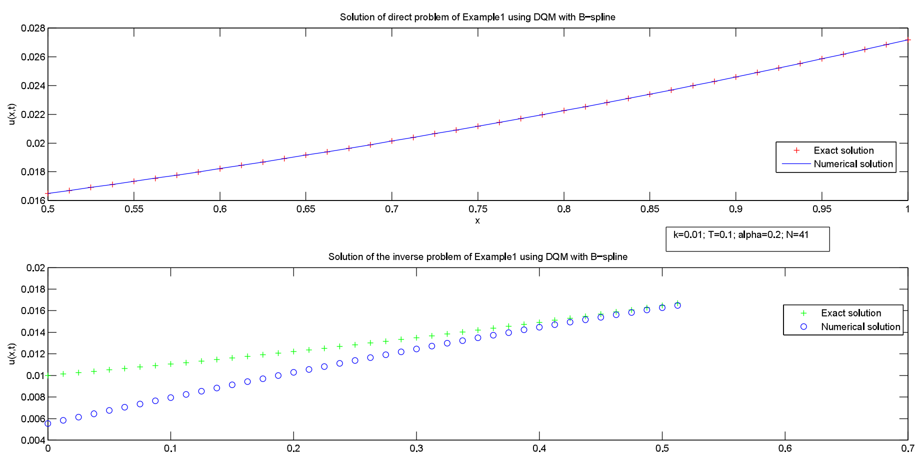


Fig. 3.8 Example 3.9.1 using DQM with B-spline

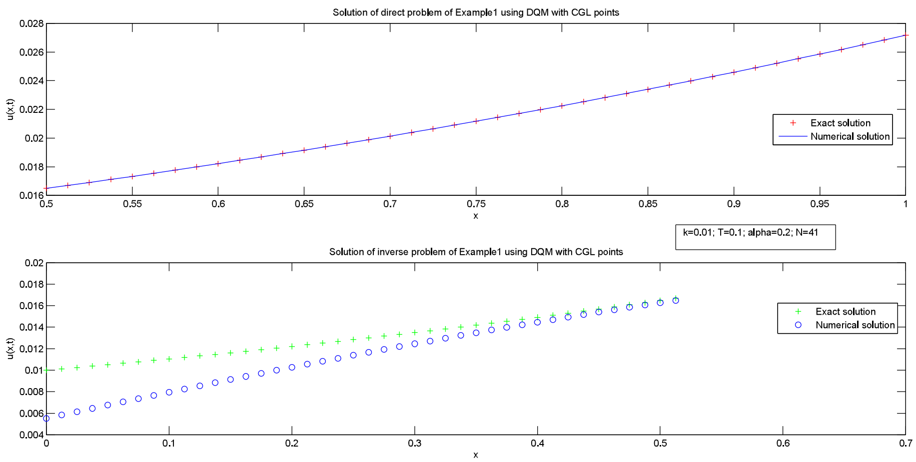


Fig. 3.9 Example 3.9.1 using DQM with CGL points

Table 3.9 Table of errors in DQM with B-spline and DQM with CGL points for different values of $\aleph$ in example 3.9.2

$\aleph$	Direct/Inverse problem	DQM with B-spline	DQM with CGL points
0.1	Direct	1.9667e-05	2.0304e-05
0.1	Inverse	0.0054	0.0690
0.25	Direct	1.9801e-05	2.0471e-05
0.25	Inverse	0.0045	0.0754
0.5	Direct	2.0688e-05	2.1297e-05
0.5	Inverse	0.0487	0.1002
0.75	Direct	2.8089e-05	2.7660e-05
0.75	Inverse	0.1429	0.1453
1.0	Direct	1.0753e-04	1.0434e-04
1.0	Inverse	0.1404	0.1553

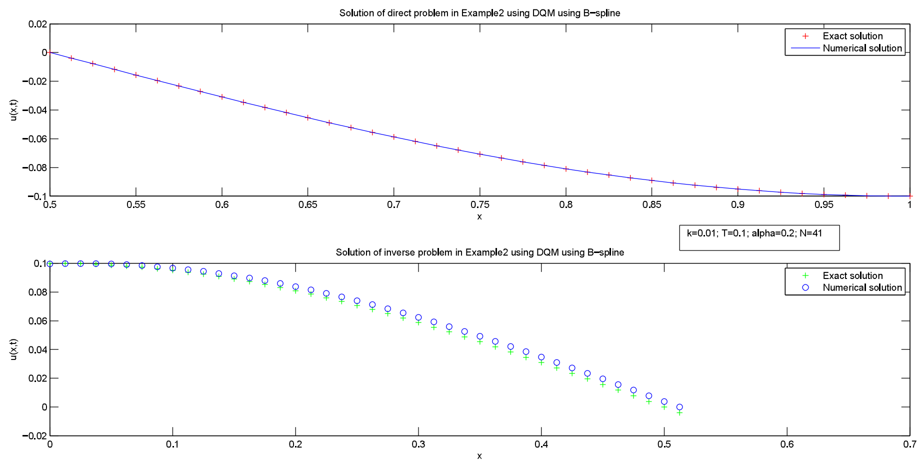


Fig. 3.10 Example 3.9.2 using DQM with B-spline

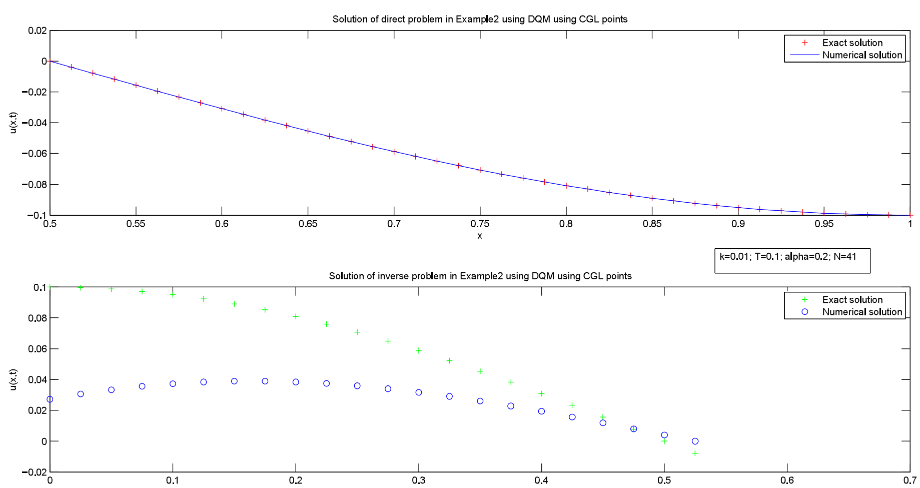


Fig. 3.11 Example 3.9.2 using DQM with CGL points

Table 3.10 Table of errors in DQM with B-spline and DQM with CGL points for different values of ℵ in example 3.9.3

ℵ	Problem	DQM with B-spline	DQM with CGL points
0.1	Direct	5.3382e-07	2.3550e-06
0.1	Inverse	4.3953e-04	0.0081
0.25	Direct	5.3334e-07	2.3465e-06
0.25	Inverse	0.0023	0.0106
0.5	Direct	5.3338e-07	2.3260e-06
0.5	Inverse	0.0138	0.0206
0.75	Direct	5.3158e-07	2.2852e-06
0.75	Inverse	0.0311	0.0318
1.0	Direct	5.1385e-07	2.1846e-06
1.0	Inverse	0.0226	0.0236

$u(x^@,t) = t^2, \quad 0 \leq t \leq T,$
$u(1,t) = -t^2, \quad 0 \leq t \leq T \qquad$ where $x^@ = 0.5$.
The exact solution of the problem is $u(x,t) = t^2(sin\ \pi x + cos\ \pi x)$ and

$$f(x,t) = (sin\ \pi x + cos\ \pi x)[\frac{2\, t^{2-\aleph}}{\Gamma(3-\aleph)} + t^2\ (\pi^2 - 1 + t^2(sin\ \pi x + \cos \pi x))].$$

The results can be seen in Table 3.10 and Figure 3.12 and 3.13.
As established in the popular book by G.D. Smith [110], the scheme

$$TU^{k+1} = Q^{n-k+1} + R^{k+1} + S^k$$

is stable if the absolute value of eigen values of the inverse of the matrix P are less than or equal to one. In the numerical scheme given by equation (3.29), The eigen values of inverse of the matrix T satistfy

$$|\frac{\tau^{\aleph}}{h^2}| \geq 1,$$
$$h \leq \tau^{\aleph/2}.$$

Thus for all suitable values of h,k and ℵ the system is stable. And in all the three examples the absolute values of the eigen values of the inverse of matrix involved for direct as well as inverse problem are less than or equal to one, hence the scheme is stable. Therefore, the methodology presented here can be used to solve inverse problems of similar nature.

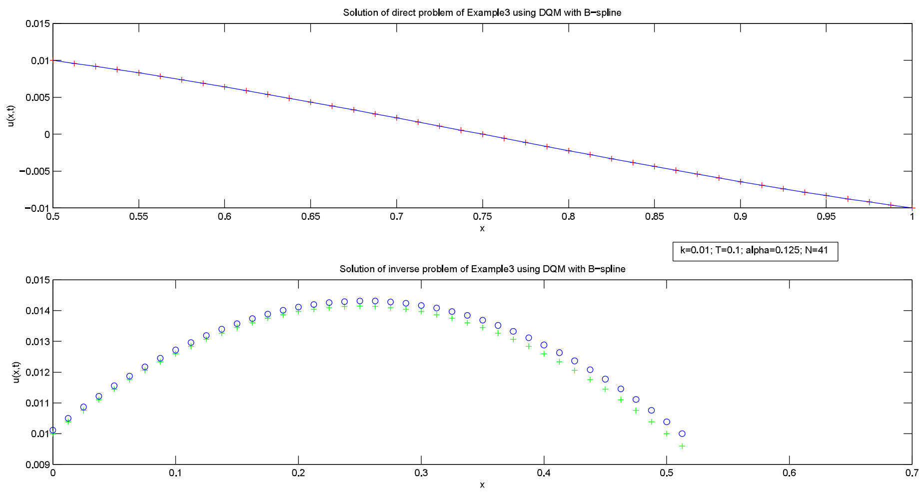


Fig. 3.12 Example 3.9.3 using DQM with B-spline

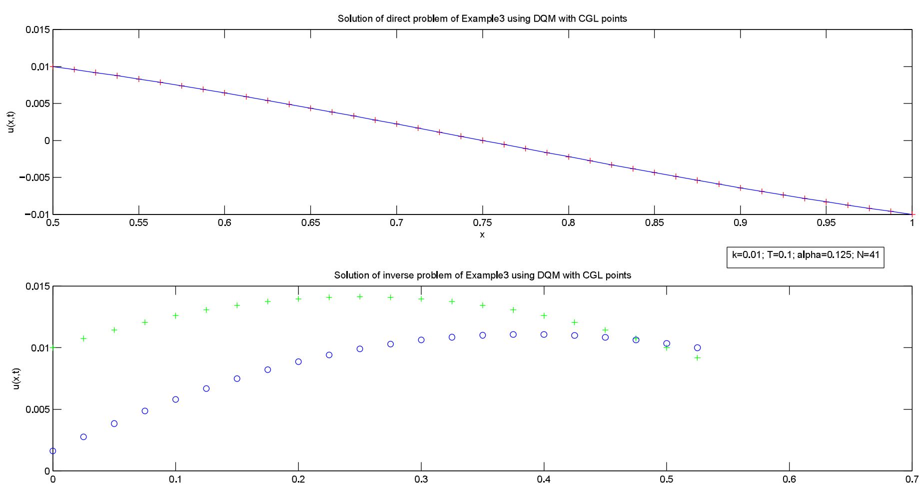


Fig. 3.13 Example 3.9.3 using DQM with CGL points

Chapter 4

Solution of two dimensional fractional differential equation using the hybrid D(TQ)M

In this chapter a two dimensional fractional diffrential equation namely the diffusion equation has been solved by using a novel hybrid of DTM and DQM. First of all the DTM is applied on the equation, and then in the transformed equation, the space derivatives are approximated using the DQM. Then by inverse DTM the solution can be found.

4.1 The fractional Diffusion Equation

Diffusion is the most commonly occuring phenomenon in our day to day life. It is the movement of a substance from a highly concentrated area to a lowly concentrated area. Needless to say that it happens in liquids and gases because their particles move randomly. The particles collide among themselves or with the vessel containing them. Thus they move in different directions and finally the particles are spread in the whole vessel. The process of diffusion happens on its own, without stirring, shaking or wafting. Osmosis and respiration are some examples of diffusion happening in living beings. Mathematically this process had been described as a partial differential equation and the model has evolved with time into a two dimensional fractional differential equation which is given as follows:

$$\frac{\partial^{\aleph} u(x,y,t)}{\partial t^{\aleph}} = a(x,y,t)\frac{\partial^2 u(x,y,t)}{\partial x^2} + b(x,y,t)\frac{\partial^2 u(x,y,t)}{\partial y^2} + c(x,y,t) \tag{4.1}$$

on a finite domain $0 \leq x \leq B_1, 0 \leq y \leq B_2$ and $0 < t \leq T$. Here $\aleph$ is the arbitrary order of the time derivative such that $0 < \aleph \leq 1$ and $a(x,y,t)$ and $b(x,y,t)$ being the non zero diffusion coefficients. The function $c(x,y,t)$ is used to represent sources and sinks.
Along with the initial condition $u(x,y,0) = F_1(x,y), \quad 0 \leq x \leq B_1, \quad 0 \leq y \leq B_2$ and the boundary conditions:

$$u(0,y,t) = F_2(y,t), u(B_1,y,t) = F_3(y,t) \quad 0 \leq y \leq B_2, 0 < t \leq T \tag{4.2}$$

$$u(x,0,t) = F_4(x,t), u(x,B_2,t) = F_5(x,t) \quad 0 \leq x \leq B_1, 0 < t \leq T \tag{4.3}$$

where $B_1, B_2, F_1(x,y), F_2(y,t), F_3(y,t), F_4(x,t)$ and $F_5(x,t)$ are given and $\frac{\partial^{\aleph} u(x,t)}{\partial t^{\aleph}}$ is the Caputo's definition of the time fractional derivative.

4.2 Implementation of the method

Applying DTM to the equation (4.1), the following expression is obtained, where the A,B,C are the differential transforms of a,b,c occuring in the diffusion equation.

$$\frac{\Gamma(k\aleph + \aleph + 1)}{\Gamma(k\aleph + 1)} U_{k+1} = \sum_{l=1}^{k} A(l) \frac{\partial^2 U(k-l+1)}{\partial x^2} + \sum_{l=1}^{k} B(l) \frac{\partial^2 U(k-l+1)}{\partial y^2} + C(k). \tag{4.4}$$

The space derivatives can be approximated by using DQM and the above relation can be written as,

$$U_{ij}^{k+1} = \frac{\Gamma(k\aleph + 1)}{\Gamma(k\aleph + \aleph + 1)} \left[\sum_{l=1}^{k} A(l) \sum_{m=1}^{M_x} a_{im} U_{mj}^{k-l+1} + \sum_{l=1}^{k} B(l) \sum_{m=1}^{M_y} b_{jm} U_{im}^{k-l+1} + C_{ij}^{k} \right]. \tag{4.5}$$

From here, $U_{ij}^2, U_{ij}^3, U_{ij}^4, \ldots$ can be calculated and then by using the inverse DTM, the approximate solution of the equation (4.1) can be written as,

$$u(x,y,t) = U_{ij}^1 + U_{ij}^2 . t^{\aleph} + U_{ij}^3 . t^{2\aleph} + \ldots. \tag{4.6}$$

4.3 Numerical examples

To check the methodology, two problems solved in [135] are considered. Throughout this section, in order to show the precision of the proposed technique, the maximum absolute error between exact and approximate solutions is given by:

$$Error = max|Exact\ value\ of\ \phi(x,y,t) - Approximate\ value\ of\ \phi(x,y,t)|.$$

4.3.1 Example

Consider the equation (4.1) with

$$a = \frac{2t^{2-\aleph}}{\pi^2\Gamma(1-\aleph)},\ b = \frac{t^{2-\aleph}}{12\pi^2\Gamma(1-\aleph)},\ c = \left[\frac{2t^{2-\aleph}}{\Gamma(3-\aleph)} + \frac{25}{12}\frac{t^{2-\aleph}(t^2+1)}{\Gamma(1-\aleph)}\right] sin\pi x sin\pi y.$$

On taking the differential transform on both sides of equation (4.1),

$$\frac{\Gamma(k\aleph+\aleph+1)}{\Gamma(k\aleph+1)}\Phi_{k+1} = \sum_{l=1}^{k} A(l)\frac{\partial^2\Phi(k-l+1)}{\partial x^2} + \sum_{l=1}^{k} B(l)\frac{\partial^2\Phi(k-l+1)}{\partial y^2} + C(k). \quad (4.7)$$

The differential transforms of a, b, c are given as,

$$A(l) = \frac{2}{\pi^2\Gamma(1-\aleph)}\delta\left(l - \frac{2-\aleph}{\aleph}\right),$$

$$B(l) = \frac{1}{12\pi^2\Gamma(1-\aleph)}\delta\left(l - \frac{2-\aleph}{\aleph}\right),$$

$$C(k) = \frac{2}{\Gamma(3-\aleph)}\delta\left(k - \frac{2-\aleph}{\aleph}\right) sin\pi x sin\pi y + \frac{25}{12\Gamma(1-\aleph)}\left[\delta\left(k - \frac{4-\aleph}{\aleph}\right) + \delta\left(k - \frac{2-\aleph}{\aleph}\right)\right] sin\pi x sin\pi y.$$

Thus, the recurrence relation from equation (4.7) can be written as

$$\Phi_{k+1} = \frac{\Gamma(k\aleph+1)}{\Gamma(k\aleph+\aleph+1)}[\sum_{l=1}^{k} A(l)\frac{\partial^2\Phi(k-l+1)}{\partial x^2} + \sum_{l=1}^{k} B(l)\frac{\partial^2\Phi(k-l+1)}{\partial y^2} + C(k)]$$

or

$$\Phi_{ij}^{k+1} = \frac{\Gamma(k\aleph+1)}{\Gamma(k\aleph+\aleph+1)}[\sum_{l=1}^{k} A(l)\sum_{m=1}^{M} a_{im}\Phi_{mj}^{k-l+1} + \sum_{l=1}^{k} B(l)\sum_{m=1}^{N} b_{jm}\Phi_{im}^{k-l+1} + C_{ij}^{k}].$$

Thus, $\Phi_{ij}^2, \Phi_{ij}^3, \Phi_{ij}^4, \ldots$ can be calculated from the above recurrence formula and then by using the inverse DTM, the approximate solution of this specific case of (4.1) can be written as,

$$\phi(x,y,t) = \Phi_{ij}^1 + \Phi_{ij}^2 . t^{\aleph} + \Phi_{ij}^3 . t^{2\aleph} + \ldots .$$

Table 4.1 Error for $M = N = 21, T = 0.1$

$\aleph$	Maximum absolute error
0.1	0.0100
0.25	0.0077
0.4	0.0148
0.5	0.0212
0.75	0.0100
1	0.0900

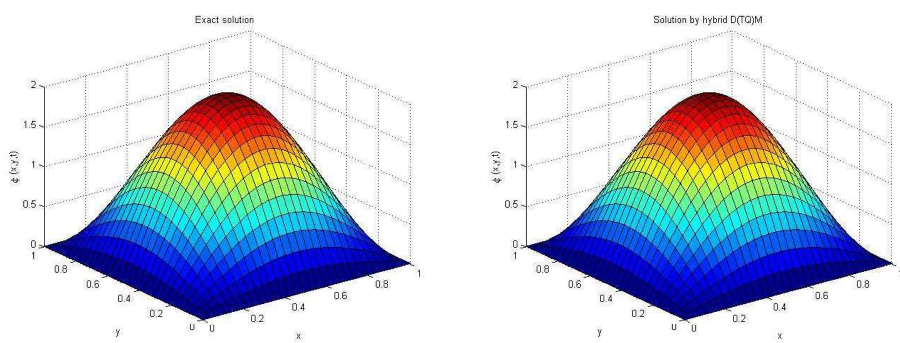


Fig. 4.1 Comparison of obtained and exact solution of diffusion equation with $M = N = 30$, $\aleph$=0.4 and T=1

It can be observed that the for $\aleph = 1$ and 0.4 the two solutions are agreeing and the error is zero. For different values of $\aleph$ the Table 4.3 shows no monotonic nature of the error.

4.3.2 Example

Consider the equation (4.1) with

$$a = 1,\ b = 1, c = 2\left[\frac{t^{2-\aleph}}{\Gamma(3-\aleph)} + t^2\right] sinxsiny.$$

On taking the differential transform on both sides of equation (4.1),

Table 4.2 The error w.r.t. 't' value and the number of terms in the approximate solution by DTM for 21*21 mesh and $\aleph = 0.4$

Number of terms	't'	Maximum absolute error with DTM
3	1	1
	0.5	0.25
	0.25	0.0625
	0.1	0.0100
	0.01	9.9e-05
4	1	1
	0.5	0.25
	0.25	0.0625
	0.1	0.0100
	0.01	9.9e-05
5	1	2.2e-16
	0.5	0.0798
	0.25	0.0463
	0.1	0.0151
	0.01	5.3e-04
6	1	2.2e-16
	0.5	0.0798
	0.25	0.0463
	0.1	0.0151
	0.01	5.3e-04

Table 4.3 The error w.r.t. ’t’ value and the number of terms in the approximate solution by the D(TQ)M method for $M = N = 21$, $\aleph = 0.4$

Number of terms	’t’	Max. absolute error with hybrid method
3	1	1
	0.5	0.25
	0.25	0.0625
	0.1	0.0100
	0.01	9.9e-05
4	1	1
	0.5	0.25
	0.25	0.0625
	0.1	0.0100
	0.01	9.9e-05
5	1	0.0081
	0.5	0.0798
	0.25	0.0463
	0.1	0.0151
	0.01	5.3e-04
6	1	0.0081
	0.5	0.0798
	0.25	0.0463
	0.1	0.0151
	0.01	5.3e-04

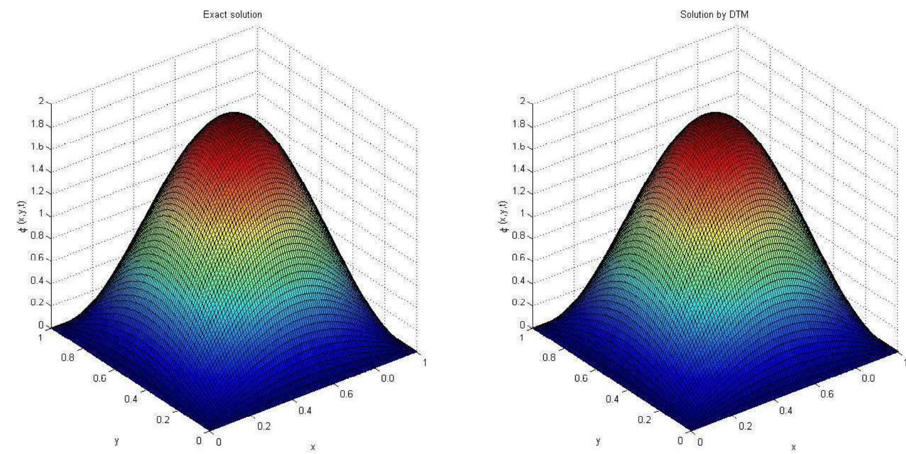


Fig. 4.2 Comparison of obtained and exact solution of diffusion equation with $M = N = 101$, $\aleph$=1 and T=1

$$\frac{\Gamma(k\aleph+\aleph+1)}{\Gamma(k\aleph+1)}\Phi_{k+1} = \sum_{l=1}^{k} A(l)\frac{\partial^2\Phi(k-l+1)}{\partial x^2} + \sum_{l=1}^{k} B(l)\frac{\partial^2\Phi(k-l+1)}{\partial y^2} + C(k). \quad (4.8)$$

The recurrence relation from equation (4.8) can be written as

$$\Phi_{ij}^{k+1} = \frac{\Gamma(k\aleph+1)}{\Gamma(k\aleph+\aleph+1)}\left[\sum_{m=1}^{M} a_{im}\Phi_{mj}^{k} + \sum_{m=1}^{N} b_{jm}\Phi_{im}^{k} + C_{ij}^{k}\right]$$

where

$$C_{ij}^{k} = 2\left[\frac{\delta\left(k-(\frac{2-\aleph}{\aleph})\right)}{\Gamma(3-\aleph)} + \delta\left(k-\frac{2}{\aleph}\right)\right] sinxsiny.$$

Thus, $\Phi_{ij}^2, \Phi_{ij}^3, \Phi_{ij}^4, \ldots$ can be calculated from the above recurrence formula and then by using the inverse DTM, the approximate solution of this specific case of equation (4.1) can be written as,

$$\phi(x,y,t) = \Phi_{ij}^1 + \Phi_{ij}^2.t^{\aleph} + \Phi_{ij}^3.t^{2\aleph} + \ldots.$$

Some error values are mentioned in Table 4.4, 4.5 and 4.6. It is observed that the series obtained due to DTM is absolutely convergent for various values of $\aleph$. However, the theoretical analysis of the convergence of D(TQ)M is to be attempted as a future work.

Table 4.4 The error for different values of ℵ and for different number of non zero terms(K) in the series solution by the DTM (at t=0.5), M= N= 15

ℵ	$K = 3$	$K = 4$	$K = 5$
1	0.2500	0.2500	0.2500
2/3	0.0707	0.0998	0.0880
1/2	0.0129	0.1307	0.0543
2/5	0.0587	0.2015	0.1963

Table 4.5 The error for different values of ℵ and for different number of non zero terms(K) in the series solution by the D(TQ)M (at t=0.5, M= N= 15) in example 4.3.2

ℵ	$K = 3$	$K = 4$	$K = 5$
1	0.75000	0.68762	0.67520
2/3	0.03872	0.35016	0.55380
1/2	0.07775	0.85338	1.5244
2/5	0.18509	1.31846	2.5550

Table 4.6 The error for different values of ℵ and for different number of non zero terms(K) in the series solution by the D(TQ)M (at t=0.01) in example 4.3.2

ℵ	$K = 3$	$K = 4$	$K = 5$
1	0.99990	0.99989	0.99989
2/3	0.93084	0.93093	0.93092
1/2	0.83427	0.83476	0.83478
2/5	0.73638	0.73260	0.73353

From the numerical examples, it can be observed that DTM is an efficient method for solving two dimensional fractional diffusion equation. The hybrid method though, approximates the solution for small values of time as can be seen in Tables 4.2 and 4.3. In the first example, the solution obtained from the hybrid method is same as exact solution for $\aleph = 1$. Moreover as the time is getting smaller, the error is also reducing; for $t < 1$.
In second example, as the value of $\aleph$ is decreasing, the errors are also decreasing in case of the D(TQ)M at time $t = 0.01$. However, this pattern is not there for the greater value of time 't' as well as for the DTM.
Thus, this work is an experiment of combining a semi analytical method with a numerical method. The experiment can be called successful as the results were near to the exact solution for small values of time. However, the method can be explored for its efficiency for large value of time, along with its theoretical analysis.

Chapter 5

Conclusion

This work is about the application of well established methods that have been used to solve the ordinary and partial differential equations, on the fractional differential equations. Though there are many ways to define the fractional derivatives, the Caputo's definition has been used, as it is the most acceptable one because of its characteristics. Differential transform method, a well known analytic method and differential quadrature method, an efficient numerical method have been used to solve the fractional differential equations in one and two dimensions. A hybrid of both these methods is also proposed.

5.1 Results

The first chapter is about various pre requisites. The details of DTM and DQM are explained for one dimensional and two dimensional form of fractional differential equations. A part of the introductory chapter is published in International Journal of Mathematical, Engineering and Management Sciences, in 2018.

In the second chapter the DTM has been used to solve fractional Bagley Torvik equation and fractional relaxation oscillation equation. The obtained numerical results are very near to the available exact solution. Paper on Bagley Torvik equation is published in AIP Conference Proceedings in 2017. The paper on fractional relaxation oscillation equation was presented in CONIAPS XX in 2017.

The analytic methods have a limitation that they can not be applied on any and every fractional differential equation, therefore a numerical method is needed! So in third chapter modified DQM is applied on the fractional Burger's equation and on fractional Fisher's equation. The results are good in terms of the absolute error. This work has been published in Nonlinear Studies in 2019 and in Journal of Physics in 2020, respectively.

The fourth chapter is about the solution of a two dimensional fractional differential equation. The trend with two dimensional fractional differential equation is different as compared to the one dimensional equations. The DTM gives the exact solution in some cases only and DQM gives good result for very small time step. The proposed hybrid method is giving good results for specific cases only for instance $\aleph = 1$ and 0.4 in one of the examples. This work is presented in ICMMAAC-21 and is accepted in Nonlinear Engineering. Modeling and Application on March 28, 2022.
Thus it can be concluded that the methods like DTM and DQM and the hybrid DTM-DQM are easy to understand, use and are effective to solve the fractional differential equations to a decent level of accuracy.

5.2 Future Scope

The future scope of this work includes:

- This work can be tested with the concept of fractional B-splines.
- The experiment with the new definitions of the fractional derivatives can be done.
- While using DQM, the weight coefficients can be found by using any other test function.
- For the discretization of the time derivative, Lubich's approach of third order was used in the third Chapter. The order can be increased and checked for its accuracy for the same differential equation.
- The hybrid method can be tested for various one dimensional fractional differential equations.
- The higher order differential equations can be tried for a solution using the quintic or higher order B-spline.
- For the space fractional differential equations, DQM can be improvised.
- The hybrid method and the DQM for two dimensional fractional differential equations worked well for small values of time only. For large value of time, some modifications can be looked into!
- This work had the experiment based results for the D(TQ)M hybrid method. Its theory and stability can be looked into detail.

Bibliography

[1] K. Miller and B. Ross. An introduction to the fractional calculus and fractional differential Equations. John Wiley and Sons Inc., New York, 1993.

[2] I. Podlubny. Fractional differential equations. Academic Press, San Diego, 1999.

[3] I. Podlubny. Geometric and physical interpretation of fractional integration and fractional differentiation. Fractional calculus and Applied Analysis, 5: 367-386, 2002.

[4] I. Podlubny. Fractional Calculus: Methods for Applications (Lecture notes). XXXVII Summer School on Mathematical Physics Ravello, Italy, September 17-29,2012

[5] B. Ross. Fractional Calculus. Mathematics Magazine, 50(3): 115-122, 1977.

[6] M. Moshrefi- Torbati and J. K. Hammond. Physical and geometrical interpretation of fractional operators. Elsevier Science, Great Britain, 1998.

[7] B. Guo, X. Pu. and F. Huang. Fractional partial differential equations and their numerical solutions. World Scientific Publishing Co., 2015.
8

[8] Y. Zhou. Basic theory of fractional differential equations. World Scientific, 1964.

[9] T. Kisela. Fractional differential equations and their applications. Diploma Thesis, Brno University of Technology, 2008.

[10] K. Diethelm. The analysis of fractional differential equations. An application oriented exposition using differential operators of Caputo type, Lecture notes in Mathematics 2004, Springer, 2010.

[11] C. Lubich. Convolution quadrature and discretized operational calculus II. Numerische Mathematik, 52: 413-425, 1988.

[12] K. Diethelm. An algorithm for the numerical solution of differential equations of fractional order. Electronic transactions on numerical analysis, 5: 1-6, 1997.

[13] K. Diethelm and Y. Luchko. Numerical solution of linear multi-term initial value problems of fractional order. Journal of Computational Analysis and Applications, 6: 243-263, 2004.

[14] L. Blank. Numerical treatment of differential equations of fractional order. Nonlinear World, 4: 473-491, 1997.

[15] R. Gorenflo. Fractional calculus: Some numerical methods. CISM lecture notes, 1996.

[16] K. Diethelm, N.J. Ford and A.D. Freed. Detailed error analysis for a fractional Adams method. Numerical Algorithms, 36: 31-52, 2004.

[17] W. H. Deng. Numerical algorithm for the time fractional Fokker- Plank equation. Journal of Computational Phyics, 227:1510-1522, 2007.

[18] L. Zhao and W. H. Deng. Jacobi-predictor-corrector approach for the fractional ordinary differential equations. arXiv:1201.5952v2, 2012.

[19] N. J. Ford, M. L. Morgado and M. Rebelo. Nonpolynomial collocation approximation of solutions to fractional differential equations. Fractional Calculus and Applied Analysis, 18: 874-891, 2013.

[20] Y. Yan, K. Pal and N. J. Ford. Higher order numerical methods for solving fractional differential equations. BIT Numerical Mathematics, 54: 555-584, 2014.

[21] W. Wyss. The fractional diffusion equation. Journal of Mathematical Physics, 27(11):2782–2785, 1986

[22] W. R. Schneider and W. Wyss. Fractional diffusion and wave equations. Journal of Mathematical Physics, 30(1):134–144, 1989

[23] R. Gorenflo, Yu. Luchko, and F. Mainardi. Wright functions as scale-invariant solutions of the diffusion-wave equation. Journal of Computational and Applied Mathematics, 118(1-2):175–191, 2000.

[24] O.P. Agrawal. Solution for a fractional diffusion-wave equation defined in a bounded domain. Nonlinear Dynamics, 29:145–155, 2002.

[25] F. Mainardi, A. Vivoli, R. Gorenflo, W. Ebeling, and B. Spagnolo. Continuous time random walk and time fractional diffusion: A numerical comparison between the fundamental solutions. Fluctuation and Noise Letters, 5(2):L291–L297, 2005.

[26] J. Chen, F. Liu, and V. Anh. Analytical solution for the time-fractional telegraph equation by the method of separating variables. Journal of Mathematical Analysis and Applications, 338(2):1364–1377, 2007.

[27] Y. Luchko and R. Gorenflo. An operational method for solving fractional differential equations with the Caputo derivatives. Acta Math. Vietnamica, 24(2):207233, 1999.

[28] R. Gorenflo and F. Mainardi. Random walk models for space-fractional diffusion processes. Fractional Calculus and Applied Analysis, 1(2):167–191, 1998

[29] Q. Liu, F. Liu, I. Turner, and V. Anh. Approximation of the L´evy-Feller advection dispersion process by random walk and finite difference method. Journal of Computational Physics, 222(1):57–70, 2007.

[30] W. Deng. Finite element method for the space and time fractional Fokker-Planck equation. SIAM J. Numer. Anal., 47(1):204–226, 2008.

[31] P. Kumar and O.P. Agrawal. An approximate method for numerical solution of fractional differential equations. Signal Processing, 86(10):2602 – 2610, 2006.

[32] H. Jafari and V. Daftardar- Gejji. Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition. Applied Mathematics and Computation, 180(2):488–497, 2006.

[33] J. Erhel, K. Burrage, and B. Pohl. Restarted GMRES preconditioned by deflation. Journal of Computational and Applied Mathematics, 69(2):303–318, 1996.

[34] D. Fulger, E. Scalas, and G. Germano. Monte Carlo simulation of uncoupled continuous time random walks yielding a stochastic solution of the space-time fractional diffusion equation. Physical Review E, 77(2):21122, 2008.

[35] M. Ili´c, F. Liu, I. Turner, and V. Anh. Numerical approximation of a fractional-in-space diffusion equation. Fractional Calculus and Applied Analysis, 8(3):323–341, 2005.

[36] R. Gorenflo and F. Mainardi. Random walk models for space-fractional diffusion processes. Fractional Calculus and Applied Analysis, 1(2):167–191, 1998.

[37] R. Gorenflo and F. Mainardi. Approximation of L´evy-Feller diffusion by random walk. Journal for Analysis and its Applications, 18(2):231–246, 1999.

[38] F. Liu, V. Anh, and I. Turner. Numerical solution of the space fractional Fokker-Planck equation. Journal of Computational and Applied Mathematics, 166(1):209–219, 2004.

[39] C. Tadjeran, M.M. Meerschaert, and H.P. Scheffler. A second-order accurate numerical approximation for the fractional diffusion equation. Journal of Computational Physics, 213(1):205–213, 2006.

[40] P. Zhuang and F. Liu. Implicit difference approximation for the time fractional diffusion equation. Journal of Applied Mathematics and Computing, 22(3):87–99, 2006.

[41] S.B. Yuste and L. Acedo. An explicit finite difference method and a new von Neumann type stability analysis for fractional diffusion equations. SIAM Journal on Numerical Analysis, 42(5):1862–1875, 2005.

[42] S.B. Yuste. Weighted average finite difference methods for fractional diffusion equations. Journal of Computational Physics, 216(1):264–274, 2006.

[43] K. Diethelm, N.J. Ford, and A.D. Freed. A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dynamics, 29(1-4):3–22, 2002.

[44] K. Diethelm, N.J. Ford, and A.D. Freed. Detailed error analysis for a fractional Adams method. Numerical Algorithms, 36(1):31–52, 2004.

[45] F. Liu, P. Zhuang, V. Anh, and I. Turner. A fractional-order implicit difference approximation for the space-time fractional diffusion equation. ANZIAM J., 47(E):48–68, 2006.

[46] F. Liu, P. Zhuang, V. Anh, I. Turner, and K. Burrage. Stability and convergence of the difference methods for the space-time fractional advection-diffusion equation. Applied Mathematics and Computation, 191(1):12–20, 2007.

[47] R. Gorenflo and E.A. Abdel-Rehim. Convergence of the Gr¨unwald-Letnikov scheme for time-fractional diffusion. Journal of Computational and Applied Mathematics, 205(2):871–881, 2007.

[48] Y. Lin and C. Xu. Finite difference/spectral approximations for the time-fractional diffusion equation. Journal of Computational Physics, 225(2):1533–1552, 2007.

[49] P. Zhuang, F. Liu, I. Turner, and V. Anh. Numerical treatment for the fractional Fokker-Planck equation. ANZIAM Journal, 48:759–774, 2007.

[50] I. Podlubny, A. Chechkin, T. Skovranek, Y.Q. Chen, and B.M. Vinagre Jara. Matrix approach to discrete fractional calculus II: Partial fractional differential equations. Journal of Computational Physics, 228(8):3137–3153, 2009.

[51] G.J. Fix and J.P. Roop. Least squares finite-element solution of a fractional order two point boundary value problem. Computers and Mathematics with Applications, 48(7-8):1017–1033, 2004.

[52] V.J. Ervin and J.P. Roop. Variational solution of the fractional advection dispersion equation. Numerical Methods for Partial Differential Equations, 22(3):558 – 576, 2006.

[53] O.P. Agrawal. A numerical scheme for initial compliance and creep response of a system. Mechanics Research Communications, 36(4):444 – 451, 2009.

[54] L. Yuan and O.P. Agrawal. A numerical scheme for dynamic systems containing fractional derivatives. Journal of Vibration and Acoustics, 124(2):321–324, 2002.

[55] H. Jafari and V. Daftardar-Gejji. Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition. Applied Mathematics and Computation, 180(2):488–497, 2006.

[56] S. Momani and Z. Odibat. Numerical solutions of the space-time fractional advection dispersion equation. Numerical Methods for Partial Differential Equations, 24(6):1416–1429, 2008.

[57] K. Al-Khaled and S. Momani. An approximate solution for a fractional diffusion-wave equation using the decomposition method. Applied Mathematics and Computation, 165(2):473–483, 2005.

[58] M. Ili´c, F. Liu, I. Turner, and V. Anh. Numerical approximation of a fractional-in-space diffusion equation. Fractional Calculus and Applied Analysis, 8(3):323–341, 2005.

[59] J. P. Roop. Computational aspects of FEM approximation of fractional advection dispersion equations on bounded domains in R2. Journal of Computational and Applied Mathematics, 193(1):243–268, 2006.

[60] M. M. Meerschaert, H. P. Scheffler, and C. Tadjeran. Finite difference methods for two-dimensional fractional dispersion equation. Journal of Computational Physics, 211(1):249–261, 2006.

[61] P. Zhuang and F. Liu. Finite difference approximation for two-dimensional time fractional diffusion equation. Journal of Algorithms and Computational Technology, 1(1):1–15, 2007.

[62] C. Tadjeran and M. M. Meerschaert. A second-order accurate numerical method for the two-dimensional fractional diffusion equation. Journal of Computational Physics, 220(2):813–823, 2007.

[63] J. K. Zhou. Differential transformation and its applications for electrical circuits(in Chinese). Huazhong University Press, Wuhan, P. R. China, 1986.

[64] A. Arikoglu and I.Ozkol. Solution of fractional differential equations by using differential transform method. Chaos, Solitons and Fractals, 34(5):1473–1481, 2007.

[65] G. Arora and Pratiksha. Solution of the Bagley Torvik equation by fractional DTM. AIP Conference Proceedings, 1860, 020032 (2017)

[66] Y. Keskin and G. Oturanc. Reduced differential transform method for partial differential equations. International Journal of Nonlinear Sciences and Numerical Simulation, 10(6): 741-749, 2009.

[67] K. Aruna and A. S. V. Ravi Kanth. Two-dimensional differential transform method and modified differential transform method for solving nonlinear fractional Klein–Gordon equation. National Academy Science Letters, 37(2): 163-171, 2014.

[68] S. Abuasad, A. Yildirim, I. Hashim, S. A. A. Karim and J.F. Gómez-Aguilar. Fractional multi-step differential transform method for approximating a fractional stochastic SIS epidemic model with imperfect vaccination. International Journal of Environmental Research and Public Health, 16, 973, 2019.

[69] V. K. Srivastava, M. K. Awasthi, S. Kumar. Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method. Egyptian Journal of Basic and Applied Sciences. Egyptian Journal of Basic and Applied Sciences, 1(1): 60-66, 2014.

[70] G. Arora and Pratiksha. A Cumulative Study on Differential Transform Method. International Journal of Mathematical, Engineering and Management Sciences, 4(1): 170–181, 2019.

[71] Yücel Çenesiz, Yıldıray Keskin and Aydın Kurnaz. The solution of the Bagley–Torvik equation with the generalized Taylor collocation method. Journal of the Franklin Institute, 347: 452-466, 2010.

[72] R. L. Bagley, P. J. Torvik. On the appearance of the fractional derivative in the behavior of real materials. Applied Mechanics, 51: 294-298, 1984.

[73] C. Leszczynski and M. Ciesielski. A numerical method for solution of ordinary differential equations of fractional order. Parallel Processing and Applied Mathematics, 2328: 695-702, 2002.

[74] V. S. Ertürk and S. Momani. Solving systems of fractional differential equations using differential transform method. Journal of Computational and Applied Mathematics, 215: 142-151, 2008.

[75] A. M. A. El-Sayed, A. E. M. El-Mesiry, H. A. A. El-Saka. Numerical solution for multi-term fractional (arbitrary) orders differential equations. Computational and Applied Mathematics, 23: 33-54, 2004.

[76] J. T. Edwards, N. J. Ford, A. C. Simpson. The numerical solution of linear multi-term fractional differential equations:systems of equations. Journal of Computational and Applied Mathematics, 148: 401-448, 2002.

[77] K. Diethelm and N. J. Ford. Numerical solution of the Bagley -Torvik equation. BIT, 42: 490–507, 2002.

[78] V. Daftardar-Gejji and H. Jafari. Adomian decomposition: A tool for solving a system of fractional differential equations. Journal of Mathematical Analysis and Applications, 301: 508–518, 2005.

[79] S. S. Ray and R. K. Bera. Analytical solution of the Bagley–Torvik equation by Adomian decomposition method. Applied Mathematics and Computation, 168: 398-410, 2005.

[80] T. Mekkaoui and Z. Hammouch. Approximate analytical solutions to the Bagley-Torvik equation by the fractional iteration method. Annals of the University of Craiova, Mathematics and Computer Science Series, 39: 251-256, 2012.

[81] B. Van Der Pol and J. Van Der Mark. The heartbeat considered as a relaxation oscillator, and an electrical model of the heart. Philosophical Magazine, 6: 763-775 1928.

[82] B. Van Der Pol. On relaxation oscillations. Philosophical Magazine, 2: 978-992, 1926.

[83] F. Mainardi. Fractional relaxation-oscillation and fractional diffusion-wave phenomena. Chaos, Solitons & Fractals, 7: 1461-1477, 1996.

[84] Y. Hu, Y. Luo and Z. Lu. Analytical solution of the linear fractional differential equation by Adomian decomposition method. Journal of Computational and Applied Mathematics, 215: 220 – 229, 2008.

[85] R. S. Chandel, A. Singh and D. Chouhan. Numerical solution of fractional relaxation oscillation equation using cubic B-spline wavelet collocation method. Italian Journal of Pure and Applied Mathematics, 36: 399-414, 2016.

[86] V. S. Ertürk, S. Momani and Z. Odibat. Application of generalized differential transform method to multi-order fractional differential equations. Communications in Nonlinear Science and Numerical Simulation, 13: 1642-1654, 2008.

[87] Z. M. Odibat, S. Kumar, N. Shawagfeha, A. Alsaedi and T. Hayat. A study on the convergence conditions of generalized differential transform method. Mathematical Methods in Applied Sciences, 40: 40–48, 2016.

[88] A. Korkmaz and I. Dag. Cubic B-spline differential quadrature methods and stability for Burgers' equation. Engineering Computations, 30(3): 320-344, 2013.

[89] A. Bashan. An effective application of differential quadrature method based on modified cubic B-splines to numerical solutions of the KdV equation. Turkish Journal of Mathematics, 42: 373-394, 2018.

[90] A. Bashan, N. M. Yagmurlu, Y. Ucar and A. Esen. An effective approach to numerical soliton solutions for the Schrödinger equation via modified cubic B-spline differential quadrature method. Chaos, Solitons and Fractals, 100: 45-56, 2017.

[91] A. Bashan, S. B. G. Karakoc and T. Geyikli, B-spline differential quadrature method for the modified Burgers' equation. Cankaya University Journal of Science and Engineering, 12(1): 001-013, 2015.

[92] A. Korkmaz and I. Dag. Polynomial based differential quadrature method for numerical solution of nonlinear Burgers' equation. Journal of the Franklin Institute, 348: 2863-2875, 2011.

[93] G. Arora and V. Joshi, A computational approach using modified trigonometric cubic B-spline for numerical solution of Burgers' equation in one and two dimensions. Alexandria Engineering Journal, 57: 1087-1098, 2018.

[94] M. P. Bonkile, A. Awasthi, C. Lakshmi, V. Mukundan and V. S. Aswin. A systematic literature review of Burgers' equation with recent advances. Pramana- Journal of Physics, 90:69, 2018.

[95] P. Miškinis. Some properties of fractional Burgers' equation. Mathematical Modelling and Analysis, 7(1): 151-158, 2002.

[96] J. Biazar and H. Aminikhah. Exact and numerical solutions for non-linear Burger's equation by VIM. Mathematical and Computer Modelling, 49(7-8): 1394-1400, 2009.

[97] K. M. Saad and E. HF Al-Sharif. Analytical study for time and time-space fractional Burgers' equation. Advances in Difference Equations, 2017:300, 2017.

[98] S. Momani. Non-perturbative analytical solutions of the space- and time-fractional Burgers' equations. Chaos, Solitons and Fractals, 28(4): 930-937, 2006.

[99] A. Esen and O. Tasbozan. Numerical solution of time fractional Burgers' equation by cubic B-spline finite elements. Mediterranean Journal of Mathematics, 13(3): 1325-1337, 2016.

[100] A. Yokus and D. Kaya. Numerical and exact solutions for time fractional Burgers' equation. Journal of Nonlinear Sciences and Applications, 10: 3419-3428, 2017.

[101] M. Chen and W. Deng. WSLD operators II: the new fourth order difference approximations for space Riemann-Liouville derivative. ArXiv e-prints, arXiv:1306.5900 (math.NA), 2013.

[102] F. Tornabene, N. Fantuzzi, F. Ubertini and E. Viola. Strong formulation finite element method based on differential quadrature: A survey. Applied Mechanics Reviews, 67: 020801, 2015.

[103] R. Bellman, B. G. Kashef and J. Casti. Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations. Journal of Computational Physics, 10(1): 40-52, 1972.

[104] I. J. Schoenberg. On trigonometric spline interpolation. Journal of Mathematics and Mechanics, 13: 795-825, 1964.

[105] X. G. Zhu, Y. F. Nie, W. W. Zhang. An efficient differential quadrature method for fractional advection-diffusion equation. Nonlinear Dynamics, 90: 1807-1827, 2017.

[106] M. K. Vaid and G. Arora. Solution of second order singular perturbed delay differential equation using trigonometric B-spline. International Journal of Mathematical, Engineering and Management Sciences, 4(2): 349-360, 2019.

[107] R. C. Mittal and R. K. Jain. Numerical solutions of non linear Burgers' equation with modified cubic B-splines collocation method. Applied Mathematics and Computation, 218(15): 7839-7855, 2012.

[108] C. W. Bert and M. Malik. Differential quadrature method in computational mechanics. Applied Mechanics Review, 49(1): 1-28, 1996.

[109] S. Tomasiello. Stability and accuracy of the iterative differential quadrature method. International Journal for Numerical Methods in Engineering, 58(9): 1277-1296, 2003.

[110] G. D. Smith. Numerical solution of partial differential equations. Oxford University Press, 1985.

[111] M. Ghasemi. High order approximations using spline-based differential quadrature method: Implementation to the multi-dimensional PDEs. Applied Mathematical Modelling, 46: 63-80, 2017.

[112] J. W. Daniel and B. K. Swartz. Extrapolated collocation for two-point boundary-value problems using cubic splines. Journal of the Institute of Mathematics and its Applications, 16: 161-174, 1975.

[113] C. Shu and B. E. Richards. Application of generalized differential quadrature to solve two-dimensional incompressible Navier Stoke's equations. International Journal for Numerical Methods in Fluids, 15: 791-798, 1992.

[114] A. Korkmaz and I. Dag. Crank-Nicolson – differential quadrature algorithms for the Kawahara equation. Chaos, Solitons and Fractals, 42: 65–73, 2009.

[115] A. Esen and O. Tasbozan. Numerical solution of time fractional Burgers' equation. Acta Universitatis Sapientiae, Mathematica, 7: 167-185, 2015.

[116] V. Isakov. Inverse Problems for Partial Differential Equations. Applied Mathematical Sciences, 127(2), 2006.

[117] H. Kheiri, A. Mojaver and S. Shahi. Analytical solutions for the fractional Fisher's equation. Sahand Communications in Mathematical Analysis, 2(1): 27-49, 2015.

[118] J. Sabatier, O. P. Agrawal and J. A. T. Machado. Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht, The Netherlands, 2007.

[119] M. M. Khader and K. M. Saad. A numerical approach for solving the problem of biological invasion (fractional Fisher equation) using Chebyshev spectral collocation method. Chaos Solitons Fractals, 110: 169-77, 2018.

[120] M. M. Khader. On the numerical solutions for the fractional diffusion equation. Communications in Nonlinear Science and Numerical Simulation, 16: 2535–2542, 2011.

[121] C. Tadjeran, M. M. Meerschaert and H. A. Scheffler. A second-order accurate numerical approximation for the fractional diffusion equation. Journal of Computational Physics. 213(1): 205-213, 2006.

[122] M. M. Meerschaert, H. Scheffler and C. Tadjeran. Finite difference methods for two-dimensional fractional dispersion equation. Journal of Computational Physics, 211(1): 249-261, 2006.

[123] G. H. Zheng and T. Wei. A new regularization method for solving a time-fractional inverse diffusion problem. Journal of Mathematical Analysis and Applications, 378(2): 418-431, 2011.

[124] M. Chen and W. Deng. WSLD operators II: the new fourth order difference approximations for space Riemann-Liouville derivative. ArXiv e-prints, arXiv:1306.5900 (math.NA)), 2013.

[125] R. Bellman, B. G. Kashef and J. Casti. Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations. Journal of Computational Physics, 10(1): 40-52, 1972.

[126] A. Korkmaz and I. Dag. Polynomial based differential quadrature method for numerical solution of nonlinear Burgers' equation. Journal of the Franklin Institute, 348: 2863-2875, 2011.

[127] P. K. Maini, D. L. S. Mcelwain and D. Leavesley. Travelling waves in a wound healing assay. Applied Mathematics Letters, 17: 575-580, 2004.

[128] H. Zeidabadi, R. Pourgholi and S. H. Tabasi. A hybrid scheme for time fractional inverse parabolic problem, Waves in Random and Complex Media, 30(2): 354-368, 2020.

[129] R. Pourgholi, S. H. Tabasi and H. Zeidabadi. Numerical techniques for solving system of nonlinear inverse problem. Engineering with Computers, 2017. https://doi.org/10.1007/s00366-017-0554-6.

[130] G. Arora and Pratiksha. Solution of fractional Burgers' equation using advanced differential quadrature method. Nonlinear Studies, 26(3): 1-16, 2019.

[131] M. Mehra and K. S. Patel. Algorithm 986: A suite of compact finite difference schemes. ACM Transactions on Mathematical Softwares, 44(2): Article 23, 2017.

[132] G. Gao, Z. Sun. Two alternating direction implicit difference schemes for two-dimensional distributed order fractional diffusion equations. Journal of Scientific Computing, 66: 1281-1312, 2016.

[133] P. Zhuang, F. Liu. Implicit difference approximation forthe two-dimensional space-time fractional diffusion equation. Journal of Applied Mathematics and Compution, 5(1–2): 269-282, 2017.

[134] X. G. Zhu, Y. F. Nie, W. W. Zhang. An efficient differential quadrature method for fractional advection-diffusion equation. Nonlinear Dynamics, 90: 1807-1827, 2017.

[135] A. M. Nagy and A. A. El-Sayed. An accurate numerical technique for solving two-dimensional time fractional order diffusion. International Journal of Modelling and Simulation, 39: 214-221, 2019.

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3.	Solution of fractional Burgers' equation using advanced differential quadrature method.	Nonlinear Studies	27 August 2019	1359-8678/ Vol. 26, No. 3
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