A STUDY ON PRIME CORDIAL LABELING AND DIVISOR CORDIAL LABELING OF GRAPHS

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By

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The dissertation is fit for the submission and the partial fulfilment of the conditions for the award of DOCTOR OF PHILOSOPHY in Mathematics.

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Abstract

A STUDY ON PRIME CORDIAL LABELING AND DIVISOR CORDIAL LABELING OF GRAPHS

by Vishally Sharma

Let $G(V(G), E(G))$ be the graph with a non empty vertex set $V(G)$ and edge set $E(G)$. An assignment of integers to the vertices (edges) of a graph G under some constraints is called a vertex labeling (edge labeling) of G. Most graph labelings finds their origins to those presented by Alexander Rosa in his famous paper titled "On certain valuations of the vertices of a graph" at International symposium held in Rome, in July 1966. Rosa identified three types of labelings; α, β, and ρ labeling. Rosa called a function f a β-valuation of G with q edges if f is an injection from $V(G)$ to the set $\{0, 1, 2, ..., q\}$ such that, when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. Golomb, later called β- labeling as graceful labeling. Graph labeling is one of the most important concept in graph theory as it plays a vital role in different domains especially, computer science and communication networks. The known applications are in coding theory, $x-$ ray crystallography, computer network security, global mobile communication, radar, circuit design, astronomy etc., yet finding the exclusive applications of a particular graph labeling is still an open area of research. There are numerous graph labeling techniques which are introduced and studied for various real life applications, one of them is a cordial labeling which is actually a weaker version of graceful and harmonious labeling. From time to time many variants of cordial labeling have been explored, a few notable ones are prime cordial labeling and divisor cordial labeling. A graph G is said to admit a prime cordial labeling if there exists a bijection $f : V(G) \rightarrow \{1, 2, 3, ..., |V(G)|\}$ defined by the induced function $f^* : E(G) \rightarrow \{0, 1\}$ such that if, $f^*(uv) = 1$ if $gcd(f(u), f(v)) = 1$ and $f^*(uv) = 0$ if $gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and 1 differ by at most 1 i.e; $|e_f(0) - e_f(1)| \leq 1$. If a graph admits a prime cordial labeling, then it is called a prime cordial graph. Similarly, a graph G is said to admit a divisor cordial labeling if there exists a bijection $f : V(G) \rightarrow \{1, 2, 3, ..., |V(G)|\}$ defined by the induced function $f^* : E(G) \rightarrow \{0, 1\}$ such that if $f^*(uv) = 1$ if $f(u)|f(v)$ or $f(v)|f(u)$ and $f^*(uv) = 0$ otherwise, then $|e_f(0) - e_f(1)| \leq 1$. If a graph admits a divisor cordial labeling, then it is called a divisor cordial graph. Though a significant work has been done concerning prime cordial labeling and divisor cordial labeling, yet a complete characterization of these labelings

is still pending and they are the area of high interest. In pursuant of obtaining the complete characterization of these labeling, researchers around the globe have established the prime cordial and divisor cordial labeling for various classes of graphs and also in the context of some graph operations such as join, union, intersection, disjoint union, subdivision, vertex switching, duplication, Cartesian product, etc.

In the proposed work, a complete characterization has been done partially concerning these labeling besides formulating some interesting conjectures and open problems as a future direction of research. Further, two new variants of divisor cordial labeling are also introduced and certain interesting results are established. Thus the thesis titled "**A Study on Prime Cordial Labeling and Divisor Cordial Labeling of Graphs**" deals with the following objectives:-

1. Deriving certain new classes of prime cordial graphs.
2. Obtaining some new classes of divisor cordial graphs.
3. Establishing the prime cordial labeling and divisor cordial labeling in the context of extension of vertices in graphs.
4. Introducing and studying new variants of divisor cordial labeling for various classes of graphs.

The thesis consists of six chapters where in chapter 1, a general overview of graph theory and graph labeling has been given. Specifically, the literature review section recalls some notable established results on prime cordial labeling and divisor cordial labeling of graphs. Based on the review of literature, research gap has been identified and some realistic objectives are proposed. This chapter also describes the relevant concepts, terminologies, and mathematical preliminaries used throughout the study undertaken.
In chapter 2, certain general results on prime cordial labeling of graphs are established. Some new results for graph operation named, corona, are investigated for prime cordial labeling. Finally, a family of tree, named, lilly graph has been investigated under different graph operations of high interest like duplication, degree splitting, subdivision, and vertex switching.
In chapter 3, prime cordial labeling has been investigated for some standard graphs like, path, cycle, wheel, gear, helm, flower, fan, double fan, star, bistar etc., in the context of graph operations namely, extension of vertex and vertex duplication.
In chapter 4, some general results on divisor cordial labeling of graphs are investigated. Some well known graphs are explored for divisor cordial labeling under the graph operation called corona. Further, lilly graph, classes of planar graphs Pl_m and $Pl_{m,n}$ are

also studied under different graph operations for divisor cordial labeling.

In chapter 5, divisor cordial labeling of certain graphs viz; path, cycle, wheel, gear, helm, flower, fan, double fan, star, bistar etc., in the context of graph operations namely, extension of vertex and vertex duplication are established.

In chapter 6, to further enrich the discipline, two new variants of divisor cordial labeling, namely, double divisor cordial labeling and average even divisor cordial labeling are introduced and investigated for various graphs.

In the end, the study undertaken has been justified by an elaborative bibliography given in the concluding part of the thesis.

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First of all I would like to thank Almighty God without whose will I couldnot able to be, what I am.
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December, 2022 Vishally Sharma

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List of Symbols and Abbreviations

$G(p,q)$ - Graph having p vertices and q edges

P_n - Path graph

C_n - Cycle graph

f_n - Fan graph

Df_n - Double fan graph

G_n - Gear graph

W_n - Wheel graph

H_n - Helm graph

Fl_n - Flower graph

$Gl(n)$ - Globe graph

F_n - Friendship graph

$\bar{G}$ - Complement of graph G

$S(G)$ - Subdivision of graph G

$\mu(G)$ - Myceilskian graph of G

$S(G)$ - Subdivision of graph G

$DS(G)$ - Degree splitting graph of G

$S'(G)$ - Splitting graph of G

$M(G)$ - Middle graph of G

$T(G)$ - Total graph of G

TL_n - Triangular Ladder graph

K_n - Complete graph

$K_{m,n}$ - Complete bipartite graph

$D_2(G)$ - Shadow graph of G

$N(v)$ - Neighbourhood of a vertex v

$N[v]$ - Closed neighbourhood of a vertex v

$G_1[G_2]$ - Composition of graphs G_1 and G_2

G^2 - Square graph of G

S_k - Stack of book graph

$T^{(n)}$ - Tree with n edges

T_n - Full binary tree having n levels

$T_{(n,m)}$ - Full $n-ary$ tree with m levels

$\delta(G)$ - Minimum degree of G

$\Delta(G)$- Maximum degree of G

$\lceil x \rceil$ - Ceiling function of x

$\lfloor x \rfloor$ - Floor function of x

$\mathbb{N}$ - Set of natural numbers

$\mathbb{Z}$ - Set of integers

$\mathbb{R}$ - Set of real numbers

$V(G)$ - Vertex/node set of G

$E(G)$ - Edge set of G

$|V(G)|$ - Cardinality of vertex set of G

$|E(G)|$ - Cardinality of Edge set of G

PCL - Prime cordial labeling

PCG - Prime cordial graph

DCL - Divisor cordial labeling

DCG - Divisor cordial graph

DDCL - Double divisor cordial labeling

DDCG - Double divisor cordial graph

AEDCL - Average even divisor cordial labeling

AEDCG - Average even divisor cordial graph

Chapter 1

Introduction

In this chapter, a brief and concise introduction to graph theory and graph labeling along with a few notable applications are given. Basic definitions and terminologies are also presented to understand the study under taken. The main theme of the thesis is presented along with a short history and broad review of literature. Based on the review of literature, the research gap is identified and consequently objectives of the present work are proposed.

1.1 Introduction to Graph Theory

A graph is informally a pictorial representation of any real life phenomena where the objects are treated as nodes and the relations among them are the edges. A study about graphs and their properties is called graph theory. One of the more fascinating uses of graph theory is in "biology and conservation where a node might represent a location where particular species live and the edges can reflect migration patterns or movements between the areas." This knowledge is crucial for knowing the breeding patterns, sickness, parasite distribution, or how changes in migration affect other species. Graph theory has a long history that dates back to 1735, when L. Euler solved the famous "Konigsberg bridge problem". It was an old puzzle in which the goal was to discover a path that walked over each of the seven bridges that were over a forked river running past an island without crossing any of them more than once. According to Euler, no such path exist. His solution to the riddle involved only references to the physical arrangement of the bridges (see Figure 1.1), yet he established graph theory's first theorem.

Though graph theory was initially used to answer fun puzzles, but now has emerged as a prominent field of multidisciplinary research due to its vast variety of applications

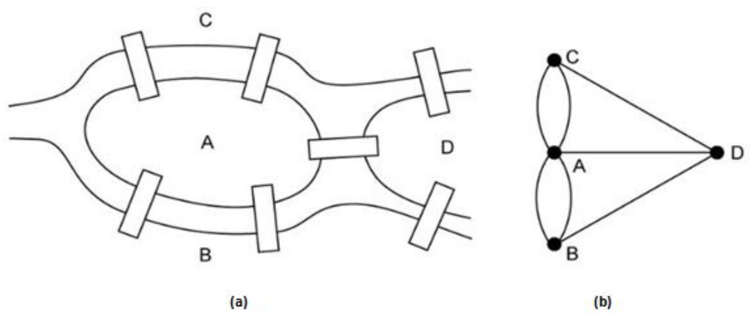

FIGURE 1.1: (a)Konigsberg bridge problem (KBG) (b)Graphical representation of KBG

in different fields. For instance, "in Computer Science, the link structure of a website can be represented by a graph in which the nodes represent web pages and directed edges represent links from one page to another. Similarly, in Chemistry, a graph is a natural representation of a molecule, with nodes representing atoms, and edges representing bonds. This technology comes in handy in computer-assisted molecular structure processing, which encompasses anything from chemical editors to database searches. Interestingly, though mathematical modeling in Organic Chemistry originates from many branches of Mathematics, yet a special emphasis is given to Chemical graph theory (CGT) [13] that applies graph theory to mathematical modeling of chemical phenomena. Moreover, in Statistical Physics, graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems. Micro-scale channels of porous medium are also represented using graphs, with nodes representing pores and edges representing smaller channels linking the pores [54]." Algebraic graph theory [12] emerged from the investigation of graphs with high symmetry. It studies various classes of graphs with reference to certain properties of automorphism groups, such as distance transitive graphs, vertex-transitive graphs, edge-transitive graphs, semi-symmetric graphs, etc. Likewise, one can see that graph theory has a numerous applications in other branches of Mathematics, Science, Engineering & Technology, Communication Networks, and real-world problems [14, 49].

1.1.1 Preliminaries

The fundamental definitions, results and concepts discussed in this subsection are very essential for the study undertaken and are mainly given by Harary [34] and Bondy and Murthy [14]. Terms vertices, nodes and points mean the same. Similarly, edges, lines and arcs mean the same.

Definition 1.1.1. "An undirected graph $G(V, E, \phi)$ consists of a non-empty set $V(G) = \{v_1, v_2, ..., v_n\}$ called nodes and another set $E(G) = \{e_1, e_2, ..., e_n\}$ called edges and an incidence function ϕ that associates with each edge of G an unordered pair of nodes of G, not necessarily distinct. ϕ is not mentioned explicitly as it can be understood from edges. If e is an edge and u, v are nodes such that $\phi(e) = uv$, then e is said to be formed by joining u and v. The ends of e are u and v.

By G, one means $G(V, E)$ or $G(V(G), E(G))$. The set of vertices and edges of G are denoted by $V(G)$ and $E(G)$ respectively. Also $|V(G)|$ and $|E(G)|$ denote the order and size of G respectively. A graph of order p and size q is oftenly called a (p, q)-graph. Graph G is called simple, if u and v are distinct, for every edge uv (see Figure 1.3). A graph $G(V, E)$ is said to be a finite graph if V and E are finite sets. An infinite graph is the one with an infinite set of V or E or both. A graph with just one node is called trivial and all other graphs non-trivial. For an edge $e = uv$ of G, u and v are adjacent and each is incident with e. If two distinct edges are incident with a common node, then they are said to be adjacent edges. Two or more edges associated with a given pair of nodes of G are called parallel edges. An edge of G associated with a node pair (v_i, v_i) is called a self-loop or loop. A graph is multigraph if no loops are allowed but parallel edges can be there. If both are permitted then the graph is called a pseudograph. For a node $v \in G$, the degree $d_G(v)$ or simply, $d(v)$ is the number of edges of G that are incident with v. The maximum degree of G denoted by $\Delta(G)$ is the degree of the node with the greatest number of edges incident to it whereas the minimum degree of G denoted by $\delta(G)$, is the degree of the node with the least number of edges incident to it. A node of degree zero and one are respectively known as an isolated node and pendant node of G. In G, if $d_G(v) = k$, $\forall\ v \in V(G)$, then G is called a $k-regular$ graph. A regular graph is a graph which is $k-regular$ for some k. Neighbourhood or open neighbourhood $N(u)$ of a node $u \in V(G)$ is the set of all the nodes which are adjacent to u. $N[u]$ denotes a closed neighbourhood of u is a set containing u and $N(u)$."

Definition 1.1.2. "A directed graph or digraph is a graph in which each edge has a direction, thus the edges v_1v_2 and v_2v_1 are not same (see Figure 1.2)."

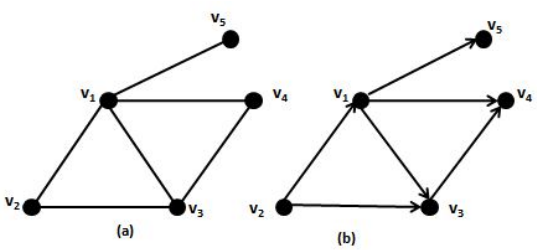

FIGURE 1.2: (a) Undirected graph (b) Directed graph

In the Figure 1.4, one can observe the following:-
(i) G is not simple.

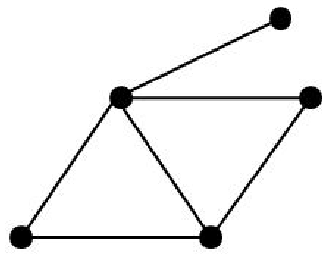

FIGURE 1.3: Simple Graph

(ii) $a = v_1v_3$, $b = v_3v_3$, $c = v_3v_4$, $e = d = v_1v_2$, $f = v_2v_3$.

(iii) $|V(G)| = 5$ and $|E(G)| = 6$.

(iv) b is a loop whereas d, e are parallel edges.

(v) v_1 and v_2 are adjacent nodes whereas v_1 and v_4 are not.

(v) a and f are incident edges (d and c are not incident).

(vi) $d(v_3) = 5$, $d(v_1) = d(v_2) = 3$, $d(v_4) = 1$ and $d(v_5) = 0$.

(vii) v_5 is an isolated node whereas v_4 is a pendant node.

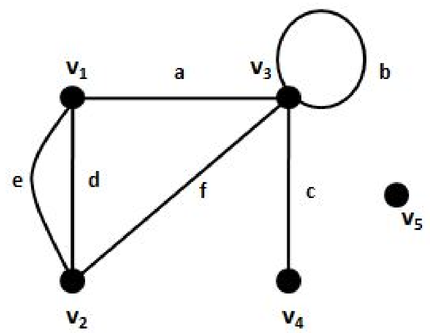

FIGURE 1.4: Graph G with self loop and multiple edges

Definition 1.1.3. "A graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a subgraph of G. A subgraph H of G is proper if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$."

Definition 1.1.4. "The subgraph obtained by deletion of a node from G is called a node deleted subgraph of G whereas subgraph obtained by deletion of an edge from G is called an edge deleted subgraph of G (see Figure 1.5)."

Definition 1.1.5. "A spanning subgraph of G is a subgraph H of G with $V(G) = V(H)$."

Definition 1.1.6. "Let $G(V, E)$ be a graph. Let V_1 be a non-empty subset of V. The subgraph $G[V_1]$ of G induced by V_1 is a graph $G[V_1](V_1, E_1)$ with edge set E_1. The set E_1 consists of those edges of G having both ends in V_1. $G[V_1]$ is referred as an induced subgraph of G.

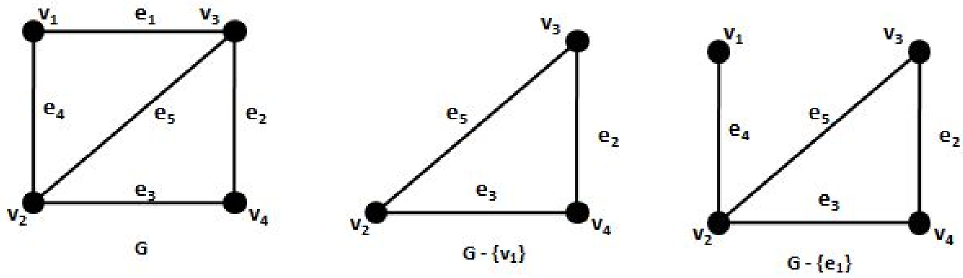

FIGURE 1.5: "Node deleted" and "edge deleted" subgraphs of G

A walk in G is an alternating sequence of nodes and edges $v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n$ beginning with v_0 and ending with v_n such that $e_i = v_{i-1}v_i$; $(i = 1, 2...n)$. It is called a $v_0 - v_n$ walk and n is called the length of the walk. A trial is a walk in which all the edges are distinct. A walk in which all the nodes are distinct is called a path denoted by P_n. If the two end points v_0 and v_n coincide in P_n, it is called a cycle C_n. Two nodes u and v of G are said to be connected if there is a $(u, v) - path$ in G. One can partition the node set V into non-empty subsets $V_1, V_2, ..., V_k$ such that the two nodes u_1 and u_2 are connected if and only if both the nodes u_1 and u_2 belong to the same set V_i. The subgraphs $G[V_1], G[V_2], ..., G[V_k]$ are called the components of G."

Definition 1.1.7. "A graph G is said to be connected if it has exactly one component; otherwise, it is called disconnected (see Figure 1.3)."

Definition 1.1.8. "A graph is considered acyclic if it without any cycle."

Definition 1.1.9. "A connected, acyclic graph is referred as a tree."

Definition 1.1.10. "Any graph without cycle is called a forest. Thus, the components of a forest are trees."

Definition 1.1.11. " A connected graph with a cycle is called a unicyclic graph."

Definition 1.1.12. "If every pair of distinct nodes of G are adjacent in G then it is known as complete graph, denoted by K_n. The size of K_n is $\frac{n(n-1)}{2}$ (see Figure 1.6)."

Definition 1.1.13. " A bipartite graph is a graph $G(V, E)$ in which $V(G)$ can be partitioned into two non-empty subsets X and Y such that each edge of G has one end in X and the other end in Y. The pair (X, Y) is called a bipartition of G. Further, if every node in X is adjacent to all the nodes of Y, then G is called a complete bipartite graph. The complete bipartite graph with bipartition (X, Y) such that $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$ (see Figure 1.7). The graph $K_{1,n}$ is called a star where the node of degree n is called central node or apex."

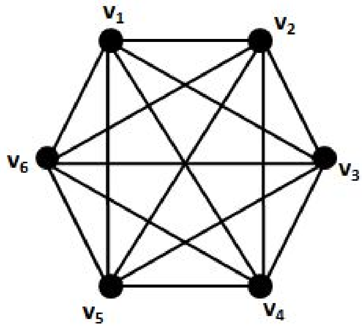

FIGURE 1.6: K_6

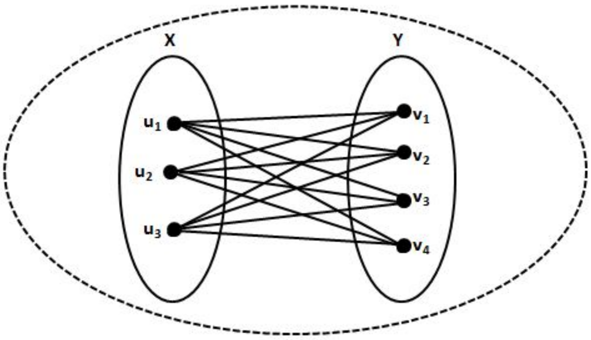

FIGURE 1.7: $K_{3,4}$

Definition 1.1.14. " Two graphs G_1 and G_2 are said to be isomorphic if there exists a bijection $\psi : V(G_1) \rightarrow V(G_2)$ such that $uv \in E(G_1)$ if and only if $\psi(u)\psi(v) \in E(G_2)$; such a function ψ is called an isomorphism from G_1 to G_2. If G_1 and G_2 are isomorphic, it is written as $G_1 \cong G_2$ (see Figure 1.8)."

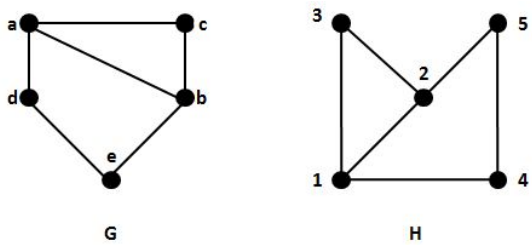

FIGURE 1.8: Isomorphic graphs G and H

Definition 1.1.15. " The complement of G denoted by $\bar{G}$ is having same node set as that of G and for each pair u, v of nodes of G, uv is an edge of $\bar{G}$ if and only if uv is not an edge of G (see Figure 1.9)."

Remark 1.1. The graph $\bar{K_n}$ has n nodes and no edges. It is called an empty graph of order n. Also K_1 and $\bar{K_1}$ represent the same graph."

Definition 1.1.16. "G_1 and G_2 are disjoint if they have no node in common and edge disjoint if they have no edge in common."

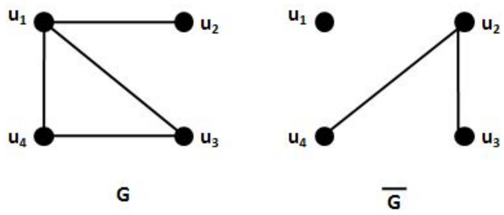

FIGURE 1.9: Graph G and $\bar{G}$

Definition 1.1.17. [77] "An edge $e = uv$ is said to be subdivided if a new node of degree 2 is inserted in it. A graph obtained by subdividing each edge of G is called the subdivision of G and is denoted by $S(G)$."

Definition 1.1.18. "Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be disjoint graphs. Join of G_1 and G_2, denoted by $G_1 + G_2$ has $V(G_1 + G_2) = V_1 \cup V_2$ and $E(G_1 + G_2) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$ (see Figure 1.10)."

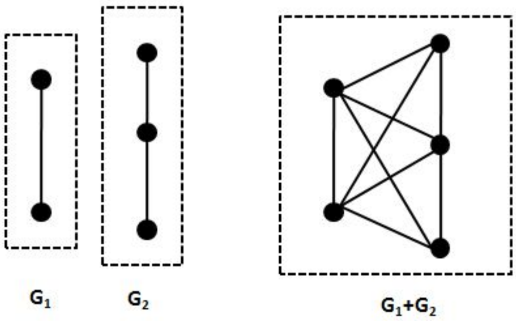

FIGURE 1.10: $G_1 + G_2$

Definition 1.1.19. [17] "The graph $f_n = P_n + K_1$ is called a fan and $Df_n = P_n + 2K_1$ is known as double fan."

Definition 1.1.20. "Wheel graph W_n is formed by joining all the nodes of C_n to K_1 (see Figure 1.11)."

Definition 1.1.21. "Gear graph G_n is obtained from W_n by subdividing each of its edge on the rim."

Definition 1.1.22. [61] Double wheel graph DW_n is a join of $2C_n$ and K_1.

Definition 1.1.23. "Helm graph H_n is obtained from W_n by attaching pendant edge to each node on the rim (see Figure 1.11)."

Definition 1.1.24. [17] "Flower graph Fl_n is obtained from H_n when each pendant node is joined to the apex node. It contains three types of nodes; an apex of degree $2n$, n nodes of degree 4 and n nodes of degree 2 (see Figure 1.11)."

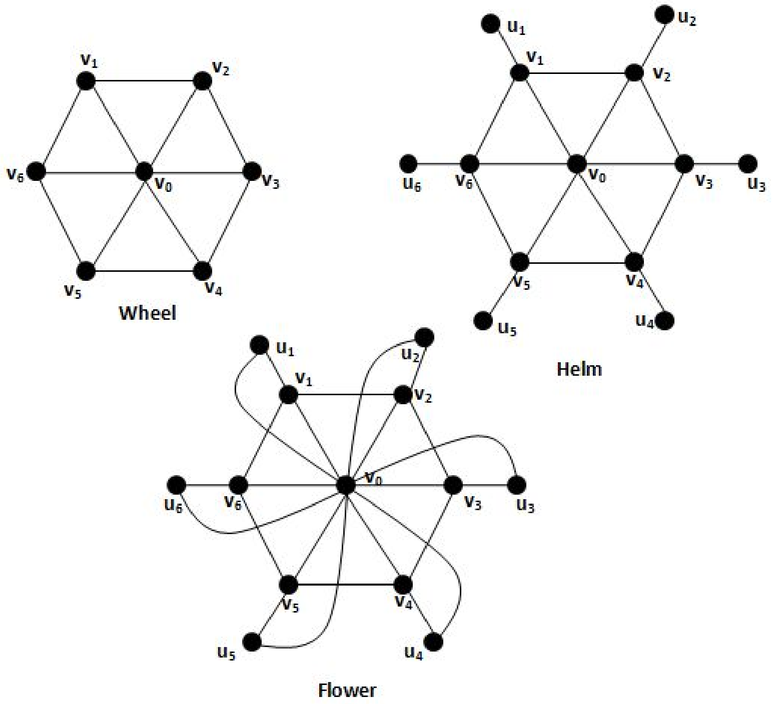

FIGURE 1.11: W_6, H_6 and Fl_6

Definition 1.1.25. [47] "A globe graph $Gl(n)$ is a join of $\bar{K}_n$ and $2K_1$."

Definition 1.1.26. [81] "Bistar $B_{m,n}$ is obtained by connecting the the apex nodes of $K_{1,m}$ and $K_{1,n}$."

Definition 1.1.27. " The union of $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is denoted by $G_1 \cup G_2$ which has node set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$ (see Figure 1.12)."

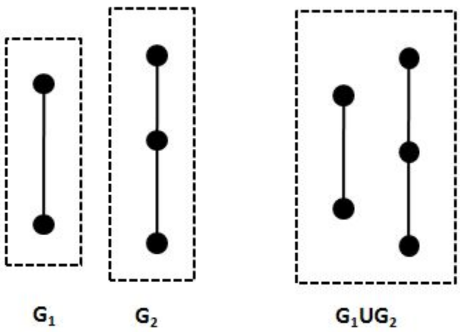

FIGURE 1.12: $G_1 \cup G_2$

Definition 1.1.28. "If $V_1 \cap V_2 \neq \phi$, then $G = (V, E)$, where $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$, is called the intersection of G_1 and G_2 and is denoted by $G_1 \cap G_2$."

Definition 1.1.29. "For Cartesian product of G_1 and G_2, consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$, then u and v are adjacent in $G_1 \times G_2$

whenever either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1. It is denoted by $G_1 \times G_2$ (see Figure 1.13)."

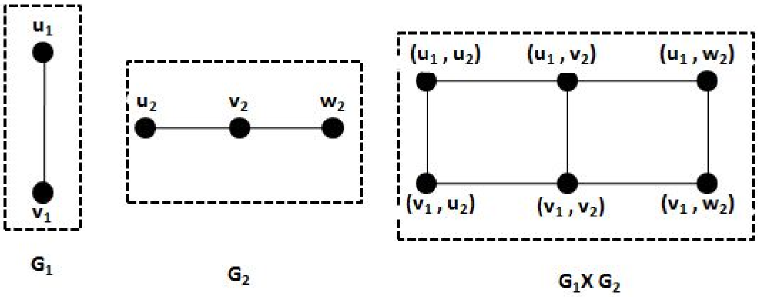

FIGURE 1.13: $G_1 \times G_2$

Definition 1.1.30. "The graph $L_n = P_n \times P_2$ is called a ladder and $P_m \times P_n$ is called a planar grid. The graph $C_n \times P_2$ is called a prism."

Definition 1.1.31. "The corona $G_1 \odot G_2$ of $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and then joining the i^{th} node of G_1 to all the nodes of G_2 in the i^{th} copy (see Figure 1.14)."

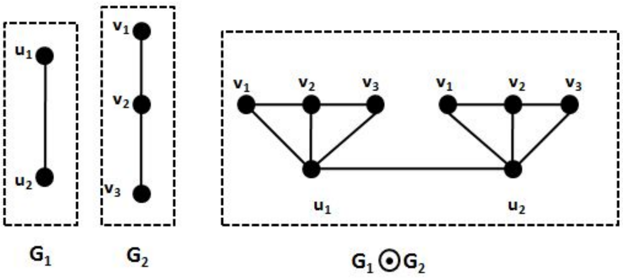

FIGURE 1.14: $G_1 \odot G_2$

Definition 1.1.32. "$P_n \odot K_1$ is called a comb whereas $C_n \odot K_1$ is called a crown."

Definition 1.1.33. [78] "The one point union of $n - copies$ of C_3 is called a friendship graph and is denoted by F_n."

Definition 1.1.34. [80] "A square graph of G denoted by G^2, is a graph having same node set as that of G and the two nodes are adjacent in G^2 if they are at a distance 1 or 2 apart in G (see Figure 1.15)."

Definition 1.1.35. [2] "The total graph $T(G)$ of G is the graph whose node set is $V(G) \cup E(G)$ and the two nodes are adjacent whenever they are either adjacent or incident in G (see Figure 1.16)."

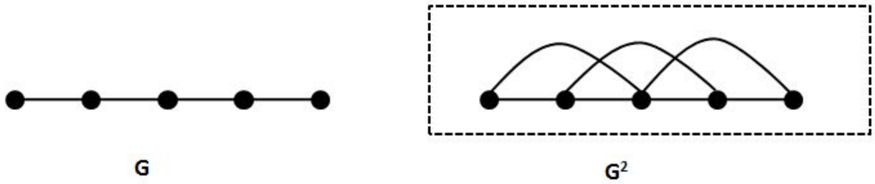

FIGURE 1.15: G and G^2

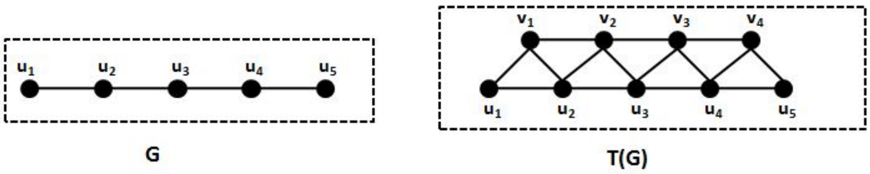

FIGURE 1.16: G and $T(G)$

Definition 1.1.36. [32] "The middle graph $M(G)$ of G is the graph whose node set is $V(G) \cup E(G)$ and in which two nodes are adjacent if and only if either they are adjacent edges of G or one is node of G and the other is an edge incident with it (see Figure 1.17)."

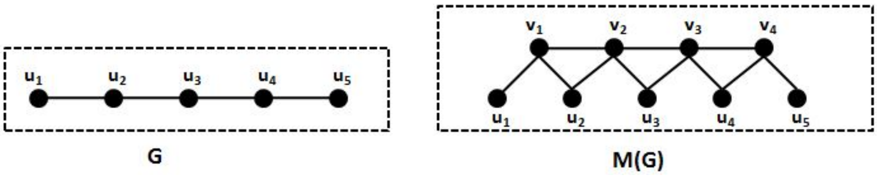

FIGURE 1.17: G and $M(G)$

Definition 1.1.37. [79] "The composition of G_1 and G_2 denoted by $G_1[G_2]$ has $V(G_1[G_2]) = V(G_1) \times V(G_2)$ and $E(G_1[G_2]) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1)$ or $[u_1 = u_2$ and $v_1v_2 \in E(G_2)]\}$ (see Figure 1.18)."

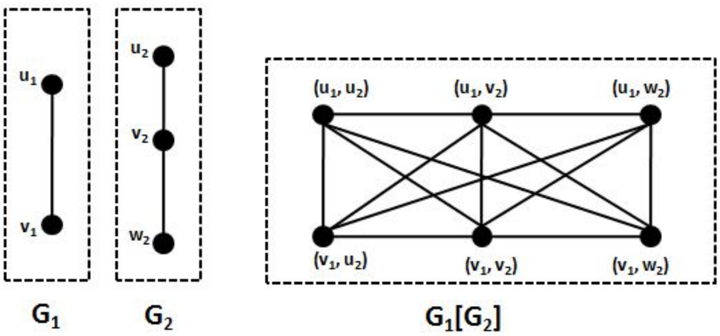

FIGURE 1.18: $G_1[G_2]$

Definition 1.1.38. [81] "The splitting graph $S'(G)$ of G is obtained by adding for each node v of G a new node v' such that v' is adjacent to every node that is adjacent to v (see Figure 1.19)."

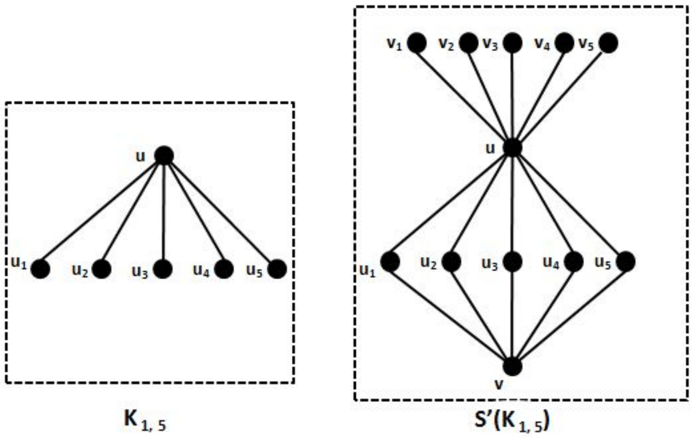

FIGURE 1.19: $S'(K_{1,5})$

Definition 1.1.39. [50] "Let G with $V(G) = S_1 \cup S_2, ... \cup S_t \cup T$, where each S_i consists of a set of nodes containing not less than two nodes and having the same degree, and $T = V - \bigcup S_i$. The degree splitting graph of G denoted by $DS(G)$ is constructed from G by inserting nodes $w_1, w_2, ..., w_t$ and joining w_i to each node of S_i; $1 \leq i \leq t$ (see Figure 1.20)."

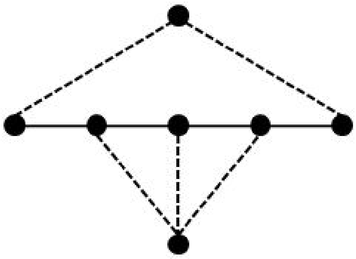

FIGURE 1.20: $DS(P_5)$

Definition 1.1.40. [66] "Duplication of a node v_k of G produces a new graph G' by inserting a node v'_k such that $N(v'_k) = N(v_k)$."

Vertex duplication operation has a lot of applications, especially when it comes to data integrity and security [9]. In any network, one can duplicate a hub of maximum degree so that if there is any fault in the hub accidentally, data may not be lost (since vertex duplication creates a parallel network).

Definition 1.1.41. [41] "Vertex switching of v in G is done by removing all the edges incident to v in G and adding edges that join v to every other node of G which are not adjacent to v in G."

Definition 1.1.42. [80] "A shadow graph $D_2(G)$ of G is obtained by taking 2-copies of G, say, G_1 and G_2, and join each node u_i in G_1 to the neighbours of the corresponding node v_i in G_2 (see Figure 1.21)."

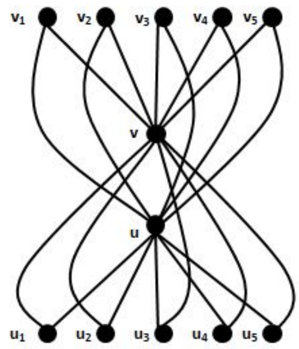

FIGURE 1.21: $D_2(K_{1,5})$

1.2 Graph Labeling

Graph labeling is informally "an allocation of labels to the nodes or edges (or both) of a graph under some conditions." The origin of graph labeling believed to dates back to the 13^{th} century when Yang Hui and others studied the labeling of geometric figures, which are today categorised as planar graphs. Efforts to solve various practical difficulties in real-world scenarios have also resulted in the creation of several graph labeling methods, for instance, "creation of some significant classes of excellent non periodic codes for pulse radar and missile guidance is similar to the labeling of complete graph with separate edge labels. The time locations at which pulses are sent are then determined by the node labels. Similarly, X-rays beam when collides with a crystal, diffracts in a variety of ways, making X-ray diffraction one of the most powerful methods for evaluating the structural characteristics of crystalline materials. In some cases, diffraction information is shared by many structures. This assignment is analogous to locating all labeling of relevant graphs that provide a given set of edge labels. Further, a communication network is composed of nodes, each of which has a tendency to compute, transmit, and receive messages over communication links. Graph labeling is also useful for allocating a node label to each user terminal, subject to the restriction that all communication links have unique labels, so that the numbers of any two communicating terminals automatically specify the link label of the connecting path, and the path label uniquely specifies the pair of user terminals it interconnects. Surprisingly, channel labeling is used in order to decide the time at which sensors communicate whereas magic labeling plays a vital role in communication field [39]." One can also use graph labeling for issues in Mobile Adhoc Networks (MANET's). Bloom and Golomb [10] linked graph labeling to a number of applications, including radar, circuit design, network design, and communication design. For more complicated families of graphs or unresolved issues, a particular type of labeling is researched in depth. Meanwhile, additional variations of labeling are discovered through breadth research by the modification of graph invariants like combining two types of labeling that already exist. Specifically most of the methods in the study of graph labeling find their inception to the labeling concept introduced by Rosa [62].

Definition 1.2.1. "A vertex labeling of G is a function f that assigns labels to the vertices of G which induces for each edge uv a label depending on labels $f(u)$ and $f(v)$ under some contraints. Similarly, an edge labeling of G is an assignment of labels to the edges of G that induces for each vertex a label depending on the labels of the edges incident to it."

Number theory and graph structures are inextricably linked in graph theory. A vast literature on graph labeling can be found in [23]. First, recall the basic graph labeling techniques that are related to the present study.

Definition 1.2.2. [29] "A graph $G(p,q)$ is graceful if there is an $1-1$ mapping $f: V(G) \to \{0,1,...,q\}$ such that the resulting difference of the node labels of all the edges is the set $\{1,2,...,q\}$ (see Figure 1.22)."

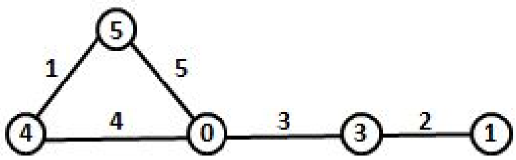

FIGURE 1.22: Graceful labeling of graph

Definition 1.2.3. [31] "A graph $G(p,q)$ is harmonious, if there is an injection f from $V(G)$ to $\mathbb{Z}_q$, the group of integers modulo q such that when each edge xy is assigned the label $(f(x)+f(y))(mod\ q)$, the resulting edge labels are distinct (see Figure 1.23)."

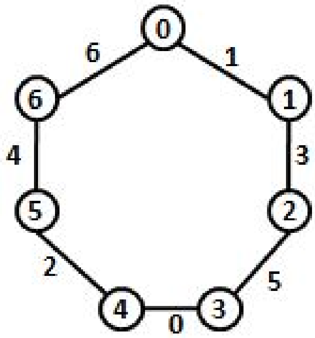

FIGURE 1.23: Harmonious labeling of C_7

Definition 1.2.4. [76] "A graph $G(p,q)$ is said to admit a prime labeling if $\exists$ a bijection $f: V(G) \to \{1,2,...,p\}$ such that for each edge $e = uv$, $\gcd(f(u), f(v)) = 1$ (see Figure 1.24)."

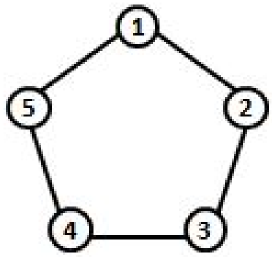

FIGURE 1.24: Prime labeling of G

Cahit [19] proposed the idea of "cordial labeling" in 1987 as a "weaker version of graceful and harmonious graphs" [10, 11, 62, 65].

Definition 1.2.5. "Let $G(V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called a binary vertex labeling of G. For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$."

Definition 1.2.6. "A binary vertex labeling f of G is called cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a cordial labeling is called cordial. Here, $v_f(0)$, $v_f(1)$ denote the number of nodes having labels 0 and 1, respectively and $e_f(0)$, $e_f(1)$ denote the number of edges with labels 0 and 1, respectively."

A significant research has been done in cordial labeling. In addition to this, a few more variants of cordial labeling are also introduced. Some of them are given here.

Definition 1.2.7. [35] "A k-cordial labeling of $G(V, E)$ is defined by a function $f : V(G) \to \mathbb{Z}_k$ so that when each edge xy is assigned the label $(f(x) + f(y))(mod\ k)$, then $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $\forall\ i, j \in \mathbb{Z}_k$."

Definition 1.2.8. [88] "For a graph $G(V, E)$, let $f : E(G) \to \{0, 1\}$. Define f on $V(G)$ by $f(v) = \sum\{f(uv) : uv \in E(G)\}(mod\ 2)$. Then f is called E-cordial if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$."

Definition 1.2.9. [73] "A product cordial labeling of $G(V, E)$ is a function $f : V(G) \to \{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, then $|e_f(0) - e_f(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$. A total product cordial labeling of $G(V, E)$ is a function $f : V(G) \to \{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$ then $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$."

1.3 Prime Cordial Labeling and Divisor Cordial Labeling

First, recall some number theoretic concepts.

Definition 1.3.1. [18] "An integer b is said to be divisible by an integer $a \neq 0$, if $\exists$ some integer k such that $b = ka$, written as $a|b$."

For example, -16 is divisible by 4, because $-16 = 4(-4)$, however 10 is not divisible by 3.

Definition 1.3.2. [86] "The divisor function of integer, $d(n)$, is defined by $d(n) = \sum 1$. That is, $d(n)$ denotes the number of all positive divisors of an integer n."

For example, for an integer 8, $d(8) = 4$ as the divisors of 8 are $1, 2, 4, 8$.

Definition 1.3.3. [86] " Let $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. The divisor summability function is defined as $D(x) = \sum_{n \leq x} d(n)$."

For example, if $x = 6.4$, then $D(6.4) = d(1) + d(2) + d(3) + d(4) + d(5) + d(6) = 1 + 2 + 2 + 3 + 2 + 4 = 14$.

Definition 1.3.4. "Let $f(z)$ and $g(z)$ be two functions defined on some subset of $\mathbb{R}$. $f(z) = O(g(z))$ as $z \to \infty$ if and only if $\exists$ a positive real number M and $z_0 \in \mathbb{R}$ such that $|f(z)| \leq M|g(z)|\ \forall\ z > z_0$."

Definition 1.3.5. "For $x \in \mathbb{R}$ and $m, n \in \mathbb{Z}$, ceiling and floor functions are defined as follows.
$\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$ and
$\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$.
It can be seen that $x - 1 < m \leq x \leq n < x + 1$."

For example, $\lceil 2.4 \rceil = 3$ and $\lfloor 2.4 \rfloor = 2$.

Definition 1.3.6. [22] "Let $a, b \in \mathbb{Z}$, (both not 0), then the greatest common divisor (gcd) of a and b denoted by (a, b) or $\gcd(a, b)$, is the $+ve$ integer d such that
(i) $d|a$ and $d|b$
(ii) if $c|a$ and $c|b$ then $c|d$. "

For example, $(36, 48) = 6$.

Theorem 1.3.1. *[22] A common divisor of m and n is a divisor of their 'gcd'.*

Theorem 1.3.2. *[22] If $g = (a, b)$, then $\exists\ x, y \in \mathbb{Z}$ such that $g = (a, b) = ax + by$.*

Definition 1.3.7. [22] "If $(\alpha, \beta) = 1$ then α and β are said to be relatively prime."

Theorem 1.3.3. *[22] If α and β are relatively prime and if $\alpha|\beta\gamma$, then α must divide γ.*

Definition 1.3.8. [18] "Let $n \in \mathbb{N}$. The Euler's phi function $\phi(n)$ denotes the number of positive integers $\leq n$ and relatively prime to n."

For example, $\phi(1) = 1$ and $\phi(10) = 4$. For a prime p, $\phi(p) = p - 1$.

Theorem 1.3.4. *For $n \in \mathbb{N}$,*
(i) If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, then $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})...(1 - \frac{1}{p_k})$.
(ii) $\phi(n)$ is even for $n \geq 3$.
(iii) $\sum\limits_{d|n} \phi(d) = n$.

Theorem 1.3.5. *[18] Every integer ≥ 1 is either a prime number or a product of prime factors, with the latter being unique up to the prime factor ordering.*

Theorem 1.3.6. *[7] For the set $F_n = \{1, 2, ..., n\}$, total number of relatively prime pairs is given by $\sum\limits_{k=2}^{n} \phi(k)$.*

Theorem 1.3.7. *[52] (i) Given odd integer m and $t \in \mathbb{N}$, $\gcd(m, m + 2^t) = 1$.*
(ii) Given odd integer m, an odd prime q and $t_1, t_2 \in \mathbb{N}$, if $m \not\equiv 0 (mod\ q)$, then $\gcd(m, m + 2^{t_1}.q^{t_2}) = 1$.

Prime numbers are unique and are rich in terms of their properties. In fact there are now many applications of prime numbers with real life significance. One such area is cryptography with the process of encryption of data as of the foremost applications of prime numbers. The difficult task of factorizing exceptionally large numbers as product of primes has turned out to be of great importance in keeping important information safe [21]. For other interesting applications of prime number one can refer to [28].
In 2005, Sundaram et al., [74] brought into being the concept of prime cordial labeling inspired by prime labeling [76] and cordial labeling [19].

Definition 1.3.9. [74] "A prime cordial labeling of $G(V, E)$ is a bijection $f : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ defined by the induced function $f^* : E(G) \to \{0, 1\}$ such that if, $f^*(uv) = 1$ if $gcd(f(u), f(v)) = 1$ and $f^*(uv) = 0$ if $gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and 1 differ by at most 1 i.e; $|e_f(0) - e_f(1)| \leq 1$. If a graph admits a prime cordial labeling, then it is called a prime cordial (see Figure 1.25)."

K_n, $n \geq 3$ is not prime cordial.
Varatharajan et al., in 2011, first presented the notion of divisor cordial labeling [86].

Definition 1.3.10. [86] " A divisor cordial labeling of $G(V, E)$ is a bijection $f : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ defined by the induced function $f^* : E(G) \to \{0, 1\}$ such that if $f^*(uv) = 1$ if $f(u)|f(v)$ or $f(v)|f(u)$ and $f^*(uv) = 0$ otherwise, then $|e_f(0) - e_f(1)| \leq 1$.

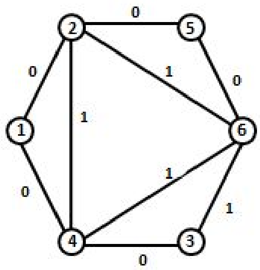

FIGURE 1.25: Prime cordial graph

If a graph admits a divisor cordial labeling, then it is called a divisor cordial graph (see Figure 1.26)."

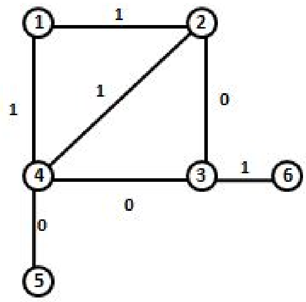

FIGURE 1.26: Divisor cordial graph

K_n, $n \geq 7$ is not divisor cordial.

Note: If G is a prime cordial graph (divisor cordial graph) and G is isomorphic to H then H also admits a prime cordial labeling (divisor cordial labeling).

1.4 Review of Literature

Throughout this thesis, "PCL, DCL, PCG and DCG" are used to denote "prime cordial labeling, divisor cordial labeling, prime cordial graph and divisor cordial graph" respectively. For prime cordial graphs and divisor cordial graphs, PCGs and DCGs are used respectively.

1.4.1 Important Results on Prime Cordial Labeling

In this subsection, a few important established results on PCL of graphs are recalled. Somasundram et al., in their introductory paper [74] proved that

- C_n is a PCG $\iff$ $n \geq 6$.

- P_n is a PCG $\iff n \neq 3, 5$.
- $S(K_{1,n})$ is a PCG $\iff n > 2$.
- The bistars, dragons, and crowns are PCGs. Triangular snakes are PCG if and only if the snake has at least three triangles.

Vaidya et al., in [78, 79] established PCL of the following.

- The "path union" of k- copies of C_n.
- F_n, $n \geq 3$.
- $T(P_n)$ and $T(C_n)$ for $n \geq 5$.
- $P_2[P_m]$ for $m \geq 5$.
- Two cycles joined by P_m, and a graph formed by "switching of an arbitrary node" in C_n except $n = 5$.

Vaidya and Shah in [80, 81, 84, 85] established the following.

- P_n^2, $n = 6$, $n \geq 8$ is a PCG.
- C_n^2, $n \geq 10$ is a PCG.
- $B_{n,n}^2$ admits a PCL.
- $D_2(B_{n,n})$ is a PCG.
- $D_2(K_{1,n})$ is a PCG for $n \geq 4$.
- $S'(K_{1,n}), S'(B_{n,n})$, and $M(P_n)$ are PCGs.
- W_n, $n \geq 8$ is a PCG.
- G_n, $n \geq 4$ is a PCG.
- H_n is a PCG $\forall$ n.
- CH_n, $n \geq 5$ is a PCG.
- Fl_n is a PCG for $n \geq 4$.
- $DS(P_n)$ and $DS(B_{n,n})$ are PCGs.
- Df_n permits a PCL for $n = 8$, $n \geq 10$.

1.4.2 Important Results on Divisor Cordial Labeling

Some notable results on DCL of graphs are presented here. Vartharajan et al., in [86, 87] proved the following results.

- P_n, C_n, W_n, $K_{1,n}$, $K_{2,n}$ and $K_{3,n}$ are DCGs.
- K_n permits a DCL only for $n = 3, 5, 6$.
- $S(K_{1,n})$ admits a DCL.
- Let G be a DCG of size m and $K_{2,n}$ be having bipartition $V = V_1 \cup V_2$ with $V_1 = \{a_1, a_2\}$ and $V_2 = \{b_1, b_2, ..., b_n\}$, then "$G \star K_{2,n}$ formed by identifying the nodes a_1 and a_2 of $K_{2,n}$ with that labeled 1 and the largest prime $p \leq m$ respectively in G is also a DCG".
- Let G be a DCG of size m and $K_{3,n}$ be having the bipartition $V = V_1 \cup V_2$ with $V_1 = \{a_1, a_2, a_3\}$ and $V_2 = \{b_1, b_2, ..., b_n\}$ and n is even, then "$G \star K_{3,n}$ formed by identifying the nodes a_1, a_2 and a_3 of $K_{3,n}$ with that labeled $1, 2$ and the largest prime $p \leq m$ respectively in G is also a DCG".

Vaidya and Shah in [82, 83] established the DCL of the following graphs.

- H_n, Fl_n, G_n.
- "Switching of a node" in C_n, "switching of a node at rim" in W_n and "switching of apex node" in H_n.
- $S'(K_{1,n})$, $S'(B_{n,n})$.
- $DS(B_{n,n})$, $D_2(B_{n,n})$, $B^2_{n,n}$.

Raj et al., in [55–57] proved the following results.

- $S'(K_{2,m})$, $S'(K_{1,n,n})$, $P_n + 2K_1$, $C_n + 2K_1$, $(\bar{K_n} \cup P_m) + 2K_1$, $(P_n \cup P_m) + 2K_1$ are DCGs.
- Corona of a graph formed by switching of any node of C_n for $n \geq 4$, with K_1 is DCG.
- Graph acquired by joining the apex nodes of 2- copies of W_n to a new node admits a DCL.

- Graph formed by joining 2 copies of W_n using a path P_k admits a DCL, where $n \geq 3$.
- Disconnected graphs $P_n \cup P_m$, $C_n \cup C_m$, $P_n \cup C_m$, $P_n \cup K_{1,m}$, $P_n \cup W_m$, $C_n \cup K_{1,m}$, $C_n \cup W_m$, $W_n \cup W_m$ are DCGs.

Maya et al., in [41] established the following results.

- Fl_n for $n \geq 3$, H_n, $n > 3$.
- Graph formed by switching a node at rim in W_n, $n \geq 4$.
- Graph formed by switching apex node in H_n.

Bosmia et al., in [15, 16] established the following results.

- $B_{m,n}$, $S'(B_{m,n})$, $DS(B_{m,n})$, $D_2(B_{m,n})$, $B^2_{m,n}$ and $S(B_{m,n})$ are DCGs.
- $S(K_{2,n})$ and $S(K_{3,n})$ admit a DCL.

Ghodsara et al., in [27] established the following.

- The ring sum of the following graphs with $K_{1,n}$ admit a DCL (i) C_n (ii) C_n with one chord (iii) C_n with triangle (iv) C_n with twin chords forming two triangles (v) P_n (vi) Df_n.

Gondalia et al., in [30] contributed the following results.

- The ring sum of the following graphs with $K_{1,n}$ admit a DCL (i) H_n (ii) G_n (iii) DW_n (iv) J_n.

K. Thirusangu et al., in [75] established the following results.

- Extended duplicated graph of $K_{1,n}$, $B_{n,n}$ and $K_{1,n,n}$ are DCGs.

As every graph need not to admit a PCL or DCL, it becomes an interesting and challenging job to see the families of graphs that admit PCL or DCL. The graphs considered in the thesis are mainly finite, simple, connected and undirected.

1.5 Research Gap and Objectives

Though an enormous work has been done concerning PCL and DCL, still there are many open problems to work on. Especially, the complete characterization of PCGs and DCGs are the main areas of high interest. The gap of establishing the PCL and DCL of some new families of graphs has been filled. Researchers have studied the PCL and DCL of certain graphs in the context of graph operations such as join, subdivision, vertex duplication, vertex switching, edge duplication etc. but establishing the PCL and DCL of graphs in context with extension of nodes is still open. Recently, researchers have explored some new variants of DCL such as "sum divisor cordial labeling, square divisor cordial labeling, cube divisor cordial labeling" etc., a few more new variants of DCL which are not explored yet, such as "average even divisor cordial labeling, double divisor cordial labeling" etc. can be introduced. Based on these research gaps, the objectives of the thesis are framed.

1. Deriving certain new classes of PCL.
2. Obtaining some new classes of DCL.
3. Establishing the PCL and DCL in the context of extension of vertices in graphs.
4. Introducing and studying new variants of DCL for various classes of graphs.

1.6 Conclusion

In this chapter, a short introduction to graph theory and graph labeling and their uses in real life situations have been given. Several graph labeling techniques specifically, prime cordial labeling and divisor cordial labeling of graphs are recalled. Further, a comprehensive review of literature along with research gap followed by the proposed objectives is also presented.

Chapter 2

Results on PCL of Graphs

2.1 Introduction

In this chapter, a few new general results concerning PCL of graphs are derived. PCL in the context of graph operation named, corona, is discussed in the second subsection which is followed by the investigation of PCL of lilly related graphs.

2.2 Certain New General Results on PCL of Graphs

This section is dedicated for obtaining some general results concerning PCL of graphs. First, a few established general results have been recalled.

Theorem 2.2.1. *[1] "$\sum_{i=2}^{n} \phi(i) \geq \frac{1}{2}\binom{n}{2} + 1$."*

Theorem 2.2.2. *[3] "If $G(p, q)$ is a PCG, then $G - e$ is also a PCG,*
(i) $\forall\ e \in E(G)$ when q is even.
(ii) for some $e \in E(G)$ when q is odd."

Theorem 2.2.3. *[3] "A maximum number of edges in a simple PCG having n nodes is $n^2 - n + 1 - 2\sum_{k=2}^{n} \phi(k)$."*

Theorem 2.2.4. *[52] "For an odd prime graph $G_1(V_1, E_1)$, and another graph $G_2(V_2, E_2)$ with $|V_1| = |V_2|$ and $|E_1| = |E_2|$, disjoint union of G_1 and G_2 is a PCG."*

Theorem 2.2.5. *[52] "Let G be an odd prime graph on n nodes and H be a graph formed by joining any pair of corresponding nodes of two copies, G_1 and G_2, of G by an edge. Then H is a PCG."*

Theorem 2.2.6. *[52] "Let G be an odd prime graph of order n and H be a graph obtained by identifying each end node of P_k with corresponding nodes of each of the two copies G_1 and G_2 of G. Then H is PCG, when one end node of P_k is identified with a node in G_1 having label 3 if*
(i) $k \geq 4$, $k \neq 5$ and $n \geq 2$
(ii) $k = 3$ and $n \geq 4$
(iii) $k = 5$ and $n \geq 3$."

Motivated by these results, a few more general results are obtained concerning PCL of graphs which are discussed as follows.

Lemma 2.2.1. *[50] $DS(K_n)$ gives rise to K_{n+1}.*

Theorem 2.2.7. *[74] K_n does not admit a PCL for $n \geq 3$.*

Theorem 2.2.8. *$DS(K_n)$ does not admit a PCL for $n \geq 2$.*

Proof. Proof is obvious from Lemma 2.2.1 and Theorem 2.2.7. □

Theorem 2.2.9. *Disjoint union of a finite copies of P_n admits a PCL.*

Proof. Let G be constructed by taking the disjoint union of k-copies of P_n with $V(G) = \{v_1, v_2, ..., v_{nk}\}$. Clearly, $|V(G)| = nk$, and $|E(G)| = k(n-1)$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., nk\}$. Now three cases arise.
Case (i) If 'k' is even.
There are exactly half even labels and half odd. Label the $\frac{k}{2}$ copies of path with even labels in any pattern and for the remaining copies assign the odd labels simultaneously from $\{1, 2, ..., nk\}$. In this case, one can note that $|e_\psi(0) - e_\psi(1)| = 0$.
Case (ii) If 'n' is even and 'k' is odd.
Here, nk is even and therefore there are exactly half even and half odd labels available for labeling. Assign all the $\frac{nk}{2}$ even labels to the first $\lfloor \frac{k}{2} \rfloor$ copies of P_n followed by $\frac{n}{2}$ nodes of $\lceil \frac{k}{2} \rceil^{th}$ copy of P_n. Next, assign the remaining $\frac{nk}{2}$ odd labels to the remaining nodes simultaneously. Clearly, $|e_\psi(0) - e_\psi(1)| = 1$.
Case (iii) If both 'n' and 'k' are odd.
Here, nk is odd and therefore there are $\lfloor \frac{nk}{2} \rfloor$ even labels and rest are odd. Labeling can be done by assigning even labels excluding least even number divisible by 3, say r, to the nodes of G begining with first copy of P_n. Let $\psi(v_{\lceil \frac{nk}{2} \rceil - 1}) = r$ and $\psi(v_{\lceil \frac{nk}{2} \rceil}) = 3$. Now assign available odd labels simultaneously to the remaining nodes of G. One can verify that $|e_\psi(0) - e_\psi(1)| = 0$.
Thus, in all the three cases, $|e_\psi(0) - e_\psi(1)| \leq 1$. Hence, G is a PCG. □

Theorem 2.2.10. *Disjoint union of finite copies of $2-regular$ graph admits a PCL.*

Proof. Suppose G is constructed by considering the disjoint union of finite copies, say k, of $2-regular$ graphs each having order n. Here $V(G)=\{g_{ij}:1\leq i\leq k,1\leq j\leq n\}$ where $g_{i1},g_{i2},...,g_{in}$ denote the nodes of i^{th} copy of $2-regular$ graph. Clearly, $|V(G)|=|E(G)|=nk$. Consider a map $\psi:V(G)\rightarrow\{1,2,...,nk\}$ defined under three conditions
Case (i) If 'k' is even.
Assign all even labels to $\frac{k}{2}$ copies of $2-regular$ graph in any order and for the remaining nodes of G assign the odd labels simultaneously from $\{1,2,...,nk\}$ in such a way that $\gcd(\psi(g_{i1}),\psi(g_{in}))=1$; $\frac{k}{2}+1\leq i\leq k$. Clearly, $|e_\psi(0)-e_\psi(1)|=0$.
Case (ii) If 'n' is even & 'k' is odd.
Here, nk is even. Label $\lfloor\frac{k}{2}\rfloor$ copies of $2-regular$ graph by using available even labels except the least even number divisible by 3, say, r. For $\lceil\frac{k}{2}\rceil^{th}$ copy, assign the unutilized even labels in such a way that $\psi(g_{\lceil\frac{k}{2}\rceil\frac{n}{2}})=r$. Now, fix $\psi(g_{\lceil\frac{k}{2}\rceil(\frac{n}{2}+1)})=3$. Label the remaining nodes with odd labels simultaneously from $\{1,2,...,nk\}$ in such a way that $\gcd(\psi(g_{i1}),\psi(g_{in}))=1$ for $\lceil\frac{k}{2}\rceil\leq i\leq k$. Clearly, $|e_\psi(0)-e_\psi(1)|=0$.
Case (iii) If both 'n' and 'k' are odd.
Label the $\lfloor\frac{k}{2}\rfloor$ copies of $2-regular$ graph by using available even labels except the least even number divisible by 3, say, s in any order. For $\lceil\frac{k}{2}\rceil^{th}$ copy, assign the remaining even labels in such a way that $\psi(g_{\lceil\frac{k}{2}\rceil\lfloor\frac{n}{2}\rfloor})=s$. Now, fix $\psi(g_{\lceil\frac{k}{2}\rceil\lceil\frac{n}{2}\rceil}))=3$. Label the remaining nodes of G with unutilized odd labels simultaneously from $\{1,2,...,nk\}$ such that $\gcd(\psi(g_{i1}),\psi(g_{in}))=1$ for $\lceil\frac{k}{2}\rceil\leq i\leq k$. Here, $|e_\psi(0)-e_\psi(1)|=1$.
In the wake of above cases, it follows that G is a PCG. □

Definition 2.2.1. [78] "Duplication of a vertex v_k by a new edge $e=v'_kv''_k$ in a graph G produces a new graph G' such that $N(v'_k)\cap N(v''_k)=v_k$."

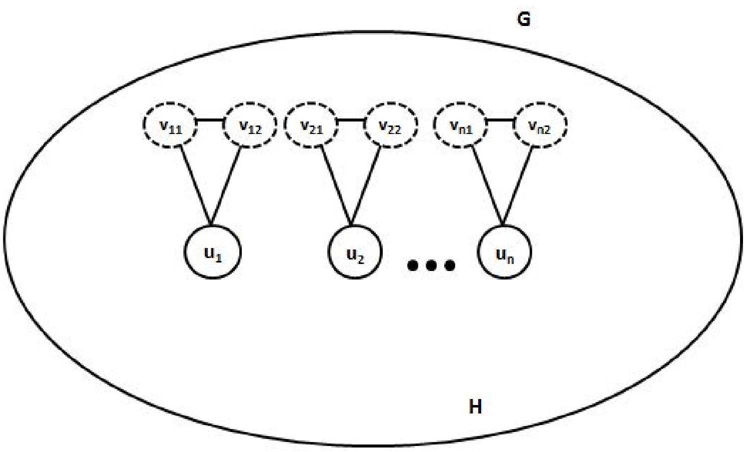

FIGURE 2.1: PCL of a graph formed by duplication of each node by an edge in H

Theorem 2.2.11. *Let H be any PCG. The graph acquired by duplicating a node by an edge at all the nodes of H admits a PCL.*

Proof. Let H be the given PCG on n nodes, namely, $u_1, u_2, ..., u_n$ and size m with labeling f. Let G be formed by duplicating a node by an edge at all nodes of H (see Figure 2.1). Let v_{i1}, v_{i2}; $1 \leq i \leq n$, denote the end nodes of an edge introduced at node u_i. Clearly, $|V(G)| = 3n$ and $|E(G)| = m + 3n$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 3n\}$ by letting $\psi(u_i) = f(u_i)$ for $1 \leq i \leq n$ and to label the remaining $2n$ nodes of G, there arise four cases.

Case (i) If both 'n' and 'm' are even.

Assign the n even labels to the end nodes of edges introduced at even labeled nodes of H. One can easily observe that $\gcd(\psi(u_i), \psi(v_{i1})) \neq 1$, $\gcd(\psi(u_i), \psi(v_{i2})) \neq 1$ and $\gcd(\psi(v_{i1}), \psi(v_{i2})) \neq 1$. Now allot the unused n odd labels to the remaining end nodes of edges introduced at odd labeled nodes in H such that $\gcd(\psi(u_i), \psi(v_{i1})) = 1$, $\gcd(\psi(u_i), \psi(v_{i2})) = 1$ and $\gcd(\psi(v_{i1}), \psi(v_{i2})) = 1$. An easy check shows that $|e_\psi(0) - e_\psi(1)| = 0$.

Case (ii) If 'n' is even and 'm' is odd.

This case is treated in similar lines with that of Case (i) for assigning the labels. One can see that $|e_\psi(0) - e_\psi(1)| = 1$.

Case (iii) If 'n' is odd and 'm' is odd.

Subcase (i) When $e_f(0) = e_f(1) + 1$.

Observe that there are $\frac{n-1}{2}$ number of even labeled nodes in H. Out of $2n$ labels, assign n even labels excluding the least even number divisible by 3, say r, to the end nodes of edges introduced at even labeled nodes of H. Assign r to one of the end nodes of an edge introduced at a node having label 3. Observe that for these edges, that $\gcd(\psi(u_i), \psi(v_{i1})) \neq 1$, $\gcd(\psi(u_i), \psi(v_{i2})) \neq 1$ and $\gcd(\psi(v_{i1}), \psi(v_{i2})) \neq 1$. Allot the unused n odd labels to the remaining end nodes of edges introduced at odd labeled nodes of H in such a way that $\gcd(\psi(u_i), \psi(v_{i2})) = 1$, $\gcd(\psi(u_i), \psi(v_{i1})) = 1$ and $\gcd(\psi(v_{i1}), \psi(v_{i2})) = 1$. It is easy to see that $|e_\psi(0) - e_\psi(1)| = 1$.

Subcase (ii) When $e_f(1) = e_f(0) + 1$.

Here there are $\frac{n-1}{2}$ number of even labels for H. Assign n even labels excluding the least even number divisible by 3, say r, to the end nodes of edges introduced at even labeled nodes of H. Assign r to one of the end nodes of an edge introduced at node having label 3 and any odd number divisible by 5, say s, to one of the end nodes of an edge introduced at the node having label 5. Observe that for the above edges $\gcd(\psi(u_i), \psi(v_{i1})) \neq 1$, $\gcd(\psi(u_i), \psi(v_{i2})) \neq 1$ and $\gcd(\psi(v_{i1}), \psi(v_{i2})) \neq 1$. Allot the unused labels to the remaining end nodes of edges introduced at odd labeled nodes of H in such a way that $\gcd(\psi(u_i), \psi(v_{i1})) = 1$, $\gcd(\psi(u_i), \psi(v_{i1})) = 1$ and $\gcd(\psi(v_{i1}), \psi(v_{i2})) = 1$. An easy check shows that $|e_\psi(0) - e_\psi(1)| = 0$.

Case (iv) If 'm' is even and 'n' is odd.

Observe that there are $\frac{n-1}{2}$ number of even labeled nodes in H. Out of $2n$ labels, assign n even labels excluding the least even number divisible by 3, say r, to the end nodes of

edges introduced at even labeled nodes of H. Assign r to one of the nodes of an edge introduced at node having label 3. Observe that for these edges, $\gcd(\psi(u_i), \psi(v_{i1})) \neq 1$, $\gcd(\psi(u_i), \psi(v_{i2})) \neq 1$ and $\gcd(\psi(v_{i1}), \psi(v_{i2})) \neq 1$. Allot the unutilized n odd labels to the remaining end nodes of edges introduced at odd labeled nodes of H in such a fashion that $\gcd(\psi(u_i), \psi(v_{i1})) = 1$, $\gcd(\psi(u_i), \psi(v_{i2})) = 1$ and $\gcd(\psi(v_{i1}), \psi(v_{i2})) = 1$. It is easy to see that $|e_\psi(0) - e_\psi(1)| = 1$.
Thus, in all the cases, ψ induces a PCL for G. □

Theorem 2.2.12. *If H is a PCG of even order, then G formed by subdividing all the edges of H admits a PCL.*

Proof. Given H a PCG, with order 'n' and size 'm' where n is even, with labeling f. Let $V_1 = \{v_i : 1 \leq i \leq \frac{n}{2}\}$ & $V_2 = \{v_i : \frac{n}{2} + 1 \leq i \leq n\}$ such that $V(H) = V_1 \cup V_2$ and $E(H) = E_1 \cup E_2$ with $f : V(H) \to \{1, 2, ..., n\}$ given by $f : V_1(H) \to \{2, 4, ..., n\}$ and $f : V_2(H) \to \{1, 3, ..., n - 1\}$ such that $|e_f(0) - e_f(1)| \leq 1$. Let G be acquired by subdividing all the edges of H. One can easily see that $|V(G)| = n+m$ and $|E(G)| = 2m$. Consider $\psi : V(G) \to \{1, 2, ..., n + m\}$. Let $\psi(v_i) = f(v_i)$; $1 \leq i \leq n$. Now arises the given conditions.
Case (i) If $G \cong G_1$, where G_1 admits a PCL, nothing remains to prove.
Case (ii) If $G \ncong G_2$, where G_2 is yet to be proved a PCG. Define a PCL for G. One can see that there are $n + m$ labels and $2m$ edges. So exactly m edges receive label 1 and the remaining edges label 0. Name the newly added nodes $w_1, w_2, ..., w_m$. The available labels are $n + 1, n + 2, ..., n + m$. Let $w_1, w_3, ..., w_{m-1}$ be the nodes which are inserted between the edges having odd end labels in H. Similarly, let $w_2, w_4, ..., w_m$ be the nodes inserted between the edges having even end labels in H. Without loss of generality, assign $\frac{m}{2}$ even labels to the nodes $w_2, w_4, ..., w_m$. Note that $\gcd(\psi(v_i), \psi(w_j)) \neq 1$, $\gcd(\psi(w_j), \psi(v_{i+1})) \neq 1$ for $1 \leq i \leq \frac{n}{2} - 1$, $2 \leq j \leq m$ (j is even) where $\psi(v_i)$, $\psi(v_{i+1})$ are even. Similarly, assign the $\frac{m}{2}$ odd labels to the nodes $w_1, w_3, ..., w_{m-1}$ in such a way that $\gcd(\psi(v_i), \psi(w_j)) = 1$, $\gcd(\psi(w_j), \psi(v_{i+1})) = 1$ for $\frac{n}{2} + 1 \leq i \leq n-1$, $1 \leq j \leq m-1$ (j is odd) where $\psi(v_i), \psi(v_{i+1})$ are odd. Thus, there is an induced function ψ' from $E(G)$ to $\{0, 1\}$ such that $|e_\psi(0) - e_\psi(1)| \leq 1$, so G is a PCG. This completes the proof. □

Notation $[n]$ and O_n denote the set of naturals $\leq n$ and set of first n odd naturals, respectively.

Definition 2.2.2. [51] "A function $\psi : V(G) \to O_n$ is said to be an odd prime labeling of G if for each $uv \in E(G)$, $\gcd(\psi(u), \psi(v)) = 1$. A graph which admits an odd prime labeling is called an odd prime."

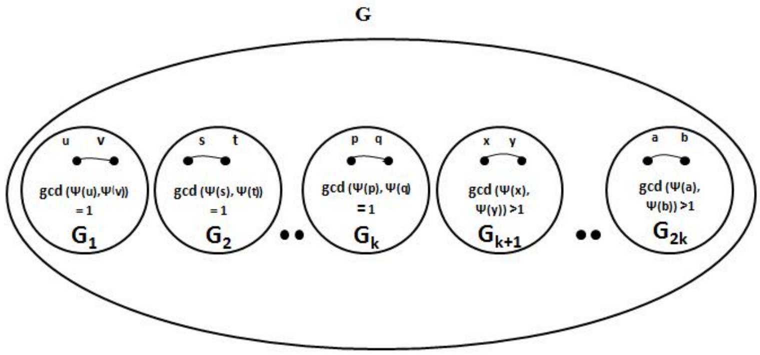

FIGURE 2.2: PCL of disjoint union of graphs

Theorem 2.2.13. *Let $G_r(V_r, E_r)$ be odd prime graphs and $G_s(V_s, E_s)$ be any graphs such that $|V_r| = |V_s|$ and $|E_r| = |E_s|$ where $r = s \in \mathbb{N}$. Let $\bigcup\limits_{r=1}^{k} (G_r(V_r, E_r))$ be odd prime graph then the disjoint union of G_r and G_s is again a PCG.*

Proof. Let G_r be odd prime graphs for $1 \leq r \leq k$ with $|V_r| = n$ and $|E_r| = m$ and $G_s(V_s, E_s)$ be graphs for $1 \leq s \leq k$ such that $|V_s| = |V_r| = n$ and $|E_s| = |E_r| = m$. Since $\bigcup\limits_{r=1}^{k} G_r(V_r, E_r)$ is odd prime graph, there exists a bijection $g_1 : \bigcup\limits_{r=1}^{k} V_r \to O_{2kn}$ such that for any edge $xy \in E_r$; $1 \leq r \leq k$, $\gcd(g_1(x), g_1(y)) = 1$. Now define another bijective function $g_2 : \bigcup\limits_{s=1}^{k} V_s \to \{1, 2, ..., 2kn\} - O_{2kn}$. Observe that $g_2(u)$ is always an even number for any $u \in V_s$; $1 \leq s \leq k$. For any edge $xy \in E_s$; $1 \leq s \leq k$, $\gcd(g_2(x), g_2(y)) \neq 1$. Let $G(V, E)$ be acquired by taking disjoint union of G_r and G_s where $1 \leq r \leq k$ and $1 \leq s \leq k$ (see Figure 2.2). See that $V(G) = (\bigcup\limits_{r=1}^{k} V_r) \bigcup (\bigcup\limits_{s=1}^{k} V_s)$ and $E(G) = (\bigcup\limits_{r=1}^{k} E_r) \bigcup (\bigcup\limits_{s=1}^{k} E_s)$ with $|V(G)| = 2kn$ and $|E(G)| = 2km$. Define $\psi : V(G) \to \{1, 2, ..., 2kn\}$ by fixing $\psi(u) = g_1(u)$, if $u \in V_r$; $1 \leq r \leq k$, $\psi(u) = g_2(u)$, if $u \in V_s$; $1 \leq s \leq k$. The induced labeling $\psi' : E(G) \to \{0, 1\}$ is obtained as follows. For any edge $e \in G$, $\psi'(e) = 1$, if $e \in \bigcup\limits_{r=1}^{k} E_r$ and $\psi'(e) = 0$, if $e \in \bigcup\limits_{s=1}^{k} E_s$. Thus, $e_\psi(1) = \sum\limits_{r=1}^{k} |E_r| = km$ and $e_\psi(0) = \sum\limits_{s=1}^{k} |E_s| = km$ establishing that $|e_\psi(0) - e_\psi(1)| \leq 1$. Hence, G admits a PCL. □

Definition 2.2.3. [67] "Let $G_1, G_2, ..., G_k$, $k \geq 2$ be the $k - $ *copies* of G. Adding an edge between G_i and G_{i+1} for $i = 1, 2, ..., k-1$ is called the path union of G_i."

Theorem 2.2.14. *Let G_i; $1 \leq i \leq k$, $k \geq 2$ be the 'k'-copies of a graph G of even size. Suppose that their disjoint union is a PCG then the graph acquired by taking the path union of G_i also admits a PCL.*

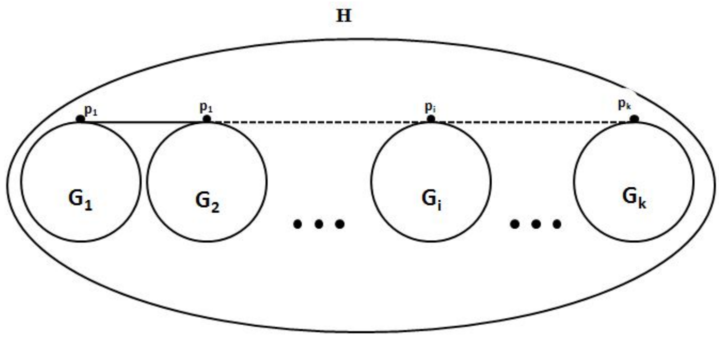

FIGURE 2.3: Path union of graphs

Proof. Let G be the given PCG of even size and G_i; $1 \leq i \leq k$, $k \geq 2$ be the 'k'-copies of G such that $\bigcup_{i=1}^{k} G_i$ is also a PCG with labeling g, thus $e_g(0) = e_g(1)$. Let H denotes the graph formed by taking the path union of k-copies of G (see Figure 2.3). Let $p_1, p_2, ..., p_k$ denote the nodes of path formed due to path union of 'k'- copies of G_i. Clearly $|E(H)| = |E(\bigcup_{i=1}^{k} G_i)| + (k-1)$. Consider $h : V(H) \rightarrow \{1, 2, ..., k|V(G_i)|\}$ defined under three conditions.
Case (i) When 'k' is even.
Superimposition of p_i with nodes of G_i can be done in such a way that $\gcd(h(p_i), h(p_{i+1})) = 1$ for $1 \leq i \leq \frac{k}{2}$ and $\gcd(h(p_j), h(p_{j+1})) \neq 1$ for $\frac{k}{2} + 1 \leq j \leq k - 1$. Clearly, $|e_h(0) - e_h(1)| = 1$.
Case (ii) When 'k' is odd.
Superimposition of p_i with nodes of G_i can be done in such a way that $\gcd(h(p_i), h(p_{i+1})) = 1$ for $1 \leq i \leq \frac{k-1}{2}$ and $\gcd(h(p_j), h(p_{j+1})) \neq 1$ for $\frac{k+1}{2} \leq j \leq k - 1$. Clearly, $|e_h(0) - e_h(1)| = 0$.
Thus, in both the cases, $|e_h(0) - e_h(1)| \leq 1$, which proves that H is a PCG. □

Corollary 2.2.1. *Let $G_r(V_r, E_r)$ be odd prime graphs and $G_s(V_s, E_s)$ are graphs such that $|V_s| = |V_r|$ and $|E_s| = |E_r|$ where $r, s \in \mathbb{N}$. Suppose that disjoint union of G_r and G_s is a PCG. Then the graph acquired by taking the path union of these graphs also admits a PCL.*

Proof. The proof is evident from Theorem 2.2.13 and Theorem 2.2.14. □

Theorem 2.2.15. *If $G_1(V_1, E_1)$ is an odd prime graph and $G_2(V_2, E_2)$ be any graph of same order and size as that of G_1, then the graph formed by joining G_1 and G_2 using a path of finite length, also admits a PCL.*

Proof. Let $G_1(V_1, E_1)$ be an odd prime graph with $|V_1| = n$ and $|E_1| = m$ and $G_2(V_2, E_2)$ be another graph such that $|V_2| = |V_1|$ and $|E_2| = |E_1|$. Clearly, $G_1 \cup G_2$ is a PCG with

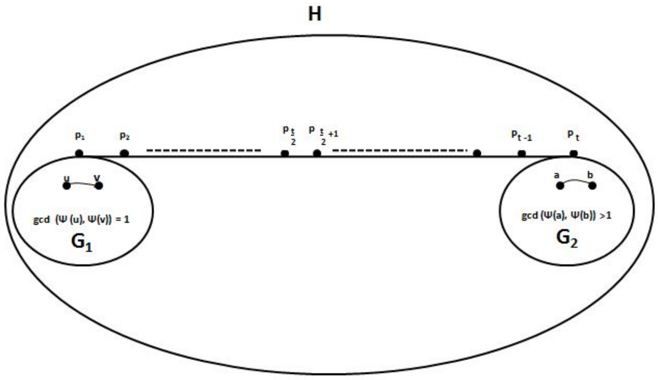

FIGURE 2.4: PCL of a graph acquired by joining two graphs by P_t

labeling f. Let H be acquired by joining these graphs by P_t (see Figure 2.4) and $p_1, p_2, ..., p_t$ denote the nodes of P_t such that p_1 is superimposed with a node of G_1, say, r having label 1 and p_t is superimposed with any node of G_2. Clearly, $|V(H)| = 2n+t-2$ and $|E(H)| = 2m+t-1$. Define $\psi : V(H) \rightarrow \{1, 2, ..., 2n+t-2\}$ by letting $\psi(u_i) = f(u_i)$ for $u_i \in G_1 \cup G_2$. Now arise two cases.

Case (i) When 't' is even.

Out of the remaining $t-2$ labels, assign odd labels to $p_2, p_3, ..., p_{\frac{t}{2}}$ and even labels to $p_{\frac{t}{2}+1}, p_{\frac{t}{2}+2}, ..., p_{t-1}$, simultaneously from $\{2n+1, 2n+2, ...2n+t-2\}$. Clearly, $|e_\psi(0) - e_\psi(1)| = 1$.

Case (ii) When 't' is odd.

Out of the remaining $t-2$ labels, assign odd labels to $p_2, p_3, ..., p_{\lfloor \frac{t}{2} \rfloor}$ and even labels to $p_{\lceil \frac{t}{2} \rceil}, ..., p_{t-1}$, simultaneously from $\{2n+1, 2n+2, ..., 2n+t-2\}$. Clearly, $|e_\psi(0) - e_\psi(1)| = 0$.

Hence, H is a PCG. □

Theorem 2.2.16. *If $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are of same order and size such that their disjoint union is a PCG, then graph acquired by joining G_1 and G_2 using a path of finite length admits a PCL.*

The proof is in similar lines of Theorem 2.2.15.

2.3 PCL in the Context of Corona

Though, a significant amount of work concerning PCL of graphs using various graph operations has been done, yet PCL of graphs in the context of an interesting graph operation called corona has to be explored. The Corona product is of utmost significance in data analytics, where enormous amounts of data must be evaluated quickly in order to make a judgement. It can be applied to forensic analysis and DNA sampling in biotechnology besides understanding compound structures in chemistry. This makes "corona" an important graph operation in graph theory. To begin with, a few notable established results in PCL are recalled here.

Definition 2.3.1. [69] "Consider $P_n^{(1)}$ and $P_n^{(2)}$ with $V(P_n^{(1)}) = \{u_1, u_2, ..., u_n\}$ and $V(P_n^{(2)}) = \{v_1, v_2, ..., v_n\}$. Join the nodes $u_{\frac{n+1}{2}}$ and $v_{\frac{n+1}{2}}$ by an edge, if n is odd, otherwise join $u_{\frac{n}{2}}$ with $v_{\frac{n}{2}+1}$. The resultant graph is called a $H - graph$ on $2n$ nodes."

Theorem 2.3.1. *[70] $P_n \odot K_{1,n-1}$ is a PCG.*

Theorem 2.3.2. *[72] Corona of $H - graph$ with K_1 admits a PCL.*

In pursuant of this, the following results are obtained.

Theorem 2.3.3. *$P_n \odot \bar{K}_1$ permits a PCL $\forall\ n \geq 2$.*

Proof. Let $V(P_n) = \{p_1, p_2, ..., p_n\}$ and $E(P_n) = \{p_j p_{j+1} : 1 \leq j \leq n-1\}$. Let G be acquired by taking the corona of P_n with $\bar{K}_1$ having $V(G) = V(P_n) \cup \{p'_1, p'_2, ..., p'_n\}$ and, $E(G) = E(P_n) \cup \{p_j p'_j : 1 \leq j \leq n\}$. Clearly, $|V(G)| = 2n$ and $|E(G)| = 2n - 1$. Consider $\psi : V(G) \to \{1, 2, ..., 2n\}$. Fix $\psi(p_1) = 2$, $\psi(p_j) = \psi(p_{j-1}) + 2$, for $2 \leq j \leq n$ and $\psi(p'_j) = \psi(p_j) - 1$; $1 \leq j \leq n$. Evidently, $e_\psi(0) = n - 1$ and $e_\psi(1) = n$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$. Hence, G is a PCG (see Figure 2.5). □

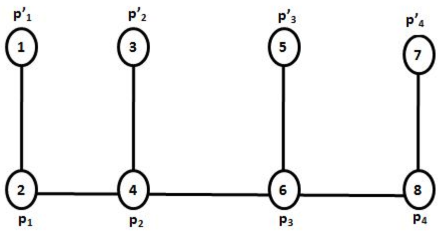

FIGURE 2.5: PCL of $P_4 \odot \bar{K}_1$

Theorem 2.3.4. *$C_n \odot \bar{K}_1$ permits a PCL $\forall\ n \geq 3$.*

Proof. Let $V(C_n) = \{c_1, c_2, ..., c_n\}$ and $E(C_n) = \{c_j c_{j+1} : 1 \leq j \leq n-1\} \cup \{c_n c_1\}$. Let $G = C_n \odot \bar{K}_1$ with $V(G) = V(C_n) \cup \{c'_1, c'_2, ..., c'_n\}$ and $E(G) = E(C_n) \cup \{c_j c'_j : 1 \leq j \leq n\}$. Clearly, $|V(G)| = 2n$ and $|E(G)| = 2n$. Consider $\psi : V(G) \to \{1, 2, ..., 2n\}$. Fix $\psi(c_1) = 2$, $\psi(c_j) = \psi(c_{j-1}) + 2$; $2 \leq j \leq n$ and $\psi(c'_j) = \psi(c_j) - 1$; $1 \leq j \leq n$. Following the above pattern, $e_\psi(0) = e_\psi(1) = n$ which shows that G is a PCG. □

Theorem 2.3.5. *$W_n \odot \bar{K}_1$ permits a PCL $\forall\ n \geq 3$.*

Proof. Let $V(W_n) = \{w_0\} \cup \{w_1, w_2, ..., w_n\}$ with w_0 the apex node, and $E(W_n) = \{w_j w_{j+1} : 1 \leq j \leq n-1\} \cup \{w_n w_1\} \cup \{w_0 w_j : 1 \leq j \leq n\}$. Let $G = W_n \odot \bar{K}_1$ with $V(G) = V(W_n) \cup \{w'_0, w'_1, w'_2, ..., w'_n\}$ & $E(G) = E(W_n) \cup \{w_j w'_j : 1 \leq j \leq n\} \cup \{w_0 w'_0\}$. Clearly, $|V(G)| = 2n+2$ and $E(G) = 3n+1$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+2\}$. Let $\psi(w_0) = 2$ and $\psi(w_1) = 4$. Now the given cases arise.

Case (i) If 'n' is even.

Fix $\psi(w'_0) = 1$, $\psi(w_{\frac{n}{2}}) = 6$, $\psi(w_{\frac{n}{2}+1}) = 3$, $\psi(w_j) = \psi(w_{j-1}) + 4$; $\frac{n}{2} + 2 \leq j \leq n$, $\psi(w'_j) = \psi(w_j) + 2$; $\frac{n}{2} + 1 \leq j \leq n$ and assign the remaining labels to unlabeled nodes (see Figure 2.6). Observe that $|e_\psi(0) - e_\psi(1)| \leq 1$.

Case (ii) If 'n' is odd.

Fix $\psi(w'_0) = 2n+1$, $\psi(w_{\frac{n+1}{2}}) = 6$, $\psi(w'_{\frac{n+1}{2}}) = 1$, $\psi(w_{\frac{n+1}{2}+1}) = 3$, $\psi(w_j) = \psi(w_{j-1}) + 4$; $\frac{n+1}{2} + 2 \leq j \leq n$ and $\psi(w'_j) = \psi(w_j) + 2$; $\frac{n+1}{2} + 1 \leq j \leq n$. Assign the remaining labels to unlabeled nodes. Here $e_\psi(0) = e_\psi(1)$.

Thus, G is a PCG. □

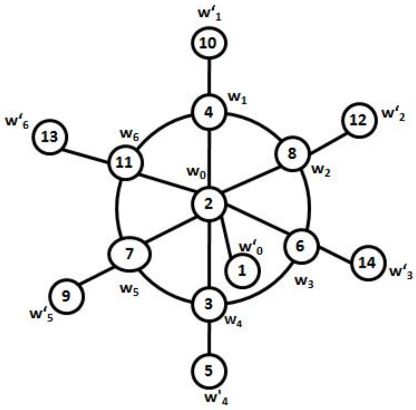

FIGURE 2.6: PCL of $W_6 \odot \bar{K}_1$

Theorem 2.3.6. *$DW_n \odot \bar{K}_1$ permits a PCL $\forall$ $n \geq 3$.*

Proof. Let $V(DW_n) = \{x_0, x_i, y_i : 1 \leq i \leq n\}$ where x_i; $1 \leq i \leq n$ and y_i; $1 \leq i \leq n$ are rim nodes of inner and outer cycles respectively. Let $G = DW_n \odot \bar{K}_1$ with $V(G) = V(W_n) \cup \{x'_0, x'_i, y'_i : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 4n+2$ and $|E(G)| = 6n+1$. Consider $\psi : V(G) \to \{1, 2, ..., 4n+2\}$. Let $\psi(x_0) = 2$, $\psi(x'_0) = 3$, $\psi(y_1) = 1$, $\psi(y'_1) = 5$, $\psi(y_2) = 7$, $\psi(y_i) = \psi(y_{i-1}) + 4$; $3 \leq i \leq n$ and $\psi(y'_i) = \psi(y_i) + 2$; $2 \leq i \leq n$. Assign the remaining labels to x_i & x'_i ; $1 \leq i \leq n$. Clearly, $|e_\psi(0) - e_\psi(1)| \leq 1$ establishing that G is a PCG. □

Theorem 2.3.7. *$G_n \odot \bar{K}_1$ permits a PCL $\forall$ $n \geq 3$.*

Proof. Let node and edge set of G_n are $\{v_0, v_j, u_j : 1 \leq j \leq n\}$ and, $\{v_0v_j : 1 \leq j \leq n\} \cup \{u_jv_j : 1 \leq j \leq n\} \cup \{u_jv_{j+1} : 1 \leq j \leq n-1\} \cup \{u_nv_1\}$ respectively. Let $G = G_n \odot \bar{K}_1$ with $V(G) = V(G_n) \cup \{v'_0, v'_j, u'_j : 1 \leq j \leq n\}$ and $E(G) = E(G_n) \cup \{u_ju'_j, v_jv'_j : 1 \leq j \leq n\} \cup \{v_0v'_0\}$. Clearly, $|V(G)| = 4n + 2$ and $|E(G)| = 5n + 1$. Consider $\psi : V(G) \to \{1, 2, ..., 4n+2\}$ defined by fixing $\psi(v_0) = 2$, $\psi(v'_0) = 1$ and $\psi(v_1) = 4$. The given cases arise.

Case (i) If 'n' is odd.

Let $\psi(v_{\frac{n+1}{2}}) = 6$, $\psi(u_{\frac{n+1}{2}}) = 3$, $\psi(v_{\frac{n+1}{2}+1}) = 7$, $\psi(v_j) = \psi(v_{j-1}) + 8$; $\frac{n+1}{2} + 2 \leq j \leq n$, $\psi(u_j) = \psi(u_{j-1}) + 8$; $\frac{n+1}{2} + 1 \leq j \leq n$, $\psi(u'_j) = \psi(u_j) + 2$; $\frac{n+1}{2} \leq j \leq n$, $\psi(v'_j) = \psi(v_j) + 2$ for $\frac{n+1}{2} + 1 \leq j \leq n$. Now assign unused even labels to unlabeled nodes.

Case (ii) If 'n' is even.

Let $\psi(u_{\frac{n}{2}}) = 6$, $\psi(v_{\frac{n}{2}+1}) = 3$, $\psi(u_{\frac{n}{2}+1}) = 7$, $\psi(v_j) = \psi(v_{j-1}) + 8$; $\frac{n}{2} + 2 \leq j \leq n$, $\psi(u_j) = \psi(u_{j-1}) + 8$; $\frac{n}{2} + 2 \leq j \leq n$, $\psi(v'_{\frac{n}{2}+1}) = 9$, $\psi(u'_{\frac{n}{2}+1}) = 5$, $\psi(u'_j) = \psi(u_j) + 2$; $\frac{n}{2} + 2 \leq j \leq n$, $\psi(v'_j) = \psi(v_j) + 2$; $\frac{n}{2} + 2 \leq j \leq n$. Now assign unused even labels to unlabeled nodes.

In both the cases, $|e_\psi(0) - e_\psi(1)| \leq 1$ showing that G is a PCG. □

Theorem 2.3.8. *$Fl_n \odot \bar{K}_1$ permits a PCL $\forall\, n \geq 3$.*

Proof. Let $V(Fl_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ and $E(Fl_n) = \{v_0v_i, v_0u_i, v_iu_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\}$, where v_i , u_i &, v_0 are nodes of degree 4, 2 and $2n$ respectively. Let $G = Fl_n \odot \bar{K}_1$ with $V(G) = V(Fl_n) \cup \{v'_0, v'_i, u'_i : 1 \leq i \leq n\}$ and $E(G) = E(Fl_n) \cup \{v_0v'_0\} \cup \{v_iv'_i : 1 \leq i \leq n\} \cup \{u_iu'_i : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 4n + 2$, and $|E(G)| = 6n + 1$. Consider $\psi : V(G) \to \{1, 2, ..., 4n + 2\}$. Let $\psi(v_0) = 2$, $\psi(v'_0) = 1$, $\psi(v_1) = 4$, $\psi(v_i) = \psi(v_{i-1}) + 4$; $2 \leq i \leq n$, $\psi(v'_i) = \psi(v_i) + 2$; $1 \leq i \leq n$, $\psi(u_i) = \psi(v_i) + 1$; $1 \leq i \leq n$ and $\psi(u'_i) = \psi(v_i) - 1$; $1 \leq i \leq n$. Following this, $|e_\psi(0) - e_\psi(1)| \leq 1$. Hence, G is a PCG (see Figure 2.7). □

Theorem 2.3.9. *$f_n \odot \bar{K}_1$ permits a PCL $\forall\, n \geq 3$.*

Proof. Let $V(f_n) = \{u_0, u_i : 1 \leq i \leq n\}$, where u_0 is apex node, and $V(f_n \odot \bar{K}_1) = V(f_n) \cup \{u'_0, u'_i : 1 \leq i \leq n\}$. Clearly, $|V(f_n \odot \bar{K}_1)| = 2n + 2$ and $|E(f_n \odot \bar{K}_1)| = 3n$. Consider $\psi : V(f_n \odot \bar{K}_1) \to \{1, 2, ..., 2n+2\}$. Fix $\psi(u_0) = 2$ and $\psi(u'_0) = 1$. There arise the given cases.

Case (i) If 'n' is even.

Fix $\psi(u_{\frac{n}{2}}) = 6$, $\psi(u_{\frac{n}{2}+1}) = 3$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $\frac{n}{2} + 2 \leq i \leq n$, $\psi(u'_i) = \psi(u_i) + 2$; $\frac{n}{2} + 1 \leq i \leq n$. Assign even labels to unlabeled nodes. Note that $e_\psi(0) = e_\psi(1)$.

Case (ii) If 'n' is odd.

Fix $\psi(u_{\lceil\frac{n}{2}\rceil}) = 6$, $\psi(u'_{\lceil\frac{n}{2}\rceil}) = 3$, $\psi(u_{\lceil\frac{n}{2}\rceil+1}) = 5$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $\lceil\frac{n}{2}\rceil + 2 \leq i \leq n$,

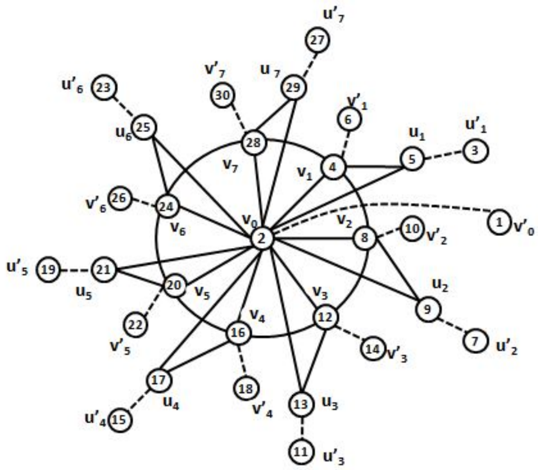

FIGURE 2.7: PCL of $Fl_7 \odot \bar{K}_1$

$\psi(u_i') = \psi(u_i) + 2$; $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$. Assigning even labels to unlabeled nodes yields, $e_\psi(0) = e_\psi(1) + 1$.

Hence, $f_n \odot \bar{K}_1$ is a PCG (see Figure 2.8). □

Theorem 2.3.10. *$Df_n \odot \bar{K}_1$ permits a PCL.*

Proof. Let $V(Df_n) = \{x_0, y_0, u_i : 1 \leq i \leq n\}$. Consider $Df_n \odot \bar{K}_1$ with $V(Df_n \odot \bar{K}_1) = V(Df_n) \cup \{x_0', y_0', u_i' : 1 \leq i \leq n\}$ and $E(Df_n \odot \bar{K}_1) = E(Df_n) \cup \{x_0x_0', y_0y_0', u_iu_i' : 1 \leq i \leq n\}$. Clearly, $|V(Df_n \odot \bar{K}_1)| = 2n + 4$ and $|E(Df_n \odot \bar{K}_1)| = 4n + 1$. Define $\psi : V(Df_n \odot \bar{K}_1) \to \{1, 2, ..., 2n + 4\}$ by fixing $\psi(x_0) = 2$, $\psi(y_0) = 1$, $\psi(x_0') = 2n + 4$, $\psi(y_0') = 2n + 3$, $\psi(u_1) = 4$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2 \leq i \leq n$ and $\psi(u_i') = \psi(u_i) - 1$; $1 \leq i \leq n$. Note that $|e_\psi(0) - e_\psi(1)| \leq 1$, which proves the theorem (see Figure 2.8). □

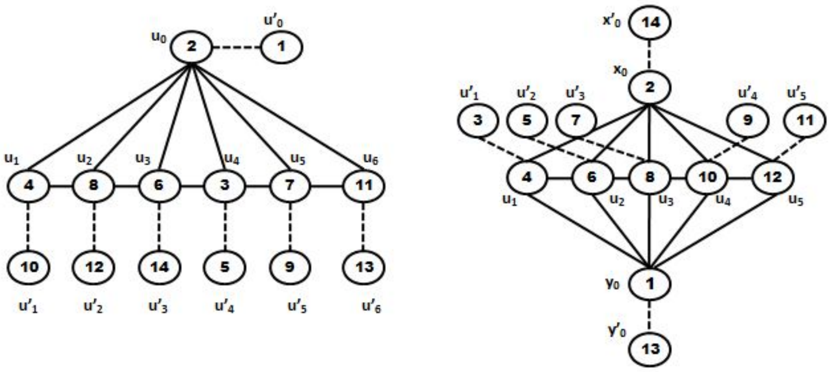

FIGURE 2.8: PCL of $f_6 \odot \bar{K}_1$ and $Df_5 \odot \bar{K}_1$

Theorem 2.3.11. *[3] "If G is a PCG then $G \pm e$ is also a PCG."*

Theorem 2.3.12. *$K_{1,n} \odot \bar{K}_1$ admits a PCL $\forall$ $n \geq 2$.*

Proof. Let $V(K_{1,n}) = \{k_0, k_i : 1 \leq i \leq n\}$. Consider corona of $K_{1,n}$ with $\bar{K}_1$ with node set $V(K_{1,n}) \cup \{k'_0, k'_i : 1 \leq i \leq n\}$ and edge set $E(K_{1,n}) \cup \{k_0k'_0, k_ik'_i : 1 \leq i \leq n\}$. Note that cardinality of node and edge set of $K_{1,n} \odot \bar{K}_1$ is respectively $2n + 2$ and $2n + 1$. Consider $\psi : V(K_{1,n} \odot \bar{K}_1) \to \{1, 2, ..., 2n + 2\}$ by fixing $\psi(k_0) = 2$, $\psi(k'_0) = 1$, $\psi(k_i) = 2i + 2; 1 \leq i \leq n$ and $\psi(k'_i) = \psi(k_i) - 1; 1 \leq i \leq n$. Clearly, $|e_\psi(0) - e_\psi(1)| \leq 1$ proving that $K_{1,n} \odot \bar{K}_1$ is a PCG. □

Remark 2.1. $S(K_{1,n})$ is obtained from $K_{1,n} \odot \bar{K}_1$ by deleting an edge. Thus, $S(K_{1,n})$ admits a PCL by Theorem 2.3.11 and Theorem 2.3.12.

Theorem 2.3.13. *$B_{n,m} \odot \bar{K}_1$ permits a PCL.*

Proof. Let $V(B_{n,m}) = \{x_0, y_0, x_i, y_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. Let $G = B_{n,m} \odot \bar{K}_1$ with $V(G) = V(B_{n,m}) \cup \{x'_0, y'_0, x'_i, y'_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. Clearly, $|V(G)| = 2n+2m+4$ and $|E(G)| = 2n + 2m + 3$. Consider $\psi : V(G) \to \{1, 2, ..., 2n + 2m + 4\}$. There arise given cases.

Case (i) If $n = m$.

Let $\psi(x_0) = 2$, $\psi(x'_0) = 2n + 2m + 4$, $\psi(y_0) = 1$, $\psi(y'_0) = 2n + 2m + 3$, $\psi(y_1) = 3$, $\psi(y_i) = \psi(y_{i-1}) + 4; 2 \leq i \leq m$ and $\psi(y'_i) = \psi(y_i) + 2; 1 \leq i \leq m$. Assign even labels to x_i & x'_i ; $1 \leq i \leq n$ in any fashion. One can see that $|e_\psi(0) - e_\psi(1)| \leq 1$ proving that $B_{n,m} \odot K_1$ is a PCG.

Case (ii) If $n \neq m$.

Assume that $n > m$. Fix $\psi(x_0) = 2$, $\psi(y_0) = 1$, $\psi(y_1) = 3$, $\psi(y_i) = \psi(y_{i-1}) + 4$; $2 \leq i \leq m$, $\psi(y'_i) = \psi(y_i) + 2$; $1 \leq i \leq m$. Assign even labels to x_i & x'_i simultaneously from available labels and once even labels are consumed, allot unused odd labels to remaining x_i and x'_i in the same way that was followed for the labeling of y_i and y'_i. Finally, allot the last unutilized label to x'_0. Here $|e_\psi(0) - e_\psi(1)| \leq 1$ and hence the result. □

Theorem 2.3.14. *$K_4 \odot P_n$ permits a PCL $\forall$ $n \geq 2$.*

Proof. Suppose $V(K_4 \odot P_n) = \{k_i, p_i^{(j)} : 1 \leq i \leq 4, 1 \leq j \leq n\}$ where $\{k_i : 1 \leq i \leq 4\}$ are nodes of K_4, and $E(K_4 \odot P_n) = E(K_4) \cup \{k_ip_i^{(j)} : 1 \leq i \leq 4, 1 \leq j \leq n\} \cup \{p_i^{(j)}p_i^{(j+1)} : 1 \leq i \leq 4, 1 \leq j \leq n - 1\}$. Note that $|V(K_4 \odot P_n)| = 4n + 4$ & $|E(K_4 \odot P_n)| = 8n + 2$. Consider $\psi : V(K_4 \odot P_n) \to \{1, 2, ..., 4n+4\}$ by taking $\psi(k_1) = 2$, $\psi(k_2) = 4$, $\psi(k_3) = 8$, $\psi(k_4) = 1$, $\psi(p_2^{(n-1)}) = 10$, $\psi(p_2^{(n)}) = 5$, $\psi(p_3^{(1)}) = 3$, $\psi(p_3^{(2)}) = 9$. Assign the remaining even labels to $p_1^{(j)}$; $1 \leq j \leq n$ and $p_2^{(j)}$; $1 \leq j \leq n - 2$. Next, assign unused odd labels simultaneously, to the remaining unlabeled nodes. Observe, $\gcd(\psi(k_1), \psi(k_2)) > 1$, $\gcd(\psi(k_2), \psi(k_3)) > 1$, $\gcd(\psi(k_1), \psi(k_3)) > 1$, $\gcd(\psi(k_1), \psi(p_1^{(j)})) > 1$; $1 \leq j \leq n$, $\gcd(\psi(k_2), \psi(p_2^{(j)})) > 1$; $1 \leq j \leq n - 1$, $\gcd(\psi(p_1^{(j)}), \psi(p_1^{(j+1)})) > 1$; $1 \leq j \leq n - 1$,

$\gcd(\psi(p_2^{(j)}), \psi(p_2^{(j+1)})) > 1$; $1 \leq j \leq n-1$, and $\gcd(\psi(p_3^{(1)}), \psi(p_3^{(2)})) > 1$. The remaining edges are labeled 1 (see Figure 2.9), resulting which $e_\psi(0) = e_\psi(1) = 4n+1$, hence the result. □

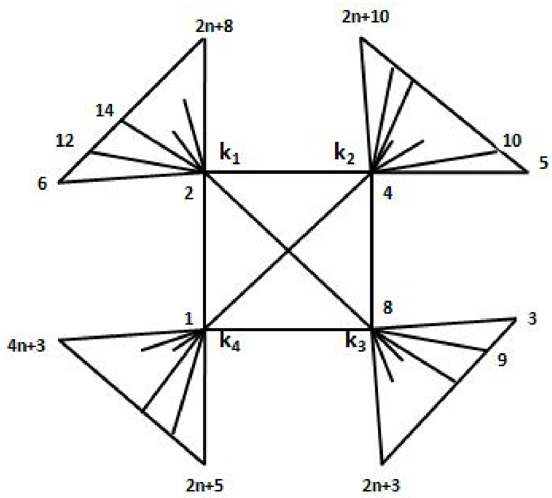

FIGURE 2.9: PCL of $K_4 \odot P_n$

Theorem 2.3.15. *$K_4 \odot \bar{K_n}$ permits a PCL $\forall\ n \geq 2$.*

Proof. Let $V(K_4 \odot \bar{K_n}) = \{k_i, u_i^{(j)} : 1 \leq i \leq 4, 1 \leq j \leq n\}$ where $\{k_i : 1 \leq i \leq 4\}$ are nodes of K_4, and $E(K_4 \odot \bar{K_n}) = E(K_4) \cup \{k_i u_i^{(j)} : 1 \leq i \leq 4, 1 \leq j \leq n\}$. Note that $|V(K_4 \odot \bar{K_n})| = 4n+4$ and $|E(K_4 \odot \bar{K_n})| = 4n+6$. Consider $\psi : V(K_4 \odot \bar{K_n}) \rightarrow \{1, 2, ..., 4n+4\}$ by taking $\psi(k_1) = 2$, $\psi(k_2) = 6$, $\psi(k_3) = 4$, $\psi(k_4) = 1$ and $\psi(u_2^{(n)}) = 3$. Assign the remaining even labels to unlabeled $u_1^{(j)}$; $1 \leq j \leq n$ and $u_2^{(j)}$; $1 \leq j \leq n-1$, and odd labels to $u_3^{(j)}$ and $u_4^{(j)}$; $1 \leq j \leq n$. Observe that $\gcd(\psi(k_1), \psi(k_2)) > 1$, $\gcd(\psi(k_2), \psi(k_3)) > 1$, $\gcd(\psi(k_1), \psi(k_3)) > 1$, $\gcd(\psi(k_1), \psi(u_1^{(j)})) > 1$; $1 \leq j \leq n$, $\gcd(\psi(k_2), \psi(u_2^{(j)})) > 1$; $1 \leq j \leq n$. Evidently, $e_\psi(0) = e_\psi(1) = 2n+3$ which proves the result. □

Theorem 2.3.16. *One point union of n-copies of K_4 allows a PCL.*

Proof. Let G be produced by taking the one point union of $n-copies$ of K_4 having $V(G) = \{k_0\} \cup \{k_{ij} : 1 \leq i \leq n, 1 \leq j \leq 3\}$ and $E(G) = \{k_0 k_{ij} : 1 \leq i \leq n, 1 \leq j \leq 3\} \cup \{k_{i1}k_{i2}, k_{i1}k_{i3}, k_{i2}k_{i3} : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 3n+1$ and $|E(G)| = 6n$. Consider $\psi : V(G) \rightarrow \{1, 2, 3, ..., 3n+1\}$ as given. Choose the largest prime p such that $3p \leq 3n+1$. Fix $\psi(k_0) = 2p$. Begining with k_{11}, allocate all even labels simultaneously to the nodes $\{k_{12}, k_{13}, k_{21}, k_{22}, k_{23}, ...\}$. There arise given cases.

Case (i) If 'n' is odd.

Allocate odd labels simultaneously from $\{1, 2, ..., 3n+1\}$ to unlabeled nodes.

Case (ii) If 'n' is even.

Fix $\psi(k_{\frac{n}{2}3}) = 1$, $\psi(k_{(\frac{n}{2}+1)1}) = 3$, $\psi(k_{(\frac{n}{2}+1)2}) = 9$ and assign unutilized labels to unlabeled nodes namely, $k_{(\frac{n}{2}+1)3}, k_{(\frac{n}{2}+2)1}, k_{(\frac{n}{2}+2)2}, ..., k_{n3}$ simultaneously from $\{5, 7, 11, 13, ..., 3n + 1\}$.

Thus, G is a PCG (see Figure 2.10). □

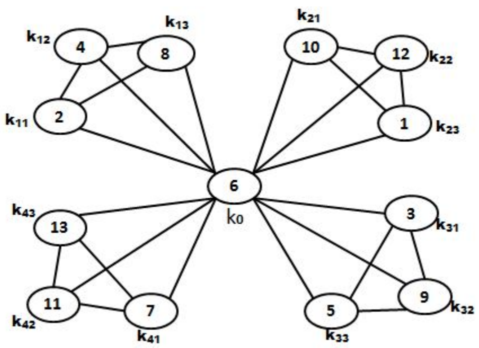

FIGURE 2.10: PCL of one point union of 4-copies of K_4

Theorem 2.3.17. *$P_n \odot \bar{K}_2$ permits a PCL.*

Proof. Let $V(P_n) = \{p_1, p_2, ..., p_n\}$ and $G = P_n \odot \bar{K}_2$ with $V(G) = V(P_n) \cup \{p'_i, p''_i : 1 \leq i \leq n\}$ and $E(G) = E(P_n) \cup \{p_ip'_i : 1 \leq i \leq n\} \cup \{p_ip''_i : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 3n$ and $|E(G)| = 3n - 1$. Consider $\psi : V(G) \to \{1, 2, ..., 3n\}$ defined by given cases.

Case (i) If 'n' is even.

Fix $\psi(p_1) = 2$, $\psi(p_{\frac{n}{2}+1}) = 1$, $\psi(p'_{\frac{n}{2}+1}) = 3$, $\psi(p''_{\frac{n}{2}+1}) = 5$. Next, $\psi(p_i) = \psi(p_{i-1}) + 2$; $2 \leq i \leq \frac{n}{2}$, $\psi(p_i) = \psi(p_{i-1}) + 6$; $\frac{n}{2} + 2 \leq i \leq n$, $\psi(p'_i) = \psi(p'_{i-1}) + 6$; $\frac{n}{2} + 2 \leq i \leq n$ and $\psi(p''_i) = \psi(p''_{i-1}) + 6$; $\frac{n}{2} + 2 \leq i \leq n$. Assign the available even labels to p'_i & p''_i where $1 \leq i \leq \frac{n}{2}$. Evidently, $|e_\psi(0) - e_\psi(1)| \leq 1$.

Case (ii) When 'n' is odd.

Fix $\psi(p_1) = 2$, $\psi(p_{\frac{n+1}{2}}) = 6$, $\psi(p'_{\frac{n+1}{2}}) = 3$, $\psi(p''_{\frac{n+1}{2}}) = 1$, $\psi(p_{\frac{n+1}{2}+1}) = 7$, $\psi(p_i) = \psi(p_{i-1}) + 6$; $\frac{n+1}{2} + 2 \leq i \leq n$, $\psi(p'_i) = \psi(p_i) + 2$; $\frac{n+1}{2} + 1 \leq i \leq n$, $\psi(p''_i) = \psi(p_i) - 2$; $\frac{n+1}{2} + 1 \leq i \leq n$. Assigning available even labels to unlabeled nodes yields $e_\psi(0) = e_\psi(1) = \frac{3n-1}{2}$.

Hence, G is a PCG. □

Theorem 2.3.18. *$C_n \odot \bar{K}_2$ permits a PCL $\forall$ $n \geq 4$.*

Proof. Let $V(C_n) = \{c_1, c_2, ..., c_n\}$ and $G = C_n \odot \bar{K}_2$ with $V(G) = V(C_n) \cup \{c'_i, c''_i : 1 \leq i \leq n\}$, $E(G) = E(C_n) \cup \{c_ic'_i, c_ic''_i : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 3n$ and $|E(G)| = 3n$. Consider $\psi : V(G) \to \{1, 2, ..., 3n\}$ defined under two cases.

Case (i) If 'n' is even.

Fix $\psi(c_1) = 2$, $\psi(c_{\frac{n}{2}}) = 6$, $\psi(c_{\frac{n}{2}+1}) = 3$, $\psi(c_{\frac{n}{2}+2}) = 7$, $\psi(c_i) = \psi(c_{i-1})+6$; $\frac{n}{2}+3 \le i \le n$, $\psi(c'_{\frac{n}{2}+1}) = 1$, $\psi(c'_{\frac{n}{2}+2}) = 9$, $\psi(c''_{\frac{n}{2}+1}) = 5$, $\psi(c'_i) = \psi(c'_{i-1}) + 6$; $\frac{n}{2} + 3 \le i \le n$ and $\psi(c''_i) = \psi(c''_{i-1}) + 6$; $\frac{n}{2} + 2 \le i \le n$. Assign the unused labels to remaining nodes in any fashion implies $|e_\psi(0) - e_\psi(1)| \le 1$.

Case (ii) If 'n' is odd.

Follow the labeling pattern of *Case* (ii) of Theorem 2.3.17.

Thus, G is a PCG. □

Theorem 2.3.19. *$P_n \odot \bar{K}_n$ permits a PCL.*

Proof. Let $V(P_n) = \{p_1, p_2, ..., p_n\}$. Let $G = P_n \odot \bar{K}_n$ with $V(G) = V(P_n) \cup \{k_{ij} : 1 \le i \le n, 1 \le j \le n\}$ and $E(G) = E(P_n) \cup \{p_i k_{ij} : 1 \le i \le n, 1 \le j \le n\}$. Clearly, $|V(G)| = n^2+n$ and $|E(G)| = n^2+n-1$. Consider $\psi : V(G) \to \{1, 2, ..., n^2+n\}$ defined under the given conditions.

Case (i) If 'n' is even.

Fix $\psi(p_1) = 2$, $\psi(p_i) = \psi(p_{i-1}) + 2$; $2 \le i \le \frac{n}{2}$, $\psi(p_{\frac{n}{2}+1}) = 1$. Consider the sequence of consecutive primes, say $q_{\frac{n}{2}+2}, q_{\frac{n}{2}+3}, ..., q_n$ such that $n^2 + n \ge q_n > q_{n-1} >, ..., > q_{\frac{n}{2}+2}$. Fix $\psi(p_i) = q_i$ for $\frac{n}{2} + 2 \le i \le n$. Assign available even labels to k_{ij} for $1 \le i \le \frac{n}{2}$, $1 \le j \le n$. Next, assign unused odd labels to k_{ij} for $\frac{n}{2} + 1 \le i \le n$, $1 \le j \le n$ simultaneously from $\{1, 2, ..., n^2 + n\}$.

Case (ii) If 'n' is odd.

Fix $\psi(p_1) = 2$, $\psi(p_i) = \psi(p_{i-1}) + 2$ for $2 \le i \le \lceil \frac{n}{2} \rceil$, $\psi(p_{\lceil \frac{n}{2} \rceil+1}) = 1$. Consider the sequence of consecutive primes, say $q_{\lceil \frac{n}{2} \rceil+2}$, $q_{\lceil \frac{n}{2} \rceil+3}, ..., q_n$ such that $n^2 + n \ge q_n > q_{n-1} > ..., > q_{\lceil \frac{n}{2} \rceil+2}$. Fix $\psi(p_i) = q_i$ for $\lceil \frac{n}{2} \rceil + 2 \le i \le n$. Assign available even labels to k_{ij} for $1 \le i \le \lfloor \frac{n}{2} \rfloor$, $1 \le j \le n$ and $k_{\lceil \frac{n}{2} \rceil j}$ for $1 \le j \le \lfloor \frac{n}{2} \rfloor$. Assign unused odd labels to $k_{\lceil \frac{n}{2} \rceil j}$ for $\lceil \frac{n}{2} \rceil \le j \le n$ and k_{ij} for $\lceil \frac{n}{2} \rceil + 1 \le i \le n$, $1 \le j \le n$ simultaneously from $\{1, 2, ..., n^2 + n\}$. Observe that $|e_\psi(0) - e_\psi(1)| \le 1$.

Hence, G is a PCG (see Figure 2.11). □

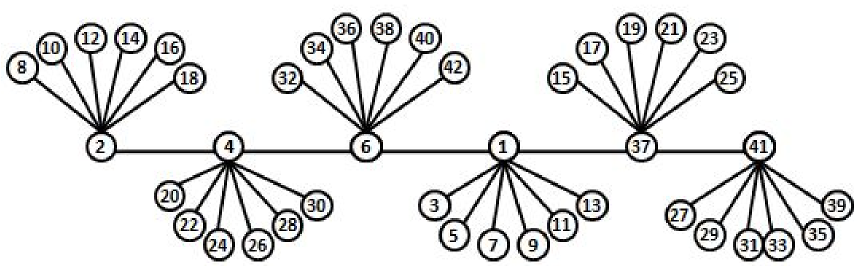

Figure 2.11: PCL of $P_6 \odot \bar{K}_6$

Remark 2.2. One can also derive the PCL of $C_n \odot \bar{K}_n$ in a similar way with the Theorem 2.3.19.

2.4 PCL of Lilly Related Graphs

A data structure is a particular way of organizing data in a computer and therefore trees constitute an important class of graphs in graph theory. Many researchers are investigating trees for different kind of graph labelings. Baskar Babujee et al., in [8] proved that the double star $K_{1,n,n}$ for $n \geq 3$ and the full binary tree admits a PCL. Motivated by [8] and [64], some results on a tree family, named, lilly graph are derived here.

Definition 2.4.1. [64] "Lilly graph I_n, $n \geq 2$ is formed by two star graphs $2K_{1,n}$, $n \geq 2$ and two path graphs $2P_n$, $n \geq 2$ which share a common node. i.e; $I_n = 2K_{1,n} \diamond 2P_n$. For illustration, refer to Figure 2.12."

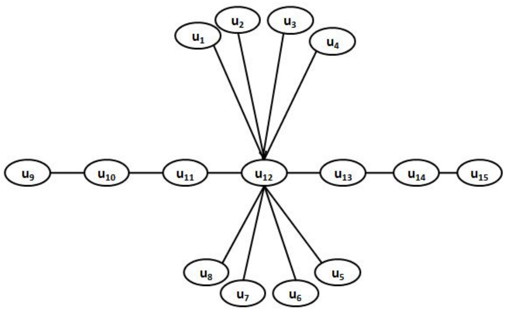

FIGURE 2.12: I_4

Throughout this section $V(I_n) = \{u_1, u_2, ..., u_{4n-1}\}$ and $E(I_n) = \{u_{3n}u_i : 1 \leq i \leq 2n\} \cup \{u_iu_{i+1} : 2n+1 \leq i \leq 4n-2\}$. Note that $\{u_1, u_2, ..., u_{2n}\}$ and $\{u_{2n+1}, u_{4n-1}\}$ are pendant nodes of I_n in which the former and latter are representing respectively the star pendant nodes and path pendant nodes in I_n. Also u_{3n} is apex node. Clearly, $|V(I_n)| = 4n-1$ and $|E(I_n)| = 4n-2$.
Note: Edward samuel [64] used + sign to represent the definition of lilly graph. Since + denotes the join operation in general, so $\diamond$ is used in this thesis.

Theorem 2.4.1. *I_n admits a PCL.*

Proof. Let $V(I_n) = \{u_1, u_2, ..., u_{4n-1}\}$ and $E(I_n) = \{u_{3n}u_i : 1 \leq i \leq 2n\} \cup \{u_iu_{i+1} : 2n+1 \leq i \leq 4n-2\}$. Clearly, $|V(I_n)| = 4n-1$ and $|E(I_n)| = 4n-2$. Consider $\psi : V(G) \to \{1, 2, ..., 4n-1\}$. Fix $\psi(u_{3n}) = 2$, $\psi(u_{2n+1}) = 4$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2n+2 \leq i \leq 3n-1$, $\psi(u_{3n+1}) = 3$, $\psi(u_{3n+2}) = 9$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $3n+3 \leq i \leq 4n-1$, $\psi(u_1) = \psi(u_{3n-1}) + 2$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2 \leq i \leq n-1$, $\psi(u_n) = 1$,

$\psi(u_{n+1}) = 5$, $\psi(u_{n+2}) = 7$, $\psi(u_{n+3}) = \psi(u_{4n-1})+2$, $\psi(u_i) = \psi(u_{i-1})+2$; $n+4 \leq i \leq 2n$. Observe that $\gcd(\psi(u_{3n}), \psi(u_i)) \neq 1$; $n-1 \geq i \geq 1$, $\gcd(\psi(u_{3n+1}), \psi(u_{3n+2})) \neq 1$ and $\gcd(\psi(u_i), \psi(u_{i+1})) \neq 1$; $2n+1 \leq i \leq 3n-1$. Evidently, $e_\psi(0) = e_\psi(1) = 2n-1$ which shows that I_n is a PCG (see Figure 2.13). □

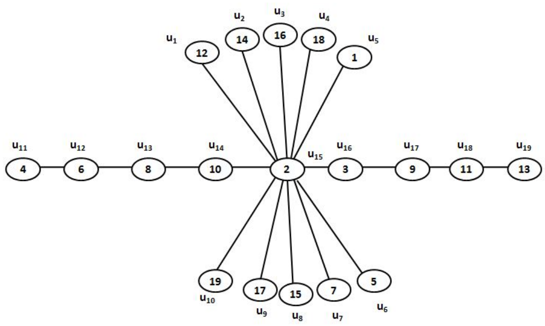

FIGURE 2.13: PCL of I_5

Theorem 2.4.2. *Switching of an arbitrary pendant node in I_n admits a PCL for $n \geq 4$.*

Proof. Let G be constructed by switching an arbitrary pendant node of I_n say, u_k, where $k \in \{1, 2, ..., 2n, 2n+1, 4n-1\}$. Clearly, $|V(G)| = 4n-1$ and $|E(G)| = 8n-6$. Consider $\psi : V(G) \to \{1, 2, ..., 4n-1\}$ defined under the given conditions.

Case (i) When $k \in \{1, 2, ..., 2n\}$.

Fix $\psi(u_k) = 2$, $\psi(u_{3n}) = 6$. Assign all available even labels out of $\{1, 2, ..., 4n-1\}$ to $u_1, u_2, ...u_{k-1}, u_{k+1}, ..., u_{2n}$ as a result of which two nodes out of $u_1, u_2, ...u_{k-1}, u_{k+1}, ..., u_{2n}$ can not be labeled (since there are exactly $\frac{4n}{2} - 1$ number of even labels available). For unlabeled star pendant nodes, assign the labels 3 and 9. Next, fix $\psi(u_{3n-1}) = 1, \psi(u_{3n+1}) = 15$ and $\psi(u_{3n+2}) = p$, where p is the largest prime $\leq 4n-1$, and assign the unused (odd) labels simultaneously to unlabeled nodes (see Figure 2.14). Observe that $\gcd(\psi(u_{3n}), \psi(u_i)) > 1$, ; $1 \leq i \leq 2n$, $\gcd(\psi(u_{3n}), \psi(u_{3n+1})) > 1$ and gcd of $\psi(u_k)$ with all pendant nodes of star except for the those that are labeled with 3 and 9, is greater than 1. Clearly there are exactly $4n-3$ edges having label 0. Evidently, $|e_\psi(0) - e_\psi(1)| \leq 1$.

Case (ii) Switching of either u_{2n+1} or u_{4n-1}.

Without loss of generality, switch u_{2n+1}. Fix $\psi(u_{2n+1}) = 2$, $\psi(u_{3n}) = 6$, $\psi(u_1) = 4$, $\psi(u_2) = 8$, $\psi(u_i) = \psi(u_{i-1})+2$; $3 \leq i \leq 2n-3$, $\psi(u_{2n-2}) = 3$, $\psi(u_{2n-1}) = 5$, $\psi(u_{2n}) = 9$, $\psi(u_{3n-1}) = 1$ & $\psi(u_{3n+1}) = p$, where p is the largest prime $\leq 4n-1$. Assign the unused labels out of $\{1, 2, ..., 4n-1\}$ simultaneously to remaining nodes. Observe that $\gcd(\psi(u_{3n}), \psi(u_i)) > 1$; $i = 1, 2, ...2n, 2n+1$; $i \neq 2n-1$, $\gcd(\psi(u_{2n+1}), \psi(u_i)) > 1$; $1 \leq i \leq 2n-3$. The edges formed using these nodes bear 0 which are $4n-3$ in count. The remaining edges bear 1. Evidently, $e_\psi(0) = e_\psi(1) = 4n-3$ which justifies

$|e_\psi(0) - e_\psi(1)| \leq 1$ (see Figure 2.15).
Hence the theorem. □

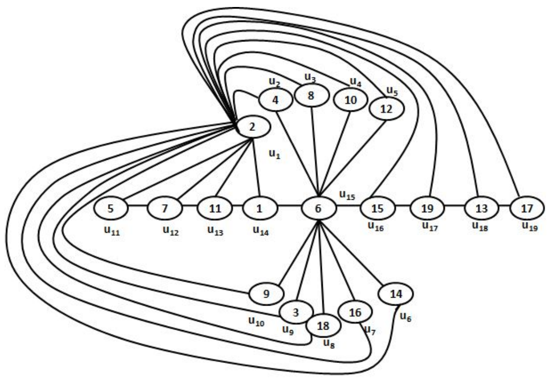

FIGURE 2.14: PCL of a graph formed by switching of u_1 in I_5

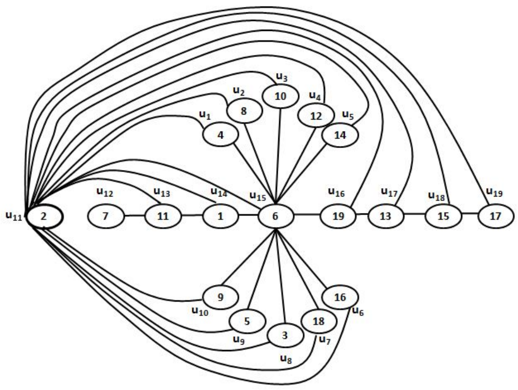

FIGURE 2.15: PCL of a graph formed by switching of u_{11} in I_5

Theorem 2.4.3. *Switching of apex node in I_n admits a PCL.*

Proof. Let G be acquired by switching u_{3n}. Clearly, $|V(G)| = 4n-1$ and $|E(G)| = 4n-8$. Consider $\psi : V(G) \to \{1, 2, ..., 4n-1\}$ as per the following algorithm. Fix $\psi(u_{3n}) = 1$, $\psi(u_{2n+1}) = 2$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2n+2 \leq i \leq 3n-1$, $\psi(u_{3n+1}) = \psi(u_{3n-1}) + 2$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $3n+2 \leq i \leq 4n-1$. Assign unused labels to the remaining nodes. Clearly, $e_\psi(0) = e_\psi(1) = 2n-4$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$. Thus, G is a PCG (see Figure 2.16). □

Theorem 2.4.4. *Duplication of apex node with a node in I_n admits a PCL for $n > 2$.*

Proof. Suppose G is formed by duplicating u_{3n} by a node, say, v. Here $V(G) = V(I_n) \cup \{v\}$ and $E(G) = E(I_n) \cup \{u_i v : 1 \leq i \leq 2n\} \cup \{u_{3n-1}v, u_{3n+1}v\}$. Clearly, $|V(G)| = 4n$ and $|E(G)| = 6n$. Consider $\psi : V(G) \to \{1, 2, ..., 4n\}$. Fix $\psi(u_{3n}) = 2$, $\psi(v) = 4$, $\psi(u_{4n-1}) = 3$, $\psi(u_{4n-2}) = 6$. Assign unused even labels to all u_i's where $i \in \{1, 2, ..., n\} \cup$

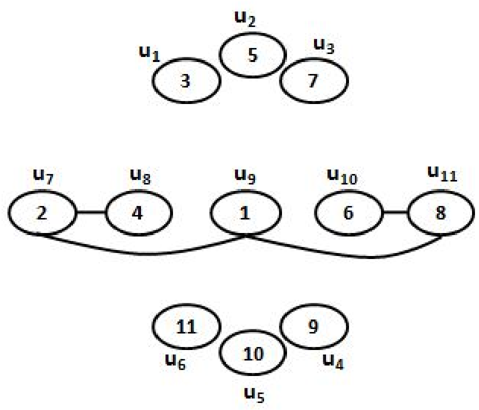

FIGURE 2.16: PCL of a graph formed by switching of u_9 in I_3

$\{3n+1\} \cup \{3n+2, ..., 4n-3\}$ in any order. Next, assign $\psi(u_{n+1}) = 1$, $\psi(u_{n+2}) = 5$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $n+3 \le i \le 3n-1$. Observe that $\gcd(\psi(u_{3n}), \psi(u_i)) > 1$; $1 \le i \le n$, $\gcd(\psi(u_i), \psi(u_{i+1})) > 1$; $3n \le i \le 4n-2$, $\gcd(\psi(v), \psi(u_i)) > 1$; $1 \le i \le n$ and $\gcd(\psi(v), \psi(u_{3n+1})) > 1$. Clearly, $e_\psi(0) = e_\psi(1) = 3n$ showing that G is a PCG (see Figure 2.17). □

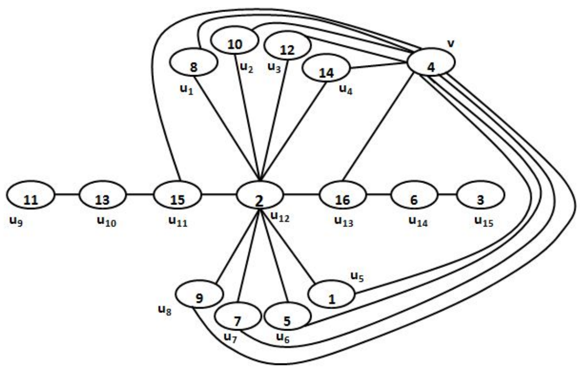

FIGURE 2.17: PCL of a graph formed by duplication of u_{3n} in I_4

Theorem 2.4.5. *Duplication of any pendant node in I_n, $n \ge 2$ permits a PCL.*

Proof. Let G be formed by duplicating any pendant node of I_n say, u_k, by a node v. Clearly, $|V(G)| = 4n$ and $|E(G)| = 4n-1$. Consider $\psi : V(G) \to \{1, 2, ..., 4n\}$ defined by letting $\psi(v) = 1$, $\psi(u_{3n}) = 2$, $\psi(u_1) = 4$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2 \le i \le n$, $\psi(u_{2n+1}) = \psi(u_n) + 2$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2n+2 \le i \le 3n-1$, $\psi(u_{n+1}) = 3$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $n+2 \le i \le 2n$, $\psi(u_{3n+1}) = \psi(u_{2n}) + 2$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $3n+2 \le i \le 4n-1$. Clearly, $e_\psi(0) = 2n-1$ and $e_\psi(1) = 2n$ which proves that G is a PCG (see Figure 2.18).

□

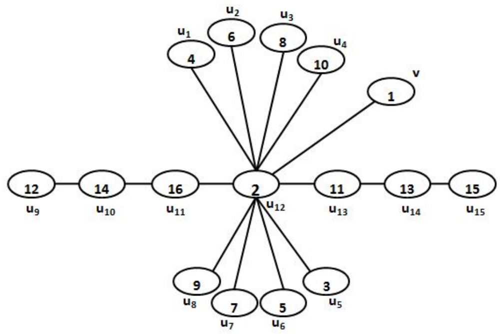

FIGURE 2.18: PCL of a graph formed by duplication of u_3 in I_4

Theorem 2.4.6. *Duplication of an arbitrary path node (except pendant and apex) in I_n permits a PCL.*

Proof. Let G be acquired by duplicating an arbitrary path node of I_n, say, u_k with a node v, where $k \in \{2n+2, 2n+3, ..., 3n-1\} \cup \{3n+1, 3n+2, ..., 4n-2\}$. Clearly, $|V(G)| = |E(G)| = 4n$. Consider $\psi : V(G) \to \{1, 2, ..., 4n\}$. Fix $\psi(u_{3n}) = 2$, $\psi(v) = 1$, $\psi(u_{3n+1}) = 3$, $\psi(u_{3n+2}) = 9$, $\psi(u_1) = 4$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2 \leq i \leq n$, $\psi(u_{2n+1}) = \psi(u_n) + 2$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2n + 2 \leq i \leq 3n - 1$. Assign unused labels simultaneously to the unlabeled nodes begining with u_{3n+3} and heading to u_{4n-1}. Next, assign the unused labels to u_i where $n + 1 \leq i \leq 2n$, in any order. Observe that $\gcd(\psi(u_{3n}), \psi(u_i)) > 1$; $1 \leq i \leq n$, $\gcd(\psi(u_i), \psi(u_{i+1})) > 1$; $2n+1 \leq i \leq 3n-1$ and $\gcd(\psi(u_{3n+1}), \psi(u_{3n+2})) > 1$. Evidently, $e_\psi(1) = 2n$ and $e_\psi(0) = 2n$ which proves that G is a PCG (see Figure 2.19). □

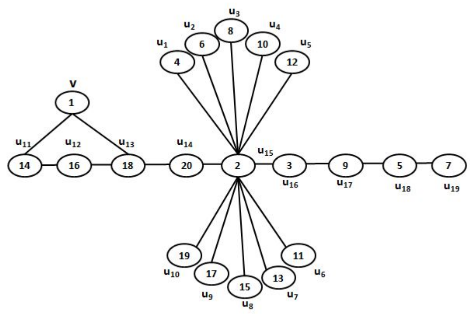

FIGURE 2.19: PCL of a graph formed by duplication of u_{12} in I_5

Theorem 2.4.7. *$DS(I_n)$, $n \geq 4$ permits a PCL.*

Proof. Let $V(DS(I_n)) = V(I_n) \cup \{v, w\}$ and $E(DS(I_n)) = E(I_n) \cup \{u_i v : 1 \leq i \leq 2n\} \cup \{u_{2n+1}v, u_{4n-1}v\} \cup \{u_i w : 2n + 2 \leq i \leq 4n - 2, i \neq 3n\}$. Clearly, $|V(DS(I_n))| = 4n + 1$ and $|E(DS(I_n))| = 8n - 4$. Define $\psi : V(DS(I_n)) \to \{1, 2, ..., 4n+1\}$ by fixing $\psi(v) = 4$, $\psi(w) = 1$, $\psi(u_{3n}) = 2$, $\psi(u_{2n+1}) = 15$, $\psi(u_{2n+2}) = 9$, $\psi(u_{2n+3}) = 3$, $\psi(u_1) = 6$ and

$\psi(u_i) = \psi(u_{i-1}) + 2$; $2 \leq i \leq 2n-2$. Assign unused labels simultaneously to unlabeled nodes. Observe that $\gcd(\psi(u_{3n}), \psi(u_i)) > 1$, ; $1 \leq i \leq 2n-2$, $\gcd(\psi(u_{2n+1}), \psi(u_{2n+2})) > 1$, $\gcd(\psi(u_{2n+2}), \psi(u_{2n+3})) > 1$ and $\gcd(\psi(v), \psi(u_i)) > 1$; $1 \leq i \leq 2n-2$. It can be found that $e_\psi(1) = 4n-2$ and $e_\psi(0) = 4n-2$ which shows that $DS(I_n)$ is a PCG (see Figure 2.20). □

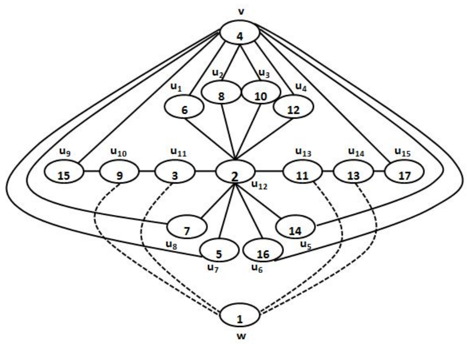

FIGURE 2.20: PCL of $DS(I_4)$

Theorem 2.4.8. *$S(I_n)$ permits a PCL.*

Proof. Let G be formed by taking the subdivision of I_n. Clearly, $V(G) = V(I_n) \cup \{v_1, v_2, ..., v_{2n}, v_{2n+1}, ..., v_{4n-2}\}$ and $E(G) = \{u_{3n}v_i : 1 \leq i \leq 2n\} \cup \{v_iu_i : 1 \leq i \leq 2n\} \cup \{u_iv_i : 2n+1 \leq i \leq 4n-2\} \cup \{v_iu_{i+1} : 2n+1 \leq i \leq 4n-2\}$. Clearly, $|V(G)| = 8n-3$ and $|E(G)| = 8n-4$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 8n-3\}$. Fix $\psi(u_{3n}) = 2$, $\psi(u_2) = 3$, $\psi(v_{2n}) = 1$, $\psi(u_{2n}) = 5$, $\psi(v_1) = 4$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq 2n-1$. Assign even labels to u_i's where $1 \leq i \leq 2n-1$, $i \neq 2$, in any order. Next, fix $\psi(u_{2n+1}) = 7$, $\psi(v_{2n+1}) = 9$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $2n+2 \leq i \leq 3n-1$, $\psi(v_i) = \psi(v_{i-1}) + 4$; $2n+2 \leq i \leq 3n-1$, $\psi(v_{3n}) = \psi(v_{3n-1}) + 2$, $\psi(u_{3n+1}) = \psi(v_{3n}) + 2$, $\psi(v_i) = \psi(v_{i-1}) + 4$; $3n+1 \leq i \leq 4n-2$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $3n+2 \leq i \leq 4n-1$. Clearly, $e_\psi(1) = 4n-2$ and $e_\psi(0) = 4n-2$ showing that G is a PCG (see Figure 2.21). □

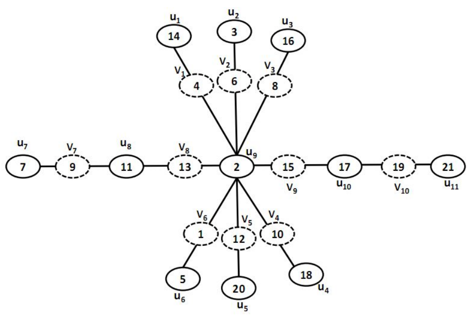

FIGURE 2.21: PCL of $S(I_3)$

Open Problems and Conjectures

Some open problems and conjectures on PCL are given here which are formulated after a keen study of literature and present work which will further fill the gap.

Conjecture 1. $P_n \odot \bar{K_m}$ permits a PCL.

This conjecture can be proved if one can find a labeling pattern. One can get the idea from Theorem 2.3.19.

Conjecture 2. If G is a k-regular PCG, then $G \odot \bar{K_1}$ is a PCG.

Since corona has been discussed for numerous graphs in this chapter, the above conjecture once proved can characterize the regular graphs for PCL.

Open problem 1. Investigate whether the graph acquired by duplicating each edge by a node, each edge by an edge and each node by a node, of a given PCG admits a PCL?

Open problem 2. Investigate whether the following graphs permit a PCL?

$C_n \odot \bar{K_m}$, $W_n \odot \bar{K_m}$, $Fl_n \odot \bar{K_m}$, $G_n \odot \bar{K_m}$

Open problem 3. If G is a k-regular PCG, then does $G \odot \bar{K_n}$ also permit a PCL? More generally, whether corona of a given PCG with any given graph, permit a PCL? The solution to open problem 3 can eventually settle open problem 2.

Open problem 4. To investigate the PCL of graphs acquired using other graph operations.

2.5 Conclusion

In this chapter, some general results for PCL of graphs are derived. Further, PCL of corona of P_n, C_n, W_n, G_n, Fl_n, $K_{1,n}$, $B_{n,m}$ etc. with $\bar{K}_1$ has been established, in addition to corona of P_n with $\bar{K_n}$. The PCL of lilly graph with various graph operations namely, switching of a node, duplication of node by a node, degree splitting graph and barycentric subdivision are discussed besides formulating some interesting conjectures and open problems for future work.

Chapter 3

PCL in the Context of Extension

3.1 Introduction

In this chapter, PCL of certain graphs in context to graph operation named, extension of a node, is explored. The concept of extension is motivated by duplication which is used in network data security. Vertex duplication acts as a foundation in Reordering Assisted Duplication/Duplication Assisted Reordering (RADAR), a method that integrates duplication and reordering into a single graph processing optimization, reaping their advantages and doing away with their disadvantages.

3.2 PCL in the Context of Extension of a node

PCL of graphs obtained by duplication operation has been investigated for various graph families. A few have been recalled with necessary definitions as follows.

Definition 3.2.1. [78] "Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$."

Definition 3.2.2. [85] "Duplication of an edge $e = uv$ by a new edge $e' = u'v'$ produces a new graph G' such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$."

Theorem 3.2.1. *[78] (i) Duplicating each edge by a node in C_n admits a PCL $\forall$ n except 4. (ii) Duplicating each node by an edge in C_n permits a PCL.*

Theorem 3.2.2. *[85] (i) Duplication of an arbitrary rim edge by an edge in W_n, $\forall$ $n \geq 6$ is a PCG. (ii) Duplication of an arbitrary spoke edge by an edge in W_n is a PCG for $n = 7$, $n \geq 9$.*

Theorem 3.2.3. *[66] Duplication of a rim node by node in H_n admits a PCL.*

Motivated by the concept of duplication, some results on duplication are presented. Moreover, an operation named, extension of a node, given by [48], is considered and a few more results on PCL are obtained.

Definition 3.2.3. [48] "An extension of a node u of H by a new node w, results in a new graph K such that $N(w) = N[u]$ (see Figure 3.1)."

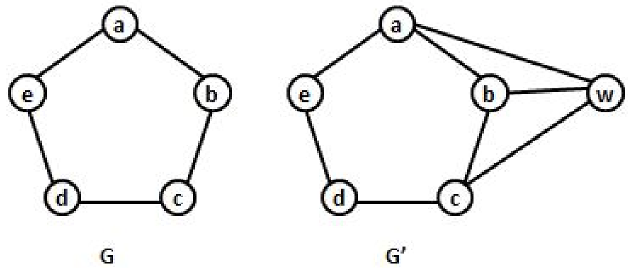

FIGURE 3.1: Graph G and extension of node b by w in G

Throughout this thesis extension means extension of node by a node.

Lemma 3.2.1. *Extension of an arbitrary node in K_n gives rise to K_{n+1}.*

Proof. Since the newly added node is joined with all the nodes of K_n including the node itself as every pair of nodes in K_n are adjacent, which eventually gives rise to K_{n+1}. □

Theorem 3.2.4. *Graph G formed by performing extension of an arbitrary node in K_n does not admit a PCL for $n \geq 2$.*

Proof. Proof is evident from Lemma 3.2.1 and Theorem 2.2.7. □

Theorem 3.2.5. *Duplicating each node with a node in P_n results in a PCG.*

Proof. Let $\{u_i : 1 \leq i \leq n\}$ denote the node set of P_n. Suppose G be formed by duplicating each node by a node in P_n with $V(G) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(G) = E(P_n) \cup \{u_{i-1}v_i : 2 \leq i \leq n\} \cup \{v_i u_{i+1} : 1 \leq i \leq n-1\}$. Clearly $|V(G)| = 2n$ and $|E(G)| = 3n - 3$. Consider $\psi : V(G) \to \{1, 2, ..., 2n\}$ defined under the given conditions.

Case (i) When 'n' is odd and $n \geq 7$.

Fix $\psi(u_{\lceil \frac{n}{2} \rceil}) = 6$, $\psi(v_{\lceil \frac{n}{2} \rceil}) = 2$, $\psi(u_{\lceil \frac{n}{2} \rceil + 1}) = 5$, $\psi(v_{\lceil \frac{n}{2} \rceil + 1}) = 3$. Allot the remaining even labels to $u_i; 1 \leq i < \lceil \frac{n}{2} \rceil$ and to $v_j; 2 \leq j < \lceil \frac{n}{2} \rceil$ in any fashion. Next, fix $\psi(u_i) = \psi(u_{i-1}) + 6; i > \lceil \frac{n}{2} \rceil + 1$, $\psi(v_i) = \psi(v_{i-1}) + 6; i > \lceil \frac{n}{2} \rceil + 1$. Assume that u_k is the farthest node that can be labeled by using above pattern. Assign the largest unused label out of $\{1, 2, ..., 2n\}$ to u_{k+1}. Assign $\psi(u_i) = \psi(u_{i-1}) - 12$; $i \geq k+2$ and for unlabeled v_i, fix $\psi(v_i) = \psi(u_i) - 6$ for $i \geq k+1$. Once this pattern ends, allot the unutilized labels to the remaining unlabeled node/nodes.

Case (ii) When 'n' is even and $n \geq 6$.

Fix $\psi(u_{\frac{n}{2}}) = 6$, $\psi(v_{\frac{n}{2}}) = 2$, $\psi(u_{\frac{n}{2}+1}) = 5$, $\psi(v_{\frac{n}{2}+1}) = 3$. Allot the available even labels to u_i's and v_i's for $1 \leq i < \frac{n}{2}$ in any fashion. Next, fix $\psi(u_i) = \psi(u_{i-1}) + 6$; $i > \frac{n}{2} + 1$, $\psi(v_i) = \psi(v_{i-1}) + 6$; $i > \frac{n}{2} + 1$. Assume that u_k and v_l are the farthest nodes that can be labeled by using above pattern. Label u_{k+1} with largest unused odd label out of $\{1, 2, ..., 2n\}$. Next, let $\psi(u_i) = \psi(u_{i-1}) - 12$; $k+2 \leq i \leq n$ and $\psi(v_i) = \psi(u_i) - 6$; $k+1 \leq i \leq n$(if choice exists). Once this pattern ends, assign the unutilized label, if any, to the unlabeled node/nodes.

In view of the above cases, it follows that $|e_\psi(0) - e_\psi(1)| \leq 1$. Hence, G is a PCG (see Figure 3.2). □

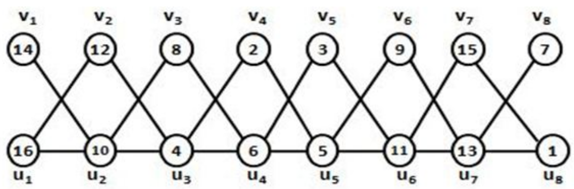

FIGURE 3.2: PCL of a graph formed by duplicating each node in P_8

Remark 3.1. It is easy to deduce the PCL of the graph formed by duplicating each node of C_n and W_n on similar lines with Theorem 3.2.5.

Theorem 3.2.6. *Extension of an arbitrary node of C_n results in a PCG for $n > 6$.*

Proof. Let $V(C_n) = \{c_i : 1 \leq i \leq n\}$ and G be produced by taking the extension of an arbitrary node of C_n. Consider the extension of c_1 and w be the freshly inserted node. Clearly, $|V(G)| = n+1$ and $|E(G)| = n+3$. Consider $\psi : V(G) \to \{1, 2, ..., n+1\}$ defined by letting $\psi(c_1) = 6$, $\psi(c_n) = 2$ and $\psi(w) = 4$ defined under two conditions.

Case (i) If 'n' is even.

Let $\psi(c_i) = \psi(c_{i-1}) + 2$; $2 \leq i \leq \frac{n}{2} - 2$, $\psi(c_{\frac{n}{2}-1}) = 1$, $\psi(c_i) = \psi(c_{i-1}) + 2$; $\frac{n}{2} \leq i \leq n-1$.

Case (ii) If 'n' is odd.

Let $\psi(c_i) = \psi(c_{i-1}) + 2$; $2 \leq i \leq \frac{n-1}{2} - 1$, $\psi(c_{\frac{n-1}{2}}) = 1$, $\psi(c_i) = \psi(c_{i-1}) + 2$; $\frac{n-1}{2} + 1 \leq i \leq n-1$.

Clearly, $|e_\psi(0) - e_\psi(1)| \leq 1$ which ensures that G is a PCG (see Figure 3.3). □

Theorem 3.2.7. *Extension of an arbitrary node at rim of W_n, $n > 7$ allows a PCL.*

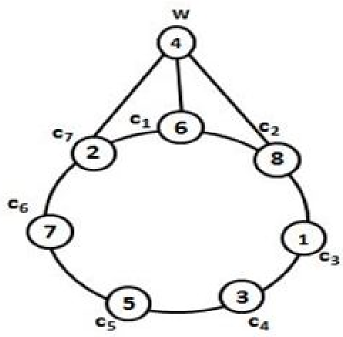

FIGURE 3.3: PCL of a graph formed by taking the extension of v_1 in C_7

Proof. Let $V(W_n) = \{w_0, w_1, w_2, ..., w_n\}$ and G be produced by taking extension of an arbitrary rim node. Without loss of generality suppose extension of w_1 is taken and x is newly added node. Clearly, $|V(G)| = n + 2$ and $|E(G)| = 2n + 4$. Consider $\psi : V(G) \to \{1, 2, ..., n + 2\}$ defined under three conditions.

Case (i) If 'n' is even.

Let $\psi(w_0) = 2$, $\psi(x) = 6$, $\psi(w_1) = 8$, $\psi(w_n) = 4$, $\psi(w_{\frac{n}{2}-1}) = 1$, $\psi(w_{\frac{n}{2}}) = 3$, $\psi(w_{\frac{n}{2}+1}) = 9$, $\psi(w_i) = \psi(w_{i-1}) + 2$; $2 \leq i \leq \frac{n}{2} - 2$. Allocate the unutilized labels to unlabeled nodes simultaneously from $\{1, 2, ..., n + 2\}$.

Case (ii) If 'n' is odd.

Subcase (i) If $n \equiv 2(mod\ 3)$.

Let $\psi(w_0) = 2$, $\psi(x) = 6$, $\psi(w_1) = 8$, $\psi(w_n) = 4$, $\psi(w_{n-1}) = 1$, $\psi(w_{\frac{n+1}{2}-2}) = 3$, $\psi(w_{\frac{n+1}{2}-1}) = 9$ and $\psi(w_i) = \psi(w_{i-1}) + 2$; $2 \leq i \leq \frac{n+1}{2} - 3$. Allocate unutilized labels to unlabeled nodes namely, $w_{\frac{n+1}{2}}, w_{\frac{n+1}{2}+1}, ..., w_{n-2}$ simultaneously from $\{1, 2, ..., n + 2\}$.

Subcase (ii) If $n \not\equiv 2(mod\ 3)$.

Let $\psi(w_0) = 2$, $\psi(x) = 8$, $\psi(w_1) = 10$, $\psi(w_n) = 4$, $\psi(w_{\frac{n+1}{2}-3}) = 6$, $\psi(w_{\frac{n+1}{2}-2}) = 3$, $\psi(w_{\frac{n+1}{2}-1}) = 9$ and $\psi(w_i) = \psi(w_{i-1}) + 2$; $2 \leq i \leq \frac{n+1}{2} - 4$, (if the case exists). Allocate unutilized labels to unlabeled nodes $w_{\frac{n+1}{2}}, w_{\frac{n+1}{2}+1}, ..., w_{n-1}$ simultaneously from $\{1, 2, 3, ..., n + 2\}$.

In all the three cases, $|e_\psi(0) - e_\psi(1)| \leq 1$, which ensures that G is a PCG (see Figure 3.4). □

Remark 3.2. Duplication of an arbitrary rim node in W_n admits a PCL $\forall\ n > 7$ and its proof is same as that of Theorem 3.2.7.

Theorem 3.2.8. *Extension of apex node of H_n admits a PCG $\forall\ n \geq 2$.*

Proof. Let $V(H_n) = \{h_0, h_i, h'_i : 1 \leq i \leq n\}$; h_0, h_i, h'_i represent the nodes of degree n, 4 and 1 respectively. Let G be produced by taking extension of apex node and w be the newly added node. Clearly, $|V(G)| = 2n+2$ and $|E(G)| = 4n+1$. Consider $\psi : V(G) \to$

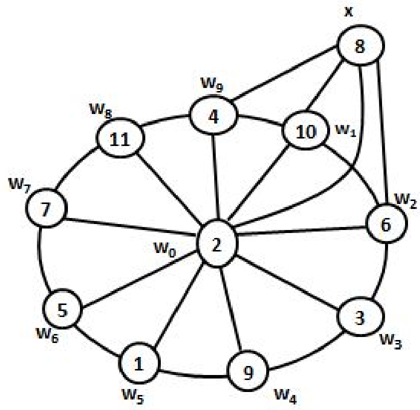

FIGURE 3.4: PCL of a graph obtained by taking the extension of rim node in W_9

$\{1, 2, ..., 2n+2\}$ as given. Let $\psi(w) = 1$, $\psi(h_0) = 2$, $\psi(h_1) = 4$, $\psi(h_i) = \psi(h_{i-1}) + 2$; $2 \leq i \leq n$ & $\psi(h'_i) = \psi(h_i) - 1$; $1 \leq i \leq n$ following which G is a PCG. □

Remark 3.3. Duplication of an apex node of H_n is a PCG $\forall$ $n \geq 2$ and the its proof is same as that of Theorem 3.2.8.

Theorem 3.2.9. *Extension of an arbitrary node of degree* 4 *in* H_n *results in a PCG* $\forall$ $n \geq 6$.

Proof. Let $V(H_n) = \{h_0, h_i, h'_i : 1 \leq i \leq n\}$; h_0, h_i, h'_i represent the nodes of degree n, 4 and *1* respectively. Let G be produced by taking extension of h_1 and w be the freshly inserted node. Clearly, $|V(G)| = 2n+2$ and $|E(G)| = 3n+5$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+2\}$. Fix $\psi(w) = 4$, $\psi(h_0) = 2$, $\psi(h_n) = 8$. The given cases arise.
Case (i) If 'n' is even.
Let $\psi(h_{\frac{n}{2}-1}) = 10$, $\psi(h'_{\frac{n}{2}-1}) = 1$, $\psi(h_{\frac{n}{2}}) = 3$, $\psi(h_i) = \psi(h_{i-1}) + 4$; $\frac{n}{2} + 1 \leq i \leq n-1$, $\psi(h'_i) = \psi(h'_{i-1}) + 4$; $\frac{n}{2} \leq i \leq n-1$. Assign remaining even labels to unlabeled nodes in any order.
Case (ii) If 'n' is odd.
Let $\psi(h_{\lfloor\frac{n}{2}\rfloor-1}) = 10$, $\psi(h_{\lfloor\frac{n}{2}\rfloor}) = 1$, $\psi(h'_{\lfloor\frac{n}{2}\rfloor}) = 3$, $\psi(h_i) = \psi(h_{i-1}) + 4$; $\lfloor\frac{n}{2}\rfloor + 1 \leq i \leq n-1$, $\psi(h'_i) = \psi(h'_{i-1}) + 4$; $\lfloor\frac{n}{2}\rfloor + 1 \leq i \leq n-1$. Allot remaining even labels to unlabeled nodes.
It follows that G is a PCG (see Figure 3.5). □

Remark 3.4. Proof of Theorem 3.2.9 holds good even if duplication of a node of degree 4 by a node in H_n is taken.

Theorem 3.2.10. *Extension of a pendant node in* H_n *results in a PCG* $\forall$ $n \geq 5$.

Proof. Let $V(H_n) = \{h_0, h_i, h'_i : 1 \leq i \leq n\}$; h_0 is apex, h'_i represent the pendant nodes of H_n. Let G be produced by taking extension of h'_1 and w be the added node. Clearly,

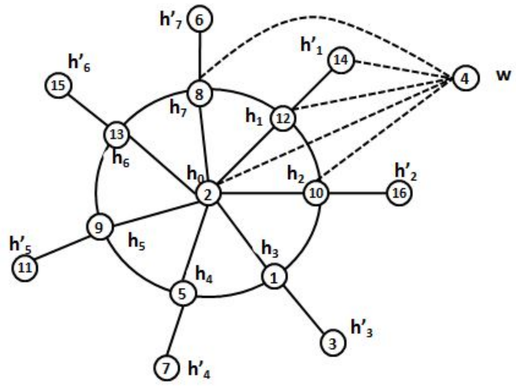

FIGURE 3.5: PCL of a graph acquired by taking extension of h_1 in H_7

$|V(G)| = 2n+2$ & $|E(G)| = 3n+2$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+2\}$ as given. Let $\psi(w) = 12$, $\psi(h_0) = 2$, $\psi(h_1) = 6$, $\psi(h_n) = 4$ and $\psi(h'_1) = 3$. The given cases arise.

Case (i) If 'n' is even.

Let $\psi(h_{\frac{n}{2}-1}) = 8$, $\psi(h_{\frac{n}{2}}) = 5$, $\psi(h'_{\frac{n}{2}}) = 1$, $\psi(h_i) = \psi(h_{i-1}) + 4$; $\frac{n}{2} + 1 \le i \le n-1$, $\psi(h'_i) = \psi(h_i) - 2$; $\frac{n}{2} + 1 \le i \le n-1$. Assign remaining even labels simultaneously to unlabeled nodes.

Case (ii) If 'n' is odd.

Let $\psi(h_{\lfloor \frac{n}{2} \rfloor}) = 8$, $\psi(h'_{\lfloor \frac{n}{2} \rfloor}) = 1$, $\psi(h_{\lfloor \frac{n}{2} \rfloor + 1}) = 5$, $\psi(h'_{\lfloor \frac{n}{2} \rfloor + 1}) = 7$, $\psi(h_i) = \psi(h_{i-1}) + 4$; $\lfloor \frac{n}{2} \rfloor + 2 \le i \le n-1$, $\psi(h'_i) = \psi(h_i) + 2$; $\lfloor \frac{n}{2} \rfloor + 2 \le i \le n-1$. The remaining labels are assigned simultaneously to unlabeled nodes.

It follows that G is a PCG. □

Remark 3.5. The proof of Theorem 3.2.10 can be used in proving the PCL of a graph acquired by duplicating a pendant node at random, by a node in H_n, $\forall\ n > 7$.

Theorem 3.2.11. *Extension of all pendant nodes in H_n results in a PCG $\forall\ n \ge 5$.*

Proof. Let $V(H_n) = \{h_0, h_i, h'_i : 1 \le i \le n\}$ where h'_i's are pendant nodes. Let G be produced by taking extension of all pendant nodes and $\{w_i : 1 \le i \le n\}$ be the freshly inserted nodes. Clearly, $|V(G)| = 3n+1$ & $|E(G)| = 5n$. Consider $\psi : V(G) \to \{1, 2, ..., 3n+1\}$. Let $\psi(h_0) = 2$ and $\psi(h_1) = 4$. There arise the given cases.

Case (i) If 'n' is even.

Let $\psi(w_n) = 1$, $\psi(h_{\frac{n}{2}}) = 6$, $\psi(h'_{\frac{n}{2}}) = 12$, $\psi(h_{\frac{n}{2}+1}) = 9$, $\psi(h_{\frac{n}{2}+2}) = 11$, $\psi(w_{\frac{n}{2}}) = 3$, $\psi(h'_{\frac{n}{2}+1}) = 5$, $\psi(w_{\frac{n}{2}+1}) = 7$, $\psi(h_i) = \psi(h_{i-1}) + 6$; $\frac{n}{2} + 3 \le i \le n$, $\psi(h'_i) = \psi(h_i) + 2$; $\frac{n}{2} + 2 \le i \le n$, $\psi(w_i) = \psi(h_i) + 4$; $\frac{n}{2} + 2 \le i \le n-1$. Assigning the remaining labels simultaneously to unlabeled nodes gives $e_\psi(0) = e_\psi(1) = \frac{5n}{2}$.

Case (ii) If 'n' is odd.

Let $\psi(h_{\lceil \frac{n}{2} \rceil}) = 6$, $\psi(h'_{\lceil \frac{n}{2} \rceil}) = 3$, $\psi(w_{\lceil \frac{n}{2} \rceil}) = 1$, $\psi(h_{\lceil \frac{n}{2} \rceil + 1}) = 5$, $\psi(h_i) = \psi(h_{i-1}) + 6$;

$\lceil \frac{n}{2} \rceil + 2 \leq i \leq n$, $\psi(h'_i) = \psi(h_i) + 2$; $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$, $\psi(w_i) = \psi(h_i) + 4$; $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$. Assign the remaining labels to unlabeled nodes results in $e_\psi(0) = \frac{5n-1}{2}$ & $e_\psi(1) = \frac{5n+1}{2}$. For both the cases, G is a PCG (see Figure 3.6). □

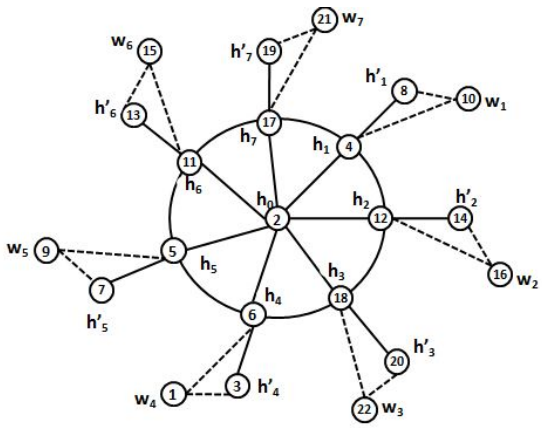

FIGURE 3.6: PCL of a graph acquired by taking extension of all pendant nodes in H_7

Remark 3.6. Graph acquired by duplicating each rim node by an edge in W_n is a PCG as it is isomorphic to a graph obtained by taking extension of each pendant node in H_n.

Remark 3.7. Proof of Theorem 3.2.11 can be used to prove that graph acquired by duplicating each pendant node by a node in H_n admits a PCL.

Theorem 3.2.12. *Extension of apex node by a node in Fl_n admits a PCL* $\forall\ n \geq 5$.

Proof. Let $V(Fl_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ where v_i & u_i have degrees 4 & 2 respectively and v_0 is apex node. Let G be produced by taking extension of apex node and w be the added node. Clearly, $|V(G)| = 2n + 2$ and $|E(G)| = 6n + 1$. Consider $\psi : V(G) \to \{1, 2, ..., 2n + 2\}$ given by assigning $\psi(v_0) = 2$, $\psi(w) = 4$, $\psi(v_1) = 6$, $\psi(u_1) = 3$, $\psi(v_2) = 10$, $\psi(u_2) = 5$, $\psi(v_n) = 9$, $\psi(u_n) = 1$ and $\psi(v_{n-1}) = 12$. Assign remaining even labels to unlabeled nodes of degree 4. Next, $\psi(u_i) = \psi(v_i) - 1$; $3 \leq i \leq n - 1$. Following this, G is a PCG. □

Remark 3.8. Duplication of an apex node of Fl_n results in a PCG $\forall\ n \geq 5$. Labeling is same as that of Theorem 3.2.12.

Theorem 3.2.13. *Extension of an arbitrary node of degree* 2 *in Fl_n results in a PCG* $\forall\ n \geq 3$.

Proof. Let $V(Fl_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ where v_i & u_i have degrees 4 & 2 respectively and v_0 is apex node. Let G be produced by taking extension of u_1 and w be the newly added node. Clearly, $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 3$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+2\}$ given by fixing $\psi(v_0) = 2$, $\psi(v_1) = 6$, $\psi(u_1) = 3$, $\psi(v_2) = 4$,

$\psi(u_2) = 5$ and $\psi(w) = 1$. Allocate the available unutilized even labels simultaneously to v_i; $3 \leq i \leq n$. Next, set $\psi(u_i) = \psi(v_i) - 1$; $3 \leq i \leq n$. Following this G is a PCG (see Figure 3.7). □

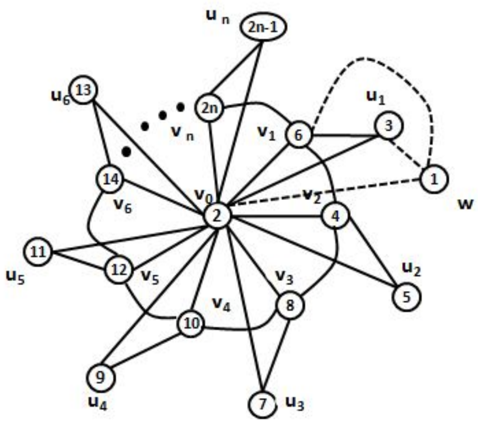

FIGURE 3.7: PCL of G acquired by taking extension of u_1 in Fl_n

Remark 3.9. Duplicating an arbitrary node of degree 2 by a node in Fl_n admits a PCL and its proof is same as that of Theorem 3.2.13.

Theorem 3.2.14. *Extension of an arbitrary node of degree* 4 *in* Fl_n *results in a PCG* $\forall\ n \geq 4$.

Proof. Let $V(Fl_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ where v_i & u_i have degrees 4 & 2 respectively. Let G be produced by taking extension of an arbitrary node of degree 4. Without loss of generality, consider the extension of v_1 and w be the added node. Clearly, $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 5$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 2n + 2\}$. Fix $\psi(v_0) = 2$, $\psi(v_1) = 6$, $\psi(v_2) = 8$, $\psi(u_1) = 3$, $\psi(v_n) = 4$, $\psi(u_n) = 5$, $\psi(v_{n-1}) = 10$, $\psi(u_{n-1}) = 1$, and $\psi(w) = 9$. Allocate available unutilized even labels to v_i; $3 \leq i \leq n-2$. Next, set $\psi(u_i) = \psi(v_i) - 1$; $2 \leq i \leq n - 2$. Clearly, G is a PCG. (see Figure 3.8). □

Remark 3.10. Duplicating an arbitrary node of by a node of degree 4 in Fl_n results in a PCG $\forall\ n \geq 4$ and its proof is same as that of Theorem 3.2.14.

Theorem 3.2.15. *Extension of apex node in* G_n *permits a PCL* $\forall\ n \geq 4$.

Proof. Let $V(G_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ and $E(G_n) = \{v_0v_i : 1 \leq i \leq n\} \cup \{v_iu_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nv_1\}$. Here v_0, v_i and u_i represent respectively the apex, node of degree 3 and 2. Let H be obtained by taking extension of v_0 in G_n by adding a new node, say, w. Clearly $|V(H)| = 2n+2$ and $|E(H)| = 4n+1$. Consider $\psi : V(H) \rightarrow \{1, 2, ..., 2n + 2\}$ as per the given algorithm. Fix $\psi(v_0) = 2$, $\psi(w) = 6$, $\psi(v_1) = 4$, $\psi(v_n) = 3$, $\psi(v_{n-1}) = 10$, $\psi(u_n) = 1$ and $\psi(u_{n-1}) = 5$. Assign even labels

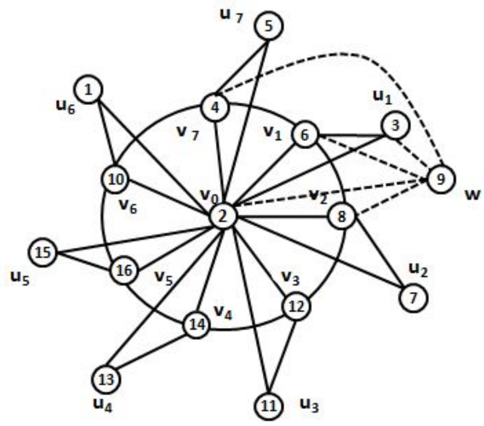

FIGURE 3.8: PCL of a graph acquired by taking extension of v_1 in Fl_7

simultaneously from unused labels to v_i; $2 \leq i \leq n-2$ and $\psi(u_i) = \psi(v_{i+1}) - 1$; $1 \leq i \leq n-2$, resulting which $|e_\psi(0) - e_\psi(1)| \leq 1$. Hence, H is a PCG. □

Remark 3.11. Duplication of an apex node by a node in G_n permits a PCL $\forall\ n \geq 4$ and its proof is same as that of Theorem 3.2.15

Theorem 3.2.16. *Extension of an arbitrary node of degree* 3 *in* G_n *permits a PCL* $\forall$ $n \geq 4$.

Proof. Let $V(G_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ where v_i and u_i are respectively the nodes of degree 3 and 2. Without loss of generality, let H be obtained by taking the extension of v_1 and w be the added node. Clearly $|V(H)| = 2n+2$ and $|E(H)| = 3n+4$. Labeling is defined by $\psi : V(H) \to \{1, 2, ..., 2n+2\}$ as per the following algorithm. Fix $\psi(v_0) = 8$, $\psi(w) = 2$ and $\psi(u_n) = 4$. Now given cases arise.

Case (i) If 'n' is even.

Let $\psi(u_{\frac{n}{2}-1}) = 6$, $\psi(v_{\frac{n}{2}}) = 3$, $\psi(u_{\frac{n}{2}}) = 1$, $\psi(v_{\frac{n}{2}+1}) = 5$, $\psi(v_i) = \psi(v_{i-1})+4$; $\frac{n}{2}+2 \leq i \leq n$ and $\psi(u_i) = \psi(v_i) + 2$; $\frac{n}{2} + 1 \leq i \leq n-1$. Next assign the remaining labels in any fashion. Note that $e_\psi(0) = e_\psi(1) = \frac{3n+4}{2}$ which proves that H is a PCG.

Case (ii) If 'n' is odd.

Let $\psi(v_{\lfloor\frac{n}{2}\rfloor}) = 6$, $\psi(u_{\lfloor\frac{n}{2}\rfloor}) = 3$, $\psi(v_{\lceil\frac{n}{2}\rceil}) = 1$, $\psi(u_{\lceil\frac{n}{2}\rceil}) = 5$, $\psi(u_i) = \psi(u_{i-1})+4$; $\lceil\frac{n}{2}\rceil+1 \leq i \leq n-1$ and $\psi(v_i) = \psi(u_{i-1})+2$; $\lceil\frac{n}{2}\rceil+1 \leq i \leq n$. Next allot the remaining even labels in any fashion. Note that $e_\psi(0) = e_\psi(1)+1$ which proves that H is a PCG. □

Remark 3.12. Duplication of an arbitrary node of degree 3 in G_n permits a PCL $\forall\ n \geq 4$. The proof is same as that of Theorem 3.2.16.

Remark 3.13. One can also establish the PCL of graphs obtained by taking extension as well as duplication of an arbitrary node of degree 2 in G_n.

Theorem 3.2.17. *Extension of all nodes of* P_n *allows a PCL.*

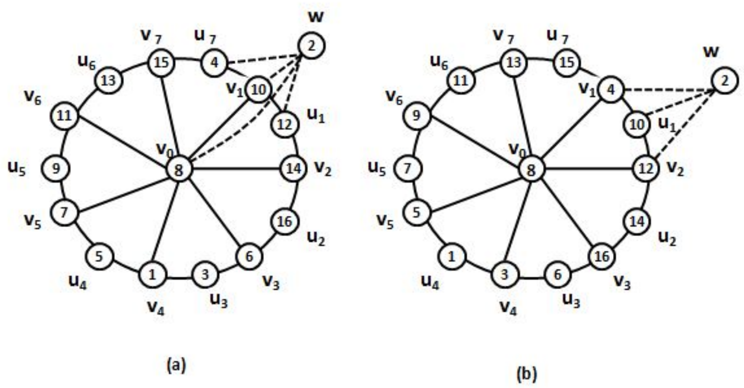

FIGURE 3.9: PCL of a graph acquired by taking extension of (a) v_1 in G_7 and (b) u_1 in G_7

Proof. Suppose node set of P_n is $\{u_i : 1 \leq i \leq n\}$ and G be acquired by taking extension of all nodes of P_n. Let $V(G) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(G) = E(P_n) \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{v_iu_{i-1} : 2 \leq i \leq n\} \cup \{v_iu_{i+1} : 1 \leq i \leq n-1\}$. The cardinality of node and edge set of G is respectively $2n$ and $4n-3$. In order to define $\psi : V(G) \to \{1, 2, ..., 2n\}$, refer to Theorem 3.2.5. □

Theorem 3.2.18. *Extension of all nodes of C_n, $n > 8$, allows a PCL.*

Proof. Suppose node set of C_n is $\{u_1, u_2, ..., u_n\}$ and G be produced by taking extension of all nodes of C_n. Note that $V(G) = V(C_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(G) = E(C_n) \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{v_iu_{i-1} : 2 \leq i \leq n\} \cup \{v_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1u_n, v_nu_1\}$. Clearly, $|V(G)| = 2n$ whereas $|E(G)| = 4n$. In order to define $\psi : V(G) \to \{1, 2, ..., 2n\}$, one can refer to Theorem 3.2.5, except for the following few changes. Replace $\psi(u_{\frac{n}{2}+1}) = 9$, $\psi(v_{\frac{n}{2}+1}) = 3$, $\psi(u_{\frac{n}{2}+2}) = 11$, $\psi(v_{\frac{n}{2}+2}) = 5$, $\psi(u_{\frac{n}{2}+3}) = 17$, $\psi(v_{\frac{n}{2}+3}) = 15$. (Similar pattern when n is odd). □

Remark 3.14. One can deduce the PCL of a graph acquired by performing the extension of all the rim nodes of W_n on similar lines.

Theorem 3.2.19. *[81] $S'(K_{1,n})$ admits a PCL.*

Theorem 3.2.20. *Extension of all nodes of $K_{1,n}$ allows a PCL.*

Proof. Let $V(K_{1,n}) = \{k_0, k_i : 1 \leq i \leq n\}$; k_0 is apex node. Let G be produced by performing extension of each node of $K_{1,n}$ and $u_0, u_1, u_2, ...u_n$ be the freshly inserted nodes with $V(G) = V(K_{1,n}) \cup \{u_0, u_1, u_2, ..., u_n\}$ and $E(G) = E(K_{1,n}) \cup \{k_iu_i : 1 \leq i \leq n\} \cup \{k_0u_0\} \cup \{k_0u_i : 1 \leq i \leq n\} \cup \{u_0k_i : 1 \leq i \leq n\}$. One can see that $|V(G)| = 2n + 2$ & $|E(G)| = 4n + 1$. Consider $\psi : V(G) \to \{1, 2, ..., 2n + 2\}$. Fix $\psi(k_0) = 2$, $\psi(k_1) = 3$, $\psi(k_2) = 4$, $\psi(k_3) = 8$, $\psi(k_i) = \psi(k_{i-1}) + 2$; $4 \leq i \leq n$, $\psi(u_0) = 6$,

$\psi(u_1) = 1$, $\psi(u_2) = 5$, $\psi(u_i) = \psi(k_i) - 1$; $3 \leq i \leq n$. Note that $\gcd(\psi(k_0), \psi(k_i)) > 1$; $2 \leq i \leq n$, $\gcd(\psi(k_0), \psi(u_0)) > 1$ and $\gcd(\psi(u_0), \psi(k_i)) > 1$; $1 \leq i \leq n$. The edges due to above observation are labeled 0 and the remaining are labeled 1. Note $e_\psi(0) = 2n$ and $e_\psi(1) = 2n + 1$ implies G is a PCG (see Figure 3.10). □

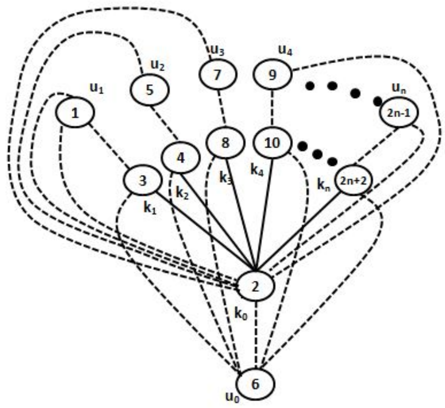

FIGURE 3.10: PCL of a graph acquired by taking extension of all nodes of $K_{1,n}$

Remark 3.15. Duplication of each node with a node in $K_{1,n}$ results in $S'(K_{1,n})$ which admits a PCL by Theorem 3.2.19.

Theorem 3.2.21. *[81] $S'(B_{n,n})$ admits a PCL.*

Theorem 3.2.22. *Extension of all nodes of $B_{n,n}$ admits a PCL.*

Proof. Let $V(B_{n,n}) = \{u_0, v_0, u_i, v_i : 1 \leq i \leq n\}$ where u_0, v_0 represent the apex nodes. Let G be produced by performing extension of each node of $B_{n,n}$ and let u'_0, v'_0, u'_i, v'_i be the freshly added nodes, $1 \leq i \leq n$. Clearly, $V(G) = V(B_{n,n}) \cup \{u'_0, v'_0, u'_i, v'_i : 1 \leq i \leq n\}$ and $E(G) = E(B_{n,n}) \cup \{u_i u'_i : 1 \leq i \leq n\} \cup \{v_i v'_i : 1 \leq i \leq n\} \cup \{u_0 v'_0, v_0 u'_0, u_0 u'_0, v_0 v'_0\} \cup \{u_0 u'_i, v_0 v'_i : 1 \leq i \leq n\} \cup \{u'_0 u_i : 1 \leq i \leq n\} \cup \{v'_0 v_i : 1 \leq i \leq n\}$. Apparently, $|V(G)| = 4n + 4$ & $|E(G)| = 8n + 5$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 4n + 4\}$. Fix $\psi(u_0) = 2$, $\psi(v_0) = 4$, $\psi(u'_0) = 6$, $\psi(v'_0) = 1$, $\psi(u_1) = 3$, $\psi(u'_1) = 12$. Allot the unutilized even labels to u_i and u'_i; $2 \leq i \leq n$ in any order. Next, $\psi(v_1) = 5$, $\psi(v_i) = \psi(v_{i-1}) + 4$; $2 \leq i \leq n$ and $\psi(v'_i) = \psi(v_i) + 2$; $1 \leq i \leq n$. Observe that $e_\psi(1) = 4n + 3$ and $e_\psi(0) = 4n + 2$ which proves that G is a PCG (see Figure 3.11). □

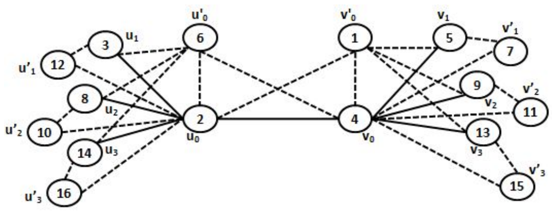

FIGURE 3.11: PCL of a graph acquired by taking extension of all nodes in $B_{3,3}$

Remark 3.16. Duplication of each node with a node in $B_{n,n}$ results in $S'(B_{n,n})$ which is a PCL by Theorem 3.2.21.

Open Problems

The following open problems arose due to the results established in this chapter.
1. If G is a PCG, then does graph obtained by performing extension of each node of G also permit a PCL?
2. Is there a characterization of graphs that do not admit a PCL but whose extension admits a PCL?

3.3 Conclusion

In this chapter, duplication and extension, which are widely used graph operations in many real life problems, permits PCL for the following graphs; arbitrary rim node in W_n, apex node in H_n, G_n & Fl_n, pendant node in H_n, node of degree 1 & 3 in H_n, node of degree 2 & 4 in Fl_n, apex node, node of degree 2 & 3 in G_n. Further, it has been established that duplication and extension of all nodes in P_n, C_n, W_n, $K_{1,n}$ and $B_{n,n}$ permit a PCL, while formulating some open problems.

Chapter 4

Results on DCL of Graphs

4.1 Introduction

In this chapter, certain general results concerning DCL of graphs are derived. The DCL of some familiar families of graphs in the frame of various notable graph operations has also been discussed.

4.2 Certain New General Results on DCL of Graphs

In this section, some general results on DCL of graphs are presented. Babitha et al., [3] studied the PCL of the construction of a new graph by using an existing PCG, say, H, and then gluing a node of some particular class of graph to one of the selected node of H. Motivated by this, some new graphs by using an existing DCGs are constructed.

Definition 4.2.1. [3] "Let $H_1(p_1, q_1)$ and $H_2(p_2, q_2)$ be two connected graphs, then $H_1 \widehat{o} H_2$ is acquired by overlaying any chosen node of H_2 on any selected node of H_1. The resultant graph $H = H_1 \widehat{o}\, H_2$ has $p_1 + p_2 - 1$ number of nodes and $q_1 + q_2$ number of edges."

Theorem 4.2.1. *If $G(p, q)$ is a DCG with labeling g, then $G\widehat{o}f_m$ admits a DCL when*
(i) q is even
(ii) q is odd with $e_g(0) = \lceil \frac{q}{2} \rceil$.

Proof. Suppose G is a DCG having $V(G) = \{u_1, u_2, ..., u_p\}$ under the labeling g. Let $u_k \in V(G)$ such that $g(u_k) = 1$. Consider f_m with $V(f_m) = \{v_0, v_i : 1 \leq i \leq m\}$ and $E(f_m) = \{v_0 v_i : 1 \leq i \leq m\} \cup \{v_i v_{i+1} : 1 \leq i \leq m - 1\}$. Let $H = G\widehat{o}f_m$ be obtained by superimposing v_0 on u_k of G (see Figure 4.1). Then $V(H) = V(G) \cup \{v_i : 1 \leq i \leq m\}$

and $E(H) = E(G) \cup \{u_k v_i : 1 \leq i \leq m\} \cup \{v_i v_{i+1} : 1 \leq i \leq m-1\}$. Consider $\psi : V(H) \rightarrow \{1, 2, ..., p, p+1, ..., p+m\}$ defined by $\psi(u_i) = g(u_i)$ for $1 \leq i \leq p$. Recall that $g(u_k) = \psi(v_0) = 1$, fix $\psi(v_i) = p+i$ for $1 \leq i \leq m$. Next to show that $G\hat{o}f_m$ is a DCG for the following cases.

Case (i) 'q' is even.

Then $e_g(0) = e_g(1) = \frac{q}{2}$. Note that $|E(H)| = q + 2m - 1$ and observing ψ, one can see that for edges $e = u_k v_i$; $1 \leq i \leq m$, $\psi(e) = 1$ and $\psi(v_i v_{i+1}) = 0$; $1 \leq i \leq m-1$. Hence, $e_\psi(1) = \frac{q}{2} + m$ and $e_\psi(0) = \frac{q}{2} + m - 1$ which justifies that $|e_\psi(0) - e_v(1)| \leq 1$.

Case (ii) 'q' is odd with $e_g(0) = \lceil \frac{q}{2} \rceil$ i.e; $e_g(0) = e_g(1) + 1$.

Keeping ψ in view, one can verify that $\psi(u_k v_i) = 1$; $1 \leq i \leq m$ and $\psi(v_i v_{i+1}) = 0$; $1 \leq i \leq m-1$. Consequently, $e_\psi(1) = \lfloor \frac{q}{2} \rfloor + m$ and $e_\psi(0) = \lceil \frac{q}{2} \rceil + m - 1$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$.

Hence, H is a DCG. □

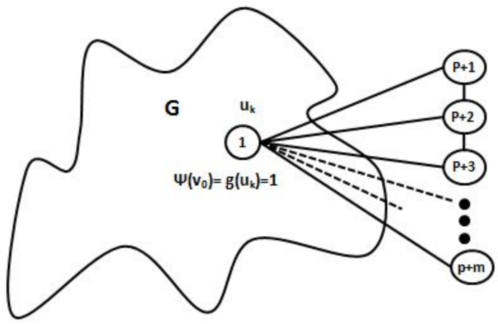

FIGURE 4.1: DCL of $G\ \hat{o}\ f_m$

Remark 4.1. If $G(p, q)$ is a DCG then, $G\hat{o}W_m$ and $G\hat{o}Fl_m$ admit a DCL. One can easily prove this in accordance with Theorem 4.2.1.

Theorem 4.2.2. *If $G(p, q)$ is a DCG then, $G\hat{o}K_{1,m}$ admits a DCL when*

(i) m is even

(ii) m is odd and

(a) q is even

(b) Both p and q are odd with $e_g(1) = \lfloor \frac{q}{2} \rfloor$

(c) p is even and q is odd with $e_g(1) = \lceil \frac{q}{2} \rceil$.

Proof. Let G be a DCG having $V(G) = \{u_1, u_2, ..., u_p\}$ and labeling g. Let $u_k \in V(G)$ such that $g(u_k) = 2$. Consider $K_{1,m}$ having $V(K_{1,m}) = \{v_0, v_i : 1 \leq i \leq m\}$ and $E(K_{1,m}) = \{v_0 v_i : 1 \leq i \leq m\}$. Let $H = G\hat{o}K_{1,m}$ be obtained by superimposing v_0 on u_k of G. Clearly, $V(H) = V(G) \cup \{v_i : 1 \leq i \leq m\}$ and $E(H) = E(G) \cup \{u_k v_i : 1 \leq i \leq m\}$. Consider $\psi : V(H) \rightarrow \{1, 2, ..., p,\ p+1, ..., p+m\}$ defined by $\psi(u_i) = g(u_i)$ for $1 \leq i \leq p$. Recall that $g(u_k) = \psi(v_0) = 2$, fix $\psi(v_i) = p+i$ for $1 \leq i \leq m$. Next is to show that

$G\hat{o}K_{1,m}$ is a DCG for the given conditions.
Case (i) 'm' is even.
If 'q' is even, then $e_g(0) = e_g(1) = \frac{q}{2}$. Note that $|E(H)| = q + m$, one can see that the aggregate count of edges bearing labels 1 and 0 are respectively $\frac{q}{2} + \frac{m}{2}$ and $\frac{q}{2} + \frac{m}{2}$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$.
If 'q' is odd, then either $e_g(0) = e_g(1) + 1$ or $e_g(1) = e_g(0) + 1$. On the other hand m being even always yield equal count of edges labeled with 1 and 0. Thus, either $e_\psi(0) = e_\psi(1) + 1$ or $e_\psi(1) = e_\psi(0) + 1$ justifiying that $|e_\psi(0) - e_\psi(1)| \leq 1$.
Case (ii) When 'm' is odd.
Subcase (i) When 'q' is even.
Since q is even, $e_g(0) = e_g(1) = \frac{q}{2}$. Note that $|E(H)| = q + m$. Following the labeling pattern ψ, one can see that either $e_\psi(1) = e_\psi(0) + 1$ or $e_\psi(0) = e_\psi(1) + 1$.
Subcase (ii) When both 'q' and 'p' are odd with $e_g(1) = \lfloor \frac{q}{2} \rfloor$. Then $e_g(0) = \lfloor \frac{q}{2} \rfloor + 1$. Following ψ, one can observe that $e_\psi(1) = \lfloor \frac{q}{2} \rfloor + \lceil \frac{m}{2} \rceil$ and $e_\psi(0) = (\lfloor \frac{q}{2} \rfloor + 1) + (\lceil \frac{m}{2} \rceil - 1)$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$.
Subcase (iii) When 'q' is odd and 'p' is even with $e_g(1) = \lceil \frac{q}{2} \rceil$. Then $e_g(0) = \lceil \frac{q}{2} \rceil - 1$. Following ψ, one can see that $e_\psi(1) = \lceil \frac{q}{2} \rceil + (\lceil \frac{m}{2} \rceil - 1)$ and $e_\psi(0) = (\lceil \frac{q}{2} \rceil - 1) + \lceil \frac{m}{2} \rceil$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$.
Hence, H is a DCG under all the mentioned conditions. □

Theorem 4.2.3. *Let $H(p_1, q_1)$ and $K(p_2, q_2)$ be two disjoint DCGs such that $H \cup K$ is a DCG with labeling ψ, then the graph formed by joining H and K by an edge also admits a DCL when*
(i) q_1 and q_2 are even.
(ii) Both q_1 and q_2 are odd with
(a) $e_\psi(1) = \lfloor \frac{q_1}{2} \rfloor + \lceil \frac{q_2}{2} \rceil$ (b) $e_\psi(1) = \lceil \frac{q_1}{2} \rceil + \lfloor \frac{q_2}{2} \rfloor$.

Proof. Given that $H \cup K$ is a DCG with labeling ψ, where union is taken over disjoint DCGs H and K. Join H with K by an edge e. Denote the newly obtained graph by G with $V(G) = V(H \cup K)$ and $E(G) = E(H) \cup E(K) \cup \{e\}$. Clearly, $|V(G)| = p_1 + p_2$ and $|E(G)| = q_1 + q_2 + 1$. Next to show that G is DCG for the following cases.
Case (i) When both 'q_1' and 'q_2' are even.
This implies $q_1 + q_2$ is even and therefore edges having labels 1 and 0 are equal. Joining H with K by an edge 'e' contribute either label 1 or 0 and in both way $|e_\psi(0) - e_\psi(1)| \leq 1$.
Case (ii) Both 'q_1' and 'q_2' are odd then $q_1 + q_2$ is even. Consider the following subcases.
Subcase (i) When $e_\psi(1) = \lfloor \frac{q_1}{2} \rfloor + \lceil \frac{q_2}{2} \rceil$ then $e_\psi(0) = (\lfloor \frac{q_1}{2} \rfloor + 1) + (\lceil \frac{q_2}{2} \rceil - 1)$. In this case adding an edge contributes edge label 1 or 0 and in either of the case $|e_\psi(0) - e_\psi(1)| \leq 1$.
Subcase (ii) When $e_\psi(1) = \lceil \frac{q_1}{2} \rceil + \lfloor \frac{q_2}{2} \rfloor$ then $e_\psi(0) = (\lceil \frac{q_1}{2} \rceil - 1) + (\lfloor \frac{q_2}{2} \rfloor + 1)$. Again, adding an edge leads to $|e_\psi(0) - e_\psi(1)| \leq 1$. Thus, G is a DCG. □

Theorem 4.2.4. *[86] K_n does not admit a DCL for $n \geq 7$.*

Theorem 4.2.5. *$DS(K_n)$ does not permit a DCL for $n \geq 6$.*

Proof. Proof is evident from Theorem 4.2.4 and Lemma 2.2.1. □

Lemma 4.2.1. *Extension of any arbitrary node of K_n yields K_{n+1}.*

Theorem 4.2.6. *The graph G produced by performing extension of any arbitrary node in K_n does not admit a DCL for $n \geq 6$.*

Proof. Proof is evident from Theorem 4.2.4 and Lemma 4.2.1. □

Lemma 4.2.2. *The graph formed by switching any arbitrary node in K_n admits a DCL for $n \leq 8$.*

Proof. Switching of any arbitary node in K_n yields a disconnected graph whose components are K_{n-1} and K_1. The result clearly follows for switching of node in K_n for $n = 3, 4, 6, 7$ (see Figure 4.2). Consider the DCL of graph produced by switching of a node in K_5 and K_8.

Case (i) When $n = 5$.

Label the isolated node with 4 and assign the remaining labels to the nodes of K_4. Clearly, $e(0) = e(1) = 3$.

Case (ii) When $n = 8$.

Label the isolated node with 7 and assign the remaining labels to the nodes of K_7. Here $e(0) = 10$ and $e(1) = 11$. □

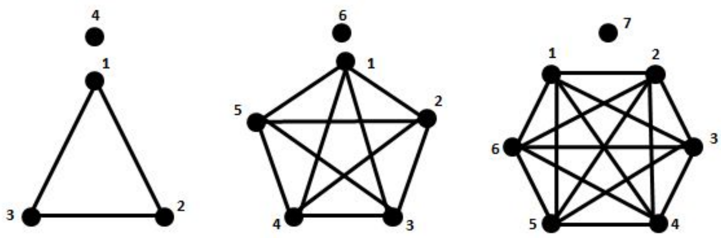

FIGURE 4.2: DCL of a graph formed by switching of a node in K_4, K_6 and K_7

Theorem 4.2.7. *Switching of an arbitrary node in K_n for $n \geq 9$ does not admit a DCL.*

Proof. Switching of an arbitrary node in K_n; $n \geq 9$ yields a disconnected graph G whose components are K_{n-1} and K_1. Consider $n = 9$ for the sake of discussion. A disconnected graph G is produced by switching a node in K_9 whose components are K_8 and K_1. A method of contradiction is used in proving this. Assume that G admits a DCL. Without loss of generality, label the isolated node with the largest prime p where $p \leq 9$ (i.e., 7) in order to get more edges having label 1, and assign the remaining labels to nodes of K_8 in any pattern. Here $e(0) = 15$, $e(1) = 13$ which results in $|e(0) - e(1)| > 1$, a contradiction. The other possibilities of assigning different labels to the isolated node can be dealt in the similar lines. The similar argument holds good for $n \geq 10$. Hence the theorem. □

4.3 DCL in the Context of Corona

In this section, DCL of corona of some known graphs has been discussed. First, some established results are recalled.

Theorem 4.3.1. *[17] "$K_{1,n} \odot K_1$, $K_{2,n} \odot K_1$, $K_{3,n} \odot K_1$, $W_n \odot K_1$, $H_n \odot K_1$, $Fl_n \odot K_1$, $f_n \odot K_1$, $Df_n \odot K_1$ and $S(K_{1,n}) \odot K_1$ admit DCL."*

Motivated by this, a few more results in the context of corona operation are derived.

Theorem 4.3.2. *$G \odot \bar{K}_1$ admits a DCG where $G(n, m)$ is a $k-regular$ DCG.*

Proof. Suppose G is a $k - regular$ graph with node set $\{v_1, v_2, ..., v_n\}$ that admits a DCL say g. Clearly, $|V(G)| = n$ and $|E(G)| = \frac{nk}{2}$. For ease of computation, fix $g(v_i) = i$; $1 \leq i \leq n$. Let $H = G \odot \bar{K}_1$ with $V(H) = V(G) \cup \{u_1, u_2, ..., u_n\}$ and $E(H) = E(G) \cup \{v_iu_i : 1 \leq i \leq n\}$. Clearly, $|V(H)| = 2n$ and $E(H) = \frac{nk}{2} + n$. Define $\psi : V(H) \to \{1, 2, ..., n, n + 1, ..., 2n\}$ as follows. Fix $\psi(v_i) = g(v_i)$ for $1 \leq i \leq n$. For labeling of u_i's the below mentioned cases arise.

Case (i) When both 'n' and 'm' are even.

For odd values of i, fix $\psi(u_i) = 2g(v_i)$; $\frac{n}{2} < i < n$, and label $\psi(u_i)$ with the largest value of $i(2^l)$ such that $i(2^l) \leq 2n$ for $1 \leq i \leq \frac{n}{2}$ and $l \in \mathbb{N}$. Now label $u_2, u_4, ..., u_n$ simultaneously from $\{n + 1, n + 3, ..., 2n\}$. Since m is even therefore, $e_g(0) = e_g(1)$. In the wake of above pattern one can find that $e_\psi(0) = e_\psi(1)$ proving that H is a DCG.

Case (ii) When 'n' is even and 'm' is odd.

Since m is odd, therefore, either $e_g(0) = e_g(1) + 1$ or $e_g(1) = e_g(0) + 1$. Following the labeling pattern of Case (i), it is an easy check that $|e_\psi(0) - e_\psi(1)| \leq 1$.

Case (iii) When 'n' is odd and 'm' is even.

For odd values of i, fix $\psi(u_i) = 2g(v_i)$; $\lceil \frac{n}{2} \rceil \leq i \leq n$ and label $\psi(u_i)$ with the largest value of $i(2^l)$ such that $i(2^l) \leq 2n$ for $1 \leq i < \lceil \frac{n}{2} \rceil$ and $l \in \mathbb{N}$. Label the remaining u_i's simultaneously with unused labels out of $\{n+1, n+2, ..., 2n\}$. Since m is even, therefore, $e_g(1) = e_g(0)$. Following the labeling pattern, $e_\psi(1) = e_\psi(0) + 1$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$.

Case (iv) When both 'n' and 'm' are odd.
Here, either $e_g(0) = e_g(1) + 1$ or $e_g(1) = e_g(0) + 1$ for G. Now three subcases arise.
Subcase (i) If $e_g(0) = e_g(1) + 1$, then follow the labeling pattern of Case (iii).
Subcase (ii) If $e_g(1) = e_g(0) + 1$, and $n \neq 5 + 4m$, where $m \in \{0, 1, 2, ...\}$. Label $\psi(u_1)$ with the largest value of $\lfloor \frac{n}{2} \rfloor 2^l$ such that $\lfloor \frac{n}{2} \rfloor 2^l \leq 2n$, $l \in \mathbb{N}$. For odd values of i, fix $\psi(u_i) = 2g(v_i)$; $\lceil \frac{n}{2} \rceil \leq i \leq n$ and label $\psi(u_i)$ with the largest value of $i(2^l)$ for $3 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$. Next, assign remaining one even label to $u_{\lfloor \frac{n}{2} \rfloor}$. For $u_2, u_4, ..., u_{n-1}$, label these nodes simultaneously with unused labels out of $\{n+1, n+2, ...2n\}$. Following above pattern, $e_\psi(1) = e_\psi(0)$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$.
Subcase (iii) If $e_g(1) = e_g(0) + 1$, and $n = 5 + 4m$, where $m \in \{0, 1, 2, ...\}$. Fix $\psi(u_1)$ with the largest value of $(\lfloor \frac{n}{2} \rfloor - 1)2^l$ such that $(\lfloor \frac{n}{2} \rfloor - 1)2^l \leq 2n$. For odd values of i, put $\psi(u_i) = 2g(v_i)$; $\lceil \frac{n}{2} \rceil \leq i \leq n$ and label $\psi(u_i)$ with the largest value of $i(2^l)$ for $3 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$ and $l \in \mathbb{N}$. Assign remaining one even label out of $\{n+1, n+2, ..., 2n\}$ to $u_{\lfloor \frac{n}{2} \rfloor - 1}$. For remaining nodes, namely, $u_2, u_4, ..., u_{n-1}$, label them simulatenously with unconsumed labels out of $\{n + 1, n + 2, ...2n\}$. Following the above pattern, one can find that $e_\psi(1) = e_\psi(0)$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$.
Thus, under all the cases, H is a DCG. □

Definition 4.3.1. [56] "Shell graph Sh_n is obtained by taking $n - 3$ concurrent chords in C_n. The node at which all chords are concurrent is called an apex node."

Theorem 4.3.3. *$Sh_n \odot \bar{K}_1$ is a DCG, $n \geq 5$.*

Proof. Suppose $\{u_1, u_2, ..., u_n\}$ and $\{u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_nu_1\} \cup \{u_1u_i : 3 \leq i \leq n - 1\}$ represent respectively the node and edge set of Sh_n. Let $G = Sh_n \odot K_1$ having $V(G) = V(Sh_n) \cup \{u'_i : 1 \leq i \leq n\}$ and $E(G) = E(Sh_n) \cup \{u_iu'_i : 1 \leq i \leq n\}$. Here, $|V(G)| = 2n$, whereas $|E(G)| = 3n - 3$. Consider $\psi : V(G) \to \{1, 2, ..., 2n\}$. The following possibilities arise.
Case (i) If 'n' is even.
Put $\psi(u_1) = 1$, $\psi(u'_1) = 2$, $\psi(u_2) = 4$, $\psi(u_3) = 6$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $4 \leq i \leq \frac{n}{2} + 1$, $\psi(u_{\frac{n}{2}+2}) = 8$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $\frac{n}{2} + 3 \leq i \leq n$, $\psi(u'_2) = \frac{\psi(u_{\frac{n}{2}+1})}{2}$, $\psi(u'_i) = \frac{\psi(u_i)}{2}$; $3 \leq i \leq \frac{n}{2}$, $\psi(u'_{\frac{n}{2}+1}) = \psi(u'_2) + 2$, $\psi(u'_i) = \psi(u'_{i-1}) + 2$; $\frac{n}{2} + 2 \leq i \leq n$. Observe that $e_\psi(1) = \frac{3n}{2} - 2$ and $e_\psi(0) = \frac{3n}{2} - 1$.
Case (ii) If 'n' is odd.
Put $\psi(u_1) = 1$, $\psi(u'_1) = 2$, $\psi(u_2) = 4$, $\psi(u_3) = 6$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $4 \leq i \leq \lceil \frac{n}{2} \rceil + 1$, $\psi(u_{\lceil \frac{n}{2} \rceil+2}) = 8$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $\lceil \frac{n}{2} \rceil + 3 \leq i \leq n$, $\psi(u'_2) = \frac{\psi(u_{\lceil \frac{n}{2} \rceil+1})}{2}$, $\psi(u'_i) = \frac{\psi(u_i)}{2}$; $3 \leq i \leq \lceil \frac{n}{2} \rceil$, $\psi(u'_{\lceil \frac{n}{2} \rceil+1}) = \psi(u'_2) + 2$, $\psi(u'_i) = \psi(u'_{i-1}) + 2$; $\lceil \frac{n}{2} \rceil + 2 \leq i \leq n$. Observe that $e_\psi(1) = e_\psi(0) = \frac{3n-3}{2}$.
This shows that G is a DCG. □

Theorem 4.3.4. *$DW_n \odot \bar{K}_1$, $n \geq 3$ is a DCG.*

Proof. Let $V(DW_n) = \{v_0, u_i, v_i : 1 \leq i \leq n\}$; u_i and v_i respectively the internal and external rim nodes of DW_n with v_0, the apex node. Let $G = DW_n \odot \bar{K}_1$ with $V(G) = V(DW_n) \cup \{v'_0, u'_i, v'_i : 1 \leq i \leq n\}$, and $E(G) = E(DW_n) \cup \{u_iu'_i, v_iv'_i, v_0v'_0 : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 4n+2$, and $|E(G)| = 6n+1$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 4n+2\}$ by fixing $\psi(v_0) = 1$, $\psi(u_1) = 6$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $2 \leq i \leq n$ such that $\psi(u_1) \not| \psi(u_n)$ (If this happens, swap the labels of u_n and u'_n). Next, let $\psi(u'_i) = \frac{\psi(u_i)}{2}$; $1 \leq i \leq n$, $\psi(v_1) = \psi(u'_n) + 2$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq n$ such that $\psi(v_1) \not| \psi(v_n)$ (If this happens, swap the labels of v_n and v'_n) $\psi(v'_1) = 2$, $\psi(v'_2) = 4$, $\psi(v'_i) = \psi(v'_{i-1}) + 4$; $3 \leq i \leq n$ and remaining even label to v'_0. It can be seen that G is a DCG (see Figure 4.3). □

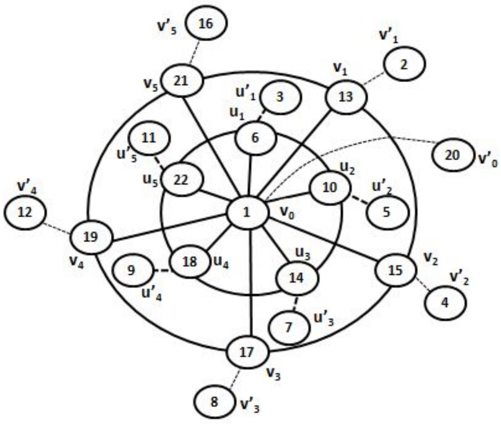

FIGURE 4.3: DCL of $DW_5 \odot \bar{K}_1$

Definition 4.3.2. [66] "The jewel graph J_n has $V(J_n) = \{x, y, u, v, z_i : 1 \leq i \leq n\}$ and $E(J_n) = \{xu, xv, yu, yv, uv, xz_i, yz_i : 1 \leq i \leq n\}$."

Theorem 4.3.5. *$J_n \odot \bar{K}_1$ is a DCG.*

Proof. Let $G = J_n \odot \bar{K}_1$ with $V(G) = V(J_n) \cup \{x', y', u', v', z'_i :\leq i \leq n\}$ and $E(G) = E(J_n) \cup \{xx', yy', uu', vv', z_iz'_i : 1 \leq i \leq n\}$. It can be seen that $|V(G)| = 2n + 8$ and $|E(G)| = 3n + 9$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 2n + 8\}$ under the given possibilities.

Case (i) If 'n' is even.

Fix $\psi(x) = 2$, $\psi(y) = 1$, $\psi(z_1) = 4$, $\psi(z_i) = \psi(z_{i-1}) + 2$; $2 \leq i \leq \frac{n}{2}$, $\psi(z'_i) = \psi(z_i) - 1$; $1 \leq i \leq \frac{n}{2}$, $\psi(z_{\frac{n}{2}+1}) = \psi(z'_{\frac{n}{2}}) + 2$, $\psi(z_i) = \psi(z_{i-1}) + 2$; $\frac{n}{2} + 2 \leq i \leq n$, $\psi(z'_i) = \psi(z_i) + 1$; $\frac{n}{2} + 1 \leq i \leq n$. Next put $\psi(x') = \psi(z_n) + 2$, $\psi(u) = \psi(z'_n) + 2$, $\psi(u') = \psi(u) + 1$, $\psi(v) = \psi(z'_n) + 4$, $\psi(v') = \psi(v) + 1$, $\psi(y') = \psi(z'_n) + 6$. One can verify that $e_\psi(1) = e_\psi(0) + 1$.

Case (ii) If 'n' is odd.

Put $\psi(x) = 2$, $\psi(y) = 1$, $\psi(z_1) = 4$, $\psi(z_i) = \psi(z_{i-1}) + 2$; $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $\psi(z'_i) =$

$\psi(z_i)-1$; $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $\psi(z_{\lceil \frac{n}{2} \rceil}) = \psi(z'_{\lfloor \frac{n}{2} \rfloor})+2$, $\psi(z_i) = \psi(z_{i-1})+2$; $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$, $\psi(z'_i) = \psi(z_i)+1$; $\lceil \frac{n}{2} \rceil \leq i \leq n$. Next, fix $\psi(x') = \psi(z_n)+2$, $\psi(u) = \psi(z'_n)+2$, $\psi(u') = \psi(u)+1$, $\psi(v) = \psi(z'_n)+4$, $\psi(v') = \psi(v)+1$, $\psi(y') = \psi(z'_n)+6$. One can observe that $e_\psi(1) = 2n - \lceil \frac{n}{2} \rceil + 5$ & $e_\psi(0) = 2n - \lfloor \frac{n}{2} \rfloor + 4$.
Thus, G admits a DCL (see Figure 4.4). □

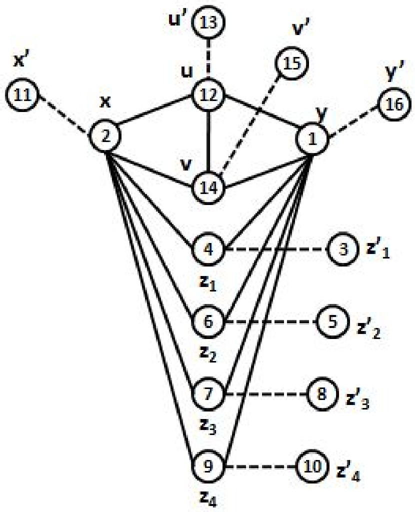

FIGURE 4.4: DCL of $J_4 \odot \bar{K}_1$

Theorem 4.3.6. *$Gl(n) \odot \bar{K}_1$, $n \geq 2$ admits a DCL.*

Proof. Suppose node and edge set of $Gl(n)$ are respectively $\{u, v, u_i : 1 \leq i \leq n\}$ and $\{uu_i, vu_i : 1 \leq i \leq n\}$. Let $G = Gl(n) \odot \bar{K}_1$ with $V(G) = V(Gl(n)) \cup \{u', v', u'_i : 1 \leq i \leq n\}$ & $E(G) = E(Gl(n)) \cup \{uu', vv', u_iu'_i : 1 \leq i \leq n\}$. Observe that $|V(G)| = 2n+4$ & $|E(G)| = 3n+2$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+4\}$ given by underlying conditions.
Case (i) When 'n' is even. Fix $\psi(u) = 1$, $\psi(u') = 2n+4$, $\psi(v) = 2$, $\psi(v') = 3$, $\psi(u_1) = 4$, $\psi(u_i) = \psi(u_{i-1})+2$; $2 \leq i \leq \frac{n}{2}$, $\psi(u_{\frac{n}{2}+1}) = \psi(u_{\frac{n}{2}})+3$, $\psi(u_i) = \psi(u_{i-1})+2$; $\frac{n}{2}+2 \leq i \leq n$, $\psi(u'_i) = \psi(u_i)+1$; $1 \leq i \leq \frac{n}{2}$ and $\psi(u'_i) = \psi(u_i)-1$; $\frac{n}{2}+1 \leq i \leq n$. Note that $e_\psi(0) = e_\psi(1) = \frac{3n+2}{2}$, hence G is a DCG (see Figure 4.5).
Case (ii) When 'n' is odd.
Put $\psi(u) = 1$, $\psi(u') = 2n+4$, $\psi(v) = 2$, $\psi(v') = 3$, $\psi(u_1) = 4$, $\psi(u_i) = \psi(u_{i-1})+2$; $2 \leq i \leq \lceil \frac{n}{2} \rceil$, $\psi(u_{\lceil \frac{n}{2} \rceil+1}) = \psi(u_{\lceil \frac{n}{2} \rceil})+3$, $\psi(u_i) = \psi(u_{i-1})+2$; $\lceil \frac{n}{2} \rceil+2 \leq i \leq n$, $\psi(u'_i) = \psi(u_i)+1$; $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and $\psi(u'_i) = \psi(u_i)-1$; $\lceil \frac{n}{2} \rceil+1 \leq i \leq n$. Note that $e_\psi(1) = e_\psi(0)+1$, establishing the DCL of G. □

Theorem 4.3.7. *$F_n \odot \bar{K}_1$ is a DCG for $n \geq 4$.*

Proof. Let $V(F_n) = \{u_0, u_{i1}, u_{i2} : 1 \leq i \leq n\}$ and $E(F_n) = \{u_0u_{i1}, u_0u_{i2}, u_{i1}u_{i2} : 1 \leq i \leq n\}$. Let $G = F_n \odot \bar{K}_1$ with $V(G) = V(F_n) \cup \{u'_0, u'_{i1}, u'_{i2} : 1 \leq i \leq n\}$ and

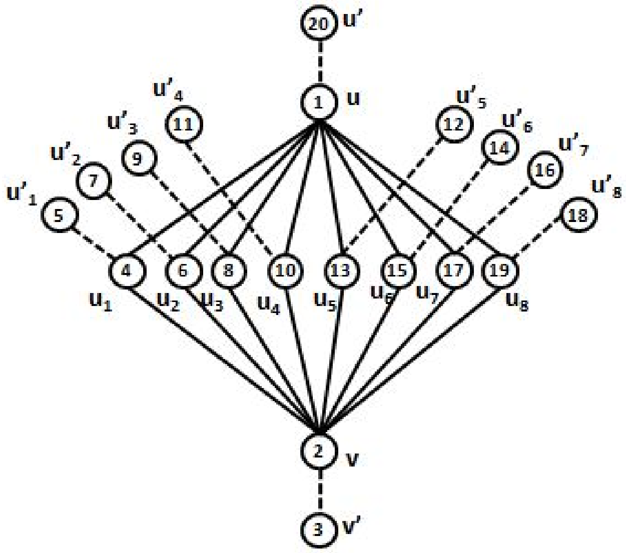

FIGURE 4.5: DCL of $Gl(8) \odot \bar{K}_1$

$E(G) = E(F_n) \cup \{u_0u'_0, u_{i1}u'_{i1}, u_{i2}u'_{i2} : 1 \leq i \leq n\}$. Observe that $|V(G)| = 4n + 2$ & $|E(G)| = 5n + 1$. Consider $\psi : V(G) \to \{1, 2, ..., 4n + 2\}$ for the under mentioned possibilities.

Case (i) When 'n' is even.

Fix $\psi(u_0) = 1$, $\psi(u'_0) = 4n+2$, $\psi(u_{11}) = 2$, $\psi(u_{i1}) = \psi(u_{(i-1)1})+4$; $2 \leq i \leq n$, $\psi(u_{12}) = 4$, $\psi(u_{i2}) = \psi(u_{(i-1)2}) + 4$; $2 \leq i \leq n$, $\psi(u'_{i1}) = \frac{\psi(u_{i1})}{2}$; $2 \leq i \leq \frac{n}{2}$, $\psi(u'_{i1}) = \psi(u_{i1}) + 1$; $\frac{n}{2} + 1 \leq i \leq n$, $\psi(u'_{i2}) = \psi(u_{i2}) + 1$; $\frac{n}{2} + 1 \leq i \leq n$. Assign the unused odd labels simultaneously to the remaining nodes. Observe that $e_\psi(1) = e_\psi(0) + 1$ which proves that G is a DCG.

Case (ii) When 'n' is odd.

Fix $\psi(u_0) = 1$, $\psi(u'_0) = 4n + 2$, $\psi(u_{11}) = 2$, $\psi(u_{i1}) = \psi(u_{(i-1)1}) + 4$; $2 \leq i \leq n$, $\psi(u_{12}) = 4$, $\psi(u_{i2}) = \psi(u_{(i-1)2}) + 4$; $2 \leq i \leq n$, $\psi(u'_{i1}) = \frac{\psi(u_{i1})}{2}$; $2 \leq i \leq \lfloor\frac{n}{2}\rfloor$, $\psi(u'_{i1}) = \psi(u_{i1}) + 1$; $\lceil\frac{n}{2}\rceil \leq i \leq n$, $\psi(u'_{i2}) = \psi(u_{i2}) + 1$; $\lceil\frac{n}{2}\rceil \leq i \leq n$. Allot the unused odd labels simultaneously to the remaining nodes gives $e_\psi(0) = e_\psi(1)$, therefore G is DCG. □

4.4 DCL of Lilly Related Graphs

Only a few results are available in the literature considering the DCL of tree related graphs except that Vartharajan et al. in [87] proved that full binary tree admits a DCL. Motivated by this, one of the families of tree called, lilly graph, for different graph operations has been investigated.

Theorem 4.4.1. *I_n admits a DCL.*

Proof. Let $V(I_n) = \{x_1, x_2, ..., x_{4n-1}\}$ and $E(I_n) = \{x_{3n}x_i : 1 \leq i \leq 2n\} \cup \{x_{3n-1}x_{3n}, x_{3n}x_{3n+1}\} \cup \{x_ix_{i+1} : 2n+1 \leq i \leq 3n-2\} \cup \{x_ix_{i+1} : 3n+1 \leq i \leq 4n-2\}$. Clearly, $|V(I_n)| = 4n-1$ & $|E(G)| = 4n-2$. Define $\psi : V(I_n) \to \{1, 2, ..., 4n-1\}$ by letting $\psi(x_{3n}) = 2$, $\psi(x_1) = 4$, $\psi(x_i) = \psi(x_{i-1})+2$; $2 \leq i \leq 2n-2$, $\psi(x_{2n-1}) = 1$, $\psi(x_i) = \psi(x_{i-1})+2$; $2n \leq i \leq 3n-1$, $\psi(x_{3n+1}) = \psi(x_{3n-1}) + 2$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $3n + 2 \leq i \leq 4n - 1$. Observe that $e_\psi(0) = e_\psi(1) = 2n - 1$ which establishes that I_n is a DCG (see Figure 4.6). □

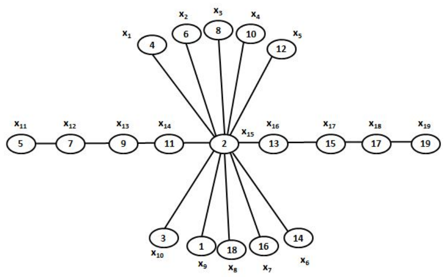

FIGURE 4.6: DCL of I_5

Theorem 4.4.2. *Switching of an arbitrary node in I_n admits a DCL.*

Proof. Let $V(I_n) = \{x_1, x_2, ..., x_{4n-1}\}$. Here, $x_1, x_2, ..., x_n, x_{n+1}, ..., x_{2n}, x_{2n+1}, x_{4n-1}$ are pendant nodes. Let G be constructed by switching arbitrary pendant node of I_n, say, x_k. Clearly, $|V(G)| = 4n - 1$ & $|E(G)| = 8n - 6$. Consider $\psi : V(G) \to \{1, 2, ..., 4n - 1\}$ by assigning $\psi(x_k) = 1$ and $\psi(x_{3n}) = p$,where p is the largest prime, $\leq 4n - 1$. Label the remaining nodes simultaneously with unutilized labels out of $\{1, 2, ..., 4n - 1\}$ gives, $|e_\psi(0) - e_\psi(1)| \leq 1$. □

Corollary 4.4.1. *Switching of any node of degree 2 in I_n admits a DCL.*

Proof. Switching a node of degree 2 in I_n results in a graph having $4n - 1$ nodes and $8n - 8$ edges. The labeling is done on the similar lines as in Theorem 4.4.2. □

Theorem 4.4.3. *Switching of the apex node in I_n, $n \geq 2$ admits a DCL.*

Proof. Let $V(I_n) = \{x_1, x_2, ..., x_{4n-1}\}$ and G be formed by switching the apex node of I_n, namely, x_{3n}. Clearly, $|V(G)| = 4n - 1$ & $|E(G)| = 4n - 8$. Define $\psi : V(G) \to \{1, 2, ..., 4n - 1\}$ by assigning $\psi(x_{3n}) = 1$, $\psi(x_{2n+1}) = 4$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $2n + 2 \leq i \leq 3n - 1$, $\psi(x_{3n+1}) = \psi(x_{3n-1}) + 2$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $3n + 2 \leq i \leq 4n - 1$. Now assigning of unutilized labels to unlabeled nodes in any order yields $|e_\psi(0) - e_\psi(1)| \leq 1$ (see Figure 4.7). □

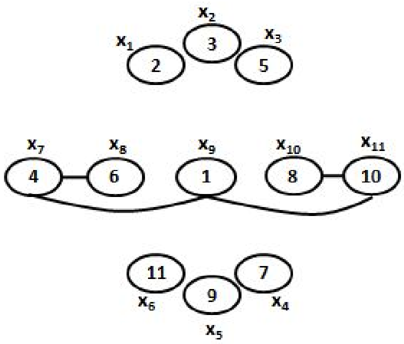

FIGURE 4.7: DCL of a graph acquired by switching of x_9 in I_3

Theorem 4.4.4. *Duplication of the apex node by a node in I_n, $n \geq 3$ admits a DCL.*

Proof. Let $V(I_n) = \{x_1, x_2, ..., x_{4n-1}\}$ and G be formed by duplicating the apex node x_{3n} of I_n by the newly added node, say, s. Clearly, $|V(G)| = 4n$ & $|E(G)| = 6n$. Define $\psi : V(G) \to \{1, 2, ..., 4n\}$ by assigning $\psi(x_{3n}) = 2$, $\psi(s) = 4$, $\psi(x_1) = 1$, $\psi(x_2) = 6$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $3 \leq i \leq 2n-1$, $\psi(x_{2n}) = 5$, $\psi(x_{2n+1}) = 3$ and $\psi(x_{2n+2}) = 9$. Now allocate the unutilized labels from $\{1, 2, ..., 4n\}$ simultaneously to the unlabeled nodes x_j ; $2n+3 \leq j \leq 4n-1$ and $j \neq 3n$. It is clear that $|e_\psi(0) - e_\psi(1)| \leq 1$ which shows that G is a DCG. □

Theorem 4.4.5. *The duplication of an arbitrary node of degree* 1 *or* 2 *by a node in I_n permits a DCL for $n \geq 3$.*

Proof. Let $V(I_n) = \{x_1, x_2, ..., x_{4n-1}\}$ and G be formed by duplication of an arbitrary node of degree 1 or 2, say, x_k of I_n by a new node s. Observe that $|V(G)| = 4n$. Now the given cases arise.
Case (i) When duplication of a node of degree 1 is taken.
In this case, $|E(G)| = 4n-1$. Define $\psi : V(G) \to \{1, 2, ..., 4n\}$ by fixing $\psi(x_{3n}) = 2$, $\psi(x_1) = 1$ and $\psi(s)$ be the largest prime $p \leq 4n$. Assign unutilized even labels to x_i; $2 \leq i \leq 2n$ and odd labels simultaneously to x_j ; $2n+1 \leq j \leq 4n-1, j \neq 3n$.
Case (ii) When duplication of a node of degree 2 is taken.
In this case, $|E(G)| = 4n$. Labeling is done by using the pattern of *Case* (i) (see Figure 4.8).
In both the cases, G is a DCG. □

Theorem 4.4.6. *$DS(I_n)$ permits a DCL.*

Proof. Let $V(I_n) = \{x_1, x_2, ..., x_{4n-1}\}$. Consider $DS(I_n)$ with $V(DS(I_n)) = V(I_n) \cup \{v, w\}$ and $E(DS(I_n)) = E(I_n) \cup \{x_i v : 1 \leq i \leq 2n\} \cup \{x_{2n+1} v, x_{4n-1}\ v\} \cup \{x_i w :$

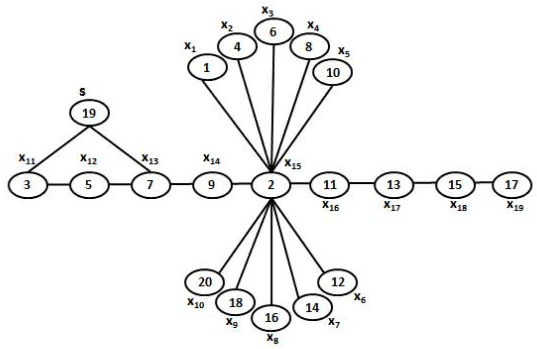

FIGURE 4.8: DCL of a graph acquired by duplication of x_{12} by s in I_5

$2n+2 \leq i \leq 4n-2, i \neq 3n\}$. Observe that $|V(DS(I_n))| = 4n+1$ & $|E(DS(I_n))| = 8n-4$. Consider $\psi : V(DS(I_n)) \to \{1, 2, ..., 4n+1\}$ determined by choosing $\psi(x_{3n}) = 1$, $\psi(v) = 4$, $\psi(w) = 2$, $\psi(x_{4n-1}) = 4n-2$, $\psi(x_1) = 3$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $2 \leq i \leq 2n$ and $\psi(x_{2n+1}) = 6$. Assign the unutilized labels simultaneously to the remaining nodes. It follows that $DS(I_n)$ is a DCG (see Figure 4.9). □

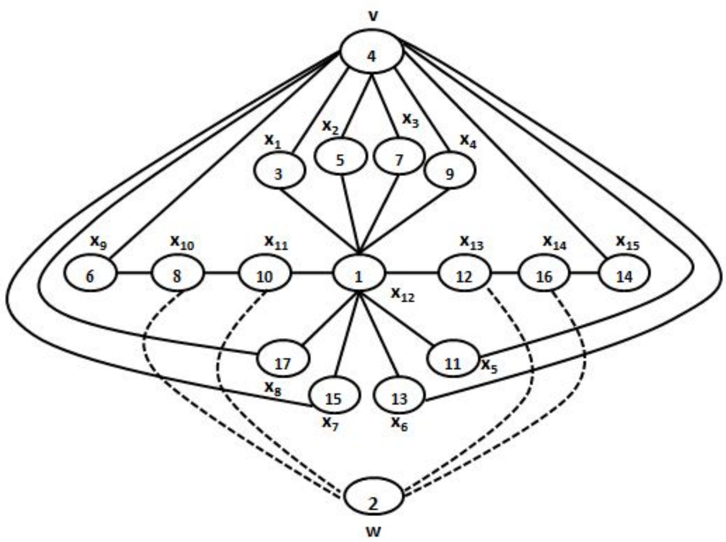

FIGURE 4.9: DCL of $DS(I_4)$

Theorem 4.4.7. *$S(I_n)$ permits a DCL.*

Proof. Let $V(I_n) = \{x_1, x_2, ..., x_{4n-1}\}$, and $E(I_n) = \{x_{3n}x_i : 1 \leq i \leq 2n\} \cup \{x_{3n-1}x_{3n}, x_{3n}x_{3n+1}\} \cup \{x_ix_{i+1} : 2n+1 \leq i \leq 3n-2\} \cup \{x_ix_{i+1} : 3n+1 \leq i \leq 4n-2\}$. Consider $S(I_n)$ with $V(S(I_n)) = V(I_n) \cup \{v_1, v_2, ..., v_{2n}, v_{2n+1}, ..., v_{4n-2}\}$ and $E(S(I_n)) = \{x_{3n}v_i : 1 \leq i \leq 2n\} \cup \{v_ix_i : 1 \leq i \leq 2n\} \cup \{x_iv_i : 2n+1 \leq i \leq 4n-2\} \cup \{v_ix_{i+1} : 2n+1 \leq i \leq 4n-2\}$. Clearly, $|V(S(I_n))| = 8n-3$ & $|E(S(I_n))| = 8n-4$. Consider $\psi : V(S(I_n)) \to \{1, 2, ..., 8n-3\}$ by choosing $\psi(x_{3n}) = 2$, $\psi(x_1) = 1$, $\psi(v_1) = 4$, $\psi(v_2) = 6$, $\psi(v_i) = \psi(v_{i-1}) + 4$; $3 \leq i \leq 2n-1$, $\psi(x_i) = \frac{\psi(v_i)}{2}$; $2 \leq i \leq 2n-1$, $\psi(v_{2n}) = \psi(x_{2n-1}) + 2$

and $\psi(x_{2n}) = \psi(v_{2n}) + 2$. Next, fix $\psi(x_{2n+1}) = 8$, $\psi(v_{2n+1}) = 12$, $\psi(x_i) = \psi(x_{i-1}) + 8$; $2n+2 \leq i \leq 3n-1$, $\psi(v_i) = \psi(v_{i-1}) + 8$; $2n+2 \leq i \leq 3n-2$, $\psi(x_{4n-1}) = \psi(x_{3n-1}) + 4$, $\psi(v_{3n-1}) = \psi(x_{2n})+2$, $\psi(v_{3n}) = \psi(v_{3n-1})+2$, $\psi(x_{3n+1}) = \psi(v_{3n})+2$, $\psi(v_i) = \psi(v_{i-1})+4$; $3n+1 \leq i \leq 4n-2$ and $\psi(x_i) = \psi(x_{i-1})+4$; $3n+2 \leq i \leq 4n-2$. Evidently, $e_\psi(1) = 4n-2$ & $e_\psi(0) = 4n-2$ which ensures that $S(I_n)$ is a DCG (see Figure 4.10). $\square$

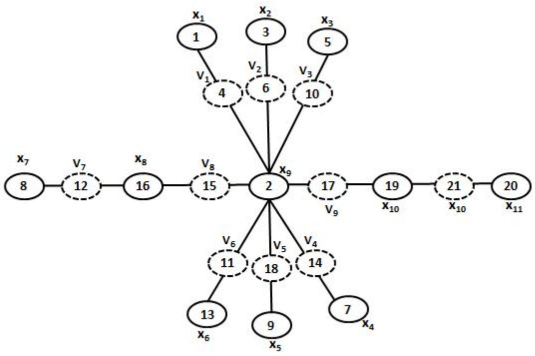

FIGURE 4.10: DCL of $S(I_3)$

Theorem 4.4.8. *Extension of all pendant nodes in I_n permits a DCL.*

Proof. Let $V(I_n) = \{x_1, x_2, ..., x_{4n-1}\}$ and G be formed by performing the extension of all pendant nodes of I_n with $V(G) = V(I_n) \cup \{v_1, v_2, ..., v_{2n}, v_{2n+1}, v_{4n-1}\}$ and $E(G) = E(I_n) \cup \{x_i v_i : 1 \leq i \leq 2n\} \cup \{v_i x_{3n} : 1 \leq i \leq 2n\} \cup \{x_{2n+1}v_{2n+1}, x_{2n+2}v_{2n+1}, x_{4n-1}v_{4n-1}, x_{4n-2}v_{4n-1}\}$. Clearly, $|V(G)| = 6n+1$ & $|E(G)| = 8n+2$. Consider $\psi : V(G) \to \{1, 2, ..., 6n+1\}$ defined by letting $\psi(x_{3n}) = 2$, $\psi(x_1) = 1$, $\psi(x_2) = 6$, $\psi(v_1) = 4$. Now the given possibilities are there.

Case (i) when 'n' is even.

Choose $\psi(x_i) = \psi(x_{i-1})+4$; $3 \leq i \leq n+\frac{n}{2}$, $\psi(v_i) = \frac{\psi(x_i)}{2}$; $2 \leq i \leq n+\frac{n}{2}$, $\psi(x_{n+\frac{n}{2}+1}) = 8$, $\psi(x_i) = \psi(x_{i-1}) + 8$; $n+\frac{n}{2}+2 \leq i \leq 2n$, $\psi(v_i) = \psi(x_i) + 4$; $n+\frac{n}{2}+1 \leq i \leq 2n$, $\psi(v_{2n+1}) = \psi(v_{n+\frac{n}{2}}) + 2$ and $\psi(v_{4n-1}) = 6n+1$. Assign all the unutilized even labels first and then odd labels simultaneously to $x_{2n+1}, x_{2n+2}, ..., x_{4n-1}$ (excluding x_{3n}).

Case (ii) When 'n' is odd.

Fix $\psi(x_i) = \psi(x_{i-1}) + 4$; $3 \leq i \leq n + \lceil\frac{n}{2}\rceil$, $\psi(v_i) = \frac{\psi(x_i)}{2}$; $2 \leq i \leq n + \lceil\frac{n}{2}\rceil$, $\psi(x_{n+\lceil\frac{n}{2}\rceil+1}) = 8$, $\psi(x_i) = \psi(x_{i-1}) + 8$; $n + \lceil\frac{n}{2}\rceil + 2 \leq i \leq 2n$, $\psi(v_i) = \psi(x_i) + 4$; $n + \lceil\frac{n}{2}\rceil + 1 \leq i \leq 2n$, $\psi(v_{2n+1}) = \psi(v_{n+\lceil\frac{n}{2}\rceil}) + 2$ and $\psi(v_{4n-1}) = 6n+1$. Allocate all unutilized even labels first and then odd labels simultaneously to $x_{2n+1}, x_{2n+2}, ..., x_{4n-1}$ (excluding x_{3n}) from $\{1, 2, ..., 6n+1\}$.

In both the cases, ψ induces a DCL for G. $\square$

Theorem 4.4.9. *Extension of the apex node in I_n permits a DCL.*

Proof. Let G be constructed by taking the extension of the apex node of I_n where $V(I_n) = \{x_1, x_2, ..., x_{4n-1}\}$. Clearly, $V(G) = V(I_n) \cup \{w\}$, and $E(G) = E(I_n) \cup \{x_i w : 1 \leq i \leq 2n\} \cup \{x_{3n}w, x_{3n-1}w, x_{3n+1}w\}$. Note here, $|V(G)| = 4n$ and $|E(G)| = 6n + 1$. Consider $\psi : V(G) \to \{1, 2, ..., 4n\}$ and fix $\psi(x_{3n}) = 2$, $\psi(w) = 4$, $\psi(x_1) = 6$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $2 \leq i \leq 2n - 2$, $\psi(x_{2n-1}) = 1$ and $\psi(x_{2n}) = 3$. Assigning unutilized odd labels simultaneously to $x_{2n+1}, x_{2n+2}, ..., x_{4n-1}$ yields G a DCG. □

4.5 DCL of Classes of Planar Graphs

Euler's polyhedral formula, which is connected to polyhedron edges, nodes, and faces, serves as the foundation for planar graph theory. This section focuses on exploring the DCL of certain classes of planar graphs obtained from K_n and $K_{m,n}$. These graphs are investigated for some graph operation. Recall, a graph is planar "if it can be embedded in the plane." Planar graphs are of great importance due to their variety of applications in circuit design, networking and cryptography [9]. A careful thought is given to the crossings especially at the crowded places. As planarity ensures the zero crossing, most of the daily life problems can be dealt by considering the planar graphs.

Though, Euler's Inequality and Kuratowski Theorem can be used to check planarity of graph, the use of planar graphics in Printed Circuit Board (PCB) manufacturing is intended to put an effort for efficient working of design process. A 2D board called a PCB contains every component and circuit that will be utilised in an electronic network. A graph can be used to represent an electrical circuit. The idea of a planar graph can then be used to create a 2D board PCB. First, recall the definitions given in [6] and [59].

Definition 4.5.1. [6] "The class of graph, denoted by Pl_n has $V(Pl_n) = \{v_1, v_2, ..., v_n\}$ and $E(Pl_n) = E(K_n) \setminus \{v_k v_l : 1 \leq k \leq n - 4, k + 2 \leq l \leq n - 2\}$."

Definition 4.5.2. [59] "Let $V_m = \{v_i : 1 \leq i \leq m\}$ and $U_n = \{u_j : 1 \leq j \leq n\}$ be the bipartition of $K_{m,n}$. The class of graph $Pl_{m,n}$ has the node set $V_m \cup U_n$ and $E(Pl_{m,n}) = E(K_{m,n}) \setminus \{v_l u_k : 3 \leq l \leq m, 2 \leq k \leq n - 1\}$."

The embedding used for Pl_n is discussed as follows. Lay $v_1, v_2, ..., v_{n-2}$ along a vertical line with v_1 at top and v_{n-2} at the bottom. Place v_{n-1} and v_n as the end points of a horizontal line perpendicular to the line having $v_1, v_2, ..., v_{n-2}$, at the bottom in such a fashion that v_{n-2}, v_{n-1}, v_n makes a triangular face, see Figure 4.11(a).

Similarly, for embedding of $Pl_{m,n}$ that is going to be used for proofs is explained here. First place $u_1, u_2, ..., u_n$ horizontally with u_1 and u_n respectively at left and right ends. Next, place $v_2, v_3, ..., v_m$ vertically above the segment $u_1, u_2, ..., u_n$, with v_2 at bottom and v_m at the top of the segment. Then place v_1 below the segment $u_1, u_2, ..., u_n$ so that

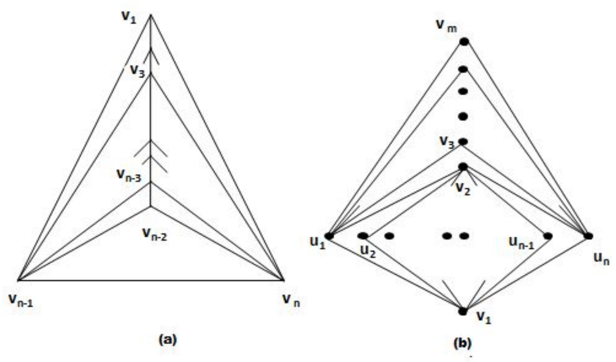

FIGURE 4.11: The class of (a) Pl_n (b) $Pl_{m,n}$

v_1, u_k, v_2, u_{k+1} forms a face of length 4 for $1 \leq k \leq n-1$. Remember, this discussion is about segment placement; no edges other than those indicated in the definitions are to be introduced, see Figure 4.11 (b).
Cordial labeling of Pl_n & $Pl_{m,n}$ has been established in [59]. DCL of Pl_n and $Pl_{m,n}$ in addition to exploring these two classes for graph operations are presented here.

Theorem 4.5.1. *Pl_n admits a DCL.*

Proof. Suppose node and edge set of Pl_n are given by $\{v_i : 1 \leq i \leq n\}$ and $\{v_iv_{i+1} : 1 \leq i \leq n-3\} \cup \{v_{n-1}v_n\} \cup \{v_nv_i, v_{n-1}v_i : 1 \leq i \leq n-2\}$ respectively. One can see that $|V(Pl_n)| = n$ and $|E(Pl_n)| = 3n-6$. Consider $\psi : V(Pl_n) \rightarrow \{1, 2, ..., n\}$ for the following three cases.
Case (i) When $n \geq 5$ is odd.
Let $\psi(v_{n-1}) = 1$, $\psi(v_n) = 2$, $\psi(v_i) = 2+i$; $1 \leq i \leq n-2$. Clearly, $e_\psi(1) = e_\psi(0)+1$.
Case (ii) When $n \geq 10$ is even such that $\frac{n}{2}$ is odd.
Let p be the largest prime $< n$. Fix $\psi(v_{n-1}) = 1$, $\psi(v_n) = p$. Label the remaining nodes begining with v_{n-2} and proceeding to v_1 in the following fashion

$$\begin{array}{ccccc} 2, & 2.2, & 2.2^2, & ..., & 2.2^{k_1}, \\ 3, & 3.2, & 3.2^2, & ..., & 3.2^{k_2}, \\ & ..., & ..., & ..., & ..., \quad ..., \end{array}$$

upto, $\frac{n}{2}-2$, $(\frac{n}{2}-2).2$,..., $(\frac{n}{2}-2)2^{k_t}$, where $(2t-1)2^{k_t} \leq n$ and $t \geq 1$, $k_t \geq 0$. Assign the unutilized labels simultaneously to the remaining nodes. Observe that $e_\psi(0) = e_\psi(1) = \frac{3n}{2}-3$.
Case (iii) When $n \geq 8$ is even such that $\frac{n}{2}$ is even.
Let p be the largest prime $< n$. Fix $\psi(v_{n-1}) = 1$, $\psi(v_n) = p$. Label the remaining nodes

begining with v_{n-2} and proceeding to v_1 in the following fashion

$$\begin{array}{ccccc} 2, & 2.2, & 2.2^2, & ..., & 2.2^{k_1}, \\ 3, & 3.2, & 3.2^2, & ..., & 3.2^{k_2}, \\ ..., & ..., & ..., & ..., & ..., \end{array}$$

upto, $\frac{n}{2}-3$, $(\frac{n}{2}-3).2$,..., $(\frac{n}{2}-3)2^{k_t}$, where $(2t-1)2^{k_t} \leq n$ and $t \geq 1$, $k_t \geq 0$. Now assigning unutilized labels simultaneously to the remaining nodes shows that $e_\psi(0) = e_\psi(1) = \frac{3n}{2} - 3$.
Hence, Pl_n is a DCG. □

Theorem 4.5.2. *$Pl_{m,n}$ admits a DCL.*

Proof. Let $V_m = \{v_i : 1 \leq i \leq m\}$ and $U_n = \{u_j : 1 \leq j \leq n\}$. Let $V(Pl_{m,n}) = V_m \cup U_n$ and $E(Pl_{m,n}) = E(K_{m,n}) \backslash \{v_l u_k : 3 \leq l \leq m, 2 \leq k \leq n-1\}$. Clearly, $|V(Pl_{m,n})| = m+n$ and $|E(Pl_{m,n})| = 2m + 2n - 4$. Consider $\psi : V(Pl_{m,n}) \to \{1, 2, ..., m+n\}$. Now three cases arise.
Case (i) When $m = n$ with $m \geq 4$, $m \neq 5$.
Let p_1, p_2 be sufficiently large primes : $p_2 < p_1 \leq m+n$. Fix $\psi(u_1) = p_1$, $\psi(u_n) = p_2$, $\psi(v_1) = 1$ and $\psi(v_2) = 2$. Assign even labels to unlabeled u_i; $2 \leq i \leq n-1$ and remaining labels simultaneously to unlabeled nodes. Observe that $e_\psi(0) = e_\psi(1) = 2m-2$, which ensures that $Pl_{m,m}$ is a DCG.
Case (ii) When $m > n$ where $m \geq, n \geq 2$.
Let p_1, p_2 be sufficiently large primes : $p_2 < p_1 \leq m+n$. Fix $\psi(u_1) = 1$, $\psi(u_n) = p_1$, $\psi(v_1) = 2$ and $\psi(v_2) = p_2$. Assign even labels to unlabeled u_i; $2 \leq i \leq n-1$ and remaining labels simultaneously to unlabeled v_j; $3 \leq j \leq m$. Observe that $e_\psi(0) = e_\psi(1) = m+n-2$, which proves that $Pl_{m,n}$ is a DCG.
Case (iii) When $m < n$ where $m \geq 2, n \geq 3$.
Let p_1, p_2 be sufficiently large primes so that $p_2 < p_1 \leq m+n$. Fix $\psi(u_1) = 2$, $\psi(u_n) = p_1$, $\psi(v_1) = 1$ and $\psi(v_2) = p_2$. Assign even labels to unlabeled v_j; $3 \leq j \leq m$ and remaining labels to unlabeled u_i; $2 \leq i \leq n-1$, simultaneously. Observe that $e_\psi(0) = e_\psi(1) = m+n-2$ showing that $Pl_{m,n}$ is a DCG.
Thus, in all the cases, $Pl_{m,n}$ admits a DCL. □

Theorem 4.5.3. *Duplicating a node by a node in Pl_n, $n \geq 4$ admits a DCL.*

Proof. Let $V(Pl_n) = \{v_i : 1 \leq i \leq n\}$ and $E(Pl_n) = \{v_i v_{i+1} : 1 \leq i \leq n-3\} \cup \{v_{n-1}v_n\} \cup \{v_n v_i, v_{n-1}v_i : 1 \leq i \leq n-2\}$. Suppose G is acquired by duplicating a node of highest degree namely, v_n, by a node, say, w. Here, $V(G) = V(Pl_n) \cup \{w\}$ and

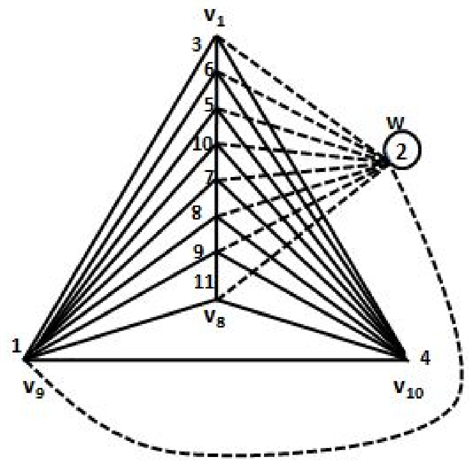

FIGURE 4.12: DCL of a graph acquired by duplication of v_{10} in Pl_{10}

$E(G) = E(Pl_n) \cup \{v_{n-1}w, v_iw : 1 \leq i \leq n-2\}$. See that $|V(G)| = n+1$, whereas $|E(G)| = 4n-7$. Consider $\psi : V(G) \to \{1, 2, ..., n+1\}$ for the under mentioned cases.

Case (i) When $n \equiv 0(mod\ 2)$ and $n \geq 4$.

Fix $\psi(v_{n-1}) = 1$, $\psi(w) = 2$, $\psi(v_n) = 4$. For odd values of i, $1 \leq i \leq \frac{n}{2} - 2$, put $\psi(v_i) = 2+i$, $\psi(v_{i+1}) = 2(\psi(v_i))$ and for the remaining nodes, assign the unused labels simultaneously. Here, $e_\psi(0) = e_\psi(1) + 1$(see Figure 4.12).

Case (ii) When $n \equiv 1(mod\ 2)$ and $n \geq 5$.

Fix $\psi(v_{n-1}) = 1$, $\psi(w) = 2$, $\psi(v_n) = 4$. For odd values of i, $1 \leq i \leq \frac{n+1}{2} - 2$, put $\psi(v_i) = 2+i$ and $\psi(v_{i+1}) = 2(\psi(v_i))$, and for remaining nodes, assign the unused labels simultaneously. Here, $e_\psi(1) = e_\psi(0) + 1$.

In both the cases, G is a DCG. □

Theorem 4.5.4. *Duplicating a node in $Pl_{m,n}$ admits a DCL $\forall\ m, n \geq 4$.*

Proof. Let $V(Pl_{m,n}) = V_m \cup U_n$ where $V_m = \{v_i : 1 \leq i \leq m\}$ and $U_n = \{u_j : 1 \leq j \leq n\}$, and $E(Pl_{m,n}) = E(K_{m,n}) \setminus \{v_lu_k : 3 \leq l \leq m, 2 \leq k \leq n-1\}$. Let G be produced by duplicating a node namely, u_n by a node, say, w. Here, $V(G) = V(Pl_{m,n}) \cup \{w\}$ and $E(G) = E(Pl_{m,n}) \cup \{v_iw : 1 \leq i \leq m\}$. Clearly, $|V(G)| = m+n+1$ and $|E(G)| = 3m+2n-4$. Consider $\psi : V(G) \to \{1, 2, ..., m+n+1\}$. Now the under mentioned cases arise.

Case (i) When $n = m$.

Let k be the largest odd integer $\leq \lfloor \frac{2n+1}{3} \rfloor$. Fix $\psi(u_1) = 1$, $\psi(v_1) = 2$, $\psi(v_2) = 4$, $\psi(u_n) = 8$ and $\psi(w) = k$. Assign even labels to u_i for $2 \leq i \leq n-2$. Next assign unused labels simultaneously to unlabeled nodes $\{u_{n-1}, v_3, v_4, ..., v_m\}$.

Case (ii) When $n > m$.

Choose p_1, p_2 sufficiently largest prime such that $p_2 < p_1 \leq m+n+1$. Fix $\psi(u_1) = 2$,

$\psi(u_n) = 4$, $\psi(v_1) = 1$, $\psi(v_2) = p_2$, $\psi(v_3) = 8$ and $\psi(w) = p_1$. There arise two subcases.
Subase (i) If m is odd.
Fix $\psi(v_i) = \psi(v_{i-1}) + 4$; $4 \leq i \leq \lceil \frac{m+1}{2} \rceil + 1$, $\psi(v_{\lceil \frac{m+1}{2} \rceil + 2}) = 6$, $\psi(v_i) = \psi(v_{i-1}) + 4$; $\lceil \frac{m+1}{2} \rceil + 3 \leq i \leq m$. Allot unused labels simultaneously to remaining unlabeled nodes.
Subcase (ii) If m is even.
Fix $\psi(v_i) = \psi(v_{i-1}) + 4$; $4 \leq i \leq \frac{m}{2} + 1$, $\psi(v_{\frac{m}{2}+2}) = 6$, $\psi(v_i) = \psi(v_{i-1}) + 4$; $\frac{m}{2} + 3 \leq i \leq m$. Allot unused labels simultaneously to unlabeled nodes. Clearly, $|e_\psi(0) - e_\psi(1)| \leq 1$.
Case (iii) When $m > n$.

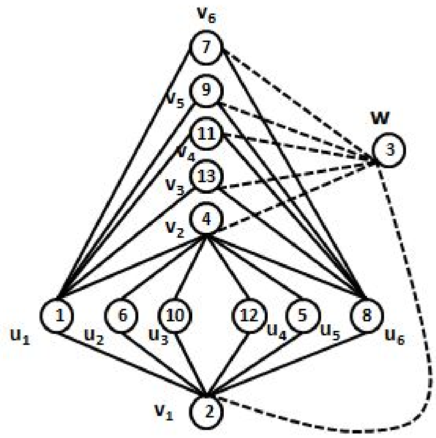

FIGURE 4.13: DCL of a graph obtained by duplicating u_6 in $Pl_{6,6}$

Let p_1 and p_2 be sufficiently large primes such that $p_2 < p_1 \leq m+n+1$. Fix $\psi(u_1) = 1$, $\psi(v_1) = 3$, $\psi(v_2) = p_2$, $\psi(u_n) = 2$, $\psi(v_n) = 4$, $\psi(u_2) = 6$, $\psi(w) = p_1$, $\psi(u_i) = \psi(u_{i-1}) + 3$; $3 \leq i \leq k < n$ such that $\psi(u_k) \leq m + n + 1$. Next assigning of unused labels simultaneously to unlabeled nodes shows that $|e_\psi(0) - e_\psi(1)| \leq 1$.
Hence, G is a DCG (see Figure 4.13). □

Theorem 4.5.5. *$Pl_n \odot K_1$ admits a DCL.*

Proof. Let $G = Pl_n \odot K_1$ with $V(G) = V(Pl_n) \cup \{u_i : 1 \leq i \leq n\}$ and $E(G) = E(Pl_n) \cup \{v_i u_i : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 2n$ and $|E(G)| = 4n - 6$. Consider $\psi : V(G) \to \{1, 2, ..., 2n\}$ defined by fixing $\psi(v_{n-1}) = 1$, $\psi(v_n) = 2$, $\psi(u_{n-1}) = 4$, $\psi(u_n) = 3$, $\psi(v_1) = 6$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq n-3$, $\psi(u_i) = \psi(v_i) - 1$; $1 \leq i \leq n-3$, $\psi(v_{n-2}) = \psi(v_{n-3}) + 1$ and $\psi(u_{n-2}) = \psi(v_{n-2}) + 1$. Observe that $e_\psi(0) = e_\psi(1) = 2n - 3$, proving that G is a DCG (see Figure 4.14). □

Theorem 4.5.6. *$Pl_{m,m} \odot K_1$ admits a DCL.*

Proof. Let $V(Pl_{m,n}) = V_m \cup U_n$ where $V_m = \{v_i : 1 \leq i \leq m\}$ and $U_n = \{u_j : 1 \leq j \leq n\}$ and $E(Pl_{m,n}) = E(K_{m,n}) \setminus \{v_l u_k : 3 \leq l \leq m, 2 \leq k \leq n-1\}$. Let

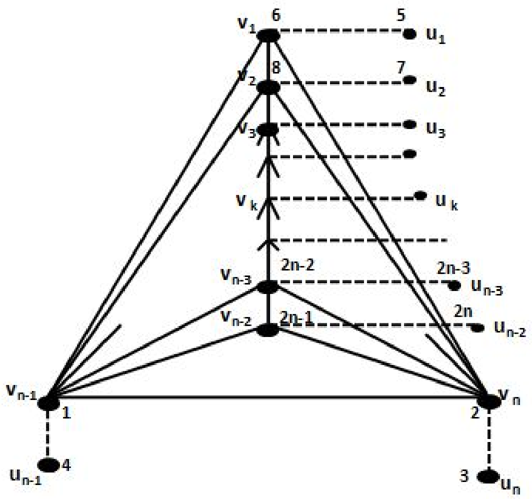

FIGURE 4.14: DCL of $Pl_n \odot K_1$

$G = Pl_{m,m} \odot K_1$ with $V(G) = V(Pl_{m,n}) \cup \{v'_i, u'_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = E(Pl_{m,n}) \cup \{v_i v'_i, u_j u'_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. The cardinality of node and edge set of G is respectively $4m$ and $6m-4$. Consider $\psi : V(G) \to \{1, 2, ..., 4m\}$ given by fixing $\psi(u_1) = 1$, $\psi(u_2) = 8$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $3 \leq i \leq n-1$, $\psi(u_n) = 6$, $\psi(v_1) = 2$, $\psi(v_2) = 4$, $\psi(v'_1) = 4m-1$, $\psi(u'_1) = 4m$, $\psi(v'_2) = \psi(v_2)-1$, $\psi(u'_i) = \psi(u_i)-1$; $2 \leq i \leq n$. Assigning available odd labels to v_i ; $3 \leq i \leq m$ and $\psi(v'_i) = \psi(v_i)+1$; $3 \leq i \leq m$ implies that $e_\psi(0) = e_\psi(1) = 3m-2$, which shows that G is a DCG. □

Considering the fact that characterization of DCGs is challenging in general, the following conjecture is formulated.

Conjecture 4.5.1. *For a given graph G, establishing a DCL of G is NP-hard.*

Remark 4.2. The conjecture can be true as there are no algorithm available in the literature and devising a particular pattern of DCG is also the hardest.

4.6 Conclusion

This chapter has dealt with certain interesting general results on DCL of graphs besides, formulating an impressive conjecture on DCL. Some new results for graph operation named, corona, for various notable graphs have been derived. Further, a class of tree named, lilly graph, has been investigated for various graph operations. Also, DCL of certain classes of planar graphs, in addition to exploring these graphs for graph operations of high interest are also established.

Chapter 5

DCL in the Context of Extension

5.1 Introduction

In this chapter, certain results concerning the DCL of graphs in the context of "duplication of a node by a node", "duplication of edge by a node", "duplication of a node by an edge" are recalled. The DCL of some well-known graphs in the frame of a graph operation known as extension of a vertex in addition to a few results on duplication operation are also been investigated.

5.2 DCL in the Context of Extension of a node

No significant work has been done concerning DCL of graphs in the context of extension operation. Motivated by this fact and inspired by [48], DCL of standard graphs in the context of extension of a node and, a few results on duplication operation are derived. First, recall some established results on duplication.

Theorem 5.2.1. *[56] "Duplicating an arbitrary node by a node in C_n, $n \geq 3$ admits a DCL."*

Theorem 5.2.2. *[41] "Duplicating an edge by an edge in C_n admits a DCL."*

Theorem 5.2.3. *[43] "Duplication of an arbitrary node by an edge in C_n and duplication of an arbitrary edge by a node in C_n permits a DCL."*

Theorem 5.2.4. *The extension of both pendant nodes of P_n, $n \geq 3$ admits a DCL.*

Proof. Suppose the node set of P_n is given by $\{v_i : 1 \leq i \leq n\}$ and G be acquired by taking extension of both pendant nodes of P_n with newly added nodes, u_1 and

u_2. Clearly, $V(G) = V(P_n) \cup \{u_1, u_2\}$ and $E(G) = E(P_n) \cup \{v_1u_1, v_2u_1, v_nu_2, v_{n-1}u_2\}$. The cardinality of node and edge set of G is respectively $n+2$ and $n+3$. Consider $\psi : V(G) \to \{1, 2, ..., n+2\}$ for the given cases.

Case (i) When 'n' is even.

Fix $\psi(u_1) = n+1$, $\psi(u_2) = n-1$ and label the remaining nodes begining with v_1 in the following fashion

$$\begin{array}{cccccc} 1, & 1.2^1, & 1.2^2, & ..., & 1.2^{k_1}, \\ 3, & 3.2^1, & 3.2^2, & ..., & 3.2^{k_2}, \\ 5, & 5.2^1, & 5.2^2, & ..., & 5.2^{k_3}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

where $(2t-1)2^{k_l} \leq n+2$ and $l \geq 1$, $k_l > 0$. Evidently, $|e_\psi(0) - e_\psi(1)| \leq 1$.

Case (ii) When 'n' is odd.

Fix $\psi(u_1) = n+2$, $\psi(u_2) = n$ and label the remaining nodes begining with v_1 in the following fashion

$$\begin{array}{cccccc} 1, & 1.2^1, & 1.2^2, & ..., & 1.2^{k_1}, \\ 3, & 3.2^1, & 3.2^2, & ..., & 3.2^{k_2}, \\ 5, & 5.2^1, & 5.2^2, & ..., & 5.2^{k_3}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

where $(2t-1)2^{k_l} \leq n+1$ and $l \geq 1$, $k_l > 0$. Observe that $|e_\psi(0) - e_\psi(1)| \leq 1$.

Thus, G admits a DCL. □

Theorem 5.2.5. *Extension of an arbitrary node of C_n, $n \geq 4$ permits a DCL.*

Proof. Suppose node set of C_n is $\{v_i : 1 \leq i \leq n\}$ and G be formed by operating extension of a random node of C_n. Without loss of generality, suppose extension of v_1 is taken and let w be the newly added node. The cardinality of node and edge set of G is respectively $n+1$ and $n+3$. Consider $\psi : V(G) \to \{1, 2, ..., n+1\}$. Fix $\psi(v_n) = 2$, $\psi(v_{n-1}) = 1$ and $\psi(w) = p$; p is the largest prime $\leq n+1$. Label the remaining nodes begining with v_1 in the following pattern

$$\begin{array}{cccccc} 2.2, & 2.2^2, & 2.2^3, & ..., & 2.2^{k_1}, \\ 3, & 3.2^1, & 3.2^2, & ..., & 3.2^{k_2}, \\ 5, & 5.2^1, & 5.2^2, & ..., & 5.2^{k_3}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

such that $(2t-1)2^{k_t} \leq n+1$ and $t \geq 1$, $k_t > 0$. Allot unused labels out of the available labels to unlabeled nodes. Obviously, $|e_\psi(0) - e_\psi(1)| \leq 1$ which proves that G is a DCG (see Figure 5.1). □

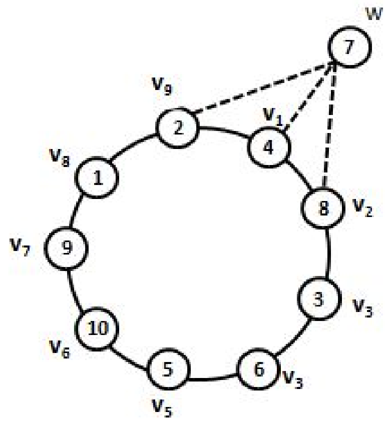

FIGURE 5.1: DCL of a graph formed by taking an extension of v_1 in C_9

Remark 5.1. For C_3, the above theorem does not hold good.

Theorem 5.2.6. *Extension of apex node in W_n, $n \geq 3$ permits a DCL.*

Proof. Let $V(W_n) = \{v_0, v_i : 1 \leq i \leq n\}$ where v_0 is apex node and G be acquired by taking extension of v_0. Let w be the newly added node. Clearly, $|V(G)| = n+2$ and $|E(G)| = 3n+1$. Define $\psi : V(G) \rightarrow \{1, 2, ..., n+2\}$ by fixing $\psi(w) = 1$, $\psi(v_0) = 2$, $\psi(v_1) = 3$ and $\psi(v_i) = \psi(v_{i-1}) + 1$; $2 \leq i \leq n$ with the condition that $\psi(v_1) \not| \psi(v_n)$ (If this occurs, swap the labels of v_n with v_{n-1}). Observe that $|e_\psi(0) - e_\psi(1)| \leq 1$ proving that G is a DCG (see Figure 5.2). □

Remark 5.2. Graph acquired by duplicating apex node by a node in W_n permits a DCL for $n \geq 3$ and the proof is same as that of Theorem 5.2.6.

Theorem 5.2.7. *Extension of an arbitrary node at rim of W_n, $n \geq 3$ permits a DCL.*

Proof. Let $V(W_n) = \{v_0, v_i : 1 \leq i \leq n\}$ where v_0 is apex node and v_i are rim nodes. Let G be acquired by taking extension of a random rim node of W_n, say, v_1 and let w be the added node. Clearly, $|V(G)| = n+2$ & $|E(G)| = 2n+4$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., n+2\}$ by taking $\psi(v_0) = 1$, $\psi(v_1) = n+1$, $\psi(v_2) = 2$ and $\psi(v_i) = \psi(v_{i-1}) + 1$; $3 \leq i \leq n$ and $\psi(w) = n+2$. Clearly, $|e_\psi(0) - e_\psi(1)| \leq 1$ which shows that G is a DCG (see Figure 5.3). □

Remark 5.3. Duplication of an arbitrary rim node by a node in W_n permits a DCL for $n \geq 3$ and its proof is same as that of Theorem 5.2.7.

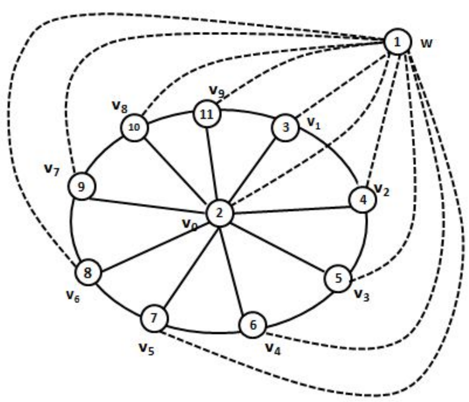

FIGURE 5.2: DCL of a graph acquired by performing the extension of w_0 in W_9

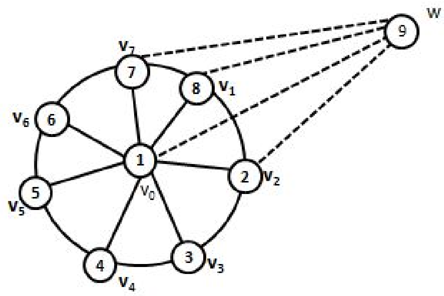

FIGURE 5.3: DCL of a graph acquired by taking an extension of v_1 in W_7

Theorem 5.2.8. *Extension of apex node in H_n permits a DCL for $n > 3$.*

Proof. Let $V(H_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ where v_0, v_i and u_i; $1 \leq i \leq n$ represent respectively the apex, rim and pendant nodes. Let G be formed by taking an extension of v_0 in H_n and w be added node. Clearly, $|V(G)| = 2n+2$ and $|E(G)| = 4n+1$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 2n+2\}$. Fix $\psi(w) = 1$, $\psi(v_0) = 2$, $\psi(v_1) = 4$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq n$, $\psi(u_i) = \psi(v_i) - 1$; $1 \leq i \leq n$, with the condition that neither $\psi(v_n)|\psi(v_1)$ nor $\psi(v_1)|\psi(v_n)$ (if such a case happens, swap the labels of v_{n-1} and v_n). Observe that $|e_\psi(0) - e_\psi(1)| \leq 1$, which shows that G is a DCG (see Theorem 5.4). □

Remark 5.4. Duplication of apex node by a node in H_n permits a DCL for $n > 3$ and its proof is same as that of Theorem 5.2.8.

Theorem 5.2.9. *Extension of an arbitrary rim node of H_n, $n \geq 3$ permits a DCL.*

Proof. Let $V(H_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ where v_0, v_i and u_i; $1 \leq i \leq n$ represent respectively the apex, rim and pendant nodes of H_n. Let G be formed by taking extension of an arbitrary rim node of H_n. Suppose extension of v_1 is taken and let w be the newly added node. Clearly, $|V(G)| = 2n + 2$ and $|E(G)| = 3n + 5$. Consider

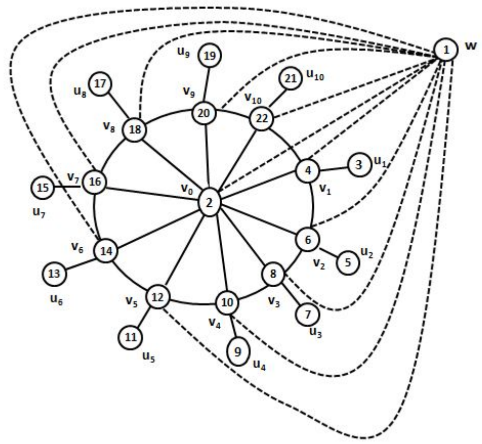

FIGURE 5.4: DCL of a graph acquired by taking extension of v_0 in H_{10}

$\psi : V(G) \to \{1, 2, ..., 2n+2\}$. Fix $\psi(w) = 2n+1$, $\psi(v_0) = 2$, $\psi(v_1) = 1$, $\psi(v_2) = 4$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $3 \leq i \leq n$, $\psi(u_n) = \psi(v_n) + 2$, $\psi(u_i) = \frac{\psi(v_i)}{2}$ such that $\frac{\psi(v_i)}{2}$ is odd. Assign the remaining labels to unlabeled u_i's simultaneously from $\{1, 2, ..., 2n+2\}$. Observe that $|e_\psi(0) - e_\psi(1)| \leq 1$ which shows that G is a DCL (see Figure 5.5). □

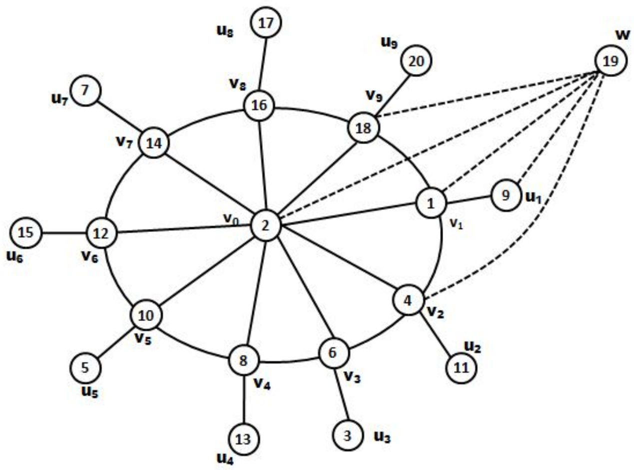

FIGURE 5.5: DCL of a graph formed by taking the extension of v_1 in H_9

Remark 5.5. Duplication of rim node by a node in H_n permits a DCL for $n > 3$ and its proof is same as that of Theorem 5.2.9.

Theorem 5.2.10. *Extension of an arbitrary node of degree* 1 *in* H_n *permits a DCL.*

Proof. Let $V(H_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ where v_0, v_i and u_i; $1 \leq i \leq n$ represent respectively the apex, rim and pendant nodes of H_n. Let G be formed by taking extension of u_1 and w be the added node. Clearly, $|V(G)| = 2n+2$ and $|E(G)| = 3n+2$. Define $\psi : V(G) \to \{1, 2, ..., 2n+2\}$ as given. Let $\psi(w) = 2$, $\psi(v_0) = 1$ & $\psi(v_1) = 3$. Now arise the under mentioned cases.

Case (i) When 'n' is even.

Let $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \le i \le n$ such that $\psi(v_1) \nmid \psi(v_n)$ (If such a case happens, swap the labels of v_n and u_n). Next, let $\psi(u_i) = 2\psi(v_i)$; $1 \le i \le \frac{n}{2}$, $\psi(u_{\frac{n}{2}+1}) = 4$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $\frac{n}{2} + 2 \le i \le n$. Observe that $|e_\psi(0) - e_\psi(1)| \le 1$.

Case (ii) When 'n' is odd.

Let $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \le i \le n$ such that $\psi(v_1) \nmid \psi(v_n)$ (If such a case happens, swap the labels of v_n and u_n). Next, fix $\psi(u_i) = 2\psi(v_i)$; $1 \le i \le \lfloor \frac{n}{2} \rfloor$, $\psi(u_{\lfloor \frac{n}{2} \rfloor+1}) = 4$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $\lfloor \frac{n}{2} \rfloor + 2 \le i \le n$. One can find that $|e_\psi(0) - e_\psi(1)| \le 1$.

Thus, G is a DCG. □

Theorem 5.2.11. *Extension of apex node of Fl_n, $n \ge 3$ permits a DCL.*

Proof. Let $V(Fl_n) = \{v_0, v_i, u_i : 1 \le i \le n\}$ where v_0 is apex node and v_i, u_i are the nodes of degree 4 and 2 respectively. Let G be acquired by taking an extension of v_0 and w be the newly introduced node. Clearly, $|V(G)| = 2n + 2$ and $|E(G)| = 6n + 1$. Consider $\psi : V(G) \to \{1, 2, ..., 2n + 2\}$ given by fixing $\psi(w) = 2$, $\psi(v_0) = 1$, $\psi(v_1) = 3$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \le i \le n$ and $\psi(u_i) = \psi(v_i) + 1$; $1 \le i \le n$ with the restriction that neither $\psi(v_1)|\psi(v_n)$ nor $\psi(v_n)|\psi(v_1)$. It can be seen that $|e_\psi(0) - e_\psi(1)| \le 1$ proving that G is a DCG. □

Remark 5.6. Duplication of apex node by a node in Fl_n, $n \ge 3$ permits a DCL and its proof is same as that of Theorem 5.2.11.

Theorem 5.2.12. *Extension of an arbitrary node of degree 2 in Fl_n permits a DCL.*

Proof. Let $\{v_0, v_i, u_i : 1 \le i \le n\}$ be the node set for Fl_n where v_0 is apex node and v_i, u_i are the nodes of degree 4 and 2 respectively. Let G be acquired by taking an extension of u_1 and w be the newly introduced node. The cardinality of node and edge set of G is respectively $2n+2$ and $4n+3$. Labeling is defined in a similar way as defined in Theorem 5.2.11. □

Remark 5.7. Duplication of an arbitrary node of degree 2 by a node in Fl_n, $n \ge 3$ permits a DCL and its proof is same as Theorem 5.2.11.

Theorem 5.2.13. *Extension of an arbitrary node of degree 4 in Fl_n, $n \ge 3$ permits a DCL.*

Proof. Proof is same as that of Theorem 5.2.11. □

Remark 5.8. Duplication of an arbitrary node of degree 4 by a node in Fl_n, $n \ge 3$ permits a DCL and its proof is same as that of Theorem 5.2.11.

Theorem 5.2.14. *Duplication of each node of degree 2 by a node in Fl_n permits a DCL.*

Proof. Let $V(Fl_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ where v_0 is apex node and v_i, u_i are the nodes of degree 4 and 2 respectively. Let G be acquired by duplicating each node of degree 2 by a node in Fl_n with $V(G) = V(Fl_n) \cup \{u'_i : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 3n+1$ and $|E(G| = 6n$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 3n+1\}$ defined by given cases.
Case (i) If 'n' is even.
Fix $\psi(v_0) = 1$, $\psi(v_1) = 4$, $\psi(v_i) = \psi(v_{i-1})+3$; $2 \leq i \leq n$, $\psi(u_1) = 3$, $\psi(u_i) = \psi(u_{i-1})+3$; $2 \leq i \leq n$, $\psi(u'_1) = 5$, $\psi(u'_i) = \psi(u'_{i-1}) + 3$; $2 \leq i \leq n-1$ and $\psi(u'_n) = 2$. Evidently, $e_\psi(0) = e_\psi(1) = 3n$.
Case (ii) If 'n' is odd.
Fix $\psi(v_0) = 1$, $\psi(v_1) = 4$, $\psi(v_i) = \psi(v_{i-1}) + 3$; $2 \leq i \leq n-1$, $\psi(u_n) = \psi(v_{n-1}) + 3$, $\psi(u_1) = 3$, $\psi(u_i) = \psi(u_{i-1})+3$; $2 \leq i \leq n-1$, $\psi(v_n) = \psi(u_{n-1})+3$, $\psi(u'_1) = 5$, $\psi(u'_i) = \psi(u'_{i-1}) + 3$; $2 \leq i \leq n-1$ and $\psi(u'_n) = 2$. One can check that $e_\psi(0) = e_\psi(1) = 3n$.
Thus in both the cases, G is a DCG (see Figure 5.6). □

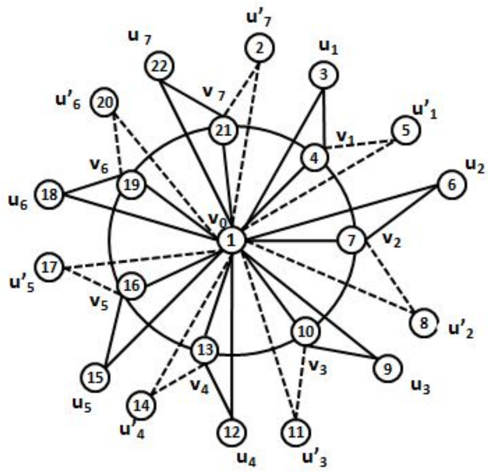

FIGURE 5.6: DCL of a graph formed by duplicating each u_i by a node in Fl_7

Theorem 5.2.15. *Extension of apex node in G_n, $n \geq 3$ permits a DCL.*

Proof. Let $V(G_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ and $E(G_n) = \{v_0v_i : 1 \leq i \leq n\} \cup \{v_iu_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nv_1\}$. Let G be obtained by taking extension of apex (v_0) of G_n by a node, say, w. Clearly, $|V(G)| = 2n+2$ and $|E(G)| = 4n+1$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 2n+2\}$ as per the given algorithm. Fix $\psi(w) = 2$, $\psi(v_0) = 1$, $\psi(v_1) = 4$, $\psi(u_n) = 3$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq n$ such that $\psi(u_n) \not| \psi(v_n)$, (if this happens then swap the labels of v_{n-1} and v_n) and $\psi(u_i) = \psi(v_i) + 1$; $1 \leq i \leq n-1$. Note that $|e_\psi(0) - e_\psi(1)| \leq 1$ which proves that G is a DCG (see Figure 5.7). □

Remark 5.9. Duplication of apex node by a node in G_n permits a DCL and the proof is same as that of Theorem 5.2.15.

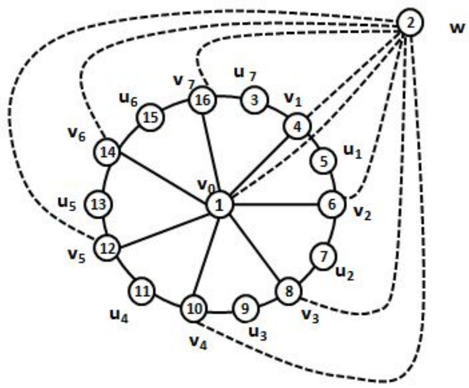

FIGURE 5.7: DCL of a graph formed by taking the extension of v_0 in G_7

Theorem 5.2.16. *Extension of an arbitrary node of degree* 2 *in* G_n, $n \geq 3$ *permits a DCL.*

Proof. Let $V(G_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ and $E(G_n) = \{v_0v_i : 1 \leq i \leq n\} \cup \{v_iu_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nv_1\}$. Let G be formed by taking extension of an arbitrary node of degree 2 say u_1 in G_n by adding a new node, say, w. Clearly, $|V(G)| = 2n+2$ and $|E(G)| = 3n+3$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+2\}$ as per the given possibilities.

Case (i) If 'n' is even.

Fix $\psi(v_0) = 1$, $\psi(w) = 2$, $\psi(v_1) = 3$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq n$ and $\psi(u_i) = 2\psi(v_i)$; $1 \leq i \leq \frac{n}{2}$. Now assign the remaning unused labels simultaneously to unlabeled nodes.

Case (ii) If 'n' is odd.

Fix $\psi(v_0) = 1$, $\psi(w) = 2$, $\psi(v_1) = 4$, $\psi(v_2) = 3$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $3 \leq i \leq n$, $\psi(u_n) = 2n+1$, $\psi(u_1) = 6$, $\psi(u_2) = 8$ and $\psi(u_i) = 2\psi(v_i)$; $3 \leq i \leq \lceil \frac{n}{2} \rceil$. Now assign the remaning unused labels simultaneously to unlabeled nodes.

In both the cases, $|e_\psi(0) - e_\psi(1)| \leq 1$ which proves that G is a DCG. □

Theorem 5.2.17. *Duplication of an arbitrary node of degree* 2 *by a node in* G_n, $n \geq 3$ *permits a DCL.*

Proof. Let $V(G_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ and $E(G_n) = \{v_0v_i : 1 \leq i \leq n\} \cup \{v_iu_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nv_1\}$. Let G be formed by duplicating an arbitrary node of degree 2 say u_1 in G_n by a node, say, w. Clearly, $|V(G)| = 2n+2$ and $|E(G)| = 3n+2$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+2\}$ as given here.

Case (i) If 'n' is even.

Fix $\psi(v_0) = 1$, $\psi(w) = 2$, $\psi(v_1) = 6$, $\psi(u_1) = 3$, $\psi(v_2) = 5$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $3 \leq i \leq n$ and $\psi(u_i) = 2\psi(v_i)$; $2 \leq i \leq \frac{n}{2}$. Assign the remaning labels simultaneously to unlabeled nodes with the condition that $\psi(u_n) \nmid \psi(v_1)$ (If such a case happens then swap the labels of v_n and u_n).

Case (ii) If 'n' is odd.

Follow the *Case* (ii) of Theorem 5.2.16. In both the cases, $|e_\psi(0) - e_\psi(1)| \leq 1$ showing that G is a DCG. □

Theorem 5.2.18. *Extension of an arbitrary node of degree 3 of G_n, $n \geq 3$ permits a DCL.*

Proof. Let $V(G_n) = \{v_0, v_i, u_i : 1 \leq i \leq n\}$ and $E(G_n) = \{v_0v_i : 1 \leq i \leq n\} \cup \{v_iu_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nv_1\}$. Let G be formed by taking extension of an arbitrary node of degree 3 of G_n, say, v_1, by adding a new node, say, w. Clearly, $|V(G)| = 2n+2$ and $|E(G)| = 3n+4$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+2\}$ as given below.

Case (i) If 'n 'is even.

Fix $\psi(v_0) = 1$, $\psi(w) = 2n+1$, $\psi(v_1) = 2$, $\psi(v_2) = 3$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $3 \leq i \leq n-1$, $\psi(v_n) = 2n$, $\psi(u_n) = 2n-1$ & $\psi(u_i) = 2\psi(v_i)$; $1 \leq i \leq \frac{n}{2}+1$. Alloting the remaining labels simultaneously to unlabeled nodes gives $|e_\psi(0) - e_\psi(1)| \leq 1$ which proves that G is a DCL.

Case (ii) If 'n' is odd.

Fix $\psi(v_0) = 1$, $\psi(w) = 2n+1$, $\psi(v_1) = 2$, $\psi(v_2) = 3$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $3 \leq i \leq n-1$, $\psi(v_n) = 2n+2$, $\psi(u_n) = 2n-1$ & $\psi(u_i) = 2\psi(v_i)$; $1 \leq i \leq \lceil \frac{n}{2} \rceil$. Assigning the remaining labels simultaneously to unlabeled nodes gives $|e_\psi(0) - e_\psi(1)| \leq 1$. Hence, G is a DCG. □

Remark 5.10. Duplication of an arbitrary node of degree 3 by a node in G_n, $n \geq 3$ permits a DCL and its proof is same as Theorem 5.2.18.

Theorem 5.2.19. *[82] $S'(K_{1,n})$ permits a DCL.*

Theorem 5.2.20. *Extension of each node in $K_{1,n}$ admits a DCL.*

Proof. Let $\{v_0, v_i : 1 \leq i \leq n\}$ be the node set of $K_{1,n}$ where v_0 is apex node and G be obtained by taking the extension of each node of $K_{1,n}$ having $V(G) = V(K_{1,n}) \cup \{u_0, u_i : 1 \leq i \leq n\}$, and $E(G) = E(K_{1,n}) \cup \{v_iu_i, v_0u_i, u_0v_i : 1 \leq i \leq n\} \cup \{v_0u_0\}$. Note that $|V(G)| = 2n+2$, whereas $|E(G)| = 4n+1$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+2\}$ defined by fixing $\psi(v_0) = 2$, $\psi(u_0) = 1$, $\psi(v_1) = 4$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq n$, $\psi(u_i) = \psi(v_i) - 1$; $1 \leq i \leq n$. Observe that $\psi(v_0)|\psi(v_i)$; $\forall\, i$, $\psi(u_0)|\psi(v_0)$ and $\psi(u_0)|\psi(v_i)$; $\forall\, i$, it implies that $e_\psi(1) = 2n+1$ and $e_\psi(0) = 2n$ which establishes that G is a DCG. □

Theorem 5.2.21. *Duplicating each node by a node in $K_{1,n}$ for $n \geq 3$ results in a DCG.*

Proof. Duplication of each node by node in $K_{1,n}$ results in $S'(K_{1,n})$ which is a DCG by Theorem 5.2.19. □

Theorem 5.2.22. *Duplicating each edge by a node in $K_{1,n}$, $n \geq 3$ admits a DCL.*

Proof. Let $V(K_{1,n}) = \{v_0, v_i : 1 \leq i \leq n\}$, and $E(K_{1,n}) = \{v_0v_i : 1 \leq i \leq n\}$ and G be formed by duplicating each edge with a node in $K_{1,n}$ having node and edge set respectively $V(K_{1,n}) \cup \{u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) \cup \{v_iu_i, v_0u_i : 1 \leq i \leq n\}$. See that the cardinality of node, and edge set of G is respectively $2n + 1$ and $3n$. Consider $\psi : V(G) \to \{1, 2, ..., 2n + 1\}$ given by fixing $\psi(v_0) = 2$, $\psi(v_1) = 1$, $\psi(v_2) = 4$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $3 \leq i \leq n$, $\psi(u_i) = \frac{\psi(v_i)}{2}$ such that $\psi(u_i)$ is odd. Assign the remaining odd labels simultaneously to the unlabeled nodes. Observe that if n is odd then $e_\psi(1) = e_\psi(0) + 1$ while $e_\psi(0) = e_\psi(1)$, if n is even. For both the cases, G is a DCG (see Figure 5.8). □

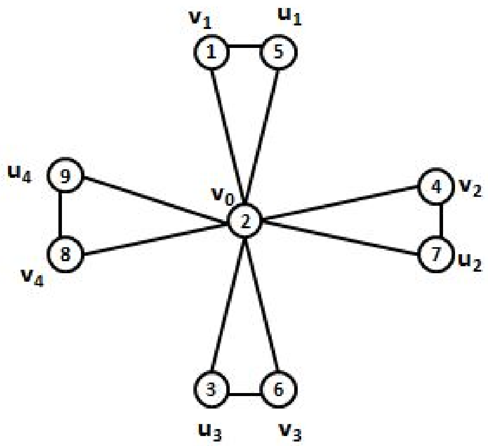

FIGURE 5.8: DCL of a graph formed by duplicating each edge by a node in $K_{1,4}$

Remark 5.11. Duplicating each edge by a node in $K_{1,n}$ results in a friendship graph which is eventually a DCG by Theorem 5.2.22.

Theorem 5.2.23. *[82] $S'(B_{n,n})$ permits a DCL.*

Theorem 5.2.24. *Extension of each node in $B_{n,n}$ admits a DCL.*

Proof. Let $V(B_{n,n}) = \{v_0, u_0, v_i, u_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{v_0v_i, u_0u_i, v_0u_0 : 1 \leq i \leq n\}$. Let G be formed by taking the extension of each node in $B_{n,n}$ with $V(G) = V(B_{n,n}) \cup \{v'_0, u'_0, v'_i, u'_i : 1 \leq i \leq n\}$ and $E(G) = E(B_{n,n}) \cup \{v_0v'_i, u_0u'_i, v_iv'_i, u_iu'_i, v'_0v_i, u'_0u_i : 1 \leq i \leq n\} \cup \{v_0v'_0, u_0u'_0, v'_0u_0, u'_0v_0\}$. See that $|V(G)| = 4n+4$ and $|E(G)| = 8n+5$. Consider $\psi : V(G) \to \{1, 2, ..., 4n + 4\}$ defined by fixing $\psi(v_0) = 1$, $\psi(v'_0) = 2$, $\psi(u'_0) = 4$, $\psi(u_0) = t$, where t is the largest odd number $\leq 4n + 4$. Next, $\psi(v_1) = 3$, $\psi(v_i) =$

$\psi(v_{i-1})+2$; $2 \leq i \leq n$, $\psi(v_i') = 2\psi(v_i)$; $1 \leq i \leq n$, $\psi(u_1) = 8$, $\psi(u_i) = \psi(u_{i-1})+4$; $2 \leq i \leq n$, $\psi(u_1') = \psi(v_n)+2$ and $\psi(u_i') = \psi(u_{i-1}')+2$; $2 \leq i \leq n$. Observe that $\psi(v_0)|\psi(v_i)$, $\psi(v_0)|\psi(v_i')$, $\psi(v_i)|\psi(v_i')$, $\psi(v_0)|\psi(v_0')$, $\psi(v_0)|\psi(u_0)$, $\psi(v_0)|\psi(u_0')$ and $\psi(u_0')|\psi(u_i)$ for all i; $1 \leq i \leq n$. Therefore, $e_\psi(1) = 4n+3$ and $e_\psi(0) = 4n+2$ resulting which G is a DCG (see Figure 5.9). □

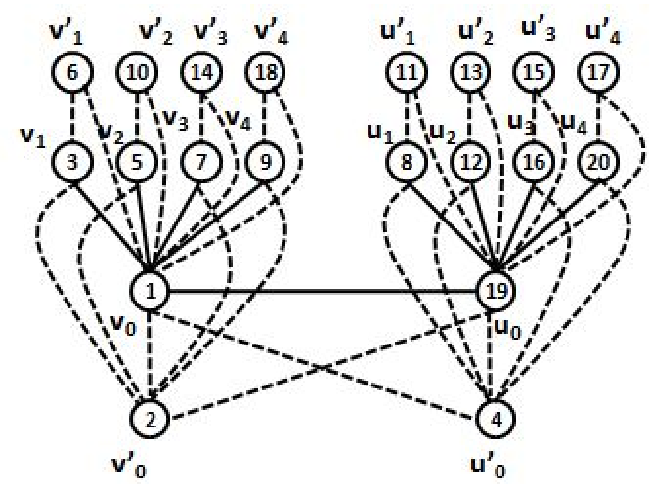

FIGURE 5.9: DCL of a graph formed by taking extension of each node in $B_{4,4}$

Remark 5.12. Duplication of each node by a node in $B_{n,n}$ results in $S'(B_{n,n})$ which is a DCG by Theorem 5.2.23.

Theorem 5.2.25. *Duplication of each edge by a node in $B_{n,n}$ admits a DCL.*

Proof. Let $V(B_{n,n}) = \{v_0, u_0, v_i, u_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{v_0v_i, u_0u_i, v_0u_0 : 1 \leq i \leq n\}$. Let G be formed by "duplicating each edge by a node" in $B_{n,n}$ having $V(G) = V(B_{n,n}) \cup \{w, v_i', u_i' : 1 \leq i \leq n\}$ and $E(G) = E(B_{n,n}) \cup \{v_0v_i', u_0u_i', u_i'u_i, v_i'v_i : 1 \leq i \leq n\} \cup \{v_0w, u_0w\}$. Clearly, $|V(G)| = 4n+3$. Define $\psi : V(G) \to \{1, 2, ..., 4n+3\}$ by fixing $\psi(v_0) = 1$, $\psi(u_0) = 2$, $\psi(v_1) = 3$, $\psi(v_i) = \psi(v_{i-1})+2$; $2 \leq i \leq n$, $\psi(v_1') = 4$, $\psi(v_i') = \psi(v_{i-1}')+2$; $2 \leq i \leq n$, $\psi(u_1) = \psi(v_n)+2$, $\psi(u_i) = \psi(u_{i-1})+2$; $2 \leq i \leq n$, $\psi(u_1') = \psi(v_n')+2$, $\psi(u_i') = \psi(v_{i-1}')+2$; $2 \leq i \leq n$ and $\psi(w) = 4n+3$. Observe, $\psi(v_0)|\psi(v_i)$, $\psi(v_0)|\psi(v_i')$, $\psi(v_0)|\psi(w)$, $\psi(v_0)|\psi(u_0)$, $\psi(u_0)|\psi(u_i')$; $1 \leq i \leq n$. Here, $e_\psi(1) = 3n+2$ and $e_\psi(0) = 3n+1$, proving that G is a DCG. □

Open Problems

The following open problems are framed due to the study done in this chapter.

1. If G is a DCG, then does graph obtained by performing extension of each node of G also permit a DCL?

2. Is there a characterization of graphs that do not admit a DCL but whose extension admits a DCL?

5.3 Conclusion

This chapter has dealt with certain results on graph operations namely, duplication of node by a node, duplication of edge by a node and extension of node for various well-known graphs viz; P_n, C_n, W_n, H_n, Fl_n, G_n, $K_{1,n}$ and $B_{n,n}$ for DCL.

Chapter 6

New Variants of DCL

6.1 Introduction

Murugesan et al., in [42] introduced a variant of DCL named, square divisor cordial labeling. Kanani et al., [37] introduced the notion of cube divisor cordial labeling and contributed some standard results. Later, Lourdusamy [40] coined a new variant of DCL named sum divisor cordial labeling. Motivated by this, two new variants of DCL namely, double divisor cordial labeling (DDCL) and average even divisor cordial labeling (AEDCL) are introduced and studied in this chapter. Throughout this chapter, DDCL, AEDCL, DDCG and AEDCG are used to denote respectively the double divisor cordial labeling, average even divisor cordial labeling, double divisor cordial graph and average even divisor cordial graph.

6.2 Double Divisor Cordial Labeling

In this section, certain general results concerning DDCL of graphs are established besides, exploring the same for some well known graphs. First, definition of DDCL is introduced.

Definition 6.2.1. "A double divisor cordial labeling (DDCL) of $G(V, E)$ is a bijection ψ from V to $\{1, 2, 3, ..., |V(G)|\}$ defined by the induced function $\psi^\star : E \to \{0, 1\}$ such that for each edge yz, $\psi^\star(yz)$ is given label 1 if $2\psi(y)|\psi(z)$ or $2\psi(z)|\psi(y)$ and label 0 otherwise, then the modulus of the difference of edges having labels 0 and labels 1 do not exceed 1 i.e; $|e_\psi(0) - e_\psi(1)| \leq 1$. If a graph permits a DDCL, then it is known as double divisor cordial graph (DDCG)."

6.2.1 DDCL of Some Well Known Graphs

In this section, DDCL of some standard graphs has been studied.

Theorem 6.2.1. *For a given $t \in \mathbb{N}$, $\exists$ a DDCG, G on t nodes.*

Proof. There can be two cases for t.

Case (i) When t is even.

Construct a path having $\frac{t}{2}+1$ nodes say $v_1, v_2, ..., v_{\frac{t}{2}+1}$. Attach $\frac{t}{2}-1$ nodes namely, $v_{\frac{t}{2}+2}$, $v_{\frac{t}{2}+3}, ..., v_t$ to v_1. Define a labeling ψ by fixing $\psi(v_i) = 2i-1$ for $1 \leq i \leq \frac{t}{2}$, $\psi(v_{\frac{t}{2}+1}) = 2$, $\psi(v_{\frac{t}{2}+i}) = \psi(v_{\frac{t}{2}+(i-1)}) + 2$ for $2 \leq i \leq \frac{t}{2}$. Observe that edges v_1v_i; $\frac{t}{2}+2 \leq i \leq t$ are labeled 1 and the remaining edges are labeled 0. Clearly, $e_\psi(1) = \frac{t}{2} - 1$ and $e_\psi(0) = \frac{t}{2}$.

Case (ii) When t is odd.

Construct a path having $\frac{t+1}{2}$ nodes say $v_1, v_2, ..., v_{\frac{t+1}{2}}$ and attach $\frac{t-1}{2}$ nodes namely, $v_{\frac{t+3}{2}}, v_{\frac{t+5}{2}}, ..., v_t$ to v_1. Define a labeling ψ by taking $\psi(v_1) = 1$, $\psi(v_i) = \psi(v_{i-1}) + 2$ for $2 \leq i \leq \frac{t+1}{2}$, and label $\{v_{\frac{t+3}{2}}, v_{\frac{t+5}{2}}, ..., v_t\}$ with unutilized even labels out of $\{1, 2, ..., t\}$. Observe that edges v_1v_i; $\frac{t+3}{2} \leq i \leq t$ are labeled 1 and the remaining edges are labeled 0. Clearly, $e_\psi(0) = e_\psi(1) = \frac{t-1}{2}$.

Hence, G is a DDCG (see Figure 6.1). □

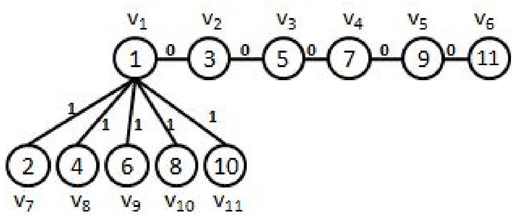

FIGURE 6.1: Construction of a DDCG on 11 nodes

Theorem 6.2.2. *If $G(p,q)$ is a DDCG with q even, then $G \pm e$ admits a DDCL.*

Proof. Since G is DDCG with labeling ψ and q is even therefore, $e_\psi(0) = e_\psi(1)$. Thus, addition or deletion of one edge yields either $e_\psi(0) = e_\psi(1) + 1$ or $e_\psi(1) = e_\psi(0) + 1$ which justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$. □

Theorem 6.2.3. *If $G(p,q)$ is a DDCG with q odd, then $G - e$ admits a DDCL.*

Proof. Let G be a DDCG with q odd. Then either $e_\psi(0) = e_\psi(1)+1$ or $e_\psi(1) = e_\psi(0)+1$. Suppose $e_\psi(0) = e_\psi(1)+1$, then removing the edge having label 0 yields $|e_\psi(0)-e_\psi(1)| \leq 1$. □

Remark 6.1. Theorem 6.2.3 holds good for $G + e$.

Definition 6.2.2. [87] “A full binary tree is a binary tree in which each internal vertex has exactly two children.”

In this chapter, by full binary tree one means a binary tree having 2^i nodes at i^{th} level, where $i = 0, 1, 2....$

Theorem 6.2.4. *Every full binary tree admits a DDCL.*

Proof. Let T_n denotes the full binary tree of n levels and t_0 is fixed as root node. It is noteworthy that $V(T_n)$ is always odd and therefore yield even count of edges. The root node is also called as zero level. Clearly, i^{th} level has 2^i nodes. Clearly, $|V(T_n)| = 2^{n+1}-1$ and $|E(T)| = 2^{n+1}-2$. Consider $\psi : V(T_n) \rightarrow \{1, 2, ..., 2^{n+1}-1\}$ and define according to the given pattern. Fix $\psi(t_0) = 1$. Next, assign the labels 2^i, 2^i+1, 2^i+2,...,$2^{i+1}-1$ to the i^{th} level nodes where $1 \leq i \leq n$. Observe that $2(2^i+m)|2^{i+1}+2m$ for $0 \leq m \leq 2^i-1$ and $2(2^i+m)$ does not divide $2^{i+1}+2m+1$ for $0 \leq m \leq 2^i-1$ resulting which $|e_\psi(0) - e_\psi(1)| = 0$ which proves that T_n is a DDCG (see Figure 6.2). □

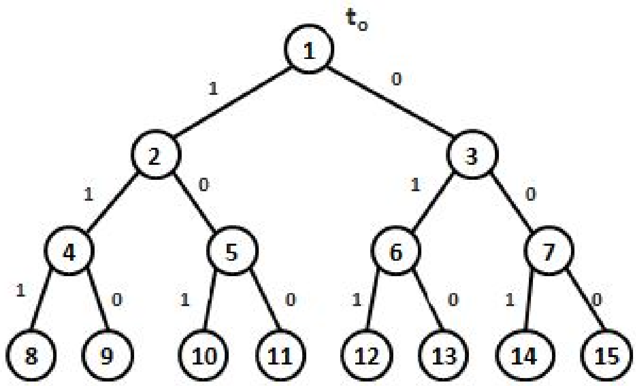

Figure 6.2: DDCL of T_3

Theorem 6.2.5. *P_n, $n \geq 3$ admits a DDCL.*

Proof. Let $V(P_n) = \{p_i : 1 \leq i \leq n\}$. Consider $\psi : V(P_n) \rightarrow \{1, 2, ..., n\}$ as given. Assign labels begining with p_1, according to the given pattern.

$$
\begin{array}{cccccc}
1, & 1.2^1, & 1.2^2, & ..., & 1.2^{k_1}, \\
3, & 3.2^1, & 3.2^2, & ..., & 3.2^{k_2}, \\
5, & 5.2^1, & 5.2^2, & ..., & 5.2^{k_3}, \\
..., & ..., & ..., & ..., & ..., & ...,
\end{array}
$$

where $(2d-1)2^{k_d} \leq n$, $d \geq 1$, $k_d \geq 0$. Note that $(2d-1)2^r|(2d-1)2^s$; $(r < s)$ and $(2d-1)2^{k_d}$ does not divide $(2d+1)$. Now, alloting unutilized labels out of $\{1, 2, ...n\}$ simultaneously to the unlabeled nodes yields $|e_\psi(0) - e_\psi(1)| \leq 1$, hence P_n is a DDCG. □

Theorem 6.2.6. *C_n, $n \geq 3$ admits a DDCL.*

Proof. Let $V(C_n) = \{c_1, c_2, c_3, ..., c_n\}$. Consider $\psi : V(C_n) \to \{1, 2, ..., n\}$ explained as follows. Allocate labels, begining with c_1, according to the given pattern.

$$\begin{array}{cccccc} 1, & 1.2^1, & 1.2^2, & ..., & 1.2^{k_1}, \\ 3, & 3.2^1, & 3.2^2, & ..., & 3.2^{k_2}, \\ 5, & 5.2^1, & 5.2^2, & ..., & 5.2^{k_3}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

where $(2d-1)2^{k_d} \leq n$, $d \neq 2$ and $d \geq 1$, $k_d \geq 0$. Alloting unutilized labels simultaneously to unlabeled nodes yields $|e_\psi(0) - e_\psi(1)| \leq 1$. Hence the result. □

Theorem 6.2.7. *W_n, $n \geq 3$ admits a DDCL for odd values n.*

Proof. Let $V(W_n) = \{w_0, w_1, w_2, ..., w_n\}$ where w_0 denotes the apex node. Consider $\psi : V(W_n) \to \{1, 2, ..., n+1\}$. Fix $\psi(w_0) = 1$. Assign labels begining with w_1 according to the given pattern.

$$\begin{array}{cccccc} 2, & 2.2^1, & 2.2^2, & ..., & 2.2^{k_1}, \\ 3, & 3.2^1, & 3.2^2, & ..., & 3.2^{k_2}, \\ 5, & 5.2^1, & 5.2^2, & ..., & 5.2^{k_3}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

where $(2d-1)2^{k_d} \leq n+1$ and $d \geq 1$, $k_d > 0$. Allocate the unutilized labels simultaneously to unlabeled nodes. Observe that $|e_\psi(0) - e_\psi(1)| = 0$, which gives W_n a DDCG (see Figure 6.3). □

Remark 6.2. Note that W_n does not admit DDCL for $n = 4, 6, 8, 10$.

Theorem 6.2.8. *W_n, $n \geq 12$ admits a DDCL for even values of n.*

Proof. Let $V(W_n) = \{w_0, w_1, w_2, ..., w_n\}$ where w_0 denotes the apex node. Consider $\psi : V(W_n) \to \{1, 2, ..., n+1\}$. Fix $\psi(w_0) = 1$, $\psi(w_n) = 12$, $\psi(w_{n-1}) = 3$, and assign the labels 3.2^1, 3.2^3, $3.2^4, ..., 3.2^{k_2}$ to $w_{n-2}, w_{n-3}, ...$. Now begining with w_1, assign the labels according to the given pattern.

$$\begin{array}{cccccc} 2, & 2.2^1, & 2.2^2, & ..., & 2.2^{k_1}, \\ 5, & 5.2^1, & 5.2^2, & ..., & 5.2^{k_3}, \\ 7, & 7.2^1, & 7.2^2, & ..., & 7.2^{k_4}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

where $(2d-1)2^{k_d} \leq n+1$ and $d \geq 1$, $k_d > 0$. Allocating the unutilized labels simultaneously to unlabeled nodes gives W_n a DDCG (see Figure 6.3). □

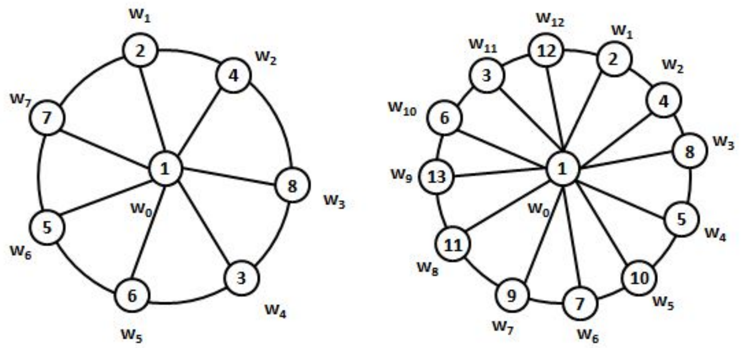

Figure 6.3: DDCL of W_7 and W_{12}

Theorem 6.2.9. *$H_n, n \geq 3$ admits a DDCL.*

Proof. Let $V(H_n) = \{x_0\} \cup \{x_1, x_2, x_3, ..., x_n\} \cup \{y_1, y_2, ..., y_n\}$;x_0, x_i and y_i represent respectively the apex, rim and pendant nodes. Clearly, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. Consider $\psi : V(H_n) \rightarrow \{1, 2, ..., 2n+1\}$ defined by letting $\psi(x_0) = 1$ and assign labels begining with x_1 according to the given pattern.

$$\begin{array}{cccccc} 2, & 2.2^1, & 2.2^2, & ..., & 2.2^{k_1}, \\ 6, & 6.2^1, & 6.2^2, & ..., & 6.2^{k_2}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

where $(4d-2)2^{k_d} \leq 2n+1$ and $d \geq 1$. Note that $2(4d-2)2^r$ divides $(4d-2)2^s$; $(r < s)$. Allocating the unutilized labels simultaneously to the remaining nodes, begining with y_1 yields $|e_\psi(0) - e_\psi(1)| \leq 1$, which proves that H_n is a DDCG. □

Theorem 6.2.10. *f_n, $n \geq 3$ admits a DDCL.*

Proof. Let $V(f_n) = \{x_0, x_1, x_2, ..., x_n\}$ where x_0 is an apex node. Clearly, $|V(f_n)| = n+1$ and $|E(f_n)| = 2n-1$. Consider $\psi : V(f_n) \rightarrow \{1, 2, ..., n+1\}$. Assign labels to

$x_0, x_1, x_2, ...,$ simultaneously, according to the given pattern.

$$\begin{array}{cccccc} 1, & 1.2^1, & 1.2^2, & ..., & 1.2^{k_1}, \\ 3, & 3.2^1, & 3.2^2, & ..., & 3.2^{k_2}, \\ 5, & 5.2^1, & 5.2^2, & ..., & 5.2^{k_3}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

where $(2d-1)2^{k_d} \leq n+1$ and $d \geq 1$. Assign unutilized labels simultaneously to the unlabeled nodes. Clearly, $|e_\psi(0) - e_\psi(1)| \leq 1$ which justifies that f_n admits a DDCL. □

Definition 6.2.3. [66] "Jelly fish graph denoted by $J(m, n)$ is obtained from a 4-cycle x_1, x_2, x_3, x_4 by joining x_1 and x_3 by an edge and adding m pendant edges to x_2 and n pendant edges to x_4."

Theorem 6.2.11. *$J(m, n)$ admits a DDCL.*

Proof. Let $V(J(m, n)) = \{x_1, x_2, x_3, x_4, y_i, z_j : 1 \leq i \leq m, i \leq j \leq n\}$ and $E(J(m, n)) = \{x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_4x_3\} \cup \{x_2y_i : 1 \leq i \leq m\} \cup \{x_4z_j : 1 \leq j \leq n\}$. Clearly, $|V(J(m, n))| = n + m + 4$ and $|E(J(m, n))| = n + m + 5$. Consider $\psi : V(J(m, n)) \rightarrow \{1, 2, ..., n + m + 4\}$ for the given cases.

Case (i) When $n = m$.

Label $\psi(x_1) = 2$, $\psi(x_2) = 1$, $\psi(x_3) = 4$, $\psi(x_4) = 3$. Now assign all the unused even labels to y_i; $1 \leq i \leq m$ and unused odd labels to z_j where $1 \leq j \leq n$.

Case (ii) When $n > m$.

Fix $\psi(x_1) = 2$, $\psi(x_2) = 3$, $\psi(x_3) = 4$, $\psi(x_4) = 1$. Now assign all unused even labels to z_j's where $1 \leq i \leq n$ and once even labels are consumed, assign unused labels (odd) to remaining z_j's if any, and to all y_i's where $1 \leq 1 \leq m$.

Case (iii) When $m > n$.

Fix $\psi(x_1) = 2$, $\psi(x_2) = 1$, $\psi(x_3) = 4$, $\psi(x_4) = 3$. Allocate all the unutilized even labels to y_i where $1 \leq i \leq m$ and once even labels are consumed, assign unused odd labels to remaining y_i's and to all z_j's where $1 \leq j \leq n$.

In all the cases, one can establish that $|e_\psi(0) - e_\psi(1)| \leq 1$ which proves the theorem. □

Theorem 6.2.12. *F_n admits a DDCL.*

Proof. Let $V(F_n) = \{k_0, k_{i1}, k_{i2} : 1 \leq i \leq n\}$ and $E(F_n) = \{k_0k_{i1}, k_{i1}k_{i2}, k_0k_{i2} : 1 \leq i \leq n\}$. Clearly, $|V(F_n)| = 2n+1$ and $|E(F_n)| = 3n$. Consider $\psi : V(F_n) \rightarrow \{1, 2, ..., 2n+1\}$. Fix $\psi(k_0) = 1$, $\psi(k_{11}) = 2$, $\psi(k_{21}) = 3$, $\psi(k_{i1}) = \psi(k_{(i-1)1}) + 2$; $3 \leq i \leq n$, $\psi(k_{i2}) = 2\psi(k_{i1})$ for $1 \leq i \leq \frac{n}{2}$. Now assign remaining labels out of $\{1, 2, ..., 2n+1\}$ simultaneously to unlabeled nodes. (If n is odd then $\psi(k_{i2}) = 2\psi(k_{i1})$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil$). Consequently, $|e_\psi(0) - e_\psi(1)| \leq 1$, which shows that F_n is a DDCG. □

Theorem 6.2.13. *K_n does not admit a DDCL for $n \geq 5$.*

Proof. Let $V(K_n) = \{k_1, k_2, ..., k_n\}$. Clearly, $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$. Suppose K_n allows a DDCL, say, ψ for $n \geq 5$. Fix $\psi(k_i) = i$; $i = 1, 2, ..., n$. Now, arise two possibilities.

Case (i) $n \equiv 0, 1(mod\ 4)$.

Since ψ is DDCL on K_n, therefore $e_\psi(0) = e_\psi(1) = \frac{n(n-1)}{4}$ should hold. Observing ψ, k_1 contributes $\lfloor \frac{n}{2.1} \rfloor$ edges having label 1, k_2 contributes $\lfloor \frac{n}{2.2} \rfloor$ edges having label 1, k_3 contributes $\lfloor \frac{n}{2.3} \rfloor$ edges having label 1 and so on, thus $e_\psi(1) = \lfloor \frac{n}{2.1} \rfloor + \lfloor \frac{n}{2.2} \rfloor + \lfloor \frac{n}{2.3} \rfloor + ... + \lfloor \frac{n}{2.m} \rfloor$, where m is the largest positive integer such that $2m \leq n$. By assumption, $e_\psi(1) = \frac{n(n-1)}{4}$, but $\frac{n(n-1)}{4} > \lfloor \frac{n}{2.1} \rfloor + \lfloor \frac{n}{2.2} \rfloor + \lfloor \frac{n}{2.3} \rfloor + ... + \lfloor \frac{n}{2.m} \rfloor$, which leads to a contradiction.

Case (ii) $n \equiv 2, 3(mod\ 4)$.

Then either $e_\psi(1) = \lceil \frac{n(n-1)}{4} \rceil$ and $e_\psi(0) = \lfloor \frac{n(n-1)}{4} \rfloor$ or $e_\psi(0) = \lceil \frac{n(n-1)}{4} \rceil$ and $e_\psi(1) = \lfloor \frac{n(n-1)}{4} \rfloor$ should hold. Observing ψ, k_1 contributes $\lfloor \frac{n}{2.1} \rfloor$ edges having label 1, k_2 contributes $\lfloor \frac{n}{2.2} \rfloor$ edges having label 1, k_3 contributes $\lfloor \frac{n}{2.3} \rfloor$ edges having label 1, and so on, thus $e_\psi(1) = \lfloor \frac{n}{2.1} \rfloor + \lfloor \frac{n}{2.2} \rfloor + \lfloor \frac{n}{2.3} \rfloor + ... + \lfloor \frac{n}{2.m} \rfloor$, where m is the largest positive integer such that $2m \leq n$. Again, $e_\psi(1) = \lfloor \frac{n}{2.1} \rfloor + \lfloor \frac{n}{2.2} \rfloor + \lfloor \frac{n}{2.3} \rfloor + ... + \lfloor \frac{n}{2.m} \rfloor < \lfloor \frac{n(n-1)}{4} \rfloor < \lceil \frac{n(n-1)}{4} \rceil$, a contradiction.

Hence, if $n \geq 5$, then K_n does not allow a DDCL (see Figure 6.4). □

Remark 6.3. One can obtain the DDCL of K_n for $n = 2, 3, 4$.

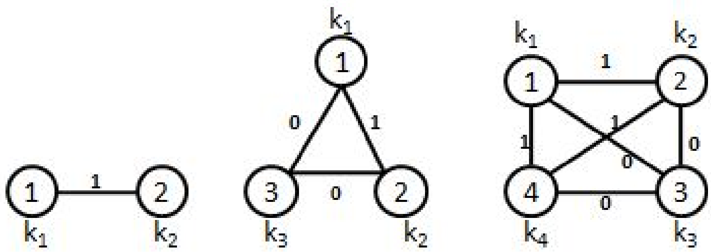

FIGURE 6.4: K_2 and K_3 are DDCG whereas K_4 is not

Remark 6.4. One can observe that not every DCG is DDCG as such it becomes very intresting to look for classes of graph that admit DCL but not DDCL. For example, K_5 and K_6 are DCG but not DDCG. Similarly, not every DDCG is DCG, it becomes equally interesting to see the families of graph that are DDCG but not DCG. For instance, K_4 is DDCG but not DCG.

One more example is flower graph. For instance, consider Fl_3, which exhibits DCL but not DDCL.

6.2.2 DDCL of Some Star Related Graphs

In this section, star related graphs are investigated for DDCL.

Theorem 6.2.14. *$K_{1,n}$ admits a DDCL.*

Proof. Let $V(K_{1,n}) = \{x_0, x_i : 1 \le i \le n\}$ with x_0 as apex node. Consider $\psi : V(K_{1,n}) \to \{1, 2, ..., n+1\}$. Fix $\psi(x_0) = 1$ and allot the unutilized labels to remaining nodes yields the result. □

Theorem 6.2.15. *$B_{n,m}$ admits a DDCL.*

Proof. Let $V(B_{n,m}) = \{x_0, y_0, x_i, y_j : 1 \le i \le n,\ 1 \le j \le m\}$ where x_0, y_0 are apex nodes. Clearly, $|V(B_{n,m})| = n + m + 2$ and $|E(B_{n,m})| = n + m + 1$. Consider a map $\psi : V(B(n, m)) \to \{1, 2, ..., n + m + 2\}$ defined by the below mentioned cases.
Case (i) When $n = m$.
Put $\psi(x_0) = 1$, $\psi(x_1) = 2$, $\psi(y_0) = q$ where q is the largest prime $\le n + m + 2$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $2 \le i \le n$ and allot the remaining labels to y_j's where $1 \le j \le m$.
Case (ii) $n > m$ or $n < m$.
First suppose $n > m$, fix $\psi(x_0) = 1$, $\psi(x_1) = 2$, $\psi(y_0) = q$ where q is the largest prime $\le n+m+2$, $\psi(x_i) = \psi(x_{i-1})+2$; $2 \le i \le n$. Once even labels are consumed fully, assign unutilized labels to remaining unlabeled nodes x_i, if any, and to all y_j's, simultaneously from $\{1, 2, ..., n + m + 2\}$.
Similar argument holds good for $n < m$ too.
Evidently, $|e_\psi(0) - e_\psi(1)| \le 1$, which shows that $B_{n,m}$ is a DDCG (see Figure 6.5). □

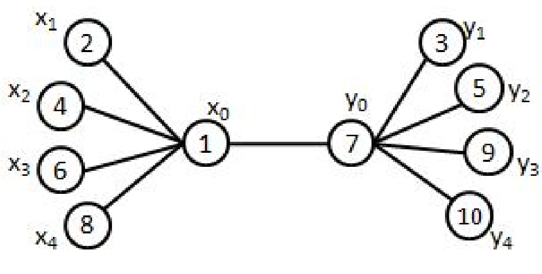

FIGURE 6.5: DDCL of $B_{4,4}$

Theorem 6.2.16. *$S(K_{1,n})$ admits a DDCL.*

Proof. Let $V(K_{1,n}) = \{x_0, x_i : 1 \le i \le n\}$ and $E(K_{1,n}) = \{x_0x_i : 1 \le i \le n\}$. For $S(K_{1,n})$, $V(S(K_{1,n})) = V(K_{1,n}) \cup \{y_i : 1 \le i \le n\}$ and $E(S(K_{1,n})) = \{x_0y_i : 1 \le i \le n\} \cup \{y_ix_i : 1 \le i \le n\}$. Clearly, $|V(S(K_{1,n}))| = 2n+1$ and $|E(S(K_{1,n}))| = 2n$. Consider $\psi : V(S(K_{1,n})) \to \{1, 2, ..., 2n + 1\}$. Fix $\psi(x_0) = 1$, $\psi(y_1) = 2$, $\psi(y_i) = \psi(y_{i-1}) + 2$;

$2 \leq i \leq n$ and $\psi(x_i) = \psi(y_i) + 1$; $1 \leq i \leq n$. Evidently, $|e_\psi(0) - e_\psi(1)| = 0$ which proves that $S(K_{1,n})$ is a DDCG (see Figure 6.6). □

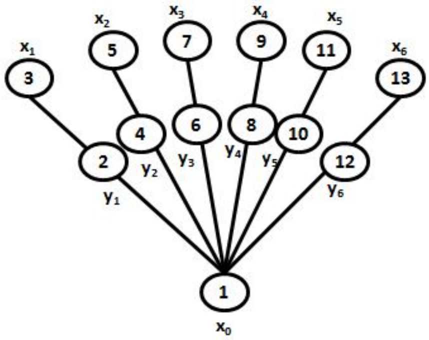

FIGURE 6.6: DDCL of $S(K_{1,6})$

Theorem 6.2.17. *$S(B_{n,n})$ admits a DDCL.*

Proof. Let $V(B_{n,n}) = \{x_0, y_0, x_i, y_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{x_0x_i : 1 \leq i \leq n\} \cup \{y_0y_i : 1 \leq i \leq n\} \cup \{x_0y_0\}$. Consider $S(B_{n,n})$ with $V(S(B_{n,n})) = V(B_{n,n}) \cup \{x'_i, y'_i : 1 \leq i \leq n\} \cup \{u\}$ and $E(S(B_{n,n})) = \{x_0x'_i : 1 \leq i \leq n\} \cup \{x'_ix_i : 1 \leq i \leq n\} \cup \{y_0y'_i : 1 \leq i \leq n\} \cup \{y'_iy_i : 1 \leq i \leq n\} \cup \{x_0u, uy_0\}$. Clearly, $|V(S(B_{n,n}))| = 4n + 3$ and $|E(S(B_{n,n}))| = 4n + 2$. Consider $\psi : V(S(B_{n,n})) \to \{1, 2, ..., 4n + 3\}$ defined by fixing $\psi(x_0) = 1$, $\psi(u) = 4$, $\psi(y_0) = 2$, $\psi(x'_1) = 6$, $\psi(y'_1) = 3$, $\psi(y'_2) = 8$, $\psi(y_1) = 5$, $\psi(x'_i) = \psi(x'_{i-1}) + 4$; $2 \leq i \leq n$, $\psi(x_i) = \psi(x'_i) + 1$; $1 \leq i \leq n$, $\psi(y'_i) = \psi(y'_{i-1}) + 4$; $3 \leq i \leq n$ and $\psi(y_i) = \psi(y'_i) + 1$; $2 \leq i \leq n$. Evidently, $|e_\psi(0) - e_\psi(1)| \leq 1$ which proves that $S(B_{n,n})$ is a DDCG. □

Remark 6.5. As subdivision of P_n and C_n yields path and cycle again, therefore $S(P_n)$ and $S(C_n)$ are DDCGs.

Theorem 6.2.18. *$S'(K_{1,n})$ permits a DDCL.*

Proof. Let $V(K_{1,n}) = \{x_0, x_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{x_0x_i : 1 \leq i \leq n\}$. Consider $S'(K_{1,n})$ with $V(S'(K_{1,n})) = V(K_{1,n}) \cup \{u_0, u_i : 1 \leq i \leq n\}$ and $E(S'(K_{1,n})) = E(K_{1,n}) \cup \{x_0u_i : 1 \leq i \leq n\} \cup \{u_0x_i : 1 \leq i \leq n\}$. Clearly, $|V(S'(K_{1,n}))| = 2n + 2$ and $|E(S'(K_{1,n}))| = 3n$. Consider $\psi : V(S'(K_{1,n})) \to \{1, 2, ..., 2n + 2\}$. Fix $\psi(x_0) = 1$, $\psi(u_0) = 2$, $\psi(x_1) = 4$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $2 \leq i \leq n$ and allocate the unutilized odd labels to u_i; $1 \leq i \leq n$. Observe that when n is even, $e_\psi(0) = n + \frac{n}{2} = \frac{3n}{2}$ and $e_\psi(1) = \frac{3n}{2}$, and when n is odd, $e_\psi(0) = n + \lfloor \frac{n}{2} \rfloor$ and $e_\psi(1) = n + \lceil \frac{n}{2} \rceil$. Thus, $|e_\psi(0) - e_\psi(1)| \leq 1$ and hence the result. □

Theorem 6.2.19. *$S'(B_{n,m})$ permits a DDCL.*

Proof. Let $V(B_{n,m}) = \{x_0, y_0, x_i, y_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(B_{n,m}) = \{x_0x_i : 1 \leq i \leq n\} \cup \{y_0y_j : 1 \leq i \leq m\} \cup \{x_0y_0\}$. Let $S'(B_{n,m})$ be having $V(S'(B_{n,m})) = V(B_{n,m}) \cup \{x'_0, y'_0, x'_i, y'_j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and $E(S'(B_{n,m})) = E(B_{n,m}) \cup \{x_0x'_i : 1 \leq i \leq n\} \cup \{x'_0x_i : 1 \leq i \leq n\} \cup \{y_0y'_j : 1 \leq j \leq m\} \cup \{y'_0y_j : 1 \leq j \leq m\} \cup \{x_0y'_0, y_0x'_0\}$. Clearly, $|V(S'(B_{n,m})| = 2n + 2m + 4$ and $|E(S'(B_{n,m})| = 3n + 3m + 3$. Consider $\psi : V(S'(B_{n,m}) \to \{1, 2, ..., 2n + 2m + 4\}$ under the below mentioned cases.

Case (i) When $n = m$.

Let $\psi(x_0) = 1$, $\psi(x'_0) = 2$, $\psi(y_0) = 4$, $\psi(x_1) = 8$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $2 \leq i \leq n$ and $\psi(y'_0) = p$; p is the largest prime $\leq 4n + 4$. Assign the unutilized even labels to x'_i; $1 \leq i \leq n$ and odd labels to y_j and y'_j; $1 \leq j \leq n$, in any order. Observe that $e_\psi(1) = 3n + 2$ and $e_\psi(0) = 3n + 1$.

Case (ii) When $n \neq m$.

Without loss of generality, suppose $n > m$. Let p_1, p_2 be sufficiently large primes and $p_2 < p_1 \leq 2n + 2m + 4$. Fix $\psi(y_0) = p_1$, $\psi(y'_0) = p_2$, $\psi(x_0) = 1$, $\psi(x'_0) = 2$, $\psi(x_1) = 4$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $i \geq 2$ such that $\psi(x_k) \leq 2n + 2m + 4$ for some $k \leq n$. Next allocate the unutilized even labels to x'_i; $i \geq 1$ in such a way that $\psi(x'_t) \leq 2n+2m+4$ for some $t \leq n$, and to unlabeled x_i, if any. Next, assign the unconsumed labels to unlabeled nodes in any fashion.

In both the cases, $|e_\psi(0)-e_\psi(1)| \leq 1$ and hence $S'(B_{n,m})$ is a DDCG (see Figure 6.7). □

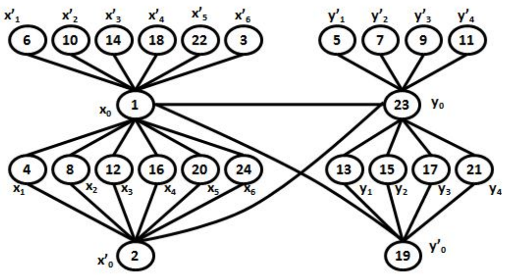

FIGURE 6.7: DDCL of $S'(B_{6,4})$

Theorem 6.2.20. *$S'(S(K_{1,n}))$ permits a DDCL.*

Proof. Let $V(S(K_{1,n})) = \{x, y_i, z_i : 1 \leq i \leq n\}$ and $V(S'(S(K_{1,n}))) = V(S(K_{1,n})) \cup \{x', y'_i, z'_i;\ 1 \leq i \leq n\}$. Clearly, $|V(S'(S(K_{1,n})))| = 4n + 2$ and $|E(S'(S(K_{1,n})))| = 6n$. Consider the function $\psi : V(S'(S(K_{1,n}))) \to \{1, 2, ..., 4n+2\}$ defined by taking $\psi(x) = 1$, $\psi(x') = 2$, $\psi(y_i) = 4i$; $1 \leq i \leq n$, $\psi(y'_i) = 4i + 2$; $1 \leq i \leq n$, $\psi(z_i) = 4i + 1$; $1 \leq i \leq n$; $\psi(z'_i) = 4i - 1$; $1 \leq i \leq n$. It follows that $e_\psi(0) = e_\psi(1) = 3n$ proving that $S'(S(K_{1,n}))$ is a DDCG. □

Definition 6.2.4. [87] "$G = \left\langle K^{(1)}_{1,n}, K^{(2)}_{1,n}, ..., K^{(m)}_{1,n} \right\rangle$ denote the graph formed by joining the apex nodes of $K^{(t-1)}_{1,n}$ and $K^{(t)}_{1,n}$, to a newly inserted node r_{t-1} where $2 \leq t \leq m$."

Theorem 6.2.21. *$G = \left\langle K_{1,n}^{(1)}, K_{1,n}^{(2)} \right\rangle$ admits a DDCL.*

Proof. Let G be formed by joining the apex nodes, say, x_0 and y_0 respectively of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$, to a new node, say, w. The cardinality of node and edge set of G is $2n+3$ and $2n+2$ respectively. Consider $\psi : V(G) \to \{1, 2, ..., 2n+3\}$ given by fixing $\psi(x_0) = 1$, $\psi(w) = 2$ and $\psi(y_0) = p$; p is the largest prime $\leq 2n+3$. Allocate all the unused even labels to the pendant nodes of $K_{1,n}^{(1)}$ and remaining labels to unlabeled nodes. Consequently, $|e_\psi(0) - e_\psi(1)| \leq 1$, establishing that G is a DDCG (see Figure 6.8). □

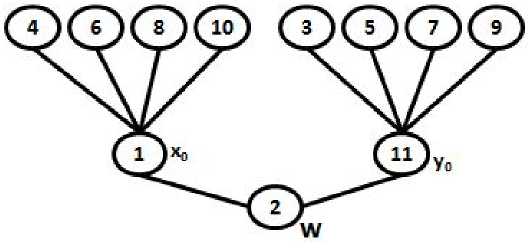

FIGURE 6.8: DDCL of $\left\langle K_{1,4}^{(1)}, K_{1,4}^{(2)} \right\rangle$

Theorem 6.2.22. *$\left\langle K_{1,n}^{(1)}, K_{1,n}^{(2)}, K_{1,n}^{(3)} \right\rangle$ permits a DDCL.*

Proof. Let $x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, ..., x_n^{(i)}$ represent the nodes of $K_{1,n}^{(i)}$ where $x_0^{(i)}$; $1 \leq i \leq 3$ stands for apex nodes. Let r_1 and r_2 be the newly introduced nodes such that $x_0^{(1)}$ and $x_0^{(2)}$ are adjacent to r_1, and $x_0^{(2)}$ and $x_0^{(3)}$ are adjacent to r_2 to form $G = \left\langle K_{1,n}^{(1)}, K_{1,n}^{(2)}, K_{1,n}^{(3)} \right\rangle$. One can see that $|V(G)| = 3n+5$ and $|E(G)| = 3n+4$. Consider $\psi : V(G) \to \{1, 2, ..., 3n+5\}$ defined by fixing $\psi(r_1) = 4$, $\psi(x_0^{(1)}) = 1$, $\psi(x_0^{(2)}) = 2$ and $\psi(x_0^{(3)}) = p$; p is the largest prime $\leq 3n+5$. Now allocate all unused even labels of the form $4n$; $n \in \mathbb{N}$ to the pendant nodes of $K_{1,n}^{(2)}$ and remaining even labels to the unlabeled nodes of $K_{1,n}^{(1)}$. Allocating the remaining labels simultaneously to the remaining unlabeled nodes yields G a DDCG. □

Inspired by definition 6.2.4, a similar construction for bistar is proposed.

Definition 6.2.5. "$G = \left\langle B_{n,n}^{(1)}, B_{n,n}^{(2)}, ..., B_{n,n}^{(m)} \right\rangle$ denotes the graph by connecting the apex nodes of $B_{n,n}^{(t-1)}$ and $B_{n,n}^{(t)}$, to new nodes r_{t-1}, s_{t-1} where $2 \leq t \leq m$ (see Figure 6.9)."

Theorem 6.2.23. *$\left\langle B_{n,n}^{(1)}, B_{n,n}^{(2)} \right\rangle$ admits a DDCL.*

Proof. Let $V(B_{n,n}^{(i)}) = \{x_0^{(i)}, y_0^{(i)}, x_j^{(i)}, y_j^{(i)} : 1 \leq j \leq n\}$ and $E(B_{n,n}^{(i)}) = \{x_0^{(i)}y_0^{(i)}, x_0^{(i)}x_j^{(i)}, y_0^{(i)}y_j^{(i)} : 1 \leq j \leq n\}$. Let $G = \left\langle B_{n,n}^{(1)}, B_{n,n}^{(2)} \right\rangle$ and r, s be newly introduced nodes such that r is

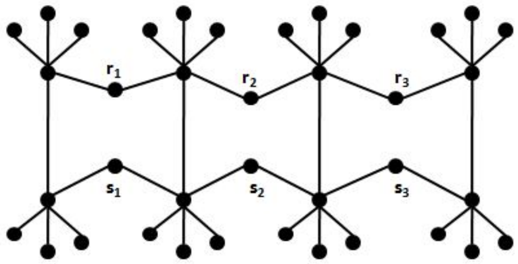

FIGURE 6.9: $\left\langle B^{(1)}_{3,3}, B^{(2)}_{3,3}, B^{(3)}_{3,3}, B^{(4)}_{3,3} \right\rangle$

adjacent to $x_0^{(1)}$ and $x_0^{(2)}$, and s is adjacent to $y_0^{(1)}$ and $y_0^{(2)}$. Clearly, $|V(G)| = 4n+6$ and $|E(G)| = 4n+6$. Consider $\psi : V(G) \to \{1, 2, ..., 4n+6\}$ defined as follows. Fix $\psi(x_0^{(1)}) = 1$, $\psi(x_0^{(2)}) = 2$, $\psi(y_0^{(1)}) = 6$, $\psi(y_0^{(2)}) = 3$, $\psi(r) = 4$, $\psi(s) = 9$. Assign even labels of the form $4t; t \in \mathbb{N}$ to $x_j^{(2)}$; $1 \leq j \leq n$ and remaining even labels to $x_j^{(1)}$; $1 \leq j \leq n$. Next, allocating the unused labels to unlabeled nodes simultaneously shows that G is a DDCG. □

6.2.3 DDCL in the Context of Ring Sum

In this section, ring sum of G with $K_{1,m}$ is discussed.

Definition 6.2.6. [27] "Ring sum of two graphs G_1 and G_2 denoted by $G_1 \oplus G_2$ have node set $V_1 \cup V_2$ and edge set $(E_1 \cup E_2) - (E_1 \cap E_2)$."

Note: The ring sum of G with $K_{1,m}$ is taken by fixing one node of G and the apex node of $K_{1,m}$ as a common node.

Theorem 6.2.24. *For a DDCG, $G(p,q)$ with labeling g, $G \oplus K_{1,m}$ admits a DDCG for the following conditions.*
(i) Even values of m
(ii) Odd values of m and
(a) q is even
(b) Both q and p are odd with $e_g(1) = \lfloor \frac{q}{2} \rfloor$
(c) q is odd and p is even with $e_g(1) = \lceil \frac{q}{2} \rceil$.

Proof. Given $G(p,q)$ a DDCG, with labeling g, and $V(G) = \{u_1, u_2, ..., u_p\}$. Choose $u_1 \in V(G)$, such that $g(u_1) = 1$. Consider $K_{1,m}$ with $V(K_{1,m}) = \{v_0 = u_1, v_i : 1 \leq i \leq m\}$ and $E(K_{1,m}) = \{v_0 v_i; 1 \leq i \leq m\}$. Let $H = G \oplus K_{1,m}$ with $V(H) = V(G) \cup \{v_i : 1 \leq i \leq m\}$ and $E(H) = E(G) \cup \{u_1 v_i : 1 \leq i \leq m\}$. Consider $\psi : V(H) \to \{1, 2, ..., p, p+1, ..., p+m\}$ defined by $\psi(u_i) = g(u_i)$ for $1 \leq i \leq p$. Recall that

$\psi(v_0) = g(u_1) = 1$, fix $\psi(v_i) = p + i$ for $1 \le i \le m$. Next is to show that $G \oplus K_{1,m}$ is a DDCL for the under mentioned conditions.

Case (i) 'm' is even.

If 'q' is even, then $e_g(0) = e_g(1) = \frac{q}{2}$. Also $|E(H)| = q + m$ and $2\psi(u_1)$ divides every even number, see that $e_\psi(1) = \frac{q}{2} + \frac{m}{2}$ and $e_\psi(0) = \frac{q}{2} + \frac{m}{2}$ which justifies that $|e_\psi(0) - e_\psi(1)| \le 1$. If q is odd, then either $e_g(0) = e_g(1) + 1$ or $e_g(1) = e_g(0) + 1$. On the other hand m being even always yields equal count of edges having labels 1 and 0 (because $2\psi(u_1)$ divides every even label that appears on pendant nodes of $K_{1,m}$). Thus, either $e_\psi(0) = e_\psi(1) + 1$ or $e_\psi(1) = e_\psi(0) + 1$, which justifies that $|e_\psi(0) - e_\psi(1)| \le 1$.

Case (ii) 'm' is odd.

The following subcases arise.

Subcase (i) 'q' is even and 'p' is even.

Since q is even, $e_g(0) = e_g(1) = \frac{q}{2}$. Also $|E(H)| = q + m$, following ψ, one can see that $e_\psi(1) = \frac{q}{2} + \lceil \frac{m}{2} \rceil - 1$ and $e_\psi(0) = \frac{q}{2} + \lceil \frac{m}{2} \rceil$ which justifies that $|e_\psi(0) - e_\psi(1)| \le 1$.

Subcase (ii) 'q' is even and 'p' is odd.

Since q is even, $e_g(0) = e_g(1) = \frac{q}{2}$. Also, $|E(H)| = q + m$. Following ψ, see that $e_\psi(1) = \frac{q}{2} + \lceil \frac{m}{2} \rceil$ and $e_\psi(0) = \frac{q}{2} + \lceil \frac{m}{2} \rceil - 1$ which justifies that $|e_\psi(0) - e_\psi(1)| \le 1$.

Subcase (iii) Both 'q' and 'p' are odd with $e_g(1) = \lfloor \frac{q}{2} \rfloor$.

Then $e_g(0) = \lfloor \frac{q}{2} \rfloor + 1$. By looking at ψ, one can find that $e_\psi(1) = \lfloor \frac{q}{2} \rfloor + \lceil \frac{m}{2} \rceil$ and $e_\psi(0) = (\lfloor \frac{q}{2} \rfloor + 1) + (\lceil \frac{m}{2} \rceil - 1)$ which justifies that $|e_\psi(0) - e_\psi(1)| \le 1$.

Subcase (iv) 'q' is odd, 'p' is even, with $e_g(1) = \lceil \frac{q}{2} \rceil$.

Then $e_g(0) = \lceil \frac{q}{2} \rceil - 1$. Observing ψ, one can find $e_\psi(1) = \lceil \frac{q}{2} \rceil + (\lceil \frac{m}{2} \rceil - 1)$ and $e_\psi(0) = (\lceil \frac{q}{2} \rceil - 1) + \lceil \frac{m}{2} \rceil$ which gives $|e_\psi(0) - e_\psi(1)| \le 1$.

Hence, $G \oplus K_{1,m}$ admits a DDCL (see Figure 6.10). □

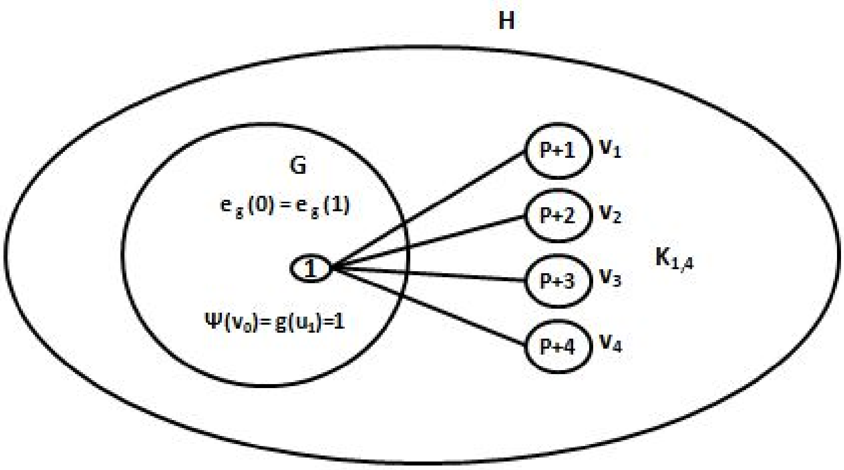

FIGURE 6.10: DDCL of $G \oplus K_{1,4}$

Corollary 6.2.1. *$P_m \oplus K_{1,m}$ is a DDCG.*

Proof. Let $V(P_m \oplus K_{1,m}) = V_1 \cup V_2$; $V_1 = \{p_1, p_2, ..., p_m\}$, $V_2 = \{v_0 = p_1, v_1, v_2, ..., v_m\}$ represent $V(P_m)$ and $V(K_{1,m})$, respectively with v_0 apex node of $K_{1,m}$. Note that $|V(P_m \oplus K_{1,m})| = 2m$ and $|E(P_m \oplus K_{1,m})| = 2m - 1$. Consider $\psi : V(P_m \oplus K_{1,m}) \to \{1, 2, ..., 2m\}$. Fix $\psi(p_1) = 1$, $\psi(v_1) = 2$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq m$, $\psi(p_i) = \psi(p_{i-1}) + 2$; $2 \leq i \leq m$. One can see that $e_\psi(0) = m - 1$ and $e_\psi(1) = m$ resulting in DDCL of $P_m \oplus K_{1,m}$. □

Corollary 6.2.2. *$C_m \oplus K_{1,m}$ is a DDCG.*

Proof. Let $V(C_m \oplus K_{1,m}) = V_1 \cup V_2$; $V_1 = \{c_1, c_2, ..., c_m\}$ and $V_2 = \{v_0 = c_1, v_1, v_2, ..., v_m\}$ represent $V(C_m)$ and $V(K_{1,m})$ respectively. Observe that $|V(C_m \oplus K_{1,m})| = 2m$ and $|E(C_m \oplus K_{1,m})| = 2m$. Consider $\psi : V(C_m \oplus K_{1,m}) \to \{1, 2, ..., 2m\}$ defined by fixing $\psi(c_1) = 1$, $\psi(v_1) = 2$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq m$, $\psi(c_i) = \psi(c_{i-1}) + 2$; $2 \leq i \leq m$. Note that $e_\psi(0) = m$ and $e_\psi(1) = m$, therefore $C_m \oplus K_{1,m}$ is a DDCG. □

Corollary 6.2.3. *$G \oplus K_{1,m}$ is a DDCG, where "G is cycle with 1 chord."*

Proof. Let $V(G \oplus K_{1,m}) = V_1 \cup V_2$; $V_1 = \{c_1, c_2, ..., c_m\}$ and $V_2 = \{v_0 = c_1, v_1, v_2, ..., v_m\}$ represent $V(C_m)$ and $V(K_{1,m})$ respectively. Also $e = c_2c_m$ be a chord of C_m and v_0 an apex node of $K_{1,m}$. Observe, $|V(G \oplus K_{1,m})| = 2m$ and $|E(G \oplus K_{1,m})| = 2m + 1$. Labeling is done by $\psi : V(G \odot K_{1,m}) \to \{1, 2, ..., 2m\}$ as defined here. Let $\psi(c_1) = 1$, $\psi(v_1) = 2$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq m$, $\psi(c_i) = \psi(c_{i-1}) + 2$; $2 \leq i \leq m$. Observe that $e_\psi(0) = m + 1$ and $e_\psi(1) = m$, thus proving that $G \oplus K_{1,m}$ is a DDCG. □

Corollary 6.2.4. *$f_m \oplus K_{1,m}$ permits a DDCL.*

Proof. Consider $f_m \oplus K_{1,m}$ with $V_1 \cup V_2$; $V_1 = \{u_0, u_1, u_2, ..., u_m\}$ and $V_2 = \{u_0 = v_0, v_1, v_2, ..., v_m\}$ representing $V(f_m)$ and $V(K_{1,m})$ respectively. Clearly, $|V(f_m \oplus K_{1,m})| = 2m + 1$ and $|E(f_m \oplus K_{1,m})| = 3m - 1$. Labeling function $\psi : V(f_m \oplus K_{1,m}) \to \{1, 2, ..., 2m + 1\}$ is defined by fixing $\psi(u_0) = 1$, and assigning the available even labels to u_i; $1 \leq i \leq m$ in the following fashion.

$$\begin{array}{cccccc} 2, & 2.2^1, & 2.2^2, & ..., & 2.2^{k_1}, \\ 6, & 6.2^1, & 6.2^2, & ..., & 6.2^{k_2}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

such that $(4t-2)2^{k_t} \leq 2m+1$ and $t \geq 1$. Assigning the remaining labels to v_i; $1 \leq i \leq m$ yields $|e_\psi(0) - e_\psi(1)| \leq 1$ establishing that $f_m \oplus K_{1,m}$ is a DDCG. □

Corollary 6.2.5. *$Df_m \oplus K_{1,m}$ permits a DDCL.*

Proof. Let $Df_m \oplus K_{1,m}$ with $V_1 \cup V_2$; $V_1 = \{x_0, y_0, u_1, u_2, ..., u_m\}$ and $V_2 = \{x_0 = v_0, v_1, v_2, ..., v_m\}$ representing $V(Df_m)$ and $V(K_{1,m})$ respectively. Clearly, $|V(Df_m \oplus K_{1,m})| = 2m + 2$ and $|E(Df_m \oplus K_{1,m})| = 4m - 1$. Consider $\psi : V(Df_m \oplus K_{1,m}) \to \{1, 2, ..., 2m + 2\}$ by fixing $\psi(x_0) = 1$, $\psi(y_0) = 2$ and assign the available even labels to u_i; $1 \leq i \leq m$ in the following fashion.

$$\begin{array}{cccccc} 2, & 2.2^1, & 2.2^2, & ..., & 2.2^{k_1}, \\ 6, & 6.2^1, & 6.2^2, & ..., & 6.2^{k_2}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

such that $(4t-2)2^{k_t} \leq 2m+2$ and $t \geq 1$. Assigning the remaining labels to v_i; $1 \leq i \leq m$ gives $|e_\psi(0) - e_\psi(1)| \leq 1$ showing that $Df_m \oplus K_{1,m}$ is a DDCG (see Figure 6.11). □

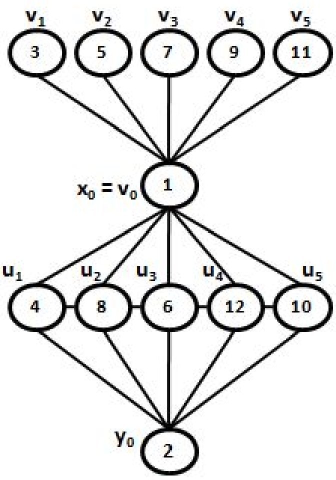

FIGURE 6.11: DDCL of $Df_5 \oplus K_{1,5}$

Corollary 6.2.6. *$K_{1,m} \oplus K_{1,m}$ permits a DDCL.*

Proof. Let $K_{1,m} \oplus K_{1,m}$ be with $V_1 \cup V_2$, where $V_1 = \{u_0, u_1, u_2, ..., u_m\}$ and $V_2 = \{v_0 = u_1, v_1, v_2, ..., v_m\}$. Clearly, $|V(K_{1,m} \oplus K_{1,m})| = 2m + 1$ and $|E(K_{1,m} \oplus K_{1,m})| = 2m$. Consider $\psi : V(K_{1,m} \oplus K_{1,m}) \to \{1, 2, ..., 2m+1\}$ defined by letting $\psi(u_0) = 1$, $\psi(v_0) = 2$. Assign the remaining even labels to u_i; $2 \leq i \leq m$ and odd labels to v_i; $1 \leq i \leq m$. Note that $|e_\psi(0) - e_\psi(1)| \leq 1$ which settles that $K_{1,m} \oplus K_{1,m}$ is a DDCG. □

6.2.4 DDCL of Corona of Certain Graphs

Here, certain results on DDCL of different graphs under a graph operation named, corona, are derived.

Theorem 6.2.25. *$W_n \odot K_1$ permits a DDCL.*

Proof. Let $V(W_n) = \{w_0, w_i : 1 \le i \le n\}$, w_0 the apex node and $G = W_n \odot K_1$ be formed with $V(G) = V(W_n) \cup \{w'_0, w'_i : 1 \le i \le n\}$. Clearly, $|V(G)| = 2n + 2$ and $|E(G)| = 3n + 1$. Consider $\psi : V(G) \to \{ 1, 2, ..., 2n + 2\}$ given by fixing $\psi(w_0) = 1$, $\psi(w'_0) = 2n + 2$ and assign the available even labels to w_i; $1 \le i \le n$ in the following fashion.

$$\begin{array}{cccccc} 2, & 2.2^1, & 2.2^2, & ..., & 2.2^{k_1}, \\ 6, & 6.2^1, & 6.2^2, & ..., & 6.2^{k_2}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

such that $(4t-2)2^{k_t} \le 2n$ and $t \ge 1$. Note that $2(4t-2)2^r | (4t-2)2^s$; $r < s$. Fix $\psi(w'_1) = 3$ and $\psi(w'_i) = \psi(w'_{i-1}) + 2$; $2 \le i \le n$. One can easily see that $|e_\psi(0) - e_\psi(1)| \le 1$ establishing that G is a DDCG. □

Theorem 6.2.26. *$DW_n \odot K_1$ permits a DDCL.*

Proof. Let $V(DW_n) = \{x_0, x_i, y_i : 1 \le i \le n\}$ where x_0, x_i, and y_i; $1 \le i \le n$ are respectively the apex, rim nodes of inner and outer circles. Let $G = DW_n \odot K_1$ with $V(G) = V(DW_n) \cup \{x'_0, x'_i, y'_i : 1 \le i \le n\}$. Clearly, $|V(G)| = 4n+2$ and $|E(G)| = 6n+1$. Labeling function $\psi : V(G) \to \{1, 2, ..., 4n + 2\}$ is given by fixing $\psi(x_0) = 1$, $\psi(x'_0) = 3$, $\psi(x_1) = 6$, $\psi(x'_1) = 2$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $2 \le i \le n$, $\psi(x'_i) = \frac{\psi(x_i)}{2}$; $2 \le i \le n$, $\psi(y_1) = 4$, $\psi(y_i) = \psi(y_{i-1}) + 4$; $2 \le i \le n$, $\psi(y'_1) = \psi(x'_n) + 2$, $\psi(y'_i) = \psi(y'_{i-1}) + 2$; $2 \le i \le n$. Clearly, $|e_\psi(0) - e_\psi(1)| \le 1$ establishing that G is a DDCG (see Figure 6.12). □

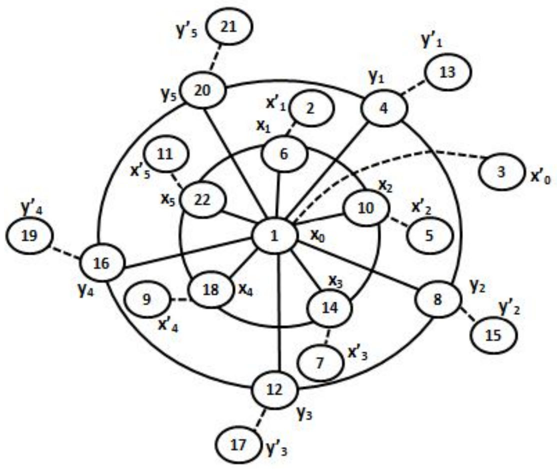

FIGURE 6.12: DDCL of $DW_5 \odot K_1$

Theorem 6.2.27. *$f_n \odot K_1$ permits a DDCL.*

Proof. Let $f_n \odot K_1$ be formed by adding $u'_0, u'_1, u'_2, ..., u'_n$ corresponding to nodes u_0, u_1, u_2,...,u_n of f_n where u_0 is apex node of f_n. Clearly, $|V(f_n \odot K_1)| = 2n+2$ and $|E(f_n \odot K_1)| = 3n$. Consider $\psi : V(f_n \odot K_1) \to \{1, 2, ..., 2n+2\}$ given by $\psi(u_0) = 1$, $\psi(u'_0) = 2n+1$, $\psi(u'_n) = 2n+2$ and assign the available even labels to u_i; $1 \leq i \leq n$ in the following fashion.

$$\begin{array}{cccccc} 2, & 2.2^1, & 2.2^2, & ..., & 2.2^{k_1}, \\ 6, & 6.2^1, & 6.2^2, & ..., & 6.2^{k_2}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

such that $(4t-2)2^{k_t} \leq 2n$ and $t \geq 1$. Note that $(4t-2)2^r|(4t-2)2^s$; $r < s$ and $2(4t-2)2^{k_j}$ does not divide $(4t+2)$ (nor $2(4t+2)$ divides $(4t-2)2^{k_j}$). Fix $\psi(u'_1) = 3$ and $\psi(u'_i) = \psi(u'_{i-1}) + 2$; $2 \leq i \leq n-1$. Note that $|e_\psi(0) - e_\psi(1)| \leq 1$ which makes $f_n \odot K_1$ a DDCG. □

Theorem 6.2.28. *$Df_n \odot K_1$ permits a DDCL.*

Proof. Let $V(Df_n) = \{x_0, y_0, u_i : 1 \leq i \leq n\}$ and $V(Df_n \odot K_1) = V(Df_n) \cup \{x'_0, y'_0, u'_i : 1 \leq i \leq n\}$. Clearly, $|V(Df_n \odot K_1)| = 2n+4$ and $|E(Df_n \odot K_1)| = 4n+1$. Consider $\psi : V(Df_n \odot K_1) \to \{1, 2, ..., 2n+4\}$. Let $\psi(x_0) = 1$, $\psi(y_0) = 2$, $\psi(x'_0) = 2n+4$, $\psi(y'_0) = 2n+3$ and assign the available even labels to u_i; $1 \leq i \leq n$ in the following fashion.

$$\begin{array}{cccccc} 2, & 2.2^1, & 2.2^2, & ..., & 2.2^{k_1}, \\ 6, & 6.2^1, & 6.2^2, & ..., & 6.2^{k_2}, \\ ..., & ..., & ..., & ..., & ..., & ..., \end{array}$$

such that $(4t-2)2^{k_t} \leq 2n+2$ and $t \geq 1$. Fix $\psi(u'_1) = 3$ and $\psi(u'_i) = \psi(u'_{i-1}) + 2$; $2 \leq i \leq n$. Observe that $|e_\psi(0) - e_\psi(1)| \leq 1$, which shows that G is a DDCG. □

Theorem 6.2.29. *$Gl(n) \odot K_1$ permits a DDCL.*

Proof. Let $V(Gl(n)) = \{x_0, y_0, u_i : 1 \leq i \leq n\}$ and $V(Gl(n) \odot K_1) = V(Gl(n)) \cup \{x'_0, y'_0, u'_i : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 2n+4$ and $|E(G)| = 3n+2$. Consider $\psi : V(G) \to \{1, 2, ..., 2n+4\}$ given by fixing $\psi(x_0) = 1$, $\psi(y_0) = 2$, $\psi(x'_0) = 2n+4$, $\psi(y'_0) = 2n+3$, $\psi(u_1) = 4$, $\psi(u_i) = \psi(u_{i-1})+2$; $2 \leq i \leq n$, $\psi(u'_i) = \psi(u_i) - 1$; $1 \leq i \leq n$. Clearly, $Df_n \odot K_1$ is a DDCG (see Figure 6.13). □

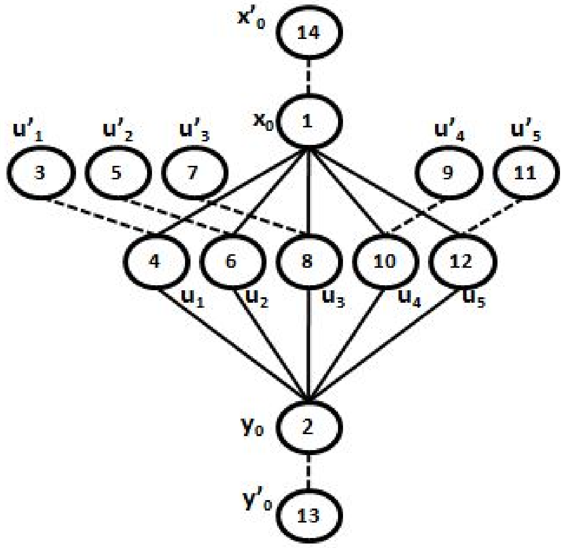

FIGURE 6.13: DDCL of $Gl(5) \odot K_1$

Theorem 6.2.30. *$Fl_n \odot K_1$ permits a DDCL.*

Proof. Let $V(Fl_n) = \{x_0, u_i, v_i;\ 1 \le i \le n\}$, where x_0, u_i and v_i represent respectively the apex, nodes of degree 4 and 2. Let $G = Fl_n \odot K_1$ with $V(G) = V(Fl_n) \cup \{x'_0, u'_i, v'_i\colon 1 \le i \le n\}$ and $E(G) = E(Fl_n) \cup \{x_0x'_0, u_iu'_i, v_iv'_i : 1 \le i \le n\}$. Clearly, $|V(G)| = 4n+2$ and $|E(G)| = 6n+1$. Consider $\psi : V(G) \to \{1, 2, ..., 4n+2\}$ given by assigning $\psi(x_0) = 1$, $\psi(x'_0) = 2$, $\psi(u_1) = 6$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $2 \le i \le n$, $\psi(u'_i) = \frac{\psi(u_i)}{2}$; $1 \le i \le n$, $\psi(v_i) = \psi(u_i) - 2$; $1 \le i \le n$, $\psi(v'_1) = \psi(u'_n) + 2$ and $\psi(v'_i) = \psi(v'_{i-1}) + 2$; $2 \le i \le n$. Note that $|e_\psi(0) - e_\psi(1)| \le 1$ establishing that $Fl_n \odot K_1$ is a DDCG. □

Theorem 6.2.31. *$K_{1,n} \odot K_1$ admits a DDCL.*

Proof. Let $V(K_{1,n}) = \{k_0, k_i : 1 \le i \le n\}$ where k_0 is apex node. Consider $K_{1,n} \odot K_1$ with $V(K_{1,n} \odot K_1) = V(K_{1,n}) \cup \{k'_0, k'_i : 1 \le i \le n\}$ and $E(K_{1,n} \odot K_1) = E(K_{1,n}) \cup \{k_0k'_0, k_ik'_i : 1 \le i \le n\}$. Clearly, $|V(K_{1,n} \odot K_1)| = 2n+2$ and $|E(K_{1,n} \odot K_1)| = 2n+1$. Let $\psi : V(K_{1,n} \odot K_1) \to \{1, 2, ..., 2n+2\}$ defined by fixing $\psi(k_0) = 1$, $\psi(k'_0) = 2n+2$, $\psi(k_i) = 2i$; $1 \le i \le n$ and $\psi(k'_i) = \psi(k_i) + 1$; $1 \le i \le n$. Observe, $|e_\psi(0) - e_\psi(1)| \le 1$ which establishes that $K_{1,n} \odot K_1$ is a DDCG. □

Theorem 6.2.32. *$K_{2,n} \odot K_1$ admits a DDCL.*

Proof. Let $U = \{x_1, x_2\}$ and $V = \{y_1, y_2, ..., y_n\}$, be the bipartition of node set of $K_{2,n}$. Let $K_{2,n} \odot K_1$ be having $V(K_{2,n} \odot K_1) = V(K_{2,n}) \cup \{x'_1, x'_2, y'_1, y'_2, ..., y'_n\}$ and $E(K_{2,n} \odot K_1) = E(K_{2,n}) \cup \{x_1x'_1, x_2x'_2, y_iy'_i : 1 \le i \le n\}$. Clearly, $|V(K_{2,n} \odot K_1)| = 2n+4$ and $|E(K_{2,n} \odot K_1)| = 3n+2$. Consider $\psi : V(K_{2,n} \odot K_1) \to \{1, 2, ..., 2n+4\}$ defined by fixing $\psi(x_1) = 1$, $\psi(x_1') = 2n+4$, $\psi(x_2) = 2$, $\psi(x_2') = 2n+3$, $\psi(y_i) = 2i+2$; $1 \leq i \leq n$ and $\psi(y_i') = \psi(y_i) - 1$; $1 \leq i \leq n$. Observe that $|e_\psi(0) - e_\psi(1)| \leq 1$, showing $K_{2,n} \odot K_1$ a DDCG. □

Theorem 6.2.33. *$K_{3,n} \odot K_1$ admits a DDCL.*

Proof. Let $U = \{x_1, x_2, x_3\}$ and $V = \{y_1, y_2, ..., y_n\}$ be the bipartition of node set of $K_{3,n}$. Let $K_{3,n} \odot K_1$ with $V(K_{3,n} \odot K_1) = V(K_{3,n}) \cup \{x_1', x_2', x_3', y_1', y_2', ..., y_n'\}$ and $E(K_{3,n} \odot K_1) = E(K_{3,n}) \cup \{x_1x_1',\ x_2x_2',\ x_3x_3',\ y_iy_i' : 1 \leq i \leq n\}$. Observe, $|V(K_{3,n} \odot K_1)| = 2n+6$ and $|E(K_{3,n} \odot K_1)| = 4n+3$. Consider $\psi : V(K_{3,n} \odot K_1) \to \{1, 2, ..., 2n+6\}$ defined by fixing $\psi(x_1) = 1$, $\psi(x_1') = 6$, $\psi(x_2) = 2$, $\psi(x_3) = 2n+5$, $\psi(x_2') = 4$, $\psi(x_3') = 2n+3$ and $\psi(y_1) = 10$. There arise below mentioned possibilities.
Case (i) When 'n' is odd.
Let $\psi(y_i) = \psi(y_{i-1}) + 4$; $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $\psi(y_{\lceil \frac{n}{2} \rceil}) = 8$, $\psi(y_i) = \psi(y_{i-1}) + 4$; $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$, $\psi(y_i') = \frac{\psi(y_i)}{2}$; $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Assign the unused labels simultaneously to unlabeled nodes.
Case (ii) When' n' is even.
Let $\psi(y_i) = \psi(y_{i-1}) + 4$; $2 \leq i \leq \frac{n}{2}$, $\psi(y_{\frac{n}{2}+1}) = 8$, $\psi(y_i) = \psi(y_{i-1}) + 4$; $\frac{n}{2} + 2 \leq i \leq n$, $\psi(y_i') = \frac{\psi(y_i)}{2}$; $1 \leq i \leq \frac{n}{2}$. Assign the unconsumed labels simultaneously to unlabeled nodes.
In both the cases, $|e_\psi(0) - e_\psi(1)| \leq 1$. Hence $K_{3,n} \odot K_1$ is a DDCG (see Figure 6.14).

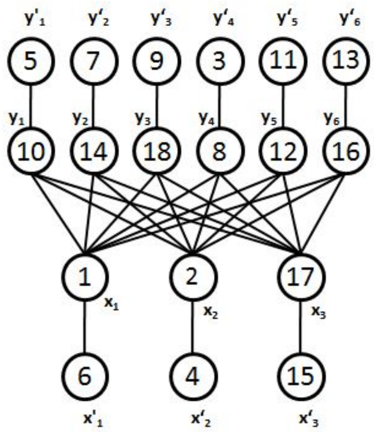

FIGURE 6.14: DDCL of $K_{3,6} \odot K_1$

□

Theorem 6.2.34. *$B_{n,n} \odot K_1$ permits a DDCL.*

Proof. Let $\{x_0', y_0', x_i', y_i';\ 1 \leq i \leq n\}$ be the added nodes corresponding to $\{x_0, y_0, x_i, y_i;$ $1 \leq i \leq n\}$ of $B_{n,n}$, for the construction of $B_{n,n} \odot K_1$. Clearly, $|V(B_{n,n} \odot K_1)| = 4n+4$ and $|E(B_{n,n} \odot K_1)| = 4n+3$. Consider $\psi : V(B_{n,n} \odot K_1) \to \{1, 2, ..., 4n+4\}$ defined by letting

$\psi(x_0) = 1$, $\psi(x'_0) = 4n+4$, $\psi(y_0) = 2$, $\psi(y'_0) = 4n+3$, $\psi(x_1) = 6$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $2 \leq i \leq n$, $\psi(y_1) = 4$, $\psi(y_i) = \psi(y_{i-1}) + 4$; $2 \leq i \leq n$, $\psi(x'_i) = \psi(x_i) - 1$; $1 \leq i \leq n$ and $\psi(y'_i) = \psi(y_i) - 1$; $1 \leq i \leq n$. One can see that $|e_\psi(0) - e_\psi(1)| \leq 1$ proving that $B_{n,n} \odot K_1$ is a DDCG. □

6.3 Average Even Divisor Cordial Labeling

In this section, a few general results concerning AEDCL of graphs are established. AEDCL of various families of graphs are investigated for different graph operations of high interest.

Definition 6.3.1. "An average even divisor cordial labeling (AEDCL) of $G(V,E)$ is a bijection $\psi : V \to \{2,4,6,...,2|V(G)|\}$ defined by the induced function $\psi^\star : E \to \{0,1\}$ such that for each edge yz, $\psi^\star(yz)$ is given label 1 if $2|\frac{\psi(y)+\psi(z)}{2}$ and label 0 otherwise, then $|e_\psi(0) - e_\psi(1)| \leq 1$. If a graph permits an AEDCL, then it is known as average even divisor cordial graph (AEDCG)."

Theorem 6.3.1. *If $G(p,q)$ admits an AEDCL with q even, then $G \pm e$ is also an AEDCG.*

Proof. Since $G(p,q)$ is an AEDCG with labeling ψ therefore $e_\psi(0) = e_\psi(1)$ (q is even). Clearly, an addition or deletion of one edge yield either $e_\psi(0) = e_\psi(1) + 1$ or $e_\psi(1) = e_\psi(0) + 1$ which in turn justifies that $|e_\psi(0) - e_\psi(1)| \leq 1$. □

Theorem 6.3.2. *If $G(p,q)$ is an AEDCG with q odd, then $G-e$ also admits an AEDCL.*

Proof. Since $G(p,q)$ is an AEDCG with labeling ψ with q odd, therefore, either $e_\psi(0) = e_\psi(1)+1$ or $e_\psi(1) = e_\psi(0)+1$. Suppose $e_\psi(0) = e_\psi(1)+1$. Removing an edge having label 0 yields $|e_\psi(0) - e_\psi(1)| \leq 1$. Similarly, if $e_\psi(1) = e_\psi(0) + 1$, then removing any edge having label 1 results in AEDCG again. □

Remark 6.6. On similar lines of proof one can observe that Theorem 6.3.2 also holds good for $G + e$.

Theorem 6.3.3. *K_n does not admit AEDCL for $n \geq 4$.*

Proof. Let $V(K_n) = \{k_i : 1 \leq i \leq n\}$. Consider $\psi : V(K_n) \to \{2,4,6,...,2n\}$ defined by fixing $\psi(k_i) = 2i$; $1 \leq i \leq n$. Now the below mentioned cases arise.
Case (i) When 'n' is even.
Observing the labeling pattern, one can see that $e_\psi(1) = e_\psi(0) - \frac{n}{2}$ which implies that

$|e_\psi(0) - e_\psi(1)| = \frac{n}{2}$ or $|e_\psi(0) - e_\psi(1)| \geq 2$.
Case (ii) When 'n' is odd.
Observing ψ, one can find that $e_\psi(1) = e_\psi(0) - \lfloor \frac{n}{2} \rfloor$ which shows that $|e_\psi(0) - e_\psi(1)| = \lfloor \frac{n}{2} \rfloor$ or $|e_\psi(0) - e_\psi(1)| \geq 2$.
Thus, for both the cases K_n, $n \geq 4$ is not an AEDCG (see Figure 6.15). □

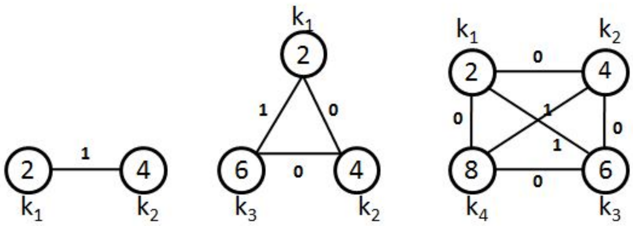

FIGURE 6.15: K_2 are K_3 admitting AEDCL and K_4 is not

Remark 6.7. For K_2 and K_3, result is obvious (see Figure 6.15).

Observation 1: If G admits an AEDCL, its supergraph need not admit AEDCL, for instance K_n is always a supergraph of a given graph on same number of nodes.
Observation 2: For a graph G admitting an AEDCL, its subgraph need not admit an AEDCL. For the sake of explanation, it is clear that C_{10} is a subgraph of W_{10} and W_{10} admits an AEDCL but C_{10} does not admit (see Figure 6.16).

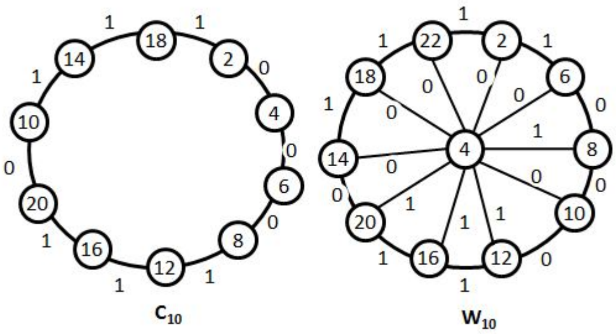

FIGURE 6.16: C_{10} is not admitting an AEDCL whereas W_{10} is admitting

Theorem 6.3.4. *$K_{m,n}$ admits an AEDCL.*

Proof. Let $V(K_{m,n}) = V_1 \cup V_2$ where $V_1 = \{x_i : 1 \leq i \leq m\}$ and $V_2 = \{y_j : 1 \leq j \leq n\}$. Consider $\psi : V(K_{m,n}) \to \{2, 4, 6, ..., 2m + 2n\}$ given by fixing $\psi(x_1) = 2$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $2 \leq i \leq m$, $\psi(y_1) = \psi(x_m) + 2$, $\psi(y_i) = \psi(y_{i-1}) + 2$; $2 \leq i \leq n$. Observe that if mn is even, then $e_\psi(0) = e_\psi(1) = \frac{mn}{2}$ and if mn is odd then $|e_\psi(0) - e_\psi(1)| = 1$, which shows that $K_{m,n}$ admits an AEDCL. □

Theorem 6.3.5. *Let G and H be isomorphic graphs. If G admits an AEDCL then H also does.*

Proof. Let G and H be isomorphic graphs with isomorphism ψ from $V(G) = \{u_1, u_2, ..., u_p\}$ to $V(H) = \{v_1, v_2, ..., v_p\}$. Let g be an AEDCL of G. If $e = u_iu_j \in E(G) \implies \psi(e = u_iu_j) \in E(H)$ for any i, j. Let $g(u_i) = r$, $g(u_j) = s$ for some $r, s \in \{2, 4, ..., 2p\}$ such that $|e_g(0) - e_g(1)| \leq 1$. Define $h : V(H) \to \{2, 4, ..., 2p\}$ such that $h(\psi(u_i)) = g(u_i)$; $1 \leq i \leq p$. Then h is a desired AEDCL of H as $|e_g(0) - e_g(1)| = |e_h(0) - e_h(1)| \leq 1$. □

Theorem 6.3.6. *All trees are AEDCG.*

Proof. Let $T^{(n)}$ denote a tree with n edges. To show that $T^{(n)}$ is an AEDCG, principle of mathematical induction is followed. Suppose $n = 2$, the $T^{(2)}$ is a path on 3 nodes which is an AEDCG. Suppose that the result holds for $n = k - 1$, i.e; $T^{(k-1)}$ is an AEDCG. Next is to show that $T^{(k)}$ is an AEDCG. Adding an edge to $T^{(k-1)}$ yields $T^{(k)}$ which is an AEDCG by Theorem 6.3.1 which completes the induction. Hence $T^{(n)}$ is an AEDCG. □

Corollary 6.3.1. *Full $n - ary$ tree admits an AEDCL, where $n = 2k, k \in \mathbb{N}$.*

Proof. Let $T_{(n,m)}$ denotes the full $n - ary$ tree having m levels. Clearly, zeroth level has one node, first level has n nodes, second level has n^2 nodes, third level has n^3 nodes and m^{th} level has n^m nodes. Define $\psi : V(T_{(n,m)}) \to \{2, 4, 6, ..., 2(n^m + n^{m-1} + n^{m-2} + ... + n + 1)\}$ such that the node of zeroth level be labeled 2. For first level, assign the labels, begining from leftmost node and proceeding to right, simultaneously from the available labels. By doing so, the last node of the first level is labeled with $2n + 2$. Similarly, for second level, the last node has label $2n^2 + 2n + 2$. Proceeding this way, one can find that the last(rightmost) node in m^{th} level has $2n^m + 2n^{m-1} + 2n^{m-2} + ... + 2n + 2$ label. Note that in every level, $e_\psi(0) = e_\psi(1)$ which means that $|e_\psi(0) - e_\psi(1)| = 0$ and hence $T_{(n,m)}$ admits an AEDCL (see Figure 6.17). □

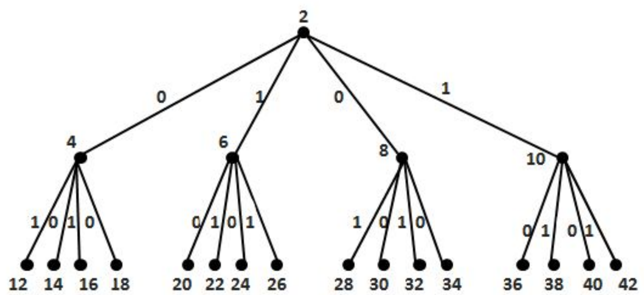

FIGURE 6.17: AEDCL of full $4 - ary$ tree with 2 levels

Corollary 6.3.2. *P_n admits an AEDCL.*

Proof. Let $V(P_n) = \{p_i : 1 \leq i \leq n\}$. Consider $\psi : V(P_n) \to \{2, 4, 6, ..., 2n\}$ defined under the below mentioned cases.
Case (i) When 'n' is even.
Let $\psi(p_1) = 2$, $\psi(p_i) = \psi(p_{i-1}) + 2$; $2 \leq i \leq \frac{n}{2}$, $\psi(p_{\frac{n}{2}+1}) = \psi(p_{\frac{n}{2}}) + 4$.
Now two subcases arise.
Subcase (i) When '$\frac{n}{2}$' is even.
Fix $\psi(p_i) = \psi(p_{i-1})+4$; $\frac{n}{2}+2 \leq i \leq \frac{n}{2}+\frac{n}{4}$, $\psi(p_{\frac{n}{2}+\frac{n}{4}+1}) = \psi(p_{\frac{n}{2}})+2$, $\psi(p_i) = \psi(p_{i-1})+4$; $\frac{n}{2} + \frac{n}{4} + 2 \leq i \leq n$. One can see that $|e_\psi(0) - e_\psi(1)| \leq 1$.
Subcase (ii) When '$\frac{n}{2}$' is odd.
Fix $\psi(p_i) = \psi(p_{i-1}) + 4$; $\frac{n}{2} + 2 \leq i \leq \frac{n}{2} + \lfloor \frac{n}{4} \rfloor$, $\psi(p_{\frac{n}{2}+\lfloor \frac{n}{4} \rfloor+1}) = \psi(p_{\frac{n}{2}}) + 2$, $\psi(p_i) = \psi(p_{i-1}) + 4$; $\frac{n}{2} + \lfloor \frac{n}{4} \rfloor + 2 \leq i \leq n$. One can see that $|e_\psi(0) - e_\psi(1)| \leq 1$.
Case (ii) When 'n' is odd.
Fix $\psi(p_1) = 2$, $\psi(p_i) = \psi(p_{i-1}) + 2$; $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $\psi(p_{\lfloor \frac{n}{2} \rfloor+1}) = \psi(p_{\lfloor \frac{n}{2} \rfloor}) + 4$, $\psi(p_i) = \psi(p_{i-1}) + 4$; $\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq k < n$, where $\psi(p_k) \leq 2n$. Next, let $\psi(p_{k+1}) = \psi(p_{\lfloor \frac{n}{2} \rfloor}) + 2$, $\psi(p_{k+2}) = \psi(p_{k+1}) + 4$, $\psi(p_i) = \psi(p_{i-1}) + 4$; $k + 3 \leq i \leq n$. An easy check shows that $|e_\psi(0) - e_\psi(1)| \leq 1$. □

Lemma 6.3.1. *C_n admits an AEDCL for all n except when $\frac{n}{2}$ is odd.*

Proof. Let $V(C_n) = \{c_i : 1 \leq i \leq n\}$. Consider $\psi : V(C_n) \to \{2, 4, 6, ..., 2n\}$ defined by the given cases.
Case (i) When 'n' is odd.
Fix $\psi(c_1) = 2$, $\psi(c_i) = \psi(c_{i-1}) + 2$; $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $\psi(c_{\lfloor \frac{n}{2} \rfloor+1}) = \psi(c_{\lfloor \frac{n}{2} \rfloor}) + 4$, $\psi(c_i) = \psi(c_{i-1}) + 4$; $\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq k < n$, where $\psi(c_k) \leq 2n$. Next, $\psi(c_{k+1}) = \psi(c_{\lfloor \frac{n}{2} \rfloor}) + 2$, $\psi(c_{k+2}) = \psi(c_{k+1}) + 4$, $\psi(c_i) = \psi(c_{i-1}) + 4$; $k + 3 \leq i \leq n$. Here, $|e_\psi(0) - e_\psi(1)| = 1$.
Case (ii) When '$\frac{n}{2}$' is even.
Fix $\psi(c_1) = 2$, $\psi(c_i) = \psi(c_{i-1})+2$; $2 \leq i \leq \frac{n}{2}$, $\psi(c_{\frac{n}{2}+1}) = \psi(c_{\frac{n}{2}})+4$, $\psi(c_i) = \psi(c_{i-1})+4$; $\frac{n}{2} + 2 \leq i \leq \frac{n}{2} + \frac{n}{4}$, $\psi(c_{\frac{n}{2}+\frac{n}{4}+1}) = \psi(c_{\frac{n}{2}}) + 2$, $\psi(c_i) = \psi(c_{i-1}) + 4$; $\frac{n}{2} + \frac{n}{4} + 2 \leq i \leq n$.
One can see that $|e_\psi(0) - e_\psi(1)| = 1$.
For both the cases, C_n is an AEDCG. □

Remark 6.8. When $\frac{n}{2}$ is odd in C_n then either $e_\psi(0) = e_\psi(1) + 2$ or $e_\psi(1) = e_\psi(0) + 2$, which means that C_n is not an AEDCL.

Definition 6.3.2. [44] "The Mycielskian of $G(V, E)$ denoted by $\mu(G)$ has $V \cup V' \cup z$ as its node set where $V' = \{v' : v \in V(G)\}$ and $E(\mu(G)) = \{uv' : uv \in E(G)\} \cup \{v'z : v' \in V'\}$."

Theorem 6.3.7. *$\mu(P_n)$ admits an AEDCL $\forall$ $n \geq 3$.*

Proof. Let $\{p_i : 1 \leq i \leq n\}$ and $\{p_ip_{i+1} : 1 \leq i \leq n-1\}$ represent respectively the node and edge set of P_n. Let $V(\mu(P_n)) = V(P_n) \cup \{p'_i : 1 \leq i \leq n\} \cup \{z\}$ and $E(\mu(P_n)) = E(P_n) \cup \{p_ip'_{i+1} : 1 \leq n-1\} \cup \{p_ip'_{i-1} : 2 \leq i \leq n\} \cup \{p'_iz : 1 \leq i \leq n\}$. Clearly, $|V(\mu(P_n))| = 2n+1$ and $|E(\mu(P_n))| = 4n-3$. Consider $\psi : V(\mu(P_n)) \rightarrow \{2, 4, ..., 2(2n+1)\}$ defined by fixing $\psi(z) = 2$, $\psi(p_1) = 4$, $\psi(p_i) = \psi(p_{i-1})+4$; $2 \leq i \leq n$, $\psi(p'_1) = 6$ and $\psi(p'_i) = \psi(p'_{i-1})+4$; $2 \leq i \leq n$. One can observe that $|e_\psi(0) - e_\psi(1)| \leq 1$, which proves the theorem (see Figure 6.18). □

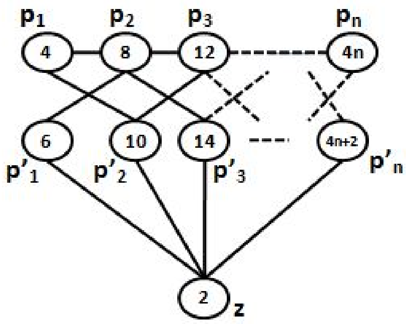

FIGURE 6.18: AEDCL of $\mu(P_n)$

Theorem 6.3.8. *$\mu(C_n)$ admits an AEDCL $\forall$ $n \geq 3$.*

Proof. Let $\{c_i : 1 \leq i \leq n\}$ and $\{c_ic_{i+1} : 1 \leq i \leq n-1\} \cup \{c_nc_1\}$ represent respectively the node and edge set of C_n. Let $V(\mu(C_n)) = V(C_n) \cup \{c'_i : 1 \leq i \leq n\} \cup \{z\}$ and $E(\mu(C_n)) = E(C_n) \cup \{c_ic'_{i+1} : 1 \leq n-1\} \cup \{c_ic'_{i-1} : 2 \leq i \leq n\} \cup \{c_1c'_n, c_nc'_1\} \cup \{c'_iz : 1 \leq i \leq n\}$. Note that $|V(\mu(C_n))| = 2n+1$ and $|E(\mu(C_n))| = 4n$. Consider $\psi : V(\mu(C_n)) \rightarrow \{2, 4, ..., 2(2n+1)\}$ defined by fixing $\psi(z) = 2$, $\psi(c_1) = 4$, $\psi(c_i) = \psi(c_{i-1})+4$; $2 \leq i \leq n$, $\psi(c'_1) = 6$ and $\psi(c'_i) = \psi(c'_{i-1}) + 4$; $2 \leq i \leq n$. Observe that $|e_\psi(0) - e_\psi(1)| \leq 1$, which proves the theorem. □

Theorem 6.3.9. *$T(P_n)$ admits an AEDCL $\forall$ $n \geq 3$.*

Proof. Let node and edge set of $T(P_n)$ be given respectively by $V(P_n) \cup \{e_i : 1 \leq i \leq n-1\}$ and $E(P_n) \cup \{p_ie_i, p_{i+1}e_i : 1 \leq i \leq n-1\} \cup \{e_ie_{i+1} : 1 \leq i \leq n-2\}$ where, $V(P_n) = \{p_i : 1 \leq i \leq n\}$ and $E(P_n) = \{e_i = p_ip_{i+1} : 1 \leq i \leq n-1\}$. One can see that $|V(T(P_n))| = 2n-1$ and $|E(T(P_n))| = 4n-5$. Consider $\psi : V(T(P_n)) \rightarrow \{2, 4, ..., 2(2n-1)\}$ defined by fixing $\psi(p_1) = 2$, $\psi(p_i) = \psi(p_{i-1}) + 4$; $2 \leq i \leq n$, $\psi(e_1) = 4$, $\psi(e_i) = \psi(e_{i-1}) + 4$; $2 \leq i \leq n-1$. Consequently, $|e_\psi(0) - e_\psi(1)| \leq 1$ (see Figure 6.19). □

Remark 6.9. $T(C_n)$ permits an AEDCL as one can define the labeling in a same way as in Theorem 6.3.9.

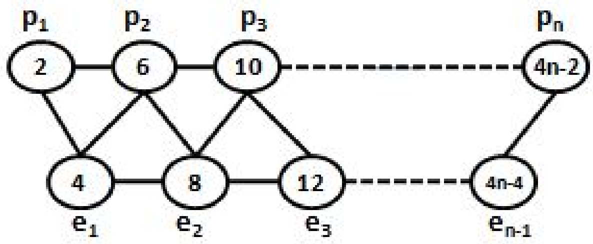

FIGURE 6.19: AEDCL of $T(P_n)$

Theorem 6.3.10. *P_n^2 admits an AEDCL $\forall\ n \geq 3$.*

Proof. Let $V(P_n) = \{p_i : 1 \leq i \leq n\}$. Consider P_n^2 having $V(P_n^2) = V(P_n)$ and $E(P_n^2) = E(P_n) \cup \{p_ip_{i+2} : 1 \leq i \leq n-2\}$. Clearly, $|V(P_n^2)| = n$ & $|E(P_n^2)| = 2n-3$. Define $\psi : V(P_n^2) \to \{2, 4, ..., 2n\}$ by fixing $\psi(p_1) = 2$, $\psi(p_i) = \psi(p_{i-1}) + 2$; $2 \leq i \leq n$. One can verify that $e_\psi(0) = e_\psi(1) + 1$ which proves that P_n^2 is an AEDCL. □

Remark 6.10. C_n^2 admits an AEDCL for even values of n and proof is similar to Theorem 6.3.10. Moreover, for odd values of n, C_n^2 does not admit an AEDCL as $|e_\psi(0) - e_\psi(1)| \geq 2$.

Lemma 6.3.2. *W_n admits an AEDCL $\forall\ n \neq 4k+3$, $k \in \mathbb{N}$.*

Proof. Let $V(W_n) = \{x_0, x_i : 1 \leq i \leq n\}$ and $E(W_n) = \{x_0x_i, x_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{x_nx_1\}$. Consider $\psi : V(W_n) \to \{2, 4, 6, ..., 2n+2\}$. Now the given cases arise.
Case (i) If $n = 4k$.
Fix $\psi(x_0) = 2$, $\psi(x_1) = 4, \psi(x_i) = \psi(x_{i-1}) + 2$; $2 \leq i \leq \frac{n}{2} - 1$, $\psi(x_{\frac{n}{2}}) = \psi(x_{\frac{n}{2}-1}) + 4$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $\frac{n}{2} + 1 \leq i \leq k < n$, such that $\psi(x_k) \leq 2n+2$. Next, $\psi(x_{k+1}) = \psi(x_{\frac{n}{2}-1}) + 2$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $k+2 \leq i \leq n$. See that $|e_\psi(0) - e_\psi(1)| = 0$.
Case (ii) If $n = 4k+2$.
Fix $\psi(x_0) = 4$, $\psi(x_1) = 2$, $\psi(x_2) = 6$, $\psi(x_i) = \psi(x_{i-1}) + 2$; $3 \leq i \leq \frac{n}{2}$, $\psi(x_{\frac{n}{2}+1}) = \psi(x_{\frac{n}{2}}) + 4$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $\frac{n}{2} + 2 \leq i \leq k < n$, such that $\psi(x_k) \leq 2n+2$. Next, fix $\psi(x_{k+1}) = \psi(x_{\frac{n}{2}}) + 2$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $k+2 \leq i \leq n$. In this case also, $|e_\psi(0) - e_\psi(1)| = 0$.
Case (iii) If $n = 4k+1$.
Fix $\psi(x_0) = 2$, $\psi(x_1) = 4, \psi(x_i) = \psi(x_{i-1}) + 2$; $2 \leq i \leq \frac{n-1}{2}$, $\psi(x_{\frac{n+1}{2}}) = \psi(x_{\frac{n-1}{2}}) + 4$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $\frac{n+1}{2} + 1 \leq i \leq k < n$, such that $\psi(x_k) \leq 2n+2$. Next, $\psi(x_{k+1}) = \psi(x_{\frac{n-1}{2}}) + 2$, $\psi(x_i) = \psi(x_{i-1}) + 4$; $k+2 \leq i \leq n$. Observe that $|e_\psi(0) - e_\psi(1)| = 0$.
Hence, W_n is an AEDCG. □

Remark 6.11. If $n = 4k+3$ in W_n, then either $e_\psi(0) = e_\psi(1)+2$ or $e_\psi(1) = e_\psi(0)+2$, which means that W_n is not an AEDCG.

Theorem 6.3.11. *If $G(p,q)$ is an AEDCG, then $G \odot \bar{K}_t$ admits an AEDCL for $t \equiv 0(mod\ 2)$.*

Proof. Given $G(p,q)$ is an AEDCG with $V(G) = \{u_i^* : 1 \le i \le p\}$, therefore $\exists$ a labeling function $\psi : V(G) \to \{2,4,6,...,2p\}$ on G such that $|e_\psi(0) - e_\psi(1)| \le 1$. Given $t \equiv 0(mod\ 2)$, fix $t = 2m$. Consider $G \odot \bar{K}_{2m}$ with $V(G \odot \bar{K}_{2m}) = V(G) \cup \{k_j^{(i)} : 1 \le i \le p, 1 \le j \le 2m\}$ and $E(G \odot \bar{K}_{2m}) = E(G) \cup \{u_i^* k_j^{(i)} : 1 \le i \le p, 1 \le j \le 2m\}$. Consider $f : V(G \odot \bar{K}_{2m}) \to \{2,4,6,...,2p,2p+2,...,2p+2p(2m)\}$ defined as here. Let $f(u_i^*) = \psi(u_i^*)$; $1 \le i \le p$. Assign $\{2p+2, 2p+4, ..., 2p+2p(2m)\}$ labels simultaneously to unlabeled nodes, begining with first copy of $\bar{K}_{2m}$ that is attached to u_1^* and then slowly proceeding to the right most copy, i.e; the one attached with u_p^*. Here are the following observations.
(i) If q is even, then $e_\psi(0) = e_\psi(1)$ and pendant nodes that appear at each u_i^* yields equal number of edges with labels 1 and 0. Thus, $e_f(0) = e_f(1)$.
(ii) If q is odd, then either $e_\psi(0) = e_\psi(1)+1$ or $e_\psi(1) = e_\psi(0)+1$. But pendant edges at each u_i^* yield same count of edges having labels 1 and 0 showing that $|e_f(0) - e_f(1)| = 1$, which proves that $G \odot \bar{K}_{2m}$ is an AEDCG (see Figure 6.20). □

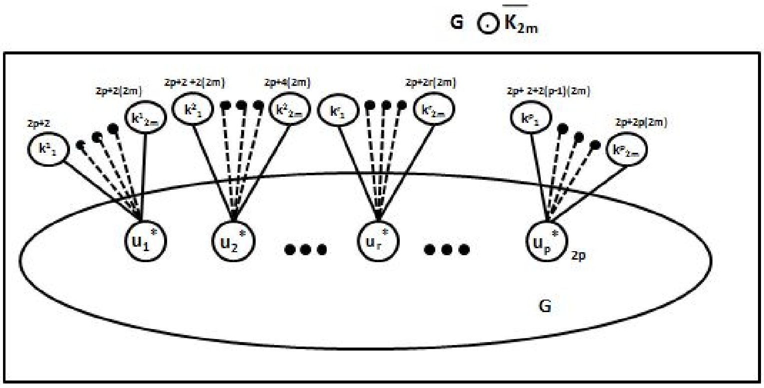

FIGURE 6.20: AEDCL of $G \odot \bar{K}_{2m}$

Corollary 6.3.3. *$P_n \odot \bar{K}_{2m}$ is an AEDCG.*

Proof. Follows directly from Corollary 6.3.2 and Theorem 6.3.11. □

Corollary 6.3.4. *$C_n \odot \bar{K}_{2m}$, $n \neq 4k+2$, $k \in \mathbb{N}$ is an AEDCG.*

Proof. The proof is evident from Lemma 6.3.1 and Theorem 6.3.11. □

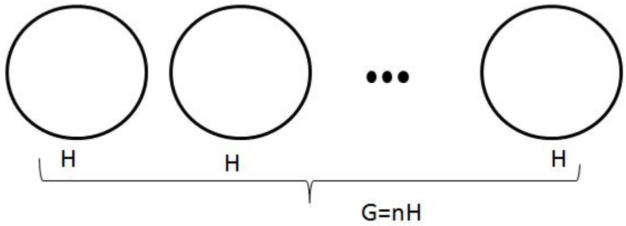

FIGURE 6.21: Disjoint union of 'n'-copies of H

Theorem 6.3.12. *The disjoint-union of 'n'-copies of $H(p, q)$ admits an AEDCL, where H is an AEDCG with q even.*

Proof. Consider $H(p, q)$, which is an AEDCG, with labeling h. Let $V(H) = \{v_1, v_2, ..., v_p\}$. Let $G = nH$ as shown in Figure 6.21 with $V(G) = \{v_j^i : 1 \leq j \leq p, 1 \leq i \leq n\}$. Define a function $\psi : V(G) \to \{2, 4, ..., 2np\}$ as follows. Let $\psi(v_j^1) = h(v_j^1)$; $1 \leq j \leq p$, $\psi(v_j^2) = \psi(v_j^1) + 2p$; $1 \leq j \leq p$, $\psi(v_j^3) = \psi(v_j^2) + 2p$; $1 \leq j \leq p$. Proceeding this way, $\psi(v_j^n) = \psi(v_j^{n-1}) + 2p$; $1 \leq j \leq p$. It can be seen that $e_\psi(0) = e_\psi(1)$ which establishes that G is an AEDCG. □

Corollary 6.3.5. *Let G be an AEDCG of even size and G^* be a copy of G. Then $G \cup G^*$ is also an AEDCG.*

Proof. Since G with $V(G) = \{u_1, u_2, ..., u_n\}$ is an AEDCG of even size, with labeling f, therefore $e_f(0) = e_f(1)$. Let G^* be a copy of G with $V(G^*) = \{u'_1, u'_2, ..., u'_n\}$. Let $H = G \cup G^*$, define labeling ψ on $V(H)$ by fixing $\psi(u_i) = f(u_i)$; $1 \leq i \leq n$ and $\psi(u'_i) = \psi(u_i) + 2n$; $1 \leq i \leq n$. This way, $e_\psi(0) = e_\psi(1)$, hence $G \cup G^*$ is an AEDCG. □

Theorem 6.3.13. *Let $G(p, q)$ be an AEDCG with q even. Then $G + G$ is also an AEDCG.*

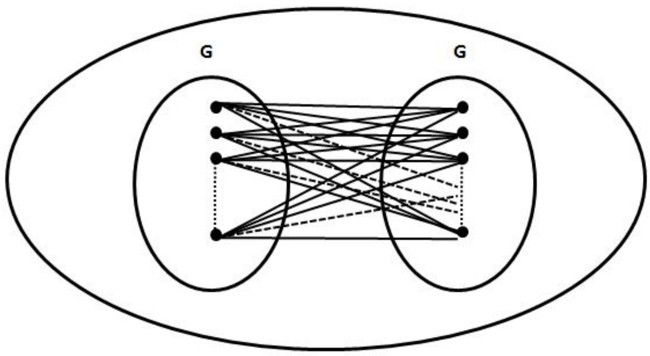

FIGURE 6.22: $G + G$

Proof. The proof is evident from Theorem 6.3.4 and Corollary 6.3.5 (see Figure 6.22). □

Theorem 6.3.14. *Ladder graph $L_n = P_n \times P_2$ is an AEDCG.*

Proof. Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$. Here, $|V(L_n)| = 2n$ and $|E(L_n)| = 3n-2$. Labeling is done by considering a $\psi : V(L_n) \to \{2, 4, 6, ..., 4n\}$ defined for under mentioned conditions.

Case (i) When $n = 4k$, $k \in \mathbb{N}$.

Let $\psi(u_1) = 2$, $\psi(u_i) = \psi(u_{i-1})+2$; $2 \leq i \leq n-1$, $\psi(u_n) = \psi(u_{n-1})+4$, $\psi(v_1) = 4n-2$, $\psi(v_i) = \psi(v_{i-1}) - 4$; $2 \leq i \leq \frac{n}{2} - 1$, $\psi(v_{\frac{n}{2}}) = 4n$, $\psi(v_i) = \psi(v_{i-1}) - 4$; $\frac{n}{2} + 1 \leq i \leq n$. Evidently, $e_\psi(0) = e_\psi(1)$.

Case (ii) When $n = 4k - 1$, $k \in \mathbb{N}$.

Let $\psi(u_1) = 2$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2 \leq i \leq n$, $\psi(v_1) = 4n - 2$, $\psi(v_i) = \psi(v_{i-1}) - 4$; $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $\psi(v_{\lceil \frac{n}{2} \rceil}) = 4n$, $\psi(v_i) = \psi(v_{i-1}) - 4$; $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$. One can verify that $e_\psi(0) = \frac{3n-1}{2}$ and $e_\psi(1) = \frac{3n-3}{2}$.

Case (iii) When $n = 4k - 3$, $k \in \mathbb{N} - \{1\}$.

Let $\psi(u_1) = 2$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2 \leq i \leq n - 2$, $\psi(u_{n-1}) = \psi(u_{n-2}) + 4$, $\psi(u_n) = \psi(u_{n-1}) + 4$, $\psi(v_1) = 4n - 2$, $\psi(v_i) = \psi(v_{i-1}) - 4$; $2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, $f(v_{\lfloor \frac{n}{2} \rfloor}) = 4n$, $\psi(v_i) = \psi(v_{i-1})-4$; $\lfloor \frac{n}{2} \rfloor+1 \leq i \leq n$. It can be seen that $e_\psi(0) = \frac{3n-3}{2}$ and $e_\psi(1) = \frac{3n-1}{2}$.

Case (iv) When $n = 4k - 2$, $k \in \mathbb{N} - \{1\}$.

Let $\psi(u_1) = 2$, $\psi(u_i) = \psi(u_{i-1}) + 2$; $2 \leq i \leq n - 2$, $\psi(u_{n-1}) = \psi(u_{n-2}) + 4$, $\psi(u_n) = \psi(u_{n-1}) + 4$, $\psi(v_1) = 4n$, $\psi(v_i) = \psi(v_{i-1}) - 4$; $2 \leq i \leq \frac{n}{2} - 1$, $\psi(v_{\frac{n}{2}}) = 4n - 2$, $\psi(v_i) = \psi(v_{i-1}) - 4$; $\frac{n}{2} + 1 \leq i \leq n$. Here, $e_\psi(0) = e_\psi(1)$.

Thus, in all the cases, $|e_\psi(0) - e_\psi(1)| \leq 1$ which proves that L_n is an AEDCG. □

Theorem 6.3.15. *Triangular ladder TL_n is an AEDCG.*

Proof. Let $V(TL_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(TL_n) = \{u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{v_iu_{i+1} : 1 \leq i \leq n - 1\}$. Consider $\psi : V(TL_n) \to \{2, 4, 6, ..., 4n\}$ defined by fixing $\psi(u_1) = 2$, $\psi(u_i) = \psi(u_{i-1})+4$; $2 \leq i \leq n$, $\psi(v_1) = 4$, $\psi(v_i) = \psi(v_{i-1})+4$; $2 \leq i \leq n$. It is noted that $|e_\psi(0)-e_\psi(1)| \leq 1$ which implies that TL_n is an AEDCG. □

Theorem 6.3.16. *$P_n \times P_n$ admits AEDCL.*

Proof. Let $V(P_n \times P_n) = \{v_i^{(j)} : 1 \leq i \leq n, 1 \leq j \leq n\}$, where $v_i^{(j)}$ represents the i^{th} node of j^{th} copy. Clearly, $|V(P_n \times P_n)| = n^2$ and $|E(P_n \times P_n)| = 2n^2 - 2n$. Consider $\psi : V(P_n \times P_n) \to \{2, 4, 6, ..., 2n^2\}$ defined by the given cases.

Case (i) When‘ n’ is even.

Let $\psi(v_1^{(1)}) = 2$, $\psi(v_i^{(1)}) = \psi(v_{i-1}^{(1)}) + 4$; $2 \leq i \leq n$, $\psi(v_1^{(2)}) = 4$, $\psi(v_i^{(2)}) = \psi(v_{i-1}^{(2)}) + 4$; $2 \leq i \leq n$, $\psi(v_1^{(3)}) = \psi(v_n^{(1)}) + 4$, $\psi(v_i^{(3)}) = \psi(v_{i-1}^{(3)}) + 4$; $2 \leq i \leq n$, $\psi(v_1^{(4)}) = \psi(v_n^{(2)}) + 4$,

$\psi(v_i^{(4)}) = \psi(v_{i-1}^{(4)}) + 4;\ 2 \leq i \leq n$, ..., ..., ..., $\psi(v_1^{(n-1)}) = \psi(v_n^{(n-3)}) + 4$, $\psi(v_i^{(n-1)}) = \psi(v_{i-1}^{(n-1)}) + 4;\ 2 \leq i \leq n$, $\psi(v_1^{(n)}) = \psi(v_n^{(n-2)}) + 4$ and $\psi(v_i^{(n)}) = \psi(v_{i-1}^{(n)}) + 4;\ 2 \leq i \leq n$.
It can be verified that $e_\psi(0) = e_\psi(1) = n^2 - n$.

Case (ii) When 'n' is odd.

For first $n-1$ rows, follow the pattern of *Case* (i). For last row, proceed with the remaining labels as per Corollary 6.3.2. In this case, $|e_\psi(0) - e_\psi(1)| \leq 1$.

Thus, $P_n \times P_n$ is an AEDCG (see Figure 6.23). □

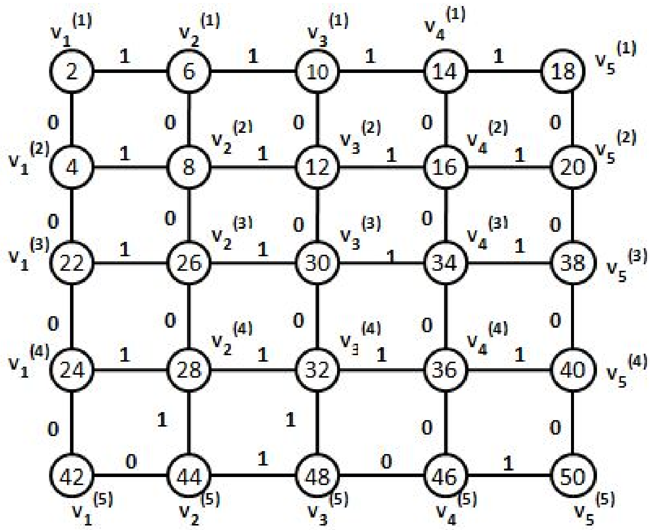

FIGURE 6.23: AEDCL of $P_5 \times P_5$

Definition 6.3.3. [38] The stack S_k of books is a union of $k-copies$ of triangular book $B_5 = K_{1,1,5}$, joined in a way that their spines form a path.

Lemma 6.3.3. *$K_{1,1,n}$ admits an AEDCG.*

Proof. Let $K_{1,1,n}$ with node set $\{x_0, x_0'\} \cup \{x_i : 1 \leq i \leq n\}$ and edge set $\{x_0x_i, x_0x_0', x_0'x_i : 1 \leq i \leq n\}$. Consider $\psi : V(K_{1,1,n}) \to \{2, 4, ..., 2n+4\}$ defined by fixing $\psi(x_0) = 2$ and $\psi(x_0') = 4$ and allocate the unused labels to remaining nodes in any fashion. □

Theorem 6.3.17. *S_k admits an AEDCG.*

Proof. Let $V(S_k) = V(P_{k+1}) \cup \{v_i^{(j)} : 1 \leq i \leq 5, 1 \leq j \leq k\}$ and $E(S_k) = E(P_{k+1}) \cup \{p_jv_i^{(j)}, p_{j+1}v_i^{(j)} : 1 \leq i \leq 5, 1 \leq j \leq k\}$ where $v_i^{(j)}$ represents the i^{th} node of j^{th} copy. Clearly, $|V(S_k)| = 6k+1$ and $|E(S_k)| = 11k$. Consider $\psi : V(S_k) \to \{2, 4, 6, ..., 2(6k+1)\}$. First label the $k+1$ nodes of P_{k+1} by using Corollary 6.3.2. This way $\{2, 4, ..., 2k+2\}$ labels are consumed. Now start assigning the remaining labels simultaneously, begining with the first node of degree 2 of first copy of B_5 and proceeding to the last node of the last copy. Clearly, $|e_\psi(0) - e_\psi(1)| \leq 1$ which shows that S_k is an AEDCG (see Figure 6.24).

□

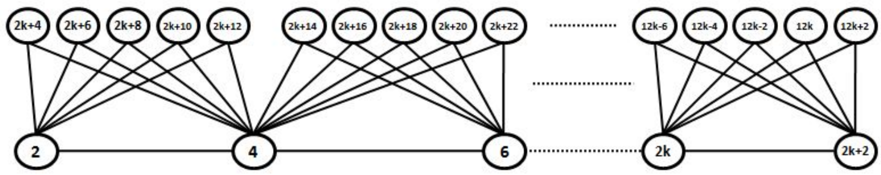

FIGURE 6.24: AEDCL of S_k

Open Problems

Since the two new variants are introduced in this chapter, a lot can be done towards characterization of these labelings. Thus, the following open problems are proposed.

1. To investigate DDCL and AEDCL of other graph families for different graph operations.

2. To investigate DDCL and AEDCL for real life apllications.

6.4 Conclusion

In this chapter, new variants of DCL, namely, double divisor cordial labeling and average even divisor cordial labeling are introduced and have been investigated for various classes of graphs. Similarly, other interesting variants of these kind can be introduced and studied to enrich the discipline. Moreover, one can investigate the necessary and sufficient condition for a graph to admit DCL, DDCL and AEDCL.

Conclusion

In this thesis, "prime cordial labeling" (PCL) and "divisor cordial labeling" (DCL) of graphs are discussed for various classes of graphs such as "path, cycle, wheel, helm, flower, fan, gear, double fan, star, bistar, regular graph, lilly graph, some classes of planar graphs" in the context of some graph operations along with some notable general results. Specifically, the PCL and DCL of some families of graphs in the context of graph operations namely, "corona, duplication of a node by a node, duplication of a node by an edge, duplication of edge by a node, subdivision, degree splitting, extension of a node" etc. are derived. The different graphs and graph operations considered in the present work are due to their utility and applications nowadays. Motivated by some of the variants of DCL, two more variants namely "double divisor cordial labeling" (DDCL) and "average even divisor cordial labeling" (AEDCL) are introduced and some remarkable results are also obtained. Moreover, a few interesting conjectures and open problems are also formulated specifically "establishing the DCL for the given graph is NP-hard". A complete characterization of PCL and DCL is yet to be done, but this thesis may serve as a path in achieving the characterization of PCL and DCL either partially or fully. One can also explore PCL, DCL, DDCL and AEDCL for other classes of graphs and graph operations which are not discussed in the thesis and this is for the future work. Though, graph labeling finds applications in numerous fields, yet discovering the exclusive applications of PCL, DCL, DDCL and AEDCL in different domains is also an interesting and open area of research.

Publications and Presentations

Papers Published from the Thesis

1. Vishally Sharma and A. Parthiban, On recent advances in divisor cordial labeling of graphs, Mathematics and Statistics, 10 (2022) 140-144. (**Scopus**)
2. Vishally Sharma and A. Parthiban, On divisor cordial labeling of certain classes of planar graphs, Journal of Algebraic Statistics, 13 (2022) 1421-1425. (**WoS**)
3. Vishally Sharma and A. Parthiban, On double divisor cordial labeling of star related graphs, Advances and Applications in Mathematical Sciences, 21 (2022) 4389-4398. (**WoS**)
4. Vishally Sharma and A. Parthiban, Some recent advances in prime cordial labeling of graphs, Turkish Online Journal of Qualitative Inquiry, 12 (2021) 833-842. (**Scopus**)
5. Vishally Sharma and A. Parthiban, Double divisor cordial labeling of graphs, Journal of Physics: Conference Series, 2267 (2022) 012026. (**Scopus**)
6. A. Parthiban and Vishally Sharma, Some results on prime cordial labeling of lilly graphs, Journal of Physics: Conference Series, 1831 (2021) 012035. (**Scopus**)
7. A. Parthiban and Vishally Sharma, A comprehensive survey on prime cordial and divisor cordial labeling of graphs, Journal of Physics: Conference Series, 1531 (2020) 012074. (**Scopus**)

Accepted Papers from the Thesis

8. Vishally Sharma and A. Parthiban, On average even divisor cordial labeling of graphs, TWMS Journal of Applied and Engineering Mathematics, (2022). (**SCOPUS**)

Papers Communicated from the Thesis

9. Vishally Sharma and A. Parthiban, Some general results on prime cordial labeling of graphs, Topology and its Applications, (2022). (**SCIE**)
10. A. Parthiban and Vishally Sharma, Some results on divisor cordial labeling of graphs, Punjab University Journal of Mathematics, (2021). (**WoS**)

Papers Presented in Conferences

1. Vishally Sharma and A. Parthiban, On divisor cordial labeling of certain classes of planar graphs, International Conference on Recent Trends in Applied Mathematics (ICRTAM-2022), March 3-4, 2022, Loyola College, Chennai, India.

2. Vishally Sharma and A. Parthiban, Some recent advances in prime cordial labeling of graphs, Integrated Approach in Science & Technology for Sustainable Future (IAST-SF 2022), February 27-28, 2022, Maulana Azad Memorial College, Jammu, India.

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4. Vishally Sharma and A. Parthiban, Double divisor cordial labeling of graphs, International Conference on Recent Advances in Fundamental and Applied Sciences (RAFAS-2021), June 25-26, 2021, Lovely Professional University, Phagwara, Punjab.

5. A. Parthiban and Vishally Sharma, Some results on prime cordial labeling of lilly graphs, International Conference on Robotics and Artificial Intelligence, December 28-29, 2020, Advanced Computing Research Society Chennai,India.

6. A. Parthiban and Vishally Sharma, Some new results on divisor cordial labeling of graphs, 10th International Conference on Soft Computing for Problem Solving (SocPros 2020), December 18-20, 2020, IIT Indore and Soft Computing Research Society, India.

7. A. Parthiban and Vishally Sharma, A comprehensive survey on prime cordial and divisor cordial labeling of graphs, International Conference on Recent Advances in Fundamental and Applied Sciences (RAFAS-2019), November 5-6, 2019, Lovely Professional University, Phagwara, Punjab.

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