THE STUDY OF SITUATION BASED FUZZY MEMBERSHIP FUNCTIONS TO IMPROVE THE FUZZY CLUSTERING AND OPTIMIZATION TECHNIQUES

A Thesis

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By

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Supervised By

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PUNJAB

2021

I would like to dedicate this thesis to God, my loving parents, teachers and friends, who make me whole.

"Education is the vaccine of violence."

—Edward James Olmos

Declaration of Authorship

I, Rakesh Kumar, declare that this thesis titled, ‘The Study of Situation Based Fuzzy Membership Functions to Refine the Fuzzy Optimization Techniques Using Improved Clustering Algorithms’ and the work presented in it are my own. I confirm that:

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Certificate

The thesis titled "The Study of Situation Based Fuzzy Membership Functions to Refine the Fuzzy Optimization Techniques Using Improved Clustering Algorithms," submitted by Mr. Rakesh Kumar for the award of the degree of Doctor of Philosophy, has been carried out under my supervision at the Department of Mathematics, Lovely Professional University, Punjab, India. The matter presented in this thesis is original and has not been submitted to any other University or Institute for an award of any degree or diploma. The work is comprehensive, complete, and fit for evaluation.

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Abstract

Operational Research (OR) is the study of how complex problems in industrial engineering can be mathematically modeled and analyzed to acquire insight into possible solutions. Optimization, dynamic programming, Markov models, simulation, and data processing are among the most commonly used solutions. Optimization is concerned with the maximization or minimization of an arbitrary function subject to constraints, which often occurs in complex high-dimensional problems. The facility location problem is the branch of the Operational Research. The facility location problem (FLP) is concerned with the optimal distribution of facilities within a given area. The FLP has perturbed both the public and private sectors. The prime concern in FLP is the number of locations to be opened/located.

The second major problem is to detect the optimal position after the discovery of the number of locations. It is also one of the most crucial decisions that service providers must take into account. The resulting challenges for public or private businesses are to provide goods or services to their respective consumers within an acceptable time and demand spectrum. Besides, most businesses are primarily aimed at maximizing revenues while minimizing overall costs.

In a real-world scenario, there might be the capacity to know about the product's performance and features. As a result, the optimization of objective functions continuously varies. This thesis is concerned with these types of circumstances and creating an elegant procedure for resolving them. At first, fuzzy cluster technology relied on different distances to determine the number of clusters through fuzzy equivalence clustering. Then, we used the newly proposed clustering algorithms based on the suggested metric space, known as an alternative generalized hard c-mean (AGHCM) and an alternative generalized fuzzy c-mean (AGHFC), to find the best place for the facility. Following that, ideal locations have been identified, and fuzzy linear programming problems (FLLPs) for optimizing the objective function using a

different fuzzy number have been suggested. We aim to investigate both the feasibility and usefulness of the method through a real/artificial data set.

Keywords: *Facility location problem, optimization, Metric space, Fuzzy numbers, Fuzzy clustering, and fuzzy linear programming problem*

Thesis Layout

The thesis is divided into the following chapters.

Chapter 1: Introduction

This chapter discusses clustering definitions and types of clustering. Discuss the two most frequently used clustering algorithms, such as k-means clustering and fuzzy C-mean clustering algorithm. We also discussed the fundamental concepts of distance, fuzzy sets, and their components. Additionally, this chapter discusses the background of fuzzy clustering and fuzzy linear programming. Finally, clarify the study gap and the thesis's primary objectives.

Chapter 2: Fuzzy Equivalence Relation Via Different Distance Measures and Its Utilizations

This chapter aims to classify the performance grades of binders for NCHRP 90-07 using fuzzy equivalence clustering via Minkowski, Mahalanobis, Cosine, Chebychev, and Correlation distance function. The performances of binders were graded in precise and equal stiffness temperatures at three different parameters. The five distance functions, namely Minkowski $(w = 2)$, Mahalanobis, Cosine, Chebychev, and Correlation, are successfully applied in the clustering methodology to achieve a better separation analysis. The clusters are discovered by all five distances and distinguished for a suitable value of membership grade. We also include a theoretical comparison between the clustering performances by these distances. The Mahalanobis distance function trailed the first time in the equivalence fuzzy clustering methodology and accomplished the desirable objectives. The core effectuations of Mahalanobis and

Chebychev distance over the other four distances on the clustering performance of binders are investigated.

Chapter 3: An Effective Generalized Exponential Metric Space Approach for C-Mean Clustering Analyzing

Wu and Yang [1] suggested substituting the actual Euclidean distance in c-means algorithms, with the Gaussian distance-dependent function, by AHCM and AFCM clustering algorithms. While it was more robust than the Euclidean distance, Zhang and Chen [2] have shown in this comment that Wu and Yang's distance is not a metric. In certain instances, the enhanced metric distance suggested by Zhang and Chen does not consider the consequence of clustering centroid as predicted due to the substantial value of b. In this chapter, we proposed a generalized new distance function by replacing the exponential constant ($\boldsymbol{e}$) with the arbitrary constant ($\boldsymbol{a}$). The initial criteria for new metrics are more stringent and precise, based on metric properties and experimental results. The focus is fogged upon clustering and developing new clustering algorithms. These algorithms are called the alternative generalized hard c-mean (AGHCM) and alternative generalized fuzzy c-mean. These alternative generalized c-mean clustering forms are faster and more robust than the alternative c-mean and other competitive algorithms. Experiments are carried out using two-and high-dimensional data such as Diamond data collection and Iris real-life data. The results rely on the demonstration of the robust simplicity and efficacy of the proposed algorithms. Furthermore, computational complexity is assessed.

Chapter 4: A Novel Approach for Fuzzy Linear Programming Using Situational Based Composite Triangular Number

In this chapter, we proposed a novel approach for fuzzy linear programming using situational-based composite triangular numbers. This model has been suggested to deal with probabilistic increment ε_j in one direction and probabilistic decrement ε_i in other direction in the basic availability β_i of classical optimization and analyzing the result with targeted membership grade. To validate the models with real-time phenomena, the Production cost data of Rail Coach Factory (RCF) Kapurthala has been taken.

Chapter 5: A Novel Approach for Fuzzy Linear Programming Using the Situational Based Trapezoidal Number

In the previous chapter, we proposed FLLP through the composite triangular number. In this chapter, the comparative interpretation of optimization and modeling of the production cost through trapezoidal FLPP is proposed and describes different incertitude situations and evolved the realistic models to reduce the production cost.

Chapter 6: An Advanced Optimization Technique for Smart Production Using $\alpha-$Cut Based Quadrilateral Fuzzy Number

In the design phase of a new smart product, production costs are unpredictable due to location, transport, and engineering design. In these situations, consequently, cost optimization becomes ambiguous. This chapter presents a methodology to obtain the optimization through fuzzy linear programming (FPL) in which fuzzy numbers signify the right-side parameters. The comparative investigation of modeling and optimizing creation cost through a new $\alpha-$cut based quadrilateral fuzzy number is proposed to solve the fuzzy linear programming and the basic operations on the proposed number. Due to the probabilistic increase and decrease in the accessibility of the various constraints, the actual expected total cost fluctuates. In this respect, a unique situation of instability is incorporated, and reasonable models to reduce the cost of eradication in the creation process are presented. The main endeavour is to look at the credibility of optimized cost employing the $\alpha-$cut based quadrilateral FLLP models, and the outcome is contrasted with its augmentation. The least lower, lower, upper, and most upper bounds are computed for each situation, and then systems of optimized FLLP are constructed. The credibility of quadrilateral FLLP concerning all situations is obtained and using this membership grade, the minimum and greatest minimum cost are illustrated.

Chapter 7: The Combined Study of Improved Fuzzy Optimization Techniques with the Analysis of the Upgraded Facility Location Center for the Covid-19 Vaccine by Fuzzy Clustering Algorithms

This chapter combines improved fuzzy optimization techniques with the analysis of upgraded cluster centers by fuzzy clustering algorithms. A smart mechanism for handling such a situation has been designed in this chapter. First, the fuzzy cluster technology offers such clustered locations so that the distance to different destinations between distribution centers is minimal. Subsequently, the ideal locations have been defined, and the fuzzy linear programming problems (FLLPs) are proposed via the composite fuzzy triangular number to calculate the highest possible distribution of the products so that transport costs can be reduced. We report on experimental studies by taking artificial data from the current warehouse to prove feasibility and showing that the proposed solution is applicable.

Chapter 8: Conclusion and Future Directions

This chapter concludes the various approaches already discussed and introduces new topics for the future. In terms of another group for fuzzy linear programming problems and clustering, this would put forward future aspirations for the area of optimization. It summarizes the inputs and identifies the areas that need attention in the near future.

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Chapter 1

Introduction

"The significant problems we have cannot be solved at the same level of thinking with which we created them."

— *Albert Einstein*

1.1 Clustering

Clustering is a data analysis method used regularly in strategy formulation, market and business system planning. Partition of commodities is a conventional issue in inventory control and management. In most industries, there are different types of materials and components of machines or other apparatus to be managed to achieve the desired goal. An expert idea to enrich the efficiency of material management is to sort different materials into groups. Clustering helps to detect normal thresholds in the data. It is a multidisciplinary analysis branch and pattern recognition unsupervised learning. Clusters are most commonly used in numerous fields of science, such as machine learning, pattern recognition, optimization, image segmentation, taxonomy, medicine, geology, industry, engineering, etc. [3][4][5][6].

1.1.1 Definitions

Clustering is a collection of data features that are somewhat close among attribute values or separate from certain groups. Unsupervised learning to discover the collection of related data items by matching characteristics is known as clustering. As several definitions have been proposed in the past, there are a plethora of definitions to pick from [7] [8] [9]. These definitions are based on identical, comparable, or specific clusters and are mainly of an ambiguous, circular sort, as indicated in [9]. It shows how difficult it is to come up with a widely agreed definition for the word cluster. Several working definitions of a cluster (see Figure 1.1) are commonly used, the most important of which are discussed below.

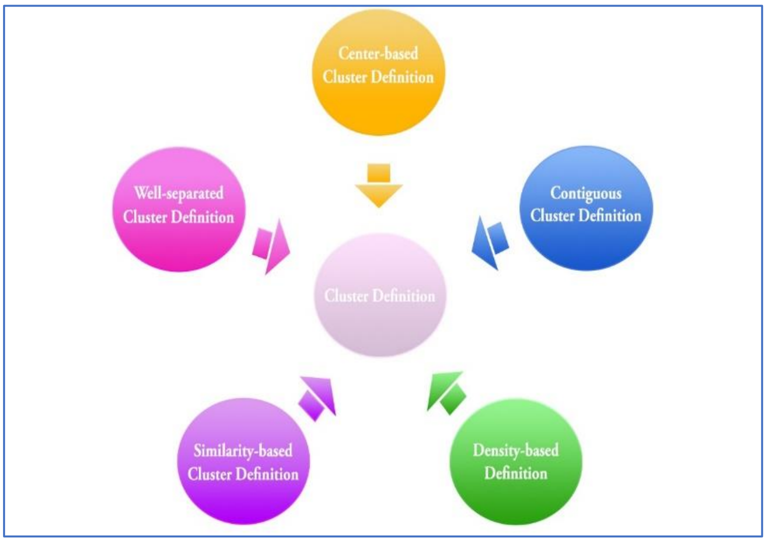


Figure 1.1: Different type of the clustering

1.1.1.1 Well-Separated Cluster Definition

A cluster is a set of identical entities to one another than any other object that is not in a cluster. A threshold is often used to classify points (see Figure 1.2) relatively close to one another. However, a certain point in datasets on the precipice of a cluster might be more similar to objects in another cluster than objects from within the cluster.

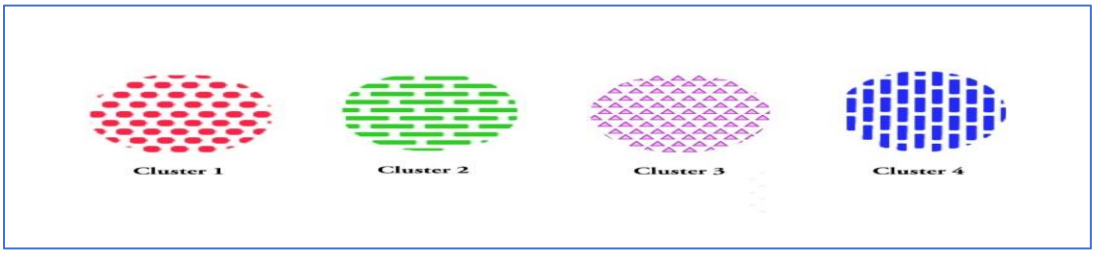


Figure 1.2: Four clusters of two-dimensional points well-separated

1.1.1.2 Center-Based Cluster Definition

Clusters are groups of objects that are more identical (closer) to the “center” of the cluster than to any other cluster in the group (see Figure 1.3). The centroid of a cluster is always located in the middle of the cluster.

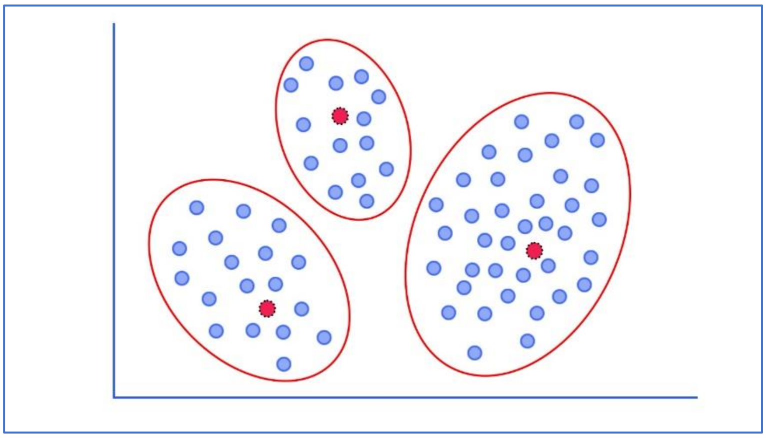


Figure 1.3: Centre-based Cluster

1.1.1.3 Contiguous Cluster Definition

A cluster is a set of points in which the proximity of one point to another is not a characteristic of the cluster (see Figure 1.4).

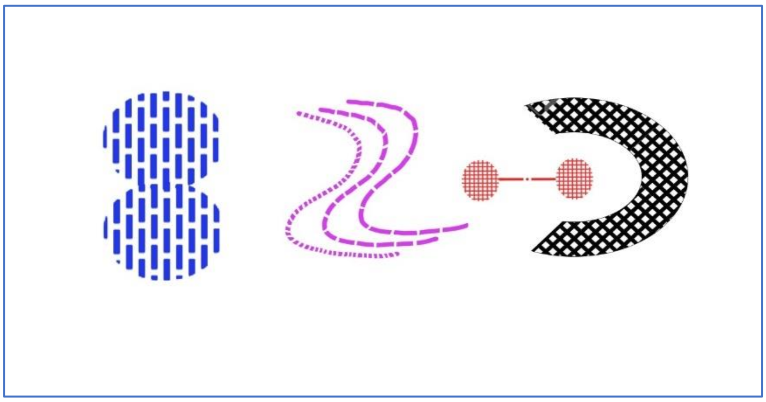


Figure 1.4: Contiguous Cluster

1.1.1.4 Density-Based Cluster Definition

Unsupervised learning approaches that identify recognizable classes or clusters (see Figure 1.5).in the data is known as density-based clustering. The concept that clusters in data space are areas of high point density isolated from several such clusters by areas of low point density. The dividing regions of low point density usually reflect noise/outliers.

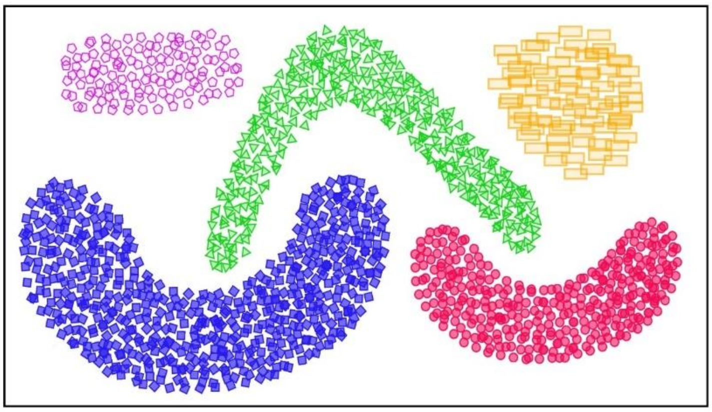


Figure 1.5: Density-based Cluster

1.1.1.5 Similarity-Based Cluster Definition

A cluster is a group of "similar" objects, and objects are not similar (see Figure 1.6) in any other cluster. Alternatively, a cluster may be described as a collection of points that form a zone with a standardized characteristic, such as density or structure.

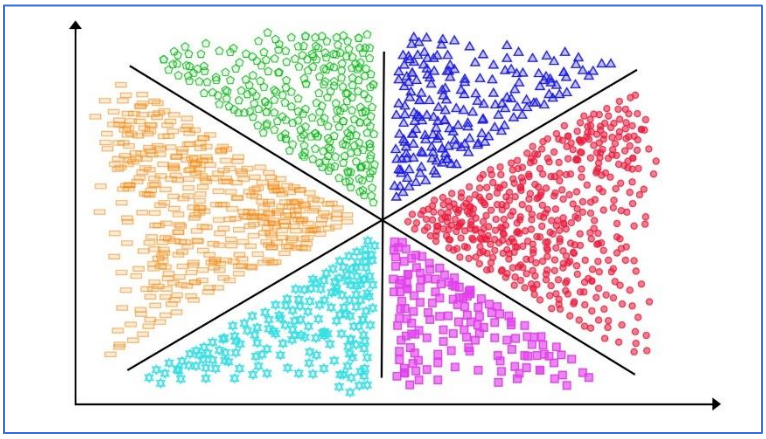


Figure 1.6: Similarity-based Cluster

1.2 Distance Measures

The aim of determining the distance between clusters is necessary to differentiate them throughout the data set. The various distances have their advantages, but not a single one is suitable for all clustering problems. In Mathematics, the phenomenon of a metric space is fundamental and significant. Various metric functions construct distinct metric spaces. In pattern recognition and machine learning, distance measurement and contrast between sample pairs play a very significant role [10][11][12][13][14][15]. With the assistance of sophisticated numerical optimization, we can acquire discriminatory characteristics through a rational description of distance function and decide whether two samples belong to the same class. In this perspective, the approaches of distance metric learning and dimensional reduction seek to learn high-level semantic distances where identical input objects are projected to close points, while distinct objects are differentiated from each other [16]. Distance metrics have been efficiently used for vast scenarios, including image selection, visual monitoring, and prototypes classification [17][18][19].

1.2.1 Minkowski Distance

Let X be a universal space and $X_{i\,k}$ and $X_{j\,k} \in X$ then the Minkowski metric distance on crisp data is defined as[20]

$$D_{w(i,j)_k} = \left[\sum_{k=1}^{n} \left|X_{i\,k} - X_{j\,k}\right|^{w}\right]^{\frac{1}{w}} \tag{1.1}$$

Minkowski's measure holds for $w \in [1, \infty)$. For the special case of w=1, it becomes Hamming distance, and when w=2, it is Euclidean distance.

1.2.2 Mahalanobis Distance

It is measured by deducting the Euclidean distance between two points through their standard deviation, which is expressed as [20]

$$M_{d(i,j)_k} = \{[(X_i - X_j)^T V^{-1} (X_i - X_j)]\}^{1/2} \tag{1.2}$$

where V is the sample covariance matrix. If the covariance matrix V is the identity matrix, then the $M_{d(i,j)}$ reduce to the Euclidean distance. If V is diagonal, then the determined distance measure is called a normalized Euclidean distance defined as

$$M_{d(i,j)_k} = \sqrt{\frac{\sum_{k=1}^{n} \left|X_{i\,k} - X_{j\,k}\right|^{2}}{V}} \tag{1.3}$$

1.2.3 Cosine Distance

The Cosine of an angle is utilized to measure the resemblance between two vectors. It is described as[21]

$$D_{cos(i,j)_k} = 1 - \frac{\sum_{k=1}^{n} X_{i\,k} . X_{j\,k}}{\sqrt{\sum_{k=1}^{n} (X_{i\,k})^2} . \sqrt{\sum_{k=1}^{n} (X_{j\,k})^2}} \tag{1.4}$$

If $X_{i\,k} . X_{j\,k} \geq 0, then\ D_{cos(i,j)_k} \in [0,1]$.

1.2.4 Correlation Distance

This distance measure is derived from the Spearman correlation coefficient(s) and is defined as[22]

$$D_{corr(i,j)_k} = 1 - \frac{\sum_{k=1}^{n} (X_{i\,k} - \overline{X_{i\,k}}) . (X_{j\,k} - \overline{X_{j\,k}})}{\sqrt{\sum_{k=1}^{n} (X_{i\,k} - \overline{X_{i\,k}})^2} . \sqrt{\sum_{k=1}^{n} (X_{j\,k} - \overline{X_{j\,k}})^2}} \tag{1.5}$$

Where $\overline{X_{i\,k}} = \frac{1}{n} \sum_{k=1}^{n} X_{i\,k}$ and $\overline{X_{j\,k}} = \frac{1}{n} \sum_{k=1}^{n} X_{j\,k}$

1.2.5 Chebyshev Distance

It measures the distance between the characteristics of a pair of data points. It is represented as[23]

$$D_{max(i,j)} = \max_k \left| X_{i\,k} - X_{j\,k} \right| \tag{1.6}$$

It would take less time to measure the distance between the data sets.

1.3 Types of Clustering

Clustering algorithms are classified into two main groups, partitional and hierarchical.

1.3.1 Hierarchical Clustering

Hierarchical approaches generate a hierarchical breakdown of the data collection. Hierarchical clustering organizes smaller clusters as extensions of the larger ones (see Figure 1.7). Every clustering level is hierarchically nested in hierarchical clustering [24]. These clusters overlap and do not serve dynamism[25] [26]. Especially over the past analysis of the data decomposes the dataset into distinct subsets. This cluster tree is called a dendrogram, and it contains clusters of varying sizes. In hierarchical clustering, there are two different types [27]: agglomerative clustering and divisive clustering.

1.3.1.1 Agglomerative Clustering

The bottom-up technique is regarded as agglomerative. It creates a separate cluster for each entity in the first instance. Related clusters are then eventually combined into longer clusters until all particles are placed in one or more clusters or are terminated.

1.3.1.2 Divisive Clustering

Similarly, the up-bottom procedure is observed as Divisive clustering. Here, every data item begins with a cluster, the cluster is subdivided in several divisions until each data element enters a self-contained cluster, resulting to the establishment of a nested clustering.

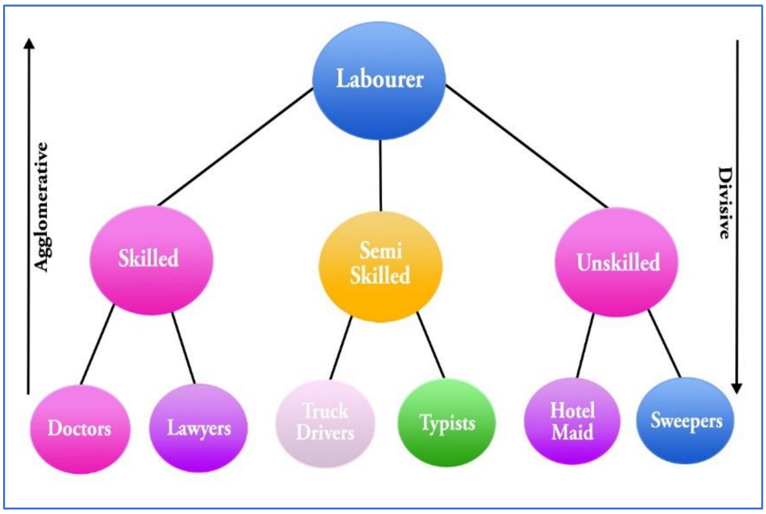


Figure 1.7: Hierarchical clustering

1.3.2 Partitional Clustering

Partitional clustering decomposes a set of disjoint clusters [28][29]. To divide a database into sub-space partitions, consider running one or more clustering algorithms on the real data. Each cluster must have at least one object because the other objects are stored within it. To discover a high-quality partitioning, the data set into k of initial partitions, and then do multiple iterations to optimize the cluster quality Any objects are transferred during the iteration through one cluster to the other, increasing the clustering efficiency. If no object can be transferred, the algorithm stops when the quality cannot be improved.

1.3.2.1 Type of the Partitional Clustering

There are two kinds of partitional clustering (a) k-means clustering and (b) Fuzzy c-means clustering (FCM). Hard c-means clustering (HCM), also known as k-mean clustering, is essentially a partitioning process that restricts each data point to precisely one cluster. Consequently, the issue of unclear boundary of the clusters can be solved by fuzzy clustering. The crisp partitioning of the data in fuzzy clustering is supplemented by a weaker partitioning requirement, where fuzzy relationships represent the connection between data. Fuzzy clustering is associated with complexity, softness, and ambiguity. The most frequently used fuzzy clustering algorithms in several essential areas is the Fuzzy c-means (FCM) algorithm [30][31]. FCM is the HCM extension, and it is shown that FCM has superior results than HCM. Bezdek 's suggested FCM algorithms [32] is perhaps the most frequently used algorithm for clustering since it has stable uncertainty characteristics and can hold much more details than hard segmentation. FCM is the extension of Dunn's [33], which proposed a fuzzy ISODATA algorithm. The traditional FCM algorithm works well for most noiseless data, but it is susceptible to noisy prototypes because of its unusual feature data.

1.3.2.1.1 The Hard C- Mean Clustering

The hard-c-mean (k-mean) clustering is the key and most popular research method for hard clusters [34]. A set-up list of overlaps is the cluster assignment, namely $\mho = \{\mho_1, \mho_2, \dots, \mho_k\}$ where $\mho_i$ is the characterizing index-vector of samples for ith cluster.

Enable $\mathcal{L}(y_i)$ to be the y_i Cluster. The objective function of the hard c-mean is the same as the sum of the fitness function square error (SSE), and it can easily be formulated to minimize the Euclidean distances in couples between points of the same cluster, i.e.,

$$min\, J_{HCM}(SSE) = \sum_{k=1}^{N}\sum_{j=1}^{c}\left\|y_k - v_j\, \right\|^2 \quad (1.7)$$

Subject to

$$v_j = \frac{\sum_{k=1}^{n} y_{jk}}{n} \quad (1.8)$$

Where y_k , $v_j \in \lambda_k$ implies that both the y_k and v_j data points are allocated to the kth cluster λ_k While clustering can be effectively applied in k −means, it is simple to converge to the local optimum, and besides, the clustering assignment relies heavily on the clustering number K, which users have to pre-assign empirically.

1.3.2.1.2 Fuzzy C-Mean Clustering

A statistical method in which experimental data sets are grouped according to similarities [16] is fuzzy clustering. The clustering technique that enables one collection of data to be used in two or more clusters is FCM. Dunn [17] introduced this technique in 1973 by and Bezdak [18] improved it in 1981. It was also widely used to identify a pattern.

The data set x_k is separated by the FCM algorithm into N clusters [16]. The Clustering of the v_j and μ_{jk}^{m} centers are by reducing the cost function. The objective function of FCM, mathematically, can be formulated as

$$\min J_{fcm} = \sum_{k=1}^{N}\sum_{j=1}^{c}\mu_{jk}^{m}\left\|y_k - v_j\right\|^2 \quad (1.9)$$

Where, μ_{jk} is the membership grade of y_k to the n^{th} clusters, the $m > 1$ parameter of fuzzification controls the membership's softness. Higher mask estimating capacity for $m = 2$ has been recognized in [17]. The fuzzy partitioning process is carried out by an iterative optimization of the objective function shown above, with the updating of membership functions μ_{jk} and cluster center v_j by:

$$v_j = \frac{\sum_{k=1}^{n} \mu_{jk}^{m} y_{jk}}{\sum_{k=1}^{n} \mu_{jk}^{m}} \qquad c \in N \tag{1.10}$$

$$\mu_{jk} = \begin{cases} \left[\sum_{i=1}^{c} \left(\frac{\|y_k - v_j\|}{\|y_k - v_i\|} \right)^{\frac{2}{m-1}} \right]^{-1} & \text{if } \|y_k - v_j\| > 0, \forall j \\ 1 & if\ \|y_k - v_j\| = 0 \\ 0 & if\ \exists\, i \neq j \quad \|y_k - v_j\| = 0 \end{cases} \tag{1.11}$$

The calculation is based on such steps:

Step-1) Initialization- U=[u_{ij}] matrix, $U^{(0)}$

Step-2) Centroid calculation – When each point in the dataset is assigned to a cluster, it is needed to recalculate the new v_i^1 , $i \in \mathbb{N}$ centroids.

Step-3) Updating of Membership function- Update $U^{(k)}$, $U^{(k+1)}$

Step-4) If || $U^{(k+1)}$ - $U^{(k)}$||<∈ then STOP; otherwise return to step 2.

1.4 Related Work of FCM Based Techniques

The traditional FCM algorithm works well for most noiseless data, but it is susceptible to noisy prototypes because of its unusual feature data. Researchers are considering different ways to accommodate this FCM limitation. Even so, one related form, Possible C-mean (PCM), suggested by Krishnapuram and Keller [35], interprets clustering as a partition of possibility. Nonetheless, clustering was impaired in one or two clusters. Pal et al. [36] implemented Possible Fuzzy C-mean(PFCM) to resolve the problem of identical clusters, which produces membership and typicality values while clustering unlabelled data. PFCM has not given the optimal results when the data set consists of the unequal size of noise clusters. Therefore, some researchers[37] [38] have adopted the so-called robust distance measures such as the L_p standards ($0 < p <$

1) to replace the L_2 norm within the FCM objective feature, to limit the effect of noise elements on clustering efficiency .

In Noise Clustering(NC), Dave [39] [40][41][42]proposed the idea of noise cluster, which distinguishes outliers in their cluster. The problem with NC is not the identity of outliers between the clusters, and the number of clusters for a certain set of data is not independent[43]. Credibility Fuzzy c-means (CFCM) [44][45][46][47] developed a credibility feature to reduce the effect of outliers on cluster centroids. Also, if CFCM reduces outliers' influence, it assigns certain outliers to several clusters most of the time. One of the advances emerging from belief function theory is the credal partition. This definition generalizes current definitions of hard, fuzzy, or possible clusters by enabling an entity to belong to many groups. To remove certain credal partitions through data and improve their robustness against outliers, a variety of algorithms have been proposed, Evidences cluster (EVCLUS) [48], Evidences c-means (ECM) [49][50], relation evidence c-means (RECM) [51] and Constrained evident evidence c means (CECM)[52] [53][54].

Various other distance measurements, such as Mahalanobis distance measurement, Kernel-based distance measurement in data space, and high-dimensional function space, can also identify non-hyper spherical/non-linear clusters[55][56].

1.5 Fuzzy Linear Programming

In a crisp situation, linear programming problems (LLP) aim to maximize or minimize a linear objective function while maintaining linear constraints. But in cases in which the objective and/constraint functions cannot be defined exactly, such as network optimization, logistics, management problems, and assignments, the decision-maker can only indicate the nature of these issues using concepts of high degrees of precision is not always possible. In certain cases, a certain kind of fuzzy linear optimization programming is ideal to ensure the decision-maker has efficient reliability. Fuzziness can occur in several different forms, and there is no standard form to describe a fuzzy linear programming problem (FLLP).

Classical LPPs are the minimum or maximum values under linear inequalities or linear function equations. The standard form of LPP is represented by

$$\text{Max /Min } Z = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq or \geq b_i$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{1.12}$$

The function to be Max Z or Min Z is called an objective function. The c_j are called cost coefficients. The A=[a_{ij}] matrix is called a restriction matrix, and the **b**= $< b_1, b_2, \dots, b_m >^T$ is called a vector on the right side. where ***y***= $< y_1, y_2, \dots, y_n >^T$ is the vector of variables. Fuzzy sets theory could be used to deal with vague and indeterminate details in LPP by generalizing the concept of membership to include ambiguity.

1.6 Fuzzy Set and its Components

The system of the fuzzy set was presented by Zadeh [57], and it was further improved by Zadeh[58]. It is an impressive technique for signifying instinctive or inaccurate evidence in dissimilar situations. The applications of fuzzy sets, including decision-making [59], probability [60], control theory [61], medical studies [62], and the characterization of complex systems [63], are currently used in the majority of scientific disciplines. We analyze some of the fuzzy set theory's fundamental principles and terminology used for the other sections of this thesis.

Definition 1.1.: A fuzzy set $\tilde{B}$ is defined on universe set Y defined as fellow:

$$\tilde{B} = \{(\text{y}, \mu_{\tilde{B}}(y) : y \epsilon Y\} \tag{1.13}$$

Where $\mu_{\tilde{B}}(y)$ represent the membership function of a fuzzy set simplifies the predictor function in crisp sets, whose range covering the interval [0,1] operating on the domain of all possible values.

Definition 1.2. A fuzzy set $\tilde{B}$ is called normal if there exists at least one element $y \epsilon Y$ with

$$\mu_{\tilde{B}}(y) = 1. \tag{1.14}$$

Definition 1.3. Let $\tilde{B}$ be a fuzzy set in Y and $\alpha \in [0,1]$. The α*-cut* of a fuzzy set $\tilde{B}$ in Y is the crisp set $\tilde{B}^{\alpha}$ given by

$$\tilde{B}^{\alpha} = \{y | \mu_{\tilde{B}}(Y) \geq \alpha\} \tag{1.15}$$

Definition 1.4. The *support* of the fuzzy set $\tilde{B}$ of Y is defined as follow:

$$supp(\tilde{B}) = \{y | \mu_{\tilde{B}}(Y) > 0\} \tag{1.16}$$

Definition 1.5. A convex fuzzy set $\tilde{B}$ of Y is defined as follows

$$\mu_{\tilde{B}}\{\lambda y_1 + (1 - \lambda) y_2\} \geq min\{\mu_{\tilde{B}}(y_1), \mu_{\tilde{B}}(y_2)\}, \text{Where } 0 \leq \lambda \leq 1 \tag{1.17}$$

It is said to be a non-convex fuzzy set if the above inequality does not hold.

Definition 1.6. A fuzzy set $\tilde{B}$ in $\mathbb{R}$ is called a fuzzy number if it satisfies the following conditions:

(i) $\tilde{B}$ *is normal,*
(ii) *It is convex fuzzy set,*
(iii) *It is closed in* $[0,1]$ *and*
(iv) *The support of* $\tilde{B}$ *is bounded.*

We denote the set of all fuzzy numbers on $\mathbb{R}$ by $F(\mathbb{R})$: It is well known that if $\tilde{B} \in F(\mathbb{R})$, then the α*-cut* of $\tilde{B}$ is a closed interval for every $\alpha \in [0,1]$, i.e., closed, bounded, and convex subset of $\mathbb{R}$. Therefore, the closed interval is denoted by $\tilde{B}_{\alpha} = [\widetilde{B_{\alpha}}^{L}, \widetilde{B_{\alpha}}^{U}]$. If $\widetilde{B_{\alpha}}^{L} \geq 0 \ \forall \ \alpha \in [0,1]$, then $\tilde{B}$ is called a non-negative fuzzy number.

Definition 1.7. A fuzzy number $\tilde{B} = (\beta_i, \beta_i + \varepsilon_i)$ is said to be a right triangular fuzzy number if its membership function is given by

$$\tilde{B}=\begin{cases} 1 & When\ y\ \leq \beta_i \\ \frac{\beta_{i_i}+\varepsilon_i-y}{\varepsilon_i} & When\ \beta_i \leq y \leq \beta_i+\varepsilon_i \\ 0 & when\ y \geq \beta_i+\varepsilon_i \end{cases} \qquad (1.18)$$

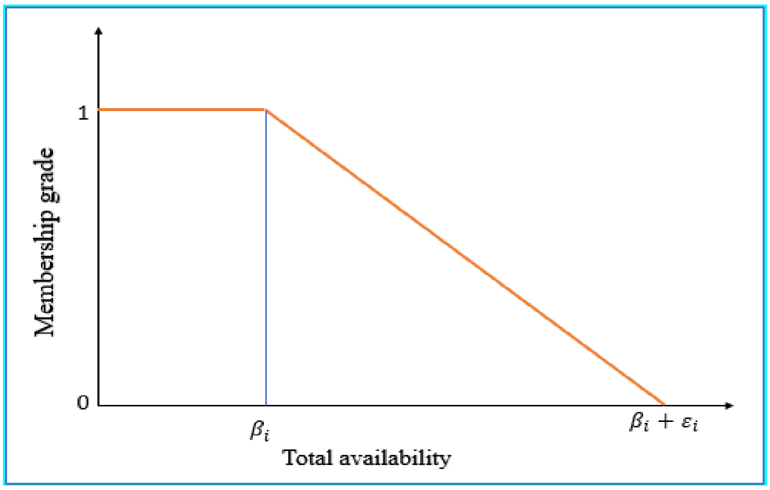


Figure 1.8: Representation of the right triangular membership function

Definition 1.7. A fuzzy number $\tilde{B} = (\beta_i - \varepsilon_i, \beta_i, \beta_i + \varepsilon_i^*)$ is said to be a composite triangular fuzzy number if its membership function is given by

$$\tilde{B} = \begin{cases} \dfrac{y-(\beta_i-\varepsilon_i)}{\varepsilon_i} & \beta_i-\varepsilon_i \leq y \leq \beta_i \\ 1 & y=\beta_i \\ \dfrac{y-(\beta_i-\varepsilon_i^*)}{\varepsilon_i^*} & \beta_i \leq y \leq \beta_i+\varepsilon_i^* \end{cases} \qquad (1.19)$$

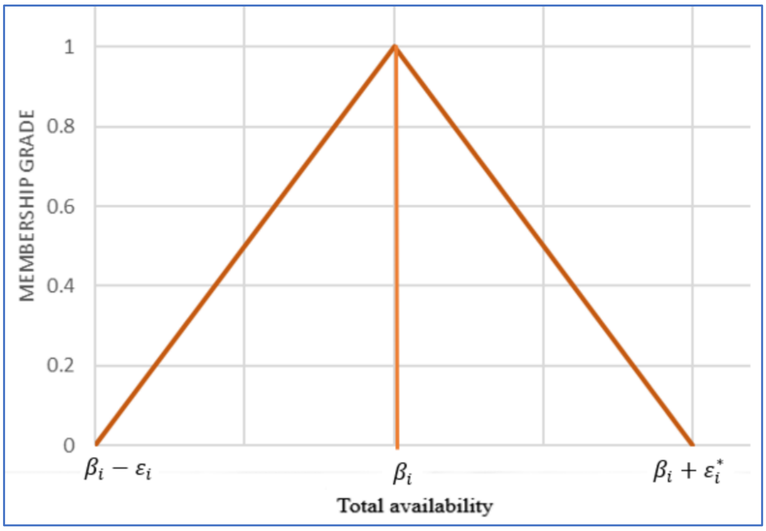


Figure 1.9: Representation of the composite triangular membership function

If $\varepsilon_i = \varepsilon_i^*$ it called a symmetric triangular fuzzy number, otherwise non- symmetric. A triangular fuzzy number$(\beta_i - \varepsilon_i, \beta_i, \beta_i + \varepsilon_i^*)$ is said to be a non-negative fuzzy number iff $\beta_i - \varepsilon_i \geq 0$.

Definition 1.8. A fuzzy number $\tilde{B}$ = $(\beta_i - \varepsilon_i, \beta_i,\ \beta_i^*, \beta_i^* + \varepsilon_i^*)$ is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{B}}(Y) = \begin{cases} \frac{y-(\beta_i-\varepsilon_i)}{\varepsilon_i} & \beta_i - \varepsilon_i \leq y \leq \beta_i \\ 1 & \beta_i \leq y \leq \beta_i^* \\ \frac{y-(\beta_i^*-\varepsilon_i^*)}{\varepsilon_i^*} & \beta_i^* \leq y \leq \beta_i^* + \varepsilon_i^* \end{cases} \tag{1.20}$$

If $\varepsilon_i = \varepsilon_i^*$ it called a symmetric trapezoidal fuzzy number, otherwise non-symmetric. A trapezoidal fuzzy number $((\beta_i - \varepsilon_i, \beta_i,\ \beta_i^*, \beta_i^* + \varepsilon_i^*)$ is said to be non-negative fuzzy number $iff\ \beta_i - \varepsilon_i \geq 0$.

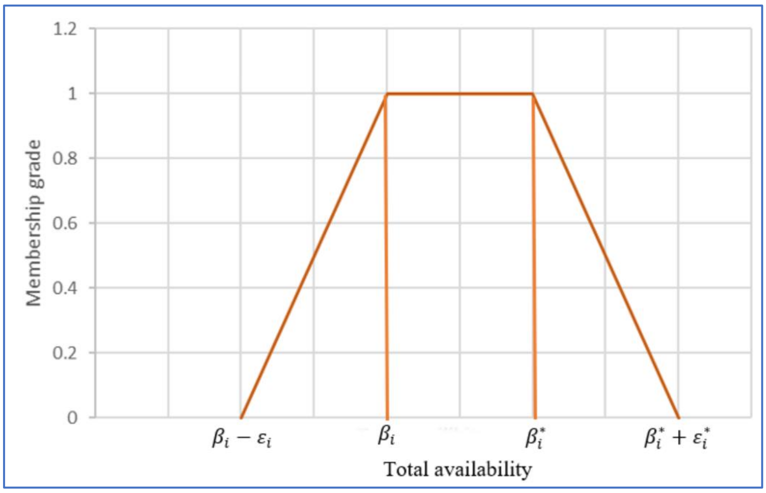


Figure 1.10: Representation of the trapezoidal membership function

where $\alpha_{\mu_B}(x)$ and $\alpha^{+}{}_{\mu_B}(x)$stands for the membership functions of fuzzy set $\tilde{B}$ and $\mu_{\tilde{B}}(X)$ represents the α -cut of fuzzy set $\tilde{B}$ at the level.

Height of a fuzzy set denoted by $h(\tilde{B})$ is defined as the largest of membership values of the elements contained in that set. For a normal fuzzy set, $h(\tilde{B})$ = 1. A fuzzy set will be convex, if $\mu_{\tilde{B}}\{ \lambda x_1+(1-\lambda) x_2\}\geq\min \{\mu_{\tilde{B}}(x_1), \mu_{\tilde{B}}(x_2)\}$,Where $0 \leq \lambda\leq 1$.

1.7 Background of Fuzzy Linear Programming Problems (FLP)

FLP problems admit imprecise restrictions, resulting in more accurate versions. They have also been employed in scientific and industrial issues[64] [65][66][67][68]. The theory of fuzzy mathematical programming was first introduced by Tanaka et al.[69] & Bellman and Zadeh [58], which was based on fuzzy decision structure to address the imprecision of LP parameters issues with fuzzy imperatives and objective functions. Zimmerman [70] has implemented an FLP formula in a crisp problem model using an existing algorithm and categorized FLP problems into symmetric and non-symmetric. Amid et al. [71] clarifies that there is no difference in the symmetric issue between the amounts of the objective and the restrictions, whereas the destinations and limitations are not identical in non-symmetric issues and have different amounts. Tanaka and Asai [72] proposed a likely formulation of LPP with crisp decision coefficient and fuzzy decision variables. Verdegay [73] proposed and used the idea of a fuzzy objective constructed on the norm of fuzzification to explain FLP problems. Herrera et al. [74] analyzed the mathematical problem as fuzzy numbers and often included fuzzy coefficients as the concept of a feasibly specified set. Ganeshan and Veeramani [75] have suggested an FLP model with symmetrical trapezoidal fuzzy numbers. They have demonstrated fuzzy analogs for some primary LP axioms without translating them into crisp LP problems. Dong et al. [76] were designed a new fuzzy linear model, with trapezoidal fuzzy numbers (TrFNs) being all target coefficients, scientific coefficients, and devices. The order relationship of the TrFNs is originally measured using the estimate of the TrFNs interval. The trapezoidal linear fuzzy system is converted into an objective interval program based on the order relationship of the TrFNs.

With the existence of the fuzzy linear programming problems, many researchers introduced some methods for solving these problems [77][78] solved fuzzy linear programming problems through the simplex method. With ample literature, we deduce from an unusual perspective the theory and approach of optimization with the fuzzy-valued objective function,[79][80][81][82][83] and their references. However, merely rare efforts recognize the feasible set and alternatives sets for the modeling, which are expressed through functional constraints. In [84] [85][86], the feasible set is described via inequalities functions specified through crisp functions, i.e., the only opportunity of survival of fuzziness in the areas is measured, and also studied that data in which the alternatives are not explained. It is very difficult to construct the models for uncertainty information, but authors try to hypothesis all possibilities according to given data. We approach to build the different fuzzy numbers for the constrains for which the achieved outcomes will be more conventional.

Meanwhile, the concerned fuzzy numbers have a realistic approach in lots of the different fields like decision making, data analysis [87][88] and also engineering problems[89][90], etc. With the help of these fuzzy numbers' assistance, we can resolve numerous optimization problems. In [91], they introduced a different process on fuzzy triangular numbers(FTN) improved subtraction and division. Also, there are a lot of modified operations that are used to enhance triangular and trapezoidal fuzzy numbers [92][93][94] [95][96][97], and these might affect optimizing FLP.

Aforementioned, many researchers had used applications of these fuzzy numbers to optimize the fuzzy linear programming problems, such as an approach of ranking a fuzzy number and fuzzy triangular number [98]to solve linear programming problems. A research paper by Chakraborty et al. [45] FLP states that a triangular fuzzy number represents all the coefficients and decision variables and all the constraints are fuzzy equality or inequality. An innovative way for solving Fully FLP by applying the Lexicography method [99]. In addition to classical linear programming, they proposed the latest pattern for solving FLP completely using (L-R) fuzzy numbers and a lexicographical method. A novel algorithm is proposed [100] built on an innovative lexicographic ordering on TFN to explain the FFLP by changing it to its correspondent a MOLP.

On the other hand, the various components of fuzzy triangular numbers[101][102] were used to design the reliability parameters to build the optimization model for the industrial systems. On the other hand, the various components of fuzzy triangular numbers[101][102] were used to design the reliability parameters to build the optimization model for the industrial systems. In these papers[103][75][104], symmetrical fuzzy number FLP problem-solving coefficients for the objective function and solution value for RHS restrictions were applied.

Shaheen et al. [105] provided an alternative approach to range estimation based on a fuzzy set theory. It has provided a technique for the extraction and processing of fuzzy numbers by experts within the fuzzy framework of the study. A fuzzy State Estimation (FSE) model is used by [106] to model the uncertainty in estimating the state of the power system based on the optimization of restricted linear programming. Uncertain measurements are expressed as fuzzy numbers with a triangular and trapezoidal membership function with medium and propagated values. In the article Jagadeeswari and Nayagam [107], efforts were made for using the distance function in terms of the α-cuts to address the problem of triangular approximations of the fuzzy parabolic numbers. A new nearest trapezoidal approach operator with expected interval survival is prescribed in [108]. Chen and Cheng [109] presented the subjective perspectives of decision-makers with trapezoidal fuzzy numbers in linguistic terms. An FLFP solution procedure where objective function, capital, and technical coefficients are fuzzy triangle numbers has been proposed [110]. Ebrahimnejad and Tavana [111] proposed an approach to address FLP problems in which symmetric fuzzy trapezoidal numbers are interpreted as objective function coefficients and right-side values, while real numbers are the components of the matrix coefficient. An approach has been suggested by [112] to solve the FFLP problem, with the symmetric trapezoidal fuzzy number representing the parameters without any conversion of crisp equivalent problem. Three cases of linear programming problems, such as real numbers, type-1 fuzzy numbers, and type-2 fuzzy sets, were discussed in [113]. A complete linear defuzzification function defined in a trapezoidal fuzzy number subsection of a fuzzy number vector space is the best way to solve a linear programming problem with real objects in the type-1 Fuzzy linear programming. The theorem α-level representation was the approach for obtaining the optimal type-2 solution. Dong and Wan [95] developed a new approach for the linear fuzzy system in which trapezoidal fuzzy numbers (TrFNs) are used to represent all objective coefficients, technology coefficients, and tools. Also, the proposed model of the paper is not only mathematically extensive as well as the degree of recognition of the fuzzy limitations is violated adequately. Karimi et al. [114] provide the best-worst method for addressing multi-attribute decision-making (MADM) issues in the fuzzy situation. Then the weight of the criterion is fully determined by a fuzzy linear mathematical model. In addition to that, all the weights are quantified by fuzzy triangular numbers. Bolos et al. [115] discussed a new hybrid model using linear schedules and fuzzy numbers to achieve tangible assets in the business. This hybrid model is suggested as the basis of decision variables, objective function coefficients, and a matrix of constraints for the resolution in the form of triangular fuzzy numbers.

1.8 Research Gaps

Based on the literature, the following research gaps have been identified:

1.8.1 Efficient Clustering Technique

The primary issue for the clustering of data is identifying groups (or clusters), depending on the given data features of the data set. It is difficult to predict the number of clusters in most real-life scenarios. However, even after implementing the case-sensitive and fuzzy clustering algorithms, there are still problems with the lack of non-overlapping clusters due to the presence of non-members and outliers. Thus, a need to establish an efficient clustering strategy determines both the number of clusters and the clustering at runtime.

1.8.2 Situational Based Fuzzy Linear Programming Problems

It is not the exact number of restrictions that matters; it is the probability that a certain constraint has a given result that needs to be incremented or decremented that the problem definition of the target is ambiguous, allowing the objective to collapse. Many researchers have attempted to address these problems, but there is still scope for improvement.

1.8.3 A Combined Study of Fuzzy Clustering and Fuzzy Linear Programming Problems

Furthermore, the enhancement of fuzzy optimization by fuzzy clustering algorithms has a lag.

1.9 Objectives of the Research

- Intensive study of fuzzy clustering algorithms and optimization techniques.
- To enhance the quality of fuzzy clustering and the optimization techniques using fuzzy membership functions/fuzzy numbers.
- Combined study of improved fuzzy optimization techniques to analyze upgraded cluster centers by fuzzy clustering algorithms.

Chapter 2

Fuzzy Equivalence Relation Via Different Distance Measures and Its Utilizations

"A statistical analysis, properly conducted, is a delicate dissection of uncertainties, a surgery of suppositions."

−Michael J. Moroney

2.1 Introduction

Cluster analysis is one of the leading approaches to acknowledge the patterns., many researchers represented the idea of fuzzy clustering to get a better classification of objects. In this direction, they made significant contributions in deciding on the existence of fuzziness, incomplete information. The first fuzzy clustering approach was initiated by Bellman et al. [116]and Ruspini[117], then Dunn [118] explained the Well-Separated Clusters and Optimal fuzzy Partitions. Tamura et al. [119] figured out an n-step procedure using max-min fuzzy compositions (max-min similarity relation) and achieved a multi-level hierarchical clustering. Some interactions between fuzzy partitions and similar relations have been studied in [120]. Several hard cluster algorithms were indeed demonstrated [121] that could be derived from the theory of maximum likelihood estimator.

Now fuzzy clustering has been extensively examined and practiced in multifarious areas [122][123] [124][125]. Groenen et al. [126] used the Minkowski distance function to get fuzzy cluster analysis. Yang and Shih [127] concentrated on Cluster analysis based on fuzzy relations, and a clustering algorithm is created for the max-t similarity relation matrix. Then three critical max-t (max-min, max-prod, and max-Δ) compositions are compared. Liang et al.[128] determined the best number of clusters using a cluster validity index by taking a suitable λ cut value. At first, the trapezoidal fuzzy numbers are defined based on the subject's attributes rating. The distance between two trapezoidal fuzzy numbers is computed subsequently to obtain the compatibility relation then the categorization of objects was done by a fuzzy equivalence relation. In articles [129][130] concentrated on fuzzy clustering analysis based on equivalence class and illustrated the desirable cluster. K. M. Bataineh et al.[131] compared the performances of fuzzy C-mean clustering algorithm and subtractive clustering algorithm according to their capabilities**.** This chapter aims to classify the binder's performances using fuzzy equivalence clustering via Minkowski Cosine, Chebyshev, Correlation, and a new (Mahalanobis) distance function. The reliability and adequacy of the Mahalanobis distance on the clustering performance of Binders are examined over Minkowski and other distance functions.

2.2 Preliminaries

In this section, we address the following basic definitions:

Definition2.1. Let $\tilde{A}$ and $\tilde{B}$ are two fuzzy sets, defined on universal spaces X and Y then a fuzzy relation on (X × Y), is defined by

$$\tilde{R} = \{[(x,y),\ \mu_{\tilde{R}}(x,y)] | (x,y) \epsilon \text{ X} \times \text{Y} \}$$

$$\text{Where, } \mu_{\tilde{R}}(x,y) \leq \min\{\ \mu_{\tilde{A}}(x),\ \mu_{\tilde{B}}(x)\} \qquad (2.1)$$

Definition2.2. Let $\tilde{R}_1$ on (X × Y) and $\tilde{R}_2$ on (Y × Z) be two fuzzy relations than the max-min composition $\tilde{R}_1 \circ \tilde{R}_2$ is defined by

$$\tilde{R}1 \circ \tilde{R}2 = \left\{ \left[(x,z),\ \max_{y \in Y} \{\min\{\ \mu_{\tilde{R}}(x,y),\ \mu_{\tilde{R}}(y,z)\}\} \right] \middle| x \epsilon \text{ X}, y \epsilon \text{ Y}, z \epsilon \text{Z} \right\} \qquad (2.2)$$

Definition2.3. Let $\tilde{R}$ be a fuzzy relation on (X × Y), then

(1) $\tilde{R}$ is called reflexive if $\mu_{\tilde{R}}(x,x) = 1, \forall\ x \in \text{X}$

(2) $\tilde{R}$ is called ε −reflexive if $\mu_{\tilde{R}}(x,x) \geq \varepsilon, \forall\ x \in \text{X}$

(3) $\tilde{R}$ is called weakly reflexive if

$\mu_{\tilde{R}}(x,y) \leq \mu_{\tilde{R}}(x,x)$ *and* $\mu_{\tilde{R}}(y,x) \leq \mu_{\tilde{R}}(x,x) \forall\, x \in$X.

Definition2.4. A fuzzy relation $\tilde{R}$ is called symmetric if

$$\mu_{\tilde{R}}(x,y) = \mu_{\tilde{R}}(y,x)\ \forall\, x,y \in \text{X}. \quad (2.3)$$

Definition2.5. A fuzzy relation $\tilde{R}$ is called transitive if

$$\mu_{\tilde{R}}(x,z) \geq \max_{y\in Y}\{\min\{\mu_{\tilde{R}}(x,y),\ \mu_{\tilde{R}}(y,z)\}\}\ \forall\, x,y,z \in \text{X} \quad (2.4)$$

Definition2.6. A fuzzy relation $\tilde{R}$ on X is said to be compatible with X if it is reflexive and symmetric.

Definition2.7. A fuzzy relation $\tilde{R}$ on X is said to be transitive on X if it is reflexive, symmetric, and transitive.

2.3 Clustering Methods Based upon Fuzzy Equivalence Relations:

The distances between the crisp data sets are required to obtain the fuzzy cluster analysis through the equivalence class. We proposed the Mahalonobis, Chebychev, Minkowski, Cosine, and Correlation metric distance on crisp data.

According to the distances, the fuzzy compatible relation matrix is yielded. For the Minkowski class *(w=2)*, the relation matrix is

$$\tilde{R}\,(X_i, X_j) = 1 - \delta\left[\sum_{k=1}^{n} \left|X_{i\,k} - X_{j\,k}\right|^2\right]^{\frac{1}{2}} \quad (2.5)$$

$$\text{Where } \delta = \left\{max\left[\sum_{k=1}^{n} \left|X_{i\,k} - X_{j\,k}\right|^2\right]^{\frac{1}{2}}\right\}^{-1} \quad (2.6)$$

and the fuzzy compatible relation matrix for Mahalanobis distance is generated by

$$\tilde{R}\,(X_i, X_j) = 1 - \lambda\left\{\sqrt{\left[(X_i - X_j)^T S^{-1}\,(X_i - X_j)\right]}\right\} \quad (2.7)$$

$$\text{Where } \lambda = \left\{max\sqrt{\left[(X_i - X_j)^T S^{-1} (X_i - X_j)\right]}\right\}^{-1} \tag{2.8}$$

The fuzzy compatible relation matrix for Cosine distance is

$$\tilde{R}\left(X_i, X_j\right) = 1 - \gamma\left\{1 - \frac{\sum_{k=1}^{n} X_{i\,k}.X_{j\,k}}{\sqrt{\sum_{k=1}^{n}(X_{i\,k})^2}.\sqrt{\sum_{k=1}^{n}(X_{j\,k})^2}}\right\} \tag{2.9}$$

$$\text{Where } \gamma = \left\{\max\left\{1 - \frac{\sum_{k=1}^{n} X_{i\,k}.X_{j\,k}}{\sqrt{\sum_{k=1}^{n}(X_{i\,k})^2}.\sqrt{\sum_{k=1}^{n}(X_{j\,k})^2}}\right\}\right\}^{-1} \tag{2.10}$$

The fuzzy compatible relation matrix for Correlation distance is

$$\tilde{R}\left(X_i, X_j\right) = 1 - \rho\left\{1 - \frac{\sum_{k=1}^{n}(X_{i\,k} - \overline{X_{i\,k}}).(X_{j\,k} - \overline{X_{j\,k}})}{\sqrt{\sum_{k=1}^{n}(X_{i\,k} - \overline{X_{i\,k}})^2}.\sqrt{\sum_{k=1}^{n}(X_{j\,k} - \overline{X_{j\,k}}))^2}}\right\} \tag{2.11}$$

$$\text{Where } \rho = \left\{\max\left\{1 - \frac{\sum_{k=1}^{n}(X_{i\,k} - \overline{X_{i\,k}}).(X_{j\,k} - \overline{X_{j\,k}})}{\sqrt{\sum_{k=1}^{n}(X_{i\,k} - \overline{X_{i\,k}})^2}.\sqrt{\sum_{k=1}^{n}(X_{j\,k} - \overline{X_{j\,k}}))^2}}\right\}\right\}^{-1} \tag{2.12}$$

The fuzzy compatible relation matrix for Chebychev distance is

$$\tilde{R}\left(X_i, X_j\right) = 1 - \sigma\left\{\max_k\left|X_{i\,k} - X_{j\,k}\right|\right\} \tag{2.13}$$

$$\text{Where } \sigma = \left\{\max\left\{\max_k\left|X_{i\,k} - X_{j\,k}\right|\right\}\right\}^{-1} \tag{2.14}$$

After the fuzzy compatible relation matrix, the fuzzy transitive closures were constructed for each matrix. If $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$ then $\tilde{R} \circ \tilde{R} = \widetilde{R}^2{}_T$ is said to be transitive closure of $\tilde{R}$ for $k = 1$. If $\tilde{R} \circ \tilde{R} \nsubseteq \tilde{R}$ then construct $\widetilde{R}^2 \circ \tilde{R}^2$. If $\widetilde{R}^2 \circ \tilde{R}^2 \subseteq \tilde{R}^2$ then $\widetilde{R}^4{}_T$ is said to be transitive closure of $\tilde{R}^2$ for $k = 2$.If there are n −elements in the universal space, then the fuzzy transitive closure is achieved until $2^k \geq n - 1$.

The α-cut relation can be obtained from a transitive fuzzy relation by taking the pairs with membership degrees no less than α.

$$\tilde{R}_{\alpha} = \{[(x, y),\ \mu_{\tilde{R}}(x, y) \geq \alpha] | (x, y) \epsilon \mathrm{X} \times \mathrm{Y}\}. \quad (2.15)$$

2.4 A Comparative Fuzzy Cluster Analysis of the Binder's Performance Grades

This chapter proposes the fuzzy equivalence class clustering using Minkowski, Mahalanobis, Cosine, Chebychev, and Correlation distance function on the performance grading of different binders used in Turner–Fairbank Highway Research Center Polymer Research Program [6]. It was observed by Aroon Shenoy in his research [6] that the super pave specification parameter $|G*|/(\frac{1}{Sin\delta})$ is not tolerable in the classification of polymer modified binders for high-temperature performance grading of paving asphalts. It was a matter of concern to subtilize this parameter to gain more consciousness in the pavement performance and detect other latent parameters that may better relate to the rutting resistance. The refined super pave specification parameter, namely, $|G*|/(1 - \frac{1}{tan\delta.Sin\delta})$ has the highest merit for possible use. It is a viable alternative for getting the high-temperature specification, such that it becomes more sensitive to field performance. Owing to the variations in the phase angleδ, the parameter $|G*|/(1 - (\frac{1}{tan\delta.sin\delta})$ can easily attain its efficiency as compared to the original super pave specification parameter. Another alternative would be to first define an equal stiffness temperature$(T_e°\mathrm{C})$, when the complex shear modulus$(|G*|)$ takes a specific value of 50 kPa. These parameters take care of the rheological contribution coming from one portion of the term $|G*|/\left[1 - \frac{1}{(tan\delta.Sin\delta)}\right]$. the result in terms of high specification temperature $(T_{TH}°\mathrm{C})$ being defined as$(T_e°\mathrm{C})/\left[1 - \frac{1}{(tan\delta.Sin\delta)}\right]$ and it is more meaningful to achieve eminent high specification temperature. To get the better discrimination between the performances of binders at different membership grades with specification parameter$|G*|/(\frac{1}{.sin\delta})$, $|G*|/(1 - (\frac{1}{tan\delta.sin\delta})$ and $(T_e°\mathrm{C})/\left[1 - \frac{1}{(tan\delta.Sin\delta)}\right]$, the fuzzy equivalence class clustering is proposed.

2.5 Experimental Data

The experimental data used was taken from the research [16] under NCHRP (National Co-operative Highway Research Program) and TFHRC (Turner–Fairbank Highway Research Center). In terms of high specification temperatures, the performance grades were computed for different polymer-modified binders that were evaluated under the NCHRP 90-7 ongoing polymer research program carried out at Turner–Fairbank Highway Research Center.

Here the performances of seventeen Binders are selected for criteria-based classification. All seventeen binders are categorized into five different sets based on super pave grades; the Pave Grade numbers shown are based on the super pave system description

Set-A consist of a binder x_1 = flux (B6224) with Pave Grade 52-28.

Set-B consist of a binder x_2 = unmodified base (B6225) with Pave Grade 64-28.

Set-C consist of thirteen binders with Pave Grade 70-28; they are described as x_3 =Unmodified high grade (B6226), x_4 =Air blown (B6227) and 8 Polymer-modified systems { x_5 =ElvaloyNo.1 (B6228) , x_6 = Styrene–Butadiene–Styrene Linear-Grafted (B6229), x_7 = Styrene–Butadiene–Styrene Linear (B6230) , x_8 = Butadiene–Styrene Radial-Grafted (B6231), x_9 = Ethylene–Vinyl– Acetate No. 1 (B6232), x_{10} = Ethylene–Vinyl–Acetate Grafted (B6233), x_{11} = Ethylene–Styrene–Inter polymer No. 1 (B6243) and x_{12} =Chemically Modified Crumb Rubber (B6251) }and x_{13} =Ethylene–Vinyl–Aacetate No. 2 (B6254) x_{14} =Elvaloy No. 2 (B6257) x_{15} =Elvaloy No. 3 (B6258).

Set D consist a binder x_{16} = polymer-modified Ethylene–Styrene–Inter polymer No. 2 (B6252) with Pave Grade 76-22

Set E consists of a binder x_{17} = polymer-modified Ethylene–Styrene–Inter polymer No. 3 (B6253) with Pave Grade 70-22.

The binder codes are the serial numbers assigned in the laboratory for logging purposes, and the numbers 1,2,3 after the name indicate that the polymer was used in different amounts in these three formulations. All the asphalts were from the same source, namely, Venezuelan crude (a blend of Boscan and Bachaquero). The air-blown grade (PG 70-28) was obtained by non-catalytic air blowing of a PG 52-28 (flux). The polymer-modified grades were obtained by addition of various amounts of different polymers to the PG 64-28 (base) or the PG 52-28 (flux) or a mixture of the PG 64-28 (base) and the PG 52-28 (flux) in different proportions to achieve the same performance grading. All these asphalts were part of the extensive NCHRP 90-07 ongoing polymer research program at the Pavement Testing Facility located at the TFHRC.

2.6 Result Analyzes of the Experimental Data

The performance grades of binders were targeted using the described methodology for Mahalanobis metric, Minkowski (w = 2) metric, Chebychev matric, Cosine matric, and Correlation metric distances. The fuzzy compatible relation matrices and transitive closure are derived for each distance. The following matrices R_1 and R_{1T} represents the fuzzy compatible and transitive relation matrix for Minkowski (w = 2) distance.

$$R_1 = \begin{bmatrix}
1.00 & 0.68 & 0.55 & 0.42 & 0.00 & 0.43 & 0.49 & 0.46 & 0.34 & 0.34 & 0.32 & 0.31 & 0.20 & 0.41 & 0.41 & 0.22 & 0.32 \\
0.68 & 1.00 & 0.85 & 0.71 & 0.25 & 0.69 & 0.75 & 0.73 & 0.62 & 0.63 & 0.61 & 0.59 & 0.49 & 0.69 & 0.66 & 0.45 & 0.58 \\
0.55 & 0.85 & 1.00 & 0.86 & 0.40 & 0.83 & 0.89 & 0.87 & 0.77 & 0.79 & 0.76 & 0.74 & 0.65 & 0.84 & 0.79 & 0.60 & 0.73 \\
0.42 & 0.71 & 0.86 & 1.00 & 0.53 & 0.92 & 0.91 & 0.93 & 0.91 & 0.92 & 0.90 & 0.88 & 0.78 & 0.97 & 0.88 & 0.71 & 0.85 \\
0.00 & 0.25 & 0.40 & 0.53 & 1.00 & 0.56 & 0.49 & 0.52 & 0.62 & 0.58 & 0.62 & 0.64 & 0.69 & 0.55 & 0.59 & 0.77 & 0.67 \\
0.43 & 0.69 & 0.83 & 0.92 & 0.56 & 1.00 & 0.93 & 0.96 & 0.91 & 0.88 & 0.88 & 0.88 & 0.76 & 0.95 & 0.95 & 0.76 & 0.89 \\
0.49 & 0.75 & 0.89 & 0.91 & 0.49 & 0.93 & 1.00 & 0.97 & 0.85 & 0.84 & 0.83 & 0.82 & 0.71 & 0.91 & 0.90 & 0.70 & 0.82 \\
0.46 & 0.73 & 0.87 & 0.93 & 0.52 & 0.96 & 0.97 & 1.00 & 0.88 & 0.87 & 0.86 & 0.85 & 0.74 & 0.94 & 0.92 & 0.72 & 0.85 \\
0.34 & 0.62 & 0.77 & 0.91 & 0.62 & 0.91 & 0.85 & 0.88 & 1.00 & 0.94 & 0.97 & 0.97 & 0.85 & 0.93 & 0.89 & 0.80 & 0.94 \\
0.34 & 0.63 & 0.79 & 0.92 & 0.58 & 0.88 & 0.84 & 0.87 & 0.94 & 1.00 & 0.96 & 0.93 & 0.86 & 0.92 & 0.85 & 0.75 & 0.88 \\
0.32 & 0.61 & 0.76 & 0.90 & 0.62 & 0.88 & 0.83 & 0.86 & 0.97 & 0.96 & 1.00 & 0.97 & 0.88 & 0.91 & 0.87 & 0.79 & 0.92 \\
0.31 & 0.59 & 0.74 & 0.88 & 0.64 & 0.88 & 0.82 & 0.85 & 0.97 & 0.93 & 0.97 & 1.00 & 0.88 & 0.90 & 0.87 & 0.82 & 0.94 \\
0.20 & 0.49 & 0.65 & 0.78 & 0.69 & 0.76 & 0.71 & 0.74 & 0.85 & 0.86 & 0.88 & 0.88 & 1.00 & 0.79 & 0.76 & 0.79 & 0.85 \\
0.41 & 0.69 & 0.84 & 0.97 & 0.55 & 0.95 & 0.91 & 0.94 & 0.93 & 0.92 & 0.91 & 0.90 & 0.79 & 1.00 & 0.91 & 0.74 & 0.88 \\
0.41 & 0.66 & 0.79 & 0.88 & 0.59 & 0.95 & 0.90 & 0.92 & 0.89 & 0.85 & 0.87 & 0.87 & 0.76 & 0.91 & 1.00 & 0.80 & 0.90 \\
0.22 & 0.45 & 0.60 & 0.71 & 0.77 & 0.76 & 0.70 & 0.72 & 0.80 & 0.75 & 0.79 & 0.82 & 0.79 & 0.74 & 0.80 & 1.00 & 0.86 \\
0.32 & 0.58 & 0.73 & 0.85 & 0.67 & 0.89 & 0.82 & 0.85 & 0.94 & 0.88 & 0.92 & 0.94 & 0.85 & 0.88 & 0.90 & 0.86 & 1.00
\end{bmatrix} \quad (2.16)$$

$$
R_{1T} = \begin{bmatrix}
1.00 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 & 0.68 \\
0.68 & 1.00 & 0.85 & 0.85 & 0.77 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 \\
0.68 & 0.85 & 1.00 & 0.89 & 0.77 & 0.89 & 0.89 & 0.89 & 0.89 & 0.89 & 0.89 & 0.89 & 0.88 & 0.89 & 0.89 & 0.86 & 0.89 \\
0.68 & 0.85 & 0.89 & 1.00 & 0.77 & 0.95 & 0.95 & 0.95 & 0.93 & 0.93 & 0.93 & 0.93 & 0.88 & 0.97 & 0.95 & 0.86 & 0.93 \\
0.68 & 0.77 & 0.77 & 0.77 & 1.00 & 0.77 & 0.77 & 0.77 & 0.77 & 0.77 & 0.77 & 0.77 & 0.77 & 0.77 & 0.77 & 0.77 & 0.77 \\
0.68 & 0.85 & 0.89 & 0.95 & 0.77 & 1.00 & 0.96 & 0.96 & 0.93 & 0.93 & 0.93 & 0.93 & 0.88 & 0.95 & 0.95 & 0.86 & 0.93 \\
0.68 & 0.85 & 0.89 & 0.95 & 0.77 & 0.96 & 1.00 & 0.97 & 0.93 & 0.93 & 0.93 & 0.93 & 0.88 & 0.95 & 0.95 & 0.86 & 0.93 \\
0.68 & 0.85 & 0.89 & 0.95 & 0.77 & 0.96 & 0.97 & 1.00 & 0.93 & 0.93 & 0.93 & 0.93 & 0.88 & 0.95 & 0.95 & 0.86 & 0.93 \\
0.68 & 0.85 & 0.89 & 0.93 & 0.77 & 0.93 & 0.93 & 0.93 & 1.00 & 0.96 & 0.97 & 0.97 & 0.88 & 0.93 & 0.93 & 0.86 & 0.94 \\
0.68 & 0.85 & 0.89 & 0.93 & 0.77 & 0.93 & 0.93 & 0.93 & 0.96 & 1.00 & 0.96 & 0.96 & 0.88 & 0.93 & 0.93 & 0.86 & 0.94 \\
0.68 & 0.85 & 0.89 & 0.93 & 0.77 & 0.93 & 0.93 & 0.93 & 0.97 & 0.96 & 1.00 & 0.97 & 0.88 & 0.93 & 0.93 & 0.86 & 0.94 \\
0.68 & 0.85 & 0.89 & 0.93 & 0.77 & 0.93 & 0.93 & 0.93 & 0.97 & 0.96 & 0.97 & 1.00 & 0.88 & 0.93 & 0.93 & 0.86 & 0.94 \\
0.68 & 0.85 & 0.88 & 0.88 & 0.77 & 0.88 & 0.88 & 0.88 & 0.88 & 0.88 & 0.88 & 0.88 & 1.00 & 0.88 & 0.88 & 0.86 & 0.88 \\
0.68 & 0.85 & 0.89 & 0.97 & 0.77 & 0.95 & 0.95 & 0.95 & 0.93 & 0.93 & 0.93 & 0.93 & 0.88 & 1.00 & 0.95 & 0.86 & 0.93 \\
0.68 & 0.85 & 0.89 & 0.95 & 0.77 & 0.95 & 0.95 & 0.95 & 0.93 & 0.93 & 0.93 & 0.93 & 0.88 & 0.95 & 1.00 & 0.86 & 0.93 \\
0.68 & 0.85 & 0.86 & 0.86 & 0.77 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 1.00 & 0.86 \\
0.68 & 0.85 & 0.89 & 0.93 & 0.77 & 0.93 & 0.93 & 0.93 & 0.94 & 0.94 & 0.94 & 0.94 & 0.88 & 0.93 & 0.93 & 0.86 & 1.00
\end{bmatrix} \tag{2.17}
$$

The following matrices R_2 and R_{2T} represents the fuzzy compatible and transitive relation matrix for Mahalanobis metric distance.

$$
R_2 = \begin{bmatrix}
1.00 & 0.37 & 0.38 & 0.32 & 0.00 & 0.45 & 0.49 & 0.45 & 0.32 & 0.23 & 0.27 & 0.29 & 0.12 & 0.35 & 0.42 & 0.26 & 0.36 \\
0.37 & 1.00 & 0.75 & 0.63 & 0.21 & 0.51 & 0.52 & 0.56 & 0.45 & 0.52 & 0.58 & 0.42 & 0.39 & 0.65 & 0.56 & 0.07 & 0.43 \\
0.38 & 0.75 & 1.00 & 0.87 & 0.23 & 0.71 & 0.72 & 0.77 & 0.70 & 0.76 & 0.79 & 0.66 & 0.62 & 0.85 & 0.62 & 0.27 & 0.64 \\
0.32 & 0.63 & 0.87 & 1.00 & 0.28 & 0.77 & 0.77 & 0.82 & 0.81 & 0.88 & 0.91 & 0.78 & 0.74 & 0.91 & 0.63 & 0.35 & 0.73 \\
0.00 & 0.21 & 0.23 & 0.28 & 1.00 & 0.33 & 0.26 & 0.31 & 0.25 & 0.21 & 0.35 & 0.23 & 0.19 & 0.36 & 0.54 & 0.16 & 0.35 \\
0.45 & 0.51 & 0.71 & 0.77 & 0.33 & 1.00 & 0.93 & 0.94 & 0.84 & 0.71 & 0.76 & 0.80 & 0.63 & 0.80 & 0.73 & 0.55 & 0.90 \\
0.49 & 0.52 & 0.72 & 0.77 & 0.26 & 0.93 & 1.00 & 0.93 & 0.83 & 0.71 & 0.73 & 0.79 & 0.62 & 0.78 & 0.68 & 0.54 & 0.84 \\
0.45 & 0.56 & 0.77 & 0.82 & 0.31 & 0.94 & 0.93 & 1.00 & 0.84 & 0.75 & 0.79 & 0.80 & 0.65 & 0.84 & 0.72 & 0.50 & 0.85 \\
0.32 & 0.45 & 0.70 & 0.81 & 0.25 & 0.84 & 0.83 & 0.84 & 1.00 & 0.83 & 0.80 & 0.96 & 0.78 & 0.77 & 0.59 & 0.53 & 0.85 \\
0.23 & 0.52 & 0.76 & 0.88 & 0.21 & 0.71 & 0.71 & 0.75 & 0.83 & 1.00 & 0.86 & 0.82 & 0.86 & 0.80 & 0.53 & 0.36 & 0.70 \\
0.27 & 0.58 & 0.79 & 0.91 & 0.35 & 0.76 & 0.73 & 0.79 & 0.80 & 0.86 & 1.00 & 0.78 & 0.77 & 0.91 & 0.66 & 0.37 & 0.75 \\
0.29 & 0.42 & 0.66 & 0.78 & 0.23 & 0.80 & 0.79 & 0.80 & 0.96 & 0.82 & 0.78 & 1.00 & 0.80 & 0.74 & 0.56 & 0.53 & 0.83 \\
0.12 & 0.39 & 0.62 & 0.74 & 0.19 & 0.63 & 0.62 & 0.65 & 0.78 & 0.86 & 0.77 & 0.80 & 1.00 & 0.68 & 0.45 & 0.36 & 0.66 \\
0.35 & 0.65 & 0.85 & 0.91 & 0.36 & 0.80 & 0.78 & 0.84 & 0.77 & 0.80 & 0.91 & 0.74 & 0.68 & 1.00 & 0.72 & 0.37 & 0.75 \\
0.42 & 0.56 & 0.62 & 0.63 & 0.54 & 0.73 & 0.68 & 0.72 & 0.59 & 0.53 & 0.66 & 0.56 & 0.45 & 0.72 & 1.00 & 0.40 & 0.69 \\
0.26 & 0.07 & 0.27 & 0.35 & 0.16 & 0.55 & 0.54 & 0.50 & 0.53 & 0.36 & 0.37 & 0.53 & 0.36 & 0.37 & 0.40 & 1.00 & 0.61 \\
0.36 & 0.43 & 0.64 & 0.73 & 0.35 & 0.90 & 0.84 & 0.85 & 0.85 & 0.70 & 0.75 & 0.83 & 0.66 & 0.75 & 0.69 & 0.61 & 1.00
\end{bmatrix} \tag{2.18}
$$

$$
R_{2T} = \begin{bmatrix}
1.00 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 & 0.49 \\
0.49 & 1.00 & 0.75 & 0.75 & 0.54 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 & 0.75 & 0.73 & 0.61 & 0.75 \\
0.49 & 0.75 & 1.00 & 0.87 & 0.54 & 0.84 & 0.84 & 0.84 & 0.84 & 0.87 & 0.87 & 0.84 & 0.86 & 0.87 & 0.73 & 0.61 & 0.84 \\
0.49 & 0.75 & 0.87 & 1.00 & 0.54 & 0.84 & 0.84 & 0.84 & 0.84 & 0.88 & 0.91 & 0.84 & 0.86 & 0.91 & 0.73 & 0.61 & 0.84 \\
0.49 & 0.54 & 0.54 & 0.54 & 1.00 & 0.54 & 0.54 & 0.54 & 0.54 & 0.54 & 0.54 & 0.54 & 0.54 & 0.54 & 0.54 & 0.54 & 0.54 \\
0.49 & 0.75 & 0.84 & 0.84 & 0.54 & 1.00 & 0.93 & 0.94 & 0.85 & 0.84 & 0.84 & 0.85 & 0.84 & 0.84 & 0.73 & 0.61 & 0.90 \\
0.49 & 0.75 & 0.84 & 0.84 & 0.54 & 0.93 & 1.00 & 0.93 & 0.85 & 0.84 & 0.84 & 0.85 & 0.84 & 0.84 & 0.73 & 0.61 & 0.90 \\
0.49 & 0.75 & 0.84 & 0.84 & 0.54 & 0.94 & 0.93 & 1.00 & 0.85 & 0.84 & 0.84 & 0.85 & 0.84 & 0.84 & 0.73 & 0.61 & 0.90 \\
0.49 & 0.75 & 0.84 & 0.84 & 0.54 & 0.85 & 0.85 & 0.85 & 1.00 & 0.84 & 0.84 & 0.96 & 0.84 & 0.84 & 0.73 & 0.61 & 0.85 \\
0.49 & 0.75 & 0.87 & 0.88 & 0.54 & 0.84 & 0.84 & 0.84 & 0.84 & 1.00 & 0.88 & 0.84 & 0.86 & 0.88 & 0.73 & 0.61 & 0.84 \\
0.49 & 0.75 & 0.87 & 0.91 & 0.54 & 0.84 & 0.84 & 0.84 & 0.84 & 0.88 & 1.00 & 0.84 & 0.86 & 0.91 & 0.73 & 0.61 & 0.84 \\
0.49 & 0.75 & 0.84 & 0.84 & 0.54 & 0.85 & 0.85 & 0.85 & 0.96 & 0.84 & 0.84 & 1.00 & 0.84 & 0.84 & 0.73 & 0.61 & 0.85 \\
0.49 & 0.75 & 0.86 & 0.86 & 0.54 & 0.84 & 0.84 & 0.84 & 0.84 & 0.86 & 0.86 & 0.84 & 1.00 & 0.86 & 0.73 & 0.61 & 0.84 \\
0.49 & 0.75 & 0.87 & 0.91 & 0.54 & 0.84 & 0.84 & 0.84 & 0.84 & 0.88 & 0.91 & 0.84 & 0.86 & 1.00 & 0.73 & 0.61 & 0.84 \\
0.49 & 0.73 & 0.73 & 0.73 & 0.54 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 1.00 & 0.61 & 0.73 \\
0.49 & 0.61 & 0.61 & 0.61 & 0.54 & 0.61 & 0.61 & 0.61 & 0.61 & 0.61 & 0.61 & 0.61 & 0.61 & 0.61 & 0.61 & 1.00 & 0.61 \\
0.49 & 0.75 & 0.84 & 0.84 & 0.54 & 0.90 & 0.90 & 0.90 & 0.85 & 0.84 & 0.84 & 0.85 & 0.84 & 0.84 & 0.73 & 0.61 & 1.00
\end{bmatrix} \quad (2.19)
$$

The following matrices R_3 and R_{3T} represents the fuzzy compatible and transitive relation matrix for Cosine distance.

$$
R_3 = \begin{bmatrix}
1.00 & 0.41 & 0.55 & 0.65 & 0.76 & 0.92 & 0.90 & 0.87 & 0.82 & 0.58 & 0.72 & 0.82 & 0.60 & 0.76 & 0.93 & 0.91 & 0.93 \\
0.41 & 1.00 & 0.99 & 0.97 & 0.00 & 0.73 & 0.78 & 0.81 & 0.87 & 0.99 & 0.93 & 0.88 & 0.98 & 0.91 & 0.50 & 0.01 & 0.67 \\
0.55 & 0.99 & 1.00 & 0.99 & 0.15 & 0.82 & 0.86 & 0.88 & 0.93 & 1.00 & 0.97 & 0.93 & 1.00 & 0.95 & 0.62 & 0.19 & 0.76 \\
0.65 & 0.97 & 0.99 & 1.00 & 0.32 & 0.89 & 0.92 & 0.94 & 0.97 & 1.00 & 0.99 & 0.97 & 1.00 & 0.99 & 0.72 & 0.34 & 0.84 \\
0.76 & 0.00 & 0.15 & 0.32 & 1.00 & 0.73 & 0.67 & 0.65 & 0.56 & 0.21 & 0.46 & 0.55 & 0.26 & 0.50 & 0.90 & 0.93 & 0.79 \\
0.92 & 0.73 & 0.82 & 0.89 & 0.73 & 1.00 & 1.00 & 0.99 & 0.97 & 0.84 & 0.94 & 0.97 & 0.86 & 0.95 & 0.96 & 0.77 & 1.00 \\
0.90 & 0.78 & 0.86 & 0.92 & 0.67 & 1.00 & 1.00 & 1.00 & 0.99 & 0.88 & 0.95 & 0.99 & 0.89 & 0.97 & 0.93 & 0.72 & 0.98 \\
0.87 & 0.81 & 0.88 & 0.94 & 0.65 & 0.99 & 1.00 & 1.00 & 0.99 & 0.90 & 0.97 & 0.99 & 0.92 & 0.98 & 0.92 & 0.69 & 0.98 \\
0.82 & 0.87 & 0.93 & 0.97 & 0.56 & 0.97 & 0.99 & 0.99 & 1.00 & 0.94 & 0.99 & 1.00 & 0.95 & 0.99 & 0.87 & 0.60 & 0.95 \\
0.58 & 0.99 & 1.00 & 1.00 & 0.21 & 0.84 & 0.88 & 0.90 & 0.94 & 1.00 & 0.98 & 0.95 & 1.00 & 0.97 & 0.65 & 0.24 & 0.79 \\
0.72 & 0.93 & 0.97 & 0.99 & 0.46 & 0.94 & 0.95 & 0.97 & 0.99 & 0.98 & 1.00 & 0.99 & 0.98 & 1.00 & 0.81 & 0.47 & 0.90 \\
0.82 & 0.88 & 0.93 & 0.97 & 0.55 & 0.97 & 0.99 & 0.99 & 1.00 & 0.95 & 0.99 & 1.00 & 0.96 & 1.00 & 0.87 & 0.59 & 0.95 \\
0.60 & 0.98 & 1.00 & 1.00 & 0.26 & 0.86 & 0.89 & 0.92 & 0.95 & 1.00 & 0.98 & 0.96 & 1.00 & 0.97 & 0.68 & 0.28 & 0.81 \\
0.76 & 0.91 & 0.95 & 0.99 & 0.50 & 0.95 & 0.97 & 0.98 & 0.99 & 0.97 & 1.00 & 1.00 & 0.97 & 1.00 & 0.84 & 0.52 & 0.92 \\
0.93 & 0.50 & 0.62 & 0.72 & 0.90 & 0.96 & 0.93 & 0.92 & 0.87 & 0.65 & 0.81 & 0.87 & 0.68 & 0.84 & 1.00 & 0.90 & 0.98 \\
0.91 & 0.01 & 0.19 & 0.34 & 0.93 & 0.77 & 0.72 & 0.69 & 0.60 & 0.24 & 0.47 & 0.59 & 0.28 & 0.52 & 0.90 & 1.00 & 0.83 \\
0.93 & 0.67 & 0.76 & 0.84 & 0.79 & 1.00 & 0.98 & 0.98 & 0.95 & 0.79 & 0.90 & 0.95 & 0.81 & 0.92 & 0.98 & 0.83 & 1.00
\end{bmatrix} \quad (2.20)
$$

$$
R_{3T} = \begin{bmatrix}
1.00 & 0.93 & 0.93 & 0.93 & 0.91 & 0.93 & 0.93 & 0.93 & 0.93 & 0.93 & 0.93 & 0.93 & 0.93 & 0.93 & 0.93 & 0.91 & 0.93 \\
0.93 & 1.00 & 0.99 & 0.99 & 0.91 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.98 & 0.91 & 0.99 \\
0.93 & 0.99 & 1.00 & 1.00 & 0.91 & 0.99 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 0.99 & 1.00 & 0.99 & 0.98 & 0.91 & 0.99 \\
0.93 & 0.99 & 1.00 & 1.00 & 0.91 & 0.99 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 0.99 & 1.00 & 0.99 & 0.98 & 0.91 & 0.99 \\
0.91 & 0.91 & 0.91 & 0.91 & 1.00 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.93 & 0.91 \\
0.93 & 0.99 & 0.99 & 0.99 & 0.91 & 1.00 & 1.00 & 1.00 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.98 & 0.91 & 1.00 \\
0.93 & 0.99 & 0.99 & 0.99 & 0.91 & 1.00 & 1.00 & 1.00 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.98 & 0.91 & 1.00 \\
0.93 & 0.99 & 0.99 & 0.99 & 0.91 & 1.00 & 1.00 & 1.00 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.98 & 0.91 & 1.00 \\
0.93 & 0.99 & 0.99 & 0.99 & 0.91 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 1.00 & 1.00 & 0.99 & 1.00 & 0.98 & 0.91 & 0.99 \\
0.93 & 0.99 & 1.00 & 1.00 & 0.91 & 0.99 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 0.99 & 1.00 & 0.99 & 0.98 & 0.91 & 0.99 \\
0.93 & 0.99 & 0.99 & 0.99 & 0.91 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 1.00 & 1.00 & 0.99 & 1.00 & 0.98 & 0.91 & 0.99 \\
0.93 & 0.99 & 0.99 & 0.99 & 0.91 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 1.00 & 1.00 & 0.99 & 1.00 & 0.98 & 0.91 & 0.99 \\
0.93 & 0.99 & 1.00 & 1.00 & 0.91 & 0.99 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 0.99 & 1.00 & 0.99 & 0.98 & 0.91 & 0.99 \\
0.93 & 0.99 & 0.99 & 0.99 & 0.91 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 1.00 & 1.00 & 0.99 & 1.00 & 0.98 & 0.91 & 0.99 \\
0.93 & 0.98 & 0.98 & 0.98 & 0.91 & 0.98 & 0.98 & 0.98 & 0.98 & 0.98 & 0.98 & 0.98 & 0.98 & 0.98 & 1.00 & 0.91 & 0.98 \\
0.91 & 0.91 & 0.91 & 0.91 & 0.93 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 0.91 & 1.00 & 0.91 \\
0.93 & 0.99 & 0.99 & 0.99 & 0.91 & 1.00 & 1.00 & 1.00 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 0.98 & 0.91 & 1.00
\end{bmatrix} \qquad (2.21)
$$

The following matrices R_4 and R_{4T} represents the fuzzy compatible and transitive relation matrix for Chebychev distance.

$$
R_4 = \begin{bmatrix}
1.00 & 0.66 & 0.54 & 0.43 & 0.00 & 0.46 & 0.52 & 0.49 & 0.38 & 0.36 & 0.34 & 0.35 & 0.24 & 0.42 & 0.44 & 0.19 & 0.37 \\
0.66 & 1.00 & 0.84 & 0.69 & 0.09 & 0.60 & 0.68 & 0.66 & 0.57 & 0.64 & 0.58 & 0.54 & 0.51 & 0.65 & 0.54 & 0.27 & 0.48 \\
0.54 & 0.84 & 1.00 & 0.86 & 0.25 & 0.76 & 0.84 & 0.82 & 0.73 & 0.80 & 0.74 & 0.70 & 0.67 & 0.81 & 0.71 & 0.43 & 0.64 \\
0.43 & 0.69 & 0.86 & 1.00 & 0.39 & 0.91 & 0.90 & 0.92 & 0.88 & 0.92 & 0.89 & 0.85 & 0.81 & 0.96 & 0.85 & 0.58 & 0.79 \\
0.00 & 0.09 & 0.25 & 0.39 & 1.00 & 0.48 & 0.41 & 0.43 & 0.52 & 0.44 & 0.50 & 0.54 & 0.58 & 0.43 & 0.54 & 0.73 & 0.60 \\
0.46 & 0.60 & 0.76 & 0.91 & 0.48 & 1.00 & 0.93 & 0.94 & 0.90 & 0.85 & 0.88 & 0.87 & 0.74 & 0.95 & 0.94 & 0.67 & 0.88 \\
0.52 & 0.68 & 0.84 & 0.90 & 0.41 & 0.93 & 1.00 & 0.97 & 0.86 & 0.82 & 0.82 & 0.83 & 0.71 & 0.90 & 0.87 & 0.60 & 0.81 \\
0.49 & 0.66 & 0.82 & 0.92 & 0.43 & 0.94 & 0.97 & 1.00 & 0.89 & 0.84 & 0.86 & 0.86 & 0.73 & 0.94 & 0.89 & 0.61 & 0.83 \\
0.38 & 0.57 & 0.73 & 0.88 & 0.52 & 0.90 & 0.86 & 0.89 & 1.00 & 0.93 & 0.96 & 0.97 & 0.84 & 0.92 & 0.85 & 0.70 & 0.91 \\
0.36 & 0.64 & 0.80 & 0.92 & 0.44 & 0.85 & 0.82 & 0.84 & 0.93 & 1.00 & 0.94 & 0.90 & 0.87 & 0.90 & 0.81 & 0.63 & 0.84 \\
0.34 & 0.58 & 0.74 & 0.89 & 0.50 & 0.88 & 0.82 & 0.86 & 0.96 & 0.94 & 1.00 & 0.96 & 0.87 & 0.92 & 0.83 & 0.69 & 0.90 \\
0.35 & 0.54 & 0.70 & 0.85 & 0.54 & 0.87 & 0.83 & 0.86 & 0.97 & 0.90 & 0.96 & 1.00 & 0.87 & 0.89 & 0.83 & 0.73 & 0.94 \\
0.24 & 0.51 & 0.67 & 0.81 & 0.58 & 0.74 & 0.71 & 0.73 & 0.84 & 0.87 & 0.87 & 0.87 & 1.00 & 0.79 & 0.70 & 0.76 & 0.81 \\
0.42 & 0.65 & 0.81 & 0.96 & 0.43 & 0.95 & 0.90 & 0.94 & 0.92 & 0.90 & 0.92 & 0.89 & 0.79 & 1.00 & 0.89 & 0.62 & 0.83 \\
0.44 & 0.54 & 0.71 & 0.85 & 0.54 & 0.94 & 0.87 & 0.89 & 0.85 & 0.81 & 0.83 & 0.83 & 0.70 & 0.89 & 1.00 & 0.73 & 0.88 \\
0.19 & 0.27 & 0.43 & 0.58 & 0.73 & 0.67 & 0.60 & 0.61 & 0.70 & 0.63 & 0.69 & 0.73 & 0.76 & 0.62 & 0.73 & 1.00 & 0.79 \\
0.37 & 0.48 & 0.64 & 0.79 & 0.60 & 0.88 & 0.81 & 0.83 & 0.91 & 0.84 & 0.90 & 0.94 & 0.81 & 0.83 & 0.88 & 0.79 & 1.00
\end{bmatrix} \qquad (2.22)
$$

$$R_{4T} = \begin{bmatrix} 1.00 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 & 0.66 \\ 0.66 & 1.00 & 0.84 & 0.84 & 0.73 & 0.84 & 0.84 & 0.84 & 0.84 & 0.84 & 0.84 & 0.84 & 0.84 & 0.84 & 0.84 & 0.79 & 0.84 \\ 0.66 & 0.84 & 1.00 & 0.86 & 0.73 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.86 & 0.79 & 0.86 \\ 0.66 & 0.84 & 0.86 & 1.00 & 0.73 & 0.95 & 0.94 & 0.94 & 0.92 & 0.92 & 0.92 & 0.92 & 0.87 & 0.96 & 0.94 & 0.79 & 0.92 \\ 0.66 & 0.73 & 0.73 & 0.73 & 1.00 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 & 0.73 \\ 0.66 & 0.84 & 0.86 & 0.95 & 0.73 & 1.00 & 0.94 & 0.94 & 0.92 & 0.92 & 0.92 & 0.92 & 0.87 & 0.95 & 0.94 & 0.79 & 0.92 \\ 0.66 & 0.84 & 0.86 & 0.94 & 0.73 & 0.94 & 1.00 & 0.97 & 0.92 & 0.92 & 0.92 & 0.92 & 0.87 & 0.94 & 0.94 & 0.79 & 0.92 \\ 0.66 & 0.84 & 0.86 & 0.94 & 0.73 & 0.94 & 0.97 & 1.00 & 0.92 & 0.92 & 0.92 & 0.92 & 0.87 & 0.94 & 0.94 & 0.79 & 0.92 \\ 0.66 & 0.84 & 0.86 & 0.92 & 0.73 & 0.92 & 0.92 & 0.92 & 1.00 & 0.94 & 0.96 & 0.97 & 0.87 & 0.92 & 0.92 & 0.79 & 0.94 \\ 0.66 & 0.84 & 0.86 & 0.92 & 0.73 & 0.92 & 0.92 & 0.92 & 0.94 & 1.00 & 0.94 & 0.94 & 0.87 & 0.92 & 0.92 & 0.79 & 0.94 \\ 0.66 & 0.84 & 0.86 & 0.92 & 0.73 & 0.92 & 0.92 & 0.92 & 0.96 & 0.94 & 1.00 & 0.96 & 0.87 & 0.92 & 0.92 & 0.79 & 0.94 \\ 0.66 & 0.84 & 0.86 & 0.92 & 0.73 & 0.92 & 0.92 & 0.92 & 0.97 & 0.94 & 0.96 & 1.00 & 0.87 & 0.92 & 0.92 & 0.79 & 0.94 \\ 0.66 & 0.84 & 0.86 & 0.87 & 0.73 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 1.00 & 0.87 & 0.87 & 0.79 & 0.87 \\ 0.66 & 0.84 & 0.86 & 0.96 & 0.73 & 0.95 & 0.94 & 0.94 & 0.92 & 0.92 & 0.92 & 0.92 & 0.87 & 1.00 & 0.94 & 0.79 & 0.92 \\ 0.66 & 0.84 & 0.86 & 0.94 & 0.73 & 0.94 & 0.94 & 0.94 & 0.92 & 0.92 & 0.92 & 0.92 & 0.87 & 0.94 & 1.00 & 0.79 & 0.92 \\ 0.66 & 0.79 & 0.79 & 0.79 & 0.73 & 0.79 & 0.79 & 0.79 & 0.79 & 0.79 & 0.79 & 0.79 & 0.79 & 0.79 & 0.79 & 1.00 & 0.79 \\ 0.66 & 0.84 & 0.86 & 0.92 & 0.73 & 0.92 & 0.92 & 0.92 & 0.94 & 0.94 & 0.94 & 0.94 & 0.87 & 0.92 & 0.92 & 0.79 & 1.00 \end{bmatrix} \quad (2.23)$$

The following matrices R_5 and R_{5T} represents the fuzzy compatible and transitive relation matrix for Correlation distance.

$$R_5 = \begin{bmatrix} 1.00 & 0.36 & 0.85 & 0.68 & 0.00 & 0.77 & 0.87 & 0.79 & 0.81 & 0.75 & 0.43 & 0.81 & 0.59 & 0.53 & 0.46 & 0.80 & 0.69 \\ 0.36 & 1.00 & 0.83 & 0.94 & 0.96 & 0.90 & 0.81 & 0.88 & 0.87 & 0.91 & 1.00 & 0.87 & 0.97 & 0.99 & 1.00 & 0.87 & 0.94 \\ 0.85 & 0.83 & 1.00 & 0.97 & 0.62 & 0.99 & 1.00 & 1.00 & 1.00 & 0.99 & 0.86 & 1.00 & 0.94 & 0.91 & 0.88 & 1.00 & 0.97 \\ 0.68 & 0.94 & 0.97 & 1.00 & 0.81 & 0.99 & 0.96 & 0.99 & 0.98 & 1.00 & 0.96 & 0.98 & 0.99 & 0.98 & 0.97 & 0.99 & 1.00 \\ 0.00 & 0.96 & 0.62 & 0.81 & 1.00 & 0.73 & 0.59 & 0.70 & 0.68 & 0.75 & 0.94 & 0.68 & 0.87 & 0.90 & 0.93 & 0.69 & 0.80 \\ 0.77 & 0.90 & 0.99 & 0.99 & 0.73 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.92 & 1.00 & 0.97 & 0.96 & 0.94 & 1.00 & 0.99 \\ 0.87 & 0.81 & 1.00 & 0.96 & 0.59 & 0.99 & 1.00 & 0.99 & 0.99 & 0.98 & 0.84 & 0.99 & 0.92 & 0.90 & 0.86 & 0.99 & 0.96 \\ 0.79 & 0.88 & 1.00 & 0.99 & 0.70 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.91 & 1.00 & 0.97 & 0.95 & 0.92 & 1.00 & 0.99 \\ 0.81 & 0.87 & 1.00 & 0.98 & 0.68 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.90 & 1.00 & 0.96 & 0.94 & 0.91 & 1.00 & 0.99 \\ 0.75 & 0.91 & 0.99 & 1.00 & 0.75 & 1.00 & 0.98 & 1.00 & 1.00 & 1.00 & 0.93 & 1.00 & 0.98 & 0.97 & 0.95 & 1.00 & 1.00 \\ 0.43 & 1.00 & 0.86 & 0.96 & 0.94 & 0.92 & 0.84 & 0.91 & 0.90 & 0.93 & 1.00 & 0.90 & 0.99 & 0.99 & 1.00 & 0.90 & 0.96 \\ 0.81 & 0.87 & 1.00 & 0.98 & 0.68 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.90 & 1.00 & 0.96 & 0.94 & 0.91 & 1.00 & 0.99 \\ 0.59 & 0.97 & 0.94 & 0.99 & 0.87 & 0.97 & 0.92 & 0.97 & 0.96 & 0.98 & 0.99 & 0.96 & 1.00 & 1.00 & 0.99 & 0.96 & 0.99 \\ 0.53 & 0.99 & 0.91 & 0.98 & 0.90 & 0.96 & 0.90 & 0.95 & 0.94 & 0.97 & 0.99 & 0.94 & 1.00 & 1.00 & 1.00 & 0.94 & 0.98 \\ 0.46 & 1.00 & 0.88 & 0.97 & 0.93 & 0.94 & 0.86 & 0.92 & 0.91 & 0.95 & 1.00 & 0.91 & 0.99 & 1.00 & 1.00 & 0.92 & 0.97 \\ 0.80 & 0.87 & 1.00 & 0.99 & 0.69 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.90 & 1.00 & 0.96 & 0.94 & 0.92 & 1.00 & 0.99 \\ 0.69 & 0.94 & 0.97 & 1.00 & 0.80 & 0.99 & 0.96 & 0.99 & 0.99 & 1.00 & 0.96 & 0.99 & 0.99 & 0.98 & 0.97 & 0.99 & 1.00 \end{bmatrix} \quad (2.24)$$

$$
R_{5T} = \begin{bmatrix}
1.00 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 & 0.87 \\
0.87 & 1.00 & 0.99 & 0.99 & 0.96 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.99 & 0.99 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00 \\
0.87 & 0.96 & 0.96 & 0.96 & 1.00 & 0.96 & 0.96 & 0.96 & 0.96 & 0.96 & 0.96 & 0.96 & 0.96 & 0.96 & 0.96 & 0.96 & 0.96 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00 \\
0.87 & 1.00 & 0.99 & 0.99 & 0.96 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.99 & 0.99 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00 \\
0.87 & 1.00 & 0.99 & 0.99 & 0.96 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.99 & 0.99 \\
0.87 & 1.00 & 0.99 & 0.99 & 0.96 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.99 & 0.99 \\
0.87 & 1.00 & 0.99 & 0.99 & 0.96 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 & 1.00 & 0.99 & 1.00 & 1.00 & 1.00 & 0.99 & 0.99 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00 \\
0.87 & 0.99 & 1.00 & 1.00 & 0.96 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 0.99 & 1.00 & 0.99 & 0.99 & 0.99 & 1.00 & 1.00
\end{bmatrix} \qquad (2.25)
$$

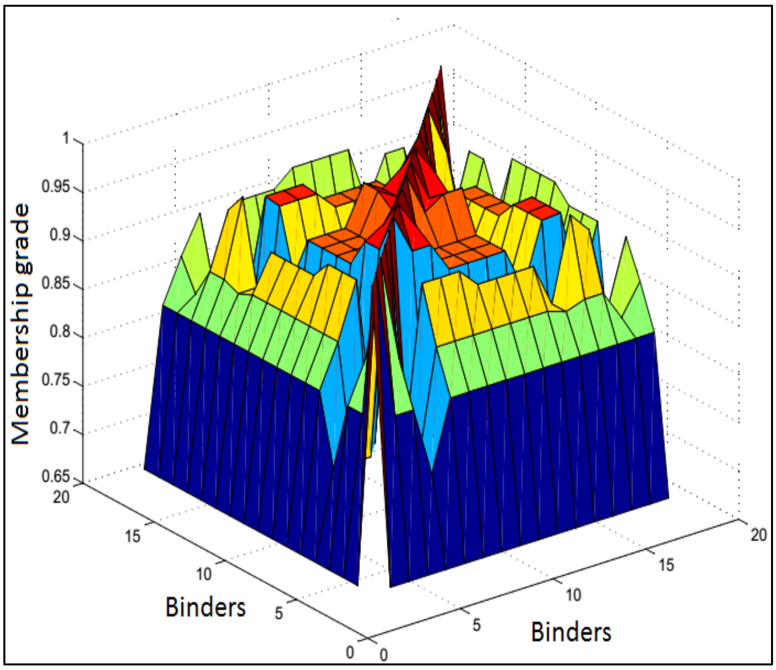


Figure 2.1. The graphical representation of results achieved by Minskowski distance.

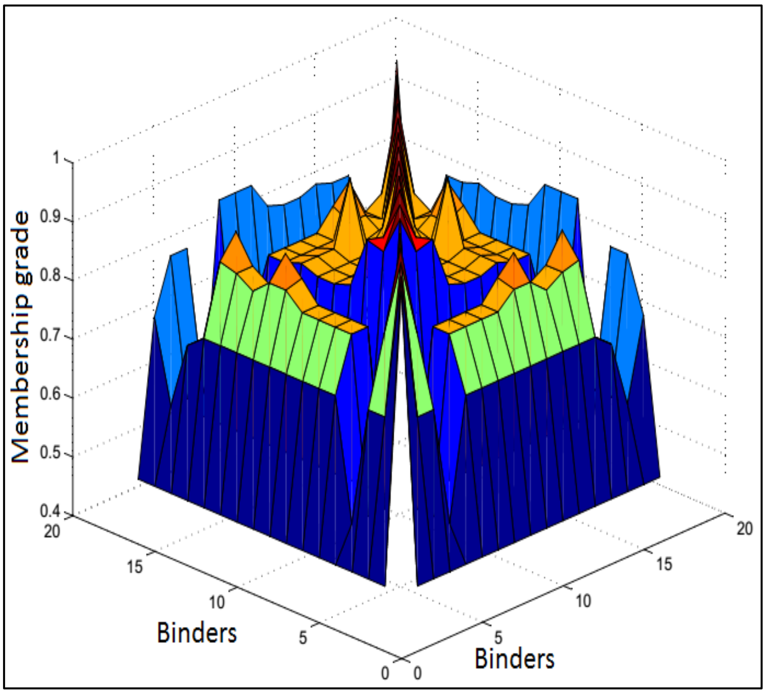


Figure 2.2. The graphical representation of results achieved by Mahalanobis distance

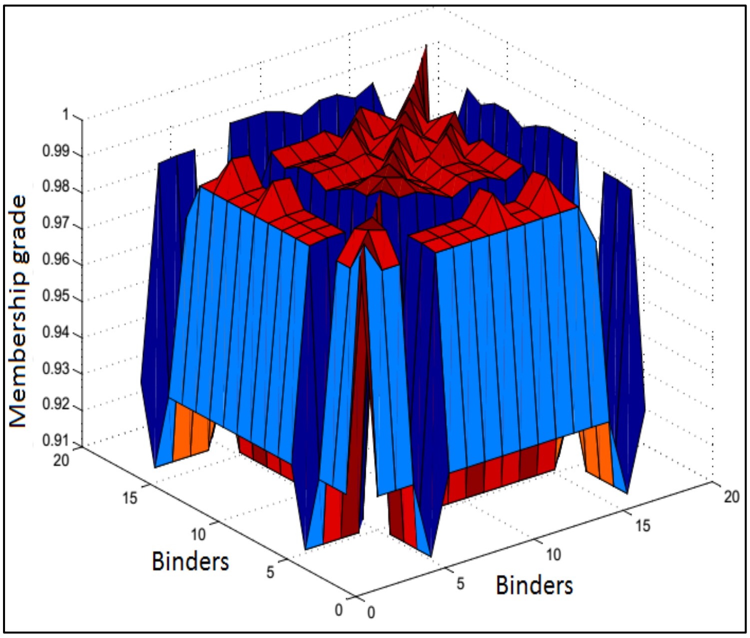


Figure 2.3. The graphical representation of results achieved by Cosine distance.

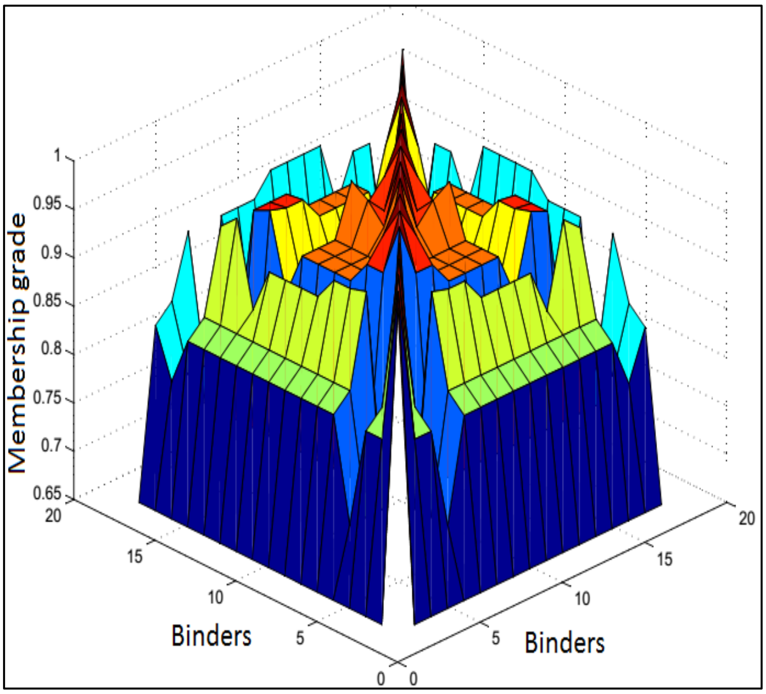


Figure 2.4. The graphical representation of results achieved by Chebychev distance.

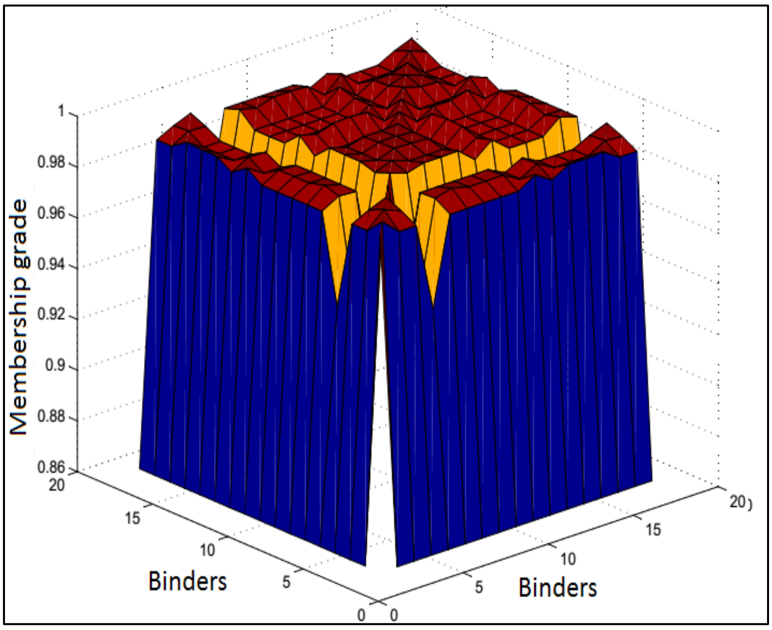


Figure 2.5. The graphical representation of results achieved by Correlation distance.

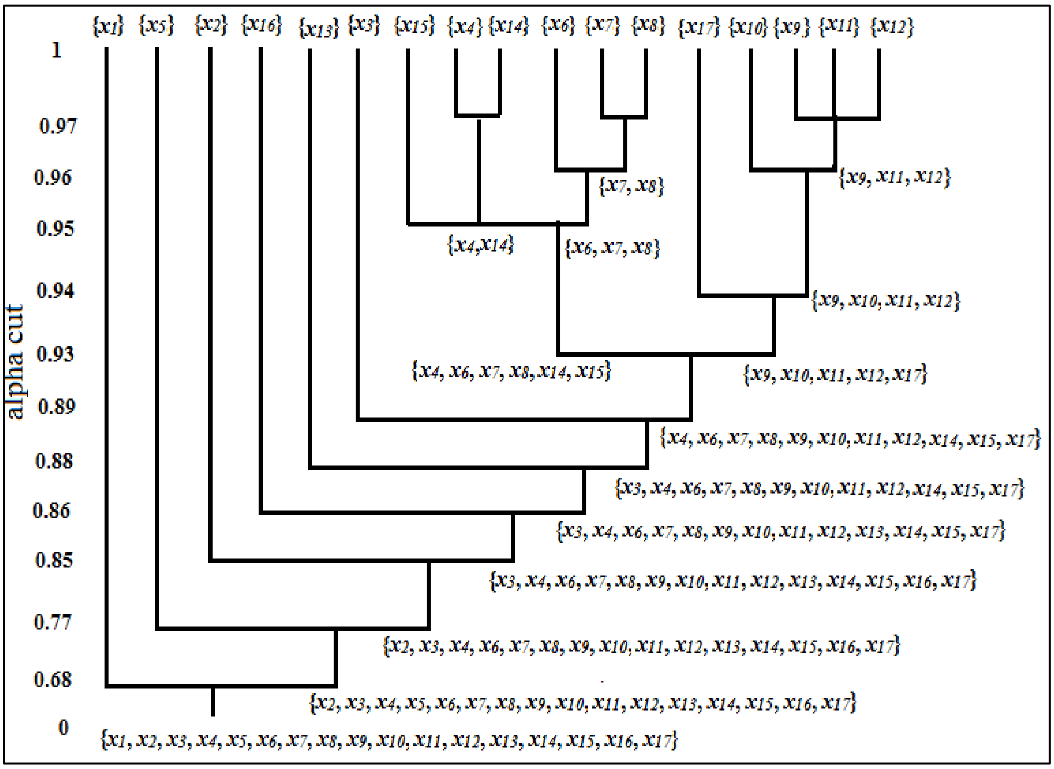


Figure 2.6: Clustering tree by Minkowski distance

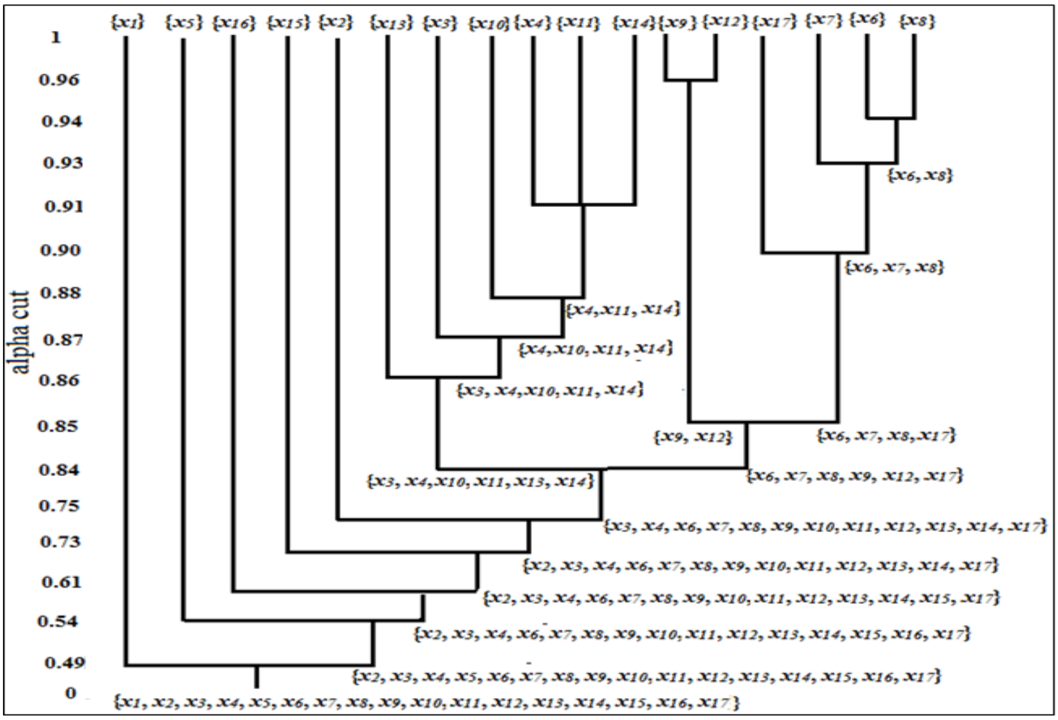


Figure 2.7: Clustering tree by Mahalanobis distance

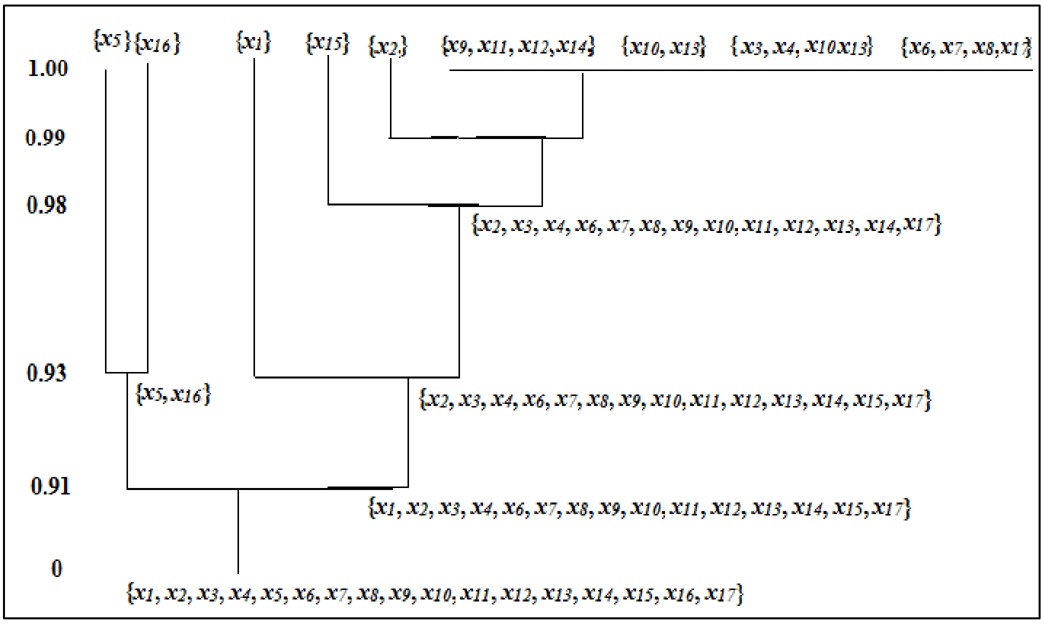


Figure 2.8: Clustering tree by Cosine distance.

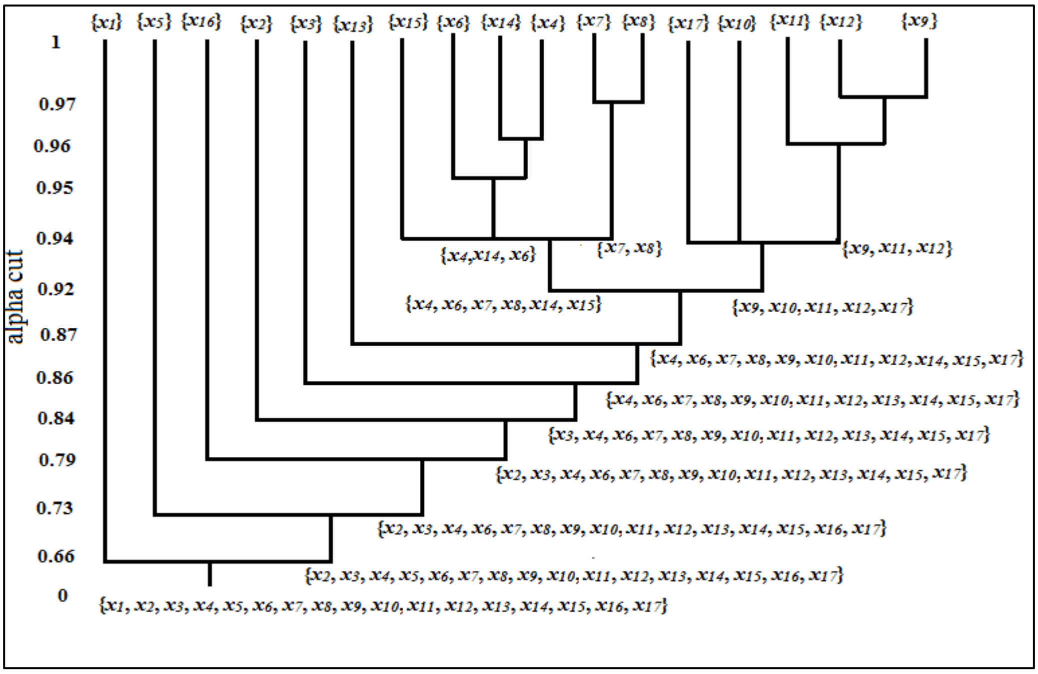


Figure 2.9: Clustering tree by Chebychev distance.

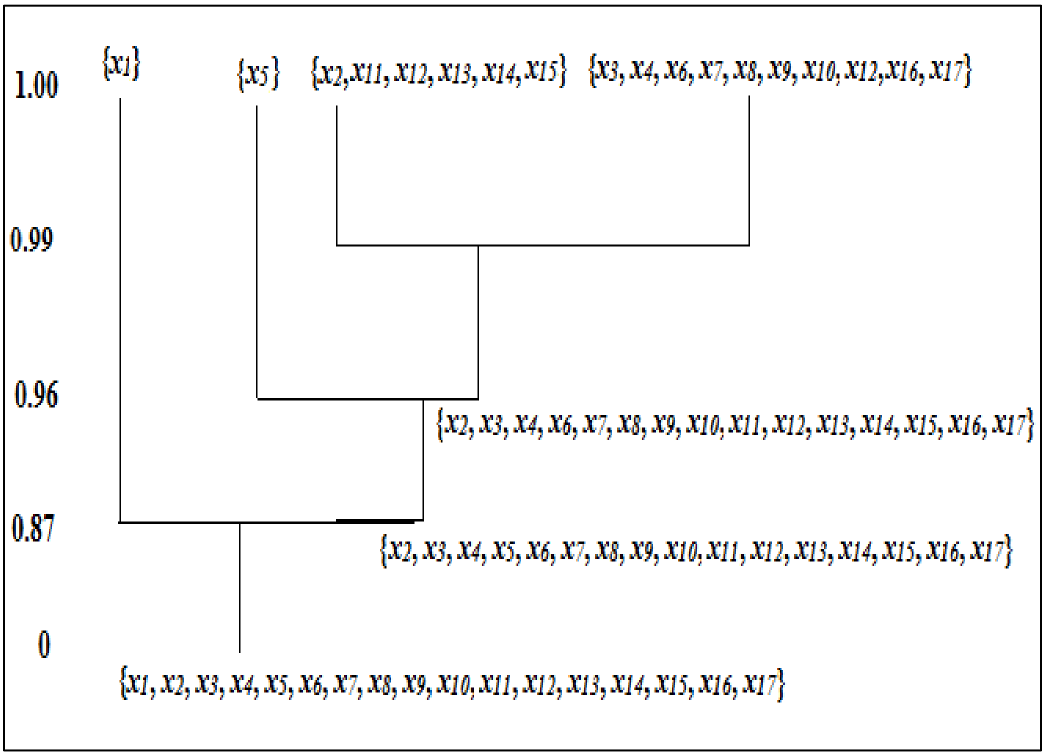


Figure 2.10: Clustering tree by Correlation distance.

2.7 Analysis

All the binders are separated according to their performances at a different level of α by using Minkowski, Mahalanobis, Cosine, Correlation, and Chebyshev distance differently. It is observed by figure 6 that Binder $x_1 =$ flux (B6224) is detached first at $\alpha = 0.68$, then Polymer-modified binder x_5 =ElvaloyNo.1 (B6228) is separated at $\alpha = 0.77$and after that $x_2 =$ unmodified base (B6225) separated at $\alpha = 0.85$. Similarly $x_{16}, x_{13} \dots, x_{12}$ are clustered successively at a different high degree of α. The desired clusters are also identified from the clustering tree for a suitable value of α for example if $\alpha = 0.90$,then the clusters are:

$$\{\{x_1\},\{x_5\},\{x_2\},\{x_{16}\},\{x_{13}\},\{x_3\},\{x_4,x_6,x_7,x_8,x_{14},x_{15}\},\{x_9,x_{10},x_{11},x_{12},x_{17}\}\}.$$

The total number of clusters can be obtained from clustering tree by Minkowski metric distance for different alpha and they are described as: If $\alpha \in [0\ 0.68]$ then N(c) = 1, If $\alpha \in (0.68\ \ 0.77]$ then N(c) =2. Similarly, if $\alpha \in (\ 0.77\ \ 0.85]$, N(c) = 3. If $\alpha \in (0.85\ \ 0.86]$, N(c) =4.

If $\alpha \in (0.86\ \ 0.88]$, N(c) = 5. If $\alpha \in (0.88\ \ 0.89]$, N(c) = 6. If $\alpha \in (0.89\ \ 0.93]$, N(c) =7.

If$\alpha \in (0.93\ \ 0.94]$, N(c) =8. If $\alpha \in (0.94\ \ 0.95]$,N(c) =9. If$\alpha \in (0.95\ \ 0.96]$, N(c) =11.

If $\alpha \in (0.96\ \ 0.97]$, N(c) =13. If $\alpha \in (0.97\ \ 1]$ N(c) =17.

According to the Clustering tree by Mahalanobis metric distances, it is observed by Figure 2.7 that the binder $x_1 =$ flux (B6224) is detached the first at$\alpha = 0.49$, then Polymer-modified binder x_5 =ElvaloyNo.1 (B6228) is separated at $\alpha = 0.54$.After that $x_{16} =$ B6252 is separated at $\alpha = 0.64$. similarly $x_1, x_2 \dots, x_8$ are clustered successively at a different high degree of α. The desired clusters are also identified for a suitable value of α. If the $\alpha = 0.90$, then the binders are separated differently as compared to Minkowski, and other distance. The clusters are:

$$\{\{x_1\},\{x_5\},\{x_{16}\},\{x_{15}\},\{x_2\},\{x_{13}\},\{x_3\},\{x_{10}\}\{x_4,x_{11},x_{14}\},\{x_9,x_{12}\},\{x_6,x_7,x_8\}\}$$

The total number of clusters can be obtained from clustering tree by Mahalonobis metric distance or different alpha and they are described as: If $\alpha \in [0\ 0.49]$ then N(c) =1. If $\alpha \in (0.49\ 0.54]$ then N(c) =2. Similarly, If $\alpha \in (0.54\ 0.61]$,N(c) = 3. If $\alpha \in (0.61\ 0.73]$, N(c) = 4.

If $\alpha \in (0.73\ 0.75]$, N(c) = 5. If$\alpha \in (0.75\ 0.84]$, N(c) = 6. If $\alpha \in (0.84\ 0.85]$, N(c) = 7. If $\alpha \in (0.85\ 0.86]$,N(c) = 8. If $\alpha \in (0.86\ 0.87]$, N(c) = 9. If $\alpha \in (0.87\ 0.88]$, N(c) =10.If $\alpha \in (0.88\ 0.90]$, N(c) =11. If $\alpha \in (0.90\ 0.91]$,N(c) = 12. If $\alpha \in (0.91\ 0.93]$, N(c) =14.If $\alpha \in (0.93\ 0.94]$, N(c) =15. If $\alpha \in (0.94\ 0.96]$, N(c) = 16. If $\alpha \in (0.96\ 1]$, N(c) = 17.

According to the Clustering tree by Correlation distances, it is observed by figure 8 that Binder $x_1 =$ flux (B6224) is detached first at $\alpha = 0.87$, then Polymer-modified binder x_5 =ElvaloyNo.1 (B6228) is separated at $\alpha = 0.96$. Similarly, the remaining binders are clustered into two groups after $\alpha = 0.99$. The desired clusters are also identified from the clustering tree for a suitable value of α for example if $\alpha = 0.90$, then the clusters are:

$$\{\{x_1\},,\{x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12},x_{13},x_{14},x_{15},x_{16},x_{17}\}\}.$$

The total number of clusters can be obtained from clustering tree by Correlation distance for different alpha and they are described as: If $\alpha \in [0\ 0.87]$ then N(c) = 1, If $\alpha \in (0.87\ 0.96]$ then N(c) =2. Similarly, if $\alpha \in (\ 0.96\ 0.99]$, N(c) = 3. If $\alpha \in (0.99\ 1]$, N(c) =4.

According to the Clustering tree by Chebychev distances, It is observed by figure 9 that Binder $x_1 =$ flux (B6224) is detached first at $\alpha = 0.66$, then Polymer-modified binder x_5 =ElvaloyNo.1 (B6228) is separated at $\alpha = 0.73$ and after that $x_{16} =$ polymer-modified Ethylene–Styrene–Inter polymer No. 2 (B6252) is separated at $\alpha = 0.79$. Similarly $x_2, x_3 \ldots, x_9$ are clustered successively at a different high degree of α. The binders are separated differently from the Mahalanobis Cosine and Correlation distances, but the separation is similar to the Minkowski metric distance. The desired clusters are also identified from the clustering tree for a suitable value of α for example if $\alpha = 0.90$, then the clusters are:

$$\{\{x_1\}, \{x_5\}, \{x_{16}\}, \{x_2\}, \{x_3\}, \{x_{13}\}, \{x_4, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{14}, x_{15}, x_{17}\}\}.$$

The total number of clusters can be obtained from clustering tree by Chebychev distance for different alpha and they are described as: If $\alpha \in [0\ 0.66]$ then N(c) = 1, If $\alpha \in (0.66\ \ 0.73]$ then N(c) =2. Similarly, if $\alpha \in (\ 0.73\ \ 0.79]$, N(c) = 3. If $\alpha \in (0.79\ \ 0.84]$, N(c) =4.

If $\alpha \in (0.84\ \ 0.86]$, N(c) = 5. If $\alpha \in (0.86\ \ 0.87]$ N(c) = 6. If $\alpha \in (0.87\ \ 0.92]$, N(c) =7.

If$\alpha \in (0.92\ \ 0.94]$, N(c) =8. If $\alpha \in (0.94\ \ 0.95]$,N(c) =12. If$\alpha \in (0.95\ \ 0.96]$, N(c) =13.

If $\alpha \in (0.96\ \ 0.97]$, N(c) =15. If $\alpha \in (0.97\ \ 1]$ N(c) =17.

According to the Clustering tree by Cosine distances, it is observed by figure 10 that no binder is separated till $\alpha = 0.91$ and after $\alpha = 0.91$ two binders x_5 =ElvaloyNo.1 (B6228) and x_{16} = polymer-modified Ethylene–Styrene–Inter polymer No. 2 (B6252) are detached together as one cluster. After the $\alpha = 0.93$ binder x_5 , x_{16} and binder $x_1 =$ flux (B6224) are detached separately. Similarly, remaining binders are clustered successively at a different high degree of α. The desired clusters are also identified from the clustering tree for a suitable value of α.

The total number of clusters can be obtained from clustering tree for different alpha and they are described as: If $\alpha \in [0\ 0.91]$ then N(c) = 1, If $\alpha \in (0.91\ \ 0.93]$ then N(c) =2. Similarly, if $\alpha \in (\ 0.93\ \ 0.98]$, N(c) = 4. If $\alpha \in (0.98\ \ 0.99]$, N(c) =5. If $\alpha \in (0.99\ \ 1)$, N(c) = 6. If $\alpha = 1$ N(c) = 9.

The following graph shows the number of clusters achieved by Mahalanobis, Chebyshev, Minkowski($w = 2$), Cosine, and Correlation distance concerning different membership grades. All distances illustrate the same number of clusters (N(c) =1) till membership grade $\alpha = 0.49$. After the $\alpha = 0.49$, there exists a significant difference in the number of clusters by all five distance functions. The Mahalanobis distance quantize a greater number of a cluster than the other four distances function for each $\alpha \in (0.49\ \ 0.97]$. The clustering performance achieved by the Chebychev distance function is quite better than Minkowski$(w = 2)$ distance and substantially finer than the Cosine and Correlation distance function. The Mahalanobis, Chebychev, and Minkowski$(w = 2)$ demonstrate the same number of clusters (N(c) =17) for each$\alpha \in (0.97\ \ 1]$. Overall, the Mahalanobis distance shows the viable feasibility compared to Chebychev, Minkowski$(w = 2)$, Cosine, and Correlation distance function in terms of the desired number of clusters.

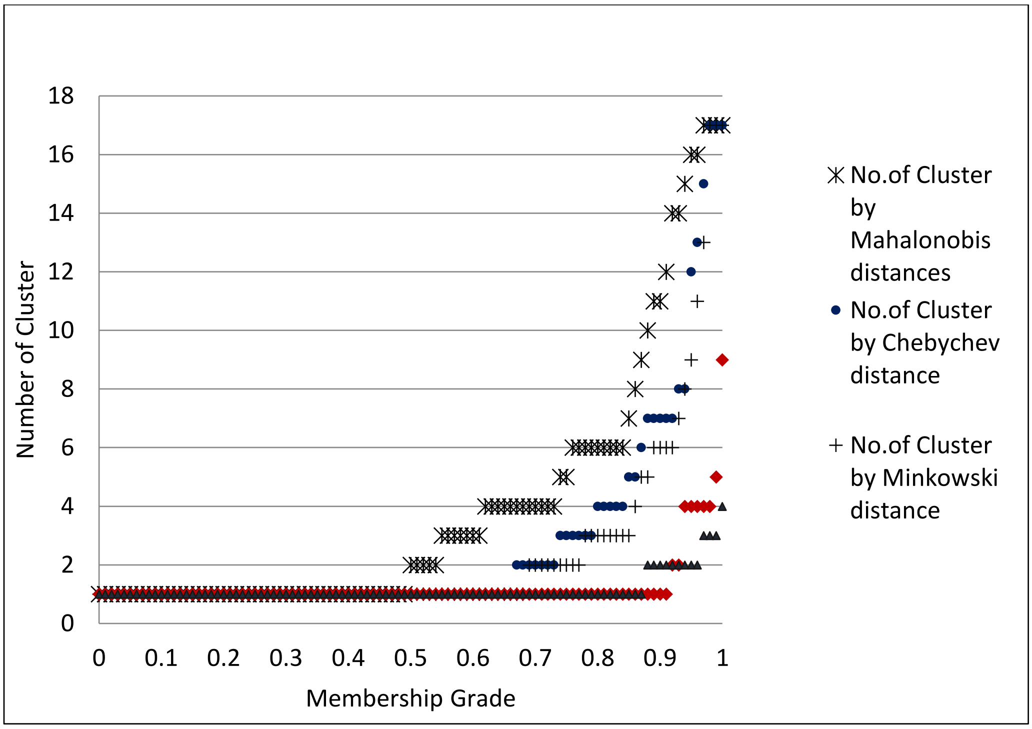


Figure 2.11: The comparison of clustering by Mahalanobis, Chebychev, Minkowski, Cosine, and Correlation distances

2.8 Summary

In this chapter, a comparative fuzzy equivalence class clustering of binders based on their performance is proposed. The performances of binders were graded in terms of high specific temperature at three different parameters. Five distance functions, namely Minkowski (w =2), Mahalanobis, Cosine, Chebychev, and Correlation, are applied in the separation methodology, and it is a first attempt where the Mahalanobis distance function is used for fuzzy equivalence clustering. The fuzzy compatible relation matrices and transitive closures are derived for each distance function. Then the separate cluster analysis is done for Minkowski, Malonobis, Cosine, Chebychev, and Correlation distance function, and the desired clusters are identified for a suitable value of membership grade. It was observed that the overall clustering performance of binders by Mahalanobis distance function is better than the performance by Minkowski and other distance functions. The overall performance achieved by the Chebychev distance function is quite better than Minkowski($w = 2$) distance and substantially finer than Cosine and Correlation distance function. Mahalanobis distance function produces a greater number of clusters at most of the $\alpha -$level. The separation stages by Mahalanobis distance are also extensively better than other distances. So, the fuzzy cluster analysis by Mahalanobis distance function can provide an effective grip in the separation analysis and strategy formulation.

Chapter 3

An Effective Generalized Exponential Metric Space Approach for C-Mean Clustering Analyzing

"The uncertainty where to look for the next opening of discovery brings the pain of conflict and the debility of indecision."

– *Alexander Bain*

3.1 Introduction

In Mathematics, the phenomenon of a metric space is fundamental and significant. Various metric functions construct distinct metric spaces. In pattern recognition and machine learning, distance measurement and contrast between sample pairs play a very significant role [10][11][12][13][14][15]. With the assistance of sophisticated numerical optimization, we can acquire discriminatory characteristics through a rational description of distance function and decide whether two samples belong to the same class. In this perspective, the approaches of distance metric learning and dimensional reduction seek to learn high-level semantic distances where identical input objects are projected to close points, while distinct objects are differentiated from each other[16]. Distance metrics have been efficiently used for vast scenarios, including image selection, visual monitoring, and prototypes classification.[17][18][19].Wu and Yang [1] had introduced a new metric that was more stable than the widely used Euclidean norm. In c-means clustering, they substituted the Euclidean norm with the new metric. They then developed two new clustering approaches called the clustering algorithms for alternative hard c-means (AHCM) and alternative fuzzy c-means (AFCM). But in the papers [2] comment that the distance used by Wu and Yang is not a metric, it was a type of squared distance, and it is not generally a metric for a squared distance.

This chapter suggests a new Gaussian function-based distance that substitutes ***e*** with the arbitrary constant and takes the square root in the proposed distance that satisfies the metric space characteristics. Therefore, the alternative generalized hard c-means (AGHCM) and alternative generalized fuzzy c-means (AGFCM) clustering algorithms have been developed as two new clustering methods. These proposed algorithms strengthen the vulnerabilities in both AHCM and AFCM.

3.2 Background Information

This section explored the fundamental concept of metric space and its variants, replacing them in c-mean clustering algorithms.

3.2.1 Matric space [132]

If Y be any sets, then mapping $\delta: Y \times Y \to \mathbb{R}$ such that

(i) $\delta(y_1, y_2) \geq 0 \;\; \forall\, y_1, y_2 \in Y$

(ii) $\delta(y_1, y_2) = \delta(y_2, y_1) \;\; \forall\, y_1, y_2 \in Y$

(iii) $\delta(y_1, y_2) = 0 \; iff \; y_1 = y_2$

(iv) $\delta(y_1, y_2) \leq \delta(y_1, y_3) + \delta(y_3, y_2) \;\; \forall\, y_1, y_2, y_3 \in Y$

is known as the metric on Y and (Y, δ) is known as metric space.

3.2.2 Euclidean Metric space

As a metric, the Euclidean norm is well known and is widely used. However, the parameter estimation based on an objective function based on a Euclidean metric cannot be robust in a noisy setting.

Suppose $X = \{y_1, y_2, \dots, y_n\}$ is a data set, where y_j is an m −dimensional Euclidean space function vector R^m. The Euclidean distance from the y_j to the center, v is defined as:

$$\text{Min} \sum_{j=1}^{n} ||y_j - v||^2 \tag{3.1}$$

Efficient center v estimation can be obtained using a minimum sum of square error (SSE) procedure. This method of finding estimators is called the least square method. Thus, the minimization of Eq.(3.1) w.r.t v can be achieved by the minimizer sample mean defined as:

$$v = \frac{\sum_{j=1}^{n} y_j}{n} \tag{3.2}$$

The data set S_1 ={3,4,4.4,4.7, 4.9,5, 5.1,5.3,5.6,5.7} taken from [133] to test and estimate v is 5 by calculating Eq.(3.2). Nevertheless, S_2 , S_3, …, S_8 defined as follows when we insert a noisy feature 20, 25 and 30 to new data sets:

S_2 ={3,4,4.4,4.7, 4.9,5, 5.1,5.3,5.6,5.7,20}

S_3 ={3,4,4.4,4.7, 4.9,5, 5.1,5.3,5.6,5.7,25}

S_4 ={3,4,4.4,4.7, 4.9,5, 5.1,5.3,5.6,5.7,30}

S_5 ={3,4,4.4,4.7, 4.9,5, 5.1,5.3,5.6,5.7,20,25}

S_6 ={3,4,4.4,4.7, 4.9,5, 5.1,5.3,5.6,5.7,20,30}

S_7 ={3,4,4.4,4.7, 4.9,5, 5.1,5.3,5.6,5.7,25,30}

S_8 ={3,4,4.4,4.7, 4.9,5, 5.1,5.3,5.6,5.7,20,25,30}

The values of v using Eq.(3.2) of these sets S_2 , S_2, …, S_8 are 6.25, 6.67,7.08,7.69,8.08,8.46, and 9.29, respectively, which are outside the range of original data. It is not stable since the noisy features have a significant effect on performance. Centered on Euclidean metric space, two standard clustering methods have been classified into two groups, i.e., hard clustering and fuzzy c-mean clustering, to discover a space division scheme for function data sets.

3.3 Exponential Function Based Metric Space

This issue [133] therefore proposed a robust exponential metric defined as:

$$\delta(y_1, y_2) = 1 - e^{-b||y_1 - y_2||^2} \tag{3.3}$$

However, the above metric function, in Eq. (3.3), was not a metric. The justification given in [2] for being a square distance form is not necessarily a metric. To resolve the issue, they, therefore, proposed a robust distance function as follows:

$$\delta(y_1, y_2) = \sqrt{1 - e^{-b||y_1 - y_2||^2}} \tag{3.4}$$

Where b is a covariance defined as

$$b = \frac{\sum_{j=1}^{n} ||y_1 - y_2||^2}{n} \text{ with } \bar{y} = \frac{\sum_{j=1}^{n} y_j}{n} \tag{3.5}$$

The distance function is a metric in Eq. (3.4), which fulfills all metric space axioms [132]. Consequently, the distance from y_j to the centre v is:

$$\text{Min } \sum_{j=1}^{n} \sqrt{1 - e^{-b||y_j - v||^2}} \tag{3.6}$$

and minimizing Eq.(3.6) w.r.t v can be achieved using the appropriate equation.

$$v = \sum_{j=1}^{n} \frac{\frac{e^{-b||y_j - v||^2}}{\sqrt{1 - e^{-b||y_j - v||^2}}} y_j}{\sum_{j=1}^{n} \frac{e^{-b||y_j - v||^2}}{\sqrt{1 - e^{-b||y_j - v||^2}}}}$$

$$= \sum_{j=1}^{n} \frac{\omega_j y_j}{\sum_{j=1}^{n} \omega_j} \tag{3.7}$$

$$\text{Where } \omega_j = \frac{e^{-b||y_j - v||^2}}{\sqrt{1 - e^{-b||y_j - v||^2}}} \quad j \in 1,2,3 \ldots, n \tag{3.8}$$

Consequently, ω_j is higher values to those y which closer to v and the data points further away from v are lower in weight. Notice v in Eq. (3.7) not resolved explicitly. But, to approximate it, use the fixed-point iterative approach.

Fixed-point iteration

Step 1) Consider the R.H.S of Eq.(3.7)*in g(v).*

Step2) By solving the Eq. (3.2), which is the optimal initial value $v^{(t-1)}$ where t=1 and fix ϵ>0.

Step 3) In step 1, take the initial value of the $v^{(0)}$ calculation to find $v^{(1)}$.

Step4) Convergence criteria – If $||v^t$-$v^{(t-1)}|| < \epsilon$,,THEN stop; ELSE t=t+1 and go to step-3.

The estimated values for the data sets S_1 , S_2, …, S_8 represented in

Table 3.2 showing the initial values of $v^{(0)}$ using Eq. (3.2) and $v^{(1)}$ by Eq. (3.7). Also, using this procedure's fixed-point iteration, obtain its estimated values $v^{(t)}$ corresponding to the number of iterations that met the condition of step-4, i.e., $||v^t$-$v^{(t-1)}|| < \epsilon$ *by fixing* $\epsilon = 0.001$.

In the following section, based on the exponential metric described in Eq.(3.4), Improved alternative hard c-mean clustering (IAHCM) and Improved alternative fuzzy c-mean clustering (IAFCM) are discussed, which give us more robust results compared to the HCM and FCM.

3.3.1 Alternative Hard C-Mean Clustering

An alternative hard c-means (AHCM) clustering objective function is proposed as:

$$Minimization\ J_{AHCM} = \sum_{i=1}^{c}\sum_{k=1}^{n} \left(1 - e^{-b||d_{ik}||^2}\right) \tag{3.9}$$

$$\text{Subject to } v_j = \frac{\sum_{k=1}^{n} e^{-b||d_{ik}||^2} y_{jk}}{\sum_{k=1}^{n} e^{-b||d_{ik}||^2}} \tag{3.10}$$

Where b is a constant which can be defined by

$$b = \left(\frac{\sum_{j=1}^{n} ||y_i - \bar{y}\,||^2}{n}\right)^{-1} \text{ with } \bar{y} = \frac{\sum_{j=1}^{n} y_j}{n} \quad \text{and } j \in I_i \tag{3.11}$$

This Eq.(3.11) shows that it is the sample covariance of the data sets (y_i).

$e^{-b||d_{ik}||^2}$ is one since b tends to be zero. The condition Eq. (3.10) appears to be necessary Eq. (3.2). So, we correlate AHCM and HCM when AHCM is typically HCM, as b tends to be zero.

3.3.2 Alternative Fuzzy C-Means Clustering

An alternative fuzzy c-means (AFCM) clustering objective function as:

$$Minimization\ J_{AHCM} = \sum_{i=1}^{c} \sum_{k=1}^{n} \tau_{jk}^{m} (1 - e^{-b||d_{ik}||^2}) \tag{3.12}$$

$where$, m>1 and $\sum_{i=1}^{c} \tau_{jk} = 1$ constraint, $j = 1, \ldots, n$. Parameter b>0 is also known as Eq.(3.11), and there are the following criteria for J_{AFCM} minimization:

$$v_j = \frac{\sum_{k=1}^{n} \tau_{jk}^{m}\, e^{-b||d_{ik}||^2} y_{jk}}{\sum_{k=1}^{n} \tau_{jk}^{m}\, e^{-b||d_{ik}||^2}} \tag{3.13}$$

$$\tau_{jk}^{m} = \frac{\left(\frac{1}{(1 - e^{-b||d_{ik}||^2}}\right)^{\frac{1}{(m-1)}}}{\sum_{j=1}^{c} \left(\frac{1}{(1 - e^{-b||d_{ij}||^2}}\right)^{\frac{1}{(m-1)}}} \tag{3.14}$$

If b is minimal, then the AFCM membership curve with this parameter is like the FCM membership curve, which is well presented for fuzzy boundaries. When b is extremely high, the AFCM membership curve with this parameter is present for the separation characteristic. But the inverse of sample covariance is a good approximation for b, as per Eq. (3.11)

3.4 The Purposed Metric Space

The robust distance function shown in Eq. (3.4) is based on exponential $\boldsymbol{e}$ whose approximation value is 2.7183. The baseline value is fixed for all types of data sets, but it is not appropriate for the same circumstances dependent value $\boldsymbol{e}$; then it takes more iterations to achieve the center, and the convergence rate could be slow. So, to increase the rate of convergence, this article proposes a new generalized exponential distance function defined as:

$$\delta(y_1, y_2) = \sqrt{1 - a^{-b\|y_1 - y_2\|^2}} \tag{3.15}$$

Where, $a > 0$, $b > 0$ is a covariance defined in Eq.(3.5). Also, every $\delta(y_1, y_2)$ distance function is a metric if the matric axiom in section 3.2.1 is fulfilled.

Theorem-3.1 If $X = \mathbb{R}^n = \{(y_1, y_2 \dots y_n): y_i \in \mathbb{R} \ \forall\, i \in \mathbb{N}\}$ and $d: Y \times Y \to \mathbb{R}$ such that $d(y_i, y_j) = \sqrt{1 - a^{-b\|y_i - y_j\|^2}}$, where $a \in (1, \infty)$ and $b \in (0, \infty)$ then (Y, d) is metric space.

Proof: Firstly, define Gaussian kernel function

$K(y_i, y_j) = a^{-b\|y_i - y_j\|^2}$, therefore $d(y_i, y_j) = \sqrt{1 - K(y_i, y_j)}$ and according to the Mercer theorem [3], there exists some nonlinear map $\varphi: Y \to F$, satisfying $K(y_i, y_j) = \varphi(y_i)^T \varphi(y_j)$, $where$ F is a compact subset of Hilbert space and $(.)^T is$ vector transpose.

We have $K(y_i, y_i) = K(y_j, y_j) = 1$ (by definition of K), Thus

$$\|\varphi(y_i) - \varphi(y_j)\|^2 = \varphi(y_i)^T \varphi(y_i) + \varphi(y_j)^T \varphi(y_j) - 2\varphi(y_i)^T \varphi(y_j)$$

$$= K(y_i, y_i) + K(y_j, y_j) - 2K(y_i, y_j)$$

$$= 2(1 - K(y_i, y_j))$$

$$\Longrightarrow d(y_i, y_j) = \sqrt{\frac{1}{2} || \varphi(y_i) - \varphi(y_j) ||} \quad (3.16)$$

From the above equation satisfied $(\boldsymbol{i})$ **to** $(\boldsymbol{iii})$ **conditions of Metric space** $\boldsymbol{i.e}$

$\delta(y_i, y_j) = \delta(y_j, y_i) > 0, \forall\, y_i \neq y_j \text{ and } \delta(y_i, y_i) = 0.$

And from Eq.(3.16)

$$\delta(y_i, y_j) = \frac{1}{\sqrt{2}} || \varphi(y_i) - \varphi(y_j) ||$$

$$\leq \frac{1}{\sqrt{2}} (|| \varphi(y_i) - \varphi(y_k) || + || \varphi(y_k) - \varphi(y_j) ||$$

$$= \delta(y_i, y_k) + \delta(y_k, y_j).$$

Thus, the triangular property (iv) is also satisfied. So, the proposed distance is a metric.

Under the metric (3.15), we have an estimate v using the same procedure with minimizing

$$\text{Min } \sum_{j=1}^{n} \sqrt{1 - a^{-b||y_j - v||^2}} \quad (3.17)$$

w.r.t v. It gives the necessary conditions with the equation (3.17)

$$v = \sum_{j=1}^{n} \frac{\frac{a^{-b||y_j - v||^2}}{\sqrt{1 - a^{-b||y_j - v||^2}}} y_j}{\sum_{j=1}^{n} \frac{a^{-b||y_j - v||^2}}{\sqrt{1 - a^{-b||y_j - v||^2}}}}$$

$$= \sum_{j=1}^{n} \frac{\kappa_j y_j}{\sum_{j=1}^{n} \kappa_j} \quad (3.18)$$

$$\text{Where } \kappa_j = \frac{a^{-b\left\|y_j - v\right\|^2}}{\sqrt{1 - a^{-b\left\|y_j - v\right\|^2}}} \quad j \in \mathbb{N} \tag{3.19}$$

The assessment of κ_j corresponding to y_j is the same as the assigned value of ω_j corresponding to y_j.Similarly, Eq. (3.18) is not resolved directly. However, the iterative approach can be used to approximate it.

Fixed- point iteration

Step1) Compute the value v by solving the Eq. (3.2), which is the optimal initial value of $v^{(t-1)}$ where $t = 1$ and fix $\epsilon > 0$.

*Step 2) Take the initial value of $v^{(0)}$ compute in step-1 to find $v^{(1)}$ for the various values of **a**.*

*Step 3) Find the optimal $v^{(1)}$ correspond to optimal value **a**.*

*Step 4) After finding the optimal value of **a**, find the values of $v^{(1)}$, $v^{(2)}$,..., $v^{(t)}$. Until step 5, a condition is satisfied.*

Step5) Convergence criteria – If $||v^t$-$v^{(t-1)}|| < \epsilon$, THEN stop; ELSE $t = t + 1$ and go to step-3.

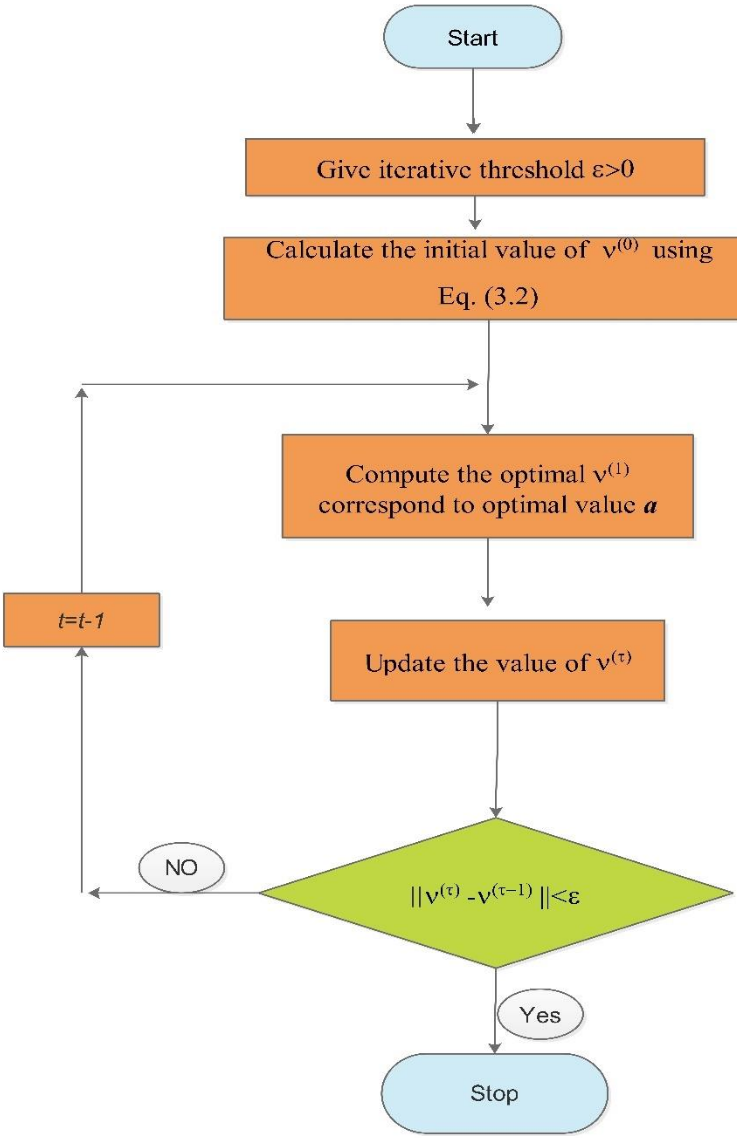


Figure 3.1: The proposed metric space distance flow chart

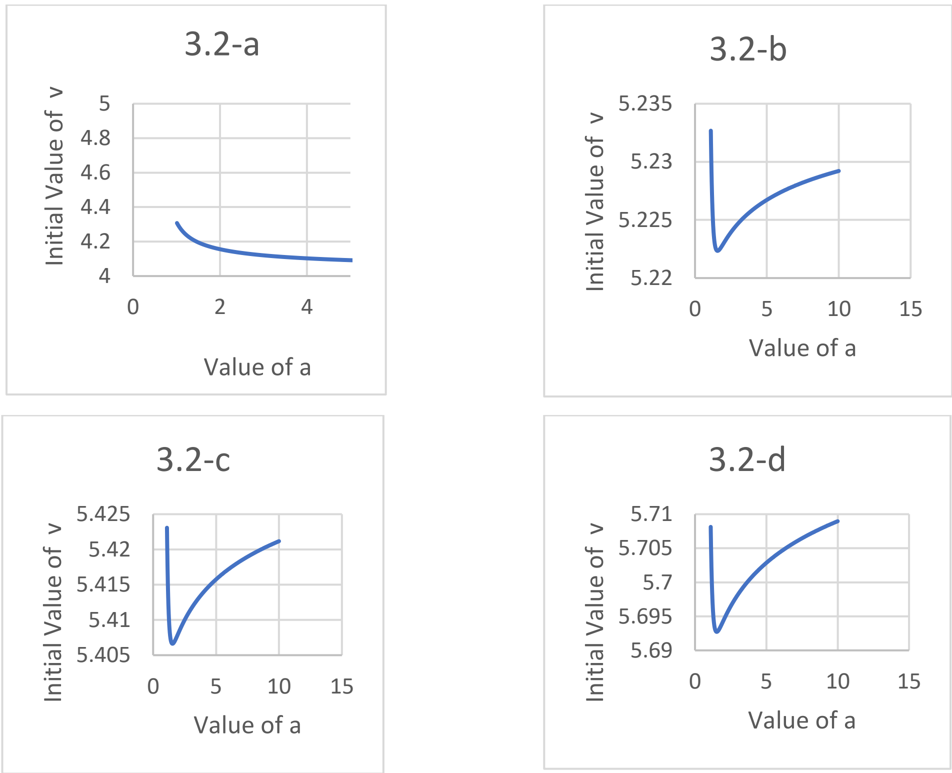
3.5 Experiment Result

The data set S_1 ={3,4,4.4,4.7, 4.9,5, 5.1,5.3,5.6,5.7} taken from [133] to test and estimate v is 5 by calculating Eq.(3.2). The values of v using Eq.(3.2) of these sets S_2 ,S_2, …, S_8 are 6.25, 6.67,7.08,7.69,8.08,8.46, and 9.29, respectively, which are outside the range of original data. It is not stable since the noisy features have a significant effect on performance. Cantered on Euclidean metric space, two standard clustering methods have been classified into two groups, i.e., hard clustering and fuzzy c-mean clustering, to discover a space division scheme for function data sets.

The usefulness and robustness of the suggested distance function are illustrated in Table3.2. Figure 3.2 indicate that $v^{(1)}$ values for the various artificial data sets S_1 , S_2, …, S_8 suggest that e is not the optimum value for the foundation. For data sets, S_1 shows the value of $v^{(1)}$=4.990981 at $\boldsymbol{a}$=0.58, which is much closer than $v^{(1)}$=4.127468 at $e = 2.7183$.Similar to the different values of a for other data sets S_2, …, S_8 the values of $v^{(1)}$ are much closer than those shown in

Table 3.2. Since these values are taken as the $v^{(t)}$ end values in Table 3.3 along with the $v^{(t)}$ end values in

Table 3.2. The number of iterations is also smaller, which means that the suggested distance function's convergence rate is greater than that of the exponential distance function [1].



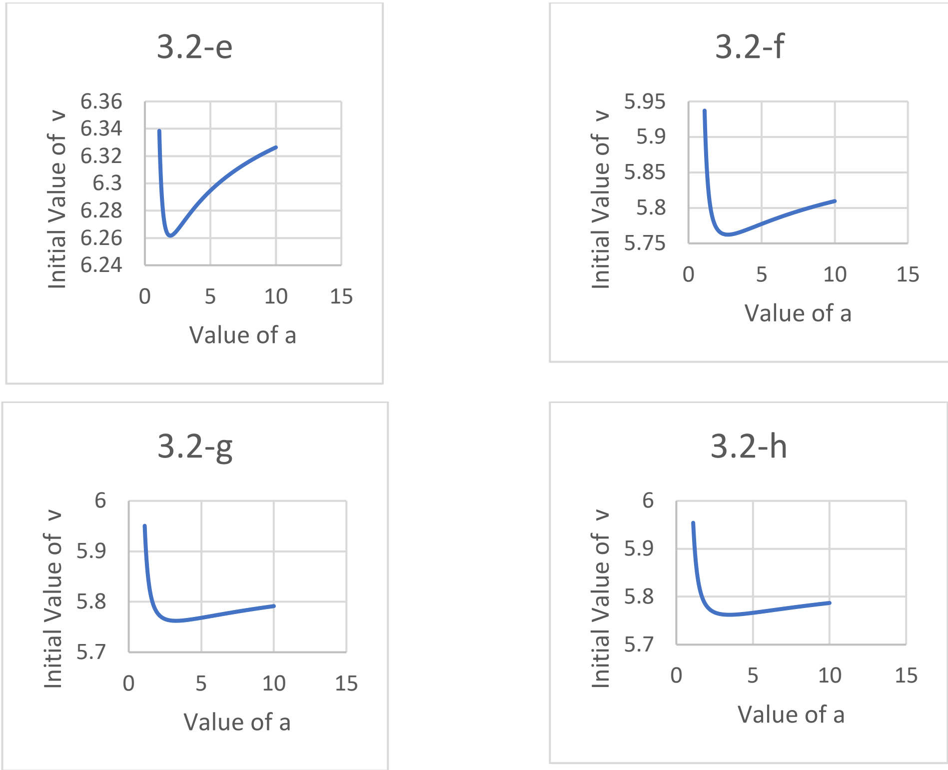


Figure 3.2: show the initial value of v corresponing to a for data sets S_1 to S_8 respectively

The metric of AHCM Eq. (3.9) and AFCM Eq. (3.12) is replaced by the proposed metric Eq.(3.15); the clustering mechanism is addressed as follows.

3.5.1 Alternative Generalized Hard C-Means Clustering

Alternative generalized hard c-mean clustering (AGHCM) based on the cluster mentioned exponential metric in Eq.(3.15). Usage of the AGHCM and AHCM connection and comparison is significant. Therefore, the objective function of the alternative generalized c-mean hard clustering algorithm leads to the following concern of optimization:

$$Minimization\ J_{AGHCM}(V;X) = \sum_{i=1}^{c}\sum_{k=1}^{n}\sqrt{(1 - a^{-b||d_{ik}||^2})}$$

Where $\boldsymbol{a} > 1, m > 0, ||d_{ik}|| = ||y_k - v_i||, b > 0.$ (3.20)

Theorem 3.2. The necessary condition for minimizing $J_{AGHCM}(V;Y)$ only if

$$v_j = \frac{\sum_{k=1}^{n} \frac{a^{-b||d_{ik}||^2}}{\sqrt{1-a^{-b||d_{ik}||^2}}} y_k}{\sum_{k=1}^{n} \frac{a^{-b||d_{ik}||^2}}{\sqrt{1-a^{-b||d_{ik}||^2}}}} \quad j \in \mathbb{N} \tag{3.21}$$

Where the $v_i I$s the cluster centers

Proof. We differentiate $J_{AGHCM}(V;Y)$ w.r.t v_i and set the derivative equal to zero. Thus, we get Eq. (3.21). The details, as given in **Appendix A.**

Algorithmic steps for **Alternative Generalized Hard C-Means Clustering** are:

Step-1) Initialization- Data-set (Y)*, number of clusters centers* v_j^0 $(j = c+1) j \in \mathbb{N}$*, fix* $\epsilon > 0$*, m=2,* $b > 0$*,* $\boldsymbol{a} > 0$*, and Number of Iterations.*

Step-2) Classification – Calculate the class with the smallest measure of distance Eq.(3.17)

Step-3) Centroid calculation – Update the cluster centroid $v_j^{(t-1)}$ *,* $t \in \mathbb{N}$

Step-4) Convergence criteria – If $||v_j^t\text{-}v_j^{(t-1)}|| < \epsilon$ *, THEN stop; ELSE* $t = t+1$ *and go to step-3.*

In AGHCM, if b=0, then Eq. (3.21) is equivalent to $||y_i - v_j||^2 = \min_k ||y_i - v_k||^2$, $k = 1,2 \ldots c$.

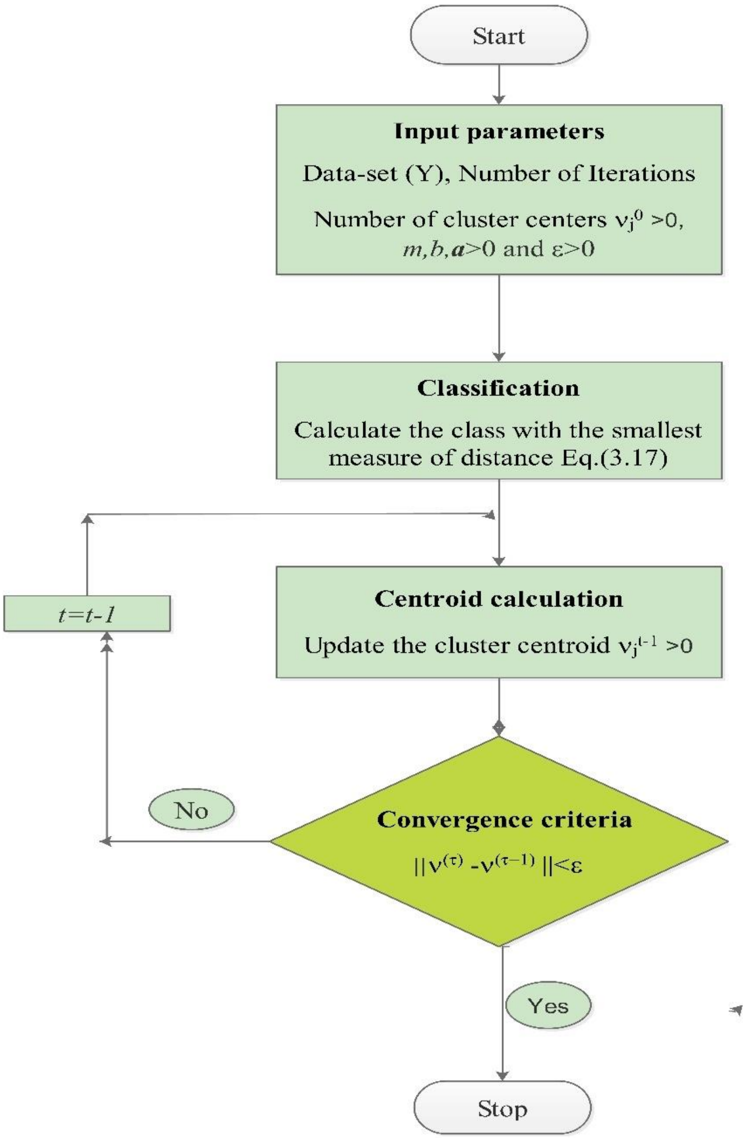


Figure 3.3: AGHCM flow chart

b is defined in Eq. (3.11) by the sample covariance of the data sets. It means that it shows the linear transformation between the various characteristics of the data set. But there are several instances in the practical scenario that occur in such a relationship, i.e., $b = 0$. If b tends to zero, then the AGHCM membership curve would be close to the HCM membership curve with this parameter, which is not stable for the noisy and outlier again. Thus, for all the possible values of $\boldsymbol{a}$, we take $b > 0$ in this chapter.

3.5.2 Alternative Generalized Fuzzy C-Means Clustering

Similarly, the objective function of Alternative Generalized fuzzy c-means clustering algorithms as follows:

$$Minimization\ J_{AGFCM}(U,V;Y) = \sum_{i=1}^{c}\sum_{k=1}^{n}\mu_{ik}^{m}(1 - a^{-b||d_{ik}||^2}) \tag{3.22}$$

$$With\ the\ constraint\ of$$

$$\sum_{i=1}^{c}\mu_{ik} = 1; k \in \mathbb{N} \tag{3.23}$$

Where $\boldsymbol{a} > 1, b, m > 0$, $\|d_{ik}|| = ||y_k - v_i||$

***Theorem* 3.3.** The necessary condition for minimizing $J_{AGFCM}(U,V;Y)$ under the constraint of Eq

(3.23)only if

$$\mu_{ik} = \frac{\left[\dfrac{1}{\sqrt{1 - a^{b||d_{ik}||^2}}}\right]^{\frac{1}{(m-1)}}}{\sum_{j=1}^{c}\left[\dfrac{1}{\sqrt{1 - a^{b||d_{ij}||^2}}}\right]^{\frac{1}{(m-1)}}} \tag{3.24}$$

$$and\ v_i = \frac{\sum_{k=1}^{n}\mu_{ik}^{m}\dfrac{a^{-b||d_{ik}||^2}}{\sqrt{1 - a^{b||d_{ik}||^2}}}\ y_k}{\sum_{k=1}^{n}\mu_{ik}^{m}\dfrac{a^{-b||d_{ik}||^2}}{\sqrt{1 - a^{b||d_{ik}||^2}}}} \tag{3.25}$$

Proof. We differentiate $J_{AGHCM}(V;Y)$ w.r.t μ_{ik} and v_i and set the derivative equal to zero. Thus, we get Eq. (3.24) and Eq. (3.25). The details, as given in **Appendix B.**

Algorithmic steps for **Alternative Generalized Fuzzy C-Means Clustering** are:

Step-1) Initialization- Data-set (Y), *number of clusters centers* v_j^0 $(j = c + 1)$ $j \in \mathbb{N}$, *fix* $\epsilon > 0$, *m=2*, $b > 0$, $\boldsymbol{a} > 0$, *and Number of Iterations.*

Step-2) Classification – Calculate the class with the smallest measure of distance Eq.(3.17)

Step-3) Membership calculation- Compute $\mu_{ik}{}_{jk}^{m}$ *by Eq. (3.24)*

Step-4) Centroid calculation – Update the cluster centroid $v_j^{(t-1)}$, $t \in \mathbb{N}$

Step-5) Convergence criteria – If $||v_j^t$-$v_j^{(t-1)}|| < \epsilon$, *THEN stop; ELSE* $t - 1 = t$ *and go to step-1.*

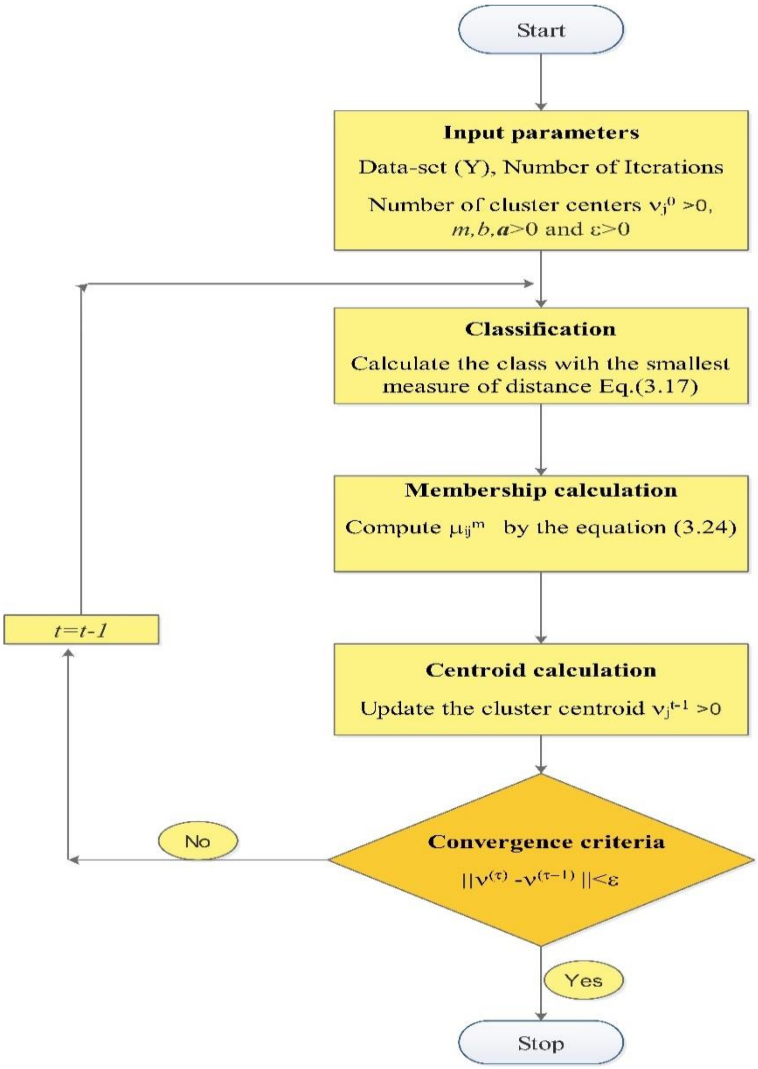


Figure 3.4: *AGFCM flow chart*

The most suitable value of the fuzziness index $m = 2$ in AGFCM plays the same role In FCM, has been discussed in [32]. But as mentioned above, we do not take b as the sample variance. In AGFCM we take the three possible value of b which is dependent on the ratio $R^*=\frac{b}{2(m-1)}$. The possible value of R are defined as follow:

$$R^* = \begin{cases} r_1 & if\ b > 2(m-1) \\ 1 & if\ b = 2(m-1) \\ r_2 & if\ b < 2(m-1) \end{cases} \tag{3.26}$$

Where $r_1 > 1$ and $r_2 < 1$.

The association of AGHCM and AGFCM with HCM, FCM, AHCM, and AFCM is shown in Figure 3.5. As described above, AGFCM is a generalized case of AFCM replacing e with $\boldsymbol{a}$ denoted by $[e/\boldsymbol{a}]$ and vice versa. Likewise, the same relation with AGHCM and AHCM. If in AFCM (or AHCM) $b \to 0$, then it coincides as FCM (HCM). The specific case of AHCM and AFCM, respectively, is HCM and FCM. . HCM, AHCM, and AGHCM are fuzzy extensions of FCM, AFCM, and AGFCM. Eventually, the numerical example, including diamond data sets $(P_{10}\ and\ P_{12})$ and Iris data set is analyzed with these six algorithms. All algorithms were started with the same initial values and stopped in the same illustrations under the same stop circumstances.

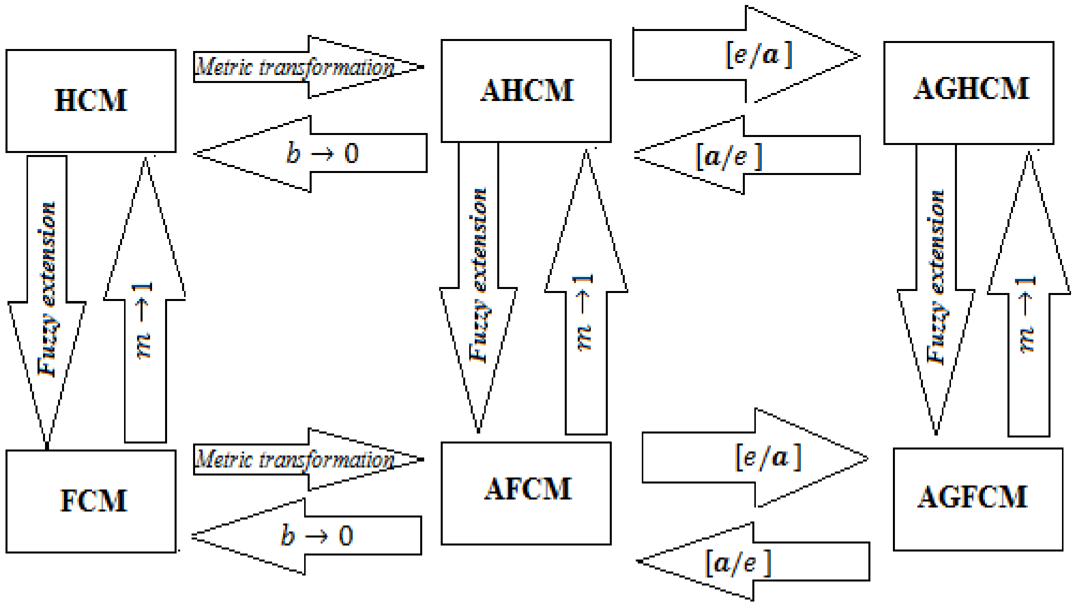


Figure 3.5: The association between HCM, FCM, AHCM, AFCM, AGHCM, and AGFCM

3.6 Result And Simulations

This section addresses the findings of the conceptual approach to two real-life test data sets, including Iris and Diamond Data Sets [36]. The analyses are measured compared to HCM, FCM, AHCM, AFCM, and other competitive algorithms. MATLAB R2015a on intel i3-370M processor, 2.40 GHz with 4 GB RAM, is used to implement and simulate experiments.

The following typical parameters have been considered: $m = 2, \epsilon = 0.00001$ maximum iterations= 100.

Example 3.1

Data sets: Diamond dataset P_{10}

Algorithms: AGHCM and AGFCM

Initialization: $V_0^1 = (-3,0.5)$ and $V_0^2 = (3,0.5)$

The initialization is accomplished by choosing two P_{10} data points randomly. The result provided by the AGFCM algorithm for the different values of $\boldsymbol{a}$ corresponding to the various values of b according to R^* is shown in Eq.(3.26)Analyse the best optimal value of $\boldsymbol{a}$ with the FCM (J_{fcm})and AGFCM (J_{AGFCM}) objective functions described in Eq.(1.8) and Eq.

(3.22), respectively. It also indicates the number of iterations (j) that have reached the ideal centroids $V_{ideal}^1 = (-3.34,0)$ and V_{ideal}^2= (3.34,0). The best optimal value for $\boldsymbol{a} = \mathbf{1.03}$ corresponds to $b = 1(R^* < 1)$ in which gives the optimal value $J_{FCM} = \mathbf{35.19895}$ and $J_{AGFCM} = \mathbf{1.685429}$. Only if $b = 1(R^* < 1)$) $\boldsymbol{a} = \boldsymbol{e}$ give the clustering center V_{ideal} with $J_{FCM} = 97.459761$ and $J_{AGFCM} = 3.935991$, does not consider the clustering center for the other value of b. Likewise, the performance AGHCM is provided in Table 3.4 ,where $R^* < 1$ the optimal value $\boldsymbol{a} = \mathbf{1.1}$ correspond to $b = 1$, and this means that the clustering center V_{ideal} with J_{AGHCM}=12.114523. has been reached. Again, the clustering center and the value of J_{AGHCM}=17.747997 for $\boldsymbol{a} = \boldsymbol{e}$ at $b(R^* < 1)$.

Example 3.2

Data sets: Diamond dataset P_{12} P11 is a noiseless data set of points $\{y_i\}_{i=1}^{11}$.P12 is the union of P11 and outlier ×12.

Algorithms: AGHCM and AGFCM $R^* < 1, b = 0.5, \boldsymbol{a} = \mathbf{1.04}$

Initialization:$V_0^1 = (-3,0.5)$ and $V_0^2 = (3,0.5)$

Now consider the P_{12}That includes two P11 and P12 outliers. Our suggested algorithms (AGHCM and AGFCM) provide the ideal centroids $V_{ideal}^1 = (-3.34,0)$ and $V_{ideal}^2 = (-3.34,0)$ for the various values seen in Table 3.6 and Table 3.7. This shows that the virtually identical centroid clusters V_{ideal} have been found by the proposed algorithms and do not impact the two outliers P11 and P12.

The consequence of AGFCM, as seen in Table 3.6, in the case of $R^* < 1$ the optimum value of $\boldsymbol{a} = \mathbf{1.04}$ corresponds to $b = 0.5$, which reached the ideal clustering core for $J_{FCM} = \mathbf{93.919702}$ and $J_{AGFCM} = \mathbf{2.103440619}$. Except in the case of $R^* < 1$ if $b = 1$, $\boldsymbol{a} = \boldsymbol{e}$ gives the clustering center V_{ideal} with $J_{FCM} = 158.61536$ and $J_{AGFCM} = 4.9359874$. and $\boldsymbol{a} = \boldsymbol{e}$ does not consider the clustering center for the other value instances. The efficiency of the AGHCM in Table 3.7 also indicates the optimum value $\boldsymbol{a} = \mathbf{2.1}$ in the case of $R^* < 1$, $b = 0.1$ with the optimal value of J_{AGHCM}=16.69017. Instead, the clustering center and the value of J_{AGHCM}=17.468363 at $b = 0.1(R^* < 1\ )$ and $J_{AGHCM} = 21.7479831\ at\ b = 1(R^* < 1)\ )$ are supplied. Otherwise, the clustering core is not recognized by $\boldsymbol{a} = e$.

Table 3.8 provides a comparison of other clustering algorithms to AGHCM and AGFCM, demonstrating that the best outcomes of our proposed algorithms are obtained after clustering centers. The result of the clustering centers of the FCM, PCM, PFCM, etc., is taken from [36] [134].

Example 3.3

Data sets: IRIS

Algorithms: AGHCM and AGFCM

Initialization:

V_0^1 =(5.01,3.42,1.42,0.25) V_0^2 =(5.83,2.79,4.29,1.34) and
V_0^3 =(6.41,3.04,5.55,2.07)

Size of clusters:50,50, 50

We presently exhibit PFCM on a real data set, IRIS [35], with three clusters centroid. It is a four-dimensional data set consisting of 50 samples of each of the three Iris flowers. One of the three (class 1) clusters is well apart from the other two, while classes 2 and 3 overlaps. We made several iterations of AGFCM and AGHCM on IRIS, showing different parameter choices in Table 3.8 and Table 3.9, respectively.

The optimal clustering centroid is shown in Table 3.8 for different values corresponding to the different value of b, depending on R^*. We use the objective functions of FCM, AGFCM, and SSE specified in equations (1.8), (3.22), and (1.7) to verify the result. Optimal value of J_{FCM}=88.3648 and J_{AGFCM}= 0.609435938 corresponding to $b = 0.1(R^* < 1)$. On the other hand, the optimal value for SSE is 112.31439 at $\boldsymbol{a}$**=2.1** corresponding to $b = 0.1(R^* < 1)$. Similarly, the optimal value for SSE for AGHCM is 112.440683 at $\boldsymbol{a}$**=8** corresponding to $b = 4(R^* > 1)$.

3.7 Summary

This chapter proposed a new, improved, generalized approach to metric space to resolve the drawback of Gaussian distance-based functionality proposed by Wu and Yang [1]. Zhang and Chen [2] have expanded it further. The proposed metric is more robust than the Euclidean and Gaussian function distance. Two clustering algorithms, called AGFCM and AGHCM clustering algorithms, are based on the suggested metric space. Diamond data sets and Iris data sets are used to assess the proposed algorithms' efficiency, compare AFCM, AHCM, and other competitive algorithms such as FCM, PCM, FPCM, and PFCM. Compared to other proven algorithms, the performance of proposed algorithms has been found to work significantly better. We suggest using the AGFCM clustering algorithm for cluster analysis applications. In the future, we can also use the proposed metric and AGFCM in a PFCM, CFCM, and other clustering algorithms where metrics play an important role.

Chapter 4

A Novel Approach for Fuzzy Linear Programming Using Situational Based Composite Triangular Number

"An optimist is someone who believes the future is uncertain."

−*Anonymous*

4.1 Introduction

Unfortunately, sometimes, the actual practical situations are often not deterministic. Certain types of dubieties in social, industrial, and economic systems, such as randomness of occurrence of events can lead to improper optimization. Such types of dubieties (Feasible uncertainties) are associated with the difficulty of making a sharp or precise decision. Feasible uncertainties deal with the situation where the information cannot be valued sharply or cannot be described clearly in a linguistic term, such as preference-related information. At a certain point of time, the availabilities of m constraints can fluctuate in terms of probabilistic increment, probabilistic decrement, or both directions then general LPP cannot make explicit the proper optimization.

The uncertainties in the realistic situation could not be overlooked in succession to create the organization's well-organized supply chain. These ambiguities are usually related to the product supply, customer demand, etc. [135]. The fuzzy numbers concerned, which have a realistic approach in many different fields like decision making, data analysis [87], and also engineering problems[81][89], etc. problem. With the assistance of these fuzzy numbers, numerous optimization problems can be resolved. In [90], they introduced the subtraction and division process with a triangular fuzzy number(FTN). Besides, many modified operations are used to promote triangular and trapezoidal fuzzy numbers [136][137][138][139][93][140][95][141][97] and may have an impact on FLP optimization. An innovative way to resolve FLP fully through the use of the lexicography method [142], in addition to the traditional linear system, using the (LR) fuzzy numbers and the lexicography method, a recent FLP resolution trend. New algorithms [99] were build based on a new lexicographic TFN to explain the FFLP by switching to its multi-objective linear programming. In terms of the vendor's implementation costs reduction, two models [143] were introduced to reduce the overall device costs. In paper [144], the results of reduction in setup costs and increases in efficiency have been established in an increasing two-echelon chain model. The goal was to reduce the overall cost of the whole SCM model by minimizing construction expense, process efficiency, number of suppliers, and lot size at the same time. In paper [145], the distribution of probabilities for consumer demand was assumed only to have a known mean and standard deviation. The retailer's costs and developing competitive distribution arrangements were suggested as an effective solution. Often included in article [146] is a supply chain network, where a single manufacturer manufactures goods in a batch phase and delivers them over several times to a variety of customers. Chandrawat et.al. [147]conducted a modelling and optimization study using FLP with symmetrical and right-angle triangle fuzzy number. To illustrate the membership grade of optimized fuzzy LPP, they used the right angle triangular fuzzy number. The various components of triangular fuzzy numbers [101][102]were used on the other hand to develop trustworthiness parameters for the extraction of the industrial system. In these papers [148][104][75][149] they used symmetric trapezoidal numbers to represent the coefficients of the restriction's objective function and solution value of the R.H.S to overcome FLP concerns.

If the fluctuation is available in increment or decrement, then the use of right angles triangular fuzzy linear programming problem proposed [70] shows the benefits in introducing the credibility for the increase or decrease. This credibility fulfills the necessities to find out the lower and upper bounds for the initial LPP.

4.2 Problem Identification

Rail Coach Factory, Kapurthala, a premier coach manufacturing unit of Indian Railways, was established in 1986. It is situated in the Kapurthala district of Punjab, India. RCF has moved on to become the largest and most modern coach manufacturing unit of Indian Railways. We visited the site and observed the data that more than 36,000 RCF built coaches are traversing our nation's length and breadth. Every year RCF is adding more than 1600 coaches to this fleet, including AC and Non-AC coaches for Broad Gauge. These coaches have higher speed potential (up to 180 kmph), higher carrying capacity, aesthetically pleasing looks, and above all, superior safety features built into their design.

Though with selective indigenization, these coaches' costs have been brought down to one-third of their original cost, still these are 50% to 80% costlier than the conventional coaches. These coaches were hitherto confined to only premium trains like Rajdhani and Shatabdi Express due to higher costs. So, till 2009, RCF was manufacturing around 100 such coaches every year. Derive the benefits of this superior technology on a broader scale; a decision was taken in 2009-10 to switch over to stainless steel coach manufacturing completely. Hence the data of the production costs of different coaches for the year 2010-11 were considered input, and the total cost has been targeted as a prime objective see Table 4.1. It was observed that owing to certain procedural changes, maybe technical shifting, the actual production cost was fluctuating or uncertain, and the uncertainty is classified by Table 4.2. Hence it is challenging to optimize the production cost in this inflexibility of creation expenses for various mentors. Therefore, the present study is carried out by proposing newly constructed composite triangular and trapezoidal FLPP models to deal with it.

4.3 Fuzzy Linear Programming using Right Angle Triangle

Classical LPPs are the minimum or maximum values under linear inequalities or linear function equations. The standard form of LPP is represented by

$$\text{Max/Min } Z = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq or \geq \beta_i$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \qquad (4.1)$$

The function to be Max Z or Min Z is called an objective function. The c_j are called cost coefficients. The A=$[a_{ij}]$ matrix is called a restriction matrix and the $\boldsymbol{\beta_i}$= $< b_1, b_2, \ldots, b_m >^T$ is called a vector on the right side. where $\boldsymbol{y}$= $< y_1, y_2, \ldots, y_n >^T$ is the vector of variables.

The standard form of fuzzy linear programming is represented by

$$\text{Max } Z = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \widetilde{B_1}$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \qquad (4.2)$$

Where $\widetilde{B_1}$ is the right triangle fuzzy number.

The coefficient on the right is the membership function, i.e., the availability of restrictions. Optimize such a problem; the optimum values' lower and upper boundaries need to be estimated. The lower bound **($\boldsymbol{Z_l}$)** value is

$$\text{Max } \boldsymbol{Z_l} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \qquad (4.3)$$

The optimal values upper bound ($\boldsymbol{Z_u}$) is as follows

$$\text{Max } \boldsymbol{Z_u} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i + \varepsilon_i$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{4.4}$$

Where, ε_i is an increase in the probabilistic availability of restrictions. In this case, the total probabilistic increase of access to restrictions are determined by the right coefficient.

The Simplex method can now be used to find a solution for the lower and upper bounds of the LPPs. Using these lower and upper bounds, the optimized FLLP is obtained as follows.

$$\text{Max Z} = \lambda$$

$$\text{Subject to } \lambda (Z_u - Z_l) - \sum_{j=1}^{n} c_j y_j \leq -Z_l$$

$$\lambda \varepsilon_i + \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i + \varepsilon_i \tag{4.5}$$

Where, $y_j \geq 0, i, j \in \mathbb{N}$ and $\lambda \in [0,1]$ is membership grade

4.4 Fuzzy Linear Programming using a Composite Fuzzy Triangular Number

According to the composite fuzzy triangular number $\tilde{B}_2 = (\beta_i - \varepsilon_i, \beta_i, \beta_i + \varepsilon_i^*)$ the general structure of the optimal values of the lower, static, and upper bounds are defined below:

The lower bound $(\boldsymbol{Z_l})$ –

$$\text{Max } \boldsymbol{Z_L} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i - \varepsilon_i$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{4.6}$$

The static bound **(Z_s)** –

$$\text{Max } \boldsymbol{Z_l} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{4.7}$$

The upper bound **(Z_u)** –

$$\text{Max } \boldsymbol{Z_u} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i + \varepsilon_i^*$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{4.8}$$

The solution for lower and upper bounds of LPP's is obtained by the Simplex method. Find the two different optimized FLPP model will be obtained by using these lower and upper bounds

4.4.1.1 Optimized Composite Triangular FLPP Model -I: -

$$Max\ Z = \lambda,$$

Subject to

$$\lambda(Z_s - Z_l) - \sum_{j=1}^{n} c_j y_j \leq -Z_l$$

$$\lambda(\varepsilon_i) + \sum_{j=1}^{n} a_{ij} y_j \leq b_i,$$

$$\lambda(Z_u - Z_s) - \sum_{j=1}^{n} c_j y_j \leq -Z_s$$

$$\lambda(\varepsilon_i^*) + \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i + \varepsilon_i^* \tag{4.9}$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N}, \lambda \in [0,1]$$

4.4.1.2 Optimized Composite Triangular FLPP Model-II:-

$$Max\ Z = \lambda, \text{ Subject to}$$

$$\lambda(Z_s - Z_l) - \sum_{j=1}^{n} c_j y_j \leq -Z_l$$

$$\lambda(\varepsilon_i) + \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i$$

$$\lambda(Z_u - Z_l) - \sum_{j=1}^{n} c_j y_j \leq -Z_l$$

$$\lambda(\varepsilon_i^* + \varepsilon_i) + \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i + \varepsilon_i^* \tag{4.10}$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N}, \lambda \in [0,1]$$

The membership grade on behalf of our primary LPP will be given by the above equations (4.9) and (4.10)fuzzy optimized LPP. Here λ signifies the membership grade and Z_u, Z_s and Z_l are the upper, static, and lower bounds. $\sum_{j=1}^{n} c_j y_j$ is the objective function of the primary LPP, ε_i^* and ε_i is the probabilistic increment and decrement respectively in the availability of the constraints.

4.4.2 Case Study and Data Identification

The data specified under is of the Rail Coach Factory (RCF), Kapurthala, Punjab, India of 2010-2011. This data indicates the built-up cost (in 'lacs' '1,00,000') of different kinds of constraints of coaches.

According to the complexity of the data, the optimization of targeted constraints might vary. Study of optimization strategies for realistic situations, the skewness and Kurtosis characteristics play a broader role. In the year 2010-11, different coaches' total production cost is taken as an objective function to be minimized concerning the other constraints. The total availability of constraints $\boldsymbol{C_{Lab.}}, \boldsymbol{C_{Mat.}}, \boldsymbol{C_{foh}}$,

C_{Aoh}, C_{Toh}, C_{Soh}, C_{Tot} , C_{Pc} and C_{Tc} are 153.2, 2328.22, 256.56, 197.13, 41.23, 18.67, 513.61, 93.83 and 3088.88 lacs respectively. Modeling for system of Optimal Solution for Case-I

Objective function

Let $y_1, y_2, \dots, y_{20}$ be variables for different constraints.

Minimize Z: $66.4y_1 + 61.58y_2 + 64.47y_3 + 264.12y_4 + 69.17y_5 + 130.48y_6 + 46.03y_7 + 164.11y_8 + 262.29y_9 + 129.41y_{10} + 52.16y_{11} + 202.79y_{12} + 206.74y_{13} + 142.17y_{14} + 236.53y_{15} + 302.08y_{16} + 234.01y_{17} + 236.54y_{18} + 98.67y_{19} + 119.13y_{20}$

Subjected to constraints: -

$4.38y_1 + 4.07y_2 + 4.04y_3 + 9.88y_4 + 4.10y_5 + 7.38y_6 + 2.5y_7 + 8.11y_8 + 14.81y_9 + 7.22y_{10} + 3.05y_{11} + 9.18y_{12} + 10.70y_{13} + 6.38y_{14} + 10.38y_{15} + 10.51y_{16} + 11.24y_{17} + 11.57y_{18} + 5.91y_{19} + 7.79y_{20} \geq B_{lab}$

$45.7y_1 + 41.99y_2 + 44.49y_3 + 211.93y_4 + 49.13y_5 + 94.06y_6 + 33.62y_7 + 124.32y_8 + 190.08y_9 + 93.76y_{10} + 37.29y_{11} + 156.57y_{12} + 153.98y_{13} + 110.86y_{14} + 184.34y_{15} + 246.6y_{16} + 178.25y_{17} + 179.28y_{18} + 70.13y_{19} + 81.84y_{20} \geq B_{mat}$

$7.33y_1 + 6.81y_2 + 6.77y_3 + 16.55y_4 + 6.87y_5 + 12.36y_6 + 4.58y_7 + 14.35y_8 + 24.4y_9 + 13.17y_{10} + 5.55y_{11} + 15.27y_{12} + 17.84y_{13} + 10.59y_{14} + 16.98y_{15} + 17.19y_{16} + 18.42y_{17} + 18.97y_{18} + 9.70y_{19} + 12.86y_{20} \geq B_{foh}$

$5.8y_1 + 5.39y_2 + 5.35y_3 + 13.09y_4 + 5.43y_5 + 9.78y_6 + 2.91y_7 + 9.14y_8 + 19.3y_9 + 8.39y_{10} + 3.54y_{11} + 12.08y_{12} + 14.11y_{13} + 8.38y_{14} + 13.43y_{15} + 13.59y_{16} + 14.57y_{17} + 15.01y_{18} + 7.67y_{19} + 10.17y_{20} \geq B_{aoh}$

$1.17y_1 + 1.09y_2 + 1.08y_3 + 2.65y_4 + 1.10y_5 + 1.98y_6 + 0.75y_7 + 2.35y_8 + 3.91y_9 + 2.16y_{10} + 0.91y_{11} + 2.45y_{12} + 2.86y_{13} + 1.70y_{14} + 2.72y_{15} + 2.75y_{16} + 2.95y_{17} + 3.04y_{18} + 1.55y_{19} + 2.06y_{20} \geq B_{tooh}$

$0.37y_1 + 0.34y_2 + 0.36y_3 + 1.74y_4 + 0.40y_5 + 0.77y_6 + 0.23y_7 + 0.85y_8 + 1.56y_9 + 0.64y_{10} + 0.25y_{11} + 1.28y_{12} + 1.26y_{13} + 0.91y_{14} + 1.51y_{15} + 2.02y_{16} + 1.46y_{17} + 1.47y_{18} + 0.58y_{19} + 0.67y_{20} \geq B_{\text{soh}}$

$14.69y_1 + 13.63y_2 + 13.57y_3 + 34.04y_4 + 13.80y_5 + 24.89y_6 + 8.47y_7 + 26.68y_8 + 49.16y_9 + 24.35y_{10} + 10.26y_{11} + 31.09y_{12} + 36.07y_{13} + 21.58y_{14} + 34.64y_{15} + 35.56y_{16} + 37.39y_{17} + 38.49y_{18} + 19.49y_{19} + 25.76y_{20} \geq B_{\text{toh}}$

$1.63y_1 + 1.89y_2 + 2.37y_3 + 8.27y_4 + 2.13y_5 + 4.14y_6 + 1.44y_7 + 5y_8 + 8.23y_9 + 4.07y_{10} + 1.57y_{11} + 5.96y_{12} + 5.99y_{13} + 3.35y_{14} + 7.17y_{15} + 9.42y_{16} + 7.12y_{17} + 7.20y_{18} + 3.14y_{19} + 3.74 \leq B_{\text{prof}}$

$$\text{For all } y_i \geq 0 \qquad (4.11)$$

4.4.3 Numerical Result:

The following cases have been identified to deal with the described situations. MATLAB R2015a on intel i3-370M processor, 2.40 GHz with 4 GB RAM, is used to implement and simulate experiments.

4.4.3.1 Case I: Unbounded Feasibility with Zero Skewness –

The production cost is targeted with at least basic availability for all constraints. It is optimized when basic availability fluctuates by incrementing average quantity in one direction and decreasing average quantity in another direction. The fluctuation is shown by Table (4.2) in one direction and intensified by the average in another direction.

4.4.3.1.1 Result Case-I

Using the described methodology, the modeling of production cost is being done, and the fuzzy numbers for all cost parameters have been derived. The lower bound, static bound, and upper bound are calculated, and the optimized fuzzy linear programming problem (OFLPP) has been constructed using the lower and upper bound.

The mathematically described membership grades of all Cases I and II constraints are shown in equations (4.12) to (4.19).

Let B_{lab} be the membership grade for labor cost, and it varies as:-

$$B_{lab} = \begin{cases} 1 & when\ y = 153.2 \\ \dfrac{y - 145.54}{7.66} & when\ \ 145.54 \leq y \leq 153.2 \\ \dfrac{160.86 - y}{7.66} & when\ \ 153.2 \leq y \leq 160.86 \\ 0 & otherwise \end{cases} \quad (4.12)$$

Let B_{mat} be the membership grade for Material cost, and it varies as:-

$$B_{mat} = \begin{cases} 1 & when\ y = 2328.22 \\ \dfrac{y - 2211.81}{116.41} & when\ \ 2211.81 \leq y \leq 2328.22 \\ \dfrac{2444.63 - y}{116.41} & when\ \ 2328.22 \leq y \leq 2444.63 \\ 0 & otherwise \end{cases} \quad (4.13)$$

Let B_{foh} be the membership grade for Factory overhead charge, and it varies as:-

$$B_{foh} = \begin{cases} 1 & when\ \ y = 256.56 \\ \dfrac{y - 243.73}{12.83} & when\ \ 243.73 \leq y \leq 256.56 \\ \dfrac{269.39 - y}{12.83} & when\ \ 256.56 \leq y \leq 269.39 \\ 0 & otherwise \end{cases} \quad (4.14)$$

Let B_{aoh} be the membership grade for Administrative, and it varies as:-

$$B_{aoh} = \begin{cases} 1 & when\ y = 197.13 \\ \dfrac{y - 187.27}{9.85} & when\ \ 187.27 \leq y \leq 197.13 \\ \dfrac{206.98 - y}{9.85} & when\ \ 197.13 \leq y \leq 206.98 \\ 0 & otherwise \end{cases} \quad (4.15)$$

Let B_{toh} be the membership grade for Township overhead charge, and it varies as: -

$$B_{toh} = \begin{cases} 1 & when y = 41.23 \\ \dfrac{y - 39.16}{2.06} when & 39.16 \leq y \leq 41.23 \\ \dfrac{43.29 - y}{2.06} when & 41.23 \leq y \leq 43.29 \\ 0 & otherwise \end{cases} \quad (4.16)$$

Let B_{soh} be the membership grade for Shop overhead charge, and it varies as: -

$$B_{soh} = \begin{cases} 1 & when y = 18.67 \\ \dfrac{y - 17.73}{0.93} when & 17.73 \leq y \leq 18.67 \\ \dfrac{19.60 - y}{0.93} when & 18.67 \leq y \leq 19.60 \\ 0 & otherwise \end{cases} \quad (4.17)$$

Let B_{toh} be the membership grade for Total overhead charge, and it varies as: -

$$B_{toh} = \begin{cases} 1 & when y = 513.61 \\ \dfrac{y - 487.92}{25.68} when & 487.92 \leq y \leq 513.61 \\ \dfrac{539.29 - y}{25.68} when & 513.61 \leq y \leq 539.29 \\ 0 & otherwise \end{cases} \quad (4.18)$$

Let B_{prof} be the membership grade for Proforma charge, and it varies as: -

$$B_{prof} = \begin{cases} 1 & when \quad y = 93.83 \\ \dfrac{y - 89.13}{4.69} & when \ 89.13 \leq y \leq 93.83 \\ \dfrac{98.52 - y}{4.69} & when \ 93.83 \leq y \leq 98.52 \\ 0 & otherwise \end{cases} \quad (4.19)$$

4.4.3.1.2 Optimized Fuzzy Linear Programming Problem of Case-I

Using the described methodology, the modeling of production cost is being done, and the fuzzy numbers for all cost parameters have been derived. The lower bound, static bound, and upper bound are calculated the value of lower, static, and upper bound are rupees **2934.3116711695**(lakhs), rupees **3088.7495985759**(lakhs), and rupees **3243.1771372067**(lakhs), respectively.

The Table 4.1 shows the solutions for the optimized value of lower, static and upper bound and for the optimized membership grade for case-I and model I and II.

4.4.3.1.3 Result Analysis Case-I

The production cost of RCF can be minimized using the cost parameter. The production's total basic cost is rupees 3088.749 (in lacs), and it can be extended and declined until rupees 3243.177, 2934.311(in lacs), respectively. The optimum production cost has been obtained to get the maximum membership grade. It shows that total production cost provides the highest credibility if the optimized cost is considered equal to rupees 3088.749 (in lacs). The credibility of production cost is being decreased if it is tending towards rupees 2934.311 and 3243.177 (in lacs).

Eq. (4.20) and figure 4.13 show the fuzzy number for optimized membership grade:

$$\boldsymbol{\lambda} = \begin{cases} 1 & when\ y = 3088.7496 \\ \dfrac{y - 2934.3116}{154.438} & when\ \ 2934.3116 \leq y \leq 3088.7496 \\ \dfrac{3243.1771 - y}{154.42} & when\ \ 3088.7496 \leq y \leq 3243.1771 \\ 0 & otherwise \end{cases} \tag{4.20}$$

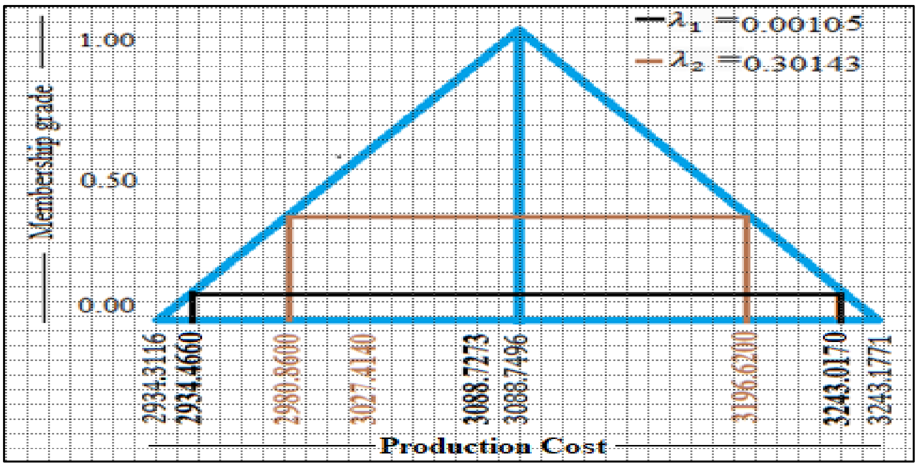


Figure 4.1: The relation between membership grade of optimized production cost with $\lambda_1$andλ_2 of case-I.

Using the structure of optimized composite triangle fuzzy LPP, Model-I illustrate the Optimized minimum cost 3088.7273 unit with the membership grade $\lambda_1 = 0.00104521$, and the minimized and greatest minimized costs 2934.466 and 3243.017 units, respectively. Similarly, optimized composite triangle fuzzy LPP, Model-II illustrate the Optimized minimum cost of 3027.414 unit with the membership grade $\lambda_2 = 0.30143$, and minimized and the greatest minimized costs 2980.86 and 3196.62 units, respectively.

4.4.3.2 Case II: Bounded Feasibility with Zero Skewness –

Case –II is like the case –I to justify the feasible bounded region, the Performa charge is included with at least availability, and all other constraints are included with at most availability. Here, the Performa charge is considered at least availability because this situation provides the bounded solution and gives the optimal value nearest to the feasible most optimum solution.

4.4.3.2.1 Optimized Fuzzy Linear Programming Problem of Case-II

The described methodology models the production cost done, and the cost parameters of fuzzy numbers have been derived and explained in equations (4.12) to (4.19). Figure 4.1 to Figure 4.8 The lower bound, static bound, and upper bound are calculated the value of lower, static, and upper bound are rupees 2943.959308 (lakhs), rupees 3098.912497 (lakhs), and rupees 3253.835047 (lakhs), respectively.

The optimized fuzzy linear programming problem (OFLPP) has been constructed using the lower, static, and upper bound.

The Table 4.2: shows the solutions for the optimized value of lower, static, and upper bound and the optimized membership grade.

4.4.3.2.2 Result Analysis Case-II

In this situation, the production cost can be increased and reduced to 3243.1770 and 2934.3110 (in lakes), respectively, and the total cost of production is 3088.7490 (in lakes). The optimal cost of production has been achieved to achieve optimum membership grades. The overall production cost is shown to have an optimum reputation if the optimized costs are substantially equal to 3088,749 (in lakes). However, if the production cost remains at 2934.3110 and 3243.1770 (in lakes), the reputation of production costs is declining.

Eq. (4.21) and Figure 4.10 show the fuzzy number for optimized membership grade:

$$\boldsymbol{\lambda} = \begin{cases} 1 & when\ y = 3098.9125 \\ \dfrac{y - 2943.9593}{154.9500} & when\ \ 2943.9593 \leq y \leq 3098.9125 \\ \dfrac{3253.8350 - y}{154.9500} & when\ \ 3098.9125 \leq y \leq 3253.8350 \\ 0 & otherwise \end{cases} \tag{4.21}$$

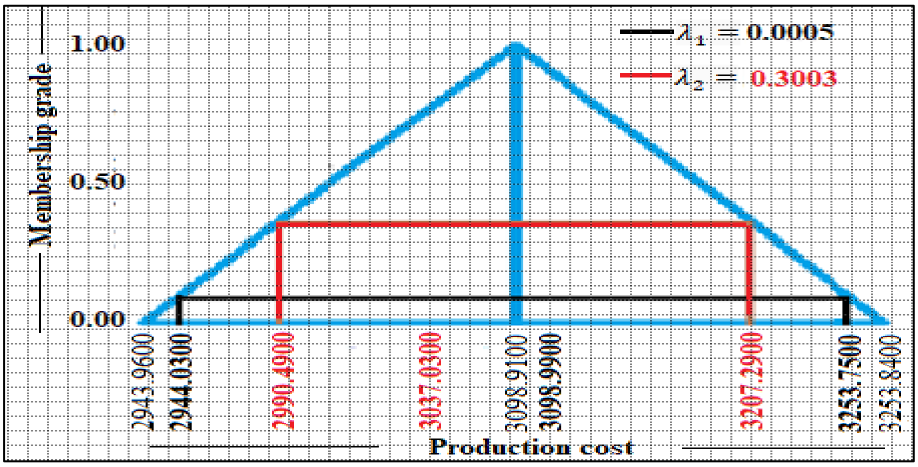


Figure 4.2: The relation between membership grade of optimized production cost with $\lambda_1$andλ_2 of case − II.

Model-I then reveals the Optimized Minimum Cost 3098.997 unit for membership grade $\lambda_1 = 0.0005434397443$, and the minimized and greatest minimized costs respectively 2944.0435 and 3253.7509 units, respectively, and Model-II, emphasizes the optimized minimum cost of 3037.033 points for membership grade$\lambda_2 = 0.3003278249$ and the minimized and greatest minimized costs of 2990.4961 and 3207.2998 units, respectively.

4.4.3.3 Case-III: Unbounded Feasibility with Positive Skewness-

The production cost is targeted with at least basic availability for all constraints. It is optimized when the basic availability of all constraints fluctuates by decrement of average quantity in one direction and by incrementing maximum quantity in another direction. The fluctuation is shown in Table 4.5.

4.4.3.3.1 Result of Case-III

Using the described methodology, the modeling of production cost is being done, and the fuzzy numbers for all cost parameters have been derived. The lower bound, static bound, and upper bound are calculated, and the optimized fuzzy linear programming problem (OFLPP) has been constructed using the lower and upper bound.

Likewise, for Case III, the mathematically defined membership grades of all constraints are seen in equations (4.22) to (4.29). Also, Figure 4.11 to Figure 4.18 show graphical representations, which are listed below:

Let B_{lab} be the membership grade for Labour cost, and it varies as: -

$$B_{lab} = \begin{cases} 1 & when \quad y = 153.2 \\ \dfrac{y - 145.54}{7.66} & when \;\; 145.54 \le y \le 153.2 \\ \dfrac{168.01 - y}{14.81} & when \;\; 153.2 \le y \le 168.01 \\ 0 & otherwise \end{cases} \qquad (4.22)$$

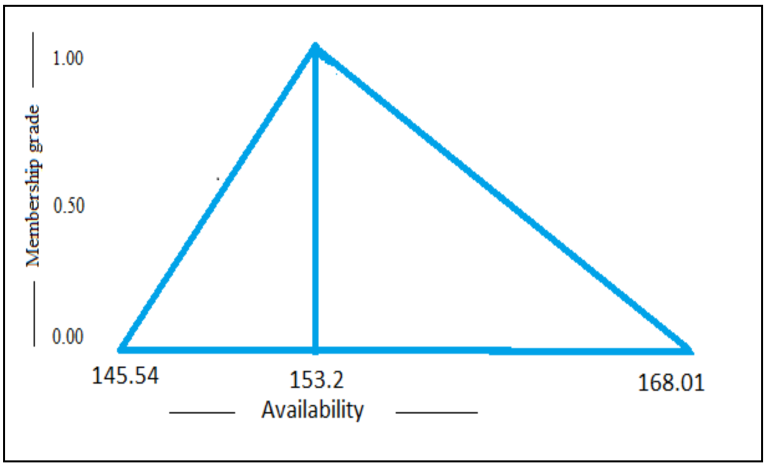


Figure 4.3: Membership grade for labor cost.

Let B_{mat} be the membership grade for Material cost, and it varies as: -

$$B_{mat} = \begin{cases} 1 & when \quad y = 2328.22 \\ \dfrac{y - 2211.81}{116.41} & when\ 2211.81 \leq y \leq 2328.22 \\ \dfrac{2574.82 - y}{246.6} & when\ 2328.22 \leq y \leq 2574.82 \\ 0 & otherwise \end{cases} \tag{4.23}$$

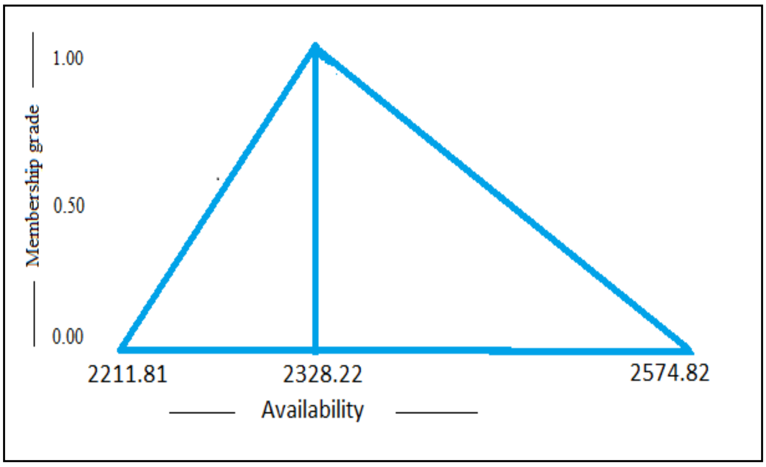


Figure 4.4: Membership grade for Material cost.

Let B_{foh} be the membership grade for Factory overhead charges, and it varies as: -

$$B_{foh} = \begin{cases} 1 & when\ y = 256.56 \\ \dfrac{y - 243.73}{12.83} & when\ \ 243.73 \leq y \leq 256.56 \\ \dfrac{280.96 - y}{24.4} & when\ \ 256.56 \leq y \leq 280.96 \\ 0 & otherwise \end{cases} \tag{4.24}$$

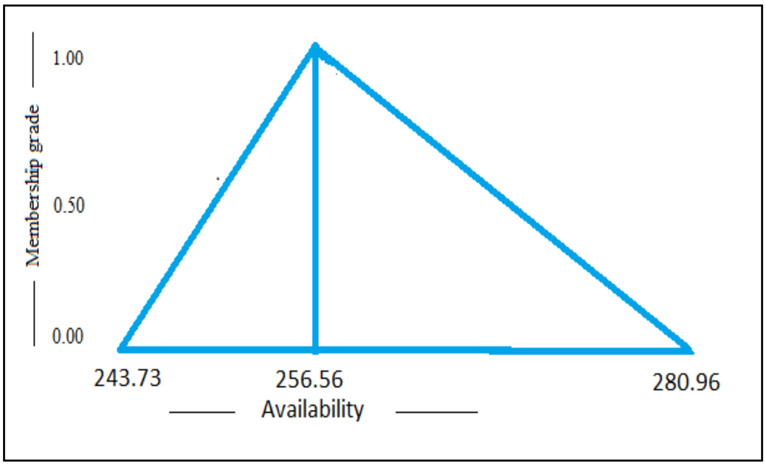


Figure 4.5: Membership grade for Factory overhead charge.

Let B_{aoh} be the membership grade for Administrative overhead charge, and it varies as: -

$$B_{aoh} = \begin{cases} 1 & when\ y = 197.13 \\ \dfrac{y - 187.27}{9.85} & when\ \ 187.27 \le y \le 197.13 \\ \dfrac{216.14 - y}{19.3} & when\ \ 197.13 \le y \le 216.14 \\ 0 & otherwise \end{cases} \quad (4.25)$$

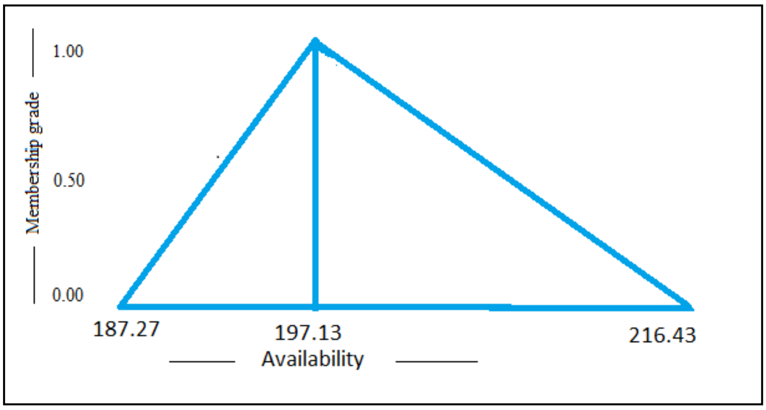


Figure 4.6: Membership grade for Administrative overhead charge.

Let B_{toh} be the membership grade for Township Overhead charge, and it varies as:-

$$B_{toh} = \begin{cases} 1 & when\ y = 41.23 \\ \dfrac{y - 39.16}{2.06} & when\ \ 39.16 \le y \le 41.23 \\ \dfrac{45.14 - y}{3.91} & when\ \ 41.23 \le y \le 45.14 \\ 0 & otherwise \end{cases} \tag{4.26}$$

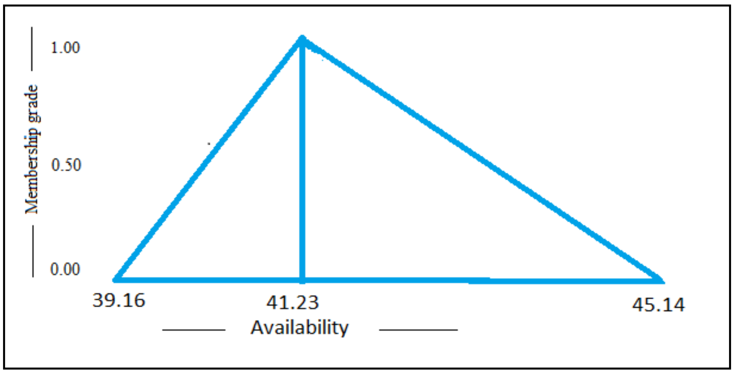


Figure 4.7: Membership grade for Township overhead charge.

Let B_{soh} be the membership grade for Shop overhead charge, and it varies as:-

$$B_{soh} = \begin{cases} 1 & when\ y = 18.67 \\ \dfrac{y - 17.73}{0.93} & when\ \ 17.73 \le y \le 18.67 \\ \dfrac{20.69 - y}{2.02} & when\ \ 18.67 \le y \le 20.69 \\ 0 & otherwise \end{cases} \tag{4.27}$$

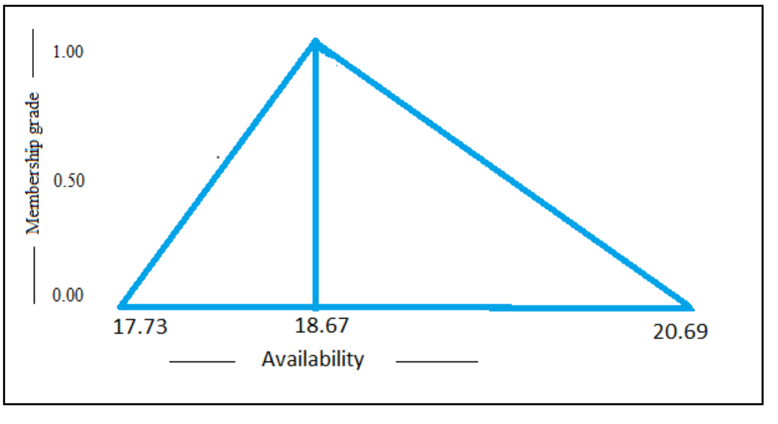


Figure 4.8: Membership grade for Shop overhead charge.

Let B_{toh} be the membership grade for Total overhead charge, and it varies as: -

$$B_{toh} = \begin{cases} 1 & when\ y = 513.61 \\ \dfrac{y - 487.92}{25.68} & when\ \ 487.92 \leq y \leq 513.61 \\ \dfrac{562.77 - y}{49.16} & when\ \ 513.61 \leq y \leq 562.77 \\ 0 & otherwise \end{cases} \tag{4.28}$$

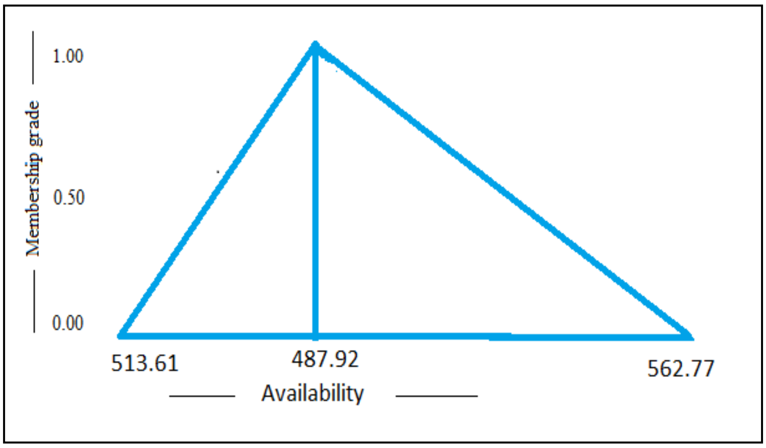


Figure 4.9: Membership grade for Total overhead charge.

Let B_{prof} be the membership grade for Proforma Charge, and it varies as: -

$$B_{prof} = \begin{cases} 1 & when\ y = 93.83 \\ \dfrac{y - 89.13}{4.69} & when\ \ 89.13 \leq y \leq 93.83 \\ \dfrac{103.25 - y}{9.42} & when\ \ 93.83 \leq y \leq 103.25 \\ 0 & otherwise \end{cases} \tag{4.29}$$

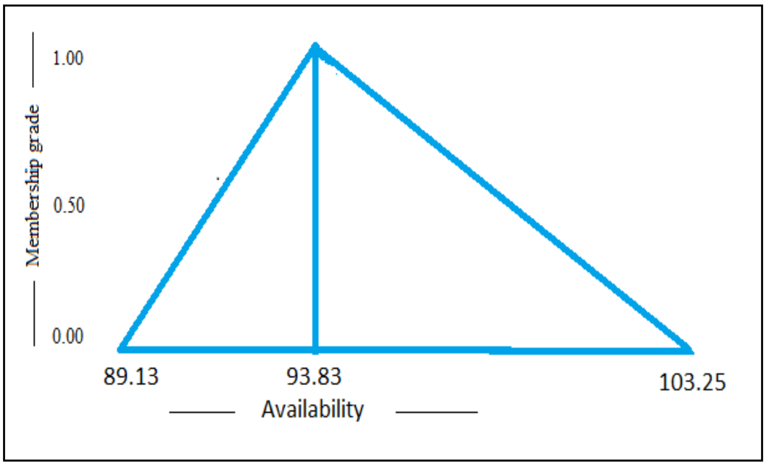


Figure 4.10: Membership grade for Proforma charge.

4.4.3.3.2 Optimized Fuzzy Linear Programming Problem of Case-III

Using the described methodology, the modeling of production cost is being done, and the fuzzy numbers for all cost parameters have been derived. The lower bound, static bound, and upper bound are calculated the value of lower, static, and upper bound are rupees **2934.3116711695** (lakhs), rupees **3088.7495985759**(lakhs), and rupees **3409.20227** (lakhs), respectively.

The optimized fuzzy linear programming problem (OFLPP) has been constructed using the lower, static, and upper bound.

The following Table 4.6 shows the solutions for the optimized value of lower, static, and upper bound and the optimized membership grade.

4.4.3.3.3 Result Analysis of Case-III

The production cost of RCF can be minimized using the cost parameter. The total basic cost of the production is rupees 3088.749(in lakhs), and it can be extended and declined to rupees 3409.20, 2934.311(in lakhs), respectively. The optimum production cost has been obtained to get the maximum membership grade. It shows that total production cost will provide the highest credibility if the optimized cost is considered equal to rupees 3088.749(in lakhs), and the credibility of production cost is being decreased if it is tending towards rupees 2934.311 and 3409.20(in lakhs). The following Eq. (4.30) and Figure 4.19 show the fuzzy number for optimized membership grade:

$$\boldsymbol{\lambda} = \begin{cases} 1 & when\ y = 3088.7496 \\ \dfrac{y - 2934.3116}{154.438} & when\ \ 2934.3116 \le y \le 3088.7496 \\ \dfrac{3409.2022 - y}{320.452} & when\ \ 3088.7496 \le y \le 3409.2022 \\ 0 & otherwise \end{cases} \qquad (4.30)$$

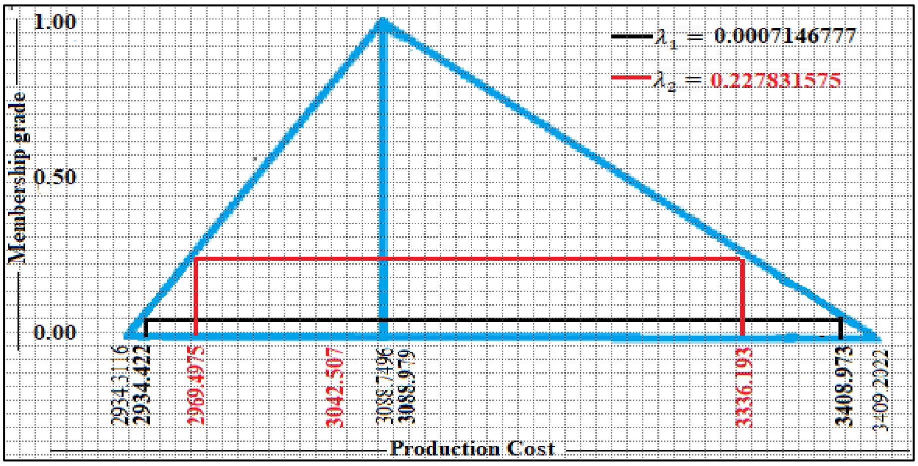


Figure 4.11: The relation between membership grade of optimized production cost with $\lambda_1$andλ_2 of case − III.

Using the structure of optimized composite triangle fuzzy LPP, Model-I illustrate the Optimized minimum cost **3088.979** unit and with the membership grade $\lambda_1 = 0.0007146777$ and the minimized and greatest minimized costs **2934.422, 3408.973** units, respectively.

Similarly, optimized composite triangle fuzzy LPP, Model-II illustrate the Optimized minimum cost of **3042.507unit** with the membership grade $\lambda_2 = 0.2278$, and the minimized and greatest minimized costs 2969.4975**,** 3336.193units, respectively.

4.5 Summary

This chapter utilizes fuzzy composite triangular to suggest a new structure for two models of fuzzy linear programming problems. We applied these two FLLP models in three cases, comparing each case to the existing FLP using due to the uncertainty of practical circumstances. After evaluating the outcomes of all the cases, it was discovered that the proposed models offered the best values for objective functions corresponding to membership functions. We propose to expand the same FLLP technique with the trapezoidal number in the next chapter. Furthermore, compare the proposed FLLP methods for fuzzy composite triangular numbers and fuzzy trapezoidal numbers.

Chapter 5

A Novel Approach for Fuzzy Linear Programming Using the Situational Based Trapezoidal Number

"The world is continuous, but the mind is discrete ".

−David Mumford

5.1 Introduction

In some certain situations, the total availability of any constrain can be inflexible from one requirement to other, and again it can be intensified and declined by any probabilistic increment and decrement. The trapezoidal fuzzy number can represent such types of problems. The first attempt is made to examine the credibility of optimized cost via different composite FLP models, and the results were compared with its extension, i.e., the trapezoidal FLP model. To validate the models with real-time phenomena, the Production cost data of Rail Coach Factory (RCF) Kapurthala has been taken. The lower, static, and upper bounds have been computed for each situation, and then systems of optimized FLP are constructed. The credibility of each model of trapezoidal FLPP concerning all situations are obtained. Using this membership grade, the minimum and the greatest minimum costs have been illustrated. The performance of each composite triangular FLPP model is compared to trapezoidal FLPP models, and the intense effects of trapezoidal on composite triangular FLPP models are investigated.

Dong et al. [76] were designed a new fuzzy linear model with trapezoidal fuzzy numbers (TrFNs) being all target coefficients, scientific coefficients, and devices. The order relationship of the TrFNs is initially measured using the estimate of the TrFNs interval. The trapezoidal linear fuzzy system was converted into an objective interval program based on the order relationship of the TrFNs. Taleshian and Rezvani [150] presented two trapezoidal fuzzy numbers with methods for solving the multiplication operation. Banerjee [151] mentioned the four basic arithmetic operations of generalized trapezoidal fuzzy numbers. A new nearest trapezoidal approach operator with expected interval survival is prescribed in [108]. Chen and Cheng [109] presented the subjective perspectives of decision-makers with trapezoidal fuzzy numbers in linguistic terms. An FLFP solution procedure where objective function, capital, and technical coefficients are fuzzy triangle numbers has been proposed [110]. Ebrahimnejad and Tavana [111] proposed an approach to address FLPP problems in which symmetric fuzzy trapezoidal numbers are interpreted as objective function coefficients and right-side values, while real numbers are the components of the matrix coefficient. An approach has been suggested by [112] to solve the FFLP problem, with a symmetric trapezoidal fuzzy number representing the parameters without any conversion of crisp equivalent problem. A complete linear defuzzification function defined in a trapezoidal fuzzy number subsection of a fuzzy number vector space is the best way to solve a linear programming problem with real objects in the type-1 fuzzy linear programming.

5.2 Fuzzy Linear Programming using the Trapezoidal Number

According to the trapezoidal fuzzy number $\tilde{B} = (\beta_i - \varepsilon_i, \beta_i,\ \beta_i^*, \beta_i^* + \varepsilon_i^*)$,the least lower, lower, upper bounds and the most upper bound of the optimal values are defined below.

The least lower bound $(\boldsymbol{Z_l^*})$ –

$$\text{Max}\boldsymbol{Z_l^*} = \sum_{j=1}^{n} \mathrm{c}_j y_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq \beta_i - \varepsilon_i$$

$$\text{Where, } \ y_j \geq 0, i, j \in \mathbb{N} \tag{5.1}$$

The lower bound $(\mathbf{Z_l})$ –

$$\text{Max } \boldsymbol{Z_l} = \sum_{j=1}^{n} \mathrm{c}_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i \tag{5.2}$$

$$\text{Where, } x_j \geq 0, i, j \in \mathbb{N}$$

The upper bound $(\mathbf{Z_u})$ –

$$\text{Max } \boldsymbol{Z_u} = \sum_{j=1}^{n} \mathrm{c}_j y$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y \leq \beta_i^*$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{5.3}$$

Now the most upper bound $(\boldsymbol{Z_u^*})$ –

$$\text{Max} \boldsymbol{Z_u^*} = \sum_{j=1}^{n} \mathrm{c}_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i^* + \varepsilon_i{}^*$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{5.4}$$

The solution for lower and upper bounds of LPP's can be obtained by using the Simplex method. To get the two different optimized FLPP model will be obtained by using these lower and upper bounds

5.2.1 Optimized Trapezoidal FLPP Model-I:-

$$Max\ Z = \lambda,$$

Subject to

$$\lambda(Z_s - Z_l) - \sum_{j=1}^{n} c_j y_j \leq -Z_l$$

$$\lambda(\varepsilon_i) + \sum_{j=1}^{n} a_{ij} y_j \leq b_i,$$

$$\lambda(Z_u^* - Z_u) - \sum_{j=1}^{n} c_j y_j \leq -Z_u$$

(5.5)

$$\lambda(\varepsilon_i^*) + \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i + \varepsilon_i^*$$

Where, $y_j \geq 0, i, j \in \mathbb{N}, \lambda \in [0,1]$

5.2.2 Optimized FLPP Model (II):-

$$MaxZ \ = \gamma$$

Subject to

$$\lambda(Z_u - Z_l^*) - \sum_{j=1}^{n} c_j y_j \leq -Z_l^*$$

$$\lambda\left(\varepsilon_i + \beta_i^* - \beta_i\right) + \sum_{j=1}^{n} \alpha_{ij}\, y_j \leq \beta_i^*$$

$$\lambda(Z_u^* - Z_l^*) - \sum_{j=1}^{n} c_j x_j \leq -Z_l^*$$

$$\lambda\left(\beta_i^* - \beta_i + \varepsilon_i{}^* + \varepsilon_i\right) + \sum_{j=1}^{n} \alpha_{ij}\, y_j \leq \beta_i^* + \varepsilon_i{}^*$$

(5.6)

$$y_j > 0, i, j \epsilon \mathbb{N}, \lambda \in [0,1]$$

This fuzzy optimized LPP will give the trapezoidal membership grade for our primary LPP. Here λ signifies the trapezoidal membership grade and Z_l^*, Z_l, Z_u *and* Z_u^* are the least lower, lower, upper, and most upper bounds, respectively. $\sum_{j=1}^{n} c_j x_j$ is the objective function of the initial LPP, $\varepsilon_i{}^*$ and ε_i is the probabilistic increment and decrement respectively in the availability of the constraints.

5.3 Numerical Result:

The following cases have been identified to deal with the described situations

5.3.1 Case IV: Bounded Feasibility with Zero Skewness

This case is an extension of the case –II of FLLP based on a composite triangular number where the trapezoidal membership grade is constant and gives a full degree of satisfaction for a small fluctuation, say minimum quantity in both directions of the basic availability of all constraints. The trapezoidal membership grade is further declined if certain increments and decrements are in the inflexible interval of basic availability. The minimum production cost is targeted with almost basic availability for all constraints and the least basic Proforma charge availability. The fluctuation is shown by the following Table 5.18.

5.3.2 Result of Case IV

Using the described methodology, the modeling of production cost is being done, and the fuzzy trapezoidal numbers for all cost parameters have been derived. The upper, lower, most upper, and least lower bounds are calculated, and the optimized fuzzy linear programming problem (OFLPP) has been constructed using the lower and upper bound.

The mathematically described membership grades of all Cases IV and V constraints are shown in Eq. (5.7) to (5.14).

Let B_{lab} be the trapezoidal membership grade for Labour cost, and it varies as:-

$$B_{lab} = \begin{cases} 1 & when \quad 150.7 \le y \le 155.7 \\ \dfrac{y - 145.54}{5.16} & when\ 145.54 < y < 150.7 \\ \dfrac{160.86 - y}{5.16} & when\ 155.7 < y < 160.86 \\ 0 & otherwise \end{cases} \qquad (5.7)$$

Let B_{mat} be the trapezoidal membership grade for Material cost, and it varies as:-

$$B_{mat} = \begin{cases} 1 & when \quad 2294.6 \le y \le 2361.84 \\ \dfrac{y - 2211.81}{82.79} & when\ 2211.81 < y < 2294.6 \\ \dfrac{2444.63 - y}{82.69} & when\ 2361.84 < y < 2444.63 \\ 0 & otherwise \end{cases} \qquad (5.8)$$

Let B_{foh} be the membership grade for Factory overhead charge, and it varies as:-

$$B_{foh} = \begin{cases} 1 & when \quad 251.98 \leq y \leq 261.14 \\ \dfrac{y - 243.73}{8.25} & when \; 243.73 < y < 251.98 \\ \dfrac{269.38 - y}{8.25} & when \; 261.14 < y < 269.38 \\ 0 & otherwise \end{cases} \qquad (5.9)$$

LetB_{aoh}be the Trapezoidal membership grade for Administrative overhead charge, and it varies as:-

$$B_{aoh} = \begin{cases} 1 & when \quad 194.22 \leq y \leq 200.04 \\ \dfrac{y - 187.27}{6.95} & when \; 187.27 < y < 194.22 \\ \dfrac{206.98 - y}{6.95} & when \; 200.04 < y < 206.98 \\ 0 & otherwise \end{cases} \qquad (5.10)$$

Let B_{toh} be the trapezoidal membership grade for Township overhead charge, and it varies as:-

$$B_{toh} = \begin{cases} 1 & when \quad 40.48 \leq y \leq 41.98 \\ \dfrac{y - 39.1685}{1.3115} & when \; 39.1685 < y < 40.48 \\ \dfrac{43.2915 - y}{1.3115} & when \; 41.98 < y < 43.2915 \\ 0 & otherwise \end{cases} \qquad (5.11)$$

Let B_{soh} be the trapezoidal membership grade for Shop overhead charge, and it varies as:-

$$B_{soh} = \begin{cases} 1 & when \quad 18.44 \leq y \leq 18.9 \\ \dfrac{y - 17.73}{0.71} & when \; 17.73 < y < 18.44 \\ \dfrac{19.60 - y}{0.71} & when \; 18.9 < y < 19.6 \\ 0 & otherwise \end{cases} \qquad (5.12)$$

Let B_{toh}be the membership grade for Total overhead charge, and it varies as:-

$$B_{toh} = \begin{cases} 1 & when \quad 505.14 \leq y \leq 522.08 \\ \dfrac{y - 487.93}{17.21} & when\ 487.93 < y < 505.14 \\ \dfrac{539.29 - y}{17.21} & when\ 522.08 < y < 539.29 \\ 0 & otherwise \end{cases} \tag{5.13}$$

Let B_{prof} be the trapezoidal membership grade for Proforma charge, and it varies as:-

$$B_{prof} = \begin{cases} 1 & when \quad 505.14 \leq y \leq 522.08 \\ \dfrac{y - 487.93}{17.21} & when\ 487.93 < y < 505.14 \\ \dfrac{539.29 - y}{17.21} & when\ 522.08 < y < 539.29 \\ 0 & otherwise \end{cases} \tag{5.14}$$

Using the described methodology, the modeling of production cost is being done, and the fuzzy numbers for all cost parameters have been derived. The least lower, lower, upper, and most upper bounds are rupees 2943.959308 (lakhs), rupees 3098.912497 (lakhs), and rupees 3253.835047 (lakhs), respectively.

The Table 5.2 shows the solutions for the optimized value of super lower, lower upper, and super upper bound and optimized membership grades.

5.3.2.1 Result Analysis of Case-IV

The production cost of RCF can be minimized using the cost parameter. The total basic cost of the product will illustrate the full degree of satisfaction, and it is from rupees 3026.4 to 3118 (in lakhs). It can be extended and declined further till rupees 3225.8, 2918.6(in lakhs), respectively. The optimum production cost has been obtained to get the maximum membership grade. It shows that total production cost will provide the highest credibility if the optimized cost is considerably near the basic cost (from 3026.4 to 3118), for which the membership grade is inflexible. The credibility of production cost decreases if it tends towards the extended costs, say rupees 2918.6 and 3225.8(in lakhs). The following equation and figure show the fuzzy number for optimized membership grade:

$$\lambda = \begin{cases} 1 & when \quad 3026.4 \le y \le 3118 \\ \dfrac{y - 2918.6}{107.8} & when\ 2918.6 < y < 3026.4 \\ \dfrac{3225.8 - y}{107.8} & when\ 3118 < y < 3225.8 \\ 0 & otherwise \end{cases} \quad (5.15)$$

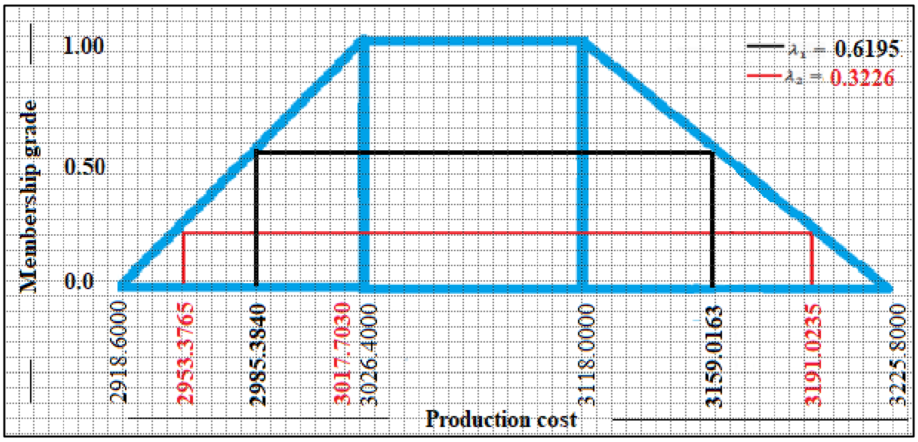


Figure 5.1: The relation between Trapezoidal membership grade of optimized production cost with $\lambda_1$and λ_2 of case-IV.

Table 5.20 displays the calculated values of Model-I and Model-II using the structure optimized trapezoidal FLPP.

5.3.3 Case V: Unbounded Feasibility with Zero Skewness

This case is an extension of the case –I of FLLP based on a composite triangular number where the Trapezoidal membership grade is constant and gives a full degree of satisfaction for a small fluctuation, say minimum quantity in both directions of the basic availability of all constraints. The membership grade is further declined if certain increments and decrements are in the inflexible interval of basic availability. The minimum production cost is targeted with at most basic availability for all constraints. The fluctuation is shown in Table 5.21.

5.3.3.1 Result and Optimized Fuzzy Linear Programming Problem of Case-V

The modeling of production cost and the fuzzy numbers for all cost parameters are derived, in equations 22 to 29 and figure from 178 to 185. The least-lower, lower, upper, and most upper bounds are 2934.320953, 3042.7297, 3134.788898, and 3243.19684rupees in lakhs, respectively. The optimized fuzzy linear programming problem (OFLPP) has been constructed using the least lower, lower, upper, and most upper bounds

The Table 5.5 shows the solutions for the optimized value of super lower, lower upper, and super upper bound and optimized membership grades.

5.3.3.2 Result Analysis of Case-V

The production cost of RCF can be minimized using the cost parameter. The total basic cost of the product will illustrate the full degree of satisfaction, and it is from rupees 3042.7297 to 3134.788898(in lakhs). It can be extended and declined further to rupees 3243.19684, 2934.320953(in lakhs), respectively. The optimum production cost has been obtained to get the maximum membership grade. It shows that total production cost will provide the highest credibility if the optimized cost is considerably neat to the basic cost [3042.7297to 3134.78889], for which the membership grade is inflexible. The credibility of production cost decreases if it is tending towards the extended costs, say rupees 2934.320953and 3243.19684(in lakhs) respectively. The following Eq.(5.16) and Figure 5.10 show the fuzzy number for optimized membership grade:

$$\lambda = \begin{cases} 1 & when \quad 3042.729 \leq y \leq 3134.788 \\ \dfrac{y - 2934.32}{108.41} & when\ 2934.32 < y < 3042.729 \\ \dfrac{3243.196 - y}{108.41} & when\ 3134.788 < y < 3243.196 \\ 0 & otherwise \end{cases} \tag{5.16}$$

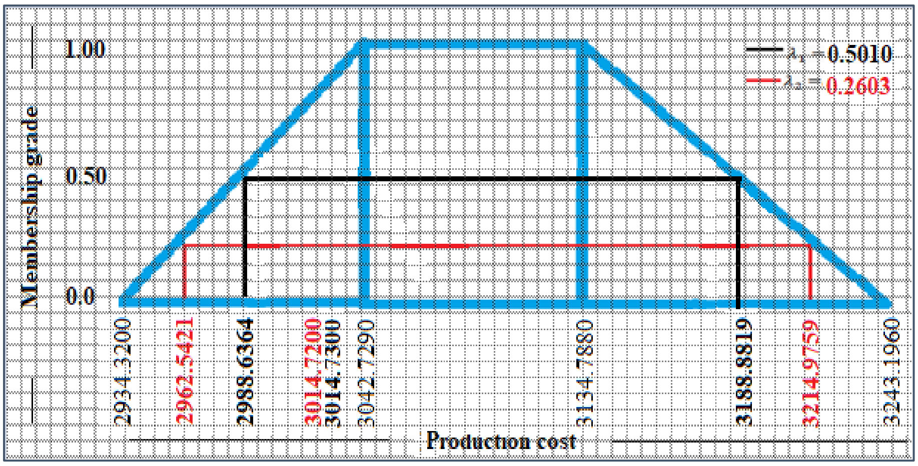


Figure 5.2: The relation between Trapezoidal membership grade of optimized production cost with $\lambda_1$and λ_2 of case-V.

Likewise, Table 5.22 displays the measured values of Model I and Model II with the structure of Trapezoidal optimized FLPP

5.4 Comparison Between All the Models with Different Cases

Figure 5.3: The performances of different models for different cases and their extension, in the form of a bar chart, show different models' performances to get optimized. It is observed that the costs obtained by model II are more appropriate as compared to a model I of composite triangular LPP, and the cost obtained by model I are more appropriate as compared to the model II of trapezoidal LPP. The overall performance of the trapezoidal LPP model – I is better than all other models. Trapezoidal LPP model – I reduced approximately 50% destruction in production cost compared to 26% of trapezoidal LPP model – II, 0.1% of the Composite triangular LPP model - I, and 30% of Composite triangular LPP model - II. The overall performance of the trapezoidal LPP model - I is better than all other models. Trapezoidal LPP model – I reduced approximately 62% production cost compared to 32% of trapezoidal LPP model – II, 0.05% of the Composite triangular LPP model - I, and 30% of Composite triangular LPP model - II.

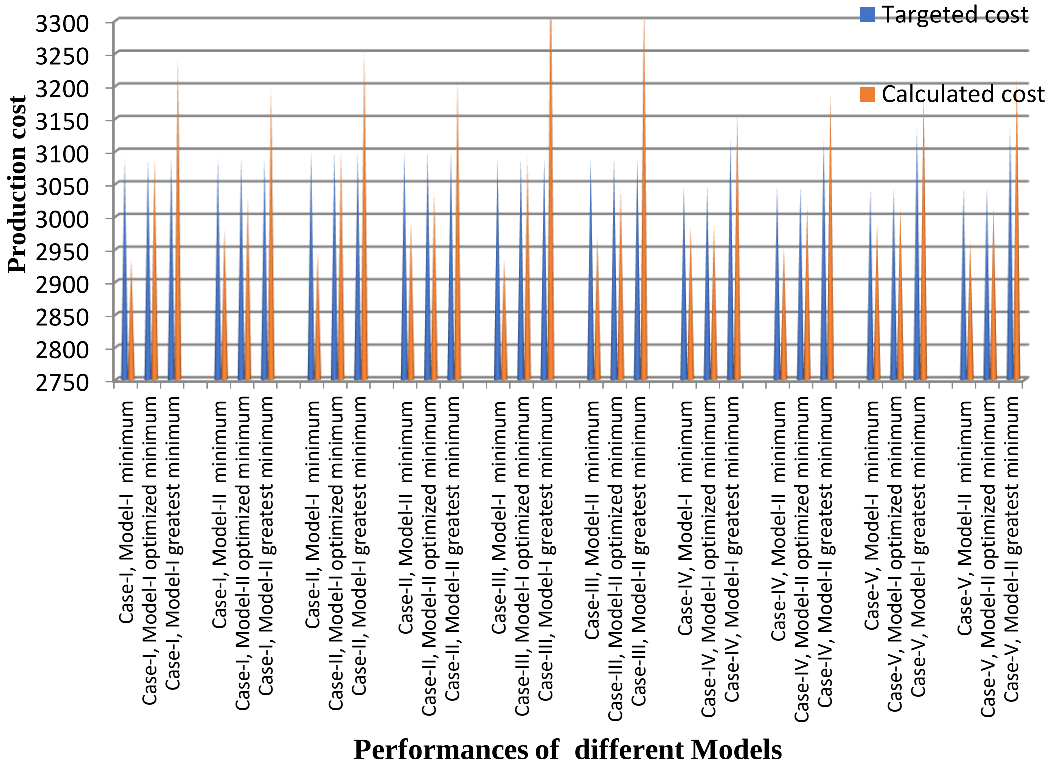


Figure 5.3: The performances of different models for different cases and their extension

5.5 Summary

In this chapter, the comparative study of modeling and optimizing the production cost of railway coaches of RCF Kapurthala via composite triangular fuzzy and trapezoidal fuzzy linear programming problem (FLPP) is proposed. Due to probabilistic increment and decrement in the availability of different constraints, the real production cost was fluctuating or uncertain. Therefore, the descriptions of five different incertitude situations are formulated, and the realistic models to extenuate the annihilation in the production cost optimization have been given in the article. Here, in the first attempt, the credibility of optimized cost via two different composites triangular FLPP models is examined, and the results were compared with its extension, i.e., trapezoidal FLPP model. The entire cost has been aimed to optimize regarding the constraints of $C_{Lab.}$, $C_{Mat.}$, C_{foh} C_{Aoh}, C_{Toh},C_{Soh}, C_{Tot} ,C_{Pc} and C_{Tc}. The lower, least lower, static, upper, and most upper bounds have been calculated for each situation, and then systems of optimized FLPP were constructed. The credibility of each model of composite triangular and trapezoidal FLPP for all situations has been obtained and using these membership grades, the minimum and greatest minimum cost have been exemplified. The performance of each model of composite triangular fuzzy linear programming to all situations was compared with the trapezoidal fuzzy linear programming problem model. In all proposed situations for the greatest lower and least upper cost, it was observed that the composite triangular FLPP model II is more appropriate than model I, and trapezoidal FLPP model I is more appropriate than model II and model I & II of composite triangular FLPP. Hence, overall, the trapezoidal FLPP model I performance is the best among all proposed models**.** It shows a better degree of conciliation than composite triangular FLPP models and trapezoidal FLPP model II.

Chapter 6

An Advanced Optimization Technique for Smart Production Using α −Cut Based Quadrilateral Fuzzy Number

"*Obvious is the most dangerous word in mathematics."*

− *E. T. Bell*

6.1 Introduction

Fuzzy number, a crucial component of fuzzy set theory, is very prominent when describing the unknown phenomena in real issues. Two unique perspectives are fuzzy numbers, the first one is their membership function, and the second being their alpha-cut. The two considerations are equivalent, and one may be superior to the other, depending on the details that we want to consider. Triangular and trapezoidal numbers are mostly used. The grade of satisfaction at β_i is 1 in the triangular fuzzy number $i.e$ $(\beta_i - \varepsilon_i, \beta_i, \beta_i + \varepsilon_i^*)$, but the grade of satisfaction from $\beta_i - \varepsilon_i$ to β_i is determined by the angle of elevation, the scale from 0 to 1, and from β_i to $\beta_i + \varepsilon_i^*$ is defined by the angle of the depression, which is from 1 to 0. Similarly, the level of satisfaction for β_i to β_i^* is 1 for a trapezoidal number $(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i^*)$. Nevertheless, the degree of satisfaction from $\beta_i - \varepsilon_i$ to β_i is shown by an elevated angle from 0 to 1 and from β_i^* to $\beta_i^* + \varepsilon_i^*$ by a slump angle of 1 to 0. In some cases, however, these fuzzy numbers do not reflect the actual description of realistic situations. In this article, a newly developed fuzzy number is suggested, ``α −cut based on a fuzzy quadrilateral number $(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i^*)$ in which the degree of satisfaction from $\beta_i - \varepsilon_i$ to β_i is expressed by an angle of elevation, the range of which is from 0 to β and between β_i to β_i^* is ß to 1, where $\beta \in [0,1]$. The depression angle with a range of 1 to 0 is also represented from β_i^* to $\beta_i^* + \varepsilon_i^*$.

The interpretation of fuzzy numbers from an alpha-cut perspective is an interval approach. In contrast, with the assistance of various arithmetic operations which have the prerequisites, the different characteristics of the fuzzy number could be found. Different features of the fuzzy number could also be identified [152] using the various arithmetic operations with the necessary criteria. To study some properties of fuzzy arithmetic operations, [153] and Guerra have analyzed the decomposition of fuzzy numbers and have compared the proposed approximation to standard fuzzy arithmetic. Taleshian and Rezvani [150] presented two trapezoidal fuzzy numbers with methods for solving the multiplication operation. Banerjee [151] mentioned the four basic arithmetic operations of generalized trapezoidal fuzzy numbers. While using the definition of distribution and complementary distribution functions, Garg, H, and Ansha [154] studied the basic arithmetic operations for two generalized positive parabolic fuzzy numbers. As a significant piece of mathematical programming, linear programming is one of the applied operation research systems. Due to the vulnerability of objective objects and the fluctuation of human muses, there are many situations in which target values, technological coefficients, and assets cannot be precisely incorporated into the linear programming model. Fuzzy numbers of techniques have been proposed to deal with fuzzy linear programming problems. Numerous researchers have extensively studied a triangular and trapezoidal fuzzy linear programming model as a typical fuzzy linear program. In this paper, we proposed a newly constructed $\alpha -$cut based quadrilateral fuzzy number on the right-hand side of the fuzzy linear programming problem, which gives a clear picture of the optimization in uncertain conditions through the general framework of FLP [155] we assess the optimal values of different situations.

6.2 Proposed Membership Function for the $\alpha - cut$ Based Quadrilateral Fuzzy Number

A fuzzy number $\widetilde{\boldsymbol{B}} = \langle(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i{}^*) \mid \beta_i, \beta_i^* \in \mathbb{R}, and\ \varepsilon_i\ , \varepsilon_i{}^* \in \mathbb{R}^+\rangle$, is said to be a $\boldsymbol{\alpha - cut}$ **based quadrilateral fuzzy number** if its membership function $B_i(y)$: $\mathbb{R} \to [0,1]$ satisfies the following properties.

(i) It is compact in $\mathbb{R}$.
(ii) It is continuous over $\mathbb{R}$.
(iii) It is monotonic increasing on $[\beta_i - \varepsilon_i\ ,\ \beta_i^*]$ and monotonic decreasing on $[\beta_i^*\ ,\ \beta_i^* + {\varepsilon_i}^*]$.
(iv) It is zero for all $y \in (-\infty\ ,\ \beta_i - \varepsilon_i)\ \cup\ (\ \beta_i^* + {\varepsilon_i}^*, \infty)$.
(v) It is normal.
(vi) It is a triangular fuzzy number when $\beta\ = 0$ and $\beta = \frac{\varepsilon_i}{\beta_i^* - (\beta_i - \varepsilon_i)}\ \forall\ \alpha\epsilon[0,1]$.
(vii) It is a trapezoidal fuzzy number when $\beta\ = 1, \forall\ \alpha\epsilon[0,1]$.
(viii) It is convex according to (vi), (vii) and $\theta_1 \geq \theta_2\ \forall\ \ \alpha \leq \beta\ , \alpha\ , \beta\ \neq 0$.
(ix) It is non-convex when$\theta_1 < \theta_2\ \forall\ \ \alpha \leq \beta\ , \alpha\ , \beta\ \neq 0.\quad where\ \theta_1 = \tan^{-1}(\frac{\beta}{\epsilon_i})\ and\ \theta_2 = \tan^{-1}(\frac{\bar{\beta}}{\beta_i^* - \beta_i})$.

A fuzzy number $\widetilde{\boldsymbol{B}} = \langle(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + {\varepsilon_i}^*)\ |\ \beta_i, \beta_i^* \in\ \mathbb{R}, and\ \varepsilon_i\ , {\varepsilon_i}^* \in \mathbb{R}^+\rangle$, is called a quadrilateral fuzzy number, and its membership function is defined by

$$B_i(y) = \begin{cases} \beta + \left(\frac{y - \beta_i}{\beta_i^* - \beta_i}\right)\bar{\beta} & When\ \beta_i \leq y \leq \beta_i^* \\ (\frac{y - \beta_i + \varepsilon_i}{\varepsilon_i})\beta & When\ \beta_i - \varepsilon_i \leq y \leq \beta_i \\ \frac{\beta_i^* + {\varepsilon_i}^* - y}{{\varepsilon_i}^*} & When\ \beta_i^* \leq y \leq \beta_i^* + {\varepsilon_i}^* \\ 0 & otherwise \end{cases} \tag{6.1}$$

Where β represents the membership grade and $\bar{\beta}$ represents the complement of β.

Meanwhile, the quadrilateral membership function shown in Figure 6.1 is most frequently used to represent α-cut of the fuzzy quadrilateral number $\widetilde{\boldsymbol{B}}$ **.**

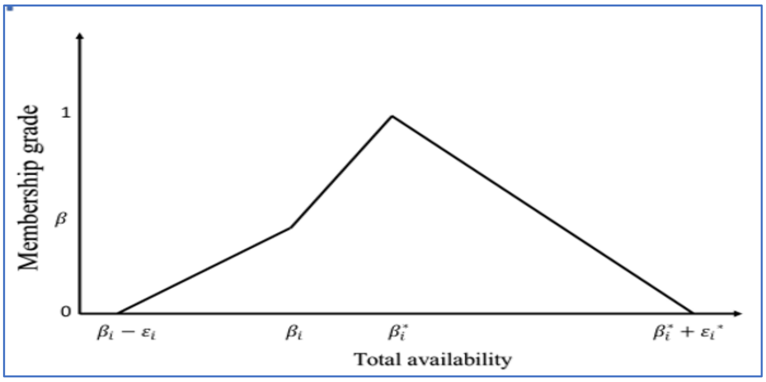


Figure 6.1: Membership grade for the fuzzy quadrilateral number

The α −cut of the fuzzy quadrilateral number $\widetilde{\boldsymbol{B}}$ is closed interval is defined below:

$$B_\alpha = \begin{cases} [\,\beta_i + \alpha(\beta_i^* - \beta_i)\,, (\beta_i^* + \varepsilon_i{}^*) - \alpha\varepsilon_i{}^*] & \beta = 0,\ \alpha\epsilon[0,1] \\ [(\beta_i - \varepsilon_i) + \alpha\varepsilon_i\,,\ (\beta_i^* + \varepsilon_i{}^*) - \alpha\varepsilon_i{}^*] & \beta = 1,\ \alpha\epsilon[0,1] \\ [(\beta_i - \varepsilon_i) + \alpha(\beta_i^* - \beta_i + \varepsilon_i), (\beta_i^* + \varepsilon_i{}^*) - \alpha\varepsilon_i{}^*] & \beta = \dfrac{\varepsilon_i}{\beta_i^* - (\beta_i - \varepsilon_i)}\,, \alpha\epsilon[0,1] \\ I_1 \cup I_2 \cup I_3 & \alpha \le \beta\,, \alpha\,, \beta\ \ne 0 \end{cases} \tag{6.2}$$

$$Where\,, I_1 = \left[\beta_i - \varepsilon_i\,, (\beta_i - \varepsilon_i) + \frac{\alpha\varepsilon_i}{\beta_1}\right) \quad , \ I_2 = \left[\beta_i\,,\ \beta_i + \frac{\bar{\alpha}-\beta}{\bar{\beta}}(\beta_i^* - \beta_i)\right) \quad and$$

$$I_3 = [\beta_i^*\,, (\beta_i^* + \varepsilon_i{}^*) - \alpha\varepsilon_i{}^*)$$

6.3 Proposed Arithmetic Operations Between $\alpha - cut$ Based Quadrilateral Fuzzy Numbers

In this section, the improved arithmetic operations have been proposed between $\alpha - cut$ based quadrilateral fuzzy numbers using $\alpha - cuts$.

Lets $\tilde{B}_i^p = (\beta_i^p - \varepsilon_i^p\,, \beta_i^p\,, \beta_i^{p*}\,,\ \beta_i^{p*} + \varepsilon_i^{p*})$ and $\tilde{B}_i^q = (\beta_i^q - \varepsilon_i^q\,, \beta_i^q\,, \beta_i^{q*}\,,\ \beta_i^{q*} + \varepsilon_i^{q*})$ be two $\alpha - cut$ based quadrilateral fuzzy numbers which membership functions B_i^p and B_i^q respectively, which can be written as

$$B_i^p(y) = \begin{cases} \beta^p + \left(\dfrac{y-\beta_i^p}{\beta_i^{p*}-\beta_i^p}\right) \times \overline{\beta^p} & When\ \beta_i^p \le y \le \beta_i^{p*} \\ \left(\dfrac{y-\beta_i^p + \varepsilon_i^p}{\varepsilon_i^p}\right) \times \beta^p & When\ \beta_i^p -\ \varepsilon_i^p \le y \le \beta_i^p \\ \dfrac{\beta_i^{p*} - \varepsilon_i^{p*} - y}{\varepsilon_i^{p*}} & When\ \beta_i^{p*} \le y \le \beta_i^{p*} +\ \varepsilon_i^{p*} \\ 0 & otherwise \end{cases} \tag{6.3}$$

Where β^p represents the membership grade and $\overline{\beta^p}$represents the complement of β^p.

$$B_i^q(y)=\begin{cases} \beta^q + \left(\frac{y-\beta_i^q}{\beta_i^{q*}-\beta_i^q}\right) \times \overline{\beta^q} & When\ \beta_i^q \le y \le \beta_i^{q*} \\ \left(\frac{y-\beta_i^q + \varepsilon_i^q}{\varepsilon_i^q}\right) \times \beta^q & When\ \beta_i^q - \varepsilon_i^q \le y \le \beta_i^q \\ \frac{\beta_i^{q*} - \varepsilon_i^{q*} - y}{\varepsilon_i^{q*}} & When\ \beta_i^{q*} \le y \le \beta_i^{q*} + \varepsilon_i^{q*} \\ 0 & otherwise \end{cases} \tag{6.4}$$

Where β^q represents the membership grade and $\overline{\beta^q}$ represents the complement of β^q.

Where β_i^p , β_i^{p*} , β_i^q *and* β_i^{q*} are real number, ε_i^p , ε_i^{p*} , ε_i^q *and* ε_i^{q*} are the positive real numbers, such that $\beta^p \le \beta^q$. Take $\beta^s \in [\beta^p, \beta^q]$, then make a $\beta^s -$ cut of fuzzy number $\tilde{B}_i^q$ such that $\tilde{B}_i^q$ will transform into a new $\alpha - cut$ based quadrilateral fuzzy number as $\tilde{B}_i^{q+}$ = ($\beta_i^q - \varepsilon_i^q$, β_i^{q+} , β_i^{q*} , $\beta_i^{q*} + \varepsilon_i^{q*}$), where $\beta_i^{q+} = \beta_i^q - \varepsilon_i^q + \frac{\beta^r}{\beta^q} \times \varepsilon_i^q$ for membership function. Clearly if $\beta^p = \beta^q$ then $\beta^r = \beta^p = \beta^q, \beta_i^{q+} = \beta_i^q$ and hence the new $\alpha - cut$ based quadrilateral fuzzy number B_i^{q+} is same as that of $\alpha - cut$ based quadrilateral fuzzy number B_i^q.

Now the a β^s −cut of fuzzy number $\tilde{B}_i^p$ and $\tilde{B}_i^q$ become the α^p −cut of the $\alpha - cut$ based quadrilateral fuzzy number $\widetilde{\boldsymbol{B^p}}$ is closed interval is defined below:

$$B_\alpha^p = \begin{cases} \left[\beta_i^p + \alpha^p(\beta_i^{p*} - \beta_i^p), (\beta_i^{p*} + \varepsilon_i{}^{p*}) - \alpha^p \varepsilon_i{}^{p*}\right] & \beta^p = 0,\ \alpha^p \epsilon [0,1] \\ \left[\left(\beta_i^p - \varepsilon_i^p\right) + \alpha^p \varepsilon_i^p,\ (\beta_i^{p*} + \varepsilon_i{}^{p*}) - \alpha^p \varepsilon_i{}^{p*}\right] & \beta^p = 1,\ \alpha^p \epsilon [0,1] \\ \left[\left(\beta_i^p - \varepsilon_i^p\right) + \alpha^p(\beta_i^{p*} - \beta_i^p + \varepsilon_i^p), (\beta_i^{p*} + \varepsilon_i{}^{p*}) - \alpha^p \varepsilon_i{}^{p*}\right] & \beta^p = \frac{\varepsilon_i^p}{\beta_i^{p*} - (\beta_i^p - \varepsilon_i^p)}, \alpha^p \epsilon [0,1] \\ I_1^p \cup I_2^p \cup I_3^p & \alpha^p \le \beta^p,\quad \alpha^p, \beta^p \ne 0 \end{cases} \tag{6.5}$$

$$Where\ , I_1^p = \left[\beta_i^p - \varepsilon_i^p , \left(\beta_i^p - \varepsilon_i^p\right) + \frac{\alpha^p \varepsilon_i^p}{\beta^p}\right), I_2^p = \left[\beta_i^p ,\ \beta_i^p + \frac{\overline{\alpha^p} - \beta^p}{\overline{\beta^p}}\left(\beta_i^{p*} - \beta_i^p\right)\right) \quad and$$

$$I_3^p = \left[\beta_i^{p*}, \left(\beta_i^{p*} + \varepsilon_i{}^{p*}\right) - \alpha^p \varepsilon_i{}^{p*}\right)$$

The α^q −cut based quadrilateral fuzzy number $\widetilde{\boldsymbol{B^q}}$ is closed interval defined below:

$$
B_{\alpha}^{q} = \begin{cases} [\beta_i^q + \alpha^q(\beta_i^{q*} - \beta_i^q), (\beta_i^{q*} + \varepsilon_i{}^{q*}) - \alpha^q \varepsilon_i{}^{q*}] & \beta^q = 0, \ \alpha^q \epsilon[0,1] \\ \left[\left(\beta_i^q - \varepsilon_i^q\right) + \alpha^q \varepsilon_i^q, \ (\beta_i^{q*} + \varepsilon_i{}^{q*}) - \alpha^q \varepsilon_i{}^{q*}\right] & \beta^q = 1, \ \alpha^q \epsilon[0,1] \\ [\left(\beta_i^q - \varepsilon_i^q\right) + \alpha^q(\beta_i^{q*} - \beta_i^q + \varepsilon_i^q), (\beta_i^{q*} + \varepsilon_i{}^{q*}) - \alpha^q \varepsilon_i{}^{q*}] & \beta^q = \dfrac{\varepsilon_i^q}{\beta_i^{q*} - (\beta_i^q - \varepsilon_i^q)}, \alpha^q \epsilon[0,1] \\ I_1^q \cup I_2^q \cup I_3^q & \alpha^q \leq \beta^q, \quad \alpha^q, \beta^q \neq 0 \end{cases} \tag{6.6}
$$

$$
Where, I_1^q = \left[\beta_i^q - \varepsilon_i^q, (\beta_i^q - \varepsilon_i^q) + \frac{\alpha^q \varepsilon_i^q}{\beta^q}\right), \ I_2^q = \left[\beta_i^q, \ \beta_i^q + \frac{\bar{\alpha^q} - \beta^q}{\overline{\beta^q}}\left(\beta_i^{q*} - \beta_i^q\right)\right) \quad and I_3^q = \left[\beta_i^{q*}, \left(\beta_i^{q*} + \varepsilon_i{}^{q*}\right) - \alpha^q \varepsilon_i{}^{q*}\right)
$$

Thus, we quantify improved arithmetical operations based on these $\beta^s -$ cuts: addition, subtraction, scalar propagation, division, etc., between the two quadrilateral fuzzy numbers.

Theorem 6.1. Addition of two $\alpha - cut$ based quadrilateral fuzzy number $\tilde{B}_i^p = (\beta_i^p - \varepsilon_i^p, \beta_i^p, \beta_i^{p*}, \beta_i^{p*} + \varepsilon_i^{p*})$ and $\tilde{B}_i^q = (\beta_i^q - \varepsilon_i^q, \beta_i^q, \beta_i^{q*}, \beta_i^{q*} + \varepsilon_i^{q*})$ with two different confidence levels generates a $\alpha - cut$ based quadrilateral fuzzy number $\tilde{B}_i^s = \tilde{B}_i^p + \tilde{B}_i^q = (\beta_i^s - \varepsilon_i^s, \beta_i^s, \beta_i^{s*}, \beta_i^{s*} + \varepsilon_i^{s*})$ where

$$
\beta_i^s - \varepsilon_i^s = (\beta_i^p - \varepsilon_i^p) + (\beta_i^q - \varepsilon_i^q) \tag{6.7}
$$

$$
\beta_i^s = \beta_i^p + \beta_i^q - \frac{\beta^s}{\beta^p}\varepsilon_i^q \tag{6.8}
$$

$$
\beta_i^{s*} = \beta_i^{p*} + \beta_i^{q*} \tag{6.9}
$$

$$
\beta_i^{s*} + \varepsilon_i^{s*} = \left(\beta_i^{p*} + \varepsilon_i^{p*}\right) + (\beta_i^{q*} + \varepsilon_i^{q*}) \tag{6.10}
$$

Proof: See Appendix-C

Theorem 6.2. If $\tilde{B} = \langle(\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i{}^*) \mid \beta_i, \beta_i^* \in \mathbb{R}, and\ \varepsilon_i, \varepsilon_i{}^* \in \mathbb{R}^+\rangle$ be a $\alpha - cut$ based quadrilateral fuzzy number then $k\tilde{B}$ is again a $\alpha - cut$ based quadrilateral fuzzy number given by

$$
k\tilde{B} = \begin{cases} (k(\beta_i - \varepsilon_i), k\beta_i, k\beta_i^*, k(\beta_i^* + \varepsilon_i{}^*)) & ; k > 0 \\ (k(\beta_i^* + \varepsilon_i{}^*), k\beta_i^*, k\beta_i, (k(\beta_i - \varepsilon_i)) & ; k < 0 \end{cases} \tag{6.11}
$$

$For\ k > 0$ the membership function is given by

$$B_{ki}(y)=\begin{cases} \beta + \left(\frac{y-k\beta_i}{\beta_i^*-\beta_i}\right) \times \bar{\beta} & When\ k\beta_i \le y \le k\beta_i^* \\ \left(\frac{y-k(\beta_i-\varepsilon_i)}{\varepsilon_i}\right) \times \beta & When\ k(\beta_i - \varepsilon_i) \le y \le k\beta_i \\ \frac{k(\beta_i^*+\varepsilon_i{}^*)-y}{\varepsilon_i{}^*} & When\ k\beta_i^* \le y \le k(\beta_i^* + \varepsilon_i{}^*) \\ 0 & otherwise \end{cases} \tag{6.12}$$

Proof: See the Appendix -C

Theorem 6.3 Subtraction of two $\alpha - cut$ based quadrilateral fuzzy number $\tilde{B}_i^p$ = $(\beta_i^p - \varepsilon_i^p\ , \beta_i^p\ , \beta_i^{p*}\ ,\ \beta_i^{p*} + \varepsilon_i^{p*}\ )$ and $\tilde{B}_i^q$= $(\beta_i^q - \varepsilon_i^q\ , \beta_i^q\ , \beta_i^{q*}\ ,\ \beta_i^{q*} + \varepsilon_i^{q*}\ )$ with two different confidence levels generates a $\alpha - cut$ based quadrilateral fuzzy number $\tilde{B}_i^s$ = $\tilde{B}_i^p - \tilde{B}_i^q$ = $(\beta_i^s - \varepsilon_i^s\ , \beta_i^s\ , \beta_i^{s*}\ ,\ \beta_i^{s*} + \varepsilon_i^{s*}\ )$ where

$$\beta_i^s - \varepsilon_i^s = (\beta_i^p - \varepsilon_i^p) - (\beta_i^{q*} + \varepsilon_i^{q*}) \tag{6.13}$$

$$\beta_i^s = \beta_i^p - (\beta_i^{q*} + \varepsilon_i^{q*}) - \frac{\beta^s}{\beta^p}\ \varepsilon_i^{q*} \tag{6.14}$$

$$\beta_i^{s*} = \beta_i^{p*} - \beta_i^{q*} \tag{6.15}$$

$$\beta_i^{s*} + \varepsilon_i^{s*} = \left(\beta_i^{p*} + \varepsilon_i^{p*}\right) - (\beta_i^q - \varepsilon_i^q) \tag{6.16}$$

$\boldsymbol{Proof}$: Follow from theorem 6.1 and 6.2, so we omit here.

Theorem 6.4. Multiplication of two $\alpha - cut$ based quadrilateral fuzzy number $\tilde{B}_i^p$ = $(\beta_i^p - \varepsilon_i^p\ , \beta_i^p\ , \beta_i^{p*}\ ,\ \beta_i^{p*} + \varepsilon_i^{p*}\ )$ and $\tilde{B}_i^q$= $(\beta_i^q - \varepsilon_i^q\ , \beta_i^q\ , \beta_i^{q*}\ ,\ \beta_i^{q*} + \varepsilon_i^{q*}\ )$ with two different confidence levels generates a quadrilateral fuzzy number $\tilde{B}_i^s$ = $\tilde{B}_i^p \times \tilde{B}_i^q$ = $(\beta_i^s - \varepsilon_i^s\ , \beta_i^s\ , \beta_i^{s*}\ ,\ \beta_i^{s*} + \varepsilon_i^{s*}\ )$ where

$$\beta_i^s - \varepsilon_i^s = (\beta_i^p - \varepsilon_i^p)(\beta_i^q - \varepsilon_i^q) \tag{6.17}$$

$$\beta_i^s = \beta_i^p(\beta_i^q - \varepsilon_i^q) + \frac{\beta^s}{\beta^q}\beta_i^p\varepsilon_i^q \tag{6.18}$$

$$\beta_i^{s*} = \beta_i^{p*}\beta_i^{q*} \tag{6.19}$$

$$\beta_i^{s*} + \varepsilon_i^{s*} = \left(\beta_i^{p*} + \varepsilon_i^{p*}\right)(\beta_i^{q*} + \varepsilon_i^{q*}) \tag{6.20}$$

$$
B_i^s(y)=\begin{cases} \beta^s + \frac{-K_1+\sqrt{K_1^2+4M_1(y-N_1)}}{2M_1} & When\ \beta_i^s \le y \le \beta_i^{s*} \\ \frac{-K_2+\sqrt{K_2^2+4M_2(y-N_2)}}{2M_2} & When\ \beta_i^s - \varepsilon_i^s \le y \le \beta_i^s \\ \frac{K_3+\sqrt{K_3^2+4M_3(y-N_3)}}{2M_3} & When\ \beta_i^{s*} \le y \le \beta_i^{s*} + \varepsilon_i^{s*} \\ 0 & otherwise \end{cases} \quad (6.21)
$$

Where,

$$
K_1 = \frac{\beta_i^p(\beta_i^{q*}-\beta_i^q)}{\overline{\beta^q}} + \frac{\beta_i^q(\beta_i^{p*}-\beta_i^p)}{\overline{\beta^s}},\ K_2 = \frac{(\beta_i^p-\varepsilon_i^p)(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^q}} + \frac{(\beta_i^q-\varepsilon_i^q)\varepsilon_i^p}{\overline{\beta^s}}
$$

$$
K_3 = (\beta_i^{p*} + \varepsilon_i^{p*})(\varepsilon_i^{q*}) + (\beta_i^{q*} + \varepsilon_i^{q*})\varepsilon_i^{p*}
$$

$$
M_1 = \frac{(\beta_i^{p*}-\beta_i^p)(\beta_i^{q*}-\beta_i^q)}{\overline{\beta^s}\,\overline{\beta^q}},\ M_2 = \frac{\varepsilon_i^p(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^s}\,\overline{\beta^q}},\ M_3 = \varepsilon_i^{p*}\varepsilon_i^{q*}
$$

$$
N_1 = \beta_i^p\beta_i^q,\ N_2 = (\beta_i^p - \varepsilon_i^p)(\beta_i^q - \varepsilon_i^q),\ N_3 = \left(\beta_i^{p*} + \varepsilon_i^{p*}\right)(\beta_i^{q*} + \varepsilon_i^{q*})
$$

Proof: See the Appendix D.

Theorem 6.5. If $\widetilde{\boldsymbol{B}} = (\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + {\varepsilon_i}^*)$ represents a $\alpha - cut$ based quadrilateral fuzzy number then the inverse of $\widetilde{\boldsymbol{B}}$ i.e $\widetilde{\boldsymbol{B}}^{-1} = ((\beta_i^* + {\varepsilon_i}^*)^{-1}$, $(\beta_i^*)^{-1}, (\beta_i)^{-1}, (\beta_i - \varepsilon_i)^{-1})$ is also a $\alpha - cut$ based quadrilateral fuzzy number whose membership function is given by

$$
B_i^{-1}(y)=\begin{cases} \beta^* + \left(\frac{y-(\beta_i^*)^{-1}}{(\beta_i)^{-1}-(\beta_i^*)^{-1}}\right)\overline{\beta^*} & When\ (\beta_i^*)^{-1} \le y \le (\beta_i)^{-1} \\ \left(\frac{y-(\beta_i^*+{\varepsilon_i}^*)^{-1}}{(\beta_i^*)^{-1}-(\beta_i^*+{\varepsilon_i}^*)^{-1}}\right)\beta^* & When\ (\beta_i^* + {\varepsilon_i}^*)^{-1} \le y \le (\beta_i^*)^{-1} \\ \frac{(\beta_i-\varepsilon_i)^{-1}-y}{(\beta_i-\varepsilon_i)^{-1}-(\beta_i)^{-1}} & When\ (\beta_i)^{-1} \le y \le (\beta_i - \varepsilon_i)^{-1} \\ 0 & otherwise \end{cases} \quad (6.22)
$$

Where β^* represents the membership grade and $\overline{\beta^*}$ represents the complement of β^*.

Theorem 6.6. Division of two $\alpha - cut$ based quadrilateral fuzzy number $\tilde{B}_i^p$ = ($\beta_i^p - \varepsilon_i^p$, β_i^p ,β_i^{p*} , $\beta_i^{p*} + \varepsilon_i^{p*}$) and $\tilde{B}_i^q$= ($\beta_i^q - \varepsilon_i^q$, β_i^q ,β_i^{q*} , $\beta_i^{q*} + \varepsilon_i^{q*}$) with two different confidence levels generates a $\alpha - cut$ based quadrilateral fuzzy number $\tilde{B}_i^s$ = $\frac{\tilde{B}_i^p}{\tilde{B}_i^q}$ = $\tilde{B}_i^p \times (\widetilde{B}_i^q)^{-1}$ =($\beta_i^s - \varepsilon_i^s$, β_i^s ,β_i^{s*} , $\beta_i^{s*} + \varepsilon_i^{s*}$), Where

$$\beta_i^s - \varepsilon_i^s = (\beta_i^p - \varepsilon_i^p) \times (\beta_i^{q*} + \varepsilon_i^{q*})^{-1} \tag{6.23}$$

$$\beta_i^s = \beta_i^p \times (\beta_i^{q*})^{-1} \tag{6.24}$$

$$\beta_i^{s*} = \beta_i^{p*} \times \left(\frac{1}{(\beta_i^q - \varepsilon_i^q) + \frac{\beta^s}{\beta^q}\varepsilon_i^q} \right) \tag{6.25}$$

$$\beta_i^{s*} + \varepsilon_i^{s*} = \left(\beta_i^{p*} + \varepsilon_i^{p*}\right) \times (\beta_i^q - \varepsilon_i^q)^{-1} \tag{6.26}$$

Proof. As $\tilde{B}_i^q$= ($\beta_i^q - \varepsilon_i^q$, β_i^q ,β_i^{q*} , $\beta_i^{q*} + \varepsilon_i^{q*}$). Thus $\frac{1}{\tilde{B}_i^q}$ = ($\frac{1}{\beta_i^{q*} + \varepsilon_i^{q*}}$,$\frac{1}{\beta_i^{q*}}$,$\frac{1}{\tilde{B}_i^q}$,$\frac{1}{\beta_i^q - \varepsilon_i^q}$) and $\tilde{B}_i^p \times \frac{1}{\tilde{B}_i^q}$ .Therefore, the proof of this theorem from theorem 6.4, so we omit.

6.4 Fuzzy Linear Programming through the $\alpha - cut$ Based Quadrilateral Fuzzy Number

The standard form of FLP in equation (4.2) is considered to be the quadrilateral fuzzy number $\widetilde{\boldsymbol{B}} = (\beta_i - \varepsilon_i, \beta_i, \beta_i^*, \beta_i^* + \varepsilon_i{}^*)$ as a consequence of increased and decreased availability of restrictions instead of the right triangle fuzzy number $\widetilde{\beta_i}$. Therefore, the general structure of the least lower, lower, upper bounds and the most upper bound of the optimal values are defined below.

The least low bound $(\boldsymbol{Z_l^*})$

$$\text{Max}\boldsymbol{Z_l^*} = \sum_{j=1}^{n} c_j y_j$$

Subject to $\sum_{j=1}^{n} a_{ij}x_j \leq \beta_i - \varepsilon_i$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{6.27}$$

The lower bound (**Z**$_l$)

$$\text{Max } \boldsymbol{Z_l} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i$$

$$\text{Where, } x_j \geq 0, i, j \in \mathbb{N} \tag{6.28}$$

The upper bound (**Z**$_u$)

$$\text{Max } \boldsymbol{Z_u} = \sum_{j=1}^{n} c_j y$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y \leq \beta_i^*$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{6.29}$$

Now the most upper bound ($\boldsymbol{Z_u^*}$)

$$\text{Max} \boldsymbol{Z_u^*} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \beta_i^* + \varepsilon_i{}^*$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{6.30}$$

Using the techniques of the simplex method to find the values of the optimization of these bounds i.e. z_u, z_l, $z_u{}^*$ and $z_l{}^*$are the upper, lower, most upper, and least lower bounds.

6.4.1 Optimized FLLP Model for $\alpha - cut$ Based Quadrilateral Fuzzy Number

Here an optimized FLLP model based on an $\alpha - cut$ based quadrilateral fuzzy number is proposed to obtain the optimized values of these limits is defined below:

$$Max\ Z = \lambda$$

Subject to

$$\lambda(z_l - z_l{}^*) - \varsigma y \leq -z_l{}^*$$

$$\lambda(\varepsilon_i) + \sum_{j=1}^{n} \alpha_{ij}\, y_j \leq \beta_i$$

$$\lambda(z_u - z_l{}^*) - \varsigma y \leq -z_l{}^*$$

$$\lambda\left(\varepsilon_i + \beta_i^* - \beta_i\right) + \sum_{j=1}^{n} \alpha_{ij}\, y_j \leq \beta_i^*$$

$$\lambda(z_u{}^* - z_l{}^*) - \varsigma y \leq -z_l{}^*$$

$$\lambda\left(\beta_i^* - \beta_i + \varepsilon_i{}^* + \varepsilon_i\right) + \sum_{j=1}^{n} \alpha_{ij}\, y_j \leq \beta_i^* + \varepsilon_i{}^*$$

$$y_j > 0, i, j \epsilon \mathbb{N} \tag{6.31}$$

This fuzzy optimized LPP will give the membership grade for our initial LPP. Here λ represents the membership grade and z_u, z_l, $z_u{}^*$ and $z_l{}^*$are the upper, lower, most upper, and least lower bounds.

ςy is the objective function of the initial LPP. The term with summation sign represents the constraints of given LPP and ε_i^* and ε_i is the probabilistic increment and decrement respectively in the availability of the constraints. In Figure 6.2 demonstrates the flow chart of the FLLP based on the proposed fuzzy number.

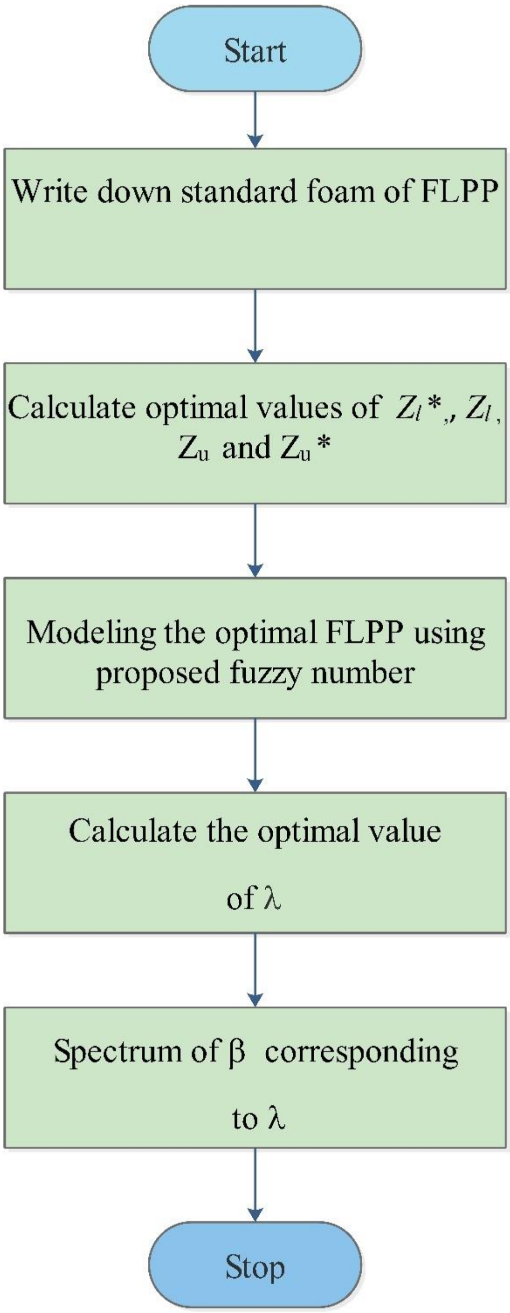


Figure 6.2:Flow chat of the FLLP based on proposed fuzzy number

6.5 Numerical Experiment

The crisp optimization techniques are not good enough to illustrate the targeted optimum result in the fluctuated situation or feasible uncertainty in the followings situations.

6.6 Data and Problem Identification

The data in Table 4. 1 are from The Railway Industry in Kapurthala for 2010-2011. This data shows the manufacturing cost ('in lacs,' i.e., 1,00,000) of different coaches' constraints. Kapurthala Railway Industry was established in 1986. It is a coach manufacturing unit of Indian Railways and manufactured more than 30000 passenger coaches of different types.

But they can be extended with some probabilistic increment, decrement, and reach to $(\beta_i - \varepsilon_i)$, β_i, β_i^* , $(\beta_i^* + \varepsilon_i{}^*)$. In this situation, we propose a newly constructed quadrilateral FLLP to minimize the total cost of production. Similarly, in certain situations, the total availability of any constraint can be inflexible from one requirement to another. Again, it can be intensified and declined by any probabilistic increment and decrement; then, we are also presenting the trapezoidal FLLP to minimize the total cost of production.

The membership grade is declined if there is certain increments and decrements in the inflexible interval of basic availability. For example - $(\beta_i - \varepsilon_i)\sim \beta_i \sim \beta_i^* \sim (\beta_i^* + \varepsilon_i{}^*)$. The minimum production cost is targeted with almost basic availability for all constraints and at least basic availability for Performa change.

The fluctuation is given in Table 6.23:

6.2 Modelling for the System of Optimal Solution

Using the values shown in Table 4.1 and Table 6.23: shows the probabilistic increments and decrements in the cost parameter, the standard form of fuzzy linear programming is defined below:

6.3. Numerical Results

Using the described methodology, the production cost modeling is being done, and the fuzzy quadrilateral numbers have been constructed. The optimized fuzzy linear programming problem has been constructed using the bounds least lower, lower, upper, and greatest upper bound, which are calculated. Demonstrating the membership grade of all constraints is represented by equations (6.32) to (6.39) and graphically summarizing the constraints shown in Figure 6.3 to Figure 6.10 at the different value of the significance level.

Let B_{lab} be the membership grade for Labor cost, and it varies as: -

$$B_{\text{lab}} = \begin{cases} \beta + \left(\frac{y-150.7}{5}\right) \times \bar{\beta} & When\ 150.70 \leq y \leq 155.7 \\ \left(\frac{y-145.54}{5.16}\right) \times \beta & When\ 145.54 \leq y \leq 150.7 \\ \frac{160.86 - y}{5.16} & When\ 155.7 \leq y \leq 160.86 \\ 0 & otherwise \end{cases} \tag{6.32}$$

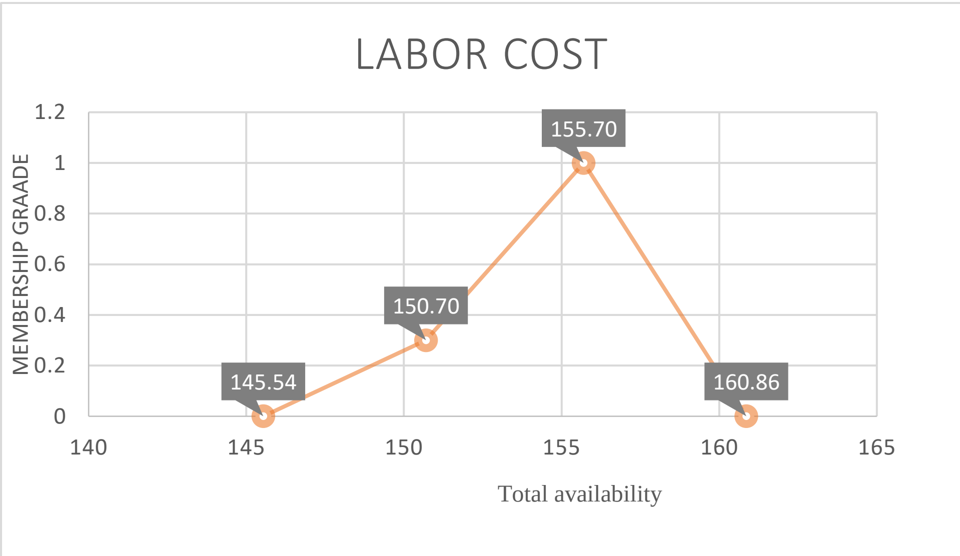


Figure 6.3: Membership grade for labor cost

Let B_{mat} be the membership grade for material cost, and it varies as: -

$$B_{\text{mat}} = \begin{cases} \beta + (\frac{y-2294.6}{67.24}) \times \bar{\beta} & \text{when } 2294.60 \le y \le 2361.84 \\ (\frac{y-2211.81}{82.79}) \times \beta & \text{when } 2211.81 \le y \le 2294.60 \\ \frac{2444.63 - y}{82.79} & \text{when} 2361.84 \le y \le 2444.63 \\ 0 & otherwise \end{cases} \tag{6.33}$$

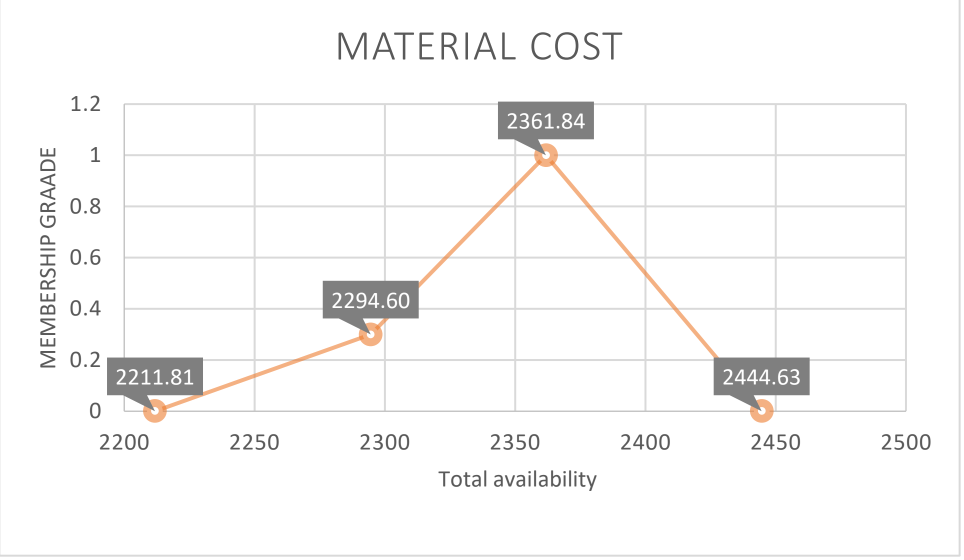


Figure 6.4: Membership grade for material cost

Let B_{foh} be the membership grade for factory overhead charges, and it varies as: -

$$B_{\text{foh}} = \begin{cases} \beta + (\frac{y-251.98}{9.16}) \times \bar{\beta} & \text{when } 251.98 \le y \le 261.14 \\ (\frac{y-243.73}{8.25}) \times \beta & \text{when } 243.73 \le y \le 251.98 \\ \frac{269.38 - y}{8.24} & \text{when } 261.14 \le y \le 269.38 \\ 0 & otherwise \end{cases} \tag{6.34}$$

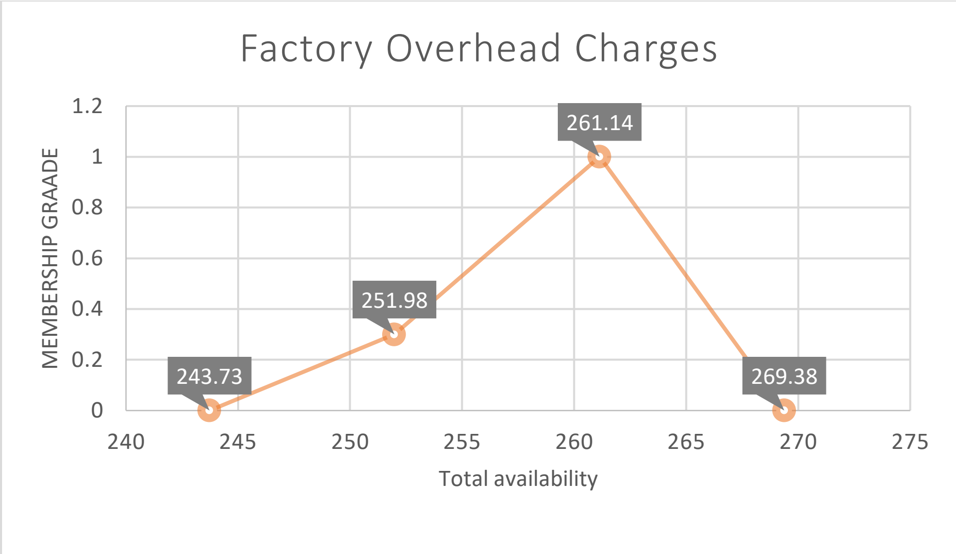


Figure 6.5: Membership grade for factory overhead charges

Let B_{aoh} be the membership grade for administrative overhead charges, and it varies as:-

$$B_{\text{aoh}} = \begin{cases} \beta + \left(\frac{y-194.22}{5.82}\right) \times \bar{\beta} & \text{when } 194.22 \leq y \leq 200.04 \\ \left(\frac{y-187.27}{6.95}\right) \times \beta & \text{when } 187.27 \leq y \leq 194.22 \\ \frac{206.98-y}{6.94} & \text{when } 200.04 \leq y \leq 206.98 \\ 0 & otherwise \end{cases} \quad (6.35)$$

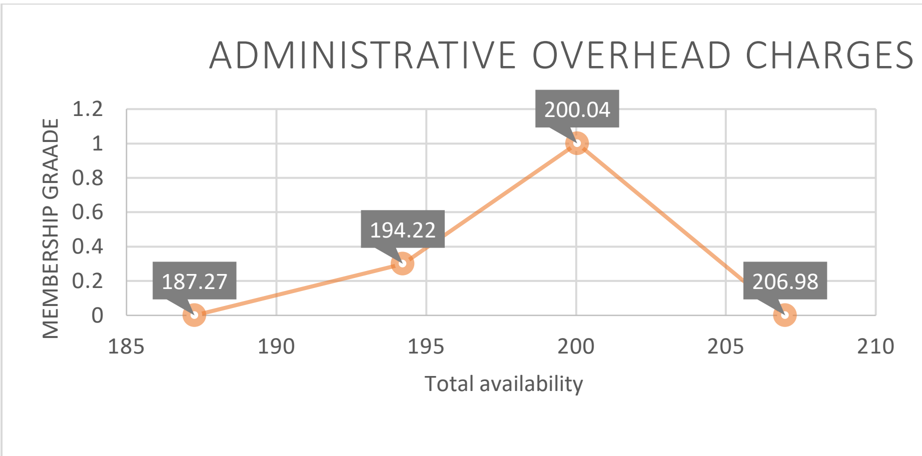


Figure 6.6: Membership grade for administrative charges

Let B_{tooh} be the membership grade for township overhead charges, and it varies as: -

$$B_{\text{tooh}} = \begin{cases} \beta + (\frac{y-40.48}{1.5}) \times \bar{\beta} & \text{when } 40.48 \le y \le 41.98 \\ (\frac{y-39.19}{1.3115}) \times \beta & \text{when } 39.17 \le y \le 40.48 \\ \frac{41.98 - y}{1.31} & \text{when } 41.98 \le y \le 43.29 \\ 0 & otherwise \end{cases} \tag{6.36}$$

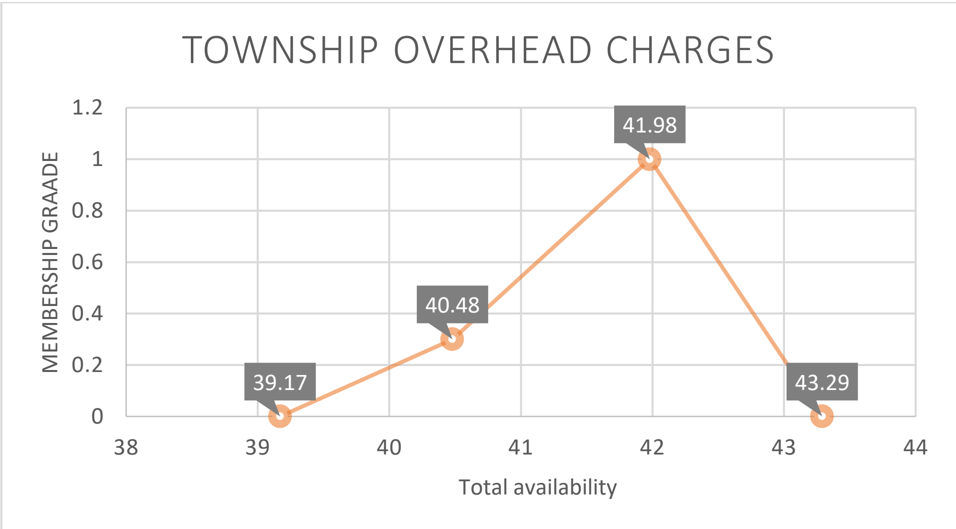


Figure 6.7: Membership grade for township overhead charges

Let B_{soh} be the membership grade for overhead shop charges, and it varies as:-

$$B_{\text{soh}} = \begin{cases} \beta + (\frac{y-18.44}{0.46}) \times \bar{\beta} & \text{when } 18.44 \le y \le 18.90 \\ (\frac{y-17.73}{0.71}) \times \beta & \text{when } 17.73 \le y \le 18.44 \\ \frac{18.90 - y}{0.70} & \text{when } 18.90 \le y \le 19.60 \\ 0 & otherwise \end{cases} \tag{6.37}$$

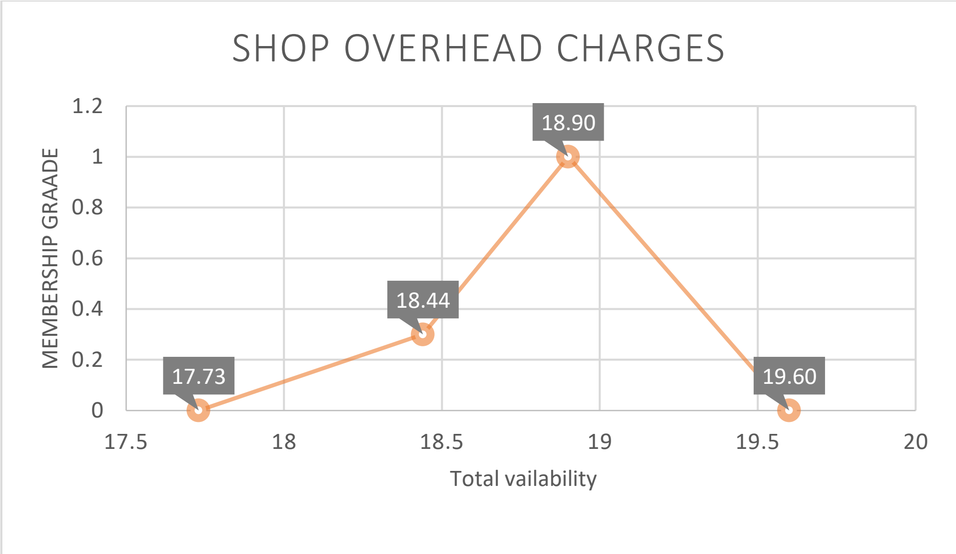


Figure 6.8: Membership grade for shop overhead charges

Let B_{toh} be the membership grade for total overhead charges, and it varies as: -

$$B_{\text{toh}} = \begin{cases} \beta + (\frac{y-505.14}{16.94}) \times \bar{\beta} & \text{when } 505.14 \le y \le 522.08 \\ (\frac{y-487093}{17.21}) \times \beta & \text{when } 487.93 \le y \le 505.14 \\ \frac{522.08-y}{17.21} & \text{when } 522.08 \le y \le 539.29 \\ 0 & otherwise \end{cases} \tag{6.38}$$

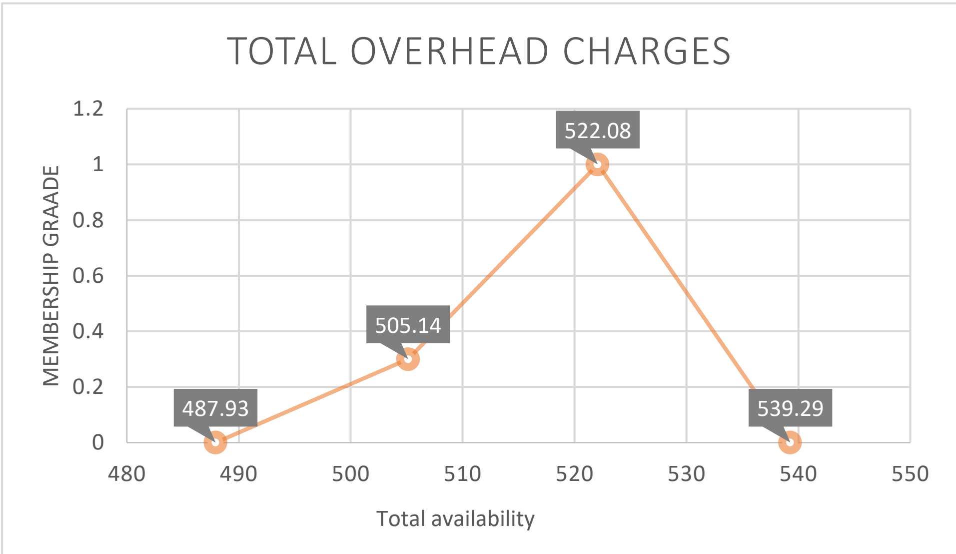


Figure 6.9: Membership grade for total overhead charges

Let $\boldsymbol{B_{prof}}$ be the membership grade for Performa charge, and it varies as: -

$$B_{\text{prof}} = \begin{cases} \beta + (\frac{y-92.39}{2.88}) \times \bar{\beta} & \text{when } 92.39 \leq y \leq 95.27 \\ (\frac{y-89.13}{3.26}) \times \beta & \text{when } 89.13 \leq y \leq 92.39 \\ \frac{95.27-y}{3.26} & \text{when } 95.27 \leq y \leq 98.52 \\ 0 & otherwise \end{cases} \tag{6.39}$$

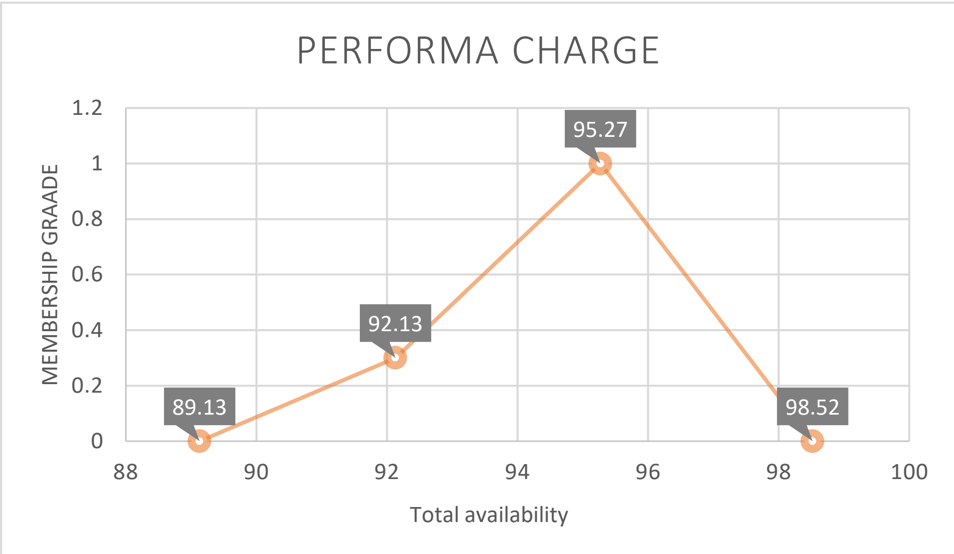


Figure 6.10: Membership grade for Performa charge

6.4 Optimal Result of Numerical Result

Using the described methodology, the modeling of production cost is being done, and quadrilateral fuzzy numbers for all cost parameters have been derived. The least lower, lower, upper, and greatest upper bounds are 2918.6, 3026.4, 3118, and 3225.8 rupees in lacs (Indian rupees), respectively. The optimized fuzzy linear programming problem (OFLPP) has been constructed using the least lower, lower, upper, and greatest upper bounds.

Table 6.24: Optimized membership grade shows the solutions for the optimized value of least low, lower, upper, and most upper bound and the optimized membership grade.

6.5 Analysis of Numerical Result

The production cost of RCF is to be minimized using the cost parameter. The optimum production cost has been obtained to get the maximum membership grade. It shows that total production cost will provide the highest credibility if the optimized cost is considered equal to the basic cost range [3026,3118]. The following equation and figure show the fuzzy number for optimized membership grade.

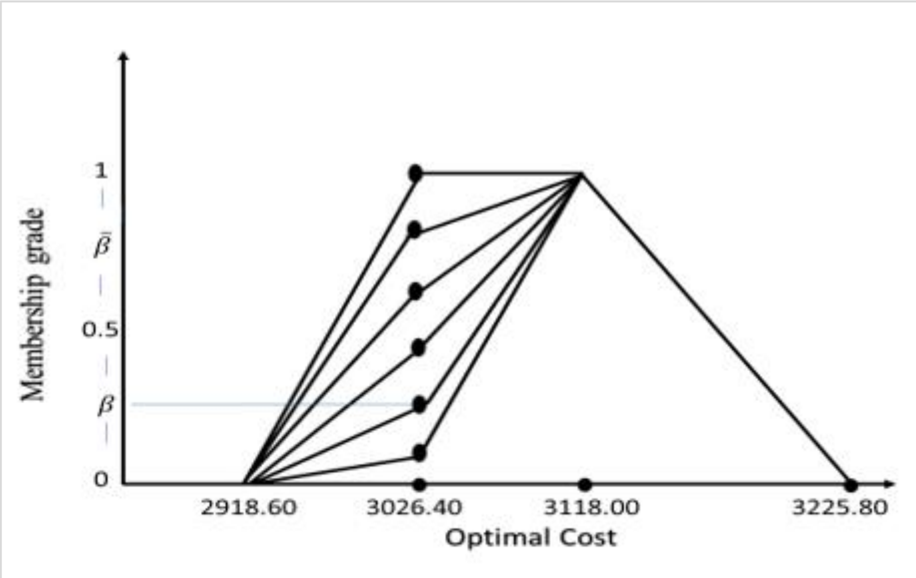


Figure 6.11: Membership grade for optimal cost

$$B_{\text{Optimized cost}-\text{I}} = \begin{cases} \beta + \left(\dfrac{y-3026.4}{91.6}\right) \times \bar{\beta} & \text{when } 3026.4 \leq y \leq 3118 \\ \left(\dfrac{y-2918.6}{107.8}\right) \times \beta & \text{when } 2918.6 \leq y \leq 3026.4 \\ \dfrac{3225.8 - y}{107.8} & \text{when } 3118 \leq y \leq 3225.8 \\ 0 & \textit{otherwise} \end{cases} \tag{6.40}$$

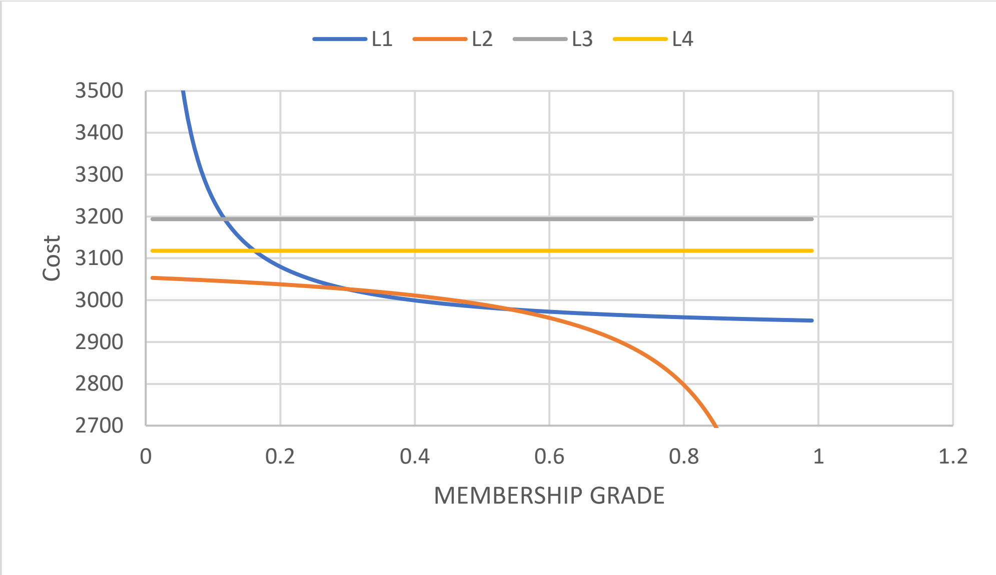


Figure 6.12: Representation of the β spectrum corresponding to the optimum value of λ

In Figure 6.11, L1 and L2 represented the range of optimal costs lies between [2918.6, 3026.4] and [3026.4,3118] respectively, which are dependent on the different value of β, while, L3 is the targeted optimal cost, i.e., 3118 and L4 the optimal cost lies between [3118, 3225.8] which are independent β.

In Figure 6.12, the line graph shows the performance in terms of optimized cost through these lines utilizing the different values of β of $\alpha - based$ quadrilateral fuzzy LPP.From the given availably of the data, the basic targeted cost is in-between lower and upper bound that is L3=3118 unit, respectively. In the proposed model using $\alpha - based$ quadrilateral fuzzy number, the performance of different values of β (degree of grade satisfaction) are observed values to achieve targeted cost, which lies between [0.17,0.30].

6.7 Summary

In this chapter, a $\alpha -$ cut based quadrilateral fuzzy number is proposed with the proof of basic mathematical operations and shows the application of it in the fuzzy linear programming problems. This fuzzy number in R.H.S of the FLLP assists in showing the uncertainty in solution values due to chances of increment and decrement in the availability of different constraints. So, the description of a case of incertitude and the realistic model to extenuate the destruction in the optimization is shown. The comparative analysis of modeling and optimization of production cost of the various coaches of RCF Kapurthala has been done through a $\alpha - \mathrm{cut\ based}$ quadrilateral fuzzy linear programming problem. The credibility of optimized cost via a $\alpha - cut\ based$ quadrilateral FLLP model is examined. The total cost has been targeted to optimize the constraints of different expenses to construct the different types of coaches. The lower, least lower, upper, and most upper bounds have been calculated for the model, and then systems of optimized FLLP were constructed. The credibility of the model has been obtained and using these memberships grade the minimum, and greatest minimum cost have been exemplified. It is observed that the performances obtained by a $\boldsymbol{\alpha} - \mathrm{cut\ based}$ quadrilateral FLLP through the different value of $\beta \in [0.17\ , 0.30]$ shows the degree of satisfaction in vague situations. Further, these numbers will be used for optimization through the other groups of FLPP, and by using different operations, the suggested methodology can be extended to include the study of uncertainty issues that can be used for further work.

Chapter 7

The Combined Study of Improved Fuzzy Optimization Techniques with the Analysis of the Upgraded Facility Location Center for the Covid-19 Vaccine by Fuzzy Clustering Algorithms

"Abstraction consists essentially in the creation and utilization of ambiguity."

—William Byers

7.1 Introduction

With the latest vaccine production after clinical trials in India, one of the next steps is to administer this vaccine to the consumer. The supply must be specifically positioned to ensure optimal distribution, and it also optimizes the transportation cost. The significant concern of the location of facilities is a major logistic extent of decision-making for the vaccine distribution. How the material is passed to customers is one of the vital characteristics of a conversion process (manufacturing system). This fact involves deciding where the building or facility should be located.

The most effective method of avoiding and/or managing infectious disease outbreaks is vaccination. This surgical technique also poses a host of technical concerns. In recent years, a growing curiosity in the conceptual implications of vaccinations has been shown to the Operations Research/Operations Managing community. We could have a Covid-19 vaccine by this year, with around eight applicants completing the completion of drug testing. The next challenge is to get the vaccines securely shipped to specific locations and finally to hospitals and clinics. Since certain Covid-19 vaccines demand varying climates and different manipulation techniques, the cold chain infrastructure, including transport and storage equipment and processes, is vital before delivered to the masses. The World Health Organization

(WHO) claims that the cold chain is a system of keeping and delivering vaccines from the production point to the usage point.

A variety of factors can cause a shortage of vaccines. Export monopoly, complicated production procedures, expanded control of processing plants, unexpected changes in demand, and decreased producers are the most commonly cited reasons for vaccine shortages [156][157][158]. The final delivery operation is administering the vaccines from the manufacturer to the customers. At the time of preparation of vaccines, store the vaccines in suitable places for monitoring. The vaccine distribution system includes an effective overall framework, an analysis of the demand rate and inventory needs, and selecting appropriate vaccine distribution sites.

Healthcare facility (HCF) is one of the major strategic problems for healthcare services, emergency relief, and humanitarian logistics received substantial interest from the working academic community over nearly four decades. The article [159] includes tables with detailed statistics on HCF position problems for ten dimensions. To address the needs of people impacted by disasters, a model [160] was developed to assess the quantity and position of the delivery centers in the aid system and the number of relief supplies to be handled at each manufacturing facility. The goal of the paper [161] was to undertake a survey on the position of facilities associated with immediate relief warehousing, concentrating on all aspects of data modeling and problem categories, and to address the pre-and post-disaster situations related to the location of facilities, such as the location of distribution centers, stores, hospitals, debris disposal sites, and medical centers. An Integrated Facility Location (IFLP) problem was described in [162] that incorporates risks of facility disruption, obstruction of en-route transport, and delay of in-facility queuing into a single issue. The primary factors considered for selecting the location of the humanitarian relief warehouse as requirements for AHP are empirically specified in [163].

In several fields of research, clusters are found [3][6]. Clustering analysis is an important method that is used when deciding the optimum position for a facility. A clustered ant colony algorithm [164] was addressed to solve the more complicated location routing problem. A complete 52 algorithm [165] for reducing the supply chain disturbances. A fuzzy integration and clustering approach were used to correctly produce position clustering based on different hierarchical validation requirements. Then a similar approach to assess and pick the best candidate for each cluster can be applied for order collection. Customers are classified according to their duties and assigned to the nearest facility available. In [166], the clustering method was contrasted with other general-purpose clustering algorithms. They also demonstrate how the combination location and riding problem can be solved using an iterative heuristic approach. The problem with several facilities is to classify the locations with medical waste. An Artificial Bee Colony (ABC) algorithm [167] was proposed for cluster analysis to solve continuous multiple facility position issues. The ABC clustering algorithm was implemented to solve the healthcare waste disposal site in Istanbul.

7.2 Problem Definition and Proposed Model

Flu viruses are associated with the cold seasons and transmission of infectious illnesses. The flu impacts millions of individuals and leads to thousands of deaths. This year's fall and winter (i.e., 2020), the world's population, along with COVID-19, will be threatened. Although there are many variants of the flu virus and the mutated virus vaccine is developed and delivered annually [167], the vaccine is available and is manufactured and sold in a limited number of countries.

Many nations are unable to have flu vaccinations due to seasonal deficiencies. It is important to establish successful national plans to distribute vaccines equitably. The distribution in which more vulnerable individuals have priority over others is as important as the optimum allocation of individuals. This chapter implements fuzzy c-mean clustering for the accurate estimate of influenza vaccines to various groups of citizens. A compromise between promoting society and customer support is required for the application. The delivery location (i.e., the distribution center) and certain demand locations are two stages of the supply chain: (i.e., city, state, etc.). To obtain an optimum supply point, the desired point has a minimum distance from the demand point. Therefore, the cost of transportation should be minimal. Health professionals classify individuals according to their predetermined preferences. This classification would calculate the vaccine prescription for each point of demand. However, in a practical situation, the supply and demand for vaccines could vary. We also suggested a fuzzy linear programming problem via a fuzzy triangular number to minimize transport costs in such circumstances. The proposed model is used to equitably distribute the vaccines across demand points according to the following assumptions:

- The number of the cluster center are pre-defined.
- The distance between the supply point and demand point should be Euclidean distance.
- To find the minimum transportation cost.
- There is a multi-period distribution model with one single product.
- The availability of the optimal location of supply points is available.
- There is a range of the availability of the supply and demand of the product.

The complete mechanism of the proposed method is shown in Figure 7.1.

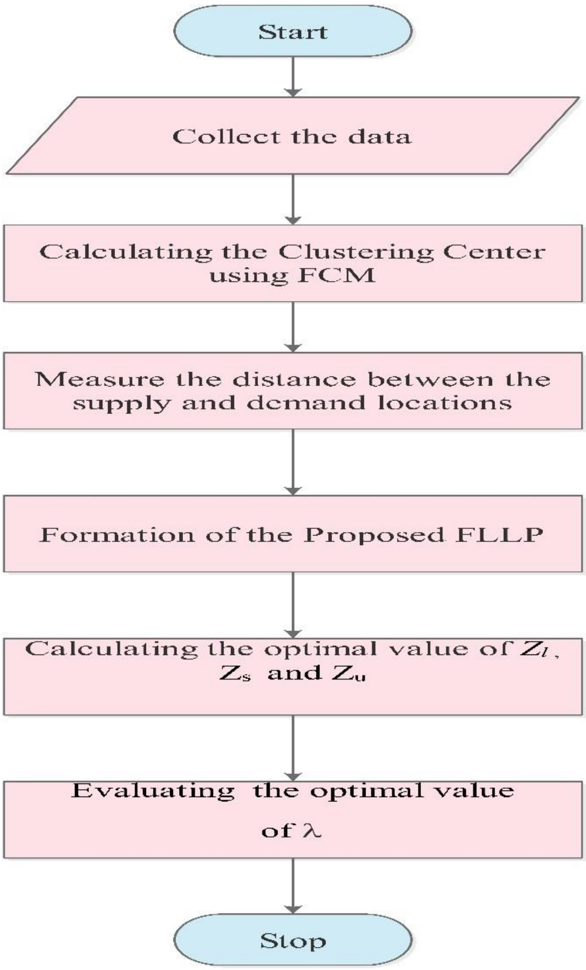


Figure 7.1: The mechanism of the proposed method

7.3 Mathematical Modeling

In this section, we first discuss the proposed fuzzy linear programming problem through a composite fuzzy triangular number

7.3.1 Solution Methodology

According to the composite fuzzy triangular number $\widetilde{\eta_i}(\eta_i - p_i \sim \eta_i \sim \eta_i + p_k)$ the general structure of the optimal values of the lower, static, and upper bounds are defined below:

The lower bound $(\mathcal{Z}_l)$ –

$$\text{Max } \mathcal{Z}_{\boldsymbol{l}} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \eta_i - p_i$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{7.1}$$

The static bound **($\mathcal{Z}$ s)** –

$$\text{Max } \mathcal{Z}_{\boldsymbol{s}} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \eta_i$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{7.2}$$

The upper bound **($\mathcal{Z}$ u)** –

$$\text{Max } \mathcal{Z}_{\boldsymbol{u}} = \sum_{j=1}^{n} c_j y_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} y_j \leq \eta_i + p_k$$

$$\text{Where, } y_j \geq 0, i, j \in \mathbb{N} \tag{7.3}$$

The solution for lower and upper bounds of LPP's is obtained by the Simplex method. Find the two different optimized FLLP model will be obtained by using these lower and upper bounds

7.3.2 Optimized Composite Triangular FLLP Model III

$$Max \ \lambda,$$

Subject to

$$\lambda(\mathcal{Z}_s - \mathcal{Z}_l) - \sum_{j=1}^{n} c_j y_j \le -\mathcal{Z}_l$$

$$\lambda(p_i) + \sum_{j=1}^{n} a_{ij} y_j \le \eta_i,$$

$$\lambda(\mathcal{Z}_u - \mathcal{Z}_s) - \sum_{j=1}^{n} c_j y_j \le -\mathcal{Z}_s$$

$$\lambda(p_i) + \sum_{j=1}^{n} a_{ij} y_j \le \eta_i + p_k,$$

$$\lambda(\mathcal{Z}_u - \mathcal{Z}_l) - \sum_{j=1}^{n} c_j y_j \le -\mathcal{Z}_l$$

$$\lambda(p_i + p_k) + \sum_{j=1}^{n} a_{ij} y_j \le \eta_i + p_k,$$

$$y_j \ge 0 \text{ and} \lambda \in [0,1] \tag{7.4}$$

7.4 Experiment Result and Discussion

In this section, we discuss a problem in which we find the proposed plant i (location centers) by the FCM clustering method. After the find, the location site j, we find the distance matrix from the plant i to the proposed potential site j**;** then we find the optimal total cost with the help of the transportation method.

Example 7.1

An organization has warehouses for life-saving drugs at 30 different locations, whose coordinates are given in Table 7.25. The organization is locating critical central warehouses that will distribute drugs to all the existing warehouses on emergency request. Find the numbers of the optimum location of the new facility (warehouse) based on the fuzzy C-means clustering concept.

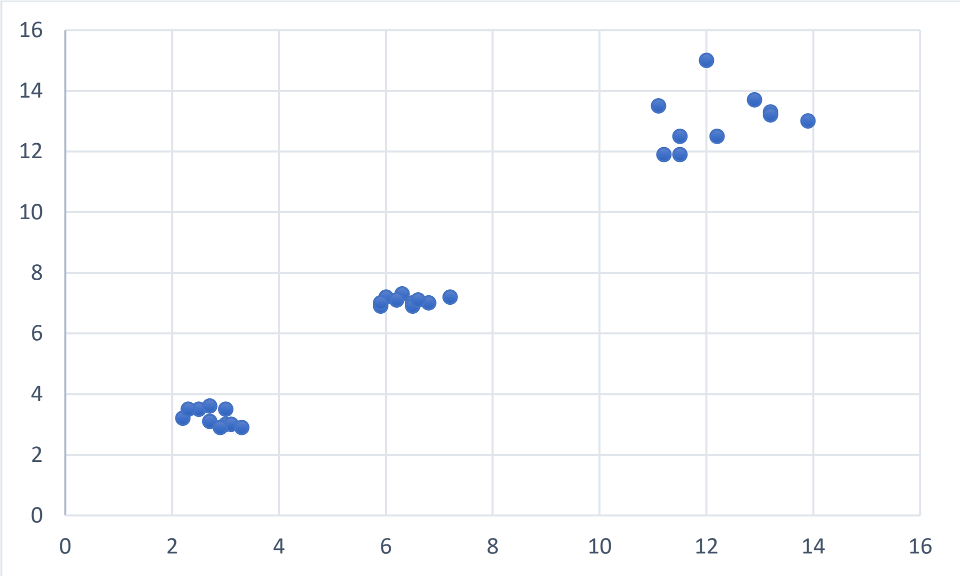


Figure 7.2: Show the geographical location of the existing warehouse

The data are also shown in Figure 7.2. Assume that we want to determine a fuzzy pseudo partition with two clusters (i.e., c =3). Assume further that we choose $m = 2, \epsilon > 0.00001$; ||. || is the Euclidean distance, and the initial fuzzy pseudo partition is $U^{(0)} = [U_1, U_2, U_3]$ with membership grade shown in Table 7.26.

Then, the algorithm stops for $k = 6$, because max $\{|| U^{(k+1)} - U^{(k)}||\} < 3.8545^{-7}$ and we obtain the pseudo partition defined in Table 7.26, the three clusters are

$\mathbf{V_1}$ = (2.77,3.22) , $\mathbf{V_2}$ = (6.39, 7.07) and V_3 = (12.29,13.05), $\boldsymbol{J_m}$ = 92.6823.

Now, In organization has 3 new facilities(warehouses) W_1, W_2 & W_3 which supply to 30 warehouses at $H_1, H_2, H_3, \ldots, H_{30}$. Due to the uncertainty of the demand and supply, the availability of drugs might vary. The fuzzy triangular number represents this variation.

The availability of the drugs of W_1, W_2 & W_3 are **(330, 400,460)** units, **(200,260,300)** units & **(273, 340,447)** units, respectively. The fuzzy triangular number also represents the monthly requirement for the warehouses, so the demand of $H_1, H_2, H_3, \ldots, H_{30}$ are

(30, 40,50) units, **(25, 30,40)** units, **(15, 20,25)** units & **(12,20,29)** units. respectively. The company wants to make sure they keep a steady, adequate flow of drugs to the existing warehouses to capitalize on the consumers' demand. Secondary, but still important to him, is to minimize the cost of transportation. The distance between the new warehouses and existing warehouses shown in Table 7.27. The average haul cost is $1 per mile for both loaded and empty trucks.

The data are shown below.

7.5 Modeling for the System of Optimal Solution

We can set up the FLP of the eq. (7.1) to eq. (7.3) for cost minimization; in such a way that to satisfy the demands of existing warehouses -

We can formulate the problem in Eq. (7.4) as: Let Y_{ij} = Transportations costs from new site i to existing j

$$i = 1, 2, 3 \ (new\ sites) j = 1, 2, \ldots, 30 \ \ (existing\ sites)$$

Objective function

$$\text{Min z} = 0.39y_{11} + \cdots + 14.81y_{130} + \cdots + 5.28y_{21} + \cdots + 9.57y_{230} + \ldots + 13.68y_{31} + \cdots + 1.67y_{330}$$

$$y_{11} + y_{12} + \cdots + y_{130} \leq \widetilde{F_1}$$

$$y_{21} + y_{22} + \cdots + y_{230} \leq \widetilde{F_2}$$

$$y_{31} + y_{32} + \cdots + y_{330} \leq \widetilde{F_3}$$

$$y_{11} + y_{21} + y_{31} \geq \widetilde{L_1}$$

$$y_{12} + y_{22} + y_{32} \geq \widetilde{L_2}$$

$$\vdots$$

$$y_{130} + y_{230} + y_{330} \geq \widetilde{L_{30}} \quad (7.5)$$

7.6 Numerical Results

Using the equation (7.1) to (7.3) of the proposed FLP, the optimal value of the lower bound **(Z_l)** is **933.22**, static bound **(Z_s)** is **1067.90** and the upper bound **(Z_u)** = **1440.50**. The value of Y_{ij} are shown in the Table 7.28 to Table 7.30.

7.6.1 Optimized Composite Triangular FLLP Model

The optimized fuzzy linear programming problem (OFLPP) has been constructed defined in Eq. (7.4) using the lower, static, and upper bound is shown in Eq.(7.5):

$$\text{Max } Z = \lambda$$

Subject to

$$372.60\,\lambda - (-0.39y_{11} - \cdots - 14.81y_{\mathbf{130}} - \cdots - 5.28y_{21} - \cdots - 9.57y_{230} - \ \ldots - 13.68y_{31} - \cdots - \ 1.67y_{330}) \leq -1067.90$$

$$60\lambda + y_{11} + \ y_{12} + \cdots + \ y_{130} \ \leq 460$$

$$40\lambda + y_{21} + \ y_{22} + \cdots + \ y_{230} \ \leq 300$$

$$107\lambda + \ y_{31} + \ y_{32} + \cdots + \ y_{330} \ \leq 447$$

$$10\lambda - y_{11} - \ y_{21} - \ y_{31} \ \leq -50$$

$$10\lambda - y_{12} - y_{22} - \ y_{32} \ \leq -40$$

$$9\lambda - \ y_{130} - \ y_{230} - \ y_{330} \ \leq -29$$

$$507.26\lambda - (-0.39y_{11} - \cdots + 14.81y_{\mathbf{130}} + \cdots + 5.28y_{21} + \cdots + 9.57y_{230} + \ \ldots + 13.68y_{31} + \cdots + \ 1.67y_{330}) \leq -933.22$$

$$130\lambda + y_{11} + \ y_{12} + \cdots + \ y_{130} \ \leq 460$$

$$100\lambda + y_{21} + \ y_{22} + \cdots + \ y_{230} \ \leq 300$$

$$174\lambda + \ y_{31} + \ y_{32} + \cdots + \ y_{330} \ \leq 447$$

$$20\lambda - y_{11} - \ y_{21} - \ y_{31} \ \leq -50$$

$$15\lambda - y_{12} - \ y_{22} - \ y_{32} \ \leq -40$$

$$\vdots$$

$$17\lambda - \ y_{130} - \ y_{230} - \ y_{330} \ \leq -29$$

$$134.68\lambda - (-0.39y_{11} - \cdots - 14.81y_{\mathbf{130}} - \cdots + 5.28y_{21} - \cdots - 9.57y_{230} - \ \ldots - 13.68y_{31} - \cdots - \ 1.67y_{330}) \leq -933.22$$

$$70\lambda + y_{11} + \ y_{12} + \cdots + \ y_{130} \ \leq 400$$

$$60\lambda + y_{21} + y_{22} + \cdots + y_{230} \leq 260$$

$$67\lambda + y_{31} + y_{32} + \ldots + y_{330} \leq 340$$

$$10\lambda - y_{11} - y_{21} - y_{31} \leq -40$$

$$5\lambda - y_{12} - y_{22} - y_{32} \leq -30$$

$$8\lambda - y_{130} - y_{230} - y_{330} \leq -20 \quad (7.6)$$

$$y_{ij} \geq 0 \text{ and } \lambda \in [0,1]$$

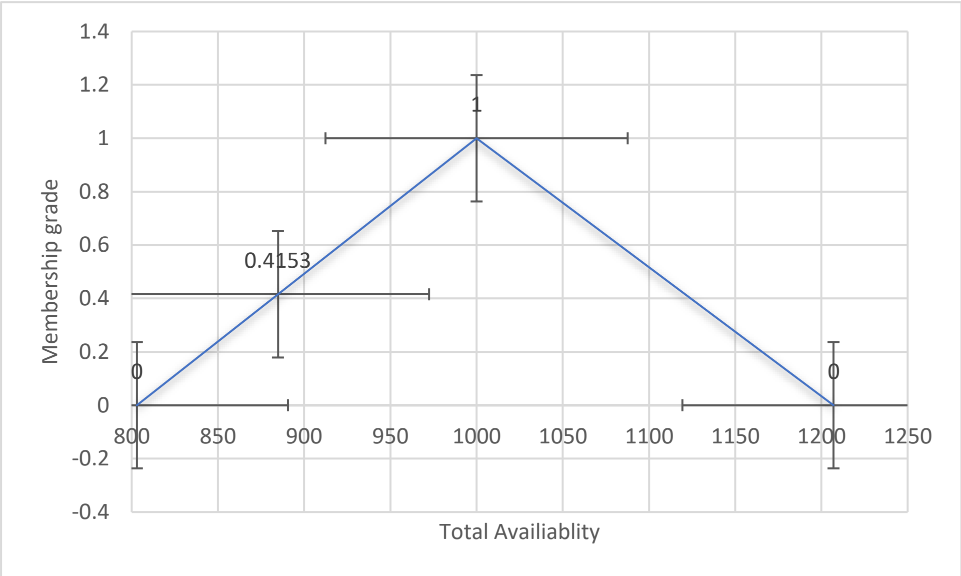


Figure 7.3: Show the optimal value of $\boldsymbol{\lambda}$ for minimal transportation cost

The proposed FLP issue the delivery of the vaccine from the warehouses W_1, W_2 & W_3 to the hospitals H_2, H_3,…, H_{30} is shown in Table 7.30. This indicates the optimum distributions of the vaccinations that are given with poor accuracy in three different scenarios. Reinforcing the proposed approach specified in Eq.(7.5), we can find the optimal degree of satisfaction value of λ to be 0.4153 which can be seen in Figure 7.3. At this amount, the optimum supply of the vaccine is (884,84,895), which gives us a transport cost of $564,37, which is at least as minimal as the availability of the vaccine.

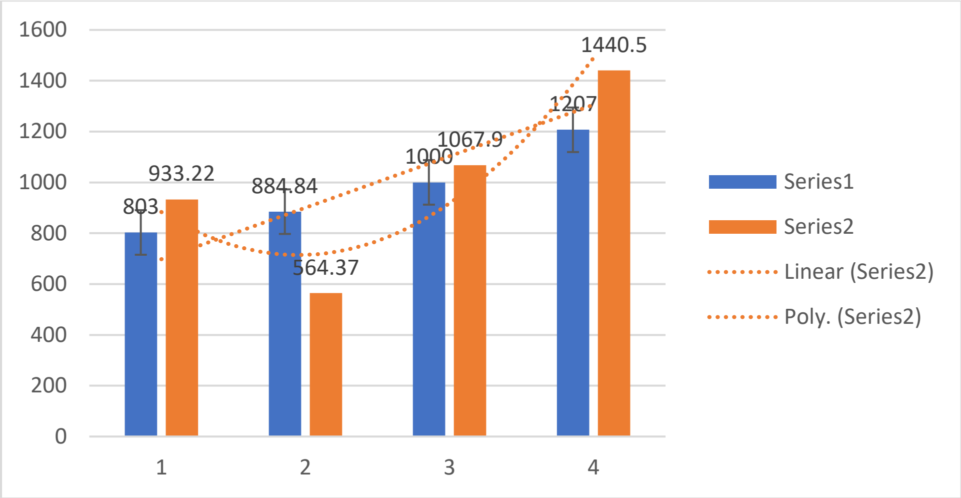


Figure 7.4:Comparison of transportation cost and availability of the vaccine

Figure 7.4 indicates the optimal costs of transport following vaccine supply. With the assistance of the conversational FLP method defined in Eq (7.6) to Eq.(7.8), we determine the cost of transport from the new warehouses to the current warehouse for the lower bound, the static bound, and the upper bound for blocks 1, 3, and 4. When the market for the commodity rises, so does the cost of shipping. From the above examples, it can be shown that block 2 indicates the correct outcome of the proposed FLLP. This implies that the cost of transport is less than the availability of the commodity. This fulfills the needs of the customer.

7.7 Summary

This chapter applies the FCM algorithms to find the optimum location of vaccine delivery warehouses in current warehouses/hospitals. Our key goal here is to minimize the cost of travel from the new site to current facilities so that the measured new sites meet the needs of existing locations. We have therefore suggested a new FLP problem, which has yielded a spectacular outcome. In the future, we will try to locate positions using better FCM algorithms using various distance metrics and apply new fuzzy linear programming problems using other fuzzy numbers that describe real-world scenarios.

Chapter 8

Conclusions and Future Scope

"Endings are thus formally unappealing to me, more than beginning or ending, in life, I enjoy continuing. Continuing is my only focus or concern."

−Brian D'Ambrosio

This research has focused on providing better and practical approaches for the FLP. We suggested finding the number of clustering centers that used fuzzy equivalence clustering via Minkowski, Mahalanobis, Cosine, Chebychev, and the Correlation distance function. The performances of binders were graded in precise and equal stiffness temperatures at three different parameters. The Mahalanobis distance function trailed first time in the equivalence fuzzy clustering methodology and accomplished the desirable objectives. To find the optimal clustering, we proposed AGFCM and AGHCM clustering algorithms, are based on the suggested metric space. Diamond data sets and Iris data sets are used to assess the proposed algorithms' efficiency, compared to AFCM, AHCM, and other competitive algorithms such as FCM, PCM, FPCM, and PFCM.

After establishing the desired locations for FLP, the availability and demands of the goods/ services might vary due to the uncertainty. So, optimizations of profit and cost are also ambiguous. To find the optimizations for these unpredictable situations, we proposed FLPP through the different fuzzy numbers, i.e., composite fuzzy triangular number, trapezoidal fuzzy number, and a $\alpha -$ cut based quadrilateral fuzzy number. To check the validation of these models with the Production cost data of Rail Coach Factory (RCF), Kapurthala has been taken. In all the different situations, we find the optimal solutions of objective functions.

The work discussed in this study has the potential to be enhanced in the following directions:

- The proposed metric is applied to fuzzy equivalence clustering algorithms to find the desired number of clustering. Also, we can use the proposed metric and AGFCM in PFCM, CFCM, and other clustering algorithms where metrics play an important role.
- An $\alpha - cut\ based$ quadrilateral fuzzy numbers will be used for optimization through the other groups of FLPP, and by using different operations, the suggested methodology can be extended to include the study of uncertainty issues that can be used for further work.
- The complete mechanism of this thesis will be applied in the same data- sets of facility location problems and compared with the existing approaches.

Appendix A

Alternative Generalized Hard C-Means Clustering

In Appendix A, we give AGHCM's proof. The problem of minimizing the objective function is described in Eq.(3.20).

Proof

By taking the partial derivatives of $J_{AGHCM}(V;Y)$ with respect to v_j

$$\frac{\partial J}{\partial v_j} = \sum_{i=1}^{c}\sum_{k=1}^{n} \frac{1}{2\sqrt{(1-a^{-b\left|\left|y_k-v_j\right|\right|^2})}} (-a^{-b\left|\left|y_k-v_j\right|\right|^2})\ \log a\ \ (-2\mathrm{b}||\,y_k - v_j\,||)\,(-1) \tag{A1}$$

Equating (A1) to zero leads to

$$\frac{\partial J}{\partial v_j} = 0$$

$$\Rightarrow \sum_{i=1}^{c}\sum_{k=1}^{n} \frac{\mathrm{b}\log a\,(-a^{-b\left|\left|y_k-v_j\right|\right|^2})\ ||\,y_k-v_j\,||)}{\sqrt{(1-a^{-b\left|\left|y_k-v_j\right|\right|^2})}} = 0 \qquad (b\ loga\ \neq 0)$$

$$\sum_{i=1}^{c}\sum_{k=1}^{n} \frac{a^{-b\left|\left|y_k-v_j\right|\right|^2} y_k}{\sqrt{(1-a^{-b\left|\left|y_k-v_j\right|\right|^2})}} - \sum_{i=1}^{c}\sum_{k=1}^{n} \frac{a^{-b\left|\left|y_k-v_j\right|\right|^2} v_j}{\sqrt{(1-a^{-b\left|\left|y_k-v_j\right|\right|^2})}} = 0$$

$$v_j = \frac{\sum_{k=1}^{n} \frac{a^{-b\left|\left|d_{ik}\right|\right|^2}}{\sqrt{1-a^{-b\left|\left|d_{ik}\right|\right|^2}}\ y_k}{\sum_{k=1}^{n} \frac{a^{-b\left|\left|d_{ik}\right|\right|^2}}{\sqrt{1-a^{-b\left|\left|d_{ik}\right|\right|^2}}} \quad j \in \mathbb{N} \tag{A2}$$

Appendix B

Alternative Generalized Fuzzy C-Means Clustering

We include AGFCM proof in Appendix B. The problem of minimizing the objective function described in Eq.(3.22) under restricted Eq.(3.23) is solved by minimizing the objective constraint defined in Eq.(B1).

Proof

$$J_{AGFCM}(U,V;Y) = \sum_{i=1}^{c}\sum_{k=1}^{n}\mu_{ik}^{m}(1-a^{-b\left\|y_k-v_j\right\|^2}) + \lambda(1-\sum_{i=1}^{c}\mu_{ik}) \quad \text{(B1)}$$

Where λ *is* Langrage multipliers.

By taking the partial derivatives of $J_{AGFCM}(U,V;Y)$ with respect to μ_{ik}

$$\frac{\partial J}{\partial \mu_{ik}} = \sum_{i=1}^{c}\sum_{k=1}^{n} m\mu_{ik}^{m-1}(1-a^{-b\left\|y_k-v_j\right\|^2}) - \lambda \quad \text{(B2)}$$

Equating (B2) to zero leads to

$$\frac{\partial J}{\partial \mu_{ik}} = 0$$

$$\mu_{ik} = \left(\frac{\lambda}{m\sqrt{(1-a^{-b\left\|y_k-v_j\right\|^2})}}\right)^{\frac{1}{m-1}} \quad \text{(B3)}$$

Using Eq.(3.23), we get

$$\left(\frac{\lambda}{m}\right)^{\frac{1}{m-1}} = \frac{1}{\left(\frac{1}{\sum_{k=1}^{c}\sqrt{(1-a^{-b\left\|y_k - v_j\right\|^2})}}\right)^{\frac{1}{m-1}}} \tag{B4}$$

From (B3) and (B4)

$$=> \mu_{ik} = \frac{\left[\frac{1}{\sqrt{1-a^{b\left\|y_k - v_j\right\|^2}}}\right]^{\frac{1}{(m-1)}}}{\sum_{j=1}^{c}\left[\frac{1}{\sqrt{1-a^{b\left\|y_k - v_j\right\|^2}}}\right]^{\frac{1}{(m-1)}}} \tag{B5}$$

By taking the partial derivatives of $J_{AGFCM}(U, V; Y)$ with respect to v_j

$$\frac{\partial J}{\partial v_j} = \sum_{i=1}^{c}\sum_{k=1}^{n} \mu_{ik}^{m} \frac{1}{2\sqrt{(1-a^{-b\left\|y_k - v_j\right\|^2})}} (-a^{-b\left\|y_k - v_j\right\|^2}) \log a \ (\text{-2b}\| y_k - v_j \|) (-1) \tag{B6}$$

Equating (B6) to zero leads to

$$\frac{\partial J}{\partial v_j} = 0$$

$$v_i = \frac{\sum_{k=1}^{n} \mu_{ik}^{m} \frac{a^{-b\left\|y_k - v_j\right\|^2}}{\sqrt{1-a^{b\left\|y_k - v_j\right\|^2}}} \ y_k}{\sum_{k=1}^{n} \mu_{ik}^{m} \frac{a^{-b\left\|y_k - v_j\right\|^2}}{\sqrt{1-a^{b\left\|y_k - v_j\right\|^2}}}} \tag{B7}$$

Appendix C

Addition of Two $\alpha - cut$ Based Quadrilateral Fuzzy Number

Proof: Let $\tilde{B}_i^p$ and $\tilde{B}_i^q$ be quadrilateral fuzzy numbers with different confidence levels such that $\beta^p \leq \beta^q$. Take $\beta^s \in [\beta^p, \beta^q]$ i.e. $\beta^s = \beta^p$ then α^s −cut of $\tilde{B}_i^p$ and $\tilde{B}_i^q$ are

When $\alpha^p \leq \beta^p$, α^p , $\beta^p \neq 0$

$$\tilde{B}_i^p = \left[\beta_i^p - \varepsilon_i^p , \beta_i^p - \varepsilon_i^p + \frac{\alpha^p \varepsilon_i^p}{\beta^p}\right) \cup \left[\beta_i^p,\ \beta_i^p + \frac{\overline{\alpha^p} - \beta^p}{\overline{\beta^p}}\left(\beta_i^{p*} - \beta_i^p\right)\right) \cup \left[\beta_i^{p*} , \alpha^p \varepsilon_i^{p*} + \left(\beta_i^{p*} - \beta_i^p\right)\right]$$

$$\tilde{B}_i^q = \left[\beta_i^q - \varepsilon_i^q , \beta_i^q - \varepsilon_i^q + \frac{\alpha^q \varepsilon_i^q}{\beta^q}\right) \cup \left[\beta_i^q,\ \beta_i^q + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}}\left(\beta_i^{q*} - \beta_i^q\right)\right) \cup \left[\beta_i^{q*} , \alpha^q \varepsilon_i^{q*} + \left(\beta_i^{q*} - \beta_i^q\right)\right]$$

$$\tilde{B}_i^{q+} = \left[\beta_i^q - \varepsilon_i^q , \beta_i^q - \varepsilon_i^q + \frac{\alpha^q \varepsilon_i^{q+}}{\beta^q}\right) \cup \left[\beta_i^q,\ \beta_i^q + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}}\left(\beta_i^{q*} - \beta_i^q\right)\right) \cup \left[\beta_i^{q*} , \alpha^q \varepsilon_i^{q*} + \left(\beta_i^{q*} - \beta_i^q\right)\right]$$

$$\beta_i^q - \varepsilon_i^{q+} = (\beta_i^q - \varepsilon_i^q) + \frac{\beta^s}{\beta^q}\varepsilon_i^q \Rightarrow \varepsilon_i^{q+} = \beta_i^q - (\beta_i^q - \varepsilon_i^q) - \frac{\beta^s}{\beta^q}\varepsilon_i^q$$

Let $\tilde{B}_i^s = \tilde{B}_i^p + \tilde{B}_i^{q+} = \{y \mid y \in \tilde{B}_\alpha^s\}\ \forall\ \alpha^s \epsilon [0,1]$. Here $\tilde{B}_\alpha^s = [\ \tilde{B}_\alpha^{s^L}(\alpha^s), \tilde{B}_\alpha^{s^U}(\alpha^s)]$be its α^s-cuts such that $\tilde{B}_\alpha^{s^L}(\alpha^s) = \tilde{B}_i^{p^L}(\alpha^p) + \tilde{B}_i^{q^L}(\alpha^q)$ and $\tilde{B}_\alpha^{s^U}(\alpha^s) = \tilde{B}_i^{p^U}(\alpha^p) + \tilde{B}_i^{q^U}(\alpha^q)$ i.e.

$$\tilde{B}_\alpha^s = [\tilde{B}_i^{p^L}(\alpha^p) + \tilde{B}_i^{q^L}(\alpha^q) , \tilde{B}_i^{p^U}(\alpha^p) + \tilde{B}_i^{q^U}(\alpha^q)]$$

$\tilde{B}_\alpha^s = I_1^s \cup I_2^s \cup I_3^s$, where

$$I_1^s = [\beta_i^p - \varepsilon_i^p + \beta_i^q - \varepsilon_i^q , \beta_i^p - \varepsilon_i^p + \frac{\alpha^p \varepsilon_i^p}{\beta^p} + \beta_i^q - \varepsilon_i^q + \frac{\alpha^q \varepsilon_i^{q+}}{\beta^q})$$

$$I_2^s = [\beta_i^p + \beta_i^q, \beta_i^p + \frac{\overline{\alpha^p} - \beta^p}{\overline{\beta^p}}(\beta_i^{p*} - \beta_i^p) + \beta_i^q + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}}(\beta_i^{q*} - \beta_i^q)]$$

$$I_3^s = [\beta_i^{p*} + \beta_i^{q*}, (\beta_i^{p*} + \varepsilon_i^{p*}) - \alpha^p \varepsilon_i^{p*} + (\beta_i^{q*} + \varepsilon_i^{q*}) - \alpha^q \varepsilon_i^{q*}]$$

Now,

$$I_1^s = [\beta_i^p + \beta_i^q - (\varepsilon_i^p + \varepsilon_i^q), \beta_i^p + \beta_i^q - (\varepsilon_i^p + \varepsilon_i^q) + \frac{\alpha^p \varepsilon_i^p}{\beta^p} + \frac{\alpha^q \varepsilon_i^{q+}}{\beta^{q+}})$$

$$\alpha^p = \alpha^q = \alpha^s$$

$$I_1^s = [\beta_i^p + \beta_i^q - (\varepsilon_i^p + \varepsilon_i^q), \beta_i^p + \beta_i^q - (\varepsilon_i^p + \varepsilon_i^q) + \alpha^s (\frac{\varepsilon_i^p}{\beta^p} + \frac{\varepsilon_i^{q+}}{\beta^q}))$$

$$\beta_i^p + \beta_i^q - (\varepsilon_i^p + \varepsilon_i^q) + \alpha^s (\frac{\varepsilon_i^p}{\beta^p} + \frac{\varepsilon_i^{q+}}{\beta^q}) - y = 0$$

$$f_{B^s}^U(y) = \frac{y - (\beta_i^p - \varepsilon_i^p) + (\beta_i^q - \varepsilon_i^q)}{(\frac{\varepsilon_i^p}{\beta^p} + \frac{\varepsilon_i^{q+}}{\beta^{q+}})}$$

$$when\ \beta^p = \beta^{q+} = \beta^s$$

$$g_{B^s}^U(y) = \beta^s \left(\frac{y - [(\beta_i^p - \varepsilon_i^p) + (\beta_i^q - \varepsilon_i^q)]}{\varepsilon_i^p + \varepsilon_i^{q+}} \right)$$

$$g_{B^s}^U(y) = \beta^s \left(\frac{y - [(\beta_i^p - \varepsilon_i^p) + (\beta_i^q - \varepsilon_i^q)]}{\varepsilon_i^p + \beta_i^q - (\beta_i^q - \varepsilon_i^q) - \frac{\beta^r}{\beta^p} \varepsilon_i^q} \right)$$

$$g_{B^s}^U(y) = \beta^s \left(\frac{y - [(\beta_i^p - \varepsilon_i^p) + (\beta_i^q - \varepsilon_i^q)]}{\beta_i^p + \beta_i^q - \frac{\beta^s}{\beta^p} \varepsilon_i^q - [(\beta_i^p - \varepsilon_i^p) + (\beta_i^q - \varepsilon_i^q)]} \right)$$

$$g^U_{B^s}(y) = \beta^s\left(\frac{y-(\beta^s_i-\varepsilon^s_i)}{\beta^p_i+\beta^q_i-\frac{\beta^s}{\beta^p}\varepsilon^q_i\ -(\beta^s_i-\varepsilon^s_i)}\right)$$

$$where\ \beta^s_i = \beta^p_i + \beta^q_i\ , \varepsilon^s_i=\varepsilon^p_i+\varepsilon^q_i$$

$$(\beta^s_i-\varepsilon^s_i) \le y \le \beta^p_i+\beta^q_i-\frac{\beta^s}{\beta^p}\varepsilon^q_i$$

Now

$$I^s_2 = [\beta^p_i+\beta^q_i, \beta^p_i+\frac{\overline{\alpha^p}-\beta^p}{\overline{\beta^p}}\left(\beta^{p*}_i-\beta^p_i\right) + \beta^q_i+\frac{\overline{\alpha^q}-\beta^q}{\overline{\beta^q}}\left(\beta^{q*}_i-\beta^q_i\right)]$$

$$I^s_2 = [\beta^p_i+\beta^q_i, \beta^p_i+\beta^q_i+\frac{\overline{\alpha^p}-\beta^p}{\overline{\beta^p}}\left(\beta^{p*}_i-\beta^p_i\right) +\frac{\overline{\alpha^q}-\beta^q}{\overline{\beta^q}}\left(\beta^{q*}_i-\beta^q_i\right)]$$

$$here\ \frac{\overline{\alpha^p}-\beta^p}{\overline{\beta^p}} = \frac{\overline{\alpha^q}-\beta^q}{\overline{\beta^q}} = \frac{\overline{\alpha^s}-\beta^s}{\overline{\beta^s}}$$

$$I^s_2 = \left[\beta^p_i+\beta^q_i, \beta^p_i+\beta^q_i+(\overline{\alpha^s}-\beta^s)\left(\frac{\beta^{p*}_i-\beta^p_i}{\overline{\beta^p}}+\frac{\beta^{q*}_i-\beta^q_i}{\overline{\beta^q}}\right)\right]$$

$$\beta^p_i+\beta^q_i+(\overline{\alpha^s}-\beta^s)\left(\frac{\beta^p_i-\beta^p_i}{\overline{\beta^p}}+\frac{\beta^q_i-\beta^q_i}{\overline{\beta^q}}\right)-y=0$$

$$when\ \overline{\beta^p} = \overline{\beta^q} = \overline{\beta^s}$$

$$g^U_{B^s}(y) = \overline{\alpha^s}-\beta^s = \left(\frac{y-(\beta^p_i+\beta^q_i)}{\beta^{p*}_i-\beta^p_i+\beta^{q*}_i-\beta^q_i}\right)\times\overline{\beta^s}$$

$$\overline{\alpha^s} = \beta^s + \left(\frac{y-(\beta^p_i+\beta^q_i)}{\beta^{p*}_i-\beta^p_i+\beta^{q*}_i-\beta^q_i}\right)\times\overline{\beta^s} \Rightarrow \overline{\alpha^s} = \beta^s + \left(\frac{y-\beta^s_i}{\beta^{s*}_i-\beta^s_i}\right)\times\overline{\beta^s}$$

$$where\ \beta^{s*}_i = \beta^{p*}_i+\beta^{q*}_i\ ,\ \beta^s_i = \beta^p_i+\beta^q_i$$

$$I^s_3 = [\beta^{p*}_i+\beta^{q*}_i, \left(\beta^{p*}_i+\varepsilon^{p*}_i\right)-\alpha^p\varepsilon^{p*}_i+\left(\beta^{q*}_i+\varepsilon^{q*}_i\right)-\alpha^q\varepsilon^{q*}_i]$$

$$\alpha^p = \alpha^q = \alpha^s$$

$$I_3^s = [\beta_i^{p*} + \beta_i^{q*}, \left(\beta_i^{p*} + \varepsilon_i^{p*}\right) + \left(\beta_i^{q*} + \varepsilon_i^{q*}\right) - \alpha^s(\varepsilon_i^{p*} + \varepsilon_i^{q*})]$$

$$y - \left(\beta_i^{p*} + \varepsilon_i^{p*}\right) - \left(\beta_i^{q*} + \varepsilon_i^{q*}\right) + \alpha^s\left(\varepsilon_i^{p*} + \varepsilon_i^{q*}\right) = 0$$

$$g_{B^s}^U(y) = \alpha^s = \left(\frac{\left(\beta_i^{p*} + \varepsilon_i^{p*}\right) + \left(\beta_i^{q*} + \varepsilon_i^{q*}\right) - y}{\varepsilon_i^{p*} + \varepsilon_i^{q*}}\right) \Rightarrow g_{B^s}^U(y) = \left(\frac{(\beta_i^{s*} + \varepsilon_i^{s*}) - y}{\varepsilon_i^{s*}}\right)$$

$$where\ , \beta_i^{s*} = \beta_i^{p*} + \beta_i^{q*}\ , \varepsilon_i^{s*} = \varepsilon_i^{p*} + \varepsilon_i^{q*}$$

$$B_i^s(y) = \begin{cases} \beta^s + \left(\frac{y - \beta_i^s}{\beta_i^{s*} - \beta_i^s}\right) \times \overline{\beta^s} & When\ \beta_i^s \le y \le \beta_i^{s*} \\ \beta^s \left(\frac{y - (\beta_i^s - \varepsilon_i^s)}{\beta_i^p + \beta_i^q - \frac{\beta^s}{\beta^p}\varepsilon_i^q - (\beta_i^s - \varepsilon_i^s)}\right) & When\ \beta_i^s - \varepsilon_i^s \le y \le \beta_i^p + \beta_i^q - \frac{\beta^s}{\beta^p}\varepsilon_i^q \\ \frac{\left(\beta_i^{s*} + \varepsilon_i^{s*}\right) - y}{\varepsilon_i^{s*}} & When\ \beta_i^{s*} \le y \le \beta_i^{s*} + \varepsilon_i^{s*} \\ 0 & otherwise \end{cases}$$

$$\beta_i^s - \varepsilon_i^s = (\beta_i^p - \varepsilon_i^p) + (\beta_i^q - \varepsilon_i^q)$$

$$\beta_i^s = \beta_i^p + \beta_i^q - \frac{\beta^s}{\beta^p}\varepsilon_i^q$$

$$\beta_i^{s*} = \beta_i^{p*} + \beta_i^{q*}$$

$$\beta_i^{s*} + \varepsilon_i^{s*} = \left(\beta_i^{p*} + \varepsilon_i^{p*}\right) + (\beta_i^{q*} + \varepsilon_i^{q*})$$

Appendix D

Multiplication of Two $\alpha - cut$ Based Quadrilateral Fuzzy Number

Proof As the quadrilateral membership functions of $\tilde{B}_i^p$ and $\tilde{B}_i^q$ are given in equation () and () respectively. Thus, To find the membership of $\tilde{B}_i^s = \tilde{B}_i^p \times \tilde{B}_i^q = (\beta_i^s - \varepsilon_i^s\ ,\ \beta_i^s\ , \beta_i^{s*}\ ,\ \beta_i^{s*} + \varepsilon_i^{s*}\ )$

Let $\tilde{B}_i^s = \tilde{B}_i^p \times \tilde{B}_i^{q+} = \{y \mid y \in \tilde{B}_\alpha^s\}\ \forall\ \alpha^s \epsilon[0,1]$. Here $\tilde{B}_\alpha^s = [\ \tilde{B}_\alpha^{s^L}(\alpha^s), \tilde{B}_\alpha^{s^U}(\alpha^s)]$ be its α^s-cuts such that $\tilde{B}_\alpha^{s^L}(\alpha^s) = \tilde{B}_i^{p^L}(\alpha^p) \times \tilde{B}_i^{q^L}(\alpha^q)$ and $\tilde{B}_\alpha^{s^U}(\alpha^s) = \tilde{B}_i^{p^U}(\alpha^p) \times \tilde{B}_i^{q^U}(\alpha^q)$ i.e.

$$\tilde{B}_\alpha^s = [\tilde{B}_i^{p^L}(\alpha^p) \times \tilde{B}_i^{q^L}(\alpha^q)\ , \tilde{B}_i^{p^U}(\alpha^p) \times \tilde{B}_i^{q^U}(\alpha^q)]$$

$\tilde{B}_\alpha^s = I_1^s \cup I_2^s \cup I_3^s$, where

$$I_1^s = [(\beta_i^p - \varepsilon_i^p)(\ \beta_i^q - \varepsilon_i^q), \{\beta_i^p - \varepsilon_i^p + \frac{\alpha^p \varepsilon_i^p}{\beta^p}\}\{\ \beta_i^q - \varepsilon_i^q + \frac{\alpha^q \varepsilon_i^{q+}}{\beta^q}\}\ )$$

$$I_2^s = [\beta_i^p\ \beta_i^q, \{\beta_i^p + \frac{\overline{\alpha^p} - \beta^p}{\overline{\beta^p}}\left(\beta_i^{p*} - \beta_i^p\right)\}\{\ \beta_i^q + \frac{\overline{\alpha^q} - \beta^q}{\overline{\beta^q}}\left(\beta_i^{q*} - \beta_i^q\right)\}\ ]$$

$$I_3^s = [\beta_i^{p*}\ \beta_i^{q*}, \{\left(\beta_i^{p*} + \varepsilon_i^{p*}\right) - \alpha^p \varepsilon_i^{p*}\}\{\left(\beta_i^{q*} + \varepsilon_i^{q*}\right) - \alpha^q \varepsilon_i^{q*}\}]$$

Now,

$$I_1^s = [(\beta_i^p - \varepsilon_i^p)(\ \beta_i^q - \varepsilon_i^q), \{\beta_i^p - \varepsilon_i^p + \frac{\alpha^p \varepsilon_i^p}{\beta^p}\}\{\ \beta_i^q - \varepsilon_i^q + \frac{\alpha^q \varepsilon_i^{q+}}{\beta^q}\}\ )$$

$$\alpha^p = \alpha^q = \alpha^s\ \ and\ \ \beta^p = \beta^{q+} = \beta^s$$

$$\{\beta_i^p - \varepsilon_i^p + \frac{\alpha^p \varepsilon_i^p}{\beta^p}\}\{\ \beta_i^q - \varepsilon_i^q + \frac{\alpha^q \varepsilon_i^{q+}}{\beta^q}\} - y = 0$$

$$(\alpha^s)^2 \frac{\varepsilon_i^p(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^s}\,\overline{\beta^q}} + \alpha^s \left\{\frac{(\beta_i^p-\varepsilon_i^p)(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^q}} + \frac{(\beta_i^q-\varepsilon_i^q)\varepsilon_i^p}{\overline{\beta^s}}\right\} + (\beta_i^p - \varepsilon_i^p)(\beta_i^q - \varepsilon_i^q) - y = 0$$

Take $M_2 = \frac{\varepsilon_i^p(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^s}\,\overline{\beta^q}}$, $K_2 = \frac{(\beta_i^p-\varepsilon_i^p)(\varepsilon_i^{q+}-\varepsilon_i^q)}{\overline{\beta^q}} + \frac{(\beta_i^q-\varepsilon_i^q)\varepsilon_i^p}{\overline{\beta^s}}$, and $N_2 = (\beta_i^p - \varepsilon_i^p)(\beta_i^q - \varepsilon_i^q)$

$(\alpha^s)^2 M_2 + \alpha^s K_2 + N_2 - y = 0.$

$$\alpha^s = \frac{-K_2 \pm \sqrt{K_2^2 + 4M_2(y - N_2)}}{2M_2}$$

Similarly, $In\ \ I_2^s = [\beta_i^p\ \beta_i^q, \{\beta_i^p + \frac{\overline{\alpha^p}-\beta^p}{\overline{\beta^p}}\left(\beta_i^{p*} - \beta_i^p\right)\}\{\beta_i^q + \frac{\overline{\alpha^q}-\beta^q}{\overline{\beta^q}}\left(\beta_i^{q*} - \beta_i^q\right)\}]$

$$\alpha^s = \beta^s + \frac{-K_1 + \sqrt{K_1^2 + 4M_1(y - N_1)}}{2M_1}$$

And $I_3^s = [\beta_i^{p*}\ \beta_i^{q*}, \{\left(\beta_i^{p*} + \varepsilon_i^{p*}\right) - \alpha^p \varepsilon_i^{p*}\}\{\left(\beta_i^{q*} + \varepsilon_i^{q*}\right) - \alpha^q \varepsilon_i^{q*}\}]$

$$\alpha^s = \frac{K_3 + \sqrt{K_3^2 + 4M_3(y - N_3)}}{2M_3}$$

Appendix E

Tables

Table 2.1: Performance Grades for NCHRP 90-07 Binders Used in Turner–Fairbank Highway Research Center Polymer Research Program

NCHRP 90-07 Binder Code	Super pave grade	T_{HS} (°C) when $\frac{\lvert G*\rvert}{\frac{1}{Sin\delta}}$ =2,200pa at ω =10rad/s	T_{HS} (°C)when $\lvert G*\rvert / \left[1-\frac{1}{(tan\delta.Sin\delta)}\right]$ = 2,200Pa at ω =10 rad/s	T_{HS} (°C) = $(T_e°C)/\left[1-\frac{1}{(tan\delta.Sin\delta)}\right]$ at ω =0.25 rad/s when $\lvert G*\rvert$ =50 Pa
B6224	PG52-28	55.1	55.7	66.0
B6225	PG64-28	67.0	67.8	69.1
B6226	PG70-28	71.3	72.1	74.9
B6227	PG70-28	74.6	76.2	80.1
B6228	PG70-28	78.3	89.1	102.0
B6229	PG70-28	72.2	75.1	83.4
B6230	PG70-28	70.9	73.0	80.8
B6231	PG70-28	71.8	74.2	81.4
B6232	PG70-28	75.9	78.0	84.6
B6233	PG70-28	77.5	78.7	82.0
B6243	PG70-28	76.7	79.4	84.1
B6251	PG70-28	76.9	79.0	85.6
B6252	PG76-22	81.5	83.2	86.8
B6253	PG70-22	73.9	76.5	81.6
B6254	PG70-28	70.6	75.9	85.5
B6257	PG70-28	74.4	79.5	95.3
B6258	PG70-28	74.8	78.5	87.7

Where $\lvert G*\rvert$ = Complex shear modulus (kPa),T = temperature (degrees), T_{HS} (°C) = high specification temperature, $(T_e°C)$ =equi- stiffness temperature, δ = phase angle (degrees or radians), , ω = frequency of oscillatory motion (rad/s).

Table 3.2: Result of Gaussian function-based distance defined in Eq. (3.4)

Data set	$v^{(0)}$	$v^{(1)}$.	b	v^t	$No. of\ iteration(j)$
S_1	5.0000	4.1275	0.9158	4.9991	19
S_2	6.2500	5.2243	0.0507	5.0010	11
S_3	6.6700	5.4115	0.0291	5.0004	13
S_4	7.0833	5.6973	0.0188	5.0021	14
S_5	7.6923	6.2687	0.0222	5.0008	14
S_6	8.0769	5.7623	0.0163	5.0007	16
S_7	8.4615	5.5649	0.0136	5.0002	16
S_8	9.2857	5.5642	0.0130	5.0004	20

Table 3.3: Result of the proposed distance metric defined in Eq. (3.17)

Dataset	$v^{(0)}$	$v^{(1)}$.	The optimal value for $\boldsymbol{a}$	b	v^t	$No. of$ $iterations\ (j)$
S_1	5.0000	4.3077	1.1000	0.9158	4.9986	14
S_2	6.2500	5.2223	1.5600	0.0507	5.0006	11
S_3	6.6700	5.4066	1.5400	0.0291	5.0001	13
S_4	7.0833	5.6927	1.5300	0.0188	5.0001	14
S_5	7.6923	6.2617	1.9400	0.0222	5.0003	15
S_6	8.0769	5.7623	2.6900	0.0163	5.0003	15
S_7	8.4615	5.7623	3.2500	0.0136	5.0003	15
S_8	9.2857	5.7623	3.4600	0.0130	5.0003	15

Table 3.4: Ideal Clustering centroid produced by AGFCM for the different values of **a** for diamond data-set $\mathbf{P_{10}}$

R	b	a	J_{FCM}	J_{AGFCM}	$No.of$ $iteration(j)$
$R < 1$	0.1	1.4	36.2183	1.7785	7
		2.7183	Does not recognize clusters		
		4	58.6216	2.8856	6
	0.5	1.06	35.0936	1.6752	7
		1.07	36.2639	1.7825	7
		2.7183	Does not recognize clusters		
		7	97.2327	3.9308	5
		8	97.7625	3.9429	5
		9	98.1519	3.9517	5
		100	100.2256	3.9983	4
	1	**1.03**	**35.1990**	**1.6854**	**7**
		2.5	96.6946	3.9185	5
		2.7183	97.4598	3.9360	5
		3	98.1519	3.9517	5
		8	100.1614	3.9969	4
		9	100.2002	3.9978	4
		10	100.2256	3.9983	4
		100	100.3002	4.0000	3
$R = 1$	2	1.07	58.0736	2.8669	6
		1.6	96.9307	3.9239	5
		1.7	97.9126	3.9463	5
		2.7183	Does not recognize clusters		
		3	100.2002	3.9978	4
		8	100.2999	4.0000	3
		9	100.3001	4.0000	3
		10	100.3002	4.0000	3
$R > 1$	4	1.7	100.1771	3.9973	4
		2.7183	Does not recognize clusters		
		3	100.3001	4.0000	3
		50	100.3003	4.0000	2
		100	100.3003	4.0000	2
	10	1.1	97.0542	3.9267	5
		1.6	100.3002	4.0000	3
		2.7183	Does not recognize clusters		
		5	100.3003	4.0000	2
		5.5	100.3003	4.0000	2
		6	100.3003	4.0000	2
	20	1.05	97.2574	3.9314	5
		2.2	100.3003	4.0000	2
		2.3	100.3003	4.0000	2

		2.5	100.3003	4.0000	2
		2.7183	Does not recognize clusters		

Table 3.5: Ideal Clustering centroid produced by AGHCM for the different values of a diamond data-set P_{10}

R	b	a	J_{AGFCM}	$No.of$ $iteration(j)$
$R < 1$	0.1	1.7	12.3927	72
		1.9	12.8576	30
		2	13.0360	24
		2.1	13.1898	21
		2.2	13.3243	18
		2.3	13.4435	17
		2.5	13.6465	14
		2.7183	Does not recognize clusters	
		3	14.0211	11
		4	14.4886	9
		5	14.7858	8
		7	15.1635	7
	0.5	1.1	12.1145	100
		1.2	13.6357	14
		1.5	15.2450	7
		2.7183	Does not recognize clusters	
		7	17.7280	6
		8	17.7747	6
		9	17.8092	6
	1	1.05	12.1764	100
		1.06	12.6257	43
		1.07	12.9804	26
		1.08	13.2711	19
		1.09	13.5166	16
		1.1	13.7289	13
		2.5	17.6806	6
		2.7183	17.7480	6
		3	17.8092	6
		5	17.9543	5
		5.5	17.9650	5
		100	18.0000	3
	2	1.03	12.6609	40
		1.04	13.3131	18
		1.05	13.7779	13
		1.07	14.4403	9
		1.08	14.6970	8

$R = 1$		1.6	17.7014	6
		1.7	17.7880	6
		2.2	17.9500	5
		2.3	17.9610	5
		2.7183	Does not recognize clusters	
		3.5	17.9962	4
		4	17.9982	4
		10	18.0000	3
$R > 1$	4	1.02	13.3344	18
		1.03	14.1697	10
		1.04	14.7349	8
		1.05	15.1693	7
		1.5	17.9559	5
		1.9	17.9968	4
		2	17.9982	4
		2.7183	Does not recognize clusters	
		3.5	18.0000	3
		50	18.0000	2
		100	18.0000	2
	10	1.02	15.1983	7
		1.1	17.7122	6
		1.6	18.0000	3
		2.7183	Does not recognize clusters	
		5	18.0000	2
		5.5	18.0000	2
		6	18.0000	2
	20	1.01	15.2081	7
		1.05	17.7302	6
		1.08	17.9444	5
		1.09	17.9668	5
		2.2	18.0000	2
		2.3	18.0000	2
		2.5	18.0000	2
		2.7183	Does not recognize clusters	

Table 3.6: Ideal Clustering centroid produced by AGFCM for the different values of a diamond data-set P_{12}

R	b	a	J_{FCM}	J_{AGFCM}	$No.of$ $iteration(j)$
	0.1	1.6	100.7136	2.8482	12
		2	106.1032	3.2032	11
		2.1	107.2254	3.2659	11
		2.7183	112.7549	3.5392	10
		4	119.7772	3.8291	9
		7	127.7734	4.1104	8

		50	144.2153	4.5873	6
	0.5	1.04	93.9197	2.1034	14
		1.1	100.8764	2.8607	12
		1.5	128.7653	4.1426	8
		1.7	135.2797	4.3418	7
		2.2	144.3814	4.5916	6
		2.3	145.5470	4.6216	6
		2.7183	Does not recognize clusters		
		7	158.3883	4.9308	5
		8	158.9181	4.9429	5
		9	159.3075	4.9517	5
		100	161.3812	4.9983	4
$R < 1$	1	1.02	93.9658	2.1112	14
		1.03	96.3545	2.4410	13
		1.05	101.1581	2.8820	12
		1.07	105.7170	3.1810	11
		1.08	107.8514	3.2996	11
		1.1	111.8018	3.4955	10
		1.2	126.2041	4.0584	8
		1.5	144.9824	4.6071	6
		2.5	157.8502	4.9185	5
		2.7183	158.6154	4.9360	5
		3	159.3075	4.9517	5
		8	161.3170	4.9969	4
		9	161.3558	4.9978	4
		10	161.3812	4.9983	4
		100	161.4558	5.0000	3
	2	1.01	93.9894	2.1151	14
		1.05	112.2660	3.5169	10
		1.07	119.2292	3.8082	9
		1.1	127.2756	4.0941	8
		1.6	158.0863	4.9239	5
		1.7	159.0682	4.9463	5
$R = 1$		2.7183	Does not recognize clusters		
		3	161.3558	4.9978	4
		8	161.4555	5.0000	3
		9	161.4557	5.0000	3
		10	161.4558	5.0000	3
	4	1.05	127.8440	4.1127	8
		1.07	135.7558	4.3557	7
		1.1	143.6551	4.5727	6
		1.7	161.3327	4.9973	4
		2.7183	Does not recognize clusters		
		3	161.4557	5.0000	3
		50	161.4559	5.0000	2
		100	161.4559	5.0000	2
	10	1.01	112.6549	3.5347	10
		1.02	128.1965	4.1242	8

$R > 1$		1.04	144.2706	4.5887	6
		1.1	158.2098	4.9267	5
		1.6	161.4558	5.0000	3
		2.7183	Does not recognize clusters		
		5	161.4559	5.0000	2
		5.5	161.4559	5.0000	2
		6	161.4559	5.0000	2
	20	1.001	94.0108	2.1187	14
		1.01	128.3161	4.1281	8
		1.02	144.4806	4.5942	6
		1.05	158.4130	4.9314	5
		2.2	161.4559	5.0000	2
		2.3	161.4559	5.0000	2
		2.5	161.4559	5.0000	2
		2.7183	Does not recognize clusters		

Table 3.7: Ideal Clustering centroid produced by AGHCM for the different values of a diamond data-set P_{12}

R	b	a	J_{AGHCM}	$No.of$ $iteration(j)$
$R < 1$	0.1	2.1	16.6902	100
		2.2	16.8539	100
		2.3	16.9992	72
		2.5	17.2467	47
		2.7183	17.4684	35
		3	17.7020	28
		3.5	18.0213	22
		4	18.2628	19
		4.5	18.4546	17
		5	18.6122	16
		5.5	18.7449	15
		6	18.8589	14
		7	19.0460	13
		8	19.1947	12
		10	19.4197	11
		100	20.7922	7
	0.5	1.2	17.2336	47
		1.4	18.7145	15
		1.6	19.4642	11
		1.7	19.7250	10
		2.1	20.4015	10
		2.5	20.7834	7
		2.7183	Does not recognize clusters	
		7	21.7279	6
		8	21.7747	6

		9	21.8092	6
	1	1.08	16.7891	100
		1.09	17.0883	60
		1.1	17.3471	41
		1.2	18.8985	42
		1.6	20.8268	7
		2.5	21.6805	6
		2.7183	21.7480	6
		3	21.8092	6
		5	21.9543	5
		5.5	21.9650	5
		100	22.0000	3
$R = 1$	2	1.04	16.8402	100
		1.05	17.4067	38
		1.06	17.8464	25
		1.07	18.2055	20
		1.08	18.5086	17
		1.09	18.7701	15
		1.1	18.9995	13
		1.2	20.3687	8
		1.6	21.7013	6
		1.7	21.7880	6
		2.2	21.9500	5
		2.3	21.9609	5
		2.7183	Does not recognize clusters	
		3.5	21.9962	4
		4	21.9982	4
		10	22.0000	3
$R > 1$	4	1.02	16.8662	100
		1.03	17.8816	24
		1.04	18.5529	16
		1.05	19.0525	13
		1.06	19.4463	11
		1.07	19.7670	10
		1.08	20.0340	9
		1.1	20.4531	8
		1.5	21.9559	5
		1.9	21.9968	4
		2	21.9982	4
		2.7183	Does not recognize clusters	
		3.5	22.0000	3
		50	22.0000	2
		100	22.0000	2
	10	1.01	17.4558	36
		1.02	19.0853	13
		1.03	19.9509	9
		1.05	20.8890	7

		1.1	21.7122	6
		1.6	22.0000	3
		2.7183	Does not recognize clusters	
		5	22.0000	2
		5.5	22.0000	2
		6	22.0000	2
	20	1.05	21.7301	6
		1.08	21.9444	5
		1.09	21.9668	5
		2.2	22.0000	2
		2.3	22.0000	2
		2.5	22.0000	2
		2.7183	Does not recognize clusters	

Table 3.8: Comparison of AGFCM and AGHCM with other clustering algorithms for diamond data-set P_{12}

Name of the algorithms	Values of Parameters	Clustering Centers
FCM	$m = 2$	$(-2.99,0.54)$ $(2.99,0.54)$
PCM	$\eta = 2$	$(-2.15,0.02)$ $(2.15,0.02)$
PFCM	$a = 1, b = 1, m = 2, \eta = 2$	$(-2.84,0.36)$ $(2.84,0.36)$
KFCM	$m = 2, h = 20, a = 1, b = 5$	$(3.252,0.003)$ $(-3.224,0.002)$
FCM-σ	$m = 2$	$(3.62,0.280)$ $(-2.14,5.249)$
KFCM-σ	$m = 2, h = 20, a = 1, b = 5$	$(3.315,0.0006)$ $(-3.073,0)$
DOFCM	$m = 2, \alpha = 0.09$	$(3.1672,0)$ $(-3.167,0)$
DKFCM-new	$m = 2, \alpha = 0.09, h = 19, a = 1, b = 6$	$(3.324,0)$ $(-3.3291,0)$
AGFCM	$\boldsymbol{R^* < 1, b = 0.5, a = 1.04}$	$\boldsymbol{(3.34, 0)}$ $\boldsymbol{(-3.34, 0)}$
AGHCM	$\boldsymbol{R^* < 1, b = 0.1, a = 2.1}$	$\boldsymbol{(3.34, 0)}$

		$(\mathbf{-3.34, 0})$

Table 3.9: Ideal Clustering centroid produced by AGFCM for the different values of a for Iris data-set

R	b	a	Cluster centroid	J_{FCM}	J_{AGFCM}	J_{HCM} (SSE)
$R < 1$	0.1	1.001	V_1 =(5.01475297, 3.393472061, 1.485005985, 0.236244565) V_2 =5.940022499, 2.835643956, 4.41071469, 1.405165193) V_3 =(6.553298701, 3.003865605, 5.407513896, 1.992010429)	88.36848	0.60943598	112.8848
		2.1	V_1 = **(5.015997456, 3.395103404,1.481501119,0.233793726)** $\boldsymbol{V_2}$ = **(5.875161645, 2.819768542, 4.341745798, 1.371643275)** V_3 =(**(6.521501312, 2.993799345, 5.365430872, 1.973036811)**	**107.56538**	**15.502268**	**112.31439**
		2.7183	V_1 =(5.016505529, 3.395658339, 1.480497649 ,0.233093163) V_2 =(5.8521626, 2.814598, 4.3174327, 1.3598113) V_3 =(6.512591947, 2.99081334, 5.351135531, 1.967379725)	114.7852097	17.600824	112.3701866
	0.5	1.001	V_1 = (5.014758982, 3.393480039, 1.484987232, 0.236230735) V_2 = (5.939810712, 2.835591577, 4.410481148, 1.405048987) V_3 = (6.553109679, 3.003809861, 5.407300302, 1.991910289)	88.45851458	1.36221893	112.8817517
		1.2	V_1 =(5.016328282, 3.395471844, 1.480823712, 0.233322462) V_2 = (5.859612595, 2.816240547, 4.325340132, 1.363663949) V_3 = (6.515492773, 2.991787749, 5.35592026, 1.969214669)	112.3143	16.93179	112.3382
		2.7183	V_1 = (5.024289736, 3.401390852, 1.474544358, 0.228295603) V_2 = (5.75227205, 2.80496812, 4.211179297, 1.309148081) V_3 = (6.468968763, 2.979276432, 5.25726459,1.94582964)	195.2740812	30.8028020	115.2126236
	1	1.001	V_1 = (5.014766496, 3.393490031, 1.484963748, 0.236213438) V_2 = (5.939543373, 2.835525356, 4.410186497, 1.40490245) V_3 = (6.552873655, 3.003740189, 5.40703313, 1.99178504)	88.57134278	1.92554756	112.8778545
		1.08	V_1 =(5.016050451, 3.395164376, 1.48138596, 0.233714013) V_2 = (5.872522777, 2.819161332, 4.338969313, 1.370294599) V_3 = (6.520482024, 2.993459165, 5.363854482, 1.972387568)	108.3386	15.75029	112.3146
		2.7183	V_1 = (5.032076096, 3.405841879, 1.47267414, 0.226464499) V_2 = (5.728250125, 2.812352852, 4.183969268, 1.298560098) V_3 = (6.474954988,2.988669846, 5.22976066, 1.962999865)	244.9658305	36.4021425	116.3911632
$R = 1$	2	1.001	V_1 = (5.014781523, 3.393510085,1.484916638,0.236178811) V_2 = (5.938999926,2.835390408, 4.409588038,1.404605076) V_3 = (6.552402453, 3.003600865, 5.406498171,1.991534291)	88.79792857	2.72053324	112.8699315
		1.04	V_1 =(5.016078966, 3.395196859, 1.481325102, 0.233671829) V_2 = (5.871126425, 2.818841363, 4.337499057, 1.369580105) V_3 = (6.519942995, 2.993279055, 5.363014819, 1.972044259)	108.75268	15.880408	112.31528

		2.7183	Does not recognize clusters			
$R > 1$	4	1.001	V_1 = (5.014811589, 3.393550467, 1.484821879, 0.236109437) V_2 = (5.937877322, 2.835110383, 4.408353898, 1.403992868) V_3 = (6.55146338, 3.003322288, 5.405425652, 1.991031722)	89.25477876	3.84006943	112.8535713
		1.02	V_1 = (5.016093823, 3.395213697, 1.481293692, 0.23365004) V_2 = (5.870405361, 2.818676501, 4.336739511, 1.369210908) V_3 = (6.519664686, 2.993186008, 5.362579647, 1.971867024)	108.96787	15.94732	112.31582
		2.7183	Does not recognize clusters			
	10	1.001	V_1 = (5.014902134,3.393673802, 1.484533817,0.235900638) V_2 = (5.934212529, 2.834187731, 4.404343576, 1.402012185) V_3 = (6.548671956,3.002486815, 5.402184392, 1.989514276)	90.65392852	6.03703545	112.8004401
		1.01	V_1 = (5.016495482, 3.395647951, 1.480515516, 0.233105778) V_2 = (5.852569167, 2.814686711, 4.317865071, 1.360022059) V_3 = (6.512750975, 2.990866768, 5.351401198, 1.967480032)	114.646	17.5644	112.368
		2.7183	Does not recognize clusters			
	20	1.001	V_1 = (5.015055712, 3.393886363, 1.484044719, 0.235552201) V_2 = (5.927072232,2.832375249,4.396602241, 1.398218513) V_3 = (6.544100981, 3.001093863, 5.396687004, 1.986947545)	93.07531406	8.45700085	112.6997986
		2.7183	Does not recognize clusters			

Table 3.10: Ideal Clustering centroid produced by AGFCM for the different values of **a** for Iris data-set

R	b	a	Cluster centroids	J_{AGFCM}	J_{HCM} (SSE)
$R < 1$	0.1	100	V_1 = (5.011930487,3.393791826,1.485613428,0.238885186) V_2 = (6.189412669,2.875467344,4.817854227,1.64730742) V_3 = (6.189412775, 2.875467356, 4.817854444,1.647307572)	329.7014	172.4445
		2.7183	V_1 = (6.094634271, 2.892675452, 4.580886928, 1.518111308) V_2 = (6.094634271, 2.892675452, 4.580886928, 1.518111308) V_3 = (6.094634271, 2.892675452, 4.580886928, 1.518111308)	224.1475741	807.2233032
	0.5	2.5	V_1 = (5.01197183, 3.393744093, 1.485795006,0.23896825) V_2 = (6.189464144,2.875464877, 4.81791004,1.647344656) V_3 = (6.189464246, 2.875464889, 4.81791025, 1.647344803)	329.4568903	172.4434976
		2.7183	V_1 = (5.011357583, 3.39446424, 1.483025132, 0.2377082) V_2 = (6.188478055, 2.875532038,4.816647305, 1.646505395) V_3 = (6.188478151, 2.875532049, 4.816647503, 1.646505534)	333.5867113	172.4703471
	1	100	V_1 = (5.020055672, 3.392520125, 1.474537983, 0.229101053) V_2 = (5.708767226, 2.794682102, 4.142707427,1.281791777) V_3 = (6.52804663, 3.011175095, 5.426000467,2.048037041)	421.3935961	112.5014197
		2.7183	V_1 = (5.009625295, 3.394923022, 1.475274796, 0.233383198) V_2 = (6.177302899, 2.877493772,4.793358345,1.62916556) (6.177302899, 2.877493772,4.793358345,1.62916556)	365.5699438	173.1670764
$R = 1$	2	10	V_1 = (5.020055672, 3.392520125,1.474537983, 0.229101053) V_2 = (5.708767226, 2.794682102,4.142707427,1.281791777) V_3 = (6.52804663, 3.011175095, 5.426000467,2.048037041)	421.3935961	112.5014197
		2.7183	V_1 = (5.009918901, 3.390831889, 1.475698828, 0.231488463) V_2 = (6.157106524, 2.886295432, 4.728320662,1.573883054)	394.5020737	175.8010882

			V_3 = (6.157998924, 2.886150764, 4.730741847, 1.575853672)		
$R > 1$	**4**	**8**	V_1 = **(5.055698026, 3.421693732, 1.465263395, 0.230916849)** V_2 = **(5.697550887, 2.802865412, 4.124742898, 1.283416894)** V_3 = **(6.492155687,2.979992958,5.463932575, 1.991995453)**	**432.1769765**	**112.440683**
		2.7183	V_1 = (5.016677713, 3.390659611, 1.475171351,0.229409264) V_2 = (5.716592997, 2.795769929, 4.154922482, 1.284109676) V_3 = (6.482214975, 2.989330209, 5.344632186, 1.990377817)	418.2288567	113.4860286
	10	2.3	V_1 = (5.055812027, 3.42180432, 1.465223604, 0.230924933) V_2 = (5.6975419, 2.802889227, 4.124728097,1.283426284) V_3 = (6.491312332, 2.979706781, 5.463889492, 1.990374926)	432.197223	112.4493339
		2.7183	V_1 = (5.070077256, 3.437199189, 1.458865561, 0.232009298) V_2 = (5.696532278, 2.806095258, 4.123292779, 1.284681498) V_3 = (6.456878051, 2.992732157, 5.486992997,1.864147983)	434.6668374	113.7502553
	20	1.6	V_1 = (5.06551018, 3.431894891, 1.461245793,0.231645538) V_2 = (5.696841274, 2.805005385, 4.12365768, 1.284261797) V_3 = (6.45842299, 2.98603903, 5.480396405, 1.887736102)	433.8849331	113.3786165
		2.7183	Does not recognize clusters		

Table 3.11: Comparison of AGFCM and AGHCM with other clustering algorithms for Iris data-set

Name of the algorithms	Values of Parameters	Cluster centroids	J_{HCM} (SSE)	Misclassi- -fication	Accuracy
FCM	$m = 2$	V_1 = (5.00, 3.41, 1.48, 0.25) V_2 = (5.89, 2.76, 4.36, 1.40) V_3 =(6.77, 3.05, 5.65, 2.05)	113.4650	16	0.8933
PCM	$\eta = 2$	V_1 = (5.00, 3.41, 1.48, 0.25) V_2 = (6.17, 2.88, 4.76, 1.61) V_3 = (6.17, 2.88, 4.76, 1.61)	174.2570	50	0.6667
PFCM	$a = 1, b = 1, m = 2, \eta = 2$	V_1 = (5.00, 3.41, 1.48, 0.25) V_2 = (5.89, 2.76, 4.36, 1.40) V_3 =((6.77, 3.05, 5.65, 2.05)	**112.2140**	13	0.9133
FPCM	$a = 1, b = 1, m = 2, \eta = 2$	V_1 = (5.00, 3.41, 1.48, 0.25) V_2 = (5.92, 2.79, 4.40, 1.41) V_3 = (6.62, 3.01, 5.46, 1.99)	126.480	13	0.9133
AFCM	$\boldsymbol{R^* < 1, b = 0.1, a = e}$	V_1 = (5.02,3.40,1.48 ,0.23) V_2 = (5.85, 2.81, 4.32, 1.36) V_3 = (6.51, 2.99, 5.35, 1.97)	112.3702	13	0.9133
AHCM	$\boldsymbol{R^* > 1, b = 4, a = e}$	V_1 = (5.02,3.39, 1.48, 0.23) V_2 = (5.72,2.80, 4.15, 1.28) V_3 =((6.48, 2.99, 5.34, 1.99)	113.4860	16	0.9067

AGFCM	$R^* < 1, b = 0.1, a = 2.1$	**V_1 = (5.02, 3.40,1.48,0.23)** **V_2 = (5.88, 2.82, 4.34, 1.37)** **V_3 = (6.52, 2.99, 5.37, 1.97)**	**112.2170**	**11**	**0.9267**
AGHCM	$R^* > 1, b = 4,, a = 8$	V_1 = (5.06, 3.42, 1.47, 0.23) V_2 = (5.70, 2.80, 4.12, 1.28) V_3 = (6.49,2.98,5.46, 1.99)	112.4407	12	0.92

Table 4.12: Coach wise different manufacturing cost for the year 2010-11

COACH TYPE	$C_{Lab.}$	$C_{Mat.}$	C_{foh}	$C_{Aoh.}$	$C_{Toh.}$	$C_{Soh.}$	$C_{Tot.}$	$C_{Pc.}$	$C_{Tc.}$
SCN/AB	4.38	45.70	7.33	5.80	1.17	0.37	14.69	1.63	66.40
SLR/AB	4.07	41.99	6.81	5.39	1.09	0.34	13.63	1.89	61.58
GS/AB	4.04	44.49	6.77	5.35	1.08	0.36	13.57	2.37	64.47
MEMU/MC	9.88	211.93	16.55	13.09	2.65	1.74	34.04	8.27	264.12
MEMU/TC	4.10	49.13	6.87	5.43	1.10	0.40	13.80	2.13	69.17
ACCN/SG	7.38	94.06	12.36	9.78	1.98	0.77	24.89	4.14	130.48
RA SHELL	2.50	33.62	4.58	2.91	0.75	0.23	8.47	1.44	46.03
LWSCZ	8.11	124.32	14.35	9.14	2.35	0.85	26.68	5.00	164.11
LWSCZDAC	14.81	190.08	24.40	19.30	3.91	1.56	49.16	8.23	262.29
WGCWNAC	7.22	93.76	13.17	8.39	2.16	0.64	24.35	4.07	129.41
VPHX	3.05	37.29	5.55	3.54	0.91	0.25	10.26	1.57	52.16
FAC (LC)	9.18	156.57	15.27	12.08	2.45	1.28	31.09	5.96	202.79
ACCW(LC)	10.70	153.98	17.84	14.11	2.86	1.26	36.07	5.99	206.74
WGCB(LC)	6.38	110.86	10.59	8.38	1.70	0.91	21.58	3.35	142.17
EOG/LHB/ACCB	10.38	184.34	16.98	13.43	2.72	1.51	34.64	7.17	236.53
EOG/LHB/ WLRRM	10.51	246.60	17.19	13.59	2.75	2.02	35.56	9.42	302.08
EOG/LHB/ ACCW	11.24	178.25	18.42	14.57	2.95	1.46	37.39	7.12	234.01
EOG/LHB/ ACCN	11.57	179.28	18.97	15.01	3.04	1.47	38.49	7.20	236.54
LGSLR	5.91	70.13	9.70	7.67	1.55	0.58	19.49	3.14	98.67
LGS(LC)	7.79	81.84	12.86	10.17	2.06	0.67	25.76	3.74	119.13
TOTAL	153.20	2328.22	256.50	197.10	41.20	18.60	513.61	93.83	3088.88

Where

$C_{Lab.}$= Labor cost of different coaches,

$C_{Mat.}$= Material cost of different coaches,

C_{Aoh}= Administrative overhead charge of different coaches,

$\boldsymbol{C_{foh}}$= Factory overhead charges of different coaches,
$\boldsymbol{C_{Toh}}$= Township overhead charges of different coaches,
$\boldsymbol{C_{Soh}}$= Shop overhead charges of different coaches,
$\boldsymbol{C_{Tot}}$ = Total overhead cost including the petty overhead of different coaches,
$\boldsymbol{C_{Pc}}$= Performa charges of different coaches
$\boldsymbol{C_{Tc}}$**=** Total cost of different coaches. All costs are in **lacs** (Indian National Rupees).

Table 4.13: show the average fluctuation in cost

Cost parameters	$\beta_i - \varepsilon_i$	β_i	$\beta_i + \varepsilon_i^*$
$\boldsymbol{C_{Lab.}}$	153.2000	145.5400	160.8600
$\boldsymbol{C_{Mat.}}$	2328.2200	2211.8090	2444.6310
$\boldsymbol{C_{foh}}$	256.5600	243.7320	269.3880
$\boldsymbol{C_{Aoh}}$	197.1300	187.2735	206.9865
$\boldsymbol{C_{Toh}}$	41.2300	39.1685	43.2915
$\boldsymbol{C_{Soh}}$	197.1300	187.2735	206.9865
$\boldsymbol{C_{Tot}}$	513.6100	487.9295	539.2905
$\boldsymbol{C_{Pc}}$	93.8300	89.1385	98.5215

Table 4.14: Lower bound, static bound, and upper bound for case I.

Parametric variable	$\beta_i - \varepsilon_i$	β_i	$\beta_i + \varepsilon_i^*$	Model-I (λ_1)	Model-II (λ_2)
y_1	0.0000	0.0000	0.0000	5.4554	0.9105
y_2	4.9800	5.0490	4.7181	0.0000	0.0000
y_3	0.0000	0.2615	0.1169	2.8959	0.0000
y_4	0.0000	0.0000	0.0000	0.0000	0.0784
y_5	0.0000	0.0000	0.0000	8.8780	10.4587
y_6	0.0000	0.0000	0.0000	0.0000	0.0000
y_7	0.0000	0.0000	0.0000	0.0000	0.0000
y_8	1.2745	1.5254	1.3331	1.7857	0.0000
y_9	0.0000	0.0000	0.0000	0.0000	0.0000
y_{10}	0.0000	0.0000	0.0000	0.8873	0.0000
y_{11}	3.1783	2.8653	3.7039	0.0000	6.6930
y_{12}	6.1425	6.7252	7.0252	0.0000	0.0000

y_{13}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{14}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{15}	0.0000	0.0000	0.0000	3.4852	0.0000
y_{16}	1.5120	1.4496	1.5572	2.2963	2.0666
y_{17}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{18}	0.0000	0.0000	0.0000	0.0000	5.2821
y_{19}	5.5772	5.6702	6.4669	0.0000	0.0000
y_{20}	0.0000	0.0000	0.0000	0.0000	0.0000
Z	**2934.3117**	**3088.7496**	**3243.1771**	λ_1=0.001045	λ_2=0.301433

Table 4.15: The optimized value of lower, static and upper bound for case II

Parametric variable	$\beta_i - \varepsilon_i$	β_i	$\beta_i + \varepsilon_i^*$	Model-I (λ_1)	Model-II (λ_2)
y_1	0.0000	0.0000	0.0000	0.0000	0.0000
y_2	0.0000	0.0000	0.0000	0.0000	0.0000
y_3	20.6343	21.7223	22.8062	21.7186	21.6790
y_4	0.0000	0.0000	0.0000	0.0000	0.0000
y_5	0.0000	0.0000	0.0000	0.0000	0.0000
y_6	0.0000	0.0000	0.0000	0.0000	0.0000
y_7	0.0000	0.0000	0.0000	0.0000	0.0000
y_8	2.4364	2.5637	2.6920	2.5636	2.5249
y_9	0.0000	0.0000	0.0000	0.0000	0.0000
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{11}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{12}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{13}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{14}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{15}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{16}	4.0182	4.2298	4.4417	4.2309	4.0553
y_{17}	0.0000	0.0000	0.0000	0.0000	0.0000

y_{18}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{19}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{20}	0.0000	0.0000	0.0000	0.0000	0.0000
Z	2943.9593	3098.9125	3253.8350	λ_1=0.0005	λ_2=0.3003

Table 4.16: Unbounded fluctuation is shown

Cost parameters	$\boldsymbol{\beta_i - \varepsilon_i}$	$\boldsymbol{\beta_i}$	$\boldsymbol{\beta_i + \varepsilon_i}^*$
$\boldsymbol{C_{Lab.}}$	153.2000	145.5400	168.0100
$\boldsymbol{C_{Mat.}}$	2328.2200	2211.8090	2574.8200
$\boldsymbol{C_{foh}}$	256.5600	243.7320	280.9600
$\boldsymbol{C_{Aoh}}$	197.1300	187.2735	216.4300
$\boldsymbol{C_{Toh}}$	41.2300	39.1685	45.1400
$\boldsymbol{C_{Soh}}$	197.1300	187.2735	216.4300
$\boldsymbol{C_{Tot}}$	513.6100	487.9295	562.7700
$\boldsymbol{C_{Pc}}$	93.8300	89.1385	103.2500

Table 4.17: The optimized value of lower, static, and upper bound

Parametric variable	$\beta_i - \varepsilon_i$	β_i	$\beta_i + \varepsilon_i^*$	Model-I (λ_1)	Model-II (λ_2)
y_1	0.0000	0.0000	0.0000	5.4563	2.2976
y_2	4.9800	5.0490	1.0089	0.0000	0.0000
y_3	0.0000	0.2615	0.0000	2.8940	0.0000
y_4	0.0000	0.0000	1.0863	0.0000	0.0000
y_5	0.0000	0.0000	0.0000	8.8892	8.8154
y_6	0.0000	0.0000	0.0000	0.0000	0.0000
y_7	0.0000	0.0000	1.4821	0.0000	0.0000
y_8	1.2745	1.5254	2.1461	1.7898	0.0000
y_9	0.0000	0.0000	0.0000	0.0000	0.0000
y_{10}	0.0000	0.0000	0.0000	0.8829	2.8219
y_{11}	3.1783	2.8653	0.0000	0.0000	0.0000
y_{12}	6.1425	6.7252	8.6262	0.0000	0.0000

y_{13}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{14}	0.0000	0.0000	0.2163	0.0000	0.0000
y_{15}	0.0000	0.0000	0.0000	3.4781	0.0000
y_{16}	1.5120	1.4496	0.0000	2.2993	2.0618
y_{17}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{18}	0.0000	0.0000	0.0000	0.0000	5.4628
y_{19}	5.5772	5.6702	8.7129	0.0000	0.0000
y_{20}	0.0000	0.0000	0.0000	0.0000	0.0000
Z	**2934.317**	**3088.7496**	**3409.2023**	$\lambda_1 = 0.0007$	$\lambda_2 = 0.2278$

Table 5.18: The probabilistic increments and decrements in the extension of total basic available cost

Cost parameters	$\beta_i - \varepsilon_i$	β_i	β_i^*	$\beta_i^* + \varepsilon_i{}^*$
$\boldsymbol{C_{Lab}}$.	145.5400	150.7000	155.7000	160.8600
$\boldsymbol{C_{Mat}}$.	2211.8090	2294.6000	2361.8400	2444.6310
$\boldsymbol{C_{foh}}$	243.7320	251.9800	261.1400	269.3880
$\boldsymbol{C_{Aoh}}$	187.2735	194.2200	200.0400	206.9865
$\boldsymbol{C_{Toh}}$	39.1685	40.4800	41.9800	43.2915
$\boldsymbol{C_{Soh}}$	17.7365	18.4400	18.9000	19.6035
$\boldsymbol{C_{Tot}}$	487.9295	505.1400	522.0800	539.2905
$\boldsymbol{C_{Pc}}$	89.1385	92.3900	95.2700	98.5215

Table 5.19: Least lower bound, lower bound, upper bound, and most upper for case IV.

Parametric variable	$\beta_i - \varepsilon_i$	β_i	β_i^*	$\beta_i^* + \varepsilon_i^*$	Model-I ($\boldsymbol{\lambda_1}$)	Model-II ($\boldsymbol{\lambda_2}$)
y_1	9.4114	9.7992	9.9621	10.3750	0.0000	0.0000
y_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_3	0.0000	0.0000	0.0000	0.0000	21.0075	21.2973
y_4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

y_6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_8	1.8638	1.5673	2.4520	2.1101	2.3617	2.3040
y_9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{11}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{12}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{13}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{14}	13.982	14.901	14.4480	15.4080	0.0000	0.0000
y_{15}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{16}	0.0000	0.0000	0.0000	0.0000	4.1163	4.1928
y_{17}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{18}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{19}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{20}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
λ	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Z	2918.6	3026.4	3118	3225.8	$\boldsymbol{\lambda_1}$=0.6195	$\boldsymbol{\lambda_2}$=0.3226

Table 5.20: Show calculated values of Model I and Model II of FLLP of case-IV.

Trapezoidal FLPP	**Model-I**	Model-II
λ_1	0.6195	–
λ_2	–	0.3226
Optimized minimum cost	2985.3840 units	3042.507 units
Minimized cost	2985.3840 units	2953.3765 units
Greatest Minimized Costs	3159.0163 units	3,191.0235 units

Table 5.21: Least lower bound, lower bound, upper bound, and most upper for case V.

Parametric variable	$\beta_i - \varepsilon_i$	β_i	β_i^*	$\beta_i^* + \varepsilon_i^*$	Model-I ($\boldsymbol{\lambda_1}$)	Model-II ($\boldsymbol{\lambda_2}$)
y_1	0.0000	0.0000	0.0000	0.0000	2.2161	2.2209
y_2	2.5638	2.5569	2.6988	2.8337	0.0000	0.0000
y_3	0.0000	0.1433	0.0000	0.0000	0.4444	0.4422
y_4	0.6561	0.4911	0.6906	0.7252	0.0000	0.0000
y_5	0.0000	0.0000	0.0000	0.0000	9.8671	10.0016
y_6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

y_7	0.4097	0.0000	1.4313	0.4528	0.0000	0.0000
y_8	1.4807	1.5098	1.5586	1.6366	0.0342	0.0349
y_9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{10}	0.0000	0.0000	0.0000	0.0000	2.5556	2.5044
y_{11}	2.2907	2.0603	2.4113	2.5318	0.0000	0.0000
y_{12}	6.7120	7.2621	7.0653	7.4186	0.0000	0.0000
y_{13}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{14}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{15}	0.0000	0.0000	0.0000	0.0000	6.9804	7.0909
y_{16}	0.6019	0.7044	0.6336	0.6653	0.4731	0.4631
y_{17}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{18}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y_{19}	6.8800	7.1514	7.2421	7.6042	0.0000	0.0000
y_{20}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Z	2934.3209	3042.7297	3134.7889	3243.1968	λ_1=0.5010	λ_2=0.2600

Table 5.22: Show calculated values of Model I and Model II of FLLP of case-V.

Trapezoidal FLPP	**Model-I**	Model-II
λ_1	0.5010	–
λ_2	–	0.2603
Optimized minimum cost	3014.7300 units	3014.7200 units
Minimized cost	2988.6364 units	2962.5421 units
Greatest Minimized Costs	3188.8819 units	3214.9759 units

Table 6.23: shows the probabilistic increments and decrements in the cost parameter

Cost parameters	$\boldsymbol{\beta_i - \varepsilon_i}$	$\boldsymbol{\beta_i}$	$\boldsymbol{\beta_i^*}$	$\boldsymbol{\beta_i^* + \varepsilon_i{}^*}$
Labour cost	145.5400	150.7000	155.7000	160.8600
Material cost	2211.8090	2294.6000	2361.8400	2444.6300
Factory overhead Charges	243.7320	251.9800	261.1400	269.3900
Administrative overhead Charge	187.2735	194.2200	200.0400	206.9900
Township overhead Charges	39.1685	40.4800	41.9800	43.2900
Shop overhead charges	17.7365	18.4400	18.9000	19.6000

Total overhead charges	487.9295	505.1400	522.0800	539.2900
Performa charges	89.1385	92.3900	95.2700	98.5215

Table 6.24: Optimized membership grade

Parametric variable	$(\beta_i - \varepsilon_i)$	β_i	β_i^*	$(\beta_i^* + \varepsilon_i{}^*)$	λ
y_1	9.4114	9.7992	9.9621	10.3750	0.0023
y_2	0.0000	0.0000	0.0000	0.0000	0.0000
y_3	0.0000	0.0000	0.0000	0.0000	0.5872
y_4	0.0000	0.0000	0.0000	0.0000	0.0000
y_5	0.0000	0.0000	0.0000	0.0000	0.0097
y_6	0.0000	0.0000	0.0000	0.0000	0.0000
y_7	0.0000	0.0000	0.0000	0.0000	0.0000
y_8	1.8638	1.5670	2.4520	2.1101	0.0722
y_9	0.0000	0.0000	0.0000	0.0000	0.0000
y_{10}	0.0000	0.0000	0.0000	0.0000	0.0024
y_{11}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{12}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{13}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{14}	13.9820	14.900	14.4480	15.4080	0.0000
y_{15}	0.0000	0.0000	0.0000	0.0000	0.0069
y_{16}	0.0000	0.0000	0.0000	0.0000	0.5196
y_{17}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{18}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{19}	0.0000	0.0000	0.0000	0.0000	0.0000
y_{20}	0.0000	0.0000	0.0000	0.0000	0.0000
λ	0.0000	0.0000	0.0000	0.0000	0.2992
Z	2918.600	3026.4	3118.0000	3225.8000	0.2992

Table 7.25:Show the geographical coordinates of the existing warehouse

Existing warehouse number	**Coordinate s of centroids**
1	(2.5,3.5)
2	(2.7,3.6)
3	(2.2,3.2)
4	(3.0,3.0)
5	(2.7,3.1)

6	(2.3,3.5)
7	(3.1,3.0)
8	(3.3,2.9)
9	(3.0,3.5)
10	(2.9, 2.9)
11	(6.0,7.2)
12	(6.5,7.0)
13	(5.9,6.9)
14	(6.3,7.3)
15	(5.9,7.0)
16	(6.8,7.0)
17	(6.5,6.9)
18	(6.2,7.1)
19	(7.2,7.2)
20	(6.6,7.1)
21	(11.1,13.5)
22	(12.2,12.5)
23	(11.5,11.9)
24	(13.2,13.3)
25	(11.5,12.5)
26	(12.0,15.0)
27	(13.2,13.2)
28	(11.2,11.9)
29	(12.9, 13.7)
30	(13.9,13.0)

Table 7.26: Initial membership grade to the input data

Input data		**Membership function for fuzzy clustering**		
X_1	X_2	U_1	U_2	U_3
2.5	3.5	.9	.05	.05
2.7	3.6	.89	.055	.055
2.2	3.2	.88	.06	.06
3.0	3.0	.87	.065	.065

2.7	3.1	.86	.07	.07
2.3	3.5	.84	.09	.07
3.1	3.0	.81	.11	.08
3.3	2.9	.8	.1	.1
3.0	3.5	.79	.11	.1
2.9	2.9	.75	.15	.1
6.0	7.2	.05	.9	.05
6.5	7.0	.055	.89	.055
5.9	6.9	.06	.88	.06
6.3	7.3	.065	.87	.065
5.9	7.0	.07	.86	.07
6.8	7.0	.09	.84	.07
6.5	6.9	.11	.81	.08
6.2	7.1	.1	.8	.1
7.2	7.2	.11	.79	.1
6.6	7.1	.15	.75	.1
11.1	13.5	.05	.05	.9
12.2	12.5	.055	.055	.89
11.5	11.9	.06	.06	.88
13.2	13.3	.065	.065	.87
11.5	12.5	.07	.07	.86
12.0	15.0	.07	.09	.84
13.2	13.2	.08	.11	.81
11.2	11.9	.1	.1	.8
12.9	13.7	.1	.11	.79
13.9	13.0	.1	.15	.75

Table 7.27:Distance between the new warehouses to existing warehouses

Distance matrix [d]				
	Factories			**Requirement**
Warehouses	**W_1**	**W_2**	**W_3**	
H_1	0.39	5.28	13.68	(30, 40,50)
H_2	0.38	5.07	13.46	(25, 30,40)
H_3	0.57	5.71	14.10	(15, 20,25)
H_4	0.32	5.30	13.69	(5, 10,15)
H_5	0.14	5.42	13.82	(40, 50, 60)
H_6	0.55	5.43	13.82	(52, 60,66)

H_7	0.40	5.24	13.62	(64, 70,76)
H_8	0.62	5.19	13.56	(7, 10,15)
H_9	0.36	4.93	13.32	(16, 20,23)
H_{10}	0.35	5.44	13.83	(26, 30, 38)
H_{11}	5.12	0.41	8.59	(31, 40,46)
H_{12}	5.31	0.13	8.37	(22, 25,31)
H_{13}	4.83	0.52	8.87	(29, 35, 42)
H_{14}	5.39	0.24	8.30	(38, 45, 52)
H_{15}	4.91	0.50	8.80	(11, 15, 23)
H_{16}	5.52	0.41	8.17	(16, 25, 32)
H_{17}	5.24	0.20	8.45	(28, 35,44)
H_{18}	5.18	0.19	8.51	(36,45,53)
H_{19}	5.95	0.82	7.75	(30, 35, 40)
H_{20}	5.45	0.21	8.23	(24,30,35)
H_{21}	13.23	7.97	1.27	(10, 20,26)
H_{22}	13.22	7.95	0.56	(27, 30, 38)
H_{23}	12.31	7.03	1.39	(13, 20, 28)
H_{24}	14.5	9.23	0.95	(50, 60,70)
H_{25}	12.74	7.45	0.96	(42, 50,57)
H_{26}	14.96	9.71	1.97	(37, 40, 44)
H_{27}	14.43	9.96	0.92	(11, 20,26)
H_{28}	12.1	6.81	1.58	(22, 30, 35)
H_{29}	14.57	9.29	0.89	(34, 40, 48)
H_{30}	14.81	9.57	1.61	(12,20,29)
Production	**(330, 400,460)**	**(200,260,300)**	**(273, 340,447)**	**(803,1001,1207)**

Table 7.28: Show the optimal value of the lower bound (Z_l) of Y_{ij}

Existing Warehouses/New Warehouses	**W_1**	**W_2**	**W_3**
H_1	30	0	0
H_2	2	23	0
H_3	0	0	15
H_4	5	0	0
H_5	0	40	0
H_6	0	0	52
H_7	64	0	0
H_8	0	0	7

H_9	0	0	16
H_{10}	26	0	0
H_{11}	0	31	0
H_{12}	0	0	22
H_{13}	29	0	0
H_{14}	0	20	18
H_{15}	0	0	11
H_{16}	16	0	0
H_{17}	0	28	0
H_{18}	0	0	36
H_{19}	30	0	0
H_{20}	0	24	0
H_{21}	0	0	10
H_{22}	27	0	0
H_{23}	0	0	13
H_{24}	0	0	50
H_{25}	42	0	0
H_{26}	37	0	0
H_{27}	0	0	11
H_{28}	22	0	0
H_{29}	0	34	0
H_{30}	0	0	12

Table 7.29: Show the optimal value of the Static bound (Zs) of Y_{ij}

Existing Warehouses/New Warehouses	W_1	W_2	W_3
H_1	40	5	0
H_2	10	0	0
H_3	70	0	0
H_4	30	0	0
H_5	35	0	0
H_6	25	0	0
H_7	35	0	0
H_8	30	0	0
H_9	50	40	0
H_{10}	30	0	0
H_{11}	0	25	0
H_{12}	0	50	0
H_{13}	0	0	0
H_{14}	0	40	0
H_{15}	0	40	0
H_{16}	0	35	0
H_{17}	0	30	0

H18	0	0	0
H19	0	0	0
H20	0	40	0
H21	0	0	20
H22	0	0	60
H23	0	10	20
H24	0	0	25
H25	0	5	15
H26	0	0	45
H27	0	0	20
H28	0	20	60
H29	0	0	20
H30	0	0	20

Table 7.30: Show the optimal value of the upper bound (Zu) of Y_{ij}

Existing Warehouses/New Warehouses	W_1	W_2	W_3
H_1	50	0	0
H_2	15	0	0
H_3	76	0	0
H_4	38	0	0
H_5	42	0	0
H_6	32	0	0
H_7	40	0	0
H_8	38	0	0
H_9	57	37	0
H10	35	0	0
H11	0	40	0
H12	0	60	0
H13	0	0	0
H14	0	46	0
H15	0	20	0
H16	0	44	0
H17	0	35	0
H18	0	0	0
H19	0	7	0
H20	0	48	0
H21	0	0	25
H22	0	0	66
H23	0	15	23
H24	0	0	31
H25	0	32	23

H_{26}	0	0	53
H_{27}	0	0	26
H_{28}	0	28	70
H_{29}	0	0	26
H_{30}	0	0	29

Table 7.31: Show the optimal value of the proposed FLP of $\boldsymbol{Y_{ij}}$

Existing Warehouses/New Warehouses	W_1	W_2	W_3
H_1	32	0	0
H_2	25	0	0
H_3	18	0	0
H_4	8	0	0
H_5	42	0	0
H_6	56	0	0
H_7	66	0	0
H_8	9	0	0
H_9	20	0	0
H_{10}	27	0	0
H_{11}	0	37	0
H_{12}	0	23	0
H_{13}	0	31	0
H_{14}	0	40	0
H_{15}	0	12	0
H_{16}	0	21	0
H_{17}	0	28	0
H_{18}	0	39	0
H_{19}	0	33	0
H_{20}	0	28	0
H_{21}	0	0	17
H_{22}	0	0	28
H_{23}	0	0	15
H_{24}	0	0	52
H_{25}	0	0	45
H_{26}	0	0	39
H_{27}	0	0	17
H_{28}	0	0	28
H_{29}	0	0	34
H_{30}	0	0	14

Bibliography

[1] K. L. Wu and M. S. Yang, “Alternative c-means clustering algorithms,” *Pattern Recognit.*, vol. 35, no. 10, pp. 2267–2278, 2002, doi: 10.1016/S0031-3203(01)00197-2.

[2] D.-Q. Zhang and S.-C. Chen, “A comment on ‘Alternative c-means clustering algorithms,’” *Pattern Recognit.*, 2004, doi: 10.1016/j.patcog.2003.08.001.

[3] R. O. Duda, P. E. Hart, and D. G. Stork, “Pattern classification,” *New York John Wiley, Sect.*, 2001.

[4] A. K. Jain, “Data clustering: 50 years beyond K-means,” *Pattern Recognit. Lett.*, 2010, doi: 10.1016/j.patrec.2009.09.011.

[5] A. S. Hadi, L. Kaufman, and P. J. Rousseeuw, “Finding Groups in Data: An Introduction to Cluster Analysis,” *Technometrics*, 1992, doi: 10.2307/1269576.

[6] G. Dhiman and V. Kumar, “Astrophysics inspired multi-objective approach for automatic clustering and feature selection in real-life environment,” *Mod. Phys. Lett. B*, 2018, doi: 10.1142/S0217984918503852.

[7] S. C. Johnson, “Hierarchical clustering schemes,” *Psychometrika*, 1967, doi: 10.1007/BF02289588.

[8] C. S. Wallace and D. M. Boulton, “An information measure for classification,” *Comput. J.*, 1968, doi: 10.1093/comjnl/11.2.185.

[9] D. S. B. Everitt, S. Landau, M. Leese, *Cluster Analysis, 5th Edition*. 2011.

[10] K. Q. Weinberger and L. K. Saul, “Distance metric learning for large margin nearest neighbor classification,” *J. Mach. Learn. Res.*, 2009.

[11] J. Liang, Q. Hu, P. Zhu, and W. Wang, “Efficient multi-modal geometric mean metric learning,” *Pattern Recognit.*, 2018, doi: 10.1016/j.patcog.2017.02.032.

[12] H. Chang and D. Y. Yeung, "Locally smooth metric learning with application to image retrieval," 2007, doi: 10.1109/ICCV.2007.4408862.

[13] F. Cakir, K. He, X. Xia, B. Kulis, and S. Sclaroff, "Deep metric learning to rank," 2019, doi: 10.1109/CVPR.2019.00196.

[14] Q. Wang, P. C. Yuen, and G. Feng, "Semi-supervised metric learning via topology preserving multiple semi-supervised assumptions," *Pattern Recognit.*, 2013, doi: 10.1016/j.patcog.2013.02.015.

[15] X. Li, Y. Bai, Y. Peng, S. Du, and S. Ying, "Nonlinear Semi-Supervised Metric Learning Via Multiple Kernels and Local Topology," *Int. J. Neural Syst.*, 2018, doi: 10.1142/S012906571750040X.

[16] H. O. Song, Y. Xiang, S. Jegelka, and S. Savarese, "Deep Metric Learning via Lifted Structured Feature Embedding," 2016, doi: 10.1109/CVPR.2016.434.

[17] H. Yan and J. Hu, "Video-based kinship verification using distance metric learning," *Pattern Recognit.*, 2018, doi: 10.1016/j.patcog.2017.03.001.

[18] D. Wang and X. Tan, "Robust Distance Metric Learning via Bayesian Inference," *IEEE Trans. Image Process.*, 2018, doi: 10.1109/TIP.2017.2782366.

[19] W. Deng, J. Hu, Z. Wu, and J. Guo, "From one to many: Pose-Aware Metric Learning for single-sample face recognition," *Pattern Recognit.*, 2018, doi: 10.1016/j.patcog.2017.10.020.

[20] R. Xu and D. Wunsch, "Survey of clustering algorithms," *IEEE Transactions on Neural Networks*. 2005, doi: 10.1109/TNN.2005.845141.

[21] P.-N. Tan, M. Steinbach, and V. Kumar, *Introduction to Data Mining (New International Editon)*. 2013.

[22] M. A. Fligner and J. S. Verducci, "Distance Based Ranking Models," *J. R. Stat. Soc. Ser. B*, 1986, doi: 10.1111/j.2517-6161.1986.tb01420.x.

[23] R. Potolea, S. Cacoveanu, and C. Lemnaru, “Meta-learning framework for prediction strategy evaluation,” 2011, doi: 10.1007/978-3-642-19802-1_20.

[24] W. Sheng, X. Liu, and M. Fairhurst, “A niching memetic algorithm for simultaneous clustering and feature selection,” *IEEE Trans. Knowl. Data Eng.*, 2008, doi: 10.1109/TKDE.2008.33.

[25] A. K. Jain, M. N. Murty, and P. J. Flynn, “Data clustering: A review,” 1999, doi: 10.1145/331499.331504.

[26] P. Contreras and F. Murtagh, “Hierarchical clustering,” in *Handbook of Cluster Analysis*, 2015.

[27] D. Sisodia, L. Singh, S. Sisodia, and K. Saxena, “Clustering Techniques: A Brief Survey of Different Clustering Algorithms,” *Int. J. Latest Trends Eng. Technol.*, 2012.

[28] M. E. Celebi, *Partitional clustering algorithms*. 2015.

[29] X. Jin and J. Han, “Partitional Clustering,” in *Encyclopedia of Machine Learning and Data Mining*, 2017.

[30] T. C. Havens, J. C. Bezdek, C. Leckie, L. O. Hall, and M. Palaniswami, “Fuzzy c-Means algorithms for very large data,” *IEEE Trans. Fuzzy Syst.*, 2012, doi: 10.1109/TFUZZ.2012.2201485.

[31] C. C. Aggarwal and C. K. Reddy, *DATA Custering Algorithms and Applications*. 2013.

[32] J. C. Bezdek, *Pattern Recognition with Fuzzy Objective Function Algorithms*. 1981.

[33] J. C. Dunn, “A fuzzy relative of the ISODATA process and its use in detecting compact well-separated clusters,” *J. Cybern.*, vol. 3, no. 3, pp. 32–57, 1973, doi: 10.1080/01969727308546046.

[34] Y.-H. Pao, “Statistical Pattern Recognition. Second edition (Keinosuke Fukunaga),” *SIAM Rev.*, 1993, doi: 10.1137/1035031.

[35] R. Krishnapuram and J. M. Keller, “A Possibilistic Approach to Clustering,” *IEEE Trans. Fuzzy Syst.*, vol. 1, no. 2, pp. 98–110, 1993, doi: 10.1109/91.227387.

[36] N. R. Pal, K. Pal, J. M. Keller, and J. C. Bezdek, “A possibilistic fuzzy c-means clustering algorithm,” *IEEE Trans. Fuzzy Syst.*, 2005, doi: 10.1109/TFUZZ.2004.840099.

[37] R. J. Hathaway, J. C. Bezdek, and Y. Hu, “Generalized fuzzy c-means clustering strategies using Lp norm distances,” *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 5, pp. 576–582, 2000, doi: 10.1109/91.873580.

[38] T. Przybyła, J. Jezewski, K. Horoba, and D. Roj, “Hybrid fuzzy clustering using LP norms,” 2011, doi: 10.1007/978-3-642-20039-7-19.

[39] R. N. Davé and S. Sen, “Generalized noise clustering as a robust fuzzy c-M-estimators model,” 1998, doi: 10.1109/NAFIPS.1998.715576.

[40] R. N. Dave, “Characterization and detection of noise in clustering,” *Pattern Recognit. Lett.*, 1991, doi: 10.1016/0167-8655(91)90002-4.

[41] R. N. Dave, “Robust fuzzy clustering algorithms,” 1993, doi: 10.1109/fuzzy.1993.327577.

[42] R. N. Davé and S. Sen, “Robust fuzzy clustering of relational data,” *IEEE Trans. Fuzzy Syst.*, 2002, doi: 10.1109/TFUZZ.2002.805899.

[43] I. Computing, “From Noisy Data,” *Computing*, vol. 3, no. 2, 2011.

[44] K. K. Chintalapudi and M. Kam, “Credibilistic fuzzy c means clustering algorithm,” 1998, doi: 10.1109/icsmc.1998.728197.

[45] D. Chakraborty, D. K. Jana, and T. K. Roy, “A New Approach to Solve Intuitionistic Fuzzy Optimization Problem Using Possibility, Necessity, and Credibility Measures,” *Int. J. Eng. Math.*, 2014, doi: 10.1155/2014/593185.

[46] J. Zhou, Q. Wang, C. C. Hung, and X. Yi, “Credibilistic Clustering: The Model and Algorithms,” *Int. J. Uncertainty, Fuzziness Knowlege-Based Syst.*, 2015, doi: 10.1142/S0218488515500245.

[47] J. Zhou, Q. Wang, C. C. Hung, and F. Yang, “Credibilistic clustering algorithms via alternating cluster estimation,” *J. Intell. Manuf.*, 2017, doi: 10.1007/s10845-014-1004-6.

[48] T. Denœux and M. H. Masson, “EVCLUS: Evidential Clustering of Proximity Data,” *IEEE Trans. Syst. Man, Cybern. Part B Cybern.*, 2004, doi: 10.1109/TSMCB.2002.806496.

[49] M. H. Masson and T. Denœux, “ECM: An evidential version of the fuzzy c-means algorithm,” *Pattern Recognit.*, 2008, doi: 10.1016/j.patcog.2007.08.014.

[50] A. J. Djiberou Mahamadou, V. Antoine, G. J. Christie, and S. Moreno, “Evidential clustering for categorical data,” 2019, doi: 10.1109/FUZZ-IEEE.2019.8858972.

[51] M. H. Masson and T. Denœux, “RECM: Relational evidential c-means algorithm,” *Pattern Recognit. Lett.*, 2009, doi: 10.1016/j.patrec.2009.04.008.

[52] V. Antoine, B. Quost, M. H. Masson, and T. Denœux, “CECM: Constrained evidential C-means algorithm,” *Comput. Stat. Data Anal.*, 2012, doi: 10.1016/j.csda.2010.09.021.

[53] P. Bradley, K. Bennett, and A. Demiriz, “Constrained k-means clustering,” 2000.

[54] R. C. De Amorim, “Constrained clustering with Minkowski Weighted K-Means,” 2012, doi: 10.1109/CINTI.2012.6496753.

[55] D. Q. Zhang and S. C. Chen, “Clustering incomplete data using kernel-based fuzzy C-means algorithm,” *Neural Process. Lett.*, 2003, doi: 10.1023/B:NEPL.0000011135.19145.1b.

[56] D. Q. Zhang and S. C. Chen, “A novel kernelized fuzzy C-means algorithm with application in medical image segmentation,” *Artif. Intell. Med.*, 2004, doi: 10.1016/j.artmed.2004.01.012.

[57] L. A. Zadeh, “Fuzzy sets,” *Inf. Control*, 1965, doi: 10.1016/S0019-9958(65)90241-X.

[58] BELLMAN RE and ZADEH LA, “Decision-Making in a Fuzzy Environment,” *Manage. Sci.*, vol. 17, no. 4, 1970, doi: 10.1142/9789812819789_0004.

[59] S. Faizi, W. Sałabun, T. Rashid, J. Watróbski, and S. Zafar, “Group decision-making for hesitant fuzzy sets based on Characteristic Objects Method,” *Symmetry (Basel).*, 2017, doi: 10.3390/sym9080136.

[60] J. Chen and X. Huang, “Dual hesitant fuzzy probability,” *Symmetry (Basel).*, 2017, doi: 10.3390/sym9040052.

[61] S. Hou, H. Wang, and S. Feng, “Attribute control chart construction based on fuzzy score number,” *Symmetry (Basel).*, 2016, doi: 10.3390/sym8120139.

[62] W. Sałabun and A. Piegat, “Comparative analysis of MCDM methods for the assessment of mortality in patients with acute coronary syndrome,” *Artif. Intell. Rev.*, 2017, doi: 10.1007/s10462-016-9511-9.

[63] M. Bucolo, L. Fortuna, and M. La Rosa, “Complex dynamics through fuzzy chains,” *IEEE Trans. Fuzzy Syst.*, 2004, doi: 10.1109/TFUZZ.2004.825969.

[64] A. Ebrahimenjad, “A New Link Between Output-oriented BCC Model with Fuzzy Data in the Present of Undesirable Outputs and MOLP,” *Fuzzy Inf. Eng.*, vol. 3, no. 2, pp. 113–125, 2011, doi: 10.1007/s12543-011-0070-0.

[65] A. Ebrahimnejad, M. Tavana, and H. Alrezaamiri, “A novel artificial bee colony algorithm for shortest path problems with fuzzy arc weights,” *Meas. J. Int. Meas. Confed.*, vol. 93, pp. 48–56, 2016, doi: 10.1016/j.measurement.2016.06.050.

[66] A. Hatami-Marbini, A. Ebrahimnejad, and S. Lozano, “Fuzzy efficiency measures in data envelopment analysis using lexicographic multiobjective approach,” *Comput. Ind. Eng.*, vol. 105, pp. 362–376, 2017, doi: 10.1016/j.cie.2017.01.009.

[67] B. Melián and J. L. Verdegay, “Fuzzy optimization models for the design of WDM networks,” *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 2, pp. 466–476, 2008, doi: 10.1109/TFUZZ.2007.895936.

[68] B. Melián and J. L. Verdegay, “Using fuzzy numbers in network design optimization problems,” *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 5, pp. 797–806, 2011, doi: 10.1109/TFUZZ.2011.2140325.

[69] H. Tanaka, T. Okuda, and K. Asai, “On fuzzy-mathematical programming,” *J. Cybern.*, vol. 3, no. 4, pp. 37–46, 1973, doi: 10.1080/01969727308545912.

[70] H. J. Zimmermann, “Fuzzy programming and linear programming with several objective functions,” *Fuzzy Sets Syst.*, 1978, doi: 10.1016/0165-0114(78)90031-3.

[71] A. Amid, S. H. Ghodsypour, and C. O’Brien, “Fuzzy multiobjective linear model for supplier selection in a supply chain,” *Int. J. Prod. Econ.*, 2006, doi: 10.1016/j.ijpe.2005.04.012.

[72] H. Tanaka and K. Asai, “Fuzzy linear programming problems with fuzzy numbers,” *Fuzzy Sets Syst.*, 1984, doi: 10.1016/0165-0114(84)90022-8.

[73] J. L. Verdegay, “A dual approach to solve the fuzzy linear programming problem,” *Fuzzy Sets Syst.*, 1984, doi: 10.1016/0165-0114(84)90096-4.

[74] F. Herrera, M. Kovács, and J. L. Verdegay, “Optimality for fuzzified

mathematical programming problems: A parametric approach," *Fuzzy Sets Syst.*, 1993, doi: 10.1016/0165-0114(93)90373-P.

[75] K. Ganesan and P. Veeramani, "Fuzzy linear programs with trapezoidal fuzzy numbers," *Ann. Oper. Res.*, vol. 143, no. 1, pp. 305–315, 2006, doi: 10.1007/s10479-006-7390-1.

[76] J. ying Dong and S. P. Wan, "A new trapezoidal fuzzy linear programming method considering the acceptance degree of fuzzy constraints violated," *Knowledge-Based Syst.*, 2018, doi: 10.1016/j.knosys.2018.02.030.

[77] C. Veeramani, C. Duraisamy, and A. Nagoorgani, "Solving fuzzy multi-objective linear programming problems with linear membership functions," *Aust. J. Basic Appl. Sci.*, vol. 5, no. 8, pp. 1163–1171, 2011.

[78] H. R. Maleki, M. Tata, and M. Mashinchi, "Linear programming with fuzzy variables," *Fuzzy Sets Syst.*, 2000, doi: 10.1016/S0165-0114(98)00066-9.

[79] Y. Chalco-Cano, A. Rufián-Lizana, H. Román-Flores, and R. Osuna-Gómez, "A note on generalized convexity for fuzzy mappings through a linear ordering," *Fuzzy Sets Syst.*, 2013, doi: 10.1016/j.fss.2013.07.001.

[80] D. H. Hong, "A convexity problem and a new proof for linearity preserving additions of fuzzy intervals," *Fuzzy Sets Syst.*, 2008, doi: 10.1016/j.fss.2008.05.020.

[81] W. A. Lodwick, K. D. Jamison, and K. A. Bachman, "Solving large-scale fuzzy and possibilistic optimization problems," *Annu. Conf. North Am. Fuzzy Inf. Process. Soc. - NAFIPS*, vol. 1, no. 3, pp. 146–150, 2004, doi: 10.1007/s10700-005-3663-4.

[82] A. Rufián-Lizana, R. Osuna-Gómez, Y. Chalco-Cano, and H. Román-Flores, "Some remarks on optimality conditions for fuzzy optimization problems," *Investig. Operacional*, 2017.

[83] H. M. Nehi, H. R. Maleki, and M. Mashinchi, "A canonical representation for the solution of fuzzy linear system and fuzzy linear programming problem," *J. Appl. Math. Comput.*, vol. 20, no. 1–2, pp. 345–354, 2006, doi: 10.1007/BF02831943.

[84] H. C. Wu, "The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function," *Eur. J. Oper. Res.*, 2007, doi: 10.1016/j.ejor.2005.09.007.

[85] Y. Chalco-Cano, W. A. Lodwick, R. Osuna-Gómez, and A. Rufián-Lizana, "The Karush–Kuhn–Tucker optimality conditions for fuzzy optimization problems," *Fuzzy Optim. Decis. Mak.*, 2016, doi: 10.1007/s10700-015-9213-9.

[86] H. C. Wu, "The optimality conditions for optimization problems with convex constraints and multiple fuzzy-valued objective functions," *Fuzzy Optim. Decis. Mak.*, 2009, doi: 10.1007/s10700-009-9061-6.

[87] M. H. Alavidoost, M. Tarimoradi, and M. H. F. Zarandi, "Fuzzy adaptive genetic algorithm for multi-objective assembly line balancing problems," *Appl. Soft Comput. J.*, 2015, doi: 10.1016/j.asoc.2015.06.001.

[88] M. A. Lubiano, A. Salas, C. Carleos, S. de la Rosa de Sáa, and M. Á. Gil, "Hypothesis testing-based comparative analysis between rating scales for intrinsically imprecise data," *Int. J. Approx. Reason.*, 2017, doi: 10.1016/j.ijar.2017.05.007.

[89] T. J. Ross, *Fuzzy Logic with Engineering Applications: Third Edition*. 2010.

[90] N. Gerami Seresht and A. R. Fayek, "Dynamic Modeling of Multifactor Construction Productivity for Equipment-Intensive Activities," *J. Constr. Eng. Manag.*, 2018, doi: 10.1061/(ASCE)CO.1943-7862.0001549.

[91] A. Nagoor Gani and S. N. Mohamed Assarudeen, "A new operation on triangular fuzzy number for solving fuzzy linear programming problem," *Appl. Math. Sci.*,

2012.

[92] Y. Yang, Z. S. Chen, Y. L. Li, and H. X. Lv, “Commentary on ‘A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems,’” *Appl. Soft Comput. J.*, 2016, doi: 10.1016/j.asoc.2016.08.050.

[93] R. Mesiar, “Shape preserving additions of fuzzy intervals,” *Fuzzy Sets Syst.*, 1997, doi: 10.1016/0165-0114(95)00401-7.

[94] M. Pandey and D. N. Khare, “New Aggregation Operator for Triangular Fuzzy Numbers based on the Geometric Means of the Slopes of the L- and R-Membership Functions,” *Int. J. Comput. Technol.*, 2003, doi: 10.24297/ijct.v2i2b.2634.

[95] J. ying Dong and S. P. Wan, “A new trapezoidal fuzzy linear programming method considering the acceptance degree of fuzzy constraints violated,” *Knowledge-Based Syst.*, vol. 148, pp. 100–114, 2018, doi: 10.1016/j.knosys.2018.02.030.

[96] D. O. Aikhuele and S. Odofin, “A generalized triangular intuitionistic fuzzy geometric averaging operator for decision-making in engineering and management,” *Inf.*, 2017, doi: 10.3390/info8030078.

[97] J. Ye, “Expected value method for intuitionistic trapezoidal fuzzy multicriteria decision-making problems,” *Expert Syst. Appl.*, vol. 38, no. 9, pp. 11730–11734, 2011, doi: 10.1016/j.eswa.2011.03.059.

[98] D. Dubey and A. Mehra, “Linear programming with triangular intuitionistic fuzzy number,” *Proc. 7th Conf. Eur. Soc. Fuzzy Log. Technol. EUSFLAT 2011 French Days Fuzzy Log. Appl. LFA 2011*, vol. 1, no. 1, pp. 563–569, 2011, doi: 10.2991/eusflat.2011.78.

[99] A. Hosseinzadeh and S. A. Edalatpanah, “A new approach for solving fully fuzzy

linear programming by using the lexicography method," *Adv. Fuzzy Syst.*, vol. 2016, 2016, doi: 10.1155/2016/1538496.

[100] B. Bhardwaj and A. Kumar, "A note on 'A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem,'" *Appl. Math. Model.*, 2015, doi: 10.1016/j.apm.2014.07.033.

[101] H. Garg, "A novel approach for analyzing the behavior of industrial systems using weakest t-norm and intuitionistic fuzzy set theory," *ISA Trans.*, 2014, doi: 10.1016/j.isatra.2014.03.014.

[102] H. Garg, M. Rani, S. P. Sharma, and Y. Vishwakarma, "Intuitionistic fuzzy optimization technique for solving multi-objective reliability optimization problems in interval environment," *Expert Syst. Appl.*, vol. 41, no. 7, pp. 3157–3167, 2014, doi: 10.1016/j.eswa.2013.11.014.

[103] A. Ebrahimnejad and M. Tavana, "A novel method for solving linear programming problems with symmetric trapezoidal fuzzy numbers," *Appl. Math. Model.*, vol. 38, no. 17–18, pp. 4388–4395, 2014, doi: 10.1016/j.apm.2014.02.024.

[104] S. H. Nasseri, A. Ebrahimnejad, and B.-Y. Cao, "Fuzzy Linear Programming: Solution Techniques and Applications," vol. 379, pp. 39–61, 2019, doi: 10.1007/978-3-030-17421-7.

[105] A. A. Shaheen, A. R. Fayek, and S. M. Abourizk, "Fuzzy numbers in cost range estimating," *J. Constr. Eng. Manag.*, 2007, doi: 10.1061/(ASCE)0733-9364(2007)133:4(325).

[106] M. Sadeghi Sarcheshmah and A. R. Seifi, "Triangular and trapezoidal fuzzy state estimation with uncertainty on measurements," *Adv. Electr. Electron. Eng.*, 2012, doi: 10.15598/aeee.v10i3.686.

[107] M. Jagadeeswari and V. Lakshmana Gomathi Nayagam, "Approximation of

parabolic fuzzy numbers," 2017, doi: 10.3233/978-1-61499-828-0-107.

[108] P. Grzegorzewski and E. Mrówka, "Trapezoidal approximations of fuzzy numbers," *Fuzzy Sets Syst.*, 2005, doi: 10.1016/j.fss.2004.02.015.

[109] C. T. Chen and H. L. Cheng, "A comprehensive model for selecting information system project under fuzzy environment," *Int. J. Proj. Manag.*, 2009, doi: 10.1016/j.ijproman.2008.04.001.

[110] C. Veeramani and M. Sumathi, "Fuzzy mathematical programming approach for solving fuzzy linear fractional programming problem," *RAIRO - Oper. Res.*, 2014, doi: 10.1051/ro/2013056.

[111] A. Ebrahimnejad and M. Tavana, "A novel method for solving linear programming problems with symmetric trapezoidal fuzzy numbers," *Appl. Math. Model.*, 2014, doi: 10.1016/j.apm.2014.02.024.

[112] D. Reza and J. Davood, "FFLP problem with symmetric trapezoidal fuzzy numbers," *Decis. Sci. Lett.*, vol. 4, no. 2, pp. 117–124, 2015, doi: 10.5267/j.dsl.2015.1.004.

[113] A. M. A. Bertone, R. S. da M. Jafelice, and M. A. da Câmara, "Fuzzy Linear Programming: Optimization of an Electric Circuit Model," *TEMA (São Carlos)*, 2018, doi: 10.5540/tema.2017.018.03.419.

[114] H. Karimi, M. Sadeghi-Dastaki, and M. Javan, "A fully fuzzy best–worst multi attribute decision making method with triangular fuzzy number: A case study of maintenance assessment in the hospitals," *Appl. Soft Comput. J.*, vol. 86, p. 105882, 2020, doi: 10.1016/j.asoc.2019.105882.

[115] M. I. Bolos, I. A. Bradea, and C. Delcea, "Linear programming and fuzzy optimization to substantiate investment decisions in tangible assets," *Entropy*, 2020, doi: 10.3390/e22010121.

[116] R. Bellman, R. Kalaba, and L. Zadeh, "Abstraction and pattern classification,"

J. Math. Anal. Appl., 1966, doi: 10.1016/0022-247X(66)90071-0.

[117] E. H. Ruspini, "A new approach to clustering," *Inf. Control*, 1969, doi: 10.1016/S0019-9958(69)90591-9.

[118] J. C. Dunn, "Well-separated clusters and optimal fuzzy partitions," *J. Cybern.*, 1974, doi: 10.1080/01969727408546059.

[119] S. Tamura, S. Higuchi, and K. Tanaka, "Pattern Classification Based on Fuzzy Relations," *IEEE Trans. Syst. Man Cybern.*, 1971, doi: 10.1109/TSMC.1971.5408605.

[120] J. C. Bezdek and J. Douglas Harris, "Fuzzy partitions and relations; an axiomatic basis for clustering," *Fuzzy Sets Syst.*, 1978, doi: 10.1016/0165-0114(78)90012-X.

[121] E. Trauwaert, L. Kaufman, and P. Rousseeuw, "Fuzzy clustering algorithms based on the maximum likelihood priciple," *Fuzzy Sets Syst.*, 1991, doi: 10.1016/0165-0114(91)90147-I.

[122] W. Peizhuang, "Pattern Recognition with Fuzzy Objective Function Algorithms (James C. Bezdek)," *SIAM Rev.*, 1983, doi: 10.1137/1025116.

[123] R. K. Blashfield, M. S. Aldenderfer, and L. C. Morey, "11 Cluster analysis software," *Handbook of Statistics*. 1982, doi: 10.1016/S0169-7161(82)02014-8.

[124] M. S. Yang, "A survey of fuzzy clustering," *Math. Comput. Model.*, 1993, doi: 10.1016/0895-7177(93)90202-A.

[125] M. S. Yang and Chen-Feng Su, "On parameter estimation for normal mixtures based on fuzzy clustering algorithms," *Fuzzy Sets Syst.*, 1994, doi: 10.1016/0165-0114(94)90270-4.

[126] P. J. F. Groenen and K. Jajuga, "Fuzzy clustering with squared Minkowski distances," *Fuzzy Sets Syst.*, 2001, doi: 10.1016/s0165-0114(98)00403-5.

[127] M. S. Yang and H. M. Shih, "Cluster analysis based on fuzzy relations," *Fuzzy Sets Syst.*, 2001, doi: 10.1016/s0165-0114(99)00146-3.

[128] G. S. Liang, T. Y. Chou, and T. C. Han, "Cluster analysis based on fuzzy equivalence relation," *Eur. J. Oper. Res.*, 2005, doi: 10.1016/j.ejor.2004.03.018.

[129] N. Gustafson, M. S. Pera, and Y. K. Ng, "Generating fuzzy equivalence classes on RSS news articles for retrieving correlated information," 2008, doi: 10.1007/978-3-540-69848-7_20.

[130] P. F. -, X. G. -, and Q. W. -, "Fuzzy Clustering Analysis of Goods Classification Based on Equivalence Relation in Inventory Management," *Int. J. Adv. Comput. Technol.*, 2013, doi: 10.4156/ijact.vol5.issue4.60.

[131] K. M. Bataineh, M. Naji, and M. Saqer, "A comparison study between various fuzzy clustering algorithms," *Jordan J. Mech. Ind. Eng.*, 2011.

[132] W. Rudin, *Principles of Mathematical Analysis 3ed*. 2004.

[133] K. Wu, M. Yang, J. Hsieh, and A. E. M. Algorithm, "Alternative Fuzzy Switching Regression," *Computer (Long. Beach. Calif).*, vol. I, 2009.

[134] P. Kaur, A. K. Soni, and A. Gosain, "Robust kernelized approach to clustering by incorporating new distance measure," *Eng. Appl. Artif. Intell.*, vol. 26, no. 2, pp. 833–847, 2013, doi: 10.1016/j.engappai.2012.07.002.

[135] J. Zhao, W. Tang, and J. Wei, "Pricing decision for substitutable products with retail competition in a fuzzy environment," *Int. J. Prod. Econ.*, vol. 135, no. 1, pp. 144–153, 2012, doi: 10.1016/j.ijpe.2010.12.024.

[136] A. Nagoor Gani and S. N. Mohamed Assarudeen, "A new operation on triangular fuzzy number for solving fuzzy linear programming problem," *Appl. Math. Sci.*, vol. 6, no. 9–12, pp. 525–532, 2012.

[137] P. K. De and D. Das, "Ranking of trapezoidal intuitionistic fuzzy numbers," *Int.*

Conf. Intell. Syst. Des. Appl. ISDA, pp. 184–188, 2012, doi: 10.1109/ISDA.2012.6416534.

[138] Y. Yang, Z. S. Chen, Y. L. Li, and H. X. Lv, "Commentary on 'A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems,'" *Appl. Soft Comput. J.*, vol. 49, pp. 611–615, 2016, doi: 10.1016/j.asoc.2016.08.050.

[139] H. Garg and Ansha, "Arithmetic Operations on Generalized Parabolic Fuzzy Numbers and Its Application," *Proc. Natl. Acad. Sci. India Sect. A - Phys. Sci.*, 2018, doi: 10.1007/s40010-016-0278-9.

[140] M. Pandey, N. Khare, and S. C. Shrivastava, "New aggregation operator for triangular fuzzy numbers based on the arithmetic means of the slopes of the L- and R-membership functions," *Int. J. Comput. Sci. Inf. Technol.*, vol. 3, no. 2, pp. 3775–3777, 2012.

[141] M. R. Seikh, M. Pal, and P. K. Nayak, "Application of triangular intuitionistic fuzzy numbers in bi-matrix games," *Int. J. Pure Appl. Math.*, 2012.

[142] F. Hosseinzadeh Lotfi, T. Allahviranloo, M. Alimardani Jondabeh, and L. Alizadeh, "Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution," *Appl. Math. Model.*, vol. 33, no. 7, pp. 3151–3156, 2009, doi: 10.1016/j.apm.2008.10.020.

[143] B. Sarkar and A. Majumder, "Integrated vendor-buyer supply chain model with vendor's setup cost reduction," *Appl. Math. Comput.*, 2013, doi: 10.1016/j.amc.2013.08.072.

[144] B. Sarkar, A. Majumder, M. Sarkar, B. K. Dey, and G. Roy, "Two-echelon supply chain model with manufacturing quality improvement and setup cost reduction," *J. Ind. Manag. Optim.*, 2017, doi: 10.3934/jimo.2016063.

[145] B. Sarkar, C. Zhang, A. Majumder, M. Sarkar, and Y. W. Seo, "A distribution

free newsvendor model with consignment policy and retailer's royalty reduction," *Int. J. Prod. Res.*, 2018, doi: 10.1080/00207543.2017.1399220.

[146] A. Majumder, C. K. Jaggi, and B. Sarkar, "A multi-retailer supply chain model with backorder and variable production cost," *RAIRO - Oper. Res.*, 2018, doi: 10.1051/ro/2017013.

[147] R. K. Chandrawat, R. Kumar, B. P. Garg, G. Dhiman, and S. Kumar, "An analysis of modeling and optimization production cost through fuzzy linear programming problem with symmetric and right angle triangular fuzzy number," 2017, doi: 10.1007/978-981-10-3322-3_18.

[148] B. Farhadinia, "Sensitivity analysis in interval-valued trapezoidal fuzzy number linear programming problems," *Appl. Math. Model.*, 2014, doi: 10.1016/j.apm.2013.05.033.

[149] S. Saati, M. Tavana, A. Hatami-Marbini, and E. Hajiakhondi, "A fuzzy linear programming model with fuzzy parameters and decision variables," *Int. J. Inf. Decis. Sci.*, vol. 7, no. 4, pp. 312–333, 2015, doi: 10.1504/IJIDS.2015.074129.

[150] a Taleshian and S. Rezvani, "Multiplication Operation on Trapezoidal Fuzzy Numbers," *J. Phys. Sci.*, 2011.

[151] S. Banerjee and T. K. Roy, "Arithmetic Operations on Generalized Trapezoidal Fuzzy Number and its Applications," *TJFS Turkish J. Fuzzy Syst. An Off. J. Turkish Fuzzy Syst. Assoc.*, 2012.

[152] A. Piegat, "A New Definition of the Fuzzy Set," *Int. J. Appl. Math. Comput. Sci.*, 2005.

[153] L. Stefanini, L. Sorini, and M. L. Guerra, "Fuzzy Numbers and Fuzzy Arithmetic," in *Handbook of Granular Computing*, 2008.

[154] H. Garg and Ansha, "Arithmetic Operations on Generalized Parabolic Fuzzy Numbers and Its Application," *Proc. Natl. Acad. Sci. India Sect. A - Phys. Sci.*,

vol. 88, no. 1, pp. 15–26, 2018, doi: 10.1007/s40010-016-0278-9.

[155] G. J. Klir, "Fuzzy arithmetic with requisite constraints," *Fuzzy Sets Syst.*, vol. 91, no. 2, pp. 165–175, 1997, doi: 10.1016/S0165-0114(97)00138-3.

[156] S. H. Jacobson, E. C. Sewell, and J. A. Jokela, "Survey of vaccine distribution and delivery issues in the USA: From pediatrics to pandemics," *Expert Review of Vaccines*. 2007, doi: 10.1586/14760584.6.6.981.

[157] W. Pisano, "Keys to strengthening the supply of routinely recommended vaccines: View from industry," 2006, doi: 10.1086/499588.

[158] B. J. Lunday and M. J. Robbins, "Collaboratively-developed vaccine pricing and stable profit sharing mechanisms," *Omega (United Kingdom)*, 2019, doi: 10.1016/j.omega.2018.04.007.

[159] A. Ahmadi-Javid, P. Seyedi, and S. S. Syam, "A survey of healthcare facility location," *Computers and Operations Research*. 2017, doi: 10.1016/j.cor.2016.05.018.

[160] B. Balcik and B. M. Beamon, "Facility location in humanitarian relief," *Int. J. Logist. Res. Appl.*, 2008, doi: 10.1080/13675560701561789.

[161] C. Boonmee, M. Arimura, and T. Asada, "Facility location optimization model for emergency humanitarian logistics," *International Journal of Disaster Risk Reduction*. 2017, doi: 10.1016/j.ijdrr.2017.01.017.

[162] S. An, N. Cui, Y. Bai, W. Xie, M. Chen, and Y. Ouyang, "Reliable emergency service facility location under facility disruption, en-route congestion and in-facility queuing," *Transp. Res. Part E Logist. Transp. Rev.*, 2015, doi: 10.1016/j.tre.2015.07.006.

[163] S. Y. Roh, H. M. Jang, and C. H. Han, "Warehouse location decision factors in humanitarian relief logistics," *Asian J. Shipp. Logist.*, 2013, doi: 10.1016/j.ajsl.2013.05.006.

[164] S. Gao, Y. Wang, J. Cheng, Y. Inazumi, and Z. Tang, "Ant colony optimization with clustering for solving the dynamic location routing problem," *Appl. Math. Comput.,* 2016, doi: 10.1016/j.amc.2016.03.035.

[165] Y. Wang, X. Ma, Y. Lao, and Z. Li, "Location Optimization of Multiple Distribution Centers Based on Fuzzy Clustering Algorithm," *Transp. Res. Board, 92nd Annu. Meet.,* 2013.

[166] A. Klose, "Using Clustering Methods in Problems of Combined Location and Routing," 1996.

[167] V. Grech and M. Borg, "Influenza vaccination in the COVID-19 era," *Early Hum. Dev.*, 2020, doi: 10.1016/j.earlhumdev.2020.105116.

Research publications/Conference attended

Research Publications:

The contribution of author's accepted publications in this thesis are mentioned below:

1. R. Kumar, R.K. Chandrawat, V. Joshi, *et al.* "An Advanced Optimization Technique for Smart Production Using α-Cut Based Quadrilateral Fuzzy Number". *Int.J.Fuzzy Syst.* **23,** 107–127 (2021). https://doi.org/10.1007/s40815-020-01002-9. **[SCI/SCIE Indexed, Impact Factor - 4.673](Published)**

2. R. Kumar, R.K. Chandrawat, V. Joshi, *et al.* "A novel approach to optimize the production cost of railway coaches of India using situational-based composite triangular and trapezoidal fuzzy LPP models". *Complex Intell. Syst.* (2021). https://doi.org/10.1007/s40747-021-00313-0. **[SCI/SCIE Indexed, Impact Factor - 4.927] (Published)**

3. R. Kumar, G.Dhiman, and V. Joshi "An Improved Exponential Metric Space Approach for C-Mean Clustering Analyzing" Expert Systems **[SCI/SCIE Indexed, Impact Factor - 2.587] (Accepted)**

4. R. Kumar, V. Joshi, R. Jhaveri *et al* "The Combined Study of Improved Fuzzy Optimization Techniques with the Analysis of the Upgraded Facility Location Center for the Covid-19 Vaccine by Fuzzy Clustering Algorithms". *International Journal of Nanotechnology (IJNT)*. **[SCI/SCIE Indexed, Impact Factor - 0.367] (Accepted)**

5. R. Kumar, R.K. Chandrawat and V. Joshi, "Profit Optimization of products at different selling prices with fuzzy linear programming problem using situational based fuzzy triangular numbers". *Journal of Physics: Conference Series.* https://doi.org/10.1088/1742-6596/1531/1/012085. **[Scopus Indexed with SJR-0.21] (Published)**

6. R. Kumar and V. Joshi, "A new approach to optimize the membership grade in fuzzy linear programming problem". *European Journal of Molecular & Clinical Medicine*, 2020, *Volume 7, Issue 7, Pages 3774-3786.* **[Scopus Indexed with SJR-0.21] (Published)**

7. R.K. Chandrawat, R.Kumar, V. Makkar, *et al.* "A Comparative Fuzzy Cluster Analysis of the Binder's Performance Grades Using Fuzzy Equivalence Relation via Different Distance Measures." *Advanced Informatics for Computing Research. ICAICR 2018. Communications in Computer and Information Science, vol 955. Springer, Singapore.* https://doi.org/10.1007/978-981-13-3140-4_11 **[Scopus Indexed with SJR-0.16] (Published)**

8. R. Kumar, and V. Joshi, "Classification of the data through fuzzy clustering based on fuzzy equivalence relation method using Euclidean distance and Mahalanobis distance", *IJRAR - International Journal of Research and Analytical Reviews (IJRAR), E-ISSN 2348-1269, P- ISSN 2349-5138, Volume.5, Issue 4, Page No pp.463-473, November 2018*. Available at: http://www.ijrar.org/IJRAR1BHP087.pdf **(Published)**

9. R. Kumar, R. Sen ,R. K.Chandrawat and V. Joshi, "Estimation of Indian Time Zone via Fuzzy Clustering Approach". *Journal of Gujarat research society.* ISSN: 0374-8588 Volume 21 Issue 6, October2019. Available at:http://www.gujaratresearchsociety.in/index.php/JGRS/article/view/2224**(Published)**

The contribution of author's communicated publications in this thesis are mentioned below:

1. R. Kumar, G.Dhiman, and V. Joshi "An Effective Generalized Exponential Metric Space Approach for C-Mean Clustering Analyzing". *Ambient*

Intelligence & Humanized Computing (AIHC) (Springer), **[SCI/SCIE Indexed, Impact Factor - 7.104] (Revision Submitted to Journal)**

Conferences:

1. An international conference *"Recent Advances in fundamental and Applied Science (RAFAS)"* was attended on 05-06th November 2019 held at Lovely Professional University and a paper titled "Profit Optimization of products at different selling prices with fuzzy linear programming problem using situational based fuzzy triangular numbers**"** was presented (Details of the paper attached).

2. The two-day international e-conference on *"Emerging Issues in Supply Chain Management: Interruption, Opportunities and Challenges"* was attended on 06-07th June 2020 held at Department of Mathematics, Kazi Nazrul University Asansol -713 340, India and a paper titled "Systematic Analysis of Various C-Means Clustering Algorithms Using Different Image Processing Data" was presented (Details of the paper attached).