PROPERTIES AND APPLICATIONS OF RECURRENCE RELATION OF SEQUENCE OF NUMBERS AND POLYNOMIALS

Thesis Submitted for the Award of the Degree of

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Mathematics

By

Mannu Arya

(Registration No. 41800117)

Supervised By

Dr. Vipin Verma

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Declaration

I, Mannu Arya, declare that the thesis entitled "PROPERTIES AND APPLICATIONS OF RECURRENCE RELATION OF SEQUENCE OF NUMBERS AND POLYNOMIALS" is a document of innovative, unique, and unbiased work consummated underneath the benign supervision of Dr. Vipin Verma, Associate Professor in Mathematics Department, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, hereby, submitted in the partial fulfillment for the award of Ph.D. degree.

I, further, declare that the thesis equipped is the original study conducted by me and has not been put forward by any other university.

Signature of Scholar

Certificate

Certified that research embodied in this thesis entitled "PROPERTIES AND APPLICATIONS OF RECURRENCE RELATION OF SEQUENCE OF NUMBERS AND POLYNOMIALS" has been done by Mr. Mannu Arya at Lovely Professional University, Phagwara for the award of Ph.D. degree. The research work has been carried out under my supervision and is to- my satisfaction.

Dated: 29 May 2024

Signature of Supervisor

Abstract

The proposed research work entitled "***PROPERTIES AND APPLICATIONS OF RECURRENCE RELATION OF SEQUENCE OF NUMBERS AND POLYNOMIALS***" is motivated by the recurrence relations of sequence of the numbers and polynomials.

The study of recurrence relations is an important part of Number Theory that has attracted attention from researchers in areas as diverse as physics, economics, and computer science. The Fibonacci sequence, the Luca sequence, the Chebyshev polynomial sequences, and the Pell numbers are all special cases of recurrence relation sequences with specified initial terms that appear in the field of number theory. An equation that describes a sequence in terms of a technique that provides the next term as a relation of the previous terms is called a recurrence relation. Because the next term in a recurrence relation depends on the previous term, they are used in mathematics as well as economics, physics, and other fields and are very helpful in solving real-world problems. Recurrence techniques allow us to compute growth in economics and many other disciplines. The recurrence relation method can be used to handle a wide variety of real-world problems that can be expressed as such. Many issues that arise in the network marketing industry can be addressed with the help of recurrence techniques, as network marketing is a specific case of recurrence relations. For a recurrence relation to yield any term of a sequence, we would first have to locate all of the terms that came before it; however, with the help of the theory presented in this thesis, we can locate any term of the sequence. The first chapter of this thesis provides a general overview of recurrence relations of numbers, which form the basis of history and have many practical uses. Some basic definitions and well-known findings that are required reading for the following chapters are also reviewed.

Chapter 2 discusses the relation between the roots and terms of recurrence relations of the first, second, third, fourth, and kth orders, as well as the results on some special kinds of recurrence relations like Fibonacci polynomials and Chebyshev

polynomials. For kth order recurrence relation, let $c_1,\ c_2, c_3, \dots, c_k$ are real numbers taken be arbitrarily and the equation is supposed

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} \dots - c_k = 0,$$

has $x_1, x_2, x_3 \dots, x_k$ roots which are distinct in nature. Then the sequence $< a_n >$ is a solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots \dots + c_{\mathrm{k}} a_{n-\mathrm{k}}\,, n > \mathrm{k},$$

$$iff$$

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \cdots + \beta_{\mathrm{k}} x_{\mathrm{k}}^n,$$

for $n =\ 0, 1, 2 \dots$, and for arbitrary constants $\beta_1, \beta_2, \beta_3 \dots \beta_{\mathrm{k}}$.

The above result is like a milestone in generalizing the concept of obtaining a recurrence relation for any polynomial.

In Chapter 3, the recurrence relation of the rational function $w_n(x)$ in the form of composition function is defined, having terms as Fibonacci numbers or Generalized Fibonacci numbers, which is defined as

$$w_n(x\,) = \frac{F_{n-1}\,x + F_n}{F_n\,x + F_{n+1}}$$

$$w_n(x) = \frac{1}{q + w_{n-1}(x)}$$

$$w_1(x) = v(x) = \frac{1}{q + x}$$

$$w_n(x) = (vovovo \dots ov)(x),$$

$$v\ o\ v = \frac{1}{1 + \frac{1}{q + x}}$$

$$v(x) = \frac{1}{q+x}$$

then a set H is considered on the basis of the above-described composition, and properties such as closure property, associate property, existence of identity, inverse, and cyclic group are verified.

In Chapter 4, the sequence of tri-diagonal matrices for Generalized Fibonacci polynomials is defined as $\{\mathrm{A(n)} = [g_{n,n}]\}$, $n \in N$:

$$[g_{i,j}] = \begin{Bmatrix} g_{i,j} = ax \quad if \ \ j = i \\ g_{i,j} = -b \ \ if \ j - 1 = i \\ g_{i,j} = 1 \ \ if \ \ j + 1 = i \\ g_{i,j} = 0 \ \ if \ otherwise \end{Bmatrix}.$$

so, that

$$A(n) = \begin{bmatrix} ax & -b & 0 & \cdots & \cdots & 0 \\ 1 & ax & -b & \cdots & \cdots & 0 \\ 0 & 1 & ax & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & ax & -b \\ 0 & 0 & \cdots & \cdots & 1 & ax \end{bmatrix}.$$

Then determinants of $A(n)$are

$$|A(n)\,| = g_{n,n}|A(n-1)\,| - g_{n,n-1}g_{n-1,n}|A(n-2)|.$$

and proving the results that the above matrices are generating matrices for obtaining corresponding recurrence relations. Also, the sequence of tri-diagonal matrices for Fibonacci numbers, Fibonacci polynomials and Chebyshev polynomials is discussed, and the corresponding theorems for obtaining the recurrence relation with the help of generating matrices are proved.

Chapter 5 mainly focuses on the sequences of complex rational functions with coefficients as Fibonacci numbers.

$$w_k(z) = \frac{f_{k-1}\, z + f_k}{f_k\, z + f_{k+1}}$$

Here z be any complex unknown and f_n Fibonacci numbers for non-negative integer n is:

$$f_n = f_{n-1} + f_{n-2},$$

with $f_0 = 0, f_1 = 1$, for $n \geq 2$.

Proving that $w_n(z)$ is a meromorphic function, $w_n(z)$ is bilinear transformations that map a unity circle into a circle center with on the real axis for all values of n, obtaining fixed points for $w_n(z)$ for any z and $w_n(z)$ is conformal functions in unit disc for all integer values of n.

In Chapter 6, the relations between the "Chebyshev polynomial of the second kind" and Hermite polynomials of two variables are discussed. If for Chebyshev polynomials $T_n(x)$, $U_n(x)$ and two-variable Hermite polynomial $H_n(x,y)$ then

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t} t^n H_n(2x, -\frac{1}{t}) dt,$$

$$T_n(x) = \frac{1}{2(n-1)!}\int_0^{\infty} e^{-t} t^n H_n(2x, -\frac{1}{t}) dt,$$

$$\frac{d}{dx} U_n(x) = n W_{n-1}(x),$$

$$U_{n+1}(x) = x W_n(x) - \frac{n}{n+1} W_{n-1}(x),$$

were

$$W_n(x) = \frac{2}{(n+1)!}\int_0^{+\infty} e^{-t} t^{n+1} H_n(2x, -\frac{1}{t}) dt.$$

Also, for the “Chebyshev polynomial of second kind” the generating function is obtained with the help of Hermite polynomial

$$\sum_{n=0}^{+\infty} \xi^n \, U_n(x) = \int_0^{+\infty} e^{-t} \sum_{n=0}^{+\infty} \frac{(t\xi)^n}{n!} H_n\left(2x, -\frac{1}{t}\right) dt.$$

In the last chapter, examples of how recurring relations can be used in NM are presented. In network marketing, people are compensated not only for the work they produce, but also for the work of those working under them. Due to its distributors and compensation structure (which may include numerous tiers), the "down line model" is a specific type of network business model. Some restrictions, like the profit-sharing percentage, are discussed; these restrictions require that the roots of the polynomial for which the recurrence relation is defined to be distinct; no employee can voluntarily leave the company; employees must be truthful; and it is assumed that each employee can only supervise one other employee.

In the later parts of Chapter 7, the application of recurrence relations, especially Fibonacci numbers, and the reproduction mechanism of honey bees are discussed. Reproduction in bees is flawlessly described by Fibonacci numbers. The Fibonacci numbers verify numerous unusual characteristics of a honeybee's family. One of the facts about honey bees is that not every honey bee has two parents. The queen is the only female in a group of honey bees. Most of the working drones are female, but all are not like the queen honey bee; no eggs are produced by them. Some male honey bees do not work; they are called automation bees. Males have only mothers and no fathers because males are created from unfertilized queen eggs. Females are made when the ruler incorporates a mate with a male; hence, a female honey bee has two guardians. Females usually end up as working drones, so the parents of female bees are of both genders, if we study male bees, they have a female bee as a parent. Based on all the above facts, relations between the reproduction mechanisms of bees and Fibonacci numbers are discussed.

Acknowledgment

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Table of Contents

List of Tables

List of Figures

List of Appendices

F_n: n^{th}Fibonacci Number

L_n: n^{th}LucasNumber

$\emptyset$: Golden Ratio

$H_n(x,y)$: n^{th}Hermite Polynomial

$\lfloor . \rfloor$: Greatest Integer

$det(A)$: Determinant of a matrix A

Chapter 1

General Introduction

1.1 Introduction

The study of recurrence relations, which is a central topic in number theory also attracts scholars from a wide variety of disciplines including mathematics, physics, economics, and computer science. There are numerous types of recurrence relation sequences in number theory. Both the Fibonacci and Lucas sequences are examples of recurrence relation sequences with specified initial terms. An equation that defines a sequence in terms of a method that provides the next term as a relation of the previous terms is called a recurrence relation. Mathematics, economics, and physics all benefit from the application of recurrence relations, which are used to solve a wide variety of practical problems. Recurrence methods allow us to estimate economic expansion. Recurrence relations are a useful tool for modeling and solving a wide variety of practical problems. Many issues that arise in network marketing can be addressed with the help of recurrence approaches because of the fact that network marketing is a special kind of recurrence relationship [6, 7]. Finding any term of a sequence in a recurrence relation requires looking up all of the terms before it, but with the theorem presented here, you only need to know the beginning of the sequence. While many significant recurrence relation identities hold only for recurrence relations of order two, the result of this paper holds for recurrence relations of any order. Fibonacci was a pseudonym for the Italian mathematician Leonardo of Pisa (1170-1240), who published The Book of the Abacus in 1202. He pioneered the study of Indian and Arabian mathematics by a European scholar. He introduced the Fibonacci sequence. He also says number theory is just advanced arithmetic. The renowned mathematician Carl Friedrich Gauss (1777-1855) was cited for some wise words on the subject of numbers: "Mathematics is the queen of all sciences, and Number Theory is the queen of Mathematics."

Integers and rational numbers are the focus of number theory, also known as "higher arithmetic," which investigates their properties beyond the reach of traditional arithmetic operations. Analyzing and posing new questions about these mathematical connections is at the heart of number theory [7, 28].

Cube numbers, odd numbers, composite numbers, prime numbers, 3 (modulo 4) numbers, 1 (modulo 4) numbers, perfect numbers, triangular numbers, Fibonacci numbers, etc., are just a few of the many types of natural numbers that have been classified since antiquity. While other types of numbers will be discussed, perfection and compositeness will be the main foci. The following table [5] provides a summary of some of the figures discussed above. Some of the more common natural numbers are listed in Table 1.1, while others are more rare. The mathematical connection between some of them is straightforward to understand, while the connection between others remains murky or has not been adequately explained. However, a few studies have shown that prime and perfect numbers are connected. Composite and perfect numbers will be investigated shortly for obvious and deducible reasons [11].

Table 1.1: Classifications of Natural Numbers.

Number Type	**Samples**
Odd	$1, 3, 5, 7, \ldots$
Cube	$1, 8, 27, \ldots.$
Prime	$2, 3, 5, \ldots$
Composite	$4, 6, 8, \ldots.$
1 (Modulo 4)	$1, 5, 9, \ldots.$
3 (Modulo 4)	$3, 7, 11, \ldots$

1.2 Basic Definitions

1.2.1 Recurrence Relation

A recurrence relation is a relation that characterizes arrangement and depends on a rule such that it generates the next term as a function of previous terms. When the next term is based only on the immediate previous term, it is the simplest case of a recurrence relation, which is of first order. If the sequence term is a non-negative integer, the first order recurrence relation [1, 2] is

$$X_{n+1} = f(X_n). \tag{1.1}$$

It is also possible to have a recurrence relation of higher order, where the term X_{n+1}depends on more than one previous term, such as X_n, X_{n-1}, $X_{n-2}, \ldots$. A 2nd order recurrence relation [2, 3] depends on two previous terms X_n and $X_{n-1}$and for integers $n \geq 1$ is

$$\mathrm{X}_{n+1} = f(X_n, X_{n-1}). \tag{1.2}$$

Also, to generate a sequence based on a recurrence relation (1.2), one needs to provide two inputs to the function f as a first step. Starting with an initial value X_0, the recurrence relation can generate all subsequent terms for a first order recursion, $X_{n+1} = f(X_n)$. Second-order recursion requires two initial values, X_0 and X_1, as in $\mathrm{X}_{n+1} = f(X_n, X_{n-1})$. More initial values are needed for recurrence relations of higher order. For example

$$a_{n+1} = a_n + 5, n \geq 1, \tag{1.3}$$

is a first order recurrence relation with initial terms $a_0 = 1$, we can find the terms

$$a_1 = 6, a_2 = 11, a_3 = 16, \ldots.$$

$$a_n = a_{n-1} + 2a_{n-2}, n \geq 2, \tag{1.4}$$

is a recurrence relation of second order with initial terms $a_0 = 0, a_1 = 1$.

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-2}, n \geq 3, \tag{1.5}$$

is a recurrence relation of third order with initial terms $a_0 = 0, a_1 = 1, a_2 = 2$.

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-2}a_{n-3}, n \geq 4, \tag{1.6}$$

is a fourth order recurrence relation with $a_0 = 0, a_1 = 1, a_2 = 2,\ a_3 = 3$.

1.2.2 Fibonacci numbers

Leonardo of Pisa, an Italian mathematician also known by the name Fibonacci, wrote the Book of the Math Device Called an Abacus in 1202. To his credit, he was the first European mathematician to study the scientific traditions of India and the Middle East. He gave the sequence of special types given by

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 2, \tag{1.7}$$

with initial terms $f_0 = 0$ and $f_1 = 1$.

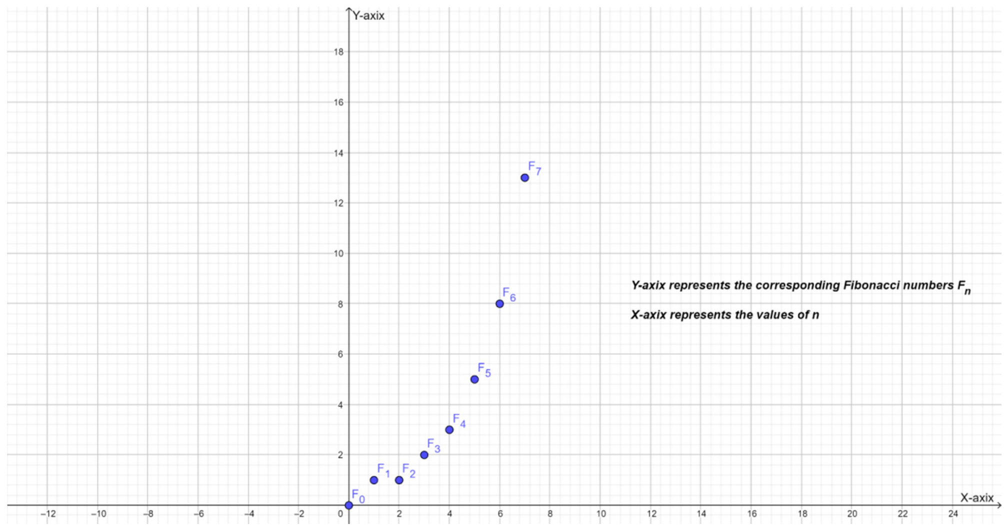

Figure 1.1: Fibonacci Numbers.

1.2.1 "Division in the Mean and Extreme Ratio (DEMR)"

"The Elements" of Euclid is one of the most famous scientific works of old science [26]. Contains the most hypotheses of old arithmetic: rudimentary geometry number hypothesis, variable based math, the hypothesis of extents and proportions, the strategy of calculation of zones and volumes etc., systematized a 300-year period of improvement in Greek arithmetic, and this work made a solid base for the assist improvement of arithmetic. The Components of Euclid surpassed all the work of his forerunner within the field of geometry for more than two millennium; "The Elements" remained the essential work for the instructing of rudimentary science. The 13 books of "The Elements" are devoted to the information of geometry and number juggling in "Euclidean space."

From "The Elements of Euclid," the taking after geometrical issue, which was named the issue of "Division in Extreme and Mean Ratio (DEMR)", was called This issue was defined in Book II of "The Elements" as takes: To bipartition line AB, a bigger AC and a littler CB, so that

$$R\ (AB, CB) = S\ (AC). \tag{1.8}$$

where $S\ (AC)$ is square area with one side AC and $R\ (AB, CB)$ is rectangle area with sides AB and CB.

so, (1.8) takes the following form:

$$(AC)^2 = AB \times CB. \tag{1.9}$$

Dividing (1.9) by AC and then by CB we get.

$$\frac{AB}{AC} = \frac{AC}{CB}. \tag{1.10}$$

"This form is well known in mathematics as the ***Golden Section.***"

We are able decipher (1.10) geometrically by partitioning a line AB at the point C in bi-sections, a bigger one AC and a littler one CB, so that the proportion of portion AC to the portion CB is equal to the proportion of AB to the AC [26].

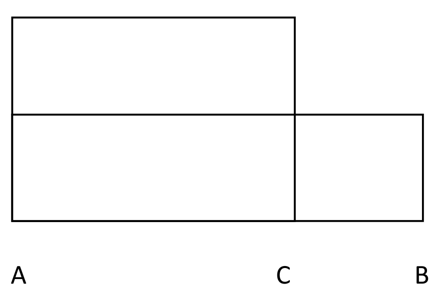

Figure 1.2: A geometrical interpretation ("The Elements of Euclid").

Denote proportion (1.10) by x, then

$$x = \frac{AB}{AC} = \frac{AC+CB}{AC} = 1 + \frac{CB}{AC} = 1 + \frac{1}{\frac{AC}{CB}} = 1 + \frac{1}{x},$$

we obtain

$$x^2 = x + 1. \tag{1.11}$$

It follows from the "geometrical meaning" of the proportion that the required solution of (1.11) has to be a positive number, it also follows that a positive root of equation is a solution of the problem. By denoting this root by φ we obtained

$$\varphi = \frac{1+\sqrt{5}}{2}. \tag{1.12}$$

"This number is called the ***Golden Proportion**, **Golden Mean***, ***Golden Number*** or ***Golden Ratio"***.

1.2.2 "Golden Mean" Remarkable Identities

"This number is called the ***Golden Proportion**, **Golden Mean***, ***Golden Number*** or ***Golden Ratio"***.

The Golden Mean is the "miracle" of nature. On the off chance that we center all of our scientific information and dive into this interesting information about science, at that point there's plausibility of getting a charge out of and understanding the superb scientific properties and excellence of this one-of-a-kind wonder – "***The Golden Mean"***.

Stakhov and Rozin[75, 76], talked in their paper about exceptionally straight-forward property of the "golden mean". On the off chance that we substitute the root φ ("The Golden mean") for x in (1.11) at that point we are going to get the following surprising identities for the golden mean:

$$\varphi^2 = \varphi + 1. \tag{1.13}$$

To prove the validity of the identity (1.13), it is fundamental to carryout basic numerical changes over LHS and RHS portion of (1.13) and indicate a match [83, 87, and 88].

RHS

$$\varphi + 1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2},$$

LHS

$$\varphi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{2} = \frac{3+\sqrt{5}}{2}.$$

So the identity (1.13) is verified.

On dividing (1.13) by φ

$$\varphi = 1 + \frac{1}{\varphi}, \tag{1.14}$$

$$\varphi - 1 = \frac{1}{\varphi}. \tag{1.15}$$

Consider the golden mean eq. (1.12)

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

This is an irrational number. Consider inverse of φ and solve

$$\frac{1}{\varphi} = \frac{2}{1+\sqrt{5}} = \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \frac{2(1-\sqrt{5})}{(1)^2-(\sqrt{5})^2} = \frac{\sqrt{5}-1}{2}.$$

Also, from the equation (1.15) the inverse number can be found in taking after way:

$$\frac{1}{\varphi} = \varphi - 1 = \frac{1+\sqrt{5}}{2} - 1 = \frac{\sqrt{5}-1}{2}.$$

Now we transform identity (1.13) by multiplying both part of the identity (1.13) by φ and then dividing by φ^2 we will get following identities:

$$\varphi^3 = \varphi^2 + \varphi, \tag{1.16}$$

and

$$\varphi = 1 + \varphi^{-1}. \tag{1.17}$$

Continuing in this way, we have

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2}, \tag{1.18}$$

Where n is an integer of the set $\{0, \pm 1, \pm 2, \pm 3, \dots\}$.

For identity (1.16) we may say that “Any number of the golden series (golden power) is the sum of previous two golden power”.

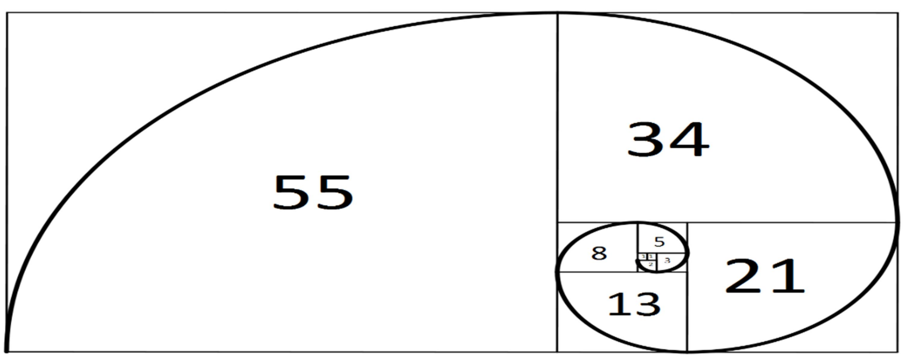

Figure 1.3: Fibonacci Spiral.

1.2.4 Lucas numbers

Between 1878 and 1891, Edouard Lucas dominated the discipline of recursive mathematics. He was the first mathematician to tie the Fibonacci number (1.8) to the arrangement, giving rise to the term "Fibonacci sequence." Lucas provides a sequence

$$L_n = L_{n-1} + L_{n-2}, n \geq 2, \tag{1.19}$$

with initial terms $L_0 = 2$ and $L_1 = 1$, Lucas number are the terms of the Lucas sequence.

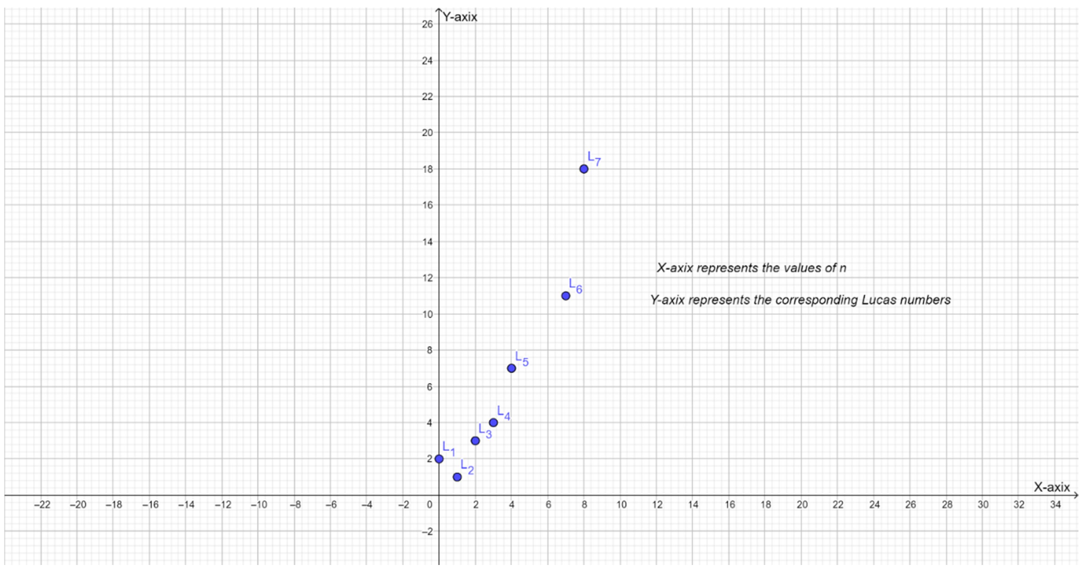

Figure 1.4: Lucas Numbers.

1.2.5 A Sequence generated by Fibonacci Numbers

If sequence R_n is

$$R_n = f_{n-1}f_{n+1} - {f_n}^2, \tag{1.20}$$

with f_n is define by equation (1.7), then by equation (1.7) and (1.12) we have

$$R_n = (-1)^n.$$

1.2.6 Generalized Fibonacci number sequence.

It is defined as

$$F_n = aF_{n-1} + bF_{n-2}, n \geq 2, F_0 = p, F_1 = q, \tag{1.21}$$

with positive integers p, q, a & b.

Consider a particular case of equation (1.21)

$$V_n = aV_{n-1} + bV_{n-2}, k \geq 2, V_0 = 0, V_1 = 1, \tag{1.22}$$

for positive integers$a$ & b.

1.2.7 Fibonacci sequences of polynomials

E.C. Catalan [56, 62, and 64] studied the Fibonacci polynomial, define by

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \tag{1.23}$$

for $n = 0,1,2,\ldots$ and with initial terms

$$F_0(x) = 0,$$

$$F_1(x) = 1.$$

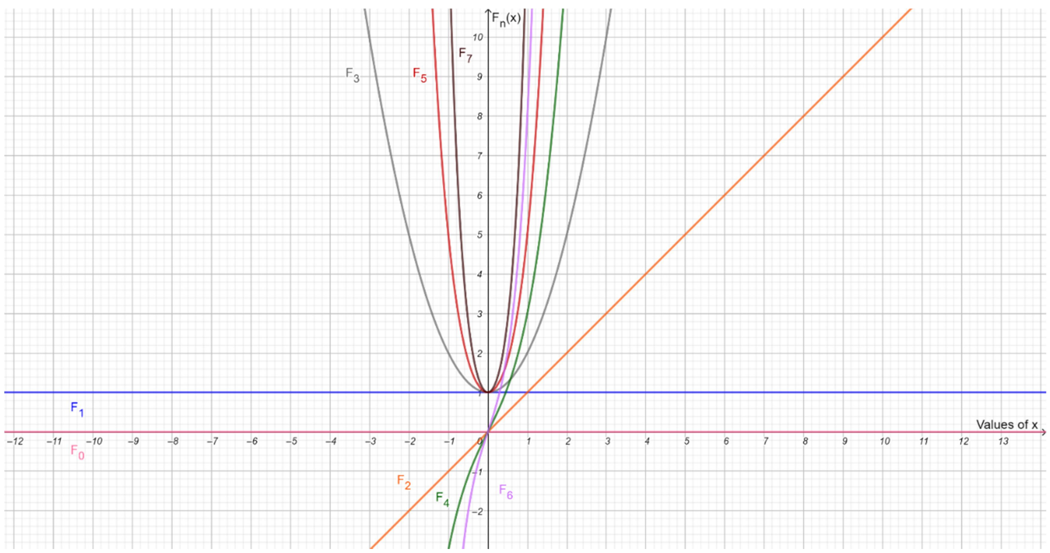

Figure 1.5: Fibonacci Polynomials.

1.2.8 Generalized Fibonacci polynomials sequences.

$G_n(x)$ Generalized Fibonacci polynomial is given by

$$G_{n+2}(x) = axG_{n+1}(x) + bG_n(x), \tag{1.24}$$

where $n = 0,1,2,\ldots$ and with initial terms $G_0(x) = 0$, $G_1(x) = 1$.

1.2.9 Sequence of tri-diagonal matrices for Generalized Fibonacci sequences of polynomials

For $n \in N$sequence of tri-diagonal matrices [26, 31, 32] $\{\mathrm{A(n)} = [g_{n,n}]\}$, is such that

$$[g_{i,j}] = \begin{Bmatrix} g_{i,j} = ax & if \;\; j = i \\ g_{i,j} = -b & if \; j-1 = i \\ g_{i,j} = 1 & if \;\; j+1 = i \\ g_{i,j} = 0 & otherwise \end{Bmatrix}, \tag{1.25}$$

so, that

$$A(n) = \begin{bmatrix} ax & -b & 0 & \cdots & \cdots & 0 \\ 1 & ax & -b & \cdots & \cdots & 0 \\ 0 & 1 & ax & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & ax & -b \\ 0 & 0 & \cdots & \cdots & 1 & ax \end{bmatrix}.$$

Then determinants of $A(n)$ is

$$|A(n)| = g_{n,n}|A(n-1)| - g_{n,n-1}g_{n-1,n}|A(n-2)|. \tag{1.26}$$

1.2.10 Sequence of tri-diagonal matrices for Fibonacci sequence of polynomial

Consider a sequence of matrices defined by (1.15) by putting, $a = 1, b = 1$, we have sequence of matrices $\{\boldsymbol{C(n)} = [h_{n,n}]\}$ [60]

$$[h_{i,j}] = \begin{Bmatrix} h_{i,j} = x & if \;\; j = i \\ h_{i,j} = -1 & if \; j-1 = i \\ h_{i,j} = 1 & if \;\; j+1 = i \\ h_{i,j} = 0 & otherwise \end{Bmatrix}. \tag{1.27}$$

$$\boldsymbol{C(n)} = \begin{bmatrix} x & -1 & 0 & \cdots & \cdots & 0 \\ 1 & x & -1 & \cdots & \cdots & 0 \\ 0 & 1 & x & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x & -1 \\ 0 & 0 & \cdots & \cdots & 1 & x \end{bmatrix}.$$

Then determinant of $\boldsymbol{C(n)}$ is

$$|C(n)| = h_{n,n}|C(n-1)| - h_{n,n-1}h_{n-1,n}|C(n-2)|. \tag{1.28}$$

1.2.11 Sequence of tri-diagonal matrices for particular case of generalized Fibonacci numbers sequence

For $n \in N$ we define a sequence of tri-diagonal matrices $\{D(n) = [q_{n,n}]\}$

$$[q_{i,j}] = \begin{Bmatrix} q_{i,j} = a & if \;\; j = i \\ q_{i,j} = -b & if \; j - 1 = i \\ q_{i,j} = 1 & if \;\; j + 1 = i \\ q_{i,j} = 0 & otherwise \end{Bmatrix}. \tag{1.29}$$

so, that

$$D(n) = \begin{bmatrix} a & -b & 0 & \cdots & \cdots & 0 \\ 1 & a & -b & \cdots & \cdots & 0 \\ 0 & 1 & a & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & a & -b \\ 0 & 0 & \cdots & \cdots & 1 & a \end{bmatrix}.$$

Then determinants of $D(n)$ is

$$|D(n)| = q_{n,n}|D(n-1)| - q_{n,n-1}q_{n-1,n}|D(n-2)|. \tag{1.30}$$

1.2.12 Chebyshev polynomials

For integers $n \geq 0$, “Chebyshev polynomials of the first kind $\{T_n(x)\}$and the second kind $\{U_n(x)\}$” are

$$T_{n+2}(x) = 2xT_n(x) - T_n(x), T_1(x) = x, T_0(x) = 1. \tag{1.31}$$

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x), U_0(x) = 1, U_1(x) = 2x. \tag{1.32}$$

The way of presenting of (1.31) and (1.32) explicitly as

$$T_n(x) = \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k \frac{(n-k-1)!}{k!\,(n-2k)!}(2x)^{n-2k}, |x| < 1. \tag{1.33}$$

and

$$U_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k \frac{(n-k-1)!}{k!\,(n-2k)!}(2x)^{n-2k}, |x| < 1. \tag{1.34}$$

On taking $x = \cos\gamma$, then

$$T_n(\cos\gamma) = \cos(n\gamma). \tag{1.35}$$

$$U_n(\cos\gamma) = \frac{\sin(n+1)\gamma}{\sin\gamma}. \tag{1.36}$$

1.2.13 Sequence of tri-diagonal matrices for Chebyshev polynomial of first kind

For integer $n \geq 0$, sequence of tri-diagonal matrix for "Chebyshev polynomial of the first kind" as $\{S(n) = [l_{i,j}]\}$ is

$$[l_{i,j}] = \begin{Bmatrix} l_{i,j} = 2x & if\ \ j = i \\ l_{i,j} = 1 & if\ j-1 = i \\ l_{i,j} = 1 & if\ \ j+1 = i \\ l_{i,j} = 0 & otherwise \end{Bmatrix}, \tag{1.37}$$

$$S(n) = \begin{bmatrix} 2x & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2x & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 2x & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 2x & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 2x \end{bmatrix}.$$

Then determinant of $S(n)$is

$$|S(n)| = l_{n,n}|S(n-1)| - l_{n,n-1}l_{n-1,n}|S(n-2)|. \tag{1.38}$$

1.2.14 Group

A group in modern algebra is a set that has the following properties with the given operation:

Closure: Let **H** be any set and $*$ be any operation on **H** if $a * b \in \mathbf{H}, \forall\, a, b \in \mathbf{H}$.

Associative: If $(a * b) * c = a * (b * c), \forall\, a, b, c \in \mathbf{H}$ then called **H** satisfied the associative property.

Existence of Identity: If there exist an element e in **H** such that $a * e = e * a = a$, $\forall\, a \in \mathbf{H}$where e is identity element.

Existence of Inverse: $\forall\, a \in \mathbf{H}, \exists\, b \in \mathbf{H}$, with condition $a * b = b * a = e$ then b is inverse element.

1.2.15Cyclic group

A group in which every element can be created by single a component of the group is cyclic group. For example, a set of integers with respect to addition is a cyclic group.

1.2.16 Rational function

If $f(x)$ is given by

$$f(x) = \frac{p(x)}{q(x)},$$

with polynomials $p(x)$and $q(x)$, $q(x) \neq 0$, then $f(x)$ is rational function.

1.2.17 Bilinear Transformation

A complex mapping

$$w(z) = \frac{az + b}{cz + d}, \tag{1.39}$$

where $ad - bc \neq 0$, then$w(z)$is "bilinear transformation mapping, a complex bilinear transformation maps a circle or line into circle or line".

1.2.18 Meromorphic functions

A complex variable a function which has no singularities other than poles called meromorphic. So we can say that a complex function is meromorphic possible singularities are only poles.

1.2.19 Conformal Mapping in Complex

A mapping that preserves the sense of rotation as well as the magnitude of Angle between images of curves, and there is a well-known result that a mapping is Conformal if it is differentiable and derivatives are non-zero.

1.2.20 Complex polynomials

A complex polynomial is one that can have constants and signs referred to as variables indeterminate to a non-negative integer power. For those terms that can be modified, one to another, if the normal characteristics of commutativity are used, the distribution with addition and multiplication distributive is considered as defining same polynomial. A complex polynomial within one indeterminate z may always have to be generated in the way

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

where a_0 , a_1 , ..., a_n are constants and z is the indeterminate. The word "intermediate" does not mean z represents is any unique value; but that any value will have to be replaced by any other value. The characterization that marks the product of this replacement for the substituted value is a feature called the complex polynomial function [4, 5].

$$\sum_{k=0}^{n} a_k z^k.$$

That is, there can be either zero polynomials or polynomials defined as the sum of non-zero amounts. Each is the product of a numerical coefficient and several indeterminate conditions multiplied by non-negative integer powers.

1.2.21 Sequence of complex rational functions

Let$z$ be any complex unknown and $u(z)$be any function of z given by

$$u(z) = \frac{1}{1+z}. \tag{1.40}$$

Then we have

$$(uou)(x) = \frac{1}{1+\frac{1}{1+z}}, \tag{1.41}$$

now, we define

$$w_n(x) = (uououo \ldots ou)(x), \tag{1.42}$$

Where $(uououo \ldots ou)$ represents n time composition.

1.2.22 Recurrence relation sequence of rational function

Recurrence relation rational function sequence defined as

$$w_1(z) = u(z) = \frac{1}{1+z}, \tag{1.43}$$

and

$$w_n(z) = \frac{1}{1+w_{n-1}(z)}, \tag{1.44}$$

for all integers $n \geq 2$.

1.2 Literature Review

The study of recurrence relations has attracted the attention of numerous scholars. In [22], [25], and [26], various summation formulae of the "Generalized Fibonacci and Gaussian Fibonacci numbers" and "Pell and Pell-Lucas numbers" are developed. The "Fibonacci," "Tribonacci," "Tetrabasic," "Pentanacci," and "Hexanacci" numbers all share similar characteristics, as described in [9, 27, 28], [29, 30], [31, 32, 33], [34, 35], and [36]. According to the Georgian Mathematical Journal, in paper [43] the author explains how Hermite polynomials have some interesting properties. An alternative generalisation is sought after in this paper

$$M_{n+1}(x) = k(x)M_n(x) + M_{n-1}(x), M_0(x) = 2, M_1(x) = m(x) + k(x)$$

for integers $n \geq 2$ and real polynomials$k(x)$ and $m(x)$. Using matrix algebra, the author generates an expanded Binet's formula for $M_n(x)$ and, as a result, identities such as Simpson's, Catalan's, and so on. In addition, they obtained sum formulas for this new generalization**.**

In [77] Nalliand Haukkanen introduced $h(x)$ −polynomials introduced a matrix whose power generates the sequence of the Fibonacci numbers.

In [78], the author presented the various summation formulae for generalized Fibonacci numbers defined as

$$W_n = rW_{n-1} + sW_{n-2}, W_0 = a, W_1 = b, \text{ for } n = 2, 3, 4, \ldots.$$

Similar work has been done for different sequences [38].

In [79], the author studied various properties of 2-Fibonacci sequences defined by

$$\alpha_{n+2} = \alpha_{n+1} + \beta_n, \text{ and } \beta_{n+2} = \beta_{n+1} + \alpha_n$$

with $\alpha_0 = a,\ \beta_0 = b, \alpha_1 = c,\ \beta_1 = d,$ and $n = 0, 1, 2, \ldots$

In [80, 81], the author introduced new schemes of 2-Fibonacci sequences defined as

$$\alpha_{n_1+2} = \frac{\alpha_{n_1+1}+\beta_{n_1+1}}{2} + \beta_{n_1}, \beta_{n_1+2} = \frac{\beta_{n_1+1}+\alpha_{n_1+1}}{2} + \alpha_{n_1},$$

and

$$\alpha_{n_1+2} = \frac{\alpha_{n_1}+\beta_{n_1}}{2} + \beta_{n_1+1}, \beta_{n_1+2} = \alpha_{n_1} + \frac{\beta_{n_1}+\alpha_{n_1}}{2},$$

with $n_1 = 0,1,2, \ldots,$ and $\alpha_0 = 2a,\ \beta_0 = 2b,\ \alpha_1 = 2c,\ \beta_1 = 2d,$where$a,\ b, c$ and d are real numbers, and he established various relationships between these sequences with a generalized Fibonacci sequence defined by

$$F_{n+2}(d_\delta, c_\gamma) = F_{n+1}(d_\delta, c_\gamma) + F_n(d_\delta, c_\gamma)$$

with

$$F_0(d_\delta, c_\gamma) = d_\delta, F_1(d_\delta, c_\gamma) = c_\gamma, n = 0, 1, 2, \ldots$$

Also, for these sequences listed various properties by an integer function σ described by

$$\sigma(j + 2) + \sigma(j) = 0; j = 0, 1, 2, \ldots,$$

with$\sigma(0) = 0$, and $\sigma(1) = 1$.

Similar work has been done by the authors [42, 43] for different schemes of sequences.

In [82], the author derived the following formulae

$$\mathcal{U}_\mathfrak{y} = \frac{\alpha_q^{\mathfrak{y}+1}}{(\alpha_q - \beta_q)(\alpha_q - \gamma_q)} + \frac{\beta_q^{\mathfrak{y}+1}}{(\beta_q - \alpha_q)(\beta_q - \gamma_q)} + \frac{\gamma_q^{\mathfrak{y}+1}}{(\gamma_q - \beta_q)(\gamma_q - \alpha_q)},$$

and

$$\mathcal{V}_{\mathfrak{y}} = \alpha_q^{\mathfrak{y}+1} + \beta_q^{\mathfrak{y}+1} + \gamma_q^{\mathfrak{y}+1},$$

were

$$\alpha_q = \frac{1 + \left(\sqrt[3]{19 + 3\sqrt{33}}\right) + \left(\sqrt[3]{19 - 3\sqrt{33}}\right)}{3},$$

$$\beta_q = \frac{1 + \omega\left(\sqrt[3]{19 + 3\sqrt{33}}\right) + \omega^2\left(\sqrt[3]{19 - 3\sqrt{33}}\right)}{3},$$

$$\gamma_q = \frac{1 + \omega^2\left(\sqrt[3]{19 + 3\sqrt{33}}\right) + \omega\left(\sqrt[3]{19 - 3\sqrt{33}}\right)}{3},$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$, for Tribonacci sequence $\{\mathcal{U}_{\mathfrak{y}}\}_{\mathfrak{y}\geq 0}$ and Tribonacci-Lucas $\{\mathcal{V}_{\mathfrak{y}}\}_{\mathfrak{y}\geq 0}$ sequence described as

$$\mathcal{U}_{\mathfrak{y}+3} = \mathcal{U}_{\mathfrak{y}+2} + \mathcal{U}_{\mathfrak{y}+1} + \mathcal{U}_{\mathfrak{y}};\ \mathcal{V}_{\mathfrak{y}+3} = \mathcal{V}_{\mathfrak{y}+2} + \mathcal{V}_{\mathfrak{y}+1} + \mathcal{V}_{\mathfrak{y}};\ \mathfrak{y} = 0, 1, 2, \ldots,$$

with$\mathcal{U}_0 = 0, \mathcal{U}_1 = 1, \mathcal{U}_2 = 1, \mathcal{V}_0 = 3, \mathcal{V}_1 = 1$, and $\mathcal{V}_2 = 3$.

Similarly, authors in [83] determined the Binet's formula by use of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

For Pentanacci sequence given by

$$\mathcal{C}_{\jmath+5} = \mathcal{C}_{\jmath+4} + \mathcal{C}_{\jmath+3} + \mathcal{C}_{\jmath+2} + \mathcal{C}_{\jmath+1} + \mathcal{C}_{\jmath}\ ; \jmath = 0, 1, 2, \ldots,$$

Fibonacci numbers and linear algebra also have a lot of connections. Many researchers have worked in this area.

In 2006, author [84] made use of the sum property of the determinant, which states that "If A, B, and C are matrices with indistinguishable elements except that one row (column) of C, say kth, is the sum of kth rows (columns) of A and B, then [A] + [B] = [C]." ...and the author validated the next property of the Fibonacci numbers by making use of the determinant's characteristic of being determinant:

$$F_{\mathrm{m}}F_n - F_{\mathrm{m}-r}F_{n+r} = (-1)^{\mathrm{m}-r}F_rF_{n+r-\mathrm{m}}$$

In [42], the author discussed how dual Bernstein polynomials bring additional differential-recurrence properties because of relationships between Jacobi polynomials and orthogonal Hahnand dual Bernstein. A fourth-order differential condition fulfilled by double Bernstein polynomials has been developed utilizing this concept. In addition, for these polynomials, a recurrence relation of fourth-order has been generated; this result can efficiently solve certain problems of computation.

In papers [45] it is seen that, based on the Schur parameters, the characteristic polynomials of some five-diagonal matrices are monic orthogonal polynomials of the unit circle. This is the result on the unit circle generated by the orthogonal Laurent polynomial, which is the result of the orthogonal Laurent polynomial's recurrence relation of five terms, as well as the one-to-one and onto mapping formed between them.

In her paper [44], the author considers a sequence of polynomials $\{P_n\}, n \geq 0$ that satisfy a special recurrence relation and have simple zeros on the real line. Eigen value problem generalized by P_n turned out to be 'the characteristic polynomial" of a simple $n \times n$, for integer $n \geq 2$. It is shown that measure (positive) on the unit circle can always be related to this recurrence relation. The property of orthogonality with respect to this calculation can also be obtained.

In Paper [54], the author studies the relationship between recurrence relations and significant statistical applications. However, only discrete distributions were covered by the initial derivation. There is a contemporary application.

In 2011, Jishe Feng [85] utilized the technique of Laplace expansions to evaluate the determinant of D_n and constructed a type of 2×2 matrix determinant to approach a new method to substantiate the following identity:

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n ,$$

were

$$D_n = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 1 & -1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

In 2017, Sümeyra [86] studied some new properties of “Generalized Fibonacci and Lucas polynomials” by using Laplace expansion of determinants and also described some new families of tri-diagonal matrices given by

$$C(n) = \begin{pmatrix} p(t_1) & i\sqrt{g(t_1)} & 0 & \dots & 0 \\ i\sqrt{g(t_1)} & p(t_1) & i\sqrt{g(t_1)} & \dots & 0 \\ 0 & i\sqrt{g(t_1)} & p(t_1) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & i\sqrt{g(t_1)} \\ 0 & 0 & 0 & i\sqrt{g(t_1)} & p(t_1) \end{pmatrix},$$

and its successive determinants generate the following sequence:

$$L_{p,g,n+1}(t_1) = p(t_1)L_{p,g,n}(t_1) + g(t_1)L_{p,g,n-1}(t_1),$$

and

$$F_{p,g,n+1}(t_1) = p(t_1)F_{p,g,n}(t_1) + g(t_1)F_{p,g,n-1}(t_1),$$

with $n = 1, 2, 3, \dots, F_{\wp,g,0}(t_1) = 0,\ F_{\wp,g,1}(t_1) = 1, L_{\wp,g,0}(t_1) = 2,$ and $L_{\wp,g,1}(t_1) = \wp(t_1)$.

In 2023, Jinseo Park [87] discusses properties and many special type identities of positive integers, which are known as Diophantine m-tuples. He related these special-type numbers to Fibonacci numbers and discussed the properties of geometry in relation to groups.

In 2023, Seán M. Stewart [88] discusses properties and many special-type identities of positive integers. In this paper, the author also related these special type numbers to Fibonacci numbers, discussed the properties of geometry, and related them to groups. Also discussed are the many special-type results of Fibonacci numbers and some very interesting new results on the Fibonacci recurrence relation sequence of numbers.

1.4 Proposed objectives of the research work

During our research work it proposed to study the following problems.

- To obtain new generalizations and extensions of the Fibonacci sequence of numbers and polynomials.
- To obtain new identities and some special representations of the Chebyshev polynomials.
- To study of relation between the group theory and the terms of recurrence relation of sequence of numbers and polynomials.
- To study the applications of recurrence relation in network marketing and in some other fields.

1.5 Proposed methodology of the research work

To achieve the proposed objective, the following methodology was used:

1. By using concepts of algebra, number theory focuses on solving polynomials, obtaining roots of polynomials and using concepts of

determinants, we have obtained the relation between polynomial roots and recurrence relations terms.

2. Concepts of group theory are used to prove theorems of cyclic groups of rational function with coefficients as Fibonacci numbers.
3. By using some properties of matrices and determinants, work will be done on finding generating matrices of recurrence relations as a sequence of tri-diagonal matrices.
4. Theorems on the sequence of complex bilinear transformations are proved using the concepts of complex algebra.
5. To prove properties and identities, we will use the methods of mathematical induction, power series, and the concepts of calculus.

1.6 Structure of Thesis

The proposed research work, entitled "***PROPERTIES AND APPLICATIONS OF RECURRENCE RELATION OF SEQUENCE OF NUMBERS AND POLYNOMIALS***" is motivated by the recurrence relations of the sequence of numbers and polynomials. The thesis is structured in the form of seven chapters in the following manner:

In the first chapter of this thesis, it provides an overview of the recurrence relations of numbers that make up history as well as their applications in a variety of different disciplines. In addition, we make a cursory review of a few key definitions and well-known results that are required to meet the bare minimum standard for the forthcoming chapters. This chapter also contains the part of the literature review that sheds light on the work done in the field of the recurrence relations of numbers and associated polynomials by a number of different researchers. This part of the review is included here. In the evaluation, the research void has been singled out, and the goals and procedures to fill in these voids have been outlined in detail.

Chapter 2, "*Relation between the Roots of Polynomials and the Term of Recurrence Relation Sequence,*" is divided into seven sections, discussing the relation between the roots and terms of "recurrence relations of first order, second order, third order, fourth order, and kth order.” Also discussed are the results on some special kinds of recurrence relations like Fibonacci polynomials and Chebyshev polynomials.

Chapter 3, “*Cyclic Group of Rational Functions with Coefficients as Fibonacci Numbers,* “is divided into four sections, starting with the basic definitions of group, Fibonacci numbers, Generalized Fibonacci sequence, and the recurrence relation of rational function in the form of composition function, which have the terms Fibonacci numbers or Generalized Fibonacci numbers. Finally, a set is considered on the basis of its defined composition, and the properties of the group are verified.

In Chapter 4, “*Generating Matrices of Recurrence Relations as Sequence of Tri-Diagonal Matrices,*” we worked on the tri-diagonal matrix sequence for generalized Fibonacci polynomials, Fibonacci numbers, and Chebyshev polynomials.

In Chapter 5, “*Sequence of Complex Bilinear Transformations with Coefficients as Fibonacci Numbers,*” the main focus is on $w_n(z)$ the sequences of complex ration functions with coefficients as Fibonacci numbers, verifying the properties of bilinear transformations for $w_n(z)$.

In Chapter 6, “Relations Between Chebyshev Polynomials and Hermite Polynomials,” we have discussed the relation between the “Chebyshev polynomial of the second kind” and Hermite polynomials of two variables; also, the generating function is obtained with the help of Hermite polynomials.

Chapter 7, “Applications of Recurrence Relations,” deals with applications of recurrence relations in network marketing with some limitations imposed on the problem. In the later parts of the chapter, the application of recurrence relations, especially Fibonacci numbers, and the reproduction mechanism of honey bees are discussed.

Chapter 2

Relation between the Roots of Polynomials and the Terms of Recurrence Relation Sequence

2.1 Introduction

We have given the identities and recurrence relations of first, second, third, fourth, and forkth order in this chapter. These are exceptionally valuable identities for obtaining any term in any order of the respective sequence. We have given an explicit formula to calculate any term of a recurrence relation sequence, which is a very important result [12, 13].

2.2 Second order Recurrence Relation

Theorem 2.2.1: If c_1 , c_2 are real numbers and let

$$x^2 - c_1 x - c_2 = 0, \tag{2.1}$$

have distinct roots x_1 and x_2. Then sequence $< a_n >$ is solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}\ , n \geq 2, \tag{2.2}$$

with initial terms $a_0 = A_1, a_1 = A_2$.

$$iff$$

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n,$$

for $n = \ 0, 1, 2, \ldots$, where β_1 and β_2 are arbitrary constants.

Proof: First suppose that $< a_n >$ is of type $a_n = \beta_1 x_1^n + \beta_2 x_2^n$ we shall prove that $< a_n >$ is a solution of the recurrence relation (2.2). Since the x_1 and x_2 roots of equation (2.1) then

$$c_1 x_1 + c_2 = x_1^2,$$

$$c_1 x_2 + c_2 = x_2^2.$$

by equation (2.2)

$c_1a_{n-1} + c_2a_{n-2} = \ c_1(\beta_1x_1^{n-1} + \beta_2x_2^{n-1}) + c_2(\beta_1x_1^{n-2} + \beta_2x_2^{n-2}) = \beta_1x_1^n + \beta_2x_2^n$

this implies

$$c_1a_{n-1} + c_2a_{n-2} = a_n.$$

which proves the result.

Converse part

Consider the sequence $< a_n >$

$$a_n = c_1a_{n-1} + c_2a_{n-2} \ \ n \geq 2,$$

with initial terms $a_0 = A_1, a_1 = A_2$.

and let

$$a_n = \beta_1x_1^n + \beta_2x_2^n.$$

So, by initial condition

$$\beta_1 + \beta_2 = \ A_1, \tag{2.3}$$

$$\beta_1x_1 + \beta_2x_2 = \ A_2, \tag{2.4}$$

By equation (2.3)

$$\beta_1 = A_1 - \beta_2,$$

putting this value in (2.4) we obtained

$$\beta_2 = \frac{A_1x_1 - x_2}{x_1 - x_2},$$

and

$$\beta_1 = \frac{A_2 - A_1x_2}{x_1 - x_2}.$$

Theorem 2.2.2: For real numbers c_1 and c_2, then the sequence $< a_n >$ is the solution to

$$a_n = c_1a_{n-1} + c_2a_{n-2}, n \geq 2, \tag{2.5}$$

with given initial terms $a_0 = A_1$ and $a_1 = A_2$. If in equation (2.5) $c_1$and c_2 are such that the roots of $x^2 - c_1x - c_2 = 0$ are distinct and greater than 1 and satisfy the conditions$A_1x_1 > A_2$ and $A_1x_2 < A_2$, then the recurrence relation sequence must be divergent.

Proof: Using theorem 2.2.1 $a_n = \beta_1x_1^n + \beta_2x_2^n$, with x_1 and x_2 are roots of the equation $x^2 - c_1x - c_2 = 0$ but according to the given condition clearly value of $x_1$and x_2 are greater than 1 so if limit of n goes to infinity then x_1^n and x_2^n both gives the value infinity

by Theorem 2.2.1

$$\beta_2 = \frac{A_1x_1 - x_2}{x_1 - x_2},$$

and

$$\beta_1 = \frac{A_2 - A_1x_2}{x_1 - x_2}.$$

According to given conditions values of $\beta_1$and β_2 are positive. As $n \to \infty$, $\beta_1x_1^n$ and $\beta_2x_2^n$ gives the value infinity. So, we can say that as $n \to \infty$, $< a_n >$ tends to infinity, so the sequence must be divergent.

2.3 Fibonacci polynomial

Theorem 2.3.1: Let$n \geq 0$ be an integer, if F_n(x), the Fibonacci polynomial, is characterized by

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x),$$

with $F_1(x) = 1$, $F_0(x) = 0$ and $x^2 > (-4)$ then

$$F_n(x) = \frac{1}{2^n\sqrt{x^2+4}}\left[x + \sqrt{x^2+4}\right]^n - \frac{1}{2^n\sqrt{x^2+4}}\left[x - \sqrt{x^2+4}\right]^n.$$

Proof: By using theorem 2.2.1 with $c_1 = x$, $c_2 = 1$ and for variable T by equation (2.1) then we have

$$T^2 - xT - 1 = 0.$$

On solving the roots are

$$\frac{x \pm \sqrt{x^2+4}}{2},$$

using theorem 2.2.1 we obtained

$$F_n(x) = \beta_1 \left[\frac{x+\sqrt{x^2+4}}{2}\right]^n + \beta_2 \left[\frac{x-\sqrt{x^2+4}}{2}\right]^n, \tag{2.6}$$

using initial condition,

$$\beta_1 + \beta_2 = 0, \tag{2.7}$$

$$\beta_1 [\frac{x+\sqrt{x^2+4}}{2}]^1 + \beta_2 [\frac{x-\sqrt{x^2+4}}{2}]^1 = 1, \tag{2.8}$$

Solving (2.7) and (2.8) we have

$$\beta_1 = \frac{1}{\sqrt{x^2+4}},$$

$$\beta_2 = \frac{-1}{\sqrt{x^2+4}},$$

on substituting values of β_1 and β_2 we get the desired result.

2.4 Chebyshev Polynomial

Theorem 2.4.1: If Chebyshev polynomial of the first kind $T_n(x)$ is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

for all $n \geq 1$, with $T_0(x) = 1$, $T_1(x) = x$, $x^2 > 1$, we can write

$$T_n(x) = \frac{1}{2}[x + \sqrt{x^2-1}]^n + \frac{1}{2}[x - \sqrt{x^2-1}]^n.$$

Proof: Using theorem 2.2.1 with $c_1 = 2x$, $c_2 = -1$ and for variable T by equation (2.1) we have polynomial

$$T^2 - 2xT + 1 = 0.$$

On solving roots obtained as

$$x \pm \sqrt{x^2 - 1},$$

by theorem 2.2.1, we obtained

$$T_n(x) = \beta_1[x + \sqrt{x^2 - 1}]^n + \beta_2[x - \sqrt{x^2 - 1}]^n. \tag{2.9}$$

Using initial conditions

$$\beta_1 + \beta_2 = 1, \tag{2.10}$$

$$\beta_1[x + \sqrt{x^2 - 1}]^1 + \beta_2[x - \sqrt{x^2 - 1}]^1 = x, \tag{2.11}$$

solving (2.10), (2.11) we have $\beta_1 = \frac{1}{2}$ and $\beta_2 = \frac{1}{2}$.

So, by equation (2.9)

$$T_n(x) = \frac{1}{2}\left[x + \sqrt{x^2 - 1}\right]^n + \frac{1}{2}\left[x - \sqrt{x^2 - 1}\right]^n.$$

2.5 Recurrence Relation of Third Order

In a recurrence relation of third order [14, 15], given the first three terms, the next term depends on the previous three terms, e.g.

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}, n \geq 3,$$

with $a_0 = 0, a_1 = 1, a_2 = 2$.

Theorem 2.5.1: If $c_1,\ c_2$ and c_3 are real numbers, let

$$x^3 - c_1x^2 - c_2x - c_3 = 0, \tag{2.10}$$

hasx_1, x_2 and $x_3$asreal and distinct roots.

Then the sequence $< a_n >$has solution

$$a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3}\,, n \geq 3, \tag{2.11}$$

with three initial terms

$$a_0 = A_1,$$

$$a_1 = A_2,$$

$$a_2 = A_3.$$

$$iff$$

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n,$$

for integers $n \geq 0$, arbitrary constants β_1, β_2, and β_3.

Proof: Suppose sequence $< a_n >$is

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n,$$

now to prove $< a_n >$ is a solution of (2.11). Ifx_1, x_2 and x_3 are roots of (2.10) then we have

$$x_1^3 = c_1 x_1^2 + c_2 x_1 + c_3,$$

$$x_2^3 = c_1 x_2^2 + c_2 x_2 + c_3,$$

$$x_3^3 = c_1 x_3^2 + c_2 x_3 + c_3.$$

Consider

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} & \\ &= c_1(\beta_1 x_1^{n-1} + \beta_2 x_2^{n-1} + \beta_3 x_3^{n-1}) \\ &+ c_2(\beta_1 x_1^{n-2} + \beta_2 x_2^{n-2} + \beta_3 x_3^{n-2}) + c_3(\beta_1 x_1^{n-3} + \beta_2 x_2^{n-3} + \beta_3 x_3^{n-3}) \\ &= \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n = a_n. \end{aligned}$$

This implies

$$c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} = a_n.$$

which prove the result.

Converse part

Suppose recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}\ , n \geq 3,$$

with

$$a_0 = A_1,$$

$$a_1 = A_2,$$

$$a_2 = A_3.$$

Let

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n$$

So, by initial conditions

$$\beta_1 + \beta_2 + \beta_3 = A_1, \quad (2.12)$$

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = A_2, \quad (2.13)$$

$$\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 = A_3. \quad (2.14)$$

Non-trivial solution of system (2.12), (2.13), and (2.14) is possible iff

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} \neq 0,$$

on expanding determinant

$$(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \neq 0. \quad (2.15)$$

As the roots are distinct equation (2.15)is always non-zero. So, non-trivial values of $\beta_1,\ \beta_2$ and β_3 can be found, therefore the result is valid.

Example 2.5.1: Let for sequence $< a_n >, a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3},\ n \geq 3$ and $a_0 = 0, a_1 = 1, a_2 = 2$. Then find a_{10}.

Solution: By theorem 2.5.1 polynomial for $< a_n >$ is

$$x^3 - 6x^2 + 11x - 6 = 0.$$

Roots of above equations are 1, 2, and 3, again by theorem 2.5.1

$$a_n = \beta_1 1^2 + \beta_2 2^2 + \beta_3 3^2. \tag{2.16}$$

Using $a_0 = 0, a_1 = 1, a_2 = 2$ in (2.16)

$$\beta_1 + \beta_2 + \beta_3 = 0, \tag{2.17}$$

$$\beta_1 + 2\beta_2 + 3\beta_3 = 1, \tag{2.18}$$

$$\beta_1 + 4\beta_2 + 9\beta_3 = 2, \tag{2.19}$$

Solving (2.17), (2.18) and (2.19) we have $\beta_1 = -\frac{3}{2}, \beta_2 = 2,\ \beta_3 = -\frac{1}{2}$.

By using (2.16)

$$a_n = -\frac{3}{2}(1^n) + 2(2^n) - \frac{1}{2}(3^n).$$

now put $n = 10$ we have $a_{10} = -27478$.

2.6 Fourth Order Recurrence Relation

In the recurrence relation of fourth order, the next term depends on the previous four terms with four initial conditions, e.g.

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-2} + a_{n-3}, n \geq 4,$$

initial terms $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3$, for integers $n \geq 0$.

Theorem 2.6.1: If $c_1,\ c_2, c_3$ and c_4 are real numbers, let

$$x^4 - c_1 x^3 - c_2 x^2 - c_3 x - c_4 = 0 \tag{2.20}$$

has distinct real roots x_1, x_2, x_3 and x_4.

Then sequence $< a_n >$ has solution

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4}, n \geq 4 \tag{2.21}$$

with

$$a_0 = A_1,$$

$$a_1 = A_2,$$

$$a_2 = A_3$$

$$a_3 = \mathrm{A}_4.$$

$$iff$$

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \beta_4 x_4^n,$$

with$n =\ 0, 1, 2, \ldots$, and for arbitrary constants $\beta_1, \beta_2, \beta_3$ and β_4 constants.

Proof: Suppose that$< a_n >$is a sequence

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \beta_4 x_4^n,$$

Now we prove $< a_n >$ is a solution of (2.21). For roots x_1, x_2, x_3, x_4 of equation (2.20) we obtained

$$x_1^4 = c_1 x_1^3 + c_2 x_1^2 + c_3 x_1 + c_4,$$

$$x_2^4 = c_1 x_2^3 + c_2 x_2^2 + c_3 x_2 + c_4,$$

$$x_3^4 = c_1 x_3^3 + c_2 x_3^2 + c_3 x_3 + c_4,$$

$$x_4^4 = c_1 x_4^3 + c_2 x_4^2 + c_3 x_4 + c_4.$$

Consider

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4} & \\ &= c_1(\beta_1 x_1^{n-1} + \beta_2 x_2^{n-1} + \beta_3 x_3^{n-1} + \beta_4 x_4^{\mathrm{n}-1}) \\ &+ c_2(\beta_1 x_1^{n-2} + \beta_2 x_2^{n-2} + \beta_3 x_3^{n-2} + \beta_4 x_4^{\mathrm{n}-2}) \\ &+ c_3(\beta_1 x_1^{n-3} + \beta_2 x_2^{n-3} + \beta_3 x_3^{n-3} + \beta_4 x_4^{\mathrm{n}-3}) \\ &+ c_4(\beta_1 x_1^{n-4} + \beta_2 x_2^{n-4} + \beta_3 x_3^{n-4} + \beta_4 x_4^{\mathrm{n}-4}) \\ &= \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \beta_4 x_4^n = a_n. \end{aligned}$$

which proves the theorem.

Converse part

Let

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4}$, $n \geq 4$ is recurrence relation with

$$a_0 = A_1,$$

$$a_1 = A_2,$$

$$a_2 = A_3,$$

$$a_3 = A_4.$$

Let $a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \beta_4 x_4^n$

So

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = A_1,$$

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 = A_2,$$

$$\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 x_4^2 = A_3,$$

$$\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_4^2 + \beta_4 x_4^3 = A_4.$$

Solution of non-trivial type of system of linear equations is possible iff

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} \neq 0$$

on expanding determinant

$$(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_1) \neq 0. \qquad (2.22)$$

Since the roots are distinct so equation (2.22)is always non-zero, therefore values of non-trivial type of β_1, β_2 β_3 and β_4 can be found and result is valid.

2.7 kth order recurrence relation

Theorem 2.7.1: If $c_1, \ c_2, c_3, \dots, c_k$ are real numbers and let

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} \dots - c_k = 0, \tag{2.23}$$

has distinct roots $x_1, x_2, x_3 \dots, x_k$.

then sequence $< a_n >$ for all non negative integers have solution.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots \dots + c_\mathrm{k} a_{n-\mathrm{k}}, n > \mathrm{k}, \tag{2.24}$$

with initial terms

$$a_0 = A_1,$$

$$a_1 = A_2,$$

$$a_2 = A_3,$$

$$\dots$$

$$a_{k-1} = \mathrm{A_k}.$$

$$iff$$

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \cdots + \beta_\mathrm{k} x_\mathrm{k}^n,$$

for $n \geq \ 0$,and for arbitrary constants$\beta_1, \beta_2, \beta_3 \dots \beta_\mathrm{k}$.

Proof: Let us suppose that the sequence $< a_n >$ as

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \cdots + \beta_\mathrm{k} x_\mathrm{k}^n,$$

Now we will prove that $< a_n >$ is a solution of (2.24). Since $x_1, x_2, x_3, \dots, x_k$ are roots of equation (2.23) so

$$x_1^k = c_1 x_1^{k-1} + c_2 x_1^{k-2} + \ \dots + c_k,$$

$$x_2^k = c_1 x_2^{k-1} + c_2 x_2^{k-2} + \ \dots + c_k,$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \ddots \qquad \vdots$$

$$x_k^k = c_1 x_k^{k-1} + c_2 x_k^{k-2} + \ldots + c_k.$$

Consider

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} \\ &= c_1\left(\beta_1 x_1^{n-1} + \beta_2 x_2^{n-1} + \beta_3 x_3^{n-1} + \cdots + \beta_k x_k^{n-1}\right) \\ &+ c_2\left(\beta_1 x_1^{n-2} + \beta_2 x_2^{n-2} + \beta_3 x_3^{n-2} + \cdots + \beta_k x_k^{n-2}\right) \\ &+ c_3\left(\beta_1 x_1^{n-3} + \beta_2 x_2^{n-3} + \beta_3 x_3^{n-3} + \ldots + \beta_k x_k^{n-3}\right) + \cdots \\ &+ c_k(\beta_1 x_1^{n-4} + \beta_2 x_2^{n-4} + \beta_3 x_3^{n-4} + \ldots + \beta_k x_k^{n-4}) \\ &= \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \cdots + \beta_k x_k^n = a_n. \end{aligned}$$

which prove the first part.

Proof of converse part

Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}\,, n > k,$$

is a sequence with k initial terms $a_0 = A_0, a_1 = A_1, a_2 = A_2, \ldots, a_k = \mathrm{A}_k$.

Let

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \cdots + \beta_k x_k^n.$$

So

$$\beta_1 + \beta_2 + \beta_3 + \cdots + \beta_k = A_0,$$

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \cdots + \beta_k x_k = A_1,$$

$$\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \cdots + \beta_k x_k^2 = A_2,$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots$$

$$\beta_1 x_1^k + \beta_2 x_2^k + \beta_3 x_3^k + \cdots + \beta_k x_k^k = A_k.$$

Solution of non-trivial type of system of linear equation is possible iff

$$\begin{vmatrix} 1 & 1 & \dots 1 \\ x_1 & x_2 & \dots x_k \\ \vdots & \vdots & \ddots \vdots \\ x_1^k & x_2^k \dots & x_k^k \end{vmatrix} \neq 0.$$

on expanding determinant

$$(x_1 - x_2)(x_2 - x_3)(x_3 - x_4) \dots (x_k - x_1) \neq 0. \tag{2.25}$$

Since the roots are different, the equation $(2.25)$is always non-zero; therefore, values of non-trivial type of $\beta_1,\ \beta_2,\ \beta_3\ , \dots , \beta_k$can be found and the result is valid.

Chapter 3

Cyclic Group of Rational functions with Coefficients as Fibonacci Numbers

3.1 Introduction

The study of group theory [17, 18, and 19] is an essential part of contemporary mathematics and is also accumulating increasing amounts of value in a wide variety of other areas. In this chapter, we demonstrate that there is a connection between group theory and number theory, describe a group of rational functions with coefficients that are the Fibonacci number in terms of the composition operation, and show that the condition of a cyclic group is satisfied. Given the recurrence relation sequence of rational functions with coefficients as Fibonacci numbers, we discuss representational relations between group properties and rational functions with Fibonacci coefficients. Furthermore, we prove that the collection of all such rational functions forms a cyclic group with respect to the composition of the function operation.

3.2 General definitions

3.2.1 Group

Group is a set that fulfills four properties with regard to the given operation. Four axioms are

Closure: Let H be any set and $*$ be any operation on **H** if $a * b \in \mathbf{H} \Rightarrow \forall\, a, b \in \mathbf{H}$.

Associative: If $(a * b) * c = a * (b * c), \forall\, a, b, c \in \mathbf{H}$.

Existence of Identity: If there exist an element e in **H** such that $a * e = e * a = a$, $\forall\, a \in \mathbf{H}$where e is identity element.

Existence of Inverse: If $\forall\, a \in \mathbf{H}, \exists\, b \in \mathbf{H}$, with $a * b = b * a = e$ then inverse of a is b.

3.2.2 Cyclic group

A cyclic group is a group in which every element can be generated by a single element of that group. e.g., a set of integers with respect to addition is a cyclic group.

3.2.3 Rational function

If $f(x)$ given by

$$f(x) = \frac{p(x)}{q(x)},$$

for polynomials $p(x)$and $q(x)$, $q(x) \neq 0$, then $f(x)$ is rational function.

3.2.4 Fibonacci numbers

Fibonacci numbers sequence is defined as

$$f_n = f_{n-1} + f_{n-2}, n \geq 2, \tag{3.1}$$

Where initial terms are $f_0 = 0$ and $f_1 = 1$. "The terms of the Fibonacci sequence are called Fibonacci numbers".

3.2.5 Lucas numbers

Lucas numbers sequence for non-negative integer n is given by

$$L_n = L_{n-1} + L_{n-2}, n \geq 2, \tag{3.2}$$

with $L_0 = 2$ and $L_1 = 1$. The terms of the Lucas sequence are called Lucas numbers.

3.2.6 Generalized Fibonacci sequences

Generalized Fibonacci sequence [20, 24], is defined as

$$F_k = pF_{k-1} + qF_{k-2}, k \geq 2, \qquad F_0 = a, F_1 = b, \tag{3.3}$$

for positive integers p, q, a & b.

3.3 Recurrence relation sequence of rational function with Fibonacci number as coefficients

If function $u: (0, \infty) \to (0,1)$is real valued defined as

$$u(x) = \frac{1}{1+x}.$$

on its domain $u(x)$ is clearly continuous. The codomain of u is a subset of the domain of u.So, considers function.

$$(u\ o\ u)(x) = \frac{1}{1+\frac{1}{1+x}},$$

now we define

$$z_n(x) = (uouou o \dots ou)(x), \tag{3.4}$$

where $(uouou o \dots ou)$ represent n time composition.

The sequence of rational functions for recurrence relations is defined as

$$z_1(x) = u(x) = \frac{1}{1+x},$$

and

$$z_n(x) = \frac{1}{1+z_{n-1}(x)},$$

for all integer $n \geq 2$.

Now, we define $z_n(x)$ such that every member of this family has Fibonacci coefficients, if the Fibonacci sequence is given by the (3.1) equation, we obtain

$$z_n(x) = \frac{f_{n-1}\,x + f_n}{f_n\,x + f_{n+1}}, \tag{3.5}$$

where f_i is ith Fibonacci number and $z_n(x)n$th term of equation (3.4) sequence of rational function. For $n \in N$, the codomain of $z_n(x)$ is

$$A_n = \left(\min\left\{\frac{f_{n-1}}{f_n}, \frac{f_n}{f_{n+1}}\right\}, \max\left\{\frac{f_{n-1}}{f_n}, \frac{f_n}{f_{n+1}}\right\}\right).$$

For example, we can say that

Codomain of $z_1(x)$, $A_1 = (0,1)$.

Codomain of $z_2(x), A_2 = \left(\frac{1}{2}, 1\right)$.

Codomain of $z_3(x)$, $A_3 = \left(\frac{1}{2}, \frac{2}{3}\right)$.

So, co-domain for all function $z_n(x)$ can be find out.

For odd n

$$A_n = \left(\frac{f_{n-1}}{f_n}, \frac{f_n}{f_{n+1}}\right).$$

For even n

$$A_n = \left(\frac{f_n}{f_{n+1}}, \frac{f_{n-1}}{f_n}\right).$$

Theorem 3.3.1: Let $I: (0, \infty) \to (0, \infty)$ such that

$$I(x) = x. \tag{3.6}$$

Let $\boldsymbol{G}$ be set of all $z_n(x)$for all $n \in N$ and including I function defined by equation (3.6) then with respect composition operation given by equation (3.4) $\boldsymbol{G}$ is cyclic group.

Proof: Closure property: let z_n and $z_m$any two functions in $\boldsymbol{G}$**,** $n, m \in N$then we according to definition (3.4)

$z_n(x) = (uouou o \dots ou)(x)$, where there are n times composition

$z_m(x) = (uouou o \dots ou)(x)$, where there are m times composition

$(z_n o z_m)\ (x) = (uouououu \dots ou)(x)$, where there are $n + m$ times composition

$$(z_n o z_m)\ (x) = z_{m+n}(x) \in \boldsymbol{G}.$$

It satisfied closure property

Associative: Associativity is clearly satisfied since all compositions are of u.

Existence of Identity: Clearly identity is existed since $\boldsymbol{G}$ is including I.

Inverse: Initially it is required to prove all functions are one-one onto.

$$z_n(x) = \frac{f_{n-1}\, x + f_n}{f_n\, x + f_{n+1}},$$

$$z_n(y) = \frac{f_{n-1}\, y \ + f_n}{f_n\, y + f_{n+1}}.$$

Consider

$$z_n(x) = z_n(y),$$

On solving we have $x = y$ which proves that all function all one-one.

Let

$$\frac{f_{n-1}\, x + f_n}{f_n\, x + f_{n+1}} = y.$$

Solving this we obtained

$$\frac{f_{n+1}\, y - f_n}{f_{n-1} \ - f_n y} = x,$$

Let, if possible

$$f_{n-1} - f_n y = 0,$$

On solving

$$y = \frac{f_{n-1}}{f_n} \notin A_n,$$

clearly $x \ > 0, \forall\ y$ in A_n.

So, we ca say that every element of A_n have pre image under z_n therefore$z_n$is onto for all n. That is every member of $\boldsymbol{G}$is one-one and onto. So, invertible property holds for every member of $\boldsymbol{G}$.

Cyclic Property: Every member can be generated by $z_1(x) = u(x) = \frac{1}{1+x}$ so by definition of cyclic group $\boldsymbol{G}$ is a cyclic group under the composition operation, which prove the theorem.

3.4 Recurrence relation sequence of rational function with Generalized Fibonacci number as coefficients

Consider function$v: (0, \infty) \rightarrow (0,1)$is real valued, defined by

$$v(x) = \frac{1}{q + x},$$

Where q in any positive integer, on its domain $v(x)$is clearly continuous and codomain of u is subset of domain of u. Now consider function

$$(v\ o\ v)(x) = \frac{1}{1 + \frac{1}{q+x}},$$

we define

$$w_n(x) = (vovovo \dots ov)(x), \tag{3.7}$$

where $(vovovo \dots ov)$ represents n time composition.

Recurrence relation sequence of rational function is given as

$$w_1(x) = \frac{1}{q + x},$$

and

$$w_n(x) = \frac{1}{q + w_{n-1}(x)}, \tag{3.8}$$

for all integer$n \geq 2$.

Now, it is required to prove that generalized Fibonacci numbers appear in the coefficients of every member of this family. For this reason, by equation (3.3), if we take $p = q,\ q = 1,\ a = 0,\ b = 1,$ we have:

$$F_n = qF_{n-1} + F_{n-2}, \forall\, n \geq 2,$$

with $F_0 = 0$ and $F_1 = 1$, where q is any positive integer.

Now to show that

$$w_n(x) = \frac{F_{n-1}\,x + F_n}{F_n\,x + F_{n+1}}, \tag{3.9}$$

for F_i is ith generalized Fibonacci number, $w_n(x)n$th term of equation (3.9) is sequence of rational functions. Using the principle of mathematical induction, (3.9) can be proved.

For $n = 1$

$$w_1(x) = v(x) = \frac{1}{q + x},$$

and

$F_0 = 0$, $F_1 = 1$ and $F_2 = q$, therefore (3.9) holds for $n = 1$.

Let the result holds for $n = k$, so by (3.9) let

$$w_k(x) = \frac{F_{k-1}\, x + F_k}{F_k\, x + F_{k+1}}.$$

To show (3.9) holds for $n = k + 1$

Consider

$$w_{k+1}(x) = \frac{1}{q + w_k(x)}.$$

by equation (3.9) we have

$$w_{k+1}(x) = \frac{1}{q + \frac{F_{k-1}\, x + F_k}{F_k\, x + F_{k+1}}},$$

$$w_{k+1}(x) = \frac{F_k x \ + \ F_{k+1}}{(qF_k \ + \ F_{k-1})x \ + (qF_{k+1} \ + \ F_k)},$$

$$w_{k+1}(x) = \frac{F_k\, x + F_{k+1}}{F_{k+1}\, x + F_{k+2}},$$

which proves the result holds for all positive integers n by the principle of mathematical induction.

Theorem 3.4.1: If $w_n(x)$is defined by equation (3.8), then $w_n(x)$ is monotonic function. If n is odd, then $w_n(x)$ is a monotonically decreasing function, and if n is even, then $w_n(x)$ is a monotonically increasing function.

Proof: It is clear from definition of $w_n(x)$, it is differentiable on given domain, so using first derivative test the theorem will be proved.

On differentiating equation (3.8) for $n = 1$

$$\frac{dw_1}{dx} = \frac{-1}{(q+x)^2} < 0.$$

So, by first derivative test $w_1(x)$ is monotonically decreasing function.

Now differentiating (3.8),

$$\frac{dw_n}{dx} = \frac{-1}{(q+w_{n-1})^2}\frac{dw_{n-1}}{dx},$$

we have

$$sgn\left[\frac{dw_n}{dx}\right] = -sgn\left[\frac{dw_{n-1}}{dx}\right],$$

for odd$n$

$$\frac{dw_n}{dx} < 0,$$

for even$n$

$$\frac{dw_n}{dx} > 0,$$

which proves our result.

Corollary 3.4.1: For$n \in N$, the range set B_n of $w_n(x)$ is $B_n = \left(\frac{F_{n-1}}{F_n}\frac{F_n}{F_{n+1}}\right)$ for odd n and $B_n = \left(\frac{F_n}{F_{n+1}}, \frac{F_{n-1}}{F_n}\right)$for even n.

Proof: By Theorem 3.4.1 for odd n, $w_n(x)$ is monotonically decreasing function by equation (3.9) $w_n(x)$ approach to its maximum value as $x \to 0$ so we can say that maximum value of $w_n(x) \to \frac{F_n}{F_{n+1}}$ and $w_n(x)$ approach to its minimum value as $x \to \infty$ so we can say that minimum value of $w_n(x) \to \frac{F_{n-1}}{F_n}$, so we can say that the range set B_n of $w_n(x)$ is

$$B_n = \left(\frac{F_{n-1}}{F_n}\frac{F_n}{F_{n+1}}\right).$$

Let n is even then $w_n(x)$ is monotonically increasing and so $w_n(x)$ approach to its minimum value as $x \to 0$ so we can say that minimum value of $w_n(x) \to \frac{F_n}{F_{n+1}}$ and $z_k(x)$ approach to its maximum value as $x \to \infty$ so we can say that maximum value of $w_n(x) \to \frac{F_{n-1}}{F_n}$, so we can say that the range set B_n of $w_n(x)$ is $B_n = \left(\frac{F_n}{F_{n+1}}, \frac{F_{n-1}}{F_n}\right)$. So, our corollary is proved.

Theorem 3.4.2: Let $I: (0, \infty) \to (0, \infty)$ such that

$$I(x) = x \tag{3.10}$$

Let for all $w_n(x), \forall\, n \in N$, the set is given by $\boldsymbol{H}$ and with including I function defined by equation (3.10), then $\boldsymbol{H}$ is cyclic group with respect composition operation given by (3.7).

Proof: Closure property

Let w_n and $w_m$are any two functions in $\boldsymbol{H}$ then by definition equation (3.7)

$w_n(x) = (vovovov \ldots ov)(x)$,where there are n times composition,

$w_m(x) = (vovovov \ldots ov)(x)$,where there are m times composition,

$(w_n o w_m)\ (x) = (vovovov \ldots ov)(x)$, where there are $n + m$ times composition,

$$(w_n o w_m)\ (x) = w_{m+n}(x) \in \boldsymbol{H},$$

Which satisfied closure property.

Associative: By definition of composition of u associative property satisfied.

Existence of Identity: Identity exists since $\boldsymbol{H}$ includes I.

Inverse: To prove inverse it is required to prove all functions are one-one and onto.

By equation (3.9)

$$w_n(x) = \frac{F_{n-1}\,x + F_n}{F_n\,x + F_{n+1}}, w_n(y) = \frac{F_{n-1}\,y + F_n}{F_n\,y + F_{n+1}}.$$

Consider

$$w_n(x) = w_n(y),$$

on solving $x = y$ so all function all one-one.

Let

$$\frac{F_{n-1}\,x + F_n}{F_n\,x + F_{n+1}} = y,$$

On solving

$$\frac{F_{n+1}\,y - F_n}{F_{n-1}\;- F_n y} = x.$$

Let if possible

$$F_{n-1} - F_n y = 0,$$

On solving

$$y = \frac{F_{n-1}}{F_n} \notin B_n\,,$$

clearly $x > 0, \forall\, y$ in B_n.

So, we can say that every element of B_n have pre-image under $w_n(x)$, therefore $w_n(x)$ is onto for all $n \in N$. So, we can say that every member of $\boldsymbol{H}$is one-one and onto, hence invariability proved by every member of $\boldsymbol{H}$.

Cyclic Property: We can generate every member by $w_1(x) = v(x) = \frac{1}{q+x}$.So, under the composition operation $\boldsymbol{G}$ is a cyclic group, which proved the theorem.

3.5. A Sequence of matrix generated by Fibonacci Numbers

Let $\quad M^1 = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $\quad M^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$

We easily prove that $(M^1)^n = M^n$

$$M^n + M^{n-1} = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} + \begin{bmatrix} f_n & f_{n-1} \\ f_{n-1} & f_{n-2} \end{bmatrix} = \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix}$$

This is an example of a recurrence relation sequence of matrix that follows the same pattern as the recurrence relation of the Fibonacci sequence [16, 25 and 26].

3.5.1. Terms of Fibonacci Numbers

f_0	0	f_8	21
f_1	1	f_9	34
f_2	1	f_{10}	55
f_3	2	f_{11}	89
f_4	3	f_{12}	134
f_5	5	f_{13}	223
f_6	8	f_{14}	367
f_7	13	f_{15}	590

3.5.2. A Sequence generated by Fibonacci Numbers

T_n	$f_{n-1}f_{n+1} - {f_n}^2$	
T_1	$f_0f_2 - {f_1}^2$	**-1**
T_2	$f_1f_3 - {f_2}^2$	**-1**
T_3	$f_2f_4 - {f_3}^2$	**-1**
T_4	$f_3f_5 - {f_4}^2$	**-1**

T_5	$f_4 f_6 - {f_5}^2$	**-1**
T_6	$f_4 f_6 - {f_5}^2$	**-1**
T_7	$f_5 f_7 - {f_6}^2$	**-1**

So we can observe $T_n = (-1)^n$

By this observation we can conclude

$$\det(M^n) = (-1)^n for\ all\ integer\ n$$

3.5.3. Set of terms of sequence of non-singular matrix

Let

$$H = \{ M^n : n \in Z \}$$

We define

$$M^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem 3.5.4: Show that H is a cyclic group with respect to matrix multiplication.

Proof: -let $M^n,\ M^m \epsilon H$

Then we have

$$M^n.\ M^m = M^{n+m}$$

So we can say that H satisfied the 1st axiom of the group.

All elements of H are matrices so must satisfied the associativity properties.

$M^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity of H

so identity exists in the set

Since

$$\det(M^n) = (-1)^n,$$

for all non-negative integers n.

So every matrix must be invertible, so we can say that the inverse of every element of a set must exist. Therefore, H must be a group with respect to matrix multiplication.

Since every element of this group can be written as a power of $M^1 = \begin{bmatrix} \mathrm{f}_2 & f_1 \\ f_1 & f_0 \end{bmatrix}$, it confirmed that H is a cyclic group with respect to matrix multiplication.

3.5.5 Generalized Fibonacci of 3rd order sequence

Let $\{f_{3,n}\}$ be the generalized Fibonacci sequence of order third which given by

$$f_{3,n} = f_{3,n+2} + f_{3,n+1} + f_{3,n},$$

where with initial terms,

$$f_{3,0} = 0\,, f_{3,1} = 0\,, f_{3,2} = 1,$$

for all non-negative integer n.

3.5.6. Special case of generalized Fibonacci of 3rd order sequence

Let $\{l_{3,n}\}$ be the special case of generalized Fibonacci sequence of order third, which given by

$$l_{3,n+3} = a l_{3,n+2} + l_{3,n+1} + l_{3,n}, \qquad a \neq 0,$$

Where given initial terms are below

$$l_{3,0} = 0\,, l_{3,1} = 0\,, l_{3,2} = 1,$$

for all non-negative integer n.

3.5.7. Sequence of non-singular matrix

Let sequence of matrix [1, 2, 3, 4 and 5]

$$P_3^1 = \begin{bmatrix} a & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_3^2 = \begin{bmatrix} l_{3,4} & l_{3,3} + l_{3,2} & l_{3,3} \\ l_{3,3} & l_{3,2} + l_{3,1} & l_{3,2} \\ l_{3,2} & l_{3,1} + l_{3,0} & l_{3,1} \end{bmatrix}$$

And continue like this we have

$$P_3^n = \begin{bmatrix} l_{3,n+2} & l_{3,n+1} + l_{3,n} & l_{3,n+1} \\ l_{3,n+1} & l_{3,2} + l_{3,n-1} & l_{3,n} \\ l_{3,n} & l_{3,n-1} + l_{3,n-2} & l_{3,n-1} \end{bmatrix}$$

In this sequence of matrix there are some special types of relations

$$P_3^n = (P_3^1)^n$$

$$\text{Let } (P_3^1)^0 = \ P_3^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(P_3^0) = 1$$

for all non-negative integer n.

3.5.8. Set of terms of sequence of non-singular matrix

Let

$$G = \{P_3^n : n \in Z\}.$$

Theorem 3.5.9: Show that G is a cyclic group with respect to matrix multiplication.

Proof: -let $P_3^n, P_3^m \epsilon G$

Then we have

$$P_3^n P_3^m = P_3^{n+m}$$

So we can say that G satisfied the 1st axiom of the group.

All elements of G are matrix so must satisfied the associativity properties.

Clearly $P_3^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity of G

So identity is exist in set

Since

$$\det(P_3^n) = 1,$$

for all integers n.

So every matrix must be invertible, so G must be a group with respect to matrix multiplication.

Since every element of this group can be written as a power of P_3^1, it has been confirmed that it is a cyclic group with respect to matrix multiplication [29, 30].

Theorem 3.5.10: Show that G is isomorphic to a group set of integers (Z) with respect to addition.

Proof: Let define a mapping $\varphi: G \to Z$ such that

$$\varphi(P_3^n) = n$$

for all integers n.

Well-defined

Let

$$P_3^n = P_3^m$$

for $n, m \in Z$.

$$P_3^{n-m} = P_3^0$$

this implies

$$n - m = 0$$

$$n = m$$

So φ is well-defined mapping.

One-one

Let

$$\varphi(P_3^n) = \varphi(P_3^m),$$

for $n, m \in Z$,

this implies

$$n = m,$$

$$n - m = 0,$$

this implies

$$P_3^{n-m} = P_3^0,$$

therefore

$$P_3^n = P_3^m,$$

So φ is one-one mapping.

Onto

Let $n \in Z$ be any integer then $P_3^n \in G$, such that we have

$$\varphi(P_3^n) = n$$

for all integers n.

So φ is onto mapping.

Homomorphism

Consider

$$\varphi(P_3^n P_3^m) = \varphi(P_3^{n+m}) = n + m = \varphi(P_3^n) + \varphi(P_3^m),$$

for $n, m \in Z$.

So, φ is homomorphism mapping. Therefore, G is isomorphic to the group set of integers (Z) with respect to addition.

Theorem 3.5.11: Show that G is a cyclic sub-group of group $\mathrm{SL}_3(\mathrm{Z})$ with respect to matrix multiplication.

Proof: Let $A \in G$ then all entries of A are integer and order is 3 also $|A| = 1$

This implies $A \in \mathrm{SL}_3(\mathrm{Z})$.

So,

$$G \subset \mathrm{SL}_3(\mathrm{Z}).$$

Also G and $\mathrm{SL}_3(\mathrm{Z})$ both are group with respect to same binary operation, so G is a sub group of $\mathrm{SL}_3(\mathrm{Z})$.

Theorem 3.5.12: Show that G is not a simple group with respect to matrix multiplication.

Proof:- We have

$$G = \{P_3^n : n \in Z\}$$

$$P_3^1 = \begin{bmatrix} a & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

If we take $a = 1$ then

$$K_3^1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We have

$$H = \{K_3^n : n \in Z\}$$

Then H is clearly a proper sub-group of G and G is a cyclic group, so H must be a normal sub-group of G. Therefore, G is not a simple group.

Theorem 3.5.13: Show that G is matrix lie-group.

Proof: We have

$$G = \{P_3^n : n \in Z\}.$$

Let

$$A_n \in G,$$

then all entries of A_n are Fibonacci numbers, so A_n not be convergent to any matrix. So, we can say that there is no convergent sequence in G. Therefore G is a matrix lie-group.

Both the definition of the 3-Generalized Fibonacci group and an analysis of its algebraic axioms can be found in this chapter. The authors of the research established a cluster with the assistance of 3-Generalized Fibonacci and a sequence of matrices. This result can be applied to a wide variety of different kinds of groups, as well as a wide variety of other sequences.

Chapter 4

Generating Matrices of Recurrence relations as Sequence of Tri-Diagonal Matrices

4.1 Introduction

Identities for generalized Fibonacci sequences of integers, Fibonacci sequences of polynomials, and Chebyshev polynomials have been presented in this chapter. The Fibonacci generalized sequence of integers, the Fibonacci sequence of polynomials, and the Chebyshev polynomial [16, 21, 22, and 23] can all be represented in the form of matrices with the assistance of these identities, which are of great use.

4.2 Definitions

4.2.1 Generalized Fibonacci sequences of numbers

Generalized Fibonacci sequence is defined as [8, 9]

$$F_n = aF_{n-1} + bF_{n-2}, n \geq 2, \tag{4.1}$$

for $F_0 = p, F_1 = q$ and positive integersp, q, a & b.

Particularly by equation (4.1) $p = 0, q = 1, F_n = V_n$

$$V_n = aV_{n-1} + bV_{n-2}, k \geq 2, \tag{4.2}$$

where $V_0 = 0, V_1 = 1$and a, b are positive integers.

4.2.2 Fibonacci sequences of polynomials

E.C. Catalan define Fibonacci polynomial $F_n(x)$as

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \tag{4.3}$$

where $F_0(x) = 0$, $F_1(x) = 1$ and $n \geq 0$.

4.2.3 Generalized Fibonacci polynomials

If $G_n(x)$ is Generalized Fibonacci polynomial define by

$$G_{n+2}(x) = axG_{n+1}(x) + bG_n(x), \tag{4.4}$$

with$G_1(x) = 1$, $G_0(x) = 0$ and $n \geq 0$.

4.2.4 Chebyshev polynomials

Chebyshev polynomial of first kind $T_n(x)$ is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \tag{4.5}$$

$T_1(x) = x$, $T_0(x) = 1$ and for all integers $n \geq 1$.

4.3 Sequence of tri-diagonal matrices

4.3.1 Sequence of tri-diagonal matrices for Generalized Fibonacci sequences of polynomials

For $n \in N$, tri-diagonal matrix sequence $\{\mathrm{A(n)} = [g_{n,n}]\}$is

$$[g_{i,j}] = \begin{cases} g_{i,j} = ax & if \;\; j = i \\ g_{i,j} = -b & if \; j-1 = i \\ g_{i,j} = 1 & if \;\; j+1 = i \\ g_{i,j} = 0 & otherwise \end{cases}, \tag{4.6}$$

So, that

$$A(n) = \begin{bmatrix} ax & -b & 0 & \cdots & \cdots & 0 \\ 1 & ax & -b & \cdots & \cdots & 0 \\ 0 & 1 & ax & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & ax & -b \\ 0 & 0 & \cdots & \cdots & 1 & ax \end{bmatrix}.$$

Then determinant of $A(n)$ is

$$|A(n)| = g_{n,n}|A(n-1)| - g_{n,n-1}g_{n-1,n}|A(n-2)|. \tag{4.7}$$

Theorem 4.3.1: Let $|A(n)| = G_{n+1} \; \forall \; n \geq 1$, where $|A(n)|$the determinant of $A(n)$ given by (4.6) then $G_{n+1}$is $(n+1)th$ term of polynomials of generalized Fibonacci sequence given by (4.4).

Proof: Principle mathematical induction is used to prove the result

For $n = 1$ by equation (4.6) we have

$$|A(1)| = ax,$$

also, for $n = 1$ by equation (4.4) we have

$$G_2 = ax,$$

so that

$$|A(1)| = G_2.$$

The result holds for $n = 1$.

Now consider for $n \leq k$result is true. So, we have

$$|A(k)| = G_{k+1}. \tag{4.8}$$

Now to show for $n = k + 1$ result is also true.

By equation (4.7)

$$|A(k+1)| = g_{k+1,k+1}|A(k)| - g_{k+1,k}g_{k,k+1}|A(k-1)|,$$

by definition (4.6)

$$g_{k+1,k+1} = ax,$$

$$g_{k+1,k}g_{k,k+1} = -b,$$

putting these two values we have

$$|A(k+1)| = ax|A(k)| + b|A(k-1)|,$$

by equation (4.8)

$$|A(k+1)| = aG_{k+1} + bG_k = G_{k+2}.$$

So, result holds for $n = k + 1$, for all $n$the theorem is proved.

4.3.2 Sequence of tri-diagonal matrices for Fibonacci sequence of polynomial

Consider a sequence of matrices defined by (4.6) by putting, $a = 1, b = 1$ we have sequence of matrices $\{C(n) = [h_{n,n}]\}$

$$[h_{i,j}] = \begin{Bmatrix} h_{i,j} = 1 \quad if \;\; i = j \\ h_{i,j=1} \quad if \; i = j+1 \\ h_{i,j} = -1 \;\; if \;\; i = j-1 \\ h_{i,j} = 0 \;\; if \; otherwise \end{Bmatrix} \tag{4.9}$$

$$C(n) = \begin{bmatrix} x & -1 & 0 & \cdots & \cdots & 0 \\ 1 & x & -1 & \cdots & \cdots & 0 \\ 0 & 1 & x & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x & -1 \\ 0 & 0 & \cdots & \cdots & 1 & x \end{bmatrix}.$$

Then determinant of $C(n)$is

$$|C(n)| = h_{n,n}|C(n-1)| - h_{n,n-1}h_{n-1,n}|C(n-2)|. \tag{4.10}$$

Theorem 4.3.2: $|C(n)| = F_{n+1}(x)$, for all integers $n \geq 1$, where $|C(n)|$ is determinant of $C(n)$ define by (4.9) sequence of matrix and $F_{n+1}(x)$ is $(n+1)th$ term Fibonacci sequence of polynomials defined by (4.3).

Proof: Principle mathematical induction can be used to prove this result

By equation (4.9) for $n = 1$

$$|C(1)| = x,$$

Also, by equation (4.3) for $n = 1$

$$F_2(x) = x,$$

which verify

$$|C(1)| = F_2(x).$$

So, the result holds for $n = 1$.

Consider for $n \leq k$ result holds therefore

$$|C(k)| = F_{k+1}(x). \tag{4.11}$$

Now show that for $n = k + 1$ theoremholds

by equation (4.10) consider

$$|C(k+1)| = h_{k+1,k+1}|C(k)| - h_{k+1,k}h_{k,k+1}|C(k-1)|.$$

by equation (4.9)

$$h_{k+1,k+1} = x,$$

$$h_{k+1,k} h_{k,k+1} = -1,$$

putting these two values we have

$$|C(k+1)| = x|C(k)| + |C(k-1)|,$$

by equation (4.11) we have

$$|C(k+1)| = xF_{k+1}(x) + F_k(x) = F_{k+2}(x).$$

Which proves the result for $n = k+1$, therefore, for all n the Theorem is true.

4.3.3 Sequence of tri-diagonal matrices for particular case of generalized Fibonacci numbers

For $n \in N$we define a sequence of tri-diagonal matrices $\{D(n) = [q_{n,n}]\}$

$$[q_{i,j}] = \begin{Bmatrix} q_{i,j} = a & if\ \ j = i \\ q_{i,j} = -b & if\ j-1 = i \\ q_{i,j} = 1 & if\ \ j+1 = i \\ q_{i,j} = 0 & otherwise \end{Bmatrix}, \tag{4.12}$$

so that

$$\boldsymbol{D(n)} = \begin{bmatrix} a & -b & 0 & \cdots & \cdots & 0 \\ 1 & a & -b & \cdots & \cdots & 0 \\ 0 & 1 & a & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & a & -b \\ 0 & 0 & \cdots & \cdots & 1 & a \end{bmatrix}.$$

Then determinants of $\boldsymbol{D(n)}$ is

$$|D(n)| = q_{n,n}|D(n-1)| - q_{n,n-1}q_{n-1,n}|D(n-2)|. \tag{4.13}$$

Theorem 4.3.3: $|D(n)| = V_{n+1}$ for every integer$n \geq 1$ where$|D(n)|$the determinant of $\boldsymbol{D(n)}$defines by (4.12) and $V_{n+1}$is $(n+1)th$ term of sequence given by (4.2).

Proof: Principle mathematical induction can be used to prove this result.

Taking $n = 1$ by equation (4.12) we have

$$|D(1)\,| = a,$$

for$n = 1$ by equation (4.13) we have

$$V_2 = a,$$

therefore

$$|D(1)\,| = V_2.$$

So, for $n = 1$result holds.

Consider for $n \leq k$ result holds therefore

$$|D(k)| = V_{k+1}. \tag{4.14}$$

Now show that for $n = k + 1$ theorem holds

by equation (4.13) consider

$$|D(k+1)\,| = q_{k+1,k+1}|D(k)\,| - q_{k+1,k}q_{k,k+1}|D(k-1)|,$$

by equation (4.12)

$$q_{k+1,k+1} = a,$$

and

$$q_{k+1,k}q_{k,k+1} = -b,$$

putting these two values we obtained

$$|D(k+1)| = a|D(k)| + b|D(k-1)|,$$

by equation we have

$$|D(k+1)| = aV_{k+1} + bV_k = V_{k+2}.$$

Which proves result for $n = k + 1$, therefore for all n the theorem is true.

4.3.4 Sequence of tri-diagonal matrices for Chebyshev polynomial

We defined a special sequence of tri-diagonal matrix $\{S(n) = [l_{i,j}]\}$ such that

$$[l_{i,j}] = \begin{Bmatrix} l_{i,j} = 2x & if \;\; j = i \\ l_{i,j} = 1 & if \; j-1 = i \\ l_{i,j} = 1 & if \;\; j+1 = i \\ l_{i,j} = 0 & otherwise \end{Bmatrix}, \tag{4.15}$$

$$S(n) = \begin{bmatrix} 2x & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2x & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 2x & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 2x & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 2x \end{bmatrix}.$$

Then determinant of $S(n)$is

$$|S(n)| = l_{n,n}|S(n-1)| - l_{n,n-1}l_{n-1,n}|S(n-2)|. \tag{4.16}$$

Theorem 4.3.4: $|S(n)| = 2T_n$for integer $n \geq 1$where$|S(n)|$ is the determinant of $S(n)$define by (4.15) and $T_n$is nth term of Chebyshev polynomials.

Proof: Principle mathematical induction can be used to prove this result.

For $n = 1$ by equation (4.15) we have

$$|S(1)| = 2x,$$

also, for $n = 1$ by equation (4.5) we have

$$2T_1 = 2x,$$

therefore

$$|S(1)| = 2T_1.$$

So, for $n = 1$result holds.

Consider for $n \leq k$result holds that is

$$|S(k)| = 2T_k. \tag{4.17}$$

Now we will show that for $n = k + 1$result is also true,

by equation(4.16) consider

$$|S(k+1)| = l_{k+1,k+1}|S(k)| - l_{k+1,k}l_{k,k+1}|S(k-1)|,$$

by equation (4.15)

$$l_{k+1,k+1} = 2x,$$

and

$$l_{k+1,k} l_{k,k+1} = 1,$$

putting these two values we obtained

$$|S(k+1)| = 2x|S(k)| - |S(k-1)|,$$

by equation (4.17)

$$|S(k+1)| = 4xT_k - 2T_{k-1} = 2T_{k+1}.$$

Which proves result for $n = k + 1$, therefore for all n the theorem is true.

4.4 Generalized k-Fibonacci sequences of numbers

Generalized k-Fibonacci sequence is defined as [6, 7, and 8].

$$F_{k,k+n} = F_{k,k+n-1} + F_{k,k+n-2} + F_{k,k+n-3} + \cdots F_{k,n+1} + F_{k,n}$$

$$where\ F_{k,0} = F_{k,1} = \cdots F_{k,k-2} = 0, F_{k,k-1} = 1$$

4.4.1. Generalized 3-Fibonaccisequences of numbers

Generalized 3-Fibonacci sequence is defined as [3, 4, 10, 11, and 12].

$$F_{3,3+n} = F_{3,n+2} + F_{k,n+1} + F_{k,n}$$

$$where\ F_{k,0} = F_{k,1} = 0, F_{k,2} = 1$$

4.4.2. Generalized 4-Fibonaccisequences of numbers

Generalized 4-Fibonacci sequence is defined as [1, 2, 8, and 9].

$$F_{4,4+n} = F_{4,n+3} + F_{4,n+2} + F_{4,n+1} + F_{4,n}$$

where

$$F_{k,0} = F_{k,1} = F_{k,2} = 0, F_{k,3} = 1$$

4.4.3 Sequence of special type for 3-fibonacci numbers

$$[k_{i,j}] = \begin{Bmatrix} k_{i,j} = 1 & if\ \ i = j \\ k_{i,j= -1} & if\ i = j+1 \\ k_{i,j} = 1 & if\ \ i = j-1 \\ k_{i,j} = 1 & if\ \ i = j-2 \\ k_{i,j} = 0 & if\ otherwise \end{Bmatrix}$$

$$\boldsymbol{K(n)} \ = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & & 0 & 0 & 0 \\ & & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & & 0 & -1 & 1 \end{bmatrix}$$

Then determinant of $\boldsymbol{K(n)}$ are of the type

$$|K(n)\,| = |K(n-1)\,| + |K(n-2)| + |K(n-3)|\ for\ all\ n > 3$$

where

$$|K(1)\,| = 1, |K(2)\,| = 2, |K(3)\,| = 4$$

4.4.4. Sequence of special type for 4-fibonacci numbers

$$[q_{i,j}] = \begin{Bmatrix} q_{i,j} = 1 & if\ \ i = j \\ q_{i,j= -1} & if\ i = j+1 \\ q_{i,j} = 1 & if\ \ i = j-1 \\ q_{i,j} = 1 & if\ \ i = j-2 \\ q_{i,j} = 1 & if\ \ i = j-3 \\ q_{i,j} = 0 & if\ otherwise \end{Bmatrix}$$

$$\boldsymbol{Q(n)} \ = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & & 0 & 0 & 0 \\ & & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & & 0 & -1 & 1 \end{bmatrix}$$

Then determinant of $\boldsymbol{Q(n)}$ are of the type

$$|Q(n)\,| = |Q(n-1)\,| + |Q(n-2)| + |Q(n-3)| + |Q(n-4)|\ \ for\ all\ n > 4$$

where

$$|Q(1)| = 1, |Q(2)| = 2, |Q(3)| = 4, |Q(4)| = 8$$

4.4.5. Sequence of special type for k-Fibonacci numbers

$$[h_{i,j}] = \left\{ \begin{array}{ll} h_{i,j} = 1 & if\ \ i = j \\ h_{i,j} = -1 & if\ i = j+1 \\ h_{i,j} = 1 & if\ \ i = j-1 \\ h_{i,j} = 1 & if\ \ i = j-2 \\ h_{i,j} = 1 & if\ \ i = j-3 \\ & \cdots \\ h_{i,j} = 1 & if\ i = j-(k-1) \\ h_{i,j} = 0 & if\ otherwise \end{array} \right\}$$

$$\boldsymbol{H(n)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & & 0 & 0 & 0 \\ & & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & & 0 & -1 & 1 \end{bmatrix}$$

Then determinant of $\boldsymbol{H(n)}$ are of the type

$$|H(n)| = |H(n-1)| + |H(n-2)| + |H(n-3)| + |H(n-4)| \ \ for\ all\ n > 4$$

Where

$$|H(1)| = 1, |H(2)| = 2, |H(3)| = 4, |H(4)| = 8 \ldots.$$

4.4.6. Statement: - $|K(n)| = F_{3,n+2}\ for\ all\ n \geq 1$, where $|K(n)|$ is the determinant of nth term of above define (1.5) sequence of matrix and $F_{3,n+2}$is $(n+2)th$ term 3-Fibonacci sequence of number.

Proof: This conclusion will be established through the application of mathematical reasoning on first principles.

For $n = 1$ we have $|K(1)| = 1$ also $F_{3,3} = 1$

So we can say $|K(1)| = F_{3,3}$

So result is true for $n = 1$

By hypothesis result is true for $n \leq m$ (our hypothesis)

So we have $|K(m)| = F_{3,m+2} \quad for\ all\ \ n \le m$

So we have $|K(m-1)\,| = F_{3,m+1}, |K(m-2)\,| = F_{3,m}, |K(m)\,| = F_{3,m+2}$

Now we will find that result is also true for $n = m + 1$

Consider $|K(m+1)\,| = |K(m)\,| + |K(m-1)| + |K(m-2)|$

So we have after the putting all above value

$$|K(m+1)\,| = F_{3,m+2} + F_{3,m+1} + F_{3,m}$$

So we get

$$|K(m+1)\,| = F_{3,m+3}$$

So result is true for $n = m + 1$

So proves the result is true $for\ all\ n$

4.4.7. Statement: - $|Q(n)\,| = F_{4,n+3}\ for\ all\ n \ge 1$, where $|Q(n)\,|$ is the determinant of nth term of above define (1.6) sequence of matrix and $F_{3,n+3}$is $(\,n+3)th$ term 4-Fibonacci sequence of number.

Proof: - We will prove this result by principle mathematical induction.

For $n = 1$ we have $|K(1)\,| = 1$ also $F_{4,4} = 1$.

So we can say $|K(1)\,| = F_{4,4}$.

So result is true for $n = 1$.

By hypothesis result is true for $n \le m$ (our hypothesis).

So we have $|Q(n)| = F_{4,n+3} \quad for\ all\ \ n \le m$

So $|Q(m)\,| = F_{4,m+3}, |Q(m-1)\,| = F_{4,m+2}, |Q(m-2)\,| = F_{4,m+1}, |Q(m-3)\,| = F_{4,m}$

Now we will find that result is also true for $n = m + 1$.

Consider

$$|Q(m+1)\,| = |Q(m)\,| + |Q(m-1)| + |Q(m-2)| + |Q(m-3)|.$$

So we have after the putting all above value

$$|Q(m+1)| = F_{4,m+3} + F_{4,m+2} + F_{4,m+1} + F_{4,m}.$$

So we get

$$|Q(m+1)| = F_{4,m+4} .$$

So result is true for $n = m + 1$.

So proves the result is true for all n.

4.4.8. Statement: - $|H(n)| = F_{k,n+k-1}\ for\ all\ n \geq 1$, where $|H(n)|$ is the determinant of nth term of above define (1.7) sequence of matrix and $F_{k,n+k-1}$is $(n + k - 1)th$ term k-Fibonacci sequence of number.

Proof: - We will prove this result by principle mathematical induction.

For $n = 1$ we have $|H(1)| = 1$ also $F_{k,k} = 1$.

So we can say $|H(1)| = F_{k,k}$.

So result is true for $n = 1$.

By hypothesis result is true for $n \leq m$ (our hypothesis).

So we have $|H(n)| = F_{4,n+3}\ \ for\ all\ \ n \leq m.$

$|H(m)| = F_{k,m+k-1}, |H(m-1)| = F_{k,m+k-2}, \dots \dots \dots . |H(m-k+1)| = F_{k,m}.$

Now we will find that result is also true for $n = m + 1$.

Consider $|H(m+1)| = |H(m)| + |H(m-1)| + \cdots + |K(m-k+1)|$.

So we have after the putting all above value

$$|H(m+1)| = F_{k,m+k-1} + F_{k,m++k-2} + \cdots + F_{k,m+1} + F_{k,m} .$$

So we get

$$|H(m+1)| = F_{k,m+k} ,$$

So result is true for $n = m + 1$.

So proves the result is true for all n.

Chapter 5

Sequence of Complex Bilinear Transformations with Coefficients as Fibonacci numbers

5.1 Introduction

This chapter deals with a sequence of complex functions of rational types with coefficients as Fibonacci numbers, then proves many properties, results, and theorems on these sequences of complex rational functions. Also, all terms of sequence in a complex rational function are forming a bilinear transformation. We show that all terms of sequence of a complex rational function are meromorphic functions and also discuss the fixed points of all terms of sequence of a complex rational function and the singularities of all terms of sequence of a complex rational function. So, this chapter represent a special relation between two main branch of mathematics Number Theory and Complex Analysis [33, 34, 36 and 40].

5.1.1 Fibonacci number sequence

Fibonacci numbers sequence for non-negative integer $n \geq 2$ is given by

$$f_n = f_{n-1} + f_{n-2}, \tag{5.1}$$

with $f_0 = 0$ and $f_1 = 1$.The terms of the Fibonacci sequence are called Fibonacci numbers.

5.1.2 A Sequence generated by Fibonacci Numbers sequence

$$R_n = f_{n-1}f_{n+1} - {f_n}^2. \tag{5.2}$$

By equation (5.2) we have

$$R_1 = f_0 f_2 - {f_1}^2 = -1,$$

$$R_2 = f_1 f_3 - {f_2}^2 = 1,$$

$$R_3 = f_2 f_4 - {f_3}^2 = -1,$$

so, we can observe

$$R_n = (-1)^n.$$

5.1.3 Bilinear Transformation

A complex mapping

$$w(z) = \frac{az + b}{cz + d}, \tag{5.3}$$

where $ad - bc \neq 0$ is called the bilinear transformation mapping, a complex bilinear transformation mapping a circle or line into circle or line.

5.1.4 Meromorphic functions

A complex variable a function which has no singularities other than poles so we can say that a complex function is meromorphic possible singularities are only poles.

5.1.5 Conformal Mapping in Complex

A mapping which preserves the sense rotation as well as the magnitude of angle between images of curves called the conformal mapping and there is a famous result a mapping is conformal if it is differentiable and derivatives is non-zero.

5.1.6 Complex polynomials

A complex polynomial is one that can have constants and signs referred to as variables or be indeterminate to a non-negative integer power. For those terms that can be modified from one to another, if the normal characteristics of commutatively are used, the distribution with addition and multiplication distributive is considered to define the same polynomial. A complex polynomial within an indeterminate z may always have to be generated in the following way [41, 42 and 44]

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

where a_0 , $a_1, \dots, a_n$ are constants and z is the indeterminate. The word intermediate does not mean z represent is any unique value; but that any value will have to be replaced by any value. The characterization that marks the product of this replacement to the substituted value is a feature called the complex polynomial function.

$$\sum_{k=0}^{n} a_k z^k.$$

So that is, there can be either zero polynomials or that can be defined as the sum of the amount of nonzero. Each is a sum of the commodity of a numerical coefficient and several indeterminate conditions brought to non-negative integer powers [46, 47 and 48].

5.2. Sequence of complex rational functions

Let$z$ be any complex unknown and $u(z)$ is any function of z given by below

$$u(z) = \frac{1}{1+z}, \tag{5.4}$$

then we have,

$$(uou)(x) = \frac{1}{1+\frac{1}{1+z}}. \tag{5.5}$$

Now we define

$$w_n(x) = (uououo \dots ou)(x), \tag{5.6}$$

Where $(uououo \dots ou)$ represent n time composition.

Rational function as recurrence relation sequence for integer $n \geq 2$, is defined by

$$w_1(z) = \frac{1}{1+z}, \tag{5.7}$$

and

$$w_n(z) = \frac{1}{1+w_{n-1}(z)}. \tag{5.8}$$

Theorem 5.2.1: If $w_n(z)$ is given by (5.8), then $w_n(z)$ represented in the form of Fibonacci coefficients by

$$w_n(z) = \frac{f_{n-1}\, z + f_n}{f_n z + f_{n+1}}. \tag{5.9}$$

Proof: Principle mathematical induction can be used to prove this result.

If $n = 1$ by equation (5.1), (5.6) and (5.8) we will get

$$w_1(z) = u(z) = \frac{1}{1+z} = \frac{f_0\, z + f_1}{f_1\, z + f_1}. \tag{5.10}$$

So, for $n = 1$ the result holds.

Now consider for $n = k$ result is true, let

$$w_k(z) = \frac{f_{k-1}\, z + f_k}{f_k z + f_{k+1}}. \tag{5.11}$$

Consider for $n = k + 1$ by equation (5.8)

$$w_{k+1}(z) = \frac{1}{1 + w_k(z)},$$

by equation (5.8) we get

$$w_{k+1}(z) = \frac{1}{1 + \frac{f_{k-1}\, z + f_k}{f_k z + f_{k+1}}},$$

on solving

$$w_{k+1}(z) = \frac{f_k z + f_{k+1}}{(f_k + f_{k-1})z + f_{k+1} + f_k},$$

we have

$$w_{k+1}(z) = \frac{f_k z + f_{k+1}}{f_{k+1}\, z + f_{k+2}}.$$

So, the result is true for $n = k + 1$, therefore the result is true for all integer n using principle of mathematical induction.

Theorem 5.2.2: If $w_n(z)$ is defined by equation (5.9) then $w_n(z)$ is meromorphic function for all integer n.

Proof: To prove $w_n(z)$are meromorphic function for all integer value of n we will prove $w_n(z)$ have singularities are poles for all $w_n(z)$, for rational function $w_n(z)$

$$w_n(z) = \frac{p(z)}{q(z)},$$

Where$p(z)$ and $q(z)$ are polynomials of degree one and z is complex variable.

If z_0 is singular point of $w_n(z)$iff$q(z_0) = 0$, by equation (5.8)

$$z_0 = -\frac{f_{k+1}}{f_k},$$

Clearly$p\left(-\frac{f_{k+1}}{f_k}\right) \neq 0andq\left(-\frac{f_{k+1}}{f_k}\right) = 0$.

Theorem 5.2.3: If$w_n(z)$is defined by equation (5.9) then $w_n(z)$ is bilinear transformation for all integers$n$.

Proof: We know that the bilinear transformation mapping is defined by (5.3) and comparing(5.3), and (5.9) we get

$$a = f_{n-1}, b = f_n, c = f_n, d = f_{n+1},$$

Using above terms

$$ad - bc = f_{n-1}f_{n+1} - {f_n}^2.$$

So, we can say $w_n(z)$ is bilinear transformation if

$$ad - bc = f_{n-1}f_{n+1} - {f_n}^2 \neq 0,$$

by equation (5.2)

$$ad - bc = f_{n-1}f_{n+1} - {f_n}^2 = R_n = (-1)^n,$$

which proves that

$$ad - bc = f_{n-1}f_{n+1} - {f_n}^2 \neq 0.$$

So $w_n(z)$is bilinear transformation for all integer values of n.

Theorem 5.2.4: If$w_n(z)$is defined by equation (5.9) then $w_n(z)$ has same fixed point for all integers$n \geq 0$.

Proof: We know that for fixed point z of $w_n(z)$, $w_n(z) = z$, by equation (5.9)

$$w_n(z) = z = \frac{f_{n-1}\,z + f_n}{f_n z + f_{n+1}},$$

$$f_n z^2 - (\,f_{n-1} - f_{n+1})z - f_n = 0,$$

$$z = \frac{(f_{n-1} - f_{n+1}) \pm \sqrt{(f_{n-1} - f_{n+1})^2 + 4{f_n}^2}}{2f_n},$$

$$z = \frac{-f_n \pm \sqrt{5{f_n}^2}}{2f_n},$$

$$z = \frac{-1 \pm \sqrt{5}}{2}.$$

So, we can say that two fixed point of $w_n(z)$ are:

$$z = \frac{-1 \pm \sqrt{5}}{2}.$$

Theorem 5.2.5: If $w_n(z)$is defined by equation (5.9) then $w_n(z)$ are conformal functions in the unit disc for all integer values of n.

Proof: We know that a mapping is conformal if and only if it is differentiable and derivatives are non-zero; by equation (5.9)

$$w_n(z) = \frac{f_{n-1}\,z + f_n}{f_n z + f_{n+1}},$$

also$w_n(z)$is

$$w_n(z) = \frac{p(z)}{q(z)},$$

where $p(z) = f_{n-1}\,z + f_n$, $q(z) = f_n z + f_{n+1}$are degree one polynomials. So $w_n(z)$ is differentiable if and only if $q(z) \neq 0$.

On differentiating equation (5.9) with respect to z

$$w_n'(z) = \frac{f_{n-1}f_{n+1} - {f_n}^2}{(f_n z + f_{n+1})^2}, \tag{5.12}$$

by equation (5.2)

$$f_{n-1}f_{n+1} - {f_n}^2 \neq 0$$

if

$$(f_n z + f_{n+1})^2 = 0,$$

so,

$$z = -\frac{f_{n+1}}{f_n},$$

since $f_n < f_{n+1}$, $|z| > 1$, therefore $w_n'(z)$ is non-zero for all every z in unit disc, which proved $w_n(z)$ is conformal in unit disc.

Theorem 5.2.6: If$w_n(z)$is defined by equation (5.9) then $w_n(z)$ is "bilinear transformation" which maps unity circle into a circle center with on real axis for all value of n.

Proof: If bilinear transformation mapping is given by (5.3) as

$$w(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$, we have pole of bilinear transformation is at

$$z = -\frac{d}{c},$$

There is a famous result in complex algebra: if the pole of bilinear transformation does not lie on the boundary of a circle, then bilinear transformation maps that circle into a circle.

So, we have pole of

$$w_n(z) = \frac{f_{n-1}\,z + f_n}{f_n z + f_{n+1}},$$

at

$$z = -\frac{f_{n+1}}{f_n},$$

Since $f_n < f_{n+1}$, so, $|z| > 1$, we can say that the pole of $w_n(z)$ not lie on the boundary of the unit circle. The result tells us that all values in the sharp image of the unit circle $|z| = 1$ for the bilinear transformation $w_n(z)$in a circle. Compute the transformation $w_n(z)$ and prove the image of the unit circle$|z| = 1$. If this is again a circle centered on the real axis, then the proof of the theorem is complete. By equation (5.8)

$$w_n(z) = \frac{f_{n-1}\, z + f_n}{f_n z + f_{n+1}},$$

Solving above equation we find value of z

$$z = \frac{f_n - f_{n+1}\, w}{f_n w - f_{n-1}}, \tag{5.13}$$

We know that equation of unity circle is$|z| = 1$. Using equation (5.13) in$|z| = 1$

$$\left|\frac{f_n - f_{n+1}\, w}{f_n w - f_{n-1}}\right| = 1,$$

$$|f_n - f_{n+1}\, w| = |f_n w - f_{n-1}|,$$

$$|f_n - f_{n+1}\,(u + iv)| = |f_n(u + iv) - f_{n-1}|,$$

$$(f_n - f_{n+1}\, u)^2 + (f_{n+1}\, v)^2 = (f_n u - f_{n-1})^2 + (f_n\, v)^2,$$

$$(f_{n+1}^2 - f_n^2)u^2 + (f_{n+1}^2 - f_n^2)v^2 + 2(f_{n-1}f_n - f_n f_{n+1})u = f_{n-1}^2 - f_n^2,$$

$$u^2 + v^2 + \frac{2(f_{n-1}f_n - f_n f_{n+1})}{(f_{n+1}^2 - f_n^2)}u = \frac{f_{n-1}^2 - f_n^2}{(f_{n+1}^2 - f_n^2)}.$$

Since in above equation coefficients are same for u^2 and v^2.So we can say that it represents a circle and coefficient of v is zero so centre must lie on real axis. Therefore, we can say that "$w_n(z)$are bilinear transformation which maps unity circle into a circle center with on real axis for all integer values of n".

Theorem 5.2.7: If $w_n(z)$is defined by equation (5.9), then "$w_n(z)$ is a bilinear transformation that maps the upper half plane into the upper half plane for all even values of n".

Proof: If bilinear transformation mapping is given by (5.3) as

$$w(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$ and $z = x + iy$.

Pole of bilinear transformation is at

$$z = -\frac{d}{c}.$$

There is a famous result in complex algebra: "if pole of bilinear transformation lies on line bilinear transformation maps that line onto line".

So, we have pole of

$$w_n(z) = \frac{f_{n-1}\, z + f_n}{f_n z + f_{n+1}},$$

at

$$z = -\frac{f_{n+1}}{f_n}.$$

Since f_n and f_{n+1} are real z are also real valued so we can say that pole lie on real axis.

We know that the equation of the upper half part of the plane is $Im(z) \geq 0$, so $\mathrm{y} \geq 0$. So, boundary of this reason is $\mathrm{y} = 0$. By using result image of boundary is boundary. Putting $\mathrm{y} = 0$ in $w_n(z) = u + iv$

$$u + iv = \frac{f_{n-1}\, x + f_n}{f_n\, x + f_{n+1}},$$

on comparing real with real part and imaginary part with imaginary part, we obtained $v=0$ therefor image of $y=0$ is $v=0$. Now there are only two possibilities for image either image of $y>0$ is $v>0$ or $v<0$.

For finding image of $y>0$, we will use trial method for this put $z=i$ in $w_n(z)$we have

$$w_n(i)=\frac{f_{n-1}\,i+f_n}{f_n i+f_{n+1}},$$

on rationalizing the denominator

$$w_n(i)=\frac{f_{n-1}\,i+f_n}{f_n i+f_{n+1}}\times\frac{f_{n+1}\ -if_n}{f_{n+1}-if_n},$$

on Solving

$$w_n(i)=\frac{(f_n f_{n+1}\ +f_{n-1}\,f_n)+(f_{n-1}\,f_{n+1}\ -f_n^2)}{(f_{n+1}^2+f_n^2)},$$

by equation (5.2)

$$R_n=(-1)^n,$$

since we working only on even value of n

$$R_n=f_{n-1}f_{n+1}-{f_n}^2=1>0.$$

So, that values of $w_n(i)$ lies in $v>0$, which proves the theorem.

Theorem 5.2.8: If$w_n(z)$is defined by equation (5.9), "then $w_n(z)$ is a bilinear transformation that maps the upper half plane into the lower half plane for all odd values of n".

Proof: If bilinear transformation mapping is given by (5.3) as

$$w(z)=\frac{az+b}{cz+d},$$

where $ad-bc\neq 0$ and $z=x+iy$.

Pole of bilinear transformation is at

$$z = -\frac{d}{c}.$$

There is a famous result in complex in complex algebra if pole of a bilinear transformation lies on line bilinear transformation maps that line onto line.

So, we have pole of

$$w_n(z) = \frac{f_{n-1}\,z + f_n}{f_n z + f_{n+1}},$$

at

$$z = -\frac{f_{n+1}}{f_n},$$

since f_n and f_{n+1} are real then we can say that z is real value, so we can say that pole lie on a real axis.

We know that the condition of the upper-half plane is $Im(z) \geq 0$, so $\mathrm{y} \geq 0$. So, boundary of this reason is $\mathrm{y} = 0$. By using result image of boundary is boundary. Putting $\mathrm{y} = 0$ in $w_n(z) = u + iv$

$$u + iv = \frac{f_{n-1}\,x + f_n}{f_n x + f_{n+1}},$$

Comparing the real and imaginary parts gives$v = 0$,so the image of $y = 0$ is $v = 0$. Now there are only two possibilities for image either image of $y > 0$ is $v > 0$or $v < 0$.

For finding image of $y > 0$, we will use trial method for this put $z = i$ in $w_n(z)$

$$w_n(i) = \frac{f_{n-1}\,i + f_n}{f_n\,i + f_{n+1}},$$

on rationalizing and solving

$$w_n(i) = \frac{(f_n f_{n+1} \ + f_{n-1}\,f_n) + (f_{n-1}\,f_{n+1} \ - f_n^2)}{(f_{n+1}^2 + f_n^2)},$$

by equation (5.2)

$$R_n = (-1)^n,$$

since we are working on only odd value of n we have

$$R_n = f_{n-1}f_{n+1} - {f_n}^2 = -1 < 0.$$

So, that values of $w_n(i)$ lies in $v < 0$, which proves the theorem.

Chapter 6

Relations between Chebyshev Polynomials and Hermite Polynomials

6.1 Introduction

In this chapter, we have obtained the relation between the "Chebyshev polynomial of the second kind" and Hermite polynomials of two variables, also the generating function is obtained with the help of the Hermite polynomial [8, 49 and 51].

6.2 Relations between Chebyshev Polynomials and Hermite Polynomials

"The Chebyshev polynomial of the first kind $\{T_n(x)\}$ and the second kind $\{U_n(x)\}$" for all integers $n \geq 0$ are given by

$$T_{n+2}(x) = 2xT_n(x) - T_n(x), T_1(x) = x, T_0(x) = 1. \tag{6.1}$$

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x), U_1(x) = 2x, U_0(x) = 1. (6.2)$$

Then the explicit representation of $T_n(x)$ and $U_n(x)$are respectively

$$T_n(x) = \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k \frac{(n-k-1)!}{k!\,(n-2k)!}(2x)^{n-2k}, |x| < 1. \tag{6.3}$$

and

$$U_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k \frac{(n-k-1)!}{k!\,(n-2k)!}(2x)^{n-2k}, |x| < 1. \tag{6.4}$$

If we take $x = \cos\gamma$, then

$$T_n(\cos\gamma) = \cos(n\gamma). \tag{6.5}$$

$$U_n(\cos\gamma) = \frac{\sin(n+1)\gamma}{\sin\gamma}. \tag{6.6}$$

In analysis, Chebyshev polynomials show integral representations of the Hermite polynomials and the generation process will add the new representations of Chebyshev polynomials.

Proposition 6.2.1. If $T_n(x)$, $U_n(x)$ are Chebyshev polynomials defined by (6.1), (6.2) and $H_n(x,y)$ two-variable Hermite polynomial then [52, 53]

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t}t^n H_n(2x, -\frac{1}{t})dt. \tag{6.7}$$

and

$$T_n(x) = \frac{1}{2(n-1)!}\int_0^{\infty} e^{-t}t^n H_n(2x, -\frac{1}{t})dt. \tag{6.8}$$

Proof: By taking note of this

$$n! = \int_0^{\infty} e^{-t}t^n dt.$$

Replacing n by $n-k$

$$(n-k)! = \int_0^{\infty} e^{-t}t^{n-k}dt. \tag{6.9}$$

For $H_n(x,y)$ and $U_n(x)$ the explicit forms are

$$H_n(x,y) = n!\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(y)^k(x)^{n-2k}}{k!\,(n-2k)!}. \tag{6.10}$$

$$U_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (n-k)!\,(2x)^{n-2k}}{k!\,(n-2k)!}. \tag{6.11}$$

In equation (6.9) replacing x by $2x$ and y replacing by $-\frac{1}{t}$ we will get

$$H_n\left(2x, -\frac{1}{t}\right) = n! \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^k (2x)^{n-2k} t^{-k}}{k!\,(n-2k)!}, \tag{6.12}$$

if $e^{-t}t^n$ is multiplied both sides of equation (6.11)and integrating in the limit $0\ to\ \infty$ we will get

$$\int_0^\infty e^{-t}t^n\, H_n\left(2x, -\frac{1}{t}\right) dt = n! \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!\,(n-2k)!} \int_0^\infty e^{-t}\, t^{n-k} dt, \tag{6.13}$$

by using equation (6.9) we will get

$$\int_0^\infty e^{-t}t^n\, H_n\left(2x, -\frac{1}{t}\right) dt = n! \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^k (2x)^{n-2k} (n-k)!}{k!\,(n-2k)!},$$

by equation (6.11) we have

$$U_n(x) = \frac{1}{n!} \int_0^{+\infty} e^{-t}t^n H_n(2x, -\frac{1}{t}) dt.$$

which proves the result.

Theorem 6.2.1: If $T_n(x)$, $U_n(x)$ are Chebyshev polynomials defined by (6.1), (6.2) then:

$$\frac{d}{dx} U_n(x) = nW_{n-1}(x),$$

$$U_{n+1}(x) = xW_n(x) - \frac{n}{n+1} W_{n-1}(x),$$

were

$$W_n(x) = \frac{2}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1}H_n(2x,-\frac{1}{t})dt.$$

Proof: From preposition 6.2.1 $H_n(x,y)$ can be costumed as follows:

$$\left[2x-\frac{1}{t}\frac{\partial}{\partial x}\right]H_n\left(2x,-\frac{1}{t}\right)=H_{n+1}\left(2x,-\frac{1}{t}\right), \tag{6.14}$$

$$\frac{1}{2}\frac{\partial}{\partial x}H_n\left(2x,-\frac{1}{t}\right)=nH_{n-1}\left(2x,-\frac{1}{t}\right). \tag{6.15}$$

First, we will prove identity (6.14) and (6.15), consider

$$\frac{\partial H_n(2x,-\frac{1}{t})}{2\partial x}=n!\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{(-1)^k(2x)^{n-2k-1}t^{-k}}{k!\,(n-2k-1)!},$$

so, by equation (6.12) we have

$$\frac{1}{2}\frac{\partial}{\partial x}H_n\left(2x,-\frac{1}{t}\right)=nH_{n-1}\left(2x,-\frac{1}{t}\right).$$

Which proves (6.15) in the same way we can prove (6.14).

It obtains integral representations in the relations of Chebyshev polynomials and by equation (6.15)

$$\frac{d}{dx}U_n(x)=\frac{2n}{n!}\int_0^{+\infty} e^{-t}t^nH_n(2x,-\frac{1}{t})dt. \tag{6.16}$$

$$\frac{d}{dx}T_n(x)=\frac{n}{(n-1)!}\int_0^{+\infty} e^{-t}t^{n-1}H_{n-1}(2x,-\frac{1}{t})dt. \tag{6.17}$$

The relation above provides a link between polynomials $U_n(x)$ and $H_n(x)$, however, as

$$U_{n-1}(x) = \frac{1}{(n-1)!}\int_{0}^{+\infty} e^{-t}t^{n-1}H_{n-1}(2x,-\frac{1}{t})dt.$$

By using the second kind of Chebyshev polynomial equation (6.7) in the first identity (6.16)

$$U_{n+1}(x) = \frac{1}{(n+1)!}\int_{0}^{+\infty} e^{-t}t^{n+1}H_{n+1}(2x,-\frac{1}{t})dt,$$

using the relation

$$\left[2x + \frac{1}{-t}\frac{\partial}{\partial x}\right] H_n\left(2x, \frac{1}{-t}\right) = H_{n+1}\left(2x, -\frac{1}{t}\right),$$

we have

$$U_{n+1}(x) = x\frac{2}{(n+1)!}\int_{0}^{+\infty} e^{-t}t^{n+1}H_n(2x,-\frac{1}{t})dt$$
$$-\frac{1}{(n+1)!}\int_{0}^{+\infty} e^{-t}t^{n}\frac{\partial}{\partial x}H_n(2\mathrm{x},-\frac{1}{t})dt,$$

using

$$\frac{1}{2}\frac{\partial}{\partial x}H_n\left(2x, -\frac{1}{t}\right) = nH_{n-1}\left(2x, -\frac{1}{t}\right),$$

We obtained

$$U_{n+1}(x) = x\frac{2}{(n+1)!}\int_{0}^{+\infty} e^{-t}t^{n+1}H_n(2x,-\frac{1}{t})dt$$
$$-\frac{2n}{(n+1)!}\int_{0}^{+\infty} e^{-t}t^{n}H_{n-1}(2x,-\frac{1}{t})dt. \qquad (6.18)$$

From R.H.S second term of (6.18) given following polynomial

$$W_n(x) = \frac{2}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1}H_n(2x, -\frac{1}{t})dt.$$

replacing$n$ by $n-1$,

$$W_{n-1}(x) = \frac{2}{(n)!}\int_0^{+\infty} e^{-t}t^{n}H_{n-1}(2x, -\frac{1}{t})dt,$$

so, we obtained

$$\frac{d}{dx}U_n(x) = nW_{n-1}(x),$$

and

$$U_{n+1}(x) = xW_n(x) - \frac{n}{n+1}W_{n-1}(x).$$

Which proves the theorem.

6.3 Generating functions of Chebyshev polynomial by Hermite polynomial

The second type of Chebyshev polynomial can draw a slightly different links from Hermite polynomial and their generating functions. Both sides of the equation (6.7) are multiplied by ξ^n, $|\xi| < 1$ and over n taking summation [57, 58, 59 and 61]

$$\sum_{n=0}^{+\infty} \xi^n\, U_n(x) = \int_0^{+\infty} e^{-t}\sum_{n=0}^{+\infty}\frac{(t\xi)^n}{n!}H_n\left(2x, -\frac{1}{t}\right)dt, \tag{6.19}$$

by remembering the polynomials

$$\sum_{n=0}^{+\infty}\frac{\mathrm{t}^{\mathrm{n}}}{\mathrm{n}!}H_n(x,y) = e^{(xt+yt^2)},$$

in (6.19) and solving we get

$$\sum_{n=0}^{+\infty} \xi^n U_n(x) = \frac{1}{1 - 2\xi x + \xi^2}.$$

which is required generating function for $U_n(x)$.

Chapter 7

Applications of Recurrence Relations

7.1 Introduction

In this chapter, applications of recurrence relations in network marketing are discussed with some limitations imposed on the problem. In the later parts of the chapter, the application of recurrence relations, especially Fibonacci numbers, and the reproduction mechanism of honey bees are verified, and Fibonacci numbers in blooms are viewed [63, 65 and 66].

7.2 Application of Recurrence Relation in Network Marketing

People are compensated in network marketing not only for the work that they produce themselves but also for the work that is generated by other employees who report to them. Because of the hierarchical structure and the network of distributors, one type of business model for networks is referred to as a "down line model." This model includes numerous levels of compensation for distributors. The problem that is being discussed is that certain limitations, such as the percentage distribution of profit, should be such that the roots of the polynomial for which the recurrence relation is defined must be distinct; any worker cannot leave the business; workers should be honest; and there is an assumption that every worker can only take one worker under him. These limitations should be met [67, 68 and 69].

In its early stages, network marketing was primarily focused on the sale of nutritional supplements, cosmetics, and household goods. The concept was first introduced in the 1950s, and by the 1980s, network marketing businesses had expanded to include companies that specialized in providing long-distance telephone services and insurance [70, 71 and 76].

There are a wide variety of business platforms available for use in network marketing. Word-of-mouth marketing and relationship referrals are two of the most common ways that employees offer products directly to customers in most of these businesses.

The majority of businesses in the network marketing industry focus on providing opportunities to people who, in other circumstances, might not have them, including individuals who are [10]

- Less certainty in running their claim trade.
- Have exceptionally little sum of cash to invest.
- With current work level, people are not happy.
- Own businesses were not running successfully.

Figure 7.1 Network Marketing Tree

7.2.1 Theorem on recurrence relation sequence

Recall the Theorem 2.2.1, we have

For real numbers c_1, c_2 and let

$$x^2 - c_1 x - c_2 = 0, \tag{7.1}$$

having distinct roots x_1 and x_2 are distinct roots.

Then the sequence $< a_n >$has solution

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} n \geq 2, \tag{7.2}$$

with initial terms $a_0 = A_1, a_1 = A_2$.

iff

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n,$$

For $n = 0, 1, 2, \ldots$,and forrandom constants β_1 and β_2.

Preposition 7.2.1: Allow a man A to begin his network business by taking on the task of a man B under him, and A must agree to share 50 percent of his profit with B. Now, B will be responsible for the labor of a man C under him, and C will receive 35% of B's profit while A receives 2% of B's profit. Now let's say that C hires D to work for him, and under this arrangement D receives 35% of C's profits while C receives 2% of D's profits. If there are one hundred individuals working in this network business, then each working man will receive 35% of the profits earned by his immediate predecessor, as well as 2% of the profits earned by his immediate predecessor's immediate predecessor. If the profit of man A is Rs. 400,000, then you need to figure out what the profit of man 10 is.

Solution: let $a_1$is the profit of the man A and a_2 is the profit of man B and a_n is the profit of the n^{th} man. Then by given condition we have $a_1 = 400000$ and $a_2 = 200000$

$$a_n = \frac{35}{100} a_{n-1} + \frac{2}{100} a_{n-2}, n > 2. \tag{7.3}$$

The characteristic equation of (7.3) is

$$x^2 - \frac{35}{100} x - \frac{2}{100} = 0, \tag{7.4}$$

Solving equation (7.2) we have $\frac{2}{5}$ and $-\frac{1}{20}$then using above theorem 7.2.1 we have

$$a_n = \beta_1 (\frac{2}{5})^n + \beta_2 (-\frac{1}{20})^n,$$

by given condition for $n = 1,2$

$$\beta_1 \left(\frac{2}{5}\right)^1 + \beta_2 \left(-\frac{1}{20}\right)^1 = 400000, \tag{7.5}$$

$$\beta_1 \left(\frac{2}{5}\right)^2 + \beta_2 \left(-\frac{1}{20}\right)^2 = 200000, \tag{7.6}$$

Solving (7.5) and (7.6) we have

$$\beta_1 = \frac{11000000}{9},$$

and

$$\beta_2 = \frac{16000000}{9},$$

By theorem 7.2.1 we can write

$$a_n = \frac{11000000}{9}\left(\frac{2}{5}\right)^n + \frac{16000000}{9}\left(-\frac{1}{20}\right)^n,$$

so put $n = 10$ we have $a_{10} = 128.15$

profit of any worker in the network line can be find out using above method.

Limitation:

- The percentage distribution of profit should be such that the roots of a polynomial must be distinct.
- Any worker cannot leave the business.
- Work should be honest.
- There is an assumption every worker can take only worker under him.

7.3 Amazing Applications of the Fibonacci Numbers

“The Fibonacci numbers” and the associated "Golden Ratio" are shown in nature and in specific show-stoppers. We study those huge numbers in nature and pursue the Fibonacci sequence. It shows up in biological settings, for example, in the fanning of trees, phyllo taxis (the course of action of leaves on a stem), the natural product sprouts of a pineapple, the blooming of an artichoke's uncurling greenery, the game plan of a pine cone's bracts, and so on. At present, Fibonacci numbers assume a significant role in coding innovation hypotheses.

7.3.1 Reproduction mechanism of Bee’s

The reproduction mechanism of Bee model is much more realistic as far as the Fibonacci numbers are concerned. The Fibonacci numbers were first uncovered by a man named Leonardo Pisano. He was notable for his Fibonacci. The Fibonacci sequence is a sequence wherein each term is the aggregate of the two numbers going before it [4]. By equation (1.7)

$$f_n = f_{n-1} + f_{n-2},$$

For integers $n \geq 2$, where$f_0 = 0$, $f_1 = 1$ andf_n represents the nth Fibonacci number.[39]

Reproduction in bees is flawlessly described by Fibonacci numbers. The Fibonacci numbers verify numerous unusual characteristics of a honeybee's family. Honey bees have some unusual facts, such as the fact that not every one of them has two parents. The queen is a unique female in the honeybee community. There are numerous working drones who are female, not at all like the queen honey bee; no eggs are produced by them. There are some male automaton bees who do not work. Unfertilized eggs from a queen's ovaries produce males, so male bees just have a mother. Females are formed when a queen bee mates with a male, so a female bee has two parents. Females usually become worker bees, so a female bee has both a male and a female parent, while a male bee has only one female bee as parent. Based on all the above facts, relations between the reproduction mechanisms of bees and Fibonacci numbers are discussed [79, 80, 81and 82].

Queen bees lay eggs only if the eggs are: Fertilized or Non fertilized then respective bees are Workers females or Drones→ males respectively.

Figure7.2: Honey bee with eggs.

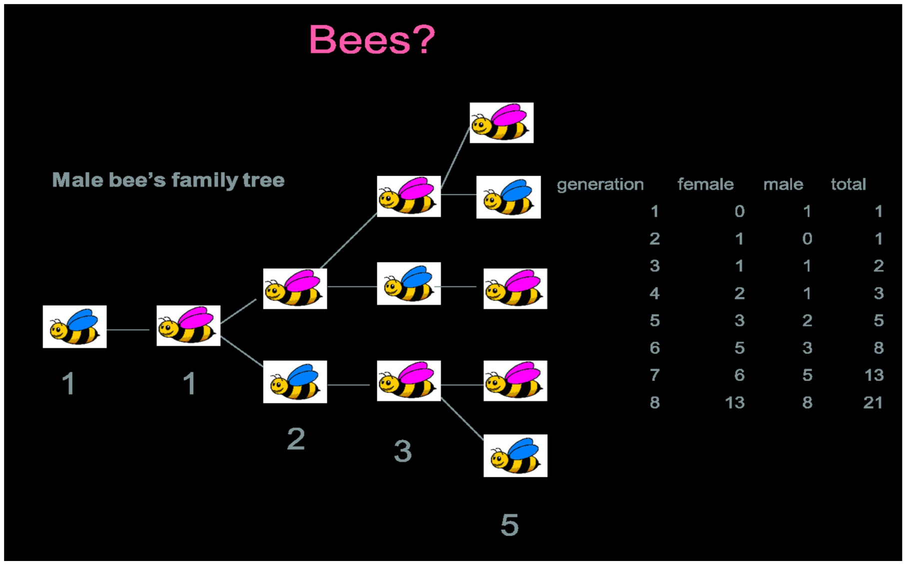

Figure 7.3 Male bee's family tree

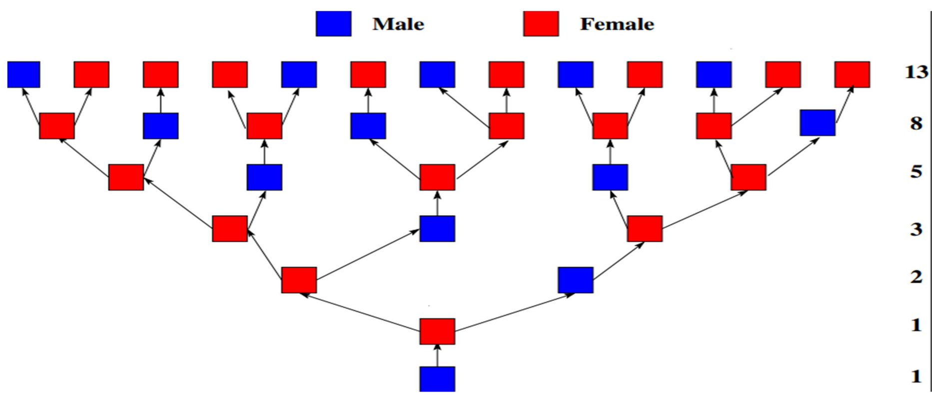

Figure 7.4: Family of Male and Female bees.

The ration to two consecutive Fibonacci numbers second divided by first is called Golden Ratio, the value 1.618 is Golden Ratio. Honey bees are shown by both Fibonacci numbers and the Golden ratio. The Fibonacci numbers are very much represented in honeybees. For instance, on the off chance that you pursue the family tree of honeybees, it follows the Fibonacci sequence splendidly [3, 8]. On the off chance that you have taken any hive and pursue this pattern, it would resemble this

Table 7.1 Honey bee and Fibonacci numbers.

Number	*Parents*	*Grand Parents*	*Great Grand Parents*	*Great - Great Grand Parents*	*Great - Great - Great Grand Parents*
Male bees	*1*	*2*	*3*	*5*	*8*
Female bees	*2*	*3*	*5*	*8*	*13*

Dividing the number of females by the number of drones yields the golden ratio of 1.618. This series of numbers works randomly for each bee hive. Usually, honeybee hives are always used to clarify the Fibonacci sequence and the Golden Ratio [5,9].

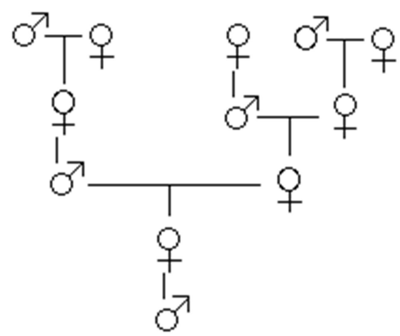

Figure 7.5 Honey Bee family Tree.

Now let's take a gander at the male honey family tree of a bee called A. A (symbol at the base of the tree for male, a hover with over a bolt) parent as one (female honey bee represent the queen honey bee symbol, a hover over across) a queen honey bee has two parents. This means grandparents of A were two. His granddad will just have one parent, while his grandma will have two, so altogether there were three great-grandparents of A. One of which will be male and, along these lines, have one parent, whereas the other two are female and, in this way, have a total of four parents. So, the total count of great-great-grandparents of A was five. Proceeding with this, one can find that the great-great-great-grandparents of A were eight, the great-great-great-great grandparents were thirteen, etc. Again, it is the Fibonacci sequence [8].

Table 7.2: Sequence of Drone and Worker or Queen Bee’s as Fibonacci numbers.

Generation	*Drone*	*Worker or Queen*
1	1	2
2	2	3
3	3	5
4	5	8
5	8	13
6	13	21

7.3.2 Fibonacci Sequence in the home garden

Fibonacci numbers can be stated in nature in lovely blossoms, on the leader of a sunflower and the seeds are pressed with a particular goal in mind so they pursue the example of the Fibonacci sequence. This winding keeps the seeds of the sun-flower from swarming themselves out consequently helping them with endurance. The petals of blossoms and different plants may likewise be identified with the Fibonacci sequence in the manner in which they make new petals [5]. Fibonacci can be originating in nature. It can be seen in the following flowers, leaf and in vegetables that daily we are consuming. God created the flowers with 3 petals,5 petals,8 petals so on. It is in the Fibonacci sequence [6].

Figure 7.6: 3 petals flowers.

Figure 7.7: 3 petals flowers.

Figure 7.8: 5 petals flowers.

Figure 7.9: 8 petals flowers.

Figure 7.10: 21 petals flowers.

Figure 7.11: 5seeds count of fruit.

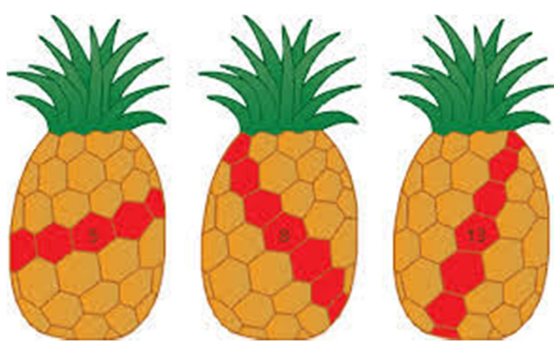

Figure 7.12: Fibonacci numbers on Pineapples.

Summary and Conclusions

In the first chapter, we provide an overview of the recurrence relations of numbers that make up history, as well as their applications in a variety of different disciplines. In addition, we make a cursory review of a few key definitions and well-known results that are required to some degree in order to proceed to the consecutive chapters.

In chapter 2, we demonstrated how the roots and terms of recurrence relations of the first, second, third, fourth, and kth orders are connected to one another. In addition, the results on particular kinds of recurrence relations, such as those involving Fibonacci polynomials and Chebyshev polynomials, have been obtained, and some of these findings are as follows:

- For arbitrary real numbers c_1 and c_2, if x_1 and x_2 are different roots of

  $$x^2 - c_1x - c_2 = 0,$$

  Then sequence $< a_n >$ has a solution

  $$a_n = c_1a_{n-1} + c_2a_{n-2}n \geq 2,$$

  with given initial terms $a_0 = A_1$ and $a_1 = A_2$, if and only if $a_n = \beta_1x_1^n + \beta_2x_2^n$,

  for arbitrary constants β_1 and β_2.

- For random real numbers c_1 and c_2 the sequence $< a_n >$ has a solution

  $$a_n = c_1a_{n-1} + c_2a_{n-2}n \geq 2,$$

  with given initial terms $a_0 = A_1$ and $a_1 = A_2$, if in equation above sequence c_1 and c_2 are such that roots of $x^2 - c_1x - c_2 = 0$ are distinct and greater than 1 and satisfied the condition $A_1x_1 > A_2$ and $A_1x_2 < A_2$ then recurrence relation sequence must be divergent.

- For real numbers c_1, c_2, and c_3 let

  $$x^3 - c_1x^2 - c_2x - c_3 = 0,$$

Has distinct roots x_1, x_2 and x_3. Then sequence $< a_n >$ has a solution

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}, n \geq 3,$$

with given initial terms $a_0 = A_1$, $a_1 = A_2$, and $a_2 = A_3$.

$$iff$$

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n,$$

for $n = 0, 1, 2, \ldots$, with arbitrary constants β_1, β_2 and β_3.

- For real numbers c_1, c_2, c_3, and c_4 and let

$$x^4 - c_1 x^3 - c_2 x^2 - c_3 x - c_4 = 0,$$

Has distinct roots x_1, x_2, x_3 and x_4. Then the sequence $< a_n >$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4}, n \geq 4,$$

with given initial terms $a_0 = A_1$, $a_1 = A_2$, $a_2 = A_3$, and $a_3 = A_4$.

iff

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \beta_4 x_4^n$$

for $n = 0, 1, 2, \ldots$, with arbitrary constants $\beta_1, \beta_2, \beta_3$ and β_4.

For real numbers $c_1, c_2, c_3, \ldots, c_k$, let

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} \ldots - c_k = 0,$$

have distinct roots $x_1, x_2, x_3 \ldots, x_k$. Then the sequence $< a_n >$ has a solution

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots \ldots + c_k a_{n-k}, n > k,$$

with given initial terms $a_0 = A_1$, $a_1 = A_2$, $a_2 = A_3, \ldots, a_{k-1} = A_k$.

$$iff$$

$$a_n = \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n + \cdots + \beta_k x_k^n$$

for $n = 0, 1, 2$ ....with arbitrary constants $\beta_1, \beta_2, \beta_3 \ldots \beta_k$.

In Chapter 3 we have worked on recurrence relation of rational function in the form of composition function have terms Fibonacci numbers or Generalized Fibonacci numbers, then we have considered a set based on the defined composition function and properties of group are verified, the main results of the research outcomes are

- If $u: (0, \infty) \rightarrow (0,1)$is real valued function given by

  $$u(x) = \frac{1}{1+x},$$

  $u(x)$ Continuous in its domain. The co-domain of u is a subset of the domain of u, so consider function

  $$(u\ o\ u)(x) = \frac{1}{1+\frac{1}{1+x}},$$

  Then $z_n(x) = (uououo \ldots ou)(x)$.
  The recurrence relation sequence of rational function is

  $$z_1(x) = u(x) = \frac{1}{1+x},$$

  and

  $$z_n(x) = \frac{1}{1+z_{n-1}(x)},$$

  For integer $n \geq 2$.

  Then, $z_n(x)$ is

  $$z_n(x) = \frac{f_{n-1}\, x + f_n}{f_n\, x + f_{n+1}},$$

  where f_i ith Fibonacci number and $z_n(x)n$th term of sequence of rational function. For $n \in N$, the codomain of $z_n(x)$ is

$$A_n = \left(\min\left\{\frac{f_{n-1}}{f_n}, \frac{f_n}{f_{n+1}}\right\}, \max\left\{\frac{f_{n-1}}{f_n}, \frac{f_n}{f_{n+1}}\right\}\right)$$

- Let $I: (0, \infty) \to (0, \infty), I(x) = x$ and Let $\boldsymbol{G}$ be set of all $z_n(x)$ for all $n \in N$ and including **I** function, then with respect to composition operation given by equation$z_n(x)\boldsymbol{G}$ is cyclic group.
- If $v: (0, \infty) \to (0,1)$a function with real value given by

$$v(x) = \frac{1}{q + x},$$

where q in any positive integer, $v(x)$ continuous on its domain and codomain of u is subset of domain of u. Considered function

$$(v\ o\ v)(x) = \frac{1}{1 + \frac{1}{q+x}},$$

And defined $w_n(x) = (vovovo \dots ov)(x)$.

Recurrence relation sequence of rational function is

$$w_1(x) = v(x) = \frac{1}{q + x},$$

$$w_n(x) = \frac{1}{q + w_{n-1}(x)},$$

for all integer$n \geq 2$.

Now, verify that each member of this family has the same coefficient as the generalized Fibonacci number. If we take $p = q,\ q = 1,\ a = 0,\ b = 1,$ by equation (3.3) we have generalized Fibonacci sequence

$$F_n = qF_{n-1} + F_{n-2},$$

$\forall\, n \geq 2, F_0 = 0\ \ F_1 = 1$,where q is any positive integer, then proved that

$$w_n(x) = \frac{F_{n-1}\, x + F_n}{F_n\, x + F_{n+1}},$$

Where F_i, ith generalized Fibonacci number, $w_n(x) nth$ term of equation sequence of rational functions.

- Then proved the result that $w_n(x)$ is monotonic function.
- For integer$n \in N$, the range set B_n of $z_n(x)$ is $B_n = \left(\frac{F_{n-1}}{F_n} \frac{F_n}{F_{n+1}}\right)$for odd$n$and $B_n = \left(\frac{F_n}{F_{n+1}}, \frac{F_{n-1}}{F_n}\right)$for n even.
- Let $I: (0, \infty) \to (0, \infty)$: $I(x) = x$, consider$\boldsymbol{H}$ set of all $w_n(x)\ \forall\ n \in N$ and including $\mathbf{I}$ function, then $\boldsymbol{H}$ is cyclic group with respect composition operation given by $w_n(x)$.

In Chapter 4, we have worked on the sequence of tri-diagonal matrices for generalized Fibonacci polynomials, Fibonacci numbers, and Chebyshev polynomials, defining the sequence of tri-diagonal matrices for different cases, and proving the results for corresponding recurrence relations. The main results are:

- For $n \in N$, sequence of tri diagonal matrix $\{\mathrm{A(n)} = [g_{n,n}]\}$ is

$$[g_{i,j}] = \begin{Bmatrix} g_{i,j} = ax & if\ \ j = i \\ g_{i,j} = -b & if\ j - 1 = i \\ g_{i,j} = 1 & if\ \ j + 1 = i \\ g_{i,j} = 0 & otherwise \end{Bmatrix},$$

so, that

$$A(n) = \begin{bmatrix} ax & -b & 0 & \cdots & \cdots & 0 \\ 1 & ax & -b & \cdots & \cdots & 0 \\ 0 & 1 & ax & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & ax & -b \\ 0 & 0 & \cdots & \cdots & 1 & ax \end{bmatrix}.$$

Then determinants of $\boldsymbol{A(n)}$ is

$$|A(n)| = g_{n,n}|A(n-1)| - g_{n,n-1}g_{n-1,n}|A(n-2)|$$

Let $|A(n)\ | = G_{n+1}\ \forall\ n \geq 1$, where $|A(n)\ |$the determinant of $A(n)$ and $G_{n+1}$is $(\ n + 1)th$ term of generalized Fibonacci sequence ofpolynomials.

- We have a sequence of tri-diagonal matrices,$\{C(n) = [h_{n,n}]\}$

$$[h_{i,j}] = \begin{Bmatrix} h_{i,j} = x & if\ \ j = i \\ h_{i,j} = -1 & if\ j-1 = i \\ h_{i,j} = 1 & if\ \ j+1 = i \\ h_{i,j} = 0 & otherwise \end{Bmatrix},$$

$$C(n) \ = \begin{bmatrix} x & -1 & 0 & \cdots & \cdots & 0 \\ 1 & x & -1 & \cdots & \cdots & 0 \\ 0 & 1 & x & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x & -1 \\ 0 & 0 & \cdots & \cdots & 1 & x \end{bmatrix}.$$

Then determinant of $C(n)$is

$|C(n)\ | = h_{n,n}|C(n-1)\ | - h_{n,n-1}h_{n-1,n}|C(n-2)|.$

Proved the result that $|C(n)\ | = F_{n+1}(x)$, $\forall$ integers $n \geq 1$, where $|C(n)\ |$ is the determinant of $C(n)$ and $F_{n+1}(x)$ is $(\ n+1)th$ term Fibonacci sequence of polynomials.

- For $n \in N$we define a sequence of tri diagonal matrices $\{D(n) = [q_{n,n}]\}$

$$[q_{i,j}] = \begin{Bmatrix} q_{i,j} = a & if\ \ j = i \\ q_{i,j} = -b & if\ j-1 = i \\ q_{i,j} = 1 & if\ \ j+1 = i \\ q_{i,j} = 0 & otherwise \end{Bmatrix},$$

so, that

$$D(n) \ = \begin{bmatrix} a & -b & 0 & \cdots & \cdots & 0 \\ 1 & a & -b & \cdots & \cdots & 0 \\ 0 & 1 & a & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & a & -b \\ 0 & 0 & \cdots & \cdots & 1 & a \end{bmatrix}.$$

Then determinants of $D(n)$is

$|D(n)\ | = q_{n,n}|D(n-1)\ | - q_{n,n-1}q_{n-1,n}|D(n-2)|.$

Proved the result $|D(n)| = V_{n+1}$ $\forall$ integers $n \geq 1$ where $|D(n)|$ the determinant of $D(n)$ and V_{n+1} is $(n+1)th$ term of a certain case of generalized Fibonacci sequence.

- For sequence of tri-diagonal matrix $\{R(n) = [l_{i,j}]\}$

$$[l_{i,j}] = \begin{Bmatrix} l_{i,j} = 2x & if\ \ j = i \\ l_{i,j} = 1 & if\ j-1 = i \\ l_{i,j} = 1 & if\ \ j+1 = i \\ l_{i,j} = 0 & otherwise \end{Bmatrix},$$

$$R(n) = \begin{bmatrix} 2x & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2x & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 2x & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 2x & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 2x \end{bmatrix}.$$

Then determinant of $R(n)$ is

$$|R(n)| = l_{n,n}|R(n-1)| - l_{n,n-1}l_{n-1,n}|R(n-2)|.$$

Proved the result $|R(n)| = 2T_n$ for integer $n \geq 1$ where $|R(n)|$ is the determinant of $R(n)$ and T_n is nth term of Chebyshev polynomials

In Chapter 5 we have mainly focuses on $w_n(z)$ the sequences of complex ration functions with coefficients as Fibonacci numbers, verifying properties of bilinear transformations for $w_n(z)$, the main research outcomes are:

- Let z be any complex unknown and $u(z)$ is any function of z given by below

$$u(z) = \frac{1}{1+z},$$

then

$$(uou)(x) = \frac{1}{1+\frac{1}{1+z}},$$

Now we define $w_n(x) = (uououo \ldots ou)(x)$. Then Recurrence relation sequence of rational function is defined as

$$w_1(z) = \frac{1}{1+z},$$

and

$$w_n(z) = \frac{1}{1+w_{n-1}(z)},$$

for all integers $n \geq 2$, then we have proved that $w_n(z)$ represented in form of Fibonacci numbers by

$$w_n(z) = \frac{f_{n-1}\, z + f_n}{f_n z + f_{n+1}}.$$

- The next result proved that $w_n(z)$ is a meromorphic function for integer $n \geq 0$.
- Then we have proved that $w_n(z)$ is bilinear transformation for integer $n \geq 0$.
- The fixed points of $w_n(z)$ for all integer values of n are discussed, and they are

$$z = \frac{-1 \pm \sqrt{5}}{2}.$$

- Then proved the result that $w_n(z)$ are conformal functions in unit disc for all integer value of n.
- Next result we have proved that "bilinear transformation $w_n(z)$ maps unity circle into a circle center with on real axis for all value of n".
- Then a "bilinear transformation, which $w_n(z)$ maps the upper half plane into the upper half plan for all even values of n" is discussed.
- Last result in this chapter is "bilinear transformation $w_n(z)$ maps upper half plane into lower half plan for all odd values of n".

In Chapter 6 we have obtained the relation between the Chebyshev polynomial of second kind and Hermite polynomials of two variables, also the generating function is obtained with the help of Hermite polynomial, main results are:

- If $T_n(x)$, $U_n(x)$ are Chebyshev polynomials and $H_n(x,y)$ two-variable Hermite polynomial then we integral characterization

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t}t^n H_n(2x,-\frac{1}{t})dt,$$

and

$$T_n(x) = \frac{1}{2(n-1)!}\int_0^{\infty} e^{-t}t^n H_n(2x,-\frac{1}{t})dt.$$

- If $T_n(x)$, $U_n(x)$ are Chebyshev polynomials then:

$$\frac{d}{dx}U_n(x) = nW_{n-1}(x),$$

$$U_{n+1}(x) = xW_n(x) - \frac{n}{n+1}W_{n-1}(x),$$

were

$$W_n(x) = \frac{2}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1} H_n(2x,-\frac{1}{t})dt.$$

- Generating functions of Chebyshev polynomial by Hermite polynomial

$$\sum_{n=0}^{+\infty} \xi^n\, U_n(x) = \int_0^{+\infty} e^{-t}\sum_{n=0}^{+\infty}\frac{(t\xi)^n}{n!}H_n\left(2x,-\frac{1}{t}\right)dt.$$

by remembering the polynomials of the

$$\sum_{n=0}^{+\infty}\frac{\mathrm{t}^{\mathrm{n}}}{\mathrm{n}!}H_n(x,y) = e^{(xt+yt^2)},$$

on solving we get

$$\sum_{n=0}^{+\infty} \xi^n U_n(x) = \frac{1}{1 - 2\xi x + \xi^2},$$

which is required generating function for $U_n(x)$.

In Chapter 7 applications of recurrence relation in network marketing with some limitation imposed on the problem is discussed. In the later parts of the chapter, applications of recurrence relations, especially Fibonacci numbers, and the reproduction mechanism of honey bees are viewed, main results are

- On the basis of the following theorem:

  For arbitrary real numbers $c_1$and c_2 , if x_1 and x_2 are distinct roots of

  $$x^2 - c_1 x - c_2 = 0.$$

  Then the sequence $< a_n >$has a solution

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} n \geq 2,$$

  if and only if $a_n = \beta_1 x_1^n + \beta_2 x_2^n$, with arbitrary constants β_1 and β_2, problem of network marketing is proposed and solved with the limitations

  - ✓ The percentage distribution of profit should be such that the roots of polynomial must be distinct.
  - ✓ Any worker cannot leave the business.
  - ✓ Work should be honest.
  - ✓ There is an assumption every worker can take only worker under him.
- The reproduction mechanism of bees was studied in view of Fibonacci numbers. There are numerous unusual features of honeybees, and we have shown how the Fibonacci numbers tally a honeybee's family.
- Then applications of Fibonacci numbers nature in lovely blossoms, on the leader of a sunflower and the seeds, petals of flowers are reviewed.

Future and Scope

(1) The relation between the roots and terms of recurrence relations of may be discussed for Jacobsthal polynomials, Pell polynomials, and orthogonal polynomial of third and fourth kind.

(2) Recurrence relation of rational function in the form of composition function have terms Lucas numbers, Generalized Lucas numbers can be defined and properties of group can be verified.

(3) $w_n(z)$ the sequences of complex ration functions with coefficients as Lucas number can be defined and try to prove some properties ion view of complex analysis.

(4) Relations between different orthogonal polynomials in the view of recurrence relations may establish.

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List of Publications

1. Vipin Verma, Mannu Arya, *Generalized Relation between the Roots of Polynomial and Term of Recurrence Relation Sequence*, Mathematics and Statistics, Vol. 9, Is. 1, ISSN2332-2071, April 2021. https://www.hrpub.org/journals/article_info.php?aid=10650
2. Vipin Verma, Mannu Arya, *Application of Recurrence relation in Network Marketing*, International journal of advanced science and technology, Vol. 29, Is. 4, pp 34-40, ISSN 2005-4238 (Print)2207-6360 (Online), February 2020. http://sersc.org/journals/index.php/IJAST/article/view/4033
3. Vipin Verma, Mannu Arya, *Recurrence Relation in Number Theory*, International journal of advanced science and technology, Vol. 29, Is. 2, pp885-893, ISSN 2005-4238 (Print)2207-6360 (Online), January 2021. http://sersc.org/journals/index.php/IJAST/article/view/3269
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6. Vipin Verma, Mannu Arya, *A Special Representation of Chebyshev Polynomials*, Journal of advance research in dynamical and control systems, Vol. 11, Is. 7, pp795-798, ISSN1943-023X, December 2019.
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https://sjcjycl.cn/ DOI: 10.5281/zenodo.7765974

Conference Presentations

1. Vipin Verma, Mannu Arya, Generating matrix for Generalized Fibonacci sequence and Fibonacci polynomials, in the International Conference on " Recent Advances in Fundamental and Applied Science "(RAFAS 2021) held on June 25-26, 2021, organized by School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab.
2. Vipin Verma, Mannu Arya, *The Fibonacci Numbers and Its Amazing Applications*, Presented in Conference: An IQAC Sponsored Two-day International Conference on RECENT TRENDS IN MODERN MATHEMATICS (RTMM-2021) 23rd & 24th September 2021.
3. Vipin Verma, Mannu Arya, *Matrix representation of Chebyshev sequence of polynomials* in the 37th Annual National Conference of The Mathematical Society Banaras Hindu University on Modern Mathematics and its Applications (MMA-2022), 29-30 January 2022, at Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi.
4. Vipin Verma, Mannu Arya, *Identities and Special Relation of Chebyshev and Hermite Polynomials*, Presented in Conference: International Conference on Mathematical and Statistical Computation (ICMSC-2022),Department of Mathematics Swami Keshvanand Institute of Technology & Management Gramothan, Jaipur (Rajasthan), India, In association with Rajasthan Academy of Physical Sciences3-5 March, 2022.
5. Vipin Verma, Mannu Arya, *Special properties of complex bilinear transformations with coefficients as Fibonacci numbers*, Presented at the International Conference on advance trends in computational Mathematics, Statistics and Operations Research (ICCMSO-2022) during April, 2-3, 2022, organized by the Department of Applied Sciences, The North Cap University, Gurugram, Haryana.
6. Vipin Verma, Mannu Arya, *Sequence of complex Bilinear transformations with coefficients as Fibonacci numbers*, Presented in Conference International

Conference on Contemporary Research on Mathematics and Computer Science (ICCRMCS-2022), SHRI RAM MURTI SMARAK COLLEGE OF ENGINEERING ANDTECHNOLOGY, BAREILLY (U.P.) INDIA, April 29-30, 2022.

Book Chapter

1. Vipin Verma, Mannu Arya, *Generating matrix for Chebyshev polynomials*, published as book chapter in the book with title Modern Research and Trends in Engineering and Multidisciplinary Studies, first Edition, High Rise Books, Amazon Publishers, ISBN: 9798782275655, pp 31-36, 2021.

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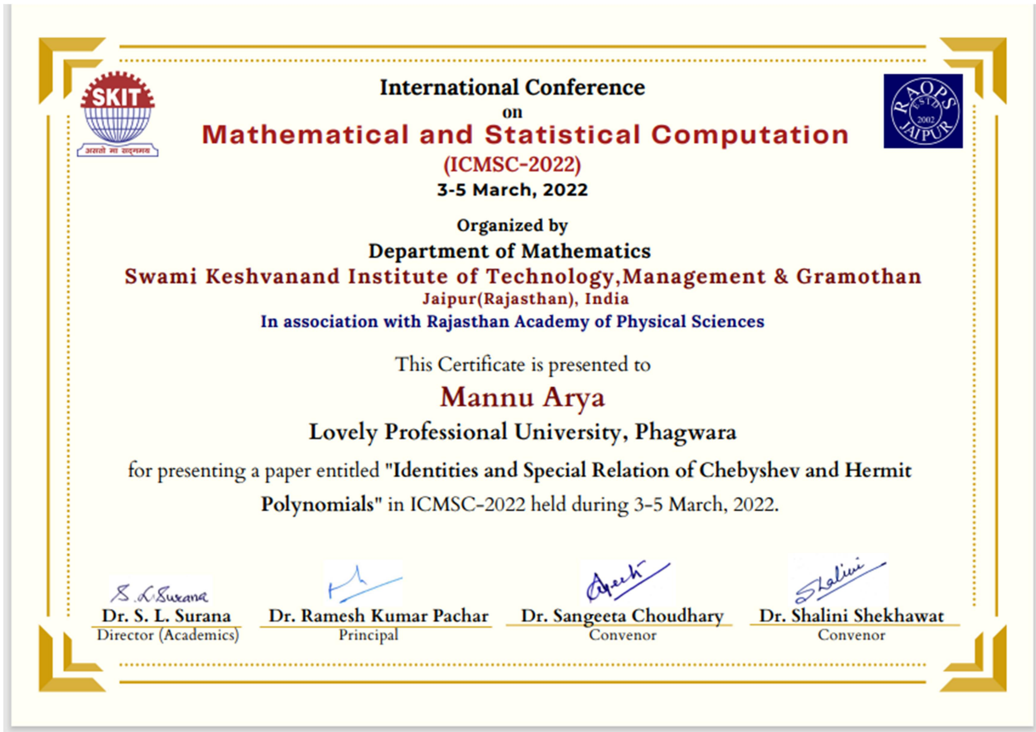

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