SOME FIXED POINT RESULTS FOR DIFFERENT MAPPINGS IN VARIOUS SPACES

Thesis submitted for the Award of Degree of

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IN
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LOVELY PROFESSIONAL UNIVERSITY

Transforming Education Transforming India

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DECLARATION

I, hereby declare that the presented work in the thesis entitled **"Some Fixed Point Results for Different Mappings in Various Spaces"** in fulfilment of the degree of Doctor of Philosophy (Ph.D.) is outcome of research work carried out by me under the supervision of Dr. Deepak Kumr, working as Professor and Assistant Dean, in the Department of Mathematics in Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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CERTIFICATE

This is to certify that the work reported in the Ph.D. thesis entitled **"Some Fixed Point Results for Different Mappings in Various Spaces"** submitted in fulfillment of the requirement for the award of degree of Doctor of Philosophy (Ph.D.) in the Department of Mathematics, is a research work carried out by Rishi, 11919643, is bonafide record of his original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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Abstract

Fixed point theory is an important tool of mathematics, primarily in developing nonlinear functional analysis. The flourishing field of fixed point theory serves as a tool in various other branches of mathematics, such as ordinary and partial differential equations, optimization, and approximation theory, for establishing the existence of the solution. This tool is the right combination of algebraic, geometric and topological properties of a mapping. With much of the development being made in this field, a broad scope of research work remains unfolded.

The objective of the present research work is to introduce a generalized approach for establishing the existence and uniqueness of fixed points and common fixed points for generalized contractions as well as expansion mappings in abstract spaces. The chapters in the thesis present results on fixed points, coincidence points, common fixed points and coupled fixed points in various spaces. Some of these spaces are well-known in the literature, while others have been introduced as a result of the research work. As an application, we claim the existence and uniqueness of the solution of the operator equation.

The first chapter briefly introduces the research work along with some notations and definitions used throughout the thesis. The chapter-wise summary of the subsequent chapters is also provided at the end of this chapter.

In the second chapter, we establish some results on the existence and uniqueness of fixed points using C_*-class function and C_*-class F-contraction in the framework of C^*-algebra valued metric space. We give the notion of $\alpha_{\mathbb{B}} - \psi_{\mathbb{B}}$-type contraction in C^*-algebra valued partial metric space and proved some fixed point results in this setting. Also, we introduce the concept of C^*-algebra valued $b_{\mathcal{R}}$-metric space, which is a generalization of C^*-algebra valued $\mathcal{R}$-metric space, and obtain some generalized fixed point results. As an application, we establish the existence and uniqueness of the solution of the operator equation.

The third chapter is concerned with the existence of common coincidence point results for compatible and weakly compatible pairs of mappings using C_*-class function in the framework of C^*-algebra valued metric space. Also, we obtain some common coincidence point results for compatible and weakly compatible pairs of self mappings using certain generalized rational type contractive condi-

tions in C^*-algebra valued metric space.

The fourth chapter deals with the existence and uniqueness of common fixed points for weakly compatible pairs of self mappings satisfying certain generalized rational type contractive conditions in a complete C^*-algebra valued metric space. We obtain some results on common fixed points using the E.A. property and CLR property. Some common fixed points result in weakly compatible pairs of mappings using generalized expansive conditions in a complete C^*-algebra valued metric space are presented. We also establish some conditions for the existence of common fixed point results for two pairs of weakly compatible mappings satisfying generalized CLR property without the condition of continuity on mappings in C^*-algebra valued metric space.

The last chapter concerns the existence and uniqueness of coupled fixed point for self mapping with mixed monotone property using C_*-class function in C^*-algebra valued metric space. We also establish some results on coupled common fixed point and coupled coincidence point for a pair of self mappings using generalized contractive conditions in the framework of C^*-algebra valued b-metric space.

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Dedicated

To

My Mother

Acknowledgement

I begin this acknowledgement with a humble prayer to the most merciful and compassionate **Almighty**, expressing my sincere gratitude for the direction and blessings illuminating my educational journey.

I would like to express my sincere gratitude to **Dr. Deepak Kumar**, whose guidance, valuable concepts, and constant encouragement have greatly influenced the direction of my thesis. His unwavering devotion to learning and striving for perfection has been a continuous source of motivation.

I owe a debt of gratitude to my hardworking colleagues **Astha Malhotra** and **Kapil Yadav**, whose joint efforts and insightful comments have elevated the calibre of this study. This academic achievement has been a genuinely satisfying experience because of their passion and sharing.

I am very grateful to my family, especially my father, whose support and selflessness have been the cornerstone of my academic endeavours. His steadfast faith in my ability has strengthened my will to overcome obstacles and achieve new heights.

Finally, but most importantly, I want to express my sincere gratitude to my wife **Nikita**, whose patience, understanding, and constant support has been my pillars of strength. This academic path has been a shared triumph because of her love and encouragement, which have been my source of drive.

Rishi

Contents

List of Abbreviations

$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{Z}$	set of integers
$\mho$	a non empty set
iff	if and only if
s.t	such that
CLR	common limit range
w.r.t	with respect to
$\mathbb{B}$	a unital C^*-algebra with unity $I_{\mathbb{B}}$
C_{seq}	Cauchy sequence
PWI	partially weakly increasing
b-MS	b-metric space
C^*_{AV}	C^*-algebra valued
C^*_{AV}-MS	C^*-algebra valued metric space
C^*_{AV}-b-MS	C^*-algebra valued b-metric space
C^*_{AV}-PMS	C^*-algebra valued partial metric space
$\mathcal{R}$-MS	$\mathcal{R}$-metric space
C^*_{AV}-$\mathcal{R}$-MS	C^*-algebra valued $\mathcal{R}$-metric space
C^*_{AV}-$b_{\mathcal{R}}$-MS	C^*-algebra valued $b_{\mathcal{R}}$-metric space

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Chapter 1

General Introduction

1.1 Introduction

Fixed point theory is a branch of mathematics that deals with the study of mathematical functions and mappings with at least one point that remains invariant under the given transformation, irrespective of the nature of the transformation. The theory is an excellent combination of algebraic, topological and geometrical aspects of mathematics. Let $\Gamma : \mho \to \mho$ be a self mapping defined on $\mho$, a point $\varkappa \in \mho$ is said to be a fixed point of Γ if $\Gamma\varkappa = \varkappa$. The existence of a fixed point of a mapping depends upon the algebraic, order theoretic and topological properties of its domain. For example, the mapping $\Gamma : \mathbb{R} \to \mathbb{R}$ defined as

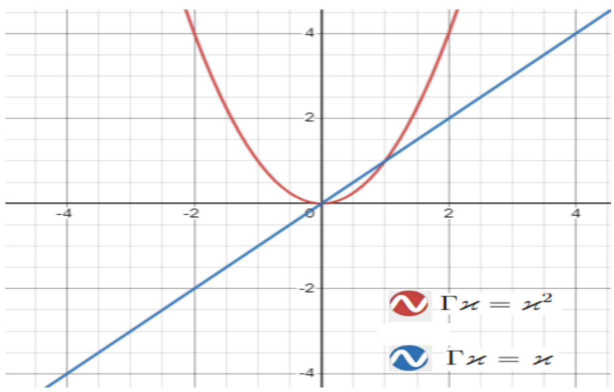

Figure 1.1: Fixed points of mapping

$\Gamma\varkappa = \varkappa^2$ has two fixed points and the translation mapping $\Gamma\varkappa = \varkappa + 4$ has no fixed point. Let $\Gamma_1, \Gamma_2 : \mho \to \mho$ be two self mappings defined on $\mho$, a point $\varkappa \in \mho$ is said to be a coincidence point of Γ_1 and Γ_2, if $\Gamma_1\varkappa = \Gamma_2\varkappa$ and a point $\varkappa \in \mho$ is said to be a common fixed point if $\Gamma_1\varkappa = \Gamma_2\varkappa = \varkappa$. For example, the

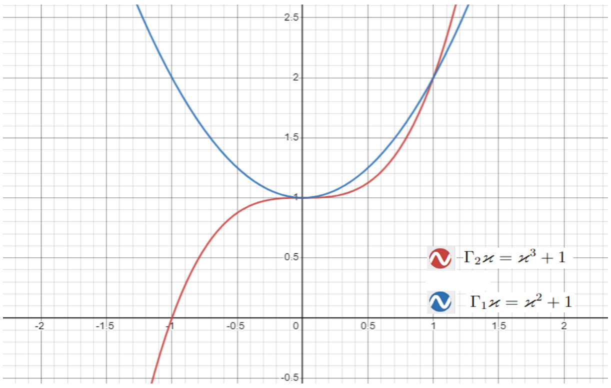

Figure 1.2: Coincidence points of the mappings

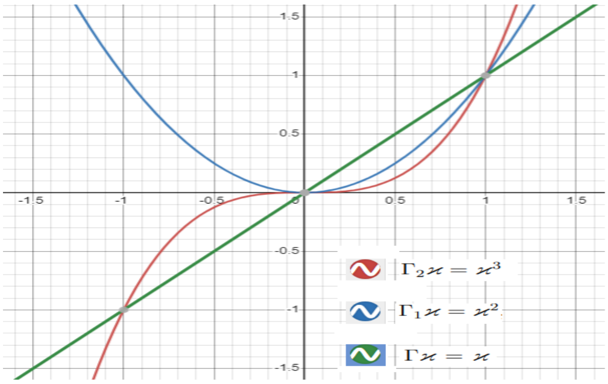

Figure 1.3: Common fixed points of the mappings

mappings $\Gamma_1, \Gamma_2 : \mathbb{R} \to \mathbb{R}$ defined as $\Gamma_1\varkappa = \varkappa^2 + 1$ and $\Gamma_2\varkappa = \varkappa^3 + 1$ have two coincidence points i.e $\Gamma_1(0) = \Gamma_2(0) = 1$ and $\Gamma_1(1) = \Gamma_2(1) = 2$. The mappings $\Gamma_1, \Gamma_2 : \mathbb{R} \to \mathbb{R}$ defined as $\Gamma_1\varkappa = \varkappa^2$ and $\Gamma_2\varkappa = \varkappa^3$ have two common fixed points i.e $\Gamma_1(0) = \Gamma_2(0) = 0$ and $\Gamma_1(1) = \Gamma_2(1) = 1$. An element $(\varkappa, \varsigma) \in \mho \times \mho$ is said to be a coupled fixed point of the mapping $\Gamma_1 : \mho \times \mho \to \mho$ if $\Gamma_1(\varkappa, \varsigma) = \varkappa$ and $\Gamma_1(\varsigma, \varkappa) = \varsigma$. An element $(\varkappa, \varsigma) \in \mho \times \mho$ is said to be a coupled coincidence point of the mappings $\Gamma_1, \Gamma_2 : \mho \times \mho \to \mho$ if $\Gamma_1(\varkappa, \varsigma) = \Gamma_2(\varkappa, \varsigma)$ and $\Gamma_1(\varsigma, \varkappa) = \Gamma_2(\varsigma, \varkappa)$. If $\Gamma_1(\varkappa, \varsigma) = \Gamma_2(\varkappa, \varsigma) = \varkappa$ and $\Gamma_1(\varsigma, \varkappa) = \Gamma_2(\varsigma, \varkappa) = \varsigma$, then $(\varkappa, \varsigma)$ is said to be a common coupled fixed point of the mappings.

Graphically, a fixed point is a point $(\varkappa, \Gamma_1(\varkappa))$ on the line $y = \varkappa$, i.e. the line $y = \varkappa$ has a point in common with the graph of Γ_1 (see figure 1.1). Let $\mho$ be a non-empty set and $\Gamma_1, \Gamma_2 : \mho \to \mho$. Graphically, a point of the intersection of the graphs of mappings Γ_1 and Γ_2 is called a coincidence point (see figure 1.2), and a point of intersection of the graphs of mappings that remain unchanged is said to be a common fixed point (see figure 1.3).

Those results which establish the existence of fixed points, subjected to certain conditions are said to be fixed point theorems. Picard (1890) presented an iterative scheme under which a sequence $\{\varpi_\jmath\}_{\jmath\in\mathbb{N}}$ in (ϖ,ς) defined by $\varpi_{\jmath+1} = \Gamma\varpi_\jmath \ \forall\ \jmath \in \mathbb{N}$ where $\Gamma : [\varpi,\varsigma] \to (-\infty,\infty)$ is continuous and differentiable on (ϖ,ς) and $|\Gamma\varpi| \leq L$ for some $L < 1$, converges to a solution of an equation $\Gamma\varpi = \varpi$. The way in which the sequence was discussed constituted one of the turning point in the history of fixed point theory and it is frequently used to establish the existence and uniqueness of a fixed point of the mappings. Brouwer (1912) explored the topological aspects of the fixed point theory with his result "*Every continuous self mapping of a closed unit ball* $\mho$ *with center at origin in* $\mathbb{R}^n$*, the n-dimensional unit Euclidean space has a fixed point*". Banach (1922) underlined the idea into an abstract framework and introduced an important result of a fixed point called the "Banach Contraction Principle" for the existence and uniqueness of self mappings in a complete metric space along with contractive conditions. Thereafter, numerous generalizations of the Banach Contraction Principle have been presented by the researchers (see Kannan (1968), Chatterjea (1972), Hardy & Rogers (1973), Jungck (1976), Jungck (1986), Guo & Lakshmikantham (1987), Pant (1999), Aamri & Moutawakil (2002), Karapınar (2010), Choudhury & Maity (2011), Aydi (2011), Abbas et al. (2011), Kumam & Sintunavarat (2011), Wardowski (2012), Wardowski & Dung (2014), Ma et al. (2014), Shukla (2014), Xin et al. (2016), Mustafa et al. (2016), Ansari & Ozturk (2017), Wu et al. (2017), Hussain et al. (2017), Hussain & Ahmad (2017), Dung et al. (2017), Huang et al. (2017), Assaf (2017), Feng (2017), Gordji & Habibi (2017), Nazam et al. (2018), Mohanta (2018), Suzuki (2018), Roy & Saha (2018), Shen et al. (2018), Nazam et al. (2019), Omran & Ozer (2019), Radenović et al. (2019), Gunaseelan et al. (2020), George et al. (2020), Khalehoghli et al. (2020), Mlaiki et al. (2020), Asim & Imdad (2020*a*), Asim & Imdad (2020*b*), Asim et al. (2020), Rao et al. (2020), Rao & Kalyani (2021), Massit & Rossafi (2021), Omran & Masmali (2021), Massit et al. (2022), Ahmad et al. (2022), Saluja (2022), Malhotra et al. (2022), Mani et al. (2022), Kim (2022), Özkan (2023), Mangapathi et al. (2023), Jain et al. (2023) and references cited therein).

Most of the results in fixed point theory are mentioned in number of books and monographs (see, Joshi & Bose (1985), Zeidler (1986), Murphy (1990), Geobel & Kirk (1990), William & Brailey (2001), Agarwal et al. (2001), Kirk & Khamsi (2001), Granas & Dugundji (2003), Agarwal et al. (2009), Chandok (2015) and references cited therein)

1.2 Notations and Definitions

To begin, we give some definitions which will be required in the subsequent chapters to establish the results.

Definition 1.2.1. (Fŕechet (1906)) On $\mho$, let $d : \mho \times \mho \to [0, \infty)$ be a mapping s.t $\forall\ \varpi, \varsigma, \vartheta \in \mho$, the following holds:

(i) $d(\varpi, \vartheta) \geq 0$ and $d(\varpi, \vartheta) = 0$ iff $\varpi = \vartheta$;

(ii) $d(\varpi, \vartheta) = d(\vartheta, \varpi)$;

(iii) $d(\varpi, \vartheta) \leq d(\varpi, \varsigma) + d(\varsigma, \vartheta)$.

Then, $(\mho, d)$ is said to be a **metric space** whereas $\boldsymbol{d}$ is a metric.

Definition 1.2.2. (see Rudin (1991)) For a vector space Θ, over the field F, **norm** is a function $\|.\| : \Theta \to [0, \infty)$ s.t

(i) $\|\varrho\| > 0$ and $\|\varrho\| = 0$ iff $\varrho = 0$;

(ii) $\|\kappa\varrho\| = |\kappa|\ \|\varrho\|$;

(iii) $\|\varrho + \vartheta\| \leq \|\varrho\| + \|\vartheta\|$;

$\forall\ \varrho, \vartheta \in \Theta$ and $\kappa \in F$.

Definition 1.2.3. (see Rosen (1991)) A binary relation '$\preceq$' is said to be a **partially ordered relation** on $\mho$ if it satisfies the following :

(i) reflexive; i.e, $\varkappa \preceq \varkappa\ \ \forall\ \varkappa \in \mho$;

(ii) antisymmetric; i.e, if $\varkappa \preceq \vartheta$ and $\vartheta \preceq \varkappa$ then $\varkappa = \vartheta\ \ \forall\ \varkappa, \vartheta \in \mho$;

(iii) transitive; i.e, if $\varkappa \preceq \vartheta$ and $\vartheta \preceq \varsigma$ then $\varkappa \preceq \varsigma\ \ \forall\ \varkappa, \vartheta, \varsigma \in \mho$.

Definition 1.2.4. (Bakhtin (1989), Czerwik (1993)) On $\mho$ and $s\ (\geq 1) \in \mathbb{R}$, let $b : \mho \times \mho \to [0, \infty)$ be a mapping s.t $\forall\ \varpi, \varsigma, \vartheta \in \mho$, the following holds :

(i) $b(\varpi, \varsigma) \geq 0$ and $b(\varpi, \varsigma) = 0$ iff $\varpi = \varsigma$;

(ii) $b(\varpi, \varsigma) = b(\varsigma, \varpi)$;

(iii) $b(\varpi, \varsigma) \leq s\Big(b(\varpi, \vartheta) + b(\vartheta, \varsigma)\Big)$.

Then, $(\mho, b)$ is said to be a **b-metric space** whereas $\boldsymbol{b}$ is a b-metric.

Definition 1.2.5. (Matthews (1992)) On $\mho$, let $p : \mho \times \mho \to [0, \infty)$ be a mapping s.t $\forall\ \varrho, \varsigma, \vartheta \in \mho$, the following holds :

(i) $p(\varrho, \varrho) = p(\varsigma, \varsigma) = p(\varrho, \varsigma)$ iff $\varrho = \varsigma$;

(ii) $p(\varrho, \varrho) \leq p(\varrho, \varsigma)$;

(iii) $p(\varrho, \varsigma) = p(\varsigma, \varrho)$

(iv) $p(\varrho, \varsigma) \leq p(\varrho, \vartheta) + p(\vartheta, \varsigma) - p(\vartheta, \vartheta)$.

Then, $(\mho, p)$ is said to be a **partial metric space** whereas $\boldsymbol{p}$ is a partial metric.

Throughout the thesis, let $\mathbb{B}$ denote a unital C^*-algebra with the unity element $I_{\mathbb{B}}$ and zero element $\theta_{\mathbb{B}}$. Let $\mathbb{B}^+ = \{\varkappa \in \mathbb{B} : \theta_{\mathbb{B}} \preceq \varkappa\}$ and $\|\varkappa\| = (\varkappa^* \varkappa)^{\frac{1}{2}}$ and $\mathbb{B}' = \{\varpi \in \mathbb{B} : \varpi\varpi' = \varpi'\varpi\ \forall \varpi' \in \mathbb{B}\}$.

Definition 1.2.6. (Ma et al. (2014)) On $\mho$, let $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ be a mapping s.t $\forall\ \varpi, \mu, \nu \in \mho$, the following holds:

(i) $d_{\mathbb{B}}(\varpi, \mu) \succeq \theta_{\mathbb{B}}$ and $d_{\mathbb{B}}(\varpi, \mu) = \theta_{\mathbb{B}}$ iff $\varpi = \mu$;

(ii) $d_{\mathbb{B}}(\varpi, \mu) = d_{\mathbb{B}}(\mu, \varpi)$;

(iii) $d_{\mathbb{B}}(\varpi, \mu) \preceq d_{\mathbb{B}}(\varpi, \nu) + d_{\mathbb{B}}(\nu, \mu)$.

Then, $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is said to be a $\mathbf{C^*_{AV} - MS}$ whereas $\boldsymbol{d}_{\mathbb{B}}$ is a C^*_{AV}-metric.

Definition 1.2.7. (Ma et al. (2014)) A sequence $\{\varpi_\jmath\}$ in $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is said to be

1. (i) **convergent** w.r.t $\mathbb{B}$, if for every $\epsilon > 0\ \exists\ k \in \mathbb{N}$ s.t $\|d_{\mathbb{B}}(\varpi_\jmath, \varpi)\| \leq \epsilon\ \forall\ \jmath \geq k$;

   (ii) a $\mathbf{C_{seq}}$ w.r.t $\mathbb{B}$, if for every $\epsilon > 0\ \exists\ k \in \mathbb{N}$ s.t $\|d_{\mathbb{B}}(\varpi_\jmath, \varpi_\imath)\| \leq \epsilon\ \forall\ \jmath, \imath \geq k$.

2. $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is said to be a **complete** $\mathbf{C^*_{AV}}$**-MS** if every C_{seq} is convergent in $\mho$.

Lemma 1.2.8. *(Murphy (1990)) In $\mathbb{B}$, the following holds:*

(*i*) *If $\varpi \in \mathbb{B}$ with $\|\varpi\| \leq 1/2$, then $(I_{\mathbb{B}} - \varpi)$ is invertible and $\|\varpi(1-\varpi)^{-1}\| \leq 1$;*

(*ii*) *For any $\vartheta \in \mathbb{B}$ and $\varpi, \varsigma \in \mathbb{B}^{+}$ such that $\varpi \preceq \varsigma$, we have $\vartheta^{*}\varpi\vartheta$ and $\vartheta^{*}\varsigma\vartheta$ are positive element and $\vartheta^{*}\varpi\vartheta \preceq \vartheta^{*}\varsigma\vartheta$;*

(*iii*) *If $\theta_{\mathbb{B}} \preceq \varpi \preceq \varsigma$ then $\|\varpi\| \leq \|\varsigma\|$;*

(*iv*) *If $\varpi, \varsigma \in \mathbb{B}^{+}$ and $\varpi\varsigma = \varsigma\varpi$ then $\varpi.\varsigma \succeq \theta_{\mathbb{B}}$;*

(*v*) *Let $\mathbb{B}'$ denote the set $\{\varpi \in \mathbb{B} : \varpi\varsigma = \varsigma\varpi$ for all $\varsigma \in \mathbb{B}\}$ and let $\varpi \in \mathbb{B}'$, if $\varsigma, \vartheta \in \mathbb{B}$ with $\varsigma \succeq \vartheta \succeq \theta_{\mathbb{B}}$ and $(I_{\mathbb{B}} - \varpi) \in (\mathbb{B}')^{+}$ is an invertible element, then $(I_{\mathbb{B}} - \varpi)^{-1}\varsigma \preceq (I_{\mathbb{B}} - \varpi)^{-1}\vartheta$.*

Lemma 1.2.9. *(Xin et al. (2016)) In $\mathbb{B}$, the following holds:*

(*i*) *If $\{\varpi_{\jmath}\}_{\jmath=1}^{\infty} \subseteq \mathbb{B}$ and $\lim_{\jmath\to\infty} \varpi_{\jmath} = \theta_{\mathbb{B}}$. Then, for any $\alpha \in \mathbb{B}$, $\lim_{\jmath\to\infty} \alpha^{*}\varpi_{\jmath}\alpha = \theta_{\mathbb{B}}$.*

(*ii*) *If $\alpha_1, \alpha_2 \in \mathbb{B}$ and $\alpha_3 \in \mathbb{B}'^{+}$, then $\alpha_1 \preceq \alpha_2$ deduces $\alpha_3\alpha_1 \preceq \alpha_3\alpha_2$, where $\mathbb{B}'^{+} = \mathbb{B}^{+} \cap \mathbb{B}'$.*

(*iii*) *Let $\{\varpi_{\jmath}\}$ be sequence in $\mho$. If $\{\varpi_{\jmath}\}$ converges to ϖ and ς respectively then $\varpi = \varsigma$.*

Definition 1.2.10. (Ma & Jiang (2015)) On $\mho$, let $b_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ be a mapping s.t $\forall\ \varpi, \varsigma, \vartheta \in \mho$ and $\alpha(\succ I_{\mathbb{B}}) \in \mathbb{B}'$, the following holds:

(*i*) $b_{\mathbb{B}}(\varpi, \varsigma) \succeq \theta_{\mathbb{B}}$ and $b_{\mathbb{B}}(\varpi, \varsigma) = \theta_{\mathbb{B}}$ iff $\varpi = \varsigma$;

(*ii*) $b_{\mathbb{B}}(\varpi, \varsigma) = b_{\mathbb{B}}(\varsigma, \varpi)$;

(*iii*) $b_{\mathbb{B}}(\varpi, \varsigma) \preceq \alpha\Big(b_{\mathbb{B}}(\varpi, \vartheta) + b_{\mathbb{B}}(\vartheta, \varsigma)\Big)$

Then, $(\mho, \mathbb{B}, b_{\mathbb{B}})$ is said to be a $\mathbf{C^{*}_{AV}}$**-b-MS** whereas $\boldsymbol{b}_{\mathbb{B}}$ is a C^{*}_{AV}-b-metric.

Definition 1.2.11. (Ma & Jiang (2015)) A sequence $\{\varpi_{\jmath}\}$ in $(\mho, \mathbb{B}, b_{\mathbb{B}})$ is said to be

1. (*i*) **convergent** w.r.t $\mathbb{B}$, if for every $\epsilon > 0\ \exists\ k \in \mathbb{N}$ s.t $\|b_{\mathbb{B}}(\varpi_{\jmath}, \varpi)\| \leq \epsilon\ \forall\ \jmath \geq k$;

   (*ii*) a $\mathbf{C_{seq}}$ w.r.t $\mathbb{B}$, if for every $\epsilon > 0\ \exists\ k \in \mathbb{N}$ s.t $\|b_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\imath})\| \leq \epsilon\ \forall\ \jmath, \imath \geq k$.

2. $(\mho, \mathbb{B}, b_{\mathbb{B}})$ is said to be a **complete $\mathbf{C^*_{AV}}$-b-MS** if every C_{seq} is convergent in $\mho$.

Definition 1.2.12. (Chandok et al. (2019)) On $\mho$, let $p_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ be a mapping s.t $\forall\ \varpi, \varsigma, \vartheta \in \mho$, the following holds:

(i) $\theta_{\mathbb{B}} \preceq p_{\mathbb{B}}(\varpi, \varsigma)$ and $p_{\mathbb{B}}(\varpi, \varpi) = p_{\mathbb{B}}(\varsigma, \varsigma) = p_{\mathbb{B}}(\varpi, \varsigma)$ iff $\varpi = \varsigma$;

(ii) $p_{\mathbb{B}}(\varpi, \varpi) \preceq p_{\mathbb{B}}(\varpi, \varsigma)$;

(iii) $p_{\mathbb{B}}(\varpi, \varsigma) = p_{\mathbb{B}}(\varsigma, \varpi)$;

(iv) $p_{\mathbb{B}}(\varpi, \varsigma) \preceq p_{\mathbb{B}}(\varpi, \vartheta) + p_{\mathbb{B}}(\vartheta, \varsigma) - p_{\mathbb{B}}(\vartheta, \vartheta)$.

Then, $(\mho, \mathbb{B}, p_{\mathbb{B}})$ is said to be a **$\mathbf{C^*_{AV}}$-PMS** whereas $\boldsymbol{p}_{\mathbb{B}}$ is a C^*_{AV}-partial metric.

Definition 1.2.13. (Chandok et al. (2019)) Let $(\mho, \mathbb{B}, p_{\mathbb{B}})$ be a C^*_{AV}-PMS, $\varpi \in \mho$ and $\{\varpi_\jmath\}_{\jmath \in \mathbb{N}} \subseteq \mho$. Then

1. (i) $\{\varpi_\jmath\}$ is **converges** to ϖ w.r.t $\mathbb{B}$, whenever for every $\epsilon > 0$ there is a $N \in \mathbb{N}$ s.t $\|p_{\mathbb{B}}(\varpi_\jmath, \varpi) - p_{\mathbb{B}}(\varpi, \varpi)\| \leq \epsilon\ \forall\ \jmath \geq N$. We denote it by

$$\lim_{\jmath \to \infty} \Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi) - p_{\mathbb{B}}(\varpi, \varpi) \Big) = \theta_{\mathbb{B}}.$$

   (ii) $\{\varpi_\jmath\}$ is a **partial $\mathbf{C_{seq}}$** w.r.t $\mathbb{B}$ whenever for every $\epsilon > 0$ there is a $N \in \mathbb{N}$ s.t

$$\Bigg(p_{\mathbb{B}}(\varpi_\jmath, \varpi_\imath) - \frac{1}{2} p_{\mathbb{B}}(\varpi_\jmath, \varpi_\jmath) - \frac{1}{2} p_{\mathbb{B}}(\varpi_\imath, \varpi_\imath) \Bigg) \Bigg(p_{\mathbb{B}}(\varpi_\jmath, \varpi_\imath) - \frac{1}{2} p_{\mathbb{B}}(\varpi_\jmath, \varpi_\jmath) - \frac{1}{2} p_{\mathbb{B}}(\varpi_\imath, \varpi_\imath) \Bigg)^* \preceq \epsilon^2\ \forall\ \imath, \jmath \geq N.$$

2. $(\mho, \mathbb{B}, p_{\mathbb{B}})$ is said to be **complete** w.r.t $\mathbb{B}$ if every partial C_{seq} converges to $\varpi \in \mho$ w.r.t $\mathbb{B}$ s.t

$$\lim_{\jmath \to \infty} \Bigg(p_{\mathbb{B}}(\varpi_\jmath, \varpi) - \frac{1}{2} p_{\mathbb{B}}(\varpi_\jmath, \varpi_\jmath) - \frac{1}{2} p_{\mathbb{B}}(\varpi, \varpi) \Bigg) = \theta_{\mathbb{B}}.$$

Definition 1.2.14. (Khalehoghli et al. (2020)) Let $(\mho, d)$ be a metric space and $\mathcal{R}$ is a relation on $\mho$. Then, $(\mho, d, \mathcal{R})$ is said to be **$\mathcal{R}$-MS**.

Definition 1.2.15. (Khalehoghli et al. (2020)) Let $(\mho, d, \mathcal{R})$ be $\mathcal{R}$-MS. Then, a sequence $\{\varpi_\jmath\}_{\jmath \in N} \subset \mho$

1. (*i*) said to be an $\mathcal{R}$**-sequence** if $\varpi_{\jmath}\mathcal{R}\varpi_{\jmath+k}$ $\forall$ $\jmath, k \in \mathbb{N}$.

   (*ii*) said to be **convergent** if for $\epsilon > 0$, $\exists$ $N \in \mathbb{Z}^+$ s.t $d(\varpi_{\jmath}, \varpi) < \epsilon$ $\forall$ $\jmath \geq N$.

   (*iii*) said to be an $\mathcal{R}$-$\mathbf{C_{seq}}$ if $\{\varpi_{\jmath}\}$ is an $\mathcal{R}$-sequence and for every $\epsilon > 0$, $\exists$ $N \in \mathbb{Z}^+$ s.t $d(\varpi_{\jmath}, \varpi_{\imath}) < \epsilon$ $\forall$ $\jmath, \imath \geq N$.

2. said to be an $\mathcal{R}$**-complete** if every $\mathcal{R}$-C_{seq} is convergent in $\mho$.

Definition 1.2.16. (Malhotra et al. (2022)) On $\mho$, let $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ be a mapping s.t the following holds:

(*i*) $d_{\mathbb{B}}$ is a C^*_{AV}-metric on $\mho$;

(*ii*) $\mathcal{R}$ is binary relation on $\mho$.

Then, $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ is said to be a $\mathbf{C^*_{AV}}$-$\mathcal{R}$-**MS** whereas $\boldsymbol{d_{\mathbb{B}}}$ is a C^*_{AV}-$\mathcal{R}$-metric.

Definition 1.2.17. (Aamri & Moutawakil (2002)) Let $(\mho, d)$ be a metric space and $\Gamma_1, \Gamma_2 : \mho \to \mho$. Then, the pair (Γ_1, Γ_2) said to satisfy **E.A. property**, if $\exists$ a sequence $\{\varpi_{\jmath}\}$ in $\mho$ s.t $\lim_{\jmath\to\infty} \Gamma_1\varpi_{\jmath} = \lim_{\jmath\to\infty} \Gamma_2\varpi_{\jmath} = \varpi$ for some $\varpi \in \mho$.

Definition 1.2.18. (Bhaskar & Lakshmikantham (2006)) A mapping $\Gamma : \mho\times\mho \to \mho$ is said to satisfy **mixed monotone property** on partially ordered set $(\mho, \leq)$, if $\forall$ $\varpi, \varsigma \in \mho$

$$\varpi_1, \varpi_2 \in \mho, \varpi_1 \leq \varpi_2 \Rightarrow \Gamma(\varpi_1, \varsigma) \leq \Gamma(\varpi_2, \varsigma)$$

and

$$\varsigma_1, \varsigma_2 \in \mho, \varsigma_1 \leq \varsigma_2 \Rightarrow \Gamma(\varpi, \varsigma_1) \geq \Gamma(\varpi, \varsigma_2).$$

Definition 1.2.19. (Altun & Simsek (2010)) Let $(\mho, d)$ be a metric space and $\Gamma_1, \Gamma_2 : \mho \to \mho$. Then, the pair (Γ_1, Γ_2) is said to be **weakly increasing** if $\Gamma_1\varpi \preceq \Gamma_2\Gamma_1\varpi$ and $\Gamma_2\varpi \preceq \Gamma_1\Gamma_2\varpi$ $\forall$ $\varpi \in \mho$.

Definition 1.2.20. (Abbas et al. (2011)) Let $(\mho, d)$ be a metric space and $\Gamma_1, \Gamma_2 : \mho \to \mho$. Then, the pair (Γ_1, Γ_2) is said to be **partial weakly increasing (PWI)** if $\Gamma_1\varpi \preceq \Gamma_2\Gamma_1\varpi$ $\forall$ $\varpi \in \mho$.

Definition 1.2.21. (Kumam & Sintunavarat (2011)) Let $(\mho, d)$ be a metric space and $\Gamma_1, \Gamma_2 : \mho \to \mho$. Then, the pair (Γ_1, Γ_2) is said to satisfy $\mathbf{CLR_{\Gamma_1}}$ **property** in a metric space $(\mho, d)$, if $\exists$ a sequence $\{\varpi_{\jmath}\}$ in $\mho$ s.t $\lim_{\jmath\to\infty} \Gamma_1\varpi_{\jmath} = \lim_{\jmath\to\infty} \Gamma_2\varpi_{\jmath} = \Gamma_1\varsigma$ for some $\varsigma \in \mho$.

Definition 1.2.22. Let $(\mho, \preceq)$ be a partially ordered set and $\Gamma_1, \Gamma_2, \Gamma_3 : \mho \to \mho$ s.t $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ and $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$. The ordered pair (Γ_1, Γ_2) is said to be

(i) (Nashine & Samet (2011)) **weakly increasing** w.r.t Γ_3 iff $\forall\, \varpi \in \mho$, $\Gamma_1\varpi \preceq \Gamma_2\vartheta \;\forall\, \vartheta \in \Gamma_3^{-1}(\Gamma_1\varpi)$ and $\Gamma_2\varpi \preceq \Gamma_1\vartheta \;\forall\, \vartheta \in \Gamma_3^{-1}(\Gamma_2\varpi)$.

(ii) (Esmaily et al. (2012)) **partially weakly increasing** w.r.t Γ_3 if $\Gamma_1\varpi \preceq \Gamma_2\vartheta \;\forall\, \vartheta \in \Gamma_3^{-1}(\Gamma_1\varpi)$.

Definition 1.2.23. (Xin et al. (2016)) Two self mappings $\Gamma_1, \Gamma_2 : \mho \to \mho$ in C^*_{AV}-MS are said to be

(i) **compatible** if for $\{\varpi_\jmath\} \subseteq \mho$ s.t $\lim_{\jmath\to\infty} \Gamma_1\varpi_\jmath = \lim_{\jmath\to\infty} \Gamma_2\varpi_\jmath = \varpi \in \Gamma$, then $d_{\mathbb{B}}(\Gamma_2\Gamma_1\varpi_\jmath, \Gamma_1\Gamma_2\varpi_\jmath) \xrightarrow{\|.\|} \theta_{\mathbb{B}}$ as $\jmath \to \infty$;

(ii) **weakly compatible mappings** if $\Gamma_1\Gamma_2\varpi = \Gamma_2\Gamma_1\varpi \;\forall\, \varpi \in \{\varpi \in \mho : \Gamma_1\varpi = \Gamma_2\varpi\}$.

Definition 1.2.24. (Chandok et al. (2019)) A continuous function $F^* : \mathbb{B}^+ \times \mathbb{B}^+ \to \mathbb{B}^+$ is said to be a **C$_*$-Class Function** if for any $\varpi, \varsigma \in \mathbb{B}^+$, the following holds:

(i) $F^*(\varkappa, \varrho) \preceq \varkappa$;

(ii) $F^*(\varkappa, \varrho) = \varkappa$ implies that either $\varkappa = \theta_{\mathbb{B}}$ or $\varrho = \theta_{\mathbb{B}}$.

If necessary, an additional condition can be applied on the function F^* s.t $F^*(\theta_{\mathbb{B}}, \theta_{\mathbb{B}}) = \theta_{\mathbb{B}}$.

Definition 1.2.25. (Isik & Türkoglu (2014)) Consider the function $\psi_{\mathbb{B}} : \mathbb{B}^+ \to \mathbb{B}^+$ satisfying:

(i) $\psi_{\mathbb{B}}$ is continuous and monotone increasing;

(ii) $\psi_{\mathbb{B}}(\varpi) = \theta_{\mathbb{B}}$ iff $\varpi = \theta_{\mathbb{B}}$;

(iii) $\psi_{\mathbb{B}}(\varpi + \varsigma) \preceq \psi_{\mathbb{B}}(\varpi) + \psi_{\mathbb{B}}(\varsigma)$

Throughout the thesis, the family of such functions is denoted by $\Psi_{\mathbb{B}}$.

Definition 1.2.26. (Khalehoghli et al. (2020)) For an $\mathcal{R}$-MS $(\mho, d, \mathcal{R})$, a self mapping $\Gamma : \mho \to \mho$ is said to be

(*i*) $\mathcal{R}$-**continuous** at $\varpi \in \mho$ if for any arbitrary $\mathcal{R}$-sequence $\{\varpi_\jmath\}_{\jmath\in\mathbb{N}} \subset \mho$ with $\lim_{\jmath\to\infty} \varpi_\jmath \to \varpi$ implies $\lim_{\jmath\to\infty} \Gamma\varpi_\jmath \to \Gamma\varpi$.

(*ii*) $\mathcal{R}$-**preserving** if for every $\varpi\mathcal{R}\varsigma$, we have $\Gamma\varpi\mathcal{R}\Gamma\varsigma$.

Remark 1.2.27. (Khalehoghli et al. (2020)) Every continuous map is $\mathcal{R}$-continuous but not conversely.

Example 1.2.28. Let $\mho = [0, 1)$ and $(\mho, d, \mathcal{R})$ be an $\mathcal{R}$-MS with $\varkappa\mathcal{R}\varsigma$ iff either $\varkappa = 0$ or $\varsigma = 0$. Define $\Gamma : \mho \to \mho$ as $\Gamma\varkappa = \begin{cases} \frac{\varkappa}{3}, & \text{if } \varkappa \leq \frac{1}{3} \\ 0, & \text{otherwise.} \end{cases}$ Then, Γ is not continuous mapping but $\mathcal{R}$-continuous.

Definition 1.2.29. (Rao et al. (2020)) Let Γ_1, $\Gamma_2 : \mho \to \mho$ be two self mappings on a partially ordered set $(\mho, \preceq)$. Then, Γ_1 is said to be **monotone $\boldsymbol{\Gamma_2}$-nondecreasing** if

$$\Gamma_2\varpi \preceq \Gamma_2\varsigma \quad \text{implies} \quad \Gamma_1\varpi \preceq \Gamma_1\varsigma \ \forall \ \varpi, \varsigma \in \mho.$$

Definition 1.2.30. (Omran & Masmali (2021)) Let $\Gamma : \mho \to \mho$ and $\alpha : \mho \times \mho \to \mathbb{B}^+$. Then, Γ is said to be α**-admissible** if $\forall\ \varpi, \varsigma \in \mho$ with $\alpha(\varpi, \varsigma) \succeq I_{\mathbb{B}}$ implies $\alpha(\Gamma\varpi, \Gamma\varsigma) \succeq I_{\mathbb{B}}$.

Definition 1.2.31. (Omran & Masmali (2021)) Suppose $\mathbb{M}$ and $\mathbb{B}$ are two C^*-algebras. A mapping $\psi : \mathbb{M} \to \mathbb{B}$ is said to be a **C*-homomorphism** if the following holds:

(*i*) $\psi_{\mathbb{B}}(\alpha_1\varpi + \alpha_2\varsigma) = \alpha_1\psi_{\mathbb{B}}(\varpi) + \alpha_2\psi_{\mathbb{B}}(\varsigma)\ \forall\ \alpha_1, \alpha_2 \in \mathbb{C}$ and $\varpi, \varsigma \in \mathbb{B}$;

(*ii*) $\psi_{\mathbb{B}}(\varpi\varsigma) = \psi_{\mathbb{B}}(\varpi)\psi_{\mathbb{B}}(\varsigma)\ \forall\ \varpi, \varsigma \in \mathbb{B}$;

(*iii*) $\psi_{\mathbb{B}}(\varpi^*) = \psi_{\mathbb{B}}(\varpi)^*$;

(*iv*) $\psi_{\mathbb{B}}$ maps unit in $\mathbb{M}$ to unit in $\mathbb{B}$.

Definition 1.2.32. (Omran & Masmali (2021)) Consider the function $\psi_{\mathbb{B}} : \mathbb{B}^+ \to \mathbb{B}^+$ satisfying :

(*i*) $\psi_{\mathbb{B}}(\varpi)$ is continuous and non decreasing;

(ii) $\psi_{\mathbb{B}}(\varpi) = \theta_{\mathbb{B}}$ iff $\varpi = \theta_{\mathbb{B}}$;

(iii) $\sum_{\jmath=1}^{\infty} \psi_{\mathbb{B}}^{\jmath}(\varpi) \prec \infty$, $\lim_{\jmath\to\infty} \psi_{\mathbb{B}}^{\jmath}(\varpi) = \theta_{\mathbb{B}} \ \forall\, \varpi \succ \theta_{\mathbb{B}}$;

(iv) The series $\sum_{\jmath=1}^{\infty} \psi_{\mathbb{B}}^{\jmath}(\varpi) \prec \infty \ \forall\, \varpi \succ \theta_{\mathbb{B}}$ is increasing and continuous at $\theta_{\mathbb{B}}$.

Throughout the thesis, the family of such functions is denoted by $\Psi_{\mathbb{B}}^{1}$.

1.3 Chapterwise Summary

In this section, we give a brief summary of the various results proved in the subsequent chapters of the thesis.

Chapter 2 deals with the existence and uniqueness of fixed points of self mappings. It has been divided into three main sections. In the first section, we establish some results for the existence and uniqueness of fixed point using C_*-class function and C_*-class F-contraction in C^*_{AV}-MS. In the second section, we establish some results on fixed points using $(\alpha_{\mathbb{B}} - \psi_{\mathbb{B}})$-type contraction mapping in C^*_{AV}-PMS. In the last section, we introduce a notion of C^*_{AV}-$b_{\mathcal{R}}$-MS which is a generalization of C^*_{AV}-R-MS and establish some adequate conditions for the presence of fixed points. The usability of the results are substantiated by using suitable illustrations. As an application of the obtained results, the existence and uniqueness of the solution of an operator equation is verified.

Chapter 3 deals with the existence of coincidence points of self mappings. It has two main sections. In the first section, we establish some results on the existence of coincidence point using C_*-class function for two pairs of compatible or weakly compatible mappings in C^*_{AV}-MS. In the second section, we present some results on the existence of coincidence point using a certain rational type contraction for two pairs of compatible or weakly compatible mappings in C^*_{AV}-MS. To support the findings, some illustrative examples are discussed.

Chapter 4 is concerned with the existence and uniqueness of common fixed points of self mappings. It has been divided into two section. In the first section, we establish some results on the existence and uniqueness of common fixed point for weakly compatible pairs of self mappings using $E.A.$ property and CLR property with certain contractive conditions in C^*_{AV}-MS. In the second section, we present some results on the existence and uniqueness of common fixed point for weakly compatible pairs of self mappings using $E.A.$ property and CLR property with expansion conditions in C^*_{AV}-MS. Some illustrative examples are also discussed to support the proved results.

Chapter 5 deals with existence and uniqueness of coupled fixed point. It has been divided into two sections. In the first section, we establish some results on the existence and uniqueness of coupled fixed point using C_*-class function in partially ordered C^*_{AV}-MS. In the second section, we present some results on coupled coincidence point and coupled common fixed point for a pair of mappings using generalized contractions in C^*_{AV}-b-MS. Appropriate illustrations are discussed to support the usability of the proved results.

The thesis completed with bibliography followed by list of publications, paper presented in conferences and workshops attended.

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Chapter 2

Some Results On Fixed Point

2.1 Introduction

The present chapter of the thesis provides the results on the existence and uniqueness of the fixed points of self mappings in various spaces. The content of this chapter is divided into three sections. In the first section, some theorems on the existence and uniqueness of fixed point using C_*-class function and C_*-class F-contraction type mappings in C^*_{AV}-MS are presented. In the second section, inspired by the work of Samet (2015), some results on fixed points using $(\alpha_{\mathbb{B}} - \psi_{\mathbb{B}})$-type contraction mapping in C^*_{AV}-PMS are presented. In the last section, the notion of C^*_{AV}-$b_{\mathcal{R}}$-MS, which is a generalization of C^*_{AV}-R-MS (Malhotra et al. (2022)) is introduced and certain results using the generalized contractions in C^*_{AV}-$b_{\mathcal{R}}$-MS are discussed. As an application of the obtained results, the existence and uniqueness of the solution of an operator equation is verified. The results of this chapter are presented in [1,2,3,4].

[1]Kumar, D., Dhariwal, R., Park, C. and Lee, J. R. (2022). On fixed point in C^*-algebra valued metric space using C_*-class function. International Journal of Nonlinear Analysis and Application, 12(2), 1157-1161.

[2]Dhariwal, R. and Kumar, D. (2023). C-Class F-Contraction in C^*-algebra valued metric space. Science and Technology Asia, 28(3), 29-36.

[3]Dhariwal, R. and Kumar, D. (2023). On existence and uniqueness of a solution of an integral equation using contractive mapping. Applied Mathematics E-Notes, 23, 412-423.

[4]Dhariwal, R. and Kumar, D. (2023). C^*-algebra valued-$b_{\mathcal{R}}$-metric space, fixed point theorems and its application. (Communicated.)

2.2 Fixed Point of Self Mappings in C^*-Algebra Valued Metric Space

In the last decade, the extension of fixed point theory to the generalized structures such as PMS, b-MS, C^*_{AV}-MS, C^*_{AV}-b-MS, C^*_{AV}-PMS etc have received a considerable attention (see, Batra & Vashistha (2014), Jhade & Khan (2014), Ma & Jiang (2015), Altun et al. (2015), Klim & Wardowski (2015), Kamran et al. (2016), Durmaz et al. (2016), Xin et al. (2016), Radenovi$\acute{c}$ et al. (2017), Dung et al. (2017), Nazam et al. (2018), Mohanta (2018), Roy & Saha (2018), Shen et al. (2018), Nazam et al. (2018), Nazam et al. (2019), Asim & Imdad (2020a), Asim & Imdad (2020b), Mlaiki et al. (2020), Williams et al. (2020), Mohan & Vijayakumaar (2020), Massit & Rossafi (2021), Massit et al. (2022) and Malhotra et al. (2022) and references cited therein). This section of the chapter is further subdivided into two subsections.

2.2.1 Fixed Point Results using C_*-Class Function

Ansari (2014) introduced the notion of C_*-class function and established some fixed point results in metric space. Later, many researchers presented fixed point results using C_*-class function in various spaces (see Ansari et al. (2016), Huang et al. (2017), Ansari & Ozturk (2017), Saluja (2022), Mangapathi et al. (2023)). In this subsection, inspired by the work of Chandok et al. (2019), some results on fixed points using C_*-class function in C^*_{AV}-MS are established.

Theorem 2.2.1. *Let $(\mho, \mathbb{B}, d_{\mathbb{B}})$ be a complete C^*_{AV}-MS and $\Gamma : \mho \to \mho$ satisfying*

$$\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma)\Big) \preceq F^*\bigg(\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big)\bigg) \;\forall\; \varpi, \varsigma \in \mho, \tag{2.2.1}$$

where $\phi_{\mathbb{B}}, \varphi_{\mathbb{B}} \in \Psi_{\mathbb{B}}$ and $F^* \in C_*$. Then, Γ has a unique fixed point.

Proof. For $\varpi_0 \in \mho$, define a sequence $\varpi_{\jmath+1} = \Gamma\varpi_\jmath = \Gamma^\jmath(\varpi_0) \;\forall\; \jmath = 1, 2, 3 \cdots$.
To prove $d_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \to \theta_{\mathbb{B}}$ as $\jmath \to \infty$. Substituting $\varpi = \varpi_{\jmath-1}$ and $\varsigma = \varpi_\jmath$ in (2.2.1), we have

$$\begin{aligned}
\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1})\Big) &= \varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \Gamma\varpi_\jmath)\Big) \\
&\preceq F^*\bigg(\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath)\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath)\Big)\bigg) \\
&\preceq \varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath)\Big). \qquad (2.2.2)
\end{aligned}$$

Since, φ is non decreasing. $\therefore d_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_{\jmath})$ is monotonically decreasing and bounded sequence in $\mathbb{B}^{+}$. Then, $\exists\ \theta_{\mathbb{B}} \preceq \varrho \in \mathbb{B}^{+}$ s.t

$$d_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1}) \to \varrho \text{ as } \jmath \to \infty.$$

Taking limit as $\jmath \to \infty$ in (2.2.2), we have

$$\varphi_{\mathbb{B}}(\varrho) \preceq F^{*}\Big(\varphi_{\mathbb{B}}(\varrho), \phi_{\mathbb{B}}(\varrho)\Big) \preceq \varphi_{\mathbb{B}}(\varrho).$$

Thus, $F^{*}\Big(\varphi_{\mathbb{B}}(\varrho), \phi_{\mathbb{B}}(\varrho)\Big) = \varphi_{\mathbb{B}}(\varrho)$ implies either $\varphi_{\mathbb{B}}(\varrho) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(\varrho) = \theta_{\mathbb{B}}$, i.e, $\varrho = \theta_{\mathbb{B}}$. Hence,

$$d_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1}) \to \theta_{\mathbb{B}}\ as\ \jmath \to \infty. \tag{2.2.3}$$

Now, to show $\{\varpi_{\jmath}\}$ is a C_{seq} in $(\mho, \mathbb{B}, d_{\mathbb{B}})$. Assume that $\{\varpi_{\jmath}\}$ is not a C_{seq} in $(\mho, \mathbb{B}, d_{\mathbb{B}})$. Then, for any $\epsilon > 0$ $\exists$ subsequences $\{\varpi_{\imath_k}\}$ and $\{\varpi_{\jmath_k}\}$ with $\jmath_k > \imath_k > k$ s.t

$$\|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k})\| \geq \epsilon. \tag{2.2.4}$$

Choose $\jmath_k$ in such a way that $\jmath_k > \imath_k$ satisfying (2.2.4) and

$$\|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k-1})\| < \epsilon. \tag{2.2.5}$$

Using (2.2.5) and (2.2.4), we have

$$\begin{aligned} \epsilon &\leq \|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k})\| \leq \|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k-1})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\jmath_k})\| \\ &\leq \epsilon + \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\jmath_k})\|. \end{aligned} \tag{2.2.6}$$

Taking limit as $k \to \infty$ in (2.2.6) and using (2.2.3), we have

$$\epsilon \leq \lim_{k\to\infty} \|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k})\| \leq \epsilon + 0,$$

or

$$\lim_{k\to\infty} \|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k})\| = \epsilon. \tag{2.2.7}$$

Again,

$$\begin{aligned} \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k})\| &\leq \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\jmath_k-1})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k})\| \\ &\leq \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\jmath_k-1})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1})\| \\ &\quad + \|d_{\mathbb{B}}(\varpi_{\imath_k-1}, \varpi_{\imath_k})\|. \end{aligned} \tag{2.2.8}$$

Also,

$$\begin{aligned}\|d_{\mathbb{B}}(\varpi_{\jmath_k-1},\varpi_{\imath_k-1})\| &\leq \|d_{\mathbb{B}}(\varpi_{\jmath_k-1},\varpi_{\jmath_k})\|+\|d_{\mathbb{B}}(\varpi_{\jmath_k},\varpi_{\imath_k-1})\|\\ &\leq \|d_{\mathbb{B}}(\varpi_{\jmath_k-1},\varpi_{\jmath_k})\|+\|d_{\mathbb{B}}(\varpi_{\jmath_k},\varpi_{\imath_k})\|\\ &\quad +\|d_{\mathbb{B}}(\varpi_{\imath_k},\varpi_{\imath_k-1})\|. \qquad (2.2.9)\end{aligned}$$

Taking limit as $k\to\infty$ in (2.2.8) & (2.2.9) and using (2.2.3) & (2.2.7), we have

$$\lim_{k\to\infty}\|d_{\mathbb{B}}(\varpi_{\jmath_k-1},\varpi_{\imath_k-1})\|=\epsilon.$$

Since, $d_{\mathbb{B}}(\varpi_{\jmath_k-1},\varpi_{\imath_k-1}), d_{\mathbb{B}}(\varpi_{\jmath_k},\varpi_{\imath_k})\in\mathbb{B}^+$ and

$$\lim_{k\to\infty}\|d_{\mathbb{B}}(\varpi_{\jmath_k-1},\varpi_{\imath_k-1})\|=\lim_{k\to\infty}\|d_{\mathbb{B}}(\varpi_{\jmath_k},\varpi_{\imath_k})\|=\epsilon.$$

$\therefore\ \exists\ \varepsilon\in\mathbb{B}^+$ with $\|\varepsilon\|=\epsilon$. Hence,

$$\lim_{k\to\infty}d_{\mathbb{B}}(\varpi_{\jmath_k-1},\varpi_{\imath_k-1})=\lim_{k\to\infty}d_{\mathbb{B}}(\varpi_{\jmath_k},\varpi_{\imath_k})=\varepsilon. \qquad (2.2.10)$$

Now, by (2.2.1), we have

$$\begin{aligned}\varphi_{\mathbb{B}}(\varepsilon) &= \lim_{k\to\infty}\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath_k},\varpi_{\imath_k})\Big)=\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma\varpi_{\jmath_k-1},\Gamma\varpi_{\imath_k-1})\Big)\\ &\preceq \lim_{k\to\infty}F^*\Bigg(\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath_k-1},\varpi_{\imath_k-1})\Big),\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath_k-1},\varpi_{\imath_k-1})\Big)\Bigg)\\ &= F^*\Big(\varphi_{\mathbb{B}}(\varepsilon),\phi_{\mathbb{B}}(\varepsilon)\Big)\preceq\varphi_{\mathbb{B}}(\varepsilon).\end{aligned}$$

Thus, $F^*\Big(\varphi_{\mathbb{B}}(\varepsilon),\phi_{\mathbb{B}}(\varepsilon)\Big)=\varphi_{\mathbb{B}}(\varepsilon)$ implies either $\varphi_{\mathbb{B}}(\varepsilon)=\theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(\varepsilon)=\theta_{\mathbb{B}}$, i.e, $\varepsilon=\theta_{\mathbb{B}}$, a contradiction. Hence, $\{\varpi_\jmath\}$ is a C_{seq} in $(\mho,\mathbb{B},d_{\mathbb{B}})$.
$\therefore\ \exists\ \varpi\in\mho$ s.t

$$\lim_{\jmath\to\infty}d_{\mathbb{B}}(\varpi_\jmath,\varpi)=\theta_{\mathbb{B}}.$$

To prove ϖ is fixed point for Γ.
Consider,

$$\begin{aligned}\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_\jmath,\Gamma\varpi)\Big) &= \varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma\varpi_{\jmath-1},\Gamma\varpi)\Big)\\ &\preceq F^*\Bigg(\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath-1},\varpi)\Big),\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath-1},\varpi)\Big)\Bigg)\\ &\preceq \varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath-1},\varpi)\Big).\end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned}\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \Gamma\varpi)\Big) &\preceq F^*\Big(\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varpi)\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varpi)\Big)\Big) \\ &\preceq \varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varpi)\Big) = \varphi_{\mathbb{B}}\Big(\theta_{\mathbb{B}}\Big).\end{aligned}$$

This implies $d_{\mathbb{B}}(\varpi, \Gamma\varpi) = \theta_{\mathbb{B}}$. Hence, $\Gamma\varpi = \varpi$.
Uniqueness: Let $\varsigma \in \mho$ be another fixed point of Γ. Then, using (2.2.1), we have

$$\begin{aligned}\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big) = \varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma)\Big) &\preceq F^*\Big(\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big)\Big) \\ &\preceq \varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big).\end{aligned}$$

Hence, $F^*\Big(\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big)\Big) = \varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big)$ implies either $\varphi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi, \varsigma)) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi, \varsigma)) = \theta_{\mathbb{B}}$. Thus, we have $d_{\mathbb{B}}(\varpi, \varsigma) = \theta_{\mathbb{B}}$, i.e, $\varpi = \varsigma$. □

For $F^*(\varpi, \varsigma) = \varpi - \varsigma$, we have

Corollary 2.2.2. *Let $(\mho, \mathbb{B}, d_{\mathbb{B}})$ be a complete C^*_{AV}-MS and $\Gamma : \mho \to \mho$ satisfying*

$$\varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma)\Big) \preceq \varphi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big) - \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big)\ \forall\ \varpi, \varsigma \in \mho,$$

where $\phi_{\mathbb{B}}, \varphi_{\mathbb{B}} \in \Psi_{\mathbb{B}}$. Then, Γ has a unique fixed point.

2.2.2 Fixed Point Results using C_*-Class F-contraction

Wardowski (2012) introduced F-contraction and proved fixed point results in metric space. Later, many researchers generalized the results in various spaces (see, Wardowski & Dung (2014), Klim & Wardowski (2015), Durmaz et al. (2016), Kamran et al. (2016), Piri et al. (2017), Nazam et al. (2018), Suzuki (2018), Massit & Rossafi (2021), Massit et al. (2022) and references cited therein). In this subsection, some fixed point results using C_*-class F-contraction in C^*_{AV}-MS are presented.

Theorem 2.2.3. *Let $(\mho, \mathbb{B}, d_{\mathbb{B}})$ be a complete C^*_{AV}-MS and $\Gamma : \mho \to \mho$. If $\exists\ \Gamma_1 : \mathbb{B}^+ \to \mathbb{B}$ satisfying :*

(*i*) *Γ_1 is continuous and nondecreasing on $\mathbb{B}^+$;*

(ii) For each sequence $\{\gamma_\jmath\} \subseteq \mathbb{B}^+$

$$\lim_{\jmath\to\infty} \gamma_\jmath = \theta_\mathbb{B} \text{ iff } \lim_{\jmath\to\infty} \Gamma_1(\gamma_\jmath) = \theta_\mathbb{B}; \tag{2.2.11}$$

and $\forall\ \varpi, \varsigma \in \mho$ *with* $\tau > 0$,

$$\tau + \Gamma_1\Big(\psi_\mathbb{B}\big(d_\mathbb{B}(\Gamma\varpi, \Gamma\varsigma)\big)\Big) \preceq F^*\Big(\Gamma_1\big(\psi_\mathbb{B}(d_\mathbb{B}(\varpi,\varsigma))\big), \Gamma_1\big(\phi_\mathbb{B}(d_\mathbb{B}(\varpi,\varsigma))\big)\Big), \tag{2.2.12}$$

where $F^* \in C_*$ *and* $\psi_\mathbb{B}, \phi_\mathbb{B} \in \Psi_\mathbb{B}$*. Then,* Γ *has a unique fixed point.*

Proof. Firstly, to prove Γ can possess at most a unique fixed point. Indeed if $\varpi_1, \varpi_2 \in \mho$ be two distinct fixed points i.e, $\Gamma\varpi_1 = \varpi_1 \neq \varpi_2 = \Gamma\varpi_2$. Then, from (2.2.12), we have

$$\begin{aligned}
\tau + \Gamma_1\Big(\psi_\mathbb{B}\big(d_\mathbb{B}(\Gamma\varpi_1, \Gamma\varpi_2)\big)\Big) &\preceq F^*\Big(\Gamma_1\big(\psi_\mathbb{B}(d_\mathbb{B}(\varpi_1,\varpi_2))\big), \Gamma_1\big(\phi_\mathbb{B}(d_\mathbb{B}(\varpi_1,\varpi_2))\big)\Big)\\
&\preceq \Gamma_1\Big(\psi_\mathbb{B}\big(d_\mathbb{B}(\varpi_1,\varpi_2)\big)\Big).
\end{aligned}$$

This implies $\tau \preceq \Gamma_1\big(\psi_\mathbb{B}(d_\mathbb{B}(\varpi_1,\varpi_2))\big) - \Gamma_1\big(\psi_\mathbb{B}(d_\mathbb{B}(\Gamma\varpi_1,\Gamma\varpi_2))\big) = \theta_\mathbb{B}$, a contradiction. Hence, Γ has atmost one fixed point in $\mho$.
For $\varpi_0 \in \mho$, define a sequence $\{\varpi_\jmath\}$ in $\mho$ s.t $\varpi_{\jmath+1} = \Gamma\varpi_\jmath\ \forall\ \jmath \in \mathbb{N}$. If $\varpi_\jmath = \varpi_{\jmath+1}$ for some $\jmath \in \mathbb{N}$, then $\varpi_\jmath$ is a fixed point of Γ.
Now, suppose that $\varpi_\jmath \neq \varpi_{\jmath+1}\ \forall\ \jmath \in \mathbb{N}$ and $d_{\mathbb{B}_\jmath} = d_\mathbb{B}(\varpi_\jmath, \varpi_{\jmath+1})$.
Consider,

$$\begin{aligned}
\Gamma_1\Big(\psi_\mathbb{B}\big(d_\mathbb{B}(\varpi_{\jmath+1},\varpi_\jmath)\big)\Big) &= \Gamma_1\Big(\psi_\mathbb{B}\big(d_\mathbb{B}(\Gamma\varpi_\jmath,\Gamma\varpi_{\jmath-1})\big)\Big)\\
&\preceq \tau + \Gamma_1\Big(\psi_\mathbb{B}\big(d_\mathbb{B}(\Gamma\varpi_\jmath,\Gamma\varpi_{\jmath-1})\big)\Big)\\
&\preceq F^*\Big(\Gamma_1\big(\psi_\mathbb{B}(d_\mathbb{B}(\varpi_\jmath,\varpi_{\jmath-1}))\big), \Gamma_1\big(\phi_\mathbb{B}(d_\mathbb{B}(\varpi_\jmath,\varpi_{\jmath-1}))\big)\Big)\\
&\preceq \Gamma_1\Big(\psi_\mathbb{B}\big(d_\mathbb{B}(\varpi_\jmath,\varpi_{\jmath-1})\big)\Big). \qquad (2.2.13)
\end{aligned}$$

Since, Γ_1 and $\psi_\mathbb{B}$ is non decreasing. $\therefore$ the sequence $\{d_{\mathbb{B}_\jmath}\}$ is monotonically decreasing and bounded in $\mathbb{B}^+$. Thus, $\exists\ \theta_\mathbb{B} \preceq \varrho \in \mathbb{B}^+$ s.t

$d_\mathbb{B}(\varpi_\jmath, \varpi_{\jmath+1}) \to \varrho$ as $\jmath \to \infty$.

Taking limit as $\jmath \to \infty$ in (2.2.13), we have

$$\Gamma_1\Big(\psi_{\mathbb{B}}(\varrho)\Big) \preceq F^*\Bigg(\Gamma_1\Big(\psi_{\mathbb{B}}(\varrho)\Big), \Gamma_1\Big(\phi_{\mathbb{B}}(\varrho)\Big)\Bigg) \preceq \Gamma_1\Big(\psi_{\mathbb{B}}(\varrho)\Big). \tag{2.2.14}$$

Thus, $F^*\Bigg(\Gamma_1\Big(\psi_{\mathbb{B}}(\varrho)\Big), \Gamma_1\Big(\phi_{\mathbb{B}}(\varrho)\Big)\Bigg) = \Gamma_1\Big(\psi_{\mathbb{B}}(\varrho)\Big)$ implies either $\Gamma_1\Big(\psi_{\mathbb{B}}(\varrho)\Big) = \theta_{\mathbb{B}}$ or $\Gamma_1\Big(\phi_{\mathbb{B}}(\varrho)\Big) = \theta_{\mathbb{B}}$. Hence, from (2.2.11), we have

$$\lim_{\jmath\to\infty} \Gamma_1\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}_\jmath})\Big) = \theta_{\mathbb{B}} \text{ implies } \lim_{\jmath\to\infty} \psi_{\mathbb{B}}(d_{\mathbb{B}_\jmath}) = \theta_{\mathbb{B}} \text{ or } \lim_{\jmath\to\infty} d_{\mathbb{B}_\jmath} = \theta_{\mathbb{B}}. \tag{2.2.15}$$

Now, to show $\{\varpi_\jmath\}$ is a C_{seq} in $(\mho, \mathbb{B}, d_{\mathbb{B}})$. Assume that $\{\varpi_\jmath\}$ is not a C_{seq} in $(\mho, \mathbb{B}, d_{\mathbb{B}})$. Then, for any $\epsilon > 0$ $\exists$ subsequences $(\varpi_{\imath_k})$ and $(\varpi_{\jmath_k})$ with $\jmath_k > \imath_k > k$ s.t

$$\|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k})\| \geq \epsilon. \tag{2.2.16}$$

Now, choose $\jmath_k > \imath_k$ in such a way that satisfy (2.2.16) and

$$\|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k - 1})\| < \epsilon. \tag{2.2.17}$$

Using (2.2.16) and (2.2.17), we have

$$\begin{aligned} \epsilon &\leq \|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k})\| \leq \|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k-1})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\jmath_k})\| \\ &\leq \epsilon + \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\jmath_k})\|. \end{aligned} \tag{2.2.18}$$

From (2.2.15), we have

$$\lim_{k\to\infty} \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\jmath_k})\| = \theta_{\mathbb{B}}. \tag{2.2.19}$$

Taking limit as $k \to \infty$ in (2.2.18) and using (2.2.19), we have

$$\lim_{k\to\infty} \|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\jmath_k})\| = \epsilon. \tag{2.2.20}$$

Again,

$$\begin{aligned} \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k})\| &\leq \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\jmath_k-1})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k})\| \\ &\leq \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\jmath_k-1})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1})\| \\ &\quad + \|d_{\mathbb{B}}(\varpi_{\imath_k-1}, \varpi_{\imath_k})\|. \end{aligned} \tag{2.2.21}$$

Also,

$$\begin{aligned} \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1})\| &\leq \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\jmath_k})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k-1})\| \\ &\leq \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\jmath_k})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k})\| \\ &\quad + \|d_{\mathbb{B}}(\varpi_{\imath_k}, \varpi_{\imath_k-1})\|. \end{aligned} \tag{2.2.22}$$

Taking limit as $k \to \infty$ in (2.2.21) & (2.2.22) and using (2.2.19) & (2.2.20), we have

$$\lim_{k\to\infty} \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1})\| = \epsilon.$$

Since, $d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1}), d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k}) \in \mathbb{B}^+$ and

$$\lim_{k\to\infty} \|d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1})\| = \lim_{k\to\infty} \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k})\| = \epsilon.$$

$\therefore \exists\, \varepsilon \in \mathbb{B}^+$ with $\|\varepsilon\| = \epsilon$ s.t

$$\lim_{k\to\infty} d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1}) = \lim_{k\to\infty} d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k}) = \varepsilon.$$

Using (2.2.12), we have

$$\begin{aligned}
\Gamma_1\Big(\psi_{\mathbb{B}}(\varepsilon)\Big) &= \lim_{k\to\infty} \Gamma_1\bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k})\Big)\bigg) = \lim_{k\to\infty} \Gamma_1\bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma\varpi_{\jmath_k-1}, \Gamma\varpi_{\imath_k-1})\Big)\bigg) \\
&\preceq \tau + \lim_{k\to\infty} \Gamma_1\bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma\varpi_{\jmath_k-1}, \Gamma\varpi_{\imath_k-1})\Big)\bigg) \\
&\preceq \lim_{k\to\infty} F^*\bigg(\Gamma_1\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1}))\Big), \Gamma_1\Big(\phi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1}))\Big)\bigg) \\
&\preceq \lim_{k\to\infty} \Gamma_1\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi_{\jmath_k-1}, \varpi_{\imath_k-1}))\Big) = \Gamma_1\Big(\psi_{\mathbb{B}}(\varepsilon)\Big). \qquad (2.2.23)
\end{aligned}$$

Thus, $F^*\Big(\Gamma_1(\psi_{\mathbb{B}}(\varepsilon)), \Gamma_1(\phi_{\mathbb{B}}(\varepsilon))\Big) = \Gamma_1(\psi_{\mathbb{B}}(\varepsilon))$ implies either $\Gamma_1(\psi_{\mathbb{B}}(\varepsilon)) = \theta_{\mathbb{B}}$ or $\Gamma_1(\phi_{\mathbb{B}}(\varepsilon)) = \theta_{\mathbb{B}}$ i.e, $\psi_{\mathbb{B}}(\varepsilon) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(\varepsilon) = \theta_{\mathbb{B}}$. Hence, $\varepsilon = \theta_{\mathbb{B}}$, a contradiction. Thus, $\{\varpi_\jmath\}$ is a C_{seq} in a complete C^*_{AV}- MS. $\therefore \exists\, \varpi \in \mho$ s.t $\varpi_\jmath \to \varpi$ as $\jmath \to \infty$. Consider,

$$\begin{aligned}
\Gamma_1\bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi, \Gamma\varpi)\Big)\bigg) &\preceq \tau + \lim_{\jmath\to\infty} \Gamma_1\bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_\jmath, \Gamma\varpi_\jmath)\Big)\bigg) \\
&\preceq \lim_{\jmath\to\infty} F^*\bigg(\Gamma_1\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath))\Big), \Gamma_1\Big(\phi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath))\Big)\bigg) \\
&\preceq \lim_{\jmath\to\infty} \Gamma_1\bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath)\Big)\bigg) \\
&= \Gamma_1\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi, \varpi))\Big) = \Gamma_1\Big(\psi_{\mathbb{B}}(\theta_{\mathbb{B}})\Big) \\
&= \Gamma_1\Big(\theta_{\mathbb{B}}\Big) = \theta_{\mathbb{B}}.
\end{aligned}$$

$\therefore \Gamma_1\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi, \Gamma\varpi))\Big) = \theta_{\mathbb{B}}$ implies $\psi_{\mathbb{B}}(d_{\mathbb{B}}(\varpi, \Gamma\varpi)) = \theta_{\mathbb{B}}$, i.e, $d_{\mathbb{B}}(\varpi, \Gamma\varpi) = \theta_{\mathbb{B}}$. Hence, $\Gamma\varpi = \varpi$. □

Consider $F^*(\varpi, \varsigma) = \varpi$ and $\psi_{\mathbb{B}}(\varrho) = \varrho = \phi_{\mathbb{B}}(\varrho)$, we have

Corollary 2.2.4. *Let* $(\mho, \mathbb{B}, d_{\mathbb{B}})$ *be a complete* C^*_{AV}*-MS and* $\Gamma : \mho \to \mho$. *If* $\exists\, \Gamma_1 : \mathbb{B}^+ \to \mathbb{B}$ *satisfying:*

(*i*) Γ_1 *is continuous and nondecreasing on* $\mathbb{B}^+$*;*

(*ii*) *For each sequence* $\{\gamma_\jmath\} \subseteq \mathbb{B}^+$

$$\lim_{\jmath\to\infty} \gamma_\jmath = \theta_{\mathbb{B}} \textit{ iff } \lim_{\jmath\to\infty} \Gamma_1(\gamma_\jmath) = \theta_{\mathbb{B}};$$

(*iii*) $\forall\, \varpi, \varsigma \in \mho$ *with* $\tau > 0$,

$$\tau + \Gamma_1\Big(d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma)\Big) \preceq \Gamma_1\Big(d_{\mathbb{B}}(\varpi, \varsigma)\Big).$$

Then, Γ *has a unique fixed point.*

Example 2.2.5. Let $\mho = [0, 2]$ and $\mathbb{B} = \mathbb{C}$. Let $\Gamma_1 : \mathbb{B}^+ \to \mathbb{B}$ defined as $\Gamma_1(a) = 25a$, $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ defined by $d_{\mathbb{B}}(\varpi, \varsigma) = \begin{cases} |\varpi| + |\varsigma| & \text{if } \varpi \neq \varsigma \\ 0 & \text{if } \varpi = \varsigma \end{cases}$ and $\Gamma\varpi = \dfrac{\varpi}{150}$. Then,

(*i*) $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is a complete C^*_{AV}- MS;

(*ii*) Γ_1 is non-decreasing;

(*iii*) Γ_1 is continuous;

(*iv*) $\lim_{\jmath\to\infty} \varsigma_\jmath = \theta_{\mathbb{B}}$ implies $\lim_{\jmath\to\infty} \Gamma_1(\varsigma_\jmath) = \theta_{\mathbb{B}}$;

(*v*) $\forall\, \varsigma, \varpi \in \mho$ with $\tau = 0.1$, $F^*(\varpi, \varsigma) = \varpi$, $\psi(\varsigma) = \phi(\varsigma) = 5\varsigma$, we have $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = \dfrac{\varpi + \varsigma}{150}$ and $\psi\Big(d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma)\Big) = \dfrac{\varpi + \varsigma}{30}$.
Hence,

$$\begin{aligned} \tau + \Gamma_1\Big(\psi(d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma))\Big) &= 0.1 + \Gamma_1\left(\frac{\varpi + \varsigma}{30}\right) \\ &= 0.1 + \frac{5(\varpi + \varsigma)}{6} \\ &\preceq 125(\varpi + \varsigma) \\ &= \Gamma_1\Big(\psi(d_{\mathbb{B}}(\varpi, \varsigma))\Big) \\ &= F^*\left(\Gamma_1\Big(\psi(d_{\mathbb{B}}(\varpi, \varsigma))\Big), \Gamma_1\Big(\phi(d_{\mathbb{B}}(\varpi, \varsigma))\Big)\right). \end{aligned}$$

Γ satisfies all the hypothesis of Theorem (2.2.3). Thus, Γ has a unique fixed point. Indeed, '0' is a fixed point.

Example 2.2.6. Let $\mho = [0,2]$ and $\mathbb{B} = \mathbb{C}$. Let $\Gamma_1 : \mathbb{B}^+ \to \mathbb{B}$ defined as $\Gamma_1(a) = 25a$, $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ defined as $d_{\mathbb{B}}(\varpi, \varsigma) = |\varpi - \varsigma|$ and $\Gamma\varpi = \begin{cases} 1/10, & \text{if } \varpi \in [0,1] \\ 1/20, & \text{otherwise} \end{cases}$.
Then,

(i) $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is a complete C^*_{AV}- MS;

(ii) Γ_1 is non-decreasing;

(iii) Γ_1 is continuous;

(iv) $\lim_{\jmath\to\infty} \varsigma_\jmath = \theta_{\mathbb{B}}$ implies $\lim_{\jmath\to\infty} \Gamma_1(\varsigma_\jmath) = \theta_{\mathbb{B}}$;

(v) $\forall\ \varsigma, \varpi \in [0,1]$ and $\forall\ \varsigma, \varpi \in (1,2]$ with $\tau = 0.1$, $F^*(\varpi, \varsigma) = \varpi$, $\psi(\varpi) = \phi(\varpi) = 5\varpi$, we have $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = 0$ and $\psi\Big(d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma)\Big) = 0$.
Hence,

$$\begin{aligned} \tau + \Gamma_1\Big(\psi(d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma))\Big) &= 0.1 + \Gamma_1(0) \\ &\preceq \Gamma_1\Big(\psi(d_{\mathbb{B}}(\varpi, \varsigma))\Big) \\ &= F^*\Bigg(\Gamma_1\Big(\psi(d_{\mathbb{B}}(\varpi, \varsigma))\Big), \Gamma_1\Big(\phi(d_{\mathbb{B}}(\varpi, \varsigma))\Big)\Bigg). \end{aligned}$$

Γ satisfies all the hypothesis of Theorem (2.2.3). Thus, Γ has a unique fixed point. Indeed, '$\frac{1}{10}$' is a fixed point.

2.3 Fixed Point of Self Mappings in C^*-Algebra Valued Partial Metric Space

Samet (2015) presented the notion of $(\alpha - \psi)$-type contraction mapping in b-MS. Omran & Masmali (2021) generalized $(\alpha - \psi)$-type contraction mapping in C^*_{AV}-b-MS and proved some fixed point results. Later, many results on $(\alpha - \psi)$-type contraction mapping in various spaces have been presented by researchers (see, Hussain & Ahmad (2017) Hussain et al. (2017), Massit et al. (2022) and reference cited therein). In this section, the notion of $(\alpha_{\mathbb{B}} - \psi_{\mathbb{B}})$-type contraction mapping and some fixed point results in C^*_{AV}-PMS are presented.

Definition 2.3.1. Let $(\mho, \mathbb{B}, p_{\mathbb{B}})$ be a C^*_{AV}-PMS and $\Gamma : \mho \to \mho$ be $(\alpha_{\mathbb{B}} - \psi_{\mathbb{B}})$ **type contractive mapping** if $\exists$ two functions $\alpha_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}^+$ and $\psi_{\mathbb{B}} \in \Psi^1_{\mathbb{B}}$ s.t

$$\alpha_{\mathbb{B}}(\varpi, \varsigma) p_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi, \varsigma)\Big)\ \forall\ \varpi, \varsigma \in \mho. \tag{2.3.1}$$

Theorem 2.3.2. *Let $(\mho, \mathbb{B}, p_{\mathbb{B}})$ be a complete C^*_{AV}-PMS and $\Gamma : \mho \to \mho$ be $(\alpha_{\mathbb{B}} - \psi_{\mathbb{B}})$-type contractive mapping satisfying:*

(i) Γ *is $\alpha_{\mathbb{B}}$-admissible;*

(ii) $\exists\, \varpi_0 \in \mho$ *s.t* $\alpha_{\mathbb{B}}(\varpi_0, \Gamma\varpi_0) \succeq I_{\mathbb{B}}$;

(iii) Γ *is continuous.*

Then, Γ *has a unique fixed point.*

Proof. Let $\varpi_0 \in \mho$ s.t $\alpha_{\mathbb{B}}(\varpi_0, \Gamma\varpi_0) \succeq I_{\mathbb{B}}$. We construct a sequence $\{\varpi_\jmath\}$ in $\mho$ as $\varpi_{\jmath+1} = \Gamma\varpi_\jmath \ \forall\ \jmath \in \mathbb{N}$. If $\varpi_\jmath = \varpi_{\jmath+1}$ for some $\jmath \in \mathbb{N}$, then $\varpi_\jmath$ is a fixed point of Γ. Now, suppose that $\varpi_\jmath \neq \varpi_{\jmath+1} \ \forall\ \jmath \in \mathbb{N}$. Since, Γ is $\alpha_{\mathbb{B}}$-admissible.

$$\therefore\ \alpha_{\mathbb{B}}(\varpi_0, \varpi_1) = \alpha_{\mathbb{B}}(\varpi_0, \Gamma\varpi_0) \succeq I_{\mathbb{B}} \Rightarrow \alpha_{\mathbb{B}}(\Gamma\varpi_0, \Gamma^2\varpi_0) = \alpha_{\mathbb{B}}(\varpi_1, \varpi_2) \succeq I_{\mathbb{B}}. \quad (2.3.2)$$

On generalizing, we have

$$\alpha_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \succeq I_{\mathbb{B}} \ \forall\ \jmath \in \mathbb{N}. \quad (2.3.3)$$

Using (2.3.1) and (2.3.3), we have

$$\begin{aligned} p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) &= p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \Gamma\varpi_\jmath) \preceq \alpha_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath) p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \Gamma\varpi_\jmath) \\ &\preceq \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath)\Big). \end{aligned} \quad (2.3.4)$$

On generalizing, we have

$$p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \preceq \psi^\jmath_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) \ \forall\ \jmath \in \mathbb{N}. \quad (2.3.5)$$

For $\imath < \jmath \in \mathbb{N}$, we have

$$\begin{aligned} p_{\mathbb{B}}(\varpi_\imath, \varpi_{\imath+\jmath}) &\preceq p_{\mathbb{B}}(\varpi_\imath, \varpi_{\imath+1}) + p_{\mathbb{B}}(\varpi_{\imath+1}, \varpi_{\imath+\jmath}) - p_{\mathbb{B}}(\varpi_{\imath+1}, \varpi_{\imath+1}) \\ &\preceq p_{\mathbb{B}}(\varpi_\imath, \varpi_{\imath+1}) + p_{\mathbb{B}}(\varpi_{\imath+1}, \varpi_{\imath+2}) + p_{\mathbb{B}}(\varpi_{\imath+2}, \varpi_{\imath+3}) \\ &\quad + \cdots + p_{\mathbb{B}}(\varpi_{\imath+\jmath-1}, \varpi_{\imath+\jmath}) - p_{\mathbb{B}}(\varpi_{\imath+1}, \varpi_{\imath+1}) \\ &\quad - p_{\mathbb{B}}(\varpi_{\imath+2}, \varpi_{\imath+2}) - \cdots - p_{\mathbb{B}}(\varpi_{\imath+\jmath-1}, \varpi_{\imath+\jmath-1}) \\ &\preceq \psi^\imath_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) + \psi^{\imath+1}_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) + \cdots \\ &\quad + \psi^{\imath+\jmath-1}_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) - \sum_{i=\imath+1}^{\imath+\jmath-1} p_{\mathbb{B}}(\varpi_i, \varpi_i) \\ &= \sum_{i=\imath}^{\imath+\jmath-1} \psi^i_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) - \sum_{i=\imath+1}^{\imath+\jmath-1} p_{\mathbb{B}}(\varpi_i, \varpi_i). \end{aligned} \quad (2.3.6)$$

Using $\theta_{\mathbb{B}} \preceq p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath}) \preceq p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1})$ and (2.3.5), we have

$$\theta_{\mathbb{B}} \preceq p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath}) \preceq p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1}) \preceq \psi_{\mathbb{B}}^{\jmath}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big). \tag{2.3.7}$$

Taking limit as $\jmath \to \infty$, we have

$$\lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath}) = \theta_{\mathbb{B}}. \tag{2.3.8}$$

Using (2.3.8) in (2.3.6), we have

$$p_{\mathbb{B}}(\varpi_{\imath}, \varpi_{\imath+\jmath}) \preceq \sum_{i=\imath}^{\imath+\jmath-1} \psi_{\mathbb{B}}^{i}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) - \sum_{i=\imath+1}^{\imath+\jmath-1} p_{\mathbb{B}}(\varpi_i, \varpi_i) \to \theta_{\mathbb{B}} \ as \ \imath \to \infty. \tag{2.3.9}$$

Hence, $\{\varpi_{\jmath}\}$ is a C_{seq} in a complete $(\mho, \mathbb{B}, p_{\mathbb{B}})$. Thus, $\exists\ \varpi \in \mho$ s.t $\varpi_{\jmath} \to \varpi$ as $\jmath \to \infty$. $\therefore$

$$\lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\imath}) = \lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_{\jmath}, \varpi) = p_{\mathbb{B}}(\varpi, \varpi).$$

To prove $\Gamma\varpi = \varpi$.
Consider,

$$\begin{aligned}
p_{\mathbb{B}}(\Gamma\varpi, \varpi) &= \lim_{\jmath\to\infty} p_{\mathbb{B}}(\Gamma\varpi, \varpi_{\jmath+1}) \\
&\preceq \lim_{\jmath\to\infty}\Big(p_{\mathbb{B}}(\Gamma\varpi, \varpi_{\jmath}) + p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1}) - p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath})\Big) \\
&= \lim_{\jmath\to\infty}\Big(p_{\mathbb{B}}(\Gamma\varpi, \Gamma\varpi_{\jmath-1}) + p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1}) - p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath})\Big) \\
&\preceq \lim_{\jmath\to\infty}\Big(\alpha_{\mathbb{B}}(\varpi, \varpi_{\jmath-1})p_{\mathbb{B}}(\Gamma\varpi, \Gamma\varpi_{\jmath-1}) + p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1}) - p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath})\Big) \\
&\preceq \lim_{\jmath\to\infty}\Big(\psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi, \varpi_{\jmath-1})\Big) + p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1}) - p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath})\Big) \\
&\preceq \lim_{\jmath\to\infty} \psi_{\mathbb{B}}^{\jmath-1}\Big(p_{\mathbb{B}}(\varpi, \varpi_0)\Big) + \lim_{\jmath\to\infty} \psi_{\mathbb{B}}^{\jmath}\Big(p_{\mathbb{B}}(\varpi_1, \varpi_0)\Big) - \lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath}) \to \theta_{\mathbb{B}}.
\end{aligned}$$

Hence, $\Gamma\varpi = \varpi$.
Uniqueness: Consider, $\forall\ \varpi, \varsigma \in \mho$, $\exists\ \varkappa \in \mho$ s.t $\alpha_{\mathbb{B}}(\varpi, \varkappa) \succeq I_{\mathbb{B}}$ and $\alpha_{\mathbb{B}}(\varsigma, \varkappa) \succeq I_{\mathbb{B}}$. Then,

$$\begin{aligned}
p_{\mathbb{B}}(\varpi, \Gamma^{\jmath}\varkappa) = p_{\mathbb{B}}(\Gamma\varpi, \Gamma(\Gamma^{\jmath-1}\varkappa)) &\preceq \alpha_{\mathbb{B}}(\varpi, \Gamma^{\jmath-1}\varkappa)p_{\mathbb{B}}\Big(\Gamma\varpi, \Gamma(\Gamma^{\jmath-1}\varkappa)\Big) \\
&\preceq \psi_{\mathbb{B}}^{\jmath}\Big(p_{\mathbb{B}}(\varpi, \varkappa)\Big) \to \theta_{\mathbb{B}} \ as \ \jmath \to \infty.
\end{aligned}$$

Thus, $\Gamma^{\jmath}\varkappa = \varpi$. Similarly, $\Gamma^{\jmath}\varkappa = \varsigma$ as $\jmath \to \infty$. Hence, $\varpi = \varsigma$. □

Theorem 2.3.3. *Let $(\mho, \mathbb{B}, p_{\mathbb{B}})$ be a complete C^*_{AV}-PMS and $\Gamma : \mho \to \mho$ satisfying*

$$\alpha_{\mathbb{B}}(\varpi, \varsigma)p_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi, \Gamma\varpi) + p_{\mathbb{B}}(\varsigma, \Gamma\varsigma)\Big)\ \forall\ \varpi, \varsigma \in \mho, \tag{2.3.10}$$

and

(i) Γ *is* $\alpha_{\mathbb{B}}$*-admissible;*

(ii) $\exists\, \varpi_0 \in \mho$ *s.t* $\alpha_{\mathbb{B}}(\varpi_0, \Gamma\varpi_0) \succeq I_{\mathbb{B}}$*;*

(iii) Γ *is continuous.*

where $\alpha_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}^+$, $\psi_{\mathbb{B}} \in \Psi^1_{\mathbb{B}}$. *Then,* Γ *has a unique fixed point.*

Proof. On the similar lines of Theorem (2.3.2), we have

$$\alpha_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \succeq I_{\mathbb{B}}\ \forall\ \jmath \in \mathbb{N}. \tag{2.3.11}$$

Using (2.3.10) and (2.3.11), we have

$$\begin{aligned}
p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) &= p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \Gamma\varpi_\jmath) \preceq \alpha_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath) p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \Gamma\varpi_\jmath) \\
&\preceq \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \varpi_{\jmath-1}) + p_{\mathbb{B}}(\Gamma\varpi_\jmath.\varpi_\jmath)\Big) \\
&= \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1}) + p_{\mathbb{B}}(\varpi_{\jmath+1}, \varpi_\jmath)\Big) \\
&= \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1})\Big) + \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1})\Big).
\end{aligned} \tag{2.3.12}$$

From (2.3.12), we have

$$(1 - \psi_{\mathbb{B}}) p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \preceq \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1})\Big),$$

$$p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \preceq \psi_{\mathbb{B}}(1 - \psi_{\mathbb{B}})^{-1} p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1}).$$

Taking $\phi_{\mathbb{B}} = \psi_{\mathbb{B}}(1 - \psi_{\mathbb{B}})^{-1} = \psi_{\mathbb{B}} \sum_{\jmath=0}^{\infty} \psi^\jmath_{\mathbb{B}} = \sum_{\jmath=0}^{\infty} \psi^{\jmath+1}_{\mathbb{B}} \prec \infty$, we have

$$p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \preceq \phi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath)\Big).$$

On generalizing, we have

$$p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \preceq \phi^\jmath_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big)\ \forall\ \jmath \in \mathbb{N}. \tag{2.3.13}$$

For $\imath < \jmath$, follow from Theorem (2.3.2), we have

$$p_{\mathbb{B}}(\varpi_\imath, \varpi_{\imath+\jmath}) \preceq \sum_{i=\imath}^{\imath+\jmath-1} \phi^i_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) - \sum_{i=\imath+1}^{\imath+\jmath-1} p_{\mathbb{B}}(\varpi_i, \varpi_i). \tag{2.3.14}$$

Using $\theta_{\mathbb{B}} \preceq p_{\mathbb{B}}(\varpi_\jmath, \varpi_\jmath) \preceq p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1})$ and (2.3.13), we have

$$\theta_{\mathbb{B}} \preceq p_{\mathbb{B}}(\varpi_\jmath, \varpi_\jmath) \preceq p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \preceq \phi^\jmath_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big). \tag{2.3.15}$$

Taking limit as $\jmath \to \infty$ in (2.3.15), we have

$$\lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_\jmath, \varpi_\jmath) = \theta_{\mathbb{B}}. \tag{2.3.16}$$

Using (2.3.16) in (2.3.14), we have

$$p_{\mathbb{B}}(\varpi_\imath, \varpi_{\imath+\jmath}) \preceq \sum_{i=\imath}^{\imath+\jmath-1} \phi_{\mathbb{B}}^{i}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) - \sum_{i=\imath+1}^{\imath+\jmath-1} p_{\mathbb{B}}(\varpi_i, \varpi_i) \to \theta_{\mathbb{B}} \text{ as } \jmath \to \infty.$$

Thus, $\{\varpi_\jmath\}$ is a C_{seq} in a complete $(\mho, \mathbb{B}, p_{\mathbb{B}})$. So, $\exists\, \varpi \in \mho$ s.t
$\varpi_\jmath \to \varpi$ as $\jmath \to \infty$ and

$$\lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_\jmath, \varpi_\imath) = \lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_\jmath, \varpi) = p_{\mathbb{B}}(\varpi, \varpi). \tag{2.3.17}$$

To prove $\Gamma\varpi = \varpi$.
Consider,

$$\begin{aligned}
p_{\mathbb{B}}(\Gamma\varpi, \varpi) &= \lim_{\jmath\to\infty} p_{\mathbb{B}}(\Gamma\varpi_\jmath, \varpi_\jmath) \\
&= \lim_{\jmath\to\infty} p_{\mathbb{B}}(\Gamma\varpi_\jmath, \Gamma\varpi_{\jmath-1}) \\
&\preceq \lim_{\jmath\to\infty} \alpha_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1}) p_{\mathbb{B}}(\Gamma\varpi_\jmath, \Gamma\varpi_{\jmath-1}) \\
&\preceq \lim_{\jmath\to\infty} \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\Gamma\varpi_\jmath, \varpi_\jmath) + p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \varpi_{\jmath-1})\Big) \\
&= \lim_{\jmath\to\infty} \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_{\jmath+1}, \varpi_\jmath) + p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1})\Big) \\
&\preceq \lim_{\jmath\to\infty} \psi_{\mathbb{B}}^{\jmath}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) + \lim_{\jmath\to\infty} \psi_{\mathbb{B}}^{\jmath-1}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) = \theta_{\mathbb{B}}.
\end{aligned}$$

Hence, $\Gamma\varpi = \varpi$.
Uniqueness: Let ς be another fixed point of Γ. On the similar lines, construct a C_{seq} $\{\varsigma_\jmath\}$ s.t $\lim_{\jmath\to\infty} \varsigma_\jmath \to \varsigma$ for $\varsigma \in \mho$.
Consider,

$$\begin{aligned}
\theta_{\mathbb{B}} \preceq p_{\mathbb{B}}(\varpi, \varsigma) &= \lim_{\jmath\to\infty} p_{\mathbb{B}}(\Gamma\varpi_\jmath, \Gamma\varsigma_\jmath) \preceq \lim_{\jmath\to\infty} \alpha_{\mathbb{B}}(\varpi_\jmath, \varsigma_\jmath) p_{\mathbb{B}}(\Gamma\varpi_\jmath, \Gamma\varsigma_\jmath) \\
&\preceq \lim_{\jmath\to\infty} \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \Gamma\varpi_\jmath) + p_{\mathbb{B}}(\varsigma_\jmath, \Gamma\varsigma_\jmath)\Big) \\
&= \lim_{\jmath\to\infty} \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) + p_{\mathbb{B}}(\varsigma_\jmath, \varsigma_{\jmath+1})\Big) \\
&= \lim_{\jmath\to\infty} \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1})\Big) + \lim_{\jmath\to\infty} \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varsigma_\jmath, \varsigma_{\jmath+1})\Big) \\
&\preceq \lim_{\jmath\to\infty} \psi_{\mathbb{B}}^{\jmath}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) + \lim_{\jmath\to\infty} \psi_{\mathbb{B}}^{\jmath}\Big(p_{\mathbb{B}}(\varsigma_0, \varsigma_1)\Big) = \theta_{\mathbb{B}}.
\end{aligned}$$

Hence, $\varpi = \varsigma$. □

Theorem 2.3.4. *Let $(\mho, \mathbb{B}, p_{\mathbb{B}})$ be a complete C^*_{AV}-PMS and $\Gamma : \mho \to \mho$ satisfying:*

$$\alpha_{\mathbb{B}}(\varpi, \varsigma)p_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \psi_{\mathbb{B}}\Big(p(\varpi, \varsigma) + p_{\mathbb{B}}(\varpi, \Gamma\varpi) + p_{\mathbb{B}}(\varsigma, \Gamma\varsigma)\Big) \ \forall\ \varpi, \varsigma \in \mho \quad (2.3.18)$$

and

(i) Γ *is $\alpha_{\mathbb{B}}$-admissible;*

(ii) $\exists\ \varpi_0 \in \mho$ *s.t* $\alpha_{\mathbb{B}}(\varpi_0, \Gamma\varpi_0) \succeq I_{\mathbb{B}}$;

(iii) Γ *is continuous.*

where $\alpha_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}^+$, $\psi_{\mathbb{B}} \in \Psi^1_{\mathbb{B}}$. Then, Γ has a unique fixed point.

Proof. On the similar lines of Theorem (2.3.2), we have

$$\alpha_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \succeq I_{\mathbb{B}}\ \forall\ \jmath \in \mathbb{N}. \quad (2.3.19)$$

Using (2.3.18) and (2.3.19), we have

$$\begin{aligned}
p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) &= p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \Gamma\varpi_\jmath) \preceq \alpha_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath)p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \Gamma\varpi_\jmath) \\
&\preceq \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath) + p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \varpi_{\jmath-1}) + p_{\mathbb{B}}(\Gamma\varpi_\jmath, \varpi_\jmath)\Big) \\
&= \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath) + p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1}) + p_{\mathbb{B}}(\varpi_{\jmath+1}, \varpi_\jmath)\Big) \\
&= \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1})\Big)2I_A + \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1})\Big) \\
(I_{\mathbb{B}} - \psi_{\mathbb{B}})p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) &\preceq \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1})\Big)2I_{\mathbb{B}} \\
p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) &\preceq \psi_{\mathbb{B}}(I_{\mathbb{B}} - \psi_{\mathbb{B}})^{-1}\Big(p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1})\Big)2I_{\mathbb{B}}. \qquad (2.3.20)
\end{aligned}$$

Taking $\dfrac{\phi_{\mathbb{B}}}{2I_{\mathbb{B}}} = \psi_{\mathbb{B}}(1 - \psi_{\mathbb{B}})^{-1}$, we have

$$p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \preceq \phi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath)\Big).$$

On generalizing, we have

$$p_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \preceq \phi^\jmath_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big)\ \forall\ \jmath \in \mathbb{N}. \quad (2.3.21)$$

For $\imath < \jmath$, by Theorem (2.3.2), we have

$$p_{\mathbb{B}}(\varpi_\imath, \varpi_{\imath+\jmath}) \preceq \sum_{i=\imath}^{\imath+\jmath-1} \phi^i_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) - \sum_{i=\imath+1}^{\imath+\jmath-1} p_{\mathbb{B}}(\varpi_i, \varpi_i). \quad (2.3.22)$$

Using $\theta_{\mathbb{B}} \preceq p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath}) \preceq p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1})$ and by (2.3.21), we have

$$\theta_{\mathbb{B}} \preceq p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath}) \preceq p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath+1}) \preceq \phi_{\mathbb{B}}^{\jmath}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big). \tag{2.3.23}$$

Taking limit as $\jmath \to \infty$ in (2.3.23), we have

$$\lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath}) = \theta_{\mathbb{B}}. \tag{2.3.24}$$

Using (2.3.24) in (2.3.22) and taking limit as $\jmath \to \infty$, we have

$$p_{\mathbb{B}}(\varpi_{\imath}, \varpi_{\imath+\jmath}) \preceq \sum_{i=\imath}^{\imath+\jmath-1} \phi_{\mathbb{B}}^{i}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) - \sum_{i=\imath+1}^{\imath+\jmath-1} p_{\mathbb{B}}(\varpi_i, \varpi_i) \to \theta_{\mathbb{B}} \tag{2.3.25}$$

Thus, $\{\varpi_{\jmath}\}$ is a C_{seq} in complete $(\mho, \mathbb{B}, p_{\mathbb{B}})$. So, $\exists\, \varpi \in \mho$ s.t $\varpi_{\jmath} \to \varpi$ as $\jmath \to \infty$. Hence,

$$\lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\imath}) = \lim_{\jmath\to\infty} p_{\mathbb{B}}(\varpi_{\jmath}, \varpi) = p_{\mathbb{B}}(\varpi, \varpi). \tag{2.3.26}$$

To prove $\Gamma\varpi = \varpi$.
Consider,

$$\begin{aligned}
p_{\mathbb{B}}(\Gamma\varpi, \varpi) &= \lim_{\jmath\to\infty} p_{\mathbb{B}}(\Gamma\varpi_{\jmath}, \varpi_{\jmath}) = \lim_{\jmath\to\infty} p_{\mathbb{B}}(\Gamma\varpi_{\jmath}, \Gamma\varpi_{\jmath-1}) \\
&\preceq \lim_{\jmath\to\infty} \alpha_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath-1}) p_{\mathbb{B}}(\Gamma\varpi_{\jmath}, \Gamma\varpi_{\jmath-1}) \\
&\preceq \lim_{\jmath\to\infty} \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath-1}) + p_{\mathbb{B}}(\Gamma\varpi_{\jmath}, \varpi_{\jmath}) + p_{\mathbb{B}}(\Gamma\varpi_{\jmath-1}, \varpi_{\jmath-1})\Big) \\
&= \lim_{\jmath\to\infty} \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath-1}) + p_{\mathbb{B}}(\varpi_{\jmath+1}, \varpi_{\jmath}) + p_{\mathbb{B}}(\varpi_{\jmath}, \varpi_{\jmath-1})\Big) \\
&= 2\psi_{\mathbb{B}}^{\jmath-1}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) + \psi_{\mathbb{B}}^{\jmath}\Big(p_{\mathbb{B}}(\varpi_0, \varpi_1)\Big) \to \theta_{\mathbb{B}}.
\end{aligned}$$

Hence, $\Gamma\varpi = \varpi$.
Uniqueness: Let ς be another fixed point of Γ. Then,

$$\begin{aligned}
\theta_{\mathbb{B}} \preceq p_{\mathbb{B}}(\varpi, \varsigma) &= p_{\mathbb{B}}(\Gamma^{\jmath}\varpi, \Gamma^{\jmath}\varsigma) \preceq \alpha_{\mathbb{B}}(\Gamma^{\jmath-1}\varpi, \Gamma^{\jmath-1}\varsigma) p_{\mathbb{B}}(\Gamma^{\jmath}\varpi, \Gamma^{\jmath}\varsigma) \\
&\preceq \psi_{\mathbb{B}}\Big(p_{\mathbb{B}}(\Gamma^{\jmath-1}\varpi, \Gamma^{\jmath-1}\varsigma) + p_{\mathbb{B}}(\Gamma^{\jmath-1}\varpi, \Gamma^{\jmath}\varpi) + p_{\mathbb{B}}(\Gamma^{\jmath-1}\varsigma, \Gamma^{\jmath}\varsigma)\Big) \\
&= \psi_{\mathbb{B}}^{\jmath-1}\Big(p_{\mathbb{B}}(\varpi, \varsigma)\Big) + \psi_{\mathbb{B}}^{\jmath-1}\Big(p_{\mathbb{B}}(\varpi, \varpi)\Big) + \psi_{\mathbb{B}}^{\jmath-1}\Big(p_{\mathbb{B}}(\varsigma, \varsigma)\Big) \\
&= \psi_{\mathbb{B}}^{\jmath-1}\Big(p_{\mathbb{B}}(\varpi, \varsigma)\Big) \to \theta_{\mathbb{B}} \ as\ \jmath \to \infty.
\end{aligned}$$

Hence, $\varpi = \varsigma$. □

On the similar lines, the following results hold

Theorem 2.3.5. *Let* $(\mho, \mathbb{B}, p_{\mathbb{B}})$ *be a complete* C^*_{AV}*-PMS and* $\Gamma : \mho \to \mho$ *satisfying*

$$\alpha_{\mathbb{B}}(\varpi, \varsigma) p_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \psi_{\mathbb{B}}\Big(p(\varpi, \varsigma) + p_{\mathbb{B}}(\varpi, \Gamma\varsigma) + p_{\mathbb{B}}(\varsigma, \Gamma\varpi)\Big) \; \forall \; \varpi, \varsigma \in \mho$$

and

(*i*) Γ *is* $\alpha_{\mathbb{B}}$*-admissible;*

(*ii*) $\exists \; \varpi_0 \in \mho$ *s.t* $\alpha_{\mathbb{B}}(\varpi_0, \Gamma\varpi_0) \succeq I_{\mathbb{B}}$*;*

(*iii*) Γ *is continuous.*

where $\alpha_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}^+$, $\psi_{\mathbb{B}} \in \Psi^1_{\mathbb{B}}$*. Then,* Γ *has a unique fixed point.*

Theorem 2.3.6. *Let* $(\mho, \mathbb{B}, p_{\mathbb{B}})$ *be a complete* C^*_{AV}*-PMS and* $\Gamma : \mho \to \mho$ *satisfying*

$$\alpha_{\mathbb{B}}(\varpi, \varsigma) p_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \psi_{\mathbb{B}}\Big(p(\varpi, \varsigma) + p_{\mathbb{B}}(\varpi, \Gamma\varpi) + p_{\mathbb{B}}(\varsigma, \Gamma\varsigma) + p_{\mathbb{B}}(\varpi, \Gamma\varsigma) + p_{\mathbb{B}}(\varsigma, \Gamma\varpi)\Big)$$

$\forall \; \varpi, \varsigma \in \mho$ *and*

(*i*) Γ *is* $\alpha_{\mathbb{B}}$*-admissible;*

(*ii*) $\exists \; \varpi_0 \in \mho$ *s.t* $\alpha_{\mathbb{B}}(\varpi_0, \Gamma\varpi_0) \succeq I_{\mathbb{B}}$*;*

(*iii*) Γ *is continuous.*

where $\alpha_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}^+$, $\psi_{\mathbb{B}} \in \Psi^1_{\mathbb{B}}$*. Then,* Γ *has a unique fixed point.*

Example 2.3.7. Let $\mho = [0, 1)$ and $\varpi \in \mathbb{B} = \mathbb{C}$ be a non zero element. Define $d_{\mathbb{B}}(\mu, \nu) = \max\{1 - \mu, 1 - \nu\}\varpi\varpi^*$ Then, $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ is a complete C^*_{AV}-PMS. But $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ is not a C^*_{AV}-MS, since $d_{\mathbb{B}}(\mu, \mu) = (1 - \mu)\varpi\varpi^* \neq \theta_{\mathbb{B}}$. Define $\Gamma : \mho \to \mho$ by $\Gamma\varpi = \varpi/3$ and $\alpha_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ s.t $\alpha_{\mathbb{B}}(\varpi, \varsigma) = I_{\mathbb{B}}$. So, $\alpha_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = I_{\mathbb{B}}$ implies that Γ is $\alpha_{\mathbb{B}}$-admissible.
Define $\psi_{\mathbb{B}} : \mathbb{C}^+ \to \mathbb{C}^+, \psi_{\mathbb{B}}(a) = 2a$. Clearly, Γ is $\alpha_{\mathbb{B}} - \psi_{\mathbb{B}}$-contractive mapping $\forall$ $\varpi, \varsigma \in \mho$. Γ satisfies all the hypothesis of Theorem (2.3.2). Thus, Γ has a unique fixed point. Indeed, '0' is a fixed point.

2.4 C^*-Algebra Valued-$b_{\mathcal{R}}$-Metric Space

Khalehoghli et al. (2020) introduced the notion of $\mathcal{R}$-MS and proved Banach contraction principle in this framework. Recently, Malhotra et al. (2022) generalized

$\mathcal{R}$-MS by introducing C^*_{AV}-$\mathcal{R}$-MS and proved some fixed point results in this framework. In this section, the notion of C^*_{AV}-$b_{\mathcal{R}}$-MS which is a generalization of C^*_{AV}-$\mathcal{R}$-MS and some results in these settings are presented.

Definition 2.4.1. $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ is said to be $\mathbf{C^*_{AV}}$**-b**$_{\mathcal{R}}$**-MS** if the following are satisfied:

(*i*) $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is C^*_{AV}-b-MS;

(*ii*) $\mathcal{R}$ is a binary relation on $\mho$.

Remark 2.4.2. Every C^*_{AV}-$\mathcal{R}$-MS is a C^*_{AV}-$b_{\mathcal{R}}$-MS but converse is not true.

Example 2.4.3. Let $\mho = \mathbb{R}$ and $\mathbb{B} = M_{\jmath}(\mathbb{R})$. Define

$$d_{\mathbb{B}}(\varpi, \varsigma) = \text{diag}\Big(c_1|\varpi - \varsigma|^p, c_2|\varpi - \varsigma|^p, \cdots c_{\jmath}|\varpi - \varsigma|^p\Big) \;\forall\; \varpi, \varsigma \in \mho,$$

where diag denotes the diagonal matrix, $c_i > 0 \;\forall\; i = 1, 2, ..., \jmath$ are constants and $p > 1$. Define $\mathcal{R}$ on $\mho$ s.t $\varpi \mathcal{R} \varsigma \Leftrightarrow (\varpi - \varsigma) < \varpi$. It is easy to verify that d is C^*_{AV}-$b_{\mathcal{R}}$-metric and $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ is a C^*_{AV}- $b_{\mathcal{R}}$-MS, where $\Lambda = 3^p I \in \mathbb{B}$ and $\Lambda \succ I$ with $3^p > 1$. But $|\varpi - \varsigma|^p \leq |\varpi - \mu|^p + |\mu - \varsigma|^p$ don't hold for $\varpi > \mu > \varsigma > 0$. Thus, $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ is not a C^*_{AV}-$\mathcal{R}$-MS.

Example 2.4.4. Assume that

$$\mathcal{P} = \begin{pmatrix} 10 & 10 \\ 12 & 14 \end{pmatrix}, \; \mathcal{Q} = \begin{pmatrix} 5 & 4 \\ 4 & 4 \end{pmatrix}, \; \mathcal{M} = \begin{pmatrix} -4 & 0 \\ -2 & -2 \end{pmatrix}.$$

Let $\mho = \{\mathcal{P}, \mathcal{Q}, \mathcal{M}\}$ and $\mathbb{B} = M_2(\mathbb{R})$. Define $d_{\mathbb{B}}(\mathcal{P}, \mathcal{Q}) = \mathcal{P} + \mathcal{Q}$, where $\mathcal{P}, \mathcal{Q}$ denotes the matrices of order 2. Define $\mathcal{R}$ on $\mho$ s.t $\mathcal{P}\mathcal{R}\mathcal{Q} \Leftrightarrow \det(\mathcal{P}) > \det(\mathcal{Q})$. Thus, $\det(\mathcal{P}) > \det(\mathcal{Q})$, $\det(\mathcal{P}) > \det(\mathcal{M})$ and $\det(\mathcal{M}) > \det(\mathcal{Q})$. Hence, $\mathcal{P}\mathcal{R}\mathcal{Q}, \mathcal{M}\mathcal{R}\mathcal{Q}$ and $\mathcal{P}\mathcal{R}\mathcal{M}$. Hence, that $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathbb{R})$ is C^*_{AV}- $b_{\mathcal{R}}$-MS, where $\Lambda = 5^p I$, $p > 1$. But, $d_{\mathbb{B}}(\mathcal{P}, \mathcal{Q}) \succeq d_{\mathbb{B}}(\mathcal{P}, \mathcal{M}) + d_{\mathbb{B}}(\mathcal{M}, \mathcal{Q})$. Thus, $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ is not a C^*_{AV}- $\mathcal{R}$-MS.

Definition 2.4.5. For a C^*_{AV}- $b_{\mathcal{R}}$-MS, a self mapping $\Gamma : \mho \to \mho$ is said to be a $\mathbf{C^*_{AV}}$**- b**$_{\mathcal{R}}$**-contractive mapping** if $\forall\; \varpi, \varsigma \in \mho$ with $(\varpi, \varsigma) \in \mathcal{R}$ $\exists$ $\alpha_1 \in \mathbb{B}$ where $\|\alpha_1\| < 1$ s.t $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1$.

Example 2.4.6. Let $\mho = \mathbb{R}$ and $\mathbb{B} = M_2(\mathbb{R})$ with involution on $\mathbb{B}$ define $\alpha_1^* = \alpha_1^t$ $\forall\; \alpha_1 \in \mathbb{B}$, where α_1^t denotes the transpose of α_1. For $\alpha_1 = [\alpha_{2_{ij}}]$, let $|\alpha_1| = \max\limits_{1 \leq i,j \leq 2} \left|\alpha_{2_{ij}}\right|$. Define

$$d_{\mathbb{B}}(\varpi, \varsigma) = \text{diag}(c_1|\varpi - \varsigma|^p, c_2|\varpi - \varsigma|^p) \;\forall\; \varpi, \varsigma \in \mho,$$

where diag denotes the diagonal matrix, $c_i > 0$ for $i = 1, 2$ are constants and $p > 1$. $\mathcal{R}$ is defined on $\mho$ s.t $\varpi\mathcal{R}\varsigma \Leftrightarrow (\varpi - \varsigma) < \varpi$. Hence, $d_{\mathbb{B}}$ is C^*_{AV}- $b_{\mathcal{R}}$-metric and $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ is a C^*_{AV}- $b_{\mathcal{R}}$-MS. To prove triangle inequality, consider following inequality

$$|\varpi - \varsigma|^p \leq 3^p(|\varpi - \mu|^p + |\mu - \varsigma|^p).$$

This implies $d_{\mathbb{B}}(\varpi, \varsigma) \preceq \Lambda\Big(d_{\mathbb{B}}(\varpi, \mu) + d_{\mathbb{B}}(\mu, \varsigma)\Big)\ \forall\ \varpi, \varsigma, \mu \in \mho$, where $\Lambda = 3^p I \in \mathbb{B}$ and $\Lambda \succ I$ with $3^p > 1$. Define $\Gamma : \mho \to \mho$ as

$$\Gamma\varpi = \begin{cases} 1/10, & \text{if } \varpi > 10 \\ 0, & \text{otherwise.} \end{cases}$$

Following cases arises :

Case (i) : If $\varpi, \varsigma < 10$, then $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = d_{\mathbb{B}}(0, 0) = \theta_{\mathbb{B}}$. For any $\alpha_1 \in \mathbb{B}^+$ with $\|\alpha_1\| < 1$, we have $\alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1 \succeq \theta_{\mathbb{B}}$. Hence, $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1$.

Case (ii) : If $\varpi > 10$ and $\varsigma < 10$, then

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = \begin{bmatrix} c_1|1/10|^p & 0 \\ 0 & c_2|1/10|^p \end{bmatrix}$$

and for $\alpha_1 = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$, we have

$$\alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1 = \alpha_1^t d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1 = \begin{bmatrix} c_1|\varpi - \varsigma|^p/2 & 0 \\ 0 & c_2|\varpi - \varsigma|^p/2 \end{bmatrix}.$$

Hence, $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1$.

Case (iii) : If $\varpi < 10$ and $\varsigma > 10$, then on the similar lines of Case (ii), we have

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1.$$

Case (iv) : If $\varpi, \varsigma > 10$, then $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = d_{\mathbb{B}}\Big(1/10, 1/10\Big) = \theta_{\mathbb{B}}$. For any $\alpha_1 \in \mathbb{B}$ with $\|\alpha_1\| < 1$, we have $\alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1 \succeq \theta_{\mathbb{B}}$. Hence, $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1$.
Thus, Γ is a C^*_{AV}- $b_{\mathcal{R}}$-contractive map.

Definition 2.4.7. A sequence $\{\varpi_\jmath\}$ in $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ is said to be a **$\mathbf{b_{\mathcal{R}}}$-sequence** if $\varpi_\jmath \mathcal{R} \varpi_{\jmath+k}\ \forall\ \jmath, k \in \mathbb{N}$.

Definition 2.4.8. Suppose $\{\varpi_\jmath\} \subset \mho$ and $\varpi \in \mho$. Then,

1. (i) a $b_{\mathcal{R}}$-sequence $\{\varpi_\jmath\}$ is **convergent** w.r.t $\mathbb{B}$, if for any $\epsilon > 0\ \exists\ \jmath_0 \in \mathbb{N}$ s.t $\|d_{\mathbb{B}}(\varpi_\jmath, \varpi)\| \leq \epsilon\ \forall\ \jmath \geq \jmath_0$. We say ϖ is a limit of $\varpi_\jmath$ denoted by $\lim_{\jmath\to\infty} \varpi_\jmath \xrightarrow{\mathcal{R}} \varpi$.

(*ii*) a $b_{\mathcal{R}}$-sequence $\{\varpi_\jmath\}$ is $\mathbf{b_{\mathcal{R}}}$**-**$\mathbf{C_{seq}}$ w.r.t $\mathbb{B}$, if for any $\epsilon > 0 \; \exists \; N \in \mathbb{N}$ s.t $\|d_{\mathbb{B}}(\varpi_\jmath, \varpi_\imath)\| \leq \epsilon \; \forall \; \jmath, \imath > N$.

2. $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ is a **complete** $\mathbf{C^*_{AV}}$**-** $\mathbf{b_{\mathcal{R}}}$**-MS** if every $b_{\mathcal{R}}$-C_{seq} w.r.t $\mathbb{B}$ is convergent in $\mho$.

Theorem 2.4.9. *Consider* $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ *be a complete* C^*_{AV}- $b_{\mathcal{R}}$*-MS and* $\Gamma : \mho \to \mho$ *be* $\mathcal{R}$*-continuous,* $b_{\mathcal{R}}$*-contraction and* $\mathcal{R}$*-preserving. Suppose* $\exists \; \varpi_0 \in \mho$ *s.t* $\varpi_0 \mathcal{R} \varsigma \; \forall \; \varsigma \in \Gamma(\mho)$*. Then,* Γ *has a unique fixed point* ϖ^**. Also,* Γ *is a Picard operator i.e,* $\lim_{\jmath \to \infty} \Gamma^\jmath(\varpi) = \varpi^* \; \forall \; \varpi \in \mho$.

Proof. Let $\varpi_1 = \Gamma(\varpi_0)$, $\varpi_2 = \Gamma(\varpi_1) = \Gamma^2(\varpi_0)$, ... , $\varpi_\jmath = \Gamma(\varpi_{\jmath-1}) = \Gamma^\jmath(\varpi_0) \; \forall \; \jmath \in \mathbb{N}$. Let $\jmath < \imath \in \mathbb{N}$. Substituting $k = \imath - \jmath \; \exists \; \varpi_0 \in \mho$ s.t $\varpi_0 \mathcal{R} \Gamma^k(\varpi_0)$. Since, Γ is $\mathcal{R}$-preserving, $\varpi_\jmath = \Gamma^\jmath(\varpi_0) \mathcal{R} \Gamma^{\jmath+k}(\varpi_0) = \varpi_\imath$. Hence, $\{\varpi_\jmath\}$ is a $b_{\mathcal{R}}$-sequence. Thus, Γ satisfies $b_{\mathcal{R}}$-contraction. $\therefore$

$$\begin{aligned}
d_{\mathbb{B}}(\varpi_{\jmath+1}, \varpi_\jmath) = d_{\mathbb{B}}(\Gamma\varpi_\jmath, \Gamma\varpi_{\jmath-1}) &\preceq \alpha_1^* d_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath-1})\alpha_1 \\
&\preceq (\alpha_1^*)^2 d_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_{\jmath-2})\alpha_1^2 \\
&\preceq \cdots \\
&\preceq (\alpha_1^*)^\jmath d_{\mathbb{B}}(\varpi_0, \varpi_1)\alpha_1^\jmath \\
&= (\alpha_1^*)^\jmath \alpha_2 \alpha_1^\jmath,
\end{aligned}$$

where $\alpha_2 = d_{\mathbb{B}}(\varpi_0, \varpi_1)$, $\alpha_1 \in \mathbb{B}$ and $\|\alpha_1\| < 1$. For any $\imath, p \in \mathbb{N}$, we have

$$\begin{aligned}
d_{\mathbb{B}}(\varpi_{\imath+p}, \varpi_\imath) &\preceq \Lambda\Big(d_{\mathbb{B}}(\varpi_{\imath+p}, \varpi_{\imath+p-1}) + d_{\mathbb{B}}(\varpi_{\imath+p-1}, \varpi_\imath)\Big) \\
&= \Lambda d_{\mathbb{B}}(\varpi_{\imath+p}, \varpi_{\imath+p-1}) + \Lambda d_{\mathbb{B}}(\varpi_{\imath+p-1}, \varpi_\imath) \\
&\preceq \Lambda d_{\mathbb{B}}(\varpi_{\imath+p}, \varpi_{\imath+p-1}) + \Lambda^2\Big(d_{\mathbb{B}}(\varpi_{\imath+p-1}, \varpi_{\imath+p-2}) + d_{\mathbb{B}}(\varpi_{\imath+p-2}, \varpi_\imath)\Big) \\
&= \Lambda d_{\mathbb{B}}(\varpi_{\imath+p}, \varpi_{\imath+p-1}) + \Lambda^2 d_{\mathbb{B}}(\varpi_{\imath+p-1}, \varpi_{\imath+p-2}) + \Lambda^2 d_{\mathbb{B}}(\varpi_{\imath+p-2}, \varpi_\imath) \\
&\preceq \Lambda d_{\mathbb{B}}(\varpi_{\imath+p}, \varpi_{\imath+p-1}) + \Lambda^2 d_{\mathbb{B}}(\varpi_{\imath+p-1}, \varpi_{\imath+p-2}) + \cdots \\
&\quad + \Lambda^{p-1} d_{\mathbb{B}}(\varpi_{\imath+2}, \varpi_{\imath+1}) + \Lambda^{p-1} d_{\mathbb{B}}(\varpi_{\imath+1}, \varpi_\imath) \\
&\preceq \Lambda(\alpha_1^*)^{\imath+p-1}\alpha_2\alpha_1^{\imath+p-1} + \Lambda^2(\alpha_1^*)^{\imath+p-2}\alpha_2\alpha_1^{\imath+p-2} + \cdots \\
&\quad + \Lambda^{p-1}(\alpha_1^*)^{\imath+1}\alpha_2\alpha_1^{\imath+1} + \Lambda^{p-1}(\alpha_1^*)^{\imath}\alpha_2\alpha_1^{\imath} \\
&= \sum_{k=1}^{p-1} \Lambda^k (\alpha_1^*)^{\imath+p-k}\alpha_2\alpha_1^{\imath+p-k} + \Lambda^{p-1}(\alpha_1^*)^{\imath}\alpha_2\alpha_1^{\imath} \\
&= \sum_{k=1}^{p-1} \Big((\alpha_1^*)^{\imath+p-k}\Lambda^{\frac{k}{2}}\sqrt{\alpha_2}\Big)\Big(\sqrt{\alpha_2}\Lambda^{\frac{k}{2}}\alpha_1^{\imath+p-k}\Big)
\end{aligned}$$

$$
\begin{aligned}
& +\Big((\alpha_1^*)^\imath \Lambda^{\frac{p-1}{2}}\sqrt{\alpha_2}\Big)\Big(\sqrt{\alpha_2}\Lambda^{\frac{p-1}{2}}\alpha_1^\imath\Big) \\
= & \sum_{k=1}^{p-1}\Big((\alpha_1)^{\imath+p-k}\Lambda^{\frac{k}{2}}\sqrt{\alpha_2}\Big)^*\Big(\sqrt{\alpha_2}\Lambda^{\frac{k}{2}}\alpha_1^{\imath+p-k}\Big) \\
& +\Big((\alpha_1)^\imath \Lambda^{\frac{p-1}{2}}\sqrt{\alpha_2}\Big)^*\Big(\sqrt{\alpha_2}\Lambda^{\frac{p-1}{2}}\alpha_1^\imath\Big) \\
= & \sum_{k=1}^{p-1}\Big|\sqrt{\alpha_2}\Lambda^{\frac{k}{2}}\alpha_1^{\imath+p-k}\Big|^2+\Big|\sqrt{\alpha_2}\Lambda^{\frac{p-1}{2}}\alpha_1^\imath\Big|^2 \\
\preceq & \sum_{k=1}^{p-1}\|\sqrt{\alpha_2}\Lambda^{\frac{k}{2}}\alpha_1^{\imath+p-k}\|^2 I_{\mathbb{B}}+\|\sqrt{\alpha_2}\Lambda^{\frac{p-1}{2}}\alpha_1^\imath\|^2 I_{\mathbb{B}} \\
\preceq & \|\sqrt{\alpha_2}\|^2\sum_{k=1}^{p-1}\|\alpha_1\|^{2(\imath+p-k)}\|\Lambda\|^k I_{\mathbb{B}}+\|\sqrt{\alpha_2}\|^2\|\Lambda\|^{p-1}\|\alpha_1^\imath\|^2 I_{\mathbb{B}} \\
= & \|\alpha_2\|\|\alpha_1\|^{2(\imath+p)}\frac{\|\Lambda\|((\|\Lambda\|\|\alpha_1\|^{-2})^{p-1})-1}{\|\Lambda\|-\|\alpha_1\|^2}I_{\mathbb{B}}+\|\alpha_2\|\|\Lambda\|^{p-1}\|\alpha_1^\imath\|^2 I_{\mathbb{B}} \\
\preceq & \|\alpha_2\|\frac{\|\Lambda\|^p\|\alpha_1\|^{2(\imath+1)}}{\|\Lambda\|-\|\alpha_1\|^2}I_{\mathbb{B}}+\|\alpha_2\|\|\Lambda\|^{p-1}\|\alpha_1\|^{2\imath}I_{\mathbb{B}}\to\theta_{\mathbb{B}}\ (\imath\to\infty).
\end{aligned}
$$

Thus, we have $\{\varpi_\jmath=\Gamma\varpi_{\jmath-1}\}_{\jmath\in\mathbb{N}}$ is a $b_{\mathcal{R}}$-C_{seq} in $\mho$. Since, $\mho$ is a complete C^*_{AV}- $b_{\mathcal{R}}$-MS. $\therefore\ \exists\ \varpi^*\in\mho$ s.t $\lim_{\jmath\to\infty}\varpi_\jmath=\varpi^*$. Since, Γ is $\mathcal{R}$-continuous. Thus, $\Gamma(\varpi_\jmath)\xrightarrow{\mathcal{R}}\Gamma\varpi^*$. Hence, $\Gamma\varpi^*=\Gamma(\lim_{\jmath\to\infty}\varpi_\jmath)=\lim_{\jmath\to\infty}\Gamma\varpi_\jmath=\lim_{\jmath\to\infty}\varpi_{\jmath+1}=\varpi^*$. Thus, ϖ^* is a fixed point of Γ.

Uniqueness: Let $\varsigma*$ be another fixed point of Γ. Then, $\exists\ \varpi_0\in\mho$ s.t $\varpi_0\mathcal{R}\Gamma\varsigma^*=\varsigma^*$. Hence, $\varpi_\jmath=\Gamma^\jmath(\varpi_0)\mathcal{R}\varsigma^*\ \forall\ \jmath\in\mathbb{N}$, we have

$$
\begin{aligned}
d_{\mathbb{B}}(\varpi^*,\varsigma^*)=d_{\mathbb{B}}(\Gamma^\jmath\varpi^*,\Gamma^\jmath\varsigma^*) &\preceq \Lambda\Big(d_{\mathbb{B}}(\Gamma^\jmath\varpi^*,\Gamma^\jmath\varpi_0)+d_{\mathbb{B}}(\Gamma^\jmath\varpi_0,\Gamma^\jmath\varsigma^*)\Big) \\
&\preceq \Lambda\Big((\alpha_1^*)^\jmath d_{\mathbb{B}}(\varpi^*,\varpi_0)\alpha_1^\jmath+(\alpha_1^*)^\jmath d_{\mathbb{B}}(\varpi_0,\varsigma^*)\alpha_1^\jmath\Big).
\end{aligned}
$$

Taking limit as $\jmath\to\infty$, we have

$$d_{\mathbb{B}}(\varpi^*,\varsigma^*)=\theta_{\mathbb{B}}.$$

Hence, $\varpi^*=\varsigma^*$.

Finally, let ϖ be an arbitrary element of $\mho$. Then, $\exists\ \varpi_0\in\mho$ s.t $\varpi_0\mathcal{R}\Gamma\varpi$. Thus, $\Gamma^\jmath(\varpi_0)\mathcal{R}\Gamma^{\jmath+1}\varpi\ \forall\ \jmath\in\mathbb{N}$. $\therefore$

$$
\begin{aligned}
d_{\mathbb{B}}(\varpi^*,\Gamma^\jmath(\varpi)) &= d_{\mathbb{B}}(\Gamma^\jmath(\varpi^*),\Gamma^\jmath(\varpi)) \\
&\preceq \Lambda\Big(d_{\mathbb{B}}(\Gamma^{\jmath-1}(\Gamma\varpi^*),\Gamma^{\jmath-1}\varpi_0)+d_{\mathbb{B}}(\Gamma^{\jmath-1}\varpi_0,\Gamma^{\jmath-1}(\Gamma\varpi))\Big) \\
&= \Lambda\Big(d_{\mathbb{B}}(\Gamma^{\jmath-1}(\varpi^*),\Gamma^{\jmath-1}\varpi_0)+d_{\mathbb{B}}(\Gamma^{\jmath-1}\varpi_0,\Gamma^{\jmath-1}(\Gamma\varpi))\Big) \\
&\preceq \Lambda\Big((\alpha_1^*)^{\jmath-1}d_{\mathbb{B}}(\varpi^*,\varpi_0)\alpha_1^{\jmath-1}+(\alpha_1^*)^{\jmath-1}d_{\mathbb{B}}(\varpi_0,\Gamma\varpi)\alpha_1^{\jmath-1}\Big) \\
&= \theta_{\mathbb{B}}\text{ as }\jmath\to\infty.
\end{aligned}
$$

Hence, $\lim_{\jmath\to\infty}\Gamma^{\jmath}(\varpi)=\varpi^*$. □

Theorem 2.4.10. *Consider* $(\mho,\mathbb{B},d_{\mathbb{B}},\mathcal{R})$ *be a complete* C^*_{AV}- $b_{\mathcal{R}}$*-MS and* $\Gamma:\mho\to\mho$ *be* $\mathcal{R}$*-continuous,* $\mathcal{R}$*-preserving and satisfying:*

$$\begin{aligned} d_{\mathbb{B}}(\Gamma\varpi,\Gamma\varsigma) \;\preceq\;& \alpha_1 d_{\mathbb{B}}(\varpi,\varsigma)+\alpha_2 d_{\mathbb{B}}(\Gamma\varsigma,\varsigma)+\alpha_3 d_{\mathbb{B}}(\varsigma,\Gamma\varpi)\\ &+\alpha_4 d_{\mathbb{B}}(\varpi,\Gamma\varsigma)+\alpha_5 d_{\mathbb{B}}(\Gamma\varpi,\varpi)\;\forall\,\varpi,\varsigma\in\mho, \end{aligned}\tag{2.4.1}$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\in\mathbb{B}'^{+}$ *and* $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\prec I_{\mathbb{B}}$. *Suppose* $\exists\,\varpi_0\in\mho$ *s.t* $\varpi_0\mathcal{R}\varsigma\;\forall\,\varsigma\in\Gamma(\mho)$. *Then,* Γ *has a unique fixed point* ϖ^*.

Proof. Let $\varpi_1=\Gamma(\varpi_0)$, $\varpi_2=\Gamma(\varpi_1)=\Gamma^2(\varpi_0)\;\cdots\;\varpi_{\jmath}=\Gamma(\varpi_{\jmath-1})=\Gamma^{\jmath}(\varpi_0)$, $\forall\jmath\in\mathbb{N}$. Let $\imath,\jmath\in\mathbb{N}$ and $\jmath<\imath$. Substituting $k=\imath-\jmath$ $\exists\,\varpi_0\in\mho$ s.t $\varpi_0\mathcal{R}\Gamma^k(\varpi_0)$. Since, Γ is $\mathcal{R}$-preserving. Thus, $\varpi_{\jmath}=\Gamma^{\jmath}(\varpi_0)\mathcal{R}\Gamma^{\jmath+k}(\varpi_0)=\varpi_{\imath}$. Hence, $\{\varpi_{\jmath}\}$ is a $b_{\mathcal{R}}$-sequence. Using (2.4.1), we have

$$\begin{aligned} d_{\mathbb{B}}(\varpi_{\jmath+1},\varpi_{\jmath}) &= d_{\mathbb{B}}(\Gamma\varpi_{\jmath},\Gamma\varpi_{\jmath-1})\\ &\preceq \alpha_1 d_{\mathbb{B}}(\varpi_{\jmath},\varpi_{\jmath-1})+\alpha_2 d_{\mathbb{B}}(\Gamma\varpi_{\jmath-1},\varpi_{\jmath-1})\\ &\quad+\alpha_3 d_{\mathbb{B}}(\varpi_{\jmath},\Gamma\varpi_{\jmath-1})+\alpha_4 d_{\mathbb{B}}(\Gamma\varpi_{\jmath},\varpi_{\jmath-1})+\alpha_5 d_{\mathbb{B}}(\Gamma\varpi_{\jmath},\varpi_{\jmath})\\ &= \alpha_1 d_{\mathbb{B}}(\varpi_{\jmath},\varpi_{\jmath-1})+\alpha_2 d_{\mathbb{B}}(\varpi_{\jmath},\varpi_{\jmath-1})\\ &\quad+\alpha_3 d_{\mathbb{B}}(\varpi_{\jmath},\varpi_{\jmath})+\alpha_4 d_{\mathbb{B}}(\varpi_{\jmath+1},\varpi_{\jmath-1})+\alpha_5 d_{\mathbb{B}}(\varpi_{\jmath+1},\varpi_{\jmath})\\ &\preceq \alpha_1 d_{\mathbb{B}}(\varpi_{\jmath},\varpi_{\jmath-1})+\alpha_2 d_{\mathbb{B}}(\varpi_{\jmath},\varpi_{\jmath-1})+\\ &\quad\alpha_4 d_{\mathbb{B}}(\varpi_{\jmath+1},\varpi_{\jmath})+\alpha_4 d_{\mathbb{B}}(\varpi_{\jmath},\varpi_{\jmath-1})+\alpha_5 d_{\mathbb{B}}(\varpi_{\jmath+1},\varpi_{\jmath}). \end{aligned}$$

Thus,

$$(I_{\mathbb{B}}-(\alpha_4+\alpha_5))d_{\mathbb{B}}(\varpi_{\jmath+1},\varpi_{\jmath})\preceq(\alpha_1+\alpha_2+\alpha_4)d_{\mathbb{B}}(\varpi_{\jmath},\varpi_{\jmath-1}),$$

or

$$d_{\mathbb{B}}(\varpi_{\jmath+1},\varpi_{\jmath})\preceq(\alpha_1+\alpha_2+\alpha_4)(I_{\mathbb{B}}-(\alpha_5+\alpha_4))^{-1}d_{\mathbb{B}}(\varpi_{\jmath},\varpi_{\jmath-1}).$$

In general, we have

$$d_{\mathbb{B}}(\varpi_{\jmath+1},\varpi_{\jmath})\preceq{\alpha_6}^2 d_{\mathbb{B}}(\varpi_{\jmath-1},\varpi_{\jmath-2})\preceq\cdots\preceq{\alpha_6}^n d_{\mathbb{B}}(\varpi_1,\varpi_0)={\alpha_6}^{\jmath}\omega,$$

where $\alpha_6=(\alpha_1+\alpha_2+\alpha_4)(I_{\mathbb{B}}-(\alpha_4+\alpha_5))^{-1}$ with $\|\alpha_6\|<1$ and $\omega=d_{\mathbb{B}}(\varpi_1,\varpi_0)$. On the similar lines of Theorem (2.4.9), $\{\varpi_{\jmath}=\Gamma\varpi_{\jmath-1}\}_{\jmath\in\mathbb{N}}$ is a $b_{\mathcal{R}}$-C_{seq} in $\mho$. Since, $\mho$ is a complete C^*_{AV}- $b_{\mathcal{R}}$-MS $\therefore$ $\exists\,\varpi^*\in\mho$ s.t $\lim_{\jmath\to\infty}\varpi_{\jmath}=\varpi^*$.

Since, Γ is $\mathcal{R}$-continuous. Thus, $\Gamma(\varpi_\jmath) \xrightarrow{\mathcal{R}} \Gamma\varpi^*$. Hence, $\Gamma\varpi^* = \Gamma(\lim_{\jmath\to\infty} \varpi_\jmath) = \lim_{\jmath\to\infty} \Gamma\varpi_\jmath = \lim_{\jmath\to\infty} \varpi_{\jmath+1} = \varpi^*$. Thus, $\Gamma\varpi^* = \varpi^*$.
Uniqueness : Let $\Gamma\varsigma^* = \varsigma*$. In fact $\Gamma^\jmath\varsigma^* = \varsigma^*$. Then, $\exists\ \varpi_0 \in \mho$ s.t $\varpi_0\mathcal{R}\Gamma\varsigma^* = \varsigma^*$. Hence, $\varpi_\jmath = \Gamma^\jmath(\varpi_0)\mathcal{R}\Gamma^\jmath\varsigma^* = \varsigma^*\ \forall\ \jmath \in \mathbb{N}$. Using (2.4.1), we have

$$\begin{aligned}
d_{\mathbb{B}}(\varpi_\jmath, \varsigma^*) = d_{\mathbb{B}}(\Gamma^\jmath\varpi_0, \Gamma^\jmath\varsigma^*) &\preceq \alpha_1 d_{\mathbb{B}}(\Gamma^{\jmath-1}\varpi_0, \Gamma^{\jmath-1}\varsigma^*) + \alpha_2 d_{\mathbb{B}}(\Gamma^\jmath\varsigma^*, \Gamma^{\jmath-1}\varsigma^*) \\
&\quad + \alpha_3 d_{\mathbb{B}}(\Gamma^{\jmath-1}\varsigma^*, \Gamma^\jmath\varpi_0) + \alpha_4 d_{\mathbb{B}}(\Gamma^{\jmath-1}\varpi_0, \Gamma^\jmath\varsigma^*) \\
&\quad + \alpha_5 d_{\mathbb{B}}(\Gamma^\jmath\varpi_0, \Gamma^{\jmath-1}\varpi_0) \\
&= \alpha_1 d_{\mathbb{B}}(\varpi_\jmath, \varsigma^*) + \alpha_2 d_{\mathbb{B}}(\varsigma^*, \varsigma^*) + \alpha_3 d_{\mathbb{B}}(\varsigma^*, \varpi_{\jmath+1}) \\
&\quad + \alpha_4 d_{\mathbb{B}}(\varpi_\jmath, \varsigma^*) + \alpha_5 d_{\mathbb{B}}(\varpi_{\jmath+1}, \varpi_\jmath) \\
(I_{\mathbb{B}} - (\alpha_1 + \alpha_4)) d_{\mathbb{B}}(\varpi_\jmath, \varsigma^*) &\preceq \alpha_3 d_{\mathbb{B}}(\varsigma^*, \varpi_{\jmath+1}) + \alpha_5 d_{\mathbb{B}}(\varpi_{\jmath+1}, \varpi_\jmath) \\
d_{\mathbb{B}}(\varpi_\jmath, \varsigma^*) &\preceq \alpha_3 (I_{\mathbb{B}} - (\alpha + \alpha_4))^{-1} d_{\mathbb{B}}(\varsigma^*, \varpi_{\jmath+1}) \\
&\quad + \alpha_5 (I_{\mathbb{B}} - (\alpha_1 + \alpha_4))^{-1} d_{\mathbb{B}}(\varpi_{\jmath+1}, \varpi_\jmath) \\
&\preceq \alpha_3 (I_{\mathbb{B}} - (\alpha_1 + \alpha_4))^{-1} d_{\mathbb{B}}(\varsigma^*, \varpi_{\jmath+1}) \\
&\quad + \alpha_5 (I_{\mathbb{B}} - (\alpha_1 + \alpha_4))^{-1} h^{\jmath-1}\omega. \qquad (2.4.2)
\end{aligned}$$

Taking limit as $\jmath \to \infty$ in (2.4.2), we have

$$d_{\mathbb{B}}(\varpi^*, \varsigma^*) \preceq \alpha_3 (I_{\mathbb{B}} - (\alpha_1 + \alpha_4))^{-1} d_{\mathbb{B}}(\varpi^*, \varsigma^*).$$

Taking norm on both side, we have

$$\|1 - \alpha_3(1 - (\alpha_1 + \alpha_4))^{-1}\| \|d_{\mathbb{B}}(\varpi^*, \varsigma^*)\| \leq 0,$$

implies $\|d_{\mathbb{B}}(\varpi^*, \varsigma^*)\| = 0$. Hence, $\varpi^* = \varsigma^*$. □

Remark 2.4.11. For different values of $\alpha_i' s$, in (2.4.1), we can extend the following version of well known results of literature for self mappings in C^*_{AV}- $b_{\mathcal{R}}$-MS.
(i) Kannan type (Kannan (1968)) There exists $\alpha_1 \in \mathbb{B}'^{+}$ and $\|\alpha_1\| < \frac{1}{2}$ s.t $\forall\ \varpi, \varsigma \in \mho$ satisfying

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1 \Big(d_{\mathbb{B}}(\Gamma\varpi, \varpi) + d_{\mathbb{B}}(\Gamma\varsigma, \varsigma) \Big).$$

(ii) Chatterjea type (Chatterjea (1972)) There exists $\alpha_1 \in \mathbb{B}'^{+}$ and $\|\alpha_1\Lambda\| < \frac{1}{2}$ s.t $\forall\ \varpi, \varsigma \in \mho$ satisfying

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1 \Big(d_{\mathbb{B}}(\Gamma\varpi, \varsigma) + d_{\mathbb{B}}(\Gamma\varsigma, \varpi) \Big).$$

(iii) Reich type (Reich (1971)) There exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{B}'^{+}$ and $\|\alpha_1\| + \|\alpha_2\| + \|\alpha_3\| < 1$ s.t $\forall\ \varpi, \varsigma \in \mho$ satisfying

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1 d_{\mathbb{B}}(\varpi, \varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma\varpi, \varpi) + \alpha_3 d_{\mathbb{B}}(\Gamma\varsigma, \varsigma).$$

(iv) Ciric type (Cirić (1971)) There exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{B}'^{+}$ and $\|\alpha_1\| + \|\alpha_2\| + \|\alpha_3\| + 2\|\alpha_4\| < 1$ s.t $\forall\, \varpi, \varsigma \in \mho$ satisfying

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1 d_{\mathbb{B}}(\varpi, \varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma\varpi, \varpi) + \alpha_3 d_{\mathbb{B}}(\Gamma\varsigma, \varsigma) + \alpha_4\Big(d_{\mathbb{B}}(\varpi, \Gamma\varsigma) + d_{\mathbb{B}}(\varsigma, \Gamma\varpi)\Big).$$

Example 2.4.12. Consider $\mho = [0, 1)$ with usual metric and let $\mathbb{B} = (-\infty, +\infty)$ together with $\|\alpha_1\| = |\alpha_1|$. Define relation $\mathcal{R}$ on $\mho$ as $\varpi\mathcal{R}\varsigma$ iff $(\varpi - \varsigma) \in \{\varpi\}$ and let $\Gamma : \mho \to \mho$ be defined as

$$\Gamma\varpi = \begin{cases} 0, & \text{if } \varpi \in [0, 1/2] \\ 1/13, & \text{otherwise.} \end{cases}$$

Then, $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is a complete C^*_{AV}- $b_{\mathcal{R}}$-MS.
For $\varpi\mathcal{R}\varsigma$, then either $\varsigma = 0$ or both are zero.
Consider, $\varsigma = 0$.
Case (i) : If $\varpi \in [0, 1/2]$, then

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = d_{\mathbb{B}}(0, 0) = \theta_{\mathbb{B}},\ d_{\mathbb{B}}(\varpi, \varsigma) = d_{\mathbb{B}}(\varpi, 0) = \varpi. \tag{2.4.3}$$

Then, for any $\alpha_1 \in \mathbb{B}^+$ with $\|\alpha_1\| < 1$ and from (2.4.3), we have

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1.$$

Case (ii) : If $\varpi \in (1/2, 1)$, then

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = d_{\mathbb{B}}(1/13, 0) = 1/13,\ d_{\mathbb{B}}(\varpi, \varsigma) = d_{\mathbb{B}}(\varpi, 0) = \varpi. \tag{2.4.4}$$

Then, for $\alpha_1 \in \mathbb{B}^+$ with $\|\alpha_1\| < 1$ s.t $\alpha_1 = 1/\sqrt{3}$ and from (2.4.4), we have

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1^* d_{\mathbb{B}}(\varpi, \varsigma)\alpha_1.$$

Thus, Γ is a C^*_{AV}- $b_{\mathcal{R}}$-contractive mapping and satisfy all the hypotheses of Theorem (2.4.9). $\therefore$ Γ possess a unique fixed point which in this case is $\varpi =$ ‘0’.

Example 2.4.13. Let $\mho = [0, 1)$ and $\mathbb{B} = M_2(\mathbb{R})$ with involution on $\mathbb{B}$ define $\alpha_1^* = \alpha_1^t\ \forall \alpha_1 \in \mathbb{B}$, where α_1^t denotes the transpose of α_1. For $\alpha_1 = [\alpha_{2_{ij}}]$, let $\|\alpha_1\| = \max\limits_{1 \le i,j \le 2} \left|\alpha_{2_{ij}}\right|$. Define

$$d_{\mathbb{B}}(\varpi, \varsigma) = \operatorname{diag}(c_1|\varpi - \varsigma|^p, c_2|\varpi - \varsigma|^p),$$

where diag denotes the diagonal matrix, $\varpi, \varsigma \in \mho$, $c_i > 0\ \forall\ (i = 1, 2)$ are constants and $p > 1$. $\mathcal{R}$ is defined on $\mho$ s.t $\varpi\mathcal{R}\varsigma \Leftrightarrow (\varpi - \varsigma) \in \{\varpi\}$. Hence, $(\mho, \mathbb{B}, d_{\mathbb{B}}, \mathcal{R})$ is a complete C^*_{AV}- $b_{\mathcal{R}}$-MS. For proving triangle inequality, i.e, $d_{\mathbb{B}}(\varpi, \varsigma) \preceq \Lambda(d_{\mathbb{B}}(\varpi, \mu)+$

$d_{\mathbb{B}}(\mu, \varsigma)) \; \forall \; \varpi, \varsigma, \mu \in \mho$ use $\Lambda = 3^p I_{\mathbb{B}} \in \mathbb{B}$.
Define a self mapping $\Gamma : \mho \to \mho$ as

$$\Gamma\varpi = \begin{cases} 0, & \text{if } \varpi \in [0, 1/2] \\ 3/25, & \text{if} \varpi \in (1/2, 1). \end{cases}$$

For $\varpi \mathcal{R} \varsigma$ then, we have either $\varsigma = 0$ or both are zero.
Consider, $\varsigma = 0$.
Case (i) : If $\varpi \in [0, 1/2]$ then $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = d_{\mathbb{B}}(0, 0) = \theta_{\mathbb{B}}$. For any $\alpha_1 \in \mathbb{B}^+$ with $\|\alpha_1\| < 1/2$, we have $\alpha_1(d_{\mathbb{B}}(\Gamma\varpi, \varpi) + d_{\mathbb{B}}(\Gamma\varsigma, \varsigma)) = \alpha_1\varpi \succeq \theta_{\mathbb{B}}$.
Hence, $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1\Big(d_{\mathbb{B}}(\Gamma\varpi, \varpi) + d_{\mathbb{B}}(\Gamma\varsigma, \varsigma)\Big)$.
Case (ii) : If $\varpi \in (1/2, 1)$ then

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = \begin{bmatrix} c_1|3/25|^p & 0 \\ 0 & c_2|3/25|^p \end{bmatrix}$$

and for $\alpha_1 = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix}$, we have

$$\alpha_1\Big(d_{\mathbb{B}}(\varpi, \Gamma\varpi) + d_{\mathbb{B}}(\varsigma, \Gamma\varsigma)\Big) = \begin{bmatrix} c_1|\varpi - 3/25|^p / \sqrt{3} & 0 \\ 0 & c_2|\varpi - 3/25|^p / \sqrt{3} \end{bmatrix}.$$

Hence, $d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1\Big(d_{\mathbb{B}}(\varpi, \Gamma\varpi) + d_{\mathbb{B}}(\varsigma, \Gamma\varsigma)\Big)$. Also, Γ satisfy $\mathcal{R}$-preserving and $\mathcal{R}$-continuous. Hence, by Kannan type contraction, Γ possess a unique fixed point which in this case is $\varpi =$ '0'.

Example 2.4.14. Consider $\mho = [0, 1)$ with usual metric and let $\mathbb{B} = (-\infty, +\infty)$ together with $\|\alpha_1\| = |\alpha_1|$ and involution given by $\alpha_1^* = \alpha_1$. Define relation $\mathcal{R}$ on $\mho$ as $\varpi \mathcal{R} \varsigma$ iff $\varpi\varsigma \in \{0\}$ and let $\Gamma : \mho \to \mho$ be defined as

$$\Gamma\varpi = \begin{cases} 0, & \text{if } \varpi \in [0, 1/2] \\ 2/33, & \text{otherwise}. \end{cases}$$

Then, $\mho$ is a complete C^*_{AV}- $b_{\mathcal{R}}$-MS since every $b_{\mathcal{R}}$-C_{seq} in $\mho$ is convergent.
For $\varpi \mathcal{R} \varsigma$ then, either $\varpi = 0$ or $\varsigma = 0$ or both are zero.
Consider $\varsigma = 0$.
Case (i) : If $\varpi \in [0, 1/2]$, then

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = \theta_{\mathbb{B}}, \; d_{\mathbb{B}}(\varpi, \Gamma\varsigma) = \varpi \text{ and } d_{\mathbb{B}}(\varsigma, \Gamma\varpi) = \theta_{\mathbb{B}}. \tag{2.4.5}$$

Then, for any $\alpha_1 \in \mathbb{B}^+$ with $\|\alpha_1\| < 1/2$ and from (2.4.5), we have

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1\Big(d_{\mathbb{B}}(\varpi, \Gamma\varsigma) + d_{\mathbb{B}}(\varsigma, \Gamma\varpi)\Big).$$

Case (ii) : If $\varpi \in (1/2, 1)$, then

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) = 2/33,\ d_{\mathbb{B}}(\varpi, \Gamma\varsigma) = \varpi \text{ and } d_{\mathbb{B}}(\varsigma, \Gamma\varpi) = 2/33. \tag{2.4.6}$$

Then, for $\alpha_1 \in \mathbb{B}^+$ with $\|\alpha_1\| < 1/2$ s.t $\alpha_1 = 1/\sqrt{3}$ and from (2.4.6), we have

$$d_{\mathbb{B}}(\Gamma\varpi, \Gamma\varsigma) \preceq \alpha_1\Big(d_{\mathbb{B}}(\varpi, \Gamma\varsigma) + d_{\mathbb{B}}(\varsigma, \Gamma\varpi)\Big).$$

Also, Γ satisfy $\mathcal{R}$-preserving and $\mathcal{R}$-continuous. Hence, by Chatterjea type contraction, Γ possess a unique fixed point which in this case is ϖ = '0'.

2.5 Application

In this section, the existence and uniqueness of a solution for the operator equation using Theorem (2.4.9) are established.

Theorem 2.5.1. *Suppose that H is a Hilbert space, $L(H)$ is the set of linear bounded operators on H. Let $\xi_1, \xi_2, ..., \xi_\jmath, ... \in L(H)$ which satisfy $\sum_{\jmath=1}^{\infty} \|\xi_\jmath\| < 1$. Then, the operator equation*

$$\varpi = \sum_{\jmath=1}^{\infty} {\xi_\jmath}^* \varpi \xi_\jmath$$

has a unique solution in $L(H)$.

Proof. Let $\alpha_1 = \left(\sum_{\jmath=1}^{\infty} \|\xi_\jmath\|\right)^p$ with $p \geq 1$, then $\|\alpha_1\| < 1$. Without loss of generality, one can suppose that $\alpha_1 \succ \theta_{\mathbb{B}}$. Choose a positive operator $\Gamma \in L(H)$. For $\varpi, \varsigma \in L(H)$ and $p \geq 1$, set $d_{\mathbb{B}}(\varpi, \varsigma) = \|\varpi - \varsigma\|^p \Gamma$. $\mathcal{R}$ is defined on $L(H)$ s.t $\varpi \mathcal{R} \varsigma \Leftrightarrow (\varpi - \varsigma) \in \{\varpi, \varsigma\}$. To prove it is a complete C^*_{AV}- $b_{\mathcal{R}}$-MS, use $\xi = 2^p I_{\mathbb{B}}$ in the triangle inequality.
Consider, $\Gamma_1 : L(H) \to L(H)$ defined as

$$\Gamma_1(\varpi) = \sum_{\jmath=1}^{\infty} \xi_\jmath^* \varpi \xi_\jmath.$$

Step (I) : Γ_1 is $\mathcal{R}$-preserving.
Proof : Let $\varpi \mathcal{R} \varsigma$. Consider,

$$\begin{aligned} \Gamma_1(\varpi) - \Gamma_1(\varsigma) &= \sum_{\jmath=1}^{\infty} \xi_\jmath^* \varpi \xi_\jmath - \sum_{\jmath=1}^{\infty} \xi_\jmath^* \varsigma \xi_\jmath \\ &= \sum_{\jmath=1}^{\infty} \xi_\jmath^* (\varpi - \varsigma) \xi_\jmath. \end{aligned}$$

Thus, if $\varpi\mathcal{R}\varsigma$ then $\Gamma_1(\varpi)\mathcal{R}\Gamma_1(\varsigma)$.
Step (II) : Γ_1 is $b_{\mathcal{R}}$-contractive mapping.
Proof : Let $\varpi\mathcal{R}\varsigma$. Consider,

$$\begin{aligned}
d_{\mathbb{B}}\Big(\Gamma_1(\varpi), \Gamma_1(\varsigma)\Big) &= \|\Gamma_1(\varpi) - \Gamma_1(\varsigma)\|^p\Gamma \\
&= \Big\|\sum_{\jmath=1}^{\infty} \xi_\jmath^*(\varpi - \varsigma)\xi_\jmath\Big\|^p\Gamma \\
&\leq \sum_{\jmath=1}^{\infty} \|\xi_\jmath\|^{2p}\|\varpi - \varsigma\|^p\Gamma \\
&= \alpha_1^2 d_{\mathbb{B}}(\varpi, \varsigma) \\
&= (\alpha_1 I_{\mathbb{B}})^* d_{\mathbb{B}}(\varpi, \varsigma)(\alpha_1 I_{\mathbb{B}}).
\end{aligned}$$

Hence, Γ_1 is a $b_{\mathcal{R}}$-contractive mapping.
Step (III) : Γ_1 is $\mathcal{R}$-continuous.
Proof : Let $\{\xi_\jmath\}$ be a $b_{\mathcal{R}}$-sequence converging to $\xi \in L(H)$. Then, $\xi_\jmath\mathcal{R}\xi$ i.e, $(\xi_\jmath - \xi) \in \{\xi_\jmath, \xi\}\ \forall\ \jmath \in \mathbb{N}$.
Using step (II), we have

$$\|\Gamma_1(\xi_\jmath) - \Gamma_1(\xi)\|^p\Gamma \leq \sum_{\jmath=1}^{\infty} \|\xi_\jmath\|^{2p}\|\xi_\jmath - \xi\|^p\Gamma.$$

Thus,

$$\|\Gamma_1(\xi_\jmath) - \Gamma_1(\xi)\| \leq {\alpha_1}^2 d_{\mathbb{B}}(\xi_\jmath, \xi) \preceq d_{\mathbb{B}}(\xi_\jmath, \xi).$$

$\therefore \Gamma_1(\xi_\jmath) \to \Gamma_1(\xi)$.
Γ_1 satisfies all the hypothesis of the Theorem (2.4.9). Thus, Γ_1 has a unique fixed point in $L(H)$. □

2.6 Conclusion

In this chapter, we have introduced a novel approach to prove the fixed point results for certain types of contraction mapping on C^*_{AV}-MS, C^*_{AV}-PMS, and C^*_{AV}-$b_{\mathcal{R}}$-MS that extends, unifies and generalizes the results on fixed points in the literature. However, under certain conditions the results proved in this chapter are reduced to some well known results of the literature.

(*i*) If in Theorem (2.2.1) we consider $\mathbb{A} = \mathbb{R}$, $F^*(r, t) = r, \psi(t) = \phi(t) = t$ then we obtain Banach (1922) Contraction principle.

(*ii*) If in Theorem (2.2.1) we consider $F^*(r,t) = r, \psi(t) = \phi(t) = \alpha^* t \alpha$ where $\|\alpha\| < 1$ then we obtain Theorem (2.1) of Ma et al. (2014).

(*iii*) If in Theorem (2.2.3) we consider $\mathbb{A} = \mathbb{R}$, $F^*(r,t) = r, \psi(t) = \phi(t) = t$ then we obtain F-contraction introduced by Wardowski (2012).

(*iv*) If in Theorem (2.4.9) we consider binary relation $\mathcal{R}$ as a universal relation (that is, relation $\mathcal{R}$ on X such that $\rho \mathcal{R} \sigma$ for all $\rho, \sigma \in X$) then we obtain Theorem (2.1) of Ma & Jiang (2015).

(*v*) If in Theorem (2.4.9) we consider $\mathbb{A} = \mathbb{R}$, $\|\alpha\| = |\alpha|$ and $\alpha^* = \alpha$ with $\Lambda = 1$ then we obtain the analogue of Theorem (3.1) of Khalehoghli et al. (2020).

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Chapter 3

Some Results On Coincidence Point

3.1 Introduction

The present chapter of the thesis deals with the results on the existence of the coincidence points for two pairs of self mappings in C^*_{AV}-MS. The content of this chapter is divided into two sections. In the first section, the existence of coincidence points using C_*-class function for two pairs of compatible or weakly compatible mappings in C^*_{AV}-MS are presented. To support the results some illustrative examples are also discussed. In the second section, the existence of coincidence points using rational type contraction for two pairs of compatible or weakly compatible mappings in C^*_{AV}-MS are established. The results of this chapter are presented in [5,6].

3.2 Coincidence Point of Self Mappings using C_*-Class Function

The concept of coincidence point originated from the evolution of fixed point theory to address the problems involving numerous mappings. Many researchers investigated the existence of coincidence points using various contractions in ab-

[5] Dhariwal, R. and Kumar, D. (2022). On A New Approach to Establish the Existence of Coincidence Point in C^* -algebra valued metric space. (Communicated).

[6] Dhariwal, R. and Kumar, D. (2023). Existence of coincidence point in C^*-algebra valued metric space. (Communicated).

stract spaces (see, Singh & Prasad (2008), Abbas & Jungck (2008), Shatanawi et al. (2011), Esmaily et al. (2012), Roldán et al. (2014), Parvaneh et al. (2015) López-de Hierro et al. (2015), Mustafa et al. (2016), Namana et al. (2022), Kalyani et al. (2022) and references cited therein). In this section, some results on the existence of coincidence points are presented in C^*_{AV}-MS.

Definition 3.2.1. $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq_{\mho})$ is said to be a partially ordered C^*_{AV}-MS iff $d_{\mathbb{B}}$ is a C^*_{AV}-metric on a partially ordered set $(\mho, \preceq_{\mho})$.

Theorem 3.2.2. *Consider a ordered complete C^*_{AV}-MS $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq_{\mho})$ and Γ_1, Γ_2, Γ_3, $\Gamma_4 : \mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$*;*

(ii) *for* $\psi_{\mathbb{B}}$, $\phi_{\mathbb{B}} \in \Psi_{\mathbb{B}}$ *and* $F^* \in C_*$ *s.t* $\forall\, \varpi, \vartheta \in \mho$,

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\vartheta)\Big) \preceq F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\varpi, \vartheta)\Big), \phi_{\mathbb{B}}\Big(\aleph(\varpi, \vartheta)\Big)\Bigg), \tag{3.2.1}$$

where

$$\aleph(\varpi, \vartheta) \in \Bigg\{d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta), d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_1\varpi), d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_2\vartheta), d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\varpi)\Bigg\}; \tag{3.2.2}$$

(iii) *the pairs* (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *are compatible and continuous;*

(iv) *the pairs* (Γ_1, Γ_2) *and* (Γ_2, Γ_1) *satisfies PWI property w.r.t* Γ_3 *and* Γ_4 *respectively.*

Then, Γ_1*,* Γ_2*,* Γ_3 *and* Γ_4 *have a coincidence point in* $\mho$*.*

Proof. Let $\varpi_0 \in \mho$ be an arbitrary point. Since, $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ and $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$. $\therefore\ \exists\ \varpi_1, \varpi_2 \in \mho$ s.t $\Gamma_1\varpi_0 = \Gamma_3\varpi_1$ and $\Gamma_2\varpi_1 = \Gamma_4\varpi_2$. Continuing this process, construct a sequence $\{\varsigma_\jmath\}$ as

$$\varsigma_{2\jmath+1} = \Gamma_1\varpi_{2\jmath} = \Gamma_3\varpi_{2\jmath+1} \quad \text{and} \quad \varsigma_{2\jmath+2} = \Gamma_2\varpi_{2\jmath+1} = \Gamma_4\varpi_{2\jmath+2}.$$

Since, $\varpi_1 \in \Gamma_3^{-1}(\Gamma_1\varpi_0)$, $\varpi_2 \in \Gamma_4^{-1}(\Gamma_2\varpi_1)$ and (Γ_2, Γ_1) & (Γ_1, Γ_2) are PWI w.r.t Γ_3 and Γ_4 respectively.

$$\begin{aligned}
\therefore\ \Gamma_3\varpi_1 = \Gamma_1\varpi_0 \quad &\preceq_{\mho} \quad \Gamma_2\varpi_1 = \Gamma_4\varpi_2 \Rightarrow \varsigma_1 \preceq_{\mho} \varsigma_2;\\
\Gamma_2\varpi_1 = \Gamma_4\varpi_2 \quad &\preceq_{\mho} \quad \Gamma_1\varpi_2 = \Gamma_3\varpi_3 \Rightarrow \varsigma_2 \preceq_{\mho} \varsigma_3;\\
\Gamma_1\varpi_2 = \Gamma_3\varpi_3 \quad &\preceq_{\mho} \quad \Gamma_2\varpi_3 = \Gamma_4\varpi_4 \Rightarrow \varsigma_3 \preceq_{\mho} \varsigma_4.
\end{aligned}$$

On generalizing, $\varsigma_{2\jmath+1} \preceq_{\mho} \varsigma_{2\jmath+2}\ \forall\ \jmath \geq 0$.
To prove the result below mentioned steps will be followed:
Step I. Firstly, to show $\lim\limits_{k\to\infty} d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1}) = \theta_{\mathbb{B}}$.
Suppose $d_{\mathbb{B}}(\varsigma_{k_0}, \varsigma_{k_0+1}) = \theta_{\mathbb{B}}$ for some k_0. Then, $\varsigma_{k_0} = \varsigma_{k_0+1}$.
In case, $k_0 = 2\jmath$ for some $\jmath$, $\varsigma_{2\jmath} = \varsigma_{2\jmath+1}$ implies $\varsigma_{2\jmath+1} = \varsigma_{2\jmath+2}$. If $\varsigma_{2\jmath+1} \neq \varsigma_{2\jmath+2}$, then

$$\begin{aligned} \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) &= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varpi_{2\jmath}, \Gamma_2\varpi_{2\jmath+1})\Big) \\ &\preceq F^*\bigg(\psi_{\mathbb{B}}\Big(\aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1})\Big), \phi_{\mathbb{B}}\Big(\aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1})\Big)\bigg), \quad (3.2.3) \end{aligned}$$

where

$$\begin{aligned} \aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1}) &\in \Big\{d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath}, \Gamma_3\varpi_{2\jmath+1}), d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath}, \Gamma_1\varpi_{2\jmath}), d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath+1}, \Gamma_2\varpi_{2\jmath+1}), \\ &\quad d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath+1}, \Gamma_1\varpi_{2\jmath})\Big\} \\ &\in \Big\{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}), d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}), d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}), d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+1})\Big\} \\ &\in \Big\{\theta_{\mathbb{B}},\ \theta_{\mathbb{B}},\ d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}),\ \theta_{\mathbb{B}}\Big\} \\ &\in \Big\{\theta_{\mathbb{B}}, d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big\}. \end{aligned}$$

In case, $\aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1}) = d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})$. Then, from (3.2.3), we have

$$\begin{aligned} \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) &\preceq F^*\bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big)\bigg) \\ &\preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}))\Big). \end{aligned}$$

$\therefore\ F^*\Big(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big)\Big) = \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big)$, implies either $\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) = \theta_{\mathbb{B}}$. Thus, $d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) = \theta_{\mathbb{B}}$ i.e, $\varsigma_{2\jmath+1} = \varsigma_{2\jmath+2}$. Similarly, if $\aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1}) = \theta_{\mathbb{B}}$, then $d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) = \theta_{\mathbb{B}}$ i.e, $\varsigma_{2\jmath+1} = \varsigma_{2\jmath+2}$.
Hence,

$$\varsigma_{2\jmath} = \Gamma_4\varpi_{2\jmath} = \varsigma_{2\jmath+1} = \Gamma_3\varpi_{2\jmath+1} = \Gamma_1\varpi_{2\jmath} = \varsigma_{2\jmath+2} = \Gamma_2\varpi_{2\jmath+1} = \Gamma_4\varpi_{2\jmath+2}$$

implies $\Gamma_4\varpi_{2\jmath} = \Gamma_1\varpi_{2\jmath}$ and $\Gamma_3\varpi_{2\jmath+1} = \Gamma_2\varpi_{2\jmath+1}$.
Similarly, if $k_0 = 2\jmath + 1$, then $\varsigma_{2\jmath+1} = \varsigma_{2\jmath+2}$ implies $\varsigma_{2\jmath+2} = \varsigma_{2\jmath+3}$ i.e, $\varsigma_{2\jmath+1} = \Gamma_3\varpi_{2\jmath+1} = \varsigma_{2\jmath+2} = \Gamma_2\varpi_{2\jmath+1} = \Gamma_4\varpi_{2\jmath+2} = \varsigma_{2\jmath+3} = \Gamma_1\varpi_{2\jmath+2} = \Gamma_3\varpi_{2\jmath+3}$ implies

$\Gamma_4\varpi_{2\jmath+2} = \Gamma_1\varpi_{2\jmath+2}$ and $\Gamma_3\varpi_{2\jmath+1} = \Gamma_2\varpi_{2\jmath+1}$.
Consequently, the sequence $\{\varsigma_k\}$ becomes constant for $k \geq k_0$ and ς_{k_0} is a coincidence point for (Γ_1, Γ_4) and (Γ_2, Γ_3).
Now, suppose

$$d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1}) \succ \theta_{\mathbb{B}} \ \forall\ k \in \mathbb{N}. \tag{3.2.4}$$

We claim that for each $k = 1, 2...$

$$d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_{k+2}) \preceq d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1}). \tag{3.2.5}$$

On the contrary, suppose for $k = 2\jmath$,

$$d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) \succ d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}) \succ \theta_{\mathbb{B}} \text{ for } \jmath \geq 0. \tag{3.2.6}$$

Using (3.2.1), we have

$$\begin{aligned} \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) &= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varpi_{2\jmath}, \Gamma_2\varpi_{2\jmath+1})\Big) \\ &\preceq F^*\bigg(\psi_{\mathbb{B}}\Big(\aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1})\Big), \phi_{\mathbb{B}}\Big(\aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1})\Big)\bigg), \end{aligned} \tag{3.2.7}$$

where

$$\begin{aligned} \aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1}) &\in \Big\{d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath}, \Gamma_3\varpi_{2\jmath+1}), d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath}, \Gamma_1\varpi_{2\jmath}), d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath+1}, \Gamma_2\varpi_{2\jmath+1}), \\ &\quad d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath+1}, \Gamma_1\varpi_{2\jmath})\Big\} \\ &\in \Big\{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}), d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}), d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}), d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+1})\Big\} \\ &\in \Big\{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}), d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}), \theta_{\mathbb{B}}\Big\}. \end{aligned} \tag{3.2.8}$$

In case, $\aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1}) = d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})$. Then, from (3.2.7), we have

$$\begin{aligned} \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) &\preceq F^*\bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big)\bigg) \\ &\preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big). \end{aligned}$$

Hence, $F^*\bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big)\bigg) = \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big)$ implies either $\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) = \theta_{\mathbb{B}}$. Thus, $d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) = \theta_{\mathbb{B}}$, a contradiction to (3.2.6). Hence,

$$d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) \preceq d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}). \tag{3.2.9}$$

On the similar lines, if $\aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1}) = \theta_{\mathbb{B}}$ or $\aleph(\varpi_{2\jmath}, \varpi_{2\jmath+1}) = d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})$, then $d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) \preceq d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})$.
Similarly, if $k = 2\jmath + 1$ then $d_{\mathbb{B}}(\varsigma_{2\jmath+2}, \varsigma_{2\jmath+3}) \preceq d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})$.
Hence, the sequence $\Big\{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})\Big\}$ is monotonically decreasing and bounded in $\mathbb{B}^+$. $\therefore \exists\, \theta_{\mathbb{B}} \preceq \varrho \in \mathbb{B}^+$ s.t

$$d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1}) \to \varrho \text{ as } \jmath \to \infty.$$

Now, using (3.2.1), we have

$$\begin{aligned}
\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_{k+2})\Big) &= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varpi_k, \Gamma_2\varpi_{k+1})\Big) \\
&\preceq F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\varpi_k, \varpi_{k+1})\Big), \phi_{\mathbb{B}}\Big(\aleph(\varpi_k, \varpi_{k+1})\Big)\Bigg),
\end{aligned}$$

where

$$\begin{aligned}
\aleph(\varpi_k, \varpi_{k+1}) &\in \Bigg\{d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1}), d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1}), d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_{k+2}), \theta_{\mathbb{B}}\Bigg\} \\
&\in \Bigg\{d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1}), d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_{k+2}), \theta_{\mathbb{B}}\Bigg\}.
\end{aligned}$$

In case, $\aleph(\varpi_k, \varpi_{k+1}) = d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1})$, then

$$\begin{aligned}
\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_{k+2})\Big) &= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varpi_k, \Gamma_2\varpi_{k+1})\Big) \\
&\preceq F^*\Bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1})\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1})\Big)\Bigg). \qquad (3.2.10)
\end{aligned}$$

Taking limit as $k \to \infty$ in (3.2.10), we have

$$\psi_{\mathbb{B}}(\varrho) \preceq F^*\Big(\psi_{\mathbb{B}}(\varrho), \phi_{\mathbb{B}}(\varrho)\Big) \preceq \psi_{\mathbb{B}}(\varrho).$$

Thus, $F^*\Big(\psi_{\mathbb{B}}(\varrho), \phi_{\mathbb{B}}(\varrho)\Big) = \psi_{\mathbb{B}}(\varrho)$, implies either $\psi_{\mathbb{B}}(\varrho) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(\varrho) = \theta_{\mathbb{B}}$. Hence, $\varrho = \theta_{\mathbb{B}}$.

$$\therefore\ d_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1}) \to \theta_{\mathbb{B}} \text{ as } \jmath \to \infty. \qquad (3.2.11)$$

Similarly, if $\aleph(\varpi_k, \varpi_{k+1}) = d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_{k+2})$ or $\aleph(\varpi_k, \varpi_{k+1}) = \theta_{\mathbb{B}}$, then $\varrho = \theta_{\mathbb{B}}$.
Step II. Now, to show $\{\varsigma_\jmath\}$ is a C_{seq} in $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq_{\mho})$. Assume $\{\varsigma_\jmath\}$ is not a C_{seq}. Then, $\exists\, \epsilon > 0$ and subsequences $\{\varsigma_{\imath_k}\}$ and $\{\varsigma_{\jmath_k}\}$ with $\jmath_k > \imath_k > k$ s.t

$$\|d_{\mathbb{B}}(\varsigma_{\imath_k}, \varsigma_{\jmath_k})\| \geq \epsilon. \qquad (3.2.12)$$

Now, $\jmath_k$ can be chosen as the smallest integer corresponding to $\imath_k$ with $\jmath_k > \imath_k$ and

$$\|d_{\mathbb{B}}(\varsigma_{\imath_k}, \varsigma_{\jmath_k - 1})\| < \epsilon. \tag{3.2.13}$$

Using (3.2.12) and (3.2.13), we have

$$\begin{aligned} \epsilon &\leq \|d_{\mathbb{B}}(\varsigma_{\imath_k}, \varsigma_{\jmath_k})\| \leq \|d_{\mathbb{B}}(\varsigma_{\imath_k}, \varsigma_{\jmath_k - 1})\| + \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\jmath_k})\| \\ &\leq \epsilon + \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\jmath_k})\|. \end{aligned} \tag{3.2.14}$$

From (3.2.11), we have

$$\lim_{k \to \infty} \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\jmath_k})\| = 0. \tag{3.2.15}$$

Using (3.2.14) & (3.2.15), we have

$$\epsilon \leq \lim_{k \to \infty} \|d_{\mathbb{B}}(\varsigma_{\imath_k}, \varsigma_{\jmath_k})\| \leq \epsilon + 0,$$

or

$$\lim_{k \to \infty} \|d_{\mathbb{B}}(\varsigma_{\imath_k}, \varsigma_{\jmath_k})\| = \epsilon. \tag{3.2.16}$$

Again,

$$\begin{aligned} \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k})\| &\leq \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\jmath_k - 1})\| + \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\imath_k})\| \\ &\leq \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\jmath_k - 1})\| + \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\imath_k - 1})\| \\ &\quad + \|d_{\mathbb{B}}(\varsigma_{\imath_k - 1}, \varsigma_{\imath_k})\|. \end{aligned} \tag{3.2.17}$$

Also,

$$\begin{aligned} \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\imath_k - 1})\| &\leq \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\jmath_k})\| + \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k - 1})\| \\ &\leq \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\jmath_k})\| + \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k})\| \\ &\quad + \|d_{\mathbb{B}}(\varsigma_{\imath_k}, \varsigma_{\imath_k - 1})\|. \end{aligned} \tag{3.2.18}$$

Taking limit as $k \to \infty$ in (3.2.17) & (3.2.18) and using (3.2.15) & (3.2.16), we have

$$\lim_{k \to \infty} \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\imath_k - 1})\| = \epsilon.$$

Since, $d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\imath_k - 1}), d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}) \in \mathbb{B}^+$ and

$$\lim_{k \to \infty} \|d_{\mathbb{B}}(\varsigma_{\jmath_k - 1}, \varsigma_{\imath_k - 1})\| = \lim_{k \to \infty} \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k})\| = \epsilon.$$

Thus, $\exists\ s \in \mathbb{B}^+$ with $\|s\| = \epsilon$ s.t

$$\lim_{k\to\infty} d_{\mathbb{B}}(\varsigma_{\jmath_k -1}, \varsigma_{\imath_k -1}) = \lim_{k\to\infty} d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}) = s.$$

Now, by (3.2.1), we have

$$\begin{aligned} \psi_{\mathbb{B}}(s) &= \lim_{k\to\infty} \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k})\Big) \\ &= \lim_{k\to\infty} \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varpi_{\jmath_k -1}, \Gamma_2\varpi_{\imath_k -1})\Big) \\ &\preceq \lim_{k\to\infty} F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\varpi_{\jmath_k -1}, \varpi_{\imath_k -1})\Big), \phi_{\mathbb{B}}\Big(\aleph(\varpi_{\jmath_k -1}, \varpi_{\imath_k -1})\Big)\Bigg), \end{aligned} \tag{3.2.19}$$

where

$$\begin{aligned} \aleph(\varpi_{\jmath_k -1}, \varpi_{\imath_k -1}) &\in \lim_{k\to\infty}\Big\{ d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath_k -1}, \Gamma_3\varpi_{\imath_k -1}), d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath_k -1}, \Gamma_1\varpi_{\jmath_k -1}), \\ &\qquad d_{\mathbb{B}}(\Gamma_3\varpi_{\imath_k -1}, \Gamma_2\varpi_{\imath_k -1}), d_{\mathbb{B}}(\Gamma_3\varpi_{\imath_k -1}, \Gamma_1\varpi_{\jmath_k -1})\Big\} \\ &\in \lim_{k\to\infty}\Big\{ d_{\mathbb{B}}(\varsigma_{\jmath_k -1}, \varsigma_{\imath_k -1}), d_{\mathbb{B}}(\varsigma_{\jmath_k -1}, \varsigma_{\jmath_k -2}), \\ &\qquad d_{\mathbb{B}}(\varsigma_{\imath_k -1}, \varsigma_{\imath_k -2}), d_{\mathbb{B}}(\varsigma_{\imath_k -1}, \varsigma_{\jmath_k -2})\Big\} \\ &\in \Big\{s, \theta_{\mathbb{B}}, \theta_{\mathbb{B}}, s\Big\} \\ &\in \Big\{s, \theta_{\mathbb{B}}\Big\}. \end{aligned}$$

If $\aleph(\varpi_{\jmath_k -1}, \varpi_{\imath_k -1}) = s$, then from (3.2.19), we have

$$\psi_{\mathbb{B}}(s) \preceq \lim_{k\to\infty} F^*\Big(\psi_{\mathbb{B}}(s), \phi_{\mathbb{B}}(s)\Big) \preceq \psi_{\mathbb{B}}(s).$$

Thus, $F^*\Big(\psi_{\mathbb{B}}(s), \phi_{\mathbb{B}}(s)\Big) = \psi_{\mathbb{B}}(s)$ implies either $\psi_{\mathbb{B}}(s) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(s) = \theta_{\mathbb{B}}$ and so $s = \theta_{\mathbb{B}}$, a contradiction. Similarly, if $\aleph(\varpi_{\jmath_k -1}, \varpi_{\imath_k -1}) = \theta_{\mathbb{B}}$, then $\psi_{\mathbb{B}}(s) = \theta_{\mathbb{B}}$ implies $s = \theta_{\mathbb{B}}$, a contradiction. Hence, $\{\varsigma_\jmath\}$ is a C_{seq} in $(\mho, \mathbb{B}, d_{\mathbb{B}})$.

Step III. Now, to prove Γ_1, Γ_2, Γ_3 and Γ_4 have a coincidence point. Since, $\{\varsigma_\jmath\}$ is a C_{seq} in a complete C^*_{AV}-MS $\therefore\ \exists\ \varsigma \in \mho$ s.t

$$\lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma)\| = \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath+1}, \varsigma)\| = \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_1\varpi_{2\jmath}, \varsigma)\| = 0$$

and

$$\lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma)\| = \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath}, \varsigma)\| = \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_2\varpi_{2\jmath-1}, \varsigma)\| = 0.$$

Hence, $\Gamma_4\varpi_{2\jmath} \to \varsigma$ and $\Gamma_1\varpi_{2\jmath} \to \varsigma$ as $\jmath \to \infty$ and the pair (Γ_1, Γ_4) is compatible, $\therefore\ \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_4\Gamma_1\varpi_{2\jmath}, \Gamma_1\Gamma_4\varpi_{2\jmath})\| = 0.$

Moreover, from $\lim_{\jmath\to\infty}\|d_{\mathbb{B}}(\Gamma_1\varpi_{2\jmath},\varsigma)\|=0=\lim_{\jmath\to\infty}\|d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath},\varsigma)\|$ and continuity of Γ_1 and Γ_4, we have

$$\lim_{\jmath\to\infty}\|d_{\mathbb{B}}(\Gamma_4\Gamma_1\ \varpi_{2\jmath},\Gamma_4\varsigma)\|=0=\lim_{\jmath\to\infty}\|d_{\mathbb{B}}(\Gamma_1\Gamma_4\varpi_{2\jmath},\Gamma_1\varsigma)\|.$$

Consider,

$$\begin{aligned}\|d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varsigma)\| &\leq \|d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_4\Gamma_1\varpi_{2\jmath})\|+\|d_{\mathbb{B}}(\Gamma_4\Gamma_1\varpi_{2\jmath},\Gamma_1\varsigma)\|\\ &\leq \|d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_4\Gamma_1\varpi_{2\jmath})\|+\|d_{\mathbb{B}}(\Gamma_4\Gamma_1\varpi_{2\jmath},\Gamma_1\Gamma_4\varpi_{2\jmath})\|\\ &\quad+\|d_{\mathbb{B}}(\Gamma_1\Gamma_4\varpi_{2\jmath},\Gamma_1\varsigma)\|. \end{aligned} \tag{3.2.20}$$

Taking limit as $\jmath\to\infty$ in (3.2.20), we have

$$\|d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varsigma)\|\leq 0.$$

Hence, $\Gamma_1\varsigma=\Gamma_4\varsigma$. Similarly, $\Gamma_2\varsigma=\Gamma_3\varsigma$.
Now, using (3.2.1), we have

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big)\preceq F^*\left(\psi_{\mathbb{B}}\Big(\aleph(\varsigma,\varsigma)\Big),\phi_{\mathbb{B}}\Big(\aleph(\varsigma,\varsigma)\Big)\right), \tag{3.2.21}$$

where

$$\begin{aligned}\aleph(\varsigma,\varsigma) &\in \Big\{d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_3\varsigma),d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varsigma),d_{\mathbb{B}}(\Gamma_3\varsigma,\Gamma_2\varsigma),d_{\mathbb{B}}(\Gamma_3\varsigma,\Gamma_1\varsigma)\Big\}\\ &\in \Big\{d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma),d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_1\varsigma),d_{\mathbb{B}}(\Gamma_2\varsigma,\Gamma_2\varsigma),d_{\mathbb{B}}(\Gamma_2\varsigma,\Gamma_1\varsigma)\Big\}\\ &\in \Big\{d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma),\theta_{\mathbb{B}},\theta_{\mathbb{B}},d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big\}\\ &\in \Big\{\theta_{\mathbb{B}},d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big\}.\end{aligned}$$

If $\aleph(\varsigma,\varsigma)=d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)$. Then, from (3.2.21), we have

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big) &\preceq F^*\left(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big),\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big)\right)\\ &\preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big).\end{aligned}$$

Thus, $F^*\left(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big),\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big)\right)=\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)\Big)$ implies either $\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_2\varsigma,\Gamma_1\varsigma)\Big)=\theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_2\varsigma,\Gamma_1\varsigma)\Big)=\theta_{\mathbb{B}}$. Hence, $d_{\mathbb{B}}(\Gamma_1\varsigma,\Gamma_2\varsigma)=\theta_{\mathbb{B}}$ i.e,, $\Gamma_1\varsigma=\Gamma_2\varsigma$. Similarly, if $\aleph(\varsigma,\varsigma)=\theta_{\mathbb{B}}$, then $\Gamma_1\varsigma=\Gamma_2\varsigma$. On combining, $\Gamma_1\varsigma=\Gamma_2\varsigma=\Gamma_4\varsigma=\Gamma_3\varsigma$. □

Consider $\psi_{\mathbb{B}}(\varrho) = \phi_{\mathbb{B}}(\varrho) = \varrho$ and $F^*(\varpi, \varrho) = \lambda\varpi$, $\lambda < 1$, we have

Corollary 3.2.3. *Consider a ordered complete C^*_{AV}-MS $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq_{\mho})$ and Γ_1, Γ_2, Γ_3, $\Gamma_4 : \mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$*;*

(ii) $\forall\ \varpi, \vartheta \in \mho\ \exists\ \aleph(\varpi, \vartheta)$ *s.t*

$$d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\vartheta) \preceq \lambda\aleph(\varpi, \vartheta),$$

where $\aleph(\varpi, \vartheta)$ *is defined in inequality (3.2.2);*

(iii) the pairs (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *are compatible and continuous;*

(iv) the pairs (Γ_1, Γ_2) *and* (Γ_2, Γ_1) *satisfies PWI property w.r.t* Γ_3 *and* Γ_4 *respectively.*

Then, Γ_1, Γ_2, Γ_3 *and* Γ_4 *have a coincidence point in* $\mho$.

Theorem 3.2.4. *Consider a ordered complete C^*_{AV}-MS $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq_{\mho})$ and Γ_1, Γ_2, Γ_3, Γ_4: $\mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$*;*

(ii) for $\psi_{\mathbb{B}}$, $\phi_{\mathbb{B}} \in \Psi_{\mathbb{B}}$ *and* $F^* \in C_*$ *s.t* $\forall\ \varpi, \vartheta \in \mho$

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\vartheta)\Big) \preceq F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\varpi, \vartheta)\Big), \phi_{\mathbb{B}}\Big(\aleph(\varpi, \vartheta)\Big)\Bigg), \tag{3.2.22}$$

where $\aleph$ *is defined in inequality (3.2.2);*

(iii) the pairs (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *are weakly compatible;*

(iv) the pairs (Γ_1, Γ_2) *and* (Γ_2, Γ_1) *satisfying PWI property w.r.t* Γ_3 *and* Γ_4 *respectively.*

Then, Γ_1, Γ_2, Γ_3 *and* Γ_4 *have a coincidence point in* $\mho$.

Proof. On the similar lines of Theorem (3.2.2), $\{\varsigma_\jmath\}$ is a C_{seq}. So, $\exists\ \varsigma \in \mho$ s.t $\lim\limits_{\jmath\to\infty} d_{\mathbb{B}}(\varsigma_\jmath, \varsigma) = \theta_{\mathbb{B}}$. Since, $\Gamma_3(\mho)$ is complete and $\{\varsigma_{2\jmath+1}\} \subseteq \Gamma_3(\mho)$. $\therefore$ for $\varsigma \in \Gamma_3(\mho)$,

$\exists\ \mu \in \mho$ s.t $\varsigma = \Gamma_3\mu$.
Also,

$$\lim_{\jmath\to\infty} d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \Gamma_3\mu) = \lim_{\jmath\to\infty} d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath+1}, \Gamma_3\mu) = \theta_{\mathbb{B}}.$$

Similarly, $\exists\ \nu \in \mho$ s.t $\varsigma = \Gamma_3\mu = \Gamma_4\nu$ and

$$\lim_{\jmath\to\infty} d_{\mathbb{B}}(\varsigma_{2\jmath}, \Gamma_4\nu) = \lim_{\jmath\to\infty} d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath}, \Gamma_4\nu) = \theta_{\mathbb{B}}.$$

Now, to show ν is a coincidence point of (Γ_1, Γ_4). Since, $\Gamma_3\varpi_{2\jmath+1} \to \varsigma = \Gamma_4\nu$ as $\jmath \to \infty$, we have

$$d_{\mathbb{B}}(\Gamma_1\nu, \varsigma) \preceq d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\varpi_{2\jmath+1}) + d_{\mathbb{B}}(\Gamma_2\varpi_{2\jmath+1}, \varsigma), \tag{3.2.23}$$

or

$$d_{\mathbb{B}}(\Gamma_1\nu, \varsigma) - d_{\mathbb{B}}(\Gamma_2\varpi_{2\jmath+1}, \varsigma) \preceq d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\varpi_{2\jmath+1}).$$

Since, $\psi_{\mathbb{B}}$ is non-decreasing sequence. $\therefore$ from (3.2.22), we have

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma) \quad &- \quad d_{\mathbb{B}}(\varsigma, \Gamma_2\varpi_{2\jmath+1})\Big) \preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\varpi_{2\jmath+1})\Big)\\ &\preceq \quad F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\nu, \varpi_{2\jmath+1})\Big), \phi_{\mathbb{B}}\Big(\aleph(\nu, \varpi_{2\jmath+1})\Big)\Bigg),\end{aligned} \tag{3.2.24}$$

where

$$\begin{aligned}\aleph(\nu, \varpi_{2\jmath+1}) \quad \in \quad &\Big\{d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_3\varpi_{2\jmath+1}), d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\nu), d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath+1}, \Gamma_2\varpi_{2\jmath+1}),\\ &d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath+1}, \Gamma_1\nu)\Big\}.\end{aligned} \tag{3.2.25}$$

Taking limit as $\jmath \to \infty$ in (3.2.25), we have

$$\aleph(\nu, \varpi_{2\jmath+1}) \in \Big\{\theta_{\mathbb{B}}, d_{\mathbb{B}}(\varsigma, \Gamma_1\nu)\Big\}.$$

If $\aleph(\nu, \varpi_{2\jmath+1}) = d_{\mathbb{B}}(\varsigma, \Gamma_1\nu)$. Then, from (3.2.24), we have

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma)\Big) \preceq F^*\Bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma)\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma)\Big)\Bigg) \preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma)\Big).$$

Thus, $F^*\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma)), \phi_{\mathbb{B}}(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma))\Big) = \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma)\Big)$ implies either $\psi_{\mathbb{B}}(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma)) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma)) = \theta_{\mathbb{B}}$. Hence, $d_{\mathbb{B}}(\Gamma_1\nu, \varsigma) = \theta_{\mathbb{B}}$ i.e, $\Gamma_1\nu = \varsigma = \Gamma_4\nu$.
Similarly, if $\aleph(\nu, \varpi_{2\jmath+1}) = \theta_{\mathbb{B}}$, then $\Gamma_1\nu = \varsigma = \Gamma_4\nu$. Since, (Γ_1, Γ_4) is weakly compatible, $\therefore\ \Gamma_1\varsigma = \Gamma_1\Gamma_4\nu = \Gamma_4\Gamma_1\nu = \Gamma_4\varsigma$. Thus, ς is a coincidence point of (Γ_1, Γ_4).
Rest proof follows on the similar lines as in Theorem (3.2.2). □

Example 3.2.5. Let $F^*(\varpi,\varsigma) = \varpi$, $\mho = [0,\infty)$, $\mathbb{B} = \mathbb{C}$ and $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ be define on $\mho$ as $d_{\mathbb{B}}(\varpi,\vartheta) = |\varpi - \vartheta|\ \forall\ \varpi,\vartheta \in \mho$. Define an ordering on $\mho$ as follow $\varpi \preceq \vartheta \Leftrightarrow \varpi \geq \vartheta\ \forall\ \varpi,\vartheta \in \mho$. Then, $(\mho,\mathbb{B},d_{\mathbb{B}},\preceq_{\mho})$ is an ordered complete C^*_{AV}-MS.
Define Γ_1, Γ_2, Γ_3, $\Gamma_4 : \mho \to \mho$ as

$$\Gamma_1\varpi = \ln\bigg(1 + \frac{\varpi}{25}\bigg), \quad \Gamma_3\varpi = e^{10\varpi} - 1. \tag{3.2.26}$$

$$\Gamma_2\varpi = \ln\bigg(1 + \frac{\varpi}{10}\bigg), \quad \Gamma_4\varpi = e^{25\varpi} - 1. \tag{3.2.27}$$

Now, to prove (Γ_1,Γ_2) satisfies PWI property w.r.t Γ_3. Let $\varpi,\vartheta \in \mho$ s.t $\vartheta \in \Gamma_3^{-1}\Gamma_1\varpi$ i.e, $\Gamma_3\vartheta = \Gamma_1\varpi$.
From (3.3.6), we have

$$\ln\bigg(1 + \frac{\varpi}{25}\bigg) = e^{10\vartheta} - 1 \Rightarrow \vartheta = \frac{\ln\Big(1 + \ln(1 + \frac{\varpi}{25})\Big)}{10}.$$

Hence,

$$\Gamma_1\varpi = \ln\bigg(1 + \frac{\varpi}{25}\bigg) \geq \ln\Bigg(1 + \frac{\ln\Big(1 + (\ln(1 + \frac{\varpi}{25}))\Big)}{100}\Bigg) = \ln\bigg(1 + \frac{\vartheta}{10}\bigg) = \Gamma_2\vartheta.$$

$\therefore\ \Gamma_1\varpi \preceq \Gamma_2\vartheta$. Hence, (Γ_1,Γ_2) satisfies PWI property w.r.t Γ_3.
Now, to prove (Γ_2,Γ_1) satisfies PWI property w.r.t Γ_4. Let $\varpi,\vartheta \in \mho$ be s.t $\vartheta \in \Gamma_4^{-1}\Gamma_2\varpi$, i.e, $\Gamma_4\vartheta = \Gamma_2\varpi$.
From (3.3.7), we have

$$e^{25\vartheta} - 1 = \ln\bigg(1 + \frac{\varpi}{10}\bigg) \Rightarrow \vartheta = \frac{\ln\Big(1 + \ln(1 + \frac{\varpi}{10})\Big)}{25}.$$

Hence,

$$\Gamma_2\varpi = \ln\bigg(1 + \frac{\varpi}{10}\bigg) \geq \ln\Bigg(1 + \frac{\ln\Big(1 + (\ln(1 + \frac{\varpi}{10}))\Big)}{625}\Bigg) = \ln\bigg(1 + \frac{\vartheta}{25}\bigg) = \Gamma_1\vartheta.$$

$\therefore\ \Gamma_2\varpi \preceq \Gamma_1\vartheta$. Hence, (Γ_2,Γ_1) satisfies PWI property w.r.t Γ_4.
Furthermore, $\Gamma_1(\mho) = \Gamma_2(\mho) = \Gamma_4(\mho) = \Gamma_3(\mho) = [0,\infty)$ and (Γ_1,Γ_4) and (Γ_2,Γ_3) are compatible.
Let $\{\varpi_\jmath\}$ be a sequence in $\mho$ s.t $\lim_{\jmath\to\infty} d_{\mathbb{B}}(\varrho,\Gamma_1\varpi_\jmath) = \lim_{\jmath\to\infty} d_{\mathbb{B}}(\varrho,\Gamma_4\varpi_\jmath) = \theta_{\mathbb{B}}$, for some $\varrho \in \mho$. $\therefore$

$$\lim_{\jmath\to\infty}\bigg|\ln\bigg(1 + \frac{\varpi_\jmath}{25}\bigg) - \varrho\bigg| = \lim_{\jmath\to\infty}\Big|e^{25\varpi_\jmath} - 1 - \varrho\Big| = \theta_{\mathbb{B}}.$$

and

$$\lim_{\jmath\to\infty}\Big|\varpi_\jmath - 25(e^{\varrho}-1)\Big| = \lim_{\jmath\to\infty}\left|\varpi_\jmath - \frac{\ln(1+\varrho)}{25}\right| = \theta_{\mathbb{B}}.$$

We have

$$25(e^{\varrho}-1) = \frac{\ln(1+\varrho)}{25}.$$

which is possible if $\varrho = \theta_{\mathbb{B}}$.
Since, Γ_1 and Γ_4 are continuous. $\therefore$

$$\lim_{\jmath\to\infty} d_{\mathbb{B}}(\Gamma_1\Gamma_4\varpi_\jmath, \Gamma_4\Gamma_1\varpi_\jmath) = \lim_{\jmath\to\infty}\Big|\Gamma_1\Gamma_4\varpi_\jmath - \Gamma_4\Gamma_1\varpi_\jmath\Big| = \theta_{\mathbb{B}}.$$

Define $\psi_{\mathbb{B}}, \phi_{\mathbb{B}} : \mathbb{B}^+ \to \mathbb{B}^+$ as $\psi_{\mathbb{B}}(\varpi) = \varpi$ and $\phi_{\mathbb{B}}(\varpi) = \varpi \ \forall\ \varpi \in \mathbb{B}^+$.
Consider,

$$\begin{aligned}
\psi_{\mathbb{B}}(d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\vartheta)) &= |\Gamma_1\varpi - \Gamma_2\vartheta| = \left|\ln\left(1+\frac{\varpi}{25}\right) - \ln\left(1+\frac{\vartheta}{10}\right)\right| \\
&\leq \Big|(e^{25\varpi}-1) - (e^{10\vartheta}-1)\Big| \\
&= d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta) \\
&= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta)\Big) \\
&= F^*\Big(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta)\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta)\Big)\Big).
\end{aligned}$$

For $\aleph(\varpi, \vartheta) = d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta)$, Γ_1, Γ_2, Γ_3 and Γ_4 satisfies all the hypothesis of the Theorem (3.2.2). Thus, Γ_1, Γ_2, Γ_3 and Γ_4 have a coincidence point. Indeed, '0' is a coincidence point of Γ_1, Γ_2, Γ_3 and Γ_4.

Example 3.2.6. Let $F^*(\varpi, \varsigma) = \varpi$, $\mho = [0, \infty)$, $\mathbb{B} = \mathbb{C}$ and $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ as $d_{\mathbb{B}}(\varpi, \vartheta) = \begin{cases} |\varpi| + |\vartheta|, & \text{if } \varpi \neq \vartheta \\ 0, & \text{otherwise.} \end{cases}$ $\forall\ \varpi, \vartheta \in \mho$. Define an ordering on $\mho$ as follow $\varpi \preceq \vartheta \Leftrightarrow \varpi \geq \vartheta\ \forall\ \varpi, \vartheta \in \mho$. Then, $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq_{\mho})$ is an ordered complete C^*_{AV}-MS.
Define Γ_1, Γ_2, Γ_3, $\Gamma_4 : \mho \to \mho$ as

$$\Gamma_1\varpi = \frac{\varpi}{2}, \quad \Gamma_3\varpi = 2\varpi. \tag{3.2.28}$$

$$\Gamma_2\varpi = \frac{\varpi}{4}, \quad \Gamma_4\varpi = 4\varpi. \tag{3.2.29}$$

Now, to prove (Γ_1, Γ_2) satisfies PWI property w.r.t Γ_3. Let $\varpi, \vartheta \in \mho$ s.t $\vartheta \in \Gamma_3^{-1}\Gamma_1\varpi$, i.e, $\Gamma_3\vartheta = \Gamma_1\varpi$.
From (3.2.28), we have

$$\frac{\varpi}{2} = 2\vartheta \Rightarrow \vartheta = \frac{\varpi}{4}$$

Hence,

$$\Gamma_1\varpi = \frac{\varpi}{2} \geq \frac{\varpi}{16} = \frac{\vartheta}{4} = \Gamma_2\vartheta.$$

$\therefore$ $\Gamma_1\varpi \preceq \Gamma_2\vartheta$. Hence, (Γ_1, Γ_2) satisfies PWI property w.r.t Γ_3.

Now, to prove (Γ_2, Γ_1) satisfies PWI property w.r.t Γ_4. Let $\varpi, \vartheta \in \mho$ be s.t $\vartheta \in \Gamma_4^{-1}\Gamma_2\varpi$, i.e, $\Gamma_4\vartheta = \Gamma_2\varpi$.

From (3.2.29), we have

$$4\vartheta = \frac{\varpi}{4} \Rightarrow \vartheta = \frac{\varpi}{16}$$

Hence,

$$\Gamma_2\varpi = \frac{\varpi}{4} \geq \frac{\varpi}{32} = \frac{\vartheta}{2} = \Gamma_1\vartheta.$$

$\therefore$ $\Gamma_2\varpi \preceq \Gamma_1\vartheta$. Hence, (Γ_2, Γ_1) satisfies PWI property w.r.t Γ_4.

Furthermore, $\Gamma_1(\mho) = \Gamma_2(\mho) = \Gamma_4(\mho) = \Gamma_3(\mho) = [0, \infty)$ and the pairs (Γ_1, Γ_4) and (Γ_2, Γ_3) are compatible.

Let $\{\varpi_\jmath\}$ be a sequence in $\mho$ s.t $\lim\limits_{\jmath\to\infty} d_{\mathbb{B}}(\varrho, \Gamma_1\varpi_\jmath) = \lim\limits_{\jmath\to\infty} d_{\mathbb{B}}(\varrho, \Gamma_4\varpi_\jmath) = \theta_{\mathbb{B}}$, for some $\varrho \in \mho$.

$$\therefore \lim_{\jmath\to\infty}\left|\frac{\varpi_\jmath}{2} + \varrho\right| = \lim_{\jmath\to\infty}\left|4\varpi_\jmath + \varrho\right| = \theta_{\mathbb{B}},$$

and

$$\lim_{\jmath\to\infty}\left|\varpi_\jmath - (-2\varrho)\right| = \lim_{\jmath\to\infty}\left|\varpi_\jmath - (-\frac{\varrho}{4})\right| = \theta_{\mathbb{B}}.$$

We have

$$2\varrho = \frac{\varrho}{4}.$$

which is possible if $\varrho = \theta_{\mathbb{B}}$.

Since Γ_1 and Γ_4 are continuous.

$$\therefore \lim_{\jmath\to\infty} d_{\mathbb{B}}(\Gamma_1\Gamma_4\varpi_\jmath, \Gamma_4\Gamma_1\varpi_\jmath) = \lim_{\jmath\to\infty}\left|\Gamma_1\Gamma_4\varpi_\jmath + \Gamma_4\Gamma_1\varpi_\jmath\right| = \theta_{\mathbb{B}}.$$

Define $\psi_{\mathbb{B}}, \phi_{\mathbb{B}} : \mathbb{B}^+ \to \mathbb{B}^+$ as $\psi_{\mathbb{B}}(\varpi) = \varpi$ and $\phi_{\mathbb{B}}(\varpi) = \varpi \ \forall \ \varpi \in \mathbb{B}^+$.

Consider,

$$\begin{aligned} \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\vartheta)\Big) &= \Big|\Gamma_1\varpi + \Gamma_2\vartheta\Big| = \left|\frac{\varpi}{2} + \frac{\vartheta}{4}\right| \\ &\preceq |4\varpi + 2\vartheta| = d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta) \\ &= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta)\Big) \\ &= F^*\Bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta)\Big), \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta)\Big)\Bigg). \end{aligned}$$

For $\aleph(\varpi, \vartheta) = d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\vartheta)$, Γ_1, Γ_2, Γ_3 and Γ_4 satisfies all the hypothesis of the Theorem (3.2.4). Thus, Γ_1, Γ_2, Γ_3 and Γ_4 have a coincidence point. Indeed, '0' is a coincidence point of Γ_1, Γ_2, Γ_3 and Γ_4.

3.3 Coincidence Point of Self Mappings using Rational Type Contraction

In this section, the existence of a coincidence point for two pairs of compatible and weakly compatible mappings for generalized rational type contraction mappings in C^*_{AV}-MS are presented.

Theorem 3.3.1. *Consider a ordered complete C^*_{AV}-MS $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq_{\mho})$ and Γ_1, Γ_2, Γ_3, $\Gamma_4 : \mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$;

(ii) for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{B}^+$ *with* $\theta_{\mathbb{B}} \preceq \alpha_1 + \alpha_2 + 2(\alpha_3 + \alpha_4) + \alpha_5 \preceq I_{\mathbb{B}}$,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq\ & \alpha_1\left(\frac{d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_4\varsigma)(1 + d_{\mathbb{B}}(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1 + d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_2\vartheta)}\right) \\ & + \alpha_2\left(\frac{d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_4\varsigma) d_{\mathbb{B}}(\Gamma_2\vartheta, \Gamma_3\vartheta)}{d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_3\vartheta)}\right) \\ & + \alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_4\varsigma) + d_{\mathbb{B}}(\Gamma_2\vartheta, \Gamma_3\vartheta)\Big) + \alpha_4\Big(d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_3\vartheta) \\ & + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\vartheta)\Big) + \alpha_5 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_3\vartheta)\ \forall\, \varsigma, \vartheta \in \mho; \quad (3.3.1) \end{aligned}$$

(iii) the pairs (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *are compatible and continuous;*

(iv) the pairs (Γ_1, Γ_2) *and* (Γ_2, Γ_1) *satisfies PWI property w.r.t* Γ_3 *and* Γ_4 *respectively.*

Then, Γ_1, Γ_2, Γ_3 *and* Γ_4 *have a coincidence point in* $\mho$.

Proof. Let $\varsigma_0 \in \mho$ be an arbitrary point. Since, $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ and $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$, $\exists\ \varsigma_1, \varsigma_2 \in \mho$ s.t $\Gamma_1\varsigma_0 = \Gamma_3\varsigma_1$ and $\Gamma_2\varsigma_1 = \Gamma_4\varsigma_2$. On continuing this process, construct a sequence $\{\varpi_\jmath\}$ as

$$\varpi_{2\jmath+1} = \Gamma_1\varsigma_{2\jmath} = \Gamma_3\varsigma_{2\jmath+1} \quad \text{and} \quad \varpi_{2\jmath+2} = \Gamma_2\varsigma_{2\jmath+1} = \Gamma_4\varsigma_{2\jmath+2}.$$

Since, $\varsigma_1 \in \Gamma_3^{-1}(\Gamma_1\varsigma_0)$, $\varsigma_2 \in \Gamma_4^{-1}(\Gamma_2\varsigma_1)$ and (Γ_2, Γ_1) & (Γ_1, Γ_2) satisfies PWI property w.r.t Γ_3 and Γ_4 respectively. $\therefore$

$$\begin{aligned} \Gamma_3\varsigma_1 = \Gamma_1\varsigma_0 &\preceq_{\mho} \Gamma_2\varsigma_1 = \Gamma_4\varsigma_2, \Rightarrow \varpi_1 \preceq_{\mho} \varpi_2; \\ \Gamma_2\varsigma_1 = \Gamma_4\varsigma_2 &\preceq_{\mho} \Gamma_1\varsigma_2 = \Gamma_3\varsigma_3, \Rightarrow \varpi_2 \preceq_{\mho} \varpi_3; \\ \Gamma_1\varsigma_2 = \Gamma_3\varsigma_3 &\preceq_{\mho} \Gamma_2\varsigma_3 = \Gamma_4\varsigma_4, \Rightarrow \varpi_3 \preceq_{\mho} \varpi_4. \end{aligned}$$

On generalizing, $\varpi_{2\jmath+1} \preceq \varpi_{2\jmath+2}\ \forall\ \jmath \geq 0$. Define $d_{\mathbb{B}_\jmath} = d_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1})$. If $d_{\mathbb{B}_{2\jmath}} = \theta_{\mathbb{B}}$, i.e, $d_{\mathbb{B}}(\varpi_{2\jmath}, \varpi_{2\jmath+1}) = \theta_{\mathbb{B}}$ for some $\jmath$. Then, $\Gamma_1\varsigma_{2\jmath} = \Gamma_3\varsigma_{2\jmath+1} = \Gamma_2\varsigma_{2\jmath-1} = \Gamma_4\varsigma_{2\jmath}$. Thus, Γ_1 and Γ_4 have coincidence point. Hence, the result.
Now, suppose that $d_{\mathbb{B}_{2\jmath}} \succ \theta_{\mathbb{B}}\ \forall\ \jmath \in \mathbb{N}$. Substituting $\varsigma = \varpi_{2\jmath+1}$ and $\vartheta = \varpi_{2\jmath+2}$ in (3.3.1), we have

$$\begin{aligned} d_{\mathbb{B}}(\varpi_{2\jmath+1}, \varpi_{2\jmath+2}) &= d_{\mathbb{B}}(\Gamma_1\varsigma_{2\jmath}, \Gamma_2\varsigma_{2\jmath+1}) \\ &\preceq \alpha_1\left(\frac{d_{\mathbb{B}}(\Gamma_1\varsigma_{2\jmath}, \Gamma_4\varsigma_{2\jmath})(1 + d_{\mathbb{B}}(\Gamma_2\varsigma_{2\jmath+1}, \Gamma_3\varsigma_{2\jmath+1}))}{1 + d_{\mathbb{B}}(\Gamma_1\varsigma_{2\jmath}, \Gamma_2\varsigma_{2\jmath+1})}\right) \\ &\quad +\alpha_2\left(\frac{d_{\mathbb{B}}(\Gamma_1\varsigma_{2\jmath}, \Gamma_4\varsigma_{2\jmath})d_{\mathbb{B}}(\Gamma_2\varsigma_{2\jmath+1}, \Gamma_3\varsigma_{2\jmath+1})}{d_{\mathbb{B}}(\Gamma_4\varsigma_{2\jmath}, \Gamma_3\varsigma_{2\jmath+1})}\right) \\ &\quad +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\varsigma_{2\jmath}, \Gamma_4\varsigma_{2\jmath}) + d_{\mathbb{B}}(\Gamma_2\varsigma_{2\jmath+1}, \Gamma_3\varsigma_{2\jmath+1})\Big) \\ &\quad +\alpha_4\Big(d_{\mathbb{B}}(\Gamma_1\varsigma_{2\jmath}, \Gamma_3\varsigma_{2\jmath+1}) + d_{\mathbb{B}}(\Gamma_4\varsigma_{2\jmath}, \Gamma_2\varsigma_{2\jmath+1})\Big) \\ &\quad +\alpha_5 d_{\mathbb{B}}(\Gamma_4\varsigma_{2\jmath}, \Gamma_3\varsigma_{2\jmath+1}) \\ &= \alpha_1\left(\frac{d_{\mathbb{B}}(\varpi_{2\jmath+1}, \varpi_{2\jmath})(1 + d_{\mathbb{B}}(\varpi_{2\jmath+2}, \varpi_{2\jmath+1}))}{1 + d_{\mathbb{B}}(\varpi_{2\jmath+1}, \varpi_{2\jmath+2})}\right) \\ &\quad +\alpha_2\left(\frac{d_{\mathbb{B}}(\varpi_{2\jmath+1}, \varpi_{2\jmath})d_{\mathbb{B}}(\varpi_{2\jmath+2}, \varpi_{2\jmath+1})}{d_{\mathbb{B}}(\varpi_{2\jmath}, \varpi_{2\jmath+1})}\right) \\ &\quad +\alpha_3\Big(d_{\mathbb{B}}(\varpi_{2\jmath+1}, \varpi_{2\jmath}) + d_{\mathbb{B}}(\varpi_{2\jmath+2}, \varpi_{2\jmath+1})\Big) \\ &\quad +\alpha_4\Big(d_{\mathbb{B}}(\varpi_{2\jmath+1}, \varpi_{2\jmath+1}) + d_{\mathbb{B}}(\varpi_{2\jmath}, \varpi_{2\jmath+2})\Big) \\ &\quad +\alpha_5 d_{\mathbb{B}}(\varpi_{2\jmath}, \varpi_{2\jmath+1}) \\ d_{\mathbb{B}_{2\jmath+1}} &\preceq \alpha_1 d_{\mathbb{B}_{2\jmath}} + \alpha_2 d_{\mathbb{B}_{2\jmath+1}} + \alpha_3(d_{\mathbb{B}_{2\jmath}} + d_{\mathbb{B}_{2\jmath+1}}) \\ &\quad +\alpha_4(d_{\mathbb{B}_{2\jmath}} + d_{\mathbb{B}_{2\jmath+1}}) + \alpha_5 d_{\mathbb{B}_{2\jmath}} \\ (I_{\mathbb{B}} - (\alpha_2 + \alpha_3 + \alpha_4)d_{\mathbb{B}_{2\jmath+1}} &\preceq (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)d_{\mathbb{B}_{2\jmath}} \\ d_{\mathbb{B}_{2\jmath+1}} &\preceq \frac{(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)}{(I_{\mathbb{B}} - (\alpha_2 + \alpha_3 + \alpha_4)}d_{\mathbb{B}_{2\jmath}} \\ &= \alpha_6 d_{\mathbb{B}_{2\jmath}}, \end{aligned} \tag{3.3.2}$$

where $\alpha_6 = \dfrac{(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)}{(I_{\mathbb{B}} - (\alpha_2 + \alpha_3 + \alpha_4)}$ with $\|\alpha_6\| \leq 1$. On the similar lines, $d_{\mathbb{B}_{2\jmath}} \preceq$

$\alpha_6 d_{\mathbb{B}_{2\jmath-1}}$, $d_{\mathbb{B}_{2\jmath-1}} \preceq \alpha_6 d_{\mathbb{B}_{2\jmath-2}} \cdots$.
In general, we have $d_{\mathbb{B}_\jmath} \preceq \alpha_6 d_{\mathbb{B}_{\jmath-1}}$ $\forall$ $\jmath \in \mathbb{N}$.
Thus,

$$d_{\mathbb{B}}(\varpi_\jmath, \varpi_{\jmath+1}) \preceq \alpha_6 d_{\mathbb{B}}(\varpi_{\jmath-1}, \varpi_\jmath) \preceq \alpha_6^2 d_{\mathbb{B}}(\varpi_{\jmath-2}, \varpi_{\jmath-1}) \preceq \cdots \preceq \alpha_6^\jmath d_{\mathbb{B}}(\varpi_0, \varpi_1).$$

For any $p \in \mathbb{N}$, we have

$$\begin{aligned} d_{\mathbb{B}}(\varpi_{\jmath+p}, \varpi_\jmath) &\preceq d_{\mathbb{B}}(\varpi_{\jmath+p}, \varpi_{\jmath+p-1}) + d_{\mathbb{B}}(\varpi_{\jmath+p-1}, \varpi_{\jmath+p-2}) + \cdots + d_{\mathbb{B}}(\varpi_{\jmath+1}, \varpi_\jmath) \\ &\preceq \alpha_6^{\jmath+p-1} d_{\mathbb{B}}(\varpi_0, \varpi_1) + \alpha_6^{\jmath+p-2} d_{\mathbb{B}}(\varpi_0, \varpi_1) + \cdots + \alpha_6^\jmath d_{\mathbb{B}}(\varpi_0, \varpi_1) \\ &= \alpha_6^\jmath d_{\mathbb{B}}(\varpi_0, \varpi_1)(1 + \alpha_6 + \alpha_6^2 + \cdots + \alpha_6^{p-1}) \\ &= \alpha_6^\jmath d_{\mathbb{B}}(\varpi_0, \varpi_1)\left(\frac{1-\alpha_6^p}{1-\alpha_6}\right). \end{aligned}$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\varpi_{\jmath+p}, \varpi_\jmath)\| \leq \|\alpha_6\|^\jmath \|\beta\| \Big\|\frac{1-\alpha_6^p}{1-\alpha_6}\Big\| \to 0 \text{ as } \jmath \to \infty,$$

where $\|\beta\| = d_{\mathbb{B}}(\varpi_0, \varpi_1)$. Hence, $\{\varpi_\jmath\}$ is a C_{seq} in $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq)$.
Now, to prove Γ_1, Γ_2, Γ_3 and Γ_4 have a coincidence point. Since, $\{\varpi_\jmath\}$ is a C_{seq} in a complete C^*_{AV}-MS $\therefore$ $\exists$ $\varpi \in \mho$ s.t

$$\lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\varpi_{2\jmath+1}, \varpi)\| = \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_3 \varsigma_{2\jmath+1}, \varpi)\| = \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_1 \varsigma_{2\jmath}, \varpi)\| = 0,$$

and

$$\lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\varpi_{2\jmath}, \varpi)\| = \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_4 \varsigma_{2\jmath}, \varpi)\| = \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_2 \varsigma_{2\jmath-1}, \varpi)\| = 0.$$

Hence, $\Gamma_4 \varsigma_{2\jmath} \to \varpi$ and $\Gamma_1 \varsigma_{2\jmath} \to \varpi$ as $\jmath \to \infty$ and the pair (Γ_1, Γ_4) is compatible, $\therefore \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_4 \Gamma_1 \varsigma_{2\jmath}, \Gamma_1 \Gamma_4 \varsigma_{2\jmath})\| = 0$.
Moreover, from $\lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_1 \varsigma_{2\jmath}, \varpi)\| = 0$, $\lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_4 \varsigma_{2\jmath}, \varpi)\| = 0$, we have

$$\lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_4 \Gamma_1 \varsigma_{2\jmath}, \Gamma_4 \varpi)\| = 0 = \lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\Gamma_1 \Gamma_4 \varsigma_{2\jmath}, \Gamma_1 \varpi)\|.$$

Consider,

$$\begin{aligned} \|d_{\mathbb{B}}(\Gamma_4 \varpi, \Gamma_1 \varpi)\| &\leq \|d_{\mathbb{B}}(\Gamma_4 \varpi, \Gamma_4 \Gamma_1 \varsigma_{2\jmath})\| + \|d_{\mathbb{B}}(\Gamma_4 \Gamma_1 \varsigma_{2\jmath}, \Gamma_1 \varpi)\| \\ &\leq \|d_{\mathbb{B}}(\Gamma_4 \varpi, \Gamma_4 \Gamma_1 \varsigma_{2\jmath})\| + \|d_{\mathbb{B}}(\Gamma_4 \Gamma_1 \varsigma_{2\jmath}, \Gamma_1 \Gamma_4 \varsigma_{2\jmath})\| \\ &\quad + \|d_{\mathbb{B}}(\Gamma_1 \Gamma_4 \varsigma_{2\jmath}, \Gamma_1 \varpi)\|. \end{aligned} \tag{3.3.3}$$

Taking limit as $\jmath \to \infty$ in (3.3.3), we have

$$\|d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_1\varpi)\| \leq 0.$$

Hence, $\Gamma_1\varpi = \Gamma_4\varpi$. Similarly, $\Gamma_2\varpi = \Gamma_3\varpi$.
From (3.3.1), we have

$$\begin{aligned}
d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varpi) \preceq\ & \alpha_1\Big(\frac{d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_4\varpi)(1 + d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_3\varpi))}{1 + d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varpi)}\Big) \\
& +\alpha_2\Big(\frac{d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_4\varpi)d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_3\varpi)}{d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\varpi)}\Big) \\
& +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_4\varpi) + d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_3\varpi)\Big) \\
& +\alpha_4\Big(d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_3\varpi) + d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_2\varpi)\Big) + \alpha_5 d_{\mathbb{B}}(\Gamma_4\varpi, \Gamma_3\varpi) \\
\preceq\ & \alpha_4\Big(d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varpi) + d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varpi)\Big) + \alpha_5 d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varpi) \\
\preceq\ & (2\alpha_4 + \alpha_5)d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varpi).
\end{aligned}$$

Taking norm on both side, we have

$$\|1 - 2\alpha_4 - \alpha_5\|\|d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varpi)\| \leq 0,$$

implies $\|d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varpi)\| = 0$. Hence, $\Gamma_1\varpi = \Gamma_2\varpi$. On combining, we have $\Gamma_1\varpi = \Gamma_2\varpi = \Gamma_3\varpi = \Gamma_4\varpi$. □

Theorem 3.3.2. *Consider a ordered complete C^*_{AV}-MS $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq_{\mho})$ and Γ_1, Γ_2, Γ_3, $\Gamma_4 : \mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_3(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_4(\mho)$*;*

(ii) for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in [0, 1)$ *with* $\theta_{\mathbb{B}} \preceq \alpha_1 + \alpha_2 + 2(\alpha_3 + \alpha_4) + \alpha_5 \preceq I_{\mathbb{B}}$,

$$\begin{aligned}
d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq\ & \alpha_1\Big(\frac{d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_4\varsigma)(1 + d_{\mathbb{B}}(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1 + d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_2\vartheta)}\Big) \\
& +\alpha_2\Big(\frac{d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_4\varsigma)d_{\mathbb{B}}(\Gamma_2\vartheta, \Gamma_3\vartheta)}{d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_3\vartheta)}\Big) \\
& +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\varsigma, \mho\varsigma) + d_{\mathbb{B}}(\Gamma_2\vartheta, \Gamma_3\vartheta)\Big) + \alpha_4\Big(d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_3\vartheta) \\
& +d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\vartheta)\Big) + \alpha_5 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_3\vartheta)\ \forall\ \vartheta, \varsigma \in \mho; \quad (3.3.4)
\end{aligned}$$

(iii) the pairs (Γ_1, Γ_4) *and* (Γ_2, Γ_3) *are weakly compatible;*

(iv) the pairs (Γ_1, Γ_2) *and* (Γ_2, Γ_1) *satisfies PWI property w.r.t* Γ_3 *and* Γ_4*.*

Then, Γ_1*,* Γ_2*,* Γ_3 *and* Γ_4 *have a coincidence point in* $\mho$*.*

Proof. On the similar lines of Theorem (3.3.1), $\{\varpi_\jmath\}$ is a C_{seq}. $\therefore \exists\ \varpi \in \mho$ s.t

$$\lim_{\jmath\to\infty} d_{\mathbb{B}}(\varpi_\jmath, \varpi) = \theta_{\mathbb{B}}.$$

Since, $\Gamma_3(\mho)$ is complete and $\{\varpi_{2\jmath+1}\} \subset \Gamma_3(\mho) \therefore\ \varpi \in \Gamma_3(\mho)$. Hence, $\exists\ \mu \in \mho$ s.t $\varpi = \Gamma_3\mu$ and

$$\lim_{\jmath\to\infty} d_{\mathbb{B}}(\varpi_{2\jmath+1}, \Gamma_3\mu) = \lim_{\jmath\to\infty} d_{\mathbb{B}}(\Gamma_3\varsigma_{2\jmath+1}, \Gamma_3\mu) = \theta_{\mathbb{B}}.$$

Similarly, $\exists\ \nu \in \mho$ s.t $\varpi = \Gamma_3\mu = \Gamma_4\nu$ and

$$\lim_{\jmath\to\infty} d_{\mathbb{B}}(\varpi_{2\jmath}, \Gamma_4\nu) = \lim_{\jmath\to\infty} d_{\mathbb{B}}(\Gamma_4\varsigma_{2\jmath}, \Gamma_4\nu) = \theta_{\mathbb{B}}.$$

Now, to show ν is a coincidence point of (Γ_1, Γ_4). Since, $\Gamma_3\varsigma_{2\jmath+1} \to \varpi = \Gamma_4\nu$ as $\jmath \to \infty$. Substituting $\varsigma = \nu$ and $\vartheta = \varsigma_{2\jmath+1}$ in (3.3.4), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\varsigma_{2\jmath+1}) \ \preceq\ & \alpha_1\left(\frac{d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_4\nu)(1 + d_{\mathbb{B}}(\Gamma_2\varsigma_{2\jmath+1}, \Gamma_3\varsigma_{2\jmath+1}))}{1 + d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\varsigma_{2\jmath+1})}\right) \\ & +\alpha_2\left(\frac{d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_4\nu)d_{\mathbb{B}}(\Gamma_2\varsigma_{2\jmath+1}, \Gamma_3\varsigma_{2\jmath+1})}{d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_3\varsigma_{2\jmath+1})}\right) \\ & +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_4\nu) + d_{\mathbb{B}}(\Gamma_2\varsigma_{2\jmath+1}, \Gamma_3\varsigma_{2\jmath+1})\Big) \\ & +\alpha_4\Big(d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_3\varsigma_{2\jmath+1}) + d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_2\varsigma_{2\jmath+1})\Big) \\ & +\alpha_5 d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_3\varsigma_{2\jmath+1}). \end{aligned} \tag{3.3.5}$$

Taking limit as $\jmath \to \infty$ in (3.3.5), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\nu, \varpi) \ \preceq\ & \alpha_1\left(\frac{d_{\mathbb{B}}(\Gamma_1\nu, \varpi)(1 + d_{\mathbb{B}}(\varpi, \varpi))}{1 + d_{\mathbb{B}}(\Gamma_1\nu, \varpi)}\right) \\ & +\alpha_2\left(\frac{d_{\mathbb{B}}(\Gamma_1\nu, \varpi)d_{\mathbb{B}}(\varpi, \varpi)}{d_{\mathbb{B}}\varpi, \varpi)}\right) \\ & +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\nu, \varpi) + d_{\mathbb{B}}(\varpi, \varpi)\Big) \\ & +\alpha_4\Big(d_{\mathbb{B}}(\Gamma_1\nu, \varpi) + d_{\mathbb{B}}(\varpi, \varpi)\Big) \\ & +\alpha_5 d_{\mathbb{B}}(\varpi, \varpi) \\ \preceq\ & (\alpha_1 + \alpha_3 + \alpha_4)d_{\mathbb{B}}(\Gamma_1\nu, \varpi) \\ (I_{\mathbb{B}} - (\alpha_1 + \alpha_3 + \alpha_4))d_{\mathbb{B}}(\Gamma_1\nu, \varpi) \ \preceq\ & \theta_{\mathbb{B}}. \end{aligned}$$

Taking norm on both side, we have

$$\|(1 - (\alpha_1 + \alpha_3 + \alpha_4))\|\|d_{\mathbb{B}}(\Gamma_1\nu, \varpi)\| \leq 0,$$

implies $\|d_{\mathbb{B}}(\Gamma_1\nu, \varpi)\| = 0$. Hence, $\Gamma_1\nu = \varpi = \Gamma_4\nu$ and (Γ_1, Γ_4) are weakly compatible, $\therefore$ $\Gamma_1\varpi = \Gamma_1\Gamma_4\nu = \Gamma_4\Gamma_1\nu = \Gamma_4\varpi$. Thus, ϖ is a coincidence point of (Γ_1, Γ_4).
Rest proof follow on the similar lines as in Theorem (3.3.1). $\square$

Remark 3.3.3. Xin et al. (2016) In metric spaces if the mappings Γ_1 and Γ_2 are compatible, then they are weakly compatible but converse is not true. The same holds for the C^*-algebra valued metric spaces.

Example 3.3.4. Let $\mho = [0, \infty)$, $\mathbb{B} = \mathbb{C}$ and $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ be define on $\mho$ as $d_{\mathbb{B}}(\varpi, \vartheta) = |\varpi - \vartheta|\ \forall\ \varpi, \vartheta \in \mho$. Define an ordering on $\mho$ as follow $\varpi \preceq \vartheta \Leftrightarrow \varpi \geq \vartheta\ \forall\ \varpi, \vartheta \in \mho$. Then, $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq_{\mho})$ is an ordered complete C^*_{AV}-MS.
Define Γ_1, Γ_2, Γ_3, $\Gamma_4 : \mho \to \mho$ as

$$\Gamma_1\varpi = \ln\left(1 + \frac{\varpi}{21}\right), \quad \Gamma_3\varpi = e^{7\varpi} - 1. \tag{3.3.6}$$

$$\Gamma_2\varpi = \ln\left(1 + \frac{\varpi}{7}\right), \quad \Gamma_4\varpi = e^{21\varpi} - 1. \tag{3.3.7}$$

Now, to prove (Γ_1, Γ_2) satisfies PWI property w.r.t Γ_3. Let $\varpi, \vartheta \in \mho$ s.t $\vartheta \in \Gamma_3^{-1}\Gamma_1\varpi$ i.e, $\Gamma_3\vartheta = \Gamma_1\varpi$.
From (3.3.6), we have

$$\ln\left(1 + \frac{\varpi}{21}\right) = e^{7\vartheta} - 1 \Rightarrow \vartheta = \frac{\ln\left(1 + \ln(1 + \frac{\varpi}{21})\right)}{7}.$$

Hence,

$$\Gamma_1\varpi = \ln\left(1 + \frac{\varpi}{21}\right) \geq \ln\left(1 + \frac{\ln\left(1 + (\ln(1 + \frac{\varpi}{21}))\right)}{70}\right) = \ln\left(1 + \frac{\vartheta}{7}\right) = \Gamma_2\vartheta.$$

$\therefore$ $\Gamma_1\varpi \preceq \Gamma_2\vartheta$. Hence, (Γ_1, Γ_2) satisfies PWI property w.r.t Γ_3.
Now, to prove (Γ_2, Γ_1) satisfies PWI property w.r.t Γ_4. Let $\varpi, \vartheta \in \mho$ be s.t $\vartheta \in \Gamma_4^{-1}\Gamma_2\varpi$, i.e, $\Gamma_4\vartheta = \Gamma_2\varpi$.
From (3.3.7), we have

$$e^{21\vartheta} - 1 = \ln\left(1 + \frac{\varpi}{7}\right) \Rightarrow \vartheta = \frac{\ln\left(1 + \ln(1 + \frac{\varpi}{7})\right)}{21}.$$

Hence,

$$\Gamma_2\varpi = \ln\left(1 + \frac{\varpi}{7}\right) \geq \ln\left(1 + \frac{\ln\left(1 + (\ln(1 + \frac{\varpi}{7}))\right)}{621}\right) = \ln\left(1 + \frac{\vartheta}{21}\right) = \Gamma_1\vartheta.$$

$\therefore\ \Gamma_2\varpi \preceq \Gamma_1\vartheta$. Hence, (Γ_2, Γ_1) satisfies PWI property w.r.t Γ_4.
Furthermore, $\Gamma_1(\mho) = \Gamma_2(\mho) = \Gamma_4(\mho) = \Gamma_3(\mho) = [0, \infty)$ and (Γ_1, Γ_4) and (Γ_2, Γ_3) are compatible.
Let $\{\varpi_\jmath\}$ be a sequence in $\mho$ s.t $\lim_{\jmath\to\infty} d_{\mathbb{B}}(\varrho, \Gamma_1\varpi_\jmath) = \lim_{\jmath\to\infty} d_{\mathbb{B}}(\varrho, \Gamma_4\varpi_\jmath) = \theta_{\mathbb{B}}$, for some $\varrho \in \mho$. $\therefore$

$$\lim_{\jmath\to\infty}\left|\ln\left(1+\frac{\varpi_\jmath}{21}\right)-\varrho\right| = \lim_{\jmath\to\infty}\left|e^{21\varpi_\jmath}-1-\varrho\right| = \theta_{\mathbb{B}}.$$

and

$$\lim_{\jmath\to\infty}\left|\varpi_\jmath - 21(e^{\varrho}-1)\right| = \lim_{\jmath\to\infty}\left|\varpi_\jmath - \frac{\ln(1+\varrho)}{21}\right| = \theta_{\mathbb{B}}.$$

We have

$$21(e^{\varrho}-1) = \frac{\ln(1+\varrho)}{21}.$$

which is possible if $\varrho = \theta_{\mathbb{B}}$.
Since, Γ_1 and Γ_4 are continuous. $\therefore$

$$\lim_{\jmath\to\infty} d_{\mathbb{B}}(\Gamma_1\Gamma_4\varpi_\jmath, \Gamma_4\Gamma_1\varpi_\jmath) = \lim_{\jmath\to\infty}\left|\Gamma_1\Gamma_4\varpi_\jmath - \Gamma_4\Gamma_1\varpi_\jmath\right| = \theta_{\mathbb{B}}.$$

Clearly,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_2\vartheta) \preceq\ & \alpha_1\left(\frac{d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_4\varsigma)(1+d_{\mathbb{B}}(\Gamma_2\vartheta, \Gamma_3\vartheta))}{1+d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_2\vartheta)}\right) \\ & +\alpha_2\left(\frac{d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_4\varsigma)d_{\mathbb{B}}(\Gamma_2\vartheta, \Gamma_3\vartheta)}{d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_3\vartheta)}\right) \\ & +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\varsigma, \mho\varsigma) + d_{\mathbb{B}}(\Gamma_2\vartheta, \Gamma_3\vartheta)\Big) + \alpha_4\Big(d_{\mathbb{B}}(\Gamma_1\varsigma, \Gamma_3\vartheta) \\ & +d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\vartheta)\Big) + \alpha_5 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_3\vartheta)\ \forall\ \vartheta, \varsigma \in \mho. \end{aligned} \tag{3.3.8}$$

Hence, (3.3.8) satisfies all the hypothesis of the Theorem (3.3.1). Thus, Γ_1, Γ_2, Γ_3 and Γ_4 have a coincidence point. Indeed, '0' is a coincidence point of Γ_1, Γ_2, Γ_3 and Γ_4.

Remark 3.3.5. The result of the example (3.3.4) can also be established by using Theorem (3.3.2).

3.4 Conclusion

In this chapter, we have introduced a novel approach to prove coincidence point results on two pairs of compatible or weakly compatible mappings using C_*-class function and certain rational type contraction mappings on a C^*_{AV}-MS that extends, unifies and generalizes the results on coincidence point in the literature. However, under certain conditions the results proved in this chapter are reduced to some well known results in the literature.

(*i*) If in Theorem (3.2.2) we consider $\mathbb{A} = \mathbb{R}$, $F^*(r,t) = r - t$ then we obtain Theorem (2.4) of Nashine & Samet (2011).

(*ii*) If in Theorem (3.2.4) we consider $\mathbb{A} = \mathbb{R}$, $F^*(r,t) = r - t$ then we obtain Theorem (2.6) of Nashine & Samet (2011).

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Chapter 4

Some Results On Common Fixed Point

4.1 Introduction

The present chapter of the thesis deals with the results on the existence and uniqueness of common fixed point in C^*_{AV}-metric space. The content of this chapter is divided into two sections. In the first section, some common fixed point theorems for weakly compatible pairs of self mappings using $E.A.$ property and CLR property with certain rational type contraction conditions without continuous mappings in C^*_{AV}-MS are established. In the last section, some theorems for weakly compatible pairs of self mappings satisfying expansion conditions in C^*_{AV}-MS are presented. The results of this chapter are presented in [7,8,9,10].

[7]Dhariwal, R., Kumar, D. (2022). On Unification of Common Fixed Point in C^*-algebra valued metric spaces. Journal of Physics – Conference Series, Article Id 2267. doi:10.1088/1742-6596/2267/1/012108.

[8]Dhariwal, R., Kumar, D. (2022). On Existence and Uniqueness of Common Fixed Point in C^*-algebra valued metric spaces. Science & Technology Asia, 27(2), 27-41.

[9] Dhariwal, R., Kumar, D. (2023). Common Fixed Point of Two Pairs of Weakly Compatible Mappings Using Rational Type Contraction in C^*-algebra value metric space. Nonlinear Studies, 30(1), 199-212.

[10] Dhariwal, R., Kumar, D. (2022). Some Results on Common Fixed point Using Expansion Mapping in C^*-algebra valued metric spaces. Iranian Journal of Mathematical Sciences and Informatics. (Accepted).

4.2 Common Fixed Point of Self Mappings using Contraction

Common fixed point results of self mappings for various contractions in the abstract spaces have been investigated broadly by many researchers (see, Jungck (1976), Sessa (1982), Jungck (1986), Aamri & Moutawakil (2002), Abbas & Jungck (2008), Vetro (2010), Kumam & Sintunavarat (2011), Chandok (2013), Manro et al. (2013), López-de Hierro & Sintunavarat (2016), Zhang et al. (2016), Xin et al. (2016), Dung et al. (2017), Ansari & Ozturk (2017), Mohanta (2018), Shen et al. (2018), Nazam et al. (2018), Nazam et al. (2019), Chandok et al. (2019), George et al. (2020), Asim et al. (2020), Mlaiki et al. (2020), Asim & Imdad (2020*b*), Asim & Imdad (2020*a*), Malhotra et al. (2022), Saluja (2022) and references cited therein). In this section, some results on common fixed points for self mappings in a C^*_{AV}-MS are presented.

Theorem 4.2.1. *Let* $(\mho, \mathbb{B}, d_{\mathbb{B}})$ *be* C^*_{AV}*-MS and* $\Gamma_1, \Gamma_2, \Gamma_3,\ \Gamma_4 : \mho \to \mho$ *satisfying:*

(*i*) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$;

(*ii*) $\forall\ \varpi, \varsigma \in \mho$, $\alpha_1 \in \mathbb{B}^+$ *with* $\|\alpha_1\| \leq 1$

$$d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma) \preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}\right)\alpha_1; \quad (4.2.1)$$

(*iii*) *either* $\Gamma_1(\mho)$, $\Gamma_2(\mho)$, $\Gamma_3(\mho)$ *or* $\Gamma_4(\mho)$ *is a complete subspace of* $\mho$.

Then, the pairs (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *have a coincidence point in* $\mho$. *Moreover, if the pairs* (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible, then* $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$.

Proof. Let $\varpi_0 \in \mho$ be any arbitrary point. From (*i*), construct a sequence $\{\varsigma_\jmath\}$ in $\mho$ as

$$\varsigma_{2\jmath+1} = \Gamma_1\varpi_{2\jmath} = \Gamma_4\varpi_{2\jmath+1} \quad \text{and} \quad \varsigma_{2\jmath+2} = \Gamma_2\varpi_{2\jmath+1} = \Gamma_3\varpi_{2\jmath+2}.$$

Define $d_{\mathbb{B}_\jmath} = d_{\mathbb{B}}(\varsigma_\jmath, \varsigma_{\jmath+1})$. Suppose that $d_{\mathbb{B}_{2\jmath}} = \theta_{\mathbb{B}}$ i.e, $d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}) = \theta_{\mathbb{B}}$ for some $\jmath$. Then, $\Gamma_1\varpi_{2\jmath} = \Gamma_4\varpi_{2\jmath+1} = \Gamma_2\varpi_{2\jmath-1} = \Gamma_3\varpi_{2\jmath}$. Thus, Γ_1 and Γ_3 have coincidence point. Hence, the result.

Now, suppose that $d_{\mathbb{B}_{2\jmath}} \succ \theta_{\mathbb{B}}\ \forall\ \jmath \in \mathbb{N}$. Then, substituting $\varpi = \varpi_{2\jmath}$ and $\varsigma = \varpi_{2\jmath+1}$ in (4.2.1), we have

$$d_{\mathbb{B}}\Big(\Gamma_1\varpi_{2\jmath}, \Gamma_2\varpi_{2\jmath+1}\Big) \preceq \alpha_1^*$$
$$\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_1\varpi_{2\jmath})d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_2\varpi_{2\jmath+1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath+1}, \Gamma_2\varpi_{2\jmath+1})d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath+1}, \Gamma_1\varpi_{2\jmath})}{1 + d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_2\varpi_{2\jmath+1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath+1}, \Gamma_1\varpi_{2\jmath})}\right)\alpha_1$$
$$d_{\mathbb{B}}\Big(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}\Big) \preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+2}) + d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+1})}{1 + d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+2}) + d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+1})}\right)\alpha_1$$

$$d_{\mathbb{B}}\Big(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}\Big) \preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+2})}{1 + d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+2})}\right)\alpha_1 \preceq \alpha_1^* d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})\alpha_1.$$

Hence, $d_{\mathbb{B}_{2\jmath+1}} \preceq \alpha_1^* d_{\mathbb{B}_{2\jmath}}\alpha_1$. Similarly, $d_{\mathbb{B}_{2\jmath}} \preceq \alpha_1^*(d_{\mathbb{B}_{2\jmath-1}})\alpha_1$, $d_{\mathbb{B}_{2\jmath-1}} \preceq \alpha_1^*(d_{\mathbb{B}_{2\jmath-2}})\alpha_1$ and so on. In general, $d_{\mathbb{B}_\jmath} \preceq \alpha_1^*(d_{\mathbb{B}_{\jmath-1}})\alpha_1\ \forall\ \jmath \in \mathbb{N}$, i.e,

$$\begin{aligned} d_{\mathbb{B}}(\varsigma_\jmath, \varsigma_{\jmath+1}) &\preceq (\alpha_1^*)d_{\mathbb{B}}(\varsigma_{\jmath-1}, \varsigma_\jmath)\alpha_1 \\ &\preceq (\alpha_1^*)^2 d_{\mathbb{B}}(\varsigma_{\jmath-2}, \varsigma_{\jmath-1})\alpha_1^2 \\ &\preceq \cdots \\ &\preceq (\alpha_1^*)^\jmath d_{\mathbb{B}}(\varsigma_0, \varsigma_1)\alpha_1^\jmath. \end{aligned}$$

For any $p \in \mathbb{N}$, we have

$$\begin{aligned} d_{\mathbb{B}}(\varsigma_{\jmath+p}, \varsigma_\jmath) &\preceq d_{\mathbb{B}}(\varsigma_{\jmath+p}, \varsigma_{\jmath+p-1}) + d_{\mathbb{B}}(\varsigma_{\jmath+p-1}, \varsigma_{\jmath+p-2}) + \cdots + d_{\mathbb{B}}(\varsigma_{\jmath+1}, \varsigma_\jmath) \\ &\preceq \sum_{\imath=\jmath}^{\jmath+p-1} (\alpha_1^*)^\imath d_{\mathbb{B}}(\varsigma_0, \varsigma_1)\alpha_1^\imath \\ &\preceq \sum_{\imath=\jmath}^{\jmath+p-1} (\alpha_2\alpha_1^\imath)^*\alpha_2\alpha_1^\imath \preceq \sum_{\imath=\jmath}^{\jmath+p-1} |\alpha_2\alpha_1^\imath|^2 \\ &\leq \sum_{\imath=\jmath}^{\jmath+p-1} \|(\alpha_2\alpha_1^\imath)^2\| I_{\mathbb{B}} \\ &\leq \|\alpha_2\|^2 I_{\mathbb{B}} \sum_{\imath=\jmath}^{\jmath+p-1} (\alpha_1^\imath)^2 \to \theta_{\mathbb{B}} \text{ as } \jmath \to \infty, \end{aligned}$$

where $|\alpha_2|^2 = d_{\mathbb{B}}(\varsigma_0, \varsigma_1)$ for some $\alpha_2 \in \mathbb{B}^+$. Hence, $\{\varsigma_\jmath\}$ is a C_{seq} in $(\mho, \mathbb{B}, d_{\mathbb{B}})$. Since, $\Gamma_3(\mho)$ is a complete subspace of $\mho$. $\therefore$ $\{\varsigma_\jmath\}$ is contained in $\Gamma_3(\mho)$ and has a limit in $\Gamma_3(\mho)$, μ (say). Let $\nu \in \Gamma_3^{-1}\mu$, then $\Gamma_3\nu = \mu$.
Next, to show $\Gamma_1\nu = \mu$. Assume that, $\Gamma_1\nu \neq \mu$. Substituting $\varpi = \nu$ and $\varsigma = \varpi_{\jmath-1}$

in (4.2.1), we have

$$d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\varpi_{\jmath-1}) \preceq$$
$$\alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_1\nu)d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_2\varpi_{\jmath-1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath-1}, \Gamma_2\varpi_{\jmath-1})d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath-1}, \Gamma_1\nu)}{1 + d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_2\varpi_{\jmath-1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath-1}, \Gamma_1\nu)}\right)\alpha_1$$
$$d_{\mathbb{B}}(\Gamma_1\nu, \varsigma_\jmath) \preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_1\nu)d_{\mathbb{B}}(\Gamma_3\nu, \varsigma_\jmath) + d_{\mathbb{B}}(\varsigma_{\jmath-1}, \varsigma_\jmath)d_{\mathbb{B}}(\varsigma_{\jmath-1}, \Gamma_1\nu)}{1 + d_{\mathbb{B}}(\Gamma_3\nu, \varsigma_\jmath) + d_{\mathbb{B}}(\varsigma_{\jmath-1}, \Gamma_1\nu)}\right)\alpha_1.$$

Taking limit as $\jmath \to \infty$, we have

$$d_{\mathbb{B}}(\Gamma_1\nu, \mu) \preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\mu, \Gamma_1\nu)d_{\mathbb{B}}(\mu, \mu) + d_{\mathbb{B}}(\mu, \mu)d_{\mathbb{B}}(\mu, \Gamma_1\nu)}{1 + d_{\mathbb{B}}(\mu, \mu) + d_{\mathbb{B}}(\mu, \Gamma_1\nu)}\right)\alpha_1.$$

Then, $d_{\mathbb{B}}(\Gamma_1\nu, \mu) \preceq \theta_{\mathbb{B}}$, hence $\Gamma_1\nu = \mu$. Thus, $\Gamma_3\nu = \mu = \Gamma_1\nu$,. Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$. $\therefore$ $\Gamma_1\nu = \mu$ implies $\mu \in \Gamma_4(\mho)$. Let $\vartheta \in \Gamma_4^{-1}\mu$, then $\Gamma_4\vartheta = \mu$.
On the similar lines, $\Gamma_2\vartheta = \Gamma_4\vartheta = \mu$. Since, (Γ_1, Γ_3) and (Γ_2, Γ_4) are weakly compatible $\therefore$ $\mu = \Gamma_1\nu = \Gamma_3\nu = \Gamma_4\vartheta = \Gamma_2\vartheta$.
Then,

$$\Gamma_4\mu = \Gamma_4\Gamma_2\vartheta = \Gamma_2\Gamma_4\vartheta = \Gamma_2\mu \text{ and } \Gamma_3\mu = \Gamma_3\Gamma_1\nu = \Gamma_1\Gamma_3\nu = \Gamma_1\mu.$$

We claim that $\Gamma_2\mu = \mu$. If possible, let $\Gamma_2\mu \neq \mu$. Then, from (4.2.1), we have

$$\begin{aligned} d_{\mathbb{B}}(\mu, \Gamma_2\mu) &= d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\mu) \\ &\preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_1\nu)d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_2\mu) + d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_2\mu)d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_1\nu)}{1 + d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_2\mu) + d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_1\nu)}\right)\alpha_1. \end{aligned}$$

Then, $\|d_{\mathbb{B}}(\mu, \Gamma_2\mu)\| \leq 0$, hence $\Gamma_2\mu = \mu$. On the similar lines, $\Gamma_1\mu = \mu$. Thus, $\Gamma_1\mu = \Gamma_3\mu = \Gamma_4\mu = \Gamma_2\mu = \mu$.
Uniqueness : Let $\varkappa$ be another common fixed point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4. Then, from (4.2.1), we have

$$\begin{aligned} d_{\mathbb{B}}(\varkappa, \mu) &= d_{\mathbb{B}}(\Gamma_1\varkappa, \Gamma_2\mu) \\ &\preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varkappa, \Gamma_1\varkappa)d_{\mathbb{B}}(\Gamma_3\varkappa, \Gamma_2\mu) + d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_2\mu)d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_1\varkappa)}{1 + d_{\mathbb{B}}(\Gamma_3\varkappa, \Gamma_2\mu) + d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_1\varkappa)}\right)\alpha_1 \\ &\preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\varkappa, \varkappa)d_{\mathbb{B}}(\varkappa, \mu) + d_{\mathbb{B}}(\mu, \mu)d_{\mathbb{B}}(\mu, \varkappa)}{1 + d_{\mathbb{B}}(\varkappa, \mu) + d_{\mathbb{B}}(\mu, \varkappa)}\right)\alpha_1. \end{aligned}$$

Then, $\|d_{\mathbb{B}}(\varkappa, \mu)\| \leq 0$, hence $\varkappa = \mu$. □

Theorem 4.2.2. *Let $(\mho, \mathbb{B}, d_{\mathbb{B}})$ be a C^*_{AV}-MS and $\Gamma_1, \Gamma_2, \Gamma_3,\ \Gamma_4 : \mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(ii) $\forall\ \varpi, \varsigma \in \mho$, $\alpha_1 \in \mathbb{B}^+$ *with* $\|\alpha_1\| \leq 1$,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma) \ \preceq\ & \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varsigma), \frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varsigma) + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)}{2}, \\ & d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi), \frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{2}\Bigg)\alpha_1 \quad (4.2.2) \end{aligned}$$

(iii) either $\Gamma_1(\mho)$, $\Gamma_2(\mho)$, $\Gamma_3(\mho)$ *or* $\Gamma_4(\mho)$ *is a complete subspace of* $\mho$.

Then, the pairs (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *have a coincidence point in* $\mho$. *Moreover, if the pairs* (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible, then* $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$.

Proof. Let $\varpi_0 \in \mho$ be any arbitrary point. From (i), construct a sequence $\{\varsigma_\jmath\}$ in $\mho$ as

$$\varsigma_{2\jmath+1} = \Gamma_1\varpi_{2\jmath} = \Gamma_4\varpi_{2\jmath+1} \quad \text{and} \quad \varsigma_{2\jmath+2} = \Gamma_2\varpi_{2\jmath+1} = \Gamma_3\varpi_{2\jmath+2}.$$

Define $d_{\mathbb{B}_\jmath} = d_{\mathbb{B}}(\varsigma_\jmath, \varsigma_{\jmath+1})$. Suppose $d_{\mathbb{B}_{2\jmath}} = \theta_{\mathbb{B}}$ i.e, $d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}) = \theta_{\mathbb{B}}$ for some $\jmath$. Then, $\Gamma_1\varpi_{2\jmath} = \Gamma_4\varpi_{2\jmath+1} = \Gamma_2\varpi_{2\jmath-1} = \Gamma_3\varpi_{2\jmath}$. Thus, Γ_1 and Γ_3 have coincidence point. Hence, the result.
Now, suppose that $d_{\mathbb{B}_{2\jmath}} \succ \theta_{\mathbb{B}}\ \forall\ \jmath \in \mathbb{N}$. Then, substituting $\varpi = \varpi_{2\jmath}$ and $\varsigma = \varpi_{2\jmath+1}$ in (4.2.2), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\varpi_{2\jmath}, \Gamma_2\varpi_{2\jmath+1}) \ \preceq\ & \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_4\varpi_{2\jmath+1}), d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath+1}, \Gamma_1\varpi_{2\jmath}), \\ & \frac{d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_4\varpi_{2\jmath+1}) + d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_1\varpi_{2\jmath})}{2}, \\ & \frac{d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_4\varpi_{2\jmath+1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath+1}, \Gamma_1\varpi_{2\jmath})}{2}\Bigg)\alpha_1 \\ d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) \ \preceq\ & \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}), \frac{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}) + d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+1})}{2} \\ & d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+1}), \frac{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}) + d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})}{2}, \Bigg)\alpha_1 \\ d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) \ \preceq\ & \alpha_1^* d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})\alpha_1. \end{aligned}$$

Thus, we have $d_{\mathbb{B}_{2\jmath+1}} \preceq \alpha_1^* d_{\mathbb{B}_{2\jmath}}\alpha_1$. Similarly, $d_{\mathbb{B}_{2\jmath}} \preceq \alpha_1^*(d_{\mathbb{B}_{2\jmath-1}})\alpha_1$, $d_{\mathbb{B}_{2\jmath-1}} \preceq \alpha_1^*(d_{\mathbb{B}_{2\jmath-2}})\alpha_1$

and so on. In general, $d_{\mathbb{B}_\jmath} \preceq \alpha_1^*(d_{\mathbb{B}_{\jmath-1}})\alpha_1 \ \forall\ \jmath \in \mathbb{N}$, i.e,

$$\begin{aligned}
d_{\mathbb{B}}(\varsigma_\jmath, \varsigma_{\jmath+1}) &\preceq (\alpha_1^*) d_{\mathbb{B}}(\varsigma_{\jmath-1}, \varsigma_\jmath)\alpha_1 \\
&\preceq (\alpha_1^*)^2 d_{\mathbb{B}}(\varsigma_{\jmath-2}, \varsigma_{\jmath-1})\alpha_1^2 \\
&\preceq \cdots \\
&\preceq (\alpha_1^*)^\jmath d_{\mathbb{B}}(\varsigma_0, \varsigma_1)\alpha_1^\jmath.
\end{aligned}$$

For any $k \in \mathbb{N}$, we have,

$$\begin{aligned}
d_{\mathbb{B}}(\varsigma_{\jmath+k}, \varsigma_\jmath) &\preceq d_{\mathbb{B}}(\varsigma_{\jmath+k}, \varsigma_{\jmath+k-1}) + d_{\mathbb{B}}(\varsigma_{\jmath+k-1}, \varsigma_{\jmath+k-2}) + \cdots + d_{\mathbb{B}}(\varsigma_{\jmath+1}, \varsigma_\jmath) \\
&\preceq \sum_{\imath=\jmath}^{\jmath+k-1} (\alpha_1^*)^\imath d_{\mathbb{B}}(\varsigma_0, \varsigma_1)\alpha_1^\imath \\
&\preceq \sum_{\imath=\jmath}^{\jmath+k-1} (\alpha_2\alpha_1^\imath)^* \alpha_2\alpha_1^\imath \\
&\preceq \sum_{\imath=\jmath}^{\jmath+k-1} |\alpha_2\alpha_1^\imath|^2 \\
&\leq \sum_{\imath=\jmath}^{\jmath+k-1} \|(\alpha_2\alpha_1^\imath)^2\| I_{\mathbb{B}} \\
&\leq \|\alpha_2\|^2 I_{\mathbb{B}} \sum_{\imath=\jmath}^{\jmath+k-1} (\alpha_1^\imath)^2 \to \theta_{\mathbb{B}} \text{ as } \jmath \to \infty,
\end{aligned}$$

where $|\alpha_2|^2 = d_{\mathbb{B}}(\varsigma_0, \varsigma_1)$ for some $\alpha_2 \in \mathbb{B}^+$. Hence, $\{\varsigma_\jmath\}$ is a C_{seq}. Since, $\Gamma_3(\mho)$ is complete subspace of $\mho$. $\therefore$ $\{\varsigma_\jmath\}$ is contained in $\Gamma_3(\mho)$ and has a limit in $\Gamma_3(\mho)$, ν (say). Let $\mu \in \Gamma_3^{-1}\nu$, then $\Gamma_3\mu = \nu$.

Next, to show $\Gamma_1\mu = \nu$. Assume $\Gamma_1\mu \neq \nu$. Substituting $\varpi = \mu$ and $\varsigma = \varpi_{\jmath-1}$ in (4.2.2), we have

$$\begin{aligned}
d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varpi_{\jmath-1}) &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\varpi_{\jmath-1}), \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\varpi_{\jmath-1}) + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu)}{2}, \\
&\qquad d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath-1}, \Gamma_1\mu), \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\varpi_{\jmath-1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath-1}, \Gamma_1\mu)}{2} \Bigg)\alpha_1 \\
d_{\mathbb{B}}(\Gamma_1\mu, \varsigma_\jmath) &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3\mu, \varsigma_{\jmath-1}), d_{\mathbb{B}}(\varsigma_{\jmath-1}, \Gamma_1\mu), \\
&\qquad \frac{d_{\mathbb{B}}(\Gamma_3\mu, \varsigma_{\jmath-1}) + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu)}{2}, \frac{d_{\mathbb{B}}(\Gamma_3\mu, \varsigma_{\jmath-1}) + d_{\mathbb{B}}(\varsigma_{\jmath-1}, \Gamma_1\mu)}{2} \Bigg)\alpha_1.
\end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\mu, \nu) &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\nu, \nu), d_{\mathbb{B}}(\nu, \Gamma_1\mu), \frac{d_{\mathbb{B}}(\nu, \nu) + d_{\mathbb{B}}(\nu, \Gamma_1\mu)}{2}, \\ &\quad \frac{d_{\mathbb{B}}(\nu, \nu) + d_{\mathbb{B}}(\nu, \Gamma_1\mu)}{2} \Bigg) \alpha_1. \end{aligned}$$

Then, $\|d_{\mathbb{B}}(\Gamma_1\mu, \nu)\| \leq \|\alpha_1\|^2 \|d_{\mathbb{B}}(\Gamma_1\mu, \nu)\|$, a contradiction. Hence $\Gamma_1\mu = \nu$. Thus, $\Gamma_3\mu = \nu = \Gamma_1\mu$. Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ and $\Gamma_1\mu = \nu$ implies $\nu \in \Gamma_4(\mho)$. Let $t \in \Gamma_4^{-1}\nu$, then $\Gamma_4 t = \nu$. Similarly, $\Gamma_2 t = \Gamma_4 t = \nu$.
Since, (Γ_1, Γ_3) and (Γ_2, Γ_4) are weakly compatible $\therefore\ \nu = \Gamma_1\mu = \Gamma_3\mu = \Gamma_4 t = \Gamma_2 t$. Then,

$$\Gamma_4\nu = \Gamma_4\Gamma_2 t = \Gamma_2\Gamma_4 t = \Gamma_2\nu \text{ and } \Gamma_3\nu = \Gamma_3\Gamma_1\mu = \Gamma_1\Gamma_3\mu = \Gamma_1\nu.$$

We claim that $\Gamma_2\nu = \nu$. If possible, let $\Gamma_2\nu \neq \nu$. Then, from (4.2.2), we have

$$\begin{aligned} d_{\mathbb{B}}(\nu, \Gamma_2\nu) = d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\nu) &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\nu), \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\nu) + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu)}{2}, \\ &\quad d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\mu), \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\nu) + d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\mu)}{2} \Bigg) \alpha_1. \end{aligned}$$

Then, $\|d_{\mathbb{B}}(\nu, \Gamma_2\nu)\| \leq 0$, hence $\Gamma_2\nu = \nu$. Similarly, $\Gamma_1\nu = \nu$. Hence, $\Gamma_1\nu = \Gamma_3\nu = \Gamma_4\nu = \Gamma_2\nu = \nu$.
Uniqueness : Let $\varkappa$ be another common fixed point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4. Then, from (4.2.2), we have

$$\begin{aligned} d_{\mathbb{B}}(\varkappa, \nu) = d_{\mathbb{B}}(\Gamma_1\varkappa, \Gamma_2\nu) &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3\varkappa, \Gamma_4\nu), \frac{d_{\mathbb{B}}(\Gamma_3\varkappa, \Gamma_4\nu) + d_{\mathbb{B}}(\Gamma_3\varkappa, \Gamma_1\varkappa)}{2}, \\ &\quad d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\varkappa), \frac{d_{\mathbb{B}}(\Gamma_3\varkappa, \Gamma_4\nu) + d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\varkappa)}{2} \Bigg) \alpha_1 \\ &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\varkappa, \nu), \frac{d_{\mathbb{B}}(\nu, \varkappa) + d_{\mathbb{B}}(\varkappa, \nu)}{2}, \\ &\quad d_{\mathbb{B}}(\nu, \varkappa), \frac{d_{\mathbb{B}}(\nu, \varkappa) + d_{\mathbb{B}}(\varkappa, \nu)}{2} \Bigg) \alpha_1 \\ &\preceq \alpha_1^* d_{\mathbb{B}}(\nu, \varkappa) \alpha_1. \end{aligned}$$

Then,

$$\|d_{\mathbb{B}}(\nu, \varkappa)\| \leq \|\alpha\|^2 \|d_{\mathbb{B}}(\nu, \varkappa)\|,$$

or

$$\Big(1-\|\alpha\|^2\Big)\|d_{\mathbb{B}}(\nu,\varkappa)\|\leq 0,$$

implies $\|d_{\mathbb{B}}(\nu,\varkappa)\|=0$. Hence, $\nu=\varkappa$. □

Theorem 4.2.3. *Let $(\mho,\mathbb{B},d_{\mathbb{B}})$ be a C^*_{AV}-MS and let $\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4:\mho\to\mho$ and $\exists\ \Gamma_5:\mathbb{B}^+\to\mathbb{B}$ satisfying:*

(*i*) $\Gamma_1(\mho)\subseteq\Gamma_4(\mho)$ *and* $\Gamma_2(\mho)\subseteq\Gamma_3(\mho)$*;*

(ii) Γ_5 *is continuous and strictly nondecreasing on* $\mathbb{B}^+$*;*

(iii) $\forall\ \varpi,\varsigma\in\mho$, $\alpha_1\in\mathbb{B}^+$ *with* $\|\alpha_1\|\leq 1$ *and* $0<\tau<1$,

$$\tau+\Gamma_5(d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_2\varsigma))\preceq$$

$$\Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma)+d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}{1+d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma)+d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}\alpha_1\right);\quad(4.2.3)$$

(*iv*) *either* $\Gamma_1(\mho)$, $\Gamma_2(\mho)$, $\Gamma_3(\mho)$ *or* $\Gamma_4(\mho)$ *is a complete subspace of* $\mho$*.*

Then, the pairs (Γ_1,Γ_3) and (Γ_2,Γ_4) have a coincidence point in $\mho$. Moreover, if the pairs (Γ_1,Γ_3) and (Γ_2,Γ_4) are weakly compatible, then $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ and Γ_4 have a unique common fixed point in $\mho$.

Proof. Let $\varpi_0\in\mho$ be any arbitrary point. From (i), construct a sequence $\{\varsigma_\jmath\}$ in $\mho$ as

$$\varsigma_{2\jmath+1}=\Gamma_1\varpi_{2\jmath}=\Gamma_4\varpi_{2\jmath+1}\quad\text{and}\quad\varsigma_{2\jmath+2}=\Gamma_2\varpi_{2\jmath+1}=\Gamma_3\varpi_{2\jmath+2}.$$

Let $d_\jmath=d_{\mathbb{B}}(\varsigma_\jmath,\varsigma_{\jmath+1})$. Suppose $d_{2\jmath}=\theta_{\mathbb{B}}$ i.e, $d_{\mathbb{B}}(\varsigma_{2\jmath},\varsigma_{2\jmath+1})=\theta_{\mathbb{B}}$ for some $\jmath$. Then, $\Gamma_1\varpi_{2\jmath}=\Gamma_4\varpi_{2\jmath+1}=\Gamma_2\varpi_{2\jmath-1}=\Gamma_3\varpi_{2\jmath}$. Thus, Γ_1 and Γ_3 have coincidence point. Hence, the result.
Now, suppose $d_{2\jmath}\succ\theta_{\mathbb{B}}\ \forall\ \jmath\in\mathbb{N}$. Substituting $\varpi=\varpi_{2\jmath}$ and $\varsigma=\varpi_{2\jmath+1}$ in (4.2.3),

we have

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\varpi_{2\jmath}, \Gamma_2\varpi_{2\jmath+1})\Big) \preceq \Gamma_5\Bigg(\alpha_1^*$$

$$\frac{d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_1\varpi_{2\jmath})d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_2\varpi_{2\jmath+1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath+1}, \Gamma_2\varpi_{2\jmath+1})d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath+1}, \Gamma_1\varpi_{2\jmath})}{1 + d_{\mathbb{B}}(\Gamma_3\varpi_{2\jmath}, \Gamma_2\varpi_{2\jmath+1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{2\jmath+1}, \Gamma_1\varpi_{2\jmath})}\alpha_1\Bigg)$$

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) \preceq$$

$$\Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+2}) + d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+1})}{1 + d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+2}) + d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+1})}\alpha_1\right),$$

$$\begin{aligned}
\tau + \Gamma_5\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) &\preceq \Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+2})}{1 + d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+2})}\alpha_1\right) \\
&\preceq \Gamma_5\Big(\alpha_1^* d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})\alpha_1\Big)
\end{aligned}$$

$$\Gamma_5\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2})\Big) \preceq \Gamma_5\Big(\alpha_1^* d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})\alpha_1\Big) - \tau \preceq \Gamma_5\Big(\alpha_1^* d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1})\alpha_1\Big).$$

Since, Γ_5 is non decreasing. $\therefore$ $d_{\mathbb{B}_{2\jmath+1}} \preceq \alpha_1^* d_{\mathbb{B}_{2\jmath}}\alpha_1$. On the similar lines, $d_{\mathbb{B}_{2\jmath}} \preceq \alpha_1^* d_{\mathbb{B}_{2\jmath-1}}\alpha_1$, $d_{2\jmath-1} \preceq \alpha_1^* d_{2\jmath-2}\alpha_1$ and so on. In general, $d_{\mathbb{B}_\jmath} \preceq \alpha_1^* d_{\mathbb{B}_{\jmath-1}}\alpha_1 \; \forall \; \jmath \in \mathbb{N}$ i.e,

$$\begin{aligned}
d_{\mathbb{B}}(\varsigma_\jmath, \varsigma_{\jmath+1}) &\preceq (\alpha_1^*) d_{\mathbb{B}}(\varsigma_{\jmath-1}, \varsigma_\jmath)\alpha_1 \\
&\preceq (\alpha_1^*)^2 d_{\mathbb{B}}(\varsigma_{\jmath-2}, \varsigma_{\jmath-1})\alpha_1^2 \preceq \cdots \\
&\preceq (\alpha_1^*)^\jmath d_{\mathbb{B}}(\varsigma_0, \varsigma_1)\alpha_1^\jmath.
\end{aligned}$$

For any $p \in \mathbb{N}$, we have

$$\begin{aligned}
d_{\mathbb{B}}(\varsigma_{\jmath+p}, \varsigma_\jmath) &\preceq d_{\mathbb{B}}(\varsigma_{\jmath+p}, \varsigma_{\jmath+p-1}) + d_{\mathbb{B}}(\varsigma_{\jmath+p-1}, \varsigma_{\jmath+p-2}) + \cdots + d_{\mathbb{B}}(\varsigma_{\jmath+1}, \varsigma_\jmath) \\
&\preceq \sum_{\imath=\jmath}^{\jmath+p-1} (\alpha_1^*)^\imath d_{\mathbb{B}}(\varsigma_0, \varsigma_1)\alpha_1^\imath \\
&\preceq \sum_{\imath=\jmath}^{\jmath+p-1} (\alpha_2\alpha_1^\imath)^*\alpha_2\alpha_1^\imath \\
&\preceq \sum_{\imath=\jmath}^{\jmath+p-1} |\alpha_2\alpha_1^\imath|^2 \\
&\leq \sum_{\imath=\jmath}^{\jmath+p-1} \|(\alpha_2\alpha_1^\imath)^2\| I_{\mathbb{B}} \\
&\leq \|\alpha_2\|^2 I_{\mathbb{B}} \sum_{\imath=\jmath}^{\jmath+p-1} (\alpha_1^\imath)^2 \to \theta_{\mathbb{B}} \quad \text{as} \quad \jmath \to \infty,
\end{aligned}$$

where $\|\alpha_2\|^2 = d_{\mathbb{B}}(\varsigma_0, \varsigma_1)$ for some $\alpha_2 \in \mathbb{B}^+$. Hence, $\{\varsigma_\jmath\}$ is a C_{seq} in $(\mho, \mathbb{B}, d_{\mathbb{B}})$. Since, $\Gamma_3(\mho)$ is a complete subspace of $\mho$. $\therefore$ $\{\varsigma_\jmath\}$ is contained in $\Gamma_3(\mho)$ and has a

limit in $\Gamma_3(\mho)$, μ (say). Let $\nu \in \Gamma^{-1}\mu$, then $\Gamma_3\nu = \mu$.
Next, to show $\Gamma_1\nu = \mu$. Assume $\Gamma_1\nu \neq \mu$. Substituting $\varpi = \nu$ and $\varsigma = \varpi_{\jmath-1}$ in (4.2.3), we have

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\varpi_{\jmath-1})\Big) \preceq$$
$$\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_1\nu)d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_2\varpi_{\jmath-1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath-1}, \Gamma_2\varpi_{\jmath-1})d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath-1}, \Gamma_1\nu)}{1 + d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_2\varpi_{\jmath-1}) + d_{\mathbb{B}}(\Gamma_4\varpi_{\jmath-1}, \Gamma_1\nu)}\alpha_1\right),$$

or

$$\tau+\Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\nu, \varsigma_\jmath)\Big) \preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_1\nu)d_{\mathbb{B}}(\Gamma_3\nu, \varsigma_\jmath) + d_{\mathbb{B}}(\varsigma_{\jmath-1}, \varsigma_\jmath)d_{\mathbb{B}}(\varsigma_{\jmath-1}, \Gamma_1\nu)}{1 + d_{\mathbb{B}}(\Gamma_3\nu, \varsigma_\jmath) + d_{\mathbb{B}}(\varsigma_{\jmath-1}, \Gamma_1\nu)}\alpha_1\right). \tag{4.2.4}$$

Taking limit as $\jmath \to \infty$ in (4.2.4), we have

$$\begin{aligned} \tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\nu, \mu)\Big) &\preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\mu, \Gamma_1\nu)d_{\mathbb{B}}(\mu, \mu) + d_{\mathbb{B}}(\mu, \mu)d_{\mathbb{B}}(\mu, \Gamma_1\nu)}{1 + d_{\mathbb{B}}(\mu, \mu) + d_{\mathbb{B}}(\mu, \Gamma_1\nu)}\alpha_1\right). \\ \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\nu, \mu)\Big) &\preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big). \end{aligned}$$

Since, Γ_5 is non decreasing. $\therefore$ $d_{\mathbb{B}}(\Gamma_1\nu, \mu) \preceq \theta_{\mathbb{B}}$. Hence, $\Gamma_1\nu = \mu$. Thus, $\Gamma_3\nu = \mu = \Gamma_1\nu$.
Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ and $\Gamma_1\nu = \mu$, then $\mu \in \Gamma_4(\mho)$. Let $\omega \in \Gamma_4^{-1}\mu$, then $\Gamma_4\omega = \mu$.
On the similar lines, $\Gamma_2\omega = \Gamma_4\omega = \mu$.
Since, the pair (Γ_1, Γ_3) and (Γ_2, Γ_4) are weakly compatible $\therefore$ $\mu = \Gamma_1\nu = \Gamma_3\nu = \Gamma_4\omega = \Gamma_2\omega$. Then,

$$\Gamma_4\mu = \Gamma_4\Gamma_2\omega = \Gamma_2\Gamma_4\omega = \Gamma_2\mu \text{ and } \Gamma_3\mu = \Gamma_3\Gamma_1\nu = \Gamma_3\Gamma\nu = \Gamma_1\mu.$$

We claim that $\Gamma_2\mu = \mu$. If possible, let $\Gamma_2\mu \neq \mu$. Then, from (4.2.3), we have

$$\begin{aligned} \tau \quad + \quad & \Gamma_5\Big(d_{\mathbb{B}}(\mu, \Gamma_2\mu)\Big) = \tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\mu)\Big) \\ \preceq \quad & \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_1\nu)d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_2\mu) + d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_2\mu)d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_1\nu)}{1 + d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_2\mu) + d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_1\nu)}\alpha_1\right). \\ & \Gamma_5\Big(d_{\mathbb{B}}(\mu, \Gamma_2\mu)\Big) \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big). \end{aligned}$$

Since, Γ_5 is non decreasing. $\therefore$ $d_{\mathbb{B}}(\mu, \Gamma_2\mu) \preceq \theta_{\mathbb{B}}$. Hence, $\Gamma_2\mu = \mu$. Thus, $\Gamma_1\mu = \Gamma_3\mu = \Gamma_4\mu = \Gamma_2\mu = \mu$.
Uniqueness : Let ϱ be another common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Then,

from (4.2.3), we have

$$\begin{aligned}
\tau+\Gamma_5\Big(d_{\mathbb{B}}(\varrho,\mu)\Big) &= \tau+\Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\varrho,\Gamma_2\mu)\Big)\\
&\preceq \Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\varrho,\Gamma_1\varrho)d_{\mathbb{B}}(\Gamma_3\varrho,\Gamma_2\mu)+d_{\mathbb{B}}(\Gamma_4\mu,\Gamma_2\mu)d_{\mathbb{B}}(\Gamma_4\mu,\Gamma_1\varrho)}{1+d_{\mathbb{B}}(\Gamma_3\varrho,\Gamma_2\mu)+d_{\mathbb{B}}(\Gamma_4\mu,\Gamma_1\varrho)}\alpha_1\right)\\
&= \Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\varrho,\varrho)d_{\mathbb{B}}(\varrho,\mu)+d_{\mathbb{B}}(\mu,\mu)d_{\mathbb{B}}(\mu,\varrho)}{1+d_{\mathbb{B}}(\varrho,\mu)+d_{\mathbb{B}}(\mu,\varrho)}\alpha_1\right).\\
\Gamma_5\Big(d_{\mathbb{B}}(\varrho,\mu)\Big) &\preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big)-\tau\preceq\Gamma_5\Big(\theta_{\mathbb{B}}\Big).
\end{aligned}$$

Since, Γ_5 is non decreasing. $\therefore$ $d_{\mathbb{B}}(\varrho,\mu)\preceq\theta_{\mathbb{B}}$. Hence, $\varrho=\mu$. □

Now, some results on common fixed point using *E.A.* property in C^*_{AV}-MS are presented.

Theorem 4.2.4. *Let* $(\mho,\mathbb{B},d_{\mathbb{B}})$ *be a* C^*_{AV}*-MS and* $\Gamma_1,\Gamma_2,\Gamma_3,\ \Gamma_4:\mho\to\mho$ *satisfying:*

(i) $\Gamma_1(\mho)\subseteq\Gamma_4(\mho)$ *and* $\Gamma_2(\mho)\subseteq\Gamma_3(\mho)$*;*

(ii) $\forall\ \varpi,\varsigma\in\mho,\ \alpha_1\in\mathbb{B}^+$ *with* $\|\alpha_1\|\le 1$

$$d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_2\varsigma)\preceq\alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma)+d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}{1+d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma)+d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}\right)\alpha_1;\tag{4.2.5}$$

(iii) the pairs (Γ_1,Γ_3) *and* (Γ_2,Γ_4) *are weakly compatible;*

(iv) either (Γ_1,Γ_3) *or* (Γ_2,Γ_4) *satisfies E.A. property.*

If the range of $\Gamma_3(\mho)$ *or* $\Gamma_4(\mho)$ *is a closed subspace of* $\mho$*. Then,* $\Gamma_1,\Gamma_2,\Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$*.*

Proof. Firstly, assume that (Γ_2,Γ_4) satisfies E.A. property. Then, $\exists$ a sequence $\{\varpi_\jmath\}$ in $\mho$ s.t $\lim_{\jmath\to\infty}\Gamma_2(\varpi_\jmath)=\lim_{\jmath\to\infty}\Gamma_4(\varpi_\jmath)=t$ for some $t\in\mho$. Further, $\Gamma_2(\mho)\subseteq\Gamma_3(\mho)$. $\therefore$ $\exists$ a sequence $\{\varsigma_\jmath\}$ in $\mho$ s.t $\Gamma_2(\varpi_\jmath)=\Gamma_3(\varsigma_\jmath)$. Hence, $\lim_{\jmath\to\infty}\Gamma_3(\varsigma_\jmath)=t$.
We claim that $\lim_{\jmath\to\infty}\Gamma_1(\varsigma_\jmath)=t$. Let if possible $\lim_{\jmath\to\infty}\Gamma_1(\varsigma_\jmath)=t_1\neq t$. Then, substituting $\varpi=\varsigma_\jmath$ and $\varsigma=\varpi_\jmath$ in (4.2.5), we have

$$d_{\mathbb{B}}(\Gamma_1\varsigma_\jmath,\Gamma_2\varpi_\jmath)\preceq\alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varsigma_\jmath,\Gamma_1\varsigma_\jmath)d_{\mathbb{B}}(\Gamma_3\varsigma_\jmath,\Gamma_2\varpi_\jmath)+d_{\mathbb{B}}(\Gamma_4\varpi_\jmath,\Gamma_2\varpi_\jmath)d_{\mathbb{B}}(\Gamma_4\varpi_\jmath,\Gamma_1\varsigma_\jmath)}{1+d_{\mathbb{B}}(\Gamma_3\varsigma_\jmath,\Gamma_2\varpi_\jmath)+d_{\mathbb{B}}(\Gamma_4\varpi_\jmath,\Gamma_1\varsigma_\jmath)}\right)\alpha_1.$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(t_1, t) &\preceq \alpha_1^* \bigg(\frac{d_{\mathbb{B}}(t, t_1) d_{\mathbb{B}}(t, t) + d_{\mathbb{B}}(t, t) d_{\mathbb{B}}(t, t_1)}{1 + d_{\mathbb{B}}(t, t) + d_{\mathbb{B}}(t, t_1)} \bigg) \alpha_1 \\ &= \theta_{\mathbb{B}}. \end{aligned}$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(t_1, t)\| \leq 0$$

implies $t_1 = t$. Hence, $\lim_{\jmath \to \infty} \Gamma_1(\varsigma_\jmath) = \lim_{\jmath \to \infty} \Gamma_2(\varpi_\jmath) = t$.
Now, suppose that $\Gamma_3(\mho)$ is closed subspace of $\mho$ and $\Gamma_3 u = t$ for some $u \in \mho$. Subsequently, we have

$$\lim_{\jmath \to \infty} \Gamma_1(\varsigma_\jmath) = \lim_{\jmath \to \infty} \Gamma_2(\varpi_\jmath) = \lim_{\jmath \to \infty} \Gamma_4(\varpi_\jmath) = \lim_{\jmath \to \infty} \Gamma_3(\varsigma_\jmath) = t = \Gamma_3 u.$$

We claim that $\Gamma_1 u = \Gamma_3 u$. If not, then substituting $\varpi = u$ and $\varsigma = \varpi_\jmath$ in (4.2.5), we have

$$\begin{aligned} &d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 \varpi_\jmath) \preceq \\ &\alpha_1^* \bigg(\frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u) d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 \varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4 \varpi_\jmath, \Gamma_2 \varpi_\jmath) d_{\mathbb{B}}(\Gamma_4 \varpi_\jmath, \Gamma_1 u)}{1 + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 \varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4 \varpi_\jmath, \Gamma_1 u)} \bigg) \alpha_1. \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 u, t) &\preceq \alpha_1^* \bigg(\frac{d_{\mathbb{B}}(t, \Gamma_1 u) d_{\mathbb{B}}(t, t) + d_{\mathbb{B}}(t, t) d_{\mathbb{B}}(t, \Gamma_1 u)}{1 + d_{\mathbb{B}}(t, t) + d_{\mathbb{B}}(t, \Gamma_1 u)} \bigg) \alpha_1 \\ &= \theta_{\mathbb{B}}. \end{aligned}$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\Gamma_1 u, t)\| \leq 0$$

implies $\Gamma_1 u = t$. Hence, $\Gamma_1 u = t = \Gamma_3 u$.
Now, the weak compatibility of (Γ_1, Γ_3) implies $\Gamma_1 \Gamma_3 u = \Gamma_3 \Gamma_1 u$ or $\Gamma_1 t = \Gamma_3 t$.
Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$, $\therefore$ $\exists\, v \in \mho$ s.t $\Gamma_1 u = \Gamma_4 v = \Gamma_3 u = t$.
Next, to prove $\Gamma_2 v = \Gamma_4 v = t$. Substitute $\varpi = u$ and $\varsigma = v$ in (4.2.5), we have

$$d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 v) \preceq \alpha_1^* \bigg(\frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u) d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 v) + d_{\mathbb{B}}(\Gamma_4 v, \Gamma_2 v) d_{\mathbb{B}}(\Gamma_4 v, \Gamma_1 u)}{1 + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 v) + d_{\mathbb{B}}(\Gamma_4 v, \Gamma_1 u)} \bigg) \alpha_1.$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(t, \Gamma_2 v)\| \leq \|\alpha_1\|^2 \bigg\| \frac{d_{\mathbb{B}}(t, t) d_{\mathbb{B}}(t, \Gamma_2 v) + d_{\mathbb{B}}(t, \Gamma_2 v) d_{\mathbb{B}}(t, t)}{1 + d_{\mathbb{B}}(t, \Gamma_2 v) + d_{\mathbb{B}}(t, t)} \bigg\|.$$

Then, $\|d_{\mathbb{B}}(\Gamma_2 v, t)\| \leq 0$, hence $\Gamma_2 v = t$. Thus, $\Gamma_2 v = \Gamma_4 v = t$.
Further, the weak compatibility of pair (Γ_2, Γ_4) implies that $\Gamma_2\Gamma_4 v = \Gamma_4\Gamma_2 v$, or $\Gamma_2 t = \Gamma_4 t$. $\therefore$ t is a common coincidence point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.
Now, to prove t is a common fixed point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4. Substitute $\varpi = u$ and $\varsigma = t$ in (4.2.5), we have

$$d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 t) \preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u) d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_4 t, \Gamma_2 t) d_{\mathbb{B}}(\Gamma_4 t, \Gamma_1 u)}{1 + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_4 t, \Gamma_1 u)} \right) \alpha_1.$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\Gamma_2 t, t)\| \leq \|\alpha_1\|^2 \left\| \frac{(d_{\mathbb{B}}(t,t) d_{\mathbb{B}}(t, \Gamma_2 t) + d_{\mathbb{B}}(t, \Gamma_2 t) d_{\mathbb{B}}(t,t))}{(1 + d_{\mathbb{B}}(t, \Gamma_2 t) + d_{\mathbb{B}}(t,t))} \right\|.$$

Thus, $\|d_{\mathbb{B}}(\Gamma_2 t, t)\| \leq 0$ implies $\Gamma_2 t = t$. Hence, $\Gamma_1 t = \Gamma_2 t = \Gamma_3 t = \Gamma_4 t = t$.
Uniqueness : Let w is another common fixed point of Γ_1,Γ_2,Γ_3 and Γ_4. Then, substituting $\varpi = w$ and $\varsigma = t$ in (4.2.5), we have

$$\begin{aligned} d_{\mathbb{B}}(w, t) &= d_{\mathbb{B}}(\Gamma_1 w, \Gamma_2 t) \\ &\preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_3 w, \Gamma_1 w) d_{\mathbb{B}}(\Gamma_3 w, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_4 t, \Gamma_2 t) d_{\mathbb{B}}(\Gamma_4 t, \Gamma_1 w)}{1 + d_{\mathbb{B}}(\Gamma_3 w, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_4 t, \Gamma_1 w)} \right) \alpha_1, \\ &\preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(w,w) d_{\mathbb{B}}(w,t) + d_{\mathbb{B}}(t,t) d_{\mathbb{B}}(t,w)}{1 + d_{\mathbb{B}}(w,t) + d_{\mathbb{B}}(t,w)} \right) \alpha_1. \end{aligned}$$

Taking norm on both side, we have $\|d_{\mathbb{B}}(w, t)\| \leq 0$, hence $w = t$. □

Theorem 4.2.5. *Let* $(\mho, \mathbb{B}, d_{\mathbb{B}})$ *be a* C^*_{AV}*-MS and* $\Gamma_1, \Gamma_2, \Gamma_3,\ \Gamma_4 : \mho \to \mho$ *satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(ii) $\forall\ \varpi, \varsigma \in \mho,\ \alpha_1 \in \mathbb{B}^+$ *with* $\|\alpha_1\| \leq 1$*,*

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma) \preceq\ & \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varsigma), \frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{2} \\ & d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi), \frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varsigma) + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)}{2} \Bigg) \alpha_1; \end{aligned}$$

(iii) the pairs (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(iv) either (Γ_1, Γ_3) *or* (Γ_2, Γ_4) *satisfies E.A. property.*

If the range of $\Gamma_3(\mho)$ *or* $\Gamma_4(\mho)$ *is a closed subspace of* $\mho$*. Then,* $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$*.*

Proof. Firstly, assume that (Γ_2, Γ_4) satisfies $E.A.$ property. $\therefore$ $\exists$ a sequence $\{\varsigma_\jmath\}$ in $\mho$ s.t $\lim_{\jmath\to\infty} \Gamma_2(\varsigma_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\varsigma_\jmath) = \varkappa$ for some $\varkappa \in \mho$.
Further, $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$. $\therefore$ $\exists$ a sequence $\{l_\jmath\}$ in $\mho$ s.t $\Gamma_2(\varsigma_\jmath) = \Gamma_3(l_\jmath)$. Hence, $\lim_{\jmath\to\infty} \Gamma_3(l_\jmath) = \varkappa$.
We claim that $\lim_{\jmath\to\infty} \Gamma_1(l_\jmath) = \varkappa$. If possible, $\lim_{\jmath\to\infty} \Gamma_1(l_\jmath) = k_1 \neq \varkappa$. Then, substituting $\varpi = l_\jmath$ and $\varsigma = \varsigma_\jmath$ in (4.2.6), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 l_\jmath, \Gamma_2\varsigma_\jmath) \;&\preceq\; \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3 l_\jmath, \Gamma_4\varsigma_\jmath), \frac{d_{\mathbb{B}}(\Gamma_3 l_\jmath, \Gamma_4\varsigma_\jmath) + d_{\mathbb{B}}(\Gamma_3 l_\jmath, \Gamma_1 l_\jmath)}{2}, \\ &\qquad d_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_1 l_\jmath), \frac{d_{\mathbb{B}}(\Gamma_3 l_\jmath, \Gamma_4\varsigma_\jmath) + d_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_1 l_\jmath)}{2}\Bigg)\alpha_1. \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(k_1, \varkappa) \;&\preceq\; \alpha_1 \max\Bigg(d_{\mathbb{B}}(\varkappa, \varkappa), d_{\mathbb{B}}(\varkappa, k_1), \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, k_1)}{2}, \\ &\qquad \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, k_1)}{2}\Bigg)\alpha_1^* \\ &\preceq\; \alpha_1 d_{\mathbb{B}}(\varkappa, k_1)\alpha_1^*. \end{aligned}$$

Taking norm on both side, we have

$$(1 - \|\alpha_1\|^2)\|d_{\mathbb{B}}(\varkappa, k_1)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varkappa, k_1)\| = 0$, hence, $k_1 = \varkappa$ i.e, $\lim_{\jmath\to\infty} \Gamma_1(l_\jmath) = \lim_{\jmath\to\infty} \Gamma_2(\varsigma_\jmath) = \varkappa$.
Now, assume that $\Gamma_3(\mho)$ is a closed subspace of $\mho$ and $\Gamma_3 u = \varkappa$ for some $u \in \mho$. Subsequently, we have

$$\lim_{\jmath\to\infty} \Gamma_1(l_\jmath) = \lim_{\jmath\to\infty} \Gamma_2(\varsigma_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\varsigma_\jmath) = \lim_{\jmath\to\infty} \Gamma_3(l_\jmath) = \varkappa = \Gamma_3 u.$$

We claim that $\Gamma_1 u = \Gamma_3 u$. If $\Gamma_1 u \neq \Gamma_3 u$, then substituting $\varpi = u$ and $\varsigma = \varsigma_\jmath$ in (4.2.6), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2\varsigma_\jmath) \;&\preceq\; \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4\varsigma_\jmath), d_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_1 u), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4\varsigma_\jmath) + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u)}{2}, \\ &\qquad \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 l_\jmath) + d_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_1 u)}{2}\Bigg)\alpha_1. \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 u, \varkappa) &\preceq \alpha_1 \max\Bigg(d_{\mathbb{B}}(\varkappa, \varkappa), d_{\mathbb{B}}(\varkappa, \Gamma_1 u), \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \Gamma_1 u)}{2}, \\ &\quad \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \Gamma_1 u)}{2} \Bigg) \alpha_1^* \\ d_{\mathbb{B}}(\Gamma_1 u, \varkappa) &\preceq \alpha_1 d_{\mathbb{B}}(\Gamma_1 u, \varkappa) \alpha_1^*. \end{aligned}$$

Taking norm on both side, we have

$$(1 - \|\alpha_1\|^2) \|d_{\mathbb{B}}(\Gamma_1 u, \varkappa)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\Gamma_1 u, \varkappa)\| \leq 0$. Hence, $\Gamma_1 u = \varkappa = \Gamma_3 u$.
Now, the weak compatibility of (Γ_1, Γ_3) implies $\Gamma_1 \Gamma_3 u = \Gamma_3 \Gamma_1 u$ or $\Gamma_1 \varkappa = \Gamma_3 \varkappa$. Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$. $\therefore$ $\exists\, \mu \in \mho$ s.t $\Gamma_1 u = \Gamma_4 \mu = \Gamma_3 u = \varkappa$.
Now, to prove $\Gamma_2 \mu = \Gamma_4 \mu = \varkappa$, substituting $\varpi = u$ and $\varsigma = \mu$ in (4.2.6), we get

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 \mu) &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 \mu), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 \mu) + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u)}{2}, \\ &\quad d_{\mathbb{B}}(\Gamma_4 \mu, \Gamma_1 u), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 \mu) + d_{\mathbb{B}}(\Gamma_4 \mu, \Gamma_1 u)}{2} \Bigg) \alpha_1. \end{aligned}$$

Taking norm on both side, we have

$$\begin{aligned} \|d_{\mathbb{B}}(\varkappa, \Gamma_2 \mu)\| &\leq \|\alpha_1\|^2 \Bigg\| \max\Bigg(d_{\mathbb{B}}(\varkappa, \varkappa), d_{\mathbb{B}}(\varkappa, \varkappa), \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)}{2}, \\ &\quad \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)}{2} \Bigg) \Bigg\|. \end{aligned}$$

Thus, $\|d_{\mathbb{B}}(\Gamma_2 \mu, \varkappa)\| \leq 0$ i.e, $\Gamma_2 \mu = \varkappa$. Hence, $\Gamma_2 \mu = \Gamma_4 \mu = \varkappa$.
Further, the weak compatibility of (Γ_2, Γ_4) implies $\Gamma_2 \Gamma_4 \mu = \Gamma_4 \Gamma_2 \mu$, or $\Gamma_2 \varkappa = \Gamma_4 \varkappa$ $\therefore$ $\varkappa$ is a common coincidence point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.
Now, to prove $\varkappa$ is a common fixed point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4. Substituting $\varpi = u$ and $\varsigma = \varkappa$ in (4.2.6), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 \varkappa) &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 \varkappa), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 \varkappa) + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u)}{2}, \\ &\quad d_{\mathbb{B}}(\Gamma_4 \varkappa, \Gamma_1 u), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 \varkappa) + d_{\mathbb{B}}(\Gamma_4 \varkappa, \Gamma_1 u)}{2} \Bigg) \alpha_1. \end{aligned}$$

Taking norm on both side, we have

$$\begin{aligned}\|d_{\mathbb{B}}(\Gamma_2\varkappa,\varkappa)\| &\leq \|\alpha_1\|^2\Bigg\|\max\Bigg(d_{\mathbb{B}}(\varkappa,\Gamma_2\varkappa),\frac{d_{\mathbb{B}}(\varkappa,\Gamma_2\varkappa)+d_{\mathbb{B}}(\varkappa,\varkappa)}{2},\\ &\quad d_{\mathbb{B}}(\Gamma_2\varkappa,\varkappa),\frac{d_{\mathbb{B}}(\varkappa,\Gamma_2\varkappa)+d_{\mathbb{B}}(\varkappa,\Gamma_2\varkappa)}{2}\Bigg)\Bigg\|\\ &= \|\alpha_1\|^2 d_{\mathbb{B}}(\Gamma_2\varkappa,\varkappa)\\ (1-\|\alpha_1\|^2)d_{\mathbb{B}}(\Gamma_2\varkappa,\varkappa) &\leq 0.\end{aligned}$$

Thus, $\|d_{\mathbb{B}}(\Gamma_2\varkappa,\varkappa)\|=0$, i.e, $\Gamma_2\varkappa=\varkappa$. Hence, $\Gamma_1\varkappa=\Gamma_2\varkappa=\Gamma_3\varkappa=\Gamma_4\varkappa=\varkappa$.
Uniqueness : Let ϑ as another common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Then, substituting $\varpi=\vartheta$ and $\varsigma=\varkappa$ in (4.2.6), we have

$$\begin{aligned}d_{\mathbb{B}}(\vartheta,\varkappa)=d_{\mathbb{B}}(\Gamma_1\vartheta,\Gamma_2\varkappa) &\preceq \alpha_1^*\max\Bigg(d_{\mathbb{B}}(\Gamma_3\vartheta,\Gamma_4\varkappa),\frac{d_{\mathbb{B}}(\Gamma_3\vartheta,\Gamma_4\varkappa)+d_{\mathbb{B}}(\Gamma_4\varkappa,\Gamma_1\vartheta)}{2}\\ &\quad d_{\mathbb{B}}(\Gamma_4\varkappa,\Gamma_1\vartheta),\frac{d_{\mathbb{B}}(\Gamma_3\vartheta,\Gamma_4\varkappa)+d_{\mathbb{B}}(\Gamma_3\vartheta,\Gamma_1\vartheta)}{2}\Bigg)\alpha_1\\ &\preceq \alpha_1^*\max\Bigg(d_{\mathbb{B}}(\vartheta,\varkappa),\frac{d_{\mathbb{B}}(\vartheta,\varkappa)+d_{\mathbb{B}}(\vartheta,\vartheta)}{2},\\ &\quad d_{\mathbb{B}}(\varkappa,\vartheta),\frac{d_{\mathbb{B}}(\vartheta,\varkappa)+d_{\mathbb{B}}(\varkappa,\vartheta)}{2}\Bigg)\alpha_1\\ &\preceq \alpha_1^* d_{\mathbb{B}}(\vartheta,\varkappa)\alpha_1.\end{aligned}$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\vartheta,\varkappa)\|\leq\|\alpha\|^2\|d_{\mathbb{B}}(\vartheta,\varkappa)\|,$$

or

$$(1-\|\alpha\|^2)\|d_{\mathbb{B}}(\vartheta,\varkappa)\|\leq 0$$

implies $\|d_{\mathbb{B}}(\vartheta,\varkappa)\|\leq 0$. Hence, $\vartheta=\varkappa$. □

Theorem 4.2.6. *Let $(\mho,\mathbb{B},d_{\mathbb{B}})$ be a C^*_{AV}-MS and $\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4:\mho\to\mho$ and $\exists$ Γ_5 : $\mathbb{B}^+\to\mathbb{B}$ satisfying:*

(i) $\Gamma_1(\mho)\subseteq\Gamma_4(\mho)$ *and* $\Gamma_2(\mho)\subseteq\Gamma_3(\mho)$*;*

(ii) Γ_5 *is continuous and strictly nondecreasing on* $\mathbb{B}^+$*;*

(iii) $\forall\ \varpi, \varsigma \in \mho$, $\alpha_1 \in \mathbb{B}^+$ *with* $\|\alpha_1\| \leq 1$ *and* $0 < \tau < 1$,

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma)\Big) \preceq$$

$$\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}\alpha_1\right); \quad (4.2.6)$$

(iv) *the pairs* (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(v) *either* (Γ_1, Γ_3) *or* (Γ_2, Γ_4) *satisfies E.A. property.*

If the range of $\Gamma_3(\mho)$ *or* $\Gamma_4(\mho)$ *is a closed subspace of* $\mho$*. Then,* $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$*.*

Proof. Firstly, assume that (Γ_2, Γ_4) satisfies $E.A.$ property. $\therefore\ \exists$ a sequence $\{\varpi_\jmath\}$ in $\mho$ s.t

$$\lim_{\jmath\to\infty} \Gamma_2(\varpi_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\varpi_\jmath) = \varkappa \text{ for some } \varkappa \in \mho.$$

Further $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$. $\therefore\ \exists$ a sequence $\{\varsigma_\jmath\}$ in $\mho$ s.t $\Gamma_2(\varpi_\jmath) = \Gamma_3(\varsigma_\jmath)$.
Hence,

$$\lim_{\jmath\to\infty} \Gamma_3(\varsigma_\jmath) = \varkappa = \lim_{\jmath\to\infty} \Gamma_2(\varpi_\jmath).$$

We claim that $\lim_{\jmath\to\infty} \Gamma_1(\varsigma_\jmath) = \varkappa$. If $\lim_{\jmath\to\infty} \Gamma_1(\varpi_\jmath) = \varkappa_1 \neq \varkappa$, then substituting $\varpi = \varsigma_\jmath$ and $\varsigma = \varpi_\jmath$ in (4.2.6), we have

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\varsigma_\jmath, \Gamma_2\varpi_\jmath)\Big) \preceq$$
$$\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\varsigma_\jmath, \Gamma_1\varsigma_\jmath)d_{\mathbb{B}}(\Gamma_3\varsigma_\jmath, \Gamma_2\varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_2\varpi_\jmath)d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_1\varsigma_\jmath)}{1 + d_{\mathbb{B}}(\Gamma_3\varsigma_\jmath, \Gamma_2\varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_1\varsigma_\jmath)}\alpha_1\right).$$

Taking limit as $\jmath \to \infty$ in (4.2.7), we have

$$\begin{aligned} \tau + \Gamma_5\Big(d_{\mathbb{B}}(\varkappa_1, \varkappa)\Big) &\preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\varkappa, \varkappa_1)d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)d_{\mathbb{B}}(\varkappa, \varkappa_1)}{1 + d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa_1)}\alpha_1\right). \\ \Gamma_5\Big(d_{\mathbb{B}}(\varkappa_1, \varkappa)\Big) &\preceq \Gamma_5(\theta_{\mathbb{B}}) - \tau \preceq \Gamma_5(\theta_{\mathbb{B}}). \end{aligned}$$

Since Γ_5 is non decreasing. $\therefore$

$$d_{\mathbb{B}}(\varkappa_1, \varkappa) \preceq \theta_{\mathbb{B}}.$$

implies $\varkappa_1 = \varkappa$. Hence, $\lim_{\jmath\to\infty} \Gamma_1(\varsigma_\jmath) = \lim_{\jmath\to\infty} \Gamma_2(\varpi_\jmath) = \varkappa$.
Now, suppose $\Gamma_3(\mho)$ is a closed subspace of $\mho$ and $\Gamma_3\mu = \varkappa$ for some $\mu \in \mho$. Subsequently, we have

$$\lim_{\jmath\to\infty} \Gamma_1(\varsigma_\jmath) = \lim_{\jmath\to\infty} \Gamma_2(\varpi_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\varpi_\jmath) = \lim_{\jmath\to\infty} \Gamma_3(\varsigma_\jmath) = \varkappa = \Gamma_3\mu.$$

We claim that $\Gamma_1\mu = \Gamma_3\mu$. If $\Gamma_1\mu \neq \Gamma_3\mu$, then substituting $\varpi = \mu$ and $\varsigma = \varpi_\jmath$ in (4.2.6), we have

$$\begin{aligned}
&\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varpi_\jmath)\Big) \preceq \\
&\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_2\varpi_\jmath) d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_1\mu)}{1 + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_1\mu)} \alpha_1\right).
\end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned}
&\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \varkappa)\Big) \preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\varkappa, \Gamma_1\mu) d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa) d_{\mathbb{B}}(\varkappa, \Gamma_1\mu)}{1 + d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \Gamma_1\mu)} \alpha_1\right) \\
&\Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \varkappa)\Big) \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big).
\end{aligned}$$

Since Γ_5 is non decreasing. $\therefore$

$$d_{\mathbb{B}}(\Gamma_1\mu, \varkappa) \preceq \theta_{\mathbb{B}}$$

implies $\Gamma_1\mu = \varkappa$. Hence, $\Gamma_1\mu = \varkappa = \Gamma_3\mu$.
Now, the weak compatibility of (Γ_1, Γ_3) implies $\Gamma_1\Gamma_3\mu = \Gamma_3\Gamma_1\mu$ or $\Gamma_1\varkappa = \Gamma_3\varkappa$.
Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$. $\therefore \exists\, \nu \in \mho$ s.t $\Gamma_1\mu = \Gamma_4\nu = \Gamma_3\mu = \varkappa$.
Now, to prove $\Gamma_2\nu = \Gamma_4\nu = \varkappa$. Substituting $\varpi = \mu$ and $\varsigma = \nu$ in (4.2.6), we have

$$\begin{aligned}
&\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\nu)\Big) \preceq \\
&\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\nu) + d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_2\nu) d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\mu)}{1 + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\nu) + d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\mu)} \alpha_1\right) \\
&\tau + \Gamma_5\Big(d_{\mathbb{B}}(\varkappa, \Gamma_2\nu)\Big) \preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\varkappa, \varkappa) d_{\mathbb{B}}(\varkappa, \Gamma_2\nu) + d_{\mathbb{B}}(\varkappa, \Gamma_2\nu) d_{\mathbb{B}}(\varkappa, \varkappa)}{1 + d_{\mathbb{B}}(\varkappa, \Gamma_2\nu) + d_{\mathbb{B}}(\varkappa, \varkappa)} \alpha_1\right) \\
&\Gamma_5\Big(d_{\mathbb{B}}(\varkappa, \Gamma_2\nu)\Big) \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big).
\end{aligned}$$

Since Γ_5 is non decreasing. $\therefore$

$$d_{\mathbb{B}}(\Gamma_2\nu, \varkappa) \preceq \theta_{\mathbb{B}}$$

implies $\Gamma_2\nu = \varkappa$. Thus, $\Gamma_2\nu = \Gamma_4\nu = \varkappa$.
Further, the weak compatibility of (Γ_2, Γ_4) implies $\Gamma_2\Gamma_4\nu = \Gamma_4\Gamma_2\nu$, or $\Gamma_2\varkappa = \Gamma_4\varkappa$.

$\therefore\ \varkappa$ is a common coincidence point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.
To prove $\varkappa$ is a common fixed point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4, substituting $\varpi = \mu$ and $\varsigma = \varkappa$ in (4.2.6), we have

$$\begin{aligned}\tau &+ \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varkappa)\Big)\\ &\preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu)d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_2\varkappa)d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\mu)}{1 + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\mu)}\alpha_1\right),\end{aligned}$$

or

$$\begin{aligned}\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_2\varkappa, \varkappa)\Big) &\preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\varkappa, \varkappa)d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa) + d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)d_{\mathbb{B}}(\varkappa, \varkappa)}{1 + d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)}\alpha_1\right).\\ \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_2\varkappa, \varkappa)\Big) &\preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big).\end{aligned}$$

Since, Γ_5 is non decreasing. $\therefore$

$$d_{\mathbb{B}}(\Gamma_2\varkappa, \varkappa) \preceq \theta_{\mathbb{B}}$$

implies $\Gamma_2\varkappa = \varkappa$.
Hence, $\Gamma_1\varkappa = \Gamma_2\varkappa = \Gamma_3\varkappa = \Gamma_4\varkappa = \varkappa$.
Uniqueness : Let ϱ be another common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Substituting $\varpi = \varrho$ and $\varsigma = \varkappa$ in (4.2.6), we have

$$\begin{aligned}\tau &+ \Gamma_5\Big(d_{\mathbb{B}}(\varrho, \varkappa)\Big) = \tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\varrho, \Gamma_2\varkappa)\Big)\\ &\preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\varrho, \Gamma_1\varrho)d_{\mathbb{B}}(\Gamma_3\varrho, \Gamma_2\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_2\varkappa)d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\varrho)}{1 + d_{\mathbb{B}}(\Gamma_3\varrho, \Gamma_2\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\varrho)}\alpha_1\right)\\ &= \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\varrho, \varrho)d_{\mathbb{B}}(\varrho, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)d_{\mathbb{B}}(\varkappa, \varrho)}{1 + d_{\mathbb{B}}(\varrho, \varkappa) + d_{\mathbb{B}}(\varkappa, \varrho)}\alpha_1\right)\\ \Gamma_5\Big(d_{\mathbb{B}}(\varrho, \varkappa)\Big) &\preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big).\end{aligned}$$

Since Γ_5 is non decreasing. $\therefore$

$$d_{\mathbb{B}}(\varrho, \varkappa) \preceq \theta_{\mathbb{B}}.$$

Hence, $\varrho = \varkappa$. □

Now, some results on common fixed points for the self mappings using (CLR) property in C^*_{AV}-MS are presented.

Theorem 4.2.7. *Let* $(\mho, \mathbb{B}, d_{\mathbb{B}})$ *be* C^*_{AV}*-MS and* $\Gamma_1, \Gamma_2, \Gamma_3,\ \Gamma_4 : \mho \to \mho$ *satisfying:*

(*i*) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(*ii*) $\forall\ \varpi, \varsigma \in \mho$

$$d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma) \preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi) d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)} \right) \alpha_1; \tag{4.2.7}$$

(*iii*) *the pairs* (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(*iv*) *either* (Γ_1, Γ_3) *or* (Γ_2, Γ_4) *satisfies* (CLR) *property.*

Then, the mappings $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$*.*

Proof. Firstly, suppose (Γ_2, Γ_4) satisfies (CLR_{Γ_2}) property. $\therefore$ $\exists$ a sequence $\{\varpi_\jmath\}$ in $\mho$ s.t

$$\lim_{\jmath\to\infty} \Gamma_2(\varpi_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\varpi_\jmath) = \Gamma_2\varpi = t \text{ for some } \varpi \in \mho.$$

Since, $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$. $\therefore$ $\Gamma_2\varpi = \Gamma_3 u$, for some $u \in \mho$.
We claim that $\Gamma_1 u = \Gamma_3 u = t$. If not, then substituting $\varpi = u$ and $\varsigma = \varpi_\jmath$ in (4.2.7), we have

$$d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2\varpi_\jmath) \preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u) d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2\varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_2\varpi_\jmath) d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_1 u)}{1 + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2\varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_1 u)} \right) \alpha_1.$$

Taking limit as $\jmath \to \infty$, we have

$$d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2\varpi) \preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_1 u) d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_2\varpi) + d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_2\varpi) d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_1 u)}{1 + d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_2\varpi) + d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_1 u)} \right).$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2\varpi)\| \leq 0.$$

Hence, $\Gamma_1 u = \Gamma_3 u = \Gamma_2\varpi = t$.
Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$, $\exists\ v \in \mho$ s.t $\Gamma_4 v = \Gamma_1 u = \Gamma_3 u = t$.
Now, to prove $\Gamma_4 v = \Gamma_2 v = t$. Substituting $\varpi = u$ and $\varsigma = v$ in (4.2.7), we have

$$d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 v) \preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u) d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 v) + d_{\mathbb{B}}(\Gamma_4 v, \Gamma_2 v) d_{\mathbb{B}}(\Gamma_4 v, \Gamma_1 u)}{1 + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 v) + d_{\mathbb{B}}(\Gamma_4 v, \Gamma_1 u)} \right) \alpha_1.$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 v)\| \leq \|\alpha_1\|^2 \left\| \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u) d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 v) + d_{\mathbb{B}}(\Gamma_4 v, \Gamma_2 v) d_{\mathbb{B}}(\Gamma_4 v, \Gamma_1 u)}{1 + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 v) + d_{\mathbb{B}}(\Gamma_4 v, \Gamma_1 u)} \right\|.$$

Thus, $\|d_{\mathbb{B}}(t,\Gamma_2 v)\| \leq 0$, i.e, $\Gamma_2 v = t$. Hence, $\Gamma_2 v = \Gamma_4 v = t$.
Further, the weak compatibility of pair (Γ_2, Γ_4) implies $\Gamma_2\Gamma_4 v = \Gamma_4\Gamma_2 v$ or $\Gamma_2 t = \Gamma_4 t$. $\therefore$ t is a common coincidence point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.
Now, to prove t is common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Substitute $\varpi = u$ and $\varsigma = t$ in (4.2.7), we have

$$d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 t) \preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u) d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_4 t, \Gamma_2 t) d_{\mathbb{B}}(\Gamma_4 t, \Gamma_1 u)}{1 + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_4 t, \Gamma_1 u)} \right) \alpha_1.$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(t, \Gamma_2 t)\| \leq \|\alpha_1\|^2 \left\| \frac{d_{\mathbb{B}}(t,t) d_{\mathbb{B}}(t, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_2 t, \Gamma_2 t) d_{\mathbb{B}}(t,t)}{1 + d_{\mathbb{B}}(t, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_2 t, t)} \right\|.$$

Thus, $\|d_{\mathbb{B}}(t, \Gamma_2 t)\| \leq 0$ implies $\Gamma_2 t = t$. Hence, $\Gamma_1 t = \Gamma_2 t = \Gamma_3 t = \Gamma_4 t = t$.
Uniqueness: Let w is another common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Then, substituting $\varpi = w$ and $\varsigma = t$ in (4.2.7), we have

$$\begin{aligned} d_{\mathbb{B}}(w,t) &= d_{\mathbb{B}}(\Gamma_1 w, \Gamma_2 t) \\ &\preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_3 w, \Gamma_1 w) d_{\mathbb{B}}(\Gamma_3 w, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_4 t, \Gamma_2 t) d_{\mathbb{B}}(\Gamma_4 t, \Gamma_1 w)}{1 + d_{\mathbb{B}}(\Gamma_3 w, \Gamma_2 t) + d_{\mathbb{B}}(\Gamma_4 t, \Gamma_1 w)} \right) \alpha_1, \\ &\preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(w,w) d_{\mathbb{B}}(w,t) + d_{\mathbb{B}}(t,t) d_{\mathbb{B}}(t,w)}{1 + d_{\mathbb{B}}(w,t) + d_{\mathbb{B}}(t,w)} \right) \alpha_1. \end{aligned}$$

Taking norm on both side, we have $\|d_{\mathbb{B}}(w,t)\| \leq 0$, i.e, $w = t$. □

Theorem 4.2.8. *Let $(\mho, \mathbb{B}, d_{\mathbb{B}})$ be a C^*_{AV}-MS and $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 : \mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$;

(ii) $\forall\ \varpi, \varsigma \in \mho$

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 \varpi, \Gamma_2 \varsigma) \preceq \alpha_1^* \max\Bigg(& d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_4 \varsigma), \frac{d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_4 \varsigma) + d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_1 \varpi)}{2}, \\ & d_{\mathbb{B}}(\Gamma_4 \varsigma, \Gamma_1 \varpi), \frac{d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_4 \varsigma) + d_{\mathbb{B}}(\Gamma_4 \varsigma, \Gamma_1 \varpi)}{2} \Bigg) \alpha_1 \end{aligned} \quad (4.2.8)$$

(iii) the pairs (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(iv) either (Γ_1, Γ_3) *or* (Γ_2, Γ_4) *satisfies CLR property.*

Then, the mappings $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$.

Proof. Firstly, suppose (Γ_2, Γ_4) satisfies CLR_{Γ_2} property. $\therefore$ $\exists$ a sequence $\{\varsigma_\jmath\}$ in $\mho$ s.t

$$\lim_{\jmath\to\infty} \Gamma_2(\varsigma_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\varsigma_\jmath) = \Gamma_2 p = \varkappa \text{ for some } p \in \mho.$$

Since, $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$, we have $\Gamma_2 p = \Gamma_3 u$, for some $u \in \mho$.
We claim that $\Gamma_1 u = \Gamma_3 u = \varkappa$. If $\Gamma_1 u \neq \Gamma_3 u$, then substituting $\varpi = u$ and $\varsigma = \varsigma_\jmath$ in (4.2.8), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2\varsigma_\jmath) \ \preceq\ & \alpha_1^* \max\bigg(d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4\varsigma_\jmath), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4\varsigma_\jmath) + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u)}{2}, \\ & d_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_1 u), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4\varsigma_\jmath) + d_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_1 u)}{2}\bigg)\alpha_1. \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 u, \varkappa) \ \preceq\ & \alpha_1^* \max\bigg(d_{\mathbb{B}}(\varkappa, \varkappa), \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \Gamma_1 u)}{2}, \\ & d_{\mathbb{B}}(\varkappa, \Gamma_1 u), \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \Gamma_1 u)}{2}\bigg)\alpha_1. \end{aligned}$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\Gamma_1 u, \varkappa)\| \leq \|\alpha_1\|^2 \|d_{\mathbb{B}}(\varkappa, \Gamma_1 u)\|,$$

or

$$(1 - \|\alpha_1\|^2)\|d_{\mathbb{B}}(\varkappa, \Gamma_1 u)\| \leq 0,$$

implies $\|d_{\mathbb{B}}(\varkappa, \Gamma_1 u)\| = 0$. Hence, $\Gamma_1 u = \Gamma_3 u = \Gamma_2 p = \varkappa$.
Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ $\therefore$ $\exists\, v \in \mho$ s.t $\Gamma_4 v = \Gamma_1 u = \Gamma_3 u = \varkappa$.
Now, to prove $\Gamma_4 v = \Gamma_2 v = \varkappa$. Substituting $\varpi = u$ and $\varsigma = v$ in (4.2.8), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 v) \ \preceq\ & \alpha_1^* \max\bigg(d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 v), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 v) + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u)}{2}, \\ & d_{\mathbb{B}}(\Gamma_4 v, \Gamma_1 u), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4 v) + d_{\mathbb{B}}(\Gamma_4 v, \Gamma_1 u)}{2}\bigg)\alpha_1. \end{aligned}$$

Taking norm on both side, we have

$$\begin{aligned} \|d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2 v)\| \ \leq\ & \|\alpha_1\|^2 \bigg\| \max\bigg(d_{\mathbb{B}}(\varkappa, \varkappa), d_{\mathbb{B}}(\varkappa, \varkappa), \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)}{2}, \\ & \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)}{2}\bigg)\bigg\|. \end{aligned}$$

Thus, $\|d_{\mathbb{B}}(\varkappa, \Gamma_2 v)\| \leq 0$ implies $\Gamma_2 v = \varkappa$. Hence, $\Gamma_2 v = \Gamma_4 v = \varkappa$.
Further, the weak compatibility of (Γ_2, Γ_4) implies $\Gamma_2\Gamma_4 v = \Gamma_4\Gamma_2 v$ or $\Gamma_2\varkappa = \Gamma_4\varkappa$ $\therefore$ $\varkappa$ is a common coincidence point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.
Now, to prove $\varkappa$ is common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Substituting $\varpi = u$ and $\varsigma = \varkappa$ in (4.2.8), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1 u, \Gamma_2\varkappa) &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4\varkappa), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4\varkappa) + d_{\mathbb{B}}(\Gamma_3 u, \Gamma_1 u)}{2}, \\ &\qquad d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1 u), \frac{d_{\mathbb{B}}(\Gamma_3 u, \Gamma_4\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1 u)}{2}\Bigg)\alpha_1. \end{aligned}$$

Taking norm on both side, we have

$$\begin{aligned} \|d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)\| &\leq \|\alpha_1\|^2 \Bigg\|\max\Bigg(d_{\mathbb{B}}(\varkappa, \varkappa), d_{\mathbb{B}}(\varkappa, \varkappa), \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)}{2}, \\ &\qquad \frac{d_{\mathbb{B}}(\varkappa, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)}{2}\Bigg)\Bigg\|. \end{aligned}$$

Thus, $\|d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)\| \leq 0$ implies $\Gamma_2\varkappa = \varkappa$.
Hence, $\Gamma_1\varkappa = \Gamma_2\varkappa = \Gamma_3\varkappa = \Gamma_4\varkappa = \varkappa$.
Uniqueness: Let ω as another common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Then, substituting $\varpi = \omega$ and $\varsigma = \varkappa$ in (4.2.8), we have

$$\begin{aligned} d_{\mathbb{B}}(\omega, \varkappa) &= d_{\mathbb{B}}(\Gamma_1\omega, \Gamma_2\varkappa) \\ &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\Gamma_3\omega, \Gamma_4\varkappa), d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\omega), \frac{d_{\mathbb{B}}(\Gamma_3\omega, \Gamma_4\varkappa) + d_{\mathbb{B}}(\Gamma_3\omega, \Gamma_1\omega)}{2}, \\ &\quad \frac{d_{\mathbb{B}}(\Gamma_3\omega, \Gamma_4\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\omega)}{2}\Bigg)\alpha_1 \\ &\preceq \alpha_1^* \max\Bigg(d_{\mathbb{B}}(\omega, \varkappa), d_{\mathbb{B}}(\varkappa, \omega), \frac{d_{\mathbb{B}}(\omega, \varkappa) + d_{\mathbb{B}}(\omega, \omega)}{2}, \frac{d_{\mathbb{B}}(\omega, \varkappa) + d_{\mathbb{B}}(\varkappa, \omega)}{2}\Bigg)\alpha_1 \\ &\preceq \alpha_1^* d_{\mathbb{B}}(\omega, \varkappa)\alpha_1. \end{aligned}$$

Taking norm on both side, we have

$$\begin{aligned} \|d_{\mathbb{B}}(\omega, \varkappa)\| &\leq \|\alpha\|^2 \|d_{\mathbb{B}}(\omega, \varkappa)\| \\ (1 - \|\alpha\|^2)\|d_{\mathbb{B}}(\omega, \varkappa)\| &\leq 0 \end{aligned}$$

implies $\|d_{\mathbb{B}}(\omega, \varkappa)\| \leq 0$. Hence, $\varkappa = \omega$. □

Theorem 4.2.9. *Let $(\mho, \mathbb{B}, d_{\mathbb{B}})$ be a C^*_{AV}-MS and $\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4 : \mho \to \mho$ and $\exists$ Γ_5 : $\mathbb{B}^+ \to \mathbb{B}$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(ii) Γ_5 *is continuous and strictly nondecreasing on* $\mathbb{B}^+$*;*

(iii) $\forall\ \varpi, \varsigma \in \mho$, $\alpha_1 \in \mathbb{B}^+$ *with* $\|\alpha_1\| \leq 1$ *and* $\tau > 0$,

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma)\Big) \preceq$$

$$\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}\alpha_1\right); \quad (4.2.9)$$

(iv) the pairs (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(v) either (Γ_1, Γ_3) *or* (Γ_2, Γ_4) *satisfies CLR property.*

Then, $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$.

Proof. Firstly, suppose (Γ_2, Γ_4) satisfies CLR_{Γ_2} property. $\therefore$ $\exists$ a sequence $\{\varpi_\jmath\}$ in $\mho$ s.t

$$\lim_{\jmath\to\infty} \Gamma_2(\varpi_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\varpi_\jmath) = \Gamma_2\varpi = \varkappa \text{ for some } \varpi \in \mho.$$

Since, $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$ $\therefore$ $\exists\ \mu \in \mho$ s.t $\Gamma_2\varpi = \Gamma_3\mu$.
We claim that $\Gamma_1\mu = \Gamma_3\mu = \varkappa$. If $\Gamma_1\mu = \Gamma_3\mu$ then, substituting $\varpi = \mu$ and $\varsigma = \varpi_\jmath$ in (4.2.9), we have

$$\begin{aligned}
&\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varpi_\jmath)\Big) \preceq \\
&\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu)d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_2\varpi_\jmath)d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_1\mu)}{1 + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\varpi_\jmath) + d_{\mathbb{B}}(\Gamma_4\varpi_\jmath, \Gamma_1\mu)}\alpha_1\right) \\
&\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varpi)\Big) \preceq \\
&\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_1\mu)d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_2\varpi) + d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_2\varpi)d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_1\mu)}{1 + d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_2\varpi) + d_{\mathbb{B}}(\Gamma_2\varpi, \Gamma_1\mu)}\alpha_1\right) \\
&\Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varpi)\Big) \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big).
\end{aligned}$$

Since, Γ_5 is non decreasing. $\therefore$ $d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varpi) \preceq \theta_{\mathbb{B}}$.
Hence, $\Gamma_1\mu = \Gamma_3\mu = \Gamma_2\varpi = \varkappa$.

Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$. $\therefore\ \exists\ \nu \in \mho$ s.t $\Gamma_4\nu = \Gamma_1\mu = \Gamma_3\mu = \varkappa$.
To prove $\Gamma_4\nu = \Gamma_2\nu = \varkappa$, then substituting $\varpi = \mu$ and $\varsigma = \nu$ in (4.2.9), we have

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\nu)\Big) \preceq$$

$$\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu)d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\nu) + d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_2\nu)d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\mu)}{1 + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\nu) + d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\mu)}\alpha_1\right),$$

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\varkappa, \Gamma_2\nu)\Big) \preceq$$

$$\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu)d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\nu) + d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_2\nu)d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\mu)}{1 + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\nu) + d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_1\mu)}\alpha_1\right).$$

$$\Gamma_5\Big(d_{\mathbb{B}}(\varkappa, \Gamma_2\nu)\Big) \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big).$$

Since, Γ_5 is non decreasing. $\therefore\ d_{\mathbb{B}}(\varkappa, \Gamma_2\nu) \preceq \theta_{\mathbb{B}}$ implies $\Gamma_2\nu = \varkappa$. Hence, $\Gamma_2\nu = \Gamma_4\nu = \varkappa$. (Γ_2, Γ_4) is weak compatible. $\therefore \Gamma_2\Gamma_4\nu = \Gamma_4\Gamma_2\nu$, or $\Gamma_2\varkappa = \Gamma_4\varkappa$. Hence, $\varkappa$ is a common coincidence point of $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ and Γ_4.
Now, to prove $\varkappa$ is common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Substituting $\varpi = \mu$ and $\varsigma = \varkappa$ in (4.2.9), we have

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varkappa)\Big) \preceq$$

$$\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu)d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_2\varkappa)d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\mu)}{1 + d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_2\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\mu)}\alpha_1\right)$$

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)\Big) \preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\varkappa, \varkappa)d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa) + d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)d_{\mathbb{B}}(\varkappa, \varkappa)}{1 + d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)}\alpha_1\right)$$

$$\Gamma_5\Big(d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)\Big) \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big).$$

Since, Γ_5 is non decreasing. $\therefore\ d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa) \preceq \theta_{\mathbb{B}}$ implies $\Gamma_2\varkappa = \varkappa$.
Hence, $\Gamma_1\varkappa = \Gamma_2\varkappa = \Gamma_3\varkappa = \Gamma_4\varkappa = \varkappa$.
Uniqueness : Let ϱ is another common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Substituting $\varpi = \varrho$ and $\varsigma = \varkappa$ in (4.2.9), we have

$$\tau + \Gamma_5\Big(d_{\mathbb{B}}(\varrho, \varkappa)\Big) = \tau + \Gamma_5(d_{\mathbb{B}}(\Gamma_1\varrho, \Gamma_2\varkappa))$$

$$\preceq \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\varrho, \Gamma_1\varrho)d_{\mathbb{B}}(\Gamma_3\varrho, \Gamma_2\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_2\varkappa)d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\varrho)}{1 + d_{\mathbb{B}}(\Gamma_3\varrho, \Gamma_2\varkappa) + d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_1\varrho)}\alpha_1\right)$$

$$= \Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\varrho, \varrho)d_{\mathbb{B}}(\varrho, \varkappa) + d_{\mathbb{B}}(\varkappa, \varkappa)d_{\mathbb{B}}(\varkappa, \varrho)}{1 + d_{\mathbb{B}}(\varrho, \varkappa) + d_{\mathbb{B}}(\varkappa, \varrho)}\alpha_1\right)$$

$$\Gamma_5\Big(d_{\mathbb{B}}(\varrho, \varkappa)\Big) \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big) - \tau \preceq \Gamma_5\Big(\theta_{\mathbb{B}}\Big).$$

Since, Γ_5 is non decreasing. $\therefore\ d_{\mathbb{B}}(\varrho, \varkappa) \preceq \theta_{\mathbb{B}}$. Hence, $\varrho = \varkappa$. □

Example 4.2.10. Let $\mho = [0,2]$ and $\mathbb{B} = \mathbb{C}$. Define $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ as

$$d_{\mathbb{B}}(\varpi, \varsigma) = \begin{cases} |\varpi| + |\varsigma| & \text{if } \varpi \neq \varsigma \\ 0 & \text{if } \varpi = \varsigma. \end{cases}$$

Then, $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is a C^*_{AV}-MS and let $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 on $\mho$ s.t

$$\Gamma_1(\varpi) = \begin{cases} \varpi & \text{if } \varpi \in [0,1] \\ 2 & \text{if } \varpi \in (1,2] \end{cases}, \Gamma_2(\varpi) = \begin{cases} \frac{\varpi}{2} & \text{if } \varpi \in [0,1] \\ 1 & \text{if } \varpi \in (1,2] \end{cases},$$

$$\Gamma_3(\varpi) = \begin{cases} 2\varpi & \text{if } \varpi \in [0,1] \\ 3 & \text{if } \varpi \in (1,2] \end{cases} \ and\ \Gamma_4(\varpi) = \begin{cases} 4\varpi & \text{if } \varpi \in [0,1] \\ 5 & \text{if } \varpi \in (1,2] \end{cases}$$

Following cases arises :
Case (i): Let $\varpi, \varsigma \in [0,1]$, clearly $\Gamma_1(\mho) \subset \Gamma_4(\mho)$ and $\Gamma_2(\mho) \subset \Gamma_3(\mho)$.
Now,

$$d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma) = \varpi + \frac{\varsigma}{2}, \quad d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi) = 3\varpi,\ d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) = 2\varpi + \frac{\varsigma}{2},$$
$$d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) = \frac{9\varsigma}{2} \text{ and } \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi) = \varpi + 4\varsigma.$$

Consider,

$$\begin{aligned}
& \alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}\right)\alpha_1 \\
= \ & \alpha_1^*\left(\frac{3\varpi(2\varpi + \frac{\varsigma}{2}) + \frac{9\varsigma}{2}(\varpi + 4\varsigma)}{1 + 3\varpi + \frac{9\varsigma}{2}}\right)\alpha_1 \\
= \ & ||\alpha_1||^2\left(\frac{12\varpi^2 + 12\varpi\varsigma + 36\varsigma^2}{6\varpi + 9\varsigma + 2}\right) \\
\geq \ & ||\alpha_1||^2\left(\frac{12\varpi^2 + 12\varpi\varsigma + 36\varsigma^2}{6\varpi + 9\varsigma + 3}\right) \\
= \ & ||\alpha_1||^2\left(\frac{4\varpi^2 + 4\varpi\varsigma + 12\varsigma^2}{2\varpi + 3\varsigma + 1}\right) \\
= \ & ||\alpha_1||^2\left(\frac{(2\varpi + 3\varsigma + 1)^2 + 3\varsigma^2 - 8\varpi\varsigma - 6\varsigma - 4\varpi - 1}{2\varpi + 3\varsigma + 1}\right) \\
= \ & ||\alpha_1||^2\left((2\varpi + 3\varsigma + 1) + \frac{3\varsigma^2 - 8\varpi\varsigma - 6\varsigma - 4\varpi - 1}{2\varpi + 3\varsigma + 1}\right) \\
\geq \ & \left(\varpi + \frac{\varsigma}{2}\right) = d_{\mathbb{B}}\Big(\Gamma_1\varpi, \Gamma_2\varsigma\Big).
\end{aligned}$$

Hence,

$$d_{\mathbb{B}}\Big(\Gamma_1\varpi, \Gamma_2\varsigma\Big) \preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}\right)\alpha_1$$
$$\forall\ \varpi, \varsigma \in [0,1].$$

Case (ii): Let $\varpi, \varsigma \in (1,2]$, clearly $\Gamma_1(\mho) \subset \Gamma_4(\mho)$ and $\Gamma_2(\mho) \subset \Gamma_3(\mho)$.
Now,

$$d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma) = 3,\ d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi) = 5, d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) = 4,$$
$$d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) = 6 \text{ and } \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi) = 7.$$

Consider,

$$\begin{aligned}
&\alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}\right)\alpha_1\\
&= \alpha_1^*\left(\frac{62}{12}\right)\alpha_1\\
&= ||\alpha_1||^2\left(\frac{62}{12}\right)\\
&\geq 3 = d_{\mathbb{B}}\Big(\Gamma_1\varpi, \Gamma_2\varsigma\Big).
\end{aligned}$$

Thus,

$$d_{\mathbb{B}}\Big(\Gamma_1\varpi, \Gamma_2\varsigma\Big) \preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi)}\right)\alpha_1$$

$\forall\ \varpi, \varsigma \in (1,2]$. Also, $\Gamma_3(\mho)$ is a complete subspace of $\mho$, (Γ_1, Γ_3) and (Γ_2, Γ_4) are weakly compatible. Hence, by Theorem (4.2.1), $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 have a unique common fixed point. Indeed, '0' is a common unique fixed point.

Example 4.2.11. Let $\mho = [0,2]$ and $\mathbb{B} = \mathbb{C}$. Define $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ by $d_{\mathbb{B}}(\varpi, \varsigma) = |\varpi - \varsigma|$. Then, $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is a C^*_{AV}-MS and let $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 on $\mho$ as

$$\Gamma_1(\varpi) = \begin{cases} 0 & \text{if } \varpi \in [0,1] \\ 1 & \text{if } \varpi \in (1,2] \end{cases},\ \Gamma_2(\varpi) = \begin{cases} 0 & \text{if } \varpi \in [0,1] \\ \frac{1}{2} & \text{if } \varpi \in (1,2] \end{cases},$$
$$\Gamma_4(\varpi) = \begin{cases} 3\varpi & \text{if } \varpi \in [0,1] \\ 4 & \text{if } \varpi \in (1,2] \end{cases} \ \textit{and}\ \Gamma_3(\varpi) = \begin{cases} \varpi & \text{if } \varpi \in [0,1] \\ 2 & \text{if } \varpi \in (1,2] \end{cases}.$$

Firstly, we shall show that (Γ_1, Γ_3) satisfies $E.A.$ property. Take a sequence $\{\varpi_\jmath\}$ in $\mho$ s.t $\{\varpi_\jmath\} = \left(\frac{1}{\sqrt{\jmath^2+2\jmath}}\right)$. Then,

$$\lim_{\jmath\to\infty} \Gamma_1\varpi_\jmath = \lim_{\jmath\to\infty} \Gamma_1 \left(\frac{1}{\sqrt{\jmath^2+2\jmath}}\right) = \lim_{\jmath\to\infty}(0) = 0$$

and

$$\lim_{\jmath\to\infty} \Gamma_3\varpi_\jmath = \lim_{\jmath\to\infty} \Gamma_3\left(\frac{1}{\sqrt{\jmath^2+2\jmath}}\right) = \lim_{\jmath\to\infty}\left(\frac{1}{\sqrt{\jmath^2+2\jmath}}\right) = 0.$$

Thus, $\lim_{\jmath\to\infty} \Gamma_1\varpi_\jmath = \lim_{\jmath\to\infty} \Gamma_3\varpi_\jmath = 0 \in \mho$. Hence, (Γ_1, Γ_3) satisfy $E.A.$ property.
Following cases arises:
Case (i) : Let $\varpi, \varsigma \in [0,1]$, clearly $\Gamma_1(\mho) \subset \Gamma_4(\mho)$ and $\Gamma_2(\mho) \subset \Gamma_3(\mho)$.
Now,

$$d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma) = 0,\ d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi) = \varpi,\ d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) = \varpi,$$
$$d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) = 3\varsigma \text{ and } \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi) = 3\varsigma.$$

Consider,

$$\begin{aligned}
& \alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}\right)\alpha_1 \\
= \ & \alpha_1^*\left(\frac{(\varpi * \varpi) + (3\varsigma * 3\varsigma)}{\varpi + 3\varsigma + 1}\right)\alpha_1 \\
= \ & ||\alpha_1||^2\left(\frac{\varpi^2 + 9\varsigma^2}{\varpi + 3\varsigma + 1}\right) \geq 0 = d_{\mathbb{B}}\Big(\Gamma_1\varpi, \Gamma_2\varsigma\Big).
\end{aligned}$$

Hence,

$$d_{\mathbb{B}}\Big(\Gamma_1\varpi, \Gamma_2\varsigma\Big) \preceq \alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}{1 + d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma) + d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}\right)\alpha_1$$

$\forall\ \varpi, \varsigma \in [0,1]$.
Case (ii): Let $\varpi, \varsigma \in (1,2]$, clearly $\Gamma_1(\mho) \subset \Gamma_4(\mho)$ and $\Gamma_2(\mho) \subset \Gamma_3(\mho)$.
Now,

$$d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varsigma) = \frac{1}{2},\ d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi) = 1, d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_2\varsigma) = \frac{3}{2},$$
$$d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) = \frac{7}{2} \text{ and } \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\varpi) = 3.$$

Consider,

$$\begin{aligned}
&\alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma)+d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}{1+d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma)+d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}\right)\alpha_1\\
&=\alpha_1^*\left(\frac{24}{11}\right)\alpha_1\\
&=||\alpha_1||^2\left(\frac{24}{11}\right)\geq\frac{1}{2}=d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_2\varsigma).
\end{aligned}$$

Thus,

$$d_{\mathbb{B}}\Big(\Gamma_1\varpi,\Gamma_2\varsigma\Big)\preceq\alpha_1^*\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma)+d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_2\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}{1+d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_2\varsigma)+d_{\mathbb{B}}(\Gamma_4\varsigma,\Gamma_1\varpi)}\right)\alpha_1$$

$\forall\ \varpi,\varsigma\in(1,2]$. Also, $\Gamma_3(\mho)$ is a closed subspace of $\mho$ and (Γ_1,Γ_3) and (Γ_2,Γ_4) are weakly compatible. Hence, by Theorem (4.2.8), $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ_4 have a unique common fixed point. Indeed, '0' is a common unique fixed point.

Example 4.2.12. Let $\mho=[0,2]$ and $\mathbb{B}=\mathbb{C}$. Define $\Gamma_5:\mathbb{B}^+\to\mathbb{B}$ as $\Gamma_5(a)=195a$ and $d_{\mathbb{B}}:\mho\times\mho\to\mathbb{B}$ as $d_{\mathbb{B}}(\rho,\sigma)=\begin{cases}|\rho|+|\sigma| & \text{if } \rho\neq\sigma\\ 0 & \text{if } \rho=\sigma\end{cases}$
Then, $(\mho,\mathbb{B},d_{\mathbb{B}})$ is a complete C^*_{AV}-MS and let $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ_4 on $\mho$ as
$\Gamma_1(\rho)=\begin{cases}\rho & \text{if } \rho\in[0,1]\\ 2\rho & \text{if } \rho\in(1,2]\end{cases},\quad \Gamma_2(\rho)=\begin{cases}0 & \text{if } \rho\in[0,1]\\ 1 & \text{if } \rho\in(1,2]\end{cases},$
$\Gamma_4(\rho)=\begin{cases}2\rho & \text{if } \rho\in[0,1]\\ 4\rho & \text{if } \rho\in(1,2]\end{cases},\quad \Gamma_3(\rho)=\begin{cases}4\rho & \text{if } \rho\in[0,1]\\ 10 & \text{if } \rho\in(1,2]\end{cases}$
Firstly, we shall show that (Γ_1,Γ_3) satisfies CLR_{Γ_1} property. Take a sequence $\{\rho_\jmath\}\subset\mho$ s.t $\{\rho_\jmath\}=\left(\frac{1}{\sqrt{\jmath^2+2\jmath}}\right)$. Then,

$$\lim_{\jmath\to\infty}\Gamma_1\rho_\jmath=\lim_{\jmath\to\infty}\Gamma_1\left(\frac{1}{\sqrt{\jmath^2+2\jmath}}\right)=\lim_{\jmath\to\infty}\frac{1}{\sqrt{\jmath^2+2\jmath}}=0$$

and

$$\lim_{\jmath\to\infty}\Gamma_3\rho_\jmath=\lim_{\jmath\to\infty}\Gamma_3\left(\frac{1}{\sqrt{\jmath^2+2\jmath}}\right)=\lim_{\jmath\to\infty}\left(\frac{7}{\sqrt{\jmath^2+2\jmath}}\right)=0.$$

Thus, $\lim_{\jmath\to\infty}\Gamma_1\rho_\jmath=\lim_{\jmath\to\infty}\Gamma_3\rho_\jmath=\Gamma_1(0)=0$ for $0\in\mho$. Hence, (Γ_1,Γ_3) satisfies CLR_{Γ_1} property.
Following cases arises :
Case (i) : Let $\rho,\sigma\in[0,1]$ and assume $0<\tau<1$, clearly $\Gamma_1(\mho)\subset\Gamma_4(\mho)$ and

$\Gamma_2(\mho) \subset \Gamma_3(\mho)$.
Now,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\rho, \Gamma_2\sigma) &= \rho, \quad d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho) = 5\rho, \quad d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) = 4\rho, \\ d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma) &= 2\sigma \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho) = \rho + 2\sigma. \end{aligned}$$

$$\begin{aligned} \therefore \quad & \Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}{1 + d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}\alpha_1\right) \\ & = \Gamma_5\left(\alpha_1^*\frac{5\rho(4\rho) + 2\sigma(\rho + 2\sigma)}{1 + \rho + 2\sigma + 4\rho}\alpha_1\right) = \Gamma_5\left(\alpha_1^*\frac{20\rho^2 + 2\rho\sigma + 4\sigma^2}{5\rho + 2\sigma + 1}\alpha_1\right) \\ & = 195\left(\alpha_1^*\frac{20\rho^2 + 2\rho\sigma + 4\sigma^2}{5\rho + 2\sigma + 1}\alpha_1\right) \succeq \tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\rho, \Gamma_2\sigma)\Big). \end{aligned}$$

Hence,

$$\begin{aligned} \tau + \Gamma_5(d_{\mathbb{B}}(\rho, \sigma)) \preceq \; & \Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}{1 + d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}\alpha_1\right) \\ & \forall\ \rho, \sigma \in [0, 1]. \end{aligned}$$

Case (ii) : Let $\rho, \sigma \in (1, 2]$ and assume $0 < \tau < 1$, clearly $\Gamma_1(\mho) \subset \Gamma_4(\mho)$ and $\Gamma_2(\mho) \subset \Gamma_3(\mho)$.
Now,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\rho, \Gamma_2\sigma) &= 2\rho + 1, \quad d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho) = 10 + 2\rho, \quad d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) = 11, \\ d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma) &= 4\sigma + 1 \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho) = 4\sigma + 2\rho. \end{aligned}$$

Consider,

$$\begin{aligned} & \Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}{1 + d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}\alpha_1\right) \\ & = \Gamma_5\left(\alpha_1^*\frac{(11(10 + 2\rho)) + (4\sigma + 1)(2\rho + 4\sigma)}{1 + 11 + 2\rho + 4\sigma}\alpha_1\right) \\ & = 195\left(\alpha_1^*\frac{(11(10 + 2\rho)) + (4\sigma + 1)(2\rho + 4\sigma)}{1 + 11 + 2\rho + 4\sigma}\alpha_1\right) \\ & \succeq \tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\rho, \Gamma_2\sigma)\Big). \end{aligned}$$

Hence,

$$\tau + \Gamma_5(d_{\mathbb{B}}(\rho, \sigma)) \preceq \Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}{1 + d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}\alpha_1\right)$$

$\forall\ \rho, \sigma \in (1, 2]$. The pairs (Γ_1, Γ_3) and (Γ_2, Γ_4) are weakly compatible. Hence, by the Theorem (4.2.9), $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 have a unique common fixed point. Indeed, '0' is common unique fixed point.

Example 4.2.13. Let $\mho = [0, 1]$ and $\mathbb{B} = \mathbb{C}$. Define $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ as $d_{\mathbb{B}}(\varpi, \varsigma) = |\varpi - \varsigma|$. Then, $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is a C^*_{AV}-MS. Let $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 on $\mho$ as

$$\Gamma_1 \varpi = \varpi, \quad \Gamma_2 \varpi = \frac{\varpi}{2}, \quad \Gamma_3 \varpi = 4\varpi\ \Gamma_4 \varpi = 2\varpi,\ \forall\ \varpi \in \mho.$$

Clearly, $\Gamma_1(\mho) \subset \Gamma_4(\mho)$ and $\Gamma_2(\mho) \subset \Gamma_3(\mho)$. Also, $\Gamma_3(\mho)$ is a complete subspace of $\mho$ and (Γ_1, Γ_3) and (Γ_2, Γ_4) are weakly compatible.
Now,

$$d_{\mathbb{B}}(\Gamma_1 \varpi, \Gamma_2 \varsigma) = \left|\varpi - \frac{\varsigma}{2}\right|,\ d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_1 \varpi) = |4\varpi - \varpi|,\ d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_2 \varsigma) = \left|4\varpi - \frac{\varsigma}{2}\right|,$$
$$d_{\mathbb{B}}(\Gamma_4 \varsigma, \Gamma_1 \varpi) = |2\varsigma - \varpi| \text{and } d_{\mathbb{B}}(\Gamma_4 \varsigma, \Gamma_2 \varsigma) = \left|2\varsigma - \frac{\varsigma}{2}\right|.$$

Observe that

$$d_{\mathbb{B}}\Big(\Gamma_1 \varpi, \Gamma_2 \varsigma\Big) \preceq \alpha_1^* \left(\frac{d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_1 \varpi) d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_2 \varsigma) + d_{\mathbb{B}}(\Gamma_4 \varsigma, \Gamma_2 \varsigma) d_{\mathbb{B}}(\Gamma_4 \varsigma, \Gamma_1 \varpi)}{1 + d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_2 \varsigma) + d_{\mathbb{B}}(\Gamma_4 \varsigma, \Gamma_1 \varpi)}\right) \alpha_1$$

$\forall\ \varpi, \varsigma \in \mho$. Hence, by Theorem (4.2.1), $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 have a unique common fixed point. Indeed, '0' is the unique common fixed point in $\mho$.

Example 4.2.14. Let $\mho = [0, 2]$ and $\mathbb{B} = \mathbb{C}$. Let $\Gamma_5 : \mathbb{B}^+ \to \mathbb{B}$ defined as $\Gamma_5(a) = \ln(a)$ and $d : \mho \times \mho \to \mathbb{B}$ defined as

$$d_{\mathbb{B}}(\rho, \sigma) = \begin{cases} |\rho| + |\sigma| & \text{if } \rho \neq \sigma \\ 0 & \text{if } \rho = \sigma. \end{cases}$$

Then, $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is a complete C^*_{AV}-MS and let $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 on $\mho$ as

$$\Gamma_1(\rho) = \begin{cases} \rho^2 & \text{if } \rho \in [0, 1] \\ 0 & \text{if } \rho \in (1, 2] \end{cases}, \qquad \Gamma_2(\rho) = \begin{cases} 0 & \text{if } \rho \in [0, 1] \\ 1 & \text{if } \rho \in (1, 2] \end{cases},$$

$$\Gamma_3(\rho) = 11\rho \quad \text{and} \quad \Gamma_4(\rho) = 8\rho \quad \forall \quad \rho \in [0, 2].$$

Following cases arises :
Case (i) : Let $\rho, \sigma \in [0, 1]$ and assume $0 < \tau < 1$, clearly $\Gamma_1(\mho) \subset \Gamma_4(\mho)$ and

$\Gamma_2(\mho) \subset \Gamma_3(\mho)$.
Now,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\rho, \Gamma_2\sigma) &= \rho^2, \quad d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho) = 11\rho + \rho^2, \quad d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) = 11\rho, \\ d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma) &= 8\sigma \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho) = \rho^2 + 8\sigma. \end{aligned}$$

Consider,

$$\begin{aligned} &\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}{1 + d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}\alpha_1\right) \\ &= \Gamma_5\left(\alpha_1^* \frac{11\rho(11\rho + \rho^2) + 8\sigma(8\sigma + \rho^2)}{1 + 11\rho + 8\sigma + \rho^2}\alpha_1\right) \\ &= \ln\left(\alpha_1^* \frac{11\rho(11\rho + \rho^2) + 8\sigma(8\sigma + \rho^2)}{1 + 11\rho + 8\sigma + \rho^2}\alpha_1\right) \\ &\succeq \tau + \ln(\rho^2) = \tau + \Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\rho, \Gamma_2\sigma)\Big). \end{aligned}$$

Hence,

$$\begin{aligned} \tau + \Gamma_5\Big(d_{\mathbb{B}}(\rho, \sigma)\Big) \ &\preceq\ \Gamma_5\left(\frac{d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}{1 + d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}\right) \\ &\quad \forall\ \rho, \sigma \in [0, 1] \text{ with } \|\alpha_1\| = 1. \end{aligned}$$

Case (ii) : Let $\rho, \sigma \in (1, 2]$ and assume $0 < \tau < 1$, clearly $\Gamma_1(\mho) \subset \Gamma_4(\mho)$ and $\Gamma_2(\mho) \subset \Gamma_3(\mho)$. Now,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\rho, \Gamma_2\sigma) &= 1, \quad d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho) = 11\rho, \quad d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) = 11\rho + 1, \\ d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma) &= 8\sigma + 1 \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho) = 8\sigma. \end{aligned}$$

Consider,

$$\begin{aligned} &\Gamma_5\left(\alpha_1^* \frac{d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}{1 + d_{\mathbb{B}}(\Gamma_3\rho, \Gamma_2\sigma) + d_{\mathbb{B}}(\Gamma_4\sigma, \Gamma_1\rho)}\alpha_1\right) \\ &= \Gamma_5\left(\alpha_1 \frac{(11\rho + 1)11\rho + (8\sigma + 1)8\sigma}{2 + 11\rho + 8\sigma}\alpha_1\right) \\ &= \ln\left(\alpha_1^* \frac{(11\rho + 1)11\rho + (8\sigma + 1)8\sigma}{2 + 11\rho + 8\sigma}\alpha_1\right) \\ &\succeq \tau + \ln(1) = \tau + \Gamma_5(d_{\mathbb{B}}(\Gamma_1\rho, \Gamma_2\sigma)). \end{aligned}$$

Hence,

$$\tau+\Gamma_5(d_{\mathbb{B}}(\rho,\sigma)) \preceq \Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}{1+d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}\alpha_1\right)$$

$\forall\ \rho,\sigma \in (1,2]$ with $\|\alpha_1\| = 1$. Also, $\Gamma_3(\mho)$ is a complete subspace of $\mho$, (Γ_1,Γ_3) and (Γ_2,Γ_4) are weakly compatible. Hence, by Theorem (4.2.3), $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ and Γ_4 have a unique common fixed point. Indeed, '0' is a common unique fixed point.

Example 4.2.15. Let $\mho = [0,2]$ and $\mathbb{B} = \mathbb{C}$. Define $\Gamma_5 : \mathbb{B}^+ \to \mathbb{B}$ as $\Gamma_5(a) = 125a$ and $d_{\mathbb{B}} : \mho\times\mho \to \mathbb{B}$ as $d_{\mathbb{B}}(\rho,\sigma) = |\rho-\sigma|$. Then, $(\mho,\mathbb{B},d_{\mathbb{B}})$ is a complete C^*_{AV}-MS and let $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ_4 on $\mho$ as

$$\Gamma_1(\rho) = \begin{cases} 0 & \text{if } \rho\in[0,1] \\ 0.5 & \text{if } \rho\in(1,2] \end{cases},\quad \Gamma_2(\rho) = \begin{cases} 0 & \text{if } \rho\in[0,1] \\ 1 & \text{if } \rho\in(1,2] \end{cases},$$

$$\Gamma_4(\rho) = \begin{cases} 5\rho & \text{if } \rho\in[0,1] \\ 8 & \text{if } \rho\in(1,2] \end{cases},\quad \text{and}\quad \Gamma_3(\rho) = \begin{cases} 4\rho & \text{if } \rho\in[0,1] \\ 20 & \text{if } \rho\in(1,2] \end{cases}$$

Firstly, we shall that (Γ_1,Γ_3) satisfies $E.A.$ property. Take a sequence $\{\rho_\jmath\}\subset\mho$ s.t $\{\rho_\jmath\} = \left(\frac{1}{\sqrt{\jmath^2+1}}\right)$. Then,

$$\lim_{\jmath\to\infty}\Gamma_1\rho_\jmath = \lim_{\jmath\to\infty}\Gamma_1\left(\frac{1}{\sqrt{\jmath^2+1}}\right) = \lim_{\jmath\to\infty}(0) = 0$$

and

$$\lim_{\jmath\to\infty}\Gamma_3\rho_\jmath = \lim_{\jmath\to\infty}\Gamma_3\left(\frac{1}{\sqrt{\jmath^2+1}}\right) = \lim_{\jmath\to\infty}\left(\frac{4}{\sqrt{\jmath^2+1}}\right) = 0.$$

Thus, $\lim_{\jmath\to\infty}\Gamma_1\rho_\jmath = \lim_{\jmath\to\infty}\Gamma_3\rho_\jmath = 0\in\mho$. Hence, (Γ_1,Γ_3) satisfies $E.A.$ property.
Following cases arises:
Case (i) : Let $\rho,\sigma\in[0,1]$ and assume $0<\tau<1$, clearly $\Gamma_1(\mho)\subset\Gamma_4(\mho)$ and $\Gamma_2(\mho)\subset\Gamma_3(\mho)$.
Now,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\rho,\Gamma_2\sigma) &= 0,\quad d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_1\rho) = 4\rho,\quad d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma) = 4\rho, \\ d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_2\sigma) &= 5\sigma \quad\text{and}\quad d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho) = 5\sigma. \end{aligned}$$

Consider,

$$\begin{aligned} &\Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}{1+d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}\alpha_1\right) \\ &= \Gamma_5\left(\alpha_1^*\frac{(4\rho*4\rho)+(5\sigma*5\sigma)}{4\rho+5\sigma+1}\alpha_1\right) \\ &= 125\left(\alpha_1^*\frac{16\rho^2+25\sigma^2}{4\rho+5\sigma+1}\alpha_1\right) \succeq \tau+\Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\rho,\Gamma_2\sigma)\Big). \end{aligned}$$

Hence,

$$\tau+\Gamma_5\Big(d_{\mathbb{B}}(\rho,\sigma)\Big) \preceq \Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}{1+d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}\alpha_1\right)$$

$\forall\ \rho,\sigma\in[0,1]$.

Case (ii) : Let $\rho,\sigma\in(1,2]$ and assume $0<\tau<1$, clearly $\Gamma_1(\mho)\subset\Gamma_4(\mho)$ and $\Gamma_2(\mho)\subset\Gamma_3(\mho)$.

Now,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\rho,\Gamma_2\sigma) &= 0.5, \quad d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_1\rho)=19.5, \quad d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)=19, \\ d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_2\sigma) &= 7 \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)=7.5. \end{aligned}$$

Consider,

$$\begin{aligned} &\Gamma_5\left(\alpha_1^*\frac{d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}{1+d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}\alpha_1\right) \\ &=\Gamma_5\left(\alpha_1^*\frac{(19*19.5+7*7.5}{1+19+7.5}\alpha_1\right) \\ &=\Gamma_5\left(\alpha_1^*\frac{423}{27.5}\alpha_1\right)=125\left(\alpha_1^*\frac{423}{27.5}\alpha_1\right) \\ &\succeq \tau+\Gamma_5\Big(d_{\mathbb{B}}(\Gamma_1\rho,\Gamma_2\sigma)\Big). \end{aligned}$$

Hence,

$$\begin{aligned} \tau+\Gamma_5\Big(d_{\mathbb{B}}(\rho,\sigma)\Big) \ \preceq\ & \Gamma_5\left(\frac{d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_1\rho)d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_2\sigma)d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}{1+d_{\mathbb{B}}(\Gamma_3\rho,\Gamma_2\sigma)+d_{\mathbb{B}}(\Gamma_4\sigma,\Gamma_1\rho)}\right) \\ & \forall\ \rho,\sigma\in(1,2]. \end{aligned}$$

Also, $\Gamma_3(\mho)$ is a closed subspace of $\mho$ and (Γ_1,Γ_3) and (Γ_2,Γ_4) are weakly compatible. Hence, by Theorem (4.2.6), $\Gamma_1,\Gamma_2,\ \Gamma_3$ and Γ_4 have a unique common fixed point. Indeed, '0' is common unique fixed point.

4.3 Common Fixed Point of Self Mappings using Expansion

Common fixed point results of self mappings for various type of expansion mappings in abstract spaces have been investigated broadly by many researchers (see, Shahi et al. (2012) Anushree & Gandhi (2013), Ma et al. (2014), Dhawan et al. (2019), Markin & Sichel (2019), Anushree & Gandhi (2021) and references cited therein). In this section, some results on common fixed point in C^*_{AV}-MS using expansion mappings are presented.

Theorem 4.3.1. *Let* $(\mho, \mathbb{B}, d_{\mathbb{B}})$ *be* C^*_{AV}*-MS and* $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 : \mho \to \mho$ *satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(ii) $\forall\ \vartheta, \varsigma \in \mho$*,* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{B}^+$ *with* $\|\alpha_1\|, \|\alpha_2\|, \|\alpha_3\| > 1$

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma); \end{aligned} \tag{4.3.1}$$

(iii) either $\Gamma_1(\mho)$*,* $\Gamma_2(\mho)$*,* $\Gamma_3(\mho)$ *or* $\Gamma_4(\mho)$ *is a complete subspace of* $\mho$*.*

Then, the pairs (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *have a coincidence point. Moreover, if the pairs* (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible, then* $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have unique common fixed point in* $\mho$*.*

Proof. Let $\vartheta_0 \in \mho$ be any arbitrary point. From (i), construct a sequence $\{\varsigma_\jmath\}$ in $\mho$ as

$$\varsigma_{2\jmath+1} = \Gamma_1\vartheta_{2\jmath} = \Gamma_4\vartheta_{2\jmath+1} \quad \text{and} \quad \varsigma_{2\jmath+2} = \Gamma_2\vartheta_{2\jmath+1} = \Gamma_3\vartheta_{2\jmath+2}.$$

Define $d_{\mathbb{B}_\jmath} = d_{\mathbb{B}}(\varsigma_\jmath, \varsigma_{\jmath+1})$. Suppose that $d_{\mathbb{B}_{2\jmath}} = \theta_{\mathbb{B}}$, i.e, $d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}) = \theta_{\mathbb{B}}$ for some $\jmath$. Then, $\Gamma_1\vartheta_{2\jmath} = \Gamma_4\vartheta_{2\jmath+1} = \Gamma_2\vartheta_{2\jmath-1} = \Gamma_3\vartheta_{2\jmath}$. Thus, Γ_1 and Γ_3 have coincidence point. Hence, the result.

Now, suppose that $d_{\mathbb{B}_{2\jmath}} \succ \theta_{\mathbb{B}}\ \forall\ \jmath \in \mathbb{N}$. On substituting $\vartheta = \vartheta_{2\jmath}$ and $\varsigma = \vartheta_{2\jmath+1}$ in (4.3.1), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta_{2\jmath}, \Gamma_4\vartheta_{2\jmath+1}) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta_{2\jmath}, \Gamma_1\vartheta_{2\jmath}) d_{\mathbb{B}}(\Gamma_1\vartheta_{2\jmath}, \Gamma_4\vartheta_{2\jmath+1}) \\ &\quad + \alpha_2 d_{\mathbb{B}}(\Gamma_4\vartheta_{2\jmath+1}, \Gamma_2\vartheta_{2\jmath+1}) + \alpha_3 d_{\mathbb{B}}(\Gamma_2\vartheta_{2\jmath+1}, \Gamma_1\vartheta_{2\jmath}) \\ d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}) &\succeq \alpha_1 d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}) d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+1}) + \alpha_2 d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) \\ d_{\mathbb{B}}(\varsigma_{2\jmath}, \varsigma_{2\jmath+1}) &\succeq \alpha_2 d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) + \alpha_3 d_{\mathbb{B}}(\varsigma_{2\jmath+1}, \varsigma_{2\jmath+2}) \\ d_{\mathbb{B}_{2\jmath+1}} &\preceq \frac{1}{\alpha_2 + \alpha_3} d_{\mathbb{B}_{2\jmath}} = h d_{\mathbb{B}_{2\jmath}}, \end{aligned}$$

where $h = \dfrac{1}{\alpha_2 + \alpha_3}$ with $\|h\| < 1$ as $\|\alpha_2 + \alpha_3\| > 1$. On the similar lines, $d_{\mathbb{B}_{2\jmath}} \preceq hd_{\mathbb{B}_{2\jmath-1}}$, $d_{\mathbb{B}_{2\jmath-1}} \preceq hd_{\mathbb{B}_{2\jmath-2}}$ and so on.
In general, $d_{\mathbb{B}_\jmath} \preceq hd_{\mathbb{B}_{\jmath-1}}$ $\forall\ \jmath \in \mathbb{N}$, i.e,

$$\begin{aligned} d_{\mathbb{B}}(\varsigma_\jmath, \varsigma_{\jmath+1}) &\preceq hd_{\mathbb{B}}(\varsigma_{\jmath-1}, \varsigma_\jmath) \\ &\preceq h^2 d_{\mathbb{B}}(\varsigma_{\jmath-2}, \varsigma_{\jmath-1}) \\ &\preceq \cdots \\ &\preceq h^\jmath d_{\mathbb{B}}(\varsigma_0, \varsigma_1). \end{aligned}$$

For any $p \in \mathbb{N}$, we have

$$\begin{aligned} d_{\mathbb{B}}(\varsigma_{\jmath+p}, \varsigma_\jmath) &\preceq d_{\mathbb{B}}(\varsigma_{\jmath+p}, \varsigma_{\jmath+p-1}) + d_{\mathbb{B}}(\varsigma_{\jmath+p-1}, \varsigma_{\jmath+p-2}) + \cdots + d_{\mathbb{B}}(\varsigma_{\jmath+1}, \varsigma_\jmath) \\ &\preceq \sum_{\imath=\jmath}^{\jmath+p-1} h^\imath d_{\mathbb{B}}(\varsigma_0, \varsigma_1) \\ &\preceq \sum_{\imath=\jmath}^{\jmath+p-1} (\alpha_2 h^{\frac{\imath}{2}})^* \alpha_2 h^{\frac{\imath}{2}} \\ &\preceq \sum_{\imath=\jmath}^{\jmath+p-1} |\alpha_2 h^\imath|^2 \\ &\leq \sum_{\imath=\jmath}^{\jmath+p-1} \|(\alpha_2 h^\imath)^2\| I_{\mathbb{B}} \\ &\leq \|\alpha_2\|^2 I_{\mathbb{B}} \sum_{\imath=\jmath}^{\jmath+p-1} (h^\imath)^2 \to \theta_{\mathbb{B}} \text{ as } \jmath \to \infty, \end{aligned}$$

where $\|\alpha_2\|^2 = d_{\mathbb{B}}(\varsigma_0, \varsigma_1)$ for some $\alpha_2 \in \mathbb{B}^+$. Hence, $\{\varsigma_\jmath\}$ is a C_{seq}.
Since, $\Gamma_3(\mho)$ is complete subspace of $\mho$. $\therefore$ $\{\varsigma_\jmath\}$ is contained in $\Gamma_3(\mho)$ and has a limit in $\Gamma_3(\mho)$, μ (say). Let $\nu \in \Gamma_3^{-1}\mu$, then $\Gamma_3\nu = \mu$.
Next, to show $\Gamma_1\nu = \mu$. Assume that, $\Gamma_1\nu \neq \mu$. Substituting $\vartheta = \nu$ and $\varsigma = \vartheta_{\jmath-1}$ in (4.3.1), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_4\vartheta_{\jmath-1}) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_1\nu) d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_4\vartheta_{\jmath-1}) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\vartheta_{\jmath-1}, \Gamma_2\vartheta_{\jmath-1}) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_2\vartheta_{\jmath-1}) \\ d_{\mathbb{B}}(\Gamma_3\nu, \varsigma_{\jmath-1}) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_1\nu) d_{\mathbb{B}}(\Gamma_1\nu, \varsigma_{\jmath-1}) + \alpha_2 d_{\mathbb{B}}(\varsigma_{\jmath-1}, \varsigma_\jmath) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\varsigma_\jmath, \Gamma_1\nu). \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$d_{\mathbb{B}}(\mu, \mu) \succeq \alpha_1 d_{\mathbb{B}}(\mu, \Gamma_1\nu) d_{\mathbb{B}}(\Gamma_1\nu, \mu) + \alpha_2 d_{\mathbb{B}}(\mu, \mu) + \alpha_3 d_{\mathbb{B}}(\mu, \Gamma_1\nu)$$

$$\theta_{\mathbb{B}} \succeq \alpha_1 d_{\mathbb{B}}(\mu, \Gamma_1\nu) d_{\mathbb{B}}(\Gamma_1\nu, \mu) + \alpha_3 d_{\mathbb{B}}(\mu, \Gamma_1\nu)$$
$$\theta_{\mathbb{B}} \succeq d_{\mathbb{B}}(\mu, \Gamma_1\nu)(\alpha_1 d_{\mathbb{B}}(\Gamma_1\nu, \mu) + \alpha_3).$$

Taking norm both side, we have

$$\|d_{\mathbb{B}}(\mu, \Gamma_1\nu)\| \|\alpha_1 d_{\mathbb{B}}(\Gamma_1\nu, \mu) + \alpha_3\| \leq 0$$

implies $\|d_{\mathbb{B}}(\mu, \Gamma_1\nu)\| \leq 0$ i.e, $\Gamma_1\nu = \mu$. Thus, $\Gamma_3\nu = \mu = \Gamma_1\nu$.
Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ and $\Gamma_1\nu = \mu$ implies $\mu \in \Gamma_4(\mho)$. Let $\varpi \in \Gamma_4^{-1}\mu$, then $\Gamma_4\varpi = \mu$.
On the similar lines, $\Gamma_2\varpi = \Gamma_4\varpi = \mu$.
Since, (Γ_1, Γ_3) and (Γ_2, Γ_4) are weakly compatible. $\therefore$ $\mu = \Gamma_1\nu = \Gamma_3\nu = \Gamma_4\varpi = \Gamma_2\varpi$. Then,

$$\Gamma_4\mu = \Gamma_4\Gamma_2\varpi = \Gamma_2\Gamma_4\varpi = \Gamma_2\mu \text{ and } \Gamma_3\mu = \Gamma_3\Gamma_1\nu = \Gamma_1\Gamma_3\nu = \Gamma_1\mu.$$

We claim that $\Gamma_2\mu = \mu$. If possible, let $\Gamma_2\mu \neq \mu$, then

$$\begin{aligned} d_{\mathbb{B}}(\mu, \Gamma_2\mu) = d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_4\mu) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\nu, \Gamma_1\nu) d_{\mathbb{B}}(\Gamma_1\nu, \Gamma_4\mu) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_2\mu) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_2\mu, \Gamma_1\nu) = \alpha_3 d_{\mathbb{B}}(\mu, \Gamma_2\mu) \\ \theta_{\mathbb{B}} &\succeq (\alpha_3 - I_{\mathbb{B}}) d_{\mathbb{B}}(\mu, \Gamma_2\mu). \end{aligned}$$

Taking norm on both side, we have

$$\|(\alpha_3 - 1)\| \|d_{\mathbb{B}}(\mu, \Gamma_2\mu)\| \leq 0,$$

implies $\|d_{\mathbb{B}}(\mu, \Gamma_2\mu)\| \leq 0$ i.e, $\Gamma_2\mu = \mu$.
Hence, $\Gamma_1\mu = \Gamma_3\mu = \Gamma_4\mu = \Gamma_2\mu = \mu$.
Uniqueness : Let $\varkappa$ be another common fixed point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4. Then,

$$\begin{aligned} d_{\mathbb{B}}(\varkappa, \mu) = d_{\mathbb{B}}(\Gamma_3\varkappa, \Gamma_4\mu) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\varkappa, \Gamma_1\varkappa) d_{\mathbb{B}}(\Gamma_1\varkappa, \Gamma_4\mu) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\mu, \Gamma_2\mu) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_2\mu, \Gamma_1\varkappa) = \alpha_3 d_{\mathbb{B}}(\mu, \varkappa) \\ \theta_{\mathbb{B}} &\succeq (\alpha_3 - I_{\mathbb{B}}) d_{\mathbb{B}}(\mu, \varkappa). \end{aligned}$$

Taking norm on both side, we have

$$\|(\alpha_3 - 1)\| \|d_{\mathbb{B}}(\mu, \varkappa)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\mu, \varkappa)\| \leq 0$ i.e, $\varkappa = \mu$. □

Theorem 4.3.2. *Let $(\mho, \mathbb{B}, d_{\mathbb{B}})$ be C^*_{AV}-MS and $\Gamma_1, \Gamma_2, \Gamma_3,\ \Gamma_4 : \mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(ii) $\forall\ \vartheta, \varsigma \in \Gamma,\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{B}^+$ *with* $\|\alpha_1\|, \|\alpha_2\|, \|\alpha_3\| > 1$,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \ &\succeq\ \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma); \end{aligned} \tag{4.3.2}$$

(iii) *the pairs* (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(iv) *either* (Γ_1, Γ_3) *or* (Γ_2, Γ_4) *satisfies E.A. property.*

If the range of one of $\Gamma_3(\mho)$ *or* $\Gamma_4(\mho)$ *is a closed subspace of* $\mho$*. Then,* $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$*.*

Proof. Firstly, assume (Γ_2, Γ_4) satisfies *E.A.* property. $\therefore$ $\exists$ a sequence $\{\vartheta_\jmath\}$ in Γ s.t $\lim_{\jmath\to\infty} \Gamma_2(\vartheta_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\vartheta_\jmath) = \varkappa$ for some $\varkappa \in \mho$.
Further, $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$. $\therefore$ $\exists$ a sequence $\{\varsigma_\jmath\}$ in $\mho$ s.t $\Gamma_2(\vartheta_\jmath) = \Gamma_3(\varsigma_\jmath)$. Hence, $\lim_{\jmath\to\infty} \Gamma_3(\varsigma_\jmath) = \varkappa$.
We claim $\lim_{\jmath\to\infty} \Gamma_1(\varsigma_\jmath) = \varkappa$. If possible $\lim_{\jmath\to\infty} \Gamma_1(\vartheta_\jmath) = \varkappa_1 \neq \varkappa$. Then, substituting $\vartheta = \varsigma_\jmath$ and $\varsigma = \vartheta_\jmath$ in (4.3.2), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\varsigma_\jmath, \Gamma_4\vartheta_\jmath) \ &\succeq\ \alpha_1 d_{\mathbb{B}}(\Gamma_3\varsigma_\jmath, \Gamma_1\varsigma_\jmath) d_{\mathbb{B}}(\Gamma_1\varsigma_\jmath, \Gamma_4\vartheta_\jmath) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\vartheta_\jmath, \Gamma_2\vartheta_\jmath) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\varsigma_\jmath, \Gamma_2\vartheta_\jmath). \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\varkappa, \varkappa) \ &\succeq\ \alpha_1 d_{\mathbb{B}}(\varkappa, \varkappa_1) d_{\mathbb{B}}(\varkappa_1, \varkappa) + \alpha_2 d_{\mathbb{B}}(\varkappa, \varkappa) + \alpha_3 d_{\mathbb{B}}(\varkappa_1, \varkappa) \\ \theta_{\mathbb{B}} \ &\succeq\ d_{\mathbb{B}}(\varkappa, \varkappa_1)(\alpha_1 d_{\mathbb{B}}(\varkappa, \varkappa_1) + \alpha_3). \end{aligned}$$

Taking norm both side, we have

$$\|d_{\mathbb{B}}(\varkappa, \varkappa_1)\| \|\alpha_1 d_{\mathbb{B}}(\varkappa, \varkappa_1) + \alpha_3\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varkappa, \varkappa_1)\| \leq 0$, i.e, $\varkappa = \varkappa_1$. Hence, $\lim_{\jmath\to\infty} \Gamma_1(\varsigma_\jmath) = \lim_{\jmath\to\infty} \Gamma_2(\vartheta_\jmath) = \lim_{\jmath\to\infty} \Gamma_3(\varsigma_\jmath) = \varkappa$.
Now, suppose $\Gamma_3(\mho)$ is closed subspace of $\mho$ and $\Gamma_3\mu = \varkappa$ for some $\mu \in \mho$. Subsequently, we have $\lim_{\jmath\to\infty} \Gamma_1(\varsigma_\jmath) = \lim_{\jmath\to\infty} \Gamma_2(\vartheta_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\vartheta_\jmath) = \lim_{\jmath\to\infty} \Gamma_3(\varsigma_\jmath) = \varkappa = \Gamma_3\mu$.

We claim that $\Gamma_1\mu = \Gamma_3\mu$. If not, substituting $\vartheta = \mu$ and $\varsigma = \vartheta_\jmath$ in (4.3.2), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\vartheta_\jmath) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_4\vartheta_\jmath) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\vartheta_\jmath, \Gamma_2\vartheta_\jmath) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\vartheta_\jmath). \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\varkappa, \varkappa) &\succeq \alpha_1 d_{\mathbb{B}}(\varkappa, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_1\mu, \varkappa) + \alpha_2 d_{\mathbb{B}}(\varkappa, \varkappa) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\mu, \varkappa) \\ \theta_{\mathbb{B}} &\succeq d_{\mathbb{B}}(\Gamma_1\mu, \varkappa)(\alpha_1 d_{\mathbb{B}}(\varkappa, \Gamma_1\mu) + \alpha_3). \end{aligned}$$

Taking norm both side, we have

$$\|\alpha_1 d_{\mathbb{B}}(\varkappa, \Gamma_1\mu) + \alpha_3\| \leq 0$$

implies $\|d_{\mathbb{B}}(\Gamma_1\mu, \varkappa)\| = 0$, i.e, $\Gamma_1\mu = \varkappa$. Hence, $\Gamma_1\mu = \Gamma_3\mu = \varkappa$.
Now, the weak compatibility of (Γ_1, Γ_3) implies $\Gamma_1\Gamma_3\mu = \Gamma_3\Gamma_1\mu$ or $\Gamma_1\varkappa = \Gamma_3\varkappa$. Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ $\exists\, \nu \in \mho$ s.t $\Gamma_1\mu = \Gamma_4\nu = \Gamma_3\mu = \varkappa$.
Now, to prove $\Gamma_2\nu = \Gamma_4\nu = \varkappa$. If $\Gamma_2\nu \neq \Gamma_4\nu$, then substituting $\vartheta = \mu$ and $\varsigma = \nu$ in (4.3.2), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\nu) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_4\nu) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_2\nu) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\nu) \\ \theta_{\mathbb{B}} &\succeq (\alpha_2 + \alpha_3) d_{\mathbb{B}}(\varkappa, \Gamma_2\nu). \end{aligned}$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\varkappa, \Gamma_2\nu)\| \leq 0$$

implies $\varkappa = \Gamma_2\nu$. Hence, $\Gamma_2\nu = \Gamma_4\nu = \varkappa$.
Further, the weak compatibility of (Γ_2, Γ_4) implies $\Gamma_2\Gamma_4\nu = \Gamma_4\Gamma_2\nu$, or $\Gamma_2\varkappa = \Gamma_4\varkappa$. $\therefore$ $\varkappa$ is a common coincidence point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.
Now, to prove $\varkappa$ is a common fixed point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4. Substituting $\vartheta = \mu$ and $\varsigma = \varkappa$ in (4.3.2), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varkappa) = d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_4\varkappa) \\ &\quad + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_2\varkappa) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varkappa) \\ &\succeq \alpha_3 d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa) \\ \theta_{\mathbb{B}} &\succeq (\alpha_3 - I_{\mathbb{B}}) d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa). \end{aligned}$$

Taking norm on both side, we have

$$\|\alpha_3 - 1\|\|d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)\| = 0$. Hence, $\Gamma_1\varkappa = \Gamma_2\varkappa = \Gamma_3\varkappa = \Gamma_4\varkappa = \varkappa$.
Uniqueness : Let ϖ is another common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Then, substituting $\vartheta = \varpi$ and $\varsigma = \varkappa$ in (4.3.2), we have

$$\begin{aligned} d_{\mathbb{B}}(\varpi, \varkappa) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi) d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_4\varkappa) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_2\varkappa) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varkappa) \\ &\succeq \alpha_3 d_{\mathbb{B}}(\varpi, \varkappa) \\ \theta_{\mathbb{B}} &\succeq (\alpha_3 - I_{\mathbb{B}}) d_{\mathbb{B}}(\varpi, \varkappa). \end{aligned}$$

Taking norm on both side, we have

$$\|\alpha_3 - 1\|\|d_{\mathbb{B}}(\varpi, \varkappa)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varpi, \varkappa)\| = 0$. Hence, $\varpi = \varkappa$. □

Theorem 4.3.3. *Let* $(\mho, \mathbb{B}, d_{\mathbb{B}})$ *be* C^*_{AV}*-MS and* $\Gamma_1, \Gamma_2, \Gamma_3,\ \Gamma_4 : \mho \to \mho$ *satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(ii) $\forall\ \vartheta, \varsigma \in \mho$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{B}^+$ *with* $\|\alpha_1\|, \|\alpha_2\|, \|\alpha_3\| > 1$,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma); \end{aligned} \tag{4.3.3}$$

(iii) the pairs (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(iv) either (Γ_1, Γ_3) *or* (Γ_2, Γ_4) *satisfies CLR property.*

Then, $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$.

Proof. Firstly, suppose (Γ_2, Γ_4) satisfies CLR_{Γ_2} property. $\therefore\ \exists$ a sequence $\{\vartheta_\jmath\}$ in $\mho$ s.t $\lim_{\jmath\to\infty} \Gamma_2(\vartheta_\jmath) = \lim_{\jmath\to\infty} \Gamma_4(\vartheta_\jmath) = \Gamma_2\vartheta = \varkappa$ for some $\vartheta \in \mho$.
Since, $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$. $\therefore\ \Gamma_2\vartheta = \Gamma_3\mu$, for some $\mu \in \mho$.
We claim that $\Gamma_1\mu = \Gamma_3\mu = \varkappa$. If not, substituting $\vartheta = \mu$ and $\varsigma = \vartheta_\jmath$ in (4.3.3), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\vartheta_\jmath) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_4\vartheta_\jmath) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\vartheta_\jmath, \Gamma_2\vartheta_\jmath) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\vartheta_\jmath). \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\varkappa, \varkappa) &\succeq \alpha_1 d_{\mathbb{B}}(\varkappa, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_1\mu, \varkappa) + \alpha_2 d_{\mathbb{B}}(\varkappa, \varkappa) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\mu, \varkappa) \\ \theta_{\mathbb{B}} &\succeq d_{\mathbb{B}}(\varkappa, \Gamma_1\mu)(\alpha_1 d_{\mathbb{B}}(\Gamma_1\mu, \varkappa) + \alpha_3). \end{aligned}$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\varkappa, \Gamma_1\mu)\| \|\alpha_1 d_{\mathbb{B}}(\Gamma_1\mu, \varkappa) + \alpha_3\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varkappa, \Gamma_1\mu)\| \leq 0$, i.e, $\varkappa = \Gamma_1\mu$. Hence, $\Gamma_1\mu = \Gamma_3\mu = \varkappa$.
Since, $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$. $\therefore\ \exists\, \nu \in \mho$ s.t $\Gamma_4\nu = \Gamma_1\mu = \Gamma_3\mu = \varkappa$.
Now, to prove $\Gamma_4\nu = \Gamma_2\nu = \varkappa$. If $\Gamma_4\nu \neq \Gamma_2\nu$, then substituting $\vartheta = \mu$ and $\varsigma = \nu$ in (4.3.3), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\nu) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_4\nu) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\nu, \Gamma_2\nu) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\nu) \\ \theta_{\mathbb{B}} &\succeq (\alpha_2 + \alpha_3) d_{\mathbb{B}}(\varkappa, \Gamma_2\nu). \end{aligned}$$

Taking norm on both side, we have

$$\|(\alpha_2 + \alpha_3)\| \|d_{\mathbb{B}}(\varkappa, \Gamma_2\nu)\| \leq 0$$

implies $\Gamma_2\nu = \varkappa$ i.e, $\Gamma_2\nu = \Gamma_4\nu = \varkappa$.
Further, the weak compatibility of (Γ_2, Γ_4) implies $\Gamma_2\Gamma_4\nu = \Gamma_4\Gamma_2\nu$, or $\Gamma_2\varkappa = \Gamma_4\varkappa$ $\therefore$ $\varkappa$ is a common coincidence point of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.
Now, to prove $\varkappa$ is common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Substituting $\vartheta = \mu$ and $\varsigma = \varkappa$ in (4.3.3), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varkappa) = d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_4\varkappa) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\mu, \Gamma_1\mu) d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_4\varkappa) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_2\varkappa) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\mu, \Gamma_2\varkappa) \\ d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa) &\succeq \alpha_3 d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa) \\ \theta_{\mathbb{B}} &\succeq (\alpha_3 - I_{\mathbb{B}}) d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa). \end{aligned}$$

Taking norm on both side, we have

$$\|\alpha_3 - 1\| \|d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varkappa, \Gamma_2\varkappa)\| = 0$. Hence, $\Gamma_2\varkappa = \varkappa$. Hence, $\Gamma_1\varkappa = \Gamma_2\varkappa = \Gamma_3\varkappa = \Gamma_4\varkappa = \varkappa$
Uniqueness : Let ϖ be another common fixed point of Γ_1, Γ_2, Γ_3 and Γ_4. Then,

substituting $\vartheta = \varpi$ and $\varsigma = \varkappa$ in (4.3.3), we have

$$\begin{aligned} d_{\mathbb{B}}(\varpi, \varkappa) = d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varkappa) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi) d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_4\varkappa) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_2\varkappa) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varkappa) = \alpha_3 d_{\mathbb{B}}(\varpi, \varkappa) \\ \theta_{\mathbb{B}} &\succeq (\alpha_3 - I_{\mathbb{B}}) d_{\mathbb{B}}(\varpi, \varkappa). \end{aligned}$$

Taking norm on both side, we have

$$\|\alpha_3 - 1\| \|d_{\mathbb{B}}(\varpi, \varkappa)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varpi, \varkappa)\| = 0$. Hence, $\varkappa = \varpi$. □

Theorem 4.3.4. *Let $(\mho, \mathbb{B}, d_{\mathbb{B}})$ be a C^*_{AV}-MS and $\Gamma_1, \Gamma_2, \Gamma_3,\ \Gamma_4 : \mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(ii) $\forall\, \vartheta, \varsigma \in \mho,\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{B}^+$ *with* $\|\alpha_1\|, \|\alpha_2\|, \|\alpha_3\| > 1$,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma); \end{aligned} \tag{4.3.4}$$

(iii) *the pairs* (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(iv) $\Gamma_1,\ \Gamma_3$ *satisfies* CLR_{Γ_4} *and* $\Gamma_2,\ \Gamma_4$ *satisfies* CLR_{Γ_3} *property.*

Then, $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$*.*

Proof. As $\Gamma_1,\ \Gamma_3$ satisfies CLR_{Γ_4} and $\Gamma_2,\ \Gamma_4$ satisfies CLR_{Γ_3} property. $\therefore \exists$ two sequences $\{\vartheta_\jmath\}$ and $\{\varsigma_\jmath\}$ in $\mho$ and $\varrho, \upsilon \in \mho$ s.t $\lim_{\jmath\to\infty} \Gamma_1\vartheta_\jmath = \lim_{\jmath\to\infty} \Gamma_3\vartheta_\jmath = \Gamma_4\varrho \in \Gamma_4(\mho)$ and $\lim_{\jmath\to\infty} \Gamma_2\varsigma_\jmath = \lim_{\jmath\to\infty} \Gamma_4\varsigma_\jmath = \Gamma_3\upsilon \in \Gamma_3(\mho)$. Substituting $\vartheta = \vartheta_\jmath$ and $\varsigma = \varsigma_\jmath$ in (4.3.4), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_4\varsigma_\jmath) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_1\vartheta_\jmath) d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_4\varsigma_\jmath) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_2\varsigma_\jmath) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_2\varsigma_\jmath). \end{aligned}$$

Taking limit $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) + \alpha_2 d_{\mathbb{B}}(\Gamma_3\upsilon, \Gamma_3\upsilon) + \alpha_3 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) \\ &\succeq \alpha_3 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) \\ \theta_{\mathbb{B}} &\succeq (\alpha_3 - I_{\mathbb{B}}) d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon). \end{aligned}$$

Taking norm on both side, we have

$$\|\alpha_3 - 1\|\|d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon)\| \leq 0$. Hence, $\Gamma_3\upsilon = \Gamma_4\varrho$.
Now, substituting $\vartheta_\jmath = \vartheta$ and $\varsigma = \varrho$ in (4.3.4), we have

$$d_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_4\varrho) \succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_1\vartheta_\jmath) d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_4\varrho) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_2\varrho) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_2\varrho).$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_2\varrho) + \alpha_3 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_2\varrho) \\ &\succeq (\alpha_2 + \alpha_3) d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_2\varrho). \end{aligned}$$

Taking norm on both side, we have

$$\|\alpha_2 + \alpha_3)\|\|d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_2\varrho)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_2\varrho)\| \leq 0$. Hence, $\Gamma_2\varrho = \Gamma_4\varrho$.
On the similar lines, $\Gamma_1\upsilon = \Gamma_3\upsilon$. Hence, $\Gamma_1\upsilon = \Gamma_3\upsilon = \Gamma_2\varrho = \Gamma_4\varrho = \varpi$.
Since, $\Gamma_1,\ \Gamma_3$ are weakly compatible mappings implies $\Gamma_1\varpi = \Gamma_1\Gamma_3\upsilon = \Gamma_3\Gamma_1\upsilon = \Gamma_3\varpi$.
Substituting $\vartheta = \varpi$ and $\varsigma = \varrho$ in (4.3.4), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varrho) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi) d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_4\varrho) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_2\varrho) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varrho) \\ d_{\mathbb{B}}(\Gamma_3\varpi, \varpi) &\succeq \alpha_3 d_{\mathbb{B}}(\Gamma_3\varpi, \varpi) \\ \theta_{\mathbb{B}} &\succeq (\alpha_3 - I_{\mathbb{B}}) d_{\mathbb{B}}(\Gamma_3\varpi, \varpi). \end{aligned}$$

Taking norm on both side, we have

$$\|\alpha_3 - 1\|\|d_{\mathbb{B}}(\Gamma_3\varpi, \varpi)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\Gamma_3\varpi, \varpi)\| = 0$. Hence, $\Gamma_3\varpi = \varpi = \Gamma_1\varpi$.
On the similar lines, $\Gamma_4\varpi = \Gamma_2\varpi = \varpi$.
Uniqueness : Let $\varkappa$ be another common fixed of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4. Substituting $\vartheta = \varpi$ and $\varsigma = \varkappa$ in (4.3.4), we have

$$\begin{aligned} d_{\mathbb{B}}(\varpi, \varkappa) = d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varkappa) &\succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi) d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_4\varkappa) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_2\varkappa) \\ &\quad + \alpha_3 d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varkappa) \\ &\succeq \alpha_3 d_{\mathbb{B}}(\varpi, \varkappa) \\ \theta_{\mathbb{B}} &\succeq (\alpha_3 - I_{\mathbb{B}}) d_{\mathbb{B}}(\varpi, \varkappa). \end{aligned}$$

Taking norm on both side, we have

$$\|\alpha_3 - 1\|\|d_{\mathbb{B}}(\varpi, \varkappa)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varpi, \varkappa)\| \leq 0$. Hence, $\varpi = \varkappa$. □

Theorem 4.3.5. *Let $(\mho, \mathbb{B}, d_{\mathbb{B}})$ be a C^*_{AV}-MS and $\Gamma_1, \Gamma_2, \Gamma_3,\ \Gamma_4 : \mho \to \mho$ satisfying:*

(i) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(i) $\forall\ \vartheta, \varsigma \in \mho,\ \alpha_1, \alpha_2 \in \mathbb{B}^+$ *with* $\|\alpha_1\|, \|\alpha_2\| > 1$,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \ &\succeq\ \alpha_1\left(\frac{d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta)d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + d_{\mathbb{B}}(\Gamma_2\varsigma, \Gamma_4\varsigma)d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_3\vartheta)}{d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma)}\right) \\ &\quad +\alpha_2 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma); \end{aligned} \tag{4.3.5}$$

(iii) the pairs (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(iv) $\Gamma_1,\ \Gamma_3$ *satisfies* CLR_{Γ_4} *and* $\Gamma_2,\ \Gamma_4$ *satisfies* CLR_{Γ_3} *property.*

Then, $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$.

Proof. As $\Gamma_1,\ \Gamma_3$ satisfies CLR_{Γ_4} and $\Gamma_2,\ \Gamma_4$ satisfies CLR_{Γ_3} property, $\exists$ two sequences $\{\vartheta_\jmath\}$ and $\{\varsigma_\jmath\}$ in $\mho$ and $\varrho, \upsilon \in \mho$ s.t $\lim\limits_{\jmath\to\infty} \Gamma_1\vartheta_\jmath = \lim\limits_{\jmath\to\infty} \Gamma_3\vartheta_\jmath = \Gamma_4\varrho \in \Gamma_4(\mho)$ and $\lim\limits_{\jmath\to\infty} \Gamma_2\varsigma_\jmath = \lim\limits_{\jmath\to\infty} \Gamma_4\varsigma_\jmath = \Gamma_3\upsilon \in \Gamma_3(\mho)$. Substituting $\vartheta = \vartheta_\jmath$ and $\varsigma = \varsigma_\jmath$ in (4.3.5), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_4\varsigma_\jmath) \ &\succeq\ \alpha_1\left(\frac{d_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_1\vartheta_\jmath)d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_4\varsigma_\jmath) + d_{\mathbb{B}}(\Gamma_2\varsigma_\jmath, \Gamma_4\varsigma_\jmath)d_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_3\vartheta_\jmath)}{d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_2\varsigma_\jmath)}\right) \\ &\quad +\alpha_2 d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_2\varsigma_\jmath). \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) \ &\succeq\ \alpha_1\left(\frac{d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho)d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) + d_{\mathbb{B}}(\Gamma_3\upsilon, \Gamma_3\upsilon)d_{\mathbb{B}}(\Gamma_3\upsilon, \Gamma_4\varrho)}{d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon)}\right) \\ &\quad +\alpha_2 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) \\ &\succeq\ \alpha_2 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) \\ \theta_{\mathbb{B}} \ &\succeq\ (\alpha_2 - I_{\mathbb{B}})d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon). \end{aligned}$$

Taking norm on both side, we have

$$\|(\alpha_2 - 1)\| \|d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_3 \upsilon)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_3 \upsilon)\| = 0$. Hence, $\Gamma_4 \varrho = \Gamma_3 \upsilon$.
Now, substituting $\vartheta_\jmath = \vartheta$ and $\varsigma = \varrho$ in (4.3.5), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3 \vartheta_\jmath, \Gamma_4 \varrho) &\succeq \alpha_1 \left(\frac{d_{\mathbb{B}}(\Gamma_3 \vartheta_\jmath, \Gamma_1 \vartheta_\jmath) d_{\mathbb{B}}(\Gamma_1 \vartheta_\jmath, \Gamma_4 \varrho) + d_{\mathbb{B}}(\Gamma_2 \varrho, \Gamma_4 \varrho) d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_3 \vartheta_\jmath)}{d_{\mathbb{B}}(\Gamma_1 \vartheta_\jmath, \Gamma_2 \varrho)} \right) \\ &\quad + \alpha_2 d_{\mathbb{B}}(\Gamma_1 \vartheta_\jmath, \Gamma_2 \varrho). \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_4 \varrho) &\succeq \alpha_1 \left(\frac{d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_4 \varrho) d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_4 \varrho) + d_{\mathbb{B}}(\Gamma_2 \varrho, \Gamma_4 \varrho) d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_4 \varrho)}{d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_2 \varrho)} \right) \\ &\quad + \alpha_2 d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_2 \varrho) \\ \theta_{\mathbb{B}} &\succeq \alpha_2 d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_2 \varrho). \end{aligned}$$

Taking norm on both side, we have

$$\|d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_2 \varrho)\| \leq 0$$

implies $\Gamma_4 \varrho = \Gamma_2 \varrho$.
On the similar lines, we have $\Gamma_1 \upsilon = \Gamma_3 \upsilon$. Hence, $\Gamma_1 \upsilon = \Gamma_3 \upsilon = \Gamma_2 \varrho = \Gamma_4 \varrho = \varpi$.
Since, Γ_1, Γ_3 are weakly compatible mapping so $\Gamma_1 \varpi = \Gamma_1 \Gamma_3 \upsilon = \Gamma_3 \Gamma_1 \upsilon = \Gamma_3 \varpi$.
Substitute $\vartheta = \varpi$ and $\varsigma = \varrho$ in (4.3.5), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_4 \varrho) &\succeq \alpha_1 \left(\frac{d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_1 \varpi) d_{\mathbb{B}}(\Gamma_1 \varpi, \Gamma_4 \varrho) + d_{\mathbb{B}}(\Gamma_2 \varrho, \Gamma_4 \varrho) d_{\mathbb{B}}(\Gamma_4 \varrho, \Gamma_3 \varpi)}{d_{\mathbb{B}}(\Gamma_1 \varpi, \Gamma_2 \varrho)} \right) \\ &\quad + \alpha_2 d_{\mathbb{B}}(\Gamma_1 \varpi, \Gamma_2 \varrho) \\ d_{\mathbb{B}}(\Gamma_3 \varpi, \varpi) &\succeq \alpha_1 \left(\frac{d_{\mathbb{B}}(\Gamma_3 \varpi, \Gamma_3 \varpi) d_{\mathbb{B}}(\Gamma_3 \varpi, \varpi) + d_{\mathbb{B}}(\varpi, \varpi) d_{\mathbb{B}}(\varpi, \Gamma_3 \varpi)}{d_{\mathbb{B}}(\Gamma_3 \varpi, \varpi)} \right) \\ &\quad + \alpha_2 d_{\mathbb{B}}(\Gamma_3 \varpi, \varpi) \\ \theta_{\mathbb{B}} &\succeq (\alpha_2 - I_{\mathbb{B}}) d_{\mathbb{B}}(\Gamma_3 \varpi, \varpi). \end{aligned}$$

Taking norm on both side, we have

$$\|(\alpha_2 - 1)\| \|d_{\mathbb{B}}(\Gamma_3 \varpi, \varpi)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\Gamma_3 \varpi, \varpi)\| = 0$. Hence, $\Gamma_3 \varpi = \varpi$ i.e, $\Gamma_3 \varpi = \varpi = \Gamma_1 \varpi$.
On the similar lines, we have $\Gamma_4 \varpi = \Gamma_2 \varpi = \varpi$. Hence, ϖ is common fixed point

of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.

Uniqueness : Let $\varkappa$ be another common fixed of $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4. Substituting $\vartheta = \varpi$ and $\varsigma = \varkappa$ in (4.3.5), we have

$$\begin{aligned} d_{\mathbb{B}}(\varpi, \varkappa) &= d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_4\varkappa) \\ &\succeq \alpha_1\left(\frac{d_{\mathbb{B}}(\Gamma_3\varpi, \Gamma_1\varpi)d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_4\varkappa) + d_{\mathbb{B}}(\Gamma_2\varkappa, \Gamma_4\varkappa)d_{\mathbb{B}}(\Gamma_4\varkappa, \Gamma_3\varpi)}{d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varkappa)}\right) \\ &\quad +\alpha_2 d_{\mathbb{B}}(\Gamma_1\varpi, \Gamma_2\varkappa) \\ \theta_{\mathbb{B}} &\succeq (\alpha_2 - I_{\mathbb{B}})d_{\mathbb{B}}(\varpi, \varkappa). \end{aligned}$$

Taking norm on both side, we have

$$\|(\alpha_2 - 1)\|\|d_{\mathbb{B}}(\varpi, \varkappa)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varpi, \varkappa)\| = 0$. Hence, $\varpi = \varkappa$. □

Theorem 4.3.6. *Let* $(\mho, \mathbb{B}, d_{\mathbb{B}})$ *be a* C^*_{AV}*-MS and* $\Gamma_1, \Gamma_2, \Gamma_3,\ \Gamma_4 : \mho \to \mho$ *satisfying:*

(*i*) $\Gamma_1(\mho) \subseteq \Gamma_4(\mho)$ *and* $\Gamma_2(\mho) \subseteq \Gamma_3(\mho)$*;*

(*ii*) $\forall\ \vartheta, \varsigma \in \mho,\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{B}^+$ *with* $\|\alpha_1\|, \|\alpha_2\|, \|\alpha_3\|, \|\alpha_4\| > 1$,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) &\succeq \alpha_1\left(\frac{d^2_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) + d^2_{\mathbb{B}}(\Gamma_2\varsigma, \Gamma_4\varsigma)}{d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + d_{\mathbb{B}}(\Gamma_2\varsigma, \Gamma_3\vartheta)}\right) \\ &\quad +\alpha_2\left(\frac{d^2_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\vartheta) + d^2_{\mathbb{B}}(\Gamma_2\varsigma, \Gamma_3\vartheta)}{d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + d_{\mathbb{B}}(\Gamma_2\varsigma, \Gamma_3\vartheta)}\right) \\ &\quad +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_3\vartheta) + d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma)\Big) \\ &\quad +\alpha_4 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma); \end{aligned} \tag{4.3.6}$$

(*iii*) *the pairs* (Γ_1, Γ_3) *and* (Γ_2, Γ_4) *are weakly compatible;*

(*iv*) $\Gamma_1,\ \Gamma_3$ *satisfies* CLR_{Γ_4} *and* $\Gamma_2,\ \Gamma_4$ *satisfies* CLR_{Γ_3} *property.*

Then, $\Gamma_1, \Gamma_2, \Gamma_3$ *and* Γ_4 *have a unique common fixed point in* $\mho$.

Proof. As $\Gamma_1,\ \Gamma_3$ satisfies CLR_{Γ_4} and $\Gamma_2,\ \Gamma_4$ satisfies CLR_{Γ_3} property. $\therefore\ \exists$ two sequences $\{\vartheta_\jmath\}$ and $\{\varsigma_\jmath\}$ in $\mho$ and $\varrho, \upsilon \in \mho$ s.t $\lim\limits_{\jmath\to\infty} \Gamma_1\vartheta_\jmath = \lim\limits_{\jmath\to\infty} \Gamma_3\vartheta_\jmath = \Gamma_4\varrho \in \Gamma_4(\mho)$

and $\lim_{\jmath\to\infty}\Gamma_2\varsigma_\jmath = \lim_{\jmath\to\infty}\Gamma_4\varsigma_\jmath = \Gamma_3\upsilon \in \Gamma_3(\mho)$. Substituting $\vartheta = \vartheta_\jmath$ and $\varsigma = \varsigma_\jmath$ in (4.3.6), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_4\varsigma_\jmath) &\succeq \alpha_1\left(\frac{d^2_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_1\vartheta_\jmath) + d^2_{\mathbb{B}}(\Gamma_2\varsigma_\jmath, \Gamma_4\varsigma_\jmath)}{d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_4\varsigma_\jmath) + d_{\mathbb{B}}(\Gamma_2\varsigma_\jmath, \Gamma_3\vartheta_\jmath)}\right) \\ &\quad +\alpha_2\left(\frac{d^2_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_1\vartheta_\jmath) + d^2_{\mathbb{B}}(\Gamma_2\varsigma_\jmath, \Gamma_3\vartheta_\jmath)}{d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_4\varsigma_\jmath) + d_{\mathbb{B}}(\Gamma_2\varsigma_\jmath, \Gamma_3\vartheta_\jmath)}\right) \\ &\quad +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_3\vartheta_\jmath) + d_{\mathbb{B}}(\Gamma_4\varsigma_\jmath, \Gamma_2\varsigma_\jmath)\Big) + \alpha_4 d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_2\varsigma_\jmath). \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) &\succeq \alpha_1\left(\frac{d^2_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) + d^2_{\mathbb{B}}(\Gamma_3\upsilon, \Gamma_3\upsilon)}{d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) + d_{\mathbb{B}}(\Gamma_3\upsilon, \Gamma_4\varrho)}\right) \\ &\quad +\alpha_2\left(\frac{d^2_{\mathbb{B}}(\Gamma_3\upsilon, \Gamma_4\varrho) + d^2_{\mathbb{B}}(\Gamma_3\upsilon, \Gamma_4\varrho)}{d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) + d_{\mathbb{B}}(\Gamma_3\upsilon, \Gamma_4\varrho)}\right) \\ &\quad +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) + d_{\mathbb{B}}(\Gamma_3\upsilon, \Gamma_3\upsilon)\Big) + \alpha_4 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) \\ &\succeq \alpha_2 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) + \alpha_4 d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon) \\ \theta_{\mathbb{B}} &\succeq (\alpha_2 + \alpha_4 - I_{\mathbb{B}}) d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon). \end{aligned}$$

Taking norm on both side, we have

$$\|(\alpha_2 + \alpha_4 - 1)\| \|d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_3\upsilon)\| = 0$. Hence, $\Gamma_4\varrho = \Gamma_3\upsilon$.
Now, substituting $\vartheta_\jmath = \vartheta$ and $\varsigma = \varrho$ in (4.3.6), we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_4\varrho) &\succeq \alpha_1\frac{d^2_{\mathbb{B}}(\Gamma_3\vartheta_\jmath, \Gamma_1\vartheta_\jmath) + d^2_{\mathbb{B}}(\Gamma_2\varrho, \Gamma_4\varrho)}{d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_4\varrho) + d_{\mathbb{B}}(\Gamma_2\varrho, \Gamma_3\vartheta_\jmath)} \\ &\quad +\alpha_2\frac{d^2_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_1\vartheta_\jmath) + d^2_{\mathbb{B}}(\Gamma_2\varrho, \Gamma_3\vartheta_\jmath)}{d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_4\varrho) + d_{\mathbb{B}}(\Gamma_2\varrho, \Gamma_3\vartheta_\jmath)} \\ &\quad +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_3\vartheta_\jmath) + d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_2\varrho)\Big) + \alpha_4 d_{\mathbb{B}}(\Gamma_1\vartheta_\jmath, \Gamma_2\varrho). \end{aligned}$$

Taking limit as $\jmath \to \infty$, we have

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) &\succeq \alpha_1\left(\frac{d^2_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) + d^2_{\mathbb{B}}(\Gamma_2\varrho, \Gamma_4\varrho)}{d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) + d_{\mathbb{B}}(\Gamma_2\varrho, \Gamma_4\varrho)}\right) \\ &\quad +\alpha_2\left(\frac{d^2_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) + d^2_{\mathbb{B}}(\Gamma_2\varrho, \Gamma_4\varrho)}{d_{\mathbb{B}}(\Gamma_4\varrho, \Gamma_4\varrho) + d_{\mathbb{B}}(\Gamma_2\varrho, \Gamma_4\varrho)}\right) \end{aligned}$$

$$
\begin{aligned}
& +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_4\varrho,\Gamma_4\varrho)+d_{\mathbb{B}}(\Gamma_2\varrho,\Gamma_4\varrho)\Big)+\alpha_4 d_{\mathbb{B}}(\Gamma_2\varrho,\Gamma_4\varrho)\\
\succeq\ & \alpha_1 d_{\mathbb{B}}(\Gamma_2\varrho,\Gamma_4\varrho)+\alpha_2 d_{\mathbb{B}}(\Gamma_4\varrho,\Gamma_3\upsilon)+\alpha_3 d_{\mathbb{B}}(\Gamma_2\varrho,\Gamma_4\varrho)+\alpha_4 d_{\mathbb{B}}(\Gamma_4\varrho,\Gamma_3\upsilon)\\
\theta_{\mathbb{B}} \succeq\ & (\alpha_1+\alpha_2+\alpha_3+\alpha_4)d_{\mathbb{B}}(\Gamma_2\varrho,\Gamma_4\varrho).
\end{aligned}
$$

Taking norm on both side, we have

$$
\|(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\|\|d_{\mathbb{B}}(\Gamma_4\varrho,\Gamma_2\varrho)\|\leq 0
$$

implies$\|d_{\mathbb{B}}(\Gamma_4\varrho,\Gamma_2\varrho)\|\leq 0$, i.e, $\Gamma_4\varrho=\Gamma_2\varrho$.
On the similar lines, we have $\Gamma_1\upsilon=\Gamma_3\upsilon$ i.e, $\Gamma_1\upsilon=\Gamma_3\upsilon=\Gamma_2\varrho=\Gamma_4\varrho=\varpi$.
Since, Γ_1,Γ_3 are weakly compatible mappings, so $\Gamma_1\varpi=\Gamma_1\Gamma_3\upsilon=\Gamma_3\Gamma_1\upsilon=\Gamma_3\varpi$.
Substituting $\vartheta=\varpi$ and $\varsigma=\varrho$ in (4.3.6), we have

$$
\begin{aligned}
d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_4\varrho) &\succeq \alpha_1\left(\frac{d^2_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_1\varpi)+d^2_{\mathbb{B}}(\Gamma_2\varrho,\Gamma_4\varrho)}{d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_4\varrho)+d_{\mathbb{B}}(\Gamma_2\varrho,\Gamma_3\varpi)}\right)\\
&\quad +\alpha_2\left(\frac{d^2_{\mathbb{B}}(\Gamma_4\varrho,\Gamma_1\varpi)+d^2_{\mathbb{B}}(\Gamma_2\varrho,\Gamma_3\varpi)}{d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_4\varrho)+d_{\mathbb{B}}(\Gamma_2\varrho,\Gamma_3\varpi)}\right)\\
&\quad +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_3\varpi)+d_{\mathbb{B}}(\Gamma_4\varrho,\Gamma_2\varrho)\Big)+\alpha_4 d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_2\varrho)\\
d_{\mathbb{B}}(\Gamma_3\varpi,\varpi) &\succeq \alpha_2 d_{\mathbb{B}}(\Gamma_3\varpi,\varpi)+\alpha_4 d_{\mathbb{B}}(\Gamma_3\varpi,\varpi)\\
\theta_{\mathbb{B}} &\succeq (\alpha_2+\alpha_4-I_{\mathbb{B}})d_{\mathbb{B}}(\Gamma_3\varpi,\varpi).
\end{aligned}
$$

Taking norm on both side, we have

$$
\|(\alpha_2+\alpha_4-1)\|\|d_{\mathbb{B}}(\varpi,\Gamma_3\varpi)\|\leq 0
$$

implies $\|d_{\mathbb{B}}(\varpi,\Gamma_3\varpi)\|=0$. Hence, $\varpi=\Gamma_3\varpi$ i.e, $\Gamma_3\varpi=\varpi=\Gamma_1\varpi$.
On the similar lines, $\Gamma_4\varpi=\Gamma_2\varpi=\varpi$. Hence, ϖ is common fixed point of $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ_4.
Uniqueness : Let $\varkappa$ be another common fixed of $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ_4. Substituting $\vartheta=\varpi$ and $\varsigma=\varkappa$ in (4.3.6), we have

$$
\begin{aligned}
d_{\mathbb{B}}(\varpi,\varkappa) &= d_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_4\varkappa)\\
&\succeq \alpha_1\left(\frac{d^2_{\mathbb{B}}(\Gamma_3\varpi,\Gamma_1\varpi)+d^2_{\mathbb{B}}(\Gamma_2\varkappa,\Gamma_4\varkappa)}{d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_4\varkappa)+d_{\mathbb{B}}(\Gamma_2\varkappa,\Gamma_3\varpi)}\right)\\
&\quad +\alpha_2\left(\frac{d^2_{\mathbb{B}}(\Gamma_4\varkappa,\Gamma_1\varpi)+d^2_{\mathbb{B}}(\Gamma_2\varkappa,\Gamma_3\varpi)}{d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_4\varkappa)+d_{\mathbb{B}}(\Gamma_2\varkappa,\Gamma_3\varpi)}\right)\\
&\quad +\alpha_3\Big(d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_3\varpi)+d_{\mathbb{B}}(\Gamma_4\varkappa,\Gamma_2\varkappa)\Big)+\alpha_4 d_{\mathbb{B}}(\Gamma_1\varpi,\Gamma_2\varkappa)\\
d_{\mathbb{B}}(\varpi,\varkappa) &\succeq \alpha_2 d_{\mathbb{B}}(\varpi,\varkappa)+\alpha_4 d_{\mathbb{B}}(\varpi,\varkappa)\\
\theta_{\mathbb{B}} &\succeq (\alpha_2+\alpha_4-I_{\mathbb{B}})d_{\mathbb{B}}(\varpi,\varkappa).
\end{aligned}
$$

Taking norm on both side, we have

$$\|(\alpha_2+\alpha_4-1)\|\|d_{\mathbb{B}}(\varpi,\varkappa)\| \leq 0$$

implies $\|d_{\mathbb{B}}(\varpi,\varkappa)\| = 0$. Hence, $\varpi = \varkappa$. □

Example 4.3.7. Let $\mho = [1,5]$ and $\mathbb{B} = \mathbb{C}$. Define $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ by $d_{\mathbb{B}}(\vartheta,\varsigma) = |\vartheta - \varsigma|$. Then, $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is C^*_{AV}-MS. Define four self mappings $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 on $\mho$ such that

$$\Gamma_1(\vartheta) = \begin{cases} 5 & \text{if } \vartheta < 3 \\ 3 & \text{if } \vartheta \geq 3 \end{cases}, \qquad \Gamma_2(\vartheta) = \begin{cases} \frac{\vartheta+3}{2} & \text{if } \vartheta \leq 3 \\ 3 & \text{if } \vartheta > 3 \end{cases}$$

$$\Gamma_3(\vartheta) = \begin{cases} 21 & \text{if } \vartheta < 3 \\ 3(4-\vartheta) & \text{if } \vartheta \geq 3. \end{cases} \qquad \Gamma_4(\vartheta) = \begin{cases} 9-2\vartheta & \text{if } \vartheta \leq 3 \\ 3 & \text{if } \vartheta > 3 \end{cases},$$

Clearly, $\Gamma_1\mho \subset \Gamma_4\mho$ and $\Gamma_2\mho \subset \Gamma_3\mho$ for all $\vartheta \in \mho$.
To prove (Γ_1, Γ_3) satisfy (CLR_{Γ_4}) and (Γ_2, Γ_4) satisfy (CLR_{Γ_3}) property, consider sequences $\{\vartheta_n\}$ and $\{\varsigma_n\}$ defined by $\varsigma_n = 3 - \dfrac{n}{n^3-2n}$ and $\vartheta_n = 3 + \dfrac{n}{n^3-2n}$.
We have $\lim_{n\to+\infty} \Gamma_1\vartheta_n = \lim_{n\to+\infty} \Gamma_3\vartheta_n = 3 = \Gamma_4 3 \in \Gamma_4\mho$ and $\lim_{n\to+\infty} \Gamma_2\varsigma_n = \lim_{n\to+\infty} \Gamma_4\varsigma_n = 3 = \Gamma_3 3 \in \Gamma_3\mho$. It can be easily proved that the pairs (Γ_1, Γ_4) and (Γ_2, Γ_3) are weakly compatible.
Following cases arises :
Case (i) : Let $\vartheta, \varsigma < 3$, $\varsigma = 2$ and $\alpha_1 = \alpha_2 = \alpha_3 = \frac{6}{5}$ $\Gamma_1\vartheta = 5$, $\Gamma_3\vartheta = 21$, $\Gamma_2\varsigma = \frac{5}{2}$ and $\Gamma_4\varsigma = 5$.

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) &= \frac{5}{2}, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) = 16, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) = 16, \\ d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) &= \frac{5}{2} \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\vartheta) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) \\ &= \alpha_1 16 * 0 + \alpha_2 \frac{5}{2} + \alpha_3 \frac{5}{2} \\ &\preceq 16 = d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma). \end{aligned}$$

Thus,

$$d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma)$$

$\forall\, \vartheta, \varsigma < 3.$
Case (ii) : Let $\vartheta, \varsigma = 3,\ \alpha_1 = \alpha_2 = \alpha_3 = \frac{6}{5}\ \Gamma_1\vartheta = 3,\ \Gamma_3\vartheta = 3,\ \Gamma_2\varsigma = 3$ and $\Gamma_4\varsigma = 3$. Then,

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) &= 0, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) = 0, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) = 0, \\ d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) &= 0 \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\vartheta) = 0. \end{aligned}$$

Therefore,

$$\alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) = 0 = d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma).$$

Thus,

$$d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma)$$

$\forall\, \vartheta, \varsigma = 3.$
Case (iii) : Let $\vartheta, \varsigma > 3,\ \vartheta = 4,\ \alpha_1 = \alpha_2 = \alpha_3 = \frac{6}{5},\ \Gamma_1\vartheta = 3,\ \Gamma_3\vartheta = 0,\ \Gamma_2\varsigma = 3$ and $\Gamma_4\varsigma = 3$.

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) &= 0, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) = 3, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) = 3, \\ d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) &= 0 \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\vartheta) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) &= \alpha_1 3 * 0 + \alpha_2 0 + \alpha_3 0 \\ &\preceq 3 = d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma). \end{aligned}$$

Thus,

$$d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma)\ \forall\, \vartheta, \varsigma > 3.$$

Hence, by the Theorem (4.3.4) the mappings $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 have a unique common fixed point. Indeed, 3 is a common unique fixed point.

Example 4.3.8. Let $\mho = [1, 6]$ and $\mathbb{B} = \mathbb{C}$. Define $d_{\mathbb{B}} : \mho \times \mho \to \mathbb{B}$ by $d_{\mathbb{B}}(\vartheta, \varsigma) = |\vartheta - \varsigma|$. Then, $(\mho, \mathbb{B}, d_{\mathbb{B}})$ is C^*_{AV}-MS. Define four self mappings $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 on $\mho$ such that

$$\Gamma_1(\vartheta) = \begin{cases} 4 & \text{if } \vartheta \leq 4 \\ 11 & \text{if } \vartheta > 4 \end{cases}, \qquad \Gamma_2(\vartheta) = \begin{cases} 4 & \text{if } \vartheta \leq 4 \\ \vartheta + 3 & \text{if } \vartheta > 4 \end{cases}$$

$$\Gamma_3(\vartheta) = \begin{cases} 16-5\vartheta & \text{if } \vartheta \le 4 \\ 0 & \text{if } \vartheta > 4. \end{cases}, \qquad \Gamma_4(\vartheta) = \begin{cases} 4 & \text{if } \vartheta < 4 \\ 7\vartheta - 24 & \text{if } \vartheta \ge 4 \end{cases}$$

Clearly, $\Gamma_1\mho \subset \Gamma_4\mho$ and $\Gamma_2\mho \subset \Gamma_3\mho$ for all $\vartheta \in \mho$.
To prove (Γ_1, Γ_3) satisfy (CLR_{Γ_1}) property, consider sequences$\{\vartheta_n\}$ defined by $\vartheta_n = 4 - \dfrac{1}{n^2 - 2n + 3}$.
We have $\lim_{n\to+\infty} \Gamma_1\vartheta_n = \lim_{n\to+\infty} \Gamma_3\vartheta_n = 4 = \Gamma_1(4)$. It can be easily proved that Γ_1, Γ_4 and Γ_2, Γ_3 are weakly compatible.
Following cases arises :
Case (i) : Let $\vartheta, \varsigma < 4$, $\vartheta = 3$ and $\alpha_1 = \alpha_2 = \alpha_3 = \frac{6}{5}$, $\Gamma_1\vartheta = 4$, $\Gamma_3\vartheta = 1$, $\Gamma_2\varsigma = 4$ and $\Gamma_4\varsigma = 4$.

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) &= 0, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) = 3, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) = 3, \\ d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) &= 0 \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\vartheta) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) \\ &= \alpha_1 16 * 0 + \alpha_2 * 0 + \alpha_3 * 0 \\ &\preceq 3 = d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \end{aligned}$$

Thus,

$$d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma)$$

$\forall\ \vartheta, \varsigma < 4$.

Case (ii) : Let $\vartheta, \varsigma = 4$, $\alpha_1 = \alpha_2 = \alpha_3 = \frac{6}{5}$, $\Gamma_1\vartheta = 4$, $\Gamma_3\vartheta = 4$, $\Gamma_2\varsigma = 4$ and $\Gamma_4\varsigma = 4$.

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) &= 0, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) = 0, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) = 0, \\ d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) &= 0 \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\vartheta) = 0. \end{aligned}$$

Therefore,

$$\alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) = 0 = d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma)$$

Thus,

$$d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma)$$

$\forall\ \vartheta, \varsigma = 4$.

Case (iii) : Let $\vartheta, \varsigma > 4,\ \varsigma = 5,\ \alpha_1 = \alpha_2 = \alpha_3 = \frac{6}{5},\ \Gamma_1\vartheta = 11,\ \Gamma_3\vartheta = 0,\ \Gamma_2\varsigma = 8$ and $\Gamma_4\varsigma = 11$.

$$\begin{aligned} d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) &= 3, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) = 11, \quad d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) = 11, \\ d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) &= 3 \quad \text{and} \quad d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_1\vartheta) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma) \\ &= \alpha_1 11 * 0 + \alpha_2 * 3 + \alpha_3 * 3 \\ &\preceq 11 = d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \end{aligned}$$

Thus,

$$d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_4\varsigma) \succeq \alpha_1 d_{\mathbb{B}}(\Gamma_3\vartheta, \Gamma_1\vartheta) d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_4\varsigma) + \alpha_2 d_{\mathbb{B}}(\Gamma_4\varsigma, \Gamma_2\varsigma) + \alpha_3 d_{\mathbb{B}}(\Gamma_1\vartheta, \Gamma_2\varsigma)$$

$\forall\ \vartheta, \varsigma > 4$. Hence, by the Theorem (4.3.3) the mappings $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 have a unique common fixed point. Indeed, 4 is a common unique fixed point.

4.4 Conclusion

In this chapter, we have introduced a novel approach to prove common fixed point results for certain types of contraction and expansion mappings on a C^*_{AV}-MS that extends, unifies and generalizes the results on common fixed point in the literature. However, under certain conditions the results proved in this chapter are reduced to some well known results of the literature.

(i) If in Theorem (4.2.2) we consider $\Gamma_1 = \Gamma_2, \Gamma_3 = \Gamma_4 = I$ then we obtain Theorem 2.1 of Ma et al. (2014).

(ii) If in Theorem (4.3.1) we consider $\Gamma_1 = \Gamma_2 = I, \Gamma_3 = \Gamma_4$ with $\alpha_1 = \alpha_2 = 0$ then we obtain Theorem (2.2) of Ma et al. (2014).

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Chapter 5

Some Results On Coupled Fixed Point, Coupled Coincidence Point and Coupled Common Fixed Point

5.1 Introduction

The present chapter of the thesis concerned with the results on the existence and uniqueness of the coupled fixed points, coupled coincidence points and coupled common fixed points. The content of this chapter is divided into two main sections. In the first section, some results on the existence and uniqueness of coupled fixed point for a mapping with mixed monotone property using C_*-class function in partially ordered C^*_{AV}-MS are established. To support our findings, some illustrative examples are discussed. In the second section, some results on coupled coincidence points and coupled common fixed points for a pair of mappings with generalized contraction in C^*_{AV}-b-MS are presented. The results of this chapter are presented in [11,12].

[11] Dhariwal, R., Kumar, D. (2022). Coupled fixed point with C_*-class function in C^*-algebra valued metric spaces. (Communicated).

[12] Dhariwal, R., Kumar, D. (2023). On Coupled common fixed point in C^*-algebra valued b-metric space. (Communicated).

5.2 Some Results in C^*-Algebra Valued Metric Space

The metric fixed point theory has experienced significant developments in the last two decades, especially within coupled fixed points. Bhaskar & Lakshmikantham (2006) introduced the mixed monotone property and established some coupled fixed point results in partially ordered metric space. After this, many researchers established coupled fixed point results using different conditions in various spaces (see, Sabetghadam et al. (2009), Ćirić & Lakshmikantham (2009), Karapınar (2010), Choudhury & Maity (2011), Alghamdi et al. (2013), Bai (2016), Qiaoling & Tianqing (2021), Mani et al. (2022), Kim (2022), Jain et al. (2023), Özkan (2023) and references cited therein). This section presents some coupled fixed point results for mixed monotone mapping using C_*-class function satisfying certain type of contraction in partially ordered complete C^*_{AV}-MS.

Definition 5.2.1. $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq)$ said to be a partially ordered C^*_{AV}-MS iff $d_{\mathbb{B}}$ is a C^*_{AV}-metric on a partially ordered set $(\mho, \preceq)$.

Theorem 5.2.2. *Consider a partially ordered complete C^*_{AV}-MS $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq)$ and $\Gamma : \mho \times \mho \to \mho$ be continuous mapping having mixed monotone property for which $\exists\ \psi_{\mathbb{B}}, \phi_{\mathbb{B}} \in \Psi_{\mathbb{B}}$ and $F^* \in C_*$ s.t $\forall\ \varsigma, \varpi, \mu, \nu \in \mho$ with $\varsigma \succeq \mu$ and $\varpi \succeq \nu$ satisfying*

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma(\varsigma,\varpi),\Gamma(\mu,\nu))\Big) \preceq \frac{1}{2}F^*\bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma,\varpi,\mu,\nu)\Big),\phi_{\mathbb{B}}\Big(\aleph(\varsigma,\varpi,\mu,\nu)\Big)\bigg), \quad (5.2.1)$$

where

$$\begin{aligned}\aleph(\varsigma,\varpi,\mu,\nu) &= \max\Big\{d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu), d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\Gamma(\nu,\mu),\nu),\\ &\quad d_{\mathbb{B}}(\Gamma(\mu,\nu),\mu)+d_{\mathbb{B}}(\varpi,\nu), d_{\mathbb{B}}(\varsigma,\Gamma(\varsigma,\varpi))+d_{\mathbb{B}}(\varpi,\Gamma(\varpi,\varsigma))\Big\}.\end{aligned}$$

If $\exists\ \varsigma_0, \varpi_0 \in \mho$ with $\varsigma_0 \preceq \Gamma(\varsigma_0,\varpi_0)$ and $\varpi_0 \succeq \Gamma(\varpi_0,\varsigma_0)$. Then, Γ has a unique coupled fixed point.

Proof. For $\varpi_0, \varsigma_0 \in \mho$, assume that $\varsigma_1 = \Gamma(\varsigma_0,\varpi_0)$ and $\varpi_1 = \Gamma(\varpi_0,\varsigma_0)$, we denote

$$\begin{aligned}\Gamma^2(\varsigma_0,\varpi_0) &= \Gamma\Big(\Gamma(\varsigma_0,\varpi_0),\Gamma(\varpi_0,\varsigma_0)\Big) = \Gamma(\varsigma_1,\varpi_1) = \varsigma_2,\\ \Gamma^2(\varpi_0,\varsigma_0) &= \Gamma\Big(\Gamma(\varpi_0,\varsigma_0),\Gamma(\varsigma_0,\varpi_0)\Big) = \Gamma(\varpi_1,\varsigma_1) = \varpi_2,\\ \Gamma^3(\varsigma_0,\varpi_0) &= \Gamma\Big(\Gamma^2(\varsigma_0,\varpi_0),\Gamma^2(\varpi_0,\varsigma_0)\Big) = \Gamma(\varsigma_2,\varpi_2) = \varsigma_3,\end{aligned}$$

$$\Gamma^3(\varpi_0,\varsigma_0)=\Gamma\Big(\Gamma^2(\varpi_0,\varsigma_0),\Gamma^2(\varsigma_0,\varpi_0)\Big)=\Gamma(\varpi_2,\varsigma_2)=\varpi_3.$$

On generalizing, we have

$$\begin{aligned}\varsigma_{\jmath+1} &= \Gamma^{\jmath+1}(\varsigma_0,\varpi_0)=\Gamma\Big(\Gamma^{\jmath}(\varsigma_0,\varpi_0),\Gamma^{\jmath}(\varpi_0,\varsigma_0)\Big),\\ \varpi_{\jmath+1} &= \Gamma^{\jmath+1}(\varpi_0,\varsigma_0)=\Gamma\Big(\Gamma^{\jmath}(\varpi_0,\varsigma_0),\Gamma^{\jmath}(\varsigma_0,\varpi_0)\Big).\end{aligned}$$

Using mixed monotone property of Γ, we have

$$\begin{aligned}\varsigma_{\jmath+1}\succeq\varsigma_{\jmath}\ \cdots\ \succeq\ \varsigma_2 &= \Gamma(\varsigma_1,\varpi_1)\succeq\Gamma(\varsigma_0,\varpi_0)=\varsigma_1\\ \varpi_{\jmath+1}\preceq\varpi_{\jmath}\ \cdots\ \preceq\ \varpi_2 &= \Gamma(\varpi_1,\varsigma_1)\preceq\Gamma(\varpi_0,\varsigma_0)=\varpi_1.\end{aligned}$$

Let $d_{\mathbb{B}_k} = d_{\mathbb{B}}(\varsigma_{k+1},\varsigma_k)+d_{\mathbb{B}}(\varpi_{k+1},\varpi_k)$. We claim that $\{d_{\mathbb{B}_k}\}$ is monotonically decreasing sequence for each $k=1,2,\cdots$. On the contrary, suppose that

$$d_{\mathbb{B}_{2\jmath+1}}\succ d_{\mathbb{B}_{2\jmath}}\succ\theta_{\mathbb{B}}\text{ for }k=2\jmath. \tag{5.2.2}$$

From (5.2.1), we have

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+2},\varsigma_{2\jmath+1})\Big) &= \psi_{\mathbb{B}}\Bigg(d_{\mathbb{B}}\Big(\Gamma(\varsigma_{2\jmath+1},\varpi_{2\jmath+1}),\Gamma(\varsigma_{2\jmath},\varpi_{2\jmath})\Big)\Bigg)\\ &\preceq \frac{1}{2}F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath})\Big),\\ &\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath})\Big)\Bigg),\end{aligned}\tag{5.2.3}$$

where

$$\begin{aligned}\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath}) &= \max\Bigg\{d_{\mathbb{B}}(\varsigma_{2\jmath+1},\varsigma_{2\jmath})+d_{\mathbb{B}}(\varpi_{2\jmath+1},\varpi_{2\jmath}),d_{\mathbb{B}}(\varsigma_{2\jmath+1},\varsigma_{2\jmath})+\\ &\quad d_{\mathbb{B}}\Big(\Gamma(\varpi_{2\jmath},\varsigma_{2\jmath}),\varpi_{2\jmath}\Big),d_{\mathbb{B}}\Big(\Gamma(\varsigma_{2\jmath},\varpi_{2\jmath}),\varsigma_{2\jmath}\Big)+d_{\mathbb{B}}(\varpi_{2\jmath+1},\varpi_{2\jmath}),\\ &\quad d_{\mathbb{B}}\Big(\varsigma_{2\jmath+1},\Gamma(\varsigma_{2\jmath+1},\varpi_{2\jmath+1})\Big)+d_{\mathbb{B}}\Big(\varpi_{2\jmath+1},\Gamma(\varpi_{2\jmath+1},\varsigma_{2\jmath+1})\Big)\Bigg\}\\ &= \max\Bigg\{d_{\mathbb{B}}(\varsigma_{2\jmath+1},\varsigma_{2\jmath})+d_{\mathbb{B}}(\varpi_{2\jmath+1},\varpi_{2\jmath}),d_{\mathbb{B}}(\varsigma_{2\jmath+1},\varsigma_{2\jmath})+\\ &\quad d_{\mathbb{B}}(\varpi_{2\jmath+1},\varpi_{2\jmath}),d_{\mathbb{B}}(\varsigma_{2\jmath+1},\varsigma_{2\jmath})+d_{\mathbb{B}}(\varpi_{2\jmath+1},\varpi_{2\jmath}),\\ &\quad d_{\mathbb{B}}(\varsigma_{2\jmath+1},\varsigma_{2\jmath+2})+d_{\mathbb{B}}(\varpi_{2\jmath+1},\varpi_{2\jmath+2})\Bigg\}\\ &= \max\Big\{d_{\mathbb{B}_{2\jmath}},d_{\mathbb{B}_{2\jmath}},d_{\mathbb{B}_{2\jmath}},d_{\mathbb{B}_{2\jmath+1}}\Big\}\\ &= \max\Big\{d_{\mathbb{B}_{2\jmath}},d_{\mathbb{B}_{2\jmath+1}}\Big\}.\end{aligned}\tag{5.2.4}$$

Similarly,

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{2\jmath+2},\varpi_{2\jmath+1})\Big) &= \psi_{\mathbb{B}}\bigg(d_{\mathbb{B}}\Big(\Gamma(\varpi_{2\jmath+1},\varsigma_{2\jmath+1}),\Gamma(\varpi_{2\jmath},\varsigma_{2\jmath})\Big)\bigg)\\ &\preceq \frac{1}{2}F^{*}\bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath})\Big),\\ &\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath})\Big)\bigg). \qquad (5.2.5)\end{aligned}$$

On adding (5.2.3) and (5.2.5), we have

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+2},\varsigma_{2\jmath+1})\Big)+\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{2\jmath+2},\varpi_{2\jmath+1})\Big) &\preceq F^{*}\bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath})\Big),\\ &\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath})\Big)\bigg).\end{aligned}$$

By using sub-additive property of $\psi_{\mathbb{B}}$, we have

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+2},\varsigma_{2\jmath+1})+d_{\mathbb{B}}(\varpi_{2\jmath+2},\varpi_{2\jmath+1})\Big) &\preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{2\jmath+2},\varpi_{2\jmath+1})\Big)\\ &\quad +\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+2},\varsigma_{2\jmath+1})\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{2\jmath+2},\varsigma_{2\jmath+1}) \;+\; d_{\mathbb{B}}(\varpi_{2\jmath+2},\varpi_{2\jmath+1})\Big) &\preceq F^{*}\bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath})\Big),\\ &\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath})\Big)\bigg). \qquad (5.2.6)\end{aligned}$$

Case (i) : If $\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath}) = d_{\mathbb{B}_{2\jmath+1}}$. From (5.2.6), we have

$$\psi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}}) \preceq F^{*}\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}}),\phi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}})\Big) \preceq \psi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}}).$$

Thus, $F^{*}\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}}),\phi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}})\Big) = \psi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}})$ implies either $\psi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}}) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}}) = \theta_{\mathbb{B}}$. Hence, $d_{\mathbb{B}_{2\jmath+1}} = \theta_{\mathbb{B}}$, a contradiction to (5.2.2).
Case (ii) : If $\aleph(\varsigma_{2\jmath+1},\varpi_{2\jmath+1},\varsigma_{2\jmath},\varpi_{2\jmath}) = d_{\mathbb{B}_{2\jmath}}$. From (5.2.6), we have

$$\psi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath+1}}) \preceq F^{*}\Big(\psi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath}}),\phi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath}})\Big) \preceq \psi_{\mathbb{B}}(d_{\mathbb{B}_{2\jmath}}).$$

Since, $\psi_{\mathbb{B}}$ is a non-decreasing function $\therefore d_{\mathbb{B}_{2\jmath+1}} \preceq d_{\mathbb{B}_{2\jmath}}$, a contradiction to (5.2.2). On the similar lines, we can prove that the result is true for $k = 2\jmath+1$. Hence, $\{d_{\mathbb{B}_k}\}$ is a monotonically decreasing and bounded sequence. Thus, $\exists\, \theta_{\mathbb{B}} \preceq \varkappa \in \mathbb{B}^{+}$

s.t $d_{\mathbb{B}_k} \to \varkappa$ as $k \to \infty$.
From (5.2.1), we have

$$\begin{aligned}
\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_k)\Big) &= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}\Big(\Gamma(\varsigma_k, \varpi_k), \Gamma(\varsigma_{k-1}, \varpi_{k-1})\Big)\Big) \\
&\preceq \frac{1}{2}F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_k, \varpi_k, \varsigma_{k-1}, \varpi_{k-1})\Big), \\
&\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_k, \varpi_k, \varsigma_{k-1}, \varpi_{k-1})\Big)\Bigg), \qquad (5.2.7)
\end{aligned}$$

and

$$\begin{aligned}
\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{k+1}, \varpi_k)\Big) &= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}\Big(\Gamma(\varpi_k, \varsigma_k), \Gamma(\varpi_{k-1}, \varsigma_{k-1})\Big)\Big) \\
&\preceq \frac{1}{2}F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_k, \varpi_k, \varsigma_{k-1}, \varpi_{k-1})\Big), \\
&\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_k, \varpi_k, \varsigma_{k-1}, \varpi_{k-1})\Big)\Bigg). \qquad (5.2.8)
\end{aligned}$$

On adding (5.2.7) and (5.2.8), we have

$$\begin{aligned}
\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_k)\Big) + \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{k+1}, \varpi_k)\Big) &\preceq F^*\Big(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_k, \varpi_k, \varsigma_{k-1}, \varpi_{k-1})\Big), \\
&\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_k, \varpi_k, \varsigma_{k-1}, \varpi_{k-1})\Big)\Big).
\end{aligned}$$

By using the sub-additive property of $\phi_{\mathbb{B}}$, we have

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_k) + d_{\mathbb{B}}(\varpi_{k+1}, \varpi_k)\Big) \preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_k)\Big) + \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{k+1}, \varpi_k)\Big).$$

Hence,

$$\begin{aligned}
\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_k) + d_{\mathbb{B}}(\varpi_{k+1}, \varpi_k)\Big) &\preceq F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_k, \varpi_k, \varsigma_{k-1}, \varpi_{k-1})\Big), \\
&\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_k, \varpi_k, \varsigma_{k-1}, \varpi_{k-1})\Big)\Bigg), \qquad (5.2.9)
\end{aligned}$$

where

$$\aleph(\varsigma_k, \varpi_k, \varsigma_{k-1}, \varpi_{k-1}) = \max\Big\{d_{\mathbb{B}_k}, d_{\mathbb{B}_{k+1}}\Big\}.$$

On taking limit as $k \to \infty$ in (5.2.9), we have

$$\psi_{\mathbb{B}}(\varkappa) \preceq F^*\Big(\psi_{\mathbb{B}}(\varkappa), \phi_{\mathbb{B}}(\varkappa)\Big) \preceq \psi_{\mathbb{B}}(\varkappa).$$

Thus, $F\Big(\psi_{\mathbb{B}}(\varkappa), \phi_{\mathbb{B}}(\varkappa)\Big) = \psi_{\mathbb{B}}(\varkappa)$ implies either $\psi_{\mathbb{B}}(\varkappa) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(\varkappa) = \theta_{\mathbb{B}}$. Hence, $\varkappa = \theta_{\mathbb{B}}$, i.e,

$$\lim_{k\to\infty}\Big(d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_k) + d_{\mathbb{B}}(\varpi_{k+1}, \varpi_k)\Big) = \theta_{\mathbb{B}},$$

implies

$$\lim_{k\to\infty} d_{\mathbb{B}}(\varsigma_{k+1}, \varsigma_k) = \lim_{k\to\infty} d_{\mathbb{B}}(\varpi_{k+1}, \varpi_k) = \theta_{\mathbb{B}}. \tag{5.2.10}$$

Now, to prove $\{\varsigma_\jmath\}$ and $\{\varpi_\jmath\}$ are $C_{seq's}$. On the contrary, suppose that $\{\varsigma_\jmath\}$ and $\{\varpi_\jmath\}$ are not C_{seq}. Then, for any $\epsilon > 0$, $\exists$ subsequences $\{\varsigma_{\imath_k}\}$, $\{\varsigma_{\jmath_k}\}$ of $\{\varsigma_\jmath\}$ and $\{\varpi_{\imath_k}\}$, $\{\varpi_{\jmath_k}\}$ of $\{\varpi_\jmath\}$ with $\jmath_k > \imath_k > k$ s.t

$$\|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}) + d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k})\| > \epsilon. \tag{5.2.11}$$

Choose $\jmath_k$ in such a way that $\jmath_k > \imath_k$ satisfying (5.2.11) and

$$\|d_{\mathbb{B}}(\varsigma_{\jmath_{k-1}}, \varsigma_{\imath_k}) + d_{\mathbb{B}}(\varpi_{\jmath_{k-1}}, \varpi_{\imath_k})\| \leq \epsilon. \tag{5.2.12}$$

Using (5.2.11) and (5.2.12), we have

$$\begin{aligned}
\epsilon &\leq \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}) + d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k})\| \leq \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\jmath_{k-1}})\| \\
&\quad + \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\jmath_{k-1}})\| + \|d_{\mathbb{B}}(\varsigma_{\jmath_{k-1}}, \varsigma_{\imath_k})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_{k-1}}, \varpi_{\imath_k})\| \\
&\leq \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\jmath_{k-1}})\| + \|d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\jmath_{k-1}})\| + \epsilon. \qquad (5.2.13)
\end{aligned}$$

Taking limit as $k \to \infty$ in (5.2.13) and using (5.2.10), we have

$$\lim_{\jmath\to\infty} \|d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}) + d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k})\| = \epsilon. \tag{5.2.14}$$

Consider,

$$\begin{aligned}
d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}) + d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k}) &\preceq d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\jmath_{k-1}}) + d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\jmath_{k-1}}) \\
&\quad + d_{\mathbb{B}}(\varsigma_{\jmath_{k-1}}, \varsigma_{\imath_{k+1}}) + d_{\mathbb{B}}(\varpi_{\jmath_{k-1}}, \varpi_{\imath_{k+1}}) \\
&\quad + d_{\mathbb{B}}(\varsigma_{\imath_{k+1}}, \varsigma_{\imath_k}) + d_{\mathbb{B}}(\varpi_{\imath_{k+1}}, \varpi_{\imath_k}) \\
&= d_{\mathbb{B}_{\jmath_{k-1}}} + d_{\mathbb{B}_{\imath_k}} + d_{\mathbb{B}}(\varsigma_{\jmath_{k-1}}, \varsigma_{\imath_{k+1}}) \\
&\quad + d_{\mathbb{B}}(\varpi_{\jmath_{k-1}}, \varpi_{\imath_{k+1}})
\end{aligned}$$

Since, $\psi_{\mathbb{B}}$ is non-decreasing. $\therefore$

$$\begin{aligned}
\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}) + d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k})\Big) &\preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}_{\jmath_{k-1}}} + d_{\mathbb{B}_{\imath_k}} + d_{\mathbb{B}}(\varsigma_{\jmath_{k-1}}, \varsigma_{\imath_{k+1}}) \\
&\quad + d_{\mathbb{B}}(\varpi_{\jmath_{k-1}}, \varpi_{\imath_{k+1}})\Big).
\end{aligned}$$

Since, $\jmath_k > \imath_k$ implies $\varsigma_{\jmath_k} \succeq \varsigma_{\imath_k}$ and $\varpi_{\jmath_k} \preceq \varpi_{\imath_k}$. From (5.2.1), we have

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{\jmath_{k+1}}, \varsigma_{\imath_{k+1}})\Big) &= \psi_{\mathbb{B}}\bigg(d_{\mathbb{B}}\Big(\Gamma(\varsigma_{\jmath_k}, \varpi_{\jmath_k}), \Gamma(\varsigma_{\imath_k}, \varpi_{\imath_k})\Big)\bigg) \\ &\preceq \frac{1}{2}F^*\bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_{\jmath_k}, \varpi_{\jmath_k}, \varsigma_{\imath_k}, \varpi_{\imath_k})\Big), \\ &\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_{\jmath_k}, \varpi_{\jmath_k}, \varsigma_{\imath_k}, \varpi_{\imath_k})\Big)\bigg), \qquad (5.2.15)\end{aligned}$$

and

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath_{k+1}}, \varpi_{\imath_{k+1}})\Big) &= \psi_{\mathbb{B}}\bigg(d_{\mathbb{B}}\Big(\Gamma(\varpi_{\jmath_k}, \varsigma_{\jmath_k}), \Gamma(\varpi_{\imath_k}, \varsigma_{\imath_k})\Big)\bigg) \\ &\preceq \frac{1}{2}F^*\bigg(\psi_{\mathbb{B}}(\aleph(\varsigma_{\jmath_k}, \varpi_{\jmath_k}, \varsigma_{\imath_k}, \varpi_{\imath_k})), \\ &\quad \phi_{\mathbb{B}}(\aleph(\varsigma_{\jmath_k}, \varpi_{\jmath_k}, \varsigma_{\imath_k}, \varpi_{\imath_k}))\bigg), \qquad (5.2.16)\end{aligned}$$

where

$$\begin{aligned}\aleph(\varsigma_{\jmath_k}, \varpi_{\jmath_k}, \varsigma_{\imath_k}, \varpi_{\imath_k}) &= \max\bigg\{d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}) + d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k}), \\ &\quad d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}) + d_{\mathbb{B}}(\Gamma(\varpi_{\imath_k}, \varsigma_{\imath_k}), \varpi_{\imath_k}), \\ &\quad d_{\mathbb{B}}(\Gamma(\varsigma_{\imath_k}, \varpi_{\imath_k}), \varsigma_{\imath_k}) + d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k}), \\ &\quad d_{\mathbb{B}}(\varsigma_{\jmath_k}, \Gamma(\varsigma_{\jmath_k}, \varpi_{\jmath_k})) + d_{\mathbb{B}}(\varpi_{\jmath_k}, \Gamma(\varpi_{\jmath_k}, \varsigma_{\jmath_k}))\bigg\}.\end{aligned}$$

Taking limit as $k \to \infty$ and using (5.2.10) & (5.2.14), we have

$$\aleph(\varsigma_{\jmath_k}, \varpi_{\jmath_k}, \varsigma_{\imath_k}, \varpi_{\imath_k}) = \max\Big\{\epsilon, d_{\mathbb{B}}(\varsigma_{\jmath_k}, \varsigma_{\imath_k}), d_{\mathbb{B}}(\varpi_{\jmath_k}, \varpi_{\imath_k}), \theta_{\mathbb{B}}\Big\}. \qquad (5.2.17)$$

By sub-additive property of $\psi_{\mathbb{B}}$, we have

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{\jmath_{k+1}}, \varsigma_{\imath_{k+1}}) + d_{\mathbb{B}}(\varpi_{\jmath_{k+1}}, \varpi_{\imath_{k+1}})\Big) \preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{\jmath_{k+1}}, \varsigma_{\imath_{k+1}}\Big) + \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi_{\jmath_{k+1}}, \varpi_{\imath_{k+1}})\Big).$$

Hence,

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma_{\jmath_{k+1}}, \varsigma_{\imath_{k+1}}) + d_{\mathbb{B}}(\varpi_{\jmath_{k+1}}, \varpi_{\imath_{k+1}})\Big) &\preceq F^*\bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma_{\jmath_k}, \varpi_{\jmath_k}, \varsigma_{\imath_k}, \varpi_{\imath_k})\Big), \\ &\quad \phi_{\mathbb{B}}\Big(\aleph(\varsigma_{\jmath_k}, \varpi_{\jmath_k}, \varsigma_{\imath_k}, \varpi_{\imath_k})\Big)\bigg) \\ &\preceq \psi_{\mathbb{B}}\Big(\aleph(\varsigma_{\jmath_k}, \varpi_{\jmath_k}, \varsigma_{\imath_k}, \varpi_{\imath_k})\Big).\end{aligned}$$

Taking limit as $k \to \infty$, we have

$$\psi_{\mathbb{B}}(\epsilon) \leq F^*\Big(\psi_{\mathbb{B}}(\epsilon), \phi_{\mathbb{B}}(\epsilon)\Big) \leq \psi_{\mathbb{B}}(\epsilon). \tag{5.2.18}$$

Thus, $F^*\Big(\psi_{\mathbb{B}}(\epsilon), \phi_{\mathbb{B}}(\epsilon)\Big) = \psi_{\mathbb{B}}(\epsilon)$ implies either $\psi_{\mathbb{B}}(\epsilon) = \theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}(\epsilon) = \theta_{\mathbb{B}}$. Hence, $\epsilon = \theta_{\mathbb{B}}$, a contradiction. So, $\{\varsigma_\jmath\}$ and $\{\varpi_\jmath\}$ are $C_{seq'}$'s. Since, $\mho$ is a complete C^*_{AV}-MS $\therefore\ \exists\, \varsigma, \varpi \in \mho$ s.t

$$\lim_{\jmath\to\infty} \varsigma_\jmath = \varsigma \quad \text{and} \quad \lim_{\jmath\to\infty} \varpi_\jmath = \varpi.$$

Also, Γ is continuous.

$$\therefore\ \varsigma = \lim_{\jmath\to\infty} \varsigma_{\jmath+1} = \lim_{\jmath\to\infty} \Gamma(\varsigma_\jmath, \varpi_\jmath) = \Gamma\Big(\lim_{\jmath\to\infty} \varsigma_\jmath, \lim_{\jmath\to\infty} \varpi_\jmath\Big) = \Gamma(\varsigma, \varpi)$$

and

$$\varpi = \lim_{\jmath\to\infty} \varpi_{\jmath+1} = \lim_{\jmath\to\infty} \Gamma(\varpi_\jmath, \varsigma_\jmath) = \Gamma\Big(\lim_{\jmath\to\infty} \varpi_\jmath, \lim_{\jmath\to\infty} \varsigma_\jmath\Big) = \Gamma(\varpi, \varsigma).$$

Hence, Γ has a coupled fixed point.
Uniqueness : Let $(\mu, \nu) \in \mho$ be another coupled fixed point of Γ. Then, from (5.2.1), we have

$$\begin{aligned} \psi_{\mathbb{B}}(d_{\mathbb{B}}(\varsigma, \mu)) &= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}\Big(\Gamma(\varsigma, \varpi), \Gamma(\mu, \nu)\Big)\Big) \\ &\preceq \frac{1}{2} F^*\Bigg(\psi_{\mathbb{B}}\Big(\aleph(\varsigma, \varpi, \mu, \nu)\Big), \phi_{\mathbb{B}}\Big(\aleph(\varsigma, \varpi, \mu, \nu)\Big)\Bigg), \end{aligned} \tag{5.2.19}$$

where

$$\begin{aligned} \aleph(\varsigma, \varpi, \mu, \nu) &= \max\Big\{ d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\varpi, \nu), d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\Gamma(\nu, \mu), \nu), \\ &\qquad d_{\mathbb{B}}(\Gamma(\mu, \nu), \mu) + d_{\mathbb{B}}(\varpi, \nu), d_{\mathbb{B}}(\varsigma, \Gamma(\varsigma, \varpi)) + d_{\mathbb{B}}(\varpi, \Gamma(\varpi, \varsigma))\Big\} \\ &= \max\Big\{ d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\varpi, \nu), d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\nu, \nu), \\ &\qquad d_{\mathbb{B}}(\mu, \mu) + d_{\mathbb{B}}(\varpi, \nu), d_{\mathbb{B}}(\varsigma, \varsigma) + d_{\mathbb{B}}(\varpi, \varpi)\Big\} \\ &= \max\Big\{ d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\varpi, \nu), d_{\mathbb{B}}(\varsigma, \mu), d_{\mathbb{B}}(\varpi, \nu), \theta_{\mathbb{B}}\Big\}. \end{aligned} \tag{5.2.20}$$

Using (5.2.20) in (5.2.19), we have

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma, \mu)\Big) = \psi_{\mathbb{B}}\Bigg(d_{\mathbb{B}}\Big(\Gamma(\varsigma, \varpi), \Gamma(\mu, \nu)\Big)\Bigg)$$

$$\preceq \frac{1}{2}F^*\Big(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big),\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)\Big) \tag{5.2.21}$$

and

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi,\nu)\Big) &= \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}\Big(\Gamma(\varpi,\varsigma),\Gamma(\nu,\mu)\Big)\Big)\\ &\preceq \frac{1}{2}F^*\Big(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big),\\ &\qquad \phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)\Big), \end{aligned} \tag{5.2.22}$$

By sub additive property of $\psi_{\mathbb{B}}$, we have

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big) \preceq \psi_{\mathbb{B}}(d_{\mathbb{B}}(\varsigma,\mu))+\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varpi,\nu)\Big).$$

Hence,

$$\begin{aligned}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big) &\preceq F^*\Big(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big),\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)\Big)\\ &\preceq \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big).\end{aligned}$$

Thus, $F^*\Big(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big),\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)\Big) = \psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)$ implies either $\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)=\theta_{\mathbb{B}}$ or $\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)=\theta_{\mathbb{B}}$ $\therefore$ $d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)=\theta_{\mathbb{B}}$. Hence, $d_{\mathbb{B}}(\varsigma,\mu)=d_{\mathbb{B}}(\varpi,\nu)=\theta_{\mathbb{B}}$ implies $\varsigma=\mu$ and $\varpi=\nu$. □

Consider $\aleph(\varsigma,\varpi,\mu,\nu)=d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)$, we have

Corollary 5.2.3. Consider a partially ordered complete C^*_{AV}-MS $(\mho,\mathbb{B},d_{\mathbb{B}},\preceq)$ and $\Gamma:\mho\times\mho\to\mho$ having mixed monotone property and $\exists\ \psi_{\mathbb{B}},\phi_{\mathbb{B}}\in\Psi_{\mathbb{B}}$ & $F^*\in C_*$ s.t $\forall\ \varsigma,\ \varpi,\ \mu,\ \nu\ \in\mho$ with $\varsigma\ \succeq\mu$ & $\varpi\ \succeq\nu$ satisfying

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}\Big(\Gamma(\varsigma,\varpi),\Gamma(\mu,\nu)\Big)\Big)\preceq\frac{1}{2}F^*\Big(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big),\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)\Big)$$

and

(i) $\{\varsigma_\jmath\}$ is non decreasing sequence with $\varsigma_\jmath\to\varsigma$, then $\varsigma_\jmath\preceq\varsigma\ \forall\ \jmath$.

(ii) $\{\varpi_\jmath\}$ is non increasing sequence with $\varpi_\jmath\to\varpi$, then $\varpi_\jmath\succeq\varpi\ \forall\ \jmath$.

If $\exists\ \varsigma_0,\varpi_0\in\mho$ with $\varsigma_0\preceq\Gamma(\varsigma_0,\varpi_0)$ and $\varpi_0\succeq\Gamma(\varpi_0,\varsigma_0)$. Then, Γ has a unique coupled fixed point.

Consider $F^*(\varkappa,\varrho)=\varkappa$ and $\psi_{\mathbb{B}}(\varkappa)=\phi_{\mathbb{B}}(\varkappa)=\varkappa$, we have

Corollary 5.2.4. Consider a partially ordered complete C^*_{AV}-MS $(\mho,\mathbb{B},d_{\mathbb{B}},\preceq)$ and $\Gamma:\mho\times\mho\to\mho$ be continuous mapping having mixed monotone property s.t $\forall\ \varsigma,\ \varpi,\ \mu,\ \nu\ \in\mho$ with $\varsigma\ \succeq\mu$ and $\varpi\ \succeq\nu$ satisfying

$$d_{\mathbb{B}}\Big(\Gamma(\varsigma,\varpi),\Gamma(\mu,\nu)\Big)\preceq\frac{1}{2}\aleph\Big(\varsigma,\varpi,\mu,\nu\Big),\tag{5.2.23}$$

where

$$\begin{aligned}\aleph\Big(\varsigma,\varpi,\mu,\nu\Big)\ &=\ max\Big\{d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu),d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\Gamma(\nu,\mu),\nu),\\ &\qquad d_{\mathbb{B}}(\Gamma(\mu,\nu),\mu)+d_{\mathbb{B}}(\varpi,\nu),d_{\mathbb{B}}(\varsigma,\Gamma(\varsigma,\varpi))+d_{\mathbb{B}}(\varpi,\Gamma(\varpi,\varsigma))\Big\}.\end{aligned}$$

If $\exists\ \varsigma_0,\varpi_0\in\mho$ with $\varsigma_0\preceq\Gamma(\varsigma_0,\varpi_0)$ and $\varpi_0\succeq\Gamma(\varpi_0,\varsigma_0)$. Then, Γ has a coupled fixed point.

Consider $F^*(\varpi,\varsigma)=k\varpi$ where $k\in[0,1)$, $\psi_{\mathbb{B}}(\varpi)=\varpi=\phi_{\mathbb{B}}(\varpi)$ and $\aleph(\varsigma,\varpi,\mu,\nu)=d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)$, we have

Corollary 5.2.5. Consider a partially ordered complete C^*_{AV}-MS $(\mho,\mathbb{B},d_{\mathbb{B}},\preceq)$ and $\Gamma:\mho\times\mho\to\mho$ having mixed monotone property s.t $\forall\ \varsigma,\ \varpi,\ \mu,\ \nu\ \in\mho$ with $\varsigma\ \succeq\mu$ and $\varpi\ \succeq\nu\ \exists$

$$d_{\mathbb{B}}\Big(\Gamma(\varsigma,\varpi),\Gamma(\mu,\nu)\Big)\preceq\frac{k}{2}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big).$$

Suppose either Γ is continuous or

(*i*) $\{\varsigma_\jmath\}$ is non decreasing sequence with $\varsigma_\jmath\to\varsigma$, then $\varsigma_\jmath\preceq\varsigma\ \forall\ \jmath$.

(*ii*) $\{\varpi_\jmath\}$ is non increasing sequence with $\varpi_\jmath\to\varpi$, then $\varpi_\jmath\succeq\varpi\ \forall\ \jmath$.

If $\exists\ \varsigma_0,\varpi_0\in\mho$ with $\varsigma_0\preceq\Gamma(\varsigma_0,\varpi_0)$ and $\varpi_0\succeq\Gamma(\varpi_0,\varsigma_0)$. Then, Γ has a coupled fixed point.

Consider $F^*(\varkappa,\varrho)=\varkappa-\varrho$ and $\aleph(\varsigma,\varpi,\mu,\nu)=d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)$, we have

Corollary 5.2.6. Consider a partially ordered complete C^*_{AV}-MS $(\mho,\mathbb{B},d_{\mathbb{B}},\preceq)$ and $\Gamma:\mho\times\mho\to\mho$ having mixed monotone property and $\exists\ \psi_{\mathbb{B}},\phi_{\mathbb{B}}\in\Psi_{\mathbb{B}}$ s.t $\forall\ \varsigma,\ \varpi,\ \mu,\ \nu\ \in\mho$ with $\varsigma\ \succeq\mu$ & $\varpi\ \succeq\nu$ satisfying

$$\psi_{\mathbb{B}}\Bigg(d_{\mathbb{B}}\Big(\Gamma(\varsigma,\varpi),\Gamma(\mu,\nu)\Big)\Bigg)\preceq\frac{1}{2}\Bigg(\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)-\phi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma,\mu)+d_{\mathbb{B}}(\varpi,\nu)\Big)\Bigg).$$

Suppose either Γ is continuous or (*i*) $\{\varsigma_\jmath\}$ is non decreasing sequence with $\varsigma_\jmath \to \varsigma$, then $\varsigma_\jmath \preceq \varsigma \ \forall \ \jmath$.

(*ii*) $\{\varpi_\jmath\}$ is non increasing sequence with $\varpi_\jmath \to \varpi$, then $\varpi_\jmath \succeq \varpi \ \forall \ \jmath$.

If $\exists \ \varsigma_0, \varpi_0 \in \mho$ with $\varsigma_0 \preceq \Gamma(\varsigma_0, \varpi_0)$ and $\varpi_0 \succeq \Gamma(\varpi_0, \varsigma_0)$. Then, Γ has a coupled fixed point.

Consider $F^*(\varkappa, \varrho) = \varkappa$ and $\aleph(\varsigma, \varpi, \mu, \nu) = d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\varpi, \nu)$, we have

Corollary 5.2.7. Consider a partially ordered complete C^*_{AV}-MS $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq)$ and $\Gamma : \mho \times \mho \to \mho$ having mixed monotone property and $\exists \ \psi_{\mathbb{B}} \in \Psi_{\mathbb{B}}$ s.t $\forall \ \varsigma, \ \varpi, \ \mu, \ \nu \ \in \mho$ with $\varsigma \ \succeq \mu$ & $\varpi \ \succeq \nu \ \exists$

$$\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\Gamma(\varsigma, \varpi), \Gamma(\mu, \nu))\Big) \preceq \frac{1}{2}\psi_{\mathbb{B}}\Big(d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\varpi, \nu)\Big). \tag{5.2.24}$$

Suppose either Γ is continuous or

(*i*) $\{\varsigma_\jmath\}$ is non decreasing sequence with $\varsigma_\jmath \to \varsigma$ then $\varsigma_\jmath \preceq \varsigma \ \forall \ \jmath$.

(*ii*) $\{\varpi_\jmath\}$ is non increasing sequence with $\varpi_\jmath \to \varpi$ then $\varpi_\jmath \succeq \varpi \ \forall \ \jmath$.

If $\exists \ \varsigma_0, \varpi_0 \in \mho$ with $\varsigma_0 \preceq \Gamma(\varsigma_0, \varpi_0)$ and $\varpi_0 \succeq \Gamma(\varpi_0, \varsigma_0)$. Then, Γ has a coupled fixed point.

Example 5.2.8. Let $\mho = [0, 1/2]$, $F^*(\varrho, \varkappa) = \varrho$, $\psi_{\mathbb{B}}(\varkappa) = \phi_{\mathbb{B}}(\varkappa) = \varkappa$ and $\aleph(\varsigma, \varpi, \mu, \nu) = d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\varpi, \nu)$. Let $d_{\mathbb{B}}(\varsigma, \varpi) = |\varsigma - \varpi| \ \forall \ \varsigma, \varpi \in \mho$. Let $\Gamma : \mho \times \mho \to \mho$ be defined as

$$\Gamma(\varsigma, \varpi) = \begin{cases} \dfrac{\varsigma^2 - \varpi^2 + 1}{4}, & \text{if } \varsigma \leq \varpi \\ \dfrac{1}{4}, & \text{otherwise.} \end{cases}$$

Then,

(*i*) $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq)$ is a partially ordered complete C^*_{AV}-MS with a natural ordering of real numbers;

(*ii*) Γ has a mixed monotone property;

(*iii*) Γ is continuous;

(*iv*) $\varsigma_0 = 0 \preceq \Gamma(0, 1/2)$ and $1/2 \succeq \Gamma(1/2, 0)$ and

$$d_{\mathbb{B}}(\Gamma(\varsigma, \varpi), \Gamma(\mu, \nu)) \preceq d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\varpi, \nu) \ \forall \ \varsigma, \varpi, \mu, \nu \in \mho.$$

Γ satisfies all the hypothesis of Theorem (5.2.2). Thus, Γ has a unique coupled fixed point. Indeed, '$(1/4, 1/4)$' is a coupled fixed point.

Example 5.2.9. Let $\mho = [0, 1]$ $F^*(\varrho, \varkappa) = \varrho$, $\psi_{\mathbb{B}}(\varkappa) = \phi_{\mathbb{B}}(\varkappa) = \varkappa$ and $\aleph(\varsigma, \varpi, \mu, \nu) = d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\varpi, \nu)$. Let $d_{\mathbb{B}}(\varsigma, \varpi) = |\varsigma - \varpi| \ \forall \ \varsigma, \varpi \in \mho$. Let $\Gamma : \mho \times \mho \to \mho$ be defined as $\Gamma(\varsigma, \varpi) = \dfrac{3\varsigma + \varpi}{4}$ Then,

(*i*) $(\mho, \mathbb{B}, d_{\mathbb{B}}, \preceq)$ is a partially ordered complete C^*_{AV}-MS with a natural ordering of real numbers;

(*ii*) Γ has a mixed monotone property;

(*iii*) Γ is continuous;

(*iv*) $\varsigma_0 = 0 \preceq \Gamma(0, 1/2)$ and $1/2 \succeq \Gamma(1/2, 0)$ and

$$d_{\mathbb{B}}(\Gamma(\varsigma, \varpi), \Gamma(\mu, \nu)) \preceq d_{\mathbb{B}}(\varsigma, \mu) + d_{\mathbb{B}}(\varpi, \nu) \ \forall \ \varsigma, \varpi, \mu, \nu \in \mho.$$

Γ satisfies all the hypothesis of Theorem (5.2.2). Thus, Γ has a unique coupled fixed point. Indeed, '$(0, 0)$' is a coupled fixed point.

5.3 Some Results in C^*-algebra valued b-metric space

Guo & Lakshmikantham (1987) introduced the notion of coupled fixed point for mapping on partially ordered sets and established some results. Later, many researchers established the results for coupled common fixed point and coupled coincidence point in various spaces (see, Ciri*ć* & Lakshmikantham (2009), Samet (2010), Choudhury & Kundu (2010), Abbas et al. (2010), Aydi (2011), Jain et al. (2014), Omran & Ozer (2019), Radenović et al. (2019), Gunaseelan et al. (2020) and references cited therein). In this section, some results on coupled common fixed points for a pair of mappings in C^*_{AV}-b-MS are presented.

Theorem 5.3.1. *Let $(\mho, \mathbb{B}, b_{\mathbb{B}})$ be a complete C^*_{AV}-b-MS and $\Gamma_1, \Gamma_2 : \mho \times \mho \to \mho$ satisfying:*

$$b_{\mathbb{B}}\Big(\Gamma_1(\varpi, \varsigma), \Gamma_2(\varkappa, \vartheta)\Big) \preceq \alpha^*\left(\frac{b_{\mathbb{B}}(\varpi, \varkappa) + b_{\mathbb{B}}(\varsigma, \vartheta)}{2}\right)\alpha, \tag{5.3.1}$$

$\forall\ \varpi, \varsigma, \varkappa, \vartheta \in \mho$ with $\alpha \in \mathbb{B}^+$ and $\|\alpha\| < 1$. Then, Γ_1 and Γ_2 have a unique coupled common fixed point.

Proof. Let ϖ_0 and ς_0 be any arbitrary points in $\mho$. Define

$$\begin{aligned}&\varpi_{2k+1} = \Gamma_1(\varpi_{2k}, \varsigma_{2k}),\ \varsigma_{2k+1} = \Gamma_1(\varsigma_{2k}, \varpi_{2k}),\ \varpi_{2k+2} = \Gamma_2(\varpi_{2k+1}, \varsigma_{2k+1}),\\ &\text{and } \varsigma_{2k+2} = \Gamma_2(\varsigma_{2k+1}, \varpi_{2k+1}) \text{ for } k = 0, 1, 2, \cdots .\end{aligned}$$

From (5.3.1), we have

$$\begin{aligned} b_{\mathbb{B}}(\varpi_{2k+1}, \varpi_{2k+2}) &= b_{\mathbb{B}}\Big(\Gamma_1(\varpi_{2k}, \varsigma_{2k}), \Gamma_2(\varpi_{2k+1}, \varsigma_{2k+1})\Big)\\ &\preceq \alpha^*\left(\frac{b_{\mathbb{B}}(\varpi_{2k}, \varpi_{2k+1}) + b_{\mathbb{B}}(\varsigma_{2k}, \varsigma_{2k+1})}{2}\right)\alpha. \end{aligned} \tag{5.3.2}$$

Similarly,

$$b_{\mathbb{B}}(\varsigma_{2k+1}, \varsigma_{2k+2}) \preceq \alpha^*\left(\frac{b_{\mathbb{B}}(\varpi_{2k}, \varpi_{2k+1}) + b_{\mathbb{B}}(\varsigma_{2k}, \varsigma_{2k+1})}{2}\right)\alpha. \tag{5.3.3}$$

On adding (5.3.2) and (5.3.3), we have

$$b_{\mathbb{B}}(\varpi_{2k+1}, \varpi_{2k+2}) + b_{\mathbb{B}}(\varsigma_{2k+1}, \varsigma_{2k+2}) \preceq \alpha^*\Big(b_{\mathbb{B}}(\varpi_{2k}, \varpi_{2k+1}) + b_{\mathbb{B}}(\varsigma_{2k}, \varsigma_{2k+1})\Big)\alpha. \tag{5.3.4}$$

Let $b_{\mathbb{B}_k} = b_{\mathbb{B}}(\varpi_k, \varpi_{k+1}) + b_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1})$. From (5.3.4), $b_{\mathbb{B}_{2k+1}} \preceq \alpha^* b_{\mathbb{B}_{2k}}\alpha$. Hence, $\{b_{\mathbb{B}_k}\}$ is monotonically decreasing sequence in $\mathbb{B}^+$.
In general, $b_{\mathbb{B}_\jmath} \preceq \alpha^*(b_{\mathbb{B}_{\jmath-1}})\alpha\ \forall\ \jmath \in \mathbb{N}$, i.e,

$$b_{\mathbb{B}_\jmath} \preceq \alpha^* b_{\mathbb{B}_{\jmath-1}}\alpha \preceq (\alpha^*)^2 b_{\mathbb{B}_{\jmath-2}}\alpha^2 \cdots \preceq (\alpha^*)^\jmath b_{\mathbb{B}_0}\alpha^\jmath = (\alpha^*)^\jmath \beta \alpha^\jmath,$$

where $\beta = b_{\mathbb{B}}(\varpi_0, \varpi_1) + b_{\mathbb{B}}(\varsigma_0, \varsigma_1)$ and $\|\alpha\| < 1$.
Then, for any $p, \imath \in \mathbb{N}$, we have

$$\begin{aligned} b_{\mathbb{B}}(\varpi_{\imath+p}, \varpi_\imath) + b_{\mathbb{B}}(\varsigma_{\imath+p}, \varsigma_\imath) \preceq\ & A\Big(b_{\mathbb{B}}(\varpi_{\imath+p}, \varpi_{\imath+p-1}) + b_{\mathbb{B}}(\varpi_{\imath+p-1}, \varpi_\imath)\\ &+ b_{\mathbb{B}}(\varsigma_{\imath+p}, \varsigma_{\imath+p-1}) + b_{\mathbb{B}}(\varsigma_{\imath+p-1}, \varsigma_\imath)\Big)\\ &\preceq A\Big(b_{\mathbb{B}}(\varpi_{\imath+p}, \varpi_{\imath+p-1}) + b_{\mathbb{B}}(\varsigma_{\imath+p}, \varsigma_{\imath+p-1})\Big)\\ &+ A^2\Big(b_{\mathbb{B}}(\varpi_{\imath+p-1}, \varpi_{\imath+p-2}) + b_{\mathbb{B}}(\varpi_{\imath+p-2}, \varpi_\imath)\end{aligned}$$

$$
\begin{aligned}
& +b_{\mathbb{B}}(\varsigma_{\imath+p-1},\varsigma_{\imath+p-2})+b_{\mathbb{B}}(\varsigma_{\imath+p-2},\varsigma_{\imath})\Big) \\
& \preceq Ab_{\mathbb{B}_{\imath+p-1}}+A^2 b_{\mathbb{B}_{\imath+p-2}}+\cdots+A^{p-1}b_{\mathbb{B}_{\imath+1}} \\
& +A^{p-1}b_{\mathbb{B}_{\imath}} \\
& \preceq A(\alpha^*)^{\imath+p-1}\beta\alpha^{\imath+p-1}+A^2(\alpha^*)^{\imath+p-2}\beta\alpha^{\imath+p-2} \\
& +\cdots+A^{p-1}(\alpha^*)^{\imath+1}\beta\alpha^{\imath+1}+A^{p-1}(\alpha^*)^{\imath}\beta\alpha^{\imath} \\
= & \sum_{k=1}^{p-1}A^k(\alpha^*)^{\imath+p-k}\beta\alpha^{\imath+p-k}+A^{p-1}(\alpha^*)^{\imath}\beta\alpha^{\imath} \\
= & \sum_{k=1}^{p-1}\Big((\alpha^*)^{\imath+p-k}A^{\frac{k}{2}}\sqrt{\beta}\Big)\Big(\sqrt{\beta}A^{\frac{k}{2}}\alpha^{\imath+p-k}\Big) \\
& +\Big((\alpha^*)^{\imath}A^{\frac{p-1}{2}}\sqrt{\beta}\Big)\Big(\sqrt{\beta}A^{\frac{p-1}{2}}\alpha^{\imath}\Big) \\
= & \sum_{k=1}^{p-1}\Big((\alpha)^{\imath+p-k}A^{\frac{k}{2}}\sqrt{\beta}\Big)^*\Big(\sqrt{\beta}A^{\frac{k}{2}}\alpha^{\imath+p-k}\Big) \\
& +\Big((\alpha)^{\imath}A^{\frac{p-1}{2}}\sqrt{\beta}\Big)^*\Big(\sqrt{\beta}A^{\frac{p-1}{2}}\alpha^{\imath}\Big) \\
= & \sum_{k=1}^{p-1}\Big|\sqrt{\beta}A^{\frac{k}{2}}\alpha^{\imath+p-k}\Big|^2+\Big|\sqrt{\beta}A^{\frac{p-1}{2}}\alpha^{\imath}\Big|^2 \\
\leq & \sum_{k=1}^{p-1}\Big\|\sqrt{\beta}A^{\frac{k}{2}}\alpha^{\imath+p-k}\Big\|^2 I_{\mathbb{B}}+\Big\|\sqrt{\beta}A^{\frac{p-1}{2}}\alpha^{\imath}\Big\|^2 I_{\mathbb{B}} \\
\leq & \|\sqrt{\beta}\|^2\sum_{k=1}^{p-1}\|\alpha\|^{2(\imath+p-k)}\|A\|^k I_{\mathbb{B}}+\|\sqrt{\beta}\|^2\|A^{\frac{p-1}{2}}\|^2\|\alpha^{\imath}\|^2 I_{\mathbb{B}} \\
= & \|\beta\|\|\alpha\|^{2(\imath+p)}\frac{\|A\|((\|A\|\|\alpha\|^{-2})^{p-1}-1)}{\|A\|-\|\alpha\|^2}I_{\mathbb{B}} \\
& +\|\beta\|\|A^{p-1}\|\|\alpha^{\imath}\|^2 I_{\mathbb{B}} \\
\preceq & \|\beta\|\frac{\|A\|^p\|\alpha\|^{2(\imath+1)}}{\|A\|-\|\alpha\|^2}I_{\mathbb{B}}+\|\beta\|\|A^{p-1}\|\|\alpha\|^{2\imath}I_{\mathbb{B}}\to\theta_{\mathbb{B}} \text{ as } \imath\to\infty.
\end{aligned}
$$

Hence, $\{\varpi_\jmath\}$ and $\{\varsigma_\jmath\}$ are $C'_{seq}s$. Since, $\mho$ is a complete C^*_{AV}-b-MS $\therefore\ \exists\ \varpi,\varsigma\in\mho$ s.t $\varpi_\jmath\to\varpi$ and $\varsigma_\jmath\to\varsigma$ as $\jmath\to\infty$.

Now, we claim that $\varpi=\Gamma_2(\varpi,\varsigma)$ and $\varsigma=\Gamma_2(\varsigma,\varpi)$. On the contrary, suppose that $\varpi\neq\Gamma_2(\varpi,\varsigma)$ and $\varsigma\neq\Gamma_2(\varsigma,\varpi)$.

$$\therefore\ b_{\mathbb{B}}(\varpi,\Gamma_2(\varpi,\varsigma))=a_1\ (say)\ \succ\theta_{\mathbb{B}}$$

Consider,

$$
\begin{aligned}
a_1 &= b_{\mathbb{B}}(\varpi,\Gamma_2(\varpi,\varsigma))\preceq b_{\mathbb{B}}(\varpi,\varpi_{2k+1})+b_{\mathbb{B}}\Big(\varpi_{2k+1},\Gamma_2(\varpi,\varsigma)\Big) \\
&= b_{\mathbb{B}}(\varpi,\varpi_{2k+1})+b_{\mathbb{B}}\Big(\Gamma_1(\varpi_{2k},\varsigma_{2k}),\Gamma_2(\varpi,\varsigma)\Big) \\
&\preceq b_{\mathbb{B}}(\varpi,\varpi_{2k+1})+\alpha^*\left(\frac{b_{\mathbb{B}}(\varpi_{2k},\varpi)+b_{\mathbb{B}}(\varsigma_{2k},\varsigma)}{2}\right)\alpha. \qquad (5.3.5)
\end{aligned}
$$

Taking limit as $k \to \infty$ in (5.3.5), we have $a_1 \preceq \theta_{\mathbb{B}}$, a contradiction. Hence, $b_{\mathbb{B}}(\varpi, \Gamma_2(\varpi, \varsigma)) = \theta_{\mathbb{B}}$, i.e, $\Gamma_2(\varpi, \varsigma) = \varpi$. On the similar lines, $\Gamma_2(\varsigma, \varpi) = \varsigma$, $\Gamma_1(\varpi, \varsigma) = \varpi$ and $\Gamma_1(\varsigma, \varpi) = \varsigma$.

Uniqueness : Let $(a, b) \in \mho \times \mho$ be another coupled common fixed point of Γ_1 and Γ_2. From (5.3.1), we have

$$b_{\mathbb{B}}(\varpi, a) = b_{\mathbb{B}}\Big(\Gamma_1(\varpi, \varsigma), \Gamma_2(a, b)\Big) \preceq \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi, a) + b_{\mathbb{B}}(\varsigma, b)}{2}\right)\alpha. \tag{5.3.6}$$

Similarly,

$$b_{\mathbb{B}}(\varsigma, b) = b_{\mathbb{B}}\Big(\Gamma_1(\varsigma, \varpi), \Gamma_2(b, a)\Big) \preceq \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi, a) + b_{\mathbb{B}}(\varsigma, b)}{2}\right)\alpha. \tag{5.3.7}$$

On adding (5.3.6) & (5.3.7) and taking norm on both side, we have

$$\|b_{\mathbb{B}}(\varpi, a) + b_{\mathbb{B}}(\varsigma, b)\| \leq \|\alpha\|^2 \|b_{\mathbb{B}}(\varpi, a) + b_{\mathbb{B}}(\varsigma, b)\|,$$

a contradiction. Hence, $(a, b) = (\varpi, \varsigma)$. □

Theorem 5.3.2. *Let* $(\mho, \mathbb{B}, b_{\mathbb{B}})$ *be a complete* C^*_{AV}*-b-MS and let* $\Gamma_1,\ \Gamma_2 : \mho \times \mho \to \mho$ *satisfying:*

$$b_{\mathbb{B}}(\Gamma_1(\varpi, \varsigma), \Gamma_2(\varkappa, \vartheta)) \preceq \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi, \Gamma_1(\varpi, \varsigma)) + b_{\mathbb{B}}(\varsigma, \Gamma_1(\varsigma, \varpi))}{2}\right)\alpha, \tag{5.3.8}$$

$\forall\ \varpi, \varsigma, \varkappa, \vartheta \in \mho$ *and* $\alpha \in \mathbb{B}^+$ *with* $\|\alpha\| < 1$. *Then,* Γ_1 *and* Γ_2 *have a unique coupled common fixed point.*

Proof. Let ϖ_0 and ς_0 be arbitrary points in $\mho$. Define

$$\begin{aligned} \varpi_{2k+1} &= \Gamma_1(\varpi_{2k}, \varsigma_{2k}), \quad \varsigma_{2k+1} = \Gamma_1(\varsigma_{2k}, \varpi_{2k}) \text{ and} \\ \varpi_{2k+2} &= \Gamma_2(\varpi_{2k+1}, \varsigma_{2k+1}), \varsigma_{2k+2} = \Gamma_2(\varsigma_{2k+1}, \varpi_{2k+1}) \text{ for } k = 0, 1, 2, \cdots. \end{aligned}$$

From (5.3.8), we have

$$\begin{aligned} b_{\mathbb{B}}(\varpi_{2k+1}, \varpi_{2k+2}) &= b_{\mathbb{B}}\Big(\Gamma_1(\varpi_{2k}, \varsigma_{2k}), \Gamma_2(\varpi_{2k+1}, \varsigma_{2k+1})\Big) \\ &\preceq \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi_{2k}, \Gamma_1(\varpi_{2k}, \varsigma_{2k}))}{2} + \frac{b_{\mathbb{B}}(\varsigma_{2k}, \Gamma_1(\varsigma_{2k}, \varpi_{2k}))}{2}\right)\alpha \\ &= \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi_{2k}, \varpi_{2k+1})) + b_{\mathbb{B}}(\varsigma_{2k}, \varsigma_{2k+1})}{2}\right)\alpha. \end{aligned} \tag{5.3.9}$$

Similarly,

$$b_{\mathbb{B}}(\varsigma_{2k+1}, \varsigma_{2k+2}) \preceq \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi_{2k}, \varpi_{2k+1}) + b_{\mathbb{B}}(\varsigma_{2k}, \varsigma_{2k+1})}{2} \right) \alpha. \tag{5.3.10}$$

On adding (5.3.9) and (5.3.10), we have

$$b_{\mathbb{B}}(\varpi_{2k+1}, \varpi_{2k+2}) + b_{\mathbb{B}}(\varsigma_{2k+1}, \varsigma_{2k+2}) \preceq \alpha^* \Big(b_{\mathbb{B}}(\varpi_{2k}, \varpi_{2k+1}) + b_{\mathbb{B}}(\varsigma_{2k}, \varsigma_{2k+1}) \Big) \alpha. \tag{5.3.11}$$

Let $b_{\mathbb{B}_k} = b_{\mathbb{B}}(\varpi_k, \varpi_{k+1}) + b_{\mathbb{B}}(\varsigma_k, \varsigma_{k+1})$. From (5.3.11), $b_{\mathbb{B}_{2k+1}} \preceq \alpha^* b_{\mathbb{B}_{2k}} \alpha$. Hence, $b_{\mathbb{B}_k}$ is monotonically decreasing sequence in $\mathbb{B}^+$.
In general, $b_{\mathbb{B}_\jmath} \preceq \alpha^* b_{\mathbb{B}_{\jmath-1}} \alpha \ \forall \ \jmath \in \mathbb{N}$, i.e,

$$b_{\mathbb{B}_\jmath} \ \preceq \ \alpha^* b_{\mathbb{B}_{\jmath-1}} \alpha \preceq (\alpha^*)^2 b_{\mathbb{B}_{\jmath-2}} \alpha^2 \cdots \preceq (\alpha^*)^\jmath b_{\mathbb{B}_0} \alpha^\jmath.$$

where $b_{\mathbb{B}_0} = b_{\mathbb{B}}(\varpi_0, \varpi_1) + b_{\mathbb{B}}(\varsigma_0, \varsigma_1)$ and $\|\alpha\| < 1$. On the similar lines of Theorem (**??**), $\{\varpi_\jmath\}$ and $\{\varsigma_\jmath\}$ are C_{seq}'s. Since, $\mho$ is a complete C^*_{AV}-b-MS $\therefore$ $\exists \ \varpi, \varsigma \in \mho$ s.t $\varpi_\jmath \to \varpi$ and $\varsigma_\jmath \to \varsigma$ as $\jmath \to \infty$.
Now, we claim that $\varpi = \Gamma_2(\varpi, \varsigma)$ and $\varsigma = \Gamma_2(\varsigma, \varpi)$. On the contrary, suppose that $\varpi \neq \Gamma_2(\varpi, \varsigma)$ and $\varsigma \neq \Gamma_2(\varsigma, \varpi)$.

$$\therefore \ b_{\mathbb{B}}(\varpi, \Gamma_2(\varpi, \varsigma)) = b_1 \ (say) \ \succ \theta_{\mathbb{B}}.$$

Consider,

$$\begin{aligned} b_1 &= b_{\mathbb{B}}(\varpi, \Gamma_2(\varpi, \varsigma)) \preceq b_{\mathbb{B}}(\varpi, \varpi_{2k+1}) + b_{\mathbb{B}}(\varpi_{2k+1}, \Gamma_2(\varpi, \varsigma)) \\ &= b_{\mathbb{B}}(\varpi, \varpi_{2k+1}) + b_{\mathbb{B}}\Big(\Gamma_1(\varpi_{2k}, \varsigma_{2k}), \Gamma_2(\varpi, \varsigma)\Big) \\ &\preceq b_{\mathbb{B}}(\varpi, \varpi_{2k+1}) + \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi_{2k}, \Gamma_1(\varpi_{2k}, \varsigma_{2k})) + b_{\mathbb{B}}(\varsigma_{2k}, \Gamma_1(\varsigma_{2k}, \varpi_{2k}))}{2} \right) \alpha \\ &= b_{\mathbb{B}}(\varpi, \varpi_{2k+1}) + \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi_{2k}, \varpi_{2k+1}) + b_{\mathbb{B}}(\varsigma_{2k}, \varsigma_{2k+1})}{2} \right) \alpha. \end{aligned} \tag{5.3.12}$$

Taking limit as $k \to \infty$ in (5.3.12), we have $b_1 \preceq \theta_{\mathbb{B}}$, a contradiction. Hence, $b_{\mathbb{B}}(\varpi, \Gamma_2(\varpi, \varsigma)) = \theta_{\mathbb{B}}$, i.e, $\Gamma_2(\varpi, \varsigma) = \varpi$. On the similar lines, we have $\Gamma_2(\varsigma, \varpi) = \varsigma$, $\Gamma_1(\varpi, \varsigma) = \varpi$ and $\Gamma_1(\varsigma, \varpi) = \varsigma$.
Uniqueness : Let $(\varpi^*, \varsigma^*) \in \mho \times \mho$ be another coupled common fixed point of Γ_1 and Γ_2. From (5.3.8), we have

$$\begin{aligned} b_{\mathbb{B}}(\varpi, \varpi^*) &= b_{\mathbb{B}}\Big(\Gamma_1(\varpi, \varsigma), \Gamma_2(\varpi^*, \varsigma^*)\Big) \\ &\preceq \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi, \Gamma_1(\varpi, \varsigma)) + b_{\mathbb{B}}(\varsigma, \Gamma_1(\varsigma, \varpi))}{2} \right) \alpha \\ &= \alpha^* \left(\frac{b_{\mathbb{B}}(\varpi, \varpi) + b_{\mathbb{B}}(\varsigma, \varsigma)}{2} \right) \alpha. \end{aligned} \tag{5.3.13}$$

Similarly,

$$\begin{aligned} b_{\mathbb{B}}(\varsigma,\varsigma^*) &= b_{\mathbb{B}}\Big(\Gamma_1(\varsigma,\varpi),\Gamma_2(\varsigma^*,\varpi^*)\Big) \\ &\preceq \alpha^*\left(\frac{b_{\mathbb{B}}(\varpi,\Gamma_1(\varpi,\varsigma))+b_{\mathbb{B}}(\varsigma,\Gamma_1(\varsigma,\varpi))}{2}\right)\alpha \\ &= \alpha^*\left(\frac{b_{\mathbb{B}}(\varpi,\varpi)+b_{\mathbb{B}}(\varsigma,\varsigma)}{2}\right)\alpha. \end{aligned} \tag{5.3.14}$$

On adding (5.3.13) & (5.3.14) and taking norm on both side, we have

$$\|b_{\mathbb{B}}(\varpi,\varpi^*)+b_{\mathbb{B}}(\varsigma,\varsigma^*)\| \leq 0.$$

Thus, $\|b_{\mathbb{B}}(\varpi,\varpi^*)+b_{\mathbb{B}}(\varsigma,\varsigma^*)\| = 0$ i.e, $\varpi = \varpi^*$ and $\varsigma = \varsigma^*$. □

Theorem 5.3.3. *Let $(\mho,\mathbb{B},b_{\mathbb{B}})$ be a complete C^*_{AV}-b-MS and $\Gamma_1,\Gamma_2 : \mho\times\mho\to\mho$ satisfying:*

$$\begin{aligned} b_{\mathbb{B}}\Big(\Gamma_1(\varpi,\varsigma),\Gamma_2(\varkappa,\vartheta)\Big) \preceq \alpha^*\Bigg(&\frac{b_{\mathbb{B}}(\varpi,\Gamma_1(\varpi,\varsigma))+b_{\mathbb{B}}(\varsigma,\Gamma_1(\varsigma,\varpi))}{2} \\ &+\frac{b_{\mathbb{B}}(\Gamma_1(\varpi,\varsigma),\varkappa)+b_{\mathbb{B}}(\Gamma_1(\varsigma,\varpi),\vartheta)}{2}\Bigg)\alpha, \end{aligned}$$

$\forall\ \varpi,\varsigma,\varkappa,\vartheta \in \mho$ and $\alpha \in \mathbb{B}^+$ with $\|\alpha\| < 1$. Then, Γ_1 and Γ_2 have a unique coupled common fixed point.

Proof. Proof follow on the similar lines as in Theorem (**??**). □

Theorem 5.3.4. *Let $(\mho,\mathbb{B},b_{\mathbb{B}})$ be a complete C^*_{AV}-b-MS and $\Gamma_1,\Gamma_2 : \mho\times\mho\to\mho$ satisfying:*

$$b_{\mathbb{B}}\Big(\Gamma_1(\varpi,\varsigma),\Gamma_1(\varkappa,\vartheta)\Big) \preceq \alpha^* b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_2(\varkappa,\vartheta)\Big)\alpha, \tag{5.3.15}$$

$\forall\ \varpi,\varsigma,\varkappa,v \in \mho$ and $\alpha \in \mathbb{B}^+$ with $\|\alpha\| < 1$. If $R(\Gamma_1) \subseteq R(\Gamma_2)$ and $R(\Gamma_2)$ is a complete subspace of $\mho$. Then, Γ_1 and Γ_2 have a unique coupled coincidence point.

Proof. Let ϖ_0 and ς_0 be any arbitrary points in $\mho$. Since, $R(\Gamma_1) \subseteq R(\Gamma_2)$ $\therefore$ $\exists\ \varpi_1,\varsigma_1 \in \mho$ s.t $\Gamma_2(\varpi_1,\varsigma_1) = \Gamma_1(\varpi_0,\varsigma_0)$ and $\Gamma_2(\varsigma_1,\varpi_1) = \Gamma_1(\varsigma_0,\varpi_0)$. Let $\varpi_2,\varsigma_2 \in \mho$ s.t $\Gamma_2(\varpi_2,\varsigma_2) = \Gamma_1(\varpi_1,\varsigma_1)$ and $\Gamma_2(\varsigma_2,\varpi_2) = \Gamma_1(\varsigma_2,\varpi_2)$. Repeating this process, we have the sequences $\{\varpi_\jmath\}$ and $\{\varsigma_\jmath\}$ s.t $\Gamma_2(\varpi_\jmath,\varsigma_\jmath) = \Gamma_1(\varpi_{\jmath-1},\varsigma_{\jmath-1})$ and $\Gamma_2(\varsigma_\jmath,\varpi_\jmath) =$

$\Gamma_1(\varsigma_{\jmath-1}, \varpi_{\jmath-1})$. Let $b_{\mathbb{B}_\jmath} = b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath+1}, \varsigma_{\jmath+1}), \Gamma_2(\varpi_\jmath, \varsigma_\jmath)\Big)$.
From (5.3.15), we have

$$\begin{aligned}
b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath+1}, \varsigma_{\jmath+1}), \Gamma_2(\varpi_\jmath, \varsigma_\jmath)\Big) &= b_{\mathbb{B}}\Big(\Gamma_1(\varpi_\jmath, \varsigma_\jmath), \Gamma_1(\varpi_{\jmath-1}, \varsigma_{\jmath-1})\Big) \\
&\preceq \alpha^* b_{\mathbb{B}}\Big(\Gamma_2(\varpi_\jmath, \varsigma_\jmath), \Gamma_2(\varpi_{\jmath-1}, \varsigma_{\jmath-1})\Big)\alpha \\
&= \alpha^* b_{\mathbb{B}}\Big(\Gamma_1(\varpi_{\jmath-1}, \varsigma_{\jmath-1}), \Gamma_1(\varpi_{\jmath-2}, \varsigma_{\jmath-2})\Big)\alpha \\
&\preceq (\alpha^*)^2 b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath-1}, \varsigma_{\jmath-1}), \Gamma_2(\varpi_{\jmath-2}, \varsigma_{\jmath-2})\Big)\alpha^2 \cdots \\
&\preceq (\alpha^*)^\jmath b_{\mathbb{B}}\Big(\Gamma_2(\varpi_1, \varsigma_1), \Gamma_2(\varpi_0, \varsigma_0)\Big)\alpha^\jmath \\
&= (\alpha^*)^\jmath b_{\mathbb{B}_0}\alpha^\jmath = (\alpha^*)^\jmath \alpha_2 \alpha^\jmath,
\end{aligned}$$

where $\alpha_2 = b_{\mathbb{B}}(\Gamma_2(\varpi_1, \varsigma_1), \Gamma_2(\varpi_0, \varsigma_0))$ and $\|\alpha\| < 1$. For any $p, \imath \in \mathbb{N}$, we have

$$\begin{aligned}
b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+p}, \varsigma_{\imath+p}), \Gamma_2(\varpi_\imath, \varsigma_\imath)\Big) &\preceq A\Big(b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+p}, \varsigma_{\imath+p}), \Gamma_2(\varpi_{\imath+p-1}, \varsigma_{\imath+p-1})\Big) \\
&\quad + b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+p-1}, \varsigma_{\imath+p-1}), \Gamma_2(\varpi_\imath, \varsigma_\imath)\Big)\Big) \\
&\preceq Ab_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+p}, \varsigma_{\imath+p}), \Gamma_2(\varpi_{\imath+p-1}, \varsigma_{\imath+p-1})\Big) \\
&\quad + A^2\Big(b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+p-1}, \varsigma_{\imath+p-1}), \Gamma_2(\varpi_{\imath+p-2}, \varsigma_{\imath+p-2})\Big) \\
&\quad + b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+p-2}, \varsigma_{\imath+p-2}), \Gamma_2(\varpi_\imath, \varsigma_\imath)\Big)\Big) \\
&\preceq Ab_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+p}, \varsigma_{\imath+p}), \Gamma_2(\varpi_{\imath+p-1}, \varsigma_{\imath+p-1})\Big) \\
&\quad + A^2\Big(b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+p-1}, \varsigma_{\imath+p-1}), \Gamma_2(\varpi_{\imath+p-2}, \varsigma_{\imath+p-2})\Big) \\
&\quad + b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+p-2}, \varsigma_{\imath+p-2}), \Gamma_2(\varpi_\imath, \varsigma_\imath)\Big)\Big) + \cdots \\
&\quad + A^{p-1}\Big(b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+2}, \varsigma_{\imath+2}), \Gamma_2(\varpi_{\imath+1}, \varsigma_{\imath+1})\Big)\Big) \\
&\quad + A^{p-1}\Big(b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\imath+1}, \varsigma_{\imath+1}), \Gamma_2(\varpi_\imath, \varsigma_\imath)\Big)\Big) \\
&\preceq Ab_{\mathbb{B}_{\imath+p-1}} + A^2 b_{\mathbb{B}_{\imath+p-2}} + \cdots + A^{p-1} b_{\mathbb{B}_{\imath+1}} \\
&\quad + A^{p-1} b_{\mathbb{B}_\imath} \\
&\preceq A(\alpha^*)^{\imath+p-1}\alpha_2\alpha^{\imath+p-1} + A^2(\alpha^*)^{\imath+p-2}\alpha_2\alpha^{\imath+p-2} \\
&\quad + \cdots + A^{p-1}(\alpha^*)^{\imath+1}\alpha_2\alpha^{\imath+1} \\
&\quad + A^{p-1}(\alpha^*)^\imath\alpha_2\alpha^\imath \\
&= \sum_{k=1}^{p-1} A^k(\alpha^*)^{\imath+p-k}\alpha_2\alpha^{\imath+p-k} + A^{p-1}(\alpha^*)^\imath\alpha_2\alpha^\imath \\
&= \sum_{k=1}^{p-1}\Big((\alpha^*)^{\imath+p-k}A^{\frac{k}{2}}\sqrt{\alpha_2}\Big)\Big(\sqrt{\alpha_2}A^{\frac{k}{2}}\alpha^{\imath+p-k}\Big) \\
&\quad + \Big((\alpha^*)^\imath A^{\frac{p-1}{2}}\sqrt{\alpha_2}\Big)\Big(\sqrt{\alpha_2}A^{\frac{p-1}{2}}\alpha^\imath\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{p-1}\Big((\alpha)^{\imath+p-k}A^{\frac{k}{2}}\sqrt{\alpha_2}\Big)^*\Big(\sqrt{\alpha_2}A^{\frac{k}{2}}\alpha^{\imath+p-k}\Big) \\
&\quad +\Big((\alpha)^{\imath}A^{\frac{p-1}{2}}\sqrt{\alpha_2}\Big)^*\Big(\sqrt{\alpha_2}A^{\frac{p-1}{2}}\alpha^{\imath}\Big) \\
&= \sum_{k=1}^{p-1}|\sqrt{\alpha_2}A^{\frac{k}{2}}\alpha^{\imath+p-k}|^2 + |\sqrt{\alpha_2}A^{\frac{p-1}{2}}\alpha^{\imath}|^2 \\
&\leq \sum_{k=1}^{p-1}\|\sqrt{\alpha_2}A^{\frac{k}{2}}\alpha^{\imath+p-k}\|^2 I_{\mathbb{B}} + \|\sqrt{\alpha_2}A^{\frac{p-1}{2}}\alpha^{\imath}\|^2 I_{\mathbb{B}} \\
&\leq \|\sqrt{\alpha_2}\|^2\sum_{k=1}^{p-1}\|\alpha\|^{2(\imath+p-k)}\|A\|^k I_{\mathbb{B}} \\
&\quad +\|\sqrt{\alpha_2}\|^2\|A^{\frac{p-1}{2}}\|^2\|\alpha^{\imath}\|^2 I_{\mathbb{B}} \\
&= \|\alpha_2\|\|\alpha\|^{2(\imath+p)}\frac{\|A\|((\|A\|\|\alpha\|^{-2})^{p-1})-1}{\|A\|-\|\alpha\|^2}I_{\mathbb{B}} + \|\alpha_2\|\|A^{p-1}\|\|\alpha^{\imath}\|^2 I_{\mathbb{B}} \\
&\leq \|\alpha_2\|\frac{\|A\|^p\|\alpha\|^{2(\imath+1)}}{\|A\|-\|\alpha\|^2}I_{\mathbb{B}} + \|\alpha_2\|\|A^{p-1}\|\|\alpha\|^{2\imath}I_{\mathbb{B}} \to \theta_{\mathbb{B}}\ (\imath\to\infty).
\end{aligned}
$$

Hence, $\{\Gamma_2(\varpi_\jmath,\varsigma_\jmath)\}$ is a C_{seq} in $R(\Gamma_2)$ and $R(\Gamma_2)$ is a complete subspace of $\mho$.
$\therefore\ \exists\, \varpi,\varsigma\in\mho$ s.t $\lim_{\jmath\to\infty}\Gamma_2(\varpi_\jmath,\varsigma_\jmath)=\Gamma_2(\varpi,\varsigma)$.
Consider,

$$
\begin{aligned}
b_{\mathbb{B}}\Big(\Gamma_2(\varpi_\jmath,\varsigma_\jmath),\Gamma_1(\varpi,\varsigma)\Big) &= b_{\mathbb{B}}\Big(\Gamma_1(\varpi_{\jmath-1},\varsigma_{\jmath-1}),\Gamma_1(\varpi,\varsigma)\Big) \\
&\preceq \alpha^* b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath-1},\varsigma_{\jmath-1}),\Gamma_2(\varpi,\varsigma)\Big)\alpha.
\end{aligned}
$$

Using $\lim_{\jmath\to\infty}\Gamma_2(\varpi_\jmath,\varsigma_\jmath)=\Gamma_2(\varpi,\varsigma)$, we have $\alpha^* b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath-1},\varsigma_{\jmath-1}),\Gamma_2(\varpi,\varsigma)\Big)\alpha\to\theta_{\mathbb{B}}$ as $\jmath\to\infty$. Hence, $\lim_{\jmath\to\infty}\Gamma_2(\varpi_\jmath,\varsigma_\jmath)=\Gamma_1(\varpi,\varsigma)$ i.e, $\Gamma_1(\varpi,\varsigma)=\Gamma_2(\varpi,\varsigma)$.
Uniqueness : Let $(\mu,\nu)\in\mho\times\mho$ s.t $\Gamma_1(\mu,\nu)=\Gamma_2(\mu,\nu)$. From (5.3.15)

$$
b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_2(\mu,\nu)\Big)=b_{\mathbb{B}}\Big(\Gamma_1(\varpi,\varsigma),\Gamma_1(\mu,\nu)\Big)\preceq\alpha^* b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_2(\mu,\nu)\Big)\alpha.
$$

Taking norm on both side, we have

$$
\Big\|b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_2(\mu,\nu)\Big)\Big\|\leq\|\alpha\|^2\Big\|b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_2(\mu,\nu)\Big)\Big\|,
$$

a contradiction. Hence, $\Gamma_2(\mu,\nu)=\Gamma_2(\varpi,\varsigma)$. □

Theorem 5.3.5. *Let $(\mho,\mathbb{B},b_{\mathbb{B}})$ be a complete C^*_{AV}-b-MS and $\Gamma_1,\Gamma_2:\mho\times\mho\to\mho$ satisfying:*

$$
b_{\mathbb{B}}\Big(\Gamma_1(\varpi,\varsigma),\Gamma_1(\varkappa,\vartheta)\Big)\preceq\alpha\bigg(b_{\mathbb{B}}\Big(\Gamma_1(\varpi,\varsigma),\Gamma_2(\varpi,\varsigma)\Big)+b_{\mathbb{B}}\Big(\Gamma_1(\varkappa,\vartheta),\Gamma_2(\varkappa,\vartheta)\Big)\bigg) \tag{5.3.16}
$$

$\forall\ \varpi, \varsigma, \varkappa, \vartheta \in \mho$ *and* $\alpha \in \mathbb{B}^+$ *with* $\|\alpha\| < \frac{1}{2}$. *If* $R(\Gamma_1) \subseteq R(\Gamma_2)$ *and* $R(\Gamma_2)$ *is a complete subspace of* $\mho$. *Then,* Γ_1 *and* Γ_2 *have a unique coupled coincidence point.*

Proof. On the similar lines of Theorem (5.3.4), we have the sequences $\{\varpi_\jmath\}$ and $\{\varsigma_\jmath\}$ s.t $\Gamma_2(\varpi_\jmath, \varsigma_\jmath) = \Gamma_1(\varpi_{\jmath-1}, \varsigma_{\jmath-1})$ and $\Gamma_2(\varsigma_\jmath, \varpi_\jmath) = \Gamma_1(\varsigma_{\jmath-1}, \varpi_{\jmath-1})$. From (5.3.16), we have

$$\begin{aligned}
b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath+1}, \varsigma_{\jmath+1}), \Gamma_2(\varpi_\jmath, \varsigma_\jmath)\Big) &= b_{\mathbb{B}}\Big(\Gamma_1(\varpi_\jmath, \varsigma_\jmath), \Gamma_1(\varpi_{\jmath-1}, \varsigma_{\jmath-1})\Big) \\
&\preceq \alpha\Bigg(b_{\mathbb{B}}\Big(\Gamma_1(\varpi_\jmath, \varsigma_\jmath), \Gamma_2(\varpi_\jmath, \varsigma_\jmath)\Big) \\
&\quad + b_{\mathbb{B}}\Big(\Gamma_1(\varpi_{\jmath-1}, \varsigma_{\jmath-1}), \Gamma_2(\varpi_{\jmath-1}, \varsigma_{\jmath-1})\Big)\Bigg) \\
&= \alpha\Bigg(b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath+1}, \varsigma_{\jmath+1}), \Gamma_2(\varpi_\jmath, \varsigma_\jmath)\Big) \\
&\quad + b_{\mathbb{B}}\Big(\Gamma_2(\varpi_\jmath, \varsigma_\jmath), \Gamma_2(\varpi_{\jmath-1}, \varsigma_{\jmath-1})\Big)\Bigg) \\
(I_{\mathbb{B}} - \alpha) b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath+1}, \varsigma_{\jmath+1}), \Gamma_2(\varpi_\jmath, \varsigma_\jmath)\Big) &\preceq \alpha b_{\mathbb{B}}\Big(\Gamma_2(\varpi_\jmath, \varsigma_\jmath), \Gamma_2(\varpi_{\jmath-1}, \varsigma_{\jmath-1})\Big).
\end{aligned}$$

Since, $\|\alpha\| \leq \frac{1}{2}$ then $(I_{\mathbb{B}} - \alpha)$ is invertible and $(I_{\mathbb{B}} - \alpha)^{-1} \in \mathbb{B}^+$. Hence,

$$b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath+1}, \varsigma_{\jmath+1}), \Gamma_2(\varpi_\jmath, \varsigma_\jmath)\Big) \preceq h b_{\mathbb{B}}\Big(\Gamma_2(\varpi_\jmath, \varsigma_\jmath), \Gamma_2(\varpi_{\jmath-1}, \varsigma_{\jmath-1})\Big), \tag{5.3.17}$$

where $h = \alpha(1-\alpha)^{-1} \in \mathbb{B}^+$ with $\|h\| < 1$. In general, we have

$$b_{\mathbb{B}}\Big(\Gamma_2(\varpi_{\jmath+1}, \varsigma_{\jmath+1}), \Gamma_2(\varpi_\jmath, \varsigma_\jmath)\Big) \preceq h^\jmath b_{\mathbb{B}}\Big(\Gamma_2(\varpi_1, \varsigma_1), \Gamma_2(\varpi_0, \varsigma_0)\Big) = h^\jmath \alpha_2,$$

where $\alpha_2 = b_{\mathbb{B}}\Big(\Gamma_2(\varpi_1, \varsigma_1), \Gamma_2(\varpi_0, \varsigma_0)\Big)$. For any $\imath \geq 1$ and $p \geq 1$, it follows from Theorem (5.3.4), $\{\Gamma_2(\varpi_\jmath, \varsigma_\jmath)\}$ is a C_{seq} in $R(\Gamma_2)$ and $R(\Gamma_2)$ is a complete subspace of $\mho$ $\therefore$ $\exists\ \varpi, \varsigma \in \mho$ s.t $\lim_{\jmath\to\infty} \Gamma_2(\varpi_\jmath, \varsigma_\jmath) = \Gamma_2(\varpi, \varsigma)$. Assume that $\lim_{\jmath\to\infty} \Gamma_2(\varpi_\jmath, \varsigma_\jmath) \neq \Gamma_1(\varpi, \varsigma)$.
Consider,

$$\begin{aligned}
b_{\mathbb{B}}\Big(\Gamma_2(\varpi_\jmath, \varsigma_\jmath), \Gamma_1(\varpi, \varsigma)\Big) &= b_{\mathbb{B}}\Big(\Gamma_1(\varpi_{\jmath-1}, \varsigma_{\jmath-1}), \Gamma_1(\varpi, \varsigma)\Big) \\
&\preceq \alpha\Bigg(b_{\mathbb{B}}\Big(\Gamma_1(\varpi_{\jmath-1}, \varsigma_{\jmath-1}), \Gamma_2(\varpi_{\jmath-1}, \varsigma_{\jmath-1})\Big) \\
&\quad + b_{\mathbb{B}}\Big(\Gamma_2(\varpi, \varsigma), \Gamma_1(\varpi, \varsigma)\Big)\Bigg) \\
&= \alpha\Bigg(b_{\mathbb{B}}\Big(\Gamma_2(\varpi_\jmath, \varsigma_\jmath), \Gamma_2(\varpi_{\jmath-1}, \varsigma_{\jmath-1})\Big)
\end{aligned}$$

$$+b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_1(\varpi,\varsigma)\Big)\Big). \tag{5.3.18}$$

Taking limit as $\jmath \to \infty$ (5.3.18), we have

$$b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_1(\varpi,\varsigma)\Big) \preceq \alpha b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_1(\varpi,\varsigma)\Big). \tag{5.3.19}$$

Taking norm on both side in (5.3.19)

$$\Big\|b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_1(\varpi,\varsigma)\Big)\Big\| \leq \|\alpha\|\Big\|b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_1(\varpi,\varsigma)\Big)\Big\|,$$

a contradiction. Hence, $\lim_{\jmath\to\infty}\Gamma_2(\varpi_\jmath,\varsigma_\jmath) = \Gamma_1(\varpi,\varsigma)$, i.e, $\Gamma_1(\varpi,\varsigma) = \Gamma_2(\varpi,\varsigma)$.
Uniqueness : Let $(\mu,\nu) \in \mho\times\mho$ s.t $\Gamma_1(\mu,\nu) = \Gamma_2(\mu,\nu)$. Using (5.3.16), we have

$$\begin{aligned} b_{\mathbb{B}}\Big(\Gamma_2(\varpi,\varsigma),\Gamma_2(\mu,\nu)\Big) &= b_{\mathbb{B}}\Big(\Gamma_1(\varpi,\varsigma),\Gamma_1(\mu,\nu)\Big) \\ &\preceq \alpha\bigg(b_{\mathbb{B}}\Big(\Gamma_1(\varpi,\varsigma),\Gamma_2(\varpi,\varsigma)\Big) + b_{\mathbb{B}}\Big(\Gamma_1(\mu,\nu),\Gamma_2(\mu,\nu)\Big)\bigg). \end{aligned}$$

Taking norm on both side, we have $\|b_{\mathbb{B}}(\Gamma_2(\varpi,\varsigma),\Gamma_2(\mu,\nu))\| = 0$. Hence, $\Gamma_2(\varpi,\varsigma) = \Gamma_2(\mu,\nu)$. □

Theorem 5.3.6. *Let $(\mho,\mathbb{B},b_{\mathbb{B}})$ be a complete C^*_{AV}-b-MS and let $\Gamma_1,\Gamma_2 : \mho\times\mho \to \mho$ satisfying:*

$$b_{\mathbb{B}}(\Gamma_1(\varpi,\varsigma),\Gamma_1(\varkappa,\vartheta)) \preceq \alpha b_{\mathbb{B}}(\Gamma_1(\varpi,\varsigma),\Gamma_2(\varkappa,\vartheta)) + \alpha b_{\mathbb{B}}(\Gamma_1(\varkappa,\vartheta),\Gamma_2(\varpi,\varsigma)),$$

$\forall\ \varpi,\varsigma,\varkappa,\vartheta \in \mho$ and $\alpha \in \mathbb{B}^+$ with $\|\alpha\| < \frac{1}{2}$. If $R(\Gamma_1) \subseteq R(\Gamma_2)$ and $R(\Gamma_2)$ is a complete subspace of $\mho$. Then, Γ_1 and Γ_2 have a unique coupled coincidence point.

Proof. Proof follow on the similar lines in Theorem (5.3.5). □

Example 5.3.7. Let $\mho = [0,1)$, $\mathbb{B} = \mathbb{C}$ and $d_{\mathbb{B}} : \mho\times\mho \to \mathbb{B}$ as $d_{\mathbb{B}}(\varpi,\vartheta) = \begin{cases} |\varpi| + |\vartheta|, & \text{if } \varpi \neq \vartheta \\ 0, & \text{otherwise.} \end{cases}$ $\forall\ \varpi,\vartheta \in \mho$. Then, $(\mho,\mathbb{B},b_{\mathbb{B}})$ is a C^*_{AV}-b-MS. Let $\Gamma_1(\varpi,\vartheta) = \dfrac{\varpi^2-\vartheta^2}{3}$ and $\Gamma_2(\varpi,\vartheta) = \dfrac{\varpi+\vartheta}{2}$.
Clearly,

$$b_{\mathbb{B}}\Big(\Gamma_1(\varpi,\varsigma),\Gamma_2(\varkappa,\vartheta)\Big) \preceq \alpha^*\bigg(\frac{b_{\mathbb{B}}(\varpi,\varkappa) + b_{\mathbb{B}}(\varsigma,\vartheta)}{2}\bigg)\alpha, \tag{5.3.20}$$

$\forall\ \varpi,\varsigma,\varkappa,\vartheta \in \mho$ with $\alpha \in \mathbb{B}^+$ and $\|\alpha\| < 1$. Hence, by Theorem (5.3.1), Γ_1 and Γ_2 have a unique coupled common fixed point. Indeed, $(0,0)$ is a unique coupled common fixed point in $\mho$.

5.4 Conclusion

In this chapter, we have introduced a novel approach to prove coupled fixed point, coupled common fixed point, and coupled coincidence point results for C_*-class function as well as particular contraction mapping in C^*_{AV}-MS and C^*_{AV}-b-MS that extends unifies and generalizes the results on coupled fixed point in the literature. However, under certain conditions the results proved in this chapter are reduced to some well known results of the literature.

(*i*) If in Theorem (5.2.2) we consider $\mathbb{A} = \mathbb{R}$, $F^*(r,t) = kr$ where $k \in [0,1)$, $\psi(t) = t = \phi(t)$ and $M(x,y,u,v) = d(x,u) + d(y,v)$ then we can obtained Theorem 2.1 in Bhaskar & Lakshmikantham (2006).

(*ii*) If in Theorem (5.2.2) we consider $\mathbb{A} = \mathbb{R}$, $F^*(r,t) = r - t$ and $M(x,y,u,v) = d(x,u) + d(y,v)$ then we can obtained Theorem 2.1 in Luong & Thuan (2011).

(*iii*) If in Theorem (5.2.2) we consider $\mathbb{A} = \mathbb{R}$, $F^*(r,t) = r$ and $M(x,y,u,v) = d(x,u) + d(y,v)$ then we can obtained Theorem 2.1 in Isik & Türkoglu (2014).

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List of Publications

1. Kumar, D., **Dhariwal, R.**, Park, C., and Lee, J. R. (2021). On fixed point in C^*-algebra valued metric spaces using C-class function. International Journal of Nonlinear Analysis and Application, 12 (2) 1157-1161.

2. Kumar, D., and **Dhariwal, R.** (2022). On existence and uniqueness of common Fixed Point in C^*-algebra valued metric spaces. Science & Technology Asia, 27 (2), 27-41. (SCOPUS, SJR 0.152)

3. **Dhariwal, R.,** and Kumar, D. (2022). On unification of common fixed point in C^*-algebra valued metric spaces. Journal of Physics: Conference Series, 2267, 012108. (SCOPUS, SJR 0.183)

4. **Dhariwal, R.,** and Kumar, D. (2022). On C_*-class F-contraction in C^*-algebra valued metric space. Science & Technology Asia, 28 (3), 29-36. (SCOPUS, SJR 0.152)

5. **Dhariwal, R.,** and Kumar, D. (2023). Common fixed point of two pairs of weakly compatible mappings using rational type contraction in C^*-algebra value metric space. Nonlinear Studies, 30 (1) (2023), 199-212. (SCOPUS, SJR 0.166)

6. **Dhariwal, R.,** and Kumar, D. (2023). Some fixed point results using $\alpha - \psi$-type contractive mapping in C^*-algebra valued partial metric space with application, Applied Mathematics E-Notes, 23, 412-423. (ESCI and Scopus SJR 0.27)

7. **Dhariwal, R.,** and Kumar, D. (2022). Some results on common fixed point using expansion mapping in C^*-algebra valued metric spaces, Iranian Journal of Mathematical Science and Informatics. (Accepted) (ESCI and Scopus SJR 0.207)

8. **Dhariwal, R.,** and Kumar, D. (2023). On C^*-algebra valued b_R-metric space, fixed point theorems and its application. (Communicated)

9. **Dhariwal, R.,** and Kumar, D. (2023). On a new approach to establish the existence of coincidence point in C^*-algebra valued metric space. (Communicated)

10. **Dhariwal, R.,** and Kumar, D. (2023). Existence of coincidence point in C^*-algebra valued metric space. (Communicated)

11. **Dhariwal, R.,** and Kumar, D. (2023). Coupled fixed point with C_*-class function in C^*-algebra valued metric space. (Communicated)

12. **Dhariwal, R.,** and Kumar, D. (2023). On coupled common fixed point in C^*-algebra valued b-metric space. (Communicated)

Papers Presented in Conferences

1. **Dhariwal, R.,** and Kumar, D. (2021). On unification of common fixed point in C^*-algebra valued metric spaces, "RAFAS-2021" held at Lovely Professional University, Phagwara, Punjab, India on June 25-26, 2021.

2. **Dhariwal, R.,** and Kumar, D. (2022). On a new approach to establish the existence of coincidence point in C^*-algebra valued metric space, "ICNAA-2022" held at Assam Don Bosco University, Sonapur, Assam, India on November 22-23, 2022.

Workshop and Conferences Attended

1. Attended National Conference on Advances in Mathematical Sciences organised by Guwahati University, Guwahati, Assam.

2. Attended International e-conference on Fixed Point Theory and its Application to Real World Problem (Dated – 27/06/2020) organised by Department of Mathamatics, Govt. Post Graduate College, Maldevta, Raipur, Uttarakhand, India.

3. Participation in Online Webinar on Nonuniqueness of Fixed Point, its Geometry and Application organised by Maitreyi College, Delhi University, New Delhi, India (16/07/2021).

4. Attended e-Workshop on Various Application of Fixed point Theory organised by King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia. (Dated – 14-15/09/2020).

5. Participation in One week International e-Faculty Development Programme on Fixed Point Theory and Its Application organised by Manipal University, Jaipur, Rajasthan, India (Dated 15-19/09/2020).