A STUDY ON 3-EQUITABLE AND DIVISOR 3-EQUITABLE LABELING OF GRAPHS

Thesis Submitted for the Award of the Degree of

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2024

Declaration of Authorship

I declare that the thesis entitled "A Study on 3-equitable and Divisor 3-equitable Labeling of Graphs" submitted for award of degree of Doctor of Philosophy in Mathematics, Lovely Professional University, Phagwara, is my own work conducted under the supervision of Dr. Arunava Majumder, Assistant Professor in Department of Mathematics at School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab and Co- Supervision of Dr. A. Parthiban, Assistant Professor Grade 2, Department of Mathematics, School of Advanced Sciences (SAS), Vellore Institute of Technology (VIT), Vellore, Tamil Nadu, India. I further declare that the work reported herein is original and does not form part of any other thesis or dissertation on the basis of which a degree or award was conferred on an earlier occasion or to any other scholar.

Signed:

Author: Sangeeta
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I hereby certify that SANGEETA, has completed the dissertation titled: "A Study on 3-equitable and Divisor 3-equitable Labeling of Graphs" under my supervision and the work done by him is worthy of consideration for the award of the degree of Doctor of Philosophy in Mathematics. I further certify that:

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Date: July 2024

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Date: July 2024

Abstract

A Study on 3-equitable and Divisor 3-equitable Labeling of Graphs

by SANGEETA

Many structures involving real world situations can be conveniently represented on paper by means of a diagram consisting of a set of points together with the lines (curves) joining some or all pairs of these points. As used in graph theory, the term Graph does not refer to data charts, such as line graphs or bar graphs instead, it refers to a set of vertices (points or nodes) and of edges (lines or links) that connect the vertices. A graph is denoted by $G(V, E)$ where V represents a non-empty set of vertices (nodes) and E denotes a set of edges. "A graph labeling is an assignment of integers to the vertices or the edges, or both, subject to certain conditions. If the domain is the set of vertices, then the labeling is called the vertex labeling. If the domain is the set of edges, then the labeling is called the edge labeling. If the labels are assigned to the vertices and also to the edges of a graph, such a labeling is called total. An enormous body of literature has grown around graph labeling in the last four decades. Labeled graphs provide mathematical models for a broad range of applications. The qualitative labeling of a graph elements has been used in diverse fields such as conflict resolutions in social psychology, energy crises etc. Quantitative labeling of graph elements has been used in missile guidance codes, radar location codes, coding theory, x-ray crystallography, astronomy, circuit design, communication network etc. For any graph $G(V, E)$ and $k > 0$, assign vertex labels from $\{0, 1, ..., k-1\}$ such that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by at most one. A graph G with such an assignment of labels is called k-equitable". When $k = 3$, it becomes a 3-equitable labeling. In 2019, Sweta Srivastav et al. introduced the notion of divisor 3-equitable labeling of graphs. "A bijection $f : V(G) \to \{1, 2, \ldots, n\}$ induces a function $f' : E(G) \to \{0, 1, 2\}$ defined by for each edge $e = xy$, (i) $f'(e) = 1\ if\ f(x)|f(y)$ or $f(y)|f(x)$, (ii) $f'(e) = 2$ if $f(x)/f(y) = 2$ or $f(y)/f(x) = 2$, and (iii) $f'(e) = 0$ otherwise such that $|e_{f'}(i) - e_{f'}(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph which admits a divisor 3- equitable

labeling is called a divisor 3-equitable graph". Thus, the thesis titled: "A Study on 3-equitable and Divisor 3-equitable Labeling of Graphs" includes the following objectives:

1. Establishing the 3-equitable labeling of some new classes of graphs.
2. Deriving the 3-equitable labeling of total graph, middle graph, central graph, degree splitting graph, and Mycielskian graphs of some graphs.
3. Obtaining the divisor 3-equitable labeling of some new graphs.
4. Deriving the divisor 3-equitable labeling of total graph, middle graph, central graph, degree splitting graph, and Mycielskian graphs of various graphs.

The present thesis is made up to analyze mainly 3-equitable labeling and divisor 3-equitable labeling of various graphs. Some new graph families have been discovered for 3-equitable labeling and divisor 3-equitable labeling. Some existing results on 3-equitable labeling and divisor 3-equitable labeling are extended by using graph operations like total graph, middle graph, central graph, degree splitting graph, and Mycielskian graph. A characterization result is introduced for the graphs satisfying the condition of 3-equitable labeling. Using both labeling with specific properties, finding the results on a particular graph labeling technique involving the conditions based on the characteristic of such specific labeling, has a good scope of research in the field of graph theory.

The thesis "A Study on 3-equitable and Divisor 3-equitable Labeling of Graphs" has been divided into five chapters of which the first chapter, gives introduction to graph theory and graph labeling along with a review of literature of graph labeling. Also, some basic preliminaries, graph operations, and applications are provided.

In chapter 2, '3-equitable Labeling of Various Graphs' the 3-equitable labeling has been established for some new graphs under two graph operations. The non-existence of 3-equitable for some graphs are also investigated by the method of contradiction.

In chapter 3, '3-equitable Labeling of Some Special Classes of Graphs' the 3-equitable labeling of total graph, middle graph, central graph, degree splitting graph, and Mycielskian graph of some graphs have been derived. The existence and non-existence of 3-equitable labeling for some families of graphs have also been investigated.

In chapter 4, 'Divisor 3-Equitable Labeling of Various Graphs' several results on divisor 3-equitable labeling have been presented with proof in detail and sufficient number of illustrations with figures of some famous named graphs. “The existence and non-existence of divisor 3-equitable labeling” for several families of graphs have also been investigated.

In chapter 5, ‘Divisor 3-Equitable Labeling of Some Classes of Graphs’ different new results on divisor 3-equitable labeling have been attained. Several families of graphs are considered and categorized on the basis of acceptance or non-acceptance of divisor 3-equitable labeling by the technique called method of contradiction.

List of publications arise from this thesis and bibliography have been put at the end of this thesis.

Acknowledgements

It gives me immense pleasure to submit my Ph.D. thesis A Study on 3-equitable and Divisor 3-equitable Labeling of Graphs. Research work is a field that involves both challenges and opportunities. This research work could not have been possible without vital assistance from various individuals. I take this opportunely to acknowledge the contribution of all who provided their support in several ways to make this research possible.

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A special gratitude goes out to my fellow research scholar Dr. Ram Dayal has provided unconditional support and guidance to me for explaining the research concepts and laying the foundation for my research.

With fond love and gratitude, I dedicate this thesis to my mother who prays for me from heaven.

July 2024 SANGEETA

Contents

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List of Symbols and Abbreviations

$N(v)$ Neighborhood of a vertex v

N Set of natural numbers

Z Set of integers

R Set of real numbers

$|V(G)|$ Cardinality of vertex set of G

$|E(G)|$Cardinality of Edge set of G

$D3EL$ Divisor 3-equitable labeling

$D3EG$ Divisor 3-equitable graph

P_n Path graph on n vertices

C_n Cycle graph on n vertices

H_n Helm graph on n vertices

W_n Wheel graph on n vertices

$B_{n,n}$ Bistar graph

AC_n Armed Crown graph

C'_n Duplicating a vertex by an edge in a cycle C_n

G_n Gear graph on n vertices

P_n^+ Comb graph on n vertices

$\mu(G)$ Myceilskian graph of G

$S(G)$ Subdivision of graph G

$DS(G)$ Degree splitting graph of G

$M(G)$ Middle graph of G

$T(G)$ Total graph of G

K_n Complete graph on n vertices

$K_{m,n}$ Complete bipartite graph

TS_n Triangular Snake graph on n vertices

$J(m,n)$ Jelly Fish graph

$U(m,n)$ Umbrella graph

$S'(B_{n,n})$ Splitting graph of Bistar graph

$D_2(B_{n,n})$ Shadow graph of Bistar graph

$B_{n,n}^2$ Square graph of Bistar graph

$SplP_m$ Splitting path graph

$L_{m,n}$ Lollipop graph

$CG(L_{m,n})$ Central Graph of Lollipop Graph

$F_{m,n}$ Fan graph

$TG(F_{m,n})$ Total Graph of Fan Graph

L^n Ladder graph on n vertices

$MG(L_n)$ Middle graph of ladder graph

F_n Friendship graph on n vertices

$DSG(F_n)$ Degree splitting graph of friendship graph

$\mu(P_n)$ Mycielski graph of path graph

S_{n+1} Star graph

$e_d(i)$ number of lines labeled with label i under *d*

$e_d(j)$ number of lines labeled with label j under *d*

$v_d(i)$ Number of vertices with vertex label i under *d*

$v_d(j)$ Number of vertices with vertex label j under *d*

$DSG(L_n)$ Degree splitting graph of ladder graph

$DSG(TS_n)$ Degree splitting graph of triangular snake graph

$DSG(L_{m,n})$ Degree splitting graph of lollipop graph

$TG(U_{m,n})$ Total graph of umbrella graph

$TG(TS_n)$ Total graph of triangular snake graph

$TG\left(W_n\right)$ Total graph of wheel graph

$\mu\left(W_n\right)$ Mycielski's graph of wheel graph

$MG\left(L_{m,n}\right) M$ iddle graph of lollipop graph

$CG(L_n)$ Central graph of ladder graph

$DSG(C_n)$ Degree splitting graph of cycle graph

$DSG(P_n)$ Degree splitting graph of path graph

$MG\left(P_n\right)$ Middle graph of path graph

$MG\left(C_n\right)$ Middle graph of cycle graph

$CG\left(TS_n\right)$ Central graph of triangular snake graph

Chapter 1

Introduction

In this chapter, a brief and concise introduction to graph theory and graph labeling along with a few notable applications are given. Basic definitions and terminologies are also presented to understand the study undertaken. The main theme of the thesis is introduced with a broad review of literature.

1.1 Graph Theory

Graph theory is discovered by the Swiss mathematician Leonhard Euler during the course of finding a solution to the famous Konigsberg bridge problem in 1736. In 1878, English mathematician James Joseph Sylvester had published a paper and introduced the term 'Graph' first time in it. Hungarian mathematician Denes Konig wrote the first textbook in the area of Graph theory in 1936. American mathematics expert Frank Harary authored a significant book on Graph theory which was published in 1969, which acts as the most important study tool of Graph theory [10, 17].

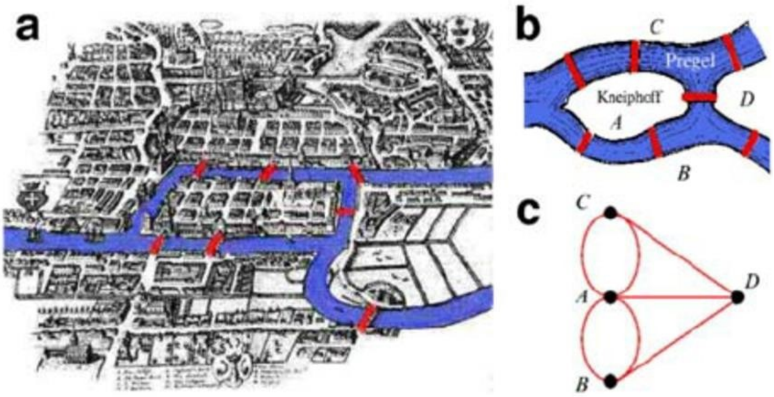

FIGURE 1.1: Konigsberg bridge

"It is possible to pinpoint the beginning of graph theory to 1735, when Swiss mathematician Leonhard Euler found an answer to the Konigsberg bridge puzzle.

An historical conundrum, known as the Königsberg Bridge Problem, included finding a way over each of the seven bridges that span a branched river that flows past an island without having to traverse any of them more than once. Such a road does not exist, according to Euler. He essentially proved the first graph theory theorem with only references to the actual arrangement of the bridges in his demonstration." [29, 33].

"Nothing more was done in the field over the following 100 years. For use in electrical networks, G. R. Kirchhoff (1824–1887) established the idea of trees in 1847. Ten years later, A. Cayley (1821-1895), while attempting to list the isomers of saturated hydrocarbons C_nH_{2n+2}, made the discovery of trees. Two more important pillars of graph theory were established during the period of Kirchhoff and Cayley. One was the Four-Color Conjecture, which postulates that four colours are more than enough to colour an atlas so that the nations with shared borders are given separate hues. The Four-Color Problem is thought to have been introduced for the first time by A. F. Mobius (1790-1868) in one of his lectures in 1840. A. De Morgan (1806-1871) examined this issue with his colleagues about ten years later." [54, 89].

1.1.1 Preliminaries

"The fundamental definitions, results and concepts discussed in this subsection are very essential for the study undertaken and are mainly given by Harary [33] and Bondy and Murthy [37]".

"A diagram made up of a set of points and the lines (curves) connecting some or all of these points can be used to conveniently describe many structures involving real-world events. In terms of graph theory, a graph is a collection of nodes (points or vertices) and lines (edges or links) that connect the nodes. It does not refer to data charts like line graphs or bar graphs. A graph is represented by the symbol $G(V, E)$, where V stands for a set of non-empty nodes and E stands for a set of lines. The number of nodes in G determines its order, while the number of lines determines its size. Two or more lines that join the same pair of distinct nodes are called parallel lines [**?**]".

Definition 1.1.1. "A caterpillar is a tree in which a single path (the spine) is incident to (or contains) every edge".

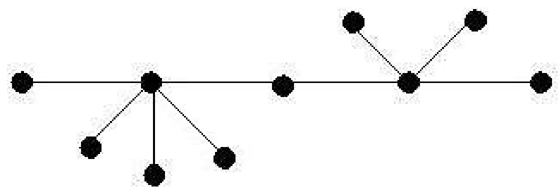

FIGURE 1.2: Caterpillar graph

Definition 1.1.2. "The cartesian product of $H_\alpha(V_\alpha, E_\alpha)$ and $H_\beta(V_\beta, E_\beta)$ denoted by $H_\alpha \times H_\beta$ is the graph with node set $V = V_\alpha \times V_\beta$ consisting of nodes $x = (x_\alpha, x_\beta)$, $y = (y_\alpha, y_\beta)$ (x_α, $y_\alpha \in V_\alpha$, x_β, $y_\beta \in V_\beta$) such that x and y are adjacent in $H_\alpha \times H_\beta$ wherever ($x_\alpha = y_\alpha$ and x_β is adjacent to y_β) or ($x_\beta = y_\beta$ and x_α is adjacent to y_α)".

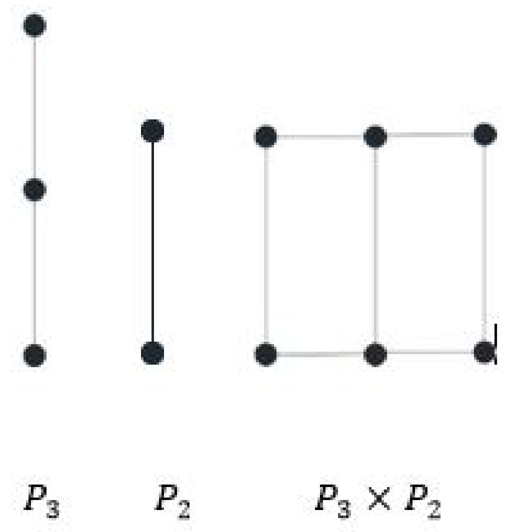

FIGURE 1.3: $P_3 \times P_2$

Definition 1.1.3. "If S and T are two graphs such that $V(S) \cap V(T) = \emptyset$, then join of S and T is denoted by $S + T$ with $V(S+T) = V(S) \cup V(T)$, $E(S+T) = E(S) \cup E(T) \cup J$, where $J = \{uv/u \in V(S), v \in V(T)\}$".

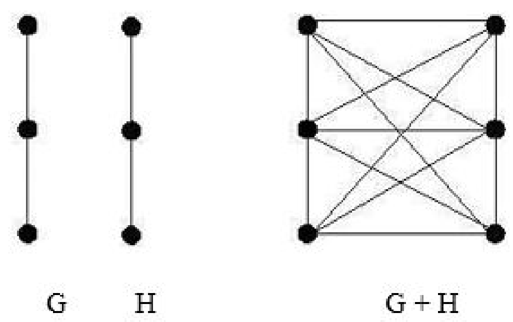

FIGURE 1.4: $G + H$

Definition 1.1.4. "A graph S is a subgraph of T if $V(S) \subseteq V(T), E(S) \subseteq E(T)$ and one can write $S \subseteq T$. When $S \subseteq T$ but $S \neq T$, written by $S \subset T$ and S is called a proper subgraph of T".

Definition 1.1.5. "A spanning subgraph is a subgraph containing all the vertices of G. A spanning subgraph need not contain all the edges in G".

Definition 1.1.6. "The subgraph of G whose vertex set is H and whose edge set is the set of those edges of G that have both ends in H is known as the subgraph of G induced by H and is denoted by $G[H]$, we say that $G[H]$ is an induced subgraph of G".

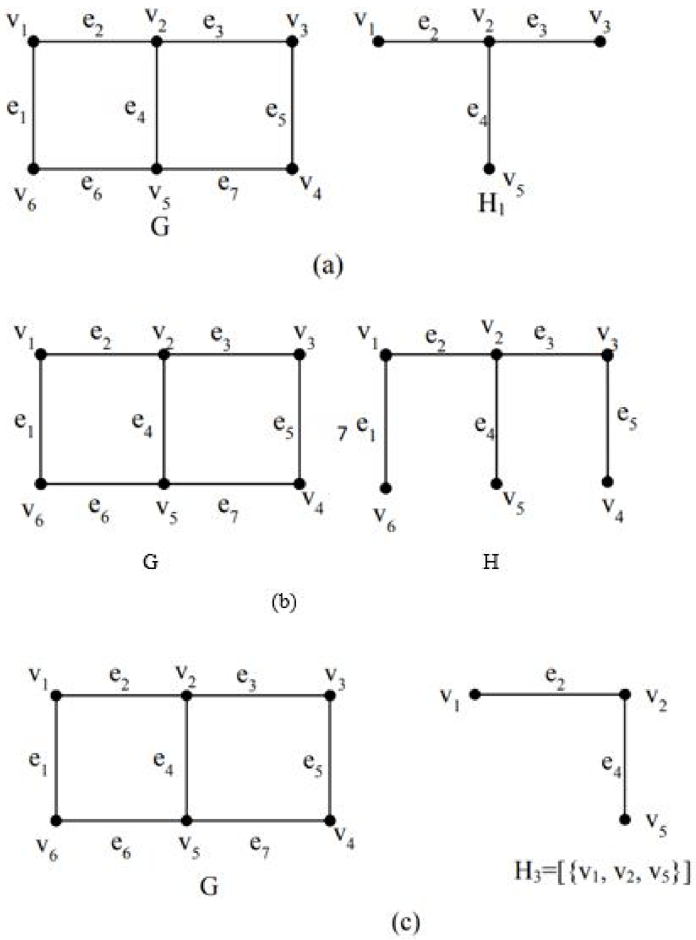

FIGURE 1.5: (a) Subgraph (b) Spanning subgraph (c) Induced subgraph

c

1.2 GRAPH LABELING

One of the intriguing subfields of graph theory with a broad range of applications is graph labelling. The 1960s saw the invention of graph labelling. Graph labeling techniques trace their origin to labeling presented by Rosa [17] in 1967. Graph labeling provides mathematical models for a wide range of applications. Over 2500 studies published over the past 50 years have examined more than 200 graph labelling strategies. [43]. "An assignment of integers to the nodes or lines or both in a graph is called labelling, provided that certain requirements are met. If the domain is the set of nodes, then its about the nodes labeling. The labelling is referred to as edge labelling if the domain is a collection of lines. Total labelling occurs when labels are assigned to both the nodes and the lines of G.

1.2.1 3-EQUITABLE LABELING

Cahit [18] first suggested dispersing the nodes and edge labels in 1990. "For $G = (V,\ E)$, a map g from $V(G)$ to $\{0,\ 1,\ 2\}$ with an induced function $g*$ from $E(G)$ to $\{0,\ 1,\ 2\}$ given by $g^*(e = xy) = |g(x) - g(y)|$ is 3-equitable labeling if the number of nodes with label s and t differ by at most 1 and in the same way the number of lines with label s and t differ by at most 1, $0 \leq s,\ t \leq 2,\ s \neq t$. Also $|v_g(s) - v_g(t)| < 1$ and $|e_g\,(s) - e_g\,(t)| < 1;\ 0 \leq s,\ t \leq 2$". A graph which admits 3-equitable labeling is called a 3-equitable graph.

1.2.2 D3EL

In 2019, Sweta Srivastav et al. introduced the notion of $D3EL$ of graphs. "A bijection $f : V\,(G) \to \{1,\ 2,\ \ldots,\ n\}$ induces a function $f' : E\,(G) \to \{0,\ 1,\ 2\}$ defined by for each edge $e = xy$, (i) $f'(e) = 1$ if $f(x)|f\,(y)\,or\,f\,(y)\,|f(x)$, (ii) $f'(e) = 2$ if $f\,(x)\,/f\,(y) = 2$ or $f\,(y)\,/f\,(x) = 2$, and (iii) $f'\,(e) = 0$ otherwise such that $|e_{f'}\,(i) - e_{f'}\,(j)\,| \leq 1$ for all $0 \leq i,\ j \leq 2$". A graph which admits $D3EL$ is called $D3EG$ [72].

1.3 LITERATURE REVIEW

Only non-trivial, simple, finite, connected, and undirected graph $G = (V(G),\ E(G))$ with p nodes and q lines are considered. In this section, a summary of the results concerning the 3-equitable and $D3EL$ of graphs are given. Further, a few relevant definitions and other necessary results which are important for the present investigations are also given.

Definition 1.3.1. [8] "A chord of $C_n$is an edge joining two non-adjacent nodes of C_n".

Definition 1.3.2. [20] "If a divide b then there is a positive integer k such that $b = ka$. It is denoted by $a \mid b$. If a does not divide b, then it is denoted by $a \nmid b$".

Definition 1.3.3. [20] "The divisor function of an integer $d(n)$ is defined by $d(n) = \Sigma 1$.

Definition 1.3.4. [20] "Let n be an integer and x be a real number. The divisor summability function is defined as $D(x) = \Sigma d(n)$.

Definition 1.3.5. [21] "A ternary node labeling of G is called a 3-equitable labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i,\ j \leq 2$. G is 3-equitable if it admits 3-equitable labeling".

Definition 1.3.6. [36] "The middle graph of G, $M(G)$ is the graph whose nodes set is $V(G) \cup E(G)$, and two nodes are adjacent if

(i) They are adjacent lines of G or

(ii) One is a node of G and the other is an edge incident with it".

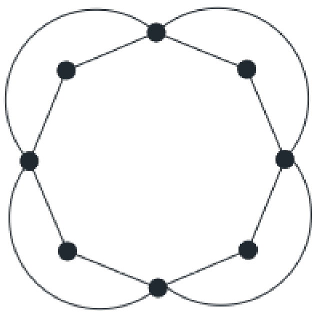

FIGURE 1.6: $M(C_4)$

Definition 1.3.7. [78] "The central graph of G, $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the non-adjacent nodes of G in $C(G)$.

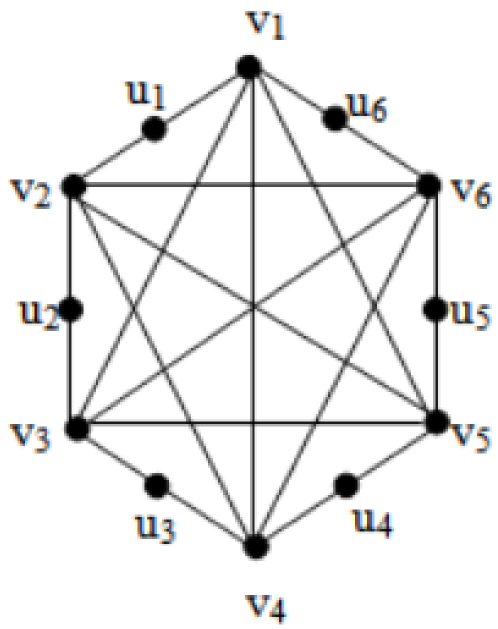

FIGURE 1.7: $C(C_6)$

Definition 1.3.8. [83] "For G, the splitting graph $S'(G)$ of G is obtained by adding new nodes v' corresponding to each nodes v of G such that $N(v) = N(v')$".

Definition 1.3.9. [83] "Let H be with $V = S_1 \cup S_2 \cup \ldots \cup S_i \cup T$, where each S_i is a set of nodes having minimum of 2 nodes of same degree and $T = V \setminus \cup S_i$. The degree splitting graph of H, $DSG(H)$, is obtained from H by adding nodes $w_1, w_2, \ldots, w_t$ and joining each node of S_i for $1 \leq i \leq t$".

Definition 1.3.10. [23] "The jelly fish graph $J(m, n)$ is obtained from C_4: v_1, v_2, v_3, v_4 by joining v_1 and v_3 with an edge and appending m pendent lines to v_2 and n pendent lines to v_4".

Definition 1.3.11. [6] "The total graph $T(G)$ of G is a graph with nodes set $V(G) \cup E(G)$ and two nodes x, y in $T(G)$ are adjacent if either

(i) $x, y \in V(G)$ and x is adjacent to y in G or

(ii) $x, y \in E(G)$ and x, y are adjacent in G or

(iii) $x \in V(G), y \in E(G)$ and x, y are incidents in G".

Definition 1.3.12. [34] "Let G be on $v_1, v_2, v_3, \ldots, v_n$. The Mycielski graph, (G), is obtained by adding to each node v_i a new node u_i such that u_i is adjacent to the neighbors of v_i. Finally, add a new node w such that w is adjacent to each and every u_i".

Definition 1.3.13. [86] "The ladder L_n with $2n$ nodes and $3n - 2$ lines, obtained as $L_n = P_n \times P_2$".

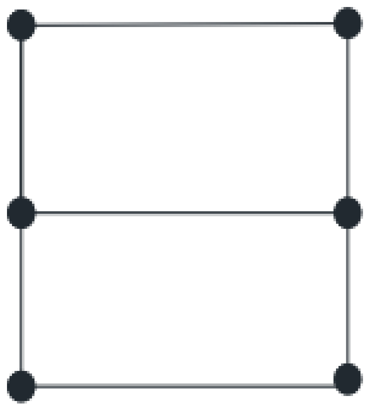

FIGURE 1.8: L_3

Definition 1.3.14. [1] "The triangular snake graph TS_n with n (odd) nodes is defined by starting with P_{n-1} and adding lines $(2i - 1, 2i + 1)$ for $i = 1, 2, ...n - 1$. Then TS_n is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} to a new node w_i for $1 \leq i \leq n - 1$".

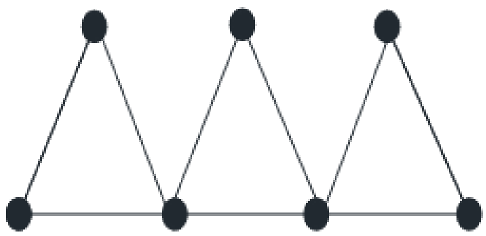

FIGURE 1.9: TS_7

Definition 1.3.15. [27] "The lollipop graph $L_{m,n}$ is a special type of graph consisting of K_m and P_n, connected with a bridge".

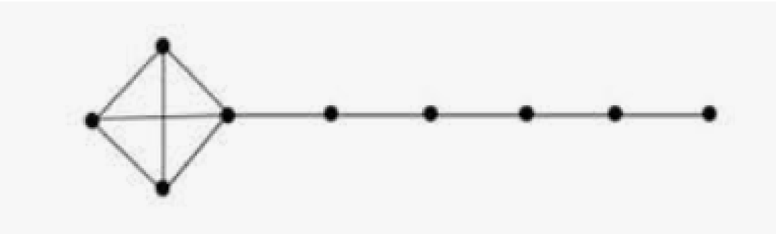

FIGURE 1.10: $L_{4,5}$

Definition 1.3.16. [23] "The fan $F_{1,n}$ $(n \geq 2)$ is obtained as $F_{1,n} = P_n + K_1$".

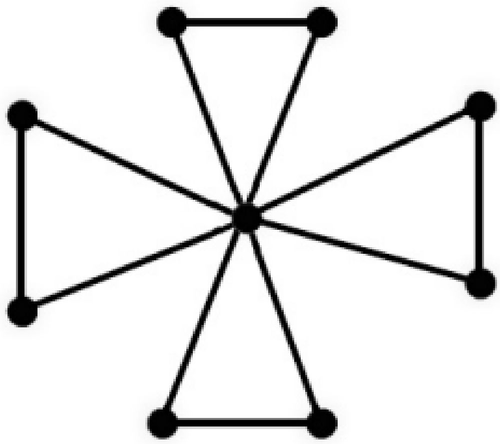

FIGURE 1.11: $F_{1,4}$

Definition 1.3.17. [23] "A friendship graph F_n consists of n triangles with a common node".

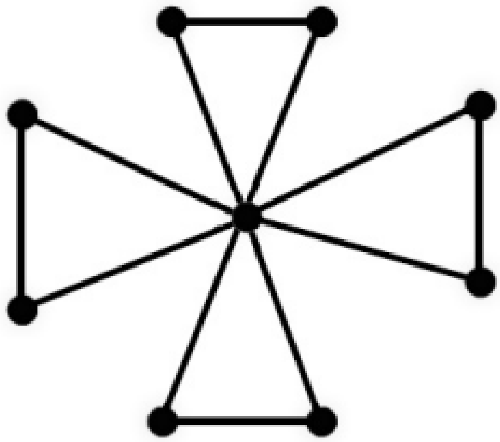

FIGURE 1.12: F_4

‘

Definition 1.3.18. [45] “For any integers $s > 2$, $t > 1$, an Umbrella graph $U(s,t)$ is obtained by identifying the end nodes of P_t with a central node of $F_{1,s}$”.

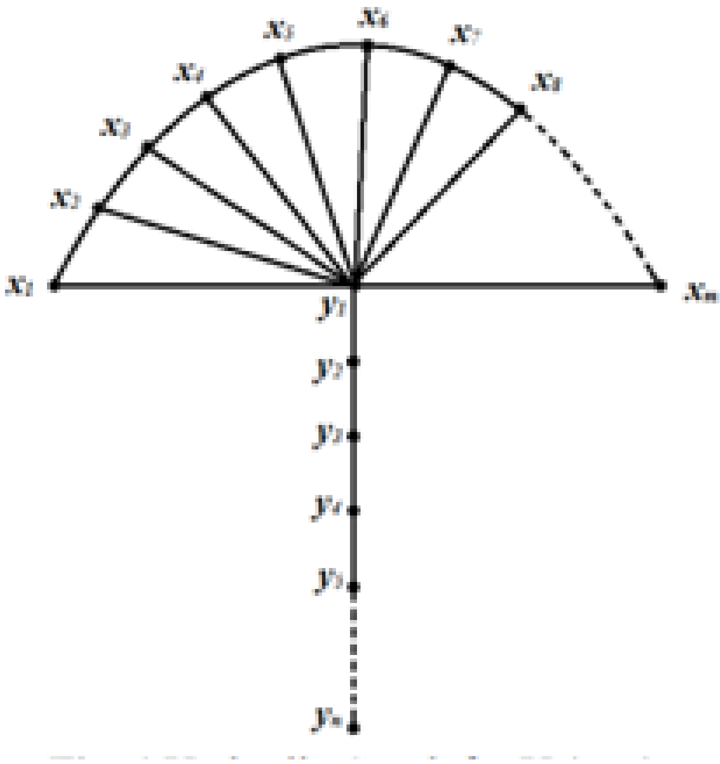

FIGURE 1.13: $U_{m,n}$

Definition 1.3.19. [69] "A wheel W_n is defined as $W_n = C_{n-1} \wedge K_1$".

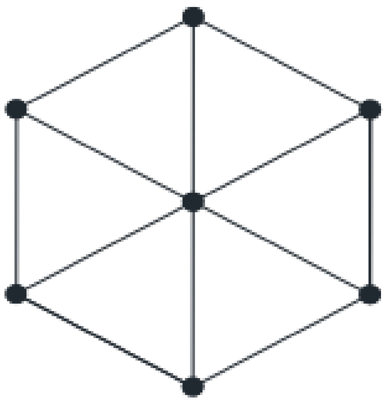

FIGURE 1.14: W_7

1.3.1 SOME KNOWN RESULTS ON 3-EQUITABLE LABELING

In this section, a few important results proved by different authors concerning the 3-Equitable Labeling of graphs are highlighted.

Cahit [14] proved the following results.

(i) C_n is 3-equitable if and only if $C_n \neq 3\ (mod\ 6)$.

(ii) An Eulerian graph with $q \equiv 3\ (mod\ 6)$ is not 3-equitable where q is the number of lines of G.

(iii) All caterpillars are 3-Equitable.

(iv) Every tree with fewer than five end nodes have a 3-equitable labeling.

Seoud and Abdel Maqsoud [67] proved the following results.

(i) A graph with p nodes and q lines in which every node has odd degree is not 3-equitable if $p \equiv 0\ (mod\ 3)$ and $q \equiv 3\ (mod\ 6)$.

(ii) All fans except $F_{1,2}$ are 3-Equitable.

(iii) P_n^2 is 3-Equitable for all n except 3.

(iv) $K_{m,n}$ (where $3 \leq m \leq n$) is 3-Equitable if and only if $(m, n) = (4, 4)$.

Bapat and Limaye [9] proved the following results on 3EL.

(i) H_n ($n \geq 4$) are 3EGs.

(ii) Flower graph admits 3EL.

(iii) The one-point union of any number of H_n is 3EG.

(iv) The one-point union of any number of copies of K_4 is a 3EG.

Youssef gave the following result in [92].

(i) W_n is 3-equitable $\forall\, n \geq 4$.

Vaidya et al. [82] have proved the following results.

(i) $SG(C_n)$ is 3EG except for $n = 3$ and 5.

(ii) $SG(P_n)$ is 3EG except for $n = 3$.

(iii) $MG(P_n)$ is 3EG.

(iv) $MG(C_n)$ is 3EG for n even and not 3-E for n odd.

Vaidya et al. have also discussed the 3EL of wheel related graphs in [80], some shell related graphs in [81], and some star related graphs in [84].

(i) All caterpillars are 3-equitable.

(ii) $S'(K_{1,n})$ is 3EG.

Illustration: 3EL of $S'(K_{1,7})$

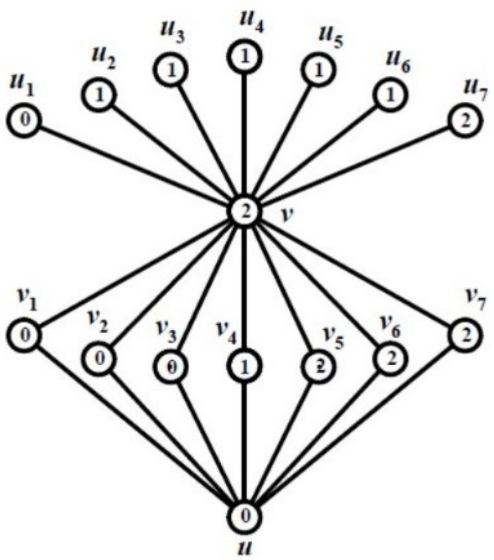

FIGURE 1.15: $S'(K_{1,7})$

(iii) $< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} : \ldots .: S_n^{(n)} >$ is 3-E.

(iv) $< W_n^{(1)}: W_n^{(2)} : W_n^{(3)} : \ldots .: W_n^{(k)} >$ is 3-E.

(v) $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : K_{1,n}^{(3)} : ...: K_{1,n}^{(k)} >$ is 3-E.

(vi) $S'(B_{n,n})$ is 3EG.

Illustration: 3EL of $S^{'}\left(B_{6,6}\right)$.

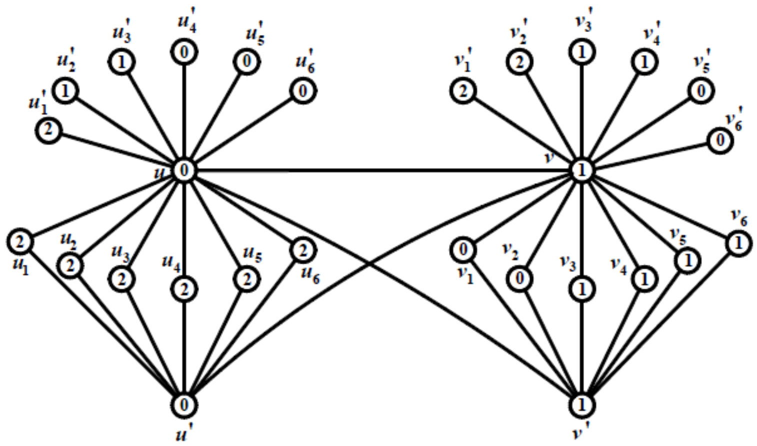

FIGURE 1.16: $S^{'}\left(B_{6,6}\right)$

(vii) $D_2(B_{n,n})$ is 3EG.

Illustration: 3EL of $D_2(B_{5,5})$.

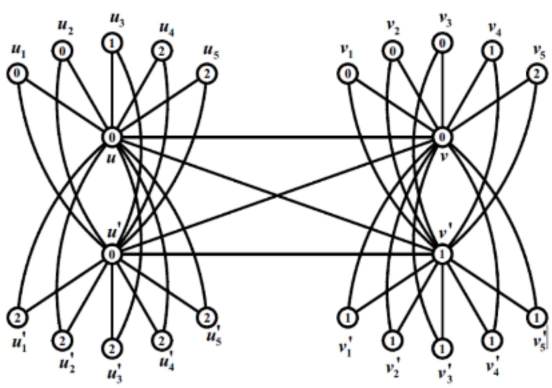

FIGURE 1.17: $D_2(B_{5,5})$

(viii) $B^2_{n,n}$ is 3EG for $n \equiv 0\ (mod\, 3)$ and $n \equiv 1\ (mod\, 3)$.

Illustration: 3EL of $B^2_{7,7}$

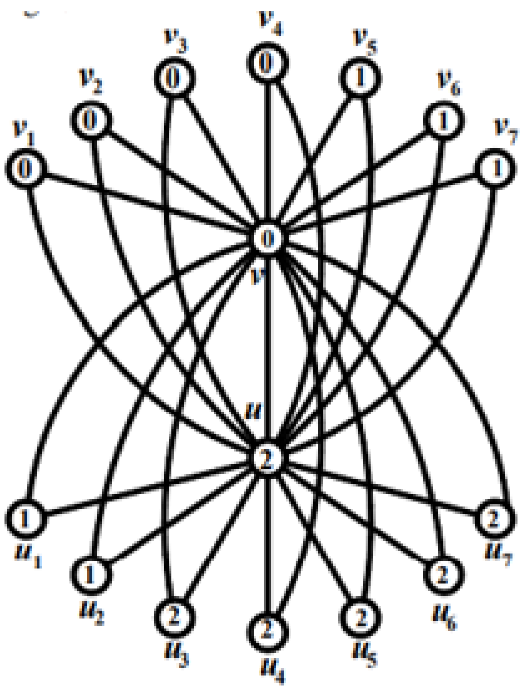

FIGURE 1.18: $B^2_{7,7}$

(ix) For $n\ \geq 6$, $C_n + \overline{K}_n$ is $3 - E$ if and only if n is even.

(x) C^2_n is $3 - E$ if and only if $n\ \geq 8$.

Cahit [14] proved that C_n is 3-E, $n\ \neq 3\ (mod\, 6)$. M. V. Bapat and N. B. Limaye in [9] proved that Helms $H_n,\ (n \geq\ 4)$ are 3-E. S.K. Vaidya and N.H. Shah in [83] have shown that $B_{n,n}$ is 3EG.

(i) Nodes switching of any rim nodes of W_n (except for $n\ \equiv 1,\ 3,\ 5\ (mod\, 6)$) is 3-E.

(ii) $Gn\ \oplus K_{1,n}$ is 3-E $\forall n$.

(iii) $G \oplus K_{1,n}$ is 3-E $\forall n$, where G is cycle having twin chords $C_{n,3}$.

(iv) The extended duplicate graph of TS_m, $m \geq 1$ admits 3EL.

(v) The extended duplicate graph of splitting path graph of P_m, $m \geq 2$ admits 3EL.

I. Jadav, G. V. Ghodasara [35] gave the following results:

(i) The barycentric subdivision of armed crown, AC_n is 3-E.

(ii) The barycentric subdivision of crown $C_n \bigodot K_1$ is 3-E.

(iii) The barycentric subdivision of double crown $C_n \bigodot 2K_1$ is 3-E.

(iv) The barycentric subdivision of $C_n^{'}$ is 3-E.

Ghodasara G V and Sonchhatra S G [31] proved the following results on 3EL.

(i) "H formed by connecting two copies of F_n by P_n of any length is 3-E".

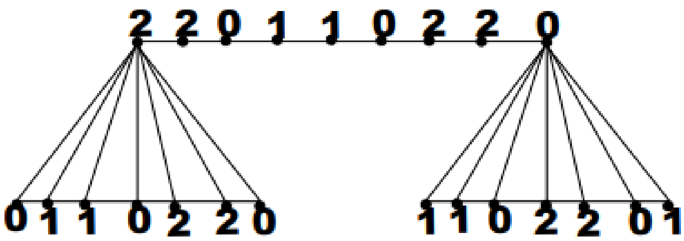

FIGURE 1.19: 3EL of H obtained by joining two copies of F_7 by P_9.

(ii) "H formed by connecting two copies of W_n by P_n of any length is 3-E".

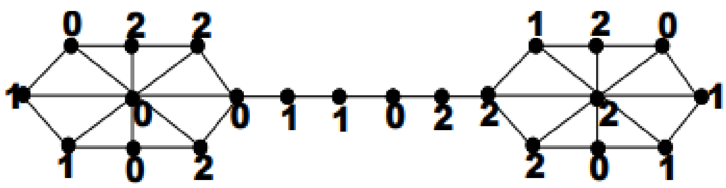

FIGURE 1.20: 3EL of H obtained by joining two copies of W_8 by P_6.

(iii) "H formed by connecting two copies of G_n by P_n of arbitrary length is 3-E".

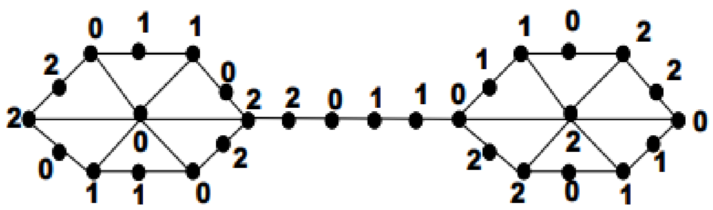

FIGURE 1.21: 3EL of H obtained by joining two copies of G_6 by P_6.

(iv) "H formed by connecting two copies of H_n by P_n of any length is 3-E".

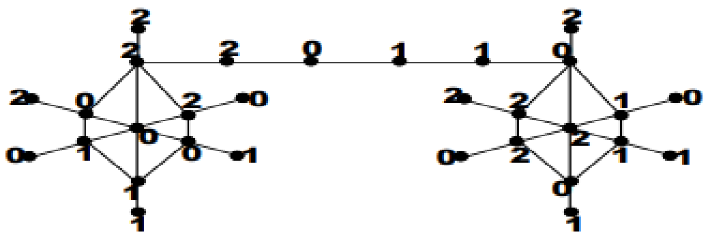

FIGURE 1.22: 3EL of H obtained by joining two copies of H_6 by P_6.

(v) "H formed by connecting two copies of C_n with one pendant edge by P_n of any length is 3-E".

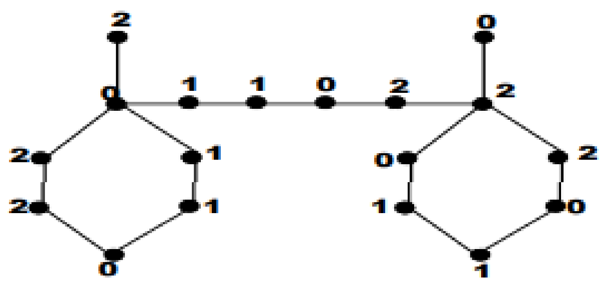

FIGURE 1.23: 3EL of H obtained by joining two copies of C_6 with one pendant edge by P_6.

S. Murugesan, & J. Shiama [64] proved the following results.

(i) $C_n \oplus K_{1,n}$ is 3-E $\forall n$.

(ii) $S_n \oplus K_{1,n}$ is 3-E $\forall n$.

Sweta Srivastav and Sangeeta Gupta [72] proved the following results on D3EL.

(i) P_n is D3EG.

Illustrations: For $n = 3$

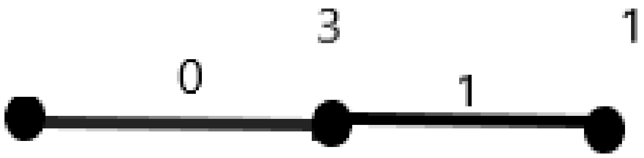

FIGURE 1.24: D3EG of P_3

For $n = 4$

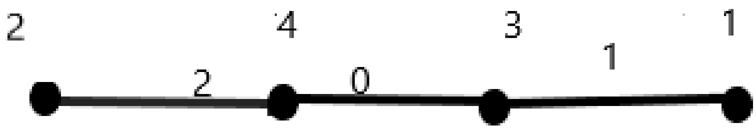

FIGURE 1.25: D3EG of P_4

For $n = 9$

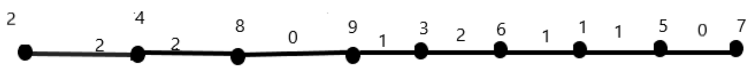

FIGURE 1.26: D3EG of P_9

For $n = 17$

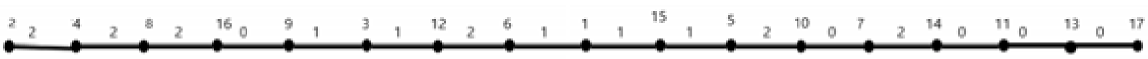

FIGURE 1.27: D3EG of P_{17}

(ii) C_n is a D3EG.

Illustrations: For $n = 12$

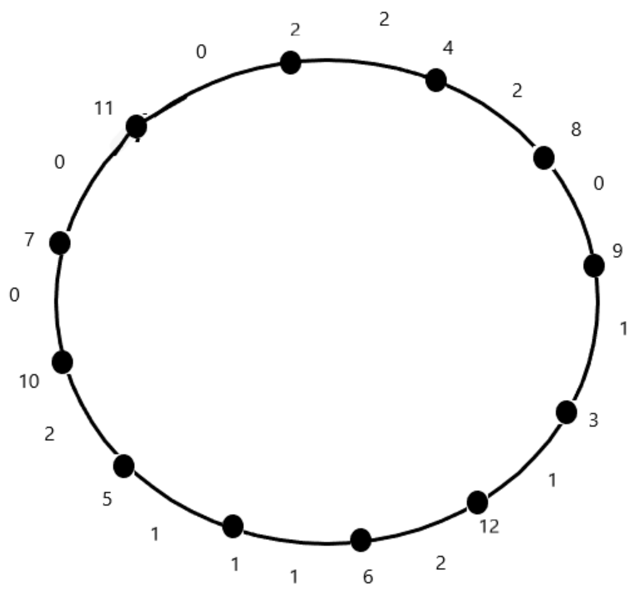

FIGURE 1.28: D3EG of C_{12}

For $n = 24$

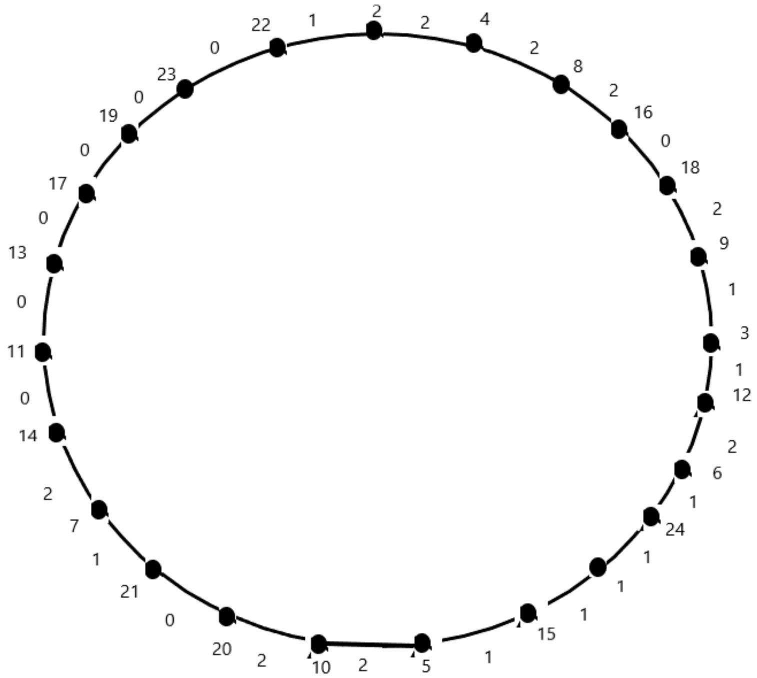

FIGURE 1.29: D3EG of C_{24}

1.3.2 Research Gap

Though a significant work has been done concerning 3-equitable and divisor 3-equitable labeling of graphs, still there are many open problems to work on. The complete characterization of 3-equitable and divisor 3-equitable labeling is still open. The divisor 3-equitable labeling of graphs in the context of graph operations such as join, subdivision, vertex duplication, vertex switching, edge duplication etc. are still open. Also the divisor 3-equitable labeling of total graph, middle graph, central graph, Mycielskian, and degree splitting graphs of various classes of graphs are also still unsolved.

1.3.3 Motivation

The study of graph labeling, particularly 3-equitable and divisor 3-equitable labeling, has garnered significant interest due to its numerous applications in various fields such as network theory, chemistry, and coding theory. Despite the progress made in understanding these concepts, there remain several open problems and unexplored areas that present both challenges and opportunities for further research. A complete characterization of 3-equitable and divisor 3-equitable labeling has not yet been achieved, leaving gaps in the existing literature. This gap underscores the need for a deeper investigation into the labeling of different classes of graphs and their operations, such as join, subdivision, vertex duplication, and edge switching. Additionally, the labeling of specific types of graphs, including total graphs, middle graphs, central graphs, and Mycielskians, as well as degree splitting graphs, remains largely unresolved. The motivation for this thesis stems from the desire to fill these gaps and contribute to the body of knowledge in graph theory. By establishing new families of graphs that can be labeled 3-equitably and by exploring the divisor 3-equitable labeling of various graph operations, this research aims to provide a comprehensive understanding of these labeling techniques. The outcomes of this study have the potential to advance theoretical insights and offer practical solutions to problems in related disciplines.

1.3.4 Contributions in the Thesis

Based on the research gap and motivation given above, the contribution of the thesis is classified into the following 3 objectives.

OBJECTIVES

1. Establishing the 3-equitable labeling of certain new classes of graphs.
2. Obtaining the divisor 3-equitable labeling of some new families of graphs.
3. Encountering the divisor 3-equitable labeling of total graph, middle graph, central graph, degree splitting graph, and Mycielskian graphs of various classes of graphs.

1.3.5 Organization of the Thesis

This thesis is organized into five sections where this chapter 1 gives a short and precise introduction to graph theory along with literature review followed by research gap and motivation of the thesis. Chapter 2 details the 3-equitable labeling of various graphs, including the Jellyfish and Umbrella graphs. Chapter 3 discusses the 3-equitable labeling of the CG of Lollipop, TG of Fan, MG of Ladder, DSG of Friendship, and Mycielski graphs of a path graphs. Chapter 4 presents the divisor 3-equitable labeling of several graphs. Chapter 5 introduces new results on the divisor 3-equitable labeling for specific classes of graphs. Further, the conclusion of the thesis is given, followed by sections on publications, presentations, and related references.

1.4 Conclusion

In summary, this chapter provided a foundational overview of graph theory and graph labeling, highlighting their essential definitions and key terminologies. The discussion also encompassed 3EL and the results related to D3EL. Notable applications were presented to underscore the relevance and utility of these concepts. This groundwork sets the stage for the more detailed explorations and analyses presented in the subsequent chapters, ensuring a clear understanding of the study's scope and objectives.

Chapter 2

3-equitable Labeling of Some Special Graphs

2.1 Introduction

The concept of 3-equitable labeling (3EL) is given by Cahit [14]. In this chapter, 3EL of some special graphs are discussed and new results are found.

2.2 Some New Results on 3EL

This section is devoted for proving the 3EL of jellyfish graph and umbrella graph.

2.2.1 3EL of Jellyfish Graph

Definition 2.2.1. "The jelly fish graph $J(m, n)$ is obtained from a $C_n : v_1, v_2, v_3, v_4$ by joining v_1 and v_3 with an edge and appending m pendent lines to v_2 and n pendent lines to v_4".

One can obtain 3EL of $J_{1,2}$ One such example is given in Figure 2.1. So, consider $J_{1,n}$ for $n \geq 3$.

Example 2.2.1. *3EL for $J_{1,2}$ is shown in the figure 2.1.*

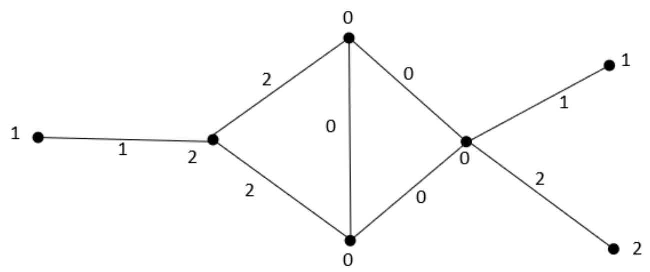

FIGURE 2.1: 3EL of $J_{1,2}$

Theorem 2.2.1. *$J_{1,m}$ does not permit 3EL $\forall\ m\ \geq 3$.*

Proof. Take m = 3 for the sake of discussion and so $|V(J_{1,m})|$ = 8 and $|E(J_{1,m})|$ = 9. The proof is by the method of contradiction. Assume that $J_{1,3}$ has 3EL f with the property that number of nodes with label s and t differ by at most 1 and in the same way the number of lines with label s and t differ by at most $1,\ 0\ \leq\ s,\ t\ \leq\ 2,\ s\ \neq\ t$. *Also* if $|v_f(s) - v_f(t)|\ \leq$ 1 and $|e_f(s) - e_f(t)|\ \leq\ 1$ for all $0\ \leq\ s,\ t\ \leq\ 2$. Observe that the number of lines labeled 0, 1, and 2 must be exactly 3 and that the number of nodes labeled 0, 1, and 2 must be at least 2 and at most 3 to satisfy the required 3EL property $|e_f\ (s) - e_f\ (t)\ |\leq\ 1$ for all $0 \leq\ \ s, t \leq 2$. But in case of lines 3 lines with label 0, there are four lines with label 2, two lines with label 1, a contradiction (see Figure 2.2). Similar argument holds good for $J_{1,m}, m > 3$. Therefore, $J_{1,m},\ \ m \geq 3$, does not admit 3EL.

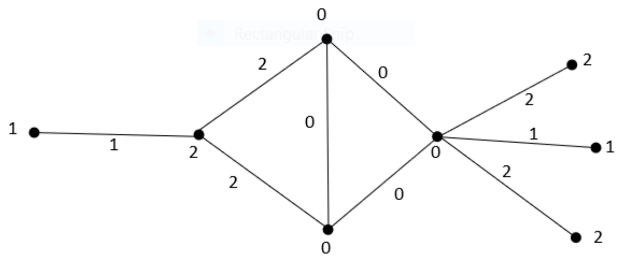

FIGURE 2.2: The non-existence of 3EL for $J_{1,3}$

2.2.2 3EL of Umbrella Graph

Definition 2.2.2. "For $s > 2; t > 1$ an umbrella graph $U(s,t)$ is obtained by identifying the end nodes of P_t with a central node of F_s".

One can obtain the 3EL of $U_{3,t}; 2 \leq t \leq 16$ One such example is given in Figure 2.3. So, consider $U_{3,t}$, for $t \geq 17$.

Example 2.2.2. *3EL for $U_{3,16}$ is shown in the figure 2.3.*

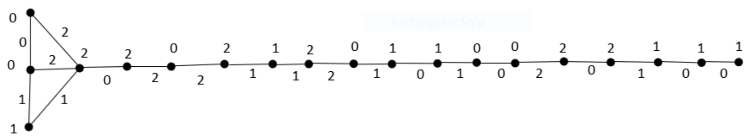

FIGURE 2.3: 3EL for $U_{3,16}$

Theorem 2.2.2. *$U_{3,n}$ does not permit 3EL $\forall\, n \geq$ 17.*

Proof. Take n = 17 for the "sake of discussion" and so $|V(U_{3,17})|$ = 20 and $|E(U_{3,17})|$ = 21. The proof is by the method of contradiction. Assume that $U_{3,17}$ has 3EL f with the property that number of nodes with label s and t differ by at most 1 and in the same way the number of lines with label s and t differ by at most 1, $0 \leq s, t \leq 2$, $s \neq t$. Also if $|v_f(s) - v_f(t)| \leq 1$ and $|e_f(s) - e_f(t)| \leq 1$ for all $0 \leq s, t \leq 2$. One can observe that the number of lines labeled 0, 1, and 2 must be exactly 7 and that the number of nodes labeled 0, 1, and 2 must be at least 6 and at most 7 to satisfy the required 3EL property $|e_f(s) - e_f(t)| \leq 1$ for all $0 \leq s, t \leq 2$. But in case of lines there are 7 lines with label 0, six lines with label 2, eight lines with label 1, a contradiction (See Figure 2.4. Similar argument holds good for $U_{3,n}$ n> 17. Therefore, $U_{3,n}$ $n \geq 17$, does not admit 3EL.

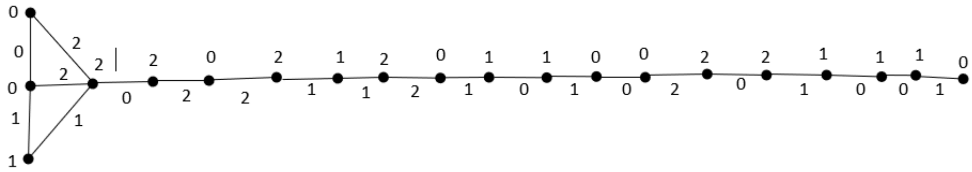

FIGURE 2.4: The non-existence of 3EL for $U_{3,17}$

2.3 Conclusion

The 3EL of some special graphs such as jelly fish graph and umbrella graph are derived.

Chapter 3

3EL of Some Special Classes of Graphs

3.1 Introduction

The aim of the present chapter is to discuss new results on deriving the 3EL of TG, MG, CG, DSG, and Mycielskian graphs of some classes of graphs.

3.2 More Results on 3EL

This section is devoted for proving 3EL of CG of lollipop graph, TG of fan graph, MG of ladder graph, DSG of friendship graph, Mycielski graph of path graph.

3.2.1 3EL of CG of Lollipop Graph

Definition 3.2.1. "The lollipop graph $L_{m,n}$ consists K_m and P_n, connected with a bridge".

One can obtain the 3EG of C($L_{3,n}$). One such example is given in Figure 3.1. So, consider C($L_{3,n}$), for n $\geq$ 2.

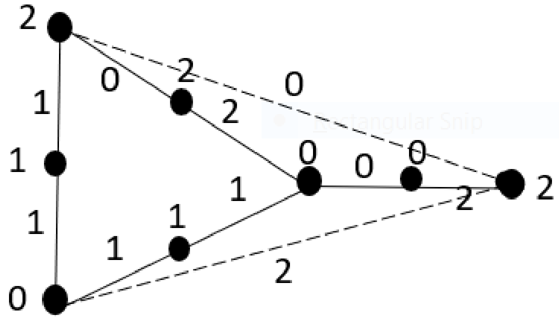

FIGURE 3.1: 3EL for $\mathbf{C(L_{3,1})}$

Theorem 3.2.1. $C(L_{3,n})$ *does not permit 3EL* $\forall\, n \geq 2$.

Proof. Take $n = 2$ for the "sake of discussion" so $|V(C\,(L_{3,2}))|$ = 10 and $|E(C(L_{3,2}))|$ = 15. Obtain C $(L_{3,n})$. "The proof is by the method of contradiction. Assume that $C(L_{3,2})$ has 3EL, f with the property that number of nodes with label i and j differ by at most 1 and in the same way the number of lines with label k and l differ by at most 1, $0 \leq k,\ l \leq 2,\ k \neq l$. Also if $|v_f(k) - v_f(l)| \leq 1$ and $|e_f(k) - e_f(l)| \leq 1$ for all $0 \leq k,\ l \leq 2$". Observe that the number of lines labeled 0, 1, and 2 must be exactly 5 and that the number of nodes labeled 0, 1, and 2 must be at least 3 and at most 4 to satisfy the required 3EL property $|e_f\,(k) - e_f\,(l)\,| \leq 1$ for all $0 \leq k, l \leq 2$. But in case of lines there are 6 lines with label 0, five lines with label 2, four lines with label 1, a contradiction (See Figure 3.2). Similar argument holds good for $C(L_{3,n}), n > 2$. Therefore, $C(L_{3,n})$ $n \geq 2$ does not admit 3EL.

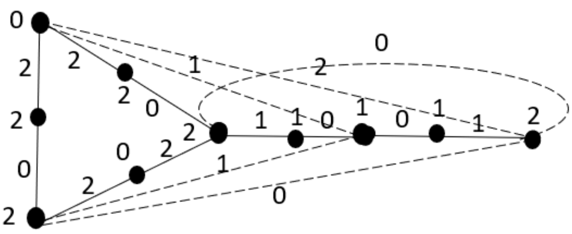

FIGURE 3.2: The non-existence of 3EL for $C(\mathbf{L_{3,2}})$

3.2.2 3EL of Total Graph of Fan Graph

Definition 3.2.2. [23] "The fan $F_{1,n}$ (n $\geq$ 2) is obtained by joining all nodes of P_n (Path of n nodes) to a further node called the center and contains n+ 1 nodes and 2n-1 lines. i.e $F_{1,n} = P_n + K_1$".

Theorem 3.2.2. *$T(F_{1,n})$ does not permit 3EL* $\forall\ n\ \geq\ 2$.

Proof. Take $n = 2$ for the "sake of discussion" and so $|V(T\ (F_{1,2}))|$ = 6 and $|E(T(F_{1,2}))|$ = 12. Obtain (T$F_{1,n}$). "The proof is by the method of contradiction. Assume that $T(F_{1,2})$ has 3EL f with the property that number of nodes with label i and j differ by at most 1 and in the same way the number of lines with label u and v differ by at most 1,0 $\leq\ u,\ v\ \leq\ 2,\ u\ \neq\ v$. Alsoif $|v_f(u) - v_f(v)|\ \leq\ 1$ and $|e_f(u) - e_f(v)|\ \leq\ 1$ for all 0 $\leq\ u,\ v\ \leq\ 2$." Note that the number of lines labeled 0, 1, and 2 must be exactly 4 and that the number of nodes labeled 0, 1, and 2 must be exactly 2 to satisfy the required 3EL property $|e_f\ (u) - e_f\ (v)\ |\leq\ 1$ for all 0 $\leq\ u, v \leq$ 2. But in case of lines three lines with label 0, there are three lines with label 2, six lines with label 1, a contradiction (See Figure 3.3). Similar argument holds good for $T(F_{1,n})$ n> 2. Therefore, $T(F_{1,n})\ \ n \geq 2$, does not admit 3EL.

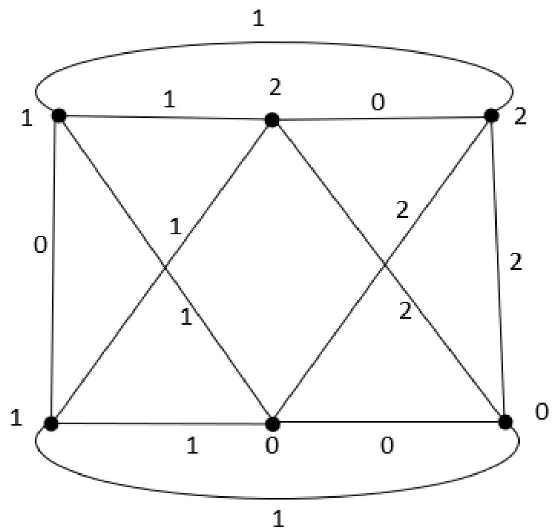

FIGURE 3.3: The non-existence of 3EL for $\mathbf{T(F_{1,2})}$

3.2.3 3EL of Middle Graph of Ladder Graph

Definition 3.2.3. "The ladder L_n is a planar graph with *2n* nodes and *3n-2* lines. The ladder is obtained as the Cartesian product of two path graphs, one of which has only one line: $L_{n,1} = P_n \times P_2$. The n-ladder graph can be defined $L_n = P_2 \Box P_n$ where P_n a path".

One can easily obtain the 3EL of $M(L_1), M(L_2), M(L_3)$. An example of this type is provided in Figure 3.4. So, consider $M(L_n)$, for $n \geq 4$.

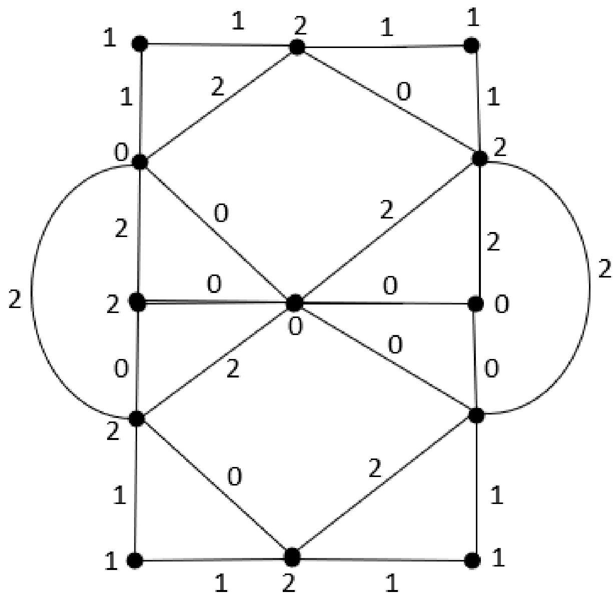

FIGURE 3.4: 3EL for$\mathbf{M(L_3)}$

Theorem 3.2.3. *$M(L_n)$ does not permit 3EL* $\forall\, n \geq 4$.

Proof. Let L_n be the given ladder graph on $n \geq 4$. "Take $n = 4$ for the sake of discussion and so $|V(M\,(L_4))|$ = 18 and $|E(M(L_4))|$ = 36. Obtain $M(L_4)$. The proof is by the method of contradiction. Assume that $M(L_4)$ has a 3EL f with the property that number of nodes with label s and t differ by at most 1 and in the same way the number of lines with label s and t differ by at most 1,0 $\leq s,\ t \leq 2,\ s \neq t$. Also if $|v_f(s) - v_f(t)| \leq 1$ and $|e_f(s) - e_f(t)| \leq 1$ for all $0 \leq s,\ t \leq 2$". Note that the number of lines labeled 0, 1, and 2 must be exactly 12 and that the number of nodes labeled 0, 1, and 2 must be exactly 6 to satisfy the required 3EL property $|e_f\,(s) - e_f\,(t)\,| \leq 1$ for all $0 \leq s, t \leq 2$. But in case of lines, there are eleven lines with label 0, eleven lines with label 2, fourteen lines with label 1, a contradiction (See Figure 3.5). Similar argument holds good for $M(L_n),\ n > 4$. Therefore, $M(L_n)\ \ n \geq 4$, does not admit 3EL.

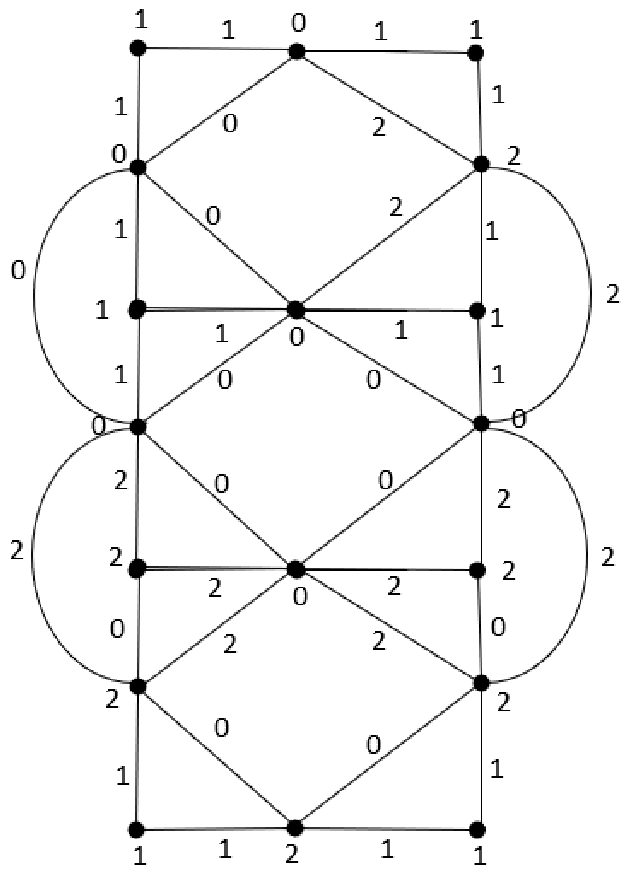

FIGURE 3.5: The non-existence of 3EL for M($\mathbf{L_4}$)

3.2.4 3EL of DSG of Friendship Graph

Definition 3.2.4. "A friendship graph F_n is a graph which consists of n triangles with a common node".

One can easily obtain the 3EL of $DS(F_1), DS(F_2)$. Figure 3.6 is one illustration of this. So, consider $DS(F_n)$, for $n \geq 3$.

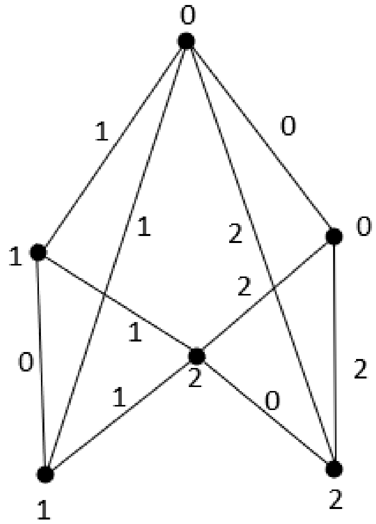

FIGURE 3.6: 3EL for $\mathbf{DS(F_2)}$

Theorem 3.2.4. *$DS(F_n)$ does not permit 3EL* $\forall\, n \;\geq\; 3$.

Proof. Take n =3 for the "sake of discussion". Obtain $DS(F_3)$ and so $|V(DS\,(F_3))|$ = 8 and $|E(DS(F_3))|$ = 15. "The proof is by the method of contradiction. Assume that $DS(F_3)$ has a 3EL f with the property that number of nodes with label k and l differ by at most 1 and in the same way the number of lines with label k and l differ by at most $1, 0 \;\leq\; k,\; l \;\leq\; 2,\; k \;\neq\; l.$ Also if $|v_f(k) - v_f(l)| \;\leq\; 1$ and $|e_f(k) - e_f(l)| \;\leq\; 1$ for all $0 \;\leq\; k,\; l \;\leq\; 2$". One can note that the number of lines labeled 0, 1, and 2 must be exactly 5 and that the number of nodes labeled 0, 1, and 2 must be at least 2 and at most 3 to satisfy the required 3EL property $|e_f\,(k) - e_f\,(l)\,| \leq\; 1$ for all $0 \leq\; k, l \leq 2$. But in case of lines, there are four lines with label 0, five lines with label 2, six lines with label 1, a contradiction (See Figure 3.7). Similar argument holds good for $DS(F_n), n > 4$. Therefore, $DS(F_n)\;\; n \geq 4$, does not admit 3EL.

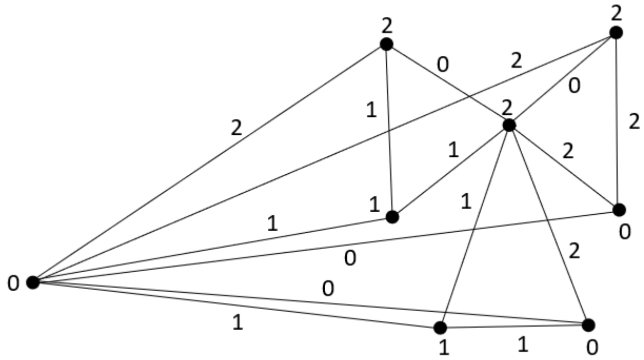

FIGURE 3.7: The non-existence of 3EL for DS($\mathbf{F_3}$)

3.2.5 3EL of Mycielski Graph of Path

One can easily obtain the 3EL of $\mu(P_n)$;$2 \leq n \leq 7$. Figure 3.8 provides one such instance. So, consider $\mu(P_n)$, for $n \geq 8$.

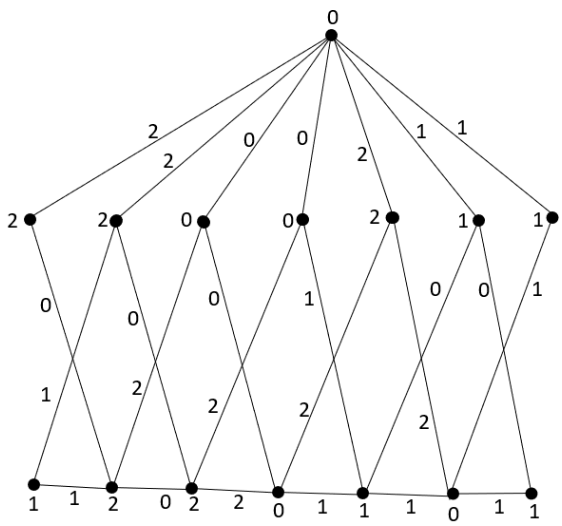

FIGURE 3.8: 3EL for $\mu(P_7)$

Theorem 3.2.5. *$\mu(P_n)$ does not permit 3EL $\forall\ n\ \geq\ 8$.*

Proof. Let P_n be the given Path on $n \geq 8$. "Take n =8 for the sake of discussion. Obtain $\mu(P_8)$ and so $|V(\,(P_8))\,|$ = 17 and $|(E\mu\,(P_8)\,|$ = 29. The proof is by the method of contradiction. Assume that $\mu(P_8)$ has a 3EL f with the property that number of nodes with label r and s differ by at most 1 and the number of lines with label r and s differ by at most $1,\ 0\ \leq\ r,\ s\ \leq\ 2,\ r\ \neq s$. Also if $|v_f(r) - v_f(s)|\ \leq\ 1$ and $|e_f(r) - e_f(s)|\ \leq\ 1$ for all $0\ \leq\ r,\ s\ \leq\ 2$". Observe that the number of lines labeled 0, 1, and 2 must be at least 9 and at most 10 and that the number of nodes labeled 0, 1, and 2 must be at least 5 and at most 6 to satisfy the required 3EL property $|e_f\,(r) - e_f\,(s)\,| \leq\ 1$ for all $0 \leq\ \ r, s \leq 2$. But in case of lines, there are nine lines with label 0, eleven lines with label 2, nine lines with label 1, a contradiction (See Figure 3.9). Similar argument holds good for $\mu(P_n)$ n> 8. Therefore, $\mu(P_n)\ \ n \geq 8$, does not admit 3EL.

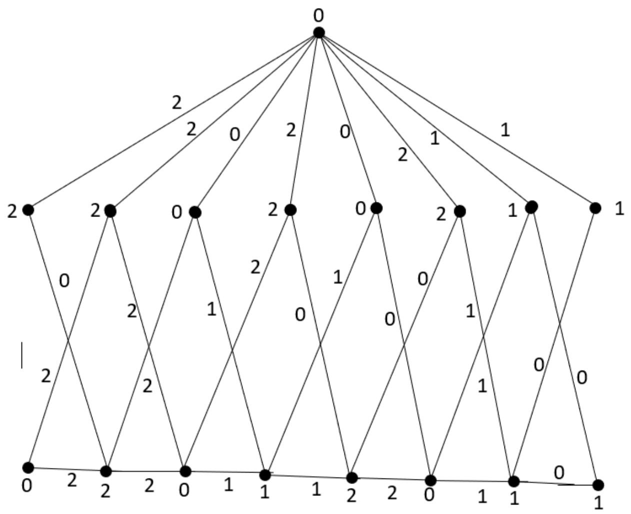

FIGURE 3.9: The non-existence of 3EL for $\mu\,(P_8)$

3.3 Conclusion

The “existence and non-existence” of 3EL of the CG of lollipop graph, TG of fan graph, MG of ladder graph, DSG of friendship graph and Mycielski graph of path are established.

Chapter 4

Divisor 3-Equitable Labeling of Various Graphs

4.1 Introduction

"In 2019, Sweta Srivastav et al. introduced the approach of divisor 3-equitable labeling (D3EL) of graphs [72]. In this chapter, D3EL of various graphs are discussed and some new results are found".

4.2 D3EL

Definition 4.2.1. [72] "A D3EL is a bijective $d : V(G) \rightarrow \{1, 2 \ldots, n\}$ such that the induced map d^* defined on the edges of G by, for any edge xy with $d(x) < d(y)$,

$$d^*\left(e = xy\right) = \begin{cases} 1,\ if\ d\left(x\right)|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

such that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1$ for all $0 \leq i, j \leq 2$, where $e_{d^*}(i)$ is the number of lines labeled with label i under d^*. A graph that permits D3EL is called divisor 3-equitable graph (D3EG)".

4.3 Some Known Results on D3EL of Some Graphs

Sweta Srivastav and Sangeeta Gupta [72] proved the following results on D3EL.

(i) P_n is D3EG.

(ii) C_n is a D3EG.

4.4 Some New Results on D3EL

Few new families of D3EL of graphs are discussed in this section.

4.4.1 D3EL of W_n

Definition 4.4.1. [69] W_n is defined as $W_n = C_{n-1} \wedge K_1$.

One can obtain the "D3EL of W_n, $1 \leq n \leq 6$." One such example is given in Figure 4.1.

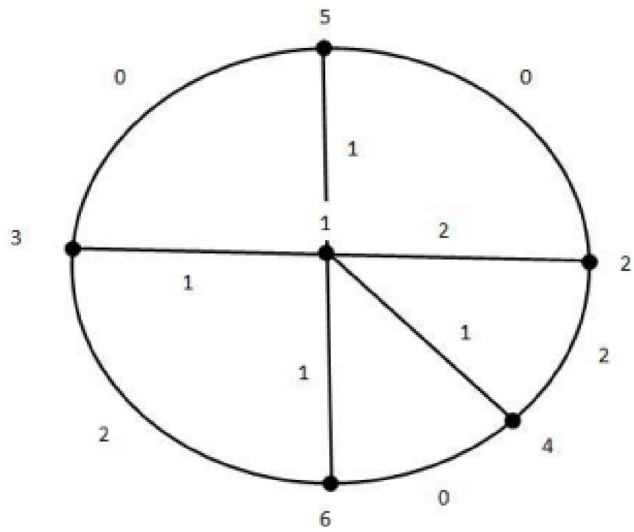

FIGURE 4.1: D3EL of W_6

Theorem 4.4.1. *W_n does not permit D3EL $\forall\, n \geq 7$.*

Proof. Let W_n = $\{v_0, v_1, v_2, \ldots, v_n\}$ be the given wheel graph on $n \geq 7$. For the sake of discussion, take $n = 7$. One can clearly observe that there are $n-1$ nodes of degree 3 on the rim and a vertex v_0 of degree 6 at the centre of a wheel as the central vertex. "Define a bijection $d : V(W_7) \to \{1, 2, ..., 7\}$ such that the induced $d^* : E(W_7) \to \{0, 1, 2\}$, for any edge xy with $d(x) < d(y), d^*\,(e = xy) = \begin{cases} 1, \; if \; d\,(x)\,|d(y) \\ 2, \; if \; \frac{d(y)}{d(x)} = 2 \\ 0, \; if \; d(x) \nmid d(y) \end{cases}$

The proof is by a method of contradiction". Suppose that W_7 has D3EL d and d^* such that "$|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq 1, \forall\, 0 \leq i, j \leq 2$, where $e_{d^*}\,(i)$ denotes the number of edges labeled with label i under d^*". Note that $|V(W_7)| = 7$ and $|E(W_7)| = 12$. So as per the definition of a D3EL, the number of lines labeled with either 0 or 1 or 2 in W_7 is at most 4. In fact, the number of lines with label 0, 1, and 2 is exactly 4. One can also see that the central vertex v_0 can take any value between 1 and 7.

The seven cases listed below now occur.

Case 1: When $d(v_0) = 1$

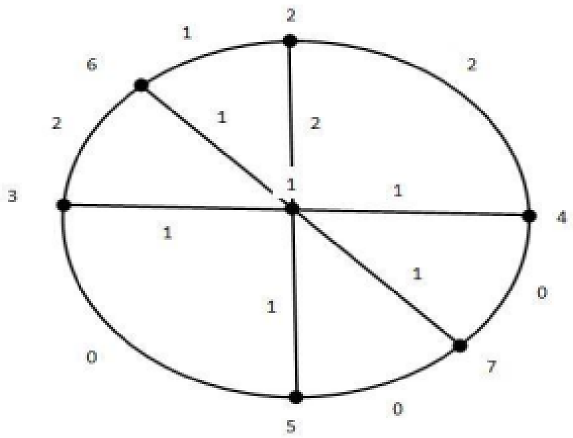

FIGURE 4.2: $d(v_0) = 1$ in W_7

"As the central vertex is adjacent to all other vertices and the label 1 divides all other labels, $d^*(v_0v_1) = 1 \; \forall \; 1 \leq i \leq 6$ except for the edge whose end vertex, say v_2, is labeled with 2 gives $d^*(v_0v_2) = 2$. So, the number of edges labeled with 1 is at least 5, a contradiction".

Case 2: When $d(v_0) = 2$

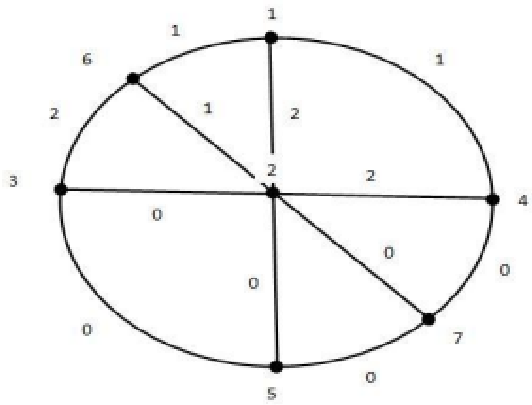

FIGURE 4.3: $d(v_0) = 2$ in W_7

One can note there are three at least five edges (with all possible assignments of numbers 1, 3, 4, 5, 6, 7 on the rim vertices) with label 0, a contradiction.

Case 3: When $d(v_0) = 3$

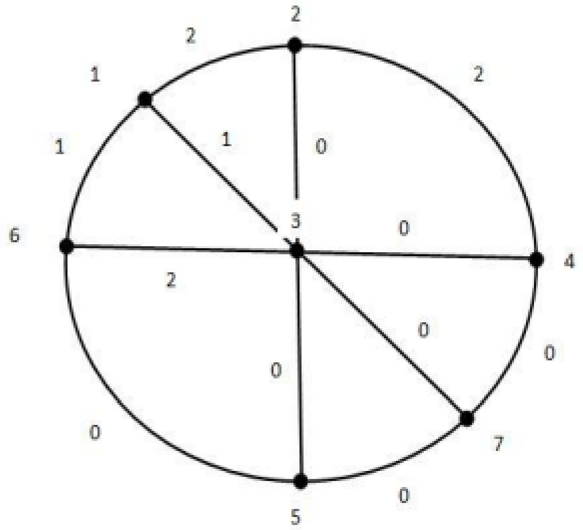

FIGURE 4.4: $d(v_0) = 3$ in W_7

"Interestingly there are more than 4 edges with label 0, and just three edges with label 2, a contradiction".

Case 4: When $d(v_0) = 4$

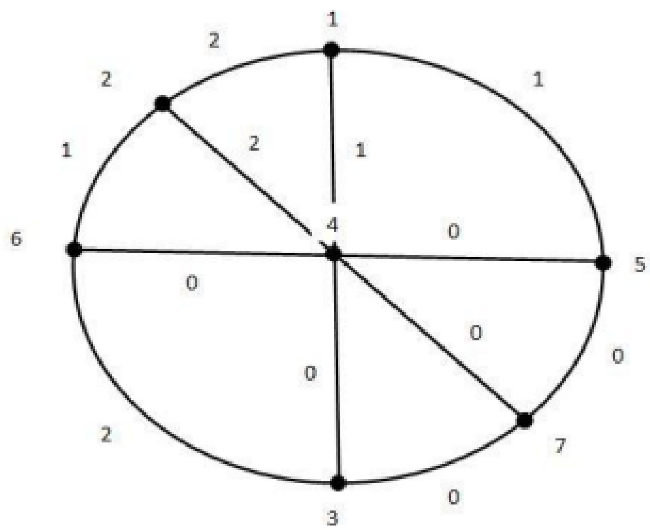

FIGURE 4.5: $d(v_0) = 4$ in W_7

The number of edges with label 2 is again 3 and the number of edges with label 0 is more than 4, a contradiction.

Case 5: When $d(v_0) = 5$

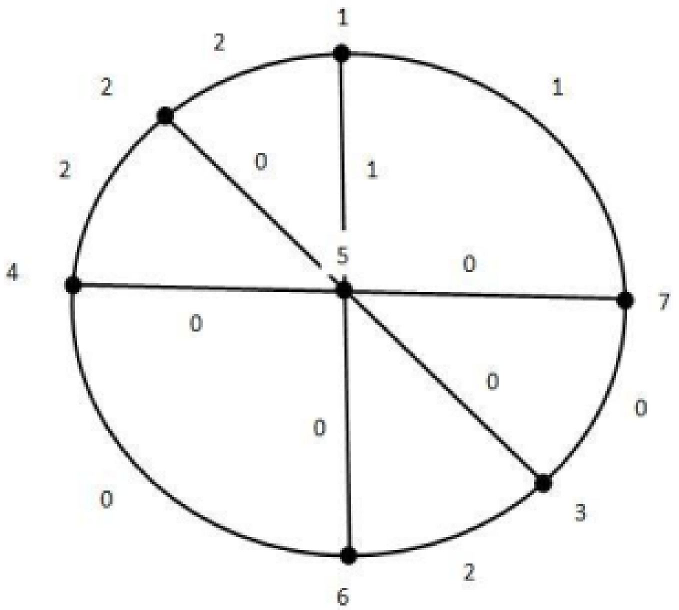

FIGURE 4.6: $d(v_0) = 5$ in W_7

Observe that the edges incident with the central vertex get the label 0 “except the one edge whose end vertex is labelled with 1, a contradiction”. This is because there are at least 5 edges with label 0.

Case 6: When $d(v_0) = 6$

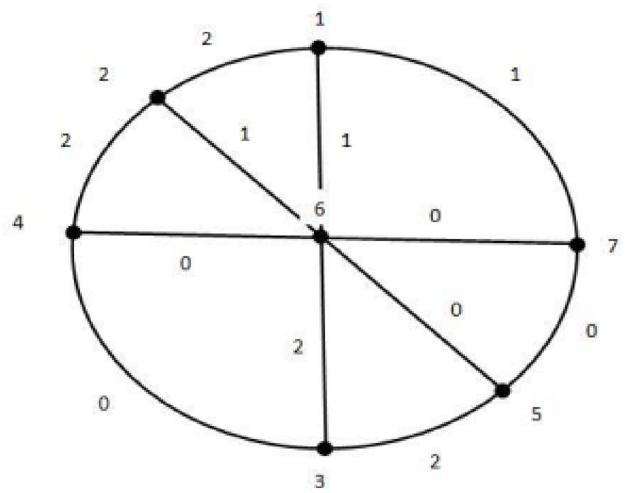

FIGURE 4.7: $d(v_0) = 6$ in W_7

"Now again there are only three edges with label 2 and more than four edges with label 0, a contradiction".

Case 7: When $d(v_0) = 7$

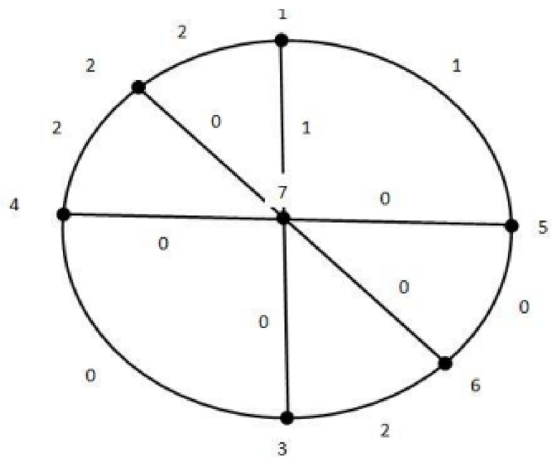

FIGURE 4.8: $d(v_0) = 7$ in W_7

This case is also rejected in a similar fashion that is of case 5.

One can also explore all other possible cases and subcases of assignment of numbers (1 to 7) to the vertices of W_7 in any possible ways (permutation and combination). These cases and subcases are treated and rejected in a similar fashion. Hence W_n, $n = 7$ does not admit D3EL. A similar argument holds good for W_n, $n \geq 8$ too.

4.4.2 D3EL of Complete Graphs

"This section is devoted for proving the non-existence of the D3EL of complete graphs"

Definition 4.4.2. The "complete graph K_n is a graph in which any two vertices are adjacent".

One can obtain the D3EL of K_1, K_2, K_3 and K_4. "One such example is given in Figure 4.9. So, consider K_n, for $n \geq 5$".

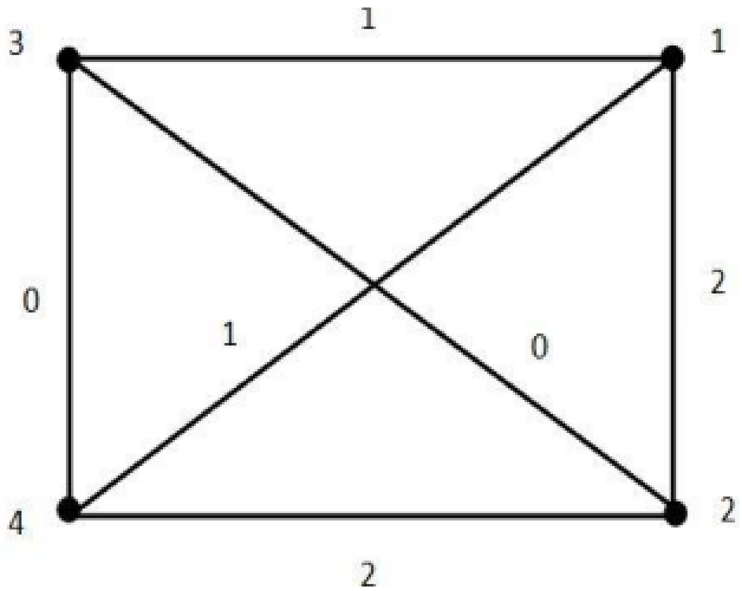

FIGURE 4.9: D3EL of K_4

Theorem 4.4.2. *K_n does not permit D3EL* $\forall n \geq 5$.

Proof. "Let K_n = $\{v_1, v_2, \ldots, v_n\}$ be the given complete graph on $n \geq 5$ vertices. One can clearly observe that any two nodes in K_n are adjacent. In other words, a vertex $v \in K_n$ is adjacent to all other vertices of K_n". Now define a bijection $d : V(K_n) \rightarrow \{1, 2, ..., n\}$ that induces $d^* : E(K_n) \to \{0,1,2\}$ for any edge xy with $d(x) < d(y)$,and defined by d^* $(e = xy) = \begin{cases} 1, \; if \; d\,(x)\,|d(y) \\ 2, \; if \; \frac{d(y)}{d(x)} = 2 \\ 0 \; if \; d(x) \nmid d(y) \end{cases}$

The proof by a method of contradiction.Assume that K_n, $n \geq 5$ has a D3EL d and induced d^* such that "$|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq \; 1 \forall \; 0 \leq \; i, j \leq 2$, where $e_{d^*}\,(i)$ is the number of lines labeled with label i under d^*. Note that a vertex labeled with 1 in K_n, say $d(v_1) = 1$, is adjacent to all other vertices, $v_i : 2 \leq i \leq n$ labeled with $2, 3, ..., n$ in any possible permutation. So as per the definition of a D3EL, if the label of a vertex v_1, say $d(v_1)$, divides the label of the adjacent vertex v_2, say $d(v_2)$, then the edge v_1v_2 must be given the label 1. As the vertex v_1 is adjacent to all other vertices in K_n, it gives 1 as the edge label to all other edges except the edge whose end vertex is labeled with 2, say v_2, (in which case the edge label is 2) in K_n which' are incident on v_1 . That is., $d^*(v_1v_i) = 1$for all $3 \leq i \leq n$, a contradiction to the fact that $|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq \; 1 \; \forall \; \; 0 \leq i, j \leq 2$". Therefore, K_n does not permit D3EL for all $n \geq 5$.

4.4.3 D3EL of Star Graphs

This section is devoted to prove "the non-existence of a D3EL of the star graphs".

Definition 4.4.3. "A star graph S_{n+1} is a $K_{1,n}$".

One can clearly obtain the "D3EL of $K_{1,n}$, $1 \leq n \leq 5$. One such example is given in Figure 4.10".

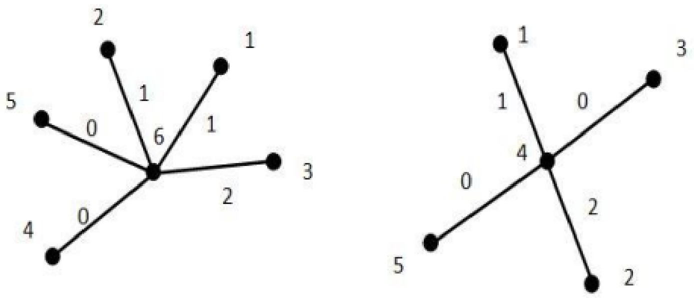

FIGURE 4.10: A D3EL of $K_{1,5}$ and $K_{1,4}$

Theorem 4.4.3. *$K_{1,n}$ does not permit D3EL $\forall\, n \geq 6$.*

Proof. "Let $K_{1,n}$ be the given star graph on $n \geq 6$ vertices. $n = 6$ for the sake of discussion and so $|V(K_{1,6})| = 7$ and $|E(K_{1,6})| = 6$. One can note that there are six vertices of degree one and a vertex of degree 6 in $K_{1,6}$". Label the central vertex as v_0 and pendant vertices as $v_1,\ v_2,\ \ldots,\ v_6$. Now define a bijective function $d\,:\,V(K_{1,6}) \rightarrow \{1, 2, ..., 7\}$ that induces $d^*\,:\,E(K_1, n) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$,and defined by $d^*\ (e\ =\ xy)\ =$

$$\begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

The proof by the method of contradiction. Assume that $K_{1,6}$ has a D3EL d and induced "d^* with the property that $|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq\ 1$ for all $\ 0\ \leq\ i, j\ \leq\ 2$. One can also observe that the number of edges labeled with label either 0 or 1 or 2 can be at most 2 (as there are exactly seven vertices in $K_{1,6}$) to satisfy the required divisor 3-equitble property$|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq\ 1$ for all $\ 0 \leq i, j \leq 2$. Now arises the following seven cases".

Case 1: When $d(v_0) = 1$

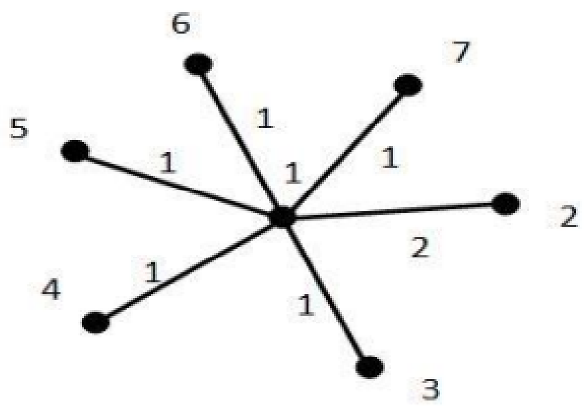

FIGURE 4.11: $d(v_0) = 1\ in\ K_1, n$

"As v_0 is adjacent to all other vertices and the label 1 divides all other labels, $d^*(v_0v_i) = 1$ for all $2 \leq i \leq 6$ and $d^*(v_0v_1) = 2$. So, the number of edges labeled with 1 is 5, a contradiction".

Case 2: When $d(v_0) = 2$

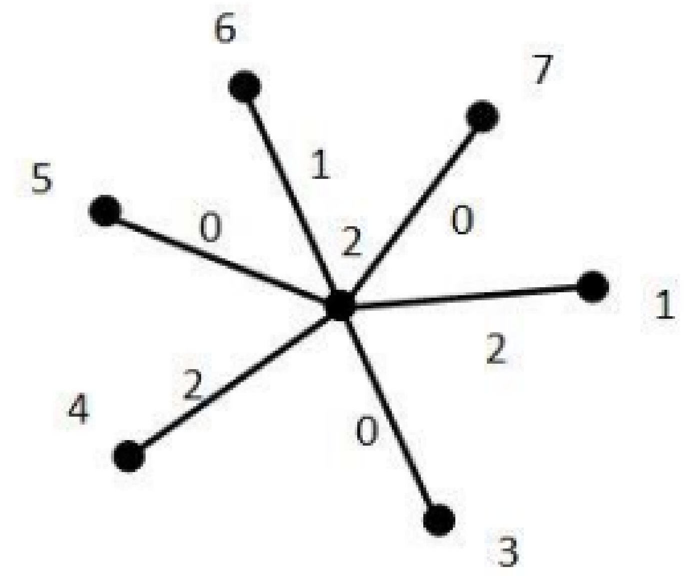

FIGURE 4.12: $d(v_0) = 2\ in\ K_1, n$

"One can note that there are three edges with label 0, two edges with label 2, and only one vertex with label 1, a contradiction. This is because $|e_{d^*}$ (0) - e_{d^*} (2) $| \leq 1$ but $|e_{d^*}$ (0) - e_{d^*} (1) | = 2".

Case 3: When $d(v_0) = 3$

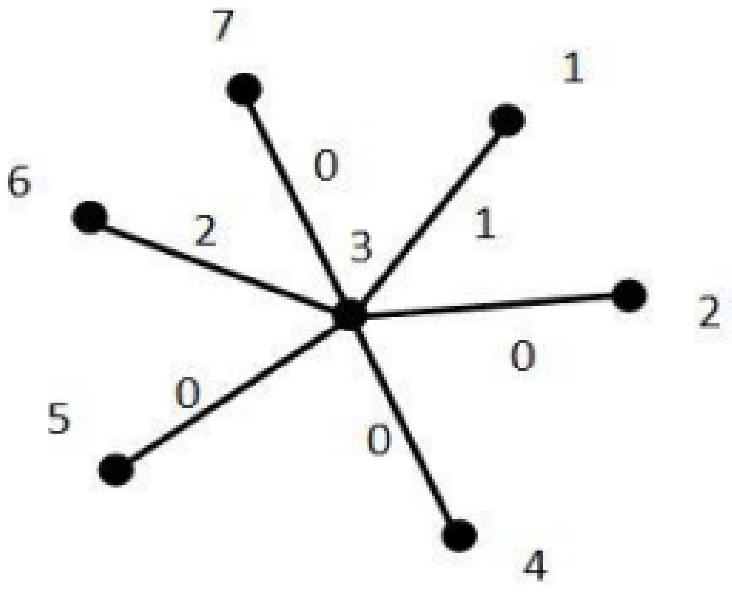

FIGURE 4.13: $d(v_0) = 3\ in\ K_{1,n}$

"Interestingly there are four edges with label 0, one edge with label 1, and an edge with label 2, a contradiction as the number of edges labeled with 0 is more than 2".

Case 4: When $d(v_0) = 4$

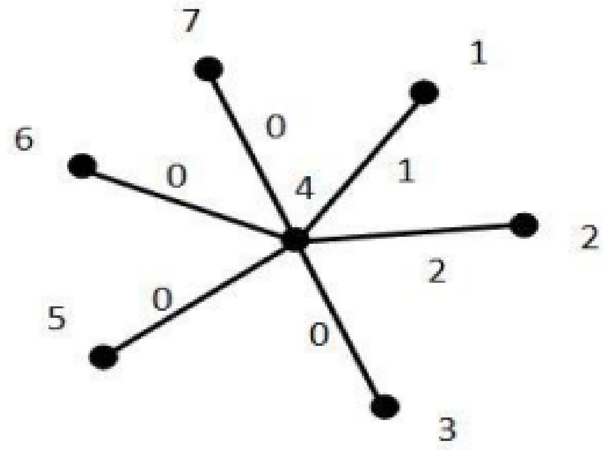

FIGURE 4.14: $d(v_0) = 4 \; in \; K_{1,n}$

"This is a similar case as Case 3".

Case 5: When $d(v_0) = 5$

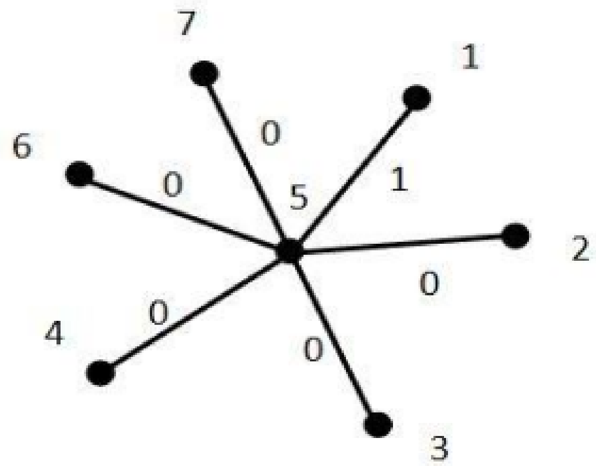

FIGURE 4.15: $d(v_0) = 5 \; in \; K_{1,n}$

"Interestingly there are five edges with label 0 and an edge with label 1, a contradiction".

Case 6: When $d(v_0) = 6$

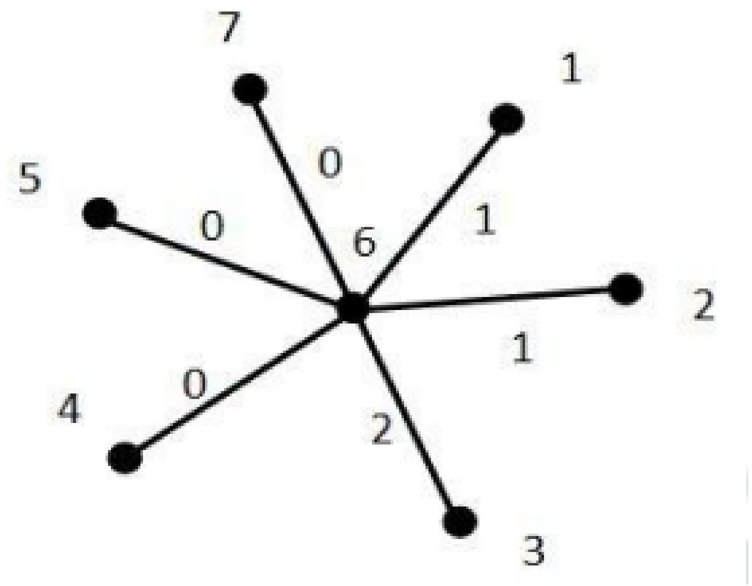

FIGURE 4.16: $d(v_0) = 6$ *in* $K_{1,n}$

"Now there are three edges with label 0, two edges with label 1, and an edge with label 2, again a contradiction because $|e_{d^*}$ (0) - e_{d^*} (1) $| \leq 1$ but $|e_{d^*}$ (0) - e_{d^*} (2) $|$ = 2".

Case 7: When $d(v_0) = 7$

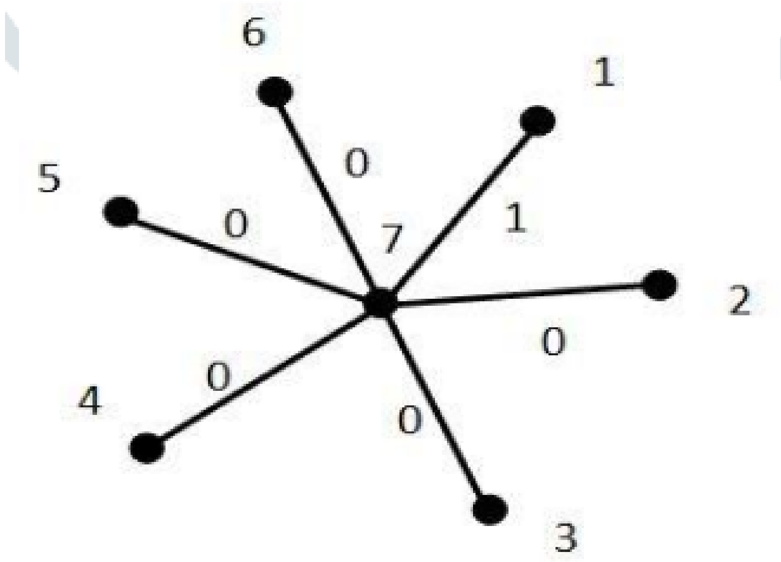

FIGURE 4.17: $d(v_0) = 7$ *in* $K_{1,n}$

"Here there are five edges of label 0 and an edge of label 1, a clear contradiction.

Hence $K_{1.n}$, when $n = 6$ does not admit D3EL". A similar argument holds good for $K_(1, n)$, $n \geq 7$.

4.4.4 D3EL of Jelly Fish Graph

"This section is devoted to prove the non-existence of a D3EL of J.F.G".

Definition 4.4.4. [23] "Jelly Fish $J(s,t)$ with order $s + t + 4$ and sizes $s + t + 5$ and formed from a 4-cycle x_1, x_2, x_3, x_4 by connecting x_1 and x_3 with a line and appending s pendent lines to x_2 and t pendent edges to x_4".

Theorem 4.4.4. *$J(m,n)$ does not admit D3EL for $m, n \geq 4$.*

Proof. One can establish the D3EL of J (0,0), J (1,1), J (2,2), J (3,3). One such example is given in Fig. 4.18. So, consider J (4,4) for $m, n > 4$.

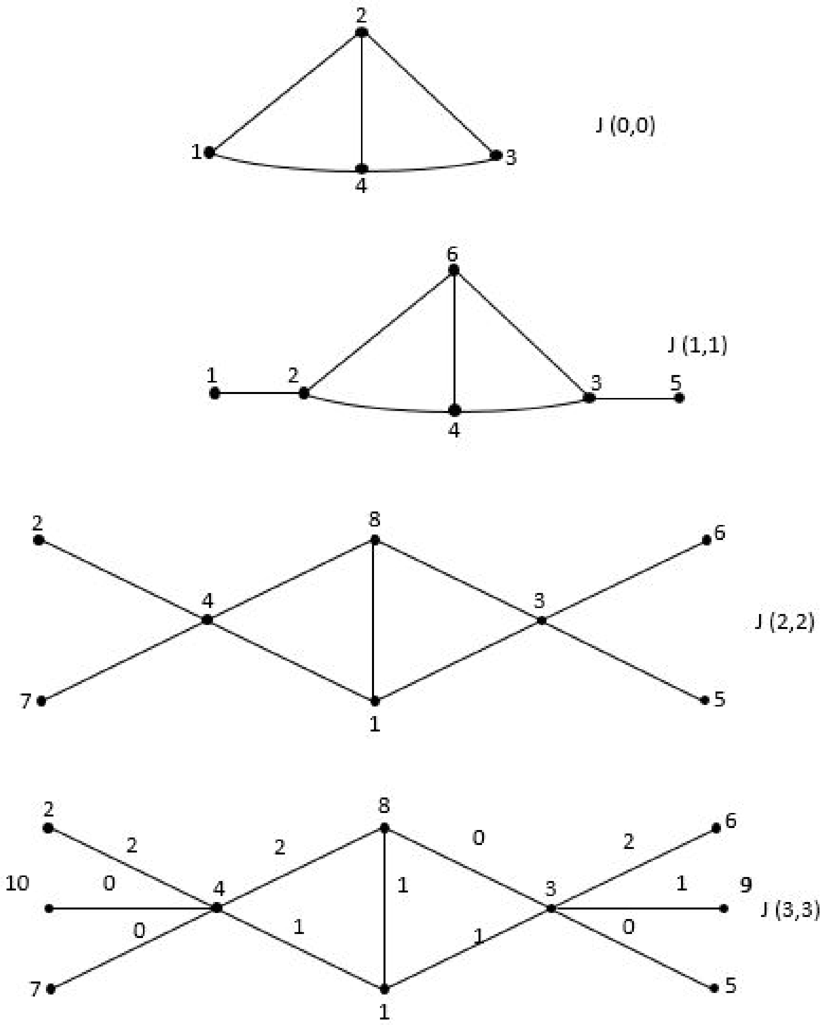

FIGURE 4.18: D3EL of$J(0,0)$,$J(1,1)$,$J(2,2)$,$J(3,3)$

Without loss of generality,consider $J(4,4)$ for the sake of discussion. One can easily see that there are 12 vertices and 13 edges. As per the definition of D3EL one must have 4 edges with label 0, 4 edges with label 1, and 5 edges with label 2 (or 4 edges with label 1, 4 edges with 2, and 5 edges with 0 or 4 edges with label 2, 4 edges with 1, and 5 edges with 0). An easy check shows that such combination and labeling is not possible. If $J(m,n)$, $m,n = 4$ does not admit D3EL, then $J(m,n)$, $m,n > 4$ also does not admit D3EL. Hence the proof.

4.5 D3EL OF SOME NAMED GRAPHS

In this section, D3EL of some special named graphs are given

4.5.1 The Cricket Graph permit D3EL

Definition 4.5.1. The cricket graph is the 5 vertex graph. "It has two nodes of degree one, two nodes of degree two" and one node of degree four.

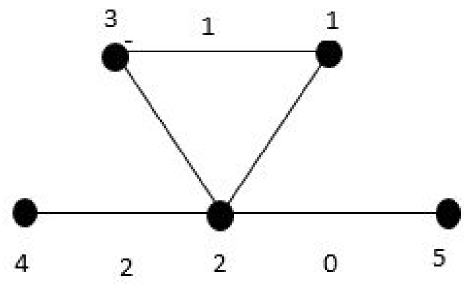

FIGURE 4.19: D3EL of Cricket graph

4.5.2 The paw Graph permit D3EL

Definition 4.5.2. The paw graph is the 3-pan graph, which is also isomorphic to the (3, 1) tadpole graph.

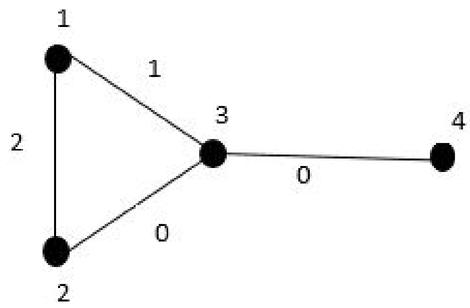

FIGURE 4.20: D3EL of Paw graph

4.5.3 The bull Graph permit D3EL

Definition 4.5.3. "The **bull graph** is a planar undirected graph with 5 vertices and 5 edges in the form of a triangle with two disjoint pendent edges".

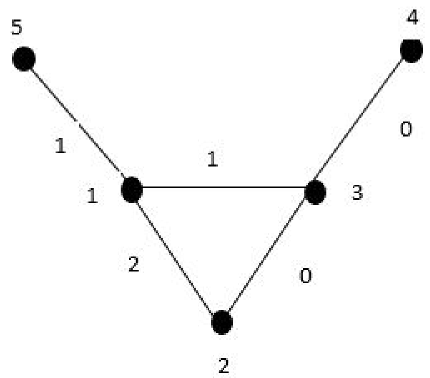

FIGURE 4.21: D3EL of Bull graph

4.5.4 The Net Graph permit D3EL

Definition 4.5.4. The net graph is the graph on 6 nodes. It has three nodes of degree one, two nodes of degree three and one node of degree two.

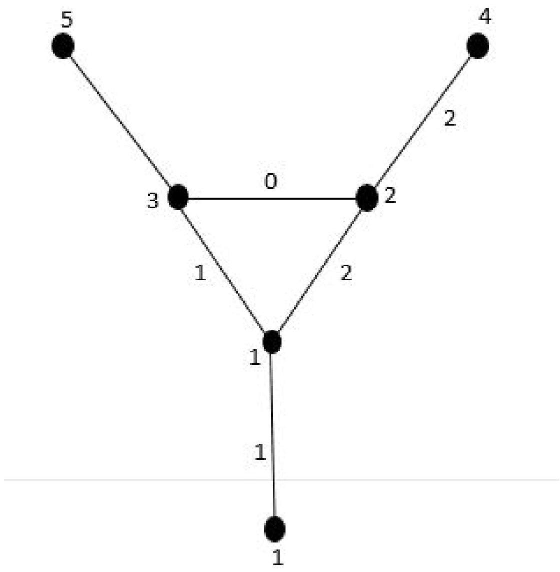

FIGURE 4.22: D3EL of Net graph

4.5.5 The House Graph permit D3EL

Definition 4.5.5. The house graph is a simple graph on 5 nodes and 6 edges. "It has three vertices of degree two, two vertices of degree three".

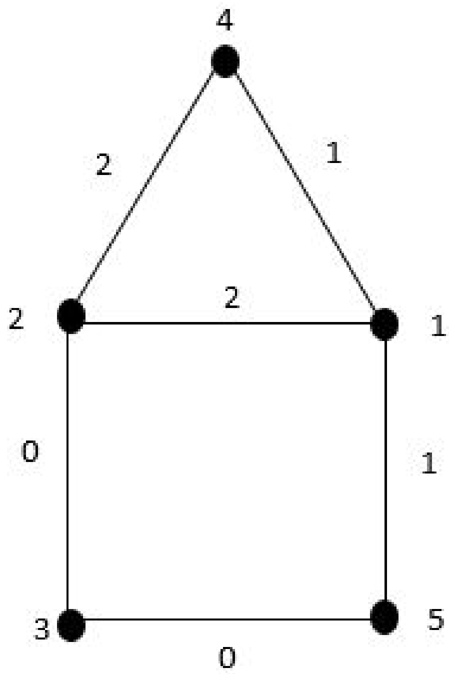

FIGURE 4.23: D3EL of House graph

4.5.6 The House X- Graph permit D3EL

Definition 4.5.6. The house X-graph is the house graph plus the two edges connecting diagonally opposite vertices of the square base.

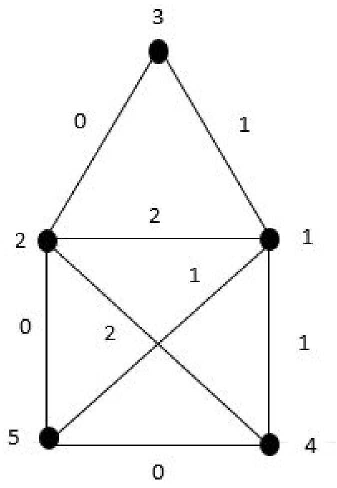

FIGURE 4.24: D3EL of House X-graph

4.5.7 The R Graph permit D3EL

Definition 4.5.7. The R graph is the graph on 6 nodes. It has three nodes of degree two, two nodes of degree one & one node of degree three.

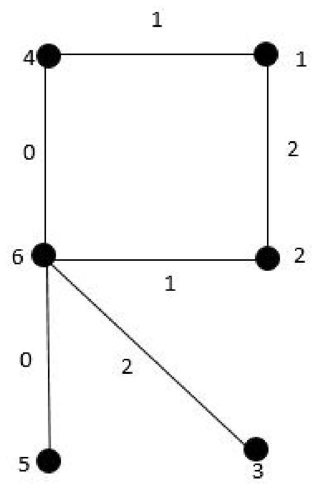

FIGURE 4.25: D3EL R-graph

4.5.8 The A Graph permit D3EL

Definition 4.5.8. The A graph is the graph on 6 nodes. "It has two nodes of degree two, two nodes of degree three & two nodes of degree one".

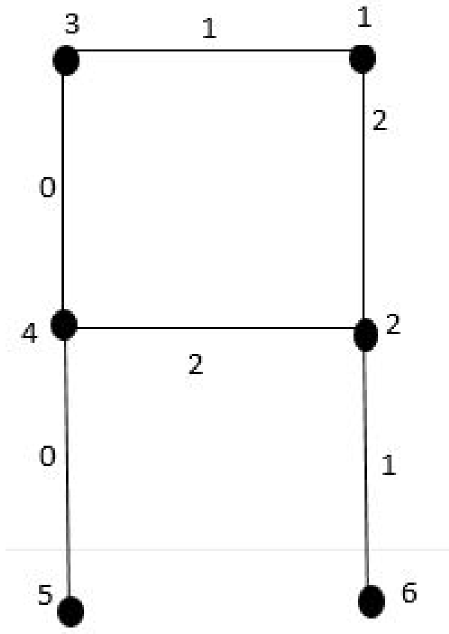

FIGURE 4.26: D3EL of A -graph

4.5.9 The Banner Graph permit D3EL

Definition 4.5.9. The **banner graph** consists of a hole on four vertices and a single vertex with precisely one neighbor on the hole.

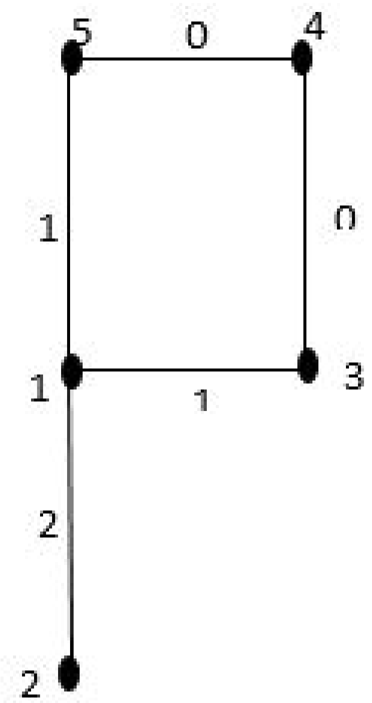

FIGURE 4.27: D3EL of the Banner graph

4.5.10 The Cross Graph permit D3EL

Definition 4.5.10. The **cross graph** is the 6 vertex tree and it has 5 edges. “It has four vertices of degree one, one vertex of degree four and one vertex of degree two”.

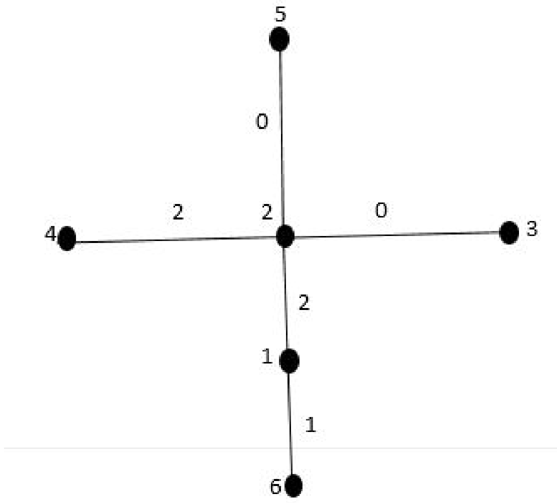

FIGURE 4.28: D3EL of Cross graph

4.5.11 The H Graph permit D3EL

Definition 4.5.11. The **H graph** is the 6 vertices tree and it has 5 edges. It has four vertices of degree one and two vertices of degree three.

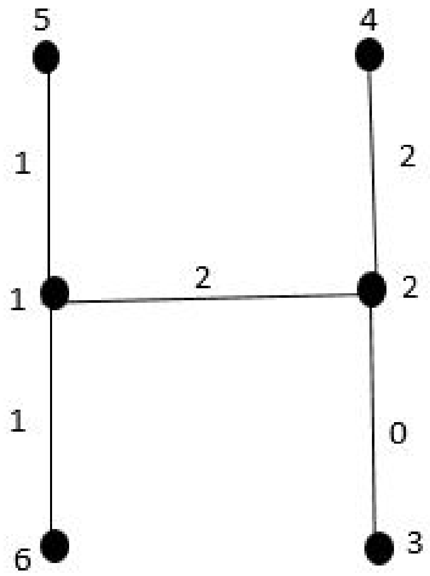

FIGURE 4.29: D3EL of H-graph

4.5.12 The Fork Graph permit D3EL.

Definition 4.5.12. The **fork graph** is the 5 vertices tree and it has 4 edges. "It has three vertices of degree one, one vertex of degree three and one vertex degree two".

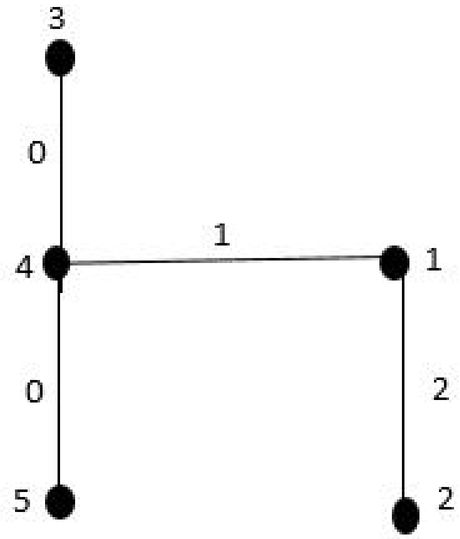

FIGURE 4.30: D3EL of Fork graph

4.5.13 Claw Permit D3EL

Definition 4.5.13. $K_{1,3}$ is known as the "claw."

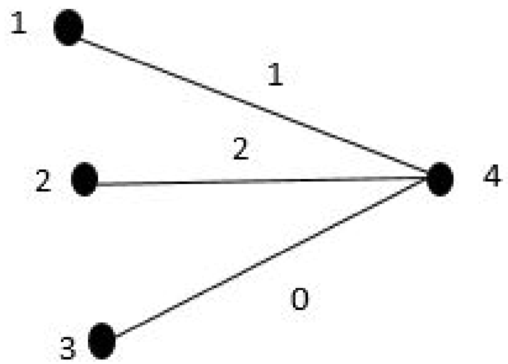

FIGURE 4.31: D3EL of Claw

4.5.14 The Dart Graph permit D3EL

Definition 4.5.14. The **dart graph** is the 5 nodes tree and it has 6 lines. “It has two nodes of degree two, one node of degree four, one node of degree one and one node of degree three”.

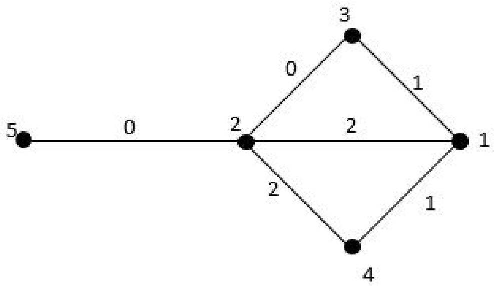

FIGURE 4.32: D3EL of Dart graph

4.5.15 The Kite Graph permit D3EL

Definition 4.5.15. The **kite graph** is the 5 vertices tree and it has 6 edges. “It has three vertices of degree three, one vertex of degree one and one vertex of degree two”.

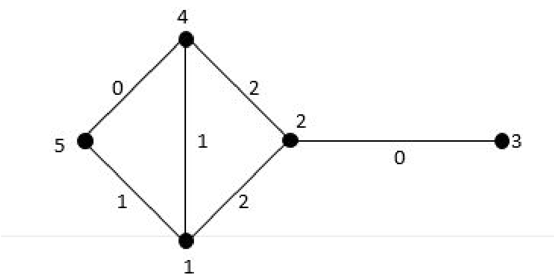

FIGURE 4.33: D3EL of Kite graph

4.5.16 The Diamond Graph permit D3EL

Definition 4.5.16. “The **diamond graph** is a planar undirected graph with 4 vertices and 5 edges. It has four vertices of degree two and two vertices of degree three”.

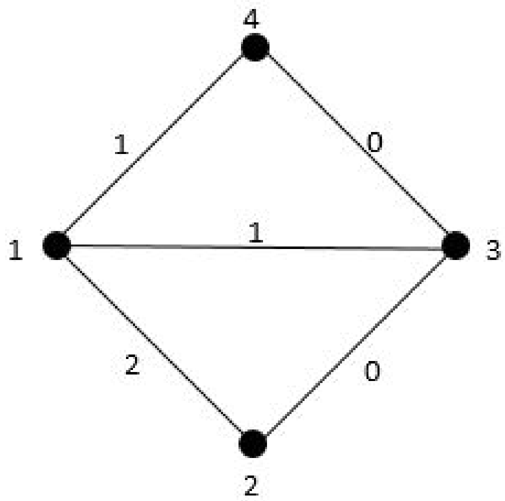

FIGURE 4.34: D3EL of Diamond graph

4.6 Conclusion

"The existence and non-existence" of D3EL of wheel graphs, complete graphs, and star graphs are established. The D3EL of jelly fish graph are also derived besides exhibiting the D3EL for other graphs such as the diamond graph, kite graph, dart graph, claw graph, fork graph etc.

Chapter 5

D3EL of Some Classes of Graphs

5.1 Introduction

In this chapter,the **D3EL of TG, MG, CG, DSG, and Mycielskian graphs** of various graphs are derived

5.1.1 D3EL OF TG OF UMBRELLA GRAPH

Definition 5.1.1. [45] "For any integers $s > 2; t > 1$, an umbrella graph $U\ (s,\ t)$ is obtained by identifying the end node of P_t with a central node of a F_s".

Theorem 5.1.1. *$T(U_{3,n})$ does not permit D3EL* $\forall\ n\ \geq\ 2$.

Proof. Let $U_{3,n}$ be the given Umbrella graph on $n \geq 2$ nodes. Take $n = 2$ and so |V($U_{3,2}$)| = 5 and |E($U_{3,2}$)| = 6. Obtain the TG of $U_{3,n},\ T(U_{3,n})$ with |V($T(U_{3,n})$)| = 11 and |E($T(U_{3,n})$)| =29. Now define a bijection "$d\ :\ V\ (T(U_{3,n}))\ \rightarrow\ \{1,\ 2,\ \ldots,\ n\}$ that induces $d^*\ :\ E(T(U_{3,n})) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$,as follows:

$$d^*\,(e\;=\;xy)\;=\begin{cases}1,\;if\;d\,(x)\,|d(y)\\ 2,\;if\;\frac{d(y)}{d(x)}=2\\ 0,\;if\;d(x)\nmid d(y)\end{cases}$$

The proof is by the method of contradiction. Assume that T($U_{3,2}$) has a D3EL d and induced d^* with "the property that $|e_{d^*}\,(i)-e_{d^*}\,(j)\,|\leq\;1,\forall\,0\leq\;i,j\;\leq 2$",. One can observe that "the number of lines labeled 0, 1, and 2 must be at least 9 and at most 10 to satisfy the required divisor 3-equitable property $|e_{d^*}\,(i)-e_{d^*}\,(j)\,|\leq\;1,\forall\,0\leq\;i,j\;\leq 2$",. "But there are only 5 lines with label 2, a contradiction" (See Figure 5.1). A similar argument holds good for T $(U_{3,n}),\;n>2$. Therefore, $T(U_{3,n}),\;n\geq 2$, does not admit D3EL.

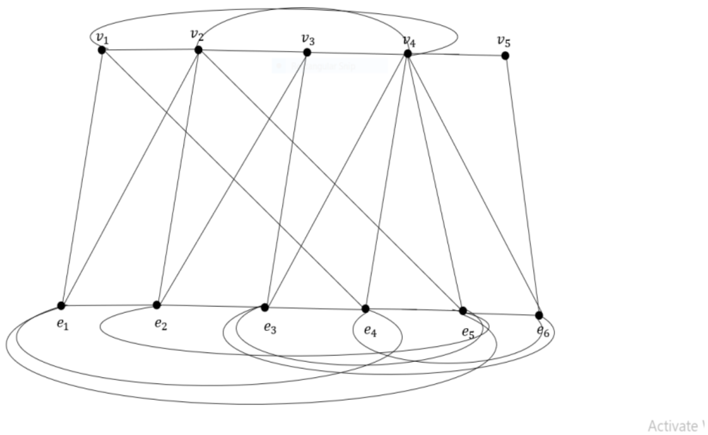

FIGURE 5.1: The non-existence of D3EL for $T(U_{3,2})$

5.1.2 D3EL OF TG OF TRIANGULAR SNAKE GRAPH

Definition 5.1.2. [1] "The triangular snake graph TS_n with n (odd) nodes is defined by starting with P_{n-1} and adding lines $(2i-1,\;2i+1)$ for $i=\;1,2,\;...,\;n-1$. A triangular snake TS_n is obtained from a path $u_1,u_2,\ldots,u_n$ by joining u_i and u_{i+1} to a new node w_i for $1\leq\;i\leq\;n{-}1$".

Theorem 5.1.2. *$T(TS_n)$ does not permit D3EL $\forall$ positive n.*

Proof. Let TS_n be the given triangular snake graph on $n \geq 1$ node. Obtain $T(TS_n)$ and take $n = 1$ and so |V $(T(TS_1))| = 6$ and |E $(T(TS_1))|$ = 12. Now define a bijection "d : V $(T(TS_n)) \rightarrow \{1, 2, \ldots, n\}$ that induces $d^* : E(T(TS_n)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$,as follows:

$$d^*\,(e = xy) = \begin{cases} 1, \; if \; d\,(x)\,|d(y) \\ 2, \; if \; \frac{d(y)}{d(x)} = 2 \\ 0, \; if \; d(x) \nmid d(y) \end{cases}$$

The proof is by the method of contradiction. Assume that $T(TS_1)$ has D3EL d and "induced d^* with the property that $|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq 1, \forall\, 0 \leq i, j \leq 2$. Observe that the number of lines labeled with label either 0 or 1 or 2 must be exactly 4 to satisfy the required divisor 3-equitable property $|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq 1, \forall\, 0 \leq i, j \leq 2$. But there are only three lines with label 2, a contradiction" (See Figure 5.2). Moreover, if $T(TS_1)$, does not admit D3EL, then $T(TS_n),\; n > 1, too$ does not admit D3EL.

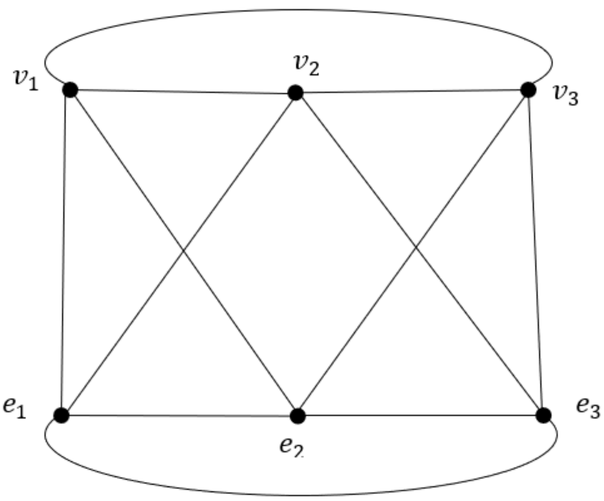

FIGURE 5.2: The non-existence of D3EL of $T(TS_1)$

5.1.3 D3EL OF TOTAL GRAPH OF $\mathbf{W_n}$

Theorem 5.1.3. *$T(W_n)$ does not permit D3EL $\forall\ n\ \geq\ 3$.*

Proof. Let W_n be the wheel on $n\ \geq\ 3$ nodes. Obtain $T(W_3)$ and so |V $(T(W_3))$| =10 and |E $(T(W_3))$| = 30. Now define a bijection "$d\ :\ V\,(T\ (W_3))\ \rightarrow\ \{1,\ 2,\ \ldots,\ n\}$ that induces $d^* : E(T(W_3)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ as follows:

$$d^*\ (e\ =\ xy)\ = \begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

The proof is by the method of contradiction". Assume that $T(W_3)$ has D3EL d with the property that $|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq\ 1, \forall\, 0 \leq\ i, j\ \leq\ 2$. One can observe that the number of lines labeled with label either 0 or 1 or 2 must be exactly 10 to satisfy the required divisor 3-equitable property $|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq\ 1, \forall\, 0 \leq\ i, j \leq 2$. But there are only five lines with label 2, a contradiction (see Figure 5.3). Moreover, if $T(W_3)$, does not admit D3EL, then $T(W_n),\ n > 3\ too$ does not admit D3EL.

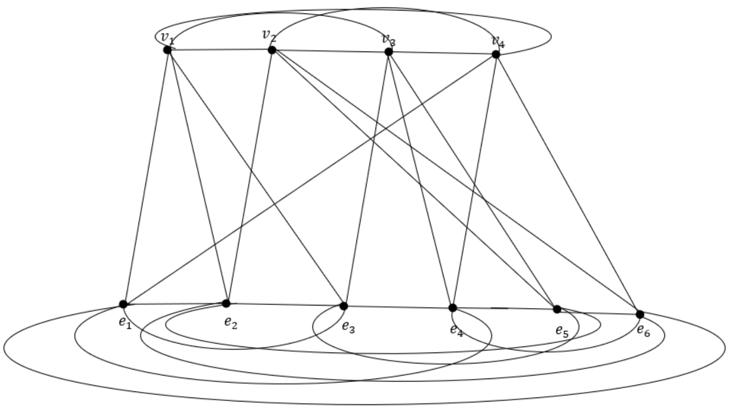

FIGURE 5.3: The non-existence of D3EL of $T(W_{3,}\ )$

5.1.4 D3EL OF MYCIELSKI'S GRAPH OF $\mathbf{W_n}$

Theorem 5.1.4. $\mu(W_n)$ *does not permit D3EL* $\forall\, n \geq 3$.

Proof. Let W_3 be wheel on $n \geq 3$ nodes. Obtain $\mu(W_3)$ and so |V $(\mu(W_3))$| =9 and |E $(\mu(W_3))$| = 22. Now define a bijection "$d : V(\mu(W_3)) \rightarrow \{1, 2, \ldots, n\}$ that induces $d^* : E(\mu(W_3)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$,as follows:

$$d^*(e = xy) = \begin{cases} 1, \ if\ d(x)\,|d(y) \\ 2, \ if\ \frac{d(y)}{d(x)} = 2 \\ 0, \ if\ d(x) \nmid d(y) \end{cases}$$

The proof is by the method of contradiction". Assume that $\mu(W_3)$ has D3EL d and induces d^* with "the property that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. One can observe that the number of lines labeled with label either 0 or 1 or 2 must be at least 7 and at most 8 to satisfy the required divisor 3-equitable property $|e_{d^*}(i) - e_{d^*}(j)| \leq 1$, $\forall\ 0 \leq i, j \leq 2$". But there are only four lines with label 2, a contradiction (see Figure 5.4). Moreover, if $\mu(W_3)$, does not admit D3EL, then $\mu(W_n)$, $n > 3$, *too* does not admit D3EL.

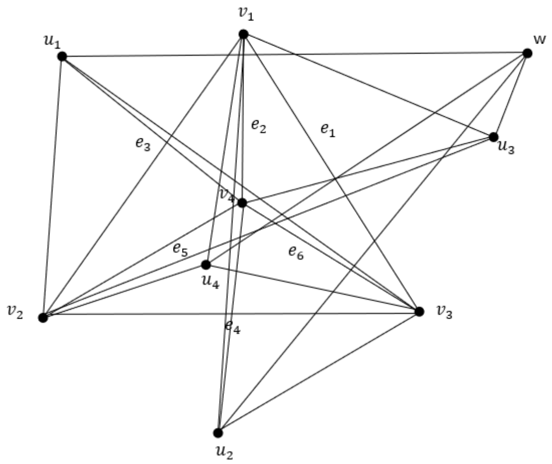

FIGURE 5.4: The non-existence of D3EL of $\mu(W_3)$

5.1.5 D3EL OF MIDDLE GRAPH OF LOLLIPOP GRAPH

Definition 5.1.3. [27] “The lollipop graph $L_{3,n}$ consisting of K_3 and P_n, connected with a bridge”

One can obtain the D3EL of One $M(L_{3,i})$:$1 \leq i \leq 7$. such example is given in Figure 5.5. So, consider $M(L_{3,n})$, for $n \geq 8$.

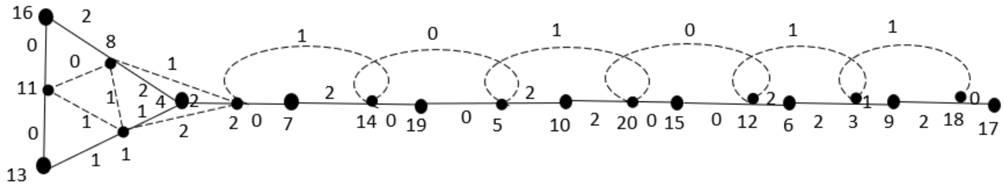

FIGURE 5.5: D3EL of the $M(L_{3,7})$

Theorem 5.1.5. *$M(L_{3,n})$ does not permit D3EL* $\forall\, n \geq 8$.

Proof. Let $L_{3,n}$ be the lollipop graph on $n \geq 8$ nodes. Take $n = 8$ and so |V($L_{3,8}$)| =11 and |E($L_{3,8}$)| = 11. Obtain, $M(L_{3,n})$ with |V($M(L_{3,8})$)| = 22 and |E($M(L_{3,8})$)| = 34. Now define a bijection ”$d : V\,(M(L_{3,n})) \rightarrow \{1,\ 2,\ \ldots,\ n\}$ that induces $d^* : E(M(L_{3,n})) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$,“as follows:

$$d^*\,(e = xy) = \begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

The proof is by the method of contradiction”. Assume that $M(L_{3,n})$ has a D3EL “$d$and induced d_* with the property that$|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq 1, \forall\, 0 \leq i, j \leq 2$. One can observe that the number of lines labeled 0, 1, and 2 must be at least 11 and at most 12 to satisfy the required divisor 3-equitable property $|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq 1, \forall\, 0 \leq i, j \leq 2$. But there are 11 lines with label 2, 9 lines with label 9, and 14 lines with label 0, a contradiction (See Figure 5.6). A similar argument holds good for M $(L_{3,n})$, $n >$ 8. Therefore, $M(L_{3,n}),\ n \geq 8$, does not admit D3EL”.

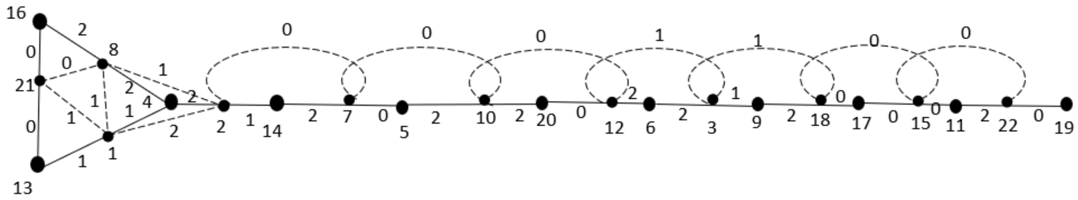

FIGURE 5.6: The non-existence of D3EL for $M(L_{3,8})$

5.1.6 D3EL OF CENTRAL GRAPH OF LOLLIPOP GRAPH

One can obtain the D3EL of $C(L_{3,1}), C(L_{3,2})$. So, consider $C(L_{3,n})$, for n $\geq$ 3.(see figure)

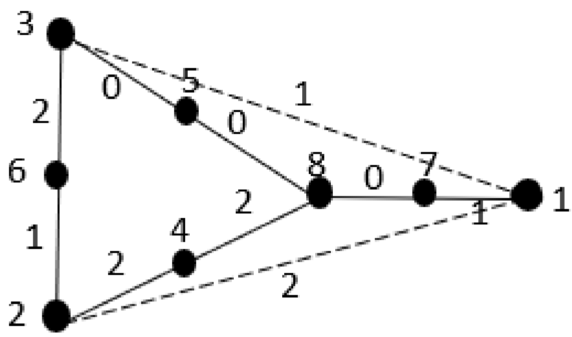

FIGURE 5.7: D3EL of C$(L_{3,1})$

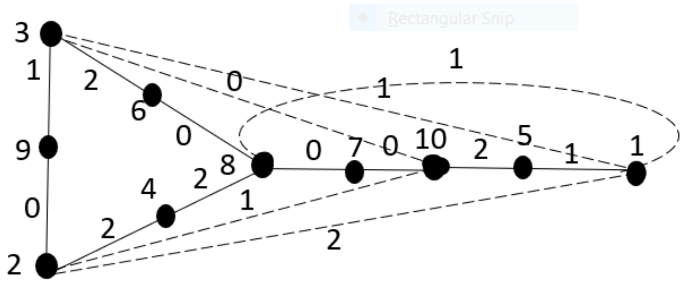

FIGURE 5.8: D3EL of C$(L_{3,2})$

Theorem 5.1.6. *$C(L_{3,n})$ does not permit D3EL* $\forall\, n \,\geq\, 3$.

Proof. Let $L_{3,n}$ be the lollipop graph on $n \,\geq\, 3$ nodes. Take $n = 3$ and so $|V(L_{3,3})| = 6$ and |E($L_{3,3}$)| = 6. Obtain the CG of $L_{3,n},\ C(L_{3,n})$ with |V($C(L_{3,3})$)| = 12 and |E($C(L_{3,3})$)| = 21. Now define a bijection ”$d\ :\ V\ (C(L_{3,3}))\ \rightarrow\ \{1,\ 2,\ \ldots,\ n\}$ that induces $d^*: E(C(L_{3,3})) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$,as follows:

$$d^*\ (e\ =\ xy)\ =\begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

The proof is by the method of contradiction. Assume that $C(L_{3,3})$ has D3EL d and induced d^* with “the property that $|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq\ 1, \forall\, 0 \leq\ i, j\ \leq\ 2$. One can observe that the number of lines labeled with label either 0 or 1 or 2 must be exactly 7 to satisfy the required divisor 3-equitable property $|e_{d^*}\,(i) - e_{d^*}\,(j)\,| \leq\ 1, \forall\, 0 \leq\ i, j\ \leq\ 2$”. But there are only six lines with label 2, a contradiction (see Figure 5.9). Moreover, if $C(L_{3,3})$, does not admit D3EL, then $C(L_{3,n}),\ n > 3,\ too$ does not admit D3EL.

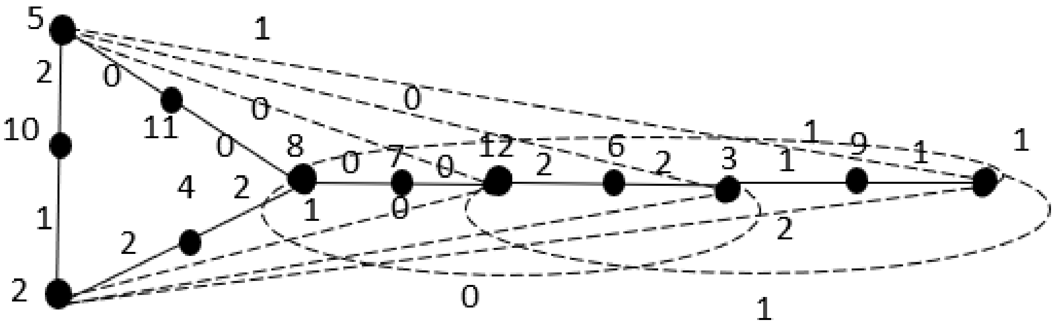

FIGURE 5.9: The non-existence of D3EL of $C(L_{3,3})$

5.1.7 D3EL of Degree Splitting Graph of Ladder Graph

Definition 5.1.4. [86] "The ladder $L_n = P_n \times P_2''$.

The D3EL of DSG (L_1), $DSG(L_2)$, and $DSG(L_3)$ are derived easily (see Fig.5.10). So, consider $DSG(L_n), \forall\, n \geq 4$.

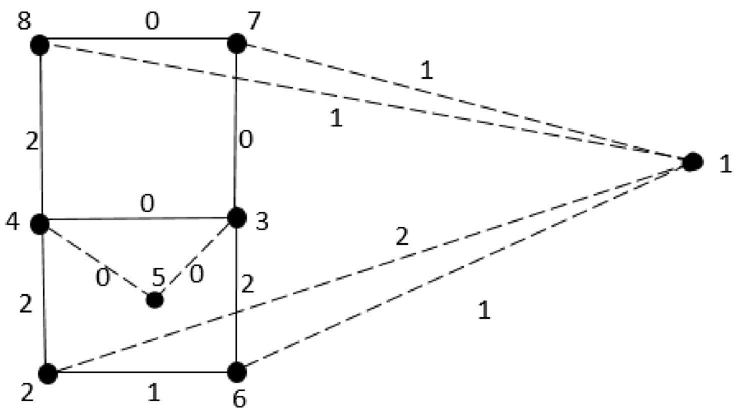

FIGURE 5.10: D3EL of $DSG(L_3)$

Theorem 5.1.7. *$DSG(L_n)$ does not allow D3EL $\forall\, n \geq 4$.*

Proof. Take L_4 for the sake of discussion with $|V(L_n)| = 8$ and $|E(L_n)| = 10$. Obtain $DSG(L_n)$ with $|V(DSG(L_n))| = 10$ and $|E(DSG(L_n))| = 18$. Now define a bijective map d : $V\ (DSG(L_n)) \rightarrow \{1,\ 2,\ \ldots,\ n\}$ that induces $d^* : E(DSG(L_n)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ "as follows:

$$d^*\ (e = xy) = \begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

The proof is by the method of contradiction. Assume that $DSG(L_4)$ has D3EL d and induced d^* with condition that$|e_{d^*}\ (i) - e_{d^*}\ (j)\,| \leq\ 1, \forall\, 0 \leq\ i, j\ \leq\ 2$". Note that the number of lines labeled 0, 1, and 2 must be 6, respectively "in order to meet the desired divisor 3-equitable property $|e_{d^*}\ (i) - e_{d^*}\ (j)\,| \leq\ 1, \forall\, 0 \leq\ i, j\ \leq\ 2$. But there are only 5 lines with 2, a contradiction (see Fig.5.11). The same argument holds good for $DSG(L_n),\ \ n > 4$. Therefore, $DSG(L_n),\ n \geq 4$, does not admit D3EL.

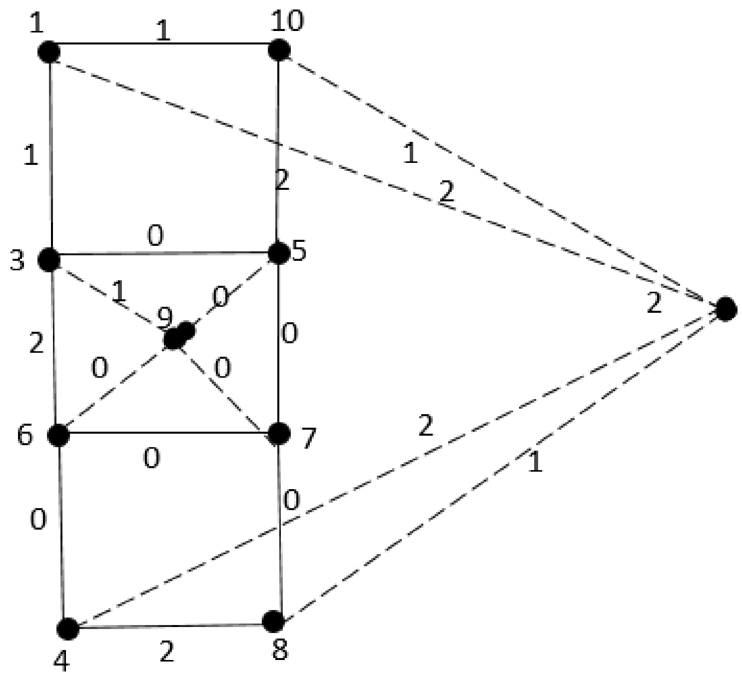

FIGURE 5.11: The non-existence of D3EL for $DSG(L_4)$

5.1.8 D3EL of DSG of Triangular Snake Graph

Definition 5.1.5. [1] graph TS_n with n (odd) nodes is defined by starting with P_{n-1} and adding lines $(2i-1,\ 2i+1)$ for $i =\ 1,\ 2,\ ...,\ n-1$".

One can derive the D3EL of $DSG(TS_n)$, $n \leq 5$. One such example is given in Fig. 5.12.

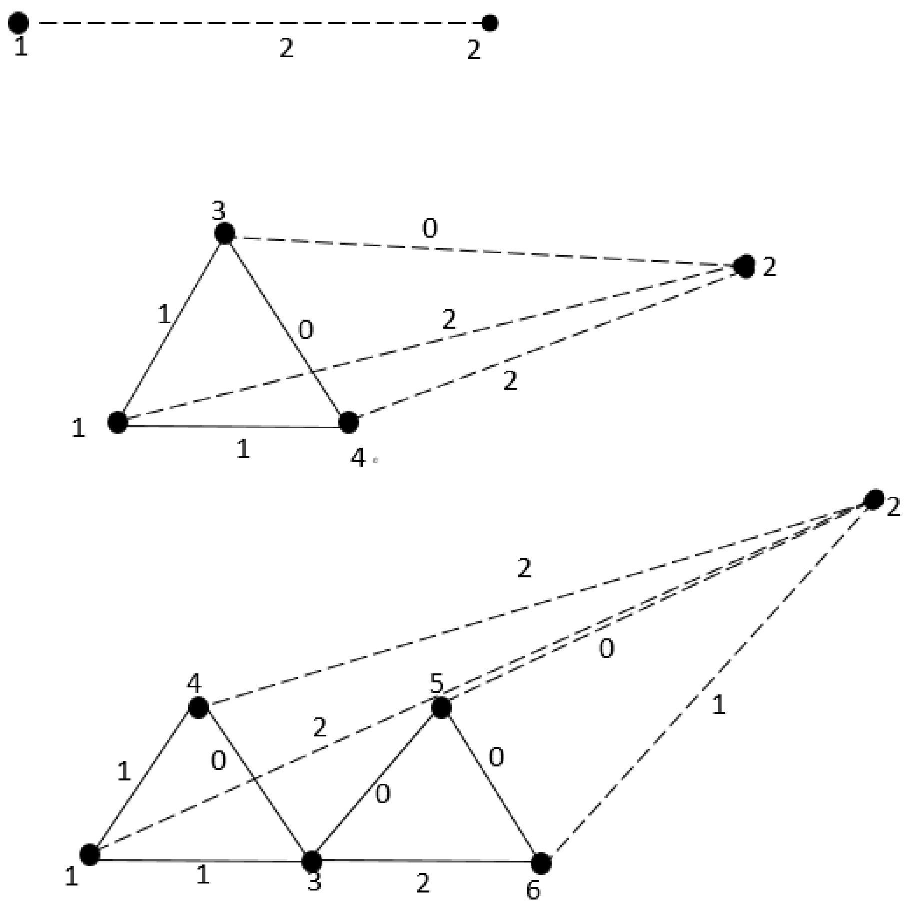

FIGURE 5.12: D3EL of $DSG(TS_1)$, $DSG(TS_3)$, **and** $DSG(TS_5)$

Theorem 5.1.8. *$DSG(TS_n)$ does not allow D3EL* $\forall\ n\ (odd)\ \geq\ 7$.

Proof. Take TS_n on $n \geq 7$ nodes and obtain $DSG(TS_n)$. Consider "$n = 7$ for the discussion purpose" and so $|V\ (DSG(TS_7))| = 9$ and $|E\ (DSG(TS_7))| = 16$. Now define a bijective map $d : V\ (DSG(TS_n)) \rightarrow \{1,\ 2,\ \ldots,\ n\}$ that induces $d^* : E(DSG(TS_n)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ "as follows:

$$d^*\ (e = xy) = \begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

The proof is by the method of contradiction. Assume that $DSG(TS_7)$ has D3EL d and induced d^* with condition that $|e_{d^*}\ (i) - e_{d^*}\ (j)\,| \leq 1, \forall\, 0 \leq i, j \leq 2$. Note that the count of lines labeled with either 0 or 1 or 2 must be at least 5 and at most 6 to meet the desired divisor 3-equitable property $|e_{d^*}\ (i) - e_{d^*}\ (j)\,| \leq 1, \forall\, 0 \leq i, j \leq 2$. But there are only four lines with label 2, a contradiction". Moreover, if $DSG(TS_7)$ does not admit D3EL, then $DSG(TS_n)$,$n > 7$ too does not admit $D3EL$.

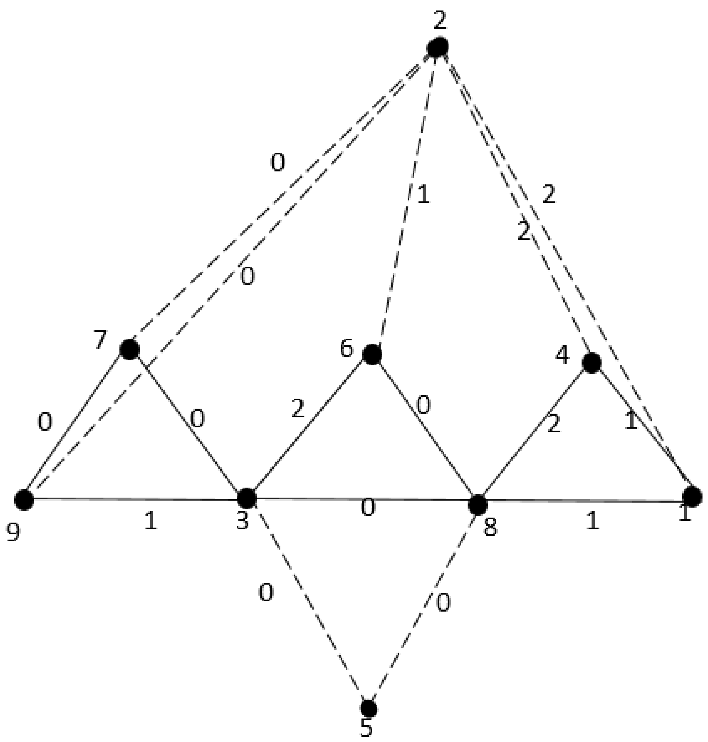

FIGURE 5.13: The non-existence of D3EL of $DSG(TS_7)$

5.1.9 D3EL of DSG of Lollipop Graph

One can establish the D3EL of $DSG(L_{3,i})$;$1 \leq i \leq 6$, Fig. 5.14 provides one such instance. So, consider $DSG(L_{3,n})$, for $n \geq 7$.

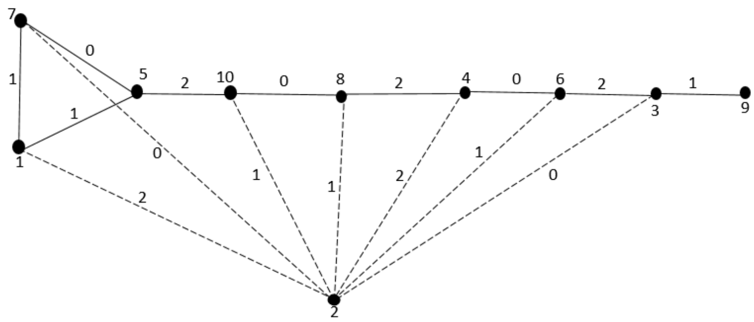

FIGURE 5.14: D3EL of $DSG(L_{3,6})$

Theorem 5.1.9. *$DSG(L_{3,n})$ does not accept D3EL* $\forall\, n \geq 7$.

Proof. "Take $L_{3,n}$ on $n \geq 7$ nodes and obtain $DS(L_{3,n})$. Take $n = 7$ for the sake of discussion, so $|V\ (DSG(L_{3,7}))| = 11$ and $|E\ (DSG(L_{3,7}))| = 18$. Define a bijective map $d : V\ (DSG(L_{3,7})) \rightarrow \{1,\ 2,\ \ldots,\ n\}$ that induces $d^* : E(DSG(L_{3,7})) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ as follows:

$$d^*\ (e = xy) = \begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

Assume that $DSG(L_{3,7})$ has D3EL d and induced d^* with the condition that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\ 0 \leq\ i, j \leq 2$. Note that the count of lines labeled with either 0 or 1 or 2 must be exactly 6 to meet the desired divisor 3-equitable property $|e_{d^*}(i) - e_{d^*}(j)| \leq\ 1, \forall\ 0 \leq\ i, j \leq 2$. But there are only five lines with label 2, a contradiction (see Fig. 5.15). Moreover, if $DSG(L_{3,7})$, does not permit D3EL, then $DSG(L_{3,n}),\ n > 7,\ too$ does not permit D3EL".

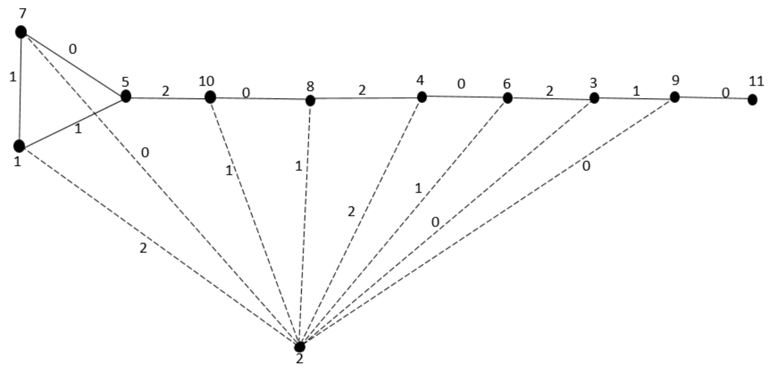

FIGURE 5.15: Non-existence of D3EL of $DSG(L_{3,7})$

5.1.10 D3EL of Degree Splitting Graph of $\mathbf{C_n}$

The D3EL of $DSG(C_i)$;$1 \leq i \leq 5$ are derived easily (see Fig. 5.16). So, consider $DSG(C_n)$, $\forall\, n \geq 6$.

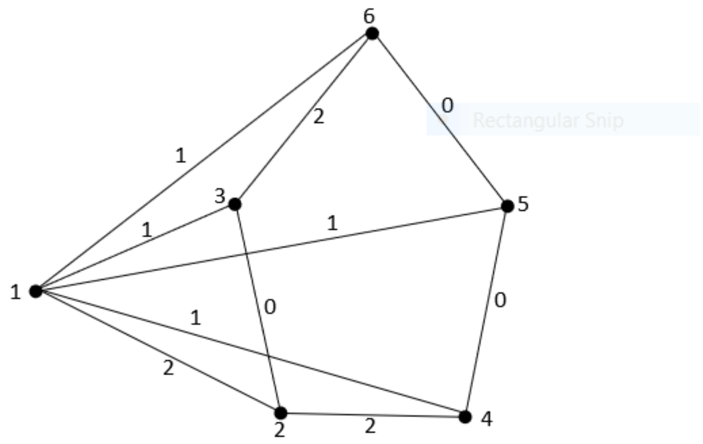

FIGURE 5.16: D3EL of $DSG(C_5)$

Theorem 5.1.10. *$DSG(C_n)$ does not allow D3EL $\forall\, n \geq 6$.*

Proof. Take C_6 for the sake of discussion with $|V(C_n)| = 6$ and $|E(C_n)| = 6$. Obtain $DSG(C_n)$ with $|V(DSG(C_n))| = 7$ and $|E(DSG(C_n))| = 12$. Now define a bijective map $d : V(DSG(C_n)) \to \{1, 2, \ldots, n\}$ that induces $d^* : E(DSG(C_n)) \to \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ "as follows:

$$d^*(e = xy) = \begin{cases} 1, \; if \; d(x) \,|d(y) \\ 2, \; if \; \frac{d(y)}{d(x)} = 2 \\ 0, \; if \; d(x) \nmid d(y) \end{cases}$$

Assume that $DSG(C_6)$ has $D3EL$ d and induced d^* with condition that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. Note that the number of lines labeled 0, 1, and 2 must be 4, respectively in order to meet the desired divisor 3-equitable property $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. But there are only 3 lines with 2, a contradiction" (see Fig.5.17). The same argument holds good for $DSG(C_n),\ n > 6$. Therefore, $DSG(C_n),\ n \geq 6$, does not admit D3EL.

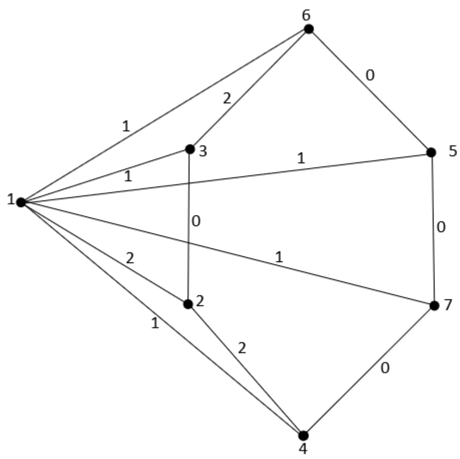

FIGURE 5.17: Non-existence of D3EL of $DSG(C_6)$

5.1.11 D3EL of Degree Splitting Graph of Friendship Graph

One can derive the D3EL of $DSG(F_n)$, $n \leq 2$. "One such example is given in Fig. 5.18 & 5.19".

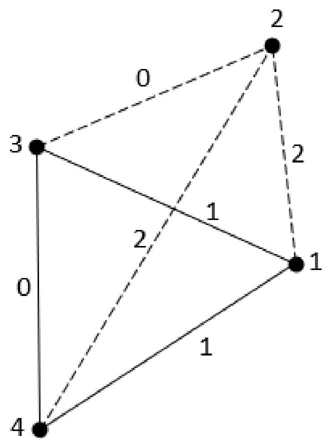

FIGURE 5.18: D3EL of $DSG(F_1)$

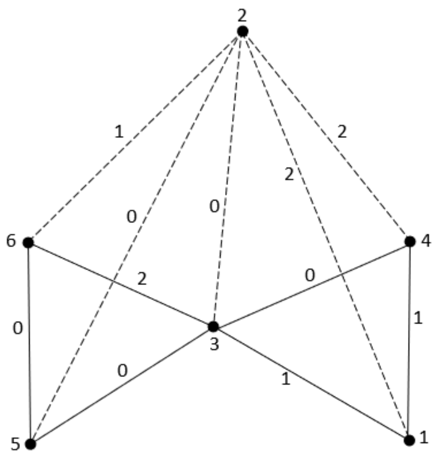

FIGURE 5.19: D3EL of $DSG(F_2)$

Theorem 5.1.11. *$DSG(F_n)$ does not allow D3EL $\forall\, n \geq 3$.*

Proof. Take F_n on $n \geq 3$ nodes and obtain $DSG(F_n)$. Consider $n = 3$ for the discussion purpose and so $|V\ (DSG(F_3))| = 8$ and $|E\ (DSG(F_3))| = 15$. Now define a bijective map

$d : V(DSG(F_n)) \rightarrow \{1, 2, \ldots, n\}$ that induces $d^* : E(DSG(F_n)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ as follows:

$$d^*(e = xy) = \begin{cases} 1, \; if \; d(x) \,|d(y) \\ 2, \; if \; \frac{d(y)}{d(x)} = 2 \\ 0, \; if \; d(x) \nmid d(y) \end{cases}$$

The proof is by the method of contradiction. Assume that $DSG(F_3)$ has $D3EL$ d and induced "d^* with condition that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. Note that the count of lines labeled with either 0 or 1 or 2 must be exactly 5 to meet the desired divisor 3-equitable property $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. But there are only four lines with label 2, a contradiction" (see Fig. 5.20). Moreover, if $DSG(F_3)$, does not admit D3EL, then $DSG(F_n)$, $n > 3$, too does not admit D3EL.

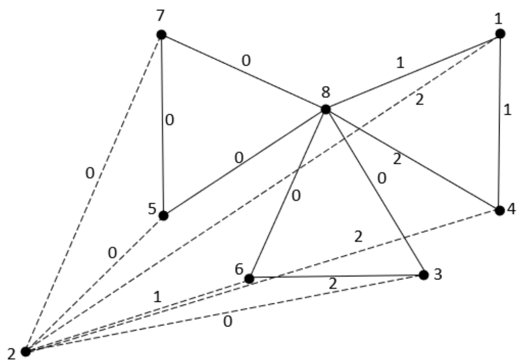

FIGURE 5.20: Non-existence of D3EL of $DSG(F_3)$

5.1.12 D3EL of Degree Splitting Graph of Path

One can establish the D3EL of $DSG(P_i)$;$1 \leq i \leq 10$. "One such example is given in Fig. 5.21".So, consider $DSG(P_n)$, for $n \geq 11$.

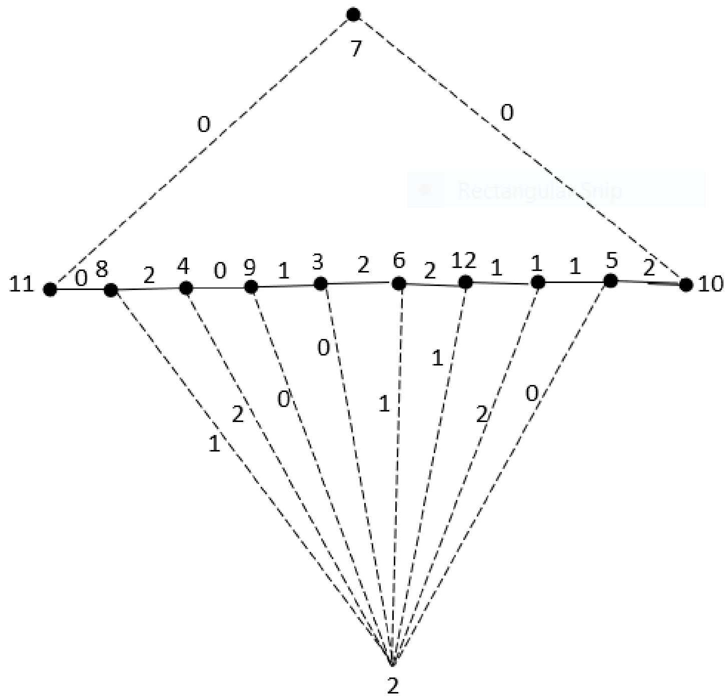

FIGURE 5.21: D3EL of $DSG(P_{10})$

Theorem 5.1.12. *$DSG(P_n)$ does not accept D3EL* $\forall\ n\ \geq\ 11$.

Proof. Take P_n on $n\ \geq\ 11$ nodes and obtain $DS(P_n)$. "Take $n\ =\ 11$ for the sake of discussion, so $|V\ (DSG(P_{11}))|\ =\ 13$ and $|E\ (DSG(P_{11}))|\ =\ 21$". Define a bijective map $d\ :\ V\ (DSG(P_{11}))\ \rightarrow\ \{1,\ 2,\ \ldots,\ n\}$ that induces $d^* : E(DSG(P_{11})) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ as follows:

$$d^*\ (e\ =\ xy)\ = \begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

Assume that $DSG(P_{11})$ has $D3EL$ d and induced "d^* with the condition that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\ 0 \leq\ i, j \leq 2$. Note that the count of lines labeled with either 0 or 1 or 2 must be exactly 7 to meet the desired divisor 3-equitable property $|e_{d^*}(i) - e_{d^*}(j)| \leq\ 1, \forall\ 0 \leq\ i, j \leq 2$. But there are only six lines with label 2, a contradiction (see Fig.5.22). Moreover, if $DSG(P_{11})$, does not permit D3EL, then $DSG(P_n),\ n > 11,\ too$ does not permit D3EL".

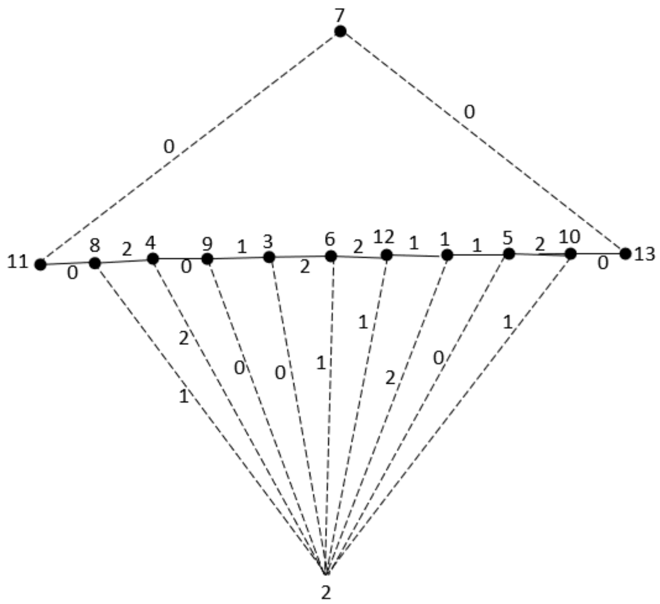

FIGURE 5.22: Non-existence of D3EL of $DSG(P_{11})$

5.1.13 D3EL of MG of Path

One can establish the D3EL of $MG(P_i)$;$1 \leq i \leq 8$. "One such example is given in Fig. 5.23". So, consider $MG(P_n)$, for $n \geq 9$.

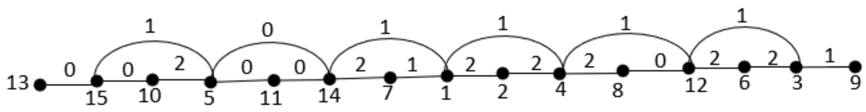

FIGURE 5.23: D3EL of $MG(P_8)$

Theorem 5.1.13. *$MG(P_n)$ does not accept D3EL* $\forall\, n \geq 9$.

Proof. Take P_n on $n \geq 9$ nodes and obtain $M(P_n)$. "We take n = 9 for the sake of discussion", so $|V\ (MG(P_9))| = 17$ and $|E\ (MG(P_9))| = 23$. Define a bijective map $d : V\ (MG(P_9)) \rightarrow \{1,\ 2,\ \ldots,\ n\}$ that induces $d^* : E(MG(P_9)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ as follows:

$$d^*\ (e = xy) = \begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

Assume that $MG(P_9)$ has $D3EL$ d and induced "d^* with the condition that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. Note that the count of lines labeled with either 0 or 1 or 2 must be at least 7 and at most 8 to meet the desired divisor 3-equitable property $|e_{d^*}\ (i) - e_{d^*}\ (j)\,| \leq 1, \forall\, 0 \leq i, j \leq 2$. But there are ten lines with label 0, seven lines with label 2 and seven lines with label 1, a contradiction" (see Fig. 5.24). Moreover, if $MG(P_9)$, does not permit D3EL, then $MG(P_n)$, $n > 9$, too does not permit D3EL.

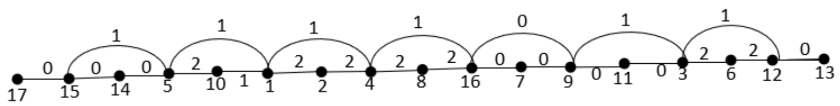

FIGURE 5.24: Non-existence of D3EL of $MG(P_9)$

5.1.14 D3EL of MG of C_n

The D3EL of $MG(C_i)$;$1 \leq i \leq 6$ are derived easily (see Fig.5.25). So, consider $MG(C_n)$, $\forall\, n \geq 7$.

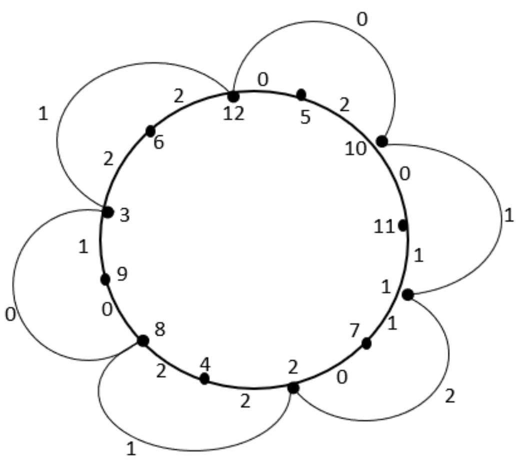

FIGURE 5.25: D3EL of $MG(C_6)$

Theorem 5.1.14. *$MG(C_n)$ does not allow D3EL* $\forall\, n \geq 7$.

Proof. Take C_7 for the sake of discussion with $|V(C_n)| = 7$ and $|E(C_n)| = 7$. Obtain $MG(C_n)$ with $|V(MG(C_n))| = 14$ and $|E(MG(C_n))| = 21$. Now define a bijective map $d : V\,(MG(C_n)) \rightarrow \{1,\, 2,\, \ldots,\, n\}$ that induces $d^* : E(MG(C_n)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ as follows:

$$d^*\,(e = xy) = \begin{cases} 1,\; if\; d\,(x)\,|d(y) \\ 2,\; if\; \frac{d(y)}{d(x)} = 2 \\ 0,\; if\; d(x) \nmid d(y) \end{cases}$$

Assume that $MG(C_7)$ has $D3EL$ d and induced "d^* with condition that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq\; i, j \leq 2$. Note that the number of lines labeled 0, 1, and 2 must be exactly 7, respectively in order to meet the desired divisor 3-equitable property $|e_{d^*}(i) - e_{d^*}(j)| \leq\; 1, \forall\, 0 \leq$

$i, j \leq 2$. But there are eight lines with label 0, seven lines with label 2 and six lines with label 1, a contradiction" (see Fig.5.26). The same argument holds good for $MG(C_n), \ n > 7$. Therefore, $MG(C_n), \ n \geq 7$, does not admit D3EL.

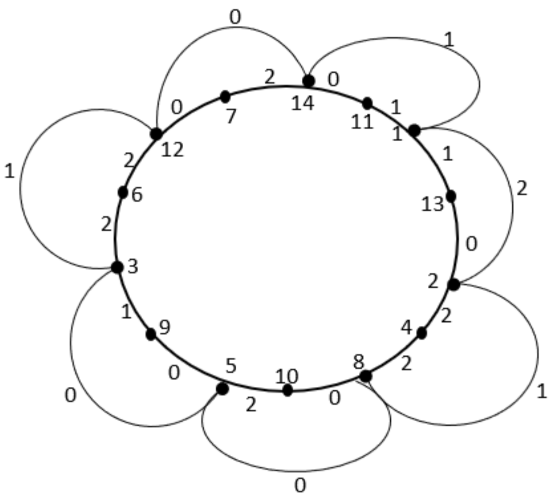

FIGURE 5.26: Non-existence of D3EL of $MG(C_7)$

5.1.15 D3EL of Central Graph of Ladder Graph

One can establish the D3EL of $CG\,(L_1)\,and\;CG(L_2)$. "One such example is given in Fig. 5.27". So, consider $CG(L_n)$, for $n \geq 3$.

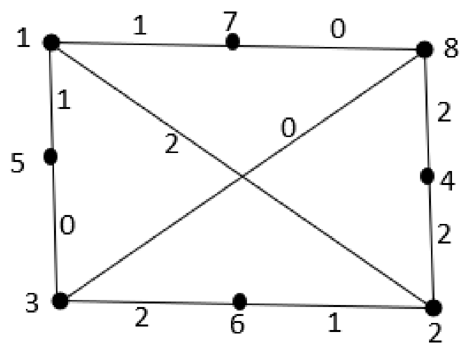

FIGURE 5.27: D3EL of $CG(L_2)$

Theorem 5.1.15. *$CG(L_n)$ does not accept D3EL $\forall\, n \geq 3$.*

Proof. Take L_n on $n \geq 3$ nodes and obtain $C(L_n)$. "Take $n = 3$ for the sake of discussion", so $|V\,(CG(L_3))| = 13$ and $|E\,(CG(L_3))| = 22$. Define a bijective map $d : V\,(CG(L_3)) \rightarrow \{1, 2, \ldots, n\}$ that induces $d^* : E(CG(L_3)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ as follows:

$$d^*\,(e = xy) = \begin{cases} 1, \; if\; d\,(x)\,|d(y) \\ 2, \; if\; \frac{d(y)}{d(x)} = 2 \\ 0, \; if\; d(x) \nmid d(y) \end{cases}$$

Assume that $CG(L_3)$ has $D3EL$ d and induced "d^* with the condition that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. Note that the count of lines labeled with either 0 or 1 or 2 must be at least 7 and at most 8 to meet the desired divisor 3-equitable property $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. But there are ten lines with label 0, six lines with label 2 and six lines with label 1, a contradiction" (see Fig. 5.28). Moreover, if $CG(L_n)$, does not permit D3EL, n >3, then $CG(L_n),\; n \geq 3,\; too$ does not permit D3EL.

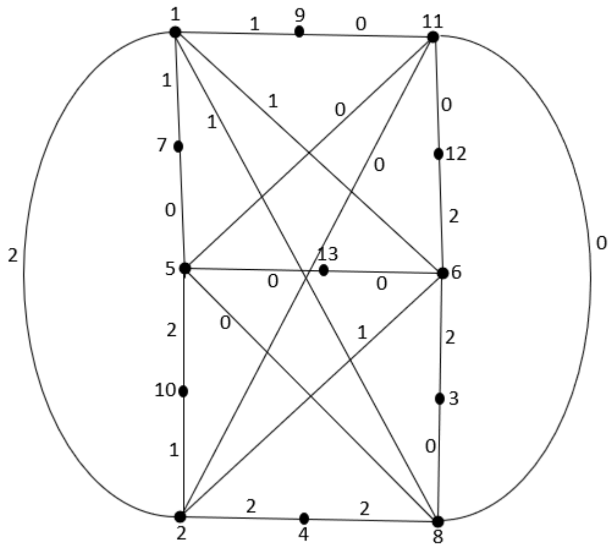

FIGURE 5.28: Non-existence of D3EL of $CG(L_3)$

5.1.16 D3EL of Central Graph of $\mathbf{TS_n}$

One can establish the D3EL of $CG\left(TS_3\right)$. "One such example is given in Fig. 5.29". So, consider $CG(TS_n)$, for $n \geq 5$.

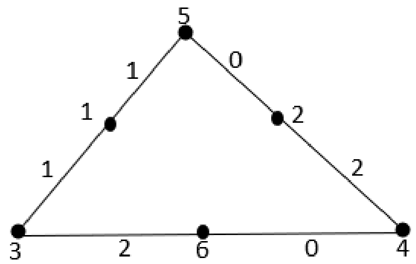

FIGURE 5.29: D3EL of $CG(TS_3)$

Theorem 5.1.16. *$CG(TS_n)$ does not accept D3EL* $\forall\, n \geq 5$.

Proof. Take TS_n on $n \geq 5$ nodes and obtain $C(TS_n)$. "Take $n = 5$ for the sake of discussion", so $|V\ (CG(TS_5))| = 11$ and $|E\ (CG(TS_5))| = 16$. Define a bijective map $d : V\ (CG(TS_5)) \rightarrow \{1,\ 2,\ \ldots,\ n\}$ that induces $d^* : E(CG(TS_5)) \rightarrow \{0, 1, 2\}$ for any edge xy with $d(x) < d(y)$ as follows:

$$d^*\ (e = xy) = \begin{cases} 1,\ if\ d\,(x)\,|d(y) \\ 2,\ if\ \frac{d(y)}{d(x)} = 2 \\ 0,\ if\ d(x) \nmid d(y) \end{cases}$$

Assume that $CG(TS_5)$ has $D3EL$ d and induced "d^* with the condition that $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. Note that the count of lines labeled with either 0 or 1 or 2 must be at least 5 and at most 6 to meet the desired divisor 3-equitable property $|e_{d^*}(i) - e_{d^*}(j)| \leq 1, \forall\, 0 \leq i, j \leq 2$. But there are seven lines with label 0, five lines with label 2 and four lines with label 1, a contradiction "(see Fig. 5.30). Moreover, if $CG(TS_n)$, does not permit $D3EL$, n >5, then $CG(TS_n),\ n \geq 5,\ too$ does not permit $D3EL$.

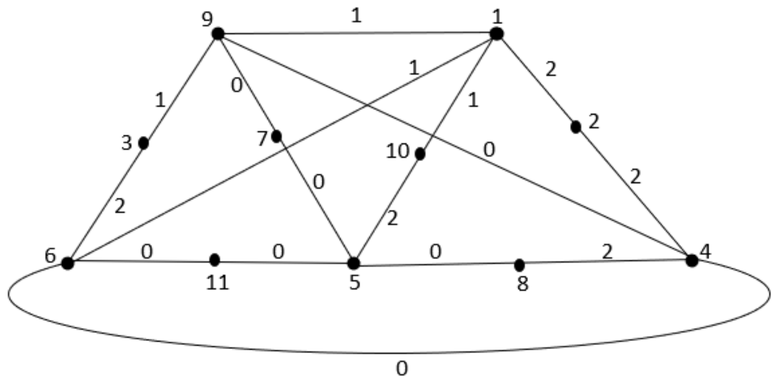

FIGURE 5.30: Non-existence of D3EL of $CG(TS_5)$

5.2 APPLICATIONS

"The concept of graph labeling in graph theory also plays a vital role in computer science and communication networks. Due to the demand for high-quality multimedia service, the International Telecommunication Union (ITU) recently gave an integrated MSS system and hybrid satellite and terrestrial system to provide broadband service. In this system, the satellite radio technique needs to match the terrestrial wireless network as much as possible in order to reduce the cost. However, if the frequency reuse factor is 1, it might cause serious inter-beam interference (IBI) because of the usage of the same subcarrier between user equipment (UE) in adjacent beams. At the same time, the bandwidth assigned to MSS is very limited, so channel using efficiency is still a prime factor.

In channel borrowing, the acceptor cell which has no more unused nominal channels can borrow free channels from donor cells. For the above channel borrowing schemes, the tasks are majorly focused on which channel to borrow and the borrowing order. The channel borrowing technique in mobile satellite communication is given in [44] and it focused on the channel borrowing between different satellites. Moreover, the schemes listed above all obey a condition: the acceptor cell can only borrow channels that are not being used in the neighboring cells. A channel from the donor cell can be borrowed only if none of the cells belonging to the same group as the donor cell is using this channel. This might lead to a great reduction in borrowable channels. If not, it may lead to severe IBI. For more applications of graph theory in communication networks and satellite communication", see [2, 17, 44]. Similarly, one can explore the exclusive applications of D3EL in the aforementioned fields which would be an interesting open problem for the future work.

5.3 Conclusion

The existence and non-existence of D3EL of the TG of umbrella graphs, triangular snake, wheel graphs, Mycielski's graph of wheel graph, MG, and CG of lollipop graph are established. "The existence and non-existence of $D3EL$ of the DSG of the ladder, triangular snake, and lollipop graphs are also derived".

Future Scope

The findings presented in this thesis open up several avenues for future research and potential applications in the field of graph theory and labeling. Below are some of the promising directions and scopes for further investigation:

1. Exploration of Additional Graph Classes: Future research could focus on applying the 3EL and D3EL to other classes of graphs not covered in this thesis. This includes but is not limited to product graphs, bipartite graphs, and various derived graph structures.

2. Algorithm Development: Developing efficient algorithms for determining 3EL and D3EL for large and complex graphs could significantly advance practical applications. This includes optimizing computational techniques for labeling and leveraging parallel computing for handling extensive datasets.

5. Mathematical Extensions: Extending the theoretical framework of 3EL and D3EL to higher dimensions and other mathematical structures such as hypergraphs and simplicial complexes. This could lead to new discoveries in combinatorial optimization and topology.

6. Software Tool Development: Creating software tools and visualization platforms to aid researchers in applying 3EL and D3EL to various types of graphs. These tools could include features for automatic labeling, visualization, and analysis of labeled graphs.

7. Educational Integration: Incorporating the concepts of 3EL and D3EL into educational curricula for graph theory and discrete mathematics. Developing teaching modules and resources to help students and educators understand and apply these labeling techniques effectively.

8. Experimental Validation: Conducting experimental studies to validate the theoretical findings and explore their practical implications in real-world scenarios. This includes collaborations with industry partners and researchers from other disciplines.

In conclusion, the potential applications and extensions of 3EL and D3EL are vast and varied. Future research in this domain promises to uncover new insights and applications, further enriching the field of graph theory and its interdisciplinary connections.

Conclusion

In this thesis, various new classes of graphs concerning $3EL$ and $D3EL$ are established. We hope that it will be very much helpful to all researchers to add more results in to the area of labeling of graph. From the above study it can be concluded that a graph can be labeled by the 3-equitable labeling and divisor 3-equitable labeling of total graph, middle graph, central graph, degree splitting graph, and Mycielskian graphs of some graphs and also some new classes of graphs. Different classes of graph of 3-equitable and divisor 3-equitable labeling have been analyzed throughout the thesis. It is very interesting to investigate graphs satisfies the various conditions of 3-equitable and divisor 3-equitable labeling and it is discussed in this thesis. It will provide a new horizon to the researchers in the area of graph labeling and graph theory for further advancements in the corresponding field. Exploring the existence and non-existence of DEL and D3EL for other other classes of graphs are the future scope of this thesis besides discovering the exclusive applications of these two labeling in various domains.

Publications and Presentations

Papers Published from the Thesis

1. Sangeeta and A. Parthiban, “3-Equitable and Prime Labeling of Some Classes of Graphs’ ’, Mathematics and Statistics, 11(3) (2023), pp. 534-540. (**Scopus Indexed**)

2. Sangeeta and A. Parthiban, “Analyzing the Applications of Graph Theory in Communication Networks through the Divisor 3-Equitable Labeling of Graphs”, Lecture Notes in Electrical Engineering, Springer, 987 (2023), pp. 105–112. (**Scopus Indexed**)

3. Sangeeta and A. Parthiban, “A Note on Divisor 3-Equitable Labeling of Graphs”, AIP Conference Proceedings, 2800(1) (2023), (**Scopus Indexed**)

4. S. Sangeeta, and A. Parthiban, ”Studying the applications of graph theory in functional materials and manufacturing through divisor 3-equitable labeling of graphs,” AIP Conference Proceedings, 2986(1) (2024). (**Scopus Indexed**)

5. Sangeeta, A. Parthiban, Arunava Majumdar and A. Felix, “Recent Advancements in Divisor 3-equitable Labeling of Graphs,” TWMS Journal of Applied and Engineering Mathematics, 13(1) (2023), pp. 433-444

6. A. Parthiban and Sangeeta, “A Comprehensive Survey on 3-equitable and Divisor 3-equitable Labeling of Graphs’ ’, Journal of Physics: Conference Series, 1531(1) (2020), pp. 1-27. **(Scopus Indexed)**

7. K. Tina Jebi Nivathitha, N. Srinivasan, A. Parthiban, and Sangeeta,“On Divisor 3-Equitable labeling of Complete and Star Graphs’ ’, Journal of Xidian University, 14(5) (2020), pp. 4448-4454.

8. K. Tina Jebi Nivathitha, N. Srinivasan, A. Parthiban, and Sangeeta, “On Divisor 3-Equitable labeling of Wheel Graphs’ ’, Journal of The Gujarat Research Society, 8(5) (2020), pp. 5480-5483. (UGC)

9. Sangeeta, Arunava Majumder and A. Parthiban, "More Results on 3-equitable Labeling of Graphs' ', has been submitted to TWMS Journal of Applied and Engineering Mathematics, (2023).

Papers Presented in Conferences

1. Presented a poster presentation titled "A Comprehensive Survey on 3-Equitable and Divisor 3-Equitable Labelling of Graphs" in RAFAS-2019 on 5^{th} – 6^{th} November, 2019 at Lovely Professional University, Punjab.

2. Participated in the NATIONAL WEBINAR "On Recent Advances in Mathematical Sciences" held on 26 June, 2021, Andhra Loyala College, Vijayawada.

3. Participated in the NATIONAL LEVEL VIRTUAL FACULTY DEVELOPMENT PROGRAM " On Graph Theory" held from 29 June 2021 to 03 July 2021, Christ College, Karnataka.

4. Participated in the NATIONAL WEB CONFERENCE "On Recent Trends in Graph Theory" held on July 12and 13, 2021, PG & Research Department of Mathematics, St. Xavier's College, Palayamkottai.

5. Presented a poster presentation titled "A Note on Divisor 3-equitable Labeling of Graphs" in RAFAS-2021, Lovely Professional University, Punjab.

6. Presented an oral presentation titled "Analyzing the Applications of Graph Theory in Communication Networks through the Divisor 3-Equitable Labeling of Graphs" in RAFAS-2022, Lovely Professional University, Punjab.

7. Presented an oral presentation titled "Studying the Applications of Graph Theory in Manufacturing through Divisor 3-Equitable Labeling of Graphs", in ICFMMP 2022, on 29-30 july,2022 at Lovely Professional University, Punjab.

8. Presented an oral presentation titled "Recent Advancements in Divisor 3-equitable Labeling of Graphs' ', in International Conference on Wavelet Analysis and Graph Theory

(ICWAGT-2022) organized by the School of Arts, Sciences, Humanities & Education (SASHE), SASTRA Deemed University, Thanjavur during September 15-16, 2022.

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