# A STUDY OF CHEBYSHEV POLYNOMIALS THEIR PROPERTIES AND APPLICATIONS

Thesis Submitted for the Award of the Degree of

**DOCTOR OF PHILOSOPHY**

**In**

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**By**

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**LOVELY PROFESSIONAL UNIVERSITY, Phagwara (Punjab)**

## DECLARATION

I, hereby declared that the presented work in the thesis entitled "**A Study of Chebyshev Polynomials their Properties and Applications**" in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of **Dr. Pankaj Pandey**, working as **Associate Professor**, in the **Department of Mathematics**, of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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**LOVELY PROFESSIONAL UNIVERSITY, Phagwara (Punjab)**

**<u>CERTIFICATE</u>**

This is to certify that the work reported in the Ph. D. thesis entitled **"<u>A Study of Chebyshev Polynomials their Properties and Applications</u>"** submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the **<u>Department of Mathematics</u>,** is a research work carried out by **<u>Anu Verma</u>, <u>42000264</u>,** is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

*Pankaj Pandey*

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Name of supervisor: Dr. Pankaj Pandey

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***This work is dedicated to my children***

***"Atharv"***

***&***

***"Advith"***

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# Abstract

"Chebyshev polynomials" have a rich historical background associated with the contributions of the Russian mathematician Pafnuty Chebyshev (1821-1894). Pafnuty Chebyshev extensively studied and introduced them in the mid of $19^{th}$ century between 1854 to 1859, with a primary focus on their properties and their application in approximation theory. Notably, Chebyshev aimed to develop methods for minimizing the maximum error in polynomial approximation, a concept now known as Chebyshev approximation. One of Chebyshev's significant contributions was demonstrating the orthogonality property of his polynomials in 1864. Beyond his work on "Chebyshev polynomials", Pafnuty Chebyshev continued to make substantial contributions to various mathematical problems. His interests spanned areas such as continued fractions, the theory of congruences, the theory of integration, and the distribution of prime numbers. "Chebyshev polynomials" found applications in control theory and signal processing during the mid-$20^{th}$ century. Their unique properties, including equi-oscillation, made them valuable tools in designing filters and addressing problems in engineering fields. Their diverse properties make them essential not only in approximation theory but also in numerical analysis, optimization, and computational mathematics. Chebyshev's pioneering efforts laid the groundwork for the development and application of these polynomials. The continued relevance underscores their importance in various mathematical and engineering disciplines today.

Fibonacci numbers, a remarkable sequence attributed to Leonardo of Pisa, stand out as a noteworthy mathematical phenomenon. They hold a special place in our world and have profound implications for various aspects of our daily existence. The genesis of these numbers can be traced back to Leonardo of Pisa's renowned "rabbit problem". Beyond their intrinsic connection to our everyday experiences, Fibonacci numbers find diverse applications in nature, music, and numerous other fields a richness that cannot be succinctly encapsulated.

The Lucas numbers constitute a sequence of integers that bears similarities to the Fibonacci numbers. These numbers are named after Édouard Lucas, a French

mathematician who introduced them during the 19th century. Similar to the Fibonacci numbers, the Lucas numbers exhibit various interesting properties and hold connections in number theory, algebra, and geometry. Édouard Lucas introduced these numbers while exploring the properties of the Fibonacci sequence, originally discovered by Leonardo Fibonacci. Lucas expanded upon Fibonacci's work and generalized the concept, leading to the Lucas sequence and the Lucas numbers. The Pell numbers derive their name from John Pell, a mathematician who contributed to the understanding of these equations. The term "Pell numbers" was later coined to acknowledge his contributions to this mathematical field.

This thesis is dedicated to exploring the profound significance of the "Chebyshev polynomials", Pell numbers, Fibonacci numbers, Lucas numbers and the accompanying polynomials that envelop them. Comprising five chapters, this work aims to provide an in-depth understanding of these divinely endowed numbers. The core of the subject matter of the manuscript grows from a series of our research papers that are cited at the end. The following overview summarizes the thesis:

The first chapter serves as an introduction, offering a concise overview of "Chebyshev polynomials" delving into their historical roots, elucidating their applications, and their polynomial expansions are presented. Additionally, here's a concise overview of definitions and well-known results related to Chebyshev polynomials, Fibonacci numbers, Fibonacci polynomials, Lucas numbers, Lucas's polynomials, Pell numbers, and Pell polynomials, which meet the minimal requirements for the evolution of the emerging chapters. It also outlines key concepts and well-established results concerning "Chebyshev polynomials" and their associated polynomials. This chapter includes a section of literature review that specifically examines the research conducted by different researchers in the field of the "Chebyshev polynomials" and their polynomial generalizations through the all four kinds of Chebyshev and similar polynomials. This review identifies a research gap, which becomes a focal point for the thesis. Additionally, the chapter outlines the objectives and methods to be employed in addressing and bridging these identified gaps. Throughout our exploration, the geogebra software has been extensively utilized for graphical representations of various sequences. This approach enhances the visual understanding of the intricate relationships between these polynomials, numbers and their associated polynomials.

The remainder of this thesis is dedicated to exploring the behavior and distinct properties of polynomials sequences analogous to "Chebyshev polynomials", Lucas numbers, Fibonacci numbers, and Pell numbers particularly focusing on their interconnections.

The primary sequences in the chapter 2, include matrix representation of "Chebyshev polynomials" of both $3^{rd}$ and $4^{th}$ kind. We will deal with matrix representation via determinant representation. Additionally, attention is given to several identities related to matrix representation with practical applications. At the end of this chapter, we deduce some identities involving the generating matrices and their determinants. Several results are developed based on their properties and interrelationships, employing diverse methodologies and techniques. We have discussed properties related to the matrix representation of both third and fourth kind, like matrix power and trace of the matrix of the Chebyshev polynomials.

In the chapter 3, we developed the concepts of generalized Chebyshev-like polynomials and discussed their properties. The thesis aims to derive explicit formulas for these generalized polynomials, accompanied by intriguing identities related to their generating matrices and corresponding determinants. The characteristic equation, characteristic roots are obtained for the generalized "Chebyshev polynomials". The sum, product, subtraction, and sum of squares of roots are discussed for the generalized version. We deduced the generating matrices and their determinants for generalized "Chebyshev polynomials", along with some identities. The generating matrix for generalized "Chebyshev polynomials" is generated with the help of the matrix algebra and deduced some related determinantal properties.

In chapter 4, we will consider the interaction between the $3^{rd}$ and $4^{th}$ kind of Chebyshev polynomials with the Lucas, Fibonacci numbers. Analogous results are obtained for the $3^{rd}$ and $4^{th}$ kind of "Chebyshev polynomials", including specific cases of these identities. We develop certain identities involving sums of their finite products. We also discussed some specific cases of these summation identities that result from different values of $r = 1, 2$. Several identities connecting summation of definite products of Lucas, Fibonacci numbers, and Chebyshev polynomials of both $3^{rd}$ and $4^{th}$ kind are investigated. The thesis then progresses to the development of identities concerning the summations of definite products of Lucas and Fibonacci numbers, presented in the form of $3^{rd}$ and $4^{th}$ kind “Chebyshev polynomials” and their derivatives.

The focus is particularly on summation representations of finite products involving diverse sequences of numbers and polynomials. Explicit formulas for these “Chebyshev polynomials” and their derivatives are obtained at certain variables along with their connections to Fibonacci, Lucas numbers.

At the end, in the chapter 5, the focus shifts to the sequence of connecting definite products of $3^{rd}$ kind of Chebyshev polynomials. New results derived on representations of definite products of the Chebyshev polynomials, Lucas, Fibonacci numbers, and Pell polynomials. The explicit formulae for $3^{rd}$ kind of Chebyshev polynomials and their derivatives with Pell, Fibonacci, and Lucas numbers are established. Further, their links with Fibonacci polynomials, Pell numbers, and Lucas numbers are also obtained. In this section some works on summations of definite products of $3^{rd}$ kind Chebyshev polynomials, Lucas numbers, Fibonacci numbers, and Pell polynomials are considered. Additional identities are explored, using computational methods. The thesis utilizes recursive methodology to establish summation representations for sequences of Lucas, Pell, Fibonacci numbers, and Chebyshev polynomials. The thesis establishes linkages between these “Chebyshev polynomials, Lucas numbers, Pell numbers, and Fibonacci numbers” including sums of finite products through elementary computations.

Future work includes the study of generalized Chebyshev polynomials, exploring their basic properties and interconnections based on the observed patterns. Finally, we lay out the brief mapping of the future research possibilities based on the content of this thesis. Further exploration involves the extension of Chebyshev polynomials, unveiling their basic properties.

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# PREFACE

The present thesis entitled **"A STUDY OF CHEBYSHEV POLYNOMIALS THEIR PROPERTIES AND APPLICATIONS"** is the outcome of investigation carried by author towards the fulfillment for the award of the degree Doctor of Philosophy under the supervision of Dr. Pankaj Pandey (Associate Professor), Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara (Punjab).

The present thesis is divided into five chapters and each chapter is sub-divided into different sections.

**Chapter 1:** The first chapter is introductory and contains a brief history of Chebyshev polynomials and some basic definitions.

**Chapter 2:** This chapter is bifurcated into two main components. Firstly, it establishes the matrix representation of third-kind Chebyshev polynomials and derives their characteristic equations. The theorem outlined in this segment establishes a significant connection between the trace of matrix power and third-kind Chebyshev polynomials, illustrated through a practical example that validates the presented theorems. The second segment focuses on the matrix representation of fourth-kind Chebyshev polynomials and deduces their characteristic equations. It delves into the exploration of interrelated identities involving matrix power and Chebyshev polynomials. The chapter further explores the relationship between the second and fourth kinds of Chebyshev polynomials and matrix power. Practical applications of these findings are discussed, underscoring the significance of the work presented in this chapter.

The whole content of this part is **In Press (For Publication)** in **"Contempory Mathematics" (Scopus Indexed).**

**Chapter 3:** This chapter presents a novel advancement in the form of a generalized version of Chebyshev-like polynomials. Additionally, the chapter explores the derivation of the Binet Formula for these Chebyshev-like polynomials. The discussion extends to the representation of generalized Chebyshev's polynomials through a matrix framework. The chapter delves into the characteristic equations associated with these generalized polynomials and their practical applications. Moreover, it emphasizes the importance and relevance of the work undertaken in this chapter. The whole content of this part is **<u>In Press (For Publication)</u> in South East Journal of Mathematics and Mathematical Sciences (Scopus Indexed).**

**Chapter 4:** Within this chapter, several straightforward lemmas are derived, accompanied by their proofs, serving as foundational steps toward attaining the main results. The chapter explores the connection between the fourth kind of Chebyshev polynomials, Fibonacci numbers, and Lucas numbers. Additionally, it examines the relationship between the third kind of Chebyshev polynomials and Lucas numbers. The practical applications are discussed, emphasizing their relevance. The chapter concludes by summarizing the findings and highlighting the significance of the work undertaken in this research endeavor.
The whole content of this part is published in **IAENG Journal of Applied Mathematics,** 53(4), 2023, 1222-1229, **(Scopus Indexed).**

**Chapter 5**: This chapter unfolds through the derivation of several lemmas, forged by connecting identities at specific variables. These connections yield a meaningful relationship among the third-kind Chebyshev polynomials, Fibonacci numbers, Lucas numbers, and Pell numbers. The rigorous proofs of the derived lemmas and overarching theorems are presented, offering a comprehensive understanding of the established connections. Furthermore, the chapter includes corollaries, logically deduced from theorems, contributing additional insights. The chapter concludes with a concise summary and conclusive remarks, highlighting the significance and implications of the established relations.

Finally, we encapsulated the research endeavor by providing a comprehensive summary and conclusion. Additionally, we delineated the potential avenues for future exploration and development in our work, identifying the future scope of our research.
The whole content of this part is **<u>In Press (For Publication)</u>** in **"AIP Conference Proceedings" (Scopus Indexed).**

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# ACKNOWLEDGMENT

This is an immense opportunity to convey my monumental gratitude to my supervisor Dr. Pankaj Pandey (Associate Professor), Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara (Punjab), for his guidance, encouragement and inspiration throughout my research work. I would like to convey my sincere respect and deep gratitudeto him for his continuous support.

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A special thanks to my family as a whole for their affection and continuous support. I wish to express my heartfelt gratitude and indebtedness to my family, words cannot express how grateful I am to my mother-in law, father-in law, my mother, and my father for their unwavering support and for always being there for me. Your sacrifices and belief in me have made this achievement possible.

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A big thanks to Almighty god for giving me the strength, knowledge, ability, and opportunity to achieving my goal in this research study.

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# Symbols and Notations

| | |
|---|---|
| $n$ | Natural number |
| $x$ | Variable |
| $T_n(x)$ | First Kind of Chebyshev Polynomials |
| $U_n(x)$ | Second Kind of Chebyshev Polynomials |
| $V_n(x)$ | Third Kind of Chebyshev Polynomials |
| $W_n(x)$ | Fourth Kind of Chebyshev Polynomials |
| $L_n$ | Lucas Numbers |
| $F_n$ | Fibonacci Numbers |
| $P_n$ | Pell Numbers |
| $L_n(x)$ | Lucas Polynomials |
| $F_n(x)$ | Fibonacci Polynomials |
| $P_n(x)$ | Pell polynomials |
| $n, q$ | Integers with $n \geq 0, q \geq 1$ |
| ${V_n}^q(x)$ | $q^{th}$ derivative of $V_n(x)$ |
| ${W_n}^q(x)$ | $q^{th}$ derivative of $W_n(x)$ |
| $\Omega_n(x)$ | $n^{th}$ Vieta-Lucas Polynomial |
| $Z_n(x)$ | $n^{th}$ Vieta-Fibonacci Polynomial |
| C. P. | Chebyshev Polynomials |
| C. E. | Characteristic Equation |
| $\lambda$ | Eigen Value |
| $G(x)$ | Generating Function |
| $\Sigma$ | Summation |
| $\prod$ | Product |
| $A$ | Non-Singular $2 \times 2$ Matrix |
| $I$ | Identity Matrix |
| $a_1$ | Trace of Matrix |

$a_2$ Determinant of Matrix

L. H. S. Left Hand Side

R. H. S. Right Hand Side

$R_n(x)$ Generalized Chebyshev-like Polynomials

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# Chapter 1

# Introduction

## 1.1 A Brief Background of Chebyshev Polynomials

Pafnuty Chebyshev, an exceptional Russian mathematician born in the small town of Okatovo in western Russia, demonstrated a distinctive perspective on the intersection of mathematical theory and its practical applications, particularly in number theory. Chebyshev is best remembered as a trailblazer in the realm of polynomials, making him a pioneering figure in the field. His insights into these polynomials, which are integral in various mathematical and scientific applications, set him apart in the history of mathematics. Born in Okatovo and leaving an indelible mark on mathematics, Chebyshev's legacy extends beyond his time, influencing subsequent generations of mathematicians and scientists who continue to draw inspiration from his unconventional and innovative thinking. Chebyshev's work not only laid the groundwork for a general theory of orthogonal polynomials but also demonstrated their vital role in solving real-world problems, leaving an enduring impact on the fields of mathematics and applied sciences. Chebyshev, with an extensive body of work comprising around 80 publications, made significant contributions across diverse mathematical domains. His research spanned various problems in analysis and practical mathematics. His influence is particularly notable in the realm of numerical analysis, where Chebyshev polynomials have gained importance with perspectives. “Chebyshev polynomials” family encompasses four kinds. While a considerable amount of literature emphasizes $1^{st}$ and $2^{nd}$ kinds of “Chebyshev polynomials” and their myriad applications, there is a notable scarcity of resources on the $3^{rd}$ and $4^{th}$ kinds. Limited attention has been given to these later types, both in terms of theoretical exploration and practical applications. This gap in literature suggests a potential avenue for further research and exploration of the $3^{rd}$ and $4^{th}$ kinds Chebyshev polynomials. which may unveil additional insights and applications in the broader landscape of mathematics and numerical analysis. We have proposed a new generalized version of the

Chebyshev kinds of polynomials. We have addressed the determinant representation of this generalized version with its characteristic equation, as well as the Binet-like formulas and the practical applications of generalized polynomials in the approximation of the functions.

## 1.2 Basic Terminologies and Preliminaries

### 1. 2. 1 Polynomial

A polynomial denoted as $P(x)$, is a mathematical function defined in the following way:

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_n x^n. \qquad (1.2.1)$$

Here, $a_0$, $a_1$, $a_2, \ldots, a_n$ are real numbers, and $x$ represents real variable. The condition for $a_n \neq 0$ indicates that $P(x)$ represents degree $n$ polynomial. This definition characterizes polynomials and establishes their structure in terms of the coefficients $a_o$, $a_1$, $a_2, \ldots,$ $a_n$ and the variable $x$.

### 1. 2. 2 Classical Orthogonal Polynomials

The most widely used classical orthogonal polynomials: Hermite polynomials, Laguerre polynomials, Jacobi polynomials (including as a special case the Gegenbauer polynomials, Chebyshev polynomials, and Legendre polynomials.

### 1. 2. 3 Chebyshev Polynomials

One method of defining these polynomials involves starting with trigonometric functions. Another approach employs recurrence relations, which express each polynomial in terms of its predecessors, establishing a recursive formula for their computation. This multifaceted definition allows for a comprehensive understanding and application of these Chebyshev polynomials in various mathematical contexts. Two sets of orthogonal polynomials that are related to De Moivre's formula associated with sine and cosine functions are represented as $T_n(x)$ and $U_n(x)$, constituting the 1st and 2nd kinds of

"Chebyshev polynomials". Additionally, there are polynomials denoted as $V_n(x)$ and $W_n(x)$, cited as the "Chebyshev polynomials" of the 3rd and 4th kinds.

## 1.2.4 Chebyshev Polynomials of the 1st Kind

"Chebyshev polynomials" (see [69]) were initially introduced by Pafnuty Chebyshev in 1853 in a paper discussing hinge mechanisms. They can be defined in several equivalent ways, one of which starts with trigonometric functions. Chebyshev polynomials $T_n(x)$ of the 1st kind, elucidate as a polynomial of degree $n$ in $x$ and stated in terms of cosine function as:

$$T_n(x) = cosn\theta, \text{ where } x = cos\theta. \qquad (1.2.2)$$

An alternative way to generate "Chebyshev polynomials of the 1st kind" is through recurrence relation:

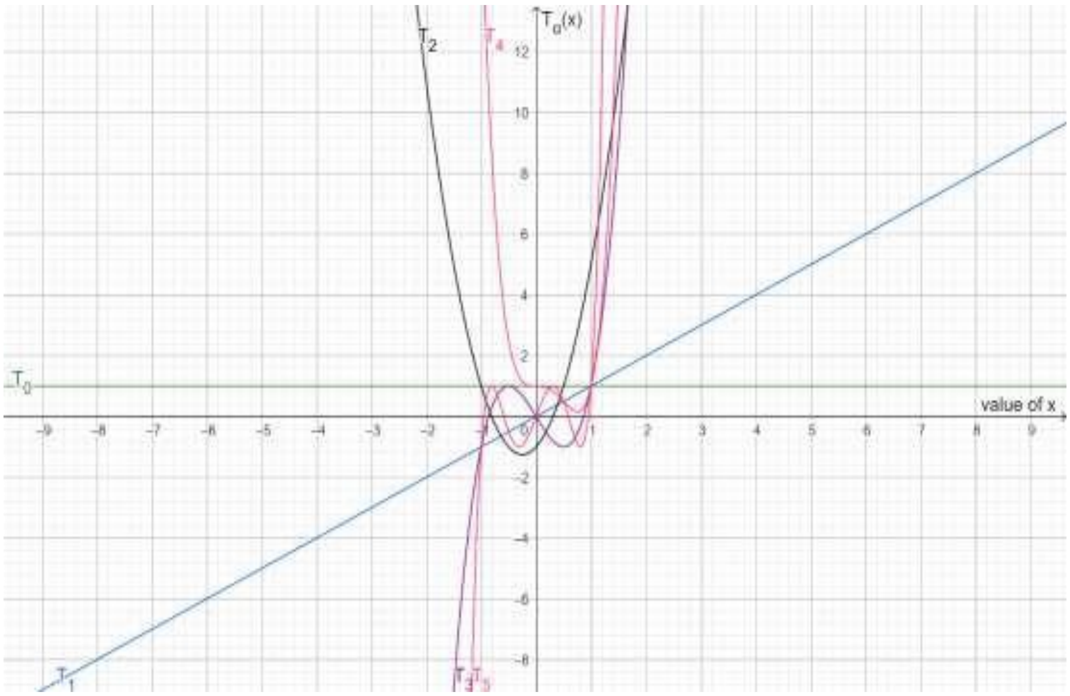
$$T_0(x) = 1,\ T_1(x) = x. \qquad (1.2.3)$$

For $n \geq 2$,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x). \qquad (1.2.4)$$

This recursive formula provides a systematic method for computing 1st kind Chebyshev polynomials based on their previous terms. The combination of the trigonometric definition and the recurrence relation offers a comprehensive understanding of these polynomials and their properties.

The graphical representation of the 1st kind Chebyshev polynomials (see fig 1.2.1)




**“Figure 1.2.1 Graph of Chebyshev Polynomials of 1st Kind”**

## 1.2.5 Chebyshev Polynomials of the 2nd Kind

“Chebyshev polynomials” $U_n(x)$ of the 2nd kind [69] is a degree $n$ polynomial in $x$ stated as:

$$U_n(x) = \frac{\sin(n+1)\theta}{sin\theta}, \text{ where } x = cos\theta. \quad (1.2.5)$$

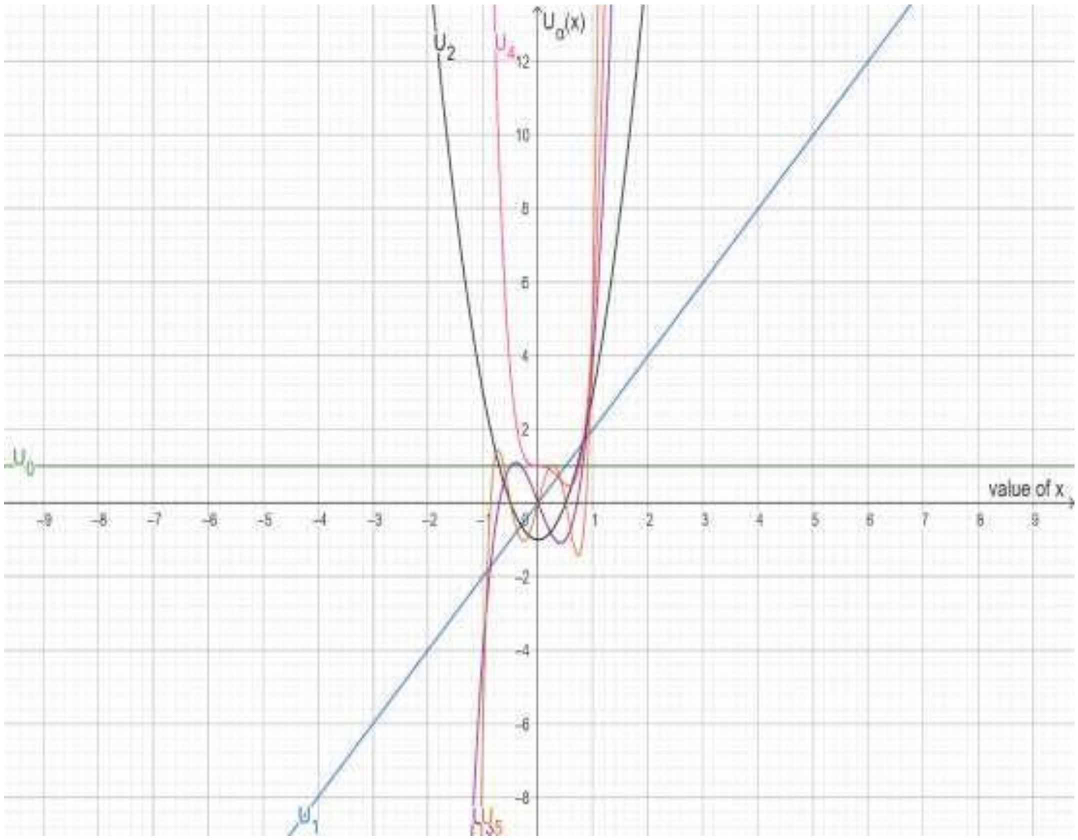
Additionally, “Chebyshev polynomials” of the 2nd kind can be originated through recurrence relation:

$$U_0(x) = 1, U_1(x) = 2x. \quad (1.2.6)$$

For $n \geq 2$,

$$U_n(x) = 2x\, U_{n-1}(x) - U_{n-2}(x). \quad (1.2.7)$$

The graphical representation of the 2nd kind Chebyshev polynomials (see fig 1.2.2)




**"Figure 1.2.2: Graph of Chebyshev Polynomials of 2nd Kind"**

## 1.2.6 "Chebyshev Polynomials of the 3rd Kind"

Chebyshev polynomials of the third kind are another sequence of orthogonal polynomials. They are also defined using trigonometric functions. "Chebyshev polynomials" of the 3rd kind [69], a set of degree $n$ polynomial, described using following expression:

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\frac{cos\theta}{2}}, \text{ where } x = cos\theta. \qquad (1.2.8)$$

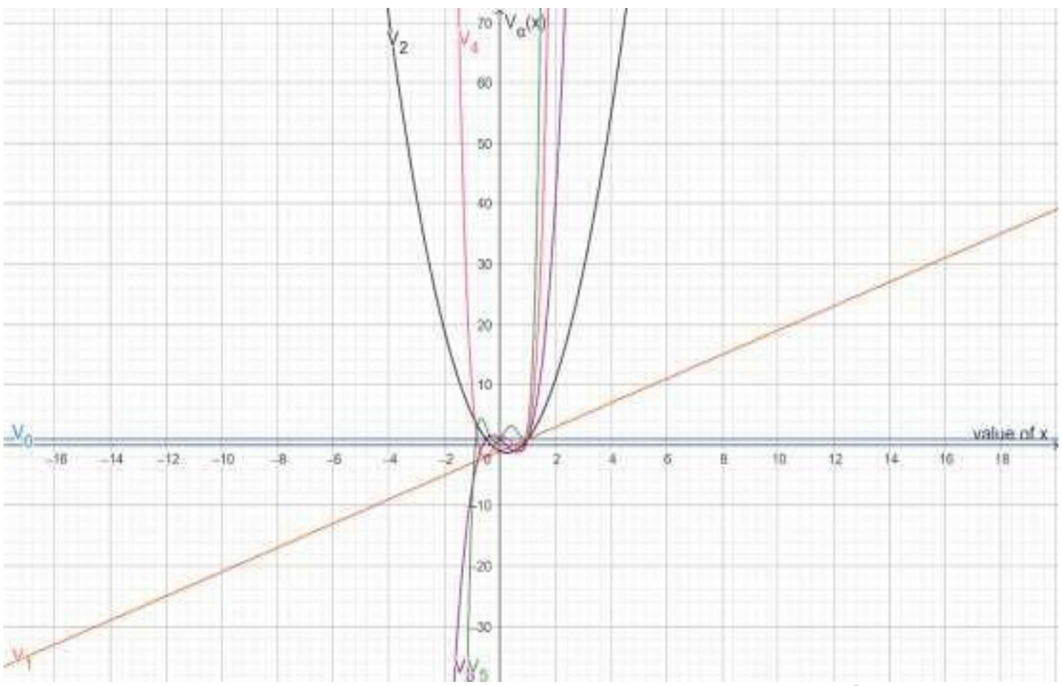
The Chebyshev polynomials $V_n(x)$ of the 3rd kind share identical recurrence relations with "Chebyshev polynomials" of the 1st and 2nd kinds. The only distinction in their generation lies in the specification of initial conditions for $n = 1$, consequently we can express this relationship as:

$$V_0(x) = 1, V_1(x) = 2x - 1, \qquad (1.2.9)$$

For $n \geq 2$,

$$V_n\ (x) = 2x\ V_{n-1}(x) - V_{n-2}(x). \qquad (1.2.10)$$

The graphical representation of the 3rd kind Chebyshev polynomials (see fig 1.2.3)




**"Figure 1.2.3: Graph of Chebyshev Polynomials of 3rd Kind"**

## 1.2.7 "The Chebyshev Polynomials of the 4th Kind"

Chebyshev polynomials of the fourth kind are another sequence of orthogonal polynomials Chebyshev polynomials of the fourth kind are less commonly discussed compared to the other kinds but they have their own unique properties and applications. They are also defined using trigonometric functions. Chebyshev polynomials $W_n(x)$ of the 4th kind [69], each of degree $n$ in $x$ can be described using following expression:

$$W_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{sin\theta/2}, \text{ where } x = cos\theta. \quad (1.2.11)$$

The recurrence relation provides a systematic way to compute $W_n(x)$ based on its previous terms, similar to the recurrence relations for the 1st, 2nd, and 3rd kinds of the Chebyshev polynomials:

$$W_0(x) = 1, W_1(x) = 2x + 1, \quad (1.2.12)$$

For $n \geq 2$,

$$W_n(x) = 2x\, W_{n-1}(x) - W_{n-2}(x). \quad (1.2.13)$$

The graphical representation of the 4th kind Chebyshev polynomials (see fig 1.2.4)

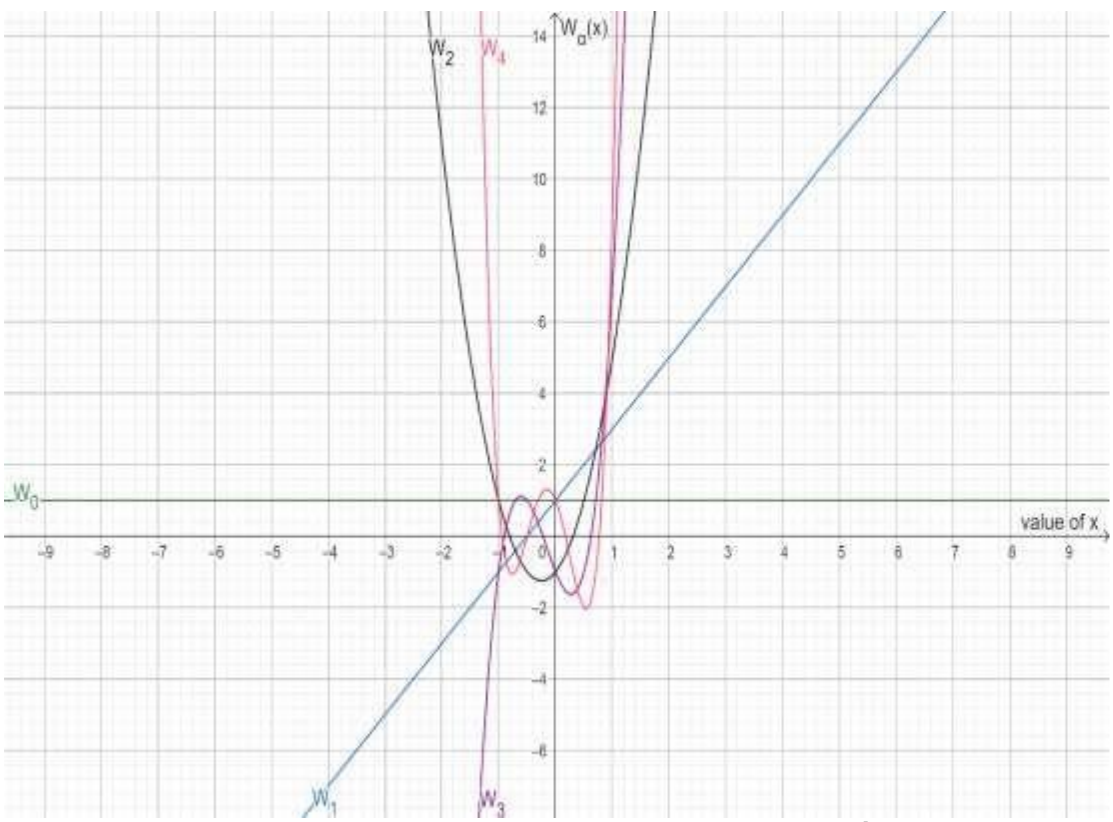



**Figure 1.2.4: Graph of Chebyshev Polynomials of 4th Kind**

### 1.2.8 Orthogonal Polynomials

Orthogonal polynomials emerged in the 19th century, originating from P.L. Chebyshev's work on continued fractions. This field was further advanced by A.A. Markov and T.J. Stieltjes. An orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product.

Two functions $f(x)$ and $g(x)$ are said to be orthogonal on the interval $[a,b]$ with respect to a given continuous and non-negative weight function $w(x)$ if

$$\int_a^b w(x)f(x)g(x)dx = 0 \tag{1.2.14}$$

If, for convenience, use the inner product notation

$$< f,g >= \int_a^b w(x)f(x)g(x)dx = 0 \tag{1.2.15}$$

Where $f,g$ are functions of $x$ on$[a,b]$, then the orthogonality conditions of the above equations are equivalent to saying that $f$ is orthogonal to $g$ if

$$< f,g >= 0 \tag{1.2.16}$$

### 1.2.9 Chebyshev Polynomials of First kind as Orthogonal Polynomials

The polynomial of the first kind is orthogonal with respect to the weight function;

$$\frac{1}{\sqrt{1-x^2}}$$

On the interval $[a,b] = [-1,1]$ i.e. we have:

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 \;, & n \neq m \\ \frac{\pi}{2}, & n = m \neq 0 \\ \pi, & n = m = 0 \end{cases} \tag{1.2.17}$$

### 1.2.10 Chebyshev Polynomials of Second kind as Orthogonal Polynomials

The polynomials $U_n(x)$ are orthogonal on the interval $[-1,1]$ with respect to the weight function;

$$\sqrt{1-x^2}$$

$$\int_{-1}^{1} U_n(x)U_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 \;, & n \neq m \\ \frac{\pi}{2}, & n = m \neq 0 \\ 0, & n = m = 0 \end{cases} \tag{1.2.18}$$

### 1.2.11 Chebyshev Polynomials of Third kind as Orthogonal Polynomials

The polynomials $V_n(x)$ are orthogonal on the interval$[-1,1]$ with respect to the weight function:

$$\frac{\sqrt{1+x}}{\sqrt{1-x}}$$

$$\int_{-1}^{1} V_n(x)V_m(x)\frac{\sqrt{1+x}}{\sqrt{1-x}} = \begin{cases} 0, & n \neq m \\ \pi, & n = m \\ 0, & n = m = 0 \end{cases} \tag{1.2.19}$$

### 1.2.12 Chebyshev Polynomials of Fourth kind as Orthogonal Polynomials

The polynomials $W_n(x)$ are orthogonal on the interval$[-1,1]$ with respect to the weight function:

$$\frac{\sqrt{1-x}}{\sqrt{1+x}}$$

$$\int_{-1}^{1} W_n(x)W_m(x)\frac{\sqrt{1-x}}{\sqrt{1+x}} = \begin{cases} 0, & n \neq m \\ \pi, & n = m \\ 0, & n = m = 0 \end{cases} \quad (1.2.20)$$

## 1.2.13 Orthogonal Series of Chebyshev Polynomials

An arbitrary function $f(x)$ which is continuous and single-valued, defined over the interval $-1 \leq x \leq 1$, can be expanded as a series of Chebyshev polynomials:

$$f(x) = A_0 T_0(x) + A_1 T_1(x) + A_2 T_2(x) + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} A_n T_n(x) \quad (1.2.21)$$

Where the coefficients of $A_n$ are given by;

$$A_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)dx}{\sqrt{1-x^2}}, \quad A_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_n(x)dx}{\sqrt{1-x^2}}, n = 1, 2, 3 \ldots \quad (1.2.22)$$

## 1.2.14 Rodrigues Formula

The Rodrigues formula applies exclusively to orthogonal polynomials. It was introduced by Olinde Rodrigues, James Ivory (1824), and Carl Gustav Jacobi (1827).

## 1.2.15 Chebyshev Polynomials $T_n(x), U_n(x), V_n(x), W_n(x)$ can be obtained by means of Rodrigues's formula

$$T_n(x) = \frac{(-2)^n n! \sqrt{1-x^2}}{(2n)!}\frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}, n = 0, 1, 2, 3 \ldots \quad (1.2.23)$$

$$U_n(x) = \frac{(-1)^n(n+1)}{1.3.5.7......(2n+1)} \frac{1}{\sqrt{1-x^2}} \frac{d^n}{dx^n}\{ 1 - x^2)^{\text{n}}\sqrt{1 - x^2}\} \quad , n = 0,1,2,3 ... \tag{1.2.24}$$

$$V_n(x) = \frac{(-1)^n}{1.3.5........(2n-1)} \sqrt{\frac{1-x}{1+X}} \frac{d^n}{dx^n}\{(1 - x)^{n-1/2}((1 + x)^{n+1/2}, n = 0,1,2,3.. \tag{1.2.25}$$

$$W_n(x) = \frac{(-1)^n}{3.5........(2n-1)} \sqrt{\frac{1+x}{1-X}} \frac{d^n}{dx^n} \{(1 - x)^{n+1/2}((1 + x)^{n-1/2}, n = 0,1,2,3 ... \tag{1.2.26}$$

## 1.2.16 Matrix Representation of "Chebyshev Polynomials of 1st Kind"

Matrix represtation provides an alternative way to generate and express "Chebyshev polynomials of the 1st kind" using determinants of matrices, offering a matrix-based perspective on their properties and relationships:

$$T_0 = 1, T_1 = x, T_n = 2xT_{n-1} - T_{n-2}, \text{for } n \geq 2. \tag{1.2.27}$$

This also can be generated as the determinant of Chebyshev matrices $T_n(x)$;

$$T_2(x) = \begin{vmatrix} x & 1 \\ 1 & 2x \end{vmatrix}, \tag{1.2.28}$$

$$T_3(x) = \begin{vmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{vmatrix}, \tag{1.2.29}$$

In general,

$$T_n(x) = \begin{vmatrix} x & 1 & 0 & \dots & \dots & 0 \\ 1 & 2x & 1 & \dots & \dots & 0 \\ 0 & 1 & 2x & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & .. \\ \dots & \dots & \dots & \dots & 2x & 1 \\ 0 & 0 & \dots & \dots & 1 & 2x \end{vmatrix}. \tag{1.2.30}$$

## 1.2.17 Matrix Representation of "Chebyshev Polynomials of the 2nd Kind"

The 2nd kind $U_n(x)$ Chebyshev polynomials represented by using a matrix approach, specifically through the determinant of Chebyshev matrices. Here, we consider recurrence relation for 2nd kind Chebyshev polynomials;

$$U_0(x) = 1,\ U_1(x) = 2x,\ U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),\ n \geq 2. \tag{1.2.31}$$

The above recurrence relation represented in matrix form as:

$$U_2(x) = \begin{vmatrix} 2x & 1 \\ 1 & 2x \end{vmatrix}, \tag{1.2.32}$$

$$U_3(x) = \begin{vmatrix} 2x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{vmatrix}, \tag{1.2.33}$$

In general;

$$U_n(x) = \begin{vmatrix} 2x & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2x & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 2x & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & .. \\ \cdots & \cdots & \cdots & \cdots & 2x & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 2x \end{vmatrix}. \tag{1.2.34}$$

## 1.2.18 Generating Function

In 1730, Abraham De Moivre initially introduced generating functions, were devised to address the problem of the general linear recurrence. They represent a closed form of an infinite series and are instrumental in solving a variety of mathematical problems. The general form of a generating function is given by:

$$G(x) = a_0 + a_1x + a_2x^2 + \ldots \tag{1.2.35}$$

where $a_0,\ a_1,\ a_2, \ldots$ are real numbers. This can also be voiced as:

$$G(x) = \sum a_n x^n. \tag{1.2.36}$$

In this context, $G(x)$ is referred to as the generating function of the series $\{a_n\}$. Generating functions provide a powerful tool for manipulating sequences and solving recurrence relations, allowing for a more compact representation of mathematical objects and relationships.

## 1.2.19 Generating Function for Chebyshev Polynomials

The four kinds of Chebyshev polynomials derived using a generating function. The generating function for $T_n(x)$ is given by:

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-tx}{1-2tx+t^2}\,. \tag{1.2.37}$$

The generating function for $U_n(x)$ is given by:

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1-2tx+t^2}\,. \tag{1.2.38}$$

The generating function for $V_n(x)$ is given by:

$$\sum_{n=0}^{\infty} V_n(x)t^n = \frac{1-t}{1-2tx+t^2}\,. \tag{1.2.39}$$

The generating function for $W_n(x)$ is given by:

$$\sum_{n=0}^{\infty} W_n(x)t^n = \frac{1+t}{1-2tx+t^2}\,. \tag{1.2.40}$$

This expression provided $n^{th}$ term for the Chebyshev polynomials of all kinds through their generating function, facilitating the efficient computation of these polynomials and their application in various mathematical contexts.

## 1.2.20 Fibonacci Numbers

The Fibonacci sequence is named after the Italian mathematician Leonardo of Pisa, who is more commonly known as Fibonacci. Fibonacci numbers commence with 0, followed by 1, and each subsequent term is obtained by adding the two preceding terms, as specified by the recurrence relation:

$$F_n = \begin{cases} 0, \ if\ n = 0 \\ 1, \ if\ n = 1 \\ F_{n-1} + F_{n-2}\ , \ if\ n \geq 2. \end{cases} \tag{1.2.41}$$


**Figure 1.2.5: Fibonacci Numbers in Nature**

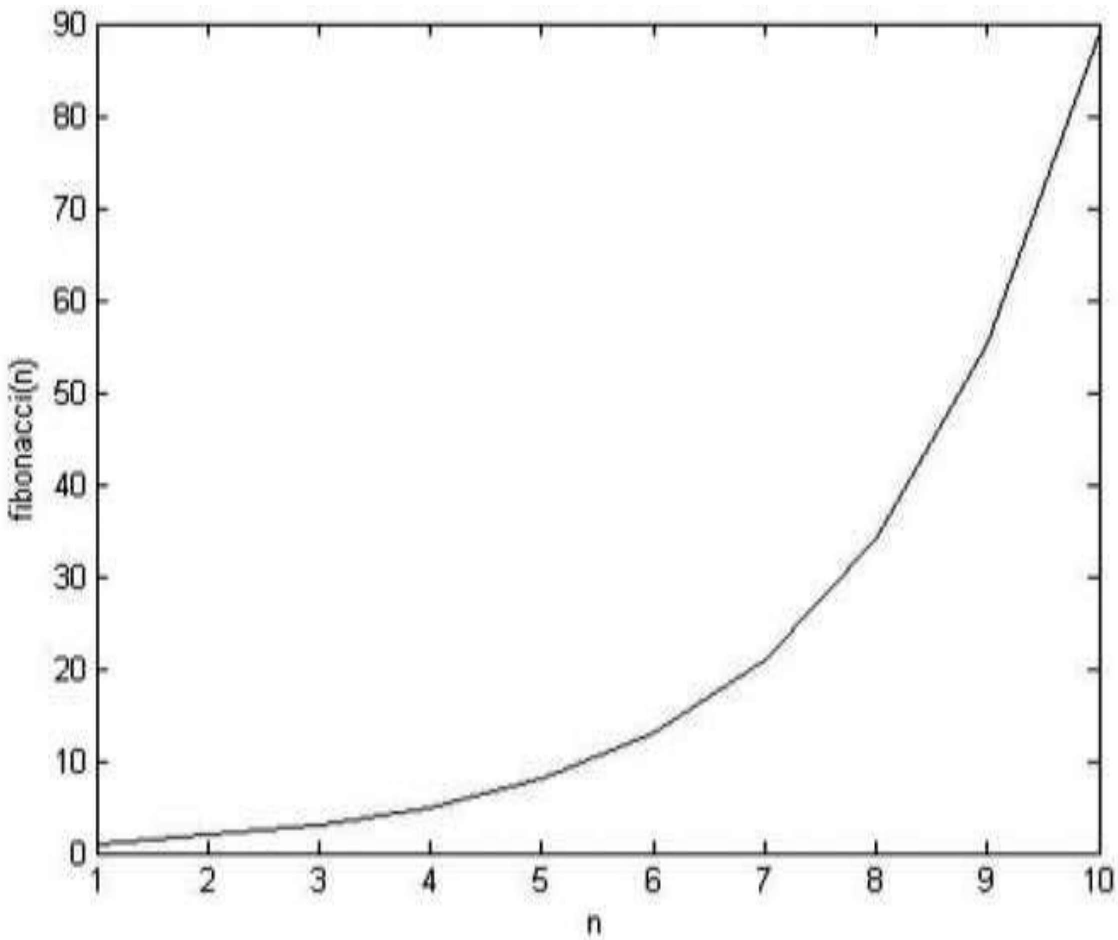



**Figure 1.2.6: Graph of Fibonacci Numbers**

### 1.2.21 Fibonacci Polynomials

In mathematics, the Fibonacci polynomials constitute a polynomial sequence that serves as a broader conceptualization of the Fibonacci numbers. Fibonacci polynomials are a generalization of the Fibonacci numbers, defined by a recurrence relation similar to the one for Fibonacci numbers.

$$F_n(x) = \begin{cases} 0\,,\ if\ n = 0 \\ 1,\ if\ n = 1 \\ xF_{n-1}(x) + F_{n-2}(x)\,,\ if\ n \geq 2. \end{cases} \tag{1.2.42}$$

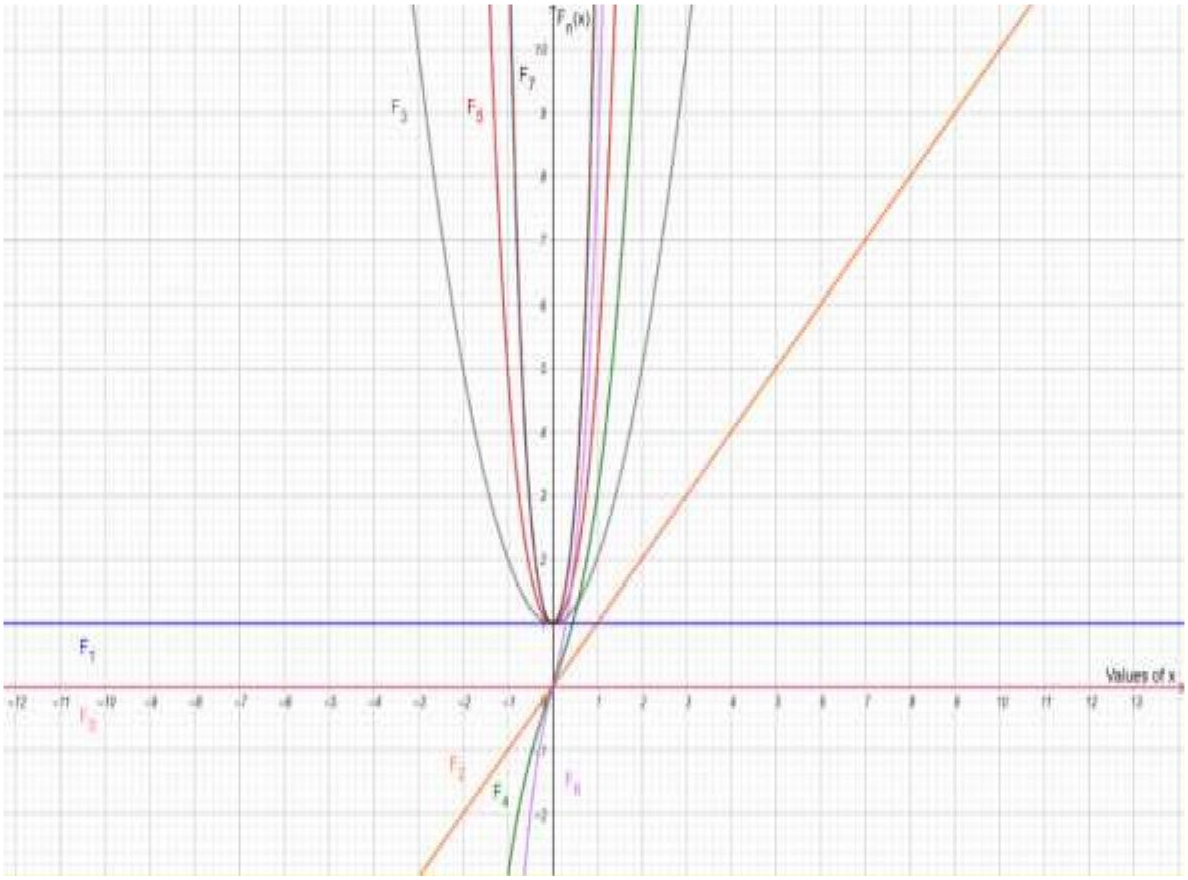


**Figure 1.2.7: Graph of Fibonacci Polynomials**

## 1.2.22 Lucas Numbers

It's an integer sequence attributed to the mathematician François Édouard Anatole Lucas. Lucas numbers are similar to the Fibonacci sequence, but with different initial values, defined as follows:

$$L_n = \begin{cases} 2\,,\ if\ n = 0 \\ 1,\ if\ n = 1 \\ L_{n-1} + L_{n-2}\,,\ if\ n \geq 2. \end{cases} \tag{1.2.43}$$

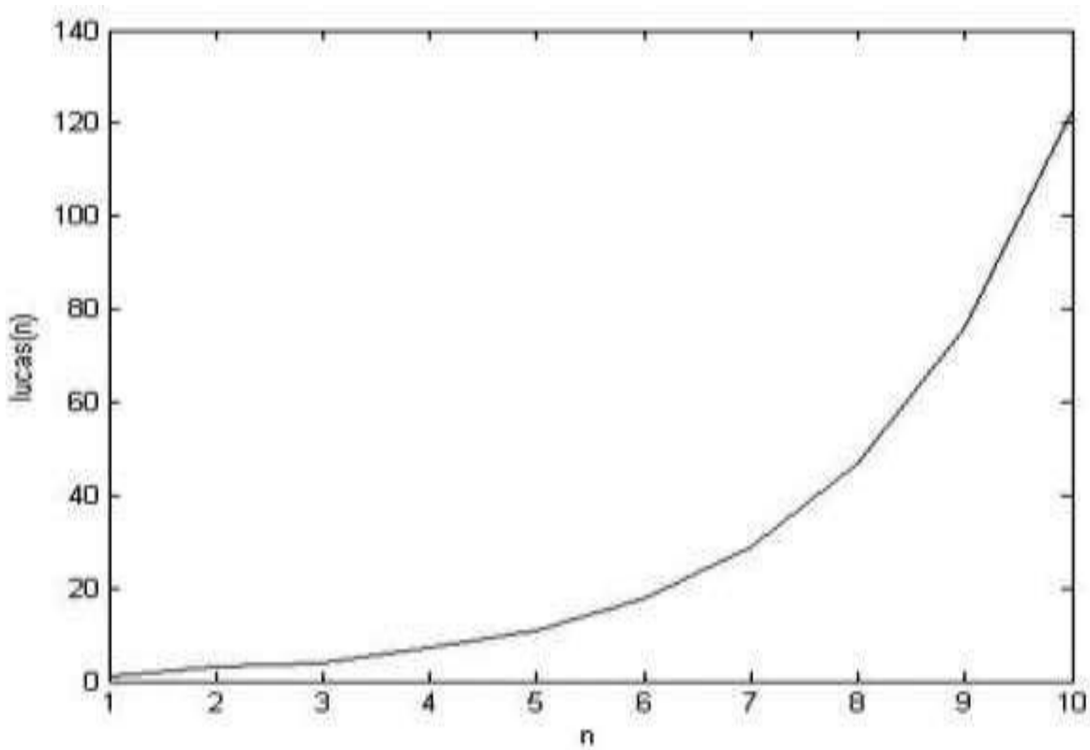



**Figure 1.2.8: Graph of Lucas Numbers**

### 1.2.23 Lucas Polynomials

The Lucas polynomials are the polynomials sequences which can be considered as a generalization of the Lucas numbers in a similar way that Fibonacci polynomials generalize Fibonacci numbers. Lucas polynomials are related to Fibonacci polynomials. They are defined by the recurrence relation:

$$L_n(x) = \begin{cases} 2\,,\ if\ n = 0 \\ x,\ if\ n = 1 \\ xL_{n-1}(x) + L_{n-2}(x)\ ,if\ n \geq 2. \end{cases} \tag{1.2.44}$$

The graphical representation is as under;

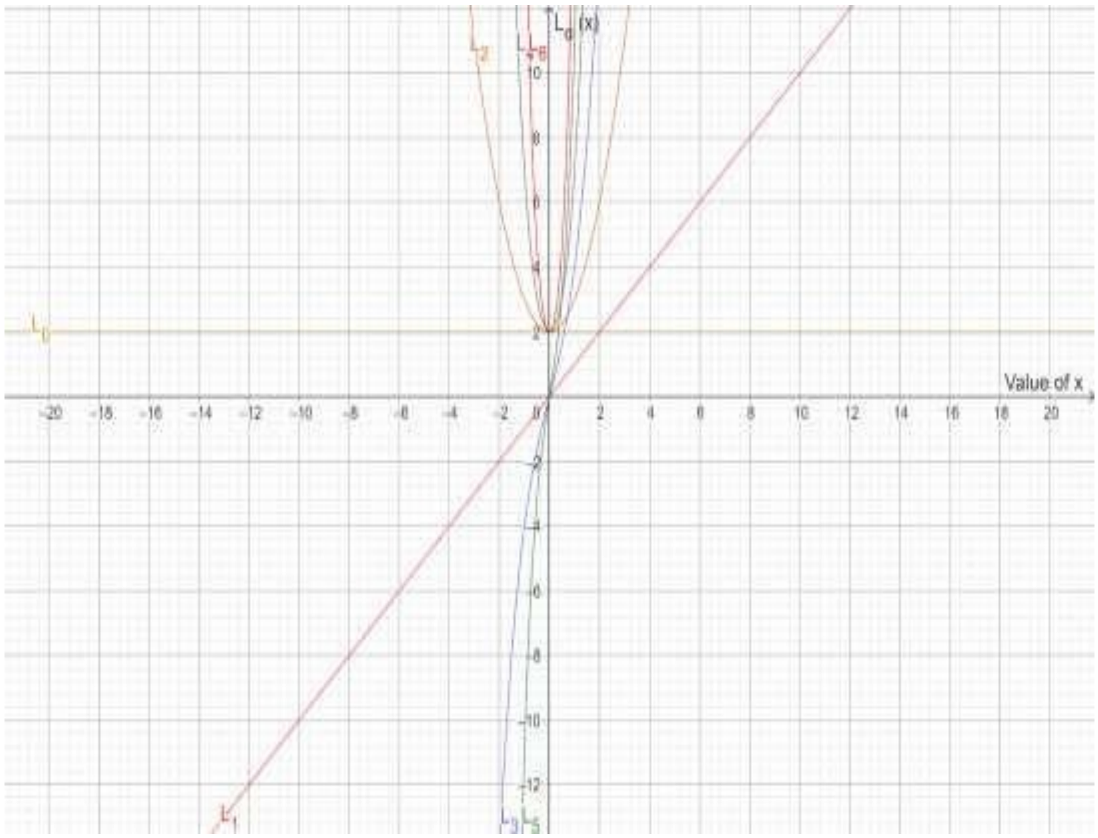



**Figure 1.2.9: Graph of Lucas Polynomials**

## 1. 2. 24 Pell Numbers

The Pell numbers are an infinite sequence of integer. The Pell numbers are the numbers that are similar to Fibonacci numbers and are generated by a recurrence relation. The Pell numbers are the sum of twice preceding "Pell number" with the Pell number immediately before it, defined as:

$$P_n = \begin{cases} 0 & , \ if\ n = 0 \\ 1 & , if\ n = 1 \\ 2P_{n-1} + P_{n-2} & , if\ n \geq 2. \end{cases} \qquad (1.2.45)$$

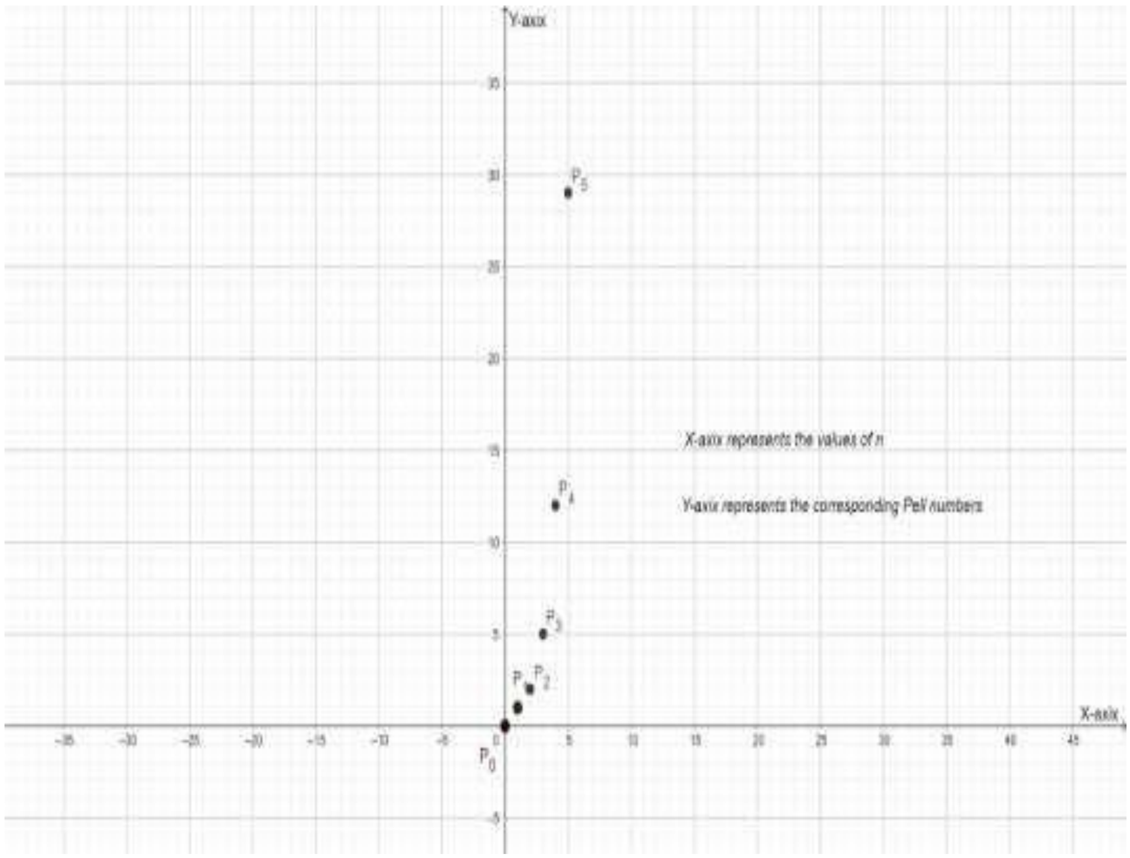



**Figure 1.2.10: The Graph of Pell Numbers**

## 1.2.25 Pell Polynomials

The Pell polynomials are a sequence of polynomials that generalize the Pell numbers. They are defined by the recurrence relation. Pell polynomials are related to Fibonacci polynomials. The Pell polynomials denoted by $P_n(x)$ are generated by the recurrence relation:

$$P_n(x) = \begin{cases} 0 & , if\ n = 0 \\ 1 & , if\ n = 1 \\ 2xP_{n-1} + P_{n-2} & , if\ n \geq 2. \end{cases} \tag{1.2.46}$$

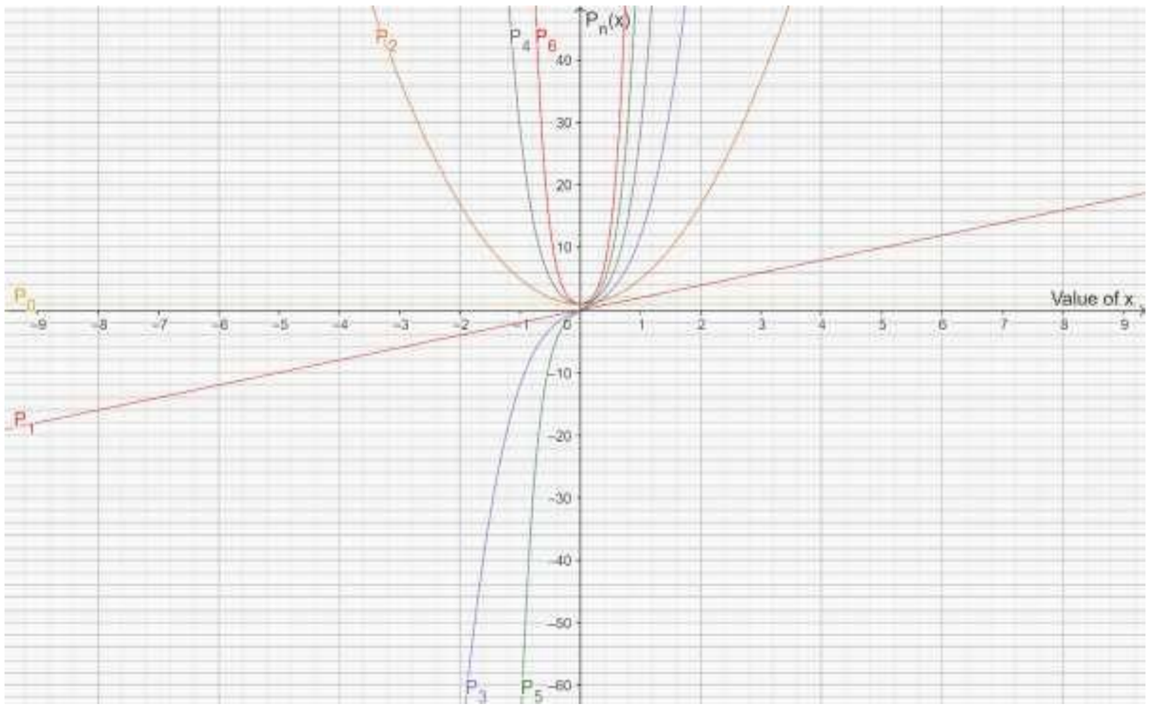



**Figure 1.2.11: Graph of Pell Polynomials**

## 1.2.26 Binet Formula

Binet's formula, recognized as Binet's closed-form expression for the Fibonacci sequence, was indeed developed by the French mathematician Jacques Philippe Marie Binet. He made this discovery in 1843. This formula offers a direct method for computing the $n^{th}$ term of the Fibonacci sequence without resorting to iterative recurrence relations, providing a more straightforward and efficient approach to calculate Fibonacci numbers. The C. E. for the Fibonacci sequence is:

$$t^2 - t - 1 = 0, \tag{1.2.47}$$

its roots are denoted as $a$ and $b$.

$$a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2}, \tag{1.2.48}$$

$$F_n = \frac{a^n - b^n}{a-b}, \text{ where } n \geq 0. \tag{1.2.49}$$

### 1.2.27 Binet Formula for Chebyshev polynomials of the 1st Kind

Binet formula allows for the direct calculation of 1st kind "Chebyshev polynomials" without relying on iterative methods. It provides an elegant and efficient way to compute these polynomials for a given $n$.

The C. E. of the Chebyshev recurrence for the 1st kind is indeed;

$$t^2 - 2xt + 1 = 0, \tag{1.2.50}$$

and the character roots of above equation are;

$$a = x + \sqrt{x^2 - 1},\ b = x - \sqrt{x^2 - 1}. \tag{1.2.51}$$

The Binet's formula $T_n(x)$ can be formulated as:

$$T_n(x) = \frac{a^n + b^n}{2},\ \text{where } n \geq 0\,. \tag{1.2.52}$$

### 1.2.28 Binet Formula for "Chebyshev polynomials" of the 2nd Kind

The C. E. of the Chebyshev recurrence for the 2nd kind is indeed;

$$t^2 - 2xt + 1 = 0, \tag{1.2.53}$$

and the characteristic roots of the above equation are;

$$a = x + \sqrt{x^2 - 1}, b = x - \sqrt{x^2 - 1}. \tag{1.2.54}$$

The Binet's formula $U_n(x)$ can be formulated as:

$$U_n(x) = \frac{a^{n+1} - b^{n+1}}{a - b}, \text{where } n \geq 0. \tag{1.2.55}$$

### 1.2.29 Binet Formula for Chebyshev Polynomials of the 3rd Kind

The characteristic equation of the Chebyshev recurrence for the 3rd kind is indeed;

$$t^2 - 2xt + 1 = 0, \quad (1.2.56)$$

and the characteristic roots of the above equation are;

$$a = x+\sqrt{x^2 - 1}, b = x-\sqrt{x^2 - 1}. \quad (1.2.57)$$

The Binet's formula for $V_n(x)$ can be formulated as;

$$V_n(x) = \frac{a^{n+1} - b^{n+1} - a^n + b^n}{a-b}, \text{ where } n \geq 0. \quad (1.2.58)$$

### 1.2.30 Binet Formula for Chebyshev Polynomials of the 4th Kind

The characteristic equation of the Chebyshev recurrence for the 4th kind is indeed;

$$t^2 - 2xt + 1 = 0, \quad (1.2.59)$$

and the characteristic roots of the above equation are;

$$a = x+\sqrt{x^2 - 1}, b = x-\sqrt{x^2 - 1}. \quad (1.2.60)$$

The Binet's formula for $W_n(x)$ can be formulated as:

$$W_n(x) = \frac{a^{n+1} - b^{n+1} + a^n + b^n}{a-b}, \text{where } n \geq 0. \quad (1.2.61)$$

### 1.2.31 Chebyshev Differential Equations

The term "Chebyshev differential equations" typically refer to a family of differential equations associated with Chebyshev polynomials. These differential equations yield solutions correspond all four kinds of "Chebyshev polynomials".

The Chebyshev differential equation of the 1st kind is given by:

$$(1-x^2)y'' - xy' + n^2y = 0. \tag{1.2.62}$$

The Chebyshev differential equation of 2nd kind is given by:

$$(1-x^2)y'' - 3xy' + n(n+2)y = 0. \tag{1.2.63}$$

The Chebyshev differential equation of the 3rd kind is given by:

$$(1-x^2)y'' + (1-2x)y' + n(n+1)y = 0. \tag{1.2.64}$$

The Chebyshev differential equation of the 4th kind is given by:

$$(1-x^2)y'' + (1+2x)y' + n(n+1)y = 0. \tag{1.2.65}$$

### 1.2.32 Shifted "Chebyshev Polynomials" of 1st, 2nd, 3rd, and 4th Kind

The shifted version of all four kind of Chebyshev polynomials can be derived through the transformations given by equation:

$$s = 2x - 1. \tag{1.2.66}$$

This transformation results in shifted version of "Chebyshev polynomials". The 1st kind of shifted Chebyshev polynomials denoted as ${T^*}_n(x)$:

$${T^*}_n(x) = T_n(s) = T_n(2x-1). \tag{1.2.67}$$

Thus,

$${T^*}_0(x) = 1, \quad {T^*}_1(x) = 2x - 1, \quad {T^*}_2(x) = 8x^2 - 8x + 1, \ldots \tag{1.2.68}$$

Similarly shifted Chebyshev polynomials ${U^*}_n, {V^*}_n, {W^*}_n$ of 2nd, 3rd, and 4th kinds defined in precisely analogous ways:

$$U^*{}_n(x) = U(2x-1),\ V^*{}_n(x) = V_n(2x-1),\ W^*{}_n(x) = W_n(2x-1). \quad (1.2.69)$$

## 1.2.33. Generalized Chebyshev Polynomials

$\Omega_n(x)$, denotes the $n^{th}$ modified Chebyshev polynomials of the first kind (called also $n^{th}$ Vieta–Lucas's polynomial)

$$\Omega_n(x) = 2T_n(x^2), \quad n \in N. \quad (1.2.70)$$

$$\Omega_1(x) = x\ , \Omega_2(x) = x^2 - 2, \Omega_{n+2}(x) = x\Omega_{n+1}(x) - \Omega_n(x), n \in N. \quad (1.2.71)$$

$Z_n(x)$ denotes the $n^{th}$ modified Chebyshev polynomials of the second kind (called also $n^{th}$ Vieta–Fibonacci polynomial)

$$Z_n(x) = Z_n(x^2), n \in N. \quad (1.2.72)$$

$$Z_1(x) = x, Z_2(x) = x^2 - 1, \quad Z_{n+2}(x) = xZ_{n+1} - Z_n(x), n \in N. \quad (1.2.73)$$

## 1.3 Literature Review

During the latter part of the 19th century, Chebyshev instigate sets of polynomials now recognized as the 1st - 2nd kinds Chebyshev polynomials. Chebyshev polynomials come under the class of classical orthogonal polynomials. The class that encompasses various types of polynomials found in literature. This class includes multiple distinct sets of orthogonal polynomials i.e. Laguerre polynomials, Jacobi polynomials, and Hermite polynomials with specific properties and applications. Many authors have studied and defined Chebyshev polynomials in the form of recurrence relations and trigonometric formulae. Chebyshev polynomials share similar properties and patterns with the Lucas and Fibonacci numbers. The foundational definitions of Chebyshev polynomials were drawn from a seminal scholarly work authored by B. G. S. Doman [30], and J. C. Manson [ 69].

Paul Butzer and François Jongmans (1999) aimed to provide a comprehensive overview of the life and contributions of Chebyshev, the founder of the largest pre-revolutionary mathematics school in Russia, based in St. Petersburg. With 80 publications to his name, Chebyshev's work spanned various domains, including, probability theory, number theory, approximation theory, and practical mathematics. Notably, he took pride in constructing diverse mechanisms, including those related to arithmetic. While the paper caters to an audience interested in approximation theory, it strives to offer a balanced exploration of Chebyshev's wide-ranging accomplishments, with particular attention to their historical context [22].

The proposition by Mason and Handscomb (2003) regarding the existence of four kinds of these polynomials significantly broadens their range of applications. It was commonly acknowledged that there are four variations of Chebyshev polynomials, each being a specific instance within the broader category of Jacobi polynomials [69]. J. C. Manson (1993) provided a comprehensive overview of the fundamental properties of all four kinds of Chebyshev polynomials. Additionally, Manson demonstrated how certain established properties of the $1^{st}$ kind polynomials could be extended to encompass the $2^{nd}$, $3^{rd}$, and $4^{th}$ kind polynomials [68].

Wenpeng Zhang et al. (2018) discovered many properties related to the derivatives of Chebyshev polynomials and gives the correspondence between them, and their derivatives [104, 105, 106, 107, 108]. Yang and Sai-nan Zheng (2013) used the Riordan array to give the determinant representation of Chebyshev polynomials, Fibonacci numbers, and Pell numbers [101]. Shannon et al. (2021) provided the connection between the Fibonacci $p$-numbers and Pell numbers [88]. Virender Singh et al. (2023) examined properties related to extension with two variables of the $2^{nd}$ kind of Chebyshev polynomials matrix, also obtained a generalized Chebyshev matrix of the $2^{nd}$ kind [91].

Brandi et al. (2020) initiated their work to identified a connection between $1^{st}$ and $2^{nd}$ kind Chebyshev polynomials and trace of successive powers of matrix through this representation. The non-singular $2 \times 2$ matrices were demonstrated to straightforwardly derive composition identities for both first and second kind Chebyshev polynomials. They employed the matrix's power, trace, and determinant, yielding the following result [20].

Let $A$ be a non-singular $2 \times 2$ matrix. The integral power of $A$, for $n \geq 2$,

$$A^n = a_2^{\frac{(n-1)}{2}} U_{n-1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) A - a_2^{\frac{n}{2}} U_{n-2}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I, \tag{1.3.1}$$

where $a_1 =$ trace of $A$, $a_2 =$ determinant of $A \neq 0$, $I$ denotes the identity matrix, and $U_n(x)$ are the 2nd kind of Chebyshev polynomials.

For any integer $n \geq 0$,

$$\text{trace } A^n = 2a_2^{\frac{n}{2}} T_n\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right), \tag{1.3.2}$$

where $T_n(x)$ are the 1st kind of Chebyshev polynomials.

Clemente Cesarano and Sandra Pinelas (2019) elucidated the fundamental characteristics exhibited by all four kinds pseudo-Chebyshev polynomials with half-integer degrees. These polynomials, constituting irrational functions, were explored for their noteworthy properties and graphical representations. Notably, Chebyshev polynomials of the 3rd and 4th kinds, in contrast to the more familiar 1st and 2nd kinds polynomials, are relatively less recognized [25]. Max A. Alekseyev (2018) studied the properties related to intersection of Lucas, Pell numbers and Fibonacci numbers [10]. Ugur Duran et al. (2018) studied Hermite polynomials, $(P, Q)$– Bernstein polynomials with their modifications and W. A. Khan et al. (2018) investigated the properties related to Laguerre polynomials. These polynomials were closely related to Chebyshev polynomials [32, 49].

Pei-Yuan Zhao et al. (2023) discussed the coupling system for electron–phonon via the product of the Chebyshev matrix [103]. Amelia Bucur et al. (2007) demonstrated that the characteristic equation of the Chebyshev matrix discloses the presence of associated polynomials of Chebyshev and provides an explicit expression for them. While the literature typically considers $T_n(x)$ as the determinant of the Chebyshev matrix and their study revealed that this matrix contains valuable hidden information that can be unveiled by examining its characteristic equation, consequently highlighting the existence of associated polynomials of Chebyshev [21].

Di Han and Xingxing Lv (2020) introduced novel identities for Chebyshev polynomials, aimed to Fibonacci polynomials and Chebyshev polynomials. They also used elementary methods, the results delved into the computational challenges of these sums, offering fresh and intriguing identities as outcomes. Additionally, the methodologies employed in this research may serve as valuable references for further investigations of general linear recursive sequences of the $2^{\text{nd}}$ order [44]. Feng Qi et al. (2019) linked tridiagonal determinant with Fibonacci polynomials, Fibonacci numbers, and Chebyshev polynomials. They also presented two formulae to calculate tridiagonal determinant [81].

Titu Andreescu and Oleg Mushkarov (2018) discussed the quadratic form and determinant representation of the Chebyshev polynomials matrix. The following result holds for all integers $n \geq 3$, [14]

$$A_n(2x) = 2T_n(x) + 2(-1)^n. \tag{1.3.3}$$

Matrix $A$ have the following eigenvalues;

$$\lambda_k = cos\left(\frac{(2k-1)\Pi}{n}\right), 1 \leqslant k \leqslant n. \tag{1.3.4}$$

Minimal and maximal eigenvalues of $A$:

$$\lambda_{min} = \begin{cases} -2cos\left(\frac{\Pi}{n}\right), if\ n\ is\ even, \\ \quad -2 \qquad\quad , if\ n\ is\ odd. \end{cases} \tag{1.3.5}$$

$$\lambda_{max} = 2cos\left(\frac{\Pi}{n}\right). \tag{1.3.6}$$

E. H. Doha (2015) introduced algorithms centered on the utilization of both 3rd and 4th kinds of Chebyshev polynomials. The key concept behind the development of these numerical algorithms of the 3rd and 4th kinds as shifted Chebyshev polynomials [29].

Mahdy and Shwayyea (2016) presented an effective computational method to work out on a non-linear fractional diffusion equation. The methodology involved a combination of their proposed scheme, utilizing the 3rd kind shifted Chebyshev polynomials [66]. E. H. Doha et al. (2014), Nigam et al. (2020) emphasized the growing significance of 3rd kind $V_n(x)$ of Chebyshev polynomials in numerical analysis. They pointed out that most literature on Chebyshev polynomials tends to focus on results related to the 1st and 2nd kinds [28, 75].

N. Gogin and M. Hirvonsalo (2017) provided a concise generating function expression for discrete Chebyshev polynomials. They achieved this by employing MacWilliams transforms of Jacobi polynomials, complemented by a binomial multiplicative factor [40]. Y. Zhang et al. (2018) obtained identities using combinatorial method connected to the Chebyshev polynomials of the 2nd kind [104]. Aghigh et al. (2008) take a look at 3rd and 4th kinds of Chebyshev polynomials, which were orthogonal functions. These sequences represented special class of Jacobi polynomials [6]. Wenpeng Zhang (2020) initiated numerous properties concerning the derivatives of Chebyshev polynomials and elucidated their interrelations;[108]

"Let $n, q$ be integers [see 59] with $n \geq 0, q \geq 1$,

$$\sum_{b_1+b_2+\cdots+b_{q+1}=l} \prod_{k=1}^{q+1} U_{b_k}(x) = \frac{1}{2^q q!} U_{n+q}{}^{(q)}(x)."\qquad (1.3.7)$$

Tom Cuchta et al. (2020) introduced and explored a novel set of difference equations associated with the classical Chebyshev differential equations of the 1st and 2nd kinds. This led to the formulation of similar properties to their continuous counterparts. These properties included representation by recurrence relations and their derivative relations [26]. Anna Tatarczak (2016) explored specifically the generalized version of Chebyshev polynomials of the 2nd kind, written as $U_n(p, q, e^{i\theta})$, and those of the 1st kind, denoted by $T_n(p, q, e^{i\theta})$. These polynomials hold particular utility in applications for distinct reasons. The second kind of generalized Chebyshev polynomials were noteworthy for their connections with generalized typically real functions and resembling the classical case [94].

Yixue Zhang and Zhuoyu Chen (2018) introduced a second-order nonlinear recursive sequence in their paper. They utilized this sequence along with combinatorial methods to conduct an in-depth investigation into computational problems related to a specific type of sums, including the Chebyshev polynomials. Their approach enabled the simplification of complex computations involving the second type Chebyshev polynomials, reducing them to a much simpler problem [104].

Xingxing Lv and Shimeng Shen (2017) aimed to utilized the characteristics of Chebyshev polynomials to explore power sum glitches related to the functions $\sin(x)$ and $\cos(x)$. Their study focused on deriving computational formulas and explicit formulas for trigonometric power sums. There was no prior study on these specific problems. The authors utilized the properties of first kind Chebyshev polynomials to obtain results [65]. Abd-Elhameed and Al-Harbi (2023) primarily concerned with the generalization of Chebyshev's 3rd kind polynomials, with contributions from different perspectives. Additionally, some new formulas were discussed [3].

In order to gain new insights into the properties of Lucas-polynomials, Abd-Elhameed and Napoli (2023) explored different approaches to obtaining results. Matrix representation was also discussed in order to identify certain properties of the polynomials [2]. S. H. Aziz et al. (2020) in their study 2nd kind of Chebyshev polynomials was used to solve differential equation with their higher order. The authors presented very promising results with the examples of this generalization. The authors reduced the actual differential equation to the solution of algebraic equation with computing programs [17].

Kizilates et al. (2019) instigates $(p, q)$1st and 2nd kinds of Chebyshev polynomials in to Fibonacci, Luca's polynomials. For any integer $n \geq 2$ and $0 < q < p \leq 1$, the $(p, q) -$ the 1st kind of Chebyshev polynomials defined as [59];

$$T_n(x, s, p, q) = (p^{n-1} + q^{n-1})x\, T_{n-1}(x, s, p, q) + (pq)^{n-1} s T_{n-2}(x, s, p, q), \qquad (1.3.8)$$

where $x, s$ are real variables, $T_0(x, s, p, q) = 1, T_1(x, s, p, q) = x.$ (1.3.9)

O. A. Taiwo et al. (2012) conducted a study on Chebyshev polynomials to solve differential equation of 4th order [95]. B. E. Kashem (2019) the primary objective of investigation was to address boundary value problems (BVPs). The approach involved transforming the infinite interval into a sufficiently large finite interval and employed the finite difference method to approximate the variable, wherein the unknowns were the shifted third kind Chebyshev coefficients [48].

Xiaoxue Li (2015) pursued the primary objective of utilized combinatorial methods to investigate sums of powers of Chebyshev polynomials, originated intriguing properties and aimed to establish divisibility properties related to Chebyshev polynomials by applying the obtained results. The significance of these identities lies in their ability to transform complex sums of powers of Chebyshev polynomials into more straightforward linear sums, thereby simplifying calculations associated with such sums [61]. M. S. Metwally et al. (2019) investigated matrix polynomials associated with the 2nd kind of polynomials of Chebyshev matrix. They found many results related to the associated Chebyshev matrix polynomials [71].

Jugal Kishore et al. (2023) presented a paper where the focus was on introducing identities related to summation of definite products of 3rd and 4th kinds of Chebyshev polynomials, Lucas, and Fibonacci numbers [50].

Z. Fan and W. Chu (2022) generating function approach was employed to establish various convolution formulae involving Chebyshev polynomials and other prominent numbers. Specifically, include relationships with numbers and polynomials of Bernoulli and Euler, as well as Fibonacci and Lucas numbers [35]. Milica et al. (2021) found a two-determinant generalized formula situated on the 2nd kind of Chebyshev polynomials. For this purpose, they utilized a tridiagonal matrix [72]. Ahmet Oteles et al. (2014) worked on a family of tridiagonal matrices relating to the 1st kind of Chebyshev polynomials and obtained eigen vectors and eigen values [78]. H. M. Ahmed (2022) derived an algorithm to solve a specific equation by employing shifted Chebyshev polynomials of the 1st kind. The algorithm provided a well approximate solution [7].

Aiyub et al. (2019) presented a binomial matrix connected to Fibonacci. They specified some results of statistical convergence and also deduced that approximation theory consolidated the statistical convergence [8]. Zoran Pucanovic and Marko Pesovic (2023) used the properties of matrices and Chebyshev polynomials. They connected matrices and Chebyshev polynomials [80]. S. Foud (2019) studied 3rd kind shifted Chebyshev polynomials for operational matrices [39].

T. Kim et al. (2018) explored the summation involving finite products of both the 3rd and 4th kinds of Chebyshev polynomials. They derived a relationship among the 4th kind of Chebyshev polynomials and their derivatives as follows: [55]

Let $n, q$ be integers with $n \geq 0, q \geq 1$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} (-1)^{n-l} \binom{q-1+n-l}{q-1} W_{b_1}(x)\, W_{b_2}(x) \ldots W_{b_{q+1}}(x)$$

$$= \frac{1}{2^q q!} W_{n+q}{}^{(q)}(x), \tag{1.3.10}$$

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} V_{b_1}(x)\, V_{b_2}(x) \ldots V_{b_{q+1}}(x)$$

$$= \frac{1}{2^q q!} V_{n+q}{}^{(q)}(x), \tag{1.3.11}$$

Where $b_1,\ b_2, \ldots, b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = l$ & $n, q$ be integers with $n \geq 0, q \geq 1$.

The work by T. He and S. Shiue (2009) a novel method was introduced for constructing explicit formulas for polynomials and numbers produced by a second-order recursive relation. The presented approach serves as a general method for establishing identities within linear recurrence relations [46]. In the research conducted by R. K. Graves et al. (2016) matrix methods are employed to derive a range of binomial summation formulas for various recursive 2nd and 3rd order sequences. The study extended several familiar summation identities to encompass formulations with negative subscripts [41].

Waleed Abd-Elhameed (2016) the primary objective was to address connectivity issues between the Chebyshev polynomials of the 3rd and 4th kinds and $(p, q)$- Fibonacci polynomials. The article also provided inversion connection formulae for these relations [1]. Sana Krioui et al. (2023) a new approach was presented for constructing the generating functions. The focus of the paper was on introducing innovative generating functions for products of $t$ - Pell numbers, $h$ - Fibonacci numbers, $s$ - Jacobsthal numbers [60].

Samundra Regmi et al. (2023) discussed the application of Chebyshev polynomials for nonlinear equations [83]. F. Chishti et al. (2021) studied a shifted 4th kind of Chebyshev polynomials for operational matrices [27]. K. Erdmanna and S. Schroll (2011) derived the results on the 2nd kind Chebyshev polynomials by using symmetric matrices [33].

M. C. Akmak and K. Uslu (2019) developed a generalized version Chebyshev polynomial of all four-kinds. Additionally, they demonstrated a Binet-style formula and demonstrated the relation between Chebyshev polynomials of all four-kinds and the generalized version of the Chebyshev polynomials [9].

T. Kim et al. (2018, 2020) studied summation of finite products of 1st kind Chebyshev with Lucas's polynomials and Chebyshev polynomials of 2nd, 3rd, and 4th kinds. [52, 53]. Clemente Cesarano (2014) introduced generalizations for the 1st kind Chebyshev polynomials and used Hermite polynomials, which served as integral representations for the generalized Chebyshev polynomials, and also applied the results obtained for the Gegenbauer polynomials to them [24].

Roman Wituła and Damian Słota (2006) presented a variation of the Chebyshev polynomials with novel outcomes and applications related to the Morgan-Voyce polynomials [98]. S. Uygun et al. (2020) the authors proposed a generalized version of some of the polynomial names; Pell, Pell Lucas, and Vieta. They identified properties like Binet-like formula, sum formula, generating function, and differentiation, as well as generating a matrix whose values were extracted from a generalized version of the Vieta-Pell -Lucas's polynomials [96].

Abdullah Altın et al. (2012) presented recurrence relations for Chebyshev matrix polynomials, especially for the 2nd kind. They also found generating matrix functions and several identities for 2nd kind of Chebyshev polynomials [13]. G. B. Djordjevi´c (2009) conducted a series of studies on the various categories of polynomials associated with Chebyshev's polynomials and the derived results associated with them [31]. A. A. Salih and S. Shihab (2020) primary objective of the research was to identify a variant of Chebyshev polynomials. Furthermore, the authors discussed their integration, derivative operational matrix [85].

M. A. Alqudah (2015) came up with a new way to look at Chebyshev's 2nd kind polynomials and used obtained results in the approximation of functions [12]. C. Kizilates et al. (2019) instigates $(p, q)$ 1st and 2nd kinds of Chebyshev polynomials in to Fibonacci, Luca's polynomials. They also talked about $n^{th}$ generalizations and properties of derivatives, which were represented by determinants of the polynomials [59]. G. S. Abed (2021) modified the 1st kind Chebyshev polynomials and employed the variable separation approach to solve the partial differential equation [5].

M. R. Eslachi et al. (2012) utilized the unique characteristics of the third, fourth kinds of Chebyshev polynomials. Their focus was specifically on determining approximations for the best uniform polynomials derived from these specific types of polynomials [34]. Mohammed Abdulhadi Sarhan et al. (2021) came up with a new way to solve Pell's polynomials problem using two variables. A significant observation was made in two variables to resolve partial differential equations. These techniques were employed to resolve the underlying issue with error-free calculations. This paper also provided examples demonstrating the rationale of the strategy [87].

Yüksel Soykan (2023) delved into a generalized form of the Fibonacci numbers and presented Simson's generating function formulas derived from matrix results, as well as calculating the infinitesimal sums of these polynomials [92]. Sarita Nemaniy et al. (2022) in their study, established the sophisticated properties of the Fibonacci sequence. Their findings concerned the divisibility of Fibonacci sequences and the representation of matrices by determinants including sequence terms [74]. B. S. Bychkov and G. B. Shabat (2021) their research focused on the generalizations of Catalan numbers and generalizations of Chebyshev polynomials [23].

Khalid K. Ali1 et al. (2022) explained the partial differential equation of space fractional using the 5th kind of shifted Chebyshev polynomials [11]. Adem and Sahin (2022) utilized a generalized form of the Fibonacci numbers to yield remarkable results with a recurrence relationship for square pyramids numbers [18]. V. Verma and Priyanka (2019) used first-order derivatives to derive a generalized version of Fibonacci polynomials. Additionally, they discussed Fibonacci polynomials with dual variables [98].

Using the Dilcher-Stolarsky approach for their study, Seon-Hong Kim (2021) modified the 2nd kind Chebyshev polynomials in various ways to determine the property of irreducibility. [51]. Sanjay Harne et al. (2014) delved into identities concerning Chebyshev polynomials, Lucas numbers, and Fibonacci numbers at specific variables, along with their derivatives [45]:

$$\sum_{b_1+b_2+\cdots+b_{q+1}=l} F_{4(2b_1+1)}\, F_{4(2b_2+1)} \ldots F_{4(2b_{q+1}+1)} = \frac{3^{q+1}}{2^q q!} U_{n+q}{}^{q}\left(\frac{7}{2}\right). \qquad (1.3.12)$$

$$\sum_{b_1+b_2+\cdots+b_{q+1}=l} F_{6(2b_1+1)}\, F_{6(2b_2+1)} \ldots F_{6(2b_{q+1}+1)} = \frac{2^{2q+3}}{q!} U_{n+q}{}^{q}(9). \qquad (1.3.13)$$

Kamal et al. (2022) discussed the Fredholm –Volterra integrodifferential equations with the first kind of shifted Chebyshev polynomials contingent on the operational matrix [47]. Shoukralla (2021) devised a numerical approach to solve the first kind of Fredholm integral equation by utilizing the matrix representation involving the second kind of Chebyshev polynomials [89]. T. Korkiatsakul et al. (2018) explored a novel method involving the Chebyshev operational matrix to address the solution of problems involving integrals of non-linear Caputo fractions [58].

Kim et al. (2021) obtained the summation of Chebyshev polynomials definite products of two different types, each one of them represented a linear combination of all other types [54]. M. Musraini et al. (2023) various modifications have been made to the Fibonacci sequence and the Lucas sequence, in some cases by maintaining the original conditions and in other cases by maintaining the recurrence relationship [73].

J. Gultekin and B. Sakiroglu (2017) introduced a method to derive generalized Chebyshev polynomials using matrices. The study establishes a novel recurrence relation for the derivatives of these polynomials and provides combinatorial expressions for these derivatives. Furthermore, tables for derivative polynomials were constructed using the derived combinatorial forms [43]. Jonny Griffiths (2016) presented many results which connected all four kinds of Chebyshev polynomials [42].

Connor M. Sanford Louisiana (2018) presented the solution to a problem from the American mathematical journal's problem section. The Chebyshev polynomial of the second kind with matrix representations was central to the problem [86]. M. Arya and V. Verma (2022) gave identities for Fibonacci polynomials and generalized Fibonacci numbers. They also obtained an exceptional representation in the form of a matrix by using obtained identities [16]. M. Arya et al. (2019) discovered a unique form of representation for Chebyshev polynomials [15].

Fonseca (2020) discussed the relationship of the Fibonacci numbers and $2^{nd}$ kind of Chebyshev polynomials. To obtain the result, he used a tridiagonal matrix determinant [37].

Fonseca (2021) provided inverses for tridiagonal-matrices using the 2nd kind of Chebyshev polynomials. They used an invertible matrix to find results related to tridiagonal matrices [38]. Waleed Mohamed Abd-Elhameed et al. (2022) objective of the research was to evolve the connection between generalized types of Lucas and Fibonacci polynomials. Hypergeometric functions were employed to link the polynomials, such as Lucas, Fermat, Pell, and Fermat Lucas [4]. Kamal Aghigh et al. [6], M. R. Eslahchi et al. [34], and Taekyun Kim et al. [53] provided numerous identities associated with both the 3rd and 4th kinds of Chebyshev polynomials.

Wenpeng Zhang (2004) introduced the fundamental concept for solving the summation of recurrence relations and also investigated several identities pertaining to Fibonacci sequences. Chebyshev polynomials have been extensively examined by researchers and are defined through various formulations such as recurrence relations and trigonometric expressions. Moreover, there exists a significant connection between Fibonacci and Lucas numbers with Chebyshev polynomials. These recursive relationships find applications in combinatorial counting [107].

Li et al. (2022) used the Chebyshev polynomials matrices to obtain numerical solutions with examples for boundary value condition of the differential equation of 4th order [64]. Yang Li (2014) determined the connection between derivatives of the Chebyshev and Fibonacci polynomials [62]. In addition, Yang Li (2015) analysed the Chebyshev polynomials of both 1st and 2nd kinds and applied elementary method to find the relationship between both kinds of Chebyshev polynomials connected to Fibonacci polynomials and finally found some results related to the Fibonacci and the Lucas numbers [63].

L. Zhang (2017) and W. Zhan (1997) used mathematical induction, to solve the problem of Chebyshev polynomials sums of powers and obtained many other properties related to Chebyshev polynomials [102, 105]. Zhang and Han (2020) provided identities for reciprocal sums of the Chebyshev polynomials through mathematical induction and leveraging identities of symmetrical polynomials [108]. J. Kishore and V. Verma (2022) used a computational method to give identities concerning products of finite sums of the Lucas, Fibonacci, and complex Fibonacci numbers [56].

Frontczak and Goy (2021) obtained the generating function of Chebyshev-Fibonacci polynomials. They explored the properties related to these polynomials, through the use of generating functions, they established novel connections and identified combinatorial identities [36]. Jerzy Kocik (2021) found a matrix representation of the Chebyshev polynomials, Fibonacci series, and Luca's polynomials by using the symmetric tensor and power of a matrix [57].

Olagunju and J. Folake (2013) proposed a problem utilizing 3rd kind Chebyshev polynomials. Despite the simplicity of the computational process, the method proves to be highly effective and appealing. The results demonstrate the applicability of this approach through numerical examples, showcasing its efficiency and simplicity. The keywords associated with this study include the collocation method, equally spaced points, and 3rd kind Chebyshev polynomials [76].

S. D. Marchi et al. (2023) familiarize themselves with the generalizations of the 1st kind Chebyshev polynomials and identify a number of properties associated with orthogonal polynomials [67]. Goksal Bilgici (2014) research on generalizing new generalized Fibonacci and Lucas sequences $F_n$ and $L_n$ based on the relationship between the recurrence relation with the basic conditions;

$$F_n = 2aF_{n-1} + (b^2 - a)F_{n-2},\ n \geq 2, \tag{1.3.14}$$

and

$$L_n = 2aL_{n-1} + (b^2 - a)L_{n-2},\ n \geq 2. \tag{1.3.15}$$

Where $a$ and $b$ are any non-zero real numbers. They were able to demonstrated Binet's formula and the generating function for these sequences [19]. M. S. Metwally et al. (2014) provide a description and examination of the 2nd kind of Chebyshev matrix polynomials. They delved into the analysis of three term recurrence relation associated with these matrix polynomials [70]. A. Patra and G. K. Panda (2022) an exploration conducted on sums involving finite products of Pell polynomials, with an emphasis on expressing these sums in relation to specific orthogonal polynomials. Additionally, each derived expression is presented as a linear combination of well-known classical polynomials [79].

A. Rababah and E. Hijazi (2019) explored the transformation among the Chebyshev polynomials basis of the $4^{th}$ kind and also includes illustrative examples to demonstrate these transformations [82]. L. T. Spelina and I. Wloch (2019) presented and examined a novel one-parameter generalization of Pell numbers. The study delves into detailing their unique properties, along with an exploration of their connections to matrix representation [93].

F. Yang and Y. Li (2021) investigated the reciprocals of infinite sums leads to the discovery of novel and intriguing identities involving Chebyshev polynomials [100]. W. A. Abd-Elhameed (2022) derived formulae encompass several well-known polynomials, including Pell, Lucas, Fibonacci, Fermat, Fermat-Lucas's polynomials, and Pell-Lucas offering new insights and generalizations beyond existing literature [4].

## 1.4 Research Gap

The thorough review of the cited literature leads to the following inferences regarding the research gap which is proposed to be bridged during tenure of our research work. This review has identified the research gap that the many researchers have worked on Chebyshev polynomial of $1^{st}$ and $2^{nd}$ kinds and very limited study on the $3^{rd}$ and $4^{th}$ kinds of the Chebyshev polynomials. The properties and applications on Chebyshev polynomial of $3^{rd}$ and $4^{th}$ kinds to be studied and new relation are to be established. The relations between third and fourth kinds Chebyshev polynomials connected to Lucas, Fibonacci, and Pell numbers are to be derived. Matrix representation of $3^{rd}$ and $4^{th}$ kinds and properties related to matrix represtation via matrix algebra are to be explored and similar concepts can be extended to the Chebyshev-like polynomials also. Their properties have been established so for, generalizations of Chebyshev polynomials are to be explored for this we may extend the recurrence relation, or the recurrence relation is preserved but the coefficients of polynomial are replaced by some new coefficients with more variables or by changing the initial conditions and established their properties. This gap in knowledge remains unaddressed in existing literature.

## 1.5 Objectives of the Research Work

1. To obtain matrix representation of Chebyshev polynomials of the third and fourth kinds and prove some properties relating to matrix representation.
2. To develop a new generalization of Chebyshev-like polynomials and to study their properties.
3. To establish relations between different Chebyshev polynomials and Fibonacci numbers, Lucas numbers, Pell numbers.

## 1.6 Methodology of the Research Work

- There are two directions of generalizing recurrence relations namely either the recurrence relation can be generalized and extended, or the recurrence relation is preserved but the coefficients of polynomials are replaced by some other coefficients with more variables. We have combined these two techniques.
- The same approach for Chebyshev third and fourth kinds polynomials.
- To express matrix representation of Chebyshev third and fourth kinds polynomials by applying matrix algebra properties.

## 1.7 Motivation of the Work

Many authors worked on the $1^{st}$ and $2^{nd}$ kinds of Chebyshev polynomials and also discovered many lemmas, identities related to these polynomials and presented many theorems with their applications [77, 80, 84, 90]. Our work is motivated by the earlier work of Wenpeng Zhag, Taekyun Kim, and the earlier work of Primo Brandi et al. The authors derived the many identities attributed to the Chebyshev polynomial of the $1^{st}$ and $2^{nd}$ kinds via non-singular matrix. Another motivation for our passion in establishing the presented results is the research of Amelia Bucur et al. The authors obtained the characteristic equations for 1st and 2nd kinds of Chebyshev polynomials. Gultekin, Betul Sakiroglu conducted a study on the analysis of Chebyshev generalized polynomials forms using matrixes and combination forms. We have prior studied S. L. Yang, S. N. Zheng, and A. Altın, B. Cekim research to generate matrix representation and characteristic equations of Chebyshev polynomials of the 3rd and 4th kinds. Chebyshev polynomials occupy prominent attention because of their substantial use in mathematics. This study is very useful in the practical and theoretical aspects of mathematics like in approximation theory.

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# Chapter 2

# Characteristic Equations of Chebyshev Polynomials and Their Generating Matrices

The core goal of the chapter is to obtain matrix representation for the Chebyshev's 3rd and 4th kinds of polynomials through the utilization of a tridiagonal matrix. We present a connection between Chebyshev polynomials of the 2nd kind, 3rd kind, and 4th kind through the concept of matrix power. We present a determinant representation by using tridiagonal matrix for both 3rd and 4th kinds of Chebyshev polynomials. We present the characteristic equations for the 3rd and 4th kinds of the Chebyshev polynomials up to degree three. We also prove some properties related to matrix representation. We elaborate the theorem and validate it through examples. The utilization of Chebyshev polynomials is explored, including a detailed discussion on the practical application of 3rd kind Chebyshev polynomials in approximation theory.

## 2.1 Introduction

C.M. Da Fonseca [37, 38] provided explicit inverse for tridiagonal matrices using the second kind of Chebyshev polynomials. They used an invertible matrix to find results related to tridiagonal matrices. Amelia Bucur et al. [21] gave the C. E. of the Chebyshev matrix of the first kind, found associated polynomials of Chebyshev, and presented an explicit formula from them. Primo Brandi et al. [20] obtained many identities for the 1st and 2nd kind of Chebyshev polynomials with a non-singular complex matrix. They employed the matrix's power, trace, and determinant to yielding the given result.

Let $A$ denoted $2 \times 2$ (non-singular) matrix. The integral power of $A$, for $n \geq 2$;

$$A^n = a_2^{\frac{(n-1)}{2}} U_{n-1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) A - a_2^{\frac{n}{2}} U_{n-2}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I,$$

where $I$ represents the identity matrix and $U_n(x)$ are the 2nd kind of Chebyshev polynomials [20].

Let $A$ denotes the $2 \times 2$ (non-singular) matrix. For any integer $n \geq 0$;

$$\text{trace } A^n = 2a_2^{\frac{n}{2}} T_n\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right),$$

where, $T_n(x)$ are the 1st kind of Chebyshev polynomials [20].

Sheng-Liang Yang and Sai-nan Zheng [101] used the Riordan array to give the determinant representation of Chebyshev polynomials, Fibonacci numbers, and Pell numbers. Virender Singh et al. [91] examined properties related to extension with two variables of the Chebyshev's 2nd kind polynomials matrix. Metwally et al. [71] explored matrix polynomials linked to the Chebyshev's 2nd kind matrix polynomials. They found many results related to the associated Chebyshev polynomials matrix.

Feng Qi et al. [81] linked tridiagonal determinants with Fibonacci polynomials, Fibonacci numbers, and Chebyshev polynomials. They also presented two formulae to calculate tridiagonal determinant. Titu Andreescu and Oleg Mushkarov [14] discussed the quadratic form and determinant representation of the Chebyshev polynomials matrix for first type of Chebyshev polynomials, obtained the following identity connected to the first kind of Chebyshev polynomials matrix.

The given result holds for integers $n \geq 3$;

$$A_n(2x) = 2T_n(x) + 2(-1)^n.$$

Matrix $A$ have the following eigen values;

$$\lambda_k = cos\left(\frac{(2k-1)\Pi}{n}\right), 1 \leqslant k \leqslant n.$$

Minimal and maximal eigenvalues of $A$;

$$\lambda_{min} = \begin{cases} -2cos\left(\frac{\Pi}{n}\right), if\ n\ is\ even, \\ -2\ , if\ n\ is\ odd. \end{cases}$$

$$\lambda_{max} = 2cos\left(\frac{\Pi}{n}\right).$$

Jerzy Kocik [57] found a matrix representation of the Chebyshev polynomials, Fibonacci series, and Luca's polynomial by using power of a particular matrix. Milica et al. [72] found determinant generalized formula situated on the second kind of Chebyshev polynomials. For this purpose, they utilized a tridiagonal matrix and a Heisenberg matrix.

Ahmet Oteles et al. [78] worked on a family of tridiagonal matrices related to the first kind of Chebyshev polynomials and obtained eigen vectors and eigen values. Abdullah Altın et al. [13] presented recursive equations for the polynomials of Chebyshev matrix, especially for the second kind. They also found generating matrix functions and several identities for this second kind of Chebyshev polynomials. Zoran Pucanovic and Marko Pesovic [80] used the properties of circulant matrices and Chebyshev polynomials. They connected circulant matrices and Chebyshev polynomials.

S. Foud [39] studied a 3rd kind shifted Chebyshev polynomials for operational matrices. F. Chishti et al. [27] studied 4th kind shifted Chebyshev polynomials for operational matrices. Our work is motivated by the earlier work of Primo Brandi et al. [20]. The authors derived the theorems and many identities correlated toward the Chebyshev 1st and 2nd kinds polynomials via $2 \times 2$ matrix.

The outline of this chapter is as follows:

The chapter is mainly divided in to four segments. In the $I^{st}$ segment, we look back at the introduction. In the next segment, we present the determinant representation for the 3rd and 4th kinds and discover characteristic equations up to degree three. In the next segment, we formulated theorems delineating the connections between matrix power, 2nd, 3rd, and 4th kinds Chebyshev polynomials. The concluding part focuses on applications associated with Chebyshev polynomials, particularly their practical utility in approximation theory.

## 2.2 Matrix Representation of Chebyshev Polynomials of Third Kind

We now introduce determinant representation and characteristic equations for the third kind of Chebyshev polynomials:

$$V_0(x) = 1, \qquad V_1(x) = 2x - 1, \qquad V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \qquad \text{for } n \geq 2, 3, \ldots$$

Tri-diagonal matrix, $[b_{q,r}]$ represents a matrix sequence for the third kind Chebyshev polynomials:

$$[b_{q,r}] = \begin{cases} b_{q,r} = ax - 1 & , if\ q = r = 1 \\ b_{q,r} = ax & , if\ q = r \geq 2 \\ b_{q,r} = 1 & , if\ q = r + 1, q = r - 1 \\ b_{q,r} = 0 & , otherwise \end{cases}$$

If $|X(n)|$ denote the determinant of Chebyshev matrices $V_n(x)$ then we have:

$$|X(n)| = \begin{vmatrix} ax - 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & ax & 1 & \cdots & \cdots & 0 \\ 0 & 1 & ax & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & .. \\ \cdots & \cdots & \cdots & \cdots & ax & 1 \\ 0 & 0 & \cdots & \cdots & 1 & ax \end{vmatrix}.$$

When $a = 2$;

$$|\mathrm{X}(n)| = \begin{vmatrix} 2x - 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2x & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 2x & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & .. \\ \cdots & \cdots & \cdots & \cdots & 2x & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 2x \end{vmatrix}.$$

For $n = 1, 2, 3, 4, 5$ etc. the determinant reduces as:

$$|X(1)| = b_{1,1} = 2x - 1 = V_1(x).$$

$$|X(2)| = b_{1,1}b_{2,2} - b_{2,1}b_{1,2} = \begin{vmatrix} 2x-1 & 1 \\ 1 & 2x \end{vmatrix}$$

$$= 4x^2 - 2x - 1 = V_2(x).$$

$$|X(3)| = b_{3,3}|X(2)| - b_{3,2}b_{2,3}|X(1)|$$

$$= 8x^3 - 4x^2 - 4x + 1$$

$$= \begin{vmatrix} 2x-1 & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{vmatrix} = V_3(x).$$

$$|X(4)| = b_{4,4}|X(3)| - b_{4,3}b_{3,4}|X(2)|$$

$$= \begin{vmatrix} 2x-1 & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{vmatrix}$$

$$= 16x^4 - 8x^3 - 12x^2 + 4x + 1 = V_4(x).$$

$$|X(5)| = b_{5,5}|X(4)| - b_{5,4}b_{4,5}|X(3)|$$

$$= \begin{vmatrix} 2x-1 & 1 & 0 & 0 & 0 \\ 1 & 2x & 1 & 0 & 0 \\ 0 & 1 & 2x & 1 & 0 \\ 0 & 0 & 1 & 2x & 1 \\ 0 & 0 & 0 & 1 & 2x \end{vmatrix}$$

$= 32x^5 - 16x^4 - 32x^3 + 12x^2 + 6x - 1 = V_5(x)$, so on.

In general,

$$|X(n)| = b_{n,n}|X(n-1)| - b_{n,n-1}b_{n-1,n}|X(n-2)|,$$

$$= \begin{vmatrix} 2x-1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 2x & 1 & \dots & \dots & 0 \\ 0 & 1 & 2x & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 2x & 1 \\ 0 & 0 & \dots & \dots & 1 & 2x \end{vmatrix} = V_n(x).$$

## 2.3 Characteristic Equations of Third Kind Chebyshev Polynomials

Here we obtain characteristic equations of third kind Chebyshev polynomials up to degree three.

1. $\lambda - V_1 = 0.$

2. $\lambda^2 - (4x-1)\lambda + V_2 = 0.$

3. $\lambda^3 - (6x-1)\lambda^2 + (12x^2 - 4x - 2)\lambda - V_3 = 0.$

## 2.4 Matrix Representation for the Chebyshev Polynomials of Fourth kind

Now we present determinant representations and characteristic equations for the Chebyshev's fourth kind polynomials. The Chebyshev's fourth kind polynomials are defined by a recursive relationship;

$$W_0(x) = 1,\ W_1(x) = 2x + 1,$$
$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x),\ \text{for } n \geq 2,\ 3,\ \dots$$

Tri-diagonal matrix, $[d_{q,r}]$ represents a matrix sequence for the fourth kind Chebyshev polynomials:

$$[d_{q,r}] = \begin{cases} d_{q,r} = ax + 1 & ,if\ q = r = 1 \\ d_{q,r} = ax & ,if\ q = r \geq 2 \\ d_{q,r} = 1\ ,if\ q = r+1\ , q = r-1 & \\ d_{q,r} = 0 & ,otherwise. \end{cases}$$

If $|Y(n)|$ denote the determinant of Chebyshev matrices $W_n(x)$ then we have;

$$|Y(n)| = \begin{vmatrix} ax+1 & 1 & 0 & \dots & \dots & 0 \\ 1 & ax & 1 & \dots & \dots & 0 \\ 0 & 1 & ax & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & .. \\ \dots & \dots & \dots & \dots & ax & 1 \\ 0 & 0 & \dots & \dots & 1 & ax \end{vmatrix}$$

When $a = 2$;

$$|Y(n)| = \begin{vmatrix} 2x+1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 2x & 1 & \dots & \dots & 0 \\ 0 & 1 & 2x & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & .. \\ \dots & \dots & \dots & \dots & 2x & 1 \\ 0 & 0 & \dots & \dots & 1 & 2x \end{vmatrix}.$$

For $n = 1, 2, 3, 4, 5$ etc. the determinant reduces as:

$$|Y(1)| = d_{1,1} = 2x + 1 = W_1(x).$$

$$|Y(2)| = d_{1,1}d_{2,2} - d_{2,1}d_{1,2} = \begin{vmatrix} 2x+1 & 1 \\ 1 & 2x \end{vmatrix},$$

$$= 4x^2 + 2x - 1 = W_2(x).$$

$$|Y(3)| = d_{3,3}|Y(2)| - d_{3,2}d_{2,3}|Y(1)|$$

$$= \begin{vmatrix} 2x+1 & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{vmatrix}$$

$$= 8x^3 + 4x^2 - 4x - 1 = W_3(x).$$

$$|Y(4)| = d_{4,4}|Y(3)| - d_{4,3}d_{3,4}|Y(2)|$$

$$= \begin{vmatrix} 2x+1 & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{vmatrix}$$

$$= 16x^4 + 8x^3 - 12x^2 - 4x + 1 = W_4(x).$$

$$|Y(5)| = d_{5,5}|Y(4)| - d_{5,4}d_{4,5}|Y(3)|$$

$$= \begin{vmatrix} 2x+1 & 1 & 0 & 0 & 0 \\ 1 & 2x & 1 & 0 & 0 \\ 0 & 1 & 2x & 1 & 0 \\ 0 & 0 & 1 & 2x & 1 \\ 0 & 0 & 0 & 1 & 2x \end{vmatrix}$$

$$= 32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1 = \;\; W_5(x), \text{so on.}$$

In general,

$$|Y(n)| = d_{n,n}|Y(n-1)| - d_{n,n-1}d_{n-1,n}|Y(n-2)|,$$

$$= \begin{vmatrix} 2x+1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 2x & 1 & \dots & \dots & 0 \\ 0 & 1 & 2x & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 2p & 1 \\ 0 & 0 & \dots & \dots & 1 & 2p \end{vmatrix} = \;\; W_n(x).$$

## 2.5 Characteristic Equations of Chebyshev Polynomials Fourth kind

Now we determine the characteristic equations up to degree three for the fourth kind Chebyshev polynomials.

1. $\lambda - W_1 = 0.$

2. $\lambda^2 - (4x+1)\lambda + W_2 = 0.$

3. $\lambda^3 - (6x+1)\lambda^2 + (12x^2 + 4x - 2)\lambda - W_3 = 0.$

## 2.6 Matrix Power and Chebyshev Polynomials: Exploring Interrelated Identities.

Here, we prove theorems involving matrix power with Chebyshev's second kind, third kind, and fourth kind polynomials.

Let $A$ be a $2 \times 2$ non-singular matrix;

$$a_1 = \text{trace } A, \;\; a_2 = \text{determinant } A \neq 0,$$

$$u = \left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) = \sqrt{\frac{1+x}{2}},$$

and the characteristic equation is;

$$\lambda^2 - a_1\lambda + a_2 = 0.$$

Here **λ** denotes the eigen values of $A$.

## 2.7 Relation Between Matrix Power and Third kind Chebyshev Polynomials

In this section, we examine the outcome linking the trace of matrix powers to the first and second kinds, as demonstrated in [20]. We also used the identity that gave a relationship between the first kind and third kind of Chebyshev polynomials.

**Theorem 1:** For any integer $n \geq 0$, it follows:

$$\text{trace } A^{2n+1} = 2ua_2^{\frac{2n+1}{2}} V_n(x).$$

$V_n(x)$ denotes the $n^{th}$degree for the third kind Chebyshev polynomials;

$$V_0(x) = 1, V_1(x) = 2x - 1,$$

$$V_2(x) = 4x^2 - 2x - 1,$$

$$V_3(x) = 8x^3 - 4x^2 - 4x + 1, \ldots$$

**Proof.** Since trace $A = a_1$, trace $A^2 = {a_1}^2 - 2a_2$. Using the formulas given by Newton–Girard, we can find the following;

$$\text{trace } A^n = a_1(\text{trace } A^{n-1}) - a_2(\text{trace } A^{n-2}).$$

Now we used the result obtained in [7],

$$\text{trace } A^n = 2a_2^{\frac{n}{2}} T_n\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right).$$

Put $n = 2n + 1$ in the above equation,

$$\text{trace } A^{2n+1} = 2a_2^{\frac{2n+1}{2}} T_{2n+1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right).$$

Using the identity in the above equation;

$$"V_n(x) = u^{-1} T_{2n+1}(u),"$$

$$\text{where } u = \left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) = \sqrt{\frac{1+x}{2}}.$$

We get,

$$\text{trace } A^{2n+1} = 2ua_2^{\frac{2n+1}{2}} V_n(x).$$

Put $n = 0$,

$$\text{trace } A^1 = 2ua_2^{\frac{1}{2}} V_0(x).$$

Put $n = 1$,

$$\text{trace } A^3 = 2ua_2^{\frac{3}{2}} V_1(x).$$

Put $n = 2$,

$$\text{trace } A^5 = 2ua_2^{\frac{5}{2}} V_2(x).$$

**Example 2**: To verify the result stated in theorem 1:

$$\text{trace } A^{2n+1} = 2ua_2^{\frac{2n+1}{2}} V_n(x).$$

**Proof:** The theorem presents the connection between trace of matrix and Chebyshev third kind polynomials. $V_n(x)$, represents $n^{th}$degree Chebyshev third kind polynomials. We demonstrated our theorem using a randomly chosen non-singular $2 \times 2$ matrix for values of $n = 0, 1, 2, \dots$

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ be a $2 \times 2$ nonsingular matrix.

$$\text{trace of matrix } A = a_1 = 9,$$

$$\text{determinant of matrix } A = a_2 = 2,$$

$$u = \left( \frac{a_1}{2a_2^{\frac{1}{2}}} \right) = \frac{9}{2\sqrt{2}} \text{ ,}$$

Hence,

$$x = \frac{77}{4}.$$

Put $n = 0$,

$$\text{trace } A^1 = 2ua_2^{\frac{1}{2}} V_0(x),$$

$$\text{L. H. S.} = \text{trace } A^1 = a_1 = 9.$$

$$\text{R. H. S.} = 2ua_2^{\frac{1}{2}} V_0(x) = 2 \times \frac{9}{2\sqrt{2}} \times 2^{\frac{1}{2}} \times 1 = 9.$$

Hence result is true for $n = 0$.

Put $n = 1$,

$$\text{trace } A^3 = 2ua_2^{\frac{3}{2}} V_1(x).$$

$$A^3 = \begin{bmatrix} 61 & 158 \\ 237 & 614 \end{bmatrix}.$$

$$\text{L. H. S.} = \text{trace } A^3 = 675.$$

$$\text{R. H. S.} = 2ua_2^{\frac{3}{2}}V_1(x) = 2 \times \frac{9}{2\sqrt{2}} \times 2^{\frac{3}{2}} \times \frac{75}{2} = 675.$$

Hence result is true for $n = 1$.

Put $n = 2$,

$$\text{trace } A^5 = 2ua_2^{\frac{5}{2}}V_2(x).$$

$$A^5 = \begin{bmatrix} 4693 & 12158 \\ 18237 & 47246 \end{bmatrix}.$$

$$\text{L. H. S.} = \text{ trace } A^5 = 51939.$$

$$\text{R. H. S.} = 2ua_2^{\frac{5}{2}}V_2(x) = 2 \times \frac{9}{2\sqrt{2}} \times 2^{\frac{5}{2}} \times \frac{5771}{4} = 51939.$$

Hence result is true for $n = 2$.

This example verified the above result for $n = 0, 1, 2$.

Similarly, we can prove our result for $n = 3, 4, 5, \ldots$

Hence, this result is hold for any non-singular matrix $A$ i.e.

$$\text{trace } A^{2n+1} = 2ua_2^{\frac{2n+1}{2}}V_n(x).$$

## 2.8 Relation Between Second, Fourth Kind of Chebyshev Polynomials and Matrix Power

In this context, we start by establishing a connection, as outlined in [20], between the matrix exponent and the Chebyshev polynomials of the 2nd kind. Furthermore, here we utilize an identity that elucidates the relationship allying Chebyshev 2nd and 4th kinds polynomials i.e.

$$A^n = a_2^{\frac{(n-1)}{2}}U_{n-1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right)A - a_2^{\frac{n}{2}}U_{n-2}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right)I.$$

Now use the identity that connects the second and fourth kinds Chebyshev polynomials with each other i.e.

$$U_{2n}(u) = W_n(x), \text{ where } u = \sqrt{\frac{1+x}{2}}.$$

**Theorem 3:** Let $n \geq 2,$ the integral power of $A$ is given by:

$$A^{2n+1} = a_2{}^n W_n(x) A - a_2{}^{\frac{2n+1}{2}} U_{2n-1}\left(\frac{a_1}{2a_2{}^{\frac{1}{2}}}\right) I\ ,$$

where $I$ denotes the identity matrix, $U_n(x)$ and $W_n(x)$ are the $n^{th}$ degree second and fourth kinds Chebyshev polynomials.

$$U_0(x) = 1, \qquad U_1(x) = 2x,\ U_2(x) = 4x^2 - 1, \qquad U_3(x) = 8x^2 - 4x, \ldots$$

$W_0(x) = 1,\ W_1(x) = 2x + 1, \quad W_2(x) = 4x^2 + 2x - 1,\ W_3(x) = 8x^3 + 4x^2 - 4x - 1, \ldots$

**Proof:** By using the above-mentioned results, we obtain,

$$A^{2n+1} = a_2{}^n U_{2n}\left(\frac{a_1}{2a_2{}^{\frac{1}{2}}}\right) A - a_2{}^{\frac{2n+1}{2}} U_{2n-1}\left(\frac{a_1}{2a_2{}^{\frac{1}{2}}}\right) I,$$

$$A^{2n+1} = a_2{}^n W_n(x) A - a_2{}^{\frac{2n+1}{2}} U_{2n-1}\left(\frac{a_1}{2a_2{}^{\frac{1}{2}}}\right) I.$$

For $n = 1,$

$$A^3 = a_2 W_1(x) A - a_2{}^{\frac{3}{2}} U_1\left(\frac{a_1}{2a_2{}^{\frac{1}{2}}}\right) I.$$

For $n = 2,$

$$A^5 = {a_2}^2 W_2(x)A - {a_2}^{\frac{5}{2}} U_3\left(\frac{a_1}{2{a_2}^{\frac{1}{2}}}\right) I.$$

For $n = 3$,

$$A^7 = {a_2}^3 W_3(x)A - {a_2}^{\frac{7}{2}} U_5\left(\frac{a_1}{2{a_2}^{\frac{1}{2}}}\right) I.$$

**Example 4:** To verify the result stated in theorem 3:

$$A^{2n+1} = {a_2}^n W_n(x)A - {a_2}^{\frac{2n+1}{2}} U_{2n-1}\left(\frac{a_1}{2{a_2}^{\frac{1}{2}}}\right) I.$$

**Proof:** The theorem presents the connection between matrix power, 2nd and 4th kind Chebyshev polynomials. The $U_n(x)$, $W_n(x)$ represents the $n^{th}$ degree Chebyshev 2nd kind and 4th kinds polynomials respectively. We have proved our theorem with a random non-singular $2 \times 2$ matrix for $n = 0, 1, 2, \dots$

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ be a $2 \times 2$ non-singular matrix.

$$\text{trace} = a_1 = 9.$$

$$\text{determinant} = a_2 = 2.$$

$$u = \left(\frac{a_1}{2{a_2}^{\frac{1}{2}}}\right) = \sqrt{\frac{1+x}{2}} = \frac{9}{2\sqrt{2}} \ .$$

Hence,

$$x = \frac{77}{4}.$$

Put $n = 1$, we get

$$A^3 = a_2 W_1(x)A - {a_2}^{\frac{3}{2}} U_1\left(\frac{a_1}{2{a_2}^{\frac{1}{2}}}\right) I.$$

$$\text{L.H.S.} = A^3 = \begin{bmatrix} 61 & 158 \\ 237 & 614 \end{bmatrix}.$$

$$\text{R. H. S.} = a_2 W_1(x) A - a_2^{\frac{3}{2}} U_1 \left( \frac{a_1}{2a_2^{\frac{1}{2}}} \right) I,$$

$$= 2 \times \frac{79}{2} \times \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} - 2^{\frac{3}{2}} \times U_1 \left( \frac{9}{2\sqrt{2}} \right) \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 79 & 158 \\ 237 & 632 \end{bmatrix} - 18 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 61 & 158 \\ 237 & 614 \end{bmatrix}.$$

Hence result is true for $n = 1$.

Put $n = 2$, we get

$$A^5 = a_2{}^2 W_2(x) A - a_2^{\frac{5}{2}} U_3 \left( \frac{a_1}{2a_2^{\frac{1}{2}}} \right) I.$$

$$\text{L. H. S.} = A^5 = \begin{bmatrix} 4693 & 12158 \\ 18237 & 47246 \end{bmatrix}.$$

$$\text{R.H.S.} = a_2{}^2 W_2(x) A - a_2^{\frac{5}{2}} U_3 \left( \frac{a_1}{2a_2^{\frac{1}{2}}} \right) I,$$

$$= 4 \times \frac{6079}{4} \times \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} - 2^{\frac{5}{2}} \times U_3 \left( \frac{9}{2\sqrt{2}} \right) \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6079 & 12158 \\ 18237 & 48632 \end{bmatrix} - 1386 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4693 & 12158 \\ 18237 & 47246 \end{bmatrix}.$$

Hence result is true for $n = 2$.

This example verified the above result for $n = 1, 2$.

Similarly, we can prove our result for $n = 3, 4, 5, ...$

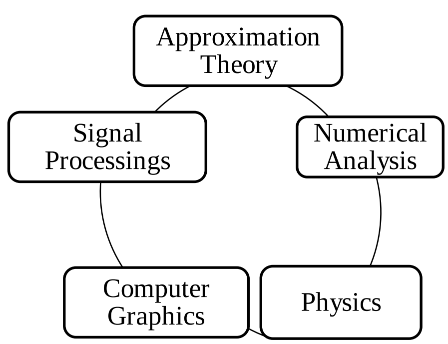
Hence, this result is hold for any nonsingular matrix $A$ i.e.

$$A^{2n+1} = {a_2}^n W_n(x)A - {a_2}^{\frac{2n+1}{2}} U_{2n-1}\left(\frac{a_1}{2{a_2}^{\frac{1}{2}}}\right) I.$$

This example verified the above result.

## 2.9 Applications

The Chebyshev polynomials are widely used to enhance the advanced technique for counting and to study the integer function. These polynomials play a vital role to solve other polynomials that are used to obtain new trigonometric identities. The approximate solution of the second order differential equations can be obtained with the help of these polynomials and the large data can be interpolated in the numerical and approximation theory. The approximate numerical solution can be derived for both differential and integral equations. These polynomials play a crucial role in computer science to obtain signal processing, where they are prominently employed in designing filters referred to as Chebyshev filters. Chebyshev polynomials are extensively utilized in computer graphics for generating various shapes, surfaces, and curves due to their versatility and effectiveness.




We can approximate every polynomial through the use of Chebyshev polynomials. In this context, we represent a polynomial using third kind Chebyshev polynomials, demonstrating the practical utility in approximation theory. The primary benefit of the given methods is the high accuracy of the approximation solution.

## Practical Application

**Express:** $f(x) = x^4 + 2x^3 - 3x - 2$ through the use of Chebyshev polynomials of 3rd kind and approximate the function by a cubic polynomial $p(x) \in P_3$ using economization.

**Solution:** First four terms of third kind Chebyshev polynomials are;

$$V_0(x) = 1\,, \qquad V_1(x) = 2x - 1, \qquad V_2(x) = 4x^2 - 2x - 1,$$

$$V_3(x) = 8x^3 - 4x^2 - 4x + 1, \qquad V_4(x) = 16x^4 - 8x^3 - 12x^2 + 4x + 1.$$

From above equations:

$$x = \frac{1}{2}[V_0(x) + V_1(x)].$$

$$x^2 = \frac{1}{4}[2V_0(x) + V_1(x) + V_2(x)].$$

$$x^3 = \frac{1}{8}[V_3(x) + V_2(x) + 3V_1(x) + 3V_0(x)].$$

$$x^4 = \frac{1}{16}[V_4(x) + V_3(x) + 4V_2(x) - 4V_1(x) + 6V_0(x)].$$

Put above values in $f(x) = x^4 + 2x^3 - 3x - 2$, we get

$$x^4 + 2x^3 - 3x - 2 = \tfrac{1}{16}[V_4(x) + V_3(x) + 4V_2(x) - 4V_1(x) + 6V_0(x)] + \tfrac{2}{8}[V_3(x) + V_2(x) + 3V_1(x) + 3V_0(x)] - \tfrac{3}{2}[V_0(x) + V_1(x)] - 2V_0(x).$$

Hence,

$$= \frac{1}{16}V_4(x) + \frac{5}{16}V_3(x) + \frac{1}{2}V_2(x) - V_1(x) - \frac{19}{8}V_0(x).$$

Here we want a cubic approximation, so we drop the $V_4$ term. This gives an error at most $\frac{1}{16}$. The best approximated polynomial is;

$$p(x) = \frac{5}{2}x^3 + \frac{3}{4}x^2 - 3x - \frac{35}{16}.$$

If we do simple approximation, we get

$$r(x) = 2x^3 - 3x - 2,$$

we would get a maximum error equal to 1.

## 2.10 Significance of the Work

Here we recapitulate the significance of the present chapter in the following points:

- Introducing matrix representation of Chebyshev polynomials of the 3rd kind and 4th kind.
- Obtaining characteristic equations of 3rd kind, and 4th kind Chebyshev polynomials.
- Identities associated to matrix power and Chebyshev polynomials.
- Obtaining relation between the trace of matrix and 3rd kind Chebyshev polynomials with example.
- Establishing relationships involving the 2nd and 4th kind of Chebyshev polynomials through matrix exponentiation.
- Approximate a polynomial by using the Chebyshev 3rd kind polynomials.

It has to be noted here that the above-obtained results and discussions are helpful. A few of their presumed uses are given below:

- The matrix representation helps to solve and acquiring algebraic outcomes for both linear and non-linear differential equations.
- The characteristic equations of Chebyshev polynomials are helpful to obtaining eigen values and eigen vectors.
- The connections between 2nd, 3rd, and 4th kind of Chebyshev polynomials with matrix power are very fruitful to obtaining identities related to them.

- The Chebyshev polynomials have a notable significance in the field of approximation theory.

## 2.11 Conclusion

It is observed that characteristic equations for Chebyshev polynomials of 3rd and 4th kind can be extended up to $n^{th}$ degree. Further, by using matrix power, we can describe more identities that connect Chebyshev polynomials. To obtain our result, we used the non-singular $2 \times 2$ matrix and utilized the trace and determinant of the matrix. The numerical examples authenticate the theoretical results.

## 2.12 Open problems

- The problems concerning invertibility, designation of eigenvalues and determinants, and other issues on -circulant matrices involving the Chebyshev polynomials will remain open for further research.
- Another open problem is the calculation of the inverse of a matrix formed by Chebyshev polynomials.
- The calculation of eigenvalues of a matrix formed by Chebyshev polynomials is also an open problem, and it is crucial for understanding the properties of these matrices.

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# Chapter 3

# New Generalization of Chebyshev-Like Polynomials and Their Applications

The study of this chapter focused on the development of a new generalized version of four known kinds of Chebyshev polynomials. Our investigation involves the development of a novel generalization encompassing four widely recognized types of Chebyshev polynomials. We have introduced distinct generalized Chebyshev polynomials by employing a modified recursive relationship with diverse initial conditions. We also get Binet's formula for generalized Chebyshev's polynomials. The Binet formula is obtained by mathematical induction. The matrix representation and the characteristic equation are presented using matrix algebra properties for these polynomials. We also talked about the sum, products, and subtraction of roots pertaining of the C.E. of generalized Chebyshev polynomials. Plus, we showed how Chebyshev-like polynomials can be used in practice with examples.

## 3.1 Introduction

Chebyshev polynomials occupy prominent attention because of their substantial use in mathematics. The authors Gultekin and Betul Sakiroglu, conducted a study on the analysis of Chebyshev generalized polynomials forms using matrixes and combination forms [ 43]. M. C. Akmak and K. Uslu, developed a generalized version of all four Chebyshev polynomials, demonstrated a Binet-style formula [9]. Generalized sequences $F_n$ and $L_n$ by Goksal Bilgici on generalizing new sequences was based on the relationship between the recurrence relation with the basic conditions;

$$F_n = 2xF_{n-1} + (b^2 - a)F_{n-2},\ n \geq 2 \text{ and}$$

$$L_n = 2xL_{n-1} + (b^2 - a)L_{n-2},\ n \geq 2.$$

They successfully derived Binet's formula and the generating function for these sequences [19]. Abd-Elhameed and Al-Harbi, primarily concerned with the generalization of Chebyshev's third-kind polynomials, with contributions from different perspectives. Additionally, some new formulas were discussed [3]. In order to gain new insights into the properties of Lucas-polynomials, W. Mohamed Abd-Elhameed and A. Napoli explored different approaches to obtaining results. Matrix representation was also discussed in order to identify certain properties of the polynomials [2].

S. Uygun et al. the authors proposed a generalized version of some of the polynomial names; Pell Lucas and Pell, Vieta and Vieta. They identified properties including a sum formula, generating function, differentiation, and Binet like formula as well as generating a matrix whose values were extracted from a generalized version of the Vieta- Pell -Lucas's polynomials [96]. Gospava B. Djordjevi´c, conducted a series of studies on the various categories of polynomials associated with Chebyshev's polynomials and the derived results associated with them [31].

Kizilates et al. instigates $(g, f)$ Chebyshev polynomials in to Fibonacci, Luca's polynomials. For any integer $l \geq 2$ and $0 < g < f \leq 1$:

$$T_l(n, d, g, f) = (g^{l-1} + f^{l-1})nT_{l-1}(n, d, g, f) + (ef)^{l-1}dT_{l-2}(n, d, g, f),$$

where $n, d$ are real variables, $(n, d, g, f) = 1, T_1(n, d, g, f) = n$. They also talked about $n^{th}$ generalizations and properties of derivatives, which were represented by determinants of the polynomials [59]. Waleed Mohamed Abd-Elhameed et al. objective of the paper was to evolve the connection between generalized types of Lucas and Fibonacci polynomials. [4]. Stefano De Marchi et al. familiarize themselves with the generalizations of the first kind Chebyshev polynomials and identify a number of properties associated with orthogonal polynomials [67].

Sarita Nemaniy et al. in their study, established the sophisticated properties of the Fibonacci sequence. Their findings concerned the divisibility of Fibonacci sequences and the representation of matrices by determinants including sequence terms [74]. The main intension of Anna Tatarczak study was to generalize the Chebyshev polynomials of two distinct types and to presented some prominent results demonstrating the relationship between these two types [94]. Anam Alwan Salih and Suha Shihab, primary objective of the research was to identify a variant of Chebyshev polynomials. $N_n(x)$ presented the $n^{th}$ modified version of Chebyshev polynomials.

$$N_n(x) = 2T_n\left(\frac{x}{2}\right), n \in N.$$

Furthermore, authors discussed their integration, derivative operational matrix, and estimation techniques to address the issue of optimal control [85]. Various modifications have been made to the Fibonacci sequence and the Lucas sequence, in some cases by maintaining the original conditions and in other cases by maintaining the recurrence relationship by M. Musraini et al. [73]. The main goals of the chapter outlined is below:

We have introduced an innovative extension of the Chebyshev polynomials, broadening their applicability and characteristics. We have addressed the determinant representation of this generalized version with its characteristic equations, as well as the Binet-like formulas and the practical application of generalized polynomials in the approximation of the functions.

## 3.2 New Generalization of Chebyshev-Like Polynomials

For $n \geq 2$,

$$R_n(x) = uxR_{n-1}(x) + vxR_{n-2}(x).$$

with initial condition;

$$R_0(x) = 1, \qquad R_1(x) = rx - s.$$

where $r, v, u, s$ are integers.

The following are the few terms of $R_n(x)$ for $n = 2, 3, 4, 5, ...$

$$R_2(x) = ux(rx - s) + vx = urx^2 - uxs + vx.$$

$$R_3(x) = ux[ux(rx - s) + vx] + vx[rx - s]$$

$$= u^2rx^3 - u^2sx^2 + uvx^2 + vrx^2 - vsx.$$

$$R_4(x) = ux[u^2rx^3 - u^2sx^2 + uvx^2 + vrx^2 - vsx] + vx[urx^2 - uxs + vx]$$

$$= u^3rx^4 - u^3sx^3 + u^2vx^3 + uvrx^3 - uvsx^2 + uvrx^3 - uvsx^2 + v^2x^2.$$

$$R_5(x) = ux[u^3rx^4 - u^3sx^3 + u^2vx^3 + uvrx^3 - uvsx^2 + uvrx^3 - uvsx^2 + v^2x^2] + vx[u^2rx^3 - u^2sx^2 + uvx^2 + vrx^2 - vsx]$$

$$= u^4rx^5 - u^4sx^4 + u^3vx^4 + u^2vrx^4 - u^2vsx^3 + u^2vrx^4 - u^2vsx^3 + uv^2x^3 + u^2vrx^4 - u^2vsx^3 + uv^2x^3 + v^2rx^3 - v^2sx^2,$$

and so on …

Characteristic equation of the generalized Chebyshev polynomials is:

$$E^n = uxE^{n-1} + vxE^{n-2},$$

$$E^2 = uxE + vx,$$

$$E^2 - uxE - vx = 0.$$

$I_1(x), I_2(x)$ denote the roots of the above equation;

$$I_1(x) = \frac{ux + \sqrt{u^2x^2 + 4vx}}{2}, \qquad I_2(x) = \frac{ux - \sqrt{u^2x^2 + 4vx}}{2}.$$

Sum of the roots is;

$$I_1(x) + I_2(x) = ux.$$

Product of roots is;

$$I_1(x)\, I_2(x) = -vx.$$

Subtraction of roots is;

$$I_1(x) - I_2(x) = \sqrt{u^2x^2 + 4vx}.$$

Sum of the squares of the roots is;

$$I_1^{\,2}(x) + I_2^{\,2}(x) = u^2x^2 + 2vx.$$

## 3.3 Binet Formula for Chebyshev-like Polynomials

To derive the Binet formula for Chebyshev-like polynomials, we start by solving the characteristic equation to find its roots. These roots are crucial for constructing the Binet formula, which provides an explicit expression for generalized Chebyshev-like polynomial.

Let general solution of above equation is;

$$R_n(x) = Z_1 I_1^{\,n+1}(x) + Z_2 I_2^{\,n+1}(x), \tag{3.1.1}$$

To find $Z_1$, $Z_2$;

$$1 = I_1(x),$$

$$rx - s = I_2(x),$$

$$1 = Z_1 I_1(x) + Z_2 I_2(x), \tag{3.1.2}$$

$$rx - s = Z_1 I_1^{\,2}(x) + Z_2 I_2^{\,2}(x). \tag{3.1.3}$$

Multiply equation (3.1.2) by $I_1(x)$,

$$I_1(x) = Z_1 I_1(x) I_1(x) + Z_2 I_1(x) I_2(x). \tag{3.1.4}$$

Now subtract from (3.1.4) to (3.1.3), we have

$$I_1(x) - (rx - s) = Z_2\left[I_1(x) I_2(x) - I_2^{\,2}(x)\right],$$

$$Z_2 = \frac{I_1(x) - (rx - s)}{[I_1(x) I_2(x) - I_2^{\,2}(x)]},$$

$$Z_2 = \frac{I_1(x) - (rx - s)}{I_2(x)[I_1(x) - I_2(x)]}.$$

Now use the value of $Z_2$ in eq (3.1.2), we get

$$1 = Z_1 I_1(x) + \frac{I_1(x) - (rx - s)}{I_2(x)[I_1(x) - I_2(x)]} I_2(x),$$

$$1 = Z_1 I_1(x) + \frac{I_1(x) - (rx - s)}{[I_1(x) - I_2(x)]},$$

$$Z_1 I_1(x) = 1 - \frac{I_1(x) - (rx - s)}{[I_1(x) - I_2(x)]},$$

$$Z_1 = \frac{(rx - s) - I_2(x)}{I_1(x)[I_1(x) - I_2(x)]}.$$

Now use the values of $Z_1, Z_2$ in (3.1.1), we get;

$$R_n(x) = \frac{(rx - s) - I_2(x)}{I_1(x)[I_1(x) - I_2(x)]} {I_1}^{n+1}(x) + \frac{I_1(x) - (rx - s)}{I_2(x)[I_1(x) - I_2(x)]} {I_2}^{n+1}(x).$$

Now further its solutions can be modified;

$$R_n(x) = \frac{(rx - s) - I_2(x)}{[I_1(x) - I_2(x)]} {I_1}^{n}(x) + \frac{I_1(x) - (rx - s)}{[I_1(x) - I_2(x)]} {I_2}^{n}(x),$$

$$(rx - s) - I_2(x) = \frac{2rx - 2s - ux + \sqrt{u^2x^2 + 4vx},}{2}$$

$$I_1(x) - I_2(x) = \sqrt{u^2x^2 + 4vx},$$

$$I_1(x) - (rx - s) = \frac{ux - 2rx + 2s + \sqrt{u^2x^2 + 4vx}}{2},$$

Hence,

$$R_n(x) = \frac{\frac{2rx - 2s - ux + \sqrt{u^2x^2 + 4vx}}{2}}{\sqrt{u^2x^2 + 4vx},} {I_1}^{n}(x) + \frac{\frac{ux - 2rx + 2s + \sqrt{u^2x^2 + 4vx}}{2}}{\sqrt{u^2x^2 + 4vx}} {I_2}^{n}(x),$$

$$R_n(x) = \frac{1}{2\sqrt{u^2x^2+4vx}}\Big[\Big(2rx-2s-ux+\sqrt{u^2x^2+4vx}\Big)I_1{}^n(x) + \Big(ux-2rx+2s+\sqrt{u^2x^2+4vx}\Big)I_2{}^n(x)\Big].$$

If we put $H = \sqrt{u^2x^2+4vx}$, we get

$$R_n(x) = \frac{1}{2H}[(2rx-2s-ux+H)I_1{}^n(x) + (ux-2rx+2s+H)I_2{}^n(x)].$$

Which is the required Binet formula of Chebyshev - like polynomials. Binet formula is an explicit formula used to find the $n^{th}$ term of Chebyshev polynomials; provide a closed form expression for Chebyshev polynomials. The recurrence relation for Chebyshev polynomials can be derived by utilizing Binet formula.

## 3.4 Generalized Chebyshev's Polynomials Through Matrix Representation

In this section, we delve in to matrix-oriented methodology for generalized Chebyshev polynomials (GCPs). The definition of these polynomials is established through a recursive relation,

$$R_0(x) = 1,\ R_1(x) = rx - s,\ R_n(x) = uxR_{n-1}(x) + vxR_{n-2}(x),\ \text{for } n \geq 2.$$

$[o_{b,c}]$ defined tri-diagonal matrix succession for the generalized Chebyshev polynomials;

$$[o_{b,c}] = \begin{cases} o_{b,c} = rx - s & if\ b = c = 1 \\ o_{b,c} = ux & if\ b = c \geq 2 \\ o_{b,c} = -vx & if\ b = c + 1 \\ o_{b,c} = 1 & if\ b = c - 1 \\ o_{b,c} = 0 & otherwise. \end{cases}$$

In general, determinant representation is given by;

$$|K(n)| = \begin{vmatrix} rx-s & 1 & 0 & \dots & \dots & 0 \\ -vx & ux & 1 & \dots & \dots & 0 \\ 0 & -vx & ux & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & .. \\ \dots & \dots & \dots & \dots & ux & 1 \\ 0 & 0 & \dots & \dots & -vx & ux \end{vmatrix},$$

$K(n) =$ determinant of Chebyshev matrices $R_n(x)$.

$$|K(1)| = o_{1,1} = rx - s = R_1(x).$$

$$|K(2)| = o_{1,1}o_{2,2} - o_{2,1}o_{1,2}$$

$$= \begin{vmatrix} rx-s & 1 \\ -vx & ux \end{vmatrix}$$

$$= urx^2 - uxs + vx = R_2(x).$$

$$|K(3)| = o_{3,3}|K(2)| - o_{3,2}o_{2,3}|K(1)|,$$

$$= \begin{vmatrix} rx-s & 1 & 0 \\ -vx & ux & 1 \\ 0 & -vx & ux \end{vmatrix},$$

$$ru^2x^3 + rvx^2 + uvx^2 - u^2x^2s - svx = R_3(x).$$

$$|K(4)| = o_{4,4}|K(3)| - o_{4,3}o_{3,4}|K(2)|,$$

$$= \begin{vmatrix} rx-s & 1 & 0 & 0 \\ -vx & ux & 1 & 0 \\ 0 & -vx & ux & 1 \\ 0 & 0 & -vx & ux \end{vmatrix},$$

$$= u^3rx^4 - u^3sx^3 + u^2vx^3 + uvrx^3 - uvsx^2 + uvrx^3 - uvsx^2 + v^2x^2 = R_4(x).$$

$$|K(5)| = o_{5,5}|K(4)| - o_{5,4}o_{4,5}|K(3)|,$$

$$= \begin{vmatrix} rx-s & 1 & 0 & 0 & 0 \\ -vx & ux & 1 & 0 & 0 \\ 0 & -vx & ux & 1 & 0 \\ 0 & 0 & -vx & ux & 1 \\ 0 & 0 & 0 & -vx & ux \end{vmatrix}$$

$$= u^4rx^5 - u^4sx^4 + u^3vx^4 + u^2vrx^4 - u^2vsx^3 + u^2vrx^4 - u^2vsx^3 + uv^2x^3 + u^2vrx^4 - u^2vsx^3 + uv^2x^3 + v^2rx^3 - v^2sx^2 = R_5(x).$$

and so on.

In general,

$$|K(n)| = o_{n,n}|K(n-1)| - o_{n,n-1}o_{n-1,n}|k(n-2)|,$$

$$= \begin{vmatrix} rx-s & 1 & 0 & \dots & \dots & 0 \\ -vx & ux & 1 & \dots & \dots & 0 \\ 0 & -vx & ux & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & ux & 1 \\ 0 & 0 & \dots & \dots & -vx & ux \end{vmatrix} = R_n(x).$$

## 3.5 Characteristic Equations of the Generalized Chebyshev Polynomials

Here we obtain the characteristic equations for the generalized Chebyshev polynomials up to fifth degree.

1. $\lambda - R_1 = 0.$
2. $\lambda^2 - (rx + ux)\lambda + R_2 = 0.$
3. $\lambda^3 - (rx - s + 2ux)\lambda^2 - (rux^2 + 2sux - u^2x^2 - 2vx)\lambda - R_3 = 0.$
4. $\lambda^4 + \lambda^3(-rx + s - 3ux) + \lambda^2(3ux^2r - 3sux + 2ux^2 + u^2x^2 + 2vx + ux) + \lambda(-3u^2x^3r + 3u^2x^3s + vxs + uxs - u^3x^3 - 4uvx^2 - rvx^2 - ux^2r) + R_4 = 0.$
5. $\lambda^5 + \lambda^4(4ux + rx - s) - \lambda^3(-4usx^2 + 4vxs - 6x^2u^2 - vs - 2vx - ux) - \lambda^2(6u^2x^3r + 2vx^2r - 6u^2x^2s - 2vxs + 4u^3x^3 + 8uvx^2 + u^2x^2 + ux^2r - uxs) - \lambda(-4u^3x^4r - 4uvx^3r - u^2x^3r + 3u^3x^3s + 5uvx^2s + u^2x^2s - u^4x^4 - 4vu^2x^3 - 3v^2x^2 - uvx^4 + u^3x^3 - u^2x^2v) - R_5 = 0.$

## 3.6 Practical Applications

**Express:** $x^3 - 3x^2 + 2x + 3$ in to generalized Chebyshev polynomials and approximate this function with a quadratic polynomial by using economization.

**Sol:** Generalized Chebyshev polynomials are well-defined by;

$$R_n(x) = uxR_{n-1}(x) + vxR_{n-2}(x), n \geq 2.$$

with initial conditions,

$$R_0(x) = 1, \qquad R_1(x) = rx - s.$$

Now from above generalized recurrence relation, we get

$$R_0(x) = 1.$$

$$R_1(x) = rx - s.$$

$$R_2(x) = urx^2 - uxs + vx.$$

$$R_3(x) = u^2rx^3 - u^2sx^2 + uvx^2 + vrx^2 - vsx.$$

Where $r, s, u, v$ are integers.

From above series we have to find out the $x,\ x^2,\ x^3$.

$$x = \frac{1}{r}[R_1(x) + s].$$

$$x^2 = \frac{1}{ur^2}[R_2(x)r + suR_1(x) + us^2 - vR_1(x) - vs].$$

$x^3 = \frac{1}{u^3r^3}[r^2uR_3(x) + u^2rsR_2(x) + u^3s^2R_1(x)r + u^3s^2 - u^2vsR_1(x) - u^2vs^2 - uvrR_2(x) - u^2vsR_1(x) - u^2vs^2 + uv^2R_1(x) + uv^2s - vR_2(x)r^2 - uvsrR_1(x) - vrus^2 + v^2rR_1(x) + v^2rs + uvsrR_1(x) + uvrs^2].$

Put the above values of $x, x^2, x^3$ in the given polynomial, we get

$$\frac{1}{u^3r^3}[r^2uR_3(x)+u^2rsR_2(x)+u^3s^2R_1(x)r+u^3s^2-u^2vsR_1(x)-u^2vs^2 \\ -uvrR_2(x)-u^2vsR_1(x)-u^2vs^2+uv^2R_1(x)+uv^2s-vR_2(x)r^2 \\ -uvsrR_1(x)-vrus^2+v^2rR_1(x)+v^2rs+uvsrR_1(x)+uvrs^2] \\ -\frac{3}{ur^2}[R_2(x)r+suR_1(x)+us^2-vR_1(x)-vs]+\frac{2}{r}[R_1(x)+s] \\ +3R_0(x).$$

Here we drop the $R_3$ term we get a maximum approximation error at most $\frac{1}{u^2r}$.

If $r=2, s=1, u=2, v=1$, $E=\frac{1}{u^2r}=\frac{1}{8}$. if we do simple approximation, we would get a maximum error equal to 1. The best approximated polynomial through generalized Chebyshev like polynomial is:

$$q(x)=\frac{19}{4}-\frac{5}{8}x-6x^2.$$

If we do simple approximation, we get

$$r(x)=3+2x-3x^2.$$

**Express:** $7x^3-2x^2+1$ in to generalized Chebyshev polynomials and approximate with a quadratic polynomial by using economization.

**Solution:** We know that,

$$R_0(x)=1.$$

$$R_1(x)=rx-s.$$

$$R_2(x)=urx^2-uxs+vx.$$

$$R_3(x)=u^2rx^3-u^2sx^2+uvx^2+vrx^2-vsx.$$

Where $r, s, u, v$ are integers.

From above, we get the values of $x,\ x^2,\ x^3\ i.e.,$

$$x = \frac{1}{r}[R_1(x) + s].$$

$$x^2 = \frac{1}{ur^2}[R_2(x)r + suR_1(x) + us^2 - vR_1(x) - vs].$$

$$x^3 = \frac{1}{u^3r^3}[r^2uR_3(x) + u^2rsR_2(x) + u^3s^2R_1(x)r + u^3s^2 - u^2vsR_1(x) - u^2vs^2 - uvrR_2(x) - u^2vsR_1(x) - u^2vs^2 + uv^2R_1(x) + uv^2s - vR_2(x)r^2 - uvsrR_1(x) - vrus^2 + v^2rR_1(x) + v^2rs + uvsrR_1(x) + uvrs^2].$$

Put the above values of $x, x^2, x^3$ in the given polynomial, we get

$$\frac{7}{u^3r^3}[r^2uR_3(x) + u^2rsR_2(x) + u^3s^2R_1(x)r + u^3s^2 - u^2vsR_1(x) - u^2vs^2 - uvrR_2(x) - u^2vsR_1(x) - u^2vs^2 + uv^2R_1(x) + uv^2s - vR_2(x)r^2 - uvsrR_1(x) - vrus^2 + v^2rR_1(x) + v^2rs + uvsrR_1(x) + uvrs^2] - \frac{2}{ur^2}[R_2(x)r + suR_1(x) + us^2 - vR_1(x) - vs] + R_0(x)$$

Here we drop the $R_3$ term, we get a maximum approximation error at most $\frac{7}{u^2r}$.

If $r = 2, s = 1, u = 2, v = 1$, $E = \frac{7}{u^2r} = \frac{7}{8} = 0.875$. if we do simple approximation, we would get a maximum error equal to 7.

The best approximated polynomial through generalized Chebyshev like polynomial is:

$$q(x) = 1 + \frac{7}{8}x - 2x^2.$$

If we do simple approximation, we get

$$r(x) = 1 - 2x^2.$$

## 3.7 Conclusion

We've come up with a new way of looking at Chebyshev-like polynomials that have three term recurrence relations. We've already looked at the generalized version using matrix algebra. In the future, we'll be looking at different types of modified and generalized Chebyshev types. We'll be looking at them from a different angle, and we'll be using matrix algebra for some basic properties.

## 3.8 Significance of the work

Following are some key points that summarize the importance of the current research:

- To obtain the generalized version of Chebyshev polynomials helps to know more about the hidden factors of Chebyshev polynomials.
- The Binet formula for the generalized version of Chebyshev polynomials presented the explicit form the current version.
- To discuss the general nature of generalized Chebyshev polynomials with characteristic equation and its roots; like sum, products, subtraction, and sum of square of roots.
- Via Matrix representation of generalized Chebyshev polynomials, we can apply matrix algebra to obtain more properties.
- Characteristic values and vectors can be obtained by utilizing the characteristic equations of generalized Chebyshev polynomials.
- In approximation theory, generalized Chebyshev polynomials helpful for approximating other polynomials.
- These polynomials can be used to calculate lower order approximations.

## 3.9 Future Scope of the work

Future work on generalized Chebyshev polynomials can be explored:

**Multivariate extensions:** Generalized Chebyshev polynomials to higher- dimensional spaces and explore their properties and applications.

**Numerical methods:** Improve computational efficiency and accuracy for evaluating and manipulating generalized Chebyshev polynomials.

**Interdisciplinary collaborations:** Combine expertise from mathematics, computer science, engineering, and other fields to tackle complex problems and develop innovative solutions.

**New applications:** Discover novel uses in the fields like data analysis, and computation.

**Connections to other areas:** Explore relationship with other mathematical structures, such as orthogonal polynomials, special functions.

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# Chapter 4

## Relationships Involving Chebyshev Polynomials, Fibonacci Numbers and Lucas Numbers

In this chapter, we demonstrate certain identities involving both third and fourth kinds of the Chebyshev polynomials taking into account the Fibonacci and Lucas numbers. This study brings to light some significant results and defines the relationship between these polynomials. We have used mathematical induction to establish the relation between these polynomials and numbers. We also used the Binet formula and the second-order differential equation to establish their relationship. We also present some results for Fibonacci and Lucas numbers, particularly by using the second-order derivative of third kind Chebyshev polynomials. We prove some results that connect the fourth kind of Chebyshev polynomials with Fibonacci and Lucas numbers. These findings will pave the way for further exploration of these polynomials. In addition, we look at the practical application of Chebyshev's polynomials in approximation theory.

These findings will definitely enrich and strengthen the existing literature on Chebyshev polynomials and their relationship with the Fibonacci and other similar and related orthogonal polynomials. This study is expected to add more depth to our understanding of the combinatorial and analytic properties of these Chebyshev polynomials and be instrumental in studying some general summation problems arising in both pure and applied mathematics involving these polynomials.

### 4.1 Introduction

This chapter comprises three primary sections. The initial section provides an introduction and definition of Chebyshev polynomials, which comprehensively cover all four kinds, including the Fibonacci and Lucas numbers. The second section encompasses the research conducted and work done on deriving six theorems and six corollaries related to Chebyshev polynomials, Lucas, and Fibonacci numbers. In the concluding sections, we have deliberated on the importance and implications of this research and delved into the vast range of applications that Chebyshev polynomials have to offer. The analysis of the initial and subsequent Chebyshev polynomials was conducted by Mason and Wenpeng Zhang, who also presented several interesting identities [68, 69,105,106,107].

Zhang [106] found a number of properties related to the Chebyshev polynomials' derivatives and showed the relationship between them as follows:

“Let $n, q$ be integers with $n \geq 0, q \geq 1$, we have:

$$\sum_{b_1+b_2+\cdots+b_{q+1}=l} \prod_{k=1}^{q+1} U_{b_k}(x) = \frac{1}{2^q q!} U_{n+q}{}^{(q)}(x)." \quad (4.1.1)$$

inside sum covers all non-negative integers $b_{1,}, b_2, \ldots, b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = l$.

Jonny Griffiths [42] presented many results which connected all four kinds of Chebyshev polynomials. Kamal Aghigh et al. [6], M.R. Eslahchi et al. [34], and Taekyun Kim et al. [52, 53, 54, 55] gave many identities attributed to both the third and fourth kinds of Chebyshev polynomials.

Wenpeng Zhang [107] gave the foundational idea to solve the summation of recurrence relations and also studied some identities related to Fibonacci sequences. Chebyshev polynomials are widely studied by researchers and defined in various forms like recurrence relations and trigonometric formulae. The Fibonacci and Lucas numbers share a close relationship with Chebyshev polynomials. These recursive relationships are employed in counting.

Sanjay Harne et al. [45] found identities related to the Chebyshev polynomials, Lucas and Fibonacci numbers at certain variables with their derivatives as follows:

“Let $n, q$ be integers with $n \geq 0, q \geq 1$, we have:

$$" \sum_{b_1+b_2+\cdots+b_{q+1}=l} F_{4(2b_1+1)} F_{4(2b_2+1)} \dots F_{4(2b_{q+1}+1)} = \frac{3^{q+1}}{2^q q!} U_{n+q}{}^{q}\left(\frac{7}{2}\right), " \quad (4.1.2)$$

$$\sum_{b_1+b_2+\cdots+b_{q+1}=l} F_{6(2b_1+1)} F_{6(2b_2+1)} \dots F_{6(2b_{q+1}+1)} = \frac{2^{2q+3}}{q!} U_{n+q}{}^{q}(9). " \quad (4.1.3)$$

inside sum covers all non-negative integers $b_{1,}, b_2, \dots, b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = l$

Kamal et al. [6] examined Chebyshev polynomials of both third and fourth kinds with their applications and obtained highly advantageous outcomes. Yang Li [62,63] determined the connection between derivatives of the Lucas, Chebyshev, and Fibonacci polynomials both first-second kinds of Chebyshev polynomials and applied the elementary method to find the relationship. T. Kim et al. [52] studied sums of finite products involving first kind Chebyshev and Lucas's polynomials represented each one in terms of various types of Chebyshev polynomials.

Zhang and Han [108] provided identities for reciprocal sums of the Chebyshev polynomials through mathematical induction and the characteristic of symmetrical polynomial sequences. T. Kim et al. [55] investigated the summation involving finite products of both third and fourth kinds of Chebyshev polynomials. They derived the subsequent relation involving the fourth kind of Chebyshev polynomials and their derivatives as:

Let $n, q$ be integers with $n \geq 0, q \geq 1$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} (-1)^{n-l} \binom{q-1+n-l}{q-1} W_{b_1}(x)\, W_{b_2}(x) \dots W_{b_{q+1}}(x)$$

$$= \frac{1}{2^q q!} W_{n+q}{}^{(q)}(x) \quad (4.1.4)$$

inside sum covers all non-negative integers $b_{1,}, b_2, \dots, b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = l$. ${W_n}^q(x)$ denoted the $q^{th}$ derivative of $W_n(x)$.

To establish the connection between the third kind of Chebyshev polynomials and Lucas numbers, we pursued the following approach obtained by Kim et. al. [55] and W. Zhang [105]: Let $n, q$ be integers with $n \geq 0, q \geq 1$;

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} V_{b_1}(x)\, V_{b_2}(x) \dots V_{b_{q+1}}(x)$$

$$= \frac{1}{2^q q!} {V_{n+q}}^{(q)}(x), \tag{4.1.5}$$

inside sum covers all non-negative integers $b_{1,},\ b_2, \dots,\ b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = l$. ${V_n}^q(x)$ denoted the $q^{th}$ derivative of $V_n(x)$.

Our work is motivated by the earlier research of Zhang and Kim, and their team. They have made significant contributions to the study of the first and second kinds of Chebyshev polynomials, Fibonacci numbers, and Lucas numbers [55, 105, 107]. In our investigation, we uncover unexpected relationships specifically combining Fibonacci and Lucas numbers in terms of derivatives of the third and fourth kinds of Chebyshev polynomials.

There are mainly four kinds of Chebyshev polynomials, each defined with distinct characteristics;

$$T_0(x) = 1,\ T_1(x) = x,\ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \text{For } n \geq 2, 3, \dots \tag{4.1.6}$$

$$U_0(x) = 1, \qquad U_1(x) = 2x,\ U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \text{For } n \geq 2, 3, \dots \tag{4.1.7}$$

$$V_0(x) = 1,\ V_1(x) = 2x - 1,\ V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x),\ \text{For } n \geq 2, 3, \dots \tag{4.1.8}$$

$W_0(x) = 1,\ W_1(x) = 2x + 1,\ W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x),$ For $n \geq 2, 3, \ldots$ (4.1.9)

Fibonacci Numbers;

$$F_0 = 0, F_1 = 1, \& F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2. \tag{4.1.10}$$

Lucas Numbers;

$$L_0 = 2,\ L_1 = 1, \& L_n = L_{n-1} + L_{n-2},\ n \geq 2. \tag{4.1.11}$$

## 4.2 Some Simple Lemmas

To prove our primary outcome, we require numerous lemmas.

For $n \geq 0$, we have following results:

1. $W_n(161) = \frac{1}{8} F_{6(2n+1)}$.
2. $W_n\left(\frac{47}{2}\right) = \frac{1}{3} F_{4(2n+1).}$
3. $W_n\left(\frac{123}{2}\right) = \frac{1}{11} L_{5(2n+1).}$
4. $V_n\left(\frac{47}{2}\right) = u^{-1} \frac{1}{2} L_{4(2n+1)}$.
5. $V_n(161) = u^{-1} \frac{1}{2} L_{6(2n+1)}$.
6. $V_n(-9) = u^{-1} \frac{(-i)^{2n+1}}{2} L_{3(2n+1)}$.

## 4.3 Proof of Lemmas

**Lemma 1:** The following result holds for all $n \geq 0$:

$$W_n(161) = \frac{1}{8} F_{6(2n+1)}.$$

**Proof:** To prove lemma 1, take $x = 161, u = \sqrt{\frac{1+x}{2}}$.

Utilizing the identity,

$$"U_n(u) = \frac{1}{8} F_{6(n+1)},"$$

$$\Rightarrow U_{2n}(u) = \frac{1}{8} F_{6(2n+1)}.$$

We also utilizing,

$$"U_{2n}(u) = W_n(x)", \text{ to get}$$

$$W_n(161) = \frac{1}{8} F_{6(2n+1).}$$

This demonstrates lemma 1.

**Lemma 2:** The following result holds true for all $n \geq 0$:

$$W_n\left(\frac{47}{2}\right) = \frac{1}{3} F_{4(2n+1).}$$

**Proof:** To prove lemma 2, take $x = \frac{47}{2}, u = \sqrt{\frac{1+x}{2}}$.

We utilize the identity,

$$"U_n(u) = \frac{1}{3} F_{4(n+1)},"$$

$$\Rightarrow U_{2n}(u) = \frac{1}{3} F_{4(2n+1)}.$$

Also utilizing,

$$"U_{2n}(u) = W_n(x)," \text{ to get}$$

$$W_n\left(\frac{47}{2}\right) = \frac{1}{3} F_{4(2n+1)}.$$

This demonstrates lemma 2.

**Lemma 3:** The following result holds true for all $n \geq 0$:

$$W_n\left(\frac{123}{2}\right) = \frac{1}{11} L_{5(2n+1).}$$

**Proof:** To prove lemma 3, take $x = \frac{123}{2}, u = \sqrt{\frac{1+x}{2}}$.

Utilizing the identity,

$$"U_n(u) = \frac{1}{11} L_{5(n+1)},"$$

$$\Rightarrow U_{2n}(u) = \frac{1}{11} L_{5(2n+1).}$$

We also utilizing,

$$"U_{2n}(u) = W_n(x)," \text{ to get}$$

$$W_n\left(\frac{123}{2}\right) = \frac{1}{11} L_{5(2n+1).}$$

This demonstrates lemma 3.

**Lemma 4:** The following result holds true for all $n \geq 0$:

$$V_n\left(\frac{47}{2}\right) = u^{-1} \frac{1}{2} L_{4(2n+1)}.$$

**Proof:** To prove lemma 4, take $x = \frac{47}{2}, u = \sqrt{\frac{1+x}{2}}$.

Using the identity,

$$"T_n(u) = \frac{1}{2} L_{4n},"$$

$$\Rightarrow T_{2n+1}(u) = \frac{1}{2} L_{4(2n+1)}.$$

We also use,

$$"V_n(x) = u^{-1} T_{2n+1}(u)," \text{ to get}$$

$$V_n\left(\frac{47}{2}\right) = u^{-1} \frac{1}{2} L_{4(2n+1).}$$

This demonstrates lemma 4.

**Lemma 5:** The following result holds true for all $n \geq 0$:

$$V_n(161) = u^{-1} \frac{1}{2} L_{6(2n+1).}$$

**Proof:** To prove lemma 5, take $x = 161, u = \sqrt{\frac{1+x}{2}}$.

We utilizing the identity,

$$"T_n(u) = \frac{1}{2} L_{6h},"$$

$$\Rightarrow T_{2n+1}(u) = \frac{1}{2} L_{6(2n+1)}.$$

We also using,

$$"V_n(x) = u^{-1} T_{2n+1}(u)," \text{ to get}$$

$$V_n(161) = u^{-1} \tfrac{1}{2} L_{6(2n+1).}$$

This demonstrates lemma 5.

**Lemma 6:** The following result holds true for all $n \geq 0$:

$$V_n(-9) = u^{-1} \frac{(-i)^{2n+1}}{2} L_{3(2n+1).}$$

**Proof:** To prove lemma 6, take $x = -9, u = \sqrt{\frac{1+x}{2}}$,

We utilizing the identity,

$$"T_n(-2i) = \frac{(-i)^n}{2} L_{3n},"$$

$$\Rightarrow T_{2n+1}(-2i) = \frac{(-i)^{2n+1}}{2} L_{3(2n+1),}$$

We also utilizing,

$$"V_n(x) = u^{-1} T_{2n+1}(u),\text{” to get}$$

$$V_n(-9) = u^{-1} \tfrac{(-i)^{2n+1}}{2} L_{3(2n+1)}\ .$$

This demonstrates lemma 6.

Thus, all six lemmas have been demonstrated here.

## 4.4 Main Results

### 4.4.1 Relation Between Fourth Kinds of the Chebyshev Polynomials, Fibonacci Numbers and Lucas Numbers.

Theorem 1 and theorem 2 present the relation among Fibonacci numbers and Chebyshev polynomials of fourth kind. While theorem 3 elucidated the relationship between Lucas numbers and fourth kind Chebyshev polynomials.

**Theorem 1.** Let $n, q$ be integers with $n \geq 0, q \geq 1, F_n$ be the $n^{th}$ Fibonacci number, we have:

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} (-1)^{n-l} \binom{q-1+n-l}{q-1} F_{6(2b_1+1)} F_{6(2b_2+1)} \dots F_{6(2b_{q+1}+1)}$$
$$= \frac{2^{2q+3}}{q!} W_{n+q}{}^{(q)}(161),$$

inside sum covers all non-negative integers $b_{1,}, b_2, \dots, b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = l$.

**Theorem 2.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, and $F_n$ be the $n^{th}$ Fibonacci number, we have:

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} (-1)^{n-l} \binom{q-1+n-l}{q-1} F_{4(2b_1+1)} F_{4(2b_2+1)} \dots F_{4(2b_{q+1}+1)}$$
$$= \frac{3^{q+1}}{2^q q!} W_{n+q}{}^{(q)}\left(\frac{47}{2}\right),$$

inside sum covers all non-negative integers $b_{1,}, b_2, \dots, b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = l$.

**Theorem 3.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, $L_n$ be the $n^{th}$ Lucas number, we have:

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} (-1)^{n-l} \binom{q-1+n-l}{q-1} L_{5(2b_1+1)} L_{5(2b_2+1)} \dots L_{5(2b_{q+1}+1)}$$
$$= \frac{11^{q+1}}{2^q q!} W_{n+q}{}^{(q)}\left(\frac{123}{2}\right),$$

inside sum covers all non-negative integers $b_1,\ b_2, \ldots, b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = l$.

### 4.4.2 Relation Between the Third Kind Chebyshev Polynomials and Lucas Numbers.

Here, we have obtained three theorems. We present the relationship between Lucas numbers and third kind Chebyshev polynomials at certain points.

**Theorem 4.** Let $n, q$ be integer with $n \geq 0,\ q \geq 1$, $L_n$ be the $n^{th}$ Lucas number, we have:

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} L_{4(2b_1+1)} L_{4(2b_2+1)} \ldots L_{4(2b_{q+1}+1)}$$

$$= \frac{2}{q!} u^{q+1} V_{n+q}{}^{(q)} \left(\frac{47}{2}\right).$$

Inside sum covers all non-negative integer, $b_1 + b_2 + \cdots + b_{q+1} = l, u = \sqrt{\frac{1+x}{2}}$.

**Theorem 5.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, $L_n$ be the $n^{th}$ Lucas number, we have:

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} L_{6(2b_1+1)} L_{6(2b_2+1)} \ldots L_{6(2b_{q+1}+1)}$$

$$= \frac{2}{q!} u^{q+1} V_{n+q}{}^{(q)} (161).$$

Inside sum covers all non-negative integers, $b_1 + b_2 + \cdots + b_{q+1} = l, u = \sqrt{\frac{1+x}{2}}$.

**Theorem 6.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, $L_n$ be the $n^{th}$ Lucas number, we have:

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} (-i)^{2b_1+1} L_{3(2b_1+1)} (-i)^{2b_2+1} L_{3(2b_2+1)} \ldots (-i)^{2b_{q+1}+1} L_{3(2b_{q+1}+1)}$$

$$= \frac{2}{q!} u^{q+1} V_{n+q}{}^{(q)} (-9).$$

Inside sum covers all non-negative integers, $b_1 + b_2 + \cdots + b_{q+1} = l,\ u = \sqrt{\frac{1+x}{2}}$.

We can draw the following six Corollary from the above six theorems:

**Corollary 1.** Let $n, q$ be integers with $n \geq 0,\ q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} (-1)^{n-l} \binom{1+n-l}{1} F_{6(2b_1+1)} F_{6(2b_2+1)} F_{6(2b_3+1)} =$$

$$\frac{[646(2n+5)\{F_{6(2n+3)} - F_{6(2n+7)}\} + 209304 F_{6(2n+5)} + 207360(n+2)(n+3) F_{6(2n+5)}]}{(25920)^2}.$$

**Corollary 2.** Let $n,\ q$ be integers with $n \geq 0,\ q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} (-1)^{n-l} \binom{1+n-l}{1} F_{4(2b_1+1)} F_{4(2b_2+1)} F_{4(2b_3+1)} =$$

$$\frac{[216(2n+5)\{F_{4(2n+3)} - F_{4(2n+7)}\} + 10584 F_{4(2n+5)} + \frac{19845}{2}(n+2)(n+3) F_{4(2n+5)}]}{(2205)^2}.$$

**Corollary 3.** Let $n, q$ be integers with $n \geq 0,\ q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} (-1)^{n-l} \binom{1+n-l}{1} L_{5(2b_1+1)} L_{5(2b_2+1)} L_{5(2b_3+1)} =$$

$$\frac{\left[7502(2n+5)\{L_{5(2n+3)} - L_{5(2n+7)}\} + 937750 L_{5(2n+5)} + \frac{20131375}{22}(n+2)(n+3) L_{5(2n+5)}\right].}{(15125)^2}$$

**Corollary 4.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} \binom{1+n-l}{1} L_{4(2b_1+1)} L_{4(2b_2+1)} L_{4(2b_3+1)} =$$

$$\frac{[u^2[92(2n+5)\{L_{4(2n+3)} - L_{4(2n+7)}\} + 4140L_{4(2n+5)} + 4410(n+2)(n+3)L_{4(2n+5)}].}{(2205)^2}$$

**Corollary 5.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} \binom{1+n-l}{1} L_{6(2b_1+1)} L_{6(2b_2+1)} L_{6(2b_3+1)} =$$

$$\frac{u^2}{(25920)^2}\left[\frac{321}{8}(2n+5)\{L_{6(2n+3)} - L_{6(2n+7)}\} + 12840L_{6(2n+5)} + 12960(n+2)(n+3)L_{6(2n+5)}\right].$$

**Corollary 6.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, the resulting output is as follows:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} \binom{1+n-l}{1} (-i)^{2b_1+1} L_{3(2b_1+1)} (-i)^{2b_2+1} L_{3(2b_2+1)} (-i)^{2b_3+1} L_{3(2b_3+1)}$$

$$= \frac{u^{-1}}{6400}\left[\frac{-19}{8}(2n+5)\{(-i)^{2n+3}L_{3(2n+3)} - (-i)^{2n+7}L_{3(2n+7)}\} + \frac{95}{2}(-i)^{2n+5} L_{3(2n+5)} + 800(n+2)(n+3)(-i)^{2n+5}L_{3(2n+5)}\right].$$

## <u>Proof of Theorems</u>

**Theorem 1: Proof.** Let $W_n(x)$ be defined by (4.1.9), then for any $n$, $q$ be integers with $n \geq 0$, $q \geq 1$ by (4.1.4);

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} (-1)^{n-l} \binom{q-1+n-l}{q-1} \prod_{k=1}^{q+1} W_{b_k}(x)$$

$$= \frac{1}{2^q q!} W_{n+q}{}^{(q)}(x), \qquad (4.4.2.1)$$

From lemma 1, the identity connected Chebyshev polynomials of fourth kind and Fibonacci numbers;

$$W_n(161) = \tfrac{1}{8} F_{6(2n+1).} \qquad (4.4.2.2)$$

From (4.4.2.2), we have

$$W_{b_1}(161) = \tfrac{1}{8} F_{6(2b_1+1)},$$

$$W_{b_2}(161) = \frac{1}{8} F_{6(2b_2+1)},$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$W_{b_{k+1}}(161) = \frac{1}{8} F_{6(2b_{k+1}+1)}.$$

Using above equations in (4.4.2.1), we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} (-1)^{n-l} \binom{q-1+n-l}{q-1} F_{6(2b_1+1)} F_{6(2b_2+1)} \dots F_{6(2b_{q+1}+1)}$$

$$= \frac{2^{2q+3}}{q!} W_{n+q}{}^{(q)}(161). \qquad (4.4.2.3)$$

Hence theorem 1 is formulated in this manner.

**Theorem 2: Proof.** Let $W_n(x)$ be defined by (4.1.9), then for any $n, q$ be integers with $n \geq 0,\ q \geq 1$, by (4.14) yields;

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots b_{q+1}=l} (-1)^{n-l} \binom{q-1+n-l}{q-1} \prod_{k=1}^{q+1} W_{b_k}(x) \quad = \frac{1}{2^q q!} W_{n+q}{}^{(q)}(x). \quad (4.4.2.4)$$

From lemma 2, the identity connected Chebyshev polynomials of fourth kind and Fibonacci numbers i.e.

$$W_n\left(\tfrac{47}{2}\right) = \tfrac{1}{3} F_{4(2n+1)}. \qquad (4.4.2.5)$$

From (4.4.2.5), we have

$$W_{b_1}\left(\frac{47}{2}\right)=\frac{1}{3}F_{4(2b_1+1)},$$

$$W_{b_2}\left(\frac{47}{2}\right)=\frac{1}{3}F_{4(2b_2+1),}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$W_{b_{k+1}}\left(\frac{47}{2}\right)=\frac{1}{3}F_{4(2b_{k+1}+1).}$$

Using above equations in (4.4.2.4), we get

$$\sum_{l=0}^{n}\sum_{b_1+b_2+\cdots+b_{q+1}=l}(-1)^{n-l}\binom{q-1+n-l}{q-1}F_{4(2b_1+1)}F_{4(2b_2+1)}\dots F_{4(2b_{q+1}+1)}$$

$$=\frac{3^{q+1}}{2^q q!}W_{n+q}{}^{(q)}\left(\frac{47}{2}\right). \quad (4.4.2.6)$$

Thus theorem 2 established through proof.

**Theorem 3: Proof.** Let $W_n(x)$ be defined by (4.1.9), then for any $n, q$ be integers with $n\geq 0,\ q\geq 1$, by using (4.1.4);

$$\sum_{l=0}^{n}\sum_{b_1+b_2+\cdots+b_{q+l}=l}(-1)^{n-l}\binom{q-1+n-l}{q-1}\prod_{k=1}^{q+1}W_{b_k}(x)\ =\frac{1}{2^q q!}W_{n+q}{}^{(q)}(x). \quad (4.4.2.7)$$

From lemma 3, identity between Chebyshev polynomials of fourth kind and Lucas numbers, i.e.

$$W_n\left(\frac{123}{2}\right)=\frac{1}{11}L_{5(2n+1).} \quad (4.4.2.8)$$

From (4.4.2.8), we have

$$W_{b_1}\left(\frac{123}{2}\right)=\frac{1}{11}L_{5(2b_1+1)},$$

$$W_{b_2}\left(\frac{123}{2}\right)=\frac{1}{11}L_{5(2b_2+1),}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$W_{b_{k+1}}\left(\frac{123}{2}\right)=\frac{1}{11}L_{5(2b_{k+1}+1).}$$

Using above equations in (4.4.2.7), we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} (-1)^{n-l} \binom{q-1+n-l}{q-1} L_{5(2b_1+1)} L_{5(2b_2+1)} \ldots L_{5(2b_{q+1}+1)}$$

$$= \frac{11^{q+1}}{2^q q!} W_{n+q}{}^{(q)}\left(\frac{123}{2}\right). \tag{4.4.2.9}$$

Hence, proof is established for theorem 3.

**Theorem 4: Proof.** Let $V_n(x)$ be defined by (4.1.8), then for any $n, q$ be integers with $n \geq 0, q \geq 1$, by using (4.1.5);

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} \prod_{k=1}^{q+1} V_{b_k}(x) = \frac{1}{2^q q!} V_{n+q}{}^{(q)}(x). \tag{4.4.2.10}$$

From lemma 4, identity between Chebyshev polynomials of third kind and Lucas numbers, i.e.

$$V_n\left(\frac{47}{2}\right) = u^{-1}\frac{1}{2} L_{4(2n+1).} \tag{4.4.2.11}$$

From (4.4.2.11), we have

$$V_{b_1}\left(\frac{47}{2}\right) = u^{-1}\frac{1}{2} L_{4(2b_1+1)},$$

$$V_{b_2}\left(\frac{47}{2}\right) = u^{-1}\frac{1}{2} L_{4(2b_2+1)},$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$V_{b_{k+1}}\left(\frac{47}{2}\right) = u^{-1}\frac{1}{2} L_{4(2b_{k+1}+1)}.$$

Using above equations in (4.4.2.10), we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} L_{4(2b_1+1)} L_{4(2b_2+1)} \ldots L_{4(2b_{q+1}+1)}$$

$$= \frac{2}{q!} u^{q+1} V_{n+q}{}^{(q)}\left(\frac{47}{2}\right). \tag{4.4.2.12}$$

Hence, theorem 4proof is now complete.

**Theorem 5: Proof.** Let $V_n(x)$ be defined by $(4.1.8)$, then for any $n, q$ be integers with $n \geq 0, q \geq 1$, by $(4.1.5)$;

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} \prod_{k=1}^{q+1} V_{b_k}(x) = \frac{1}{2^q q!} V_{n+q}{}^{(q)}(x). \qquad (4.4.2.13)$$

From lemma 5, identity between Chebyshev polynomials of third kind and Lucas numbers, i.e.

$$V_n(161) = u^{-1}\frac{1}{2}L_{6(2n+1)}. \qquad (4.4.2.14)$$

From (4.4.2.14), we have

$$V_{b_1}(161) = u^{-1}\frac{1}{2}L_{6(2b_1+1)},$$

$$V_{b_2}(161) = u^{-1}\frac{1}{2}L_{6(2b_2+1)},$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$V_{b_{k+1}}(161) = u^{-1}\frac{1}{2}L_{6(2b_{k+1}+1)}.$$

Using above equation in $(4.4.2.13)$, we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} L_{6(2b_1+1)} L_{6(2b_2+1)} \ldots L_{6(2b_{q+1}+1)}$$

$$= \frac{2}{q!} u^{q+1} V_{n+q}{}^{(q)}(161). \qquad (4.4.2.15)$$

The 5 theorem proof is now complete.

**Theorem 6: Proof.** Let $V_n(x)$ be defined by $(4.1.8)$, then for any $n, q$ be integers with $n \geq 0, q \geq 1$, by $(4.1.5)$ yields;

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} \prod_{k=1}^{q+1} V_{b_k}(x) = \frac{1}{2^q q!} V_{n+q}{}^{(q)}(x). \qquad (4.4.2.16)$$

From lemma 6, identity between Chebyshev polynomials of third kind and Lucas numbers, i.e.

$$V_n(-9) = u^{-1}\frac{(-i)^{2n+1}}{2}L_{3(2n+1)}. \qquad (4.4.2.17)$$

From (4.4.2.17), we have

$$V_{b_1}(-9) = u^{-1}\frac{(-i)^{2b_1+1}}{2}L_{3(2b_1+1)},$$

$$V_{b_2}(-9) = u^{-1}\frac{(-i)^{2b_2+1}}{2}L_{3(2b_2+1)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$V_{b_{k+1}}(-9) = u^{-1}\frac{(-i)^{2b_{k+1}+1}}{2}L_{3(2b_{k+1}+1)},$$

Using above equation in (4.4.2.16), we get

$$\sum_{l=0}^{n}\sum_{b_1+b_2+\cdots+b_{q+1}=l}\binom{q-1+n-l}{q-1}(-i)^{2b_1+1}L_{3(2b_1+1)}(-i)^{2b_2+1}L_{3(2b_2+1)}\cdots$$

$$(-i)^{2b_{q+1}+1}L_{3(2b_{q+1}+1)} = \frac{2}{q!}u^{q+1}V_{n+q}{}^{(q)}(-9), \qquad (4.4.2.18)$$

All of our theorem now has through proof.

## Proof of the Corollaries

## Proof of Corollary 1:

$$(1-x^2)W_n{}'(x) = \frac{1}{2}\left(n+\frac{1}{2}\right)\left(W_{n-1}(x)-W_{n+1}(x)\right)+\frac{1}{2}(1+x)W_n(x). \qquad (4.4.2.19)$$

$$(1-x^2)W_n{}''(x) = (1+2x)W_n{}'(x)-n(n+1)W_n(x). \qquad (4.4.2.20)$$

Putting $q = 2$ in (4.4.2.3), by (4.1.10), (4.4.2.19), (4.4.2.20), we get,

$$\sum_{l=0}^{n}\sum_{b_1+b_2+b_3=l}(-1)^{n-l}\binom{1+n-l}{1}F_{6(2b_1+1)}F_{6(2b_2+1)}F_{6(2b_3+1)} =$$

$$64\left[\frac{323}{(25920)^2}\left\{\frac{(2n+5)}{4}\left(\frac{1}{8}F_{6(2n+3)}-\frac{1}{8}F_{6(2n+7)}\right)+81(\frac{1}{8}F_{6(2n+5)}\right\}\right.$$

$$\left.+\frac{(n+2)(n+3)\frac{1}{8}F_{6(2n+5)}}{25920}\right]$$

$$=\frac{646(2n+5)\{F_{6(2n+3)}-F_{6(2n+7)}\}+209304F_{6(2n+5)}+207360(n+2)(n+3)F_{6(2n+5)}}{(25920)^2}.$$

## Proof of Corollary 2:

Put $q=2$ in $(4.4.2.6)$, and by $(4.1.10)$, $(4.4.2.19)$, $(4.4.2.20)$, we get,

$$\sum_{l=0}^{n}\sum_{b_1+b_2+b_3=l}(-1)^{n-l}\binom{1+n-l}{1}F_{4(2b_1+1)}F_{4(2b_2+1)}F_{4(2b_3+1)}$$

$$=\frac{27}{8}\left[\frac{192}{(2205)^2}\left\{(2n+5)\left(\frac{1}{3}F_{4(2n+3)}-\frac{1}{3}F_{4(2n+7)}\right)+49\left(\frac{1}{3}F_{4(2n+5)}\right)\right\}\right.$$

$$\left.+\frac{4(n+2)(n+3)\frac{1}{3}F_{4(2n+5)}}{2205}\right]$$

$$=\frac{\left[216(2n+5)\{F_{4(2n+3)}-F_{4(2n+7)}\}+10584F_{4(2n+5)}+\frac{19845}{2}(n+2)(n+3)F_{4(2n+5)}\right]}{(2205)^2}.$$

## Proof of Corollary 3:

Put $q=2$ in $(4.4.2.9)$ and by $(4.1.11)$, $(4.4.2.19)$, $(4.4.2.20)$, we get

$$\sum_{l=0}^{n}\sum_{b_1+b_2+b_3=l}(-1)^{n-l}\binom{1+n-l}{1}L_{5(2b_1+1)}L_{5(2b_2+1)}L_{5(2b_3+1)}=$$

$$\frac{1331}{8}\left[\frac{496}{(15125)^2}\left\{(2n+5)\left(\frac{1}{11}L_{5(2n+3)}-\frac{1}{11}L_{5(2n+7)}\right)+125(\frac{1}{11}L_{5(2n+5)}\right\}\right.$$

$$\left.+\frac{4}{15125}(n+2)(n+3)\frac{1}{11}L_{5(2n+5)}\right]=$$

$$\frac{\left[7502(2n+5)\left\{L_{5(2n+3)}-L_{5(2n+7)}\right\}+937750L_{5(2n+5)}+\frac{20131375}{22}(n+2)(n+3)L_{5(2n+5)}\right]}{(15125)^2}.$$

**Proof of Corollary 4:**

$$(1-x^2)V_n{}'(x)=\frac{1}{2}\left(n+\frac{1}{2}\right)\left(V_{n-1}(x)-V_{n+1}(x)\right)-\frac{1}{2}(1-x)V_n(x). \qquad (4.4.2.21)$$

$$(1-x^2)V_n{}''(x)=-(1-2x)V_n{}'(x)-n(n+1)V_n(x). \qquad (4.4.2.22)$$

Put $q=2$ in (4.4.2.12), and by (4.1.11), (4.4.2.21), (4.4.2.22), we get,

$$\sum_{l=0}^{n}\sum_{b_1+b_2+b_3=l}\binom{1+n-l}{1}L_{4(2b_1+1)}L_{4(2b_2+1)}L_{4(2b_3+1)}=$$

$$\frac{[u^2[92(2n+5)\left\{L_{4(2n+3)}-L_{4(2n+7)}\right\}+4140L_{4(2n+5)}+4410(n+2)(n+3)L_{4(2n+5)}]}{(2205)^2}.$$

**Proof of Corollary 5:**

Put $q=2$ in (4.4.2.15), and by (4.1.11), (4.4.2.21), (4.4.2.22), we get

$$\sum_{l=0}^{n}\sum_{b_1+b_2+b_3=l}\binom{1+n-l}{1}L_{6(2b_1+1)}L_{6(2b_2+1)}L_{6(2b_3+1)}=$$

$$\frac{u^2\left[\frac{321}{8}(2n+5)\left\{L_{6(2n+3)}-L_{6(2n+7)}\right\}+12840L_{6(2n+5)}+12960(n+2)(n+3)L_{6(2n+5)}\right]}{(25920)^2}.$$

**Proof of Corollary 6:**

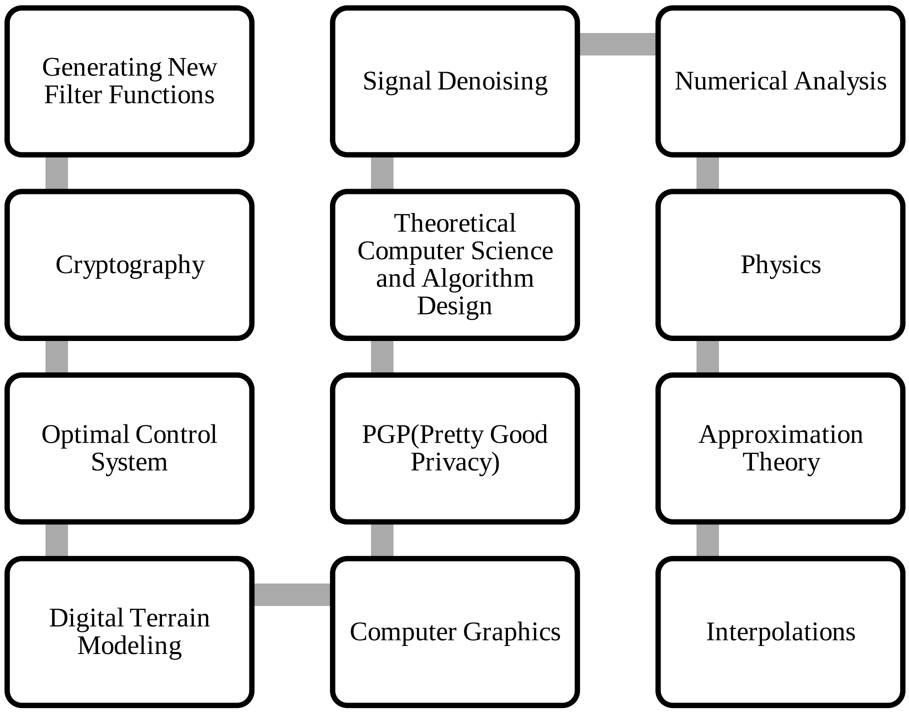
Put $q = 2$ in $(4.4.2.18)$, and by $(4.1.11), (4.4.2.21), (4.4.2.22)$, we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} \binom{1+n-l}{1} (-i)^{2b_1+1} L_{3(2b_1+1)} (-i)^{2b_2+1} L_{3(2b_2+1)} (-i)^{2b_3+1} L_{3(2b_3+1)} =$$

$$\frac{u^{-1}}{6400}\left[\begin{array}{c} u^{-1}[\frac{-19}{8}(2n+5)\{(-i)^{2n+3}L_{3(2n+3)} - (-i)^{2n+7}L_{3(2n+7)}\} + \frac{95}{2}(-i)^{2n+5}L_{3(2n+5)} \\ +800(n+2)(n+3)(-i)^{2n+5}L_{3(2n+5)}] \end{array}\right].$$

## 4.5 Applications

Chebyshev polynomials fall under the category of recurrence relations and are extensively used to improve the advanced techniques for counting. These polynomials are used to study the integer function that allows us to establish new relationships among other polynomials. Chebyshev polynomials are of significant importance when it comes to solving other polynomials for obtaining novel trigonometric identities, finding the solutions to second-order differential equations, interpolating large data and in approximation theory. With the help of Chebyshev polynomials, approximate numerical solutions can be obtained for differential and integral equations. Chebyshev polynomials play a key role in signal processing, primarily in the design of filters known as Chebyshev filters. They are in high demand in the field of computer graphics to generate a variety of shapes, surfaces, and curves.




## 4.6 Practical Applications

**Express:** $x^4 - 4x^3 - 2x^2 + 3x - 1$ in terms of third kind of the Chebyshev polynomials and approximated this function with a cubic polynomial using economization.

**Solution:** The first four terms of third kinds Chebyshev polynomials are:

$$V_0(x) = 1\,,$$
$$V_1(x) = 2x - 1,$$
$$V_2(x) = 4x^2 - 2x - 1,$$
$$V_3(x) = 8x^3 - 4x^2 - 4x + 1,$$
$$V_4(x) = 16x^4 - 8x^3 - 12x^2 + 4x + 1,$$

From above equations, we get

$$x = \frac{1}{2}[V_0(x) + V_1(x)],$$

$$x^2 = \frac{1}{4}[2V_0(x) + V_1(x) + V_2(x)],$$

$$x^3 = \frac{1}{8}[3V_0(x) + 3V_1(x) + V_2(x) + V_3(x)],$$

$$x^4 = \frac{1}{16}[6V_0(x) - 4V_1(x) + 4V_2(x) + V_3(x)+V_4(x)].$$

Putting above values in $x^4 - 4x^3 - 2x^2 + 3x - 1$, we get,

$\frac{1}{16}[6V_0(x) - 4V_1(x) + 4V_2(x) + V_3(x)+V_4(x)] - \frac{1}{2}[3V_0(x) + 3V_1(x) + V_2(x) + V_3] - \frac{1}{2}[2V_0(x) + V_1(x) + V_2(x)] + \frac{3}{2}[V_0(x) + V_1(x)] - V_0(x).$

Hence,

$$-\frac{13}{8}V_0(x) - \frac{3}{4}V_1(x) - \frac{3}{4}V_2(x) - \frac{7}{16}V_3(x) + \frac{1}{16}V_4(x).$$

Here we want a cubic approximation, so we drop the $V_4$ term. The best approximated polynomial is:

$$p(x) = \frac{9}{16} + \frac{7}{4}x - \frac{5}{4}x^2 - \frac{7}{2}x^3 .$$

This gives an error at most $\frac{1}{16}$ . If we do simple approximation i.e.

$$r(x) = -\,4x^3 - 2x^2 + 3x - 1,$$

Here we would get a maximum error equal to 1.

## 4.7 Conclusion

In relation to both the 3rd and 4th kinds of Chebyshev polynomials with Fibonacci numbers and Lucas numbers, we have discovered six theorems and six corollaries. The first two theorems present the relationship between fourth kind Chebyshev polynomials and Fibonacci numbers. Theorem 3 clearly demonstrates the strong correlation between the Lucas numbers and fourth kind Chebyshev polynomials, emphasizing the significance of the relationship. Theorems $4, 5$, and 6 provided the link between Lucas numbers and third kind Chebyshev polynomials at certain points. The six corollaries are merely particular cases associated with our six theorems. These findings are useful for understanding the properties and identities of Chebyshev polynomials with Fibonacci and Lucas numbers.

## 4.8 Significance of the Work

Here, we illustrate the significance of the present work in the following elements:

- Obtaining connections among fourth kind of the Chebyshev polynomials and Fibonacci numbers with some variables: $x = 161, \frac{47}{2}$.
- Establishing relationships between the fourth-kind Chebyshev polynomials and Lucas numbers at a specific variable is a key aspect of our investigation: $x = \frac{123}{2}$.
- Obtaining connections among third kind of the Chebyshev polynomials and Lucas numbers with some variables:

$$x = 161, \frac{47}{2}, -9.$$

It is worth mentioning here that the above-achieved results and analysis are fruitful. Some of their presumed uses are given below:

- These results strengthen the correlation of Chebyshev polynomials to Fibonacci and Lucas.
- They are also beneficial in studying problems connected to calculating general summations.
- They help study integer sequences.
- These polynomials are fruitful in solving convolution sum problems.
- These polynomials can be used to solve differential equations, whether they are linear or non-linear.
- To acquire numerical answers to differential equations, whether linear or nonlinear.
- The interconnections among Chebyshev polynomials, Fibonacci numbers, and Lucas numbers play a crucial role in deriving meaningful identities related to these mathematical concepts.
- The Chebyshev polynomials are fruitful in approximation theory;

In this example we approximate the polynomial $p(x) = x^4 + x^3 - x^2 + x$ by a cubic polynomial. For this we have to use the technique of economization i.e. write each power of $x$ in terms of Chebyshev polynomials. Here we use Chebyshev polynomial of third kind;

$x^4+x^3-x^2+x=\frac{1}{16}[6V_0(x)-4V_1(x)+4V_2(x)+V_3(x)+V_4(x)]+\frac{1}{8}[3V_0(x)+3V_1(x)+V_2(x)+V_3(x)]-\frac{1}{4}[2V_0(x)+V_1(x)+V_2(x)]+\frac{1}{2}[V_0(x)+V_1(x)]$

$$=\frac{3}{4}V_0+\frac{3}{8}V_1+\frac{1}{8}V_2+\frac{3}{16}V_3+\frac{1}{16}V_4.$$

Now we drop the $V_4$ terms for cubic approximation. Hence the best approximating polynomial is;

$f(x)=\frac{7}{16}+\frac{1}{4}x-\frac{1}{4}x^2+\frac{3}{2}x^3$, and at most error is $\frac{1}{16}$ by Chebyshev approximation otherwise by ordinary cubic approximation i. e.

$q(x)=x^3-x^2+x$, we would get error equal to 1.

*****

# Chapter 5

# Relation of Chebyshev Polynomials to the Fibonacci, Pell, and Lucas Numbers

The primary intent of this chapter is to demonstrate various identities that involve Chebyshev polynomials of the $3^{rd}$ kind in context of Fibonacci and Lucas numbers. We have successfully established a connection between Pell numbers and Fibonacci numbers. We utilize the Binet formula, method of mathematical induction and second order differential equations to obtain the results. We find a relation between $3^{rd}$ kind of Chebyshev polynomials and Lucas numbers as well as with the Fibonacci numbers. Additionally, we derive several identities involving Pell and Fibonacci numbers.

## 5.1 Introduction

Zhang discovered many properties affiliated to the derivatives of Chebyshev polynomials and gives the correspondence among them, and their derivatives [105, 106, 107]. Y. Zhang and Z. Chen gave some results on Chebyshev polynomials [104]. Max A. Alekseyev studied the properties related to intersection of Lucas, Pell numbers, and Fibonacci numbers [1]. Han and Lv provided some new identities for Chebyshev polynomials [10]. Anthony G. Shannon et al. provide the connection between the Fibonacci $p-$ numbers and Pell numbers [88]. A. Patra and G.K. Panda studied Pell polynomials and also obtained some results on sums of finite products [79]. To substantiate the result presented in this chapter, a brief overview of fundamental definitions is required. Specifically, the Chebyshev polynomials of the third kind is essential.

($3^{rd}$ kind of Chebyshev polynomials) The $n^{th}$ degree of Chebyshev polynomials of third kind are denoted as:

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), V_0(x) = 1 \text{ and } V_1(x) = 2x - 1. \quad (5.1.1)$$

(Fibonacci sequence) The $n^{th}$ Fibonacci sequence of numbers is denoted as:

$$F_n = F_{n-1} + F_{n-2}\,, \qquad F_0 = 0 \text{ and } F_1 = 1\,. \tag{5.1.2}$$

(Lucas's sequence) The $n^{th}$ Lucas's sequence of numbers is denoted as:

$$L_n = L_{n-1} + L_{n-2}\,, \qquad L_0 = 2 \text{ and } L_1 = 1. \tag{5.1.3}$$

(Pell sequence) The $n^{th}$ Pell sequence of numbers is denoted as:

$$P_n = 2P_{n-1} + P_{n-2}, \qquad P_0 = 0 \text{ and } P_1 = 1\,. \tag{5.1.4}$$

(Pell polynomials) The $n^{th}$ Pell polynomials is denoted as:

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}\,(x), \qquad P_0 = 0 \text{ and } P_1 = 1\,. \tag{5.1.5}$$

In this chapter, we combine the ideas of Taekyun Kim and Wenpeng Zhang to prove our theorems. We used following result to obtain the relationship between these polynomials. Let $n, q$ be integers with $n \geq 0,\ q \geq 1,$ then the identities from [53, 105]:

$$\sum_{l=0}^{n} \sum_{b_{1+b_{2+\dots+b_{q+1}=l}}} \binom{q-1+n-l}{q-1} V_{b_1}(x)V_{b_2}(x)\dots V_{b_{q+1}}(x)$$
$$= \frac{1}{2^q q!} V_{n+q}{}^{(q)}(x). \tag{5.1.6}$$

the inner sum rounds concluded non-negative integers $b_1,\ b_2, \dots, b_{q+1}$ with

$$b_1 + b_2 + \cdots + b_{q+1} = l \text{ [104].}$$

The aforementioned outcome establishes a connection between the 3rd kind Chebyshev polynomials, Lucas numbers, and Fibonacci numbers. We also used the following result to obtain the relationship between Fibonacci and Pell polynomials. Let $n, q$ be integers with $n \geq 0,\ q \geq 1,$

$$\sum_{l=0}^{n} \sum_{b_{1+b_{2+\dots+b_{q+1}=l}}} P_{b_1+1}(x)\, P_{b_2+1}(x) \dots P_{b_{q+1}}(x) = \frac{1}{2^q q!} P_{n+q+1}{}^{(q)}(x)\,, \tag{5.1.7}$$

with $b_1, b_2, \dots, b_{q+1} = l$ [79].

Our work is motivated by the earlier work of Harne et al. [45] Zhag and Kim [53, 105]. The authors derived the many identities attributed to the Chebyshev polynomials of the 1st and 2nd kinds, Fibonacci and Lucas numbers. By exploring these identities i.e. combining Fibonacci, Lucas, and Pell numbers with Chebyshev polynomials of third kind reveals unexpected relationship.

## 5.2. Main Results

### 5.2.1 Some Lemmas

These lemmas serve as essential building blocks, each contributing a specific and foundational aspect to the overall demonstration of our major consequence. Let $n$ be integers with $n \geq 0$, we have these lemmas:

1. $V_n\left(\frac{-3}{2}\right) = u^{-1}\frac{i^{2n+1}}{2}L_{(2n+1).}$
2. $(-1)^n V_n\left(-\frac{123}{2}\right) = \frac{1}{11}L_{5(2n+1)}$.
3. $(-1)^n V_n(-161) = \frac{1}{8}F_{6(2n+1)}$.
4. $(-1)^n V_n\left(-\frac{47}{2}\right) = \frac{1}{3}F_{4(2n+1).}$
5. $P_{n+1}\left(\frac{1}{2}\right) = F_{n+1.}$

### 5.2.2 Relation Between Third Kind of the Chebyshev Polynomials, Fibonacci Numbers, Lucas Numbers, and Pell Numbers.

**Theorem 1:** Let $n, q$ be integers with $n \geq 0, q \geq 1$, $L_n$ be the $n^{th}$ Lucas number, then the following result holds;

$$\sum_{l=0}^{n} \sum_{b_{1+b_{2+\dots+b_{q+1}=l}}} \binom{q-1+n-l}{q-1} (i)^{2b_1+1} L_{(2b_1+1)} (i)^{2b_2+1} L_{(2b_2+1)} \cdots$$

$$(i)^{2b_{q+1}+1}L_{(2b_{q+1}+1)} = \frac{2}{q!}u^{q+1}V_{n+q}{}^{(q)}\left(\frac{-3}{2}\right),$$

with $b_1 + b_2 + \cdots + b_{q+1} = l$ and $u = \sqrt{\frac{1+x}{2}}, i = \sqrt{-1}.$

**Theorem 2:** Let $n, q$ be integers with $n \geq 0, q \geq 1$then the following result holds:

$$\sum_{l=0}^{n} \sum_{b_{1+b_2 \ldots + b_{q+1}=l}} \binom{q-1+n-l}{q-1} (-1)^{b_1} L_{5(2b_1+1)} (-1)^{b_2} L_{5(2b_2+1)} \ldots (-1)^{b_{q+1}} L_{5(2b_{q+1}+1)}$$

$$= \frac{11^{q+1}}{2^q q!} V_{n+q}{}^{(q)}\left(-\frac{123}{2}\right).$$

with $b_1 + b_2 + \cdots + b_{q+1} = l$ and $u = \sqrt{\frac{1+x}{2}}.$

**Theorem 3:** Let $n, q$ be integers with $n \geq 0, q \geq 1$then the following result holds:

$$\sum_{l=0}^{n} \sum_{b_{1+b_{2+\ldots+b_{q+1}=l}}} \binom{q-1+n-l}{q-1} (-1)^{b_1} F_{6(2b_1+1)} (-1)^{b_2} F_{6(2b_2+1)} \ldots (-1)^{b_{q+1}} F_{6(2b_{q+1}+1)}$$

$$= \frac{2^{2q+3}}{q!} V_{n+q}{}^{(q)}(-161).$$

with $b_1 + b_2 + \cdots + b_{q+1} = l$ and $u = \sqrt{\frac{1+\mathrm{x}}{2}}.$

**Theorem 4:** Let $n, q$ be integers with $n \geq 0, q \geq 1$then the following result holds:

$$\sum_{l=0}^{n} \sum_{b_{1+b_{2+\ldots+b_{q+1}=l}}} \binom{1+n-l}{q-1} (-1)^{b_1} F_{4(2b_1+1)} (-1)^{b_2} F_{4(2b_2+1)} \ldots (-1)^{b_{q+1}} F_{4(2b_{q+1}+1)}$$

$$= \frac{1}{2^q q!} V_{n+q}{}^{(q)}\left(-\frac{47}{2}\right).$$

With $b_1 + b_2 + \cdots + b_{q+1} = l$ and $u = \sqrt{\frac{1+x}{2}}.$

**Theorem 5:** Let $n, q$ be integers with $n \geq 0, q \geq 1$, then the following result holds:

$$\sum_{l=0}^{n} \sum_{b_{1+b_{2+\cdots+b_{q+1}}=l}} F_{b_1+1} F_{b_2+1} \ldots F_{b_{q+1}} = \frac{1}{2^q q!} P_{n+q+1}{}^{(q)} \left(\frac{1}{2}\right).$$

With $b_1 + b_2 + \cdots + b_{q+1} = l$.

Now we draw the following five corollary from above five theorems.

**Corollary 1.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, then the following result holds:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} \binom{1+n-l}{1} (i)^{2b_1+1} L_{(2b_1+1)} (i)^{2b_2+1} L_{(2b_2+1)} (i)^{2b_3+1} L_{(2b_3+1)}$$

$$= \frac{16u^{-1}}{50} \begin{bmatrix} -(2n+5)\{i^{2n+3} L_{2n+3} - i^{2n+7} L_{2n+7}\} + 5i^{2n+5} L_{2n+5} + \frac{5}{4}(n+2) \\ (n+3) i^{2n+5} L_{2n+5} \end{bmatrix}.$$

**Corollary 2.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, then the following result holds:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_{1+b_{2+b_3=l}}} \binom{1+n-l}{1} (-1)^{b_1} L_{5(2b_1+1)} (-1)^{b_2} L_{5(2b_2+1)} (-1)^{b_3} L_{5(2b_3+1)} =$$

$$\frac{1}{(15125)^2} \begin{bmatrix} 7502(2n+5)\left\{-(-1)^{(n+1)} L_{5(2n+3)} + (-1)^{(n+3)} L_{5(2n+7)}\right\} + 937750 \\ (-1)^{(n+2)} L_{5(2n+5)} + \frac{20131375}{22} (n+2)(n+3)(-1)^{(n+2)} L_{5(2n+5)} \end{bmatrix}.$$

**Corollary 3.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, then the following result holds:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} \binom{1+n-l}{1} (-1)^{b_1} F_{6(2b_1+1)} (-1)^{b_2} F_{6(2b_2+1)} (-1)^{b_3} F_{6(2b_3+1)} =$$

$$\frac{1}{(25920)^2} \begin{bmatrix} 646(2n+5)\left\{-(-1)^{(n+1)} F_{6(2n+3)} + (-1)^{(n+3)} F_{6(2n+7)}\right\} + 209304 \\ (-1)^{(n+2)} F_{6(2n+5)} + 207360(n+2)(n+3)(-1)^{(n+2)} F_{6(2n+5)} \end{bmatrix}.$$

**Corollary 4.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, then the following result holds:

For $q = 2$,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} \binom{1+n-l}{1} (-1)^{b_1} F_{4(2b_1+1)} (-1)^{b_2} F_{4(2b_2+1)} (-1)^{b_3} F_{4(2b_3+1)} =$$

$$\frac{1}{(2205)^2} \begin{bmatrix} 216(2n+5)\left\{-(-1)^{(n+1)} F_{4(2n+3)} + (-1)^{(n+3)} F_{4(2n+7)}\right\} + 10584 \\ (-1)^{(n+2)} F_{4(2n+5)} + \frac{19845}{2}(n+2)(n+3)(-1)^{(n+2)} F_{4(2n+5)} \end{bmatrix}.$$

**Corollary 5.** Let $n, q$ be integers with $n \geq 0, q \geq 1$, then the following result holds:

For $q = 2$

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} F_{b_1+1} F_{b_2+1} F_{b_3+1} = \frac{1}{8} P_{n+3}'' \left(\frac{1}{2}\right).$$

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} F_{b_1+1} F_{b_2+1} F_{b_3+1} = F_{n+1} + 2F_n + 5.$$

### 5.2.3 Proof of Lemmas

**Lemma1:** The following result holds true for all $n$ be integers with $n \geq 0$,

$$V_n\left(\frac{-3}{2}\right) = u^{-1}\frac{i^{2n+1}}{2}L_{(2n+1).}$$

**Proof:** For proving lemma 1, take $x = \frac{-3}{2}$, $u = \sqrt{\frac{1+x}{2}}$.

Utilizing the identity,

$$V_n(x) = u^{-1}\, T_{2n+1}(u)$$

We also utilizing,

$$T_n\left(\frac{i}{2}\right) = \frac{i^n}{2}\, L_n,$$

$$T_{2n+1}\left(\frac{i}{2}\right) = \frac{i^{2n+1}}{2}\, L_{2n+1}$$

$$T_{2n+1}\left(\frac{i}{2}\right) = \; T_{2n+1}(u) = \frac{i^{2n+1}}{2}L_{2n+1}$$

$$V_n\left(\frac{-3}{2}\right) = u^{-1}\frac{i^{2n+1}}{2}L_{(2n+1).}$$

This validates lemma 1.

**Lemma 2:** The following result holds true for all $n$ be integers with $n \geq 0$,

$$(-1)^n V_n\left(-\frac{123}{2}\right) = \frac{1}{11}L_{5(2n+1)\,.}$$

**Proof:** For proving lemma 2, take $x = \left(-\frac{123}{2}\right)$, $u = \sqrt{\frac{1+x}{2}}$,

Utilizing the identity,

$$U_{2n}(u) = \frac{1}{11}L_{5(2n+1)}.$$

We also utilizing,

$$U_{2n}(u) = (-1)^n V_n(-x), \text{ to get}$$

$$(-1)^n V_n\left(-\frac{123}{2}\right) = \frac{1}{11}L_{5(2n+1)\,.}$$

This validates lemma 2.

**Lemma 3:** The following result holds true for all $n$ be integers with $n \geq 0$,

$$(-1)^n V_n(-161) = \frac{1}{8} F_{6(2n+1)}.$$

**Proof:** For proving lemma 3, take $x = (-161)$, $u = \sqrt{\frac{1+x}{2}}$.

Utilizing the identity,

$$U_{2n}(u) = \frac{1}{8} F_{6(2n+1)}.$$

We also utilizing,

$$U_{2n}(u) = (-1)^n V_n(-x), \text{ to get}$$

$$(-1)^n V_n(-x) = \frac{1}{8} F_{6(2n+1)}.$$

This validates lemma 3.

**Lemma 4:** The following result holds true for all $n$ be integers with $n \geq 0$,

$$(-1)^n V_n\left(-\frac{47}{2}\right) = \frac{1}{3} F_{4(2n+1).}$$

**Proof:** For proving lemma 4, take $x = \left(-\frac{47}{2}\right)$, and $u = \sqrt{\frac{1+x}{2}}$.

Utilizing the identity,

$$\Rightarrow U_{2n}(u) = \frac{1}{3} F_{4_{(2n+1)}}.$$

Also utilizing,

$$U_{2n}(u) = (-1)^n V_n(-x), \text{ to get}$$

$$(-1)^n V_n\left(-\frac{47}{2}\right) = \frac{1}{3} F_{4(2n+1).}$$

This demonstrates lemma 4.

**Lemma 5:** The following result holds true for all $n$ be integers with $n \geq 0$,

$$P_{n+1}\left(\frac{1}{2}\right) = F_{n+1.}$$

**Proof:** For proving lemma 5, take $x = \left(\frac{1}{2}\right)$, $u = \sqrt{\frac{1+x}{2}}$.

Utilizing the identity,

$$U_n\left(\frac{i \times 1}{2}\right) = i^n F_{n+1.}$$

We also utilizing,

$$\frac{1}{(i)^n} U_n(ix) = P_{n+1}(\,x)$$

$$P_{n+1}\left(\frac{1}{2}\right) = F_{n+1.}$$

This validates lemma 5.

### 5.2.4 Proof of Theorems

**Proof of Theorem1:** Let $V_n(x)$ be defined as in equation $(5.1.1)$, by $(5.1.6)$;

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} \prod_{k=1}^{q+1} V_{b_k}(x) = \frac{1}{2^q q!} V_{n+q}{}^{(q)}\left(\frac{-3}{2}\right). \qquad (5.2.4.1)$$

Taking $x = \frac{-3}{2}$

$$V_n\left(\frac{-3}{2}\right) = u^{-1}\frac{i^{2n+1}}{2} L_{(2n+1).} \qquad (5.2.4.2)$$

From equation (5.2.4.1), (5.2.4.2) we get,

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\cdots+b_{q+1}=l} \binom{q-1+n-l}{q-1} (i)^{2b_1+1} L_{(2b_1+1)} (i)^{2b_2+1} L_{(2b_2+1)} \ldots (i)^{2b_{q+1}+1} L_{(2b_{q+1}+1)}$$

$$= \frac{2}{q!} u^{q+1} V_{n+q}{}^{(q)}\left(\frac{-3}{2}\right). \qquad (5.2.4.3)$$

**Proof of Theorem 2:** Let $V_n(x)$ be defined as in equation $(5.1.1)$, by $(5.1.6)$;

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\dots+b_{q+1}=l} \binom{q-1+n-l}{q-1} \prod_{k=1}^{q+1} V_{b_k}(-x) = \frac{1}{2^q q!} V_{n+q}{}^{(q)}(-x). \tag{5.2.4.4}$$

Taking $x = \frac{123}{2}$

$$(-1)^n V_n\left(-\frac{123}{2}\right) = \frac{1}{11} L_{5(2n+1)}\,. \tag{5.2.4.5}$$

From equation (5.2.4.4), (5.2.4.5), we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\dots+b_{q+1}=l} \binom{q-1+n-l}{q-1} (-1)^{b_1} L_{5(2b_1+1)} (-1)^{b_2} L_{5(2b_2+1)} \dots (-1)^{b_{q+1}} L_{5(2b_{q+1}+1)}$$

$$= \frac{11^{q+1}}{2^q q!} V_{n+q}{}^{(q)}\left(-\frac{123}{2}\right). \tag{5.2.4.6}$$

**Proof of Theorem 3:** Let $V_n(x)$ be defined as in (5.1.1), by (5.1.6);

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\dots+b_{q+1}=l} \binom{q-1+n-l}{q-1} \prod_{k=1}^{q+1} V_{b_k}(-x) = \frac{1}{2^q q!} V_{n+q}{}^{(q)}(-x). \tag{5.2.4.7}$$

Taking $x = 161$

$$(-1)^n V_n(-161) = \frac{1}{8} F_{6(2n+1)}$$

$$\Rightarrow V_n(-161) = (-1)^{-n} \frac{1}{8} F_{6(2n+1).} \tag{5.2.4.8}$$

By equation (5.2.4.7), (5.2.4.8), we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\dots+b_{q+1}=l} \binom{q-1+n-l}{q-1} (-1)^{b_1} F_{6(2b_1+1)} (-1)^{b_2} F_{6(2b_2+1)} \dots (-1)^{b_{q+1}} F_{6(2b_{q+1}+1)}$$

$$= \frac{2^{2q+3}}{q!} V_{n+q}{}^{(q)}(-161). \tag{5.2.4.9}$$

**Proof of Theorem 4:** Let $V_n(x)$ be defined as in equation (5.1.1), by (5.1.6);

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\dots+b_{q+1}=l} \binom{q-1+n-l}{q-1} \prod_{k=1}^{q+1} V_{b_k}(-x) = \frac{1}{2^q q!} V_{n+q}{}^{(q)}(-x). \qquad (5.2.4.10)$$

Taking $x = \frac{47}{2}$

$$(-1)^n V_n\left(-\frac{47}{2}\right) = \frac{1}{3} F_{4(2n+1).} \qquad (5.2.4.11)$$

By equation (5.2.4.10), (5.2.4.11), we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\dots+b_{q+1}=l} \binom{q-1+n-l}{q-1} (-1)^{b_1} F_{4(2b_1+1)} (-1)^{b_2} F_{4(2b_2+1)} \dots (-1)^{b_{q+1}} F_{4(2b_{q+1}+1)}$$

$$= \frac{1}{2^q q!} V_{n+q}{}^{(q)} \left(-\frac{47}{2}\right). \qquad (5.2.4.12)$$

**Proof of Theorem 5:** Let $P_n(x)$ be defined as in equation (5.1.5), by (5.1.7)

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\dots+b_{q+1}=l} P_{b_1+1}(x)\, P_{b_2+1}(x) \dots P_{b_{q+1}}(x) = \frac{1}{2^q q!} P_{n+q+1}{}^{(q)}(x)\,. \qquad (5.2.4.13)$$

Taking $x = \frac{1}{2}$

$$P_{n+1}\left(\frac{1}{2}\right) = F_{n+1} \qquad (5.2.4.14)$$

By equation (5.2.4.13), (5.2.4.14), we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+\dots+b_{q+1}=l} F_{b_1+1}\, F_{b_2+1} \dots F_{b_{q+1}+1} = \frac{1}{2^q q!} P_{n+q+1}{}^{(q)} \left(\frac{1}{2}\right). \qquad (5.2.4.15)$$

### 5.2.5 Proof of Corollaries

### Proof of Corollary 1:

$$(1-x^2){V_n}'(x)=\frac{1}{2}\left(n+\frac{1}{2}\right)\left(V_{n-1}(x)-V_{n+1}(x)\right)-\frac{1}{2}(1-x)V_n(x). \qquad (5.2.4.16)$$

$$(1-x^2){V_n}''(x)+(1-2x){V_n}'(x)+n(n+1)V_n(x)=0. \qquad (5.2.4.17)$$

Put $q=2$ in (5.2.4.3) and by $(5.1.3),(5.2.4.16),(5.2.4.17)$, we get

$$\sum_{l=0}^{n}\sum_{b_1+b_2+b_3=l}\binom{1+n-l}{q-1}(i)^{2b_1+1}L_{(2b_1+1)}(i)^{2b_2+1}L_{(2b_2+1)}(i)^{2b_3+1}L_{(2b_3+1)}$$

$$=\frac{16u^{-1}}{50}\left[\begin{matrix}-(2n+5)\{i^{2n+3}L_{2n+3}-i^{2n+7}L_{2n+7}\}+5i^{2n+5}L_{2n+5}+\frac{5}{4}(n+2)\\(n+3)i^{2n+5}L_{2n+5}\end{matrix}\right].$$

### Proof of Corollary 2:

Put $q=2$ in $(5.2.4.6)$ , by $(5.1.3),(5.2.4.16),(5.2.4.17)$, we get,

$$\sum_{l=0}^{n}\sum_{b_{1+b_{2+}b_3=l}}\binom{1+n-l}{1}(-1)^{b_1}L_{5(2b_1+1)}(-1)^{b_2}L_{5(2b_2+1)}(-1)^{b_3}L_{5(2b_3+1)}$$

$$=\frac{1331}{8}\left[\frac{496}{(15125)^2}\left\{(2n+5)\left(-(-1)^{(n+1)}\frac{1}{11}L_{5(2n+3)}\right.\right.\right.$$

$$\left.\left.+(-1)^{(n+3)}\frac{1}{11}L_{5(2n+7)}\right)+125(-1)^{(n+2)}\frac{1}{11}L_{5(2n+5)}\right\}$$

$$\left.+\frac{4(n+2)(n+3)(-1)^{(n+2)}\frac{1}{11}L_{5(2n+5)}}{15125}\right]$$

$$= \frac{1}{(15125)^2}\left[\begin{array}{c} 7502(2n+5)\left\{-(-1)^{(n+1)}L_{5(2n+3)} + (-1)^{(n+3)}L_{5(2n+7)}\right\} + 937750 \\ (-1)^{(n+2)}L_{5(2n+5)} + \frac{20131375}{22}(n+2)(n+3)(-1)^{(n+2)}L_{5(2n+5)} \end{array}\right].$$

## Proof of Corollary 3:

Put $q = 2$ in (5.2.4.9) and by (5.1.2), (5.2.4.16), (5.2.4.17), we get

We have

$$\sum_{l=0}^{n}\sum_{b_1+b_2+b_3=l}\binom{1+n-l}{1}(-1)^{b_1}F_{6(2b_1+1)}(-1)^{b_2}F_{6(2b_2+1)}(-1)^{b_3}F_{6(2b_3+1)}$$

$$= 64\left[\frac{323}{(25920)^2}\left\{\frac{(2n+5)}{4}\left(-(-1)^{(n+1)}\frac{1}{8}F_{6(2n+3)} + (-1)^{(n+3)}\frac{1}{8}F_{6(2n+7)}\right)\right.\right.$$
$$\left.\left. + 81(-1)^{(n+2)}\frac{1}{8}F_{6(2n+5)}\right\} + \frac{(n+2)(n+3)}{25920}(-1)^{(n+2)}\frac{1}{8}F_{6(2n+5)}\right]$$

$$= \frac{1}{(25920)^2}\left[\begin{array}{c} 646(2n+5)\left\{-(-1)^{(n+1)}F_{6(2n+3)} + (-1)^{(n+3)}F_{6(2n+7)}\right\} + 209304 \\ (-1)^{(n+2)}F_{6(2n+5)} + 207360(n+2)(n+3)(-1)^{(n+2)}F_{6(2n+5)} \end{array}\right].$$

## Proof of Corollary 4:

Put $q = 2$ in (5.2.4.12) and by (5.1.2), (5.2.4.16), (5.2.4.17) we get,

$$\sum_{l=0}^{n}\sum_{b_1+b_2+b_3=l}\binom{1+n-l}{1}(-1)^{b_1}F_{4(2b_1+1)}(-1)^{b_2}F_{4(2b_2+1)}(-1)^{b_3}F_{4(2b_3+1)} =$$

$$\frac{27}{8}\left[\frac{768}{(2205)^2}\left\{\frac{2n+5}{4}\left(-(-1)^{(n+1)}\frac{1}{3}F_{4(2n+3)} + (-1)^{(n+3)}\frac{1}{3}(F_{4(2n+7)}\right)\right.\right.$$
$$\left.\left. + \frac{49}{4}(-1)^{n+2}\frac{1}{3}F_{4(2n+5)}\right\} + \frac{4(n+2)(n+3)(-1)^{(n+2)}\frac{1}{3}F_{4(2n+5)}}{2205}\right]$$

$$= \frac{1}{(2205)^2}\begin{bmatrix} 216(2n+5)\{-(-1)^{(n+1)}F_{4(2n+3)} + (-1)^{(n+3)}F_{4(2n+7)}\} + 10584 \\ (-1)^{(n+2)}F_{4(2n+5)} + \dfrac{19845}{2}(n+2)(n+3)(-1)^{(n+2)}F_{4(2n+5)} \end{bmatrix}.$$

**Proof of Corollary 5:**

Put $q = 2$ in $(5.2.4.15)$, we get

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} F_{b_1+1} F_{b_2+1} F_{b_3+1} = \frac{1}{8} P_{n+3}{}^{(q)}(\frac{1}{2})$$

$$\sum_{l=0}^{n} \sum_{b_1+b_2+b_3=l} F_{b_1+1} F_{b_2+1} F_{b_3+1} = F_{n+1} + 2F_n + 5.$$

## 5.3 Conclusion

In this chapter, we obtain five theorems associated third kind Chebyshev polynomials, Fibonacci numbers, Pell numbers, and Lucas's sequence. We also discuss applications of Chebyshev polynomials. Theorems 1 and 2 relate the third kind Chebyshev polynomials with Lucas numbers, whereas theorems 3 and 4 connect the third kind Chebyshev polynomials with Fibonacci numbers. Theorem 5 connects the Fibonacci numbers with Pell polynomials.

To prove theorems 1 and 2, we first discover the identity between the third kind Chebyshev polynomials and Lucas numbers at certain variables. Then we used that identity to obtain our theorem, which gave the relationship between the third kind of Chebyshev polynomial and Lucas numbers. Similarly, to prove theorems 3 and 4, we obtain the identity that connected the 3rd kind Chebyshev polynomials with Fibonacci numbers at some variables. We used that identity to obtain our theorem that gave a relationship between the Chebyshev polynomial of third the kind with Fibonacci numbers. To prove theorem 5 firstly we obtain an identity between Fibonacci and Pell polynomials for a certain variable. Then we used that identity to obtain a relationship between Pell polynomials and Fibonacci numbers.

## 5.4 Utility of the work

The $q^{th}$ derivatives of Chebyshev polynomials to solve some calculating problems of the general summations, presented many formulas and relations between polynomials and their derivatives. This fact allows them to present a family of integer sequences in a new and direct way. These results strengthen the connections of two kinds of polynomials. They are also helpful in dealing with some calculating problems of the general summations or studying some integer sequences. The identities on sums of finite products in terms of Pell polynomials, however, have not been investigated, so identities primarily in terms of Pell polynomials are obtained.

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# Summary and Conclusion of the Work

In this thesis, we studied the Chebyshev polynomials their properties and applications. We first looked at the matrix representation of Chebyshev polynomials of both third and fourth kinds. We then looked at generalized version of Chebyshev polynomials and identities connecting Chebyshev polynomials, with Fibonacci, Lucas, and Pell numbers. The first chapter serves as an introduction.

In the second chapter, we formulated a matrix representation for both the $3^{rd}$ and $4^{th}$ kinds of Chebyshev polynomials. The motivation behind extending the matrix representation to third and fourth kind Chebyshev polynomials is to generalize the existing theory and lead to a deeper understanding of their behavior. The initial section focused on deriving the matrix representation for the Chebyshev polynomials of the $3^{rd}$ kind and establishing up to $3^{rd}$ degree characteristic equations. The subsequent section extended this analysis to the $4^{th}$ kind, deducing characteristic equations for degrees up to three as well. Notably, analogous identities were discovered for both the $3^{rd}$ and $4^{th}$ kinds of Chebyshev polynomials. Furthermore, we established a relationship between the trace and matrix power specifically pertaining to various types of Chebyshev polynomials. The practical applications and significance of these findings were also discussed.

In the third chapter, we introduced the notion of generalized Chebyshev polynomials and explored their inherent characteristics. The motivation for studying the generalized version of Chebyshev polynomials is to uncover new results in this area and provide a more comprehensive framework. Leveraging these properties, we formulated an explicit expression for generalized Chebyshev polynomials and derived specific outcomes regarding the generating matrices and their determinants. The determination of the characteristic equation was extended up to the fifth degree for generalized Chebyshev polynomials. Additionally, we delved into the practical implications and applications of these generalized Chebyshev polynomials.

In the fourth chapter, we established findings concerning the summation of definite products involving the third and fourth kinds of "Chebyshev polynomials, Fibonacci numbers, and Lucas numbers", articulated in terms of Chebyshev polynomials and their derivatives. The motivations behind obtaining these identities are to broadens our understanding and provide new insights. Additionally, we explored specific instances of these results through corollaries, considering various values of $r$ such as $r = 1$ and $r = 2$. Through elementary methods, we identified a relationship connecting the third and fourth kinds of Chebyshev polynomials with Lucas and Fibonacci numbers. We also explored the importance and relevance of our findings.

Concluding in Chapter 5, we obtained identities that articulate sums of finite products involving Fibonacci $(F_n)$ Lucas $(L_n)$, and Pell numbers $(P_n)$ with derivatives of the 3rd kind of Chebyshev polynomials $V_n(x)$ through straightforward computations. The motivation to study these identities is to uncover new connections and new properties of these polynomials. We presented additional results, expressing summations of finite products of Lucas, Pell, and Fibonacci numbers as linear combinations of their derivatives, utilizing their fundamental properties via elementary calculations. We talked about the utility of Chebyshev polynomials in various field. We discussed the versatility of Chebyshev polynomials across multiple fields.

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# Future Scope of the Work

The future scope of Chebyshev polynomials is promising and encompasses various avenues of research and application. The results concerning the sums of finite products of Chebyshev polynomials of both the third and fourth kind express each of them in terms of other polynomials. In future research, we can explore congruences involving Chebyshev polynomials, Fibonacci numbers, and Lucas numbers. Chebyshev polynomials have been used to study the power sum problem involving Fibonacci and Lucas polynomials, as well as to prove new divisible properties related to these polynomials. Some other potential areas of exploration include:

## Advanced Generalizations:

Further extending and generalizing Chebyshev polynomials to create new families of orthogonal polynomials with unique properties and applications.

## Matrix Representation:

Exploring and refining matrix representations of Chebyshev polynomials for applications in linear algebra, numerical analysis, and related fields.

## Higher-Order Matrices:

Investigating the use of higher-order matrices in the context of Chebyshev polynomials to derive more sophisticated results, particularly in terms of matrix powers and traces.

## Trace and Matrix Power Results:

Developing advanced results and theorems related to traces and matrix powers of Chebyshev polynomials, potentially leading to practical applications in signal processing, quantum computing, and control theory.

## Relations with Numbers Sequences:

Establishing and exploring deeper connections between Chebyshev polynomials and various number sequences, such as Pell, Lucas, and Fibonacci numbers. This could lead to novel insights into the interplay between polynomial theory and number theory.

## Hybrid polynomial Systems:

Integrating Chebyshev polynomials into hybrid polynomial systems, combining them with other families of polynomials to create versatile mathematical frameworks with applications in diverse scientific and engineering disciplines.

In conclusion, there is potential for exploring a novel class of Chebyshev polynomials, involving generalizations and innovative matrix representations. By extending the scope to higher order matrices, we can derive insightful results related to traces and matrix powers. Additionally, there is an opportunity to establish deeper connections between various Chebyshev polynomials and sequences such as Pell, Lucas, and Fibonacci numbers, unveiling new relationships. Overall, the future of Chebyshev polynomials holds great potential for both theoretical advancements and practical applications across multiple scientific and engineering domains.

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## List of Publications

| Sr.No. | Title | Name of Journal | Indexed | Status |
|---|---|---|---|---|
| 1. | Some identities Involving Chebyshev Polynomial of third kind, Lucas Numbers and Fourth kind, Fibonacci Numbers. | IAENG International Journal of Applied Mathematics | Scopus | Published |
| 2. | Characteristic Equations of Chebyshev Polynomials of Third and Fourth Kinds and their Generating Matrices. | Contemporary Mathematics | Scopus | In Press |
| 3. | Relation of Chebyshev Polynomials to the Fibonacci, Lucas, and Pell Numbers. | AIP Conference Proceeding | Scopus | In Press |
| 4. | To Develop New Generalization of Chebyshev-Like Polynomials and Their Applications. | South East Asian Journal of Mathematics and Mathematical Sciences | Scopus | In Press |

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## List of Conferences

| Sr.No. | Title | Name of Conference | Date |
|---|---|---|---|
| 1. | Relation of Chebyshev Polynomials to the Fibonacci, Lucas, and Pell Numbers. | 7th international Joint Conference on Computing Sciences (ICCS-2023) KILBY100. | 05-05-2023. |
| 2. | New identities Connecting Chebyshev polynomials of the third kind, Fibonacci Numbers and Lucas Numbers. | International Conference on Research Trends in Contemporary Mathematics. | 03-02-2023 to 04-02-2023. |
| 3. | Some identities Involving Chebyshev Polynomial of third kind, Fibonacci Numbers, Lucas Numbers. | Multidisciplinary international Web conference on Emerging Trends & Challenges in Humanities, Education and Social Sciences. | 26-03-2022 to 27-03-2022. |

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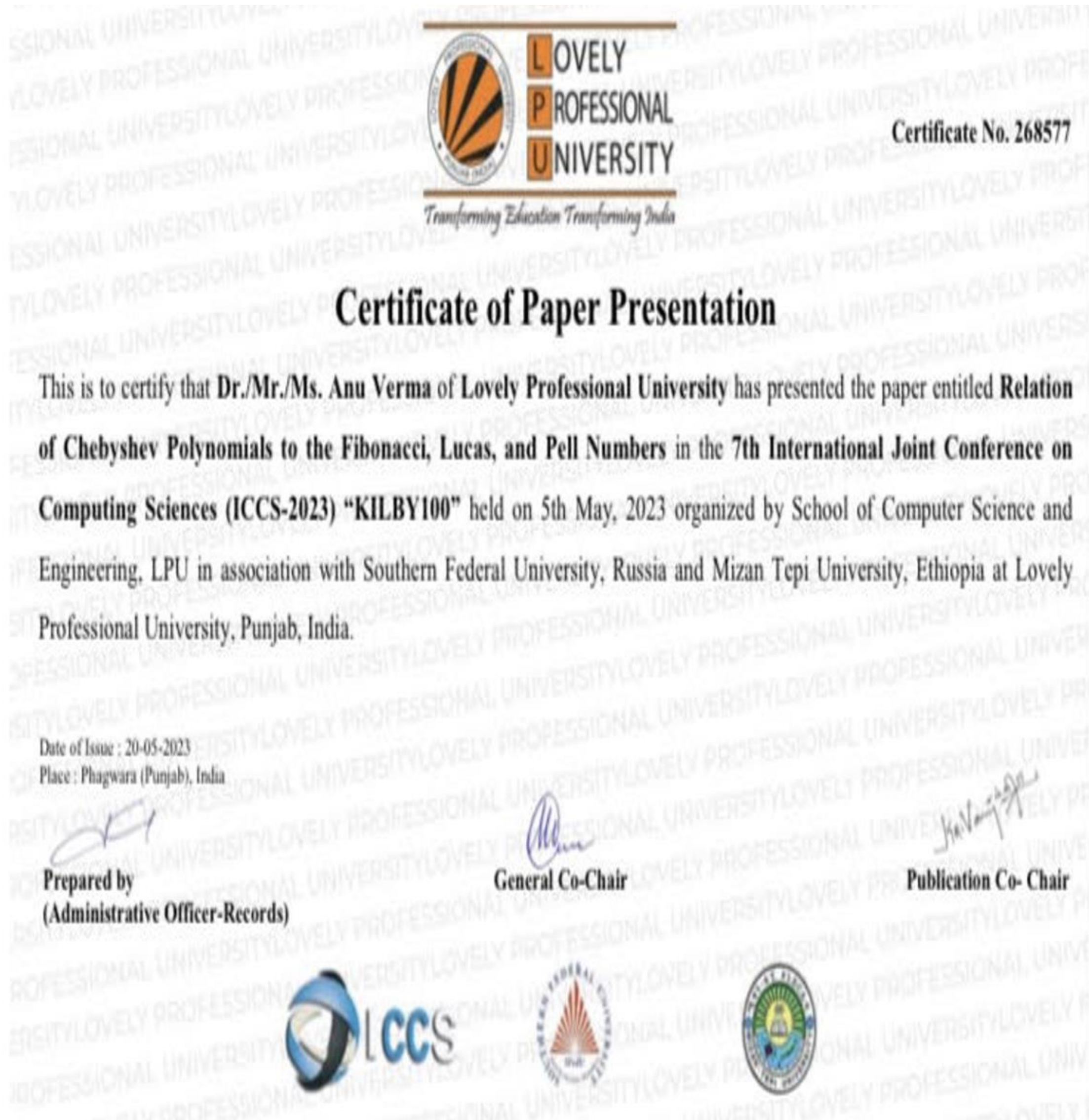

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