RESIDUAL POWER SERIES METHOD FOR SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS

Thesis Submitted for the Award of the Degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

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2024

DECLARATION

I, hereby declared that the presented work in the thesis entitled "**RESIDUAL POWER SERIES METHOD FOR SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS**" in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of **Dr. Geeta Arora**, working as **Professor**, in the **Department of Mathematics, School of Chemical Engineering and Physical Sciences, of Lovely Professional University, Punjab, India.** In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled "**RESIDUAL POWER SERIES METHOD FOR SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS**" submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the **Department of Mathematics, School of Chemical Engineering and Physical Sciences, of Lovely Professional University, Punjab, India**, is a research work carried out by **Rajendra Pant**, **12106076,** is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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Acknowledgements

I would want to express my gratitude to my son Suwarna Kumar Pant for his unwavering support in encouraging me to pursue a Ph. D. in mathematics, either in our home country of Nepal or in a neighbouring country, India, or overseas. I would express gratitude towards my working Far-western University's authorities who have provided study leave to me for academic enrichment.

I would express my gratitude towards my respected supervisor Dr. Geeta Arora who had been extra ordinary, live of dedication and diligence towards research. She gave her best to guide me and my work as a guardian.

Again I mention HOS, HOD and my colleagues of School of Chemical Engineering and Physical Sciences at Department of Mathematics, Lovely Professional University Punjab, India who helped me at various stages directly and indirectly.

Abstract

The area of fractional calculus known as "Applied Mathematics" examines integrals and derivatives of arbitrary orders and how they are used in a variety of fields, including mathematics, computer science, and physics. Since then, Calculus of rational order has been referred to as "Fractional Calculus". Mathematicians such as Cauchy Riemann have been exploring this strategy since the late 1800s. Many mathematicians have examined it since then. In the fields of science and engineering, fractional calculus is used to study natural and physical events that correspond to mathematical models with differential equation solutions. Much attention has been paid to the solutions of FDEs, integral equations, PDEs, and other problems. Since most FDEs are known to have an approximate analytical solution, approximations and numerical approaches are used so often.

There are numerous analytical methods that can be used to solve linear and non-linear FDEs. Among the more important techniques are the variational iteration method, the Fourier transform method, the Laplace transforms method, the green function method, the Homotopy perturbation method, the Adomian decomposition method, the differential transform method, the Homotopy analysis method, and the residual power series method. Numerous academics have looked into the numerical methods for solving FDEs. Much work has been done in the last year to develop robust numerical analytical methods for solving FDEs of physical interest.

This work is about exploring the application of residual power series method for the numerical solution of fractional order differential equations as well as numerical solution of these equations by using Laplace and Elzaki transforms with residual power series method. The relaxation-oscillation equations are solved by residual power series method. The Laplace residual power series method is used to solve one-dimensional and two-dimensional fractional order differential equations with simulations and graphs. The Elzaki residual power series method is used to solve two-dimensional diffusion equation. Also comparative study of solutions of diffusion equation by both ERPSM and LRPSM are also observed in this thesis. These methods of numerical solutions of such fractional order differential equations are very useful, effective and reliable. Some common techniques for solving fractional differential equations in time numerically are explained in this Thesis widely which verifies the importance of such methods. Separate discussions and convincing numerical examples are provided for the discretization techniques for the fractal, Riemann–Liouville, Caputo, and positive time-fractional derivatives. We refer to equations involving time fractional derivatives as time fractional differential equations. In Fractional Calculus, the solutions to such equations are very significant.

Table of Contents

List of figures

List of tables

Chapter 1

Introduction

1. Fractional Calculus

A branch of Applied Mathematics known as Fractional Calculus studies integrals and derivatives of arbitrary orders and their applications in Physics, Computer Science, Engineering, and Mathematics, among other subjects. Since then, the term "Fractional Calculus" has been used to refer to Calculus of rational order. This method is not new; mathematicians like as Cauchy Riemann began studying it around the end of the 1800s. After that several mathematicians have studied about it [1- 3]. Physical and natural phenomena of Fractional Calculus are studied in the field of engineering and science which represent mathematical models having solutions of differential equations. The solutions to fractional differential equations (FDEs), integral equations, partial differential equations (PDEs), and other problems of physical significance have received an attention of great deal. The majority of FDEs are known to have an approximate analytical solution that is why numerical methods and approximations are so frequently employed.

The solution of linear or non-linear FDEs can be achieved by a variety of analytical techniques. Variational iteration method [9], Fourier transform method [10], Laplace transforms method [7, 8], & green function methods [4-6] are a few of the more significant techniques. Many researchers [11, 12] have examined the numerical approaches used to solve FDEs. Over the past year, a great deal of work has been done to establish reliable numerical analytical approaches to solve FDEs of physical interest. Homotopy perturbation method [13], Adomian decomposition method [14], differential transform method [15], & Homotopy analysis method [13] are few of novel techniques for the analytical solution of FDEs both linear as well as non-linear.

Because FDEs concern requires complex mathematical solution methodologies, they are not only significant but also highly tough. It is very difficult to obtain exact solutions for FDEs and other equations; hence a reliable and very efficient numerical method is required to solve such FDEs.

Without discretization, linearization, or disturbance, residual power series method (RPSM) offers a quick and effective way to generate power series solutions for both linear and non-linear FDEs. This method does not necessitate a recursive relation of the coefficients of the connected terms, in contrast to the traditional power series approach. The RPSM determines the coefficients of power series using a collection of algebraic equations in one or more variables. This method is also used to find analytical solutions to FDEs of higher order.

This RPSM was developed to give a rapid and efficient method for finding the values of the coefficients in the power series solution for fuzzy differential equations [16]. This method has been successfully applied to,

-numerical solution of the highly nonlinear singular differential equation known as the generalised Lane-Emden problem [17].

-numerical solution of normal differential equations of greater order [18]

-nonlinear KdV-Burgers equation's approximate solution [21],

-solution of composite and non-composite FDEs [19],

-prediction and representation of variety of solutions of boundary value problems [20], and in many other applications as well [20, 22, 23].

-this method has recently led to the discovery of analytic solution of second order two-component evolutionary scheme. [24].

1.2 Some Useful Results

1.2.1 The Beta function

The definite integral defined by $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \; for \; m>0, n>0$ is called the beta function. Beta function is also known as the Eulerian integral of first kind and it is denoted by $\beta(m,n)$.

$$\therefore \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \; for \; m>0, n>0 \qquad (1.1)$$

1.2.2 The Gamma function

The definite integral defined by $\int_0^\infty e^{-x}\,x^{n-1}dx \;\; for \; n>0$ is called gamma function.

Gamma function is also known as the Eulerian integral of second kind and is denoted by $\Gamma(n)$.

$$\therefore \Gamma\,(n) = \int_0^\infty e^{-x}\,x^{n-1}dx \;\; for \; n>0 \qquad (1.2)$$

The integrals (1.1) and (1.2) are convergent for these values of m and n since it is evident that the former is valid only for m >0, n >0, while the latter is valid only for n >0.

Properties

i) $\Gamma(1) = 1$.

ii) $\Gamma(n+1) = n\Gamma(n) \quad n>0$.

iii) If n is a positive integer then $\Gamma(n+1) = n!$.

iv) $\Gamma(n) = \infty$ if n is zero or a negative integer.

v) An important result $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

vi) Symmetrical property of beta function, $\beta(m,n) = \beta(n,m)$.

Remark 1: The relation between beta and gamma function is, $\beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$ $, for\ m > 0, n > 0.$

1.2.3 Power Series

An infinite series, $\sum_{n=0}^{\infty} c_n\, (x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots$ is known as a general power series in $x - x_0$. In particular, the infinite series $\sum_{n=0}^{\infty} c_n\, x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ is known as power series in x.

Convergence

The series $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ is an absolutely convergent series for $|x| < R$, here $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$, provided that limit exists.

Radius of convergence and interval of convergence

The radius of convergence, denoted by a definite number R > 0, occurs when a given power series does not converge everywhere or nowhere, meaning that it is absolutely convergent for every |x| < R and divergent for every |x| > R. An interval of convergence is denoted by the open interval (-R, R).

Remarks

i) Power expansion of e^x is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

ii) A power series represents a continuous function inside its convergence interval.

iii) A power series is differentiable term by term inside its interval of convergence.

iv) Power series is considered nowhere convergent if it does not converges to any value other than x = 0. That is the power series $\sum_{n=0}^{\infty} n^n x^n$ is nowhere convergent.

v) A particular power series is everywhere convergent if it converges for all values of x. That is, the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent everywhere.

vi) The set of all values of x for which the given power series is convergent is referred to as its area of convergence if it converges for certain x & diverges for other x.

1.2.4 Analytic function

A function $f(x)$ defined on the period containing $x = x_0$ is known as an analytic function at x_0 if its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x - x_0)^n$ exists and converges to $f(x)$ for all x in that interval of convergence of its Taylor series.

Remarks

i) In short a function which is defined and differentiable at a point is also known as an analytic function at that point.

ii) All the polynomial functions, $Sin\, x, Cos\, x, Sinh\, x\; and\; Cosh\, x$ are analytic functions everywhere.

iii) A rational function is analytic except when its denominator is zero, which occurs at certain values of x. Such function given by $\frac{3x+1}{x^2-5x+6}$ is analytic everywhere except $x = 2\; and\; x = 3$.

1.2.5 Ordinary and singular points

i) A point $x = x_0$ is known as an **ordinary point** of differential equation

$y'' + P(x)y' + Q(x)y = 0$ if $P(x)$ & $Q(x)$ are both analytic at $x = x_0$.

ii) The point $x = x_0$in the differential equation above is referred to as a singular point if it is not an ordinary point. The singular points are of two types:

a) Regular singular point

A singular point $x = x_0$ of an equation is known as **regular singular point** if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$.

b) Irregular singular point

A singular point that is not regular is an irregular singular.

For example $2x^2 \frac{d^2y}{dx^2} + 7x(x+1)\frac{dy}{dx} - 3y = 0$ is a differential equation of order two. We may identify $x = 0$ is an ordinary point.

The above equation can be written as,

$$\frac{d^2y}{dx^2} + \frac{7(x+1)}{2x}\frac{dy}{dx} - \frac{3}{2x^2}y = 0.$$

Comparing this equation with $y'' + P(x)y' + Q(x)y = 0$, we get

$$P(x) = \frac{7(x+1)}{2x}\; and\; Q(x) = -\frac{3}{2x^2}$$

It is evident that in the case of $x = 0$ $neither P(x)\ nor\ Q(x)$ is analytic as they are both undefined. Consequently, since $x = 0$ is not ordinary point for equation provided, it is a **singular point**.

1.2.6 Laplace Transform

Assume that a function $f(t)$is defined for every $t > 0$. Then the Laplace transform of $f(t)$ is defined as the integral $\int_0^\infty e^{-pt} f(t)dt$, provided that it exists with parameter t and is denoted by $\mathcal{L}\{f(t)\}$ and written as $\bar{f}(\mathrm{p})$. The Laplace transform is a function of p.

$\therefore \bar{f}(\mathrm{p}) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-pt} f(t)dt.$

Again, function $f(t)$ is called inverse Laplace transform of $\bar{f}(\mathrm{p})$.

Laplace transform of function $f(t)\ for\ t > 0$ exists if,

i) $f(t)$ is continuous and

ii) $\lim_{t\to\infty} e^{-pt} f(t)$ is finite.

It is remarkable that conditions are sufficient but not necessary.

1.2.7 Laplace transforms of some standard functions

i) $\mathcal{L}(\mathrm{t^n}) = \frac{1}{\mathrm{p}}$**,** for ($p > 0$)

ii) $\mathcal{L}(\mathrm{t^n}) = \frac{\mathrm{n!}}{\mathrm{p^{n+1}}} = \frac{\Gamma(\mathrm{n+1})}{\mathrm{p^{n+1}}}$ where $n = 0,1,2 \ldots \ldots$

iii) $\mathcal{L}(e^{at}) = \frac{1}{p-a}$ $(p > 0)$

iv) $\mathcal{L}(\sin at) = \frac{a}{p^2+a^2}$ $(p > 0)$

v) $\mathcal{L}(\cos at) = \frac{p}{p^2+a^2}$ $(p > 0)$

vi) $\mathcal{L}(\sinh at) = \frac{a}{p^2-a^2}$ $(p > |a|)$

vii) $\mathcal{L}(\cosh at) = \frac{p}{p^2-a^2}$ $(p > |a|)$

1.2.8 Properties of Laplace transform

i) Linearity Property

If a, b, c are constants and f, g, h are functions of t then

$\mathcal{L}\{af(t) + bg(t) - ch(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} - c\mathcal{L}\{h(t)\}.$

ii) First shifting property

If $\mathcal{L}\{f(t)\} = \bar{f}(p)$ then $\mathcal{L}\{e^{at}f(t)\} = \bar{f}(p-a)$.

With the help of this property it can be determined that if $\bar{f}(p)$ is Laplace of $f(t)$ then the transform of $e^{at}f(t)$ can be simply written by replacing p by $p-a$ to get $\bar{f}(p-a)$. Again this property gives us the following useful formulae:

i) $\mathcal{L}(e^{at}) = \frac{1}{p-a} \quad (p > 0)$

ii) $\mathcal{L}(e^{at}t^n) = \frac{n!}{(p-a)^{n+1}} = \frac{\Gamma(n+1)}{(p-a)^{n+1}}$ *, where n=0, 1, 2, … …*

iii) $\mathcal{L}(e^{at}\sin bt) = \frac{b}{(p-a)^2+b^2}$

iv) $\mathcal{L}(e^{at}\cos bt) = \frac{p-a}{(p-a)^2+b^2}$

v) $\mathcal{L}(e^{at}\sinh bt) = \frac{b}{(p-a)^2-b^2}$

vi) $\mathcal{L}(e^{at}\cosh bt) = \frac{p-a}{(p-a)^2-b^2}$, where $p > a$ in each case.

1.3 Review of Literature

FDEs can be found in many fields of mathematics and statistics as well as in the investigation of numerous scientific and biological phenomena. These equations have extensive applications in Physics, Mechanics, Fluid Mechanics, Optics, Visco-elasticity, Electric Networks, Signal Processing, Image Processing, Fractional Dynamical approach, and numerous other fields of Mathematics including Physics. The analytical technique is a challenging approach to solve these non-linear fractional problems, and in certain cases, it is not feasible. Determining the coefficients in series form is a challenging task when using an analytical approach.

Numerous methods, including differential transform method, homotopy perturbation method, Adomian decomposition method, homotopy analysis method, Laplace transform method, modified least squares homotopy perturbation method, and other methods, are available in the literature for the analytical solutions [25] of non-linear FDEs and ordinary PDEs.

The analytical technique RPSM is based on power series extension without any linearization, without any perturbation, and without discretization. The error function of the residual power series and the generalised Taylor series formula serve as the broad foundation for this approach. This is an alternate method for handling linear or non-linear differential equation solutions. This method used for solution of FDEs that are linear or non-linear. The Sharma-Tasso-Oliver equation is solved using the RPSM in the work suggested by researcher [27]. Literature demonstrates this method is dependable and effective analytical technique for solving many non-linear fractional order differential equations.

When a non-linear FDE is solved using this method, a power series solution for a truncated series is obtained. One way to find the coefficients of power series is to solve a set of one or more variable algebraic equations. Consequently, we have a series solution for the non-linear FDE under consideration. By choosing an approximate sufficient initial guess, this method's primary advantage over other ways is its ability to be applied quickly to the current scenario.

The error role of residual series and generalised Taylor series formula serve as the broad foundation for this approach. The approach presented in this study, according to the authors, is effective, simple to use, and trustworthy when applied to various non-linear FDEs that arise in Science, Technology, and Mathematics.

The least squares method combined with RPSM is provided in Zhang et al.'s work [28]. This method's numerical computations rely on the idea of Caputo derivative. First, the analytical answer can be found through the use of the conventional RPSM. Subsequently, the notion of Wronskian is employed to verify the functions' linear independence. In addition, a linear grouping of first few terms which have unknown coefficients is utilised as an approximate solution. Compared to the classical RPSM, this method requires fewer term expansions to obtain the estimated solutions and unknown coefficients.

The non-linear FDEs that occur in a number of fields of Mathematics as well as Physics are solved using the boosted RPSM with Laplace transform method [29]. The temporal-fractional derivatives in this work represent physical applications with certain memory properties. These memories are recognised as a homotopy mapping that maintains the physical nature of the fractional solutions while mapping them into the integer solution. This paper introduces a novel strategy that combines the RPSM with the Laplace transform approach. It has been used with the time-space Benney-Lin equation that arises in falling film situations as well as the temporal-fractional Newell-Whitehead-Segel approach [30].

The generic residual power series solutions are used to derive analytical solutions for non-linear FDEs with variable coefficients in the study Chen et. al. [31]. It is discovered that the approach is straightforward, potent, and successful.

FDEs with variable coefficients, parabolic non-homogeneous equations, wave equations in space, and heat equations in planes can all have their analytical series solutions found [32]. Additionally, Klein-Gordon Schrodinger problem [33] may be successfully and efficiently solved using the RPSM. The approximate solution of non-linear FDEs using the RPSM was also described in this study by Khadar et al. [34]. Fractional integro-differential equations can be solved using Homotopy perturbation method and variational iteration method [35].

The authors successfully present a unique method for getting the approximate solution of the biological population diffusion equation. In this study, Alquran et al. [36] generalised the Taylor series and established the foundation for residual power series, also which is used to solve multi-pantograph delay differential equations [22] and be expanded to handle two-dimensional models. The RPSM can also be used to discover the analytical solution of the gas dynamic equations [38] and Schrodinger equations [37].

The fractional initial Emden-Folwer equation [39] can be solved by applying RPSM. The series solution of non-linear higher PDEs can be found quickly and easily using this method. This method is more useful for solving non-linear FDEs.

Numerous well-known equations have been investigated in literature to answer FDEs using RPSM, however additional work can be done to solve FDEs utilising different fractional derivative approaches. To solve such FDEs by RPSM, the applications of Laplace transform and other available transforms can also be implemented.

1.4 Difference between analytic and numerical approximate method for solution of FDEs

The difference between analytic & numerical approximate methods of solution of FDEs is identical in pure and applied mathematics. In practice both involve each other so there is no fine difference between these two. According as the literature the researchers have used this categorization particularly. Analytic approach provides unending nearby of the problem most important to the exact solution or an analytical solution where as numerical solution is all about reaching the best approximate solution. Analytic method applies the methods of analysis and numerical method applies the methods of numerical analysis. First method gives a comprehensive general solution where as the second method gives mainly a problem based distinct solution. As a result solving a differential equation using transformation or special functions is considered analytical and using approximate derivatives/ initial solutions or other applications to solve the differential equation comes in the numerical methods' type. In this thesis, the FDEs have been solved using both ideologies. Out of many analytic and numerical methods residual power series approach with different transforms used to solve such FDEs. These methods have a vast literature on the solution of FDEs. They have been used to solve such various FDEs in this thesis.

1.5 Objectives of the Research Work

The specific objectives of this research work approved in the state of the art of seminar (SOTA) are given as follows:

1) To apply the concept of residual power series method to solve fractional differential equations.

2) To solve the fractional differential equations in one-dimension by Laplace transform with residual power series approach.

3) To solve the fractional differential equations in two dimensions by Laplace transform with residual power series approach.

4) To implement residual power series approach by other available transforms in fractional differential equations.

1.6 Organisation of the thesis

This work aims and determines to find a suitable and efficient technique for numerical solutions of the FDEs. In particular residual power series method (RPSM), Laplace transform with residual power series method (LRPSM) as well as Elzaki transform with residual power series methods (ERPSM) are implemented for numerical solutions of FDEs with their programming in this research work. In present work, these methods have been applied to solve some different fractional order differential equations.

The first chapter provides an introduction to Fractional Calculus along with preliminaries' information, basic concepts, and an overview of FDEs with literature.

In the second chapter, the relaxation-oscillation equations of fractional order have been solved by using RPSM.

In the third chapter, one-dimensional FDEs such as BBM-Burger, Schrödinger, Fisher's and logistic differential equations are solved by using LRPSM. One part of it is also presented in an international conference conducted by IRDCP-2022.

In the fourth chapter, LRPSM is used to solve two dimensional FDEs such as diffusion and biological population equations.

In the fifth chapter, ERPSM is used to solve two dimensional fractional order diffusion equation and it is also presented in an international conference conducted by RAMSA-2022.

In the sixth chapter, the conclusion of the research work and the future scopes of this research work are outlined.

Chapter 2

Solution of Relaxation-Oscillation Equations by Residual Power Series Method

In this chapter, the solution of relaxation-oscillation equations via residual power series method (RPSM) is explained.

FDEs are coming to the attention of researchers nowadays as they are able to discuss the phenomena happening in the real world with deep insight. Recently, authors have presented a mathematical model involving a FDE for optimal control of a pandemic, investigating the disease dynamics. It is obvious that the concept of a fractional derivative will be traced back to the genesis of integral calculus [40]. These equations have received valuable attention in recent years, including various fractional forms. Researchers have investigated the fractional order derivatives of various forms, including Riemann-Liouville, Caputo-Fabrizio, and Atangana-Baleanu integrals. Because most FDEs do not have analytical solutions, we must use an approximate approach to solve them. To tackle and solve FDEs, there are approximate techniques such as Variational iteration method [41], Adomian decomposition method [42], operational matrix method [43], collocation method [44], and Tau method [45]. Researchers are continuously working to enhance the methods for solving fractional equations, with a focus on existence and uniqueness of the equations [46].

Many mathematicians have recently acknowledged that the fractional models can explain natural phenomena in a systematic and accurate way as compared to the classic integer-order time-derivatives. Researchers are currently using fractional calculus to explain many complex fractional biological systems, such as the zooplankton-phytoplankton system [47] and the study of a non-linear doffing oscillator. FDEs also exist in the chemical and physical studies of applications involving solute transport models [48] and many more.

Relaxation-oscillator is a type of oscillator that depends on behaviour of physical phenomena that return to equilibrium after having been distributed. Relaxation-oscillation models are of many types, involving fractional derivatives [49], appropriate fractional derivatives [50], and fractal derivatives [51]. Relaxation-oscillation equation is major equation of relaxation and oscillation processes.

Standard relaxation equation is given as follows:

$$\frac{du}{dt} + Pu = f(t), \tag{2.1}$$

where P denotes the elastic modulus and $f(t)$ denotes strain rate multiplied by P. If $f(t) = 0$ then the analytic solution of this equation is $u(t) = Ae^{-Pt}$, where A is a constant obtained by using initial conditions.

Also, standard oscillation equation be given as:

$$\frac{d^2u}{dt^2} + Pu = f(t), \tag{2.2}$$

where $P = \frac{k}{m}$, k denotes stiffness coefficient and m is the mass. When $f(t) = 0$ then the analytic solution of this equation is $u(t) = Ccos\sqrt{P}t + Dsin\sqrt{P}t$, where C and D are the constants to be obtained by using initial conditions.

To represent slow relaxation damping oscillation [52], fractional derivatives have been applied in the relaxation-oscillation modules. The fractional relaxation-oscillation model may be written as:

$$D_t^{\beta} u(t) + Au(t) = f(t), \qquad t > 0,$$

$$u(0) = a \; if \; 0 < \beta \leq 1,$$

$$u(0) = \lambda \; and \; u'(0) = \mu \; if \; 1 < \beta \leq 2, \tag{2.3}$$

where A, a +ve constant.

When $0 < \beta \leq 2$ then this equation is known as a relaxation-oscillation.

When $0 < \beta < 1$ then this relation gives the relaxation by means of power law attenuation.

When $1 < \beta < 2$ then this relation gives the damped oscillation having viscoelastic essential damping of oscillator [53].

Such relation has been applied for modelling cardiac pacemakers [54], spruce-budworm interactions [55], predator-prey system [56] and electrical form of heart and signal processing [57]. Due to their wide applicability, fractional relaxation-oscillation equations have been solved by various approaches, including the cubic B-spline wavelet collocation method [58], the reproducing kernel Hilbert space method [59], Adomian's method [60], the differential transform method [61], and so on.

The problems in FDEs are not only important but also quite challenging to solve using mathematical techniques. Therefore, a reliable as well as efficient method for the solution of such equations is required. One of the most reliable and efficient methods for solving FDEs is the residual power series approach. This method is a capable technique for finding solutions of such equations. This approach is applied to work out various FDEs, such as the Sharma-Tasso-Olever equation in fractional form [27], approximate solutions of fractional IVP's [62], fractional population diffusion equations [63], fractional fuzzy delay differential equation [64], fractional Bousinesq equation [65], time-fractional Schrodinger equations [38], etc. The current effort aims to solve relaxation-oscillation equations via RPSM. The method has the capability to solve fractional order differential equations and has the potential to be hybridised with other distinguished transforms as Laplace [66], Elzaki [63] & Sumudu [67]. The stability and convergence analysis of fractional order differential equation solutions is dependent on the variables, order of fractional derivatives, and domain.

2.1 Methodology for Implementation

RPSM is a very powerful and widely used one-of-a-kind method for obtaining approximate solutions to ODE & FDEs. This methodology has the benefit of allowing one to find the solution in terms of a power series, where the coefficients are found through a sequence of algebraic procedures. This method has implemented efficiently to find approximate analytical solutions of various linear and non-linear FDEs of higher order [62–65]. The basic steps for implementing RPSM can be demonstrated as follows:

Step 1:

Consider a fractional differential equation with the considered Caputo-fractional derivative,

$$D_t^{n\alpha} z(x,t) + L[x]z(x,t) + N[x]\, z(x,t) = \emptyset(x,t), \tag{2.4}$$

For $t > 0$, $x \in R$, $n\text{-}1 < n\alpha \leq n$, with initial condition, $\emptyset_0(x) = z(x,0)$,

and $\emptyset_{n-1}(x) = D_t^{(n-1)\alpha} z(x,0) = \mu(x)$, (2.5)

where $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$,

$L[x]$ = linear function in x, $N[x]$ = universal non-linear function in x, and $\emptyset(x, t)$ = continuous operator. For RPSM, solution of FDEs (2.4) & (2.5) is written as non-integer power series form with primary value t = 0.

Step 2:

Suppose the solution has expansion relation of the form,

$$z(x,t) = \sum_{n=0}^{\infty} \emptyset_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \quad 0 < \alpha \leq 1, x \in I, 0 \leq t < R. \tag{2.6}$$

Step 3:

The k[th] truncated chain for $z(x, t)$ denoted as $z_k(x,t)$ has been defined as follows,

$$z_k(x,t) = \sum_{n=0}^{k} \emptyset_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{2.7}$$

for, $k = 1,2,3, \dots$. Evidently $z(x,t)$ satisfy initial condition (2.5) and hence the equation results in $z(x,0) = \emptyset_0(x)$. (2.8)

Step 4:

The initial guess or first RPS approximation from equation (2.7) must be,

$$z_1(x,t) = \emptyset(x) + \emptyset_1(x) \frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.9}$$

Hence, it may reformulate expansion of equation (2.7) as given below,

$$z_k(x,t) = \emptyset(x) + \emptyset_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \sum_{n=2}^{k} \emptyset_n(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \text{ with } k = 2,3,4,\dots\dots, \quad (2.10)$$

Step 5:

Now for finding coefficients $\emptyset_n(x)$ (i.e. values) by RPS techniques $n = 1, 2, 3, \dots, k$ in series form of equation (2.10) residual function is defined as follows:

$$\mathcal{R}es(x,t) = D_t^{n\alpha} z(x,t) + L[x]z(x,t) + N[x]\, z(x,t) - \emptyset(x,t). \quad (2.11)$$

and then k^{th} residual function $\mathcal{R}es_k$ in desirable form is as follows:

$$\mathcal{R}es_k(x,t) = D_t^{n\alpha} z_k(x,t) + L[x]z_k(x,t) + NL[x]z_k(x,t) - \emptyset(x,t), k = 1, 2, 3, \dots \quad (2.12)$$

It is well known that, $\mathcal{R}es(x, t) = 0$, and $\lim_{k\to\infty} \mathcal{R}es_k(x,t) = \mathcal{R}es(x, t)$, for $x \in I$ and $t \geq 0$.

Step 6:

From step 5 we get the results,

$D_t^{(n-1)\alpha} \mathcal{R}es_n(x, t_0), = 0, n = 1, 2, 3, \dots\dots k,$ so that fractional derivatives $D_t^{(n-1)\alpha}$ of $\mathcal{R}es(x, t)$ and $\mathcal{R}es_n(x, t)$ are identical at $t = 0$ for every $n = 1, 2, 3, \dots$, k,

i.e. $D_t^{(n-1)\alpha} \mathcal{R}es\,(x, 0) = D_t^{(n-1)\alpha} \mathcal{R}es_n(x, 0) = 0, n = 1, 2, 3, \dots$, k.

Step 7:

For finding the coefficients, $\emptyset_n(x), n = 1, 2, 3, \dots$, k, Substitute n^{th} truncated series for z(x, t) on equation (2.12), and operate the fractional derivative procedure $D_t^{(n-1)\alpha}$ *on* $\mathcal{R}es_n(x,t), n = 1, 2, 3, \dots$, k, on substituting $t = 0$ and solving the obtained numerical equation results in the required type of the other coefficients.

Step 8:

At last, to solve the following fractional differential algebraic equations completely put residue equal to zero as,

$$D_t^{(n-1)\alpha} \mathcal{R}es_k(x,t) = 0, 0 < \alpha \leq 1, x \in I, 0 \leq t < R, n = 1, 2, 3, \dots\dots, \text{k}. \quad (2.13)$$

Hence this is the way of finding all the necessary coefficients of numerous power series of FDEs (2.4) & (2.5).

Convergence of RPSM has been discussed in detail by Momani et. al. [68] showing residual power series as a Taylor's series expansion that converges to the exact solution with the increase in the number of terms.

Suppose $z_i(t)$ is the exact solution of the considered equation and $z_i^k(t)$ is the approximate solution with k-terms of that equation. Then the difference between these two solutions $z_i(t)$ and $z_i^k(t)$ is denoted by $Rem_i^k(t)$ and is given by

$Rem_i^k(t) = z_i(t) - z_i^k(t)$

$Rem_i^k(t) = z_i = \sum_{m=k+1}^{\infty} \frac{z_i^m(t_0)}{m!}(t-t_0)^m$

which is k[th] –remainder of Taylor's series of $z_i(t)$ that approaches to zero with increase in number of terms in approximate analytical solution of considered equation.

2.2 Related Examples

Example 2.2.1 Consider the relaxation-oscillation equation in fractional form as,

$D_t^\alpha u(t) + u(t) = 0$ with the primary situation $u(0) = 1$ (2.14)
Exact solution of equation when $\alpha = 1$ is $u(t) = e^{-t}$.

Now for $k = 1$, first truncated series approximation is of the form,

$$u_1(t) = 1 + c_1 \frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{2.15}$$

and first residual function is

$$Resu_1(t) = D_t^\alpha u_1(t) + u_1(t) \tag{2.16}$$

$$= D_t^\alpha \left\{1 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha\right\} + 1 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha$$

$$= \frac{c_1}{\alpha!}\alpha! + 1 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha$$

$$= c_1\left\{1 + \frac{t^\alpha}{\Gamma(1+\alpha)}\right\} + 1$$

Again for residue $Resu_1(0) = 0$ gives us $c_1\left\{1 + \frac{0^\alpha}{\Gamma(1+\alpha)}\right\} + 1 = 0$

$or, c_1 + 1 = 0 \; or, c_1 = -1.$

Hence from (2.15) first residual power series is given as, $u_1(t) = 1 - \frac{t^\alpha}{\Gamma(1+\alpha)}$.

For $k = 2$, second truncated approximation is of the form,

$$u_2(t) = 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + c_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{2.17}$$

and second residual function is given as,

$$D_t^\alpha Resu_2 = D_t^\alpha\{D_t^\alpha\left(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + c_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) + 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + c_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\} \tag{2.18}$$

$$= c_2 \frac{(2\alpha)!}{(2\alpha)!} - \frac{\alpha!}{\alpha!} + c_2 \frac{\alpha!}{(2\alpha)!} t^{\alpha}$$

$$= c_2 - 1 + c_2 \frac{\alpha!}{(2\alpha)!} t^{\alpha}$$

But $D_t^{\alpha} Resu_2(t) = 0 \; for \; t = 0$ gives us $c_2 - 1 = 0 \; or, c_2 = 1$.

Hence, second series solution from (2.17) is written as,

$$u_2(t) = 1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

For $k = 3$, third truncated series approximation is,

$$u_3(t) = 1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \tag{2.19}$$

and third residual function is

$$D_t^{2\alpha} Resu_3(t) = D_t^{2\alpha} \left\{ D_t^{\alpha} \left(1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) + 1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right\} \tag{2.20}$$

$$= c_3 + 1 + c_3 \frac{(2\alpha)!}{(3\alpha)!} t^{\alpha}$$

But $D_t^{2\alpha} Resu_3(t) = 0$ for $t = 0$ gives that $c_3 + 1 = 0 \quad or, c_3 = -1$.

Hence from (2.19), third residual series solution is given by,

$$u_3(t) = 1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}.$$

In the same way, fourth residual power series solution is,

$$u_4(t) = 1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}.$$

Therefore, in general series solution of the given equation is,

$$u(t) = 1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} - \cdots. \tag{2.21}$$

The both solutions of the equation for different values of t are calculated & compared. The comparison of exact with approximate solution taken as 5 to 10 terms are presented in Table 2.1 when value of $\alpha = 1$. The point-wise errors are also shown in Table 2.2 with the exact solution taking eleven terms. The particular solution for different values of t in exact solution and absolute errors containing different number of terms of equation 1 for $\alpha = 0.5$ are given in Table 2.3 to observe reliability of this approach. The graph of exact solution of equation 2.3.1 and its approximation solutions with different number of terms are shown in Figure 2.1. Also the graph of errors of RPSM with respect to exact solution of equation 2.3.1 are as shown in Figure 2.2, which shows that the reliability and efficiency of this method.

Table 2.1 Comparison of both solutions with different number of terms of example 2.3.1

		Number of terms					
t	Exact solution	5	6	7	8	9	10
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.9048374	0.9047542	0.9048374	0.9048374	0.9048374	0.9048374	0.9048374
0.2	0.8187308	0.8193500	0.8187306	0.8187307	0.8187307	0.8187307	0.8187307
0.3	0.7408182	0.7408375	0.7408341	0.7408351	0.7408182	0.7408182	0.7408182
0.4	0.6703200	0.6704000	0.6703146	0.6703203	0.6703200	0.6703200	0.6703200
0.5	0.6065307	0.5234375	0.6065104	0.6065321	0.6065305	0.6065306	0.6065306
0.6	0.5488160	0.5494000	0.5487520	0.5488168	0.5488112	0.5488116	0.5488116
0.7	0.4965853	0.4978375	0.4964369	0.4966003	0.4965839	0.4965854	0.4965852
0.8	0.4493290	0.4517333	0.4490026	0.4493667	0.4493251	0.4493293	0.4493389
0.9	0.4065697	0.4108375	0.4059167	0.4066548	0.4065599	0.4065704	0.4065691
1.0	0.3678794	0.3750000	0.3666666	0.3680555	0.3678571	0.3678819	0.3678791

Table 2.2 Point-wise absolute errors with 11 terms for example 2.3.1

T	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
Errors	0	0	0	0	0	1.0e-11	9.0e-11	4.7e-10	2.0e-09	7.31e-09	2.31e-08

Table 2.3 Absolute errors for $\alpha = 0.5$ at t=0 to 0.8 of equation 2.3.1 in comparison with OHAM [69] and GTMM [52].

		Absolute Errors			
t	Exact solution	RPS(n=10)	RPS(n=20)	OHAM[69]	GTMM[52]
0.0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.7288934	9.7467e-09	0.0000000	0.1508e-03	0.2355e-04
0.2	0.6394073	4.2132e-07	3.5527e-15	0.6078e-04	0.2523e-03
0.3	0.5782652	3.7879e-06	2.3370e-13	0.7485e-04	0.1002e-02
0.4	0.5312856	1.7927e-05	4.6866e-12	0.1313e-03	0.2655e-02
0.5	0.4930686	5.9724e-05	4.7896e-11	0.1000e-03	0.5638e-02
0.6	0.4608896	1.5939e-04	3.1952e-10	0.1357e-04	0.1043e-01
0.7	0.4331548	3.6510e-04	1.5882e-09	0.7979e-04	0.1746e-01
0.8	0.4088417	7.4784e-04	6.3656e-08	0.1256e-03	0.2732e-01

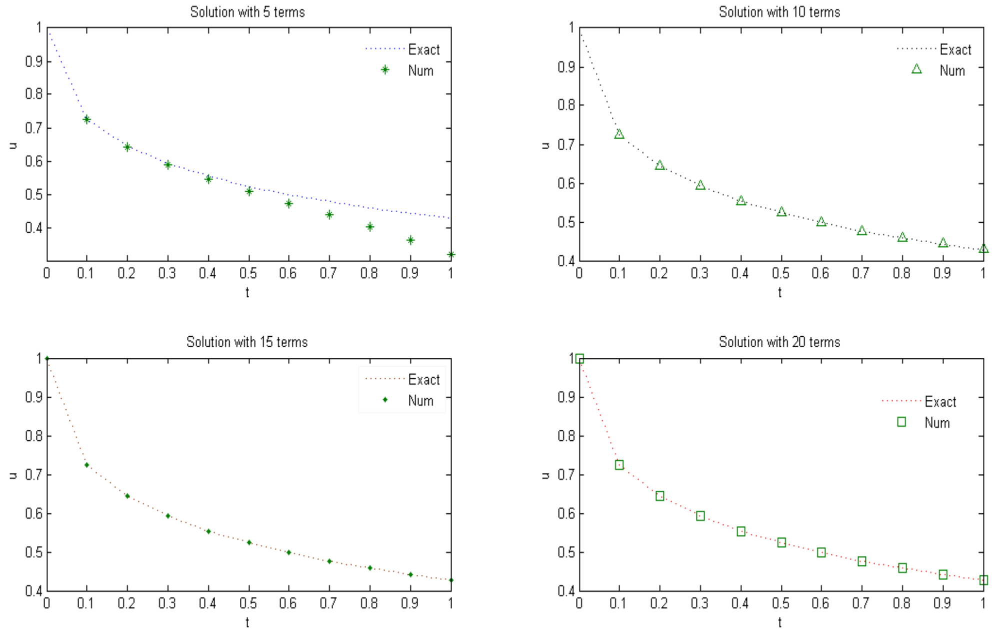


Figure 2.1 Exact solution and approximation solution of example 1 with different number of terms.

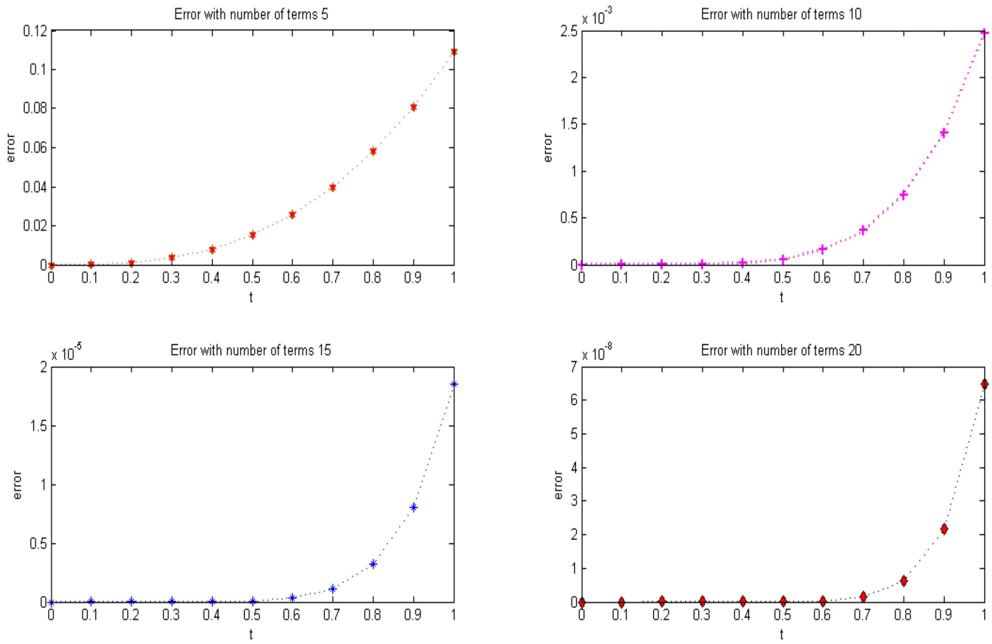


Figure 2.2 Error of both solutions of equation 2.3.1 with different number of terms.

Example 2.2.2 Consider following fractional equation,

$$D_t^{\alpha} u(t) - 4u(t) = 0 \text{ having primary condition } u(0) = 1. \tag{2.22}$$

Exact solution of equation when $\alpha = 1$ is $u(t) = e^{4t}$.

Now for $k = 1$, first truncated approximation is,

$$u_1(t) = 1 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha, \tag{2.23}$$

and first residual function is

$$Resu_1(t) = D_t^\alpha u_1(t) - 4u_1(t) \tag{2.24}$$

$$= D_t^\alpha \left\{1 + \frac{c_1}{\Gamma(1+\alpha)} t^\alpha\right\} - 4 - 4\frac{c_1}{\Gamma(1+\alpha)} t^\alpha$$

$$= \frac{c_1}{\alpha!}\alpha! - 4 - 4\frac{c_1}{\Gamma(1+\alpha)} t^\alpha$$

$$= c_1\left\{1 - 4\frac{t^\alpha}{\Gamma(1+\alpha)}\right\} - 4$$

Again for residue $Resu_1(0) = 0$ gives us $c_1\left\{1 - \frac{0^\alpha}{\Gamma(1+\alpha)}\right\} - 4 = 0\ or, c_1 = 4.$

Hence from (2.23) first residual power series is, $u_1(t) = 1 + 4\frac{t^\alpha}{\Gamma(1+\alpha)}$

For $k = 2$, second truncated approximation is,

$$u_2(t) = 1 + 4\frac{t^\alpha}{\Gamma(1+\alpha)} + c_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{2.25}$$

and second residual function is

$$D_t^\alpha Resu_2(t) = D_t^\alpha\{D_t^\alpha u_2(t) - 4u_2(t)\} \tag{2.26}$$

$$= D_t^\alpha\{D_t^\alpha\left(1 + 4\frac{t^\alpha}{\Gamma(1+\alpha)} + c_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) - 4 - 16\frac{t^\alpha}{\Gamma(1+\alpha)} - 4c_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\}$$

$$= c_2\frac{(2\alpha)!}{(2\alpha)!} - 16\frac{\alpha!}{\alpha!} - 4c_2\frac{\alpha!}{(2\alpha)!}t^\alpha$$

$$= c_2 - 16 - 4c_2\frac{\alpha!}{(2\alpha)!}t^\alpha$$

But $D_t^\propto Resu_2(t) = 0\ fort = 0$ gives us $c_2 - 16 = 0\ or, c_2 = 16.$

Hence from (2.25), second residual power series solution is,

$$u_2(t) = 1 + 4\frac{t^\alpha}{\Gamma(1+\alpha)} + 16\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

For $k = 3$, third truncated approximation is,

$$u_3(t) = 1 + 4\frac{t^\alpha}{\Gamma(1+\alpha)} + 16\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \tag{2.27}$$

and the third residual function is

$$D_t^{2\alpha} Resu_3(t) = D_t^{2\alpha} \left\{ D_t^{\alpha} \left(1 + 4\frac{t^{\alpha}}{\Gamma(1+\alpha)} + 16\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3\, \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right) - 4 - 16\frac{t^{\alpha}}{\Gamma(1+\alpha)} - 64\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - 4c_3\, \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right\} \quad (2.28)$$

$$= c_3 - 64 - c_3 \frac{(2\alpha)!}{(3\alpha)!} t^{\alpha}.$$

But $D_t^{2\alpha} Resu_3(t) = 0\; for t = 0$ gives that $c_3 - 64 = 0\;\; or, c_3 = 64$.

Hence from (2.27), third residual power series solution is given as,

$$u_3(t) = 1 + 4\frac{t^{\alpha}}{\Gamma(1+\alpha)} + 16\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + 64\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}.$$

In the same way fourth residual power series solution is,

$$u_4(t) = 1 + 4\frac{t^{\alpha}}{\Gamma(1+\alpha)} + 16\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + 64\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 256\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}.$$

Therefore, power series solution of given equation is

$$u(t) = 1 + 4\frac{t^{\alpha}}{\Gamma(1+\alpha)} + 16\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + 64\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 256\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots. \quad (2.29)$$

Both solutions of this equation for different values of t is calculated compared. The comparison of both solutions taken as 10, 15, and 20, which are presented in Table 2.4, when value of $\alpha = 1$. The point-wise errors are also shown in Table 2.4 with the exact solution taking eleven terms.

Table 2.4 Point-wise values & errors of both solutions of example 2.2.2.

		Number of terms					
		10		15		20	
t	**Exact solution**	Value	Error	Value	Error	Value	Error
0.0	1.0000000	1.0000000	0	1.0000000	0	1.0000000	0
0.1	1.4918246	1.4918246	1.08e-12	1.4918246	2.22e-13	1.4918246	2.22e-16
0.2	2.2255409	2.2255409	2.30e-09	2.2255409	1.33e-12	2.2255409	0
0.3	3.3201169	3.3201167	2.06e-07	3.3201169	9.49e-10	3.3201169	4.44e-16
0.4	4.9530324	4.9530273	5.07e-06	4.9530324	9.72e-08	4.9530324	0
0.5	7.3890560	7.3889947	6.13e-05	7.3890560	3.54e-06	7.3890560	4.61e-14
0.6	11.0231763	11.0227019	4.74e-04	11.0231763	6.73e-05	11.0231763	2.11e-12
0.7	16.4446467	16.4419542	2.69e-03	16.4446459	8.15e-04	16.4446467	5.50e-11
0.8	24.5325301	24.5203334	1.21e-02	24.5325230	7.09e-06	24.5325301	9.27e-10
0.9	36.5982344	36.5517073	4.65e-02	36.5981863	4.80e-05	36.5982344	1.12e-08
1.0	54.5981500	54.4431040	1.55e-01	54.5978829	2.67e-04	54.5981499	1.04e-07

Example 2.2.3 Consider following fractional equation,

$$D_t^{\alpha} u(t) - u(t) - 1 = 0 \text{ with primary condition } u(0) = 0. \quad (2.30)$$

Exact solution of equation when $\alpha = 1$ is $u(t) = e^t - 1$.

Now for $k = 1$, first truncated approximation is,

$$u_1(t) = \frac{c_1}{\Gamma(1+\alpha)}\alpha, \ (\because \ u(0) = 0) \tag{2.31}$$

and first residual function is,

$$Resu_1(t) = D_t^\alpha u_1(t) - u_1(t) - 1 \tag{2.32}$$
$$= D_t^\alpha \left\{\frac{c_1}{\Gamma(1+\alpha)} t^\alpha\right\} - \frac{c_1}{\Gamma(1+\alpha)} t^\alpha - 1$$

$$= \frac{c_1}{\alpha!}\alpha! - \frac{c_1}{\Gamma(1+\alpha)} t^\alpha - 1$$

$$= c_1 \left\{1 - \frac{t^\alpha}{\Gamma(1+\alpha)}\right\} - 1.$$

Again residue $Resu_1(0) = 0$ gives us $c_1 \left\{1 - \frac{0^\alpha}{\Gamma(1+\alpha)}\right\} - 1 = 0 \ \ or, c_1 - 1 = 0 \ \ or, c_1 = 1$.

Hence from (2.31), first residual power series is, $u_1(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$.

For $k = 2$, second truncated approximation is,

$$u_2(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} + c_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. (\because \ u(0) = 0) \tag{2.33}$$

And second residual function is,

$$D_t^\alpha Resu_2(t) = D_t^\alpha \{D_t^\alpha \left(\frac{t^\alpha}{\Gamma(1+\alpha)} + c_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) - \frac{t^\alpha}{\Gamma(1+\alpha)} - c_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - 1\} \tag{2.34}$$

$$= c_2 \frac{(2\alpha)!}{(2\alpha)!} - \frac{\alpha!}{\alpha!} + c_2 \frac{\alpha!}{(2\alpha)!} t^\alpha$$

$$= c_2 - 1 + c_2 \frac{\alpha!}{(2\alpha)!} t^\alpha.$$

But $D_t^\alpha Resu_2(t) = 0 \, for \ t = 0$ gives us $c_2 - 1 + c_2 \frac{\alpha!}{(2\alpha)!} 0^\alpha = 0 \ or, c_2 - 1 = or, c_2 = 1$.

Hence from (2.33), second residual power series solution is,

$$u_2(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

Therefore in general, power series solution of given equation is,

$$u(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots. \tag{2.35}$$

Both solutions of this equation for different values of t is compared. The comparison of the exact solution with numerical values of approximate solutions of the different number of terms, taken 10, 15 and 20 terms presented in Table 2.5 when the value of $\alpha = 1$. The point-wise errors are also shown in Table 5 with the exact solution taking eleven terms.

Table 2.5 Point-wise values & errors of both solutions of example 2.2.3 for different number of terms.

		Number of terms					
t	Exact solution	10		15		20	
		Value	Error	Value	Error	Value	Error
0.0	0.0000000	0.0000000	0	0.0000000	0	0.0000000	0
0.1	0.1051709	0.1051709	8.32e-17	0.1051709	8.32e-17	0.1051709	8.32e-17
0.2	0.2214027	0.2214027	4.99e-16	0.2214027	0	0.2214027	0
0.3	0.3498588	0.3498588	4.55e-14	0.3498588	5.55e-17	0.3498588	5.55e-17
0.4	0.4918246	0.4918246	1.08e-12	0.4918246	0	0.4918246	0
0.5	0.6487212	0.6487212	1.27e-11	0.6487212	1.11e-16	0.6487212	1.11e-16
0.6	0.8221188	0.8221188	9.56e-11	0.8221188	1.11e-16	0.8221188	1.11e-16
0.7	1.0137527	1.0137527	5.25e-10	1.0137527	8.88e-16	1.0137527	6.66e-16
0.8	1.2255409	1.2255409	2.30e-09	1.2255409	1.99e-15	1.2255409	6.66e-16
0.9	1.4596031	1.4596031	8.49e-09	1.4596031	1.99e-15	1.4596031	2.22e-16
1.0	1.7182818	1.7182818	2.73e-08	1.7182818	5.08e-14	1.7182818	0

2.3 Conclusion

The residual power series method is used in this paper to obtain approximate solutions to the relaxation-oscillation equations and to compare different numerical values of approximate solutions with numerical values of the exact solution at various values of α. Also, the errors of both the solutions of different numbers of terms are observed for relaxation-oscillation equations. In this paper, the approximate values are very close to the exact values, indicating that RPSM is one of the best efficient methods for solving FDEs. The presented tables and graphs show comparisons of both the solutions with errors. This method for solving FDEs like relaxation-oscillation equations is a very prevailing and accurate method that is considered an additive instrument in the area of fractional theory and its computations. It has been demonstrated that the accuracy and efficiency of RPSM designed for the solution of such differential equations are reliable.

Chapter 3

Solutions of One-dimensional Fractional Differential Equations by Laplace Transform with Residual Power Series Method

In this chapter, the solutions of one-dimensional FDEs by using LRPSM are discussed. In particular, time-fractional BBM-Burger (BBMB), Fisher's, logistic and Schrödinger differential equations are solved by this method.

FDEs are highly helpful nowadays in a variety of domains, including dynamic systems, engineering, and mathematics. Leibniz and L'Hospital were first proposed FDEs in 1695. FDEs were used by Lakshmikantham and Vatsala [70–71] to describe the fundamental theory of initial value problems relating Riemann-Liouville differential operators. Diethelm and Ford raised the analytical issues of FDEs existence as well as distinctiveness of solutions [72]. Numerous scholars have examined various ideas related to such FDEs. Numerous numerical techniques have been discovered to solve these FDEs; however the majority of the problems lack an analytical solution.

FDEs are a generalised version of classical differential equations that have seen significant use in a variety of scientific fields in recent years. The BBMB, Fisher's, logistic and Schrödinger differential equations are all solved numerically in this chapter using LRPSM, one of the numerous analytical techniques available for FDEs solutions.

Variational iteration method [73], homotopy analysis transform [74], G′/G expansion method [75], cubic B-spline functions [76], cubic B-spline method [77], homotopy analysis method [78], unified method [79], Runge Kutta method [80], Hermite wavelet technique along by Newton-Raphson iteration method [81], Lie symmetry method [82], differential transform method [83], Adomian decomposition method [84] and many more are used to find the analytical solution of FDEs.

The well-known BBMB equation has been applied to the study of long-wavelength surface waves into liquids, acoustic-gravity release into compressible fluids, and hydro-magnetic waves within cold plasma [85]. Given that solving the water wave shape analytically is renowned for being difficult. Therefore, several studies have conducted recently to ascertain the water wave model's numerical solution. The BBMB equation has applications in science and engineering, including acoustic-gravity waves in fluids [86], thermodynamics, cracked rock, and acoustic waves in anharmonic crystals.

To obtain the logical solution to BBMB equation, homotopy analysis method has been devised. Furthermore, homotopy analysis method [87] has established the precise solution to the time-fractional order BBMB equation. Researchers have also looked into adopting a linearised difference approach to solve the BBMB equation. The equation containing a time-fractional non-local viscous factor has been handled using the linear difference technique with Crank Nicolson. Cubic B-spline technique was utilised by Majeed et al. [83] to estimate the solution of the temporal fractional non-homogeneous BBMB equation. By employing method based on Laplace transform & Adomian decomposition method, Ostrovsky and Degasperis-Procesi have determined analytical solution of the BBMB [88].

In, the researchers have also derived the BBMB and K-dV equations [89] on water signal model. The space-time fractional BBMB equation [90] has solved analytically using ADM. Again, energy method [91] is applied to verify the distinctiveness as well as reality of BBMB form.

In recent times, many analytical techniques utilising power series expansion have been discovered and effectively used to various types of FDEs that arise in non-linear dynamical systems. These techniques do not involve linearization, discretization, or perturbation. LRPSM is one approach that is used to solve these types of FDEs. In many different sectors, the RPSM has been widely applied. It has been suggested and demonstrated by numerous writers that the differential algebraic equation systems can also be solved using the RPSM [92]. In this chapter, the BBMB equation is solved using the LRPSM.

FDEs are a generalised version of classical differential equations [38] that have found extensive use in various scientific fields in recent years. A multitude of books have provided definitions and basic information about Fractional Calculus, and various other applications [93–95]. Nonetheless, a plethora of analytical techniques exist for the numerical resolution of FDEs; the most widely used and relevant techniques are documented in the literature [96–103]. FDEs are manually solved using the LRPSM.

One of most effective and dependable methods for solving linear or non-linear FDEs in closed form is the RPSM [104]. Sometimes it is impossible to discover such solutions for linear FDEs, and finding the series coefficients can also be exceedingly challenging. Next, utilising transformed functions, a RPSM is introduced to obtain the coefficients in sequential form as a recurrence relation. To obtain the n^{th} ordered coefficients of a power series, one dependable but uncommon method is to differentiate its n^{th} partial sum in (n-1) times. In this technique, ordinary derivatives are sometimes upgraded to fractional order derivatives in order to handle certain fractional linear situations. For solving such FDEs, the LRPSM is established.

As a result, LRPSM is used analytically to solve some of the significant models that arise in various departments of mathematics, physics, and engineering. In this chapter, fractional Fisher's, logistic as well as Schrödinger differential equations are solved with numerical simulations and graphs by using the LRPSM. The aim of this work is to enhance the accuracy and reliability of the RPSM by applying Laplace to the problem's approach.

In mid-19th century, Fractional Calculus was first created as a mathematical model. Subsequently, it began to emerge in an increasing number of engineering and science fields. FDEs have drawn lots of consideration due to their continued rise in multiple applications in biology and fluid dynamics. Fractional order behaviour appears to be exhibited by many physical processes and can vary in space and time. Quite accurate numerical approaches can be applied, since FDEs usually lack exact solutions. The numerical solution of such issues is very desirable and has large variety of applications. It is essential to employ numerical methods to generate approximations of solutions because only a portion of FDEs encountered in practise can be solved explicitly. Among its particular uses is the modelling of cancer development in medicine using logistic curves. This use within the context of environmental research can be viewed as an extension of the use that was previously discussed.

One effective and dependable method for solving linear or non-linear FDEs in closed form using the solution of well-known functions is the RPSM. The FDEs are the standard differential equations expanded from integer order to fractional order. Recently, a large number of fresh researchers have taken up learning of FDEs theory. The existence of these equations' solutions is one of their most important qualitative characteristics. The discovery of solutions confirms that these equations have the necessary conditions for a solution. Fractional power series are used to express the solutions to FDEs. By truncating the series into the first term, the RPSM employs the n^{th} residual function to determine the n^{th} coefficients of power series form.

For non-linear FDEs, there are no such solutions, and finding series coefficients is quite difficult. Then coefficients are found in sequential form as a recurrence relation using a RPSM and transformed functions. One can differentiate the n^{th} partial sum of the power series in (n-1) times to find the n^{th} ordered coefficients. To handle fractional non-linear equations, the ordinary derivatives are typically upgraded to fractional order derivatives. The LRPSM is defined for fractional non-linear problems. Consequently, LRPSM finds use in the analytical solution of fundamental models that arise in several fields of mathematics, physics, and engineering. This work employs the Laplace transform as part of its technique to improve the accuracy and reliability of RPSM.

3.1 Methodology to solve one-dimensional FDEs

The following are the steps that comprise the LRPSM methodology for solving one-dimensional FDEs:

Consider the following one-dimensional FDE, in general as,

$$D_t^{n\alpha}u(x,t) + Lu(x,t) + Nu^q(x,t) = g(x,t) \quad n-1 < n\alpha \leq n \tag{3.1}$$

With initial condition, $u(x,0) = h(x)$ (3.2)

Where $D_t^{n\alpha}$= Caputo fractional derivative, $g(x,t)$= continuous function, L= linear operator and N = non-linear operators.

Step 1 Using the Laplace transform with an equation (3.1) as

$$\mathcal{L}\{D_t^{n\alpha}u(x,t) + Lu(x,t) + Nu^q(x,t)\} = \mathcal{L}\{g(x,t)\} \tag{3.3}$$

From Laplace transform of fractional derivatives using the relation

$\mathcal{L}[D_t^{\alpha}u] = s^{\alpha}\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)$ on equation (3.3), then it can be re-framed as,

$$U(x,s) = \frac{1}{s}f_0(x) - \frac{1}{s^{\alpha}}[LU(x,s) + N\mathcal{L}(\{\mathcal{L}^{-1}U(x,s)\}^q)] + \mathcal{L}\{g(x,t)\} \tag{3.4}$$

where $f_0(x) = u(x,0),\ U(x,s) = \mathcal{L}[u(x,t)]$

Step 2 It is possible to express the transformed function $U(x,s)$as,

$$U(x,s) = \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\alpha+1}} \tag{3.5}$$

The k[th]-truncated series of this equation (3.5) can also be expressed as,

$$U_k(x,s) = \sum_{n=0}^{k} \frac{f_n(x)}{s^{n\alpha+1}}$$

$$i.e.\ U_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} \tag{3.6}$$

Again the k^{th} −Laplace residual function is,

$$\mathcal{L}Res_k(x,s) = U_k(x,s) - \frac{1}{s}f_0(x) - \frac{1}{s^{\alpha}}[LU_k(x,s) + N\mathcal{L}(\{\mathcal{L}^{-1}U_k(x,s)\}^q)] + \mathcal{L}\{g(x,t)\} \tag{3.7}$$

To find the values of $f_k(x), k = 1, 2, 3, \dots\dots\dots$ substitute $k^{th}-$ truncated series (3.6) in k^{th} −Laplace residual function (3.7).

Step 3 By solving the following relation recursively the coefficients $f_n(x)$ can be obtained,

$$\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res_k(x,s) = 0\ for\ 0 < \alpha \le 1, k = 1,2,3, \dots \tag{3.8}$$

Following are some useful relations which are used in LRPSM;

$i)$ $\mathcal{L}Res(x,s) = 0\ and\ \lim_{k\to\infty} \mathcal{L}Res_k(x,s) = \mathcal{L}Res(x,s)$, for $s > 0$.

$ii)$ $\lim_{s\to\infty} s\mathcal{L}Res(x,s) = 0\ gives\ \lim_{s\to\infty} s\mathcal{L}Res_k(x,s) = 0$.

$iii)$ $\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res(x,s) = \lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res_k(x,s) = 0\ for\ 0 < \alpha \le 1$.

Step 4 At last applying the inverse Laplace transform to $U_k(x,s)$ for obtaining the k^{th} approximate supportive solution $u_k(x,t)$.

3.2 Implementation of method

The numerical solutions of one-dimensional FDEs by LRPSM can be done as follows:

Example 3. 2. 1 BBMB equation

Consider the one-dimensional BBMB equation defined as,

$$D_t^{\alpha}u - u_{xxt} + u_x + (\frac{u^2}{2})_x = 0, t > 0, 0 < \alpha \le 1 \tag{3.9}$$

with initial situation $u(x,0) = f_0(x) = sech^2(\frac{x}{4})$ (3.10)

and exact solution for $\alpha = 1$ is $u(x,t) = sech^2(\frac{x}{4} - \frac{t}{4})$ (3.11)

This equation is a non-linear FDE and is solved by LRPSM.

Applying Laplace transform on equation (3.9) we get,

$$\mathcal{L}\{D_t^\alpha u - u_{xxt} + u_x + (\frac{u^2}{2})_x\} = 0 \quad (3.12)$$

$$or, \mathcal{L}(D_t^\alpha u) = \mathcal{L}(u_{xxt}) - \mathcal{L}(u_x) - \mathcal{L}(\frac{u^2}{2})_x$$

From Laplace transform of fractional derivatives using the relation

$\mathcal{L}[D_t^\alpha u(x,t)] = s^\alpha \mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)$ on equation (3.12), then it can be re-framed as,

$$s^\alpha \mathcal{L}[u] - s^{\alpha-1}u(x,0) = \mathcal{L}(u_{xxt}) - \mathcal{L}(u_x) - \mathcal{L}(\frac{u^2}{2})_x$$

$$or, \mathcal{L}[u] = \frac{1}{s}u(x,0) + \frac{1}{s^\alpha}\{\mathcal{L}(u_{xxt}) - \mathcal{L}(u_x) - \mathcal{L}\left(\frac{u^2}{2}\right)_x\}$$

$$or, U(x,s) = \frac{1}{s}f_0(x) + \frac{1}{s^\alpha}[\{U(x,s)\}_{xxt} - \{U(x,s)\}_x - \frac{1}{2}[\mathcal{L}\{\mathcal{L}^{-1}[U(x,s)^2]\}]_x \quad (3.13)$$

where $\mathcal{L}[u(x,t)] = U(x,s)$ and $u(x,0) = f_0(x)$

The transformed function $U(x,s)$ can be written as

$$U(x,s) = \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\alpha+1}} \quad (3.14)$$

Also the $k^{th} -$ truncated series of this relation (3.14) can be written as

$$U_k(x,s) = \sum_{n=0}^{k} \frac{f_n(x)}{s^{n\alpha+1}}$$

$$i.e. U_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} \quad (3.15)$$

Again the k^{th} –Laplace residual function of (3.15) is

$$\mathcal{L}Res_k(x,s) = U_k(x,s) - \frac{1}{s}f_0(x) - \frac{1}{s^\alpha}[\Big\{U_k(x,s)\}_{xxt} - \{U_k(x,s)\}_x - \frac{1}{2}[\mathcal{L}\{\mathcal{L}^{-1}[U_k(x,s)^2]\}]_x\Big] \quad (3.16)$$

To find the values of $f_k(x), k = 1,2,3, \ldots\ldots\ldots$ substitute the $k^{th} -$ truncated series (3.15) in k^{th} –Laplace residual function (3.16) we get,

$$\mathcal{L}Res_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{f_0(x)}{s} - \frac{1}{s^\alpha}[\Big\{\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_{xxt} - \Big\{\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\Big\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}})^2\right\}\right]_x\Big]$$

$= \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}[\left\{\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt} - \left\{\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}})^2\right\}\right]_x]$

$= \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}[\left\{\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt} - \left\{\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}})^2\right\}\right]_x]$

$= \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}\left[\{\sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_{xxt} + \frac{1}{2s} sech^2\frac{x}{4} tanh\frac{x}{4} - \{\sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_x - \frac{1}{2}\left[\mathcal{L}\{\mathcal{L}^{-1}(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}})^2\}\right]_x\right]$

Or, $\mathcal{L}Res_k(x,s) = \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}\left[\{\sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_{xxt} + \frac{1}{2s} sech^2\frac{x}{4} tanh\frac{x}{4} - \{\sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}})^2\right\}\right]_x\right]$ (3.17)

For $k = 1$ from (3.17) the first Laplace residual function is,

$\mathcal{L}Res_1(x,s) = \frac{f_1(x)}{s^{\alpha+1}} - \frac{1}{s^\alpha}\left[\{f_1(x)\}_{xxt}\frac{1}{s^{\alpha+1}} + \frac{1}{2s} sech^2\frac{x}{4} tanh\frac{x}{4} - \{f_1(x)\}_x\frac{1}{s^{\alpha+1}} - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2\frac{x}{4}}{s} + \frac{f_1(x)}{s^{\alpha+1}})^2\right\}\right]_x\right]$

(3.18)

$= \frac{f_1(x)}{s^{\alpha+1}} - [\{f_1(x)\}_{xxt}\frac{1}{s^{2\alpha+1}} + \frac{1}{2s^{\alpha+1}} sech^2\frac{x}{4} tanh\frac{x}{4} - \{f_1(x)\}_x\frac{1}{s^{2\alpha+1}} - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^4\frac{x}{4}}{s^2} + \frac{2sech^2\frac{x}{4}f_1(x)}{s^{\alpha+2}} + \frac{\{f_1(x)\}^2}{s^{2\alpha+2}}\right\}\right]_x]$

$= \frac{f_1(x)}{s^{\alpha+1}} - \{f_1(x)\}_{xxt}\frac{1}{s^{2\alpha+1}} - \frac{1}{2s^{\alpha+1}} sech^2\frac{x}{4} tanh\frac{x}{4} + \{f_1(x)\}_x\frac{1}{s^{2\alpha+1}} + \frac{1}{2s^\alpha}\mathcal{L}[sech^4\frac{x}{4}\frac{t}{1!} + 2sech^2\frac{x}{4}f_1(x)\frac{t^{\alpha+1}}{(\alpha+1)!} + \frac{\{f_1(x)\}^2}{(2\alpha+1)!}\, t^{2\alpha+1}]$

$= \frac{f_1(x)}{s^{\alpha+1}} - \{f_1(x)\}_{xxt}\frac{1}{s^{2\alpha+1}} - \frac{1}{2s^{\alpha+1}} sech^2\frac{x}{4} tanh\frac{x}{4} + \{f_1(x)\}_x\frac{1}{s^{2\alpha+1}} + \frac{1}{2s^\alpha}[sech^4\frac{x}{4}\frac{1}{s^2} + 2sech^2\frac{x}{4}f_1(x)\frac{1}{(\alpha+1)!}\frac{(\alpha+1)!}{s^{\alpha+2}} + \frac{\{f_1(x)\}^2}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}}]$

$= \frac{f_1(x)}{s^{\alpha+1}} - \{f_1(x)\}_{xxt}\frac{1}{s^{2\alpha+1}} - \frac{1}{2s^{\alpha+1}} sech^2\frac{x}{4} tanh\frac{x}{4} + \{f_1(x)\}_x\frac{1}{s^{2\alpha+1}} + \frac{1}{2} sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} + \frac{\{f_1(x)\}^2}{s^{3\alpha+2}}$

Now, the relation $\lim_{s\to\infty}(s^{\alpha+1}\,\mathcal{L}Res_1(x,s)) = 0\ for\ k = 1$ gives that,

$f_1(x) - \frac{1}{2} sech^2\frac{x}{4} tanh\frac{x}{4} = 0$

i.e. $f_1(x) = \frac{1}{2} sech^2 \frac{x}{4} tanh \frac{x}{4}$

For $k = 2$ from (3.17) the second Laplace residual function is,

$$\mathcal{L}Res_2(x,s) = \sum_{n=1}^{2} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}[\left\{\sum_{n=1}^{2} \frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt} + \frac{1}{2s} sech^2 \frac{x}{4} tanh \frac{x}{4} - \left\{\sum_{n=1}^{2} \frac{f_n(x)}{s^{n\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{2} \frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}\right]_x] \quad (3.19)$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{s^\alpha}[\left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\right\}_{xxt} + \frac{1}{2s} sech^2 \frac{x}{4} tanh \frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\right\}_x - \frac{1}{2}[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}})\right\}^2]_x]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{s^\alpha}[\left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_{xxt} + \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_{xxt} + \frac{1}{2s} sech^2 \frac{x}{4} tanh \frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_x - \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_x - \frac{1}{2}[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}})\right\}^2]_x]$$

$$= \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{1}{2s^\alpha}[\mathcal{L}\{\mathcal{L}^{-1}(\frac{sech^4\frac{x}{4}}{s^2} + \frac{\{f_1(x)\}^2}{s^{2\alpha+2}} + \frac{\{f_2(x)\}^2}{s^{4\alpha+2}} + \frac{2sech^2\frac{x}{4}f_1(x)}{s^{\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{3\alpha+2}} + \frac{2sech^2\frac{x}{4}f_2(x)}{s^{2\alpha+2}})]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{1}{2s^\alpha}[\mathcal{L}\{sech^4 \frac{x}{4}\frac{t}{1!} + \{f_1(x)\}^2 \frac{t^{2\alpha+1}}{(2\alpha+1)!} + \{f_2(x)\}^2 \frac{t^{4\alpha+1}}{(4\alpha+1)!} + 2sech^2 \frac{x}{4} f_1(x) \frac{t^{\alpha+1}}{(\alpha+1)!} + 2f_1(x)f_2(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2sech^2 \frac{x}{4} f_2(x) \frac{t^{2\alpha+1}}{(2\alpha+1)!}]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{1}{2s^\alpha}[sech^4 \frac{x}{4}\frac{1}{s^2} + \{f_1(x)\}^2 \frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + \{f_2(x)\}^2 \frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + 2sech^2 \frac{x}{4} f_1(x) \frac{1}{(\alpha+1)!}\frac{(\alpha+1)!}{s^{\alpha+2}} + 2f_1(x)f_2(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} + 2sech^2 \frac{x}{4} f_2(x) \frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}}]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + [sech^4 \frac{x}{4}\frac{1}{s^{\alpha+2}} + \frac{1}{2}\{f_1(x)\}^2 \frac{1}{s^{3\alpha+2}} + \frac{1}{2}\{f_2(x)\}^2 \frac{1}{s^{5\alpha+2}} + sech^2 \frac{x}{4} f_1(x) \frac{1}{s^{2\alpha+2}} + f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}} + sech^2 \frac{x}{4} f_2(x) \frac{1}{s^{3\alpha+2}}]_x$$

Now, the relation $\lim_{s\to\infty}(s^{2\alpha+1}\,\mathcal{L}Res_2(x,s)) = 0 \; for \; k = 2$, gives us that

$f_2(x) + \{f_1(x)\}_x = 0$

$f_2(x) = -\{f_1(x)\}_x$

$f_2(x) = -\frac{1}{8} sech^4 \frac{x}{4} + \frac{1}{4} sech^2 \frac{x}{4} tanh^2 \frac{x}{4}$

For $k = 3$ from (3.17) the third Laplace residual function is,

$$\mathcal{L}Res_3(x,s) = \sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}\Big[\Big\{\sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}}\Big\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \Big\{\sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}}\Big\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}\right]_x\Big] \quad (3.20)$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{1}{s^\alpha}\Big[\Big\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\Big\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \Big\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\Big\}_x - \frac{1}{2}\Big[\mathcal{L}\Big\{\mathcal{L}^{-1}\Big(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\Big)\Big\}^2\Big]_x\Big]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{1}{s^\alpha}\Big[\Big\{\frac{f_1(x)}{s^{\alpha+1}}\Big\}_{xxt} + \Big\{\frac{f_2(x)}{s^{2\alpha+1}}\Big\}_{xxt} + + \Big\{\frac{f_3(x)}{s^{3\alpha+1}}\Big\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \Big\{\frac{f_1(x)}{s^{\alpha+1}}\Big\}_x - \Big\{\frac{f_2(x)}{s^{2\alpha+1}}\Big\}_x - \Big\{\frac{f_3(x)}{s^{3\alpha+1}}\Big\}_x - \frac{1}{2}\Big[\mathcal{L}\Big\{\mathcal{L}^{-1}\Big(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\Big)\Big\}^2\Big]_x\Big]$$

$$= \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{1}{2s^\alpha}\Big[\mathcal{L}\Big\{\mathcal{L}^{-1}\Big(\frac{sech^4\frac{x}{4}}{s^2} + \frac{\{f_1(x)\}^2}{s^{2\alpha+2}} + \frac{\{f_2(x)\}^2}{s^{4\alpha+2}} + \frac{\{f_3(x)\}^2}{s^{6\alpha+2}} + \frac{2sech^2\frac{x}{4}f_1(x)}{s^{\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{3\alpha+2}} + \frac{2sech^2\frac{x}{4}f_2(x)}{s^{2\alpha+2}} + \frac{2sech^2\frac{x}{4}f_3(x)}{s^{3\alpha+2}} + \frac{2f_1(x)f_3(x)}{s^{4\alpha+2}} + \frac{2f_2(x)f_3(x)}{s^{5\alpha+2}}\Big)\Big\}\Big]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{1}{2s^\alpha}\Big[\mathcal{L}\Big\{sech^4\frac{x}{4}\frac{t}{1!} + \{f_1(x)\}^2\frac{t^{2\alpha+1}}{(2\alpha+1)!} + \{f_2(x)\}^2\frac{t^{4\alpha+1}}{(4\alpha+1)!} + \{f_3(x)\}^2\frac{t^{6\alpha+1}}{(6\alpha+1)!} + 2sech^2\frac{x}{4}f_1(x)\frac{t^{\alpha+1}}{(\alpha+1)!} + 2f_1(x)f_2(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2sech^2\frac{x}{4}f_2(x)\frac{t^{2\alpha+1}}{(2\alpha+1)!} + 2sech^2\frac{x}{4}f_3(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2f_1(x)f_3(x)\frac{t^{4\alpha+1}}{(4\alpha+1)!} + 2f_2(x)f_3(x)\frac{t^{5\alpha+1}}{(5\alpha+1)!}\Big]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{1}{2s^\alpha}\Big[sech^4\frac{x}{4}\frac{1}{s^2} + \{f_1(x)\}^2\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + \{f_2(x)\}^2\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + \{f_3(x)\}^2\frac{1}{(6\alpha+1)!}\frac{(6\alpha+1)!}{s^{6\alpha+2}} + 2sech^2\frac{x}{4}f_1(x)\frac{1}{(\alpha+1)!}\frac{(\alpha+1)!}{s^{\alpha+2}} + 2f_1(x)f_2(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} + 2sech^2\frac{x}{4}f_2(x)\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + 2sech^2\frac{x}{4}f_3(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} + 2f_1(x)f_3(x)\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + 2f_2(x)f_3(x)\frac{1}{(5\alpha+1)!}\frac{(5\alpha+1)!}{s^{5\alpha+2}}\Big]_x$$

$$= \frac{-\frac{1}{8}sech^4\frac{x}{4} + \frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} + \frac{\frac{1}{8}sech^4\frac{x}{4} - \frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \Big[sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + \frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}} + \frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}} + \frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} + f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}} + sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}} + f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}} + f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}}\Big]_x$$

$$= \frac{f_3(x)}{s^{3\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \Big[sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + \frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}} + \frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}} + \frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} +$$

$f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}} + sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}} + f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}} +$
$f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}}]_x$

Now, the relation $\lim_{s\to\infty}(s^{3\alpha+1}\,\mathcal{L}Res_3(x,s)) = 0\; for\; k = 3$, gives us that

$f_3(x) + \{f_2(x)\}_x = 0$

i.e. $f_3(x) = -\{f_2(x)\}_x$

i.e. $f_3(x) = -\frac{1}{12}sech^4\frac{x}{4}tanh\frac{x}{4} + \frac{1}{4}sech^2\frac{x}{4}tanh^3\frac{x}{4}$

For $k = 4$ from (3.17) the fourth Laplace residual function is,

$$\mathcal{L}Res_4(x,s) = \sum_{n=1}^{4}\frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}[\left\{\sum_{n=1}^{4}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\sum_{n=1}^{4}\frac{f_n(x)}{s^{n\alpha+1}}\right\}_x - \frac{1}{2}\left[\mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{sech^2\frac{x}{4}}{s} + \sum_{n=1}^{4}\frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}\right]_x] \quad (3.21)$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{1}{s^\alpha}[\left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}}\right\}_x - \frac{1}{2}[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}})\right\}^2]_x]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{1}{s^\alpha}[\left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_{xxt} + \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_{xxt} + \left\{\frac{f_3(x)}{s^{3\alpha+1}}\right\}_{xxt} + \left\{\frac{f_4(x)}{s^{4\alpha+1}}\right\}_{xxt} + \frac{1}{2s}sech^2\frac{x}{4}tanh\frac{x}{4} - \left\{\frac{f_1(x)}{s^{\alpha+1}}\right\}_x - \left\{\frac{f_2(x)}{s^{2\alpha+1}}\right\}_x - \left\{\frac{f_3(x)}{s^{3\alpha+1}}\right\}_x - \left\{\frac{f_4(x)}{s^{4\alpha+1}}\right\}_x - \frac{1}{2}[\mathcal{L}\left\{\mathcal{L}^{-1}(\frac{sech^2(\frac{x}{4})}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}})\right\}^2]_x]$$

$$= \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}} - \frac{sech^2\frac{x}{4}tanh\frac{x}{4}}{2s^{\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{\{f_4(x)\}_x}{s^{5\alpha+1}} + \frac{1}{2s^\alpha}[\mathcal{L}\{\mathcal{L}^{-1}\left(\frac{sech^4\frac{x}{4}}{s^2} + \frac{\{f_1(x)\}^2}{s^{2\alpha+2}} + \frac{\{f_2(x)\}^2}{s^{4\alpha+2}} + \frac{\{f_3(x)\}^2}{s^{6\alpha+2}} + \frac{\{f_4(x)\}^2}{s^{8\alpha+2}} + \frac{2sech^2\frac{x}{4}f_1(x)}{s^{\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{3\alpha+2}} + \frac{2sech^2\frac{x}{4}f_2(x)}{s^{2\alpha+2}} + \frac{2sech^2\frac{x}{4}f_3(x)}{s^{3\alpha+2}} + \frac{2f_1(x)f_3(x)}{s^{4\alpha+2}} + \frac{2f_2(x)f_3(x)}{s^{5\alpha+2}} + \frac{2sech^2\frac{x}{4}f_4(x)}{s^{4\alpha+2}} + \frac{2f_1(x)f_4(x)}{s^{5\alpha+2}} + \frac{2f_2(x)f_4(x)}{s^{6\alpha+2}} + \frac{2f_3(x)f_4(x)}{s^{7\alpha+2}}\right)\}]_x$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{\{f_4(x)\}_x}{s^{5\alpha+1}} + \frac{1}{2s^\alpha}[\mathcal{L}\{sech^4\frac{x}{4}\frac{t}{1!} + \{f_1(x)\}^2\frac{t^{2\alpha+1}}{(2\alpha+1)!} + \{f_2(x)\}^2\frac{t^{4\alpha+1}}{(4\alpha+1)!} + \{f_3(x)\}^2\frac{t^{6\alpha+1}}{(6\alpha+1)!} + \{f_4(x)\}^2\frac{t^{8\alpha+1}}{(8\alpha+1)!} + 2sech^2\frac{x}{4}f_1(x)\frac{t^{\alpha+1}}{(\alpha+1)!} + 2f_1(x)f_2(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2sech^2\frac{x}{4}f_2(x)\frac{t^{2\alpha+1}}{(2\alpha+1)!} + 2sech^2\frac{x}{4}f_3(x)\frac{t^{3\alpha+1}}{(3\alpha+1)!} + 2f_1(x)f_3(x)\frac{t^{4\alpha+1}}{(4\alpha+1)!} + 2f_2(x)f_3(x)\frac{t^{5\alpha+1}}{(5\alpha+1)!} + 2sech^2\frac{x}{4}f_4(x)\frac{t^{4\alpha+1}}{(4\alpha+1)!} + 2f_1(x)f_4(x)\frac{t^{5\alpha+1}}{(5\alpha+1)!} + 2f_2(x)f_4(x)\frac{t^{6\alpha+1}}{(6\alpha+1)!} + 2f_3(x)f_4(x)\frac{t^{7\alpha+1}}{(7\alpha+1)!}]_x$$

$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}} + \frac{\{f_1(x)\}_x}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{\{f_4(x)\}_x}{s^{5\alpha+1}} + \frac{1}{2s^\alpha}[sech^4\frac{x}{4}\frac{1}{s^2} + \{f_1(x)\}^2\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + \{f_2(x)\}^2\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + \{f_3(x)\}^2\frac{1}{(6\alpha+1)!}\frac{(6\alpha+1)!}{s^{6\alpha+2}} + \{f_4(x)\}^2\frac{1}{(8\alpha+1)!}\frac{(8\alpha+1)!}{s^{8\alpha+2}} + 2sech^2\frac{x}{4}f_1(x)\frac{1}{(\alpha+1)!}\frac{(\alpha+1)!}{s^{\alpha+2}} + 2f_1(x)f_2(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} + 2sech^2\frac{x}{4}f_2(x)\frac{1}{(2\alpha+1)!}\frac{(2\alpha+1)!}{s^{2\alpha+2}} + 2sech^2\frac{x}{4}f_3(x)\frac{1}{(3\alpha+1)!}\frac{(3\alpha+1)!}{s^{3\alpha+2}} + 2sech^2\frac{x}{4}f_4(x)\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + 2f_1(x)f_3(x)\frac{1}{(4\alpha+1)!}\frac{(4\alpha+1)!}{s^{4\alpha+2}} + 2f_2(x)f_3(x)\frac{1}{(5\alpha+1)!}\frac{(5\alpha+1)!}{s^{5\alpha+2}} + 2f_1(x)f_4(x)\frac{1}{(5\alpha+1)!}\frac{(5\alpha+1)!}{s^{5\alpha+2}} + 2f_2(x)f_4(x)\frac{1}{(6\alpha+1)!}\frac{(6\alpha+1)!}{s^{6\alpha+2}} + 2f_3(x)f_4(x)\frac{1}{(7\alpha+1)!}\frac{(7\alpha+1)!}{s^{7\alpha+2}}]_x$

$= \frac{-\frac{1}{8}sech^4\frac{x}{4}+\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}} + \frac{\frac{1}{8}sech^4\frac{x}{4}-\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4}}{s^{2\alpha+1}} + \frac{\{f_2(x)\}_x}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{\{f_4(x)\}_x}{s^{5\alpha+1}} + [sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + \frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}} + \frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}} + \frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}} + \frac{1}{2}\{f_4(x)\}^2\frac{1}{s^{9\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} + f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}} + sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_4(x)\frac{1}{s^{5\alpha+2}} + f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}} + f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}} + f_1(x)f_4(x)\frac{1}{s^{6\alpha+2}} + f_2(x)f_4(x)\frac{1}{s^{7\alpha+2}} + f_3(x)f_4(x)\frac{1}{s^{8\alpha+2}}]_x$

$= -\frac{\frac{1}{4}sech^4\frac{x}{4}tanh\frac{x}{4}-\frac{1}{8}sech^2\frac{x}{4}tanh^3\frac{x}{4}}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}} + \frac{\frac{1}{4}sech^4\frac{x}{4}tanh\frac{x}{4}-\frac{1}{8}sech^2\frac{x}{4}tanh^3\frac{x}{4}}{s^{3\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{\{f_4(x)\}_x}{s^{5\alpha+1}} + [sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + \frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}} + \frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}} + \frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}} + \frac{1}{2}\{f_4(x)\}^2\frac{1}{s^{9\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} + f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}} + sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_4(x)\frac{1}{s^{5\alpha+2}} + f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}} + f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}} + f_1(x)f_4(x)\frac{1}{s^{6\alpha+2}} + f_2(x)f_4(x)\frac{1}{s^{7\alpha+2}} + f_3(x)f_4(x)\frac{1}{s^{8\alpha+2}}]_x$

$= \frac{f_4(x)}{s^{4\alpha+1}} - \frac{(f_1(x))_{xxt}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xxt}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xxt}}{s^{4\alpha+1}} - \frac{\{f_4(x)\}_{xxt}}{s^{5\alpha+1}} + \frac{\{f_3(x)\}_x}{s^{4\alpha+1}} + \frac{\{f_4(x)\}_x}{s^{5\alpha+1}} + [sech^4\frac{x}{4}\frac{1}{s^{\alpha+2}} + \frac{1}{2}\{f_1(x)\}^2\frac{1}{s^{3\alpha+2}} + \frac{1}{2}\{f_2(x)\}^2\frac{1}{s^{5\alpha+2}} + \frac{1}{2}\{f_3(x)\}^2\frac{1}{s^{7\alpha+2}} + sech^2\frac{x}{4}f_1(x)\frac{1}{s^{2\alpha+2}} + f_1(x)f_2(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_2(x)\frac{1}{s^{3\alpha+2}} + sech^2\frac{x}{4}f_3(x)\frac{1}{s^{4\alpha+2}} + sech^2\frac{x}{4}f_4(x)\frac{1}{s^{5\alpha+2}} + f_1(x)f_3(x)\frac{1}{s^{5\alpha+2}} + f_2(x)f_3(x)\frac{1}{s^{6\alpha+2}} + f_1(x)f_4(x)\frac{1}{s^{6\alpha+2}} + f_2(x)f_4(x)\frac{1}{s^{7\alpha+2}} + f_3(x)f_4(x)\frac{1}{s^{8\alpha+2}}]_x$

Now, the relation $\lim_{s\to\infty}(s^{4\alpha+1}\,\mathcal{L}Res_4(x,s)) = 0\ for\ k = 4$, gives us that

$f_4(x) + \{f_3(x)\}_x = 0$

i.e. $f_4(x) = -\{f_3(x)\}_x$

i.e. $f_4(x) = \frac{1}{48}sech^6\frac{x}{4} - \frac{13}{48}sech^4\frac{x}{4}tanh^2\frac{x}{4} + \frac{1}{8}sech^2\frac{x}{4}tanh^4\frac{x}{4}$

Hence by Laplace residual power series solution of given equation in infinite form is, $U(x,s) = sech^2\frac{x}{4}\frac{1}{s} + \frac{1}{2}sech^2\frac{x}{4}tanh\frac{x}{4}\frac{1}{s^{\alpha+1}} + \left(\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4} - \frac{1}{8}sech^4\frac{x}{4}\right)\frac{1}{s^{2\alpha+1}} + \left(-\frac{1}{12}sech^4\frac{x}{4}tanh\frac{x}{4} + \frac{1}{4}sech^2\frac{x}{4}tanh^3\frac{x}{4}\right)\frac{1}{s^{3\alpha+1}} + (\frac{1}{48}sech^6\frac{x}{4} - \frac{13}{48}sech^4\frac{x}{4}tanh^2\frac{x}{4} + \frac{1}{8}sech^2\frac{x}{4}tanh^4\frac{x}{4})\frac{1}{s^{4\alpha+1}} \ldots$ (3.22)

At last taking inverse Laplace in equation (3.22) then the required solution of given equation by LRPSM is,

$$u(x,t) = sech^2\frac{x}{4} + \frac{1}{2}sech^2\frac{x}{4}tanh\frac{x}{4}\frac{t^\alpha}{\alpha!} + \left\{\frac{1}{4}sech^2\frac{x}{4}tanh^2\frac{x}{4} - \frac{1}{8}sech^4\frac{x}{4}\right\}\frac{t^{2\alpha}}{(2\alpha)!} + \left(-\frac{1}{12}sech^4\frac{x}{4}tanh\frac{x}{4} + \frac{1}{4}sech^2\frac{x}{4}tanh^3\frac{x}{4}\right)\frac{t^{3\alpha}}{(3\alpha)!} + (\frac{1}{48}sech^6\frac{x}{4} - \frac{13}{48}sech^4\frac{x}{4}tanh^2\frac{x}{4} + \frac{1}{8}sech^2\frac{x}{4}tanh^4\frac{x}{4})\frac{t^{4\alpha}}{(4\alpha)!} \ldots \quad (3.23)$$

Example 3. 2. 2 Fisher's equation

Consider one-dimensional non-linear Fisher's equation [116] defined as,

$$D_t^\alpha u(x,t) = u_{xx}(x,t) + 6\,u(x,t)\{1 - u(x,t)\}, t \geq 0, 0 < \alpha \leq 1 \quad (3.24)$$

with initial condition $u\,(x,0) \;=\; \frac{1}{(1+e^x)^2}$ (3.25)

& exact solution at α = 1 is $u\,(x,t) \;=\; \frac{1}{(1+e^{x-5t})^2}$ (3.26)

Applying Laplace transform on equation (3.24) then,

$$\mathcal{L}[D_t^\alpha u(x,t)] = \mathcal{L}[u_{xx}(x,t) + 6u(x,t)\{1 - u(x,t)\}]$$

i.e. $\mathcal{L}[D_t^\alpha u(x,t)] = \mathcal{L}[u_{xx}(x,t)] + 6[\mathcal{L}(u(x,t)) - \mathcal{L}(u^2(x,t))]$ (3.27)

From famous Laplace transform of fractional order derivatives using the relation $\mathcal{L}[D_t^\alpha u(x,t)] = s^\alpha\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)$ on equation (3.27) then it is framed as,

$$s^\alpha\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0) = \mathcal{L}[u_{xx}(x,t)] + 6[\mathcal{L}(u(x,t)) - \mathcal{L}(u^2(x,t))]$$

$$s^\alpha U(x,s) - s^{\alpha-1}f_0(x) = \{U(x,s)\}_{xx} + 6[U(x,s) - \mathcal{L}[\mathcal{L}^{-1}\{(U(x,s))^2\}]$$

i.e. $U(x,s) = \frac{f_0(x)}{s} + \frac{1}{s^\alpha}\{U(x,s)\}_{xx} + \frac{6}{s^\alpha}[U(x,s) - \mathcal{L}[\mathcal{L}^{-1}\{(U(x,s))^2\}]$ (3.28)

where $U(x,s) = \mathcal{L}[u(x,t)]$

Transformed function $U(x,s)$ can be written as,

$$U(x,s) = \sum_{n=0}^{\infty}\frac{f_n(x)}{s^{n\alpha+1}} \quad (3.29)$$

Also, $k^{th}-$ truncated series of this relation (3.29) is written as,

$U_k(x,s) = \sum_{n=0}^{k} \frac{f_n(x)}{s^{n\alpha+1}}$

$i.e. U_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}$ (3.30)

Again, k^{th} −Laplace residual function is

$\mathcal{L}Res_k(x,s) = U_k(x,s) - \frac{f_0(x)}{s} - \frac{1}{s^\alpha}\{U_k(x,s)\}_{xx} - \frac{6}{s^\alpha}[U_k(x,s) - \mathcal{L}[\mathcal{L}^{-1}\left\{\left(U_k(x,s)\right)^2\right\}]$(3.31)

To find the values of $f_k(x), k = 1,2,3, \ldots\ldots\ldots$ substitute $k^{th}-$ truncated series (3.30) in k^{th} −Laplace residual function (3.31) we get,

$\mathcal{L}Res_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{f_0(x)}{s} - \frac{1}{s^\alpha}\{\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_{xx} - \frac{6}{s^\alpha}[\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \mathcal{L}\{\mathcal{L}^{-1}(\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}})^2\}]$.

$\mathcal{L}Res_k(x,s) = \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}\{\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_{xx} - \frac{6}{s^\alpha}[\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \mathcal{L}\{\mathcal{L}^{-1}(\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}})^2\}]$ (3.32)

For $k = 1$ from (3.32) the first Laplace residual function is,

$\mathcal{L}Res_1(x,s) = \frac{f_1(x)}{s^{\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{6}{s^\alpha}[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} - \mathcal{L}\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}}\right)^2\}]$(3.33)

$= \frac{f_1(x)}{s^{\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{6}{s^\alpha}[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} - \mathcal{L}\{\mathcal{L}^{-1}\left(\frac{(f_0(x))^2}{s^2} + 2\frac{f_0(x)f_1(x)}{s^{\alpha+2}} + \frac{f_1^2(x)}{s^{2\alpha+2}}\right)\}]$

$= \frac{f_1(x)}{s^{\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{6}{s^\alpha}[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} - \mathcal{L}\{(f_0(x))^2 t + \frac{2f_0(x)f_1(x)t^{\alpha+1}}{(\alpha+1)!} + \frac{(f_1(x))^2 t^{2\alpha+1}}{(2\alpha+1)!}\}]$

$= \frac{f_1(x)}{s^{\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{6}{s^\alpha}[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} - \frac{(f_0(x))^2}{s^2} - \frac{2(\alpha+1)!f_0(x)f_1(x)}{(\alpha+1)!s^{\alpha+2}} - \frac{(2\alpha+1)!(f_1(x))^2}{(2\alpha+1)!s^{2\alpha+2}}]$

$= \frac{f_1(x)}{s^{\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - 6[\frac{f_0(x)}{s^{\alpha+1}} + \frac{f_1(x)}{s^{2\alpha+1}} - \frac{(f_0(x))^2}{s^{\alpha+2}} - \frac{2f_0(x)f_1(x)}{s^{2\alpha+2}} - \frac{((f_1(x))^2}{s^{3\alpha+2}}]$

Now, the relation $\lim_{s\to\infty}(s^{\alpha+1}\,\mathcal{L}Res_1(x,s)) = 0\ for\ k = 1$, gives that

$f_1(x) - \{f_0(x)\}_{xx} - 6f_0(x) = 0$

$f_1(x) = \{f_0(x)\}_{xx} + 6f_0(x) = \frac{10e^{2x}+10e^x+6}{(1+e^x)^4}$ where $u\,(x,0) = f_0(x) = \frac{1}{(1+e^x)^2}$

For $k = 2$ from (3.32) the second Laplace residual function is,

$\mathcal{L}Res_2(x,s) = \sum_{n=1}^{2} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{6}{s^\alpha}[\frac{f_0(x)}{s} + \sum_{n=1}^{2} \frac{f_n(x)}{s^{n\alpha+1}} - \mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s} + \sum_{n=1}^{2} \frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}]$ (3.34)

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{6}{s^\alpha}\Big[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\right)^2\right\}\Big]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{6}{s^\alpha}\Big[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{(f_0(x))^2}{s^2} + \frac{(f_1(x))^2}{s^{2\alpha+2}} + \frac{(f_2(x))^2}{s^{4\alpha+2}} + \frac{2f_0(x)f_1(x)}{s^{\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{3\alpha+2}} + \frac{2f_0(x)f_2(x)}{s^{2\alpha+2}}\right)\right\}\Big]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{6}{s^\alpha}\Big[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \mathcal{L}\left\{(f_0(x))^2 t + \frac{(f_1(x))^2 t^{2\alpha+1}}{(2\alpha+1)!} + \frac{(f_2(x))^2 t^{4\alpha+1}}{(4\alpha+1)!} + \frac{2f_0(x)f_1(x)t^{\alpha+1}}{(\alpha+1)!} + \frac{2f_1(x)f_2(x)t^{3\alpha+1}}{(3\alpha+1)!} + \frac{2f_0(x)f_2(x)t^{2\alpha+1}}{(2\alpha+1)!}\right\}\Big]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{6}{s^\alpha}\Big[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \left\{\frac{(f_0(x))^2}{s^2} + \frac{(2\alpha+1)!(f_1(x))^2}{(2\alpha+1)!s^{2\alpha+2}} + \frac{(4\alpha+1)!(f_2(x))^2}{(4\alpha+1)!s^{4\alpha+2}} + \frac{2(\alpha+1)!f_0(x)f_1(x)}{(\alpha+1)!s^{\alpha+2}} + \frac{2(3\alpha+1)!f_1(x)f_2(x)}{(3\alpha+1)!s^{3\alpha+2}} + \frac{2(2\alpha+1)!f_0(x)f_2(x)}{(2\alpha+1)!s^{2\alpha+2}}\right\}\Big]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - 6\Big[\frac{f_0(x)}{s^{\alpha+1}} + \frac{f_1(x)}{s^{2\alpha+1}} + \frac{f_2(x)}{s^{3\alpha+1}} - \left\{\frac{(f_0(x))^2}{s^{\alpha+2}} + \frac{(f_1(x))^2}{s^{3\alpha+2}} + \frac{(f_2(x))^2}{s^{5\alpha+2}} + \frac{2f_0(x)f_1(x)}{s^{2\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{4\alpha+2}} + \frac{2f_0(x)f_2(x)}{s^{3\alpha+2}}\right\}\Big]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{6f_0(x)}{s^{\alpha+1}} - \frac{6f_1(x)}{s^{2\alpha+1}} - \frac{6f_2(x)}{s^{3\alpha+1}} + \frac{6(f_0(x))^2}{s^{\alpha+2}} + \frac{6(f_1(x))^2}{s^{3\alpha+2}} + \frac{6(f_2(x))^2}{s^{5\alpha+2}} + \frac{12f_0(x)f_1(x)}{s^{2\alpha+2}} + \frac{12f_1(x)f_2(x)}{s^{4\alpha+2}} + \frac{12f_0(x)f_2(x)}{s^{3\alpha+2}}$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{6f_1(x)}{s^{2\alpha+1}} - \frac{6f_2(x)}{s^{3\alpha+1}} + \frac{6(f_0(x))^2}{s^{\alpha+2}} + \frac{6(f_1(x))^2}{s^{3\alpha+2}} + \frac{6(f_2(x))^2}{s^{5\alpha+2}} + \frac{12f_0(x)f_1(x)}{s^{2\alpha+2}} + \frac{12f_1(x)f_2(x)}{s^{4\alpha+2}} + \frac{12f_0(x)f_2(x)}{s^{3\alpha+2}}$$

Now, the relation $\lim_{s\to\infty}(s^{2\alpha+1}\,\mathcal{L}Res_2(x,s)) = 0 \; for \; k = 2$, gives us that

$$f_2(x) - \{f_1(x)\}_{xx} - 6f_1(x) = 0 \; i.e. f_2(x) = \{f_1(x)\}_{xx} + 6f_1(x)$$

$$i.e. f_2(x) = \frac{100e^{4x} + 150e^{3x} + 252e^{2x} + 118e^x + 36}{(1+e^x)^6}$$

For $k = 3$ from (3.32) the third Laplace residual function is,

$$\mathcal{L}Res_2(x,s) = \sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xx}}{s^{4\alpha+1}} - \frac{6}{s^\alpha}\Big[\frac{f_0(x)}{s} + \sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}} - \mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s} + \sum_{n=1}^{3}\frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}\Big] \quad (3.35)$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xx}}{s^{4\alpha+1}} - \frac{6}{s^\alpha}\Big[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\right)^2\right\}\Big]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xx}}{s^{4\alpha+1}} - \frac{6}{s^{\alpha}}\Big[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \mathcal{L}\Big\{\mathcal{L}^{-1}\Big(\frac{{f_0}^2(x)}{s^2} + \frac{{f_1}^2(x)}{s^{2\alpha+2}} + \frac{{f_2}^2(x)}{s^{4\alpha+2}} + \frac{{f_3}^2(x)}{s^{6\alpha+2}} + \frac{2f_0(x)f_1(x)}{s^{\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{3\alpha+2}} + \frac{2f_0(x)f_2(x)}{s^{2\alpha+2}} + \frac{2f_1(x)f_3(x)}{s^{4\alpha+2}} + \frac{2f_2(x)f_3(x)}{s^{5\alpha+2}} + \frac{2f_0(x)f_3(x)}{s^{3\alpha+2}}\Big)\Big\}\Big]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xx}}{s^{4\alpha+1}} - \frac{6f_0(x)}{s^{\alpha+1}} - \frac{6f_1(x)}{s^{2\alpha+1}} - \frac{6f_2(x)}{s^{3\alpha+1}} - \frac{6f_3(x)}{s^{4\alpha+1}} + \frac{6}{s^{\alpha}}\mathcal{L}\Big\{\frac{{f_0}^2(x)t}{1!} + \frac{{f_1}^2(x)t^{2\alpha+1}}{(2\alpha+1)!} + \frac{{f_2}^2(x)t^{4\alpha+1}}{(4\alpha+1)!} + \frac{{f_3}^2(x)t^{6\alpha+1}}{(6\alpha+1)!} + \frac{2f_0(x)f_1(x)t^{\alpha+1}}{(\alpha+1)!} + \frac{2f_1(x)f_2(x)t^{3\alpha+1}}{(3\alpha+1)!} + \frac{2f_0(x)f_2(x)t^{2\alpha+1}}{(2\alpha+1)!} + \frac{2f_1(x)f_3(x)t^{4\alpha+1}}{(4\alpha+1)!} + \frac{2f_2(x)f_3(x)t^{5\alpha+1}}{(5\alpha+1)!} + \frac{2f_0(x)f_3(x)t^{3\alpha+1}}{(3\alpha+1)!}\Big\}$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xx}}{s^{4\alpha+1}} - \frac{6f_0(x)}{s^{\alpha+1}} - \frac{6f_1(x)}{s^{2\alpha+1}} - \frac{6f_2(x)}{s^{3\alpha+1}} - \frac{6f_3(x)}{s^{4\alpha+1}} + \frac{6}{s^{\alpha}}\Big\{\frac{{f_0}^2(x)1!}{1!s^2} + \frac{{f_1}^2(x)(2\alpha+1)!}{(2\alpha+1)!s^{2\alpha+2}} + \frac{{f_2}^2(x)(4\alpha+1)!}{(4\alpha+1)!s^{4\alpha+2}} + \frac{{f_3}^2(x)(6\alpha+1)!}{(6\alpha+1)!s^{4\alpha+2}} + \frac{2f_0(x)f_1(x)(\alpha+1)!}{(\alpha+1)!s^{\alpha+2}} + \frac{2f_1(x)f_2(x)(3\alpha+1)!}{(3\alpha+1)!s^{3\alpha+2}} + \frac{2f_0(x)f_2(x)(2\alpha+1)!}{(2\alpha+1)!s^{2\alpha+2}} + \frac{2f_1(x)f_3(x)(4\alpha+1)!}{(4\alpha+1)!s^{4\alpha+2}} + \frac{2f_2(x)f_3(x)(5\alpha+1)!}{(5\alpha+1)!s^{5\alpha+2}} + \frac{2f_0(x)f_3(x)(3\alpha+1)!}{(3\alpha+1)!s^{3\alpha+2}}\Big\}$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{\{f_0(x)\}_{xx}}{s^{\alpha+1}} - \frac{\{f_1(x)\}_{xx}}{s^{2\alpha+1}} - \frac{\{f_2(x)\}_{xx}}{s^{3\alpha+1}} - \frac{\{f_3(x)\}_{xx}}{s^{4\alpha+1}} - \frac{6f_0(x)}{s^{\alpha+1}} - \frac{6f_1(x)}{s^{2\alpha+1}} - \frac{6f_2(x)}{s^{3\alpha+1}} - \frac{6f_3(x)}{s^{4\alpha+1}} + \frac{6{f_0}^2(x)}{s^{\alpha+2}} + \frac{6{f_1}^2(x)}{s^{3\alpha+2}} + \frac{6{f_2}^2(x)}{s^{5\alpha+2}} + \frac{6{f_3}^2(x)}{s^{5\alpha+2}} + \frac{12f_0(x)f_1(x)}{s^{2\alpha+2}} + \frac{12f_1(x)f_2(x)}{s^{4\alpha+2}} + \frac{12f_0(x)f_2(x)}{s^{3\alpha+2}} + \frac{12f_1(x)f_3(x)}{s^{5\alpha+2}} + \frac{12f_2(x)f_3(x)}{s^{6\alpha+2}} + \frac{12f_0(x)f_3(x)}{s^{4\alpha+2}}$$

Now, the relation $\lim_{s\to\infty}(s^{3\alpha+1}\,\mathcal{L}Res_3(x,s)) = 0\ for\ k = 3$, gives us that

$$f_3(x) - \{f_2(x)\}_{xx} - 6f_2(x) = 0\ i.e.\ f_3(x) = \{f_2(x)\}_{xx} + 6f_2(x)$$

$$\therefore f_3(x) = \frac{1000e^{6x}+1250e^{5x}+5944e^{4x}+3388e^{3x}+3560e^{2x}+1042e^{x}+216}{(1+e^{x})^8}$$

Hence, the power series solution of given Fisher's equation in infinite form is,

$$U(x,s) = \frac{1}{(1+e^{x})^{2}}\frac{1}{s} + \frac{10e^{2x}+10e^{x}+6}{(1+e^{x})^4}\frac{1}{s^{\alpha+1}} + \frac{100e^{4x}+150e^{3x}+252e^{2x}+118e^{x}+36}{(1+e^{x})^6}\frac{1}{s^{2\alpha+1}} + \frac{1000e^{6x}+1250e^{5x}+5944e^{4x}+3388e^{3x}+3560e^{2x}+1042e^{x}+216}{(1+e^{x})^8}\frac{1}{s^{3\alpha+1}} + \cdots \quad (3.36)$$

At last taking inverse Laplace in (3.36) then the required solution of Fisher's equation via LRPSM is,

$$u(x,t) = \frac{1}{(1+e^{x})^{2}} + \frac{10e^{2x}+10e^{x}+6}{(1+e^{x})^4}\frac{t^{\alpha}}{\alpha!} + \frac{100e^{4x}+150e^{3x}+252e^{2x}+118e^{x}+36}{(1+e^{x})^6}\frac{t^{2\alpha}}{(2\alpha)!} + \frac{1000e^{6x}+1250e^{5x}+5944e^{4x}+3388e^{3x}+3560e^{2x}+1042e^{x}+216}{(1+e^{x})^8}\frac{t^{3\alpha}}{(3\alpha)!} + \cdots \quad (3.37)$$

3.3 Numerical Simulations and Graphs

i) The both solutions of BBMB equation for various values of t are computed and compared. The graph of both the solutions with values of t=0.02, 0.04, 0.06, 0.08 and 0.10 and alpha=0.5 are shown in Figure 3.1, Figure 3.2, Figure 3.3, Figure 3.4 and Figure 3.5 when x varies from -20 to 20.

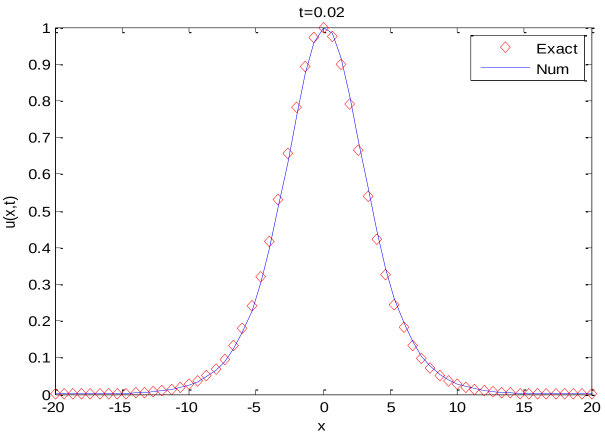


Figure 3.1 When t=0.02 and alpha=0.5

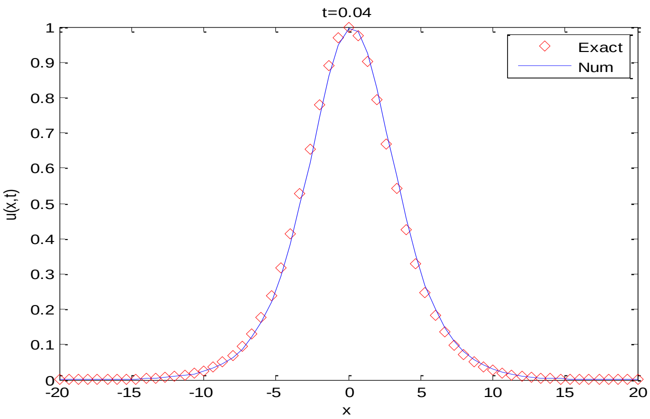


Figure 3.2 When t=0.04 and alpha=0.5

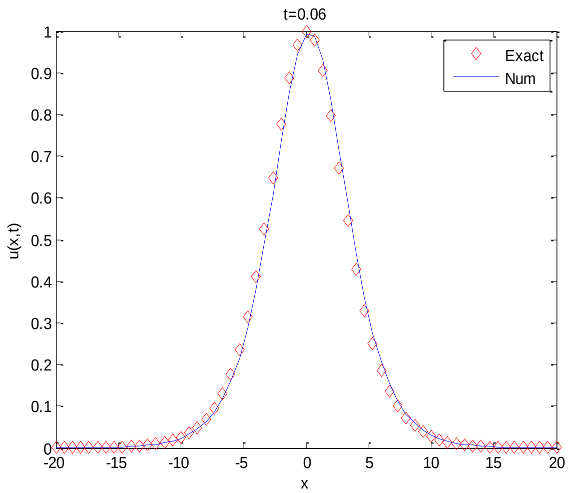


Figure 3.3 When t=0.06 and alpha=0.5

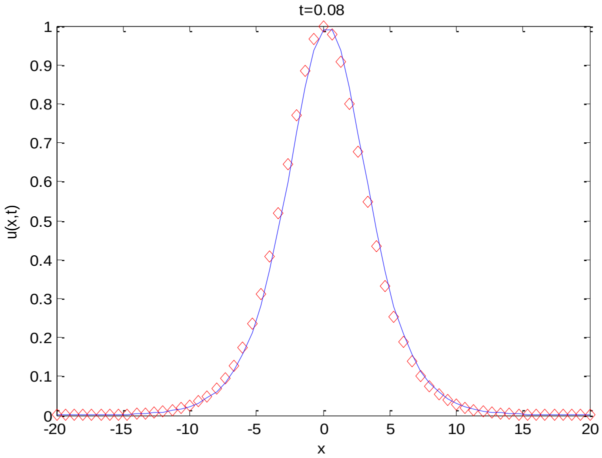


Figure 3.4 When t=0.08 and alpha=0.5

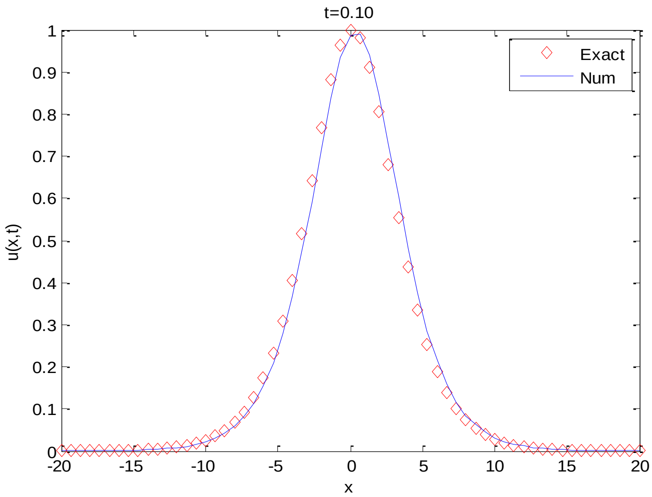


Figure 3.5 When t=0.10 and alpha=0.5

Table 3.1 Absolute errors of solutions of BBMB equation as prescribed points when alpha=0.5.

X	t=0.02	t=0.04	t=0.06	t=0.08	t=0.10
-20	1.1865e-05	1.5332e-05	1.7464e-05	1.8924e-05	1.9966e-05
-16	8.7574e-05	1.1317e-04	1.2891e-04	1.3969e-04	1.4739e-04
-12	6.4193e-04	8.2977e-04	9.4538e-04	1.0246e-03	1.0813e-03
-8	4.4721e-03	5.7918e-03	6.6090e-03	7.1731e-03	7.5798e-03
-4	2.1629e-02	2.8390e-02	3.2751e-02	3.5890e-02	3.8263e-02
4	2.3164e-02	3.1422e-02	3.7242e-02	4.1803e-02	4.5560e-02
8	5.1000e-03	7.0400e-03	8.4701e-03	9.6394e-03	1.0644e-02
12	7.3863e-04	1.0222e-03	1.2325e-03	1.4055e-03	1.5548e-03
16	1.0089e-04	1.3967e-04	1.6845e-04	1.9214e-04	2.1262e-04
20	1.3671e-05	1.8926e-05	2.2828e-05	2.6040e-05	2.8816e-05

Table 3.2 Absolute errors of solutions of BBMB equation at point t=0.02 for prescribed values of alpha.

x/alpha	0.25	0.50	0.75	1.0
-20	3.0789e-05	1.1865e-05	3.3528e-06	3.0187e-11
-16	2.2728e-04	8.7574e-05	2.4746e-05	2.2288e-10
-12	1.6676e-03	6.4193e-04	1.8133e-04	1.6375e-09
-8	1.1696e-02	4.4721e-03	1.2600e-03	1.1591e-08
-4	5.9167e-02	2.1629e-02	5.9830e-03	6.0683e-08
4	7.1211e-02	2.3164e-02	6.1328e-03	6.0859e-08
8	1.6985e-02	5.1000e-03	1.3209e-03	1.1646e-08
12	2.4895e-03	7.3863e-04	1.9070e-04	1.6457e-09
16	3.4060e-04	1.0089e-04	2.6036e-05	2.2400e-10
20	4.6162e-05	1.3671e-05	3.5278e-06	3.0338e-11

ii) A comparison is made between both the solutions for varying numbers of terms in the one-dimensional non-linear Fisher's equation for a range of t values. The graph of both solutions with values of t=0.002, 0.004, 0.006, 0.008 and 0.010 and alpha=0.5 are shown in Figure 3.6, Figure 3.7, Figure 3.8, Figure 3.9, Figure 3.10 where x varies from -20 to 20.

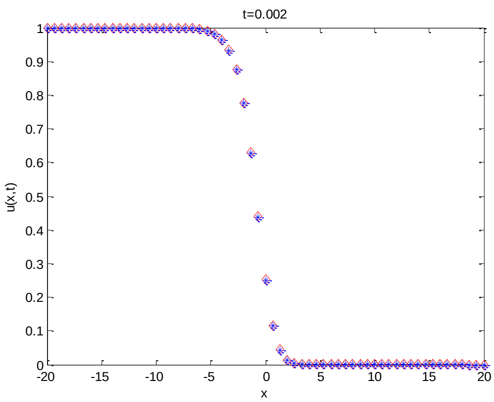


Figure 3.6 When t=0.002 and alpha=0.5

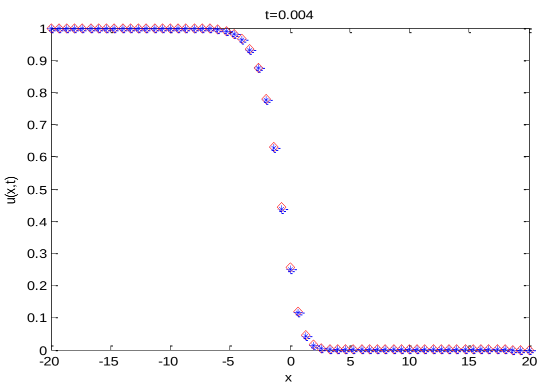


Figure 3.7 When t=0.004 and alpha=0.5

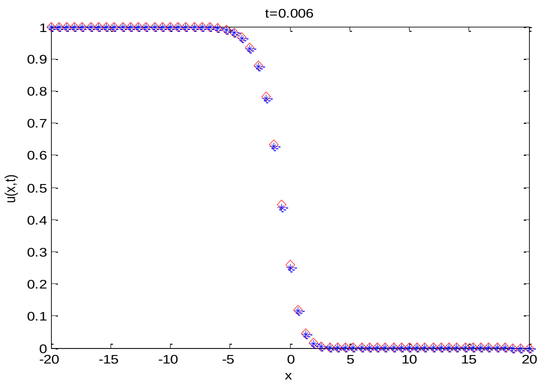


Figure 3.8 When t=0.006 and alpha=0.5

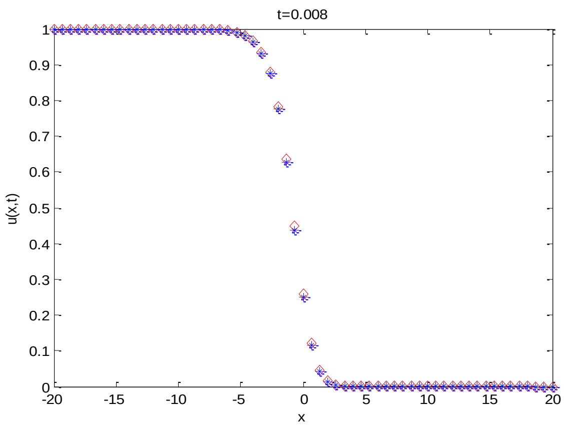


Figure 3.9 When t=0.008 and alpha=0.5

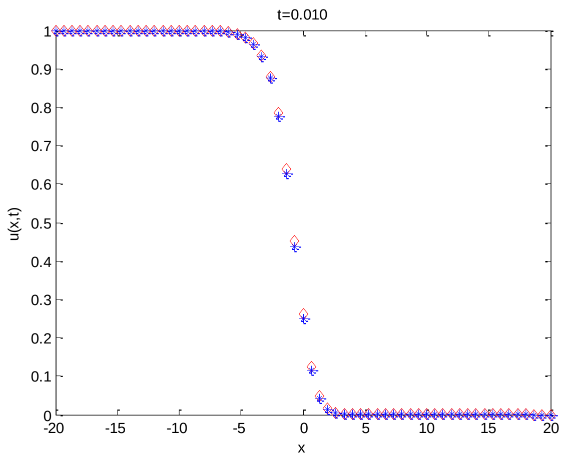


Figure 3.10 When t=0.010 and alpha=0.5

Table 3.3 Absolute errors of solutions of Fisher's equation as prescribed points when alpha=0.5.

X	t=0.002	t=0.004	t=0.006	t=0.008	t=0.010
-20	4.1018e-11	8.1627e-11	1.2183e-10	1.6164e-10	2.0105e-10
-16	2.2395e-09	4.4567e-09	6.6518e-09	8.8251e-09	1.0977e-08

-12	1.2227e-07	2.4332e-07	3.6317e-07	4.8183e-07	5.9930e-07
-8	6.6691e-06	1.3272e-05	1.9809e-05	2.6281e-05	3.2689e-05
-4	3.4526e-04	6.8728e-04	1.0261e-03	1.3617e-03	1.6941e-03
4	6.4159e-06	1.2958e-05	1.9628e-05	2.6430e-05	3.3365e-05
8	2.2711e-09	4.5880e-09	6.9517e-09	9.3631e-09	1.1823e-08
12	7.6261e-13	1.5406e-12	2.3344e-12	3.1441e-12	3.9703e-12
16	2.5186e-16	5.1013e-16	7.7375e-16	1.0427e-15	1.3171e-15
20	3.8857e-18	6.5262e-18	9.0985e-18	1.1689e-17	1.4323e-17

Table 3.4 Absolute errors of solutions of Fisher's equation at point t=0.002 for prescribed values of alpha

x/alpha	0.25	0.50	0.75	1.0
-20	4.1018e-11	4.1018e-11	4.1018e-11	4.1018e-11
-16	2.2395e-09	2.2395e-09	2.2395e-09	2.2395e-09
-12	1.2227e-07	1.2227e-07	1.2227e-07	1.2227e-07
-8	6.6691e-06	6.6691e-06	6.6691e-06	6.6691e-06
-4	3.4526e-04	3.4526e-04	3.4526e-04	3.4526e-04
4	6.4159e-06	6.4159e-06	6.4159e-06	6.4159e-06
8	2.2711e-09	2.2711e-09	2.2711e-09	2.2711e-09
12	7.6252e-13	7.6261e-13	7.6261e-13	7.6261e-13
16	1.6556e-16	2.5186e-16	2.5527e-16	2.5573e-16
20	9.0186e-17	3.8857e-18	4.8201e-19	1.8566e-20

Example 3.2.3 The Schrödinger equation $D_t^{\alpha}u + iu_{xx} = 0, 0 < \alpha \leq 1$ (3.38)

with primary condition $u(x,0) = 1 + \cosh(2x)$ (3.39)

and exact solution $u(x,t) = 1 + E_{\alpha}(-4it^{\alpha})\cosh(2x)$

where $E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}, z = -4it^{\alpha}$

and $u(x,t) = 1 + \cosh(2x)\, e^{-4it} for\ \alpha = 1$ with $i^2 = -1$. (3.40)

Taking Laplace transform on equation (3.38), we get

$\mathcal{L}[D_t^{\alpha}u] + \mathcal{L}[iu_{xx}] = 0$ (3.41)

From Laplace transform of fractional derivatives using the relation,

$\mathcal{L}[D_t^{\alpha}u(x,t)] = s^{\alpha}\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)$ on equation (3.41), then it can be framed as,

$s^{\alpha}\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0) + \mathcal{L}[iu_{xx}] = 0$

$s^{\alpha}\mathcal{L}[u(x,t)] = s^{\alpha-1}u(x,0) - i\mathcal{L}[u_{xx}]$

$U(x,s) = \frac{f_0(x)}{s} - i\frac{1}{s^{\alpha}}\{U(x,s)\}_{xx}$ (3.42) where $U(x,s) = \mathcal{L}[u(x,t)]$ and $u(x,0) = f_0(x)$

The transformed function $U(x,s)$ written as,

$$U(x,s) = \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\alpha+1}} \tag{3.43}$$

Also the $k^{th}-$ truncated series of this relation (3.43) is

$$U_k(x,s) = \sum_{n=0}^{k} \frac{f_n(x)}{s^{n\alpha+1}}$$

$$i.e. U_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} \tag{3.44}$$

Again the k^{th} $-$Laplace residual function of (3.42) is

$$\mathcal{L}Res_k(x,s) = U_k(x,s) - \frac{f_0(x)}{s} - i\frac{1}{s^{\alpha}}\{U_k(x,s)\}_{xx} \tag{3.45}$$

To find the values of $f_k(x), k = 1,2,3, \ldots\ldots\ldots$ substitute $k^{th}-$ truncated series (3.44) in k^{th} $-$Laplace residual function (3.45) we get,

$$\mathcal{L}Res_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{f_0(x)}{s} - i\frac{1}{s^{\alpha}}\{\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_{xx}$$

$$= \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_{xx} \tag{3.46}$$

For $k = 1$ from (3.46) first Laplace residual function is,

$$\mathcal{L}Res_1(x,s) = \frac{f_1(x)}{s^{\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{1+\cosh(2x)}{s} + \frac{f_1(x)}{s^{\alpha+1}}\}_{xx} \quad [\because u(x,0) = 1 + \cosh(2x) = f_0(x)]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} - \frac{4i\cosh(2x)}{s^{\alpha+1}} - i\frac{(f_1(x))_{xx}}{s^{2\alpha+1}}$$

Now, the relation $\lim_{s\to\infty}(s^{\alpha+1}\,\mathcal{L}Res_1(x,s)) = 0 \; for \; k = 1$ gives that,

$$f_1(x) - 4i\cosh(2x) = 0$$

$$i.e. f_1(x) = 4i\cosh(2x)$$

For $k = 2$ from (3.46) the second Laplace residual function is,

$$\mathcal{L}Res_2(x,s) = \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{1+\cosh(2x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\}_{xx}$$

$$= \frac{4i\cosh(2x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{1+\cosh(2x)}{s} + \frac{4i\cosh(2x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\}_{xx}$$

$$= \frac{4i\cosh(2x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{4i\cosh(2x)}{s^{\alpha+1}} + \frac{16\cosh(2x)}{s^{2\alpha+1}} - \frac{i(f_2(x))_{xx}}{s^{3\alpha+1}}$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{16\cosh(2x)}{s^{2\alpha+1}} - \frac{i(f_2(x))_{xx}}{s^{3\alpha+1}}$$

Now the relation $\lim_{s\to\infty}(s^{2\alpha+1}\,\mathcal{L}Res_2(x,s)) = 0\; for\; k = 2$, gives us that

$f_2(x) + 16\cosh(2x) = 0$

$i.e. f_2(x) = -16\cosh(2x)$

For $k = 3$ from (3.46) the third Laplace residual function is,

$$\mathcal{L}Res_2(x,s) = \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - i\frac{1}{s^{\alpha}}\left\{\frac{1+\cosh(2x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\right\}_{xx}$$

$$=\frac{4i\cosh(2x)}{s^{\alpha+1}} - \frac{16\cosh(2x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - i\frac{1}{s^{\alpha}}\left\{\frac{1+\cosh(2x)}{s} + \frac{4i\cosh(2x)}{s^{\alpha+1}} - \frac{16\cosh(2x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\right\}_{xx}$$

$$= \frac{4i\cosh(2x)}{s^{\alpha+1}} - \frac{16\cosh(2x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - i\frac{1}{s^{\alpha}}\left\{\frac{4\cosh(2x)}{s} + \frac{16i\cosh(2x)}{s^{\alpha+1}} - \frac{64\cosh(2x)}{s^{2\alpha+1}} + \frac{(f_3(x))_{xx}}{s^{3\alpha+1}}\right\}$$

$$= \frac{4i\cosh(2x)}{s^{\alpha+1}} - \frac{16\cosh(2x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - \frac{4i\cosh(2x)}{s^{\alpha+1}} + \frac{16\cosh(2x)}{s^{2\alpha+1}} + \frac{64i\cosh(2x)}{s^{3\alpha+1}} - \frac{i(f_3(x))_{xx}}{s^{4\alpha+1}}$$

$$= \frac{f_3(x)}{s^{3\alpha+1}} + \frac{64i\cosh(2x)}{s^{3\alpha+1}} - \frac{i(f_3(x))_{xx}}{s^{4\alpha+1}}$$

Now the relation $\lim_{s\to\infty}(s^{3\alpha+1}\,\mathcal{L}Res_3(x,s)) = 0\; for\; k = 3$, gives us that

$f_3(x) + 64i\cosh(2x) = 0$

$i.e. f_3(x) = -64i\cosh(2x)$

Therefore, the solution to the above equation in infinite form in the Laplace residual power series form is,

$$U(x,s) = \frac{1+\cosh(2x)}{s} + 4i\cosh(2x)\frac{1}{s^{\alpha+1}} - 16\cosh(2x)\frac{1}{s^{2\alpha+1}} - 64i\cosh(2x)\frac{1}{s^{3\alpha+1}} - \cdots \quad (3.47)$$

At last by taking inverse Laplace in (3.47) then solution of given equation via LRPSM as,

$$u(x,t) = 1 + \cosh(2x) + 4i\cosh(2x)\frac{t^{\alpha}}{\alpha!} - 16\cosh(2x)\frac{t^{2\alpha}}{(2\alpha)!} - 64i\cosh(2x)\frac{t^{3\alpha}}{(3\alpha)!} - \ldots \quad (3.48)$$

Example 3.2.4 The Schrödinger equation $D_t^{\alpha}u + iu_{xx} = 0, 0 < \alpha \le 1$ (3.49)

with primary condition $u(x,0) = e^{3ix}, i^2 = -1$ (3.50)

and exact solution is $u(x,t) = 1 + E_{\alpha}(-4it^{\alpha})\cosh(2x)$ (3.51)

where $E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}, z = -4it^{\alpha}$

and $u(x,t) = 1 + \cosh(2x)\,e^{-4it} for\; \alpha = 1$ with $i^2 = -1$.

Taking Laplace transform on equation (3.49), we get

$$\mathcal{L}[D_t^{\alpha}u] + \mathcal{L}[iu_{xx}] = 0 \quad (3.52)$$

From Laplace transform of fractional derivatives using the relation,

$\mathcal{L}[D_t^{\alpha} u(x,t)] = s^{\alpha}\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)$ on equation (3.52), then it can be framed as,

$s^{\alpha}\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0) + \mathcal{L}[iu_{xx}] = 0$

$s^{\alpha}\mathcal{L}[u(x,t)] = s^{\alpha-1}u(x,0) - i\mathcal{L}[u_{xx}]$

$$U(x,s) = \frac{f_0(x)}{s} - i\frac{1}{s^{\alpha}}\{U(x,s)\}_{xx} \tag{3.53}$$

where $U(x,s) = \mathcal{L}[u(x,t)]$ and $u(x,0) = f_0(x)$

The transformed function $U(x,s)$ is written as

$$U(x,s) = \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\alpha+1}} \tag{3.54}$$

Also the $k^{th}-$ truncated series of this relation (3.44) is

$U_k(x,s) = \sum_{n=0}^{k} \frac{f_n(x)}{s^{n\alpha+1}}$

$$i.e.\, U_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} \tag{3.55}$$

Again the $k^{th}-$Laplace residual function of (3.53) is

$$\mathcal{L}Res_k(x,s) = U_k(x,s) - \frac{f_0(x)}{s} - i\frac{1}{s^{\alpha}}\{U_k(x,s)\}_{xx} \tag{3.56}$$

To find the values of $f_k(x), k = 1,2,3,\ldots\ldots\ldots$ substitute $k^{th}-$ truncated series (3.55) in $k^{th}-$Laplace residual function (3.56) we get,

$$\mathcal{L}Res_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{f_0(x)}{s} - i\frac{1}{s^{\alpha}}\{\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_{xx}$$

$$= \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\}_{xx} \tag{3.57}$$

For $k = 1$ from (3.57) the first Laplace residual function is,

$$\mathcal{L}Res_1(x,s) = \frac{f_1(x)}{s^{\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{e^{3ix}}{s} + \frac{f_1(x)}{s^{\alpha+1}}\}_{xx} \quad [\because u(x,0) = e^{3ix} = f_0(x)]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} - \frac{9i^3e^{3ix}}{s^{\alpha+1}} - i\frac{(f_1(x))_{xx}}{s^{2\alpha+1}}$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{9ie^{3ix}}{s^{\alpha+1}} - i\frac{(f_1(x))_{xx}}{s^{2\alpha+1}}$$

Now, the relation $\lim_{s\to\infty}(s^{\alpha+1}\,\mathcal{L}Res_1(x,s)) = 0\ for\ k = 1$ gives that,

$f_1(x) + 9ie^{3ix} = 0$

$i.e.\, f_1(x) = -9ie^{3ix}$

For $k = 2$ from (3.57) the second Laplace residual function is,

$$\mathcal{L}Res_2(x,s) = \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{e^{3ix}}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\}_{xx}$$

$$= -\frac{9ie^{3ix}}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{e^{3ix}}{s} + \frac{-9ie^{3ix}}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\}_{xx}$$

$$= -\frac{9ie^{3ix}}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{9ie^{3ix}}{s^{\alpha+1}} + \frac{81e^{3ix}}{s^{2\alpha+1}} - \frac{i(f_2(x))_{xx}}{s^{3\alpha+1}}$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} + \frac{81e^{3ix}}{s^{2\alpha+1}} - \frac{i(f_2(x))_{xx}}{s^{3\alpha+1}}$$

Now the relation $\lim_{s\to\infty}(s^{2\alpha+1}\,\mathcal{L}Res_2(x,s)) = 0\ for\ k = 2$, gives us that

$f_2(x) + 81e^{3ix} = 0$

$i.e. f_2(x) = -81e^{3ix}$

For $k = 3$ from (3.57) the third Laplace residual function is,

$$\mathcal{L}Res_2(x,s) = \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{e^{3ix}}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\}_{xx}$$

$$=-\frac{9ie^{3ix}}{s^{\alpha+1}} - \frac{81e^{3ix}}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{e^{3ix}}{s} - \frac{9ie^{3ix}}{s^{\alpha+1}} - \frac{81e^{3ix}}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}}\}_{xx}$$

$$= -\frac{9ie^{3ix}}{s^{\alpha+1}} - \frac{81e^{3ix}}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} - i\frac{1}{s^{\alpha}}\{\frac{9i^2e^{3ix}}{s} + \frac{81ie^{3ix}}{s^{\alpha+1}} + \frac{729e^{3ix}}{s^{2\alpha+1}} + \frac{(f_3(x))_{xx}}{s^{3\alpha+1}}\}$$

$$= -\frac{9ie^{3ix}}{s^{\alpha+1}} - \frac{81e^{3ix}}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{9ie^{3ix}}{s^{\alpha+1}} + \frac{81e^{3ix}}{s^{2\alpha+1}} - \frac{729ie^{3ix}}{s^{3\alpha+1}} - \frac{i(f_3(x))_{xx}}{s^{4\alpha+1}}$$

$$= \frac{f_3(x)}{s^{3\alpha+1}} - \frac{729ie^{3ix}}{s^{3\alpha+1}} - \frac{i(f_3(x))_{xx}}{s^{4\alpha+1}}$$

Now the relation $\lim_{s\to\infty}(s^{3\alpha+1}\,\mathcal{L}Res_3(x,s)) = 0\ for\ k = 3$, gives us that

$f_3(x) - 729ie^{3ix} = 0$

$i.e. f_3(x) = 729ie^{3ix}$

Therefore, the solution to the above equation in infinite form in the Laplace residual power series form is,

$$U(x,s) = \frac{e^{3ix}}{s} - 9ie^{3ix}\frac{1}{s^{\alpha+1}} - 81e^{3ix}\frac{1}{s^{2\alpha+1}} + 729ie^{3ix}\ \frac{1}{s^{3\alpha+1}} - \cdots \qquad (3.58)$$

At last taking inverse Laplace in (3.58) then required solution of considered equation via LRPSM is,

$$u(x,t) = e^{3ix} - 9ie^{3ix}\frac{t^{\alpha}}{\alpha!} - 81e^{3ix}\frac{t^{2\alpha}}{(2\alpha)!} + 729ie^{3ix}\ \frac{t^{3\alpha}}{(3\alpha)!} - \cdots$$

$$= e^{3ix}\{1 - 9i\frac{t^{\alpha}}{\alpha!} - 81\frac{t^{2\alpha}}{(2\alpha)!} + 729i\,\frac{t^{3\alpha}}{(3\alpha)!} - \cdots\} \tag{3.59}$$

Example 3.2.5 Logistic equation

The logistic equation is defined as,

$$D_t^{\alpha}u(x,t) = \frac{1}{a^{\alpha}}u(x,t)\{1 - u(x,t)\}, t \geq 0, 0 < \alpha \leq 1 \tag{3.60}$$

with primary condition $u(x,0) = f_0(x) = \mu$ (3.61)

and exact solution is $u(x,t) = \frac{u(x,0)}{u(x,0)+\{1-u(x,0)\}e^{-\frac{t}{a}}}$

$$i.e. u(x,t) = \frac{\mu}{\mu+(1-\mu)e^{-\frac{t}{a}}} \text{ , where } \mu = u(x,0) \tag{3.62}$$

This is a non-linear FDE that can be solved with LRPSM.

3.2.5.1 Methodology for solution of logistic equation

Logistic equation is solved with LRPSM by taking the following subsequent steps:

Step 1: Applying Laplace on fractional logistic equation (3.60) as,

$$\mathcal{L}[D_t^{\alpha}u(x,t)] = \frac{1}{a^{\alpha}}[\mathcal{L}\big(u(x,t)\big) - \mathcal{L}(u^2(x,t))] \tag{3.63}$$

From Laplace transform of fractional derivatives using the relation,

$\mathcal{L}[D_t^{\alpha}u(x,t)] = s^{\alpha}\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)$ on equation (3.63), then it can be framed as,

$$s^{\alpha}\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0) = \frac{1}{a^{\alpha}}[\mathcal{L}\big(u(x,t)\big) - \mathcal{L}(u^2(x,t))]$$

$$i.e. s^{\alpha}\mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0) = \frac{1}{a^{\alpha}}[\mathcal{L}\big(u(x,t)\big) - \mathcal{L}(u^2(x,t))]$$

$$U(x,s) = \frac{f_0(x)}{s} + \frac{1}{a^{\alpha}s^{\alpha}}[U(x,s) - \mathcal{L}[\mathcal{L}^{-1}\{(U(x,s))^2\}] \tag{3.64}$$

where $U(x,s) = \mathcal{L}[u(x,t)]$

Step 2: The transformed function $U(x,s)$ is written as,

$$U(x,s) = \sum_{n=0}^{\infty}\frac{f_n(x)}{s^{n\alpha+1}} \tag{3.65}$$

Also the $k^{th} -$ truncated series of this relation (3.64) is,

$$U_k(x,s) = \sum_{n=0}^{k}\frac{f_n(x)}{s^{n\alpha+1}}$$

$$i.e. U_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}} \tag{3.66}$$

Again the k^{th} −Laplace residual function is,

$$\mathcal{L}Res_k(x,s) = U_k(x,s) - \frac{f_0(x)}{s} - \frac{1}{a^\alpha s^\alpha}[U_k(x,s) - \mathcal{L}[\mathcal{L}^{-1}\{\left(U_k(x,s)\right)^2\}] \tag{3.67}$$

Substitute the k^{th} − truncated series (3.66) in k^{th} −Laplace residual function (3.67) then it becomes,

$$\mathcal{L}Res_k(x,s) = \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{f_0(x)}{s} - \frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \mathcal{L}\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\right)^2\}].$$

$$\mathcal{L}Res_k(x,s) = \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}} - \mathcal{L}\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}\right)^2\}] \tag{3.68}$$

Step 3: By solving the following relation recursively the coefficients $f_n(x)$ can be obtained,

$$\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res_k(x,s) = 0 \; for \; 0 < \alpha \le 1, k = 1,2,3,\ldots \tag{3.69}$$

Following are some useful relations which are used in standard RPSM:

$i)$ $\mathcal{L}Res(x,s) = 0 \; and \; \lim_{k\to\infty} \mathcal{L}Res_k(x,s) = \mathcal{L}Res(x,s)$, for $s > 0$.

$ii)$ $\lim_{s\to\infty} s\mathcal{L}Res(x,s) = 0 \; gives \; \lim_{s\to\infty} s\mathcal{L}Res_k(x,s) = 0$.

$iii)$ $\lim_{s\to\infty} s^{k\alpha+1}\,\mathcal{L}Res(x,s) = \lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res_k(x,s) = 0 \; for \; 0 < \alpha \le 1$.

Step 4: At last applying the inverse Laplace transform to $U_k(x,s)$ for obtaining the k^{th} approximate solution $u_k(x,t)$ of logistic equation.

3.2.5.2 Numerical solution

The LRPSM is used to calculate the logistic FDE's numerical solution, which is as follows. Applying Laplace transform on equation (3.60) then,

$$\mathcal{L}[D_t^\alpha u(x,t)] = \mathcal{L}[\frac{1}{a^\alpha} u(x,t)\{1 - u(x,t)\}]$$

i.e. $$\mathcal{L}[D_t^\alpha u(x,t)] = \frac{1}{a^\alpha}[\mathcal{L}\left(u(x,t)\right) - \mathcal{L}(u^2(x,t))] \tag{3.70}$$

From famous Laplace transform of fractional order derivatives using the relation $\mathcal{L}[D_t^\alpha u(x,t)] = s^\alpha \mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0)$ on equation (3.70) then it is framed as,

$$s^\alpha \mathcal{L}[u(x,t)] - s^{\alpha-1}u(x,0) = \frac{1}{a^\alpha}[\mathcal{L}\left(u(x,t)\right) - \mathcal{L}(u^2(x,t))]$$

$$s^\alpha U(x,s) - s^{\alpha-1}f_0(x) = \frac{1}{a^\alpha}[U(x,s) - \mathcal{L}[\mathcal{L}^{-1}\{(U(x,s))^2\}]$$

i.e. $U(x,s)=\frac{f_0(x)}{s}+\frac{1}{a^\alpha s^\alpha}[U(x,s)-\mathcal{L}[\mathcal{L}^{-1}\{(U(x,s))^2\}]$ (3.71)

where $U(x,s)=\mathcal{L}[u(x,t)]$

The transformed function $U(x,s)$ is,

$$U(x,s)=\sum_{n=0}^{\infty}\frac{f_n(x)}{s^{n\alpha+1}} \quad (3.72)$$

Also the $k^{th}-$ truncated series of this relation (3.72) is given as,

$$U_k(x,s)=\sum_{n=0}^{k}\frac{f_n(x)}{s^{n\alpha+1}}$$

$$i.e.\, U_k(x,s)=\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}} \quad (3.73)$$

Again the $k^{th}-$Laplace residual function is

$$\mathcal{L}Res_k(x,s)=U_k(x,s)-\frac{f_0(x)}{s}-\frac{1}{a^\alpha s^\alpha}[U_k(x,s)-\mathcal{L}[\mathcal{L}^{-1}\{\left(U_k(x,s)\right)^2\}] \quad (3.74)$$

To find the values of $f_k(x), k=1,2,3,\dots\dots\dots$ substitute the $k^{th}-$ truncated series (3.73) in $k^{th}-$Laplace residual function (3.74) we get,

$$\mathcal{L}Res_k(x,s)=\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{f_0(x)}{s}-\frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\mathcal{L}\{\mathcal{L}^{-1}(\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}})^2\}].$$

$$\mathcal{L}Res_k(x,s)=\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}}-\mathcal{L}\{\mathcal{L}^{-1}(\frac{f_0(x)}{s}+\sum_{n=1}^{k}\frac{f_n(x)}{s^{n\alpha+1}})^2\}] \quad (3.75)$$

For $k=1$ from (3.75) the first Laplace residual function is,

$$\mathcal{L}Res_1(x,s)=\frac{f_1(x)}{s^{\alpha+1}}-\frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s}+\frac{f_1(x)}{s^{\alpha+1}}-\mathcal{L}\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s}+\frac{f_1(x)}{s^{\alpha+1}}\right)^2\}] \quad (3.76)$$

$$=\frac{f_1(x)}{s^{\alpha+1}}-\frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s}+\frac{f_1(x)}{s^{\alpha+1}}-\mathcal{L}\{\mathcal{L}^{-1}\left(\frac{(f_0(x))^2}{s^2}+2\frac{f_0(x)f_1(x)}{s^{\alpha+2}}+\frac{f_1^2(x)}{s^{2\alpha+2}}\right)\}]$$

$$=\frac{f_1(x)}{s^{\alpha+1}}-\frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s}+\frac{f_1(x)}{s^{\alpha+1}}-\mathcal{L}\{(f_0(x))^2t+\frac{2f_0(x)f_1(x)t^{\alpha+1}}{(\alpha+1)!}+\frac{(f_1(x))^2t^{2\alpha+1}}{(2\alpha+1)!}\}]$$

$$=\frac{f_1(x)}{s^{\alpha+1}}-\frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s}+\frac{f_1(x)}{s^{\alpha+1}}-\frac{(f_0(x))^2}{s^2}-\frac{2(\alpha+1)!f_0(x)f_1(x)}{(\alpha+1)!s^{\alpha+2}}-\frac{(2\alpha+1)!(f_1(x))^2}{(2\alpha+1)!s^{2\alpha+2}}]$$

$$=\frac{f_1(x)}{s^{\alpha+1}}-\frac{1}{a^\alpha}[\frac{f_0(x)}{s^{\alpha+1}}+\frac{f_1(x)}{s^{2\alpha+1}}-\frac{(f_0(x))^2}{s^{\alpha+2}}-\frac{2f_0(x)f_1(x)}{s^{2\alpha+2}}-\frac{((f_1(x))^2}{s^{3\alpha+2}}]$$

Now, the relation $\lim_{s\to\infty}(s^{\alpha+1}\,\mathcal{L}Res_1(x,s))=0\ for\ k=1$, gives that

$f_1(x)-\frac{f_0(x)}{a^\alpha}=0$ i.e. $f_1(x)=\frac{f_0(x)}{a^\alpha}$ i.e. $f_1(x)=\frac{\mu}{a^\alpha}$ $(\because f_0(x)=\mu)$

For $k=2$ from (3.75) the second Laplace residual function is,

$$\mathcal{L}Res_2(x,s) = \sum_{n=1}^{2}\frac{f_n(x)}{s^{n\alpha+1}} - \frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s} + \sum_{n=1}^{2}\frac{f_n(x)}{s^{n\alpha+1}} - \mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s} + \sum_{n=1}^{2}\frac{f_n(x)}{s^{n\alpha+1}}\right)^2\right\}] \quad (3.77)$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}}\right)^2\right\}]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{(f_0(x))^2}{s^2} + \frac{(f_1(x))^2}{s^{2\alpha+2}} + \frac{(f_2(x))^2}{s^{4\alpha+2}} + \frac{2f_0(x)f_1(x)}{s^{\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{3\alpha+2}} + \frac{2f_0(x)f_2(x)}{s^{2\alpha+2}}\right)\right\}]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \mathcal{L}\left\{(f_0(x))^2 t + \frac{(f_1(x))^2 t^{2\alpha+1}}{(2\alpha+1)!} + \frac{(f_2(x))^2 t^{4\alpha+1}}{(4\alpha+1)!} + \frac{2f_0(x)f_1(x)t^{\alpha+1}}{(\alpha+1)!} + \frac{2f_1(x)f_2(x)t^{3\alpha+1}}{(3\alpha+1)!} + \frac{2f_0(x)f_2(x)t^{2\alpha+1}}{(2\alpha+1)!}\right\}]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{a^\alpha s^\alpha}[\frac{f_0(x)}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \left\{\frac{(f_0(x))^2}{s^2} + \frac{(2\alpha+1)!(f_1(x))^2}{(2\alpha+1)!s^{2\alpha+2}} + \frac{(4\alpha+1)!(f_2(x))^2}{(4\alpha+1)!s^{4\alpha+2}} + \frac{2(\alpha+1)!f_0(x)f_1(x)}{(\alpha+1)!s^{\alpha+2}} + \frac{2(3\alpha+1)!f_1(x)f_2(x)}{(3\alpha+1)!s^{3\alpha+2}} + \frac{2(2\alpha+1)!f_0(x)f_2(x)}{(2\alpha+1)!s^{2\alpha+2}}\right\}]$$

$$= \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{a^\alpha}[\frac{f_0(x)}{s^{\alpha+1}} + \frac{f_1(x)}{s^{2\alpha+1}} + \frac{f_2(x)}{s^{3\alpha+1}} - \left\{\frac{(f_0(x))^2}{s^{\alpha+2}} + \frac{(f_1(x))^2}{s^{3\alpha+2}} + \frac{(f_2(x))^2}{s^{5\alpha+2}} + \frac{2f_0(x)f_1(x)}{s^{2\alpha+2}} + \frac{2f_1(x)f_2(x)}{s^{4\alpha+2}} + \frac{2f_0(x)f_2(x)}{s^{3\alpha+2}}\right\}]$$

$$= \frac{f_0(x)}{a^\alpha s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} - \frac{1}{a^\alpha}\frac{f_0(x)}{s^{\alpha+1}} - \frac{f_1(x)}{a^\alpha s^{2\alpha+1}} - \frac{f_2(x)}{a^\alpha s^{3\alpha+1}} + \frac{(f_0(x))^2}{a^\alpha s^{\alpha+2}} + \frac{(f_1(x))^2}{a^\alpha s^{3\alpha+2}} + \frac{(f_2(x))^2}{a^\alpha s^{5\alpha+2}} + \frac{2f_0(x)f_1(x)}{a^\alpha s^{2\alpha+2}} + \frac{2f_1(x)f_2(x)}{a^\alpha s^{4\alpha+2}} + \frac{2f_0(x)f_2(x)}{a^\alpha s^{3\alpha+2}}$$

$$= \frac{f_2(x)}{s^{2\alpha+1}} - \frac{f_1(x)}{a^\alpha s^{2\alpha+1}} - \frac{f_2(x)}{a^\alpha s^{3\alpha+1}} + \frac{(f_0(x))^2}{a^\alpha s^{\alpha+2}} + \frac{(f_1(x))^2}{a^\alpha s^{3\alpha+2}} + \frac{(f_2(x))^2}{a^\alpha s^{5\alpha+2}} + \frac{2f_0(x)f_1(x)}{a^\alpha s^{2\alpha+2}} + \frac{2f_1(x)f_2(x)}{a^\alpha s^{4\alpha+2}} + \frac{2f_0(x)f_2(x)}{a^\alpha s^{3\alpha+2}}$$

Now, the relation $\lim_{s\to\infty}(s^{2\alpha+1}\,\mathcal{L}Res_2(x,s)) = 0 \; for \; k = 2$, gives us that

$$f_2(x) - \frac{f_1(x)}{a^\alpha} = 0 \; i.e. f_2(x) = \frac{f_1(x)}{a^\alpha} \; i.e. f_2(x) = \frac{\mu}{a^{2\alpha}}$$

Hence power series solution of considered logistic equation in infinite form is,

$$U(x,s) = \frac{\mu}{s} + \frac{\mu}{a^\alpha}\frac{1}{s^{\alpha+1}} + \frac{\mu}{a^{2\alpha}}\frac{1}{s^{2\alpha+1}} + \frac{\mu}{a^{3\alpha}}\frac{1}{s^{3\alpha+1}} + \cdots \quad (3.78)$$

At last by taking inverse Laplace transform in (3.78) then the required solution of considered equation (3.60) via LRPSM is,

$$u(x,t) = \mu + \frac{\mu}{a^\alpha}\frac{t^\alpha}{\alpha!} + \frac{\mu}{a^{2\alpha}}\frac{t^{2\alpha}}{(2\alpha)!} + \frac{\mu}{a^{3\alpha}}\frac{t^{3\alpha}}{(3\alpha)!} + \cdots \quad (3.79)$$

3.2.5.3 Numerical simulations and graphs

Graphs of both solutions are compared with values of alpha = 1.0, 0.95, and 0.9, and t = 0:0.01:0.1, also displayed in Figure 3.11, Figure 3.12, and Figure 3.13 respectively.

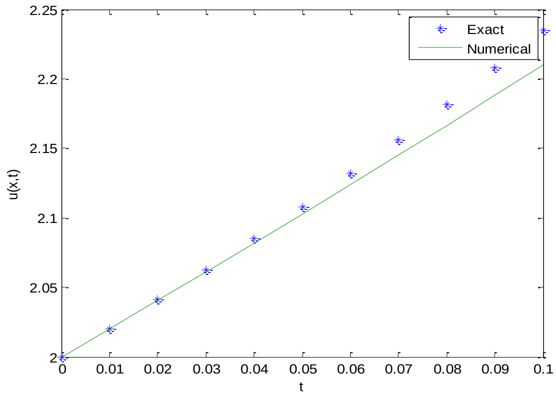


Figure 3.11 solutions when alpha=1.0.

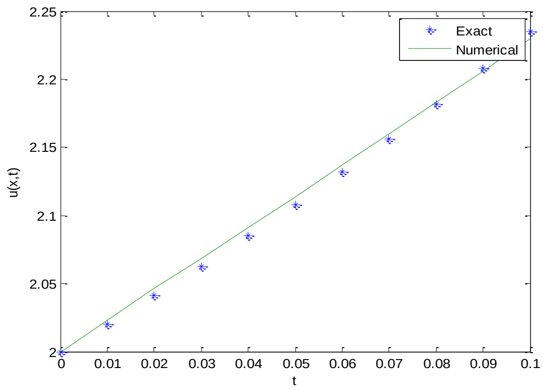


Figure 3.12 Solutions when alpha=0.97.

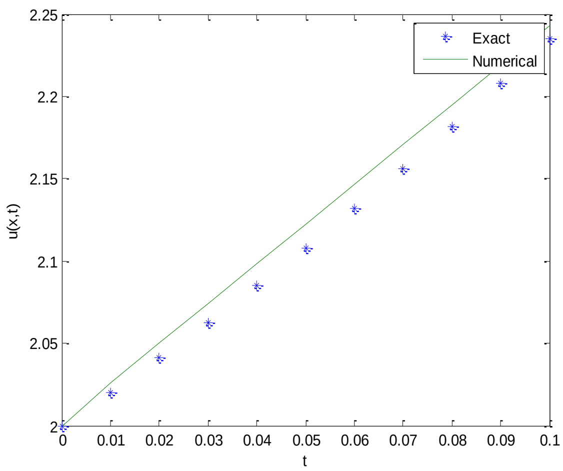


Figure 3.13 Solutions when alpha=0.95.

3.4 Conclusion

This Chapter develops a novel and reliable method for solving one-dimensional time-fractional order logistic, BBMB, Fisher's, and Schrödinger differential equations using LRPSM. The advantage of considered technique is to decrease computational effort required for finding the solutions in residual power series form after applying Laplace transform. The coefficients of power series solution form are determined after applying LRPSM in the above successive steps. These differential equations are solved by using the LRPSM which proved its ability to work out linear or non-linear FDEs with sufficient accuracy and reliable calculation steps. When number of terms increased in numerical solutions then the solution becomes closer towards the exact solution of these equations. This demonstrated the exactness and consistency of the considered method's solution of one-dimensional FDEs.

The LRPSM is a practical, reliable and efficient technique for locating analytical approximations of one dimensional time-fractional BBMB, Fisher's, Schrödinger, and logistic differential equations.

Chapter 4

Solutions of Two-dimensional Fractional Differential Equations by Laplace Transform with Residual Power Series Method

The use of LRPSM to solve two-dimensional FDEs, specifically the diffusion and biological population equations are explained in this chapter. Due to their broad implications and numerous applications in different types of problems arising in signal processing systems, diffusion-reaction processes, electrical network systems, and some other technical issues, FDEs have piqued the interest of researchers studying differential calculus [109].

The extended versions of the classical differential equations are the FDEs [52, 99], which have been widely applied in a variety of scientific domains during the past few decades. There are various applications in science that describe the importance of the principles of fractional calculus. Although there are many available approximate analytical techniques for solving FDEs mathematically, researchers are putting their efforts in developing new technique that can lead to more accurate solution of the fractional equations.

There are many trustworthy and effective numerical and analytical procedures that can be used to address time-fractional problems with greater accuracy. FDEs have solved with variety of methods, including the homotopy analysis method [110], Laplace transform [103], Adomian decomposition method [38], variational iteration method [111], promoted residual power series method [69], homotopy perturbation method [119], differential transform method [112], iterative method [120], and others. Local fractional integral transforms [101] are also applied to find the solutions of such equations in numerical forms. Recently, a well-known relaxation oscillation equation is solved by RPSM [113] and got highly reliable and efficient results.

For the principle of solving systems of ODEs, PDEs & FDEs, the RPSM is a well-liked semi-analytical method [103]. The method yields a polynomial as the answer, but the analytical approximation gives a convergent power series with easily calculable components. There are several ways why RPSM is not traditional superior order Taylor chain approach. This approach differs from Taylor's series approximation, among other things, in that it is simple to hybridise the RPSM with the transform. A Schrödinger equation [105], a relaxation-oscillation equation [133], a foam drainage model [114], a fractional Boussinesq equation [115], coupled physical equations occurring in fluid flow [116], and many more linear and nonlinear equations are solved using RPSM.

Many other well-known equations can also be solved with the help of transform in addition to RPSM. For example, the temporal-fractional NWS equation, Burger's equation, Drinfeld-Sokolov-Wilson system, Riccati differential equation, reaction diffusion Brusselator model, and Burger's equation can all be solved using the LRPSM [114-119]. The two-dimensional diffusion and biological population diffusion equations [65, 119] are two examples of FDEs that can be successfully solved using the LRPSM in this chapter.

In the last several decades, numerous fractional generalisations of the diffusion equation have been presented and have been the subject of significant discussion in both the academic literature and various diffusion model applications [119]. By using the notions of fractional calculus, the diffusion equation is a PDE that describes the temporal evolution of a quantity, such as heat, mass, or particles. In contrast to the conventional diffusion equation that employs integer-order derivatives, the diffusion equation incorporates fractional derivatives in the temporal domain. This feature enables the model to effectively capture non-local and memory-dependent behaviours in diffusion processes, making it particularly advantageous for modelling phenomena characterized by long-range interactions or intricate temporal dependencies. Although the fractional diffusion equations has been solved by many numerical and analytical approaches. Here an attempt is made to implement the LPRSM to solve the equation for various values of the fractional power.

Another equation which is presented in this work is a biological population equation. The biological population equation is a mathematical model that integrates fractional calculus concepts to clarify the time dynamics of a biological population [65]. Within the framework of a biological population equation, the fractional order is commonly utilized to denote a level of memory or a more intricate reliance on previous occurrences compared to conventional differential equations with integer orders. The utilization of these equations in the field of ecology and population dynamics serves to effectively capture much behaviour, including but not limited to population growth, competition, predation, and the influence of environmental factors, with enhanced precision.

Consider the general FDE in two dimensions of the form,

$$D_t^{\alpha}\Psi(x,y,t) + L[x,y]\Psi(x,y,t) + NL[x,y]\,\Psi(x,y,t) = \varphi(x,y,t) \tag{4.1}$$

for $t > 0, n-1 < n\alpha \leq n$

with initial condition, $\Psi(x,y,0) = f_0(x,y)$ (4.2)

and $f_{n-1}(x,y) = D_t^{\alpha-1}\Psi(x,y,0) = \mu(x,y)$ (4.3)

where $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, $L[x,y]$ = linear function in x and y,

$NL[x,y]$ = general non-linear function in x & y,

and $\varphi(x,y,t)$ = continuous function.

The solutions to equations (4.1) and (4.2) in relation to the primary point $t = 0$ is written in power series form by using LRPSM.

4.1 Methodology for Implementation

The steps below outline the methods used to solve two-dimensional FDEs using LRPSM: **Step 1:** Performing Laplace transform in two-dimensional diffusion equation, it takes the form,

$$\mathcal{L}D_t^{\alpha}\Psi(x,y,t) = \mathcal{L}\Psi_{xx}(x,y,t) + \mathcal{L}\Psi_{yy}(x,y,t) \quad (4.4)$$

By the property of Laplace transform,

$$\mathcal{L}[D_t^{\alpha}\Psi(x,y,t)] = s^{\alpha}\mathcal{L}[\Psi(x,y,t)] - s^{\alpha-1}\Psi(x,y,0) \quad (4.5)$$

Hence equation (4.4) can be written as,

$$s^{\alpha}\mathcal{L}[\Psi(x,y,t)] - s^{\alpha-1}\Psi(x,y,0) = \mathcal{L}(\Psi(x,y,t))_{xx} + \mathcal{L}(\Psi(x,y,t))_{yy}$$

$$\Psi(x,y,s) = \frac{\Psi(x,y,0)}{s} + \frac{1}{s^{\alpha}}\Psi_{xx}(x,y,s) + \frac{1}{s^{\alpha}}\Psi_{yy}(x,y,s) \quad (4.6)$$

Step 2: The solutions in LRPS form can be written as,

$$\Psi(x,y,s) = \sum_{k=0}^{\infty}\frac{h_k(x,y)}{s^{\alpha k+1}} \quad (4.7)$$

$$\Psi_{xx}(x,y,s) = \sum_{k=0}^{\infty}\frac{\partial_x^2 h_k(x,y)}{s^{\alpha k+1}} \quad (4.8)$$

$$\Psi_{yy}(x,y,s) = \sum_{k=0}^{\infty}\frac{\partial_y^2 h_k(x,y)}{s^{\alpha k+1}} \quad (4.9)$$

Again the k^{th} −Laplace residual function of (4.6) is,

$$\mathcal{L}Res_k(x,y,s) = \Psi_k(x,y,s) - \frac{1}{s}\Psi(x,y,0) - \frac{1}{s^{\alpha}}(\Psi_k(x,y,s))_{xx} - \frac{1}{s^{\alpha}}(\Psi_k(x,y,s))_{yy} \quad (4.10)$$

$$= \sum_{k=0}^{\infty}\frac{h_k(x,y)}{s^{\alpha k+1}} - \frac{1}{s}\Psi(x,y,0) - \sum_{k=1}^{\infty}\frac{\nabla^2 h_{k-1}(x,y)}{s^{\alpha k+1}}$$

$$= \frac{h_0 - \Psi(x,y,0)}{s} + \sum_{k=1}^{\infty}\frac{h_k - \nabla^2 h_{k-1}}{s^{\alpha k+1}}$$

Step 3: Using adopting properties of LRPS,

$h_0 = \Psi(x,y,0)$ and $h_k - \nabla^2 h_{k-1} = 0$ or $h_k = \nabla^2 h_{k-1}$

Following are some useful relations which are used in standard RPSM:

$i)$ $\mathcal{L}Res(x,y,s) = 0$ and $\lim_{k\to\infty}\mathcal{L}Res_k(x,y,s) = \mathcal{L}Res(x,y,s)$, for $s > 0$.

$ii)$ $\lim_{s\to\infty} s\mathcal{L}Res(x,y,s) = 0$ $gives$ $\lim_{s\to\infty} s\mathcal{L}Res_k(x,y,s) = 0$.

$iii)$ $\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res(x,y,s) = \lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}Res_k(x,y,s) = 0$ for $0 < \alpha \leq 1$.

Step 4: At last performing inverse Laplace transform to $\Psi'(x,y,s)$ then the k^{th} − approximate solution $\Psi(x,y,t)$ can be obtained.

4.2 Numerical Experiments

This section computes numerical solution of FDEs in two dimensions using LRPSM. Two well-known equations are taken as numerical solution in order to demonstrate the effectiveness of the method. A **diffusion equation** is the first equation under consideration, and a **biological population equation** is the second.

Example 4.2.1 The **diffusion equation** is given by,

$$D_t^{\alpha}\Psi(x,y,t) = \Psi_{xx}(x,y,t) + \Psi_{yy}(x,y,t) \text{ with } 0 < \alpha \leq 1 \tag{4.11}$$

with primary condition, $\Psi(x,y,0) = \sin x \sin y$ (4.12)

and exact solution is, $\Psi(x,y,t) = e^{-2t} \sin x \sin y for\ \alpha = 1$ (4.13)

& for $0 < \alpha < 1$, is $\Psi(x,y,t) = E_{\alpha}(z) \sin x \sin y$

where $E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}, z = -4it^{\alpha}$.

To obtain the solution, applying Laplace transform on equation (4.1),

$$\mathcal{L}D_t^{\alpha}\Psi(x,y,t) = \mathcal{L}\Psi_{xx}(x,y,t) + \mathcal{L}\Psi_{yy}(x,y,t) \tag{4.14}$$

By property of Laplace transform, $\mathcal{L}[D_t^{\alpha}\Psi(x,y,t)] = s^{\alpha}\mathcal{L}[\Psi(x,y,t)] - s^{\alpha-1}\Psi(x,y,0)$, then equation (4.14) can be written as,

$$s^{\alpha}\mathcal{L}[\Psi(x,y,t)] - s^{\alpha-1}\Psi(x,y,0) = \mathcal{L}(\Psi(x,y,t))_{xx} + \mathcal{L}(\Psi(x,y,t))_{yy}$$

$$\Psi(x,y,s) = \frac{\Psi(x,y,0)}{s} + \frac{1}{s^{\alpha}}\Psi_{xx}(x,y,s) + \frac{1}{s^{\alpha}}\Psi_{yy}(x,y,s) \tag{4.15}$$

The solutions in LRPS form is written as,

$$\Psi(x,y,s) = \sum_{k=0}^{\infty} \frac{h_k(x,y)}{s^{\alpha k+1}} \tag{4.16}$$

$$\Psi_{xx}(x,y,s) = \sum_{k=0}^{\infty} \frac{\partial_x^2 h_k(x,y)}{s^{\alpha k+1}}$$

$$\Psi_{yy}(x,y,s) = \sum_{k=0}^{\infty} \frac{\partial_y^2 h_k(x,y)}{s^{\alpha k+1}}$$

Then we get,

$$\sum_{k=0}^{\infty} \frac{h_k(x,y)}{s^{\alpha k+1}} = \frac{1}{s^{\alpha}}\sum_{k=0}^{\infty} \frac{\partial_x^2 h_k(x,y) + \partial_y^2 h_k(x,y)}{s^{\alpha k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{\nabla^2 h_k(x,y)}{s^{(k+1)\alpha+1}}$$

$$= \sum_{k=1}^{\infty} \frac{\nabla^2 h_{k-1}(x,y)}{s^{\alpha k+1}}$$

Therefore, $h_0 = \Psi(x,y,0)$, $h_1 = \nabla^2 h_0$ and $h_2 = \nabla^2 h_1$

Again the k^{th} −Laplace residual function of (4.15) is,

$$\mathcal{L}Res_k(x,y,s) = \Psi_k(x,y,s) - \frac{1}{s}\Psi(x,y,0) - \frac{1}{s^\alpha}(\Psi_k(x,y,s))_{xx} - \frac{1}{s^\alpha}(\Psi_k(x,y,s))_{yy} \quad (4.17)$$

$$= \sum_{k=0}^{\infty}\frac{h_k(x,y)}{s^{\alpha k+1}} - \frac{1}{s}\Psi(x,y,0) - \sum_{k=1}^{\infty}\frac{\nabla^2 h_{k-1}(x,y)}{s^{\alpha k+1}}$$

$$= \frac{h_0 - \Psi(x,y,0)}{s} + \sum_{k=1}^{\infty}\frac{h_k - \nabla^2 h_{k-1}}{s^{\alpha k+1}}$$

By adopting properties of LRPS,

$h_0 = \Psi(x,y,0)$ and $h_k - \nabla^2 h_{k-1} = 0$ or $h_k = \nabla^2 h_{k-1}$

Hence,

$$h_0 = sin\,x\,sin\,y$$

$$h_1 = \nabla^2(sin\,x\,sin\,y) = -2sin\,x\,sin\,y$$

$$h_2 = \nabla^2 h_1 = \nabla^2(-2sin\,x\,sin\,y) = 4sin\,x\,sin\,y$$

$\dots\ \dots\ \dots$

$$h_k = \nabla^k(-2sin\,x\,sin\,y) = (-2)^k sin\,x\,sin\,y$$

Therefore by LRPSM solution of given equation in infinite form is,

$$\Psi'(x,y,s) = sin\,x\,sin\,y\frac{1}{s} - 2sin\,x\,sin\,y\frac{1}{s^{\alpha+1}} + 4sin\,x\,sin\,y\frac{1}{s^{2\alpha+1}} - 8sin\,x\,sin\,y\frac{1}{s^{3\alpha+1}} + \cdots$$
(4.18)

Finally, by taking inverse Laplace in (4.18), to obtain required solution of considered equation via LRPSM as,

$$\Psi(x,y,t) = sin\,x\,sin\,y - 2sin\,x\,sin\,y\frac{t^\alpha}{\alpha!} + 4sin\,x\,sin\,y\frac{t^{2\alpha}}{(2\alpha)!} - 8sin\,x\,sin\,y\frac{t^{3\alpha}}{(3\alpha)!} + \cdots$$
(4.19)

Example 4.2.2 The **biological population equation** is given by,

$D_t^\alpha\Psi(x,y,t) = (\Psi^2(x,y,t))_{xx} + (\Psi^2(x,y,t))_{yy} + h\Psi(x,y,t)$, where h is constant,(4.20)

with primary condition $\Psi(x,y,0) = \sqrt{xy}$ (4.21)

and exact solution for $\alpha = 1$ is $\Psi(x,y,t) = \sqrt{xy}e^{ht}$ (4.22)

To obtain the solution, performing Laplace transform on equation (4.20) results in

$$\mathcal{L}\left(D_t^\alpha\Psi(x,y,t)\right) = \mathcal{L}((\Psi^2(x,y,t))_{xx}) + \mathcal{L}((\Psi^2(x,y,t))_{yy}) + h\mathcal{L}(\Psi(x,y,t)) \quad (4.23)$$

But it is clear that, $\mathcal{L}[D_t^\alpha\Psi(x,y,t)] = s^\alpha\mathcal{L}[\Psi(x,y,t)] - s^{\alpha-1}\Psi(x,y,0)$, so we may write,

$s^{\alpha}\mathcal{L}[\Psi(x,y,t)] - s^{\alpha-1}\Psi(x,y,0) = \mathcal{L}((\Psi^2(x,y,t))_{xx}) + \mathcal{L}((\Psi^2(x,y,t))_{yy}) + h\mathcal{L}(\Psi(x,y,t))$

or, $s^{\alpha}\Psi'(x,y,s) - s^{\alpha-1}\sqrt{xy} = \mathcal{L}[\{\mathcal{L}^{-1}(\Psi'(x,y,s)_{xx})\}^2] + \mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\Psi'(x,y,s)_{yy}\right)\right\}^2\right] + h\Psi'(x,y,s)$

where $\Psi(x,y,0) = \sqrt{xy}$

or, $\Psi'(x,y,s) = \frac{1}{s}\sqrt{xy} + \frac{1}{s^{\alpha}}\mathcal{L}[\{\mathcal{L}^{-1}(\Psi'(x,y,s)_{xx})\}^2] + \frac{1}{s^{\alpha}}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\Psi'(x,y,s)_{yy}\right)\right\}^2\right] + h\,\frac{1}{s^{\alpha}}\Psi'(x,y,s)$ (4.24)

where $\Psi'(x,y,s) = \mathcal{L}[\Psi(x,y,t)]$

Now, the transformed function $\Psi'(x,y,s)$ in expansion is,

$$\Psi'(x,y,s) = \sum_{n=0}^{\infty}\frac{f_n(x,y)}{s^{n\alpha+1}} \quad (4.25)$$

The k^{th} −truncated series of (4.25) is,

$$\Psi'_k(x,y,s) = \sum_{n=0}^{k}\frac{f_n(x,y)}{s^{n\alpha+1}} \quad (4.26)$$

Then by the Laplace residual function of (4.24) is,

$\mathcal{L}Res(x,s) = \Psi'(x,y,s) - \frac{1}{s}\sqrt{xy} - \frac{1}{s^{\alpha}}\mathcal{L}[\{\mathcal{L}^{-1}(\Psi'(x,y,s)_{xx})\}^2] - \frac{1}{s^{\alpha}}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\Psi'(x,y,s)_{yy}\right)\right\}^2\right] - h\,\frac{1}{s^{\alpha}}\Psi'(x,y,s)$
(4.27)

Again the k^{th} −Laplace residual function of (4.27) is

$\mathcal{L}Res_k(x,s) = \Psi'_k(x,y,s) - \frac{1}{s}\sqrt{xy} - \frac{1}{s^{\alpha}}\mathcal{L}[\{\mathcal{L}^{-1}(\Psi'_k(x,y,s)_{xx})\}^2] - \frac{1}{s^{\alpha}}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\Psi'_k(x,y,s)_{yy}\right)\right\}^2\right] - h\,\frac{1}{s^{\alpha}}\Psi'_k(x,y,s)$
(4.28)

To find the values of $f_k(x,y), k = 1,2,3,\ldots\ldots$ putting $k^{th}-$ truncated series (4.26) in k^{th} −Laplace residual function (4.28) we get,

$$\mathcal{L}Res_k(x,y,s) = \sum_{n=0}^{k}\frac{f_n(x,y)}{s^{n\alpha+1}} - \frac{1}{s}\sqrt{xy} - \frac{1}{s^{\alpha}}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\left(\sum_{n=0}^{k}\frac{f_n(x,y)}{s^{n\alpha+1}}\right)_{xx}\right)\right\}^2\right] - \frac{1}{s^{\alpha}}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\left(\sum_{n=0}^{k}\frac{f_n(x,y)}{s^{n\alpha+1}}\right)_{yy}\right)\right\}^2\right] - h\,\frac{1}{s^{\alpha}}\sum_{n=0}^{k}\frac{f_n(x,y)}{s^{n\alpha+1}}$$

$$= \frac{f_0(x,y)}{s} + \sum_{n=1}^{k}\frac{f_n(x,y)}{s^{n\alpha+1}} - \frac{1}{s}\sqrt{xy} - \frac{1}{s^{\alpha}}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\left(\frac{f_0(x,y)}{s} + \sum_{n=1}^{k}\frac{f_n(x,y)}{s^{n\alpha+1}}\right)_{xx}\right)\right\}^2\right] - \frac{1}{s^{\alpha}}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\left(\frac{f_0(x,y)}{s} + \sum_{n=1}^{k}\frac{f_n(x,y)}{s^{n\alpha+1}}\right)_{yy}\right)\right\}^2\right] - \frac{h}{s^{\alpha}}\left(\frac{f_0(x,y)}{s} + \sum_{n=1}^{k}\frac{f_n(x,y)}{s^{n\alpha+1}}\right)$$

$$= \frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}} - \frac{1}{s}\sqrt{xy} - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left((\frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}})_{xx}\right)\right\}^2\right] - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left((\frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}})_{yy}\right)\right\}^2\right] - \frac{h}{s^\alpha}(\frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}})$$

$$= \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left((\frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}})_{xx}\right)\right\}^2\right] - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left((\frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}})_{yy}\right)\right\}^2\right] - \frac{h}{s^\alpha}(\frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}})$$

$$\mathcal{L}Res_k(x,y,s) = \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}} - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left((\frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}})_{xx}\right)\right\}^2\right] - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left((\frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}})_{yy}\right)\right\}^2\right] - \frac{h}{s^\alpha}(\frac{1}{s}\sqrt{xy} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}}) \quad (4.29)$$

For $k = 1$ from (4.29), the first Laplace residual function is,

$$\mathcal{L}Res_1(x,y,s) = \frac{f_1(x,y)}{s^{\alpha+1}} - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left((\frac{1}{s}\sqrt{xy} + \frac{f_1(x,y)}{s^{\alpha+1}})_{xx}\right)\right\}^2\right] - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left((\frac{1}{s}\sqrt{xy} + \frac{f_1(x,y)}{s^{\alpha+1}})_{yy}\right)\right\}^2\right] - \frac{h}{s^\alpha}(\frac{1}{s}\sqrt{xy} + \frac{f_1(x,y)}{s^{\alpha+1}}) \quad (4.30)$$

$$= \frac{f_1(x,y)}{s^{\alpha+1}} - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\left(1 * \sqrt{xy} + \frac{f_1(x,y)t^\alpha}{\alpha!}\right)_{xx}\right\}^2\right] - \frac{1}{s^\alpha}\mathcal{L}\left[\left\{\left(1 * \sqrt{xy} + \frac{f_1(x,y)t^\alpha}{\alpha!}\right)_{yy}\right\}^2\right] - \frac{h}{s^{\alpha+1}}\sqrt{xy} - \frac{hf_1(x,y)}{s^{2\alpha+1}}$$

$$= \frac{f_1(x,y)}{s^{\alpha+1}} - \frac{1}{s^\alpha}\mathcal{L}[\left\{-\frac{1}{4}x^{-\frac{3}{2}}y^{\frac{1}{2}} + \{f_1(x,y)\}_{xx}\frac{t^\alpha}{\alpha!}\right\}^2] - \frac{1}{s^\alpha}\mathcal{L}[\left\{-\frac{1}{4}x^{\frac{1}{2}}y^{\frac{-3}{2}} + \{f_1(x,y)\}_{yy}\frac{t^\alpha}{\alpha!}\right\}^2] - \frac{h}{s^{\alpha+1}}\sqrt{xy} - \frac{hf_1(x,y)}{s^{2\alpha+1}}$$

$$= \frac{f_1(x,y)}{s^{\alpha+1}} - \frac{1}{s^\alpha}\ \mathcal{L}\left[\frac{1}{16}x^{-3}y - \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}\{f_1(x,y)\}_{xx}\frac{t^\alpha}{\alpha!} + [\{f_1(x,y)\}_{xx}]^2\frac{t^{2\alpha}}{(\alpha!)^2}\right] - \frac{1}{s^\alpha}\mathcal{L}\left[\frac{1}{16}xy^{-3} - \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}\{f_1(x,y)\}_{yy}\frac{t^\alpha}{\alpha!} + [\{f_1(x,y)\}_{yy}]^2\frac{t^{2\alpha}}{(\alpha!)^2}\right] - \frac{h}{s^{\alpha+1}}\sqrt{xy} - \frac{hf_1(x,y)}{s^{2\alpha+1}}$$

$$= \frac{f_1(x,y)}{s^{\alpha+1}} - \frac{1}{s^\alpha}\ \left[\frac{1}{16}x^{-3}y\frac{1}{s} - \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}\{f_1(x,y)\}_{xx}\frac{1}{\alpha!}\frac{\alpha!}{s^{\alpha+1}} + [\{f_1(x,y)\}_{xx}]^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{2\alpha+1}}\right] - \frac{1}{s^\alpha}\left[\frac{1}{16}xy^{-3}\frac{1}{s} - \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}\{f_1(x,y)\}_{yy}\frac{1}{\alpha!}\frac{\alpha!}{s^{\alpha+1}} + [\{f_1(x,y)\}_{yy}]^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{2\alpha+1}}\right] - \frac{h}{s^{\alpha+1}}\sqrt{xy} - \frac{hf_1(x,y)}{s^{2\alpha+1}}$$

$$= \frac{f_1(x,y)}{s^{\alpha+1}} - \ \frac{1}{16}x^{-3}y\frac{1}{s^{\alpha+1}} + \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}\{f_1(x,y)\}_{xx}\frac{1}{s^{2\alpha+1}} - [\{f_1(x,y)\}_{xx}]^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{3\alpha+1}} - \frac{1}{16}xy^{-3}\frac{1}{s^{\alpha+1}} + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}\{f_1(x,y)\}_{yy}\frac{1}{\alpha!}\frac{\alpha!}{s^{2\alpha+1}} - [\{f_1(x,y)\}_{yy}]^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{3\alpha+1}} - \frac{h}{s^{\alpha+1}}\sqrt{xy} - \frac{hf_1(x,y)}{s^{2\alpha+1}}$$

Now, the relation $\lim_{s\to\infty}(s^{\alpha+1}\,\mathcal{L}Res_1(x,y,s)) = 0\ for\ k = 1$, gives that

$$f_1(x,y) - \ \frac{1}{16}x^{-3}y - \frac{1}{16}xy^{-3} - h\sqrt{xy} = 0$$

$$\therefore f_1(x,y) = \frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} + h\sqrt{xy}$$

For $k = 2$, from (4.29) the second Laplace residual function is,

$$\mathcal{L}Res_2(x,y,s)=\sum_{n=1}^{2}\frac{f_n(x,y)}{s^{n\alpha+1}}-\frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\left(\frac{1}{s}\sqrt{xy}+\sum_{n=1}^{2}\frac{f_n(x,y)}{s^{n\alpha+1}}\right)_{xx}\right)\right\}^2\right]-\frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\left(\frac{1}{s}\sqrt{xy}+\sum_{n=1}^{2}\frac{f_n(x,y)}{s^{n\alpha+1}}\right)_{yy}\right)\right\}^2\right]-\frac{h}{s^\alpha}\left(\frac{1}{s}\sqrt{xy}+\sum_{n=1}^{2}\frac{f_n(x,y)}{s^{n\alpha+1}}\right) \tag{4.31}$$

$$=\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}-\frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\left(\frac{1}{s}\sqrt{xy}+\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}\right)_{xx}\right)\right\}^2\right]-\frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(\left(\frac{1}{s}\sqrt{xy}+\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}\right)_{yy}\right)\right\}^2\right]-\frac{h}{s^\alpha}\left(\frac{1}{s}\sqrt{xy}+\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}\right)$$

$$=\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}-\frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(-\frac{1}{4}x^{-\frac{3}{2}}y^{\frac{1}{2}}\frac{1}{s}+(f_1(x,y))_{xx}\frac{1}{s^{\alpha+1}}+(f_2(x,y))_{xx}\frac{1}{s^{2\alpha+1}}\right)\right\}^2\right]-\frac{1}{s^\alpha}\mathcal{L}\left[\left\{\mathcal{L}^{-1}\left(-\frac{1}{4}x^{\frac{1}{2}}y^{\frac{-3}{2}}\frac{1}{s}+(f_1(x,y))_{yy}\frac{1}{s^{\alpha+1}}+(f_2(x,y))_{yy}\frac{1}{s^{2\alpha+1}}\right)\right\}^2\right]-\frac{h}{s^\alpha}\left(\frac{1}{s}\sqrt{xy}+\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}\right)$$

$$=\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}-\frac{1}{s^\alpha}\mathcal{L}\left[\left\{-\frac{1}{4}x^{-\frac{3}{2}}y^{\frac{1}{2}}*1+(f_1(x,y))_{xx}\frac{t^\alpha}{\alpha!}+(f_2(x,y))_{xx}\frac{t^{2\alpha}}{(2\alpha)!}\right\}^2\right]-\frac{1}{s^\alpha}\mathcal{L}\left[\left\{-\frac{1}{4}x^{\frac{1}{2}}y^{\frac{-3}{2}}*1+(f_1(x,y))_{yy}\frac{t^\alpha}{\alpha!}+(f_2(x,y))_{yy}\frac{t^{2\alpha}}{(2\alpha)!}\right\}^2\right]-\frac{h}{s^\alpha}\left(\frac{1}{s}\sqrt{xy}+\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}\right)$$

$$=\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}-\frac{1}{s^\alpha}\mathcal{L}\left[\frac{1}{16}x^{-3}y+\{(f_1(x,y))_{xx}\}^2\frac{t^{2\alpha}}{(\alpha!)^2}+\{(f_2(x,y))_{xx}\}^2\frac{t^{4\alpha}}{((2\alpha)!)^2}-\frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_1(x,y))_{xx}\frac{t^\alpha}{\alpha!}-\frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_2(x,y))_{xx}\frac{t^{2\alpha}}{(2\alpha)!}+2(f_1(x,y))_{xx}(f_2(x,y))_{xx}\frac{t^{3\alpha}}{\alpha!(2\alpha)!}\right]-\frac{1}{s^\alpha}\mathcal{L}\left[\frac{1}{16}xy^{-3}+\{(f_1(x,y))_{yy}\}^2\frac{t^{2\alpha}}{(\alpha!)^2}+\{(f_2(x,y))_{yy}\}^2\frac{t^{4\alpha}}{((2\alpha)!)^2}-\frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_1(x,y))_{yy}\frac{t^\alpha}{\alpha!}-\frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_2(x,y))_{yy}\frac{t^{2\alpha}}{(2\alpha)!}+2(f_1(x,y))_{yy}(f_2(x,y))_{yy}\frac{t^{3\alpha}}{\alpha!(2\alpha)!}\right]]-\frac{h}{s^{\alpha+1}}\sqrt{xy}-\frac{hf_1(x,y)}{s^{2\alpha+1}}-\frac{hf_2(x,y)}{s^{3\alpha+1}}$$

$$=\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}-\frac{1}{s^\alpha}\left[\frac{1}{16}x^{-3}y\frac{1}{s}+\{(f_1(x,y))_{xx}\}^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{2\alpha+1}}+\{(f_2(x,y))_{xx}\}^2\frac{1}{((2\alpha)!)^2}\frac{(4\alpha)!}{s^{4\alpha+1}}-\frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_1(x,y))_{xx}\frac{1}{\alpha!}\frac{\alpha!}{s^{\alpha+1}}-\frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_2(x,y))_{xx}\frac{1}{(2\alpha)!}\frac{(2\alpha)!}{s^{2\alpha+1}}+2(f_1(x,y))_{xx}(f_2(x,y))_{xx}\frac{1}{\alpha!(2\alpha)!}\frac{(3\alpha)!}{s^{3\alpha+1}}\right]-\frac{1}{s^\alpha}\left[\frac{1}{16}xy^{-3}\frac{1}{s}+\{(f_1(x,y))_{yy}\}^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{2\alpha+1}}+\{(f_2(x,y))_{yy}\}^2\frac{1}{((2\alpha)!)^2}\frac{(4\alpha)!}{s^{4\alpha+1}}-\frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_1(x,y))_{yy}\frac{1}{\alpha!}\frac{\alpha!}{s^{\alpha+1}}-\frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_2(x,y))_{yy}\frac{1}{(2\alpha)!}\frac{(2\alpha)!}{s^{2\alpha+1}}+2(f_1(x,y))_{yy}(f_2(x,y))_{yy}\frac{1}{\alpha!(2\alpha)!}\frac{(3\alpha)!}{s^{3\alpha+1}}\right]]-\frac{h}{s^{\alpha+1}}\sqrt{xy}-\frac{hf_1(x,y)}{s^{2\alpha+1}}-\frac{hf_2(x,y)}{s^{3\alpha+1}}$$

$$=\frac{f_1(x,y)}{s^{\alpha+1}}+\frac{f_2(x,y)}{s^{2\alpha+1}}-\frac{1}{16}x^{-3}y\frac{1}{s^{\alpha+1}}-\{(f_1(x,y))_{xx}\}^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{3\alpha+1}}-\{(f_2(x,y))_{xx}\}^2\frac{1}{((2\alpha)!)^2}\frac{(4\alpha)!}{s^{5\alpha+1}}+\frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_1(x,y))_{xx}\frac{1}{s^{2\alpha+1}}+\frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_2(x,y))_{xx}\frac{1}{s^{3\alpha+1}}-2(f_1(x,y))_{xx}(f_2(x,y))_{xx}\frac{1}{\alpha!(2\alpha)!}\frac{(3\alpha)!}{s^{4\alpha+1}}-\frac{1}{16}xy^{-3}\frac{1}{s^{\alpha+1}}-\{(f_1(x,y))_{yy}\}^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{3\alpha+1}}-\{(f_2(x,y))_{yy}\}^2\frac{1}{((2\alpha)!)^2}\frac{(4\alpha)!}{s^{5\alpha+1}}+\frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_1(x,y))_{yy}\frac{1}{s^{2\alpha+1}}+\frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_2(x,y))_{yy}\frac{1}{s^{3\alpha+1}}-2(f_1(x,y))_{yy}(f_2(x,y))_{yy}\frac{1}{\alpha!(2\alpha)!}\frac{(3\alpha)!}{s^{4\alpha+1}}-\frac{h}{s^{\alpha+1}}\sqrt{xy}-\frac{hf_1(x,y)}{s^{2\alpha+1}}-\frac{hf_2(x,y)}{s^{3\alpha+1}}$$

$= \frac{f_1(x,y)}{s^{\alpha+1}} + \frac{f_2(x,y)}{s^{2\alpha+1}} - \{\frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} + h\sqrt{xy}\}\frac{1}{s^{\alpha+1}} - \{(f_1(x,y))_{xx}\}^2 \frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{3\alpha+1}} -$
$\{(f_2(x,y))_{xx}\}^2 \frac{1}{((2\alpha)!)^2}\frac{(4\alpha)!}{s^{5\alpha+1}} + \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_1(x,y))_{xx}\frac{1}{s^{2\alpha+1}} + \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_2(x,y))_{xx}\frac{1}{s^{3\alpha+1}} -$
$2(f_1(x,y))_{xx}(f_2(x,y))_{xx}\frac{1}{\alpha!(2\alpha)!}\frac{(3\alpha)!}{s^{4\alpha+1}} - \{(f_1(x,y))_{yy}\}^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{3\alpha+1}} -$
$\{(f_2(x,y))_{yy}\}^2\frac{1}{((2\alpha)!)^2}\frac{(4\alpha)!}{s^{5\alpha+1}} + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_1(x,y))_{yy}\frac{1}{s^{2\alpha+1}} + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_2(x,y))_{yy}\frac{1}{s^{3\alpha+1}} -$
$2(f_1(x,y))_{yy}(f_2(x,y))_{yy}\frac{1}{\alpha!(2\alpha)!}\frac{(3\alpha)!}{s^{4\alpha+1}} - \frac{hf_1(x,y)}{s^{2\alpha+1}} - \frac{hf_2(x,y)}{s^{3\alpha+1}}$

$= \frac{f_1(x,y)}{s^{\alpha+1}} + \frac{f_2(x,y)}{s^{2\alpha+1}} - f_1(x,y)\frac{1}{s^{\alpha+1}} - \{(f_1(x,y))_{xx}\}^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{3\alpha+1}} - \{(f_2(x,y))_{xx}\}^2\frac{1}{((2\alpha)!)^2}\frac{(4\alpha)!}{s^{5\alpha+1}} +$
$\frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_1(x,y))_{xx}\frac{1}{s^{2\alpha+1}} + \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_2(x,y))_{xx}\frac{1}{s^{3\alpha+1}} -$
$2(f_1(x,y))_{xx}(f_2(x,y))_{xx}\frac{1}{\alpha!(2\alpha)!}\frac{(3\alpha)!}{s^{4\alpha+1}} - \{(f_1(x,y))_{yy}\}^2\frac{1}{(\alpha!)^2}\frac{(2\alpha)!}{s^{3\alpha+1}} -$
$\{(f_2(x,y))_{yy}\}^2\frac{1}{((2\alpha)!)^2}\frac{(4\alpha)!}{s^{5\alpha+1}} + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_1(x,y))_{yy}\frac{1}{s^{2\alpha+1}} + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_2(x,y))_{yy}\frac{1}{s^{3\alpha+1}} -$
$2(f_1(x,y))_{yy}(f_2(x,y))_{yy}\frac{1}{\alpha!(2\alpha)!}\frac{(3\alpha)!}{s^{4\alpha+1}} - \frac{hf_1(x,y)}{s^{2\alpha+1}} - \frac{hf_2(x,y)}{s^{3\alpha+1}}$

Now, the relation $\lim_{s\to\infty}(s^{2\alpha+1}\,\mathcal{L}Res_2(x,y,s)) = 0$ for $k = 2$, gives us that,

$f_2(x,y) + \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(f_1(x,y))_{xx} + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(f_1(x,y))_{yy} - hf_1(x,y) = 0$

$or, f_2(x,y) + \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(\frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} + h\sqrt{xy})_{xx} + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(\frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} +$
$h\sqrt{xy})_{yy} - h(\frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} + h\sqrt{xy}) = 0$

$or, f_2(x,y) + \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(\frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} + hx^{\frac{1}{2}}y^{\frac{1}{2}})_{xx} + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(\frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} +$
$hx^{\frac{1}{2}}y^{\frac{1}{2}})_{yy} - h(\frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} + hx^{\frac{1}{2}}y^{\frac{1}{2}}) = 0$

$or, f_2(x,y) + \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}(-3\frac{1}{16}x^{-4}y + \frac{1}{16}y^{-3} + \frac{1}{2}hx^{-\frac{1}{2}}y^{\frac{1}{2}})_x + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}(\frac{1}{16}x^{-3} - 3\frac{1}{16}xy^{-4} +$
$hx^{\frac{1}{2}}y^{\frac{-1}{2}})_y - h\frac{1}{16}x^{-3}y - h\frac{1}{16}xy^{-3} - h^2x^{\frac{1}{2}}y^{\frac{1}{2}} = 0$

$or, f_2(x,y) + \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}\{-3(-4)\frac{1}{16}x^{-5}y + 0 + \frac{1}{2}\left(-\frac{1}{2}\right)hx^{-\frac{3}{2}}y^{\frac{1}{2}}\} + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{-3}{2}}\{0 -$
$3(-4)\frac{1}{16}xy^{-5} + \frac{1}{2}\left(-\frac{1}{2}\right)hx^{\frac{1}{2}}y^{\frac{-3}{2}}\} - \frac{1}{16}hx^{-3}y - \frac{1}{16}hxy^{-3} - h^2x^{\frac{1}{2}}y^{\frac{1}{2}} = 0$

$or, f_2(x,y) + \frac{3}{8}x^{-\frac{13}{2}}y^{\frac{3}{2}} - \frac{1}{8}hx^{-3}y + \frac{3}{8}x^{\frac{3}{2}}y^{\frac{-13}{2}} - \frac{1}{8}hxy^{-3} - \frac{1}{16}hx^{-3}y - \frac{1}{16}hxy^{-3} -$
$h^2x^{\frac{1}{2}}y^{\frac{1}{2}} = 0$

$or, f_2(x,y) + \frac{3}{8}x^{-\frac{13}{2}}y^{\frac{3}{2}} + \frac{3}{8}x^{\frac{3}{2}}y^{\frac{-13}{2}} - \frac{3}{16}hx^{-3}y - \frac{3}{16}hxy^{-3} - h^2x^{\frac{1}{2}}y^{\frac{1}{2}} = 0$

$$or, f_2(x,y) = -\frac{3}{8}x^{-\frac{13}{2}}y^{\frac{3}{2}} - \frac{3}{8}x^{\frac{3}{2}}y^{\frac{-13}{2}} + \frac{3}{16}hx^{-3}y + \frac{3}{16}hxy^{-3} + h^2x^{\frac{1}{2}}y^{\frac{1}{2}}$$

Hence by LRPSM the solution of given equation in infinite form is,

$$\Psi'(x,y,s) = \sqrt{xy}\frac{1}{s} + \left\{\frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} + h\sqrt{xy}\right\}\frac{1}{s^{\alpha+1}} + \left\{-\frac{3}{8}x^{-\frac{13}{2}}y^{\frac{3}{2}} - \frac{3}{8}x^{\frac{3}{2}}y^{\frac{-13}{2}} + \frac{3}{16}hx^{-3}y + \frac{3}{16}hxy^{-3} + h^2x^{\frac{1}{2}}y^{\frac{1}{2}}\right\}\frac{1}{s^{2\alpha+1}} + \cdots \tag{4.32}$$

Finally, by performing an inverse Laplace transform in (4.32), the required solution by utilising LRPSM is,

$$\Psi(x,y,t) = \sqrt{xy} + \left\{\frac{1}{16}x^{-3}y + \frac{1}{16}xy^{-3} + h\sqrt{xy}\right\}\frac{t^\alpha}{\alpha!} + \left\{-\frac{3}{8}x^{-\frac{13}{2}}y^{\frac{3}{2}} - \frac{3}{8}x^{\frac{3}{2}}y^{\frac{-13}{2}} + \frac{3}{16}hx^{-3}y + \frac{3}{16}hxy^{-3} + h^2x^{\frac{1}{2}}y^{\frac{1}{2}}\right\}\frac{t^{2\alpha}}{(2\alpha)!} + \cdots \tag{4.33}$$

4.3 Explanation and Results

In this part, the obtained findings for diffusion and biological population equations are used to discuss the trustworthiness and effectiveness of the LRPSM.

Figure 4.1 compares the behaviour of the solutions of diffusion equation at different values of t when $\alpha = 0.5, 0.7, 1.0,$ with exact solution. This shows that solutions are reliable for alpha less than equal to one.

The absolute errors of number of terms 6, 8, 10, and 12 of solutions of diffusion equation at values of t = 0.2, 0.4, 0.6, 0.8, 1.0 and alpha equal to 0.7 with exact solution are presented in **Figure 4.2.** This shows that when number of terms are increased then the approximate solution approaches the numerical solution and hence errors s reduced.

The absolute errors of number of terms 6, 8, 10, and 12 of solutions of diffusion equation at values of t = 0.2, 0.4, 0.6, 0.8, 1.0 and alpha = 0.9 with exact solution are presented in **Figure 4.3**. This shows that when numbers of terms are increased then errors are decreased.

In **Figure 4.4** the absolute errors of number of terms 6, 8, 10, and 12 of solutions of diffusion equation at values of t = 0.2, 0.4, 0.6, 0.8, 1.0 and alpha = 1.0 with exact solution are presented. It is evident from the results that the errors are reducing with the addition of more terms in the solution.

Figures 4.5(a) & 4.5(b) show the errors for various values of t in two dimensions. The first graph is drawn for $t = 0.5$ and alpha as 0.5 while the second graph is presenting the errors in solution for alpha as one at $t = 1$. This shows that when value of alpha is approaching to 1 the errors are improved even at higher time levels.

Table 4.1 displays the numerical values of (L_∞), the maximum errors for prescribed values of t as 0.25, 0.50, 0.75, and 1.0 at various alpha values. This shows that maximum errors are decreasing as alpha is approaching to 1.

Table 4.2 and Table 4.3 displays both solutions of diffusion equation at predetermined points in two dimensions where x varies from 0.0 to 1.0 as well as y varies from 0.1 to 0.2 and 0.3 to 0.4 respectively. It is evident that both the solutions are very close enough.

Table 4.4 and Table 4.5 demonstrates both solutions of diffusion equation at predetermined sites in two dimensions where x ranges from 0.0 to 1.0 as well as y changes from 0.6 to 0.7 and 0.8 to 0.9 respectively which also verifies that both the solutions are close enough and reliable. This demonstrates the precision of the diffusion equation's mathematical solutions using the LRPSM.

In **Figure 4.6** the solution behaviour of biological population equation at various values of t= 0.0, 0.2, 0.4, 0.6, 0.8, 1.0 is presented when $\alpha = 0.5, 0.7, 1.0$. From the diagram it is observed that the lines representing both solutions are almost overlapping hence, there is no significant difference between the solutions at different alpha.

Figure 4.7 displays the absolute errors of the two-dimensional biological population equation at various values of t = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0 for a fixed value of alpha taken as 0.7. This shows that when number of terms are increased then errors of both solutions decreased.

This demonstrates how effective and innovative the **LRPSM** is for solving two-dimensional FDEs and obtaining analytical approximate of solutions.

Table 4.1 Maximum errors of **fractional order diffusion equation** at various **t** and α.

t/α	**0.5**	**0.7**	**1.0**
0.25	4.5786688e-08	3.2196467e-15	0
0.50	6.0499380e-05	8.1960438e-11	0
0.75	4.0032692e-03	3.0248714e-08	8.3266726e-17
1.0	7.7937435e-02	1.9896927e-06	2.6645352e-14

Table 4.2 The **fractional order diffusion equation** solutions at the designated points.

x/y	**0.1**		**0.2**	
	Exact	**Numerical**	**Exact**	**Numerical**
0.0	0.0	0.0	0.0	0.0
0.1	0.003419805307053	0.003419805307054	0.006805441049916	0.006805441049918
0.2	0.006805441049916	0.006805441049918	0.013542884382441	0.013542884382444
0.3	0.010123079075388	0.010123079075390	0.020145011690899	0.020145011690904
0.4	0.013339570640983	0.013339570640986	0.026545856701597	0.026545856701604
0.5	0.016422777626210	0.016422777626214	0.032681464287027	0.032681464287035
0.6	0.019341893646044	0.019341893646048	0.038490529484356	0.038490529484365
0.7	0.022067751858146	0.022067751858152	0.043915010034355	0.043915010034365
0.8	0.024573116388311	0.024573116388317	0.048900706320463	0.048900706320475
0.9	0.026832954462317	0.026832954462323	0.053397802913441	0.053397802913454
1.0	0.028824686525130	0.028824686525137	0.057361366310675	0.057361366310689

Table 4.3 The **fractional order diffusion equation** solutions at the designated points.

x/y	**0.3**	**0.4**

	Exact	Numerical	Exact	Numerical
0.0	0.0	0.0	0.0	0.0
0.1	0.010123079075388	0.010123079075388	0.013339570640983	0.013339570640986
0.2	0.020145011690899	0.020145011690904	0.026545856701597	0.026545856701604
0.3	0.029965662008651	0.029965662008658	0.039486905336943	0.039486905336952
0.4	0.039486905336943	0.039486905336952	0.052033413866797	0.052033413866810
0.5	0.048613608559744	0.048613608559755	0.064060021725254	0.064060021725270
0.6	0.057254580675338	0.057254580675352	0.075446563022060	0.075446563022078
0.7	0.065323483946673	0.065323483946688	0.086079267200468	0.086079267200489
0.8	0.072739696559485	0.072739696559502	0.095851895795031	0.095851895795054
0.9	0.079429118168822	0.079429118168841	0.104666803931235	0.104666803931260
1.0	0.085324910285191	0.085324910285212	0.112435915960803	0.112435915960830

Table 4.4 The **fractional order diffusion equation** solutions at the designated points.

x/y	0.6		0.7	
	Exact	Numerical	Exact	Numerical
0.0	0.0	0.0	0.0	0.0
0.1	0.0193418936460436	0.0193418936460483	0.0220677518581463	0.0220677518581516
0.2	0.0384905294843560	0.0384905294843652	0.0439150100343549	0.0439150100343654
0.3	0.0572545806753381	0.0572545806753518	0.0653234839466726	0.0653234839466882
0.4	0.0754465630220603	0.0754465630220784	0.0860792672004680	0.0860792672004886
0.5	0.0928847082503841	0.0928847082504064	0.1059749748704190	0.1059749748704440
0.6	0.1093947801774730	0.1093947801774990	0.1248118156221340	0.1248118156221640
0.7	0.1248118156221340	0.1248118156221640	0.1424015779694540	0.1424015779694880
0.8	0.1389817726624000	0.1389817726624340	0.1585685108214060	0.1585685108214440
0.9	0.1517630697714900	0.1517630697715260	0.1731510795290120	0.1731510795290530
1.0	0.1630280004536240	0.1630280004536630	0.1860035798861000	0.1860035798861450

Table 4.5 The **fractional order diffusion equation** solutions at the designated points.

x/y	0.8		0.9	
	Exact	Numerical	Exact	Numerical
0.0	0.0	0.0	0.0	0.0
0.1	0.0245731163883113	0.0245731163883172	0.0268329544623166	0.0268329544623230
0.2	0.0489007063204629	0.0489007063204746	0.0533978029134413	0.0533978029134540
0.3	0.0727396965594849	0.0727396965595023	0.0794291181688217	0.0794291181688407
0.4	0.0958518957950313	0.0958518957950543	0.1046668039312350	0.1046668039312600
0.5	0.1180063745722180	0.1180063745722460	0.1288586935870130	0.1288586935870440
0.6	0.1389817726624000	0.1389817726624340	0.1517630697714900	0.1517630697715260
0.7	0.1585685108214060	0.1585685108214440	0.1731510795290120	0.1731510795290530
0.8	0.1765708848360650	0.1765708848361070	0.1928090209360170	0.1928090209360630
0.9	0.1928090209360170	0.1928090209360630	0.2105404783400180	0.2105404783400680
1.0	0.2071206730329640	0.2071206730330140	0.2261682848798740	0.2261682848799280

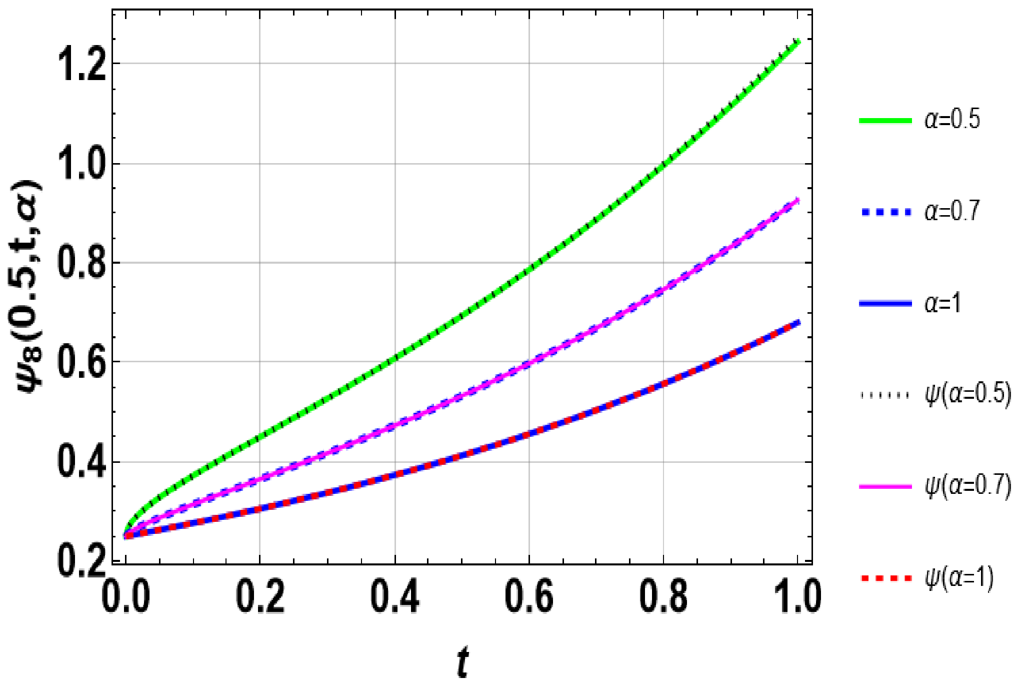


Fig 4.1 Comparison of solution behaviour of **diffusion equation** at different t when α=0.5, 0.7, 1.

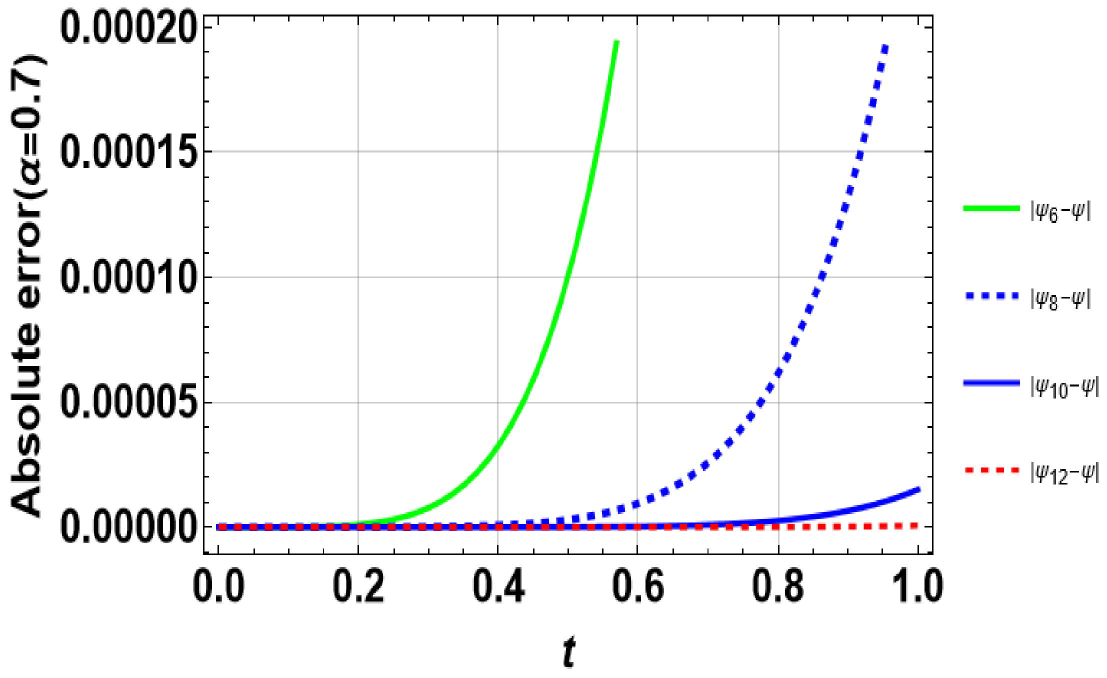


Fig 4.2 Errors of **diffusion equation** when value of α=0.7.

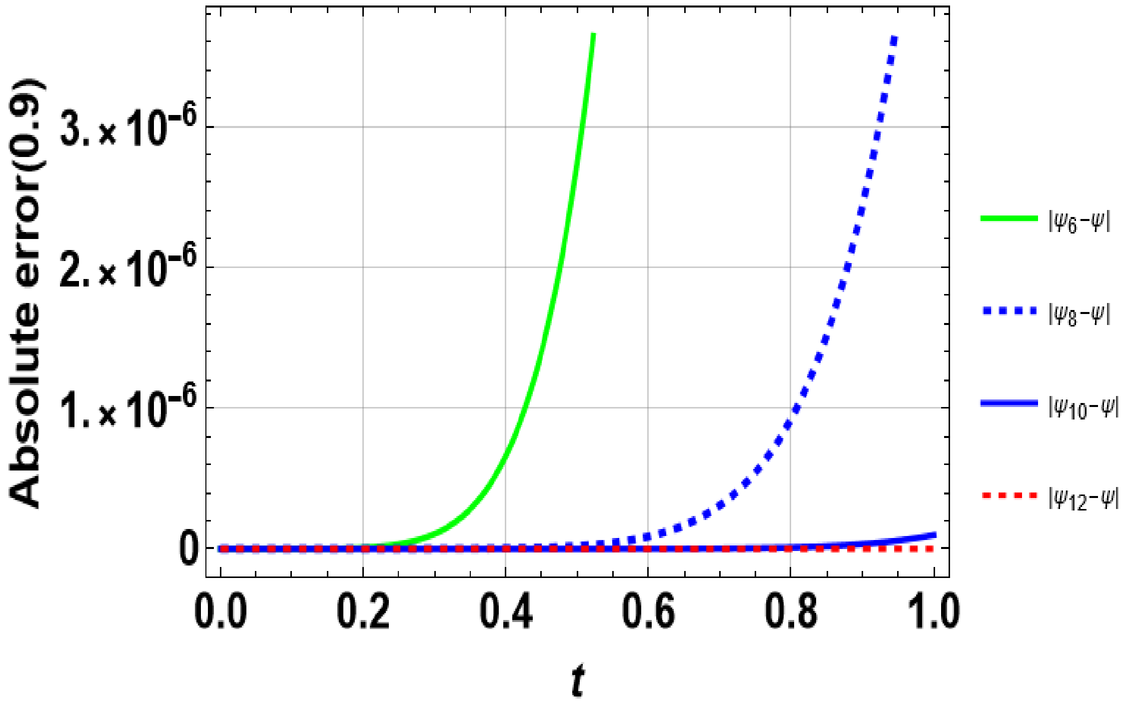


Fig 4.3 Errors of **diffusion equation** when α= 0.9.

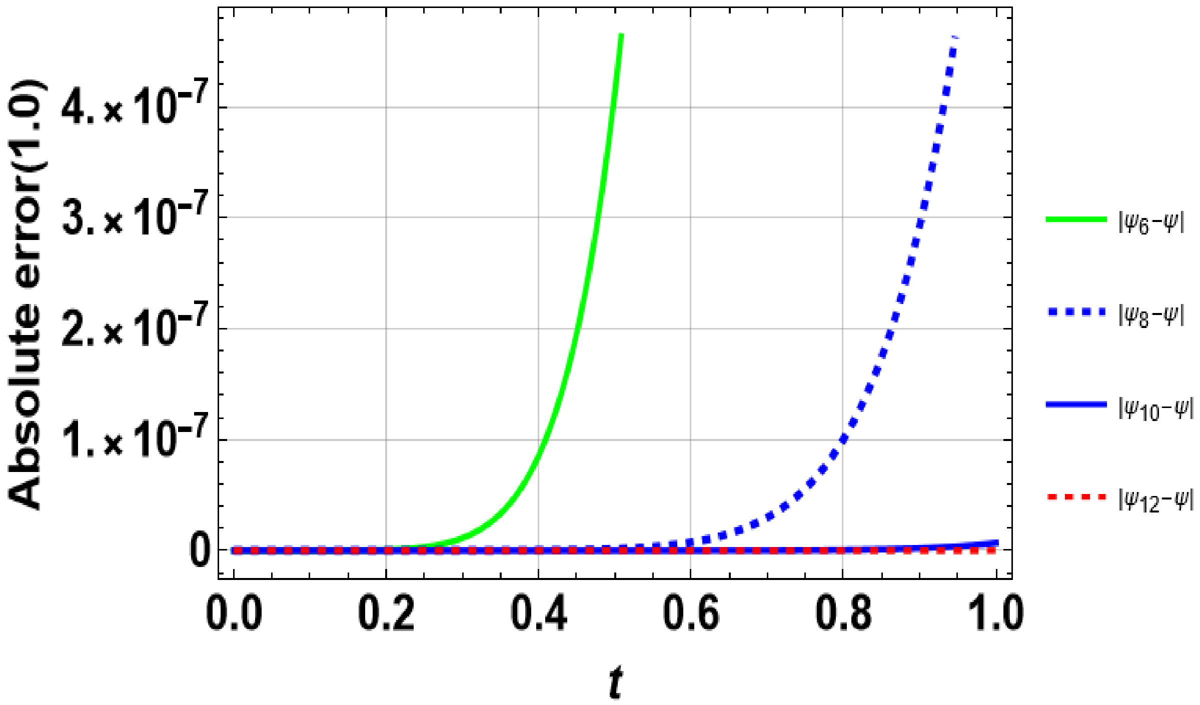
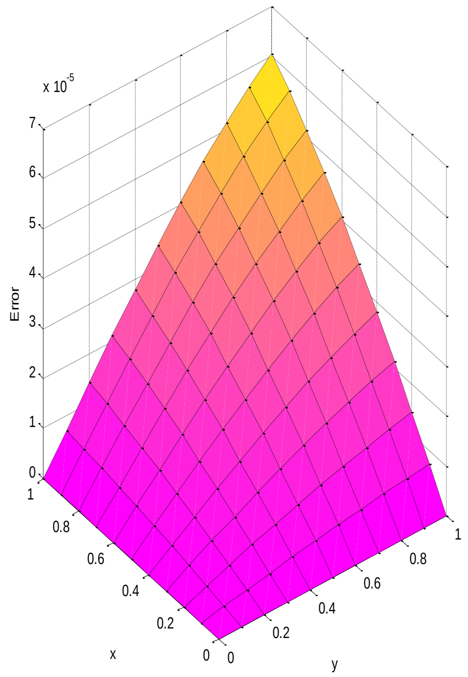


Fig 4.4 Errors of **diffusion equation** when α=1.0.



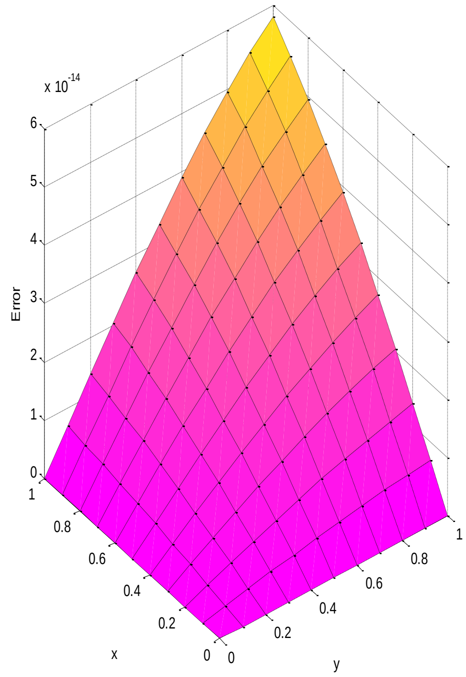


Fig 4.5 (a) **Fig 4.5 (b)**

Figure 4.5 (a) The absolute errors for **α = 0.5 at t = 0.5**

Figure 4.5 (b) absolute errors for **α = 1 for t = 1**.

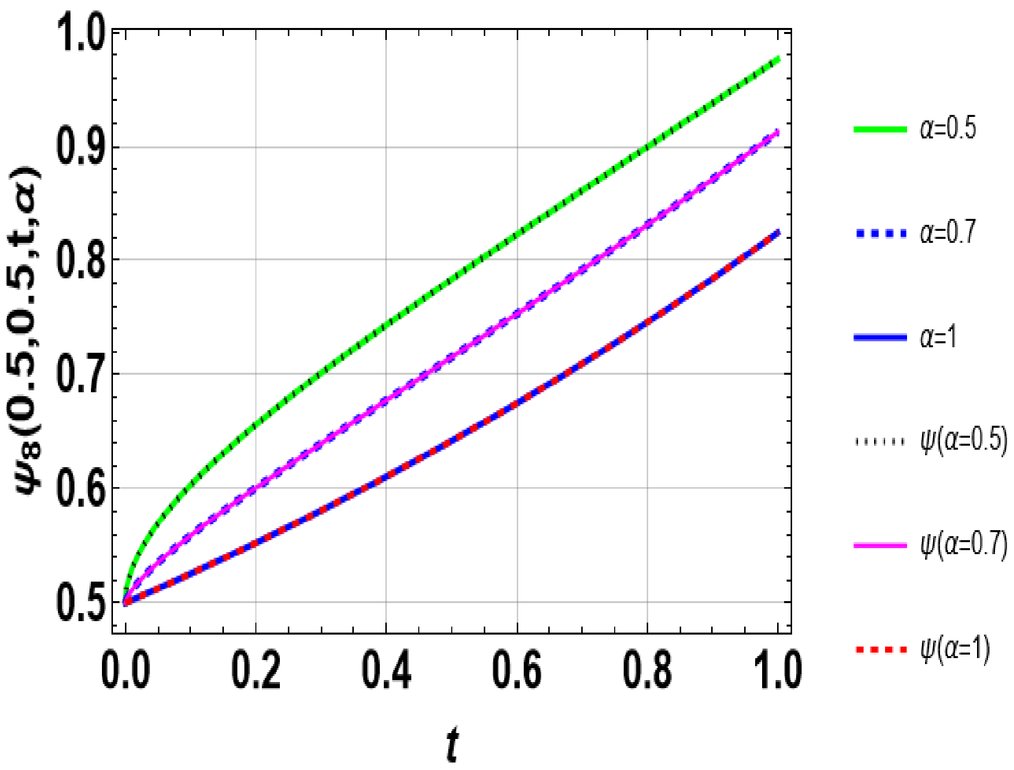


Fig 4.6 Comparison of solution behaviour of **fractional biological population equation** at different t if α=0.5, 0.7, 1.0.

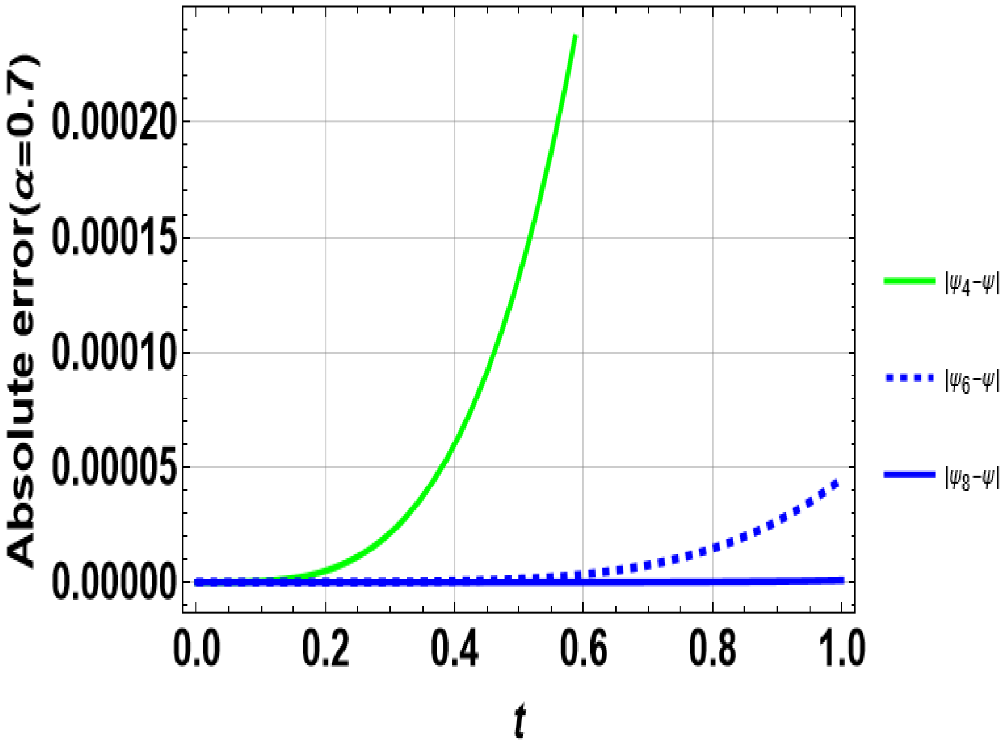


Fig 4.7 Absolute error of **fractional biological population equation** when α=0.7.

4.4 Conclusion

In order to use LRPSM to analytically approximate the solutions of these FDEs in two dimensions, a semi-analytical approach is used to two significant equations in this chapter. This method has the advantage of requiring less computation to obtain numerical solutions in power series form following the application of Laplace in a manner where the coefficients of those solutions are determined in the previously specified sequential algebraic phases. It is consequently possible to work out the two-dimensional FDEs precisely and efficiently using the LRPSM. It is shown that the method can solve two-dimensional FDEs with sufficient accuracy. Thus, this approach yields reliable results with fewer mistakes.

Chapter 5

Solutions of two-dimensional Fractional Differential Equations by Elzaki Transform with Residual Power Series Method

The most reliable and effective method of solving fractional differential equations (FDEs) is Elzaki transform with residual power series method (ERPSM), which is used to illustrate solution of two-dimensional temporal-fractional diffusion equation in current chapter.
FDEs are widely applicable and have wide-ranging consequences for a variety of problems in electrical network systems, diffusion-reaction processes, and signal processing systems, among other areas [120]. Researchers are interested in examining the impact of the fractional derivative included in these equations. Extensive variants of the classical differential equations, the FDEs [52] have widely functional in abundant scientific domains in last few decades. Although there are many analytical techniques for solving FDEs, scientists are constantly striving to create new techniques that can lead to a more precise solution of the fractional equations.

For determining the solutions to FDEs, there are numerous trustworthy, well-liked, and effective analytical techniques available. Various methods have employed to solve FDEs, including residual power series method [121], homotopy perturbation and analysis methods [119], differential transform method [122], iterative method [123], Adomian decomposition method [124], various forms of non-integer power series version [120], biological engineering image dispensation [125], physical model [126], risk analysis [127], Taylor's method [114], novel hybrid D(TQ) method [128], and numerous others.

The residual power series method (RPSM) is one of the reliable and effective ways to solve FDEs. It is complicated and impossible to find the series solutions and coefficients for non-linear FDEs [129]. Solutions can be achieved with a residual power series technique [99] to extract the coefficients in sequential form, using transformed functions as recurrence relations. By using RPSM, the nth ordered coefficients of the power series solutions are found by differentiating n^{th} partial sum of series (n-1) times. In order to solve non-integer non-linear problems, ordinary derivatives are typically upgraded to fractional derivatives [69].

Many transforms, including the Laplace, Sumudu, Elzaki, and many more transforms, can be used to solve the FDEs [111]. One transform that has been used to determine the numerical solution of well-known differential equations that serve as mathematical representations of scientific and technological phenomena is the Elzaki transform [130].

Researchers have used a variety of transforms along with well-known methodologies to solve differential equations, including non-integer order logistic differential models [131], non-integer order BBM-Burger equations [132], non-integer order ordinary differential equations [133], and more. The ERPSM has successfully used to solve some well-known FDEs. The ERPSM was developed to provide for the analytical solutions of FDEs that exists for a variety of applications in mathematics, engineering and physics. Using this method to solve two-dimensional equation is the main goal of the current effort, which aims to increase method's accuracy [65]. A modified form of the Sumudu and Laplace transforms is the Elzaki transform.

Over the past few decades, a number of fractional generalisations of diffusion equation have proposed. These have caused a great deal of discussion in the academic literature as well as in numerous diffusion model implementations. A PDE named diffusion equation uses fractional calculus ideas to explain the temporal evolution of a quantity such as heat, mass, or particles. Unlike the standard diffusion equation, which employs integer-order derivatives, diffusion equation includes fractional derivatives [121] in the temporal domain. Because it enables the model to effectively represent non-local and memory-dependent behaviours in diffusion processes, this trait makes it especially helpful for researching phenomena with complicated temporal dependencies or long-range interactions [134]. Nevertheless, numbers of analytical techniques have used to diffusion equations. An attempt is made to apply the ERPSM to solve the problems for various values of the fractional power in this chapter.

The diffusion equation of non-integer order in two dimensions is,

$D_t^\alpha u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t)$ with $0 < \alpha \leq 1$ (5.1)

with initial condition, $u(x,y,0) = sin\, x\, sin\, y$ (5.2)

and exact solution is, $u(x,y,t) = e^{-2t} sin\, x\, sin\, y$ for α =1

For $0 < \alpha \leq 1$ exact solution is, $u(x,y,t) = E_\alpha(z)\, sin\, x\, sin\, y$ (5.3)

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}, z = -4it^\alpha$

ERPSM is used to solve equation (5.1), which is a fractional order differential equation.

5.1 Methodology for Implementation

The following steps are included in the process for solving the diffusion equation [136] using ERPSM [137]:

Step 1 Applying Elzaki transform on equation (5.1) as,

$E(D_t^\alpha u(x,y,t)) = E(u_{xx}(x,y,t) + u_{yy}(x,y,t))$ (5.4)

Applying the differentiation property of Elzaki transform,

$E[D_t^\alpha u(x,y,t)] = \frac{1}{v^\alpha}\{E\big(u(x,y,t)\big) - g(x,y,t)\}$ on equation (5.4), we get

$\frac{1}{v^\alpha}\{E(u(x,y,t)) - g(x,y,t)\} = E\{(u(x,y,t))_{xx} + (u(x,y,t))_{yy}\}$

$i.e. E(u(x,y,t)) = g(x,y,t) + v^\alpha E\{(u(x,y,t))_{xx} + (u(x,y,t))_{yy}\}$ (5.5)

Step 2 Taking inverse Elzaki transform in equation (5.5), we get

$u(x,y,t) = G(x,y,t) + E^{-1}[v^\alpha E\{(u(x,y,t))_{xx} + (u(x,y,t))_{yy}\}]$ (5.6)

here G(x, y, t) is the primary circumstance of given problem.

Step 3 By this method algorithm of $u(x,y,t)$ is proposed as,

$u(x,y,t) = \sum_{n=0}^{\infty} f_n(x,y) \frac{t^{n\alpha}}{(n\alpha)!}$ (5.7)

To find the mathematical solutions of (5.7), $u_i(x,y,t)$ is written as,

$s_i = \sum_{n=0}^{i} u_n(x,y,t) = \sum_{n=0}^{i} f_n(x,y) \frac{t^{n\alpha}}{(n\alpha)!}$ (5.8)

Step 4 The Elzaki residual function from (5.6) can be written as,

$Res_i(x,y,t) = u_i(x,y,t) - G(x,y,t) - E^{-1}[v^\alpha E\{u_{i-1}(x,y,t))_{xx} + (u_{i-1}(x,y,t))_{yy}\}]$(5.9)

and hence the values of $f_n(x,y)$ may be obtained by putting $n = 0,1,2, \dots \dots$in the relation,

$t^{-n\alpha} Res_n(x,y,t)/_{t=0} = 0$ (5.10)

Substituting these values of $f_n(x,y)$ obtained from equation (5.10) in equation (5.7) the approximate solution of diffusion equation [135] is obtained analytically by using ERPSM.

5.2 Numerical Experiment

Using Elzaki transform on equation (5.1), then

$E(D_t^\alpha u(x,y,t)) = E(u_{xx}(x,y,t) + u_{yy}(x,y,t))$ (5.11)

Applying the differentiation property of Elzaki transform,

$E[D_t^\alpha u(x,y,t)] = \frac{1}{v^\alpha}\{E(u(x,y,t)) - g(x,y,t)\}$ on equation (5.11), we get

$\frac{1}{v^\alpha}\{E(u(x,y,t)) - g(x,y,t)\} = E\{(u(x,y,t))_{xx} + (u(x,y,t))_{yy}\}$

$i.e. E(u(x,y,t)) = g(x,y,t) + v^\alpha E\{(u(x,y,t))_{xx} + (u(x,y,t))_{yy}\}$ (5.12)

Taking inverse Elzaki transform in equation (5.12), we get

$u(x,y,t) = G(x,y,t) + E^{-1}[v^\alpha E\{(u(x,y,t))_{xx} + (u(x,y,t))_{yy}\}]$ (5.13)

Now $u_i(x,y,t)$ may be written as,

$$s_i = \sum_{n=0}^{i} u_n(x,y,t) = \sum_{n=0}^{i} f_n(x,y)\frac{t^{n\alpha}}{(n\alpha)!} \tag{5.14}$$

Then the values of $f_n(x,y)$ can be obtained by using

$Res_i(x,y,t) = u_i(x,y,t) - G(x,y,t) - E^{-1}\left[v^{\alpha}E\{u_{i-1}(x,y,t))_{xx} + (u_{i-1}(x,y,t))_{yy}\}\right]$(5.15)

When $i = 0$ from equation (5.15),

$Res_0(x,y,t) = u_0(x,y,t) - G(x,y,t)$ and from equation (5.14),

$0 = u_0(x,y,t) - G(x,y,t)$ i.e. $u_0(x,y,t) = G(x,y,t)$

$u_0(x,y,t) = f_0(x,y) = \sin x\, sin\, y$
(5.16)

When $i = 1$ from equation (5.15),

$Res_1(x,y,t) = u_1(x,y,t) - G(x,y,t) - E^{-1}[v^{\alpha}E\{(u_0(x,y,t))_{xx} + (u_0(x,y,t))_{yy}\}]$ with the conditions $u_1(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^{\alpha}}{\alpha!}$ then we can obtain,

$$Res_1(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^{\alpha}}{\alpha!} - G(x,y,t) - E^{-1}[v^{\alpha}E\{f_0(x,y))_{xx} + f_0(x,y))_{yy}\}]$$

$$= sin\ x\ sin\ y + f_1(x,y)\frac{t^{\alpha}}{\alpha!} - sin\ x\ sin\ y - E^{-1}[v^{\alpha}E\{(sin\ x\ sin\ y)_{xx} + (sin\ x\ sin\ y)_{yy}\}]$$

$$= f_1(x,y)\frac{t^{\alpha}}{\alpha!} - E^{-1}[v^{\alpha}E(-sin\ x\ sin\ y - sin\ x\ sin\ y)]$$

$$= f_1(x,y)\frac{t^{\alpha}}{\alpha!} - E^{-1}[v^{\alpha}E(-2sin\ x\ sin\ y)]$$

$$= f_1(x,y)\frac{t^{\alpha}}{\alpha!} + 2sin\ x\ sin\ y\ E^{-1}[v^{\alpha}E(1)]$$

$$= f_1(x,y)\frac{t^{\alpha}}{\alpha!} + 2sin\ x\ sin\ yE^{-1}[v^{\alpha+2}]$$

$$= f_1(x,y)\frac{t^{\alpha}}{\alpha!} + 2sin\ x\ sin\ y\frac{t^{\alpha}}{\alpha!}$$

$$= \{f_1(x,y) + 2sin\ x\ sin\ y\}\frac{t^{\alpha}}{\alpha!}$$

Then after solving $t^{-\alpha}Res_1(x,y,t)_{t=0} = 0$ gives that

$$f_1(x,y) + 2sin\ x\ sin\ y = 0\ i.e. f_1(x,y) = -2sin\ x\ sin\ y \tag{5.17}$$

When $i = 2$ from equation (5.15)

$Res_2(x,y,t) = u_2(x,y,t) - G(x,y,t) - E^{-1}[v^\alpha E\{(u_1(x,y,t))_{xx} + (u_1(x,y,t))_{yy}\}]$, with conditions $u_1(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^\alpha}{\alpha!}$ and

$u_2(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!}$, we get

$$Res_2(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} - f_0(x,y) - E^{-1}\left[v^\alpha E\left\{(f_0(x,y) + f_1(x,y)\frac{t^\alpha}{\alpha!})_{xx} + (f_0(x,y) + f_1(x,y)\frac{t^\alpha}{\alpha!})_{yy}\right\}\right]$$

$$= f_1(x,y)\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}\left[v^\alpha E\left\{(\sin x \sin y - 2\sin x \sin y\frac{t^\alpha}{\alpha!})_{xx} + (\sin x \sin y - 2\sin x \sin y\frac{t^\alpha}{\alpha!})_{yy}\right\}\right]$$

$$= -2\sin x \sin y\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}\left[v^\alpha E\left\{-\sin x \sin y + 2\sin x \sin y\frac{t^\alpha}{\alpha!} + -\sin x \sin y + 2\sin x \sin y\frac{t^\alpha}{\alpha!}\right\}\right]$$

$$= -2\sin x \sin y\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}\left[v^\alpha E\left\{-2\sin x \sin y + 4\sin x \sin y\frac{t^\alpha}{\alpha!}\right\}\right]$$

$$= -2\sin x \sin y\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}\left[v^\alpha E\left\{-2\sin x \sin y + 4\sin x \sin y\frac{t^\alpha}{\alpha!}\right\}\right]$$

$$= -2\sin x \sin y\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}\left[v^\alpha\{-2\sin x \sin y\, E(1) + 4\sin x \sin y\, E(\frac{t^\alpha}{\alpha!})\right]$$

$$= -2\sin x \sin y\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}\left[v^\alpha\{-2\sin x \sin y\, v^2 + 4\sin x \sin y\frac{\alpha! v^{\alpha+2}}{\alpha!}\right]$$

$$= -2\sin x \sin y\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} + 2\sin x \sin y\, E^{-1}(v^{\alpha+2}) - 4\sin x \sin y\, E^{-1}[v^{2\alpha+2}]$$

$$= -2\sin x \sin y\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} + 2\sin x \sin y\frac{t^\alpha}{\alpha!} - 4\sin x \sin y\ \frac{t^{2\alpha}}{(2\alpha)!}$$

$$= \{f_2(x,y) - 4\sin x \sin y\}\ \frac{t^{2\alpha}}{(2\alpha)!}$$

Therefore, from $t^{-2\alpha}Res_2(x,y,t)/_{t=0} = 0$ we have,

$$f_2(x,y) - 4\sin x \sin y = 0 \ \ i.e. f_2(x,y) = 4\sin x \sin y \qquad (5.18)$$

Now, the second approximate solution is,

$$u_2(x,y,t) = \sin x \sin y - 2\sin x \sin y\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\frac{t^{2\alpha}}{(2\alpha)!}$$

When $i = 3$ from equation (5.15)

$Res_3(x,y,t) = u_3(x,y,t) - G(x,y,t) - E^{-1}[v^\alpha E\{(u_2(x,y,t))_{xx} + (u_2(x,y,t))_{yy}\}]$, with conditions $u_2(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!}$ and

$u_3(x,y,t) = f_0(x,y) + f_1(x,y)\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!}$ we get,

$$Res_3(x,y,t) = f_0(x,y) + f_1(x,y)\,\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} - f_0(x,y) - E^{-1}\left[v^\alpha E\left\{(f_0(x,y) + f_1(x,y)\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!})_{xx} + (f_0(x,y) + f_1(x,y)\frac{t^\alpha}{\alpha!} + f_2(x,y)\frac{t^{2\alpha}}{(2\alpha)!})_{yy}\right\}\right]$$

$$= -2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} - E^{-1}\left[v^\alpha E\left\{(\sin x \sin y - 2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!})_{xx} + (\sin x \sin y - 2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!})_{yy}\right\}\right]$$

$$= -2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} - E^{-1}\left[v^\alpha E\left\{-\sin x \sin y + 2\sin x \sin y\,\frac{t^\alpha}{\alpha!} - 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!} - \sin x \sin y + 2\sin x \sin y\,\frac{t^\alpha}{\alpha!} - 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!}\right\}\right]$$

$$= -2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} - E^{-1}\left[v^\alpha E\left\{-2\sin x \sin y + 4\sin x \sin y\,\frac{t^\alpha}{\alpha!} - 8\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!}\right\}\right]$$

$$= -2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} - E^{-1}\left[v^\alpha\left\{-2\sin x \sin y\,E(1) + 4\sin x \sin y\,\frac{E(t^\alpha)}{\alpha!} - 8\sin x \sin y\,\frac{E(t^{2\alpha})}{(2\alpha)!}\right\}\right]$$

$$= -2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} - E^{-1}\left[v^\alpha\left\{-2\sin x \sin y\,v^2 + 4\sin x \sin y\,\frac{\alpha!v^{\alpha+2}}{\alpha!} - 8\sin x \sin y\,\frac{(2\alpha)!v^{2\alpha+2}}{(2\alpha)!}\right\}\right]$$

$$= -2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} - E^{-1}[-2\sin x \sin y\,v^{\alpha+2} + 4\sin x \sin y v^{2\alpha+2} - 8\sin x \sin y\,v^{3\alpha+2}]$$

$$= -2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} + 2\sin x \sin y\,E^{-1}(v^{\alpha+2}) - 4\sin x \sin y\,E^{-1}(v^{2\alpha+2}) + 8\sin x \sin y\,E^{-1}(v^{3\alpha+2})$$

$$= -2\sin x \sin y\,\frac{t^\alpha}{\alpha!} + 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} + 2\sin x \sin y\,\frac{t^\alpha}{\alpha!} - 4\sin x \sin y\,\frac{t^{2\alpha}}{(2\alpha)!} + 8\sin x \sin y\,\frac{t^{3\alpha}}{(3\alpha)!}$$

$= f_3(x,y)\frac{t^{3\alpha}}{(3\alpha)!} + 8 sin\, x\, sin\, y\ \frac{t^{3\alpha}}{(3\alpha)!}$

$= \{f_3(x,y) + 8 sin\, x\, sin\, y\}\frac{t^{3\alpha}}{(3\alpha)!}$

Therefore, from $t^{-3\alpha} Res_3(x,y,t)/_{t=0} = 0$ we have,

$$f_3(x,y) + 8 sin\, x\, sin\, y = 0\ \ i.e.\ f_3(x,y) = -8 sin\, x\, sin\, y \quad (5.19)$$

Now, the third approximate solution is,

$$u_3(x,y,t) = sin\, x\, sin\, y - 2\, sin\, x\, sin\, y\frac{t^{\alpha}}{\alpha!} + 4 sin\, x\, sin\, y\frac{t^{2\alpha}}{(2\alpha)!} - 8 sin\, x\, sin\, y\frac{t^{3\alpha}}{(3\alpha)!}$$

Similarly, the n^{th} coefficient of $u(x,y,t)$ is $f_n(x,y) = (-2)^n sin\, x\, sin\, y$

At last the n^{th} ERPSM approximate solutions of $u(x,y,t)$ is

$$u_n(x,y,t) = sin\, x\, sin\, y \sum_{n=0}^{i} \frac{(-2t^{\alpha})^n}{(n\alpha)!} \quad (5.20)$$

5.3 Numerical Simulations and graphs

The numerical solution of this equation has been obtained for the **domain [0, 1] for both x and y a**nd the results are presented at **t=0.5 for α=0.5, 0.8 and 1.0** in **Table 5.1-5.3 and Figure 5.1-5.3.**

Table 5.1 Solution when value of **α** =0.5

y/x	0.2	0.4	0.6	0.8	1.0
0.1	0.006805	0.01334	0.019342	0.024573	0.028825
0.2	0.013543	0.026546	0.038491	0.048901	0.057361
0.3	0.020145	0.039487	0.057255	0.07274	0.085325
0.4	0.026546	0.052033	0.075447	0.095852	0.112436
0.5	0.032681	0.06406	0.092885	0.118006	0.138423
0.6	0.038491	0.075447	0.109395	0.138982	0.163028
0.7	0.043915	0.086079	0.124812	0.158569	0.186004
0.8	0.048901	0.095852	0.138982	0.176571	0.207121
0.9	0.053398	0.104667	0.151763	0.192809	0.226168
1.0	0.057361	0.112436	0.163028	0.207121	0.242956

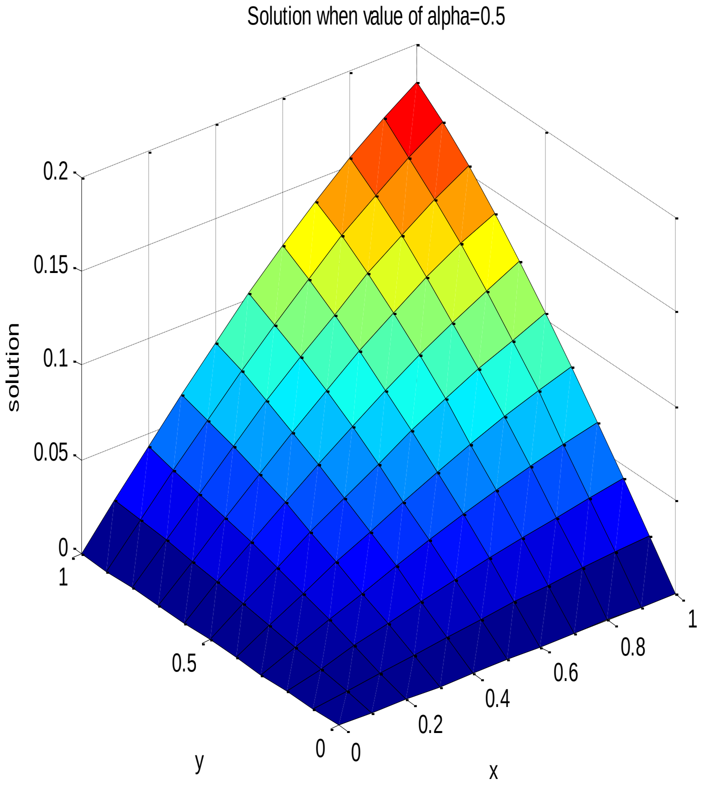


Figure 5.1 Graph when value of **α** =0.5

Table 5.2 Solution when value of **α** =0.8

y/x	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
0.1	0.003764	0.007379	0.010699	0.013593	0.015944
0.2	0.007491	0.014684	0.021291	0.027049	0.031729
0.3	0.011143	0.021842	0.03167	0.040236	0.047197
0.4	0.014684	0.028782	0.041733	0.053020	0.062193
0.5	0.018078	0.035434	0.051379	0.065275	0.076568
0.6	0.021291	0.041733	0.060511	0.076877	0.090178
0.7	0.024291	0.047614	0.069039	0.087711	0.102887
0.8	0.027049	0.053020	0.076877	0.097669	0.114568
0.9	0.029537	0.057896	0.083947	0.106651	0.125104
1.0	0.031729	0.062193	0.090178	0.114568	0.134390

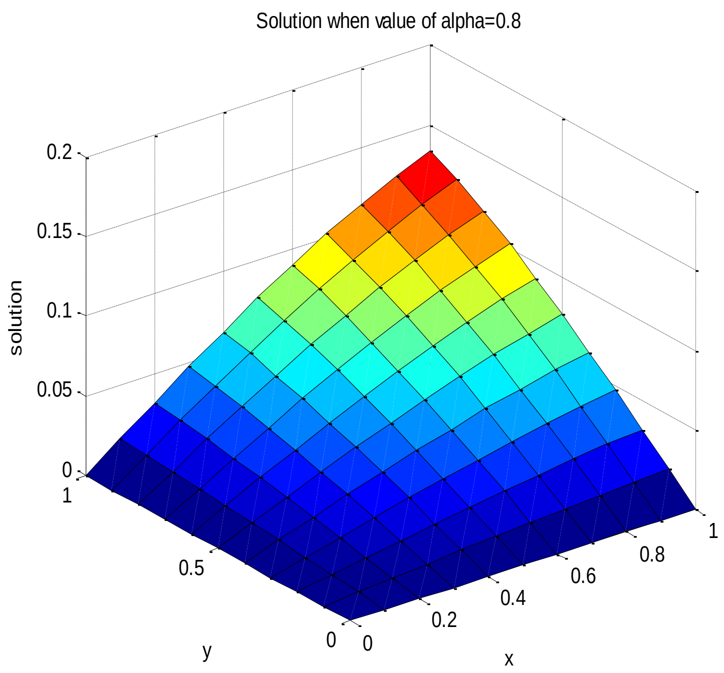


Figure 5.2 Graph when value of **α** =0.8

Table 5.3 Solution when value of **α** =1.0

y/x	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
0.1	0.002684	0.005261	0.007629	0.009692	0.011369
0.2	0.005342	0.01047	0.015182	0.019288	0.022625
0.3	0.007946	0.015575	0.022582	0.02869	0.033654
0.4	0.01047	0.020523	0.029758	0.037806	0.044347
0.5	0.01289	0.025267	0.036636	0.046544	0.054597
0.6	0.015182	0.029758	0.043148	0.054818	0.064302
0.7	0.017321	0.033952	0.049229	0.062543	0.073364
0.8	0.019288	0.037806	0.054818	0.069644	0.081693
0.9	0.021061	0.041283	0.059859	0.076048	0.089206
1.0	0.022625	0.044347	0.064302	0.081693	0.095827

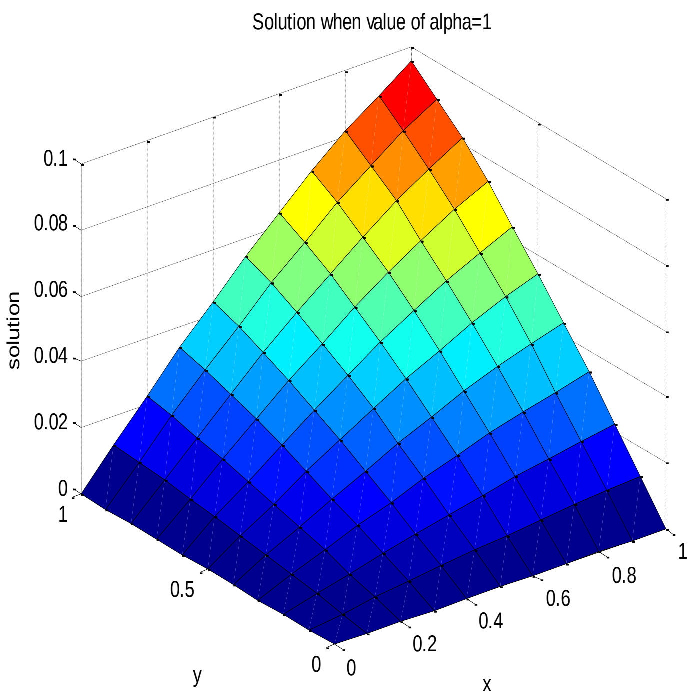


Figure 5.3 Graph when value of **α** =1.0

Table 5.4 For **α** =1, maximum errors (l_∞) with different time levels

t	l_∞
0.1	4.4408e-16
0.2	7.1981e-13
0.3	6.1280e-11
0.4	1.4280e-09
0.5	1.6366e-08
0.6	1.1974e-07
0.7	6.4273e-07
0.8	2.7504e-06
0.9	9.8999e-06
1.0	3.1088e-05

Table 5.5 Comparison of maximum errors of diffusion equation by ERPSM and RPSM for α =1.

t	**ERPSM with different number of terms**			**RPSM** [137]
	10	12	14	10
0.2	7.1981e-13	6.1062e-16	0	0.5e-07
0.4	1.4280e-09	5.9121e-12	1.8152e-14	1.0e-07
0.6	1.1974e-07	1.1200e-09	7.7582e-12	1.5.e-07
0.8	2.7504e-06	4.5924e-08	5.6724e-10	2.5e-06
1.0	3.1088e-05	8.1419e-07	1.5759e-08	3.0e-05

5.4 Conclusion

In this chapter, the current research presents the implementation of a novel and trustworthy method ERPSM to solve the non-integer order diffusion equation. This strategy combines the RPSM, an improvement of the conventional RPSM, with the Elzaki transform. This method has a benefit of requires less calculation and give less error in the solution. The aforementioned sequential procedures lead to the determination of the coefficients of this power series solution. ERPSM demonstrated its capacity to solve non-integer order differential equations with sufficient correctness and dependable computing steps for two-dimensional non-integer order diffusion equations. From the table of comparison in numerical example it can be concluded that this novel approach is advantageous. This approach also offers straightforward and precise algorithms for estimating solutions of diffusion equation.

Chapter 6

Summary and Future work

The work completed in this thesis is about the application of newly established methods having applications to solve one-dimensional as well as two-dimensional FDEs. There are plentiful approaches to define non-integer derivatives; Caputo's explanation has been used because it is the most acceptable one due to its characteristics. RPSM has applied to solve the relaxation-oscillation differential equations and same method is also used to solve FDEs in one and two dimensions by the use of Laplace transforms as well as Elzaki transforms separately. The introduction of prerequisites is covered in the first chapter. The relaxation-oscillation equation has solved by RPSM in the second chapter. The numerical solutions that were produced are extremely close to the exact answer that is currently accessible, meaning that the errors in both solutions are negligible. Similar to this, chapter three's numerical solutions of one-dimensional FDEs using LRPSM produce significant and trustworthy results.

Reliable and productive results are also obtained by using LRPSM to solve diffusion and fractional biological population equations in two-dimension of temporal fractional order in chapter four. Similarly, chapter five of this thesis explains solution of diffusion equation of temporal fractional order using ERPSM as well as its comparison with solution of same equation by LRPSM.

6.1 Future Scope

The future scope of this work includes:

• This work can be tested for the solution of FDEs by using different solution methods.

• The applications and experiments with the new definitions of the fractional derivatives can be done.

• The RPSM with different transforms can be tested for various one and two dimensional FDEs.

• The higher order differential equations can be tried for a solution using the RPSM with different transforms.

• For the space FDEs, the RPSM with different transforms can be improvised and implemented.

• This work had the experiment based results for RPSM with different transforms. Furthermore the theory and stability can be observed into details in future if possible.

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