Mathematical Modelling of Plant Herbivore Dynamics under Allee Effect using Delay Differential Equations

Thesis Submitted for the Award of the Degree of

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in
Mathematics

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2024

DECLARATION

I, hereby declared that the presented work in the thesis entitled "**Mathematical Modelling of Plant Herbivore Dynamics under Allee Effect using Delay Differential Equations**" in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of **Dr. Pankaj Kumar (11893)**, working as Associate Professor, in the Department of Mathematics at School of Chemical Engineering and Physical Sciences of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

Rupali Verma

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CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled "**Mathematical Modelling of Plant Herbivore Dynamics under Allee Effect using Delay Differential Equations**" submitted in fulfillment of the requirement for the reward of degree of **Doctor of Philosophy (Ph.D.)** in the Department of Mathematics, is a research work carried out by **Rupali Verma, 12014919,** is bonafide record of her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

Pankaj Kumar | 11893

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DEDICATED

TO

LORD SHIVA

AND

MY PAPA

ABSTRACT

The proposed work examines plant-herbivore dynamics considering the delay in plant population, infectious plant population, herbivore population, and logistic growth. The availability of nutrients, favourable resources, and the Allee effect all play a significant role in plant growth. Plant growth can be impacted by the Allee effect. The delay parameter, which is used as the basis for this entire study, is this impedance in plant growth. In the proposed work, models including strong or weak Allee effects are developed, considering various kinds of functional responses incorporating the delay in growth dynamics under the Allee effect. The proposed models have been mathematically analyzed, and the results have been numerically verified.

The comparison theorem is a mathematical tool used to prove the positivity and boundedness of analytical solutions. The possible and feasible interior equilibrium is calculated. The local stability analysis of interior equilibrium is established. When the delay parameter is considered during the stability analysis of the interior equilibrium, Hopf bifurcation occurs, which illustrates the complex dynamical behaviour. Using Rouche's theorem and Routh Hurwitz's criteria, the nature of the roots has been thoroughly investigated. The "Direct Method" is used to do a sensitivity analysis of state variables with respect to model parameters for almost all models. MATLAB is used for numerical simulation, and distinct numerical values have been assigned to each model parameter. This helped in identifying the delay parameter's critical value, below this critical value, the system exhibited stability, and above this, the system lost stability and Hopf-bifurcation occurred.

In Chapter-1, The general overview of plant growth dynamics under the Allee effect is given. Moreover, a broad overview of plant-herbivore dynamics is described. The remarkable work done by the researchers is cited and the gaps have been identified through an extensive literature review. All the important concepts of plant physiology including the Allee effect and the necessary mathematical concepts required for their study have also

been described. It also includes the proposed objectives of the study and a summary of all the chapters.

In Chapter-2, A mathematical framework is constructed to examine the dynamics of the interaction between plants and herbivores, considering the Allee effect by using delay differential equations. An analysis is conducted to check the influence of the delay parameter on plants and herbivores. It is proved that the solutions are bounded and positive. The feasible non-zero equilibrium is determined for both the Allee effects. It is indicated that plant and herbivore population goes to extinction due to the strong Allee effect. The stability of the system around the non-zero equilibrium point, considering the weak Allee effect is examined. When the delay is negligible, it can be observed that the equilibrium point is absolutely stable. The system destabilizes if the delay parameter is below the critical value $\tau = 3.25$, indicating that the equilibrium point is asymptotically stable. The equilibrium point shows complex behaviour and Hopf- bifurcation occurs when the delay exceeds the critical value. Numerical simulation is utilized to support analytical findings using MATLAB code.

In Chapter-3, A model of plant-herbivore interactions, considering the strong Allee effect with intraspecific competition among herbivores is proposed. The Holling type- II functional response is utilized which defines that an individual predator consumes food till it feels famished. The loss in stability is shown by utilizing the time delay (τ) using delay differential equations. The non-trivial equilibrium is determined for the strong Allee effect. The value of time delay $\tau = 2.7$ is the crucial value that shifts the behaviour of the system from stability to complex one, leading to Hopf-bifurcation. Moreover, it is identified that the entire system becomes more and more stable while the intraspecific competition rate of herbivores (δ) keeps on increasing. The same fact is represented in time series graphs with the help of MATLAB software.

In Chapter-4, A delayed modified Leslie-Gower plant-herbivore model is analyzed under the Allee effect. Holling type-II functional response is used to modify the model. The non-trivial equilibrium of the proposed model is calculated. Moreover, it is shown that the system represents absolute stability when the system has no delay $(\tau = 0)$. The insertion of the delay parameter disturbs the stability of the non-trivial equilibrium and the system becomes asymptotically stable when the delay parameter is below the threshold value which is $\tau = 3.2$. Hopf bifurcation is seen at the threshold value of the delay parameter. The time series graphs are also represented to demonstrate that the system becomes more stable with the maximum rate of predation. MATLAB software is incorporated to perform the graphs to justify the theoretical results.

In Chapter-5, An eco-epidemiological model which is composed of susceptible plants, infectious plants, and herbivores, under the weak Allee effect accompanied by Holling type- I functional response is studied. The supposition is that infectious plants are generated by the infection of susceptible plants and the herbivores consume only infectious plants. The delay differential equations are considered to make the model more realistic. Furthermore, stability at the non-trivial equilibrium is determined. It is observed that the system shows asymptotic stability when time lag is lower than the threshold point i.e., $\tau = 10.6$ and the system undergoes Hopf-bifurcation when time lag surpasses the threshold point. Sensitivity analysis of the state variables with respect to the herbivores gain by consuming infectious plants and the mortality rate of herbivores is represented with time series graphs. Numerical examples are shown with the help of MATLAB code.

In Chapter-6, A delayed Gauss-type Plant-herbivore model is modelled with Holling type-III functional response which states as a matter of fact that the predation on plant population by herbivores becomes more intensive when there is an increment in plant population. The growth of plant population is influenced by the Allee effect. By assuming that the delay parameter (τ) is a bifurcation parameter, the stability analysis of the feasible non-trivial equilibrium is investigated. The change of behaviour of the system from being absolutely stable to asymptotically stable and then finally Hopf-bifurcation at equilibrium for distinct values of time parameter (τ) is represented graphically where the threshold value of the time parameter is $\tau = 3.2$. A graphic representation of the change in system behaviour

triggered by variations in the model parameters at variance with the time parameter is shown. MATLAB software is used to perform numerical simulations.

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LIST OF PUBLISHED AND COMMUNICATED PAPERS FROM THE THESIS

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4. Pankaj Kumar and Rupali Verma, “Analyzing the Dynamics of Plant-Herbivore Interactions under Allee Effect using Delay Differential Equation”, *Journal of Applied Mathematics and Computing*, Submission ID: JAMC-D-23-01797. **(Communicated).**
5. Pankaj Kumar and Rupali Verma, “Mathematical Analysis of Delayed Gauss-Type Plant-Herbivore Model with Holling Type-III Functional Response”, *Mathematics and Computers in Simulation*. **(Communicated).**

LIST OF CONFERENCES

1. “3rd International Conference on Recent Advances in Fundamental and Applied Sciences (RAFAS 2021)” Organized by School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology & Sciences, Lovely Professional University, Phagwara, Punjab, India on 25-26 June, 2021.
2. “International Conference on Fractional Calculus: Theory, Applications and Numerics (ICFCTAN 2023)” Organized by Department of Mathematics, National Institute of Technology, Puducherry, Karaikal, India on January 27-29, 2023.
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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

General Introduction

1.1 Introduction

In the kingdom Plantae, plants are multicellular organisms that use photosynthesis to produce their sustenance. There are over 300,000 plant species. Grasses, trees, and shrubs are some examples of plants. Most of the oxygen in the environment is produced by plants, and they are crucial to the food chain since many organisms consume plants or organisms that consume plants. As a result, they are crucial to the ecosystems of the world. The scientific study of plants is called botany.

In the Animal Kingdom, multicellular eukaryotes are animals. All animals are heterotrophs. They cannot produce their food as plants; consequently, they eat other living things. Additionally, all animals have specialized cells that can do distinct kinds of work. Higher levels of organization can also be found in most animals. They might possess specialized organ systems, tissues, and organs. Animals are able to accomplish an array of difficult tasks because of their higher levels of organization. The kingdom of Animalia, or the Metazoa, includes a significant group of species that are known as animals. They often are multicellular, capable of locomotion and responsive to their surroundings, and obtain their food by eating other living things. Animals distinguish themselves from other living things by an assortment of characteristics. Animals differ from bacteria and the majority of protists in that they are eukaryotic and generally multicellular. They are heterotrophic, which sets them apart from plants and algae since they usually digest food in an interior chamber. Their absence of cell walls makes them distinct from plants, algae, and fungi. Several distinct animal species, each of which eats a specific type of food, are included in the food chain.

Herbivores, omnivores, and carnivores are the three main categories of animals. Animals are classified as herbivores that only consume plants. Animals that only eat meat are called carnivores. Animals that consume both plants and meat are termed omnivores. An animal's diet is not influenced by its size. Even very small animals can be carnivores, and some of the biggest animals solely consume plants. The kind of food an animal consumes will determine how its digestive system is designed. Moreover, depending on the kind of food an animal consumes, its teeth will also be uniquely designed.

Herbivores are animals that only consume plants. Frequently, herbivores only consume the fruits or seeds of plants and end up leaving the stems, leaves, and roots alone. Herbivores have unique digestive systems that are made to handle the various kinds of plants they may consume. Large front teeth, or incisors, are typically present in herbivores. Plants can be grasped and chopped using these teeth. Moreover, herbivores have molars in the back that they use to crush the plants in their mouths. Herbivores, including cows, deer, and elk, are frequently quite large animals. A large herbivore needs a lot of sustenance to provide it with the energy it requires. Consequently, herbivores may spend a significant part of their day eating. Medium-sized animals like sheep or goats can also be herbivores. Squirrels and chipmunks are a few examples of small herbivores. Herbivores require a lot of food to satisfy their appetites. All herbivores consume plants, but some specialize in one kind of plant or a particular component of plants.

Herbivores and plants have a cyclic relationship, according to predator-prey interaction theory. When prey (plants) are abundant, their predators (herbivores) increase in number, reducing the prey population and causing the predator population to decrease. The prey population subsequently recovers, and a new cycle begins. This implies that the herbivore population fluctuates in relation to the carrying capacity of the plant. Several factors influence these fluctuating populations and aid in the stabilization of predator-prey dynamics. This stabilizing dynamic is incredibly significant for specialist herbivores that feed on a single plant species, as it prevents these specialists from eradicating their food source. Prey defenses also aid in the stabilization of predator-prey dynamics. Eating a second prey type aids in the stabilization of herbivore populations. The herbivore's population is stable when it alternates between two or more plant types, whereas plant populations fluctuate. This is crucial for generalist herbivores that eat a wide variety of plants.

The Allee effect is known as a biological phenomenon where the relationship between the density of a population and its average individual fitness is established, often measured as the rate of population growth per individual. The Allee effect can be divided into two distinct categories: Strong Allee effect and Weak Allee effect.

A demographic Allee effect with a critical population density is named the strong Allee effect. The demographic Allee effect is known as the weak Allee effect due to the non-existence of the critical population density. A population that is characterized by a weak Allee effect, declines the per capita growth rate as the population size decreases, which is directly associated with population fitness. There will always be a positive per capita growth rate within the population, regardless of the small population size. A population exhibiting a strong Allee effect will experience a critical size, below which the population growth rate becomes negative. That population will therefore be destined for extinction when its size declines below this threshold. It is easier to identify the density when the per-individual growth rate becomes negative using time series data, resulting in an easier empirical determination of the strong Allee effect.

Numerous mechanisms that affect reproduction and survival can result in a positive relationship between fitness and population growth. As a well-known example, mate limitation, which requires interaction between male and female gametes for sexual reproduction, may lead to under-crowding in species that reproduce sexually. Mate limitation decreases reproduction when males and females are unable to locate one another or when plants or animals release gametes into the environment. In larger social groupings, cooperative actions like breeding, feeding, and defense are more efficient, which increases reproductive survivability. Although social vertebrates like birds, ungulates, and prairie dogs are among those where cooperative behaviour is most obvious. Moreover, aquatic creatures like cichlid fish and insects like bark beetles may experience adverse impacts from group feeding or defense.

However, many mechanisms only depend on the presence of conspecific individuals and do not require behavioural cooperation. For instance, large prey populations have a lower per capita predation risk in contrast to small prey populations. It is also well known that having a large population might improve biotic or environmental conditions. Examples of this type include eliminating competitors through allelopathy. Finally, Allee effects may result from genetic and demographic mechanisms. Active dispersal away from lower-density populations may have a negative impact on the growth rates of animal populations. Inbreeding depression can cause an Allee effect on many organisms when the population size is small by decreasing average fitness as population size decreases. Hence, small populations generally have decreased average individual fitness.

Modelling often serves purposes like knowledge integration or quantitative testing of hypotheses in research environments. The system of interest needs to be specified for modelling. The system of interest in horticulture and agricultural sciences is generally a plant, and very often a collection of interacting plants, such as a row of plants or a homogeneous crop canopy.

Many researchers have incorporated time delays of distinct kinds into biological models. In the real world, plant population growth rates frequently have a time lag or delay in responding to fluctuations in their population rather than reacting instantaneously. A type of differential equation known as a delay differential equation describes the derivative of an unknown function at a specific time by incorporating the function's values at previous times. The terms time-delay systems, aftereffect or dead-time systems, equations with diverging arguments, or differential-difference equations are also used for referring to DDEs.

The factors that introduce time delay may include plant population, herbivore population, and logistic growth. In comparison to the dynamics described by ordinary differential equations, the dynamics exhibited by delay differential equations are generally much more intricate. A stable equilibrium is compelled by the time delay to lose stability, become unstable, and cause fluctuations. The influence of delay on plant population experiencing the Allee effect is a relatively recent area of research.

Mathematical modelling for a wide range of biological problems is considered an interesting topic in the field of applied mathematics. An accurate representation of the environmental and ecological data is provided by mathematical models. It is impossible to incorporate all uncountable variables and factors into a mathematical model of a natural phenomenon; only relevant data is considered in order to have a logical concept of the dynamics. The proposed research work will be carried out by using mathematical models to examine the impact of delay on the dynamics of interactions between plants and herbivores in the presence of the Allee effect.

1.2 Review of literature

Studies on plant growth have their origins in antiquity. Nomadic early humans noticed that crops and plants planted during specific seasons produced more fruit and provided more food than those planted during other seasons. Leonardo da Vinci was the first person in the middle ages to systematically observe the seasonal periodicity of growth and some characteristics of plant forms. In the 17th century, theories are conducted on how leaves should be arranged on an axis or stem started to develop. The key environmental elements that influence plant growth are temperature, humidity, radiation input, respiration, transpiration, photosynthesis, carbon dioxide, etc. The first to propose mathematical models incorporating each of these variables separately and combined was Thornley [1]. In his novel theoretical model, Watkinson [2] included the following assumptions: that there was variability between the individual plants in a population; that competition, which is particularly for the smaller plants in a large population, reduced the growth rates, possibly leading to mortality; and that there was a dynamic growth process whose rate was influenced by the size of the plant, reducing as a plant matured towards an upper size limit. In the study of the age-dependent plant growth model, Pugliese [3] proposed a continuous-time model of growth and reproduction. Paxson and Simon [4] demonstrated how computers can directly control and monitor plant growth. According to Jaremo and Palmqvist [5], the compensatory plant type has the ability to accelerate its inherent rate of biomass growth as a response to damage. A study of this system revealed that when a highly effective herbivore is present, the compensatory capability such as this provides it a competitive benefit over other plants, even though it will not increase equilibrium plant density. Five distinct consumer-prey models were proposed by O'Connor et al. [6]. They observed that plant and herbivore abundances are influenced by environmental elements in response to temperature.

The Allee effect, which allows for a decrease in fitness due to decreasing population sizes, was taken into consideration by Ferreira et al. [7] in a predator-prey system. They examined the weak Allee effect, which is a component in the equation describing prey dynamics, and identified the necessary conditions for the occurrence of Hopf bifurcation. The findings indicated that the prey population is constrained by the carrying capacity of the environment, whereas the growth rate of the predator is influenced by the past quantities of the prey. The inducible changes harm the growth and reproduction of individual herbivores [8], [9]. These induced changes are commonly referred to as induced defenses in plant quality, and they frequently have an important effect on the dynamics of herbivore populations [10]. Some herbivore populations experienced population declines as a result of inducible defenses [11], [12], while cyclical fluctuations occur in other populations [13], [14]. Models that depict predator-prey interactions and make the assumption that fluctuations in the prey abundance cause fluctuations in the predator abundance support many of these arguments [15], [16]. There is evidence that diseases have shown significant effects on the extinction of species [17]. For instance, there would be negative growth in the predators if they consumed infectious prey [18].

Models of plant-herbivore interactions are one of the most fundamental processes and are crucial to ecology. The most vital component in ecology is the relationship between plants and herbivores, which is influenced by environmental variables like rainfall, temperature, and altitude. It will be easier to predict outcomes and recommend possible conservation strategies if we can recognize patterns in these environment-dependent interactions. It is frequently defined by complex, nonlinear relationships that might result in thresholds by which small scales processes experience weak and slight changes, while larger scales experience sudden and dramatic changes [19]–[21]. Several factors besides the size of the population or biomass alone can affect the interaction between plants and herbivores. The plant-herbivore system has been classically modelled using the predator-prey system [22]–[24]. Several ecologists examined models of plant-herbivore interactions using various functional responses and Allee effects [19], [22], [23], [25]–[31]. The interaction between plants and herbivores is a major factor in determining community structure all over the world. They play a vital role in lowland tropical rainforests where herbivory rates are higher, plants are better protected physically and chemically, and herbivore diets are more specialized [32]. Asfaw et al. [30] focused on the reformulation of a mathematical plant-herbivore model with the Allee effect such that the mortality rate of herbivores is influenced by the plant population and the threshold conditions for the existence of the herbivore population are also demonstrated. A mathematical model developed by Asfaw et al. [33] incorporates variations in some parameter values caused by changes in temperature and rainfall. They demonstrated how the coexistence of the population is greatly influenced by environmental changes and collected real data to verify the outcomes. A model by Edelstein-Keshet [22] was proposed that includes the impact of plant quality. The key finding is that the frequency distribution of plant quality in the vegetation can satisfy a conservation equation, and three scenarios are discussed to explain the qualitative aspects of the system. It was also highlighted that stable equilibrium and stable periodic oscillations can be obtained depending on the response of herbivores and plant quality. A general mathematical framework was utilized by Edelstein-Keshet and Rausher [34] to predominantly concentrate on inducible plant defenses and explain how variations in plant quality affect herbivore population dynamics. Anttila et al. [35] investigated the combined impacts of UV-B and herbivory on the defense mechanisms of the mountain birch as well as the influence of increased UV-B radiation on an autumnal moth. Model [28] examined how plants and herbivores interact when the herbivore is at risk such as disease and harvesting, while the plant has a refuge available. The study discussed the conditions of feasibility and stability for the equilibrium points of the system.

To evaluate the theoretical findings and examine the system's properties, the numerical simulations were validated. To investigate the interactions between plants and herbivores, specifically focusing on their scaling up to landscape scale, Oene et al. [25] proposed an ecosystem model which is used to determine how herbivory will affect the growth of vegetation. Li [36] developed a plant-herbivore model where plant toxicity results in herbivore SIS parasite infection. It was shown that the dynamics of both populations are significantly impacted by their interactions. Kartal [37] has explored the boundedness, periodicity, and stability of the plant-herbivore model using both difference and differential equations. By incorporating the center manifold theorem and the bifurcation theory, Yousef and Yousef [29] utilized fractional-order differential equations to analyze the plant-herbivore model and demonstrate the occurrence of flip bifurcation.

Nowadays, many ecologists are very curiously studying the topic of mathematical modelling. Using graphs, diagrams, equations, etc., aids in the theoretical and experimental solution of real-world problems. It provides precise problem-solving techniques. As a result, the interest of ecologists in this subject is constantly increasing, and various attempts have been made to theoretically predict solutions to issues that arise in the real world. The researchers have developed several kinds of mathematical models for issues encountered in the real world. The theory of predator-prey mathematical modelling was established in 1920. Predator-prey systems that may show interactions between prey-predator species are crucial in the biological field. The first mathematically realistic models with two or more interacting species were developed by Lotka [38] and Volterra [39]. When the ecological law of nature was discovered, Lotka and Volterra developed the first predator-prey model with the differential equation which is referred to as the Lotka-Volterra equation. Their model is now considered the basis for several models and research. Many models are modified versions of the prey-predator model [40]–[44]. The modified versions of predator-prey models include plant-herbivore models, Leslie-Gower predator-prey models, and Gauss-type predator-prey models. Modified versions of the Lotka-Volterra predator-prey model are frequently used to represent the dynamics of plant-herbivore interactions as well [45]–[47].

Initially, the biologist Warder Allee [48] defined the term Allee effect in the 1930s, which describes that certain factors related to an individual's fitness are positively influenced by the density of the population. In general terms, the Allee effect is a phenomenon observed within a population where the initial rate of growth per individual increases when the population is sparse, or when the growth rate at low density is positive in the first instance [30], [49]. Since conservation concerns and issues with rarity have increasingly attained prominence over the past few decades, this phenomenon has become the focus of interest [49]. The Allee effect is responsible for the increase in the risk of extinction when populations exist at low densities, introducing a population threshold that must be exceeded under certain circumstances for a population to successfully grow [50]. In the fisheries sciences, it is known as depensation [50], [51], and in epidemiology, it is recognized as the population threshold of susceptible individuals or eradication threshold, and an infectious disease is effectively eliminated from a population below this threshold [52]. The negative competition effect is an alternate term used in population dynamics to describe this phenomenon [53]. In more precise terms, the (component) Allee effect refers to the occurrence of a positive association between a specific aspect of individual fitness and the density of a population [54]–[56]. The Allee effect is triggered by several mechanisms. Mate limitation is the fundamental factor that leads to the occurrence of the Allee effect in populations of both plants and animals. Environmental factors, inbreeding depression, demographic genetics, feeding habits, interacting with society, predator satiation, cooperative defense, etc. are some of the other mechanisms [27], [49], [55]–[58]. Natural populations of plants [59], birds [60], insects [61], and animals [62] all exhibit the Allee effect. In recent years, there has been growing importance placed on comprehending the impact of the Allee effect on behaviour of the species, reproduction, and conservation efforts. The investigation and comprehension of the Allee effect have significant ecological and applied engineering applications, including Agropecuary, Fishing, and Forestry Industries. If plant populations are sparsely scattered and low in density, plants can experience the Allee effect [63]–[65]. Populations are more vulnerable to extinction due to Allee effects, particularly in situations of severe predation or harvesting [66]. The Allee effect can be classified into two distinct categories: Strong and Weak. The strong Allee effect [54] or critical depensation [51] implies that a threshold population level, also known as the Allee threshold [51], [52], exists and the growth rate becomes non-positive when the density is extremely low. Conversely, when the growth rate at zero density is non-negative and there is no threshold population level, the weak Allee effect [56] or noncritical depensation [51] occurs [65], [67]–[76]. Species with strong Allee effects are more likely to exhibit less susceptibility to additional factors causing mortality, exhibit slower recovery rates, and increase their risk of extinction in comparison to other species [56]. The majority of predation models consider the influence of the Allee effect on the prey, without incorporating the functional response that signifies the variations in predation rates due to changes in density of prey. The quantitative assumption is that the functional response directly influences the extension of the region exhibiting bistability [77], [78]. Various types of population dynamics models, incorporating Allee effects were developed by Elaydi and Sacker [69]. Based on the biological supposition of the Allee effect, they developed several fitness functions, leading to corresponding models incorporating the Allee effect. Additionally, they demonstrated the existence of carrying capacity with asymptotically stable 2-periodic systems. Dennis [50] developed mathematical models that represent the risk of extinction, critical density, and growth in sparse populations suffering Allee effects. Both the prey and predator populations demonstrate herd behaviour and are prone to harvesting in a prey-predator system incorporating the Allee effect in the prey, according to Biswas et al. [79]. To support the mathematical conclusions, numerical examples and sensitivity analysis of the key parameters are provided. The Glanville fritillary, an endangered butterfly, was studied by Kuussaari et al. [61] for evidence of the Allee effect in natural populations. They revealed a relationship between the ability to find a mate and the rate at which a population grows and demonstrated that the emigration rate increases and the proportion of mated females declines when local density decreases. To analyze the dynamics of competitive models involving intraspecific and two-species interactions incorporating the Allee effect, Wang et al. [53] utilized phase plane analysis and simulation. They proved that the Allee effect alone could result in alternate stable states in a two-species competitive system and that even in high population densities, both species may become extinct if interspecific competition is intense. Gonzalez-Olivares et al. [80] presented a set of ordinary differential equations with the Allee effect. They established that there exists topological equivalency between the majority of them, but they can have uncommon properties. A discrete-time predator-prey system with the Allee effect was considered by Wang et al [81]. It is investigated how the Allee effect stabilizes the populations of both prey and predators. Li et al. [82] demonstrated the existence and direction of a Hopf bifurcation, as well as the stability of the system using an aquatic diffusive predator-prey model, considering a double Allee effect on prey and pH-dependent capture rate. Aguirre et al. [83] investigated a predator-prey model that incorporates the Allee effect, supposing that environmental randomness is described by noise factors impacting both populations. It was demonstrated that the long-term survival of both populations could occur if there was a weak Allee effect. When a significant Allee effect is taken into consideration, it becomes possible for both species to extinct. Green paramecia and cyanobacteria are two examples of photo-autotrophic organisms that Ohkawa et al. [84] reported may exhibit the Allee. The finding is that the sensitivity of an autotrophic organism to the Allee effect can change as a result of a single-gene mutation. According to Asfaw et al. [27], the dynamics change more when noise is added to the herbivores than to the plants in the extended stochastic plant-herbivore model, considering the Allee effect. The investigation of the behaviour of a predator-prey model that incorporates a weak Allee effect and a non-monotonic functional response was studied by Lin et al. [85]. A nonlinear discretized predator-prey model, considering the nonlinear Allee effect in the prey as well as in both populations was the subject of Song's [86] investigation. He proved the stability and existence of the fixed points of the model through his findings. Furthermore, the dynamical behaviour can be stabilized by a moderate Allee effect in predators. According to Fang et al. [87], who examined a Leslie-Gower model with a weak Allee effect, both the prey and predator populations will tend towards abundance if the Allee effect is intense. Additionally, they demonstrated that the Allee effect alone is able to change the stability and trigger supercritical Hopf-bifurcation. Recent ecological studies presented the possibility that two or more Allee effects provide mechanisms that act on a single population at the same time and the combined impact of certain occurrences has been termed multiple Allee effects [54], [78].

The logistic equation is frequently used in ecology and population biology to simulate the population of organisms. Many mathematical models have been proposed by combining the logistic model with the Allee effect and functional response. In 1959, Holling [88] introduced the functional response theory, which defines the rate at which a predator consumes prey and the size of the prey [26], [88], [89]. Holling categorized the functional response into three distinct kinds. The Holling type-I or rectilinear functional response is represented as: $P(x) = kx$, where $x(t)$ represents the prey biomass and k is a positive constant. A saturation function is employed to define the amount of consumption because the Holling type-II or hyperbolic response assumes that a single predator consumes food until it feels famished. Thus, the type-II functional response is: $P(x) = kx/(1 + T_h kx)$; where $x(t)$ is the biomass of the prey, k is the positive searching efficiency, and T_h called the average time for each prey [57], [90]. Herbivores feeding in patches where plants are concentrated in space usually exhibit type II functional response [91]. Predators who increase their search activity with increasing prey density exhibit Holling type-III or sigmoidal functional response. It is given by: $Y = \frac{aT_t X^k}{1 + abX^k}$, where a is the discovery rate and $k > 1$ [92]. The plant-herbivore model under the Allee effect with Holling type-I functional response was presented by Kumar and Verma [59]. The presence of a strong Allee effect was seen in relation to the decline and disappearance of plant and herbivore populations. It was proved that the system is stable around the non-trivial point and becomes unstable when the threshold value crosses the delay parameter, and Hopf-bifurcation occurs when there is a weak Allee effect. The theory of Holling type- II response was utilized by Dupke et al. [93] for the selection of herbivore habitats. A toxin-determined functional response resulting from chemical defenses was established by Liu et al. [94] by modifying the traditional Holling type- II functional response in the plant-herbivore model, allowing herbivores to consume fewer plants. The Leslie-Gower predator-prey model, incorporating a Holling type-II response and considering a weak Allee effect was analyzed by Arancibia-Ibarra et al. [95] and reveals that both populations are able to coexist and exhibit oscillations. Moreover, they demonstrated the model's bistability for some fixed subset of parameters. Aziz-Alaoui and Okiye [96] developed a modification of the Leslie-Gower predator-prey model, incorporating a Holling type-II response and exhibits boundedness and stability about the interior point. Sengupta and Das [97] investigated the interpretation of the dynamical characteristics of the stochastic prey-predator model and non-autonomous deterministic using sigmoid functional response and derived the permanence, persistence, non-persistence, and positivity. The Gauss-type predator-prey model, considering a sigmoid response as well as Allee effect was examined by Rojas-Palma and Gonzalez-Olivares [98]. They presented a description of the optimal harvest policy and fishing effort by solving the autonomous ordinary differential equations. The behaviour of the Eco-epidemiological model with SI-type disease in the prey and a nonlinear incidence rate was examined by Naji and Mustafa [99]. A Leslie-Gower predator-prey model was studied by Mishra et al. [100], who concluded that the model's instability could be caused by the prey only. Zhang et al. [101] investigated the Leslie-Gower predator-prey model with the harvesting system. The findings of harvesting showed that the concentration of the predator population is strictly declining and that, according to certain limitations, prey size has no impact. Yue [102] used a modified Leslie-Gower model to study a prey refuge. The stability and the global attractivity of a non-zero equilibrium were investigated. Farajzadeh et al. [103] examined the stability behaviour of the interacting species by investigating the Gauss predator-prey model with one prey and two predators.

Delay differential equations (DDEs) have a historical background of over two centuries. The first applications of delay differential equations can be found in geometry and number theory. However, the topic gained prominence after 1940 as a result of its application in engineering systems and control. A general theory of DDE was presented in Chapter II of the book "Delay-Differential Equations" [104]. Research has been conducted regarding the problem of the solution existence and the uniqueness characteristics of those solutions for delay-differential equations. First-order nonlinear differential-delay equations were utilized by Mackey and Glass [105], which show the limit cycle oscillations and chaotic solutions to characterize dynamical respiratory systems. Cooke and Grossman [106] discussed how crucial it is to consider the time delays that are part of biological, physical, and social phenomena. A summary of recent advances in the stability and oscillation of autonomous delay differential equations was provided by Gopalsamy [107] in his book. The stability and bifurcation phenomenon exhibited by DDEs involving two delays was investigated by Li et al. [108]. Numerical methods and a software programme for performing bifurcation analysis of delay differential equations were introduced by Engelborghs et al. [109]. Bocharov and Rihan [110] studied models of biological phenomena whose dynamics are better described by delay differential equations, and the tools for their solutions are numerical approaches. The distribution and nature of the roots of a transcendental polynomial of third degree were examined by Ruan and Wei [111]. A MATLAB code called DDE23 was written by Shampine and Thompson [112] to solve delay differential equations with constant delays. A MATLAB package called DDE-BIFTOOL was described by Engelborghs et al. [113], specifically designed for analyzing bifurcation behaviour in systems of DDEs that incorporate several fixed, discrete delays. An analysis of population dynamics through the examination of non-linear DDEs was conducted by Kubiaczyk and Saker [114] for oscillation and stability. By altering the parameters' values, Kuznetsov [115] investigated the nonlinear dynamics system and its bifurcation. Nonlinear delay differential equations involving population growth were studied by Lenbury and Giang [116]. The distribution and nature of the roots of an exponential polynomial of the fourth degree were discussed by Li and Wei [117]. In biological models developed by several researchers dealing with the dynamics of single species populations, Ruan [118] incorporated time delays of one type or another. Roose and Szalai [119] examined the continuation and bifurcation phenomenon of DDEs. The distribution and nature of zeros of a fifth-degree transcendental polynomial were discussed in detail by Zhang et al. [120]. Smith [121] emphasized the key tools in his book to understand the background of delay differential equations and their application in different models. Kuang [122] demonstrated how the well-known MATLAB-based dde23 solver for delay differential equations developed by Shampine and Thompson can be used for numerically solving the most of delay differential equations and stability analysis. Huang et al. [123] conducted an analysis of the global stability in population dynamics, considering non-linear DDEs. To investigate the influence of time delay on plant biomass, Kalra and Kumar [124] proposed a model involving toxic metals. Li et al. [125] explored a herbivore-plant system incorporating time delay with reaction-diffusion equations and demonstrated that this model exhibits the characteristic of Hopf bifurcation. Sun et al. [126] constructed a mathematical model that illustrates the dual impact of time delay on herbivore populations. Their observations revealed that when time delay is coupled with space, it promotes the average density of herbivore populations during outbreaks, suggesting that time delay might influence the resilience of herbivore populations. Gazi and Das [127] examined a nutrient-based model depicting the interaction between plants and herbivores, incorporating delayed nutrient recycling. The model incorporated the number of herbivores attacking plants following Holling type-III functional response. They interpreted all analytical findings in ecological terms and compared them with computer simulations.

In 1942, Hopf published his basic paper. As a parameter reaches a critical point, Hopf [128] introduced the concept of Hopf bifurcation, defining it as the occurrence of local birth or mortality of a periodic solution, which may be a self-excited oscillation, from an equilibrium. A complex conjugate pair of eigenvalues undergoes Hopf bifurcation when they become purely imaginary.

As a result, Hopf bifurcation can only be observed in systems with two dimensions or more. Hopf bifurcation was studied by Marsden et al. [129] with applications to particular issues, such as stability calculations. Hsu and Hwang [130] investigated the Hopf-bifurcation for the well-known prey-predator ecological model, Holling-Tanner. A comprehensive investigation of the impact of time on the collective dynamics of coupled limit-cycle oscillators on Hopf bifurcation was reported by Reddy et al. [131]. The relationships between stability theorems and the notions of simple and general Hopf bifurcations were elucidated by Manfredi and Fanti [132]. In Nicholson's blowflies equation, Wei and Li [133] demonstrated that an occurrence of Hopf bifurcation was observed as the delay parameter exceeded a certain threshold point. In the presence of non-linear prey harvesting, Gupta and Chandra [134] explored the analysis of Hopf-bifurcation, considering a modified Leslie-Gower prey-predator model. Zhang and Guo [135] investigated the analysis of stability and direction of Hopf-bifurcation by utilizing the application of the center manifold theorem and the well-known Van der Pol equation. Wang et al. [136] used the theory of normal-form and the center manifold theorem to study the stability analysis and direction of Hopf bifurcation within a model concerning phytoplankton-zooplankton dynamics. Pal et al. [137] conducted an analysis of a predator-prey system characterized by intraspecific predator competition and the presence of a strong Allee effect in the prey population. They demonstrated that Hopf-bifurcation could occur when the discrete delay exceeded a critical value and used the normal form method and center manifold theory to investigate the stability and direction of the Hopf-bifurcating periodic solution. Kalra and Kumar [138] established a model for the study of plant growth that includes compartments for the shoot and roots. The inclusion of a delay parameter in the system disrupts its stability and simultaneously leads to the occurrence of Hopf bifurcation. Singh [139] examined the SIR epidemic model and demonstrated that Hopf-bifurcation occurs under some circumstances.

Dickinson and Galinas [140] developed the "Direct method" for the purpose of analyzing the sensitivity of ordinary differential equations with inaccurately known parameters. The sensitivity of the solution is measured by the partial derivatives with respect to model parameters. Baker and Rihan [141] developed a novel approach for estimating the sensitivity of variables to model parameters and non-linearity effects in DDEs. In order to prepare consideration merit for application, Frey and Patil [142] discovered and compared all of the sensitivity methods utilized in various fields. Using adjoint equations and direct methods, Rihan [143] established a comprehensive theory for conducting sensitivity analysis in mathematical models incorporating delays. The model parameters were taken in this study to be constants. Kepler [144] employed both the adjoint method and the direct method to investigate the sensitivity of mathematical models. Based on Perumal and Gunawan method [145], if parametric sensitivity analysis is used indiscriminately, it may not only produce inaccurate results but also fail to accurately depict dynamics. The utilization of delay differential equations in dynamical systems, as well as their computational tools, parameter estimation, and sensitivity analysis, were examined by Rihan [146]. Wu [147] established a framework for functional structural plant models to be subjected to sensitivity analysis in order to understand the underlying biological processes.

In plant-herbivore interactions, delay differential equations are not very prominent and widely used. Only a few instances of plant-herbivore dynamics, influenced by the Allee effect have been presented by utilizing DDEs. Due to herbivore damage and the use of inducible defenses, delay differential equations may frequently occur in interactions between plants and herbivores [23], [24], [148], [149]. Sun et al. [149] examined how time lags affect herbivore cycles. The necessary conditions for the existence and global attractiveness of a positive periodic solution, considering an impulsive delay differential equation incorporating the Allee effect were discovered by Yan et al. [150]. The findings improve and extend the existing theorems. Liz and Ruiz-Herrera [66] demonstrated how the interaction of Allee effects and harvest strength causes collapses as a result of overexploitation. Moreover, when population densities are at levels that would ensure survival with smaller time delays, the interaction between large delays and Allee effects can lead to extinction. Furthermore, they revealed that when harvesting effort increases, a stable equilibrium loses stability, leading to sustained oscillations, whereas equilibrium again stabilizes due to increased death rates. A set of non-linear DDEs was employed to investigate the effect of time lag on the extinction of toxicant-affected plant populations [124], [151], [152]. The aspect of time delay in the plant population has been investigated under the inhibitory and stimulatory allelopathic effects [153]. When predators consume infectious plants, there is a time lag for the gestation period. Delay differential equations thus play an important part in realistic models [138], [154]–[156].

1.3 Motivation and Background (Research Gap)

From the study of the exclusive literature survey, the following research gaps have been identified:

1. The earlier studies on Plant- Herbivore interaction reveal that a lot of work has been reported statistically in this field.
2. There is some research has been done by using Ordinary differential equations on the Plant- Herbivore interaction.
3. From the available literature, it can be seen that a limited work has been attempted on Plant- Herbivore dynamics using Ordinary differential equations under the Allee effect.
4. There is a lack of information in the existing literature regarding Plant-Herbivore dynamics where Delay differential equations are used under the Allee effect.

1.4 Hypothesis (Scope of Study)

Nowadays, a lot of researchers are interested in the dynamics of plants and herbivores. Plant growth dynamics is significantly impacted by the Allee effect. Several mathematical models that describe the dynamics of plants and herbivores were proposed by the researchers. In competing mathematical models, ordinary differential equations are very useful for studying the dynamics of plant-herbivore interactions under the Allee effect with various kinds of functional responses. Moreover, a delay is incorporated into the model because the Allee effect periodically takes time to manifest; this time period is known as the Allee effect's incubation period. Therefore, the delay differential equations and delay parameter are used to describe the incubation period.

1.5 Proposed Objectives of the study

After an exhaustive literature survey, the following objectives have been selected to fill the research gaps for the present research work:

1. To model Plant- Herbivore dynamics with strong Allee effect in plant population using Delay differential equations.
2. To model Plant- Herbivore dynamics with weak Allee effect using Delay differential equations.
3. To perform stability analysis, Hopf bifurcation and sensitivity analysis of the proposed model.

1.6 Basic concepts of general plant physiology used in the thesis

1.6.1 Structure and Storage

Plant models are always based on a substantially oversimplified representation of the real system. Plant material is classified into two kinds: structure and storage, which is a simplification with what seems to be some physiological importance [1].

Let W stand for the dry weight of the plant under consideration, and let W_G and W_S represent the structural and storage components of the total dry weight, respectively, so that

$$W = W_G + W_S \tag{1.1}$$

Using a simple definition of "growth" as the dry matter increment and differentiating equation (1.1) with respect to time t, it follows that the total growth rate is composed of two components:

$$\frac{dW}{dt} = \frac{W_G}{dt} + \frac{W_S}{dt} \tag{1.2}$$

The total growth rate $\frac{dW}{dt}$, is composed of the structural growth rate $\frac{W_G}{dt}$, and the storage growth rate $\frac{W_S}{dt}$. With a positive structural growth rate and a negative storage growth rate, a plant can therefore have a negative total growth rate. An illustration of this is overnight growth, during which the storage material of many plants is significantly reduced, significant structural growth occurs, and the overall dry weight decreases as a result of respiration. The relative growth rate (RGR) or specific growth rate, which is denoted by the sign R_W in physiology, is a quantity that is determined by:

$$R_G = \frac{1}{W}\frac{dW}{dt} \tag{1.3}$$

In terms of the total dry weight, which includes both structure and storage, a specific growth rate is defined. Calculating a specific structural growth rate μ_G is more beneficial, which is defined by:

$$\mu_G = \frac{1}{W_G}\frac{dW_G}{dt} \tag{1.4}$$

1.6.2 Plant Growth Curve

The plant grows slowly and sluggishly at the beginning (lag phase). Following then, the plant's growth increases quickly (exponential phase). Due to nutrient restriction, the growth rate now gradually decreases (stationary phase). A plant growth curve refers to the usual sigmoid or S-shaped curve generated by plotting growth and time (Figure 1.1) [157].

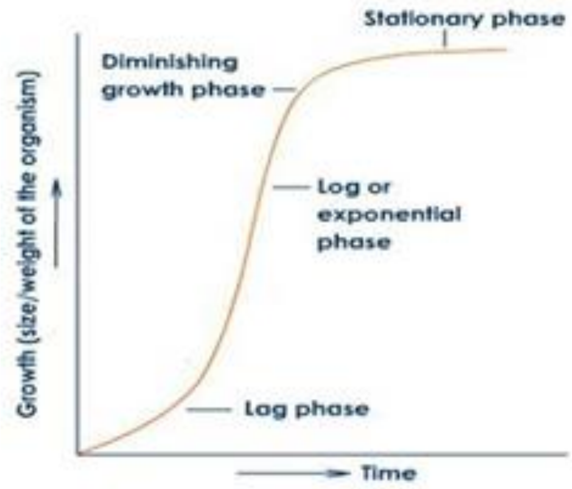

Figure 1.1 Plant Growth Curve

1.6.3 Plant Growth Rate

The impact of a limiting resource on plants at the individual, population, and ecosystem levels of an organization is known as the plant-resource interaction. A growth-rate dependence on nutrition availability at the individual level can be expressed as [158]:

$$r(R) = \eta \mu_m W_r \frac{R}{k_R + R} \tag{1.5}$$

In the above equation, R stands for nutrient availability, η for nutrient use efficiency, W_r for the proportion of total biomass that is allocated to root mass, μ_m for the resource-saturated rate of resource uptake per unit of root mass, and k_R for the half-saturation constant for nutrient uptake.

1.6.4 Allee effect

A biological phenomenon, known as the Allee effect is defined by a correspondence between the density of a population and the average fitness of its individuals, which is typically measured as the per-individual population growth rate [48]. There are two distinct kinds of Allee effects: Strong Allee effect and Weak Allee effect, which are shown in figure 1.2 and figure 1.3 [159]. A weak Allee effect refers to situations where the density of a population is low and the per capita growth rate increases with an increasing population, although it is always positive.

When population density is low, the Allee effect is strong, meaning that the per capita growth rate increases with the increasing population. However, when density is zero, the Allee effect exhibits a negative impact below a certain threshold referred to as the Allee threshold.

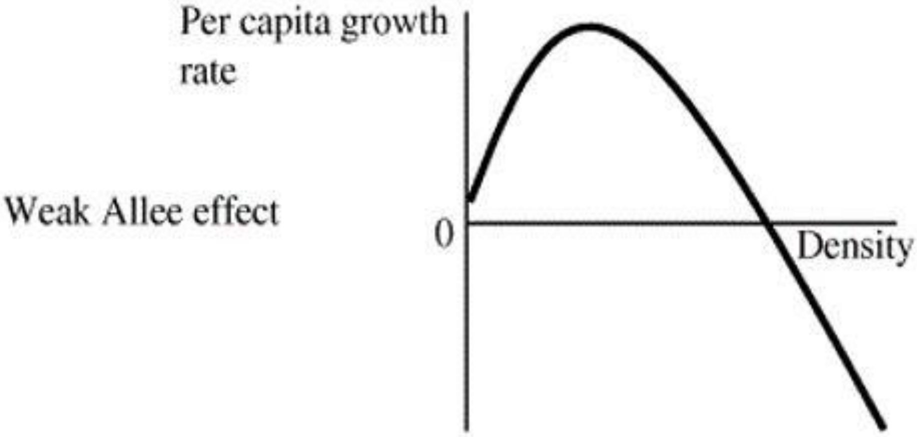

Figure 1.2 Weak Allee effect

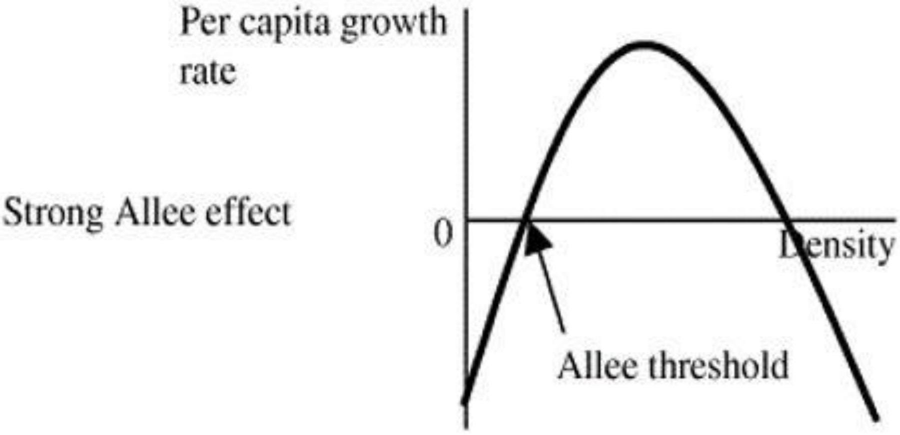

Figure 1.3 Strong Allee effect

For example, if $x = x(t)$ refers to the density of a population, the following is the prominent continuous growth equation to describe the Allee effect:

$$\frac{dx}{dt} = r\left(1 - \frac{x}{K}\right)(x - m)x \tag{1.6}$$

known as the multiplicative Allee effect.

Consider the natural growth function provided by the following equation:

$$\frac{dx}{dt} = r\left(1 - \frac{x}{K} - \frac{m}{x+b}\right)x \tag{1.7}$$

Here it is additive Allee effect. It is clear that if $m \leq 0$, then there is a weak Allee effect, and if $m > 0$, then there is a strong Allee effect [78].

1.6.5 Functional Response

According to Holling's [88] research, the predation rate increases as the density of the prey population increases. Two factors contribute to this:

a) Whenever a predator is exposed to a higher prey density, their consumption rate increases.
b) The density of predators increases as prey density increases.

Holling considered these impacts as two distinct kinds of predator population responses to prey density: Functional response and Numerical response.

The functional response is the term used to describe the relationship between the density of a prey population and the number of successfully attacked prey per predator. It explains how a predator responds to varying prey density. Functional responses are typically divided into three categories, known as Holling's type I, II, and III functional responses. Figure 1.4 represents these functional responses [160].

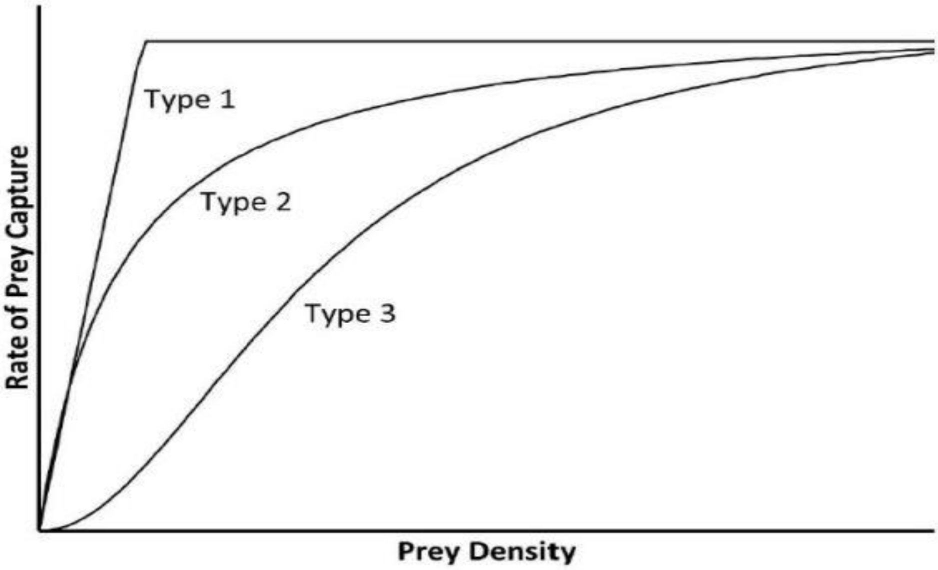

Figure 1.4 Holling's type I, II, and III functional responses

1.6.5.1 Type-I Functional response

The first kind is the most basic of the three functional responses. Type-I is assumed by the Lotka-Volterra equations, which state that the consumption rate increases linearly with prey density. The linear increase is based on the supposition that the processing time for food items by consumers is negligible. Figure 1.3 depicts the Type-I functional response, which exhibits a linear relationship between the quantity of prey consumed and the density of prey. The Type-I functional response cane be expressed as:

$$C = \alpha N \tag{1.8}$$

where C is the capturing rate, α is a proportionality constant determined by the rate of encounters between predators and prey and N is the density of prey [88], [92], [160].

1.6.5.2 Type-II Functional response

According to the assumption that the consumer is constrained by its capacity to process food, the Type-II functional response shows that the rate of consumption increases with density of prey but eventually declines until it reaches a plateau while consumption rate remains constant regardless of density of prey. Type-II functional response is given by:

$$C = \frac{\alpha N}{1 + \alpha t N} \tag{1.9}$$

where t is handling time [92], [160].

1.6.5.3 Type- III Functional response

The functional response of type- III is comparable to type- II at high prey densities. However, at low levels of prey density, when there is an increment in the prey density, the rate of predation increases, but as prey density gets higher even further, it abruptly declines. A type- III functional response is S-shaped or sigmoidal. Therefore, type-III functional response is of the form [92], [160]:

$$C = \frac{\alpha N^2}{1 + \alpha t N^2} \tag{1.10}$$

1.7 Biological Significance of the study

The study describes distinct mathematical models analyzing the impact of the delay parameter on plant and herbivore populations under the Allee effect. The findings provide insights into how plant-herbivore dynamics may be influenced by ecological and biological factors:

i. **Allee Effect:** The Allee effect may result from cooperative behaviors, increased mating success, or other mechanisms.

ii. **Strong Allee Effect and Extinction:** The study reveals that under a strong Allee effect, both plant and herbivore populations can experience extinction. This extinction could be due to mechanisms such as cooperative behaviors for finding mates or protecting against predators, which are more effective at higher population densities. If the population falls below a critical threshold, these cooperative behaviors may become ineffective, leading to extinction. This suggests that the positive effects on individual fitness at low population densities may lead to insufficient reproduction and survival when populations are too small.

iii. **Stability Analysis with Delay $(\boldsymbol{\tau})$:** The delay parameter (τ) represents a time delay in the response of populations to changes in density or environmental conditions. When there is no delay $(\tau = 0)$, the equilibrium point E^* is shown to be absolutely stable, indicating that the populations reach a stable state without fluctuations. As the delay increases but remains below a critical value, the equilibrium point E^* becomes asymptotically stable. This means that the populations eventually stabilize.

iv. **Hopf Bifurcation and Delay:** When the delay surpasses a critical value, the stability of the equilibrium point E^* becomes unstable, leading to the observation of Hopf bifurcation. The occurrence of Hopf bifurcation suggests that the introduction of a time delay in the system, beyond a critical threshold, can destabilize the populations. This could be interpreted in a biological context as the delayed response of populations to changes in the environment, resource availability, or other factors.

In summary, the results suggest that the presence of the Allee effect, coupled with a time delay has significant implications for the stability and dynamics of plant and herbivore populations in the model which can lead to complex dynamics including extinction, stable equilibrium, and bifurcations in plant and herbivore populations. The critical value of the delay parameter plays a key role in determining the stability or instability of the system.

1.8 Mathematical Preliminaries

1.8.1 Existence of Unique, Bounded and Positive Solution of Delay differential equation

A delay differential equation is a differential equation in which the present-time derivative depends on the solution and derivatives of earlier times. Here an initial history function, rather than an initial condition, needs to be defined. A delayed state variable can be used to demonstrate the past dependence of a differential equation. The derivative of the state variable is not required in this case. The corresponding delay differential equation, which includes a single delay $\tau > 0$, can be expressed as follows [121]:

$$\dot{x}(t) = f(x, x(t), x(t - \tau)) \tag{1.11}$$

Assume that $f(t, x, y)$ and $f_x(t, x, y)$ are continuous on R^3. Let $s \in R$ and $\emptyset: [s - \tau, s] \to R$ be continuous. We require a solution $x(t)$ of equation (1.11) satisfying:

$$x(t) = \emptyset(t), t \in [s - \tau, s], x(0) = x_0 \tag{1.12}$$

And satisfying equation (1.11) on $t \in [s,\ s + \sigma]$ for some $\sigma > 0$.

Theorem 1.7.1.1 (Existence of unique solution). Let $f(t, x, y)$ and $f_x(t, x, y)$ are continuous on R^3. Let $s \in R$ and $\emptyset: [s - \tau, s] \to R$ be continuous. Then $\exists\ \sigma > s$ and a single solution that is unique to the initial-value problem (1.11)-(1.12) on $[s - \tau, \sigma]$.

Theorem 1.7.1.2 (Boundedness of solution). Let f satisfy the hypothesis of theorem 1.7.1.1 and assuming $x: [s - \tau, \sigma) \to R$ is the solution to the initial value problem (1.11)- (1.12), that is not continuous. If $\sigma < \infty$ then $\lim_{t \to \sigma -} |x(t)| = \infty$.

Remark 1.7.1.3 Theorems 1.7.1.1 and 1.7.1.2 can be expanded to include the scenario where $x \in R^n$ and $f: R \times R^n \times R^n \to R^n$. Additionally, these theorems can also be applied to situations involving multiple discrete delays $\tau_0 < \tau_1 < \cdots < \tau_m$ where $f = f(t, y(t), y(t - \tau_0), y(t - \tau_1), \ldots, y(t - \tau_m))$.

Theorem 1.7.1.4 (Positivity of solution). Suppose that $f: R \times R_+^n \times R_+^n \to R^n$satisfies the hypothesis of theorem 1.7.1.1 and remark 1.7.1.3 and for all i, t and for all $x, y \in R_+^n$:

$$x_i = 0 \Rightarrow f_i(t, x, y) \geq 0$$

When the initial data $\emptyset$ in equation (1.12) satisfies $\emptyset \geq 0$, the resulting solution $x(t)$ of equation (1.11) will also satisfy $x(t) \geq 0\ \forall\ t \geq s$, where the solution is defined.

1.8.2 Stability by Variational matrix method

Let an autonomous system of equations be

$$\frac{dy}{dt} = f(y) \tag{1.13}$$

where y is an n-tuple vector i.e. $y = (y_1, y_2, - - - y_n)$. Let $\phi(t)$ be the solution of system (1.13). The linear part of the expansion of the system (1.13) about $\phi(t)$ is given by the variational equation (1.13) with respect to $\phi(t)$, written as:

$$\frac{dx}{dt} = f_y(\phi(t))x \tag{1.14}$$

where $f_y(\phi(t)) = \frac{df_i}{(dy_j)_{n\times n}}$ at $\phi(t)$. Since the stability of the variational system depicts the stability of any solution of a non-linear system governed by it, the stability of $x = 0$ of (1.14) determines the stability of $y = \phi(t)$ of (1.13). Particularly, when $\phi(t) = \phi_0$, a constant, the system (1.13) becomes:

$$\frac{dx}{dt} = Ax \tag{1.15}$$

where $A = f_y(\phi_0)$. Since a small perturbation of the system (1.13) is represented by the system (1.14), the stability of $y = \phi_0$ of (1.15) gives the stability of the solution of $x = 0$ of (1.14). The description of the stability of every solution of $x' = Ax$ is given by following theorems [161].

Theorem 1.7.2.1 If every characteristic root of A has $-ve$ real part, then every solution of the system $x' = Ax$, where $A = (a_{ij})$ is a constant matrix, and is asymptotically stable.

Theorem 1.7.2.2 If every characteristic root of A with multiplicity grater than one has $-ve$ real part and all its roots with multiplicity one have non-positive real parts, then every solution of the system $x' = Ax$ is bounded and therefore stable.

By referring to the theorem [161], one can determine the sign of the real parts of the roots found in the characteristic equation.

Theorem 1.7.2.3 Hurwitz's Theorem. An essential and adequate prerequisite for ensuring the negativity of the real parts of every root of the polynomial

$L(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ----+a_n$ with real coefficients representing the positivity of every principal diagonal of the minors of the Hurwitz matrix

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

& With the help of Routh- Hurwitz we can calculate the stability of the system, whether the system is stable or not stable. For stability we check the first column of Hurwitz must be positive & $a_1 > 1$. If any value of the first column is negative & $a_1 < 1$, then the system is unstable. The Hurwitz is called an auxiliary polynomial. The order of the auxiliary polynomial is always even.

Theorem 1.7.2.4 Let $\varsigma_{1,}\varsigma_2 \ldots . \varsigma_m$ are all positive and 0 is included and $\zeta_i{}^j(j = 0,1,2, \ldots m\colon i = 1,2, \ldots n)$ remains constant. The alteration of the summation of the orders of the zeros of exponential polynomial $P(\chi, e^{-\chi\varsigma_1}, \ldots ., e^{-\chi\varsigma_m})$ on the open right half plane is only possible if a zero emerges on the imaginary axis or crosses it, as $(\varsigma_1, \varsigma_2, \ldots, \varsigma_m)$ undergo variations, where

$$P(\chi, e^{-\chi\varsigma_1}, \ldots ., e^{-\chi\varsigma_m}) = \chi^n + \zeta_1{}^0\chi^{n-1} + \cdots + \zeta_{n-1}{}^0\chi^n + \zeta_n{}^0 + [\zeta_1{}^1\chi^{n-1} + \cdots + \zeta_{n-1}{}^1\chi^n + \zeta_n{}^1]e^{-\chi\varsigma_1} + \cdots + [\zeta_1{}^m\chi^{n-1} + \cdots + \zeta_{n-1}{}^m\chi^n + \zeta_n{}^m]e^{-\chi\varsigma_m}$$

The proof of this theorem, initially established by Ruan and Wei [111], [162], relies on the application of Rouche's theorem [163].

1.8.3 Hopf-Bifurcation

Hopf's crucial contribution was the extension from two dimensions to higher dimensions. Sometimes Hopf bifurcation is also called as "Poincaré-Andronov-Hopf bifurcation"[164]. The Hopf-bifurcation theorem explains how a topological characteristic of a flow changes when one or more parameters are modified. The key observation is that if the stationary point is hyperbolic, that means if eigenvalues of the linearized flow at the stationary point, all possess non-zero real parts, then the local behaviour of the flow is entirely determined by the linearized flow. Consequently, bifurcations of stationary points can only be observed at parameter values where the stationary point becomes non-hyperbolic. More, precisely, the bifurcation value of a parameter is a value at which the qualitative nature of the flow changes.

Analyzing the Hopf bifurcation is significantly more challenging due to the presence of a non-hyperbolic stationary point with linearized eigenvalues $\mp i\omega$, resulting in a two-dimensional center manifold. In addition, the bifurcating solutions associated with the Hopf bifurcation are periodic in nature rather than stationary. As a result, the complexity of analyzing the Hopf bifurcation is several orders of magnitude higher.

Theorem 1.7.3.1 Hopf-Bifurcation Theorem.

Let us consider a one-parameter family of delay equations

$$x'(t) = F(x_t, \mu) \tag{1.16}$$

Considering that $F: C \times R \to R^n$ is twice continuously differentiable function with respect to its arguments and $x = 0$ represents a steady state for all values of μ: $F(0, \mu) \equiv 0$.

Linearize the function F around $\emptyset = 0$ in the following manner:

$$F(\emptyset, \mu) = L(\mu)\emptyset + f(\emptyset, \mu)$$

where $L(\mu): C \to R^n$ represents a bounded linear operator and the function f is of higher order:

$$\lim_{\emptyset \to 0} \frac{|f(\emptyset, \mu)|}{||\emptyset||} = 0$$

The characteristic equation concerning L can be expressed as given below:

$$|\lambda I - A(\mu, \lambda)| = 0, A_{ij}(\mu) = L(\mu)_i\left(e_\lambda e_j\right)$$

The roots of this equation constitute the main assumption.

(H) The characteristic equation will be having two simple roots $\mp i\omega$, where $\omega_0 \neq 0$ and there will be no other roots that are integer multiples of $i\omega_0$ for $\mu = 0$.

Here a root of order one means [165] a simple root.

Let $h(\mu, \lambda) = 0$ represents the characteristic equation, then **(H)** implies $h_\lambda(0, i\omega_0) \neq 0$. So, by the implicit function theorem, $\exists$ a continuously differentiable family of roots $\lambda = \lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ for small μ which satisfies $\lambda(0) = i\omega_0$. Specifically, $\alpha(0) = 0$ and $\omega(0) = \omega_0$. The next supposition is that as μ increases through zero, the line of the imaginary axis is crossed transversally by these roots. The assumption is:

$$\alpha'(0) > 0 \tag{1.17}$$

In case $\alpha'(0) < 0$, we always ensure that equation (1.17) holds by changing the sign of the parameter i.e. we take parameter $v = -\mu$. Thus, the $+ve$ sign is a normalization that shows if $\mu < 0$, then two roots have a $-ve$ real part, and if $\mu > 0$, then they have a $+ve$ real part.

Theorem 1.7.3.2 Let **(H)** and equation (1.17) is true. Then $\exists\ \varepsilon_0 > 0$, real-valued even function $\mu(\varepsilon)$ and $T(\varepsilon) > 0$ which satisfies $\mu(0) = 0$ and $T(\varepsilon) = {}^{2\pi}/_{\omega_0}$, and a non-constant $T(\varepsilon)$- periodic function $p(t,\varepsilon)$ with all functions being continuously differentiable in ε for $|\varepsilon| < \varepsilon_0$, such that $p(t,\varepsilon)$ is a solution of equation (1.16) and $p(t,\varepsilon) = \varepsilon\, q(t,\varepsilon)$ where $q(t,0)$ is a ${}^{2\pi}/_{\omega_0}$-periodic solution of $q' = L(0)q$.

Additionally, $\exists\ \mu_0, \beta_0, \delta > 0$ such that if equation (1.16) has a non-constant periodic solution $x(t)$ of period P for some μ satisfying $|\mu| < \mu_0$ with $max_t|x_t| < \beta_0$ and $\left|P - {}^{2\pi}/_{\omega_0}\right| < \delta$, then $\mu = \mu(\varepsilon)$ and $x(t) = p(t+\theta,\varepsilon)$ for some $|\varepsilon| < \varepsilon_0$ and θ.

Assuming that F is a function that is five times continuously differentiable, the following holds:

$$\mu(\varepsilon) = \mu_1\varepsilon^2 + O(\varepsilon^4) \tag{1.18}$$

$$T(\varepsilon) = \frac{2\pi}{\omega_0}[1 + \tau_1\varepsilon^2 + O(\varepsilon^4)] \tag{1.19}$$

If, for $\mu = 0$, all the remaining characteristic roots exhibit strictly negative real parts, other than $\mp i\omega$, then $p(t,\varepsilon)$ will be asymptotically stable when $\mu_1 > 0$ and unstable when $\mu_1 < 0$.

1.8.4 Sensitivity Analysis of State Variables with respect to Model Parameters

Systematic evaluation of the effects of model parameters on system solutions is called sensitivity analysis. There are a number of methods to do a sensitivity analysis of systems without delay, but there are only a few methods for sensitivity analysis of systems involving delays. The knowledge of how a small change in model parameters can bring change in the state variable can be a great help in the modelling process. It helps in the elimination of ineffective and irrelevant parameters. It gives a complete insight into the overall behaviour of the proposed model.

If all the parameters in the given system (1.11)- (1.12) are considered constants, the sensitivity analysis involves the calculation of partial derivatives of the solution with respect to parameters [143]. The following is the form of a matrix of sensitivity functions:

$$S(t) \equiv S(t,\alpha) = \left[\frac{\partial}{\partial \alpha}\right]^T x(t,\alpha) \tag{1.20}$$

Its jth column is:

$$S_j(t,\alpha) = \left[\frac{\partial x_j(t,\alpha)}{\partial \alpha_1}, \frac{\partial x_j(t,\alpha)}{\partial \alpha_2}, \dots, \frac{\partial x_j(t,\alpha)}{\partial \alpha_n}\right]^T$$

This column vector gives the sensitivity of the solution $x_j(t,\alpha)$ for a small change in parameter $\alpha_i, i = 1,2,3,\dots,n$.

Theorem 1.7.4.1 $S(t)$ satisfies the delay differential equation:

$$S'(t) = J(t)S(t) + J_\tau(t)S(t-\tau) + B(t), t \geq 0 \tag{1.21}$$

where $J(t) = \frac{\partial}{\partial x} f(t,x,x_\tau),\ J_\tau(t) = \frac{\partial}{\partial x_\tau} f(t,x,x_\tau),\ B(t) = \frac{\partial}{\partial \alpha} f(t,x,x_\tau)$

1.9 Summary

This thesis consists of six chapters whose detail is as follows:

In Chapter-1, The general introduction of the subject matter is given. Some important concepts of plant physiology, the Allee effect, and Functional responses are mentioned. The descriptions of all the necessary mathematical concepts are given, which have also made it possible to conduct analytical and numerical analysis of all the mathematical models that have been proposed. Additionally, a thorough analysis of earlier research in the area of plant-herbivore dynamics with the Allee effect is included in the form of a literature review.

In Chapter-2, A mathematical model is utilized to examine the dynamics of the interaction between plants and herbivores, considering the Allee effect to analyze the influence of delay parameter on both populations. It is demonstrated that solutions are positive and bounded. The non-trivial equilibrium point is determined for both, the strong and weak Allee effects. Under the strong Allee effect, it is determined that both populations experience extinction. The stability of the system is examined with respect to the interior equilibrium point under the influence of the weak Allee effect. The analysis demonstrates that when there is a negligible delay, the equilibrium point is proven to be absolutely stable, and when the time delay is lower than a threshold value, it becomes asymptotically stable. Additionally, the equilibrium point becomes unstable, leading to the occurrence of Hopf- bifurcation when the time delay exceeds the threshold value. The numerical examples have been presented by utilizing MATLAB code dde23 to justify the analytical results.

In Chapter-3, A mathematical model of delayed plant-herbivore consolidating the strong Allee effect with intraspecific competition among herbivores is analyzed. The Holling type- II functional response is utilized which describes that an individual predator consumes food till it reaches the saturating value of the prey consumption rate. The time delay (τ) is also considered in the model due to which the equilibrium loses stability. The non-trivial equilibrium point is determined for the strong Allee effect. The observation of Hopf- bifurcation is shown at the crucial value of the time delay. Furthermore, it is found that the entire system becomes more and more stable while the intraspecific competition rate of herbivores (δ) keeps on increasing as represented in time series graphs. MATLAB software is utilized to graphically present the analytical findings.

In Chapter-4, An eco-epidemiological model under the weak Allee effect accompanied by Holling type- I functional response is studied. The mathematical model is composed of susceptible plants, infectious plants, and herbivores. It is assumed that herbivores consume only infectious plants and infectious plants are generated by the infection of susceptible plants. The use of delay differential equations plays a vital role to make the model more realistic. Furthermore, stability at the non-zero equilibrium is carried out. The behaviour of the system becomes unstable and exhibits asymptotic stability when the value of time lag is below the threshold point and the system undergoes Hopf-bifurcation when the value of time lag crosses the threshold point. Sensitivity analysis of the state variables with respect to the herbivores gain by consuming infectious plants and the mortality rate of herbivores is represented graphically. Numerical examples are shown with the help of MATLAB software.

In Chapter-5, A modified Leslie-Gower plant-herbivore model is studied under the Allee effect. The model is modified by incorporating Holling type- II functional response and delay differential equations. The non-trivial equilibrium of the proposed model is calculated. Moreover, the stability and instability of the state variables are described graphically. It is shown that the system represents absolute stability when the system has no time parameter (τ). At a specific threshold value of the time parameter, Hopf bifurcation is observed, leading to the emergence of periodic solutions. The time series graphs are also represented to demonstrate that the system becomes more stable with the maximum rate of predation. MATLAB software is used to perform the graphs to justify the theoretical results.

In Chapter-6, A delayed Gauss-type Plant-herbivore model is proposed with Holling type-III functional response which describes the matter of fact that the predation on plant population by herbivores becomes more intensive when there is an increment in plant population. The Allee effect influences the growth of plants. The insertion of the time parameter (τ) is demonstrated to change the behaviour of the system which leads to introduce complex dynamics with limit cycles, periodic solutions, and the bifurcation occurrence. By considering the time parameter as a bifurcation parameter, the stability analysis of the feasible non-trivial equilibrium is investigated. The change of behaviour of the system from being absolutely stable to asymptotically stable and then finally Hopf-bifurcation at equilibrium for distinct values of time parameter (τ) is represented graphically. The change in system behaviour triggered by changes in the model parameters that differ from the time parameter is depicted graphically. MATLAB software is used to perform numerical simulations.

In the end, a bibliography is included to justify the problems that were encountered during the study of this thesis.

Chapter 2

Impact of Delay parameter on Plant-Herbivore dynamics under Allee effect

2.1 Introduction

Ecology is the branch of science that studies the relationships between living things and their environments. Plant ecology is a branch of ecology that concentrates on the study of the dispersal, abundance, and interactions of plants with both the biotic and abiotic components of the environment. The plant-herbivore dynamics is one of the most fundamental processes in ecology. It is often characterized by non-linear and complex relationships that accelerate thresholds by which slight and weak changes in processes operate at smaller scales and can effectively lead to significant and sudden changes at larger scales [19]–[21]. The interaction between plants and herbivores can be influenced by factors except population sizes or biomass. The predator-prey system is the classical approach to modelling the plant-herbivore system [22], [24], [37]. Lotka [38] and Volterra [39] formulated the first differential equation model of the predator-prey type, known as the Lotka-Volterra equation during the 1920s when the discovery of ecological laws of nature was undertaken. The impact of herbivores on plants leads to changes in the physical, chemical, and nutritional composition of the leaves [10]. Individual herbivores' growth and reproduction are negatively affected by these inducible changes; for example, see [8], [9]. These inducible changes are often termed inducible defenses in plant quality, and these inducible defenses can frequently have a significant influence on herbivore population dynamics [10]. Some herbivore populations experience population crashes as a result of inducible defenses [11], [12] and cyclic fluctuations in others [13], [14]. Numerous models supporting these arguments describe interactions between predator and prey, assuming that the variations in the predator abundance are caused by variations in the prey abundance [15], [16]. O' Connor et al. [6] proposed five different consumer-prey models. They discovered that environmental factors affect herbivores and plant abundances in response to temperature. Delay differential equations may frequently occur in plant-herbivore interactions due to damage to herbivores and the deployment of inducible defenses [24], [37], [148]. Sun et al. [149] investigated how the herbivore cycles are effected by time lags. Allee [48] introduced the concept of the Allee effect, which defines a positive association between individual fitness components and density of a population. There are two classifications of Allee effects: Strong Allee effect and Weak Allee effect. A strong Allee effect occurs when per individual growth rate becomes negative and there is a low population density. In contrast, a weak Allee effect occurs when there is a positive per individual population growth rate is observed and its density is zero [67]. Lin et al. [85] examined the behaviours of a predator-prey model incorporating a non-monotonic functional response and weak Allee effects specifically on the prey population. Asfaw et al. [27] considered a stochastic extension of a model in their research. They reported that noise adding to the plant population caused less change in the dynamics than noise adding to the herbivore population.

Although a lot of research has been conducted on plant-herbivore dynamics, the utilization of delay differential equations is not commonly employed in this field. Considering this fact, a mathematical model is presented to examine the dynamics of the plants and herbivores under the Allee effect by introducing a delay parameter.

2.2 Mathematical Model

The dynamics of Plant population and Herbivores is governed by the following equations:

$$\frac{\mathrm{d}P}{\mathrm{d}T} = rP\left(1-\frac{P}{K}\right)(P-a) - bPH \tag{2.1}$$

$$\frac{\mathrm{d}H}{\mathrm{d}T} = cP(t-\tau)H - lH \tag{2.2}$$

With initial conditions $P(0) > 0, H > 0 \; for \; all \; T \geq 0$, and $P(t-\tau) = \varepsilon,$ Contant for all $t \in [0, \tau]$.

Also, $P \leq K,$ and all the parameters $r, K, b, c, and\ l$ are taken as positive constants. Furthermore, $a > 0$ for strong Allee effect and $a \leq 0$ for weak Allee effect.

Variables/Parameters	Description
$\boldsymbol{P}(\boldsymbol{T})$	Plant population
$\boldsymbol{H}(\boldsymbol{T})$	Herbivores
$\boldsymbol{r}$	Plant intrinsic growth rate
$\boldsymbol{K}$	Carrying capacity of the environment
$\boldsymbol{a}$	Allee threshold
$\boldsymbol{b}$	Capturing rate/Harvesting rate
$\boldsymbol{c} = \boldsymbol{eb}$	Conversion rate of plants into herbivores
$\boldsymbol{l}$	Death rate of Herbivores
$\boldsymbol{\tau}$	Delay Parameter

Table 2.1 Description of the parameters/variables of the system.

2.3 Non-dimensionalization

2.3.1 For Strong Allee effect

For non- dimensionalization, let $x = \frac{P}{K},\ y = H, t = KrT$. The re-scaled system becomes:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x)(x-\alpha) - \beta xy \tag{2.3}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \gamma x(t-\tau)y - \delta y \tag{2.4}$$

where $\beta = \frac{b}{Kr}, \ \alpha = \frac{a}{K}, \ \gamma = \frac{cb}{r} \ and \ \delta = \frac{l}{Kr}$

2.3.2 For Weak Allee effect

When considering the weak Allee effect, the system of equations (2.3)-(2.4) becomes:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x)(x+\alpha) - \beta xy \tag{2.5}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \gamma x(t-\tau)y - \delta y \tag{2.6}$$

2.4 Boundedness of Solutions

Theorem 2.4.1 Solutions of the model (2.3)-(2.4) are uniformly bounded in R_+^2.

Proof: Consider,

$$\chi = x + \frac{\beta}{\gamma-\delta+\eta} y.$$

$$\Rightarrow \frac{\mathrm{d}\chi}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\beta}{\gamma-\delta+\eta}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$= -x^3 + (1+\alpha)x^2 - \alpha x - \beta xy + \frac{\beta}{\gamma-\delta+\eta}(\gamma xy - \delta y)$$

Now for each $\eta > 0 \ and \ 0 \leq x \leq 1$, we have

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} + \eta\chi = -x^3 + (1+\alpha)x^2 - \alpha x - \beta xy + \frac{\beta}{\gamma-\delta+\eta}(\gamma xy - \delta y) + \eta x + \frac{\eta\beta}{\gamma-\delta+\eta} y$$

$$= -x^3 + (1+\alpha)x^2 - \alpha x + \eta x - \beta xy + \frac{\beta\gamma}{\gamma-\delta+\eta} xy - \frac{\beta\delta}{\gamma-\delta+\eta} y + \frac{\eta\beta}{\gamma-\delta+\eta} y$$

$$\leq -x^3 + (1+\alpha)x^2 - \alpha x + \eta x - \beta y + \frac{\beta\gamma}{\gamma-\delta+\eta} y - \frac{\beta\delta}{\gamma-\delta+\eta} y + \frac{\eta\beta}{\gamma-\delta+\eta} y$$

$$\leq -x^3 + (1+\alpha)x^2 - \alpha x + \eta x$$

$$\leq (1+\alpha)x^2 - \alpha x + \eta x$$

$$\leq 1 + \eta$$

Therefore, it is possible to determine a positive value ω such that

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} + \eta\chi = \omega.$$

To summarize, we have the following:

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} \leq -\eta\chi + \omega,$$

This implies that

$$\chi(t) \leq \mathrm{e}^{-\eta t}\chi(0) + \frac{\omega}{\eta}(1 - \mathrm{e}^{-\eta t})$$

$$\leq \max\left(\chi(0), \frac{\omega}{\eta}\right)$$

Moreover, we have lim sup $\chi(t) \leq \frac{\omega}{\eta} < M$ as $t \to \infty$, and it is not related to the initial conditions.

2.5 Positivity of Solutions

From equation (2.3):

$$\frac{\mathrm{d}x}{\mathrm{d}t} \geq -x^3 - \alpha x - \beta xy$$

$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} \geq -x(x^2 + \alpha + \beta y)$$

$$\Rightarrow x \geq \mathrm{e}^{-(\alpha+(1+\eta))t}$$

From equation (2.4):

$$\frac{\mathrm{d}y}{\mathrm{d}t} \geq -\delta y$$

$$\Rightarrow y \geq \mathrm{e}^{-\delta t}$$

So, $x \geq 0, y \geq 0 \ \forall\ t > 0$.

As a result, the solution set of the proposed system maintains positivity for all time t, ensuring the persistence of the system.

2.6 Equilibrium point of the model

2.6.1 For Strong Allee effect

At non-trivial equilibrium $E^*(x^* \neq 0, y^* \neq 0)$, $x^*(t-\tau) \approx x^*(t)$

From equation (2.4): $\frac{dy}{\mathrm{d}t} = 0$

$\Rightarrow x^* = \frac{\delta}{\gamma}$

From equation (2.3): $\frac{dx}{\mathrm{d}t} = 0$

$\Rightarrow y^* = \frac{-\delta^2+(1+\alpha)\gamma\delta-\alpha\gamma^2}{\beta\gamma^2}$; provided $(1+\alpha)\gamma\delta > \alpha\gamma^2 + \delta^2$

Therefore, the non-trivial equilibrium is expressed as $E^*\left(\frac{\delta}{\gamma}, \frac{-\delta^2+(1+\alpha)\gamma\delta-\alpha\gamma^2}{\beta\gamma^2}\right)$.

2.6.2 For Weak Allee effect

At non-trivial equilibrium $E^*(x^* \neq 0, y^* \neq 0)$, $x^*(t-\tau) \approx x^*(t)$

From equation (2.4): $\frac{dy}{\mathrm{d}t} = 0$

$\Rightarrow x^* = \frac{\delta}{\gamma}$

From equation (2.3): $\frac{dx}{\mathrm{d}t} = 0$

$\Rightarrow y^* = \frac{-\delta^2+(1+\alpha)\gamma\delta-\alpha\gamma^2}{\beta\gamma^2}$; provided $(1+\alpha)\gamma\delta > \alpha\gamma^2 + \delta^2$

Therefore, the non-trivial equilibrium is expressed as $E^*\left(\frac{\delta}{\gamma}, \frac{-\delta^2+(1+\alpha)\gamma\delta-\alpha\gamma^2}{\beta\gamma^2}\right)$

2.7 Stability Analysis and Hopf-Bifurcation

The equations (2.5)-(2.6) about the equilibrium point $E^*(x^*, y^*)$ becomes:

$$\frac{\mathrm{d}x^*}{\mathrm{d}t} = x^*(1-x^*)(x^*+\alpha) - \beta x^* y^* \tag{2.7}$$

$$\frac{\mathrm{d}y^*}{\mathrm{d}t} = \gamma x^*(t-\tau)y^* - \delta y^* \tag{2.8}$$

The characteristic equation of the system of equations (2.7)-(2.8) is:

$$(\lambda^2 + p\lambda + q) + \mathrm{e}^{-\lambda\tau}s = 0 \tag{2.9}$$

where $p = \left(\alpha - 2x^* + 3{x^*}^2 - 2\alpha x^* + \beta y^* + \delta\right)$,

$$q = \delta\left(\alpha - 2x^* + 3{x^*}^2 - 2\alpha x^* + \beta y^*\right),$$
$$s = (-\beta\gamma x^* y^*)$$

When $\tau = 0$, the equation (2.9) becomes:

$$\lambda^2 + p\lambda + (q + s) = 0 \tag{2.10}$$

According to the Routh-Hurwitz criteria, the roots of the equation (2.10) will have a negative real part, indicating that the system is stable if:

$$(\boldsymbol{R_1}): p > 0;$$
$$(\boldsymbol{R_2}): (q + s) > 0$$

That clearly holds.

Next, we aim to examine how the negative real parts of the roots shift to positive real parts as the values of τ vary.

Assume that $\lambda = \mathrm{i}\omega$ is a root of equation (2.9). Consequently, equation (2.9) can be expressed as:

$$(\mathrm{i}\omega)^2 + p(\mathrm{i}\omega) + q + s\mathrm{e}^{-(\mathrm{i}\omega)\tau} = 0$$
$$\Rightarrow -\omega^2 + p(\mathrm{i}\omega) + q + \ s(\cos\omega\tau - \mathrm{i}\sin\omega\tau\,) = 0$$

Separating real parts from imaginary:

$$\omega^2 - q = s\cos\omega\tau \tag{2.11}$$
$$p\omega = s\sin\omega\tau \tag{2.12}$$

Squaring and adding:

$$\omega^4 + (p^2 - 2q)\omega^2 + (p^2 - s^2) = 0 \tag{2.13}$$

The two roots of equation (2.13) are:

$$\omega_{1,2}^2 = \frac{(2q - p^2) \pm [(p^2 - 2q)^2 - 4(p^2 - s^2)]^{1/2}}{2} \tag{2.14}$$

None of the two roots $\omega_{1,2}^2$ is positive if:

$$(\boldsymbol{R_3}): (2q - p^2) < 0 \ and\ (p^2 - s^2) > 0 \ or\ (p^2 - 2q) < 4(p^2 - s^2)$$

This implies that equation (2.14) will not have positive roots if the condition (R_3) is satisfied. We can now introduce the following lemma (Ruan [166]).

Lemma 2.7.1 For any $\tau \geq 0$, if the conditions $(R_1) - (R_2)$ are satisfied, then all the roots of equation (2.9) possess negative real parts.

Conversely, if:

$(\boldsymbol{R_4})$: $(p^2 - s^2) < 0\ or\ (2q - p^2) > 0\ and\ (p^2 - 2q)^2 = 4(p^2 - s^2)$

In that case, the $+ve$ root of equation (2.11) corresponds to ω_1^2.

Correspondingly, if:

$(\boldsymbol{R_5})$: $(p^2 - s^2) > 0\ or\ (2q - p^2) > 0\ and\ (p^2 - 2q)^2 > 4(p^2 - s^2)$

Therefore, the equation (2.11) will have two $+ve$ roots, known as $\omega_{1,2}^2$.

Both conditions $(R_4)$and (R_5) lead to the equation (2.9) having purely imaginary roots for specific values of τ. The critical values of τ, denoted as $\tau_j^{\pm}$, can be determined by solving the system of equations (2.11)-(2.12), as follows:

$$\tau_j^{\pm} = \frac{1}{\omega_{1,2}} \cos^{-1}\left(\frac{\omega_{1,2}^2 - q}{s}\right) + \frac{2j\pi}{\omega_{1,2}}, j = 0,1,2, \dots \qquad (2.15)$$

The aforementioned discussion can be succinctly summarized in the subsequent lemma (Ruan [166]).

Lemma 2.7.2 (A) When conditions $(R_1) - (R_2)\ and\ (R_4)$ are satisfied, and $\tau = \tau_j^{+}$, equation (2.9) has two roots that are purely imaginary, specifically $\pm i\omega_1$.

(B) When conditions $(R_1) - (R_2)\ and\ (R_5)$ are satisfied, and $\tau = \tau_j^{+}\big(\tau = \tau_j^{-}\ respectively\big)$, equation (2.9) will exhibit two roots that are purely imaginary, denoted as $\pm \mathrm{i}\omega_1(\pm \mathrm{i}\omega_2\ respectively)$.

We aim that the negative real part of certain roots of equation (2.9) will shift to the positive real part when $\tau > \tau_j^{+}$ and $\tau < \tau_j^{-}$. In order to investigate this scenario, let us define:

$$\tau_j^{\pm} = \mu_j^{\pm}(\tau) + \mathrm{i}\omega_j^{\pm}(\tau); j = 0,1,2,3, \dots$$

The characteristics of the roots of equation (2.9) can be described using the following conditions:

$$\mu_j^{\pm}\big(\tau_j^{\pm}\big) = 0, \omega_j^{\pm}\big(\tau_j^{\pm}\big) = \omega_{1,2}$$

It can be confirmed that the following transversality condition is valid:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(Re\ \lambda_j^{+}\big(\tau_j^{+}\big)\Big) > 0\ and\ \frac{d}{d\tau}\Big(Re\ \lambda_j^{-}\big(\tau_j^{-}\big)\Big) < 0$$

This implies that $\tau_j^{\pm}$ represent bifurcation values (Ruan [166]).

2.8 Numerical Example

The dynamics characterized by the system of equations (2.1)-(2.2) are represented by considering the following set of parameter values:

$$\alpha = 0.2, \beta = 0.5, \gamma = 0.36, \delta = 0.2$$

2.8.1 For Strong Allee effect

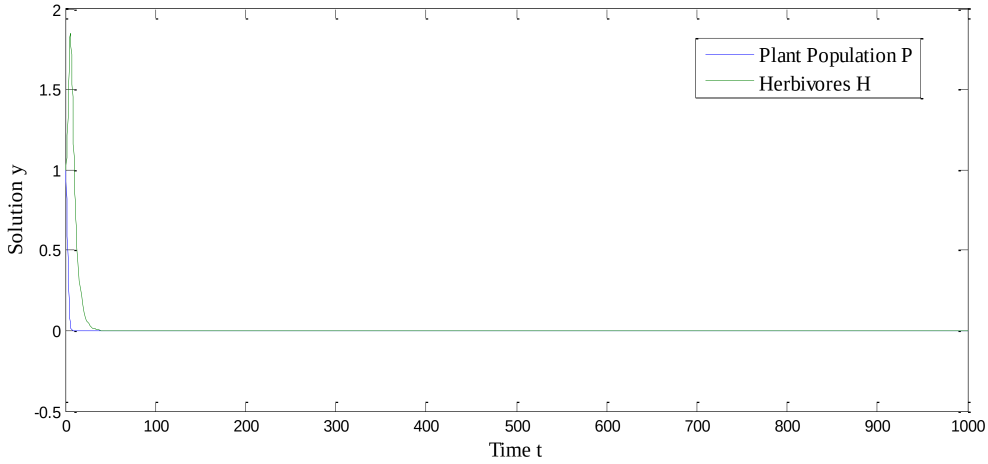

Figure 2.1 The graph shows the perishing of both plant population and herbivores under the strong Allee effect.

2.8.2 For Weak Allee effect

The variation of the system of equations (2.1)-(2.2) from stable behaviour to complex dynamics around the equilibrium point $E^*(x^*, y^*)$ under the influence of the weak Allee effect is illustrated below for different values of τ and parameters:

$$\alpha = -0.2, \beta = 0.5, \gamma = 0.36, \delta = 0.2$$

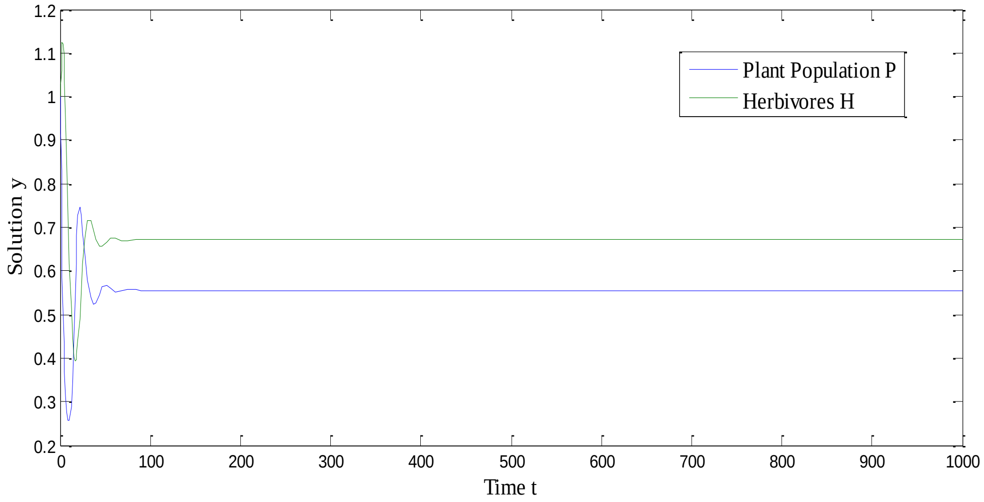

Figure 2.2 Equilibrium $E^*(0.5555,0.6717)$ is absolutely stable, when the delay is not present i.e., $\tau = 0$.

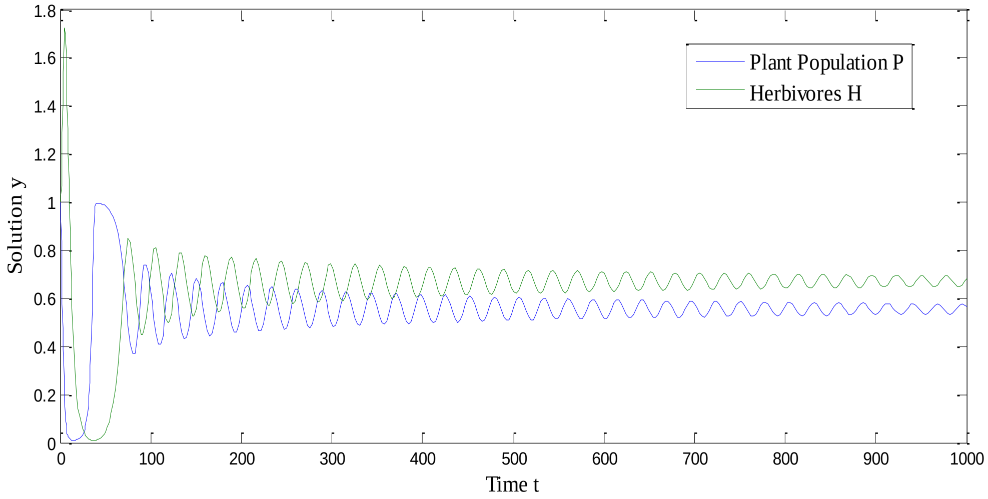

Figure 2.3 Equilibrium $E^*(0.5555,0.6716)$ is asymptotically stable, when delay is lesser than the critical value i.e., $\tau < 3.25$.

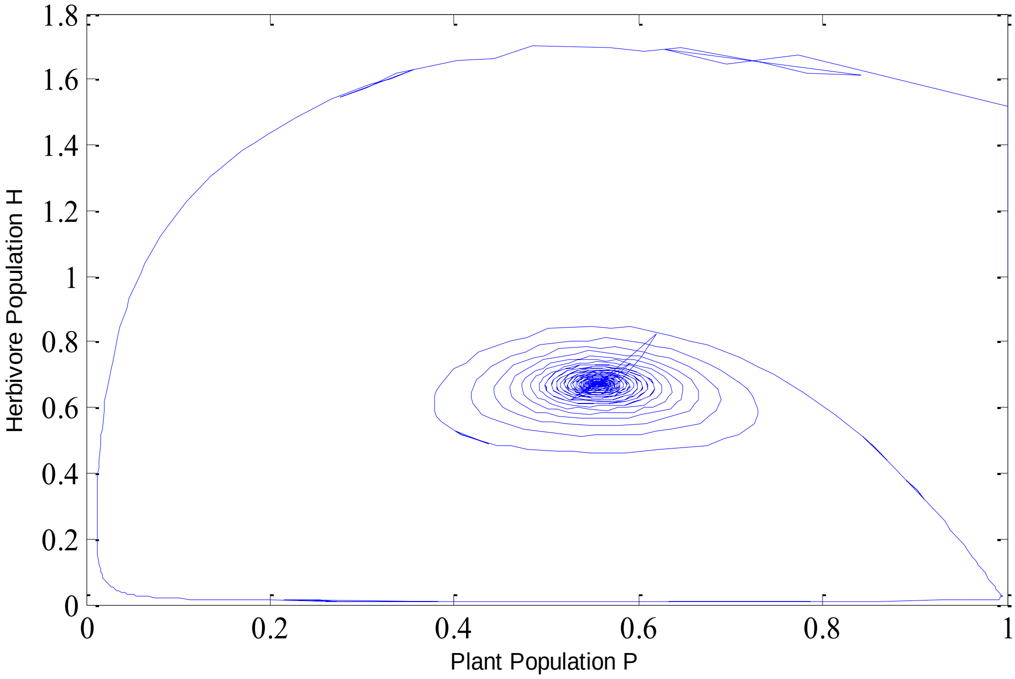

Figure 2.4 Phase space diagram showing the asymptotic stability of equilibrium $E^{*}(0.5555,0.6716)$, when delay is lesser than the critical value i.e., $\tau < 3.25$.

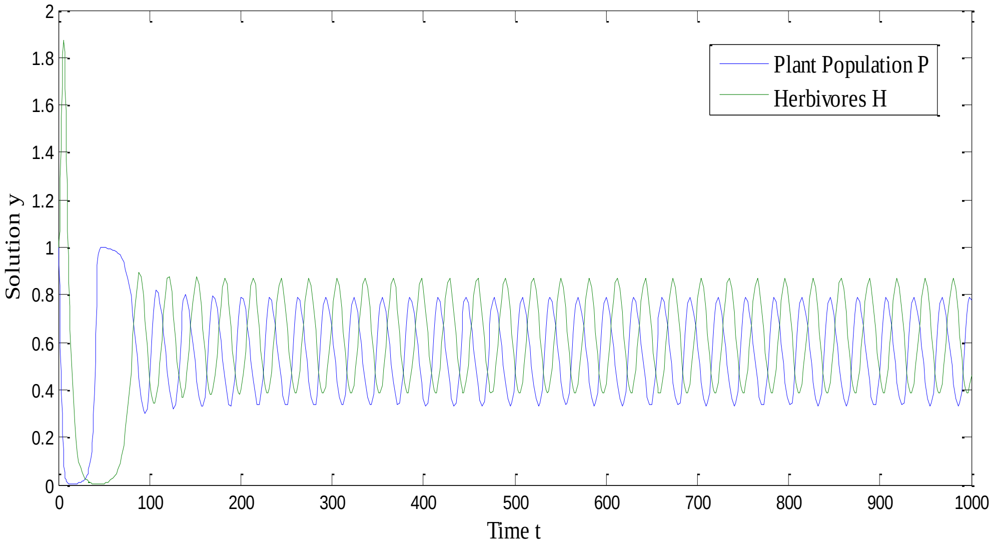

Figure 2.5 Equilibrium $E^{*}(0.7701,0.3894)$ becomes unstable and Hopf- bifurcation occurs when delay surpasses the critical value i.e., $\tau \geq 3.25$.

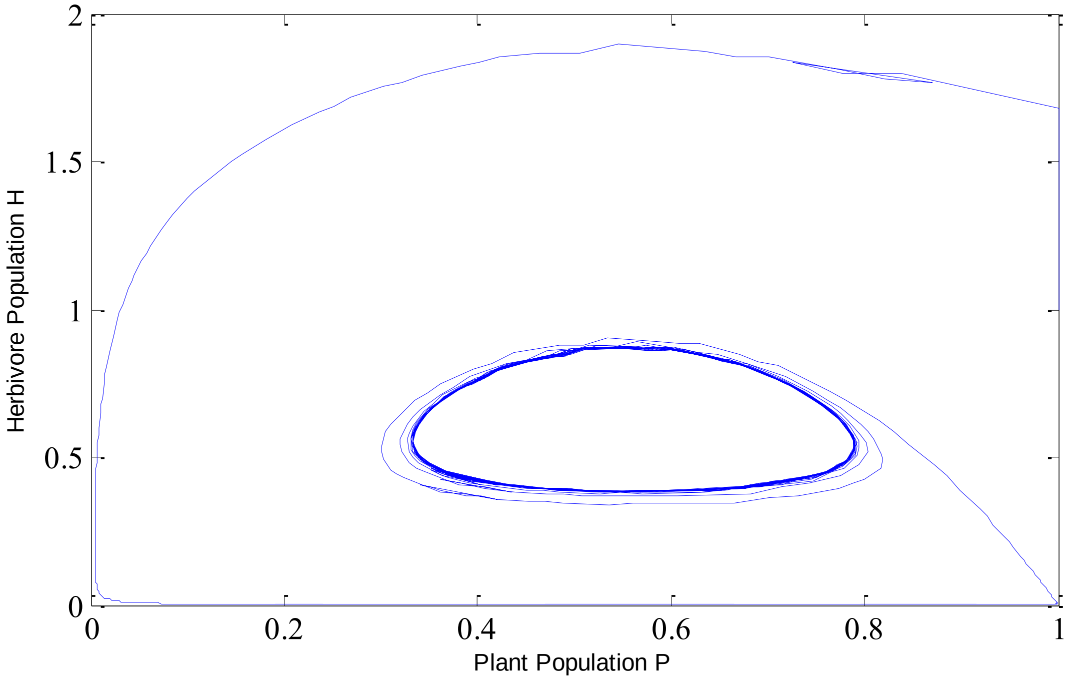

Figure 2.6 Phase space diagram showing the Hopf Bifurcation of equilibrium $E^*(0.7701,0.3894)$ when delay surpasses the critical value i.e., $\tau \geq 3.25$.

2.9 Conclusion

The impact of the delay parameter on plant population and Herbivores under the Allee effect is analyzed with the proposed model. The boundedness and positivity of the system are proved. The feasible interior equilibrium points E^* for both Allee effects is calculated. The study revealed that in the case of a strong Allee effect, both populations experience extinction, as depicted in Figure 2.1. The stability of the system about the non-trivial equilibrium E^* is examined under the influence of the weak Allee effect. It is demonstrated that when the delay is not present ($\tau = 0$), the equilibrium point E^* is absolutely stable as represented in figure 2.2. Furthermore, when the delay is lesser than a critical value ($\tau < 3.25$), the equilibrium point E^* becomes asymptotically stable, as illustrated in figure 2.3 and figure 2.4. However, when the delay surpasses the critical value ($\tau \geq 3.25$), the stability of the equilibrium point E^* becomes unstable, leading to the observation of Hopf bifurcation, as depicted in figure 2.5 and figure 2.6.

Chapter 3

A delayed Plant-Herbivore model with a Strong Allee effect in Plant population

3.1 Introduction

The domain of applied mathematics considers mathematical modelling for diverse biological issues to be an interesting research area. In the biological field, predator-prey systems that can exhibit interactions between prey-predator species play a vital role. In 1920, the theory of predator-prey mathematical modelling was first developed. Lotka [38], from the United States, and Volterra [39], were the first to develop the predator-prey model. Plant-herbivore dynamics are frequently depicted by modified versions of the Lotka-Volterra predator-prey model [45]–[47]. Warder Allee invented the Allee effect in the 1930s [48]. If the growth rate at low density is positive in the first instance, then the population has an impact on the Allee effect [30], [49]. The Allee effect usually occurs due to the challenges in searching for mates, feeding habits, interaction with society, predator satiation, and cooperative defense, among other factors [27], [57]. The categories of the Allee effect are the Strong and Weak Allee effects. A strong Allee effect introduces a threshold value of the population size and it has a negative per individual growth rate which is lower than the threshold value. However, if there is a positive population growth rate but decreases when the population densities are low, then it is categorized as the weak Allee effect [65], [68], [76]. The behaviour of the plant-herbivores under both Allee effects with Holling-type I functional response is proposed. Under the strong Allee effect, the plants and herbivores population vanishes. Under the weak Allee effect, Hopf-bifurcation is seen [59]. In 1959, Holling introduced the theory of functional response. A functional response is an immediate change in the rate at which it consumes prey in response to prey abundance [26], [88], [89]. $P(x) = kx$ gives the Holling type-I response, which includes $x(t)$ as the prey biomass and k as positive constant. A saturation is applicable for explaining the food consumption because Holling type-II response incorporates that an individual predator consumes food till it feels famished. Therefore, the functional response is: $P(x) = kx/(1 + T_h kx)$; here $x(t)$ is the prey biomass and k is searching efficiency which is always positive and T_h is average time for each prey [57]. The occurrence of delay differential equations in the plant–herbivore dynamics is a result of herbivore damage [24], [148], [149]. Using a system of non-linear delay differential equations, the impact of time lag on the extinction of toxicant-affected plant population has been investigated [124], [151], [152]. Under the inhibitory and stimulatory allelopathic effects, the aspect of delay parameter in the plant population has been examined [153], [167].

Many researchers have studied the plant-herbivore dynamics with or without Allee effect. However, the utilization of delay differential equations imposed by the Allee effect has been studied rarely. Therefore, in this article, Plant herbivore dynamics with Holling type-II response imposed by strong Allee effect is considered.

3.2 Mathematical Model

In this paper, the associate variables $P(t)$ and $H(t)$ stands for Plant population and Herbivore population respectively. The variables used in the model are positive i.e., $P(t) > 0$ and $H(t) > 0$ and $0 < b < C$ for strong Allee effect. The reproduction of herbivores does not occur immediately after consuming plants. There is an essential delay in gestation. Therefore, time delay (τ) has been introduced in the proposed model.

The initial conditions are given as: $P(0) > 0, H(0) > 0\ for\ all\ t \geq 0$, and $P(t-\tau) = \varepsilon$, Contant for all $t \in [0, \tau]$.

Also, $P \leq C$, and all the parameters $r, C, a_1, a_2, m, d\ and\ l$ are taken as positive constants.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{C}\right)(P - b) - \frac{a_1 PH}{P + m} \tag{3.1}$$

$$\frac{dH}{dt} = \frac{a_2 P(t-\tau)H}{P(t-\tau)+m} - dH - lH^2 \tag{3.2}$$

Parameters	Description
$\boldsymbol{r}$	The internal growth rate of plants
$\boldsymbol{C}$	Carrying capacity
$\boldsymbol{b}$	Survival threshold of the plants
$\boldsymbol{a_1}$	Capturing rate
$\boldsymbol{a_2 = (ea_1)}$	Conversion rate
$\boldsymbol{m}$	Half-capturing saturation constant
$\boldsymbol{d}$	The death rate of herbivores
$\boldsymbol{l}$	Intraspecific competition rate of herbivores
$\boldsymbol{\tau}$	Time delay

Table 3.1 Details of the system parameters (3.1)-(3.2).

3.3 Non-dimensionalization

To curtail the number of parameters, let

$$\frac{rt'}{b} = t\,, \frac{P}{C} = x'\,, \frac{a_{1H}}{a_{2C}} = y'$$

and then dropping the dashes for simplicity, the system is simplified as follows:

$$\frac{dx}{dt} = x(1-x)\left(1-\frac{x}{M}\right) - \frac{\alpha xy}{x+\gamma} \tag{3.3}$$

$$\frac{dy}{dt} = \frac{\alpha x(t-\tau)y}{x(t-\tau)+\gamma} - \beta y - \delta y^2 \tag{3.4}$$

where the dimensionless parameters are

$$M = \frac{b}{c}\,, = \frac{m}{c}\,, \alpha = \frac{a_2 b}{r}\,, \beta = \frac{db}{r} \text{ and } \delta = \frac{la_2}{rba_1}.$$

3.4 Equilibrium point of the model

This section accommodates the non-trivial equilibrium which depicts the co-existence of the state variables (P^*, H^*), where these state variables are not equal to zero. The stability of the model represents that there is no influence of delay on any of the other equilibrium points, therefore only non-trivial equilibrium is calculated. To calculate the feasible non-trivial equilibrium, equate the system of equations (3.1) and (3.2) to zero at $E^*(x^* \neq 0, y^* \neq 0)$, $x^*(t-\tau) \approx x^*(t)$:

$$\frac{dx}{dt} = 0 \Rightarrow y^* = \frac{(1-x^*)(M-x^*)(x^*+\gamma)}{\alpha M}, \text{ provided } 1 > x^*, M > x^*$$

In this scenario, there exists a positive root x^* to the quartic equation.

$$\text{And } \frac{dy}{dt} = 0$$

$$\Rightarrow {x^*}^4 + k_1 {x^*}^3 + k_2 {x^*}^2 + k_3 x^* + k_4 = 0$$

where $k_1 = 2\gamma + M - 1$, provided $2\gamma + M > 1$;

$$k_2 = \gamma^2 - 2\gamma - 2\gamma M + M, \text{ provided } M < \frac{\gamma^2 - 2\gamma}{2\gamma - 1};$$

$$k_3 = \frac{M[\alpha(\alpha-\beta)+\gamma\delta(2-\gamma)]-\delta\gamma^2}{\delta},$$

provided $\alpha > \beta, \gamma < 2 \; and \; M[\alpha(\alpha-\beta)+\gamma\delta(2-\gamma)] > \delta\gamma^2$;

and $\quad k_4 = \frac{\gamma M(\delta\gamma - \alpha\beta)}{\delta}$, provided $\gamma > \frac{\alpha}{\delta}\beta$

3.5 Stability Analysis and Hopf-Bifurcation

The set of equations (3.3)-(3.4) of a delayed plant-herbivore model under the strong Allee effect about $E^*(x^*, y^*)$ are equivalent to:

$$\frac{dx^*}{dt} = x^*(1-x^*)\left(1-\frac{x^*}{M}\right) - \frac{\alpha x^* y^*}{x^*+\gamma} \tag{3.5}$$

$$\frac{dy^*}{dt} = \frac{\alpha x^*(t-\tau) y^*}{x^*(t-\tau)+\gamma} - \beta y^* - \delta {y^*}^2 \tag{3.6}$$

The exponential equation of the above system is

$$(\lambda^2 + l\lambda + m) + e^{-\lambda\tau} n = 0 \tag{3.7}$$

$$\text{where} \quad l = \left[\frac{\alpha\gamma y^*}{(x^*+\gamma)^2} - \frac{x^*(3x^*-2M-2)}{M} + 2\delta y^* + \beta - 1\right];$$

$$m = (\beta + 2\delta y^*)\left[\frac{\alpha\gamma y^*}{(x^*+\gamma)^2} - \frac{x^*(3x^*-2M-2)}{M} - 1\right];$$

$$\text{And} \quad n = \frac{-\alpha^2 \gamma x^* y^*}{(x^*+\gamma)[x^*(t-\tau)+\gamma]^2}$$

If $\tau = 0$, then equation (3.7) becomes:

$$\lambda^2 + l\lambda + (m+n) = 0 \tag{3.8}$$

By the criteria [166], the system is stable, i.e., roots of equation (3.8) will have its real part as negative when:

$$(\boldsymbol{R_1}): l > 0;$$

$$(\boldsymbol{R_2}): (m+n) > 0$$

which clearly holds.

Further, we investigate the transference of the real part of the roots which is negative to the real part which is positive with alteration in the values of time delay (τ).

Consider, equation (3.7) has a root $\lambda = \mathrm{i}\omega$:

$$(\mathrm{i}\omega)^2 + l(\mathrm{i}\omega) + m + n\mathrm{e}^{-(\mathrm{i}\omega)\tau} = 0$$

$$\Rightarrow -\omega^2 + l(\mathrm{i}\omega) + m + \ n(\cos\omega\tau - \mathrm{i}\sin\omega\tau\,) = 0$$

Segregating real and imaginary parts:

$$\omega^2 - m = n\cos\omega\tau \tag{3.9}$$

$$l\omega = n\sin\omega\tau \tag{3.10}$$

Squaring and adding:

$$\omega^4 + (l^2 - 2m)\omega^2 + (l^2 - n^2) = 0 \tag{3.11}$$

The two roots of equation (3.11) are:

$$\omega_{1,2}^2 = \frac{(2m - l^2) \pm [(l^2 - 2m)^2 - 4(l^2 - n^2)]^{1/2}}{2} \tag{3.12}$$

The two roots $\omega_{1,2}^2$ are not positive if:

$$(\boldsymbol{R_3}): (2m - l^2) < 0 \; and \; (l^2 - n^2) > 0 \; or \; (l^2 - 2m) < 4(l^2 - n^2)$$

Therefore, if the condition (R_3) holds, then equation (3.12) does not have positive roots.

There are the following two lemmas [166].

Lemma 3.5.1 If $(R_1) - (R_2)$ is true, then each root of the equation (3.7) has real parts which are negative for all $\tau \geq 0$.

On the contrary, when:

$$(\boldsymbol{R_4}): (l^2 - n^2) < 0 \; or \; \ (2m - l^2) > 0 \; and \; (l^2 - 2m)^2 = 4(l^2 - n^2)$$

Then, the positive root of equation (3.9) is ω_1^2.

$$(\boldsymbol{R_5}): (l^2 - n^2) > 0 \; or \; (2m - l^2) > 0 \; and \; (l^2 - 2m)^2 > 4(l^2 - n^2)$$

Then, two positive roots of equation (3.9) are $\omega_{1,2}^2$.

For some specific values of τ, roots of equation (3.7) are purely imaginary in both- $(R_4)$and (R_5). The crucial values $\tau_j^{\pm}$ of τ can be enumerated from the set of equations (3.9)-(3.10):

$$\tau_j^{\pm} = \frac{1}{\omega_{1,2}} \cos^{-1}\left(\frac{\omega_{1,2}^2 - m}{n}\right) + \frac{2j\pi}{\omega_{1,2}}, j = 0,1,2,3,4, \dots \qquad (3.13)$$

The above preceding can be summarised in the following lemma [166].

Lemma 3.5.2 (A) There are purely imaginary roots $\pm i\omega_1$ of equation (3.7), when $(R_1) - (R_2) \; and \; (R_4)$ is true and $\tau = \tau_j^+$.

(B) There exists purely imaginary roots $\pm \mathrm{i}\omega_1(\pm \mathrm{i}\omega_2 \; respectively)$ of equation (3.7), when $(R_1) - (R_2) \; and \; (R_5)$ is true and $\tau = \tau_j^+ (\tau = \tau_j^- \; respectively)$.

The expectancy is that the displacement of the real part of some roots of equation (3.7) which are negative to the real part which is positive if $\tau > \tau_j^+$ and $\tau < \tau_j^-$. Consider the following to investigate the option:

$$\tau_j^{\pm} = \mu_j^{\pm}(\tau) + \mathrm{i}\omega_j^{\pm}(\tau); j = 0,1,2,3,4, \dots$$

$$\mu_j^{\pm}(\tau_j^{\pm}) = 0, \omega_j^{\pm}(\tau_j^{\pm}) = \omega_{1,2}$$

is satisfied by the roots of equation (3.7).

The following transversality conditions can be verified:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(Re \; \lambda_j^+(\tau_j^+)\right) > 0 \; and \; \frac{d}{d\tau}\left(Re \; \lambda_j^-(\tau_j^-)\right) < 0$$

Hence, the nature of the bifurcating values $\tau_j^{\pm}$ given by the above expression is discussed in detail by [166].

3.6 Numerical Example

The dynamics graphs are represented using equations (3.1)- (3.2), considering the given set of parametric values:

$$\alpha = 1.65, \gamma = 0.5, \beta = 0.8, \delta = 0.2, M = 4.$$

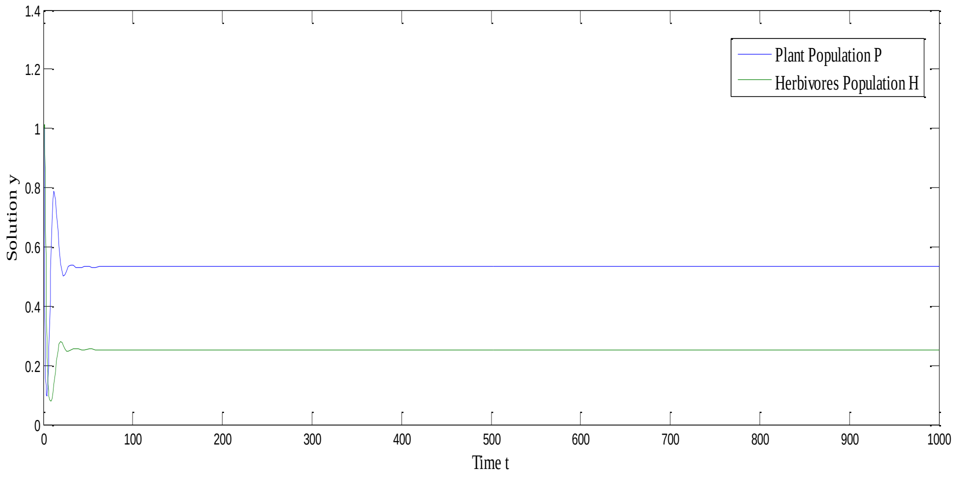

Figure 3.1 When the system is without time delay $(i.e., \tau = 0)$, then E* (0.5322, 0.2537) is absolutely stable.

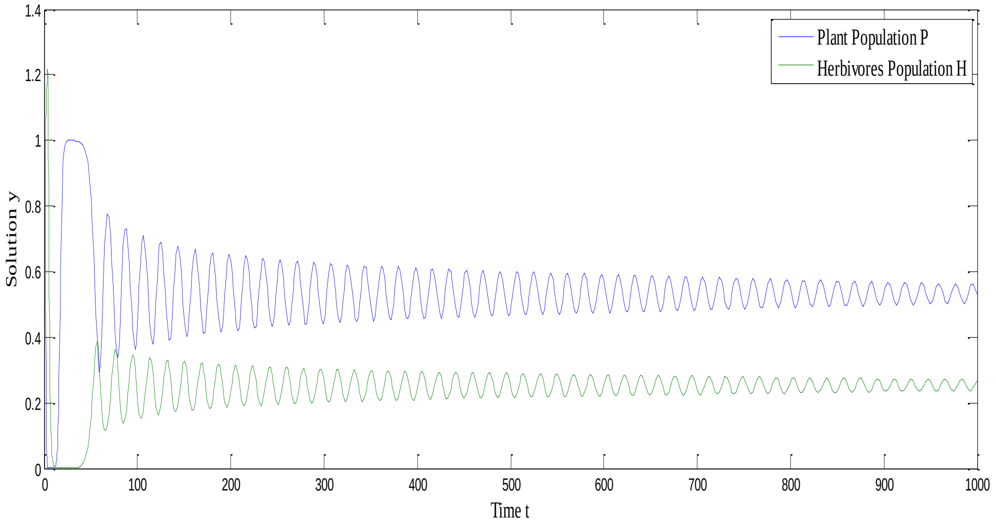

Figure 3.2 When the time parameter is low than the crucial value which is 2.7 $(i.e., \tau < 2.7)$, then E*(0.5520, 0.2553) is asymptotically stable.

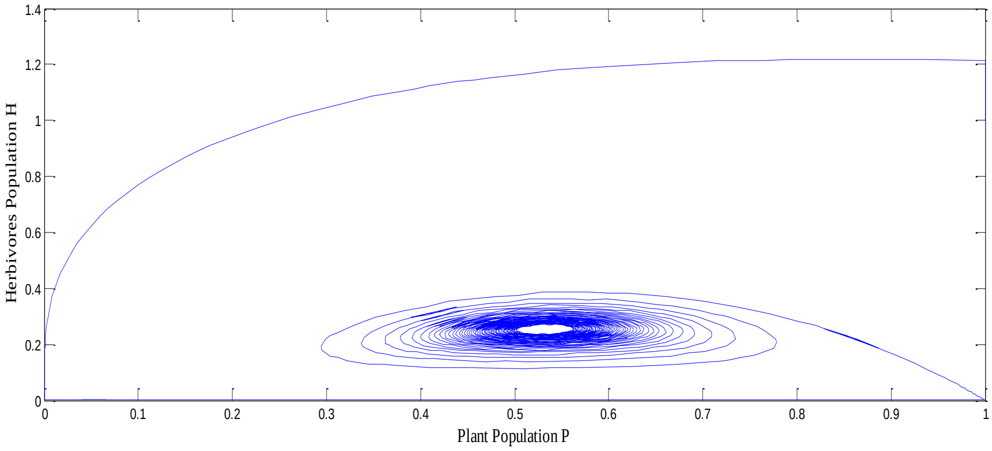

Figure 3.3 Phase space diagram showing the asymptotic stability of equilibrium E* (0.5520,0.2553), when the time parameter is low than the crucial value which is 2.7 $(i.e., \tau < 2.7)$.

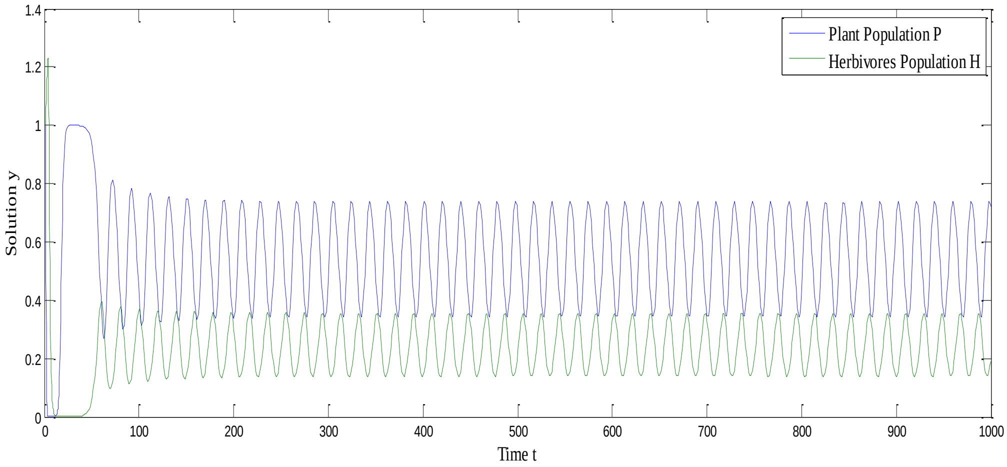

Figure 3.4 When the time delay surpasses the crucial value which is 2.7 $(i.e., \tau \geq 2.7)$, then E* (0.7276,0.1444) surrenders its stability, and limit cycles are seen, and hence Hopf- bifurcation.

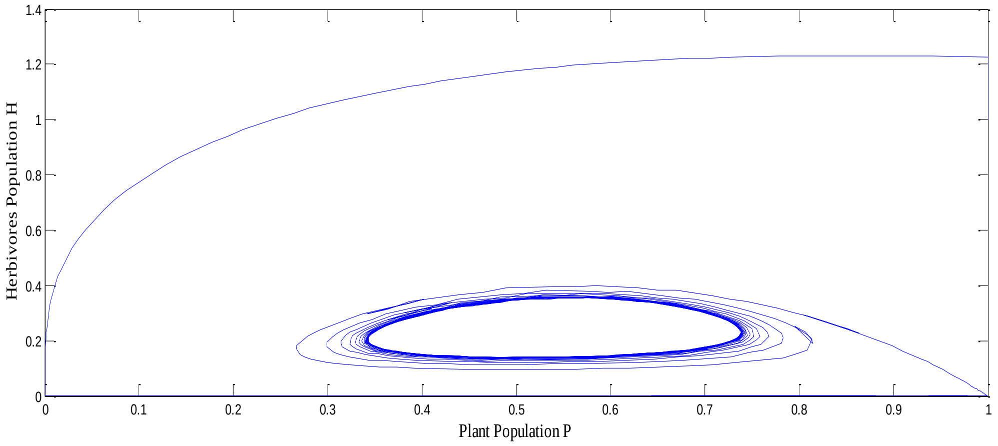

Figure 3.5 Phase space diagram showing the Hopf- bifurcation of equilibrium E* (0.7276, 0.1444) when the time delay surpasses the crucial value which is 2.7 $(i.e., \tau \geq 2.7)$.

3.7 Sensitivity Analysis

The research that represents the stability behaviour of the system while triggering the model parameters other than the time delay (τ) is known as Sensitivity analysis. The Direct method is used for this purpose. For an illustration, the sensitivity equations of the partial derivatives of the solution (P, H) with respect to the intraspecific competition rate of herbivores (δ) are given below:

$$\frac{dS_1}{dt} = \left[1 - 2\left(1 + \frac{1}{M}\right)x - \frac{3}{M}x^2 - \frac{\alpha xy}{(x+\gamma)^2}\right]S_1 - \frac{\alpha x}{x+\gamma}S_2 \tag{3.14}$$

$$\frac{dS_2}{dt} = \left[\frac{\alpha x(t-\tau)}{[x(t-\tau)+\gamma]} - \beta - 2\delta y\right]S_2 + \frac{\alpha xy}{[x(t-\tau)+\gamma]^2}S_1(t-\tau) \tag{3.15}$$

where $S_1 = \frac{\partial x}{\partial \delta}$ and $S_2 = \frac{\partial y}{\partial \delta}$

3.7.1 Sensitivity of Variables to Parameter δ

When we increase the value of the intraspecific competition rate of herbivores (δ) from $\delta = 0.2$ to $\delta = 0.23$, then the state variables (P, H) shift its behaviour from Hopf-bifurcation to asymptotic stability. Again, when we increase $\delta = 0.23$ to $\delta = 0.25$, eventually, as depicted in figures 3.6 and 3.7, the entire system shifts its dynamics from asymptotic stability to absolute stability.

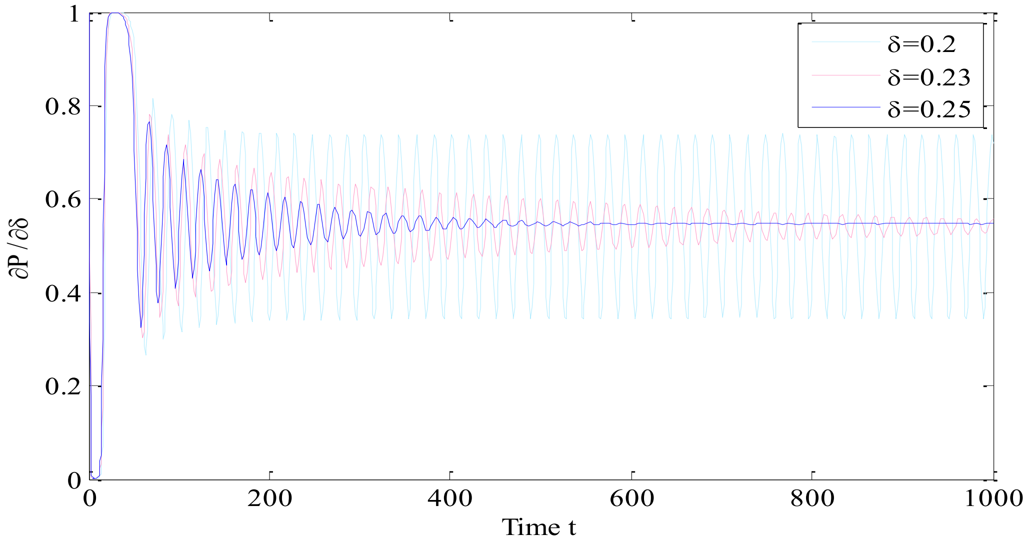

Figure 3.6 Time series graph of change in the size of plant population P with respect to changes in the intraspecific competition rate of herbivores (δ).

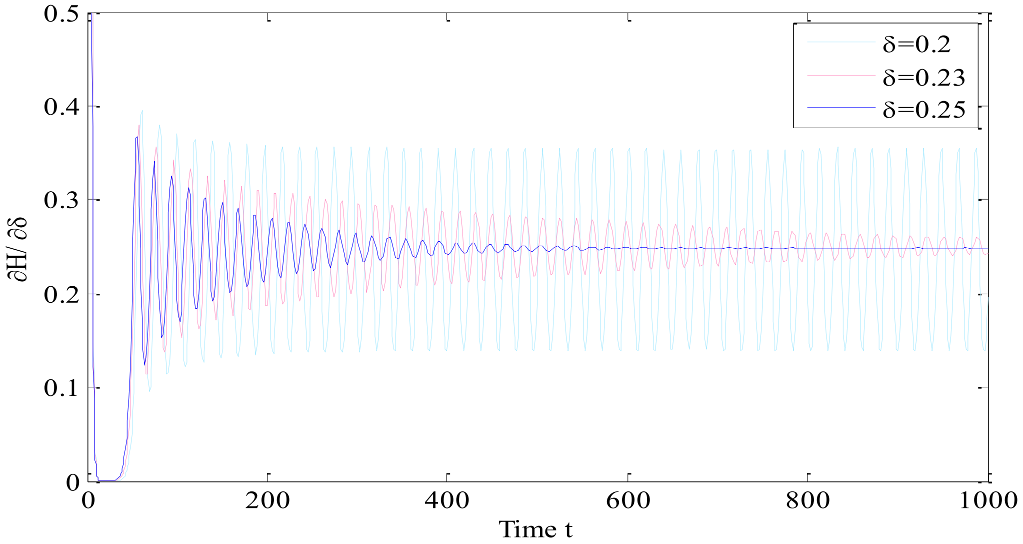

Figure 3.7 Time series graph of change in the size of herbivore population H with respect to changes in the intraspecific competition rate of herbivores (δ).

3.8 Conclusion

The proposed mathematical model represented the plant-herbivore dynamics using Holling Type-II functional response with intraspecific competition among herbivores. The strong Allee effect and time delay (τ) are incorporated into the model. The non-trivial equilibrium points E^* and stability of the model about

E^* incorporating the strong Allee effect are examined. Figure 3.1 indicates that $E^*(0.5322, 0.2537)$ is absolutely stable, without time delay ($\tau = 0$). When the time delay is below 2.7 ($\tau < 2.7$), where 2.7 is a crucial value and keeping all the other parameters the same, then $E^*(0.5520, 0.2553)$ is asymptotically stable which is shown in figure 3.2 and figure 3.3. When the time parameter surpasses the crucial value ($\tau \geq 2.7$), then $E^*(0.7276, 0.1444)$ surrenders its stability and Hopf- bifurcation occurred which is depicted in figure 3.4 and figure 3.5.

Additionally, figure 3.6 and figure 3.7 indicate the change in the model parameters with respect to changes in the intraspecific competition rate of herbivores (δ) using time series graphs. It is found that when there is an increment in the value of the intraspecific competition rate of herbivores (δ) from $\delta = 0.2$ to $\delta = 0.23$, then the state variables which include the size of plant population and size of herbivore population (P, H) changes its behaviour from Hopf-bifurcation to asymptotic stability. Similarly, when there is an increment in the value of $\delta = 0.23$ to $\delta = 0.25$, the entire system again shifts its dynamics from asymptotic stability to absolute stability. Hence, when we keep on increasing the value of the intraspecific competition rate of herbivores, it is favourable for both the populations and the entire system tends to become more and more stable.

Chapter 4

The study of Stability analysis of Modified Leslie-Gower Herbivore model with Allee effect in Plants

4.1 Introduction

Plant-herbivore models play an essential role in ecology. Many ecologists analyzed plant-herbivore models with different types of functional responses and Allee effects [19], [22], [23], [25]–[31]. Functional responses are stated by Holling [88], which describes the rate of consumption of prey by the predator and the size of the prey. The effect which occurs when the density of plants is very small, is called the Allee effect. It can be divided into two categories which include the Strong Allee effect and the Weak Allee effect [48]. The Allee effect is called strong when the growth rate is non-positive in the low limit of low density and it has a population threshold as well. Conversely, the Allee effect is called weak when the growth rate at zero density is non-negative [67], [70]–[75]. Kumar and Verma [59] presented the plant-herbivore model incorporated under the Allee effect with Holling type-I functional response. It is shown that the system is stable about the interior point and unstable when the threshold value crosses the delay parameter and Hopf-bifurcation occurs. Dupke et. al. [93] applied Holling's concept of Holling type- II functional response and applied it to the habitat selection of herbivores. Liu et. al. [94] modified the traditional Holling type- II functional response for the plant-herbivore model and formed a toxin-determined functional response due to chemical defenses so that the herbivores can consume less amount of plants. The interpretation of dynamical properties of the Stochastic prey-predator model and non-autonomous deterministic is studied by Sengupta and Das [97] with a sigmoid functional response. Persistence, non-persistence, permanence, and positivity are derived. The interest of the authors is continuously increasing in the study of functional responses and they are modifying the types of functional responses to make them more difficult to analyze. These types of mathematical models can be utilized to demonstrate the stability behaviour of the system along with time series behaviour [124], [139], [151]–[153], [168]. Arancibia-Ibarra et al. [95] studied the Leslie-Gower predator-prey model with Holling type- II functional response under the weak Allee effect and revealed the coexistence and oscillations of both the predator and prey populations. They also proved the bistability of the model for some fixed subset of parameters. Mishra et al. [100] examined a Leslie-Gower predator-prey model which conclude that the prey alone can be responsible for the instability of the model. A prey population can release chemical substance which can be dangerous for the predator population and it may lead to the predator's death. The predation rate may also become less due to the inability of the predators. Zhang et al. [101] studied Leslie-Gower predator-prey model with the harvesting system. The results due to harvesting declare that the concentration of the predator population is strictly decreasing and the size of prey has no impact under some restrictions. They used Pontryagin's maximal principle to attain the optimal harvesting policy and illustrate a case to represent that the optimal harvesting policy is realizable. Yue [102] examined a prey refuge with modified Leslie-Gower model. The global attractivity of a positive equilibrium and stability of the system is studied. Fang et al. [87] analyzed Leslie-Gower model with weak Allee effect and concluded that if the intensity of the Allee effect is strong, then both prey and predator population will tend to abundance. It is also shown that Allee effect alone can change the stability of the system and supercritical Hopf-bifurcation occurred.

The motivation for this work lies in the investigation of a crucial ecological problem, namely, the dynamics of a modified Leslie-Gower plant-herbivore model incorporating the Allee effect. Although previous research has extensively investigated different aspects of population dynamics, the application of delay differential equations in combination with the proposed model has yet to be explored. To address this gap, a novel mathematical model is proposed that incorporates delay differential equations to account for past historical interactions between the plant and herbivore populations, with τ as the time parameter. Furthermore, the inclusion of a Holling type- II functional response adds another layer of complexity to the model, allowing it to examine more realistic and intricate ecological interactions. Through this work, the aim is to enhance the understanding of ecological systems and contribute valuable insights to the field of population dynamics with implications for conservation and ecosystem management.

The combination of the Allee effect and the Holling type-II functional response results in increased vulnerability of prey populations to predation at low prey densities. Furthermore, the inclusion of a time delay introduces additional complexity to the dynamics of the system. In general, the combined presence of these factors in a biological system can have significant implications for population dynamics, including aspects such as population persistence, stability, as well as the risk of extinction.

4.2 Mathematical Model

Let $P(t)$ represent the population density of the plants and $H(t)$ be the population density of the herbivores at time t, respectively. Both the plant and herbivore populations grow logistically with internal growth rates r and u respectively. K is known as the carrying capacity of the plant population. The carrying capacity of the herbivore population depends on the plant population and γ is a measure of the quality of the plant as food for the herbivore. The Holling type-II functional response is used where α is called the per capita rate of maximum predation and β is half of the saturated response level. $(P-b)$ is the Allee effect term where $b>0$ is the Allee threshold and τ is the time parameter that characterizes the past history. A set of non-linear delay differential equations using the above notations is given by:

$$\frac{dP}{dt} = rP\left(1-\frac{P(t-\tau)}{K}\right)(P-b)-\frac{\alpha PH}{P+\beta} \tag{4.1}$$

$$\frac{dH}{dt} = uH\left(1-\frac{H}{\gamma P}\right) \tag{4.2}$$

The given model is characterized by the following initial value conditions:

$$P(t)>0, H(t)>0\ \forall\ t\geq 0.$$

Also, $P(t-\tau)=\varepsilon$, constant $\forall\ t\in[0,\tau]$.

Also, $P\leq K,$ and $b\leq 0$ for weak Allee effect. All the parameters $r, K, b, \alpha, \beta, u, and\ \gamma$ are taken as positive constants.

4.3 Equilibrium point of the model

In this section, non-trivial equilibrium is considered which characterizes the co-existence of all the state variables (P^*, H^*), where none of them is zero, because the stability of the system shows no impact of delay on any of the other equilibrium points. The non-trivial equilibrium $E^*(P^*\neq 0, H^*\neq 0)$ is as follows:

Let $P(t-\tau)\cong P(t)$ and equate the system of equations (4.1) and (4.2) to zero:

$$\frac{dH}{dt}=0\Rightarrow P^*=\frac{H^*}{\gamma}$$

$$\frac{dP}{dt}=0$$

$$\Rightarrow A_1{H^*}^3+A_2{H^*}^2+A_3H^*-A_4=0$$

where $A_1=1,$

$$A_2=[(\beta-K)\gamma-b];\text{ provided }\beta>K\ and\ (\beta-K)\gamma>b,$$

$$A_3=\left\{\left[\left(\frac{\alpha\gamma}{r}-\beta\right)K\gamma+(K-\beta)b\right]\gamma\right\};\text{ provided }\alpha>\frac{\beta r}{\gamma}\ and\ K>\beta,$$

$$A_4=\ Kb\beta\gamma^2$$

4.4 Stability Analysis and Hopf-Bifurcation

A modified Leslie-Gower plant-herbivore model at the equilibrium $E^*$is as follows:

$$\frac{dP^*}{dt}=rP^*\left(1-\frac{P^*(t-\tau)}{K}\right)(P^*-b)-\frac{\alpha P^*H^*}{P^*+\beta} \tag{4.3}$$

$$\frac{dH^*}{dt}=uH^*\left(1-\frac{H^*}{\gamma P^*}\right) \tag{4.4}$$

The exponential characteristic equation for the set of equations (4.3)-(4.4) is represented by:

$$\lambda^2 + K_1\lambda + K_2 + (K_3\lambda + K_4)e^{-\lambda\tau} = 0 \tag{4.5}$$

Here $K_1 = -2rP^* + rb + \frac{\alpha\beta H^*}{(P^*+\beta)^2} - m_4$,

$$K_2 = m_4\left[2rP^* - rb - \frac{\alpha\beta H^*}{(P^*+\beta)^2}\right] - m_2m_3,$$

$$K_3 = \frac{rP^*}{K}(P^* - b), K_4 = m_4\left[\frac{-rP^*}{K}(P^* - b)\right]$$

where $m_1 = 2rP^* - rb - \frac{\alpha\beta H^*}{(P^*+\beta)^2} - \frac{rP^*}{K}(P^* - b)e^{-\lambda\tau}$,

$$m_2 = \frac{u}{\gamma}\left(\frac{H^*}{P^*}\right)^2, m_3 = -\frac{\alpha P^*}{(P^*+\beta)}, m_4 = u\left(1 - \frac{2H^*}{\gamma P^*}\right)$$

When $\tau = 0$, the equation (4.5) is represented as:

$$\lambda^2 + (K_1 + K_3)\lambda + (K_2 + K_4) = 0 \tag{4.6}$$

Using Hurwitz's criteria, the system shows stability i.e. the zeros of equation (4.6) will contain a real part that is negative if:

$(\boldsymbol{R_1})$: $(K_1 + K_3) > 0$;

$(\boldsymbol{R_2})$: $(K_2 + K_4) > 0$

When we change the values of τ, the roots of the negative real parts shift to positive real parts as follows:

Let the root of equation (4.5) be $\lambda = i\omega$, then the equation (4.5) is given by:

$$(i\omega)^2 + K_1(i\omega) + K_2 + (K_3(i\omega) + K_4)e^{-(i\omega)\tau} = 0$$

$$\Rightarrow -\omega^2 + K_1(i\omega) + K_2 + \ (K_3(i\omega) + K_4)(\cos\omega\tau - i sin\,\omega\tau) = 0$$

Segregating real parts from the imaginary parts:

$$-\omega^2 + K_2 = -K_4\cos\omega\tau - K_3\omega sin\,\omega\tau \tag{4.7}$$

$$K_1\omega = -K_3\cos\omega\tau + K_4 sin\,\omega\tau \tag{4.8}$$

$\therefore$ ω is true for:

$$\omega^4 - \left(K_3{}^2 - K_1{}^2 + 2K_2\right)\omega^2 + \left(K_2{}^2 - K_4{}^2\right) = 0 \tag{4.9}$$

The two roots of equation (4.9) are:

$$\omega_{1,2}^2 = \frac{({K_3}^2-{K_1}^2+2K_2)\pm\sqrt{({K_3}^2-{K_1}^2+2K_2)^2-4({K_2}^2-{K_4}^2)}}{2} \qquad (4.10)$$

$\omega_{1,2}^2$ has no positive roots if:

$$(\boldsymbol{R_3}): ({K_3}^2-{K_1}^2+2K_2) < 0 \; and \; ({K_2}^2-{K_4}^2) > 0 \; or \; ({K_3}^2-{K_1}^2+2K_2)^2 < 4({K_2}^2-{K_4}^2)$$

It gives that if the condition (R_3) holds, equation (4.10) has no positive roots.

There are the following lemmas [162].

Lemma 4.4.1 Every root of equation (4.5) has real parts which are negative $\forall\, \tau \geq 0$, if $(R_1)-(R_2)$ hold.

In contrast, if:

$$(\boldsymbol{R_4}): ({K_2}^2-{K_4}^2) < 0 \; or \; ({K_3}^2-{K_1}^2+2K_2) > 0 \; and \; ({K_3}^2-{K_1}^2+2K_2)^2 = 4({K_2}^2-{K_4}^2)$$

Then, ω_1^2 is the $+ve$ root of equation (4.7).

Similarly, if:

$$(\boldsymbol{R_5}): ({K_2}^2-{K_4}^2) > 0 \; or \; ({K_3}^2-{K_1}^2+2K_2) > 0 \; and \; ({K_3}^2-{K_1}^2+2K_2)^2 > 4({K_2}^2-{K_4}^2)$$

Then, $\omega_{1,2}^2$ are positive roots of equation (4.7).

When τ takes certain values, then equation (4.5) includes purely imaginary roots in both- $(R_4)$and (R_5). The system of equations (4.5)-(4.6) gives the threshold values $\tau_j^{\pm}$ of τ:

$$\tau_j^{\pm} = \frac{1}{\omega_{1,2}} cos^{-1}\left[\frac{K_4(\omega_{1,2}^2-K_2)-K_1K_3\omega_{1,2}^2}{{K_3}^2\omega_{1,2}^2+{K_4}^2}\right]+\frac{2j\pi}{\omega_{1,2}}, j = 0,1,2,\ldots \qquad (4.11)$$

The succeeding lemma can condense the above discussion [162].

Lemma 4.4.2 (A) If $(R_1)-(R_2) \; and \; (R_4)$ is true and $\tau = \tau_j^{+}$, then there are two purely imaginary roots $\pm i\omega_1$ of equation (4.5).

(B) There is a pair of purely imaginary roots $\pm i\omega_1(\pm i\omega_2 \; respectively)$ of equation (4.5), if $(R_1)-(R_2) \; and \; (R_5)$ is true and $\tau = \tau_j^{+}(\tau = \tau_j^{-} \; respectively)$.

Let $\tau_j^{\pm} = \mu_j^{\pm}(\tau) + i\omega_j^{\pm}(\tau); j = 0,1,2,3,\ldots$ to shift the negative real part of some roots to the real part of the equation (4.5) which is positive if $\tau > \tau_j^{+}$ and $\tau < \tau_j^{-}$.

$\mu_j^{\pm}(\tau_j^{\pm}) = 0, \omega_j^{\pm}(\tau_j^{\pm})$ $=\omega_{1,2}$ is satisfied by the roots of equation (4.5).

The following transversality condition holds for the above criteria:

$$\frac{d}{d\tau}\left(Re\ \lambda_j^{+}(\tau_j^{+})\right) > 0\ and\ \frac{d}{d\tau}\left(Re\ \lambda_j^{-}(\tau_j^{-})\right) < 0$$

It represents that $\tau_j^{\pm}$ are bifurcating values. The scattering of the zeros of equation (4.5) can be described by the following theorem [162].

Theorem 4.1 Consider, $\tau_j^{+}(j = 0,1,2,3, \dots)$ which is deducted from equation (4.11).

(A) Every root has $-ve$ real part $\forall\ \tau \geq 0$ in equation (4.5), if $(R_1), (R_2)$ hold.

(B) Every root has $-ve$ real part in equation (4.5), if $(R_1), (R_2)\ and\ (R_4)$ hold and when $\tau \in [0, \tau_0^{+})$. Equation (4.5) has two roots which are purely imaginary $\pm i\omega_1$, when $\tau = \tau_0^{+}$. There exists at least one root, including $+ve$ real part in equation (4.3) when $\tau > \tau_0^{+}$.

(C) When $(R_1), (R_2)\ and\ (R_5)$ is true, then there is a positive integer m such that

$0 < \tau_0^{+} < \tau_0^{-} < \tau_1^{+} < \tau_1^{-} - - - < \tau_{m-1}^{-} < \tau_m^{+}$ and there are m fluctuations between stability and instability. That is, every root has a negative real part in equation (4.5) when $\tau \in [0, \tau_0^{+}), (\tau_0^{-}, \tau_1^{+}), - - -, (\tau_{m-1}^{-}, \tau_m^{+})$. There is at least one root with $+ve$ real part in equation (4.5) if $\tau \in (\tau_0^{+}, \tau_0^{-}), (\tau_1^{+}, \tau_1^{-}), - - -, (\tau_{m-1}^{+}, \tau_{m-1}^{-})$ and $\tau > \tau_m^{+}$.

4.5 Numerical Example

The graphical examples are presented by the following parametric values to depict the modified Leslie-Gower plant-herbivore dynamics of a set of equations (4.1)-(4.2):

$$r = 0.8, K = 0.4, b = -0.3, \alpha = 0.8, \beta = 0.4, u = 0.2, \gamma = 0.1$$

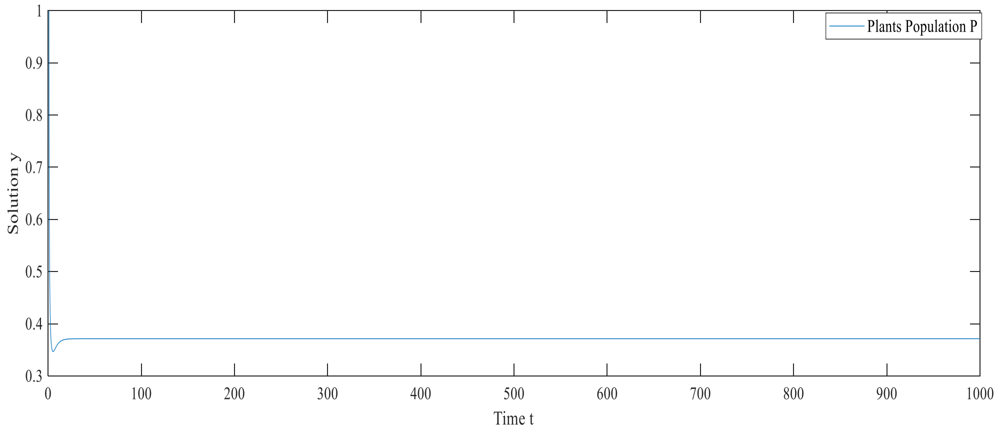

Figure 4.1 The plant population P is absolutely stable in the absence of time parameter i.e. when $\tau = 0$.

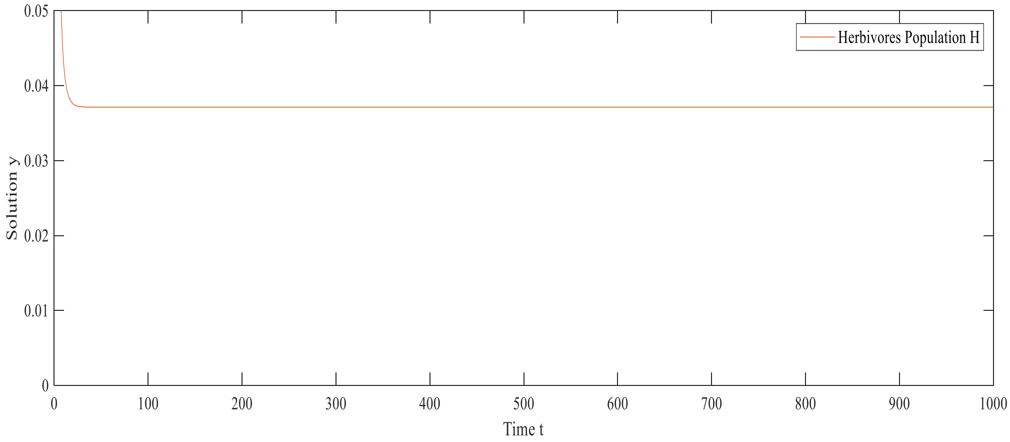

Figure 4.2 The herbivore population H is absolutely stable in the absence of time parameter i.e. when $\tau = 0$.

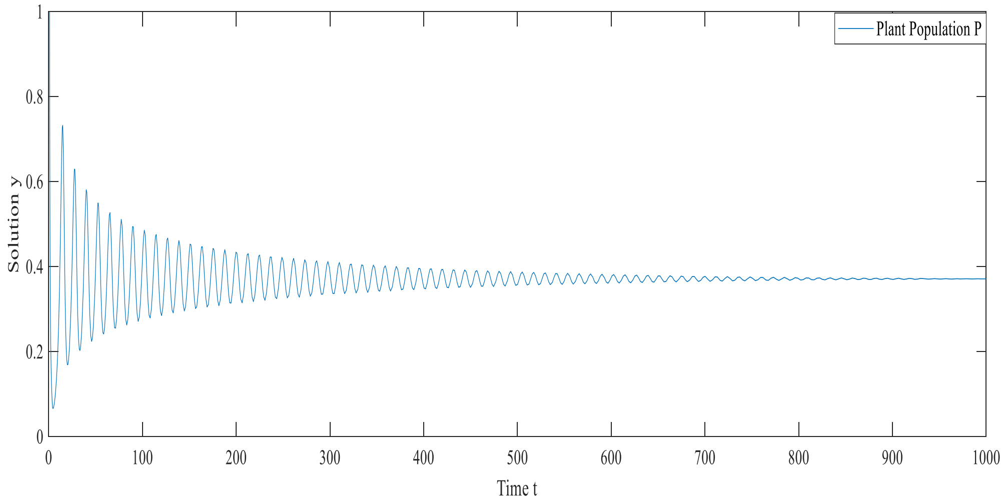

Figure 4.3 The plant population P is asymptotically stable when the time parameter is below the threshold value i.e. when $\tau < 3.2$.

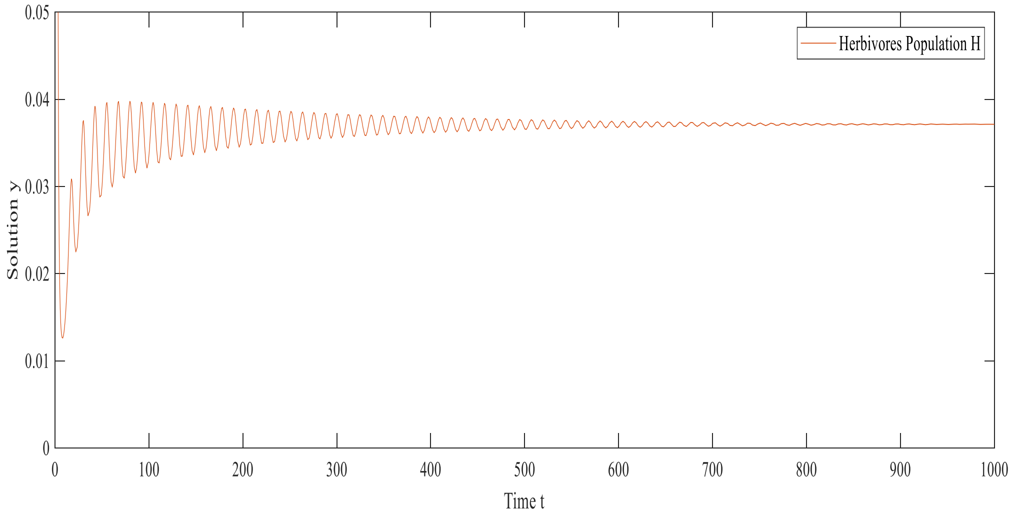

Figure 4.4 The herbivore population H is asymptotically stable when the time parameter is below the threshold value i.e. when $\tau < 3.2$.

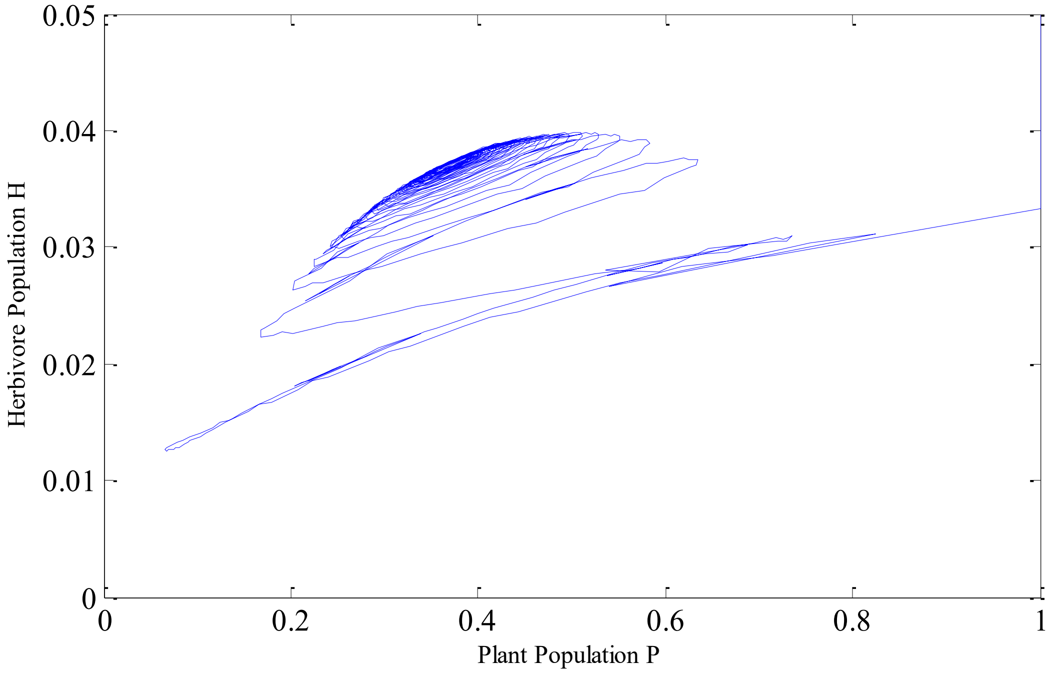

Figure 4.5 Equilibrium E^* demonstrate phase space graph of asymptotic stability when the time parameter is below the threshold value i.e. when $\tau < 3.2$.

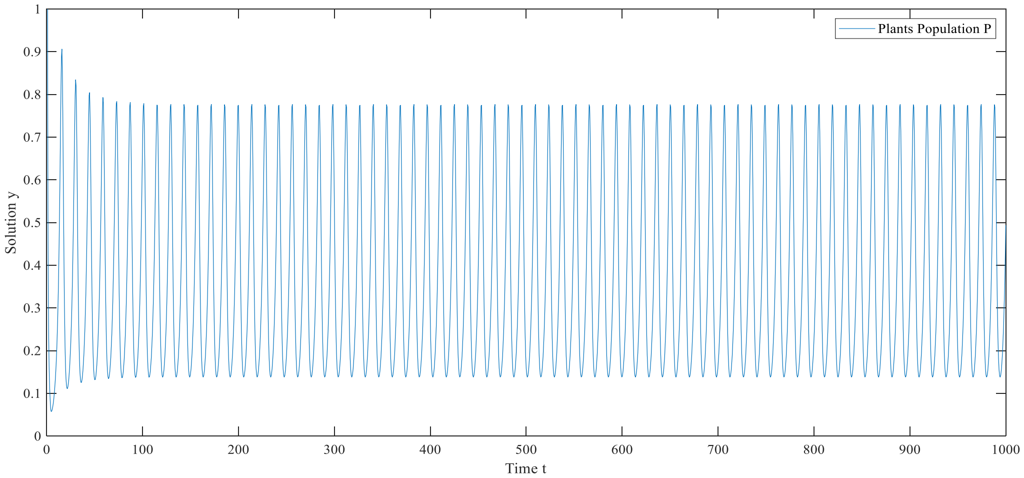

Figure 4.6 The plant population P represents Hopf-bifurcation when the time parameter surpasses the threshold value i.e. when $\tau \geq 3.2$.

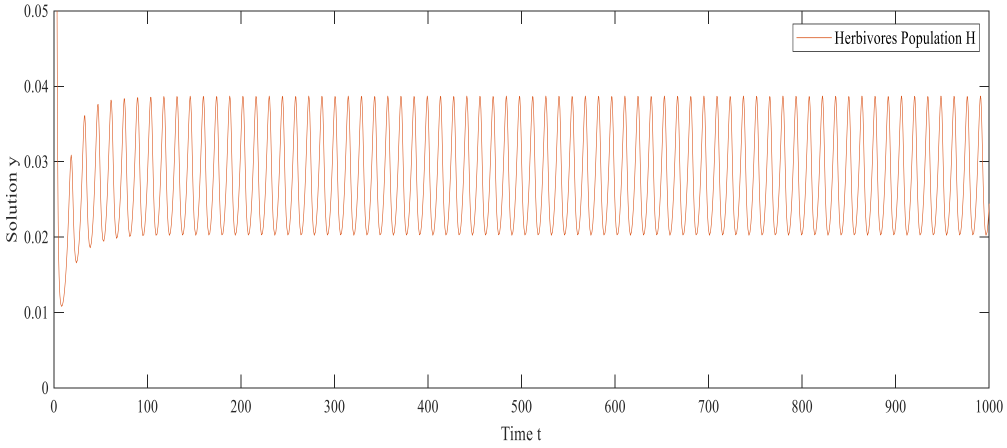

Figure 4.7 The herbivore population H represents Hopf-bifurcation when the time parameter surpasses the threshold value i.e. when $\tau \geq 3.2$.

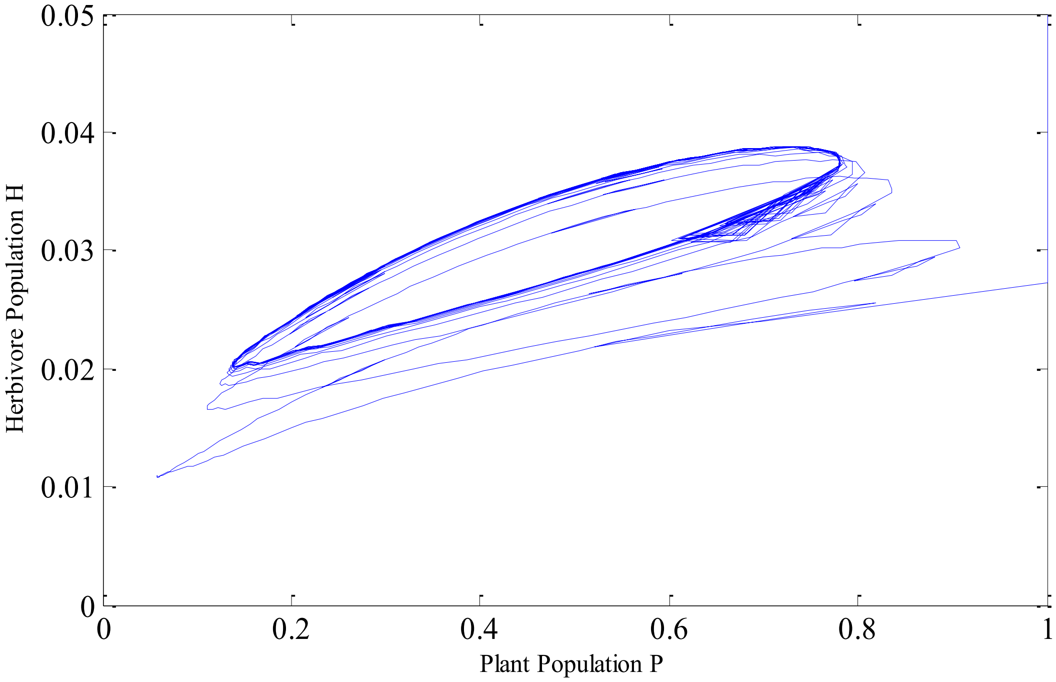

Figure 4.8 Equilibrium E^* demonstrate phase space graph of Hopf-bifurcation when the time parameter surpasses the threshold value i.e. when $\tau \geq 3.2$.

4.6 Sensitivity Analysis

Sensitivity analysis is a study that represents the behaviour of the stability of the proposed state variables by varying the parameter values and taking the delay parameter (τ) as constant. For instance, the sensitivity equations of the partial derivatives of the solution (P, H) with respect to the per capita rate of maximum predation α are as follows:

$$\frac{dS_1}{dt} = \left[r(2P - b) - \frac{\alpha\beta H}{(P+\beta)^2} + r\left(\frac{b-2P}{K}\right)P(t-\tau)\right]S_1 - \alpha\left[\frac{P^2+\beta P}{(P+\beta)^2}\right]S_2 + rP\frac{(b-P)}{K}S_1(t-\tau) \tag{4.12}$$

$$\frac{dS_2}{dt} = \frac{uH^2}{\gamma P^2}S_1 + \left(u - \frac{2uPH}{\gamma P^2}\right)S_2 \tag{4.13}$$

where $S_1 = \frac{\partial P}{\partial \alpha}$ and $S_2 = \frac{\partial H}{\partial \alpha}$

4.6.1 Sensitivity of Variables to Parameter α

When we increase the value of the per capita rate of maximum predation from $\alpha = 0.8$ to $\alpha = 2.3$, the system (P, H) changes its behaviour from Hopf-bifurcation to asymptotic stability. After that, when we further increase the per capita rate of maximum predation from $\alpha = 2.3$ to $\alpha = 2.9$, the system (P, H) again shifts its equilibrium from asymptotic stable to absolutely stable as shown in figure 4.9 and figure 4.10.

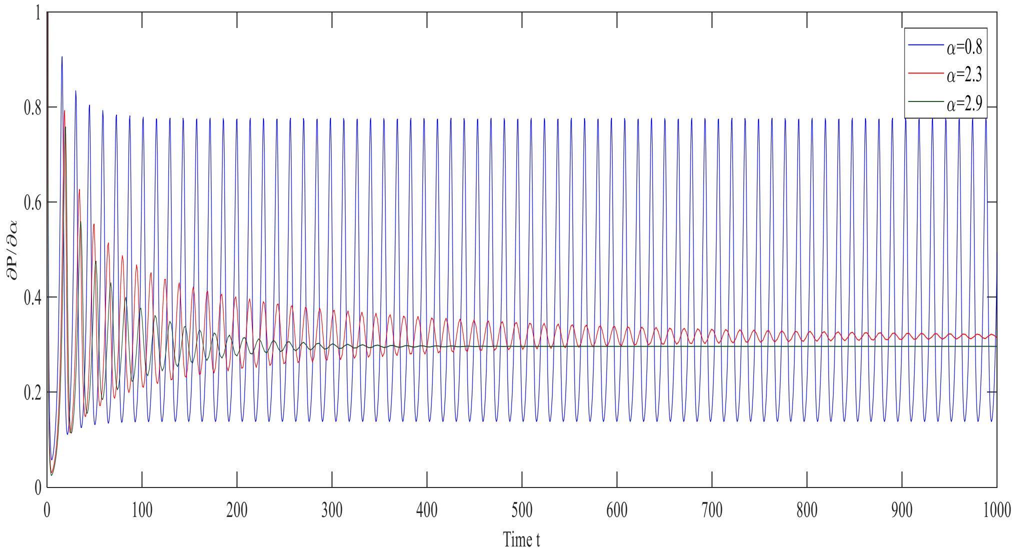

Figure 4.9 Time series graph of change in density of plant population P with respect to changes in the per capita rate of maximum predation α.

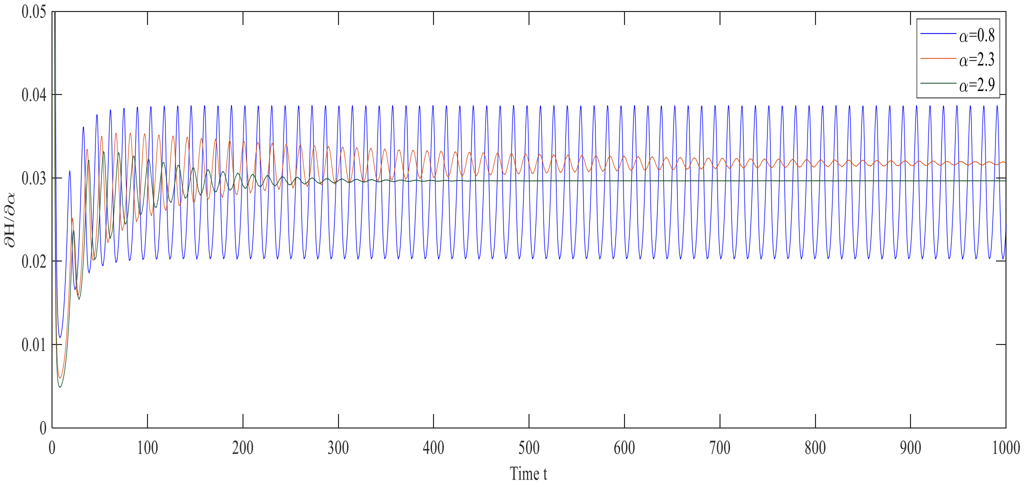

Figure 4.10 Time series graph of change in density of herbivore population H with respect to changes in the per capita rate of maximum predation α.

4.7 Discussion and Outcome of Sensitivity Analysis

The relationship between the per capita rate of maximum predation (α) and stability is explained by the dynamics of plant-herbivore interactions. In the absence of significant predation (low α), the system exhibits complex dynamics, including Hopf-bifurcation for both the plant and herbivore populations. This implies oscillatory behavior and instability in the population dynamics.

As the per capita rate of maximum predation (α) increases, the plant population (P) shifts its behavior from Hopf-bifurcation to asymptotic stability. This suggests that higher predation pressure initially leads to more stable plant population dynamics. The herbivore population (H) also shifts from Hopf-bifurcation to asymptotic stability with increased α. The explanation provided is that as herbivores consume more plants, competition among herbivores increases, leading to a decrease in the herbivore population size. The system becomes more stable due to this scenario.

With a continued increase in the per capita rate of maximum predation, both the plant and herbivore populations become more stable. The ecological interpretation is that as predation pressure intensifies (higher α), the herbivores consume more plants, leading to a decrease in both plant and herbivore populations. This intensified predation reduces competition among herbivores, contributing to the stability of the entire system.

In summary, the sensitivity parameter α, representing the per capita rate of maximum predation, plays a crucial role in shaping the dynamics of the plant-herbivore system. Increasing predation pressure not only affects the plant population directly but also has cascading effects on herbivore populations, ultimately influencing the stability of the entire ecological system.

4.8 Conclusion

A modified Leslie-Gower plant-herbivore model is taken into consideration by incorporating the Holling-type-II response under the Allee effect. The time parameter τ is utilized to depict the past history. Both the populations (P, H) grow logistically with different intrinsic growth rates. The carrying capacity of the herbivore population H depends on the plant population P. The non-trivial equilibrium $E^*(P^* \neq 0, H^* \neq 0)$ is calculated. The stability behaviour of the system and time series graphs are performed graphically. The observation in figure 4.1 and figure 4.2 is that plant population P and herbivore population H are absolutely stable when the time parameter $\tau = 0$. Both the populations are asymptotically stable when the time parameter $\tau < 3.2$, as shown in figure 4.3 and figure 4.4. Furthermore, both populations are unstable and demonstrate Hopf-bifurcation when the time parameter $\tau \geq 3.2$, which is represented by figure 4.6 and figure 4.7. The phase space graphs representing the asymptotic stability and Hopf-bifurcation are indicated in figure 4.5 and figure 4.8.

Time series graphs of change in density of plant population P and herbivore population H with respect to changes in the per capita rate of maximum predation α are given in figure 4.9 and figure 4.10, respectively.

Chapter 5

Analyzing the Dynamics of Plant-Herbivore Interactions under Allee Effect using Delay Differential Equation

5.1 Introduction

The study of infectious disease is known as Ecological epidemiology. Eco-epidemiological models are models where the predator-prey population suffers from an infectious disease [18]. Lotka [38] and Volterra [169] were the first who started to create realistic models mathematically which include two or more interacting species. Initially, the eco-epidemiological model is established by Hadeler and Freedman [170]. Thereafter, many researchers worked on eco-epidemiology models. Different researchers focused on the spread of disease in different populations such as prey population only, predator population only, or both prey and predator populations [171], [172]. There are some epidemic models present in literature where authors supposed that the predators consume infectious prey, some supposed that the predator population does not consume the infectious prey population, and some authors also supposed that the predator population consumes both the susceptible and infectious plants [173]–[175]. Some researchers incorporated the Allee effect accompanied by infectious disease in the population [176], [177]. Initially, the biologist, Allee [48] defined the term Allee effect which describes the correspondence between density of a population and mean individual fitness. The Allee effect is observed in natural populations such as plants [59], birds [60], insects [61], and animals [62]. There are many mechanisms that cause the Allee effect. Mate limitation is the primary factor responsible for the occurrence of the Allee effect in both plants and animals. The other mechanisms include environmental mechanisms, inbreeding depression, demographic genetics, etc. [49], [56]. Plants can suffer from the Allee effect if the density of plants is low and sparsely distributed. The Allee effect falls into two categories. When the population density is low below a specific threshold known as the Allee threshold, the presence of a strong Allee effect leads to a significant decrease in the population size. Adversely, the weak Allee effect means the increment of the growth rate with the increment of population size but there is no threshold value [65], [69]. The large effects of the extinction of species are documented due to the diseases [50]. For instance, if predators consume infectious prey, then there is negative growth of the predators [18]. There is a required time delay for the gestation period when predators eat infectious plants. Hence, delay differential equations have a significant role in realistic models [124], [138], [153].

The use of the weak Allee effect with infectious disease in plants and time lag is very rare in eco-epidemiological models. Therefore, the purpose of this contribution is to study an eco-epidemiological model accompanied by a weak Allee effect. The susceptible, as well as infectious plants, are incorporated with herbivores. Additionally, delay differential equations are used as there is an essential time lag for the gestation of the infectious plants by herbivores.

5.2 Mathematical Model

5.2.1 Assumptions of the model

There are some assumptions for the Allee effect as well as for plants:

1. Reproduction is not possible without mates.
2. The increment in population density causes the decrement of the Allee effect.
3. When population density is large, i.e., if population density approaches infinity, then the Allee function approaches one.
4. The predation of susceptible plants by herbivores is ignored in this article.
5. The assumption is that the infectious plants are generated by the infection of susceptible plants. Furthermore, infectious plants cannot grow, recover, or reproduce. Therefore, $\gamma > 0$ is the mortality rate of the infectious plant.
6. **6.** The supposition for the weak infectious plant is that the handling time of herbivore for the infectious plant is zero, so there is a functional response with a linear mass-action βP_I where $\beta > 0$ is the attack rate of herbivores.

5.2.2 Mathematical Formulation

The plant-herbivore model using the delay differential equations is as follows:

$$\frac{dP_S}{dt} = P_S\{[1-(P_S+P_I)](P_S-b)-\alpha P_I\} \tag{5.1}$$

$$\frac{dP_I}{dt} = P_I(\alpha P_S - \beta H - \gamma) \tag{5.2}$$

$$\frac{dH}{dt} = \delta H(t-\tau)P_I(t-\tau) - \mu H \tag{5.3}$$

where $P_S(t) > 0, P_I(t) > 0, H(t) > 0 \; \forall \; t \geq 0$ and $b \leq 0$ for weak Allee effect and $P(t-\tau) = \varepsilon,$ Constant for all $t \in [0,\tau]$.

Here, all the parameters $b, \alpha, \beta, \gamma, \delta, and \; \mu$ are positive constants.

Variables/Parameters	**Description**
$\boldsymbol{P_S}$	Density of Susceptible Plant
$\boldsymbol{P_I}$	Density of Infectious Plant
$\boldsymbol{H}$	Density of Herbivores
$\boldsymbol{b}$	Allee threshold
$\boldsymbol{\alpha}$	Rate of infection
$\boldsymbol{\delta}$	Herbivores gain by consuming Infectious Plant
$\boldsymbol{\gamma}$	Mortality rate of Infectious Plant
$\boldsymbol{\beta}$	Attack rate of Herbivores
$\boldsymbol{\mu}$	Mortality rate of Herbivores
$\boldsymbol{\tau}$	Gestation period of Herbivores

Table 5.1 Description of Variables/Parameters.

5.3 Equilibrium point of the model

There are eight equilibrium points in the system, each with distinct non-negativity criteria [178]. However, for the more comprehensive study, non-zero equilibrium is discussed which represents the co-existence of all the state variables $\left({P_S}^*, {P_I}^*, H^*\right)$ where none of them is zero, because the stability of the system shows no impact of delay on any of the other seven equilibrium points.

To calculate the non-zero equilibrium points of the model, equate the equations (5.1)-(5.3) to zero.

$$\text{At } E^*\left({P_S}^*, {P_I}^*, H^*\right), {P_I}^*(t-\tau) \approx {P_I}^*(t) \text{ and } H^*(t-\tau) \approx H^*(t).$$

$$\frac{dP_I}{dt} = 0 \Rightarrow H^* = \frac{\alpha {P_S}^* - \gamma}{\beta}, \text{ provided } {P_S}^* > \frac{\gamma}{\alpha}$$

$$\frac{dH}{dt} = 0 \Rightarrow {P_I}^* = \frac{\mu}{\delta}, \text{ provided } \mu > 0$$

$$\text{and } \frac{dP_S}{dt} = 0$$

$$\Rightarrow {P_S}^{*2} - \left(1 + b - \frac{\mu}{\delta}\right){P_S}^* + \left[b + (\alpha - b)\frac{\mu}{\delta}\right] = 0$$

P_S has real and distinct roots if the discriminant is greater than 0, i.e.,

$$\text{if } \left(1 + b - \frac{\mu}{\delta}\right)^2 > 4\left[b + (\alpha - b)\frac{\mu}{\delta}\right]$$

5.4 Stability Analysis and Hopf-Bifurcation

The original system (5.1)-(5.3) in terms of the non-zero equilibrium $E^*({P_S}^*, {P_I}^*, H^*)$ can be rewritten as given below:

$$\frac{d{P_S}^*}{dt} = {P_S}^*\{[1 - ({P_S}^* + {P_I}^*)]({P_S}^* - b) - \alpha {P_I}^*\} \tag{5.4}$$

$$\frac{d{P_I}^*}{dt} = {P_I}^*(\alpha {P_S}^* - \beta H^* - \gamma) \tag{5.5}$$

$$\frac{dH^*}{dt} = \delta H^*(t-\tau){P_I}^*(t-\tau) - \mu H^* \tag{5.6}$$

The exponential characteristic equation at equilibrium $E^*({P_S}^*, {P_I}^*, H^*)$ is given by

$$\begin{vmatrix} \lambda - m_1 & -m_2 & -m_3 \\ -m_4 & \lambda - m_5 & -m_6 \\ -m_7 & -m_8 & \lambda - m_9 \end{vmatrix} = 0$$

$$\text{where } m_1 = -3{P_S}^{*2} + 2(1+b){P_S}^* - 2{P_S}^*{P_I}^* + (b - \alpha){P_I}^* - b,$$

$$m_2 = \alpha {P_I}^*, m_3 = 0, m_4 = -{P_S}^{*2} + b{P_S}^* - \alpha {P_S}^*, m_5 = \alpha {P_S}^* - \beta H^* - \gamma,$$

$$m_6 = \delta H^* e^{-\lambda\tau}, m_7 = 0, m_8 = -\beta {P_I}^*, m_9 = \delta {P_I}^* e^{-\lambda\tau} - \mu$$

$$\therefore (\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3) + (B_1\lambda^2 + B_2\lambda + B_3)e^{-\lambda\tau} = 0 \qquad (5.7)$$

Here $A_1 = -m_1 - m_5 + \mu$, $A_2 = m_1m_5 - m_2m_4 - \mu(m_1 + m_5)$,

$$A_3 = \mu(m_1m_5 - m_2m_4),$$

$$B_1 = -\delta {P_I}^*, B_2 = (m_1 + m_5)\delta {P_I}^* - m_6m_8,$$

$$B_3 = (-m_1m_5 + m_2m_4)\delta {P_I}^* + m_1m_6m_8$$

Here, $\lambda = i\omega$ is a solution of equation (5.7) if and only if:

$$(i\omega)^3 + A_1(i\omega)^2 + A_2(i\omega) + A_3 + (B_1(i\omega)^2 + B_2(i\omega) + B_3)e^{-i\omega\tau} = 0 \quad (5.8)$$

Segregating real and imaginary parts:

$$A_3 - A_1\omega^2 + (B_3 - B_1\omega^2)cos\ \omega\tau + B_2\omega \sin \omega\tau = 0 \qquad (5.9)$$

$$A_2\omega - \omega^3 + B_2\omega\ cos\ \omega\tau - (B_3 - B_1\omega^2) \sin \omega\tau = 0 \qquad (5.10)$$

And it gives:

$$\begin{aligned} \omega^6 + \left({A_1}^2 - {B_1}^2 - 2A_2\right)\omega^4 \\ + \left({A_2}^2 - {B_2}^2 + 2B_1B_3 - 2A_1A_3\right)\omega^2 \\ + \left({A_3}^2 - {B_3}^2\right) = 0 \end{aligned} \qquad (5.11)$$

Let $c_1 = \left({A_1}^2 - {B_1}^2 - 2A_2\right), c_2 = \left({A_2}^2 - {B_2}^2 + 2B_1B_3 - 2A_1A_3\right),$

$c_3 = \left({A_3}^2 - {B_3}^2\right).$

Let $\omega^2 = x$, then equation (5.11) becomes:

$$x^3 + c_1x^2 + c_2x + c_3 = 0 \qquad (5.12)$$

Lemma 5.4.1 When $c_3 < 0$, Equation (5.12) has at least one real zero which is positive.

Proof: Let

$$l(x) = x^3 + c_1x^2 + c_2x + c_3$$

Here $l(0) = c_3$, which is less than zero, $\lim\limits_{y\to\infty} l(x) = \infty$. therefore, $\exists\ x_0 \in (0, \infty)$ such that $l(x_0)$ is zero. Hence proved.

Lemma 5.4.2 When $c_3 \geq 0$, an essential condition for equation (5.12) to have real zeros which are positive, is $D = {c_1}^2 - 3c_2$ and it is greater than and equal to zero.

Proof: Because

$$l(x) = x^3 + c_1x^2 + c_2x + c_3,$$

therefore

$$l'(x) = 3x^2 + 2c_1x + c_2$$

The derivative is equated equal to zero to discuss the nature of roots. We reduce the cubic equation to a quadratic equation so that is it easier to check the roots.

$$l'(x) = 0$$

$$\Rightarrow 3x^2 + 2c_1x + c_2 = 0 \tag{5.13}$$

The zeros of equation (5.13) can be represented as

$$x_{1,2} = \frac{-2c_1 \mp \sqrt{4{c_1}^2 - 12c_2}}{6} = \frac{-c_1 \mp \sqrt{D}}{3} \tag{5.14}$$

When D is less than zero, equation (5.13) has no zero which is real. Therefore, $l(x)$ is a monotone-increasing function in x. The equation (5.12) does not have any real zero which is positive because $l(0) = c_3 \geq 0$. Hence proved.

When D is greater than and equal to zero, $x_1 = \frac{-c_1+\sqrt{D}}{3}$ is local minima of $l(x)$.

Lemma 5.4.3 When c_3 is greater than and equal to zero, equation (5.12) has zeros which are positive if and only if $x_1 > 0$ and $l(x_1) \leq 0$.

Proof: Consider, $x_1 \leq 0$ or $x_1 > 0$ and $l(x_1) > 0$. When $x_1 \leq 0$, $l(x)$ does not have any real zeros which are positive because $l(x)$ is increasing for $x \geq x_1$ and $l(0) = c_3 \geq 0$. When $x_1 > 0$ and $l(x_1) > 0$, $l(x_1) \leq l(x_2)$ because $x_2 = \frac{-c_1-\sqrt{D}}{3}$ is the local maxima value. Therefore, $l(0) = c_3 \geq 0$, because $l(x)$ has no real zeros which are positive. Hence proved.

Lemma 5.4.4 Suppose equation (5.14) defines x_1.

(I) When $c_3 < 0$, Equation (5.12) has at least one real zero which is positive.

(II) When $c_3 \geq 0$ and $D = {c_1}^2 - 3c_2 < 0$, equation (5.12) does not have any zero which is positive.

(III) When $c_3 \geq 0$, equation (5.12) has zeros which are positive if and only if $x_1 > 0$ and $l(x_1) \leq 0$.

Proof: Assume that equation (5.12) has zeros that are positive. With no loss of generality, let x_1, x_2, x_3 are positive zeros. Then equation (5.11) has three zeros that are positive, say $\omega_1 = \sqrt{x_1}\,, \omega_2 = \sqrt{x_2}\,, \omega_3 = \sqrt{x_3}$.

From (5.10),

$$\sin \omega\tau = \frac{A_2\omega - \omega^3}{d}$$

Which gives

$$\tau = \frac{1}{\omega}\left[sin^{-1}\left(\frac{A_2\omega - \omega^3}{d}\right) + 2(j-1)\pi\right] \; ; j = 1,2,3, \dots$$

Consider,

$$\tau_k{}^{(j)} = \frac{1}{\omega_k}\left[sin^{-1}\left(\frac{A_2\omega_k - {\omega_k}^3}{d}\right) + 2(j-1)\pi\right] \; ; k = 1,2,3. ; j = 0,1,2, \dots$$

Therefore, $\mp i\omega_k$ is a pair of zeros of equation (5.11) that are purely imaginary.

Where

$$\tau = \tau_k{}^{(j)}, k = 1,2,3. ; j = 0,1,2, \dots., \lim_{j \to \infty} \tau_k{}^{(j)} = \infty, k = 1,2,3.$$

Thus, define

$$\tau_0 = \tau_{k_0}{}^{(j_0)} = \min_{1 \leq k \leq 3, j \geq 1}\left[\tau_k{}^{(j)}\right], \qquad \omega_0 = \omega_{k_0}, \qquad x_0 = x_{k_0} \qquad (5.15)$$

Lemma 5.4.5 Suppose that $A_1 > 0, (A_3 + d) > 0, A_1A_2 - (A_3 + d) > 0$.

(A) When $c_3 \geq 0$ and $D = {c_1}^2 - 3c_2 < 0$, then every zero of equation (5.7) includes real parts which are negative $\forall\, \tau \geq 0$.

(B) When $c_3 < 0$ or $c_3 \geq 0$, $x_1 > 0$ and $l(x_1) \leq 0$, then every zero of equation (5.7) includes real parts which are negative $\forall\, \tau \in [0, \tau_0)$.

Proof: If $\tau = 0$, equation (5.7) becomes

$$\lambda^3 + (A_1 + B_1)\lambda^2 + (A_2 + B_2)\lambda + (A_3 + B_3) = 0 \qquad (5.16)$$

By Hurwitz's condition, $(\boldsymbol{R_1})$**:** Every zero of equation (5.7) includes real parts which are $-ve$ if and only if

$$(A_3 + B_3) > 0, (A_1 + B_1)(A_2 + B_2) - (A_3 + B_3) > 0.$$

When $c_3 \geq 0$ and $D = {c_1}^2 - 3c_2 < 0$, Lemma 5.4.4 (B) represents that equation (5.7) does not have any root with zero real part $\forall\, \tau \geq 0$. If $c_3 < 0$ or $c_3 \geq 0$, $x_1 > 0$ and $l(x_1) \leq 0$, Lemma 5.4.4 (A) and (C) implies that if $\tau \neq {\tau_k}^{(j)}, k = 1,2,3.; j \geq 1$, equation (5.7) does not include any roots with a real part of zero and τ_0 is the minimum value of τ, therefore equation (5.7) has zeros that are purely imaginary.

Consider,

$$\lambda(\tau) = \psi(\tau) + i\omega(\tau) \qquad (5.17)$$

be the zeros of equation (5.7) which satisfy:

$$\psi(\tau_0) = 0, \qquad \omega(\tau_0) = \omega_0$$

Let $l'(x_0) \neq 0$ to prove that equation (5.7) has zeros $\mp\omega_0$ which are purely imaginary, that includes $\tau = \tau_0$ and $\lambda(\tau)$ satisfies the transversality condition.

Lemma 5.4.6 Consider $x_0 = {\omega_0}^2$. If $\tau = \tau_0$, Then Sign $[\psi'(\tau_0)]$ =Sign $[l'(x_0)]$

Proof: Put $\lambda(\tau)$ in equation (5.7) and differentiate w.r.t τ,

$$\frac{d\lambda}{d\tau}\left[3\lambda^2 + 2A_1\lambda + A_2 + \left((B_1\lambda^2 + B_2\lambda + B_3)(-\tau) + (2B_1\lambda + B_2)\right)e^{-\lambda\tau}\right]$$

$$= \lambda(B_1\lambda^2 + B_2\lambda + B_3)e^{-\lambda\tau}$$

$$\text{Then } \left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2A_1\lambda + A_2)e^{\lambda\tau}}{\lambda(B_1\lambda^2 + B_2\lambda + B_3)} + \frac{(2B_1\lambda + B_2)}{\lambda(B_1\lambda^2 + B_2\lambda + B_3)} - \frac{\tau}{\lambda}$$

From equations (5.9)-(5.11):

$$\mu'(\tau_0) = Re\left[\frac{(3\lambda^2+2A_1\lambda+A_2)e^{\lambda\tau}}{\lambda(B_1\lambda^2+B_2\lambda+B_3)}\right] + Re\left[\frac{(2B_1\lambda+B_2)}{\lambda(B_1\lambda^2+B_2\lambda+B_3)}\right] = \frac{1}{\Delta}[3{\omega_0}^6 + 2c_1{\omega_0}^4 + b{\omega_0}^2]$$

$$\text{Here } \Delta = [(B_3 - B_1\omega^2)^2 + (B_2\omega)^2]$$

$$\text{where } \Delta > 0 \text{ and } \omega_0 > 0.$$

Therefore,

$$\text{Sign }[\psi'(\tau_0)] = \text{Sign }[l'(x_0)].$$

Hence proved.

5.5 Numerical Example

The following parameter values are incorporated to represent the stability behaviour of the entire system:

$$\alpha = 0.5, \beta = 0.1, \gamma = 0.05, \delta = 0.55, \mu = 0.1, b = -5$$

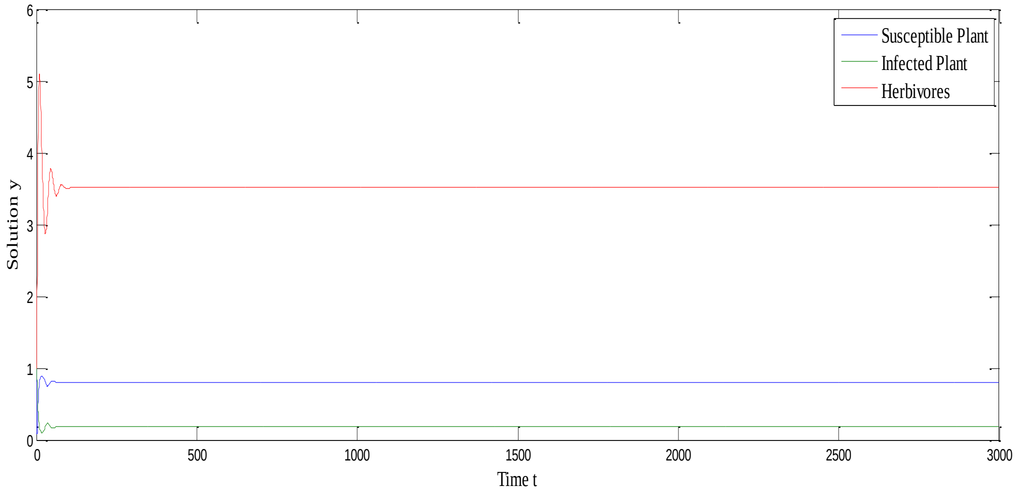

Figure 5.1 $E^*(0.8013, 0.1818, 3.5127)$ is absolutely stable when there is no time lag that is when $\tau = 0$.

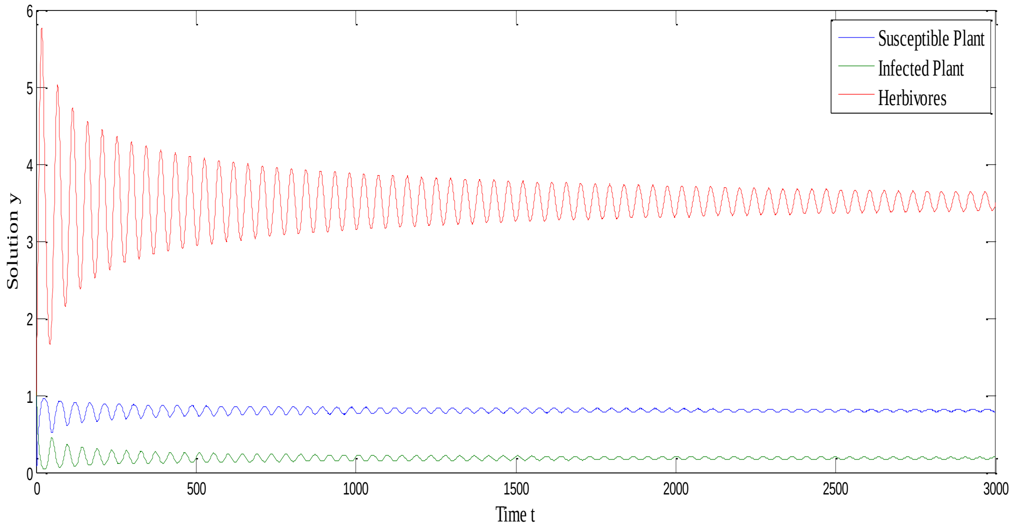

Figure 5.2 $E^*(0.7888, 0.1949, 3.4142)$ is asymptotic stable when the time lag is lower than the threshold value that is $\tau < 10.6$.

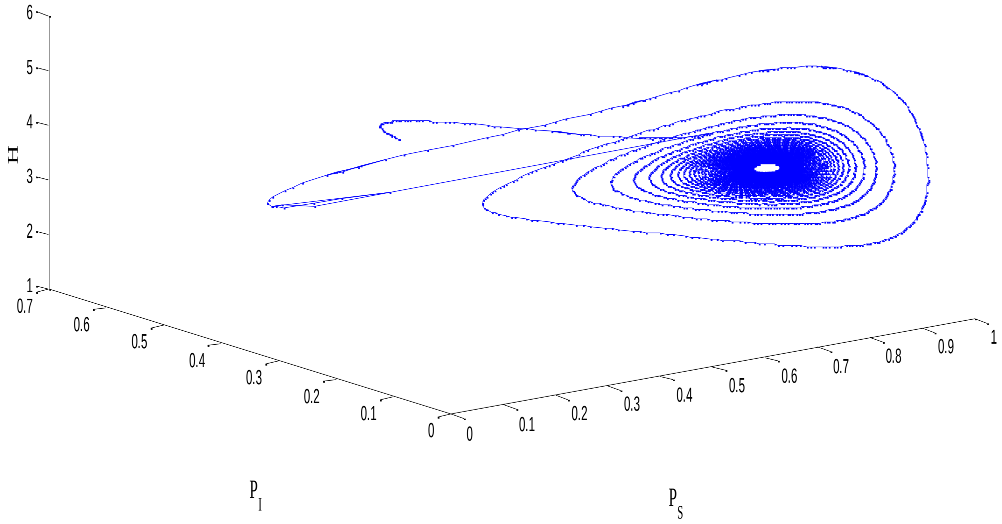

Figure 5.3 Phase space diagram showing the asymptotic stability of $E^*(0.7888, 0.1949, 3.4142)$ when the time lag is lower than the threshold value that is $\tau < 10.6$.

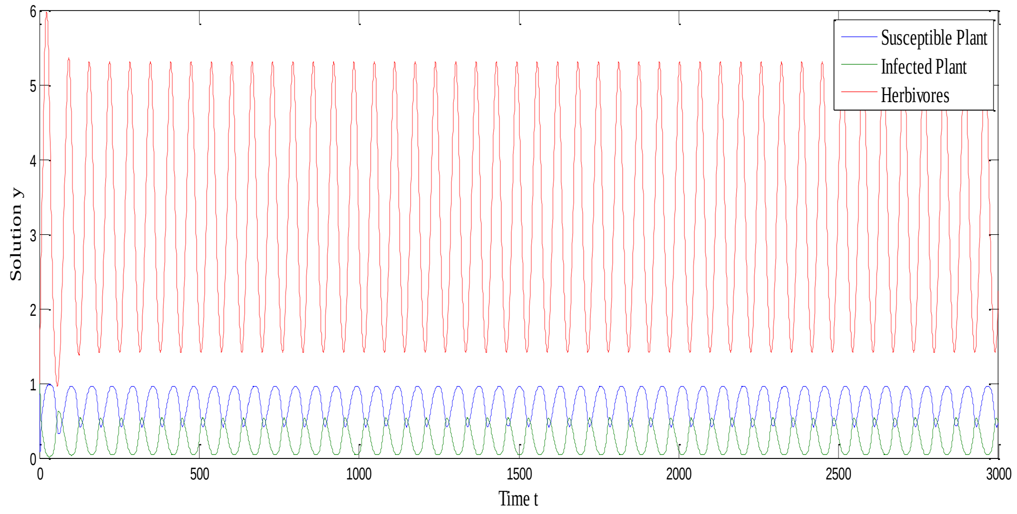

Figure 5.4 $E^*(0.5014, 0.4673, 1.4156)$ loses stability, and Hopf-bifurcation observes when time lag surpasses the threshold point that is $\tau \geq 10.6$.

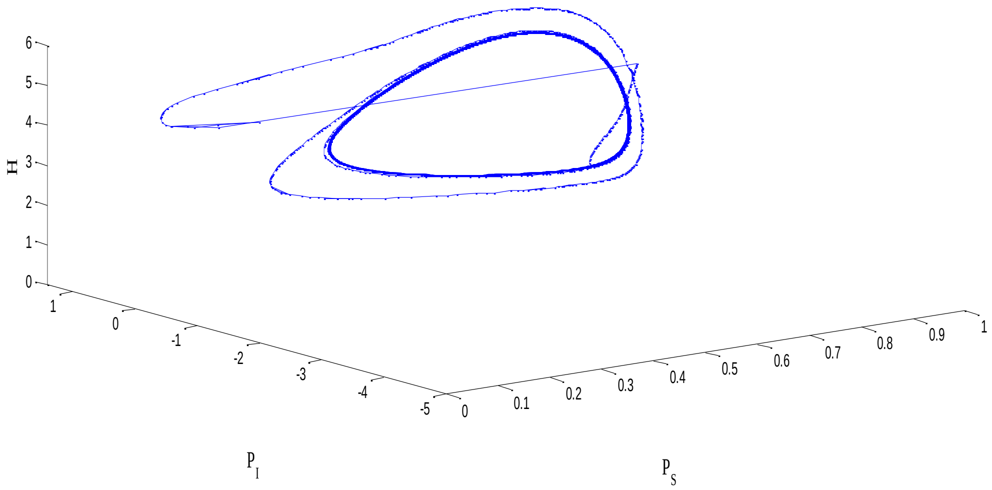

Figure 5.5 Phase space diagram showing the Hopf- bifurcation of $E^*(0.5014, 0.4673, 1.4156)$ when time lag crosses the threshold value that is $\tau \geq 10.6$.

5.6 Sensitivity Analysis

Sensitivity analysis represents the impact on the stability behaviour of the state variables when we vary the parameter values at variance with the time lag that is τ. For instance**,** the partial derivatives of the system (P_S, P_I, H) w.r.t. δ

(herbivores gain by consuming infectious plant) has the following set of sensitivity equations:

$$\frac{dS_1}{dt} = \left[-3{P_S}^2 + 2(b+1)P_S + (b-\alpha)P_I - 2P_SP_I\right]S_1 + \left[-{P_S}^2 + (b-\alpha)P_S\right]S_2 \tag{5.18}$$

$$\frac{dS_2}{dt} = \alpha P_I S_1 + (\alpha P_S - \beta H - \gamma)S_2 - \beta P_I S_3 \tag{5.19}$$

$$\frac{dS_3}{dt} = -\mu S_3 + \delta H(t-\tau)S_2(t-\tau) + \delta P_I(t-\tau)S_3(t-\tau) \tag{5.20}$$

$$\text{where } S_1 = \frac{\partial P_S}{\partial \delta}, S_2 = \frac{\partial P_I}{\partial \delta}, S_3 = \frac{\partial H}{\partial \delta}$$

5.6.1 Sensitivity of Variables to Parameter δ

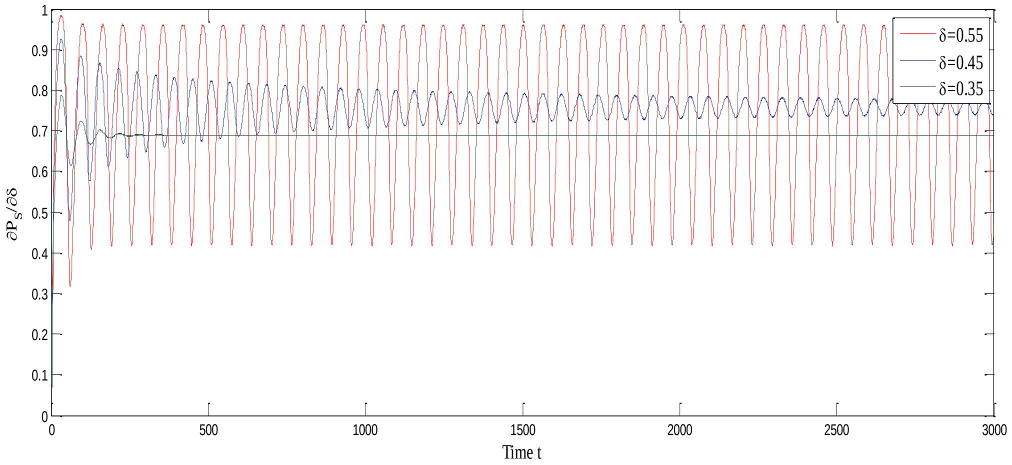

Figure 5.6 Time series graph of change in density of susceptible plant P_S with respect to change in the herbivores gain by consuming infectious plants (δ).

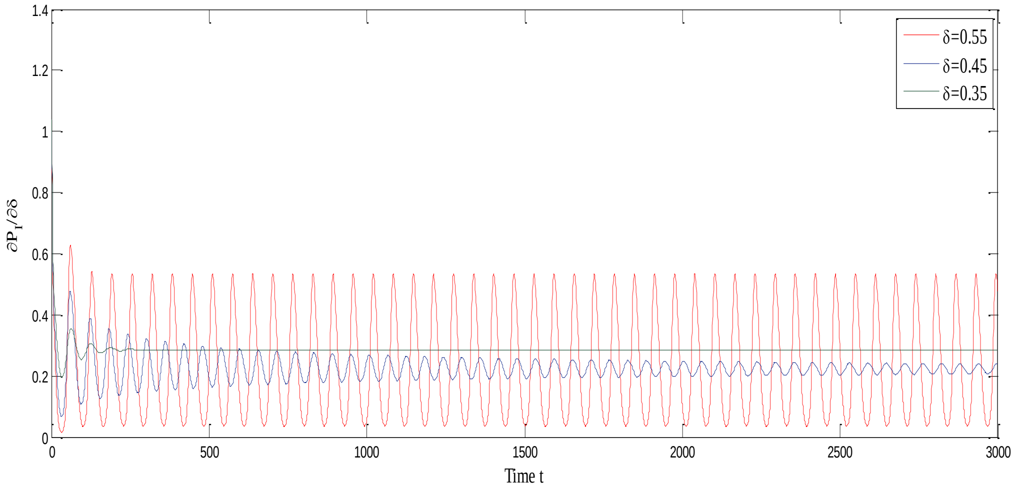

Figure 5.7 Time series graph of change in density of infectious plant P_I with respect to change in the herbivores gain by consuming infectious plant (δ).

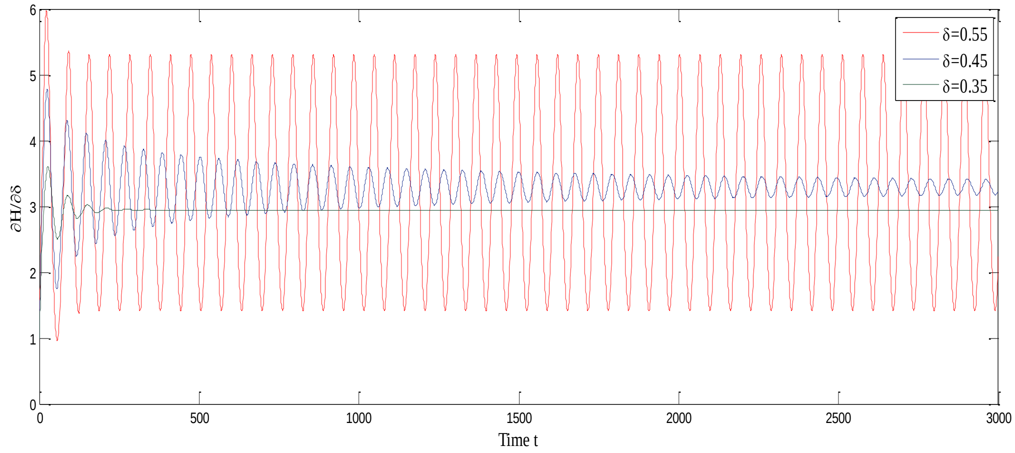

Figure 5.8 Time series graph of change in density of herbivores H with respect to change in the herbivores gain by consuming infectious plant (δ).

5.6.2 Sensitivity of Variables to Parameter μ

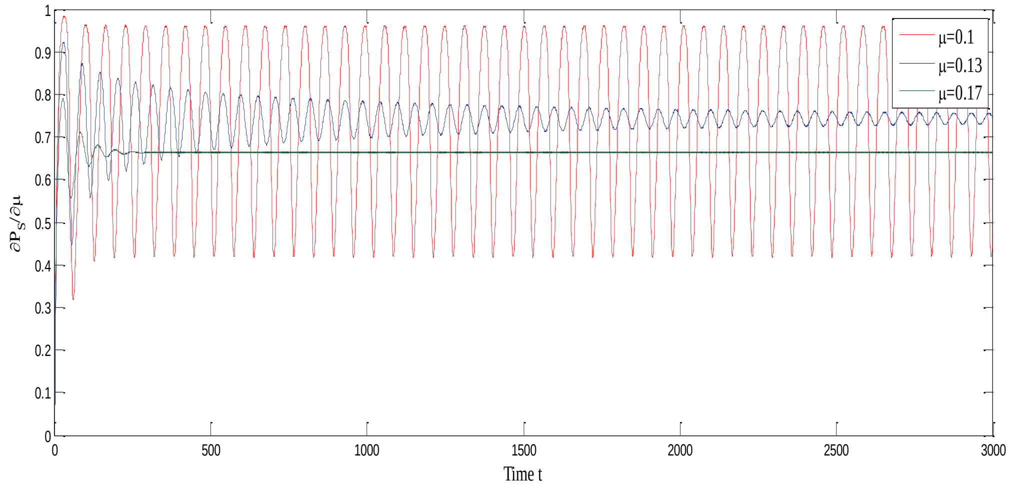

Figure 5.9 Time series graph of change in density of susceptible plant P_S with respect to change in the mortality rate of herbivores (μ).

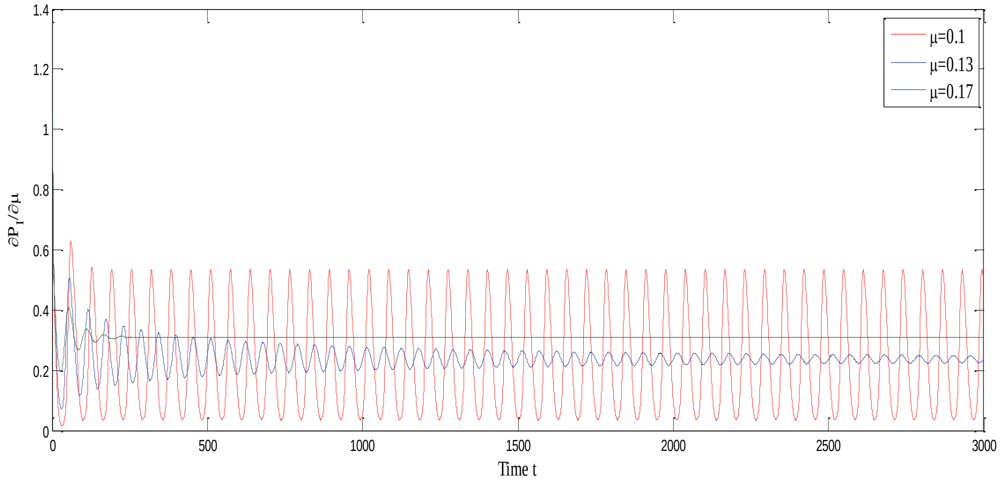

Figure 5.10 Time series graph of change in density of infectious plant P_I with respect to change in the mortality rate of herbivores (μ).

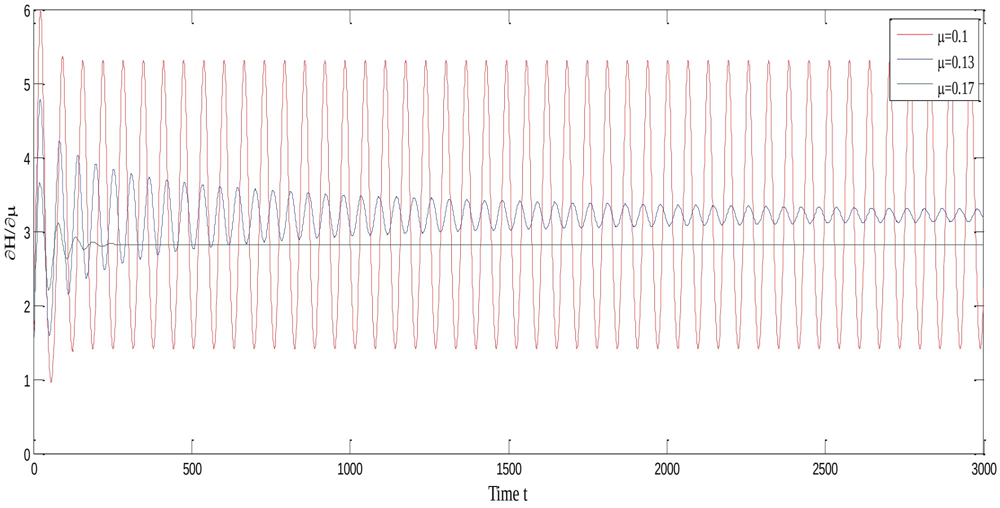

Figure 5.11 Time series graph of change in density of herbivores H with respect to change in the mortality rate of herbivores (μ).

5.7 Results and Discussion

Figure 5.1 indicates that the equilibrium $E^*(0.8013, 0.1818, 3.5127)$ is absolutely stable when the time lag is absent that is $\tau = 0$. The system destabilizes and shows asymptotic stability when the time lag is lesser than the threshold point which is $\tau < 10.6$ as represented in figure 5.2. When the time lag crosses the threshold point as shown in figure 5.4, then the system undergoes Hopf-bifurcation that is $\tau \geq 10.6$. Figures 5.3 and 5.5 indicate the phase space diagrams for the asymptotic stability and Hopf-bifurcation.

The time series graphs are indicated in figures 5.6 to 5.11. As represented in figure 5.6, figure 5.7, and figure 5.8, when we decrease the value of δ (herbivores gain by consuming infectious plants), the entire system which comprises the densities of susceptible plants, infectious plants, and herbivores (P_S, P_I, H), changes its behaviour from Hopf bifurcation to asymptotic and then asymptotic to absolutely stable. When δ (herbivores gain by consuming infectious plants) changes from $\delta = 0.55$ to $\delta = 0.45$, the equilibrium shifts its behaviour from Hopf bifurcation to asymptotic. Furthermore, when we decrease the value of δ (herbivores gain by consuming infectious plant) from $\delta = 0.45$ to $\delta = 0.35$, the system again changes its behaviour to absolute stability.

Specifically, initially, when herbivores consume infectious plants, then it is beneficial for them as it can satisfy their hunger and facilitate reproduction. However, as the herbivore population increases, they start competing with each other for food, leading to a decline in the plant population. Consequently, the system experiences initial fluctuations. Subsequently, when herbivores gain by consuming infectious plants start decreasing, the herbivore population starts declining, and the entire system tends towards absolute stability. Moreover, figure 5.9, figure 5.10, and figure 5.11 indicates that as we increase the mortality rate of herbivores μ, the state variables which include the densities of susceptible plant, infectious plant, and herbivores (P_S, P_I, H) change its behaviour from Hopf-bifurcation to asymptotic and finally absolute stability. When there is an increment in the value of the mortality rate of herbivores μ from $\mu = 0.1$ to $\mu = 0.13$, the equilibrium shifts its behaviour from Hopf-bifurcation to asymptotic. Similarly, when there is an increment in the value of the mortality rate of herbivores μ from $\mu = 0.13$ to $\mu = 0.17$, every state variable tends to an equilibrium point that is stable. It means that as the mortality rate of herbivores increases by consuming infectious plants, the infectious plant decreases. This implies an increase in susceptible plants. Consequently, this scenario leads the entire system towards stability.

5.8 Conclusion

The eco-epidemiological model incorporates the susceptible plant, infectious plant, and herbivores, and the weak Allee effect is analyzed. The Holling type-I response is used. The main assumption is that susceptible plants generate infectious plants. The delay differential equations are used as many processes depend on the history and it is important for the gestation period. The non-zero equilibrium points are calculated. It is represented that the system is stable when there is a negligible time lag that is when $\tau = 0$. When time lag is below the threshold value that is when $\tau < 10.6$, then the system demonstrates asymptotic stability. Additionally, when time lag surpasses the threshold value that is when $\tau \geq 10.6$, then the system becomes unstable, and Hopf- bifurcation is observed. The phase space diagrams for the asymptotic stability and Hopf-bifurcation are also represented. The time series graphs which represent the dynamics of the system are also included.

Chapter 6

Mathematical Analysis of Delayed Gauss-type Plant-Herbivore Model with Holling type-III Functional response

6.1 Introduction

Nowadays, the topic of mathematical modelling is curiously studied by many ecologists. It helps us to solve real-world problems theoretically as well as experimentally by using graphs, diagrams, equations, etc. It gives accurate strategies to solve problems. As a consequence of this, the interest of ecologists is increasing gradually in this topic, and several attempts have been made to predict the solutions for real-world problems mathematically. The authors generated distinct types of mathematical models to solve real-life problems. Lotka [38] and Volterra [39] were the developers of the predator-prey model in 1927. Now, their model has been considered the base of several models and studies. Canale [179] developed a model of interaction between predator and prey as host population using nonlinear ordinary differential equations. It is proved that the periodic solution may either depend on the initial conditions or limit cycles and it is not necessary that an aperiodic solution will exhibit oscillatory behaviour. Many models are modified versions of the prey-predator model [40]–[44]. Plant-herbivore models, Leslie-Gower predator-prey models, and Gauss-type predator-prey models are some examples of the modified versions of predator-prey models. Aziz-Alaoui and Okiye [96] proposed a modification of the Leslie-Gower predator-prey and presented boundedness and stability about the interior point. Farajzadeh et al. [103] analyzed Gauss predator-prey model which includes one prey and two predators and they analyzed the stability behaviour of the interacting species. Han et al. [180] proposed a predator-prey model with SIS or SIR parasitic infection. They concluded that if the prey population is infected and the predator population has feeding effectiveness to survive, even then the predator population becomes infected. Rojas-Palma and Gonzalez-Olivares [98] analyzed Gauss-type predator-prey model accompanied by Allee effect and sigmoid response. The optimal harvest policy and the fishing effort are described by solving the autonomous ordinary differential equations. Naji and Mustafa [99] analyzed the behaviour of the Eco-epidemiological model with SI-type disease in prey and a non-linear incidence rate. Kumar and Verma [59] proposed the plant-herbivore model with both Allee effects. The perishing of both plants and herbivore populations is presented by considering the strong Allee effect. The stability and instability of the system at non-zero equilibrium are also presented. A model with the shoot and root compartments is proposed by Kalra and Kumar [138] to study the growth of plants. A delay parameter is included which disturbs the stability of the system and shows Hopf-bifurcation. Kalra and Kumar [124] developed a model with toxic metals and studied the impact of time delay on plant biomass. Singh [139] investigated the SIR epidemic model and proved that Hopf-bifurcation occurred under certain conditions. Asfaw et al. [27] considered the extended stochastic plant-herbivore model, incorporating the Allee effect, and found that the dynamics produce more changes when the noise is added to the herbivores as compared to the plant population. Asfaw et al. [30] reformulated the plant-herbivore model with Allee effect that mortality rate of herbivores is dependent on the plant population and the threshold conditions for the non-extinction of the herbivores are also demonstrated. Asymptotically stable 2-periodic systems were used by Elaydi and Sacker [69] to show the occurrence of carrying capacity in a mathematical model with the Allee effect. Li [36] established plant-herbivore model with toxicity in plants which causes SIS parasitic infection in herbivores. It is demonstrated that the interaction between both populations has a dramatic impact on population dynamics. The boundedness, periodic nature, and stability of plant-herbivore model, considering both difference and differential equations is studied by Kartal [37]. Yousef and Yousef [29] conducted a study on the plant-herbivore model, utilizing fractional-order differential equations to demonstrate the occurrence of flip bifurcation. This analysis involved the application of bifurcation theory and the center manifold theorem. Kumar and Verma [181] conducted a study on plant-herbivore model, considering a strong Allee effect and concluded that Hopf-bifurcation is seen when time parameter surpasses the critical value. Chan et al. [182] explored the necessity of comprehending the complex patterns of predator foraging behavior in natural systems through an investigation into how alternate prey and predator interference affect density-dependent predation. Huang et al. [183] examined a Leslie-type predator-prey system with a generalized Holling type-III functional response. They demonstrated that the model simultaneously undergoes subcritical Hopf bifurcation and Bogdanov–Takens bifurcation within the small neighbourhoods of the two degenerate equilibria. Morozov [184] suggested a generic model and demonstrated that changes in the vertical distribution of food, along with the active foraging behavior of zooplankton, can alter the nature of the functional response.

Several types of mathematical models are introduced and examined by different authors in the literature. The authors also modified the existing models and then examined them. In the current work, a delayed Gauss-type Plant-herbivore model under the Allee effect is examined by consolidating delay as a time parameter. The Holling type-III functional response is utilized. The stability and sensitivity analysis to demonstrate the variations in the dynamics of the system are plotted in the graph.

6.2 Mathematical Model

Let $P = P(t)$ and $H = H(t)$ be the plants and herbivores populations respectively with respect to time t. Therefore, a Gauss-type Plant-herbivore model, considering the Allee effect and time parameter τ is presented below:

$$\frac{dP}{dt} = rP\left(1 - \frac{P(t-\tau)}{K}\right)(P - b) - \frac{\alpha P^2}{P^2 + \beta^2}H \tag{6.1}$$

$$\frac{dH}{dt} = \left(\frac{\gamma P^2}{P^2 + \beta^2} - d\right)H \tag{6.2}$$

The initial conditions of the above system are supplemented as follows:

$$P(t) > 0,\ H(t) > 0\ \forall\ t \geq 0$$

$$\text{and } P(t-\tau) = \varepsilon, \text{ constant } \forall\ t \in [0,\tau].$$

Also, $P \leq K,$ and $b \leq 0$ for weak Allee effect. Moreover, the parameters $r, K, b, \alpha, \beta, \gamma, and\ d$ are taken to be positive.

The proposed model provides the following ecological interpretations for the notations or parameters used within it:

r is called internal growth rate of plants, K represents the carrying capacity of the plants, b is called the Allee constant, α is known as the maximum per capita consumption rate, β is called the capturing rate, $\gamma = e\beta$ stands for the conversion effectiveness of consumed plants, d is the mortality rate of herbivores and τ is called time parameter which represents the past history. The parameters $r, K, b, \beta, \gamma, and\ d$ are positive constants.

6.3 Equilibrium point of the model

In this section, non-trivial equilibrium is considered which characterizes the co-existence of all the state variables (P^*, H^*), where none of them is zero, because the stability of the system shows no impact of delay on any of the other equilibrium points. For feasible non-trivial equilibrium, consider $P(t-\tau) \cong P(t)$ and equate the system of equations (6.1) and (6.2) to zero:

$$\therefore \frac{dH}{dt} = 0$$

$$\Rightarrow P^* = \sqrt{\frac{d}{\gamma - d}}\beta, \text{provided } \gamma > d \tag{6.3}$$

$$\text{and } \frac{dP}{dt} = 0$$

$$\Rightarrow H^* = \frac{r\beta\gamma}{\alpha\sqrt{d(\gamma-d)}}\left(1 - \frac{\beta}{K}\sqrt{\frac{d}{\gamma-d}}\right)\left(\beta\sqrt{\frac{d}{\gamma-d}} - b\right), \tag{6.4}$$

$$\text{provided } \gamma > d$$

Thus, the non-trivial equilibrium is given by:

$$E^*(P^*, H^*) = E^*\left(\sqrt{\frac{d}{\gamma - d}}\beta, \frac{r\beta\gamma}{\alpha\sqrt{d(\gamma - d)}}\left(1 - \frac{\beta}{K}\sqrt{\frac{d}{\gamma - d}}\right)\left(\beta\sqrt{\frac{d}{\gamma - d}} - b\right)\right)$$

6.4 Stability Analysis and Hopf-Bifurcation

The following system of equations is used to examine the dynamics of the equilibrium $E^*(P^*, H^*)$ of the delayed Gauss-type Plant-herbivore model under the Allee effect:

$$\frac{dP^*}{dt} = rP^*\left(1 - \frac{P^*(t-\tau)}{K}\right)(P^* - b) - \frac{\alpha {P^*}^2}{{P^*}^2 + \beta^2}H^* \tag{6.5}$$

$$\frac{dH^*}{dt} = \left(\frac{\gamma {P^*}^2}{{P^*}^2 + \beta^2} - d\right)H^* \tag{6.6}$$

Differentiating the system of equations (6.5)-(6.6) w.r.t P^*,

$$m_1 = r(2P^* - b) + \frac{rP^*}{K}(b - P^*)e^{-\lambda\tau} - \frac{2\alpha\beta^2}{\left({P^*}^2 + \beta^2\right)^2}P^*H^*;$$

$$m_2 = \frac{2\gamma\beta^2}{\left({P^*}^2 + \beta^2\right)^2}P^*H^*.$$

Differentiating the system of equations (6.5)-(6.6) w.r.t H^*,

$$m_3 = -\frac{\alpha {P^*}^2}{\left({P^*}^2 + \beta^2\right)};$$

$$m_4 = \frac{\gamma {P^*}^2}{\left({P^*}^2 + \beta^2\right)} - d.$$

Equation (6.7) is the exponential characteristic equation for the system of equations (6.5)–(6.6):

$$\begin{vmatrix} \lambda - m_4 & m_2 \\ m_3 & \lambda - m_4 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + A_1\lambda + A_2 + e^{-\lambda\tau}(A_3\lambda + A_4) = 0 \tag{6.7}$$

$$\text{where } A_1 = -r(2P^* - b) + \frac{2\alpha\beta^2}{\left(P^{*2}+\beta^2\right)^2} P^*H^* - m_4;$$

$$A_2 = \left[r(2P^* - b) - \frac{2\alpha\beta^2}{\left(P^{*2} + \beta^2\right)^2} P^*H^* \right] m_4 - m_2m_3;$$

$$A_3 = -\frac{rP^*}{K}(b - P^*);$$

$$A_4 = \frac{rP^*}{K}(b - P^*)m_4.$$

The equation (6.7) at $\tau = 0$ is :

$$\lambda^2 + (A_1 + A_3)\lambda + (A_2 + A_4) = 0 \tag{6.8}$$

The entire system is stable which means the equation (6.8) will have $-ve$ real roots by Routh-Hurwitz's criteria if:

$$(\boldsymbol{R_1}): (A_1 + A_3) > 0 \text{ and } (\boldsymbol{R_2}): (A_2 + A_4) > 0$$

Now, with changes in the values of the time parameter (τ), verify the switching of the roots of $-ve$ real parts to $+ve$ real part:

Consider, the root of equation (6.7) to be $\lambda = i\omega$,

then the following result:

$$(i\omega)^2 + A_1(i\omega) + A_2 + [A_3(i\omega) + A_4]e^{-(i\omega)\tau} = 0$$

$$\Rightarrow -\omega^2 + A_1(i\omega) + A_2 + \ [A_3(i\omega) + A_4](\cos\omega\tau - i\sin\omega\tau\,) = 0$$

We get the following equation for separating the real parts from the imaginary part:

$$-\omega^2 + A_2 = -A_4\cos\omega\tau - A_3\omega\sin\omega\tau \tag{6.9}$$

$$A_1\omega = -A_3\cos\omega\tau + A_4\sin\omega\tau \tag{6.10}$$

$\therefore$ ω satisfies the following:

$$\omega^4 - \left({A_3}^2 - {A_1}^2 + 2A_2\right)\omega^2 + \left({A_2}^2 - {A_4}^2\right) = 0 \tag{6.11}$$

Equation (6.11) has the following roots:

$$\omega_{1,2}^2 = \frac{({A_3}^2 - {A_1}^2 + 2A_2) \pm \sqrt{({A_3}^2 - {A_1}^2 + 2A_2)^2 - 4({A_2}^2 - {A_4}^2)}}{2} \tag{6.12}$$

$\omega_{1,2}^2$ has no $+ve$ roots if:

$$(\boldsymbol{R_3}): ({A_3}^2 - {A_1}^2 + 2A_2) < 0 \ and \ ({A_2}^2 - {A_4}^2) > 0$$

$$\text{Or } ({A_3}^2 - {A_1}^2 + 2A_2)^2 < 4({A_2}^2 - {A_4}^2)$$

This implies that if the condition (R_3) is true, there is no $+ve$ root of equation (6.12).

Now, there are some lemmas [162]:

Lemma 6.4.1 Every root of equation (6.7) has $-ve$ real parts $\forall \, \tau \geq 0$ when $(R_1) - (R_2)$ is true.

On the contrary, if:

$$(\boldsymbol{R_4}): ({A_2}^2 - {A_4}^2) < 0 \ or \ ({A_3}^2 - {A_1}^2 + 2A_2) > 0$$

And

$$({A_3}^2 - {A_1}^2 + 2A_2)^2 = 4({A_2}^2 - {A_4}^2)$$

Then, ω_1^2 is the $+ve$ root of equation (6.9).

Also, if:

$$(\boldsymbol{R_5}): ({A_2}^2 - {A_4}^2) > 0 \ or \ ({A_3}^2 - {A_1}^2 + 2A_2) > 0$$

$$\text{And } ({A_3}^2 - {A_1}^2 + 2A_2)^2 > 4({A_2}^2 - {A_4}^2)$$

$\therefore \ \omega_{1,2}^2$ are the two $+ve$ roots of equation (6.9).

When the time parameter (τ) takes certain values, equation (6.7) exhibits purely imaginary roots in both- (R_4) and (R_5).

The system of equations (6.7)-(6.8) gives the critical values $\tau_j^{\pm}$ of time parameter (τ):

$$\tau_j^{\pm} = \frac{1}{\omega_{1,2}} cos^{-1}\left[\frac{A_4(\omega_{1,2}^2 - A_2) - A_1 A_3 \omega_{1,2}^2}{{A_3}^2 \omega_{1,2}^2 + {A_4}^2}\right] + \frac{2j\pi}{\omega_{1,2}}, j = 0,1,2, \dots \quad (6.13)$$

The succeeding lemma condenses the aforementioned discussion [162]:

Lemma 6.4.2 (A) If conditions $(R_1) - (R_2)\ and\ (R_4)$ are satisfied, equation (6.7) will have two purely imaginary roots $\pm i\omega_1$.

(B) Equation (6.7) exhibits a pair of purely imaginary roots $\pm i\omega_1(\pm i\omega_2\ respectively)$ when

$(R_1) - (R_2)\ and\ (R_5)$ is true
and $\tau = \tau_j^+ \left(\tau = \tau_j^-\ respectively\right)$.

According to our assumption, when $\tau > \tau_j^+$ and $\tau < \tau_j^-$, the $-ve$ real part of certain roots of equation (6.7) will shift to the $+ve$ real part.

$$\therefore \text{Let } \tau_j^{\pm} = \mu_j^{\pm}(\tau) + i\omega_j^{\pm}(\tau); j = 0,1,2,3, \dots$$

$$\mu_j^{\pm}(\tau_j^{\pm}) = 0, \omega_j^{\pm}(\tau_j^{\pm}) = \omega_{1,2}$$

is satisfied by the roots of the equation (6.7).

We can demonstrate that the transversality criterion stated below is true:

$$\frac{d}{d\tau}\left(Re\ \lambda_j^+(\tau_j^+)\right) > 0 \text{ and } \frac{d}{d\tau}\left(Re\ \lambda_j^-(\tau_j^-)\right) < 0$$

$\therefore$ $\tau_j^{\pm}$ are known as the bifurcating values.

The scattering of the roots of equation (6.7) is explained by the following theorem [162].

Theorem 6.1 Consider equation (6.13) defines $\tau_j^+ (j = 0,1,2,3, \dots)$.

(A) Every root of equation (6.7) has $-ve$ real part $\forall\ \tau \geq 0$ when $(R_1)\ and\ (R_2)$ is true.

(B) Every root of equation (6.7) has $-ve$ real part when $(R_1), (R_2)\ and\ (R_4)$ are true and $\tau \in [0, \tau_0^+)$.
There are two purely imaginary roots $\pm i\omega_1$ of equation (6.7) if $\tau = \tau_0^+$. There is at least one root with $+ve$ real part of the equation (6.5) if:

$$\tau > \tau_0^+.$$

(C) When $(R_1), (R_2)\ and\ (R_5)$ is true, then there exists a $+ve$ integer m such that

$$0 < \tau_0^+ < \tau_0^- < \tau_1^+ < \tau_1^- - -\!-< \tau_{m-1}^- < \tau_m^+$$

and there are m fluctuations between stability and instability. That is, every root has $-ve$ real part in equation (6.7) if $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), - - -, (\tau_{m-1}^-, \tau_m^+)$. There is at least one root with a $+ve$ real part in equation (6.7) when

$$\tau \in (\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), - - -, (\tau_{m-1}^+, \tau_{m-1}^-) \text{ and } \tau > \tau_m^+.$$

6.5 Numerical Example

The results of the change of the behaviour of the system graphically using MATLAB for distinct values of time parameter (τ) are described by the following values:

$$r = 0.8, K = 0.4, b = -0.3, \alpha = 0.1, \beta = 0.1, \gamma = 0.9, d = 0.4$$

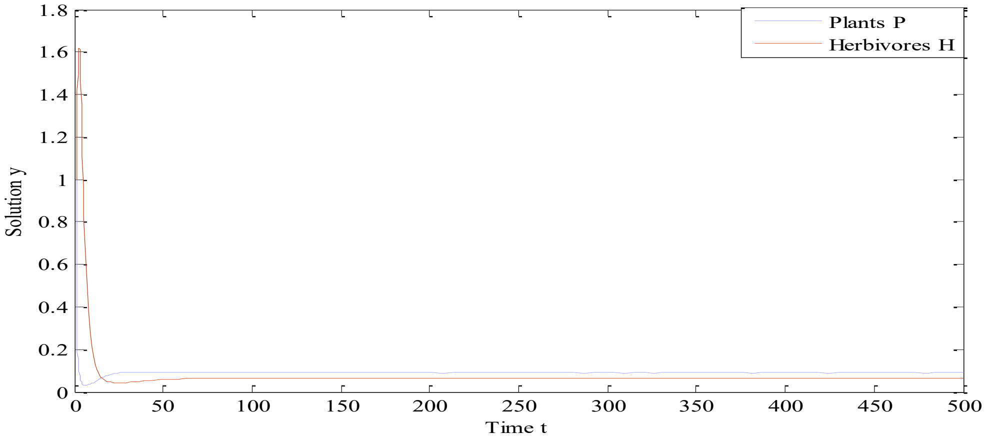

Figure 6.1 Equilibrium $E^*(0.0895, 0.4864)$ demonstrate absolute stability when the system has a negligible delay which means when $\tau = 0$.

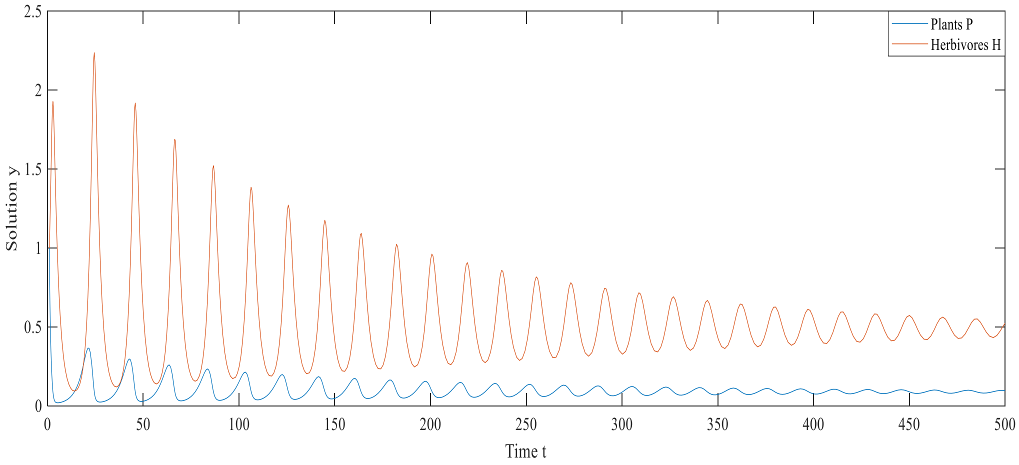

Figure 6.2 Equilibrium $E^*(0.0977,0.4691)$ demonstrate asymptotic stability when the system has a time parameter that is lesser than the threshold value 3.2 which means when $\tau < 3.2$.

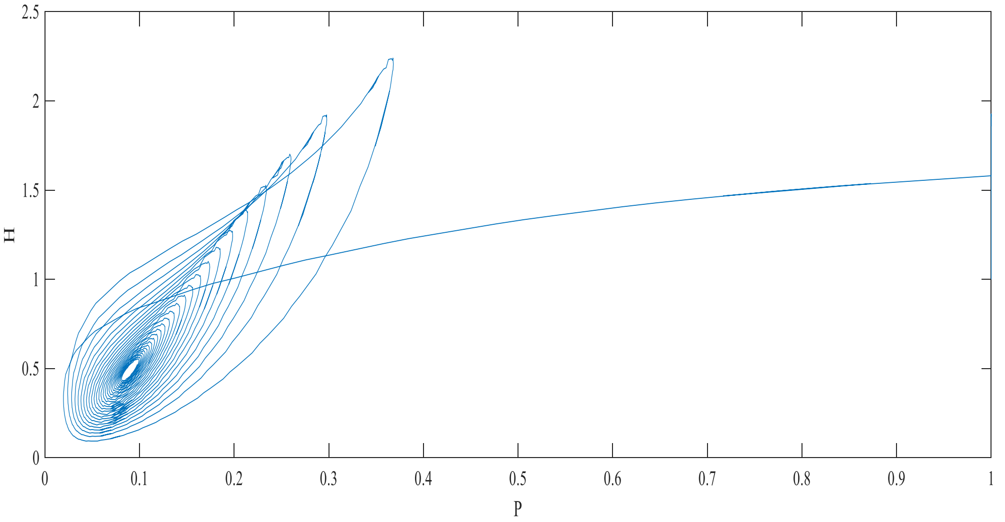

Figure 6.3 Equilibrium $E^*(0.0977,0.4691)$ demonstrate phase plane graph of asymptotic stability when the system has a time parameter that is lesser than the threshold value 3.2 which means when $\tau < 3.2$.

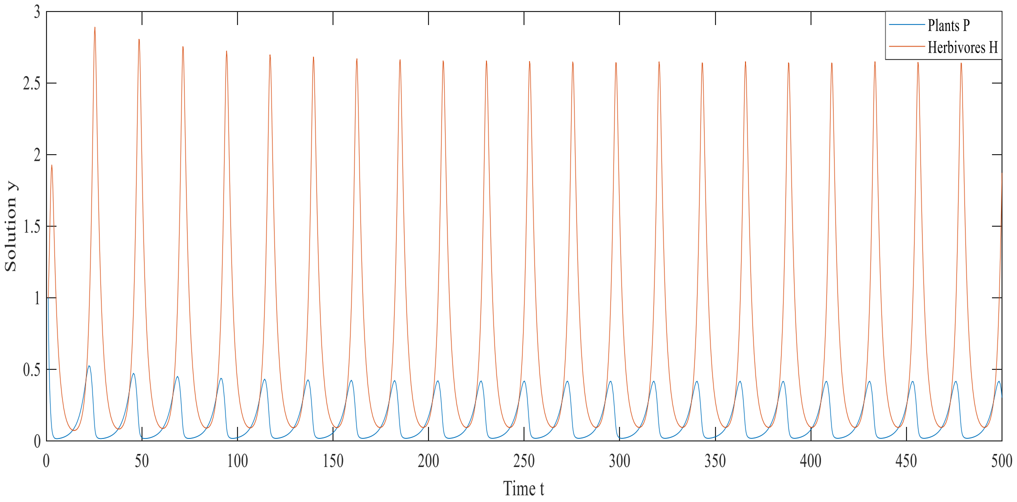

Figure 6.4 Equilibrium $E^*(0.3526,0.4541)$ demonstrate Hopf-bifurcation when the system has a time parameter that surpasses the threshold value 3.2 which means when $\tau \geq 3.2$.

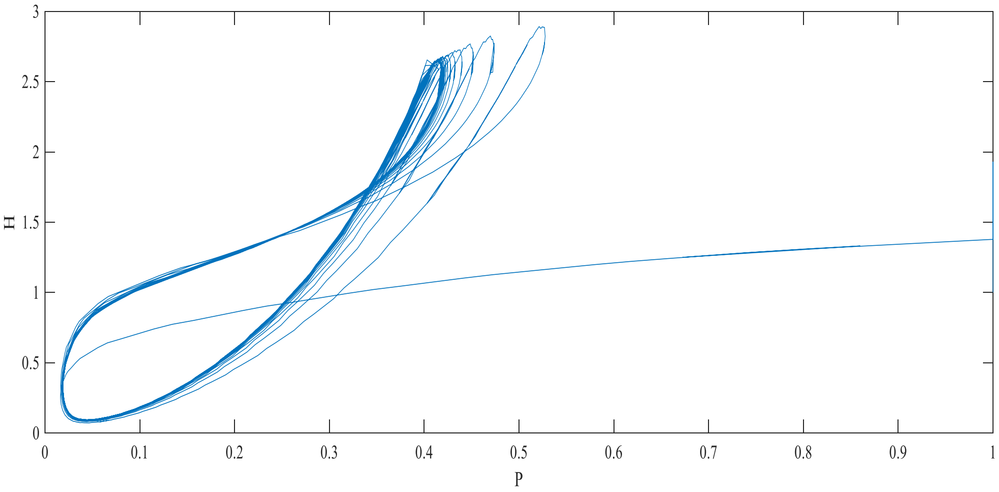

Figure 6.5 Equilibrium $E^*(0.3526,0.4541)$ demonstrate phase plane graph of Hopf-bifurcation when the system has a time parameter that surpasses the threshold value 3.2 which means when $\tau \geq 3.2$.

6.6 Sensitivity Analysis

Sensitivity analysis is research that uses the "Direct method" to examine how the system responds to variations in the model parameters that differ from changes in the time parameter (τ). For an illustration, the sensitivity equations of the partial derivatives of the solution (P, H) with respect to the conversion effectiveness of consumed plants (γ) are given below:

$$\frac{dS_1}{dt} = \left[r(2P - b) - \frac{2\alpha\beta^2}{(P^2 + \beta^2)^2} PH + r\left(\frac{b - 2P}{K}\right) P(t - \tau) \right] S_1 - \frac{\alpha P^2}{P^2 + \beta^2} S_2 + \frac{rP(b - P)}{K} S_1(t - \tau) \tag{6.14}$$

$$\frac{dS_2}{dt} = 2\beta^2 \gamma \frac{PH}{(P^2 + \beta^2)^2} S_1 + \left(\frac{\gamma P^2}{P^2 + \beta^2} - d\right) S_2 \tag{6.15}$$

where $S_1 = \frac{\partial P}{\partial \gamma}$ and $S_2 = \frac{\partial H}{\partial \gamma}$

6.6.1 Sensitivity of Variables to Parameters γ

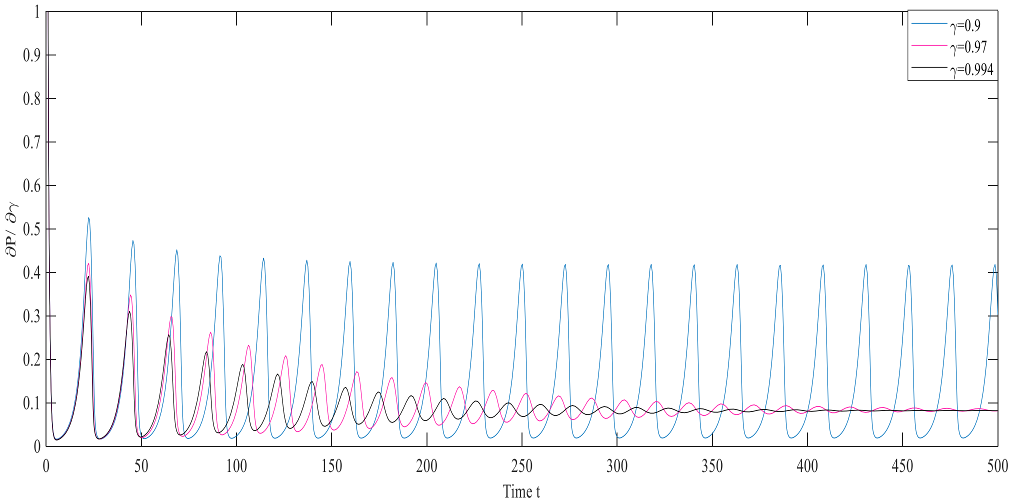

Figure 6.6 Time series graph of change in the size of plant population P with respect to changes in the conversion effectiveness of consumed plants (γ).

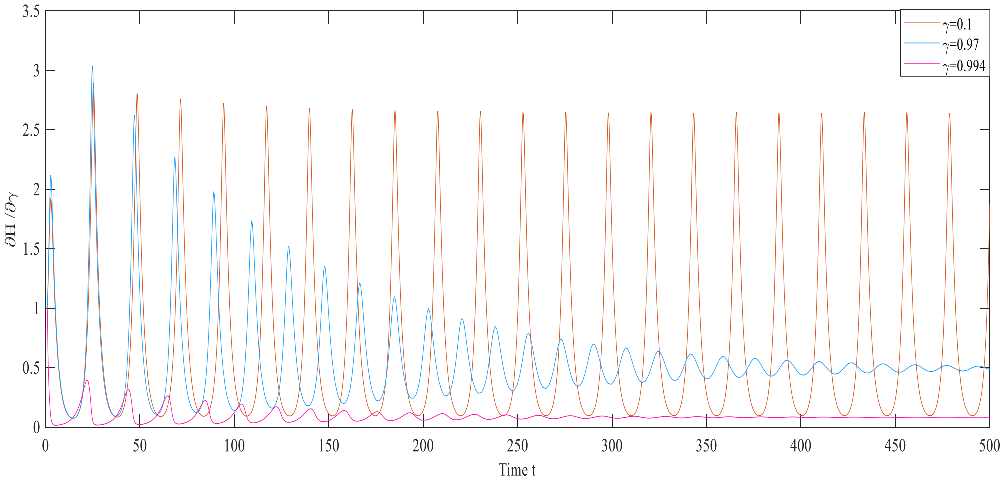

Figure 6.7 Time series graph of change in the size of herbivore population H with respect to changes in the conversion effectiveness of consumed plants (γ).

6.6.2 Sensitivity of Variables to Parameter β

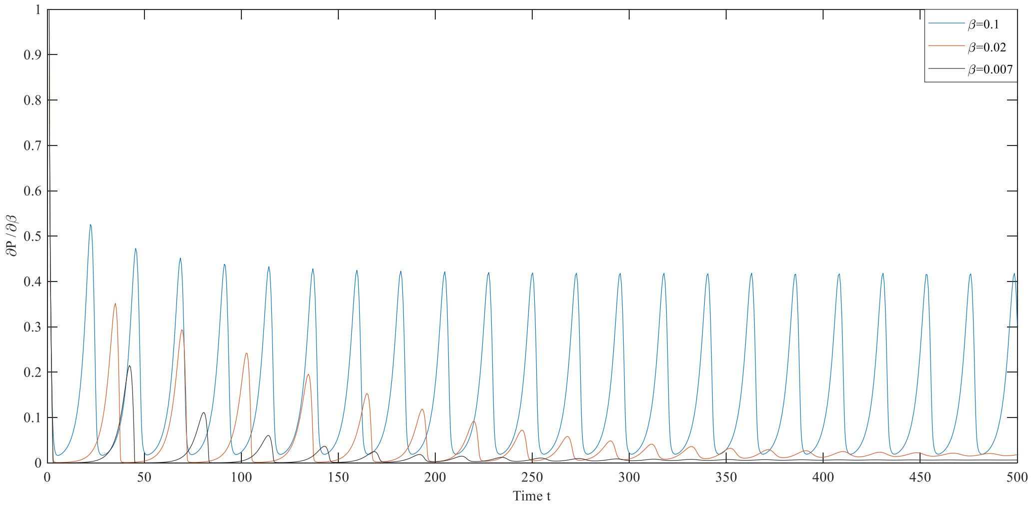

Figure 6.8 Time series graph of change in the size of plant population P with respect to changes in the number of plants where the maximum rate of predation of the herbivores occur (β).

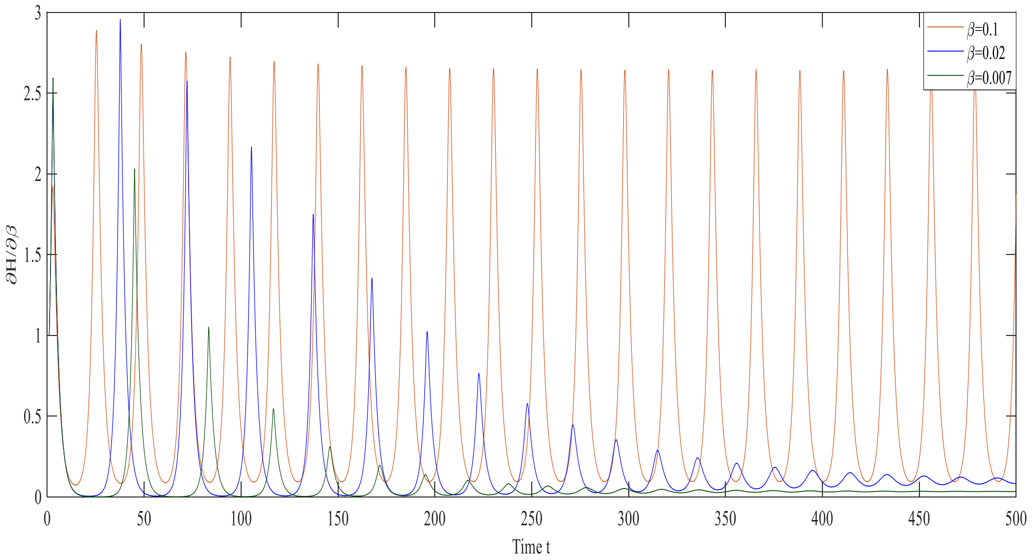

Figure 6.9 Time series graph of change in the size of herbivore population H with respect to changes in the number of plants where the maximum rate of predation of the herbivores occur (β).

6.7 Results and Discussion

Figure 6.1 indicates that after initial fluctuations, the equilibrium $E^*(0.0895,0.4864)$ tends to be absolutely stable when there is a negligible delay which means when $\tau = 0$. The observation of this fact is analytically demonstrated by Routh-Hurwitz's criteria $(K_1) - (K_2)$ as in lemma 6.4.1. Figure 6.2 represents that the fluctuations are initially noticeable, but they diminish over time. Therefore, the equilibrium $E^*(0.0977,0.4691)$ demonstrate asymptotic stability when the time parameter is below the threshold value which means when $\tau < 3.2$. Figure 6.4 reveals that the fluctuations are infinite and it is represented that the periodic solutions have large amplitude and limit cycle trajectory. Due to Hopf-bifurcation, the equilibrium $E^*(0.3526,0.4541)$ exhibits complex behaviour and limit cycles emerge when the time parameter exceeds the threshold value that is when $\tau \geq 3.2$. The same fact of asymptotic stability and Hopf-bifurcation is supported by $(K_4) - (K_5)$ as in lemma 6.4.2. Figure 6.3 and figure 6.5 are the phase plane graphs of asymptotic stability and Hopf-bifurcation respectively.

Moreover, the variation in the system behaviour is shown by the variations in the model parameters at variance with the time parameter (τ). That means, when we increase the value of the conversion effectiveness of consumed plants (γ) from $\gamma = 0.9$ to $\gamma = 0.97$, then the system (P, H) shifts its behaviour from Hopf-bifurcation to asymptotic stability. Again, when we increase $\gamma = 0.97$ to $\gamma = 0.994$, eventually, as depicted in figures 6.6 and 6.7, the state variables of the system start to converge into a stable equilibrium. Also, figure 6.8 and figure 6.9 indicate that when we decrease the number of plants where the maximum rate of predation of the herbivores occur (β) from $\beta = 0.1$ to $\beta = 0.02$, the state variables (P, H) changes its behaviour from Hopf-bifurcation to asymptotic. Similarly, when we decrease $\beta = 0.02$ to $\beta = 0.007$, then the entire system shifts its dynamics from asymptotic stability to absolute stability.

6.8 Conclusion

A delayed Gauss-type Plant-herbivore model is considered in this article. The growth of the plant population has an impact on the Allee effect. The Holling type- III response is used which characterizes the fact that when there is an increment in the plant population, then the herbivore predation rate will also increase. The time parameter τ is incorporated which leads to introduce complex behaviour with limit cycles, periodic solutions, and the bifurcation occurrence. The feasible non-trivial equilibrium point is calculated. The change of behaviour of the proposed model from being absolutely stable to asymptotically stable and then asymptotically stable to complex dynamics results in the occurrence of Hopf-bifurcation at equilibrium $E^*(P^*, H^*)$ for distinct values of time parameter τ is represented analytically as well as graphically.

Additionally, the variation in the behaviour of the system is represented using time series graphs by the variations in the model parameters at variance with the time parameter (τ). It is concluded that when we increase the value of the conversion effectiveness of consumed plants or decrease the number of plants where the maximum rate of predation of the herbivores occurs, then the entire system changes its behaviour and starts converging to a stable equilibrium.

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APPENDICES

Units of Parameters used in Chapter 2

Parameters	Description	Units
$\boldsymbol{r}$	Plant intrinsic growth rate	$days^{-1}$
$\boldsymbol{K}$	Carrying capacity of the environment	$days^{-1}$
$\boldsymbol{a}$	Allee threshold	$days^{-1}$
$\boldsymbol{b}$	Capturing rate/Harvesting rate	$days^{-1}$
$\boldsymbol{c = eb}$	Conversion rate of plants into herbivores	$days^{-1}$
$\boldsymbol{l}$	Death rate of Herbivores	$days^{-1}$
$\boldsymbol{\tau}$	Delay Parameter	$days^{-1}$

Units of Parameters used in Chapter 3

Parameters	Description	Units
$\boldsymbol{r}$	The internal growth rate of plants	$days^{-1}$
$\boldsymbol{C}$	Carrying capacity	$days^{-1}$
$\boldsymbol{b}$	Survival threshold of the plants	$days^{-1}$
$\boldsymbol{a_1}$	Capturing rate	$days^{-1}$
$\boldsymbol{a_2 = (ea_1)}$	Conversion rate	$days^{-1}$
$\boldsymbol{m}$	Half-capturing saturation constant	$days^{-1}$

$\boldsymbol{d}$	The death rate of herbivores	$days^{-1}$
$\boldsymbol{l}$	Intraspecific competition rate of herbivores	$days^{-1}$
$\boldsymbol{\tau}$	Time delay	$days^{-1}$

Units of Parameters used in Chapter 4

Parameters	Description	Units
$\boldsymbol{r}$	The internal growth rate of plants	$days^{-1}$
$\boldsymbol{u}$	The internal growth rate of herbivores	$days^{-1}$
$\boldsymbol{K}$	Carrying capacity of the plant population	$days^{-1}$
$\boldsymbol{\gamma}$	Measure of the quality of the plant as food for the herbivore	$days^{-1}$
$\boldsymbol{\alpha}$	The per capita rate of maximum predation	$days^{-1}$
$\boldsymbol{\beta}$	Half of the saturated response level	$days^{-1}$
$\boldsymbol{b}$	Allee threshold	$days^{-1}$
$\boldsymbol{\tau}$	Time parameter	$days^{-1}$

Units of Parameters used in Chapter 5

Parameters	Description	Units
$\boldsymbol{b}$	Allee threshold	$days^{-1}$
$\boldsymbol{\alpha}$	Rate of infection	$days^{-1}$

$\boldsymbol{\delta}$	Herbivores gain by consuming Infectious Plant	$days^{-1}$
$\boldsymbol{\gamma}$	Mortality rate of Infectious Plant	$days^{-1}$
$\boldsymbol{\beta}$	Attack rate of Herbivores	$days^{-1}$
$\boldsymbol{\mu}$	Mortality rate of Herbivores	$days^{-1}$
$\boldsymbol{\tau}$	Gestation period of Herbivores	$days^{-1}$

Units of Parameters used in Chapter 6

Parameters	**Description**	**Units**
$\boldsymbol{r}$	Internal growth rate of plants	$days^{-1}$
$\boldsymbol{K}$	Carrying capacity of the plants	$days^{-1}$
$\boldsymbol{b}$	Allee constant	$days^{-1}$
$\boldsymbol{\alpha}$	Maximum per capita consumption rate	$days^{-1}$
$\boldsymbol{\beta}$	Capturing rate	$days^{-1}$
$\boldsymbol{\gamma = e\beta}$	Conversion effectiveness of consumed plants	$days^{-1}$
$\boldsymbol{d}$	Mortality rate of herbivores	$days^{-1}$
$\boldsymbol{\tau}$	Time parameter	$days^{-1}$