STUDY ON THE SPACE OF QUASICONTINUOUS FUNCTIONS UNDER DIFFERENT TOPOLOGIES

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By

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<u>DECLARATION</u>

I, hereby declared that the presented work in the thesis entitled "**STUDY ON THE SPACE OF QUASICONTINUOUS FUNCTIONS UNDER DIFFERENT TOPOLOGIES**" in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of Dr. Sanjay Mishra, working as Professor in the Department of Mathematics of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgments have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled "**STUDY ON THE SPACE OF QUASICONTINUOUS FUNCTIONS UNDER DIFFERENT TOPOLOGIES**" submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the Department of Mathematics, is a research work carried out Chander Mohan, 11919689 is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

(Signature of Supervisor)

Name of supervisor: Dr. Sanjay Mishra

Designation: Professor

Department/school: Department of Mathematics

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Abstract

Throughout this thesis, the research work on $Q(X,Y)$ is a space of quasicontinuous function from a topological space X to topological space Y. Function spaces from a topological perspective have acquired great eminence from ages. Initiated from the set of continuous functions, mathematicians have studied many different topologies and attained tremendous benchmarks as applied in many different ways. The quite famous functions space of continuous functions has been studied concerning numerous topologies such as; Point-open topology, compact-open topology, uniform topology, fine topology, regular topology, etc. However, quite good literature is available regarding the analysis of such topologies on a slightly larger set containing quasicontinuous functions. The two most important reasons are: The first reason is the fairly strong relationship between continuity and quasicontinuity, despite the broad applicability of the latter concept. The second reason pertains to the significant connection between quasi-continuity and the fields of mathematical analysis and topology. In this regard, we study the set of quasicontinuous functions corresponding to pointwise and compact convergence topology in this thesis. To begin with the quasicontinuous functions and its basic properties, we explore the results regarding the preservation of some strong forms of connectedness under the quasicontinuous function. Following the same result, we are able to prove a general form of the Intermediate Value Theorem for the quasicontinuous function. Nonetheless, we explore $Q(X,Y)$ under the pointwise topology as we study some of the cardinal invariants corresponding to the same. In particular, we evaluate tightness, network weight, and pseudocharacter of $Q_p(X,Y)$. Moreover, we obtained a condition on X for which the network weight and weight of $Q_p(X)$ coincide and provided a condition for the separability of a regular space in terms of the cardinal functions of a compact subset of $Q_p(X,Y)$. Furthermore, we elucidated maps such as the openness of the restriction map and studied the denseness of the image of the induced map on $Q_p(X,Y)$ corresponding to the same topology. Moreover, we also elucidate the space $Q_C(X)$ under the compact convergence topology and study some properties. Specifically, we explore the cardinal invariants such as density and various types of tightness (i.e. Density-tightness, fan-tightness and strongly fan-tightness), also we obtained a condition of coincidence of tightness of $Q_C(X)$ and compact Lindelof number of space X. Moreover, we also prove that if X be locally compact Hausdorff, then the space $Q_C(X)$ is Frechet-Urysohn, has countable tightness coincides with the σ-compactness of X.

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Contents

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Abbreviations

$\exists$	**T**here exist.
$\in$	**B**elongs to.
$\notin$	**N**ot belongs to.
$\subset$	**S**ubset.
$\supset$	**S**uperset.
$\bigcup$	**U**nion.
$\bigcap$	**I**ntersection.

Symbols

Symbol	Name
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
(M, d)	Metric space M with metric d
(M, τ)	Topological space M with topology τ
$\cong$	Homeomorphism
$T \backslash A$	Complement of A in T
$supp(f)$	support of a function f
$P(T)$	The collection of all subset of a set T
$\mathcal{K}(T)$	The collection of all compact subset of a set T
$\aleph_0$	The first infinite ordinal
$\aleph_1$	The first uncountable ordinal
$f \mid A$	A restriction of function $f \colon T \to M$ to $A \subset T$
$\pi_{\mid Y}$	Restriction map from $C(T)$ to $C(M)$, where $\pi_M(f) = f_{\mid M}$, $f \in C(T)$ and $M \subset T$
$\overline{Z}$ or $Cl(Z)$	Closure of Z
$(Z)^\circ$ or $int(Z)$	Interior of Z
$\lvert A \rvert$	Cardinality of set A
$\chi(Z)$	Character of Z
$w(Z)$	Weight of Z
$w(Z)$	i-Weight of Z
$wc(Z)$	Weak covering number of Z
$d(Z)$	Density of Z
$c(Z)$	Cellularity of Z
$\psi(Z)$	Pseudocharacter of Z
$e(Z)$	Extent of Z
$nw(Z)$	Network weight of Z

$\pi_\chi(Z)$	π-character of Z
$t(Z)$	Tightness of Z
$s(Z)$	Suslin number of Z
$u(Z)$	Uniform weight of Z
$L(Z)$	Lindelöf number of Z
$F_p(T, M)$	Set of functions from a set T to a set M with pointwise convergence topology
$C_p(T, M)$	Set of continuous functions from T to M with pointwise convergence topology
$C_C(T, M)$	Set of continuous functions from T to M with compact convergence topology
$Q_p(T, M)$	Set of quasicontinuous functions from T to M with pointwise convergence topology
$Q_p(T)$	Set of real quasicontinuous functions on T with pointwise convergence topology
$Q_C(T, M)$	Set of quasicontinuous functions from T to M with compact convergence topology
$Q_C(T)$	Set of real quasicontinuous functions on T with compact convergence topology

Chapter 1

Introduction

The domain of topology consists of the study of surfaces and the name "topology" bears a mix of two greek origin words "topos (surface) and "logos"(study). Topology provides a structure on a set in terms of the subsets of given set and then study its properties based on the considered structure. The study of set of functions under topological structure was not present till early $19th$ century. Although, the notion of pointwise convergence of a sequence of real-valued functions was present in calculus, specifically in the study of trigonometric and power series. After that, Balzano $(1781 - 1848)$ and Cauchy $(1781 - 1848)$, who considered investigating the uniform convergence of a function's sequence and explained the notion of convergence of sequence, series, and continuous function. Earlier in this development, it resulted in contradictions and disagreement when convergence and divergence were not considered.

In 1817, Bolzano provided an understanding of the requirement for convergence of a sequence of maps. Cauchy followed, who misinterpreted the term-by-term integration of maps in 1821 and the study of the limit of a convergent series of maps. As a result, Cauchy overlooked the necessity for further uniform convergence conditions. Interestingly, Cauchy's proof was erroneous which Abel$(1802 - 1829)$ pointed out. Further, in 1826 Abel's paper demonstrate that the sum of a continuous series that is uniformly converging is continuous inside the convergence interval. This is recognized as the initial stage of uniform convergence, which paved the way for Stokes and Philipp L. Seidel to separately identify and emphasize the concept of uniform convergence in 1847–1848 and Cauchy in 1853.

Although Weierstrass (1815–1897) proposed the idea of uniform convergence pretty precisely in 1842, his work on the subject was not published until much later in 1894. Heine$(1821 - 1881)$, Weierstrass, and others have worked on the same concept in the last half of the $19th$ century. They paved a route to the concept of uniform convergence to be considered for application purposes, for example in Fourier series and integration theory. It appears that during the final twenty years of the nineteenth century, the work of Ascoli [1], Arzela [2], and Hadamards [3] influenced the emergence of the theory of function space. Roughly, we can infer a topological space in which functions are treated as points. It is noteworthy to centralize Riemann as the creator of the field of topology because he was the first who came forward with the concept of topology.

The notion of pointwise and uniform convergence of series of continuous maps inspired the idea of topologizing the set of continuous maps. Point-open topology and uniform topology are the terms used to describe the topologies resulting from pointwise and uniform convergence, respectively. In 1906 Frechet [4] studied supremum metric topology, in fact which is the uniform topology. The point-open and uniform topology are the first two topologies on the function space in general topology. After that search to find some topology that lies between point-open topology and the uniform topology. This gave birth to the compact convergence topology. Since all these topologies (except point-open) involve metric, so they started to look for a new topology stronger than point-open topology that does not involve metric. In 1945, R.H.Fox [5] introduced compact-open topology on set of continuous functions.

We would like to adopt some symbols and what they signify which we will use throughout the thesis. The symbols $\mathbb{R}$, $\mathbb{N}$, J, and ω stand for sets of real numbers, natural numbers, index sets, and countable sets of numbers, respectively. A metric space is denoted by (M, d) where d is metric and M is arbitrary space. Next, we define a collection of functions that we use in our thesis. The family of all functions from set T to M is represented by the symbol $F(T, M)$ (or M^T). We will use the expressions $C(T, M)$, $C(T)$, and $Q(T, M)$, where $C(T, M) = \{f \colon f \colon T \to M \text{ and } f \text{ is continuous}\}$, $C(T) = \{f \colon f \colon T \to \mathbb{R} \text{ and } f \text{ is continuous}\}$ and $Q(T, M) = \{f \colon f \colon T \to M \text{is quasicontinuous}\}$, respectively. The function space of quasicontinuous functions from set T to $\mathbb{R}$ is denoted by $Q(T)$, which is equal to $\{f \colon f \colon T \to \mathbb{R}, \text{ and } f \text{ is quasicontinuous}\}$ and $QS(T, M) = \{f \colon f \colon T \to M \text{ and} f \text{ is quasi-subcontinuous}\}$. In the same way that different letters are used in subscript to $C(T)$ to denote different topology on functions space, such as k, d, g, f, r, and w signify compact-open, uniform, graph, fine topology, and weak topology, respectively. The $C_p(T, M)$ stand for $C(T, M)$ is equipped with a pointwise convergence topology. $C_p(T, M) \subseteq C_k(T, M)$ indicates that $C_p(T, M)$ is less powerful than $C_k(T, M)$. Throughout this thesis the term 'space' denotes a 'topological space'.

1.1 Topologizing a Function Space

The motive behind topologizing a function space is to study the convergence of the sequence of functions. By changing the topology of function space we can change means for convergence of the sequence of functions. So topologizing the function space differently and examining the properties of function space M^T (i.e. set of all functions from set T to M) is a generalization of finite product space M^n (i.e a set of all function $f\colon \{1, 2, \ldots, n\} \to M$). Firstly, the finite power set is replaced by indexed set T then by any arbitrary set T and denoted as $M^T = \prod_{y \in T} M$, whose elements are infinite tuples of point of M. Then define the product topology on it and point-open topology on it. Then both are generated by the same sub-basis(i.e. convergence in product topology is the same as point-wise convergence). The point-open topology is defined as;“ T is set and M is space. Given point $y \in T$, $S(y, U) = \{f : f \in M^T \text{and} f(y) \in U\}$, where U is open in M, is sub-basis element that generate topology on M^T.” In point-open topology, we see some examples which shows that the sequence of continuous maps is not necessarily converge to a continuous map. So some strong concepts were needed and this idea lead to uniform convergence that generate uniform topology. The concept of uniform convergence is stronger than pointwise convergence so uniform topology is stronger than point-open topology. As it is seen that the point-open topology is defined on any set M and any space T but uniform topology involves uniform structure (as Metric space). Now take a look on the metric and metric space such as, let (M, d) be metric space and defines “standard bounded metric $\bar{d}(a, b) = min\{d(a, b), 1\}$ on M is induced by d. Let $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ are elements of M^J then $\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) | \alpha \in J\}$ is uniform metric on M^J.” Then by generalizing the index set J with any set T, we get the uniform metric on M^T. This metric induces the uniform topology on M^T, which is defined as; “T be space and (M, d) is a metric space, for $f \in M^T$ and given $\epsilon > 0$ the set $B_\rho(f, \epsilon) = \{g \in M^T \colon \bar{\rho}(f, g) < \epsilon\}$ is basic set to generate the topology on M^T.” The sequence of continuous function in uniform topology converges to continuous function, but it may not be converges to continuous function in point-open topology. Then, a question arise that “Is there any topology exists which lies between these point-open topology and uniform topology, that ensure that the sequence of continuous maps converges to a continuous map”. For that answer is: Yes, there exists a topology named as compact convergence topology in which if T is compactly generated and (M, d) is metric space, then a sequence of continuous functions $\{f_n\}$ converges to a continuous function f. This topology is defined as;“ Any $f \in M^T$, compact set K in M and given $\epsilon > 0$, then the sets $B_K(f, \epsilon) = \{g \in M^T : \sup\{d(f(x), g(x))\colon x \in K\} < \epsilon\}$ generate topology on M^T which is also known as topology of uniform convergence on a compact set.” A limit of the convergent sequence $\{f_n\}$ in $C_C(T)$ is a function f, then $f \in C_C(T)$ if and only if the restricted function sequence $\{f_n|C\}$ uniformly converges $f|C$ for each compact set C in

T. Next we see that if we take M is compact then uniform topology and compact convergence topology coincide and if we assume that T is a discrete space, then point-open topology and compact convergence topology are identical. Since the point-open topology does not depends on the metric but the other two depend on the metric. So again arises the question " Is there any topology exist that consider M is any space instead of a metric space and stronger than point-open topology". There is no sufficient answer for this on M^T. However, it can be demonstrated for the subspace $C(T, M)$ under the topology known as compact-open topology. It is defined as: "For any space T and M, if C is compact subset in M and U is open set in M then $S(C, U) = \{f \in C(T, M) \colon f(C) \subset U\}$ sets form a subbasis for topology on $C(T, M)$". Compact-open topology is stronger than point-open topology, as may be seen from the definition. Assuming that M is a metric space, we can state that the topology of compact convergence and compact-open topology coincide, or that the topology of compact convergence is independent of metric. Now, let us now examine the evaluation map $e : T \times C(T, M) \to M$, which is defined as follows; for every $x \in T$ and every $f \in C(T, M)$, $e(x, f) = f(x)$. When $C(T, M)$ has the pointwise convergence topology, this map is not continuous. However, in this case, the evaluation map becomes continuous if we take M to be a locally compact Hausdorff space and $C(T, M)$ equipped with the compact-open topology.

A general notion of uniform convergence in real analysis was provided by E.H. Moore in 1911–1912 when he established the idea of roughly uniform convergence relative to a scale function. Unlike the traditional definition using a positive constant, Moore generalized the concept by incorporating a positive function, denoted as $\epsilon(x)$ on $\mathbb{R}$, known as a scale function. Hewitt then created the m-topology on the function space $C(T)$. This was in 1948. Similar to Moore's concept of approximate regularity of convergence relative to a scale function, this topology, called the fine topology, expands the notion of convergence in $C(T)$. Hewitt highlighted the relationship between the m-topology for $C(\mathbb{R})$ and E.H. Moore's work.

In the current state of function space, the traditional topologies such as uniform topology and compact-open topology are not robust enough to be used. As fine topology is a sincere illustration of this. Therefore, it makes sense to discover a different topology on $C(T, M)$ which outweigh uniform topologies in strength, one of them is the graph topology [6] on $C(T, M)$. Also, many more topology are defined on $C(T, M)$ like σ-compact open topology [7], open-open Topology [8], bounded-open topology [9], support-open topology [10], C-compact topology [11] etc. Study the properties of function space under above mentioned topologies.

The $Q(T, M)$ is a set of quasicontinuous functions from set T to M, which is a superset of $C(T, M)$. Several topologies on $C(T, M)$ are studied, as previously specified. Since the $Q(T, M)$ is richer space than $C(T, M)$. It took researcher's attention to study convergence of a sequence of functions in $Q(T, M)$ and L. Hola and D. Holy investigated the idea of pointwise convergence in $Q(T, M)$ in 2011, and they determined the requirements for the sequence of a quasicontinuous map to converge to a quasicontinuous map. All topologies on $Q(T, M)$ and their properties we will study in brief in chapter"Review of literature".

1.2 Impact of Function Spaces

The function space, although has arisen from general topology, have importance in many other branches. Firstly we take hyperspace, for any space T the hyperspace of real valued multifunction can be considered as subset of $T \times \mathbb{R}$. Due to this reason hyperspace is denoted as $2^{T\times\mathbb{R}}$. There are two traditional topologies named as Hausdorff topology [12] and Vietoris topology [13], which have spacial importance. A specific reason behind its importance is that $C(X)$ under the graph topology is same as $C(X)$ as a subspace of set of all non-empty closed subset of $T \times \mathbb{R}$ equipped with Vietoris topology. For more see [14].

From the space T and M, a larger space appears, which is the function space $C(T, M)$ and there is an inherent connection between $C(T, M)$ and the properties of T or M. The function space $C(T, M)$ equipped with the uniform topology is complete if and only if M is a complete metric space, as is shown in a specific conclusion. Also, there is another important result is called Nagata theorem [15] gives a strong result to check duality in $C(M)$ with point-open topology and the result is " Any two Tychonoff spaces T and M are homeomorphic if and only if $C_p(T)$ and $C_p(M)$ are ring isomorphic".

The function space plays an important role in Approximation Theory. For the Stone-Weierstrass Approximation theorem, Weierstrass initially proved that the sequence of polynomials in a closed bounded interval converges to a real-valued map defined on that interval. Then, M.H.Stone made its generalization by the sequence of polynomials on a compact Hausdorff space converges to a real-valued or complex-valued map.

The function space is by some means the inspiration for Homotopy Theory. First, in 1930 Hurewicz started the study of Homotopy group. The base for Homotopy theory may functions and Homotopy is just a map that study the transformation of a function to other. Also, the compact-open topology possesses an appropriate value in Homotopy theory because it is observed that Homotopy between two functions corresponds absolutely to a path in space $C(T, M)$. In 1946, G.W.Whithead [16] introduced a problem to categorizing the Homotopy types consisting of path components of function, while focusing on his idea of the case map (s_1, s_2). Also, the Homotopy theory is studied at its own importance as a separate subject.

The compact-open topology on function space has got quite much importance in mathematics. The Arzela-Ascoli theorem is among one of the important results of compact-open topology which provides the equivalent condition for compact subspace and equicontinuity in $C(T, M)$. That condition is, " a set $F \subset C(T, M)$ in compact-open topology has compact closure if and only if it is closed, bounded and equicontinuous". Arzela and Ascoli, respectively, provide the necessary and enough conditions. Since $Q(T, M)$ is a space, it is a larger class than $C(T, M)$. The notion of topologizing the space $Q(T, M)$ in many ways has recently come to light, and its features, such as the closedness of $Q(T, M)$ in $F(T, M)$ under various topologies, have been examined in [17, 18].

Last but not the least, function spaces $C(T, M)$ and $Q(T, M)$ can behave as a vector space, ring, group, etc. in combination with some topology, replacing the objects in general topology and transforming it into algebraic topology. There is a wide area in which the topological function space is studied under an algebraic structure, as whatever operations can be put on a simple space can be also put on a topological function space. For example, the function space along with some topology carries two algebraic operations, addition and multiplication, which makes it a topological ring and under uniform structure it becomes an uniform space. Then study the algebraic topological properties, ring properties, and μ-properties.

1.3 Thesis organisation

The principal focus of our thesis is to study the space $Q(T, M)$ endowed with different topologies. Firstly we go through the comprehensive study of quasicontinuity. Next, we examine the behavior of $Q(T, M)$ under different topologies and study different topological proprieties. We formulate different equivalent results and their interdependencies with each other. Our thesis is distributed into six chapters; (1) Introduction, (2) Review of literature, (3) Quasicontinuity, (4) The topology of pointwise convergence on the space $Q(T, M)$, (5) The space $Q(T)$ endowed with compact convergence topology and (6) Conclusion and future works.

In the Chapter (1), delves into a detailed exploration of the origin, significance, and operational principles governing function spaces and their topologies. Additionally, it elucidates diverse approaches to studying function spaces, aiming to contribute original insights to the field and propose solutions to unresolved challenges. The chapter further offers a concise overview of the applications of function space topologies in various other mathematical domains.

In the second Chapter (2), first, we briefly go through the existing literature of quasicontinuity. Next, a comprehensive literature review unfolds, systematically presenting function space $Q(T, M)$ under various topologies in chronological order, ranging from older to more contemporary ones. This examination delves into the reasons behind introducing each topology, conducts comparisons with others, and explores both topological and non-topological properties associated with the respective spaces. Notably, this Chapter highlights pivotal findings from influential papers that have left a lasting impact on the field.

In the third Chapter (3), we initiate our exploration by examining quasicontinuous functions and their equivalent representations. Subsequently, we conduct a comparative analysis of quasicontinuity with other types of continuity. Our investigation extended to exploring the construction of quasicontinuous functions under various constraints on their domain and range. Additionally, we delve into the behavior of quasicontinuous functions under composition, product, and different algebraic operations. Furthermore, we investigate the preservation of certain properties of spaces under quasicontinuity. Finally, we examine the Intermediate Value Theorem in the context of quasicontinuous function.

In the Chapter (4), we have systematically examine the cardinal invariants, including pseudocharacter, network weight, weight, and tightness, within the context of the space $Q_p(T, M)$. Our findings establish a dominant relationship, demonstrating that the pseudocharacter of $Q_p(T, M)$ surpasses the network weight, density, and weak covering number of a regular space T. Furthermore, we have obtain a set of sufficient and necessary criteria on T such that the weight and network weight of $Q_p(T)$ are the same. Additionally, we have establish a condition that expresses a regular space's separability in terms of pseudocharacter of a compact subset of $Q_p(T, M)$. Our exploration about the openness of the restriction map on $Q_p(T)$ and demonstration that $Q_p(T, Z)$ is densely embedded in the image of $Q_p(T, M)$ under the induced map.

In the Chapter (5), we delve into the examination of the density and various types of tightness within the space $Q_C(T)$. Our proofs establish the equivalence between the tightness of $Q_C(T)$ and the compact Lindelöf number of the underlying Hausdorff space T. Furthermore, we demonstrate that the density of $Q_C(T)$ is bounded by the k-cofinality of T, and we identify conditions on T that lead to the confluence of density tightness and tightness in $Q_C(T)$. We proceeded to characterize fan tightness and strong fan-tightness in terms of k-covers of T. We also prove that, given a locally compact Hausdorff space T, σ-compactness of T, countable tightness, and Frechet-Urysohn properties of $Q_C(T)$ are mutually equivalent. Furthermore, we prove that every k_f-open covering of T has a countable subcover that converges to T if $Q_C(T)$ is a Frechet-Urysohn space.

Chapter 2

Review of literature

The space of functions has been used since the 19th century to make the framework for the study of convergence of the sequence of functions. Till now it's an intense and active research area. G. Ascoli [1], C.Arezla [2], and J.Hadamard [3] on functions space marked a good contribution to the theory of function space. Topology of pointwise convergence (also known as point-open topology) is a type of topology on function space that was first studied in 1935 by Tychnoff. He discovered that topology on M^T could be created using the condition of pointwise convergence in product topology. In his opinion, the topology of pointwise convergence is all that product topology on M^T is. We define the point-open topology for any space T and M on M^T. It comes from the result, " A sequence of function f_n converges to function f in the topology of pointwise convergence if and only if for each point $x \in T$ the sequence $f_n(x)$ of points in M converges to $f(x)$".

The idea of uniform convergence in all circumstances is stronger than that of pointwise convergence. Uniform topology on M^T, thus, involves the investigation of uniform convergence of a sequence of functions such as: " for any space T and (M, d) is metric space, sequence $f_n \in M^T$ converges to uniformly to $f \in M^T$; if for given $\epsilon > 0 \; \exists \; n_0 \in \mathbb{N}$ s.t. $d(f_n(x), f(x)) < \epsilon$ for all $n \geq n_0$ and $x \in T$". In uniform topology, the series of continuous functions in M^T converges to a continuous function. Then the question arises that, is there any other topology weaker than uniform topology and stronger than point open topology in which subspace $C(T, M)$ is closed in M^T? For that answer is: Yes, the concept of uniform convergence of the sequence on a compact subspace is part of the topology of compact convergence. The sequence of functions $f_n \in M^T$ (where T is any space and M is metric space) converges to a function $f \in M^T$ if and only if the sequence $f_n|K$ uniformly converges to $f|K$ for every compact subset K of T. This is the result known as the topology of compact convergence. In addition, T must be compactly generated and (M, d) can be any metric space for $C(T, M)$ to be closed in M^T under the topology of compact convergence.

After going through the topology of pointwise convergence, compact convergence, and uniform convergence on $C(T, M)$. One question arises "Is there any other topology that is stronger than point-open in which M is any space instead of metric space". For that answer is: Yes the compact-open topology on $C(T, M)$ exists. It was introduced by R.H. Fox in 1945 and improved by Arens and Dugundji [19]. Finally, it is explored by J.R. Jackson [20] by studying the convergence of a sequence of functions on compact subsets. That is defined as; "let T and M be any space, let $K \subset T$ is compact and U is open set in M then the set $S(K, U) = \{f \in C(T, M) : f(C) \subset U\}$ is sub basic set to form topology on $C(T, M)$". When assuming that M is metric space then compact-open topology and topology of compact convergence coincide on $C(T, M)$, or then the topology of compact convergence is independent of metric. The compact-open topology is more useful than point-open as we can see in an example that an evaluation map $e : T \times C(T, M) \to M$ as $e(x, f) = f(x)$ is continuous in compact-open topology.

There are various of topologies defined on $C(T, M)$ or $C(T)$ to study the convergence of a sequence of functions. Some of them are named fine topology, graph topology, bounded-open topology, open-open topology, support-open topology, compact G_δ topology, bi-point open topology etc. Then they studied the topological properties (like cardinality, separation axioms, countability, connectedness, metrizibility, and compactness) of their functions spaces and also studied duality theory on it.

2.1 Quasicontinuity on topological spaces

In 1932, S. Kempisty [21] introduced the concept of the quasicontinuous map for the real-valued function of several variables. But the first time use of conditions of quasi-continuity found in R.Baire's paper [22]. There are many reasons to study the quasicontinuity. But the best two of them are, the first is a deep connection between continuity and quasicontinuity despite generality and the second is a good connection between quasicontinuity with mathematical analysis and topology. Quasicontinuity was studied for both single-valued and multi-valued maps. Let T and M be space, $f : T \to M$ is a single-valued map, and $F : T \to M$ is a multi-valued map. If F is multi-valued map for $A \subset M$ we denote $F^+(A) = \{x \in T : F(x) \subset A\}$ and $F^-(A) = \{x \in T : F(x) \cap A \neq \phi\}$.

The definition of quasicontinuity of a function $f : T \to M$ was given for $T = \mathbb{R}^n$ and $M = \mathbb{R}$ by Kempisty [21]. However, the function of two variables being quasicontinuous under consideration that it is continuous in each variable separately mentioned by Volterra [22].

This definition was reformulated for the space as; a map $f : T \to M$ is quasicontinuous at $p \in T$ if for any open sets U and V such that $p \in U$ and $f(p) \in V$, there exists a non-empty subset G of U such that $f(G) \subset V$. It is said to be quasicontinuous if it is quasicontinuous at any $x \in T$. Every continuous map is quasicontinuous but not conversely. For example every monotone left(right) continuous map is quasicontinuous $f : \mathbb{R} \to \mathbb{R}$, but not continuous. Next, take a look at multi-valued quasicontinuous, which is defined as; a multi-valued map $F : T \to M$ is upper(lower) continuous at $p \in T$ if for any open set V, $F(p) \subset V$ $(F(p) \cap V \neq \phi)$ $\exists$ a neighborhood U of p such that $F(x) \subset V(F(x) \cap V \neq \phi)$ for all $x \in U$. It is continuous at p if it is both upper and lower continuous. A multi-valued map F is upper(lower) quasicontinuous at $x \in T$ [23] if any open $V \subset M$ such that $F(p) \subset V(F(p) \cap V \neq \phi)$ and for any open set U containing p $\exists$ nonempty subset G of U, such that $F(x) \subset V$ $(F(x) \cap V \neq \phi)$ for all $x \in G$. It is said to be upper(lower) quasicontinuous if it is upper(lower) quasicontinuous for all $x \in T$. Any multi-valued map is upper(lower) continuous then it is upper(lower) quasicontinuous but not conversely.

Now we study some concepts that are equivalent to quasicontinuity in various situations. The first one is the neighbourly function, which was introduced by W.W. Bledose in his paper [24] and defined as; let (T, ρ) and $(M, \rho^{'})$ are metric spaces. A map $f : T \to M$ is a neighbourly function if for given $\epsilon > 0$, $\exists$ an open sphere $S \subset T$ such that $\rho(x, y) + \rho^{'}(f(x), f(y)) < \epsilon$ for all $y \in S$. S.Marculs [25] proved that the concept of neighbourly and quasicontinuity was equivalent for T and M is metric space. Another equivalent concept is semi-continuity, it was introduced by N.Levine [26] by using the notion of semi-open. A subset A of T is called semi-open if it is contained in the closure of the interior of itself and semi-continuity is defined as; a map $f : T \to M$ is semi-continuous if the inverse image of open in M is semi-open in T. In [27] proved that " for single-valued map the concept of semi-continuity and quasicontinuity are equivalent". Also, they proved for upper (lower) quasicontinuity of multifunction as, a multifunction $F : T \to M$ is upper(lower)quasicontinuous if and only if $F^{+}(V)(F^{-}(V))$ is semi-open set for every open set $V \subset M$.

The concept of somewhat continuity is closely related to quasicontinuity. In 1971, K.R.Gentry and H.B.Hoyle, [28] introduced the concept of somewhat continuous map, defined as; a map $f : T \to M$ is somewhat continuous if for any open set V in M the inverse image and interior of inverse image $f^{-1}(V) \neq \phi$ and $(f^{-1}(V))^{\circ} \neq \phi$. Clearly, by definition, every quasicontinuous map is somewhat continuous but not conversely. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 0$ if $x < 0$, x is rational or $x \in [0, 1]$and $f(x) = 1$ if $x < 0$, x is irrational or $x \in (0, \infty)$. Then f is somewhat but not quasicontinuous. Also, the restriction of a somewhat continuous map need not be somewhat continuous. But restriction is useful and a connection between somewhat continuity and quasicontinuity as seen in the result" a map $f : T \to M$ is quasicontinuous if and only if $\exists$ basis $\mathcal{B}$ of open set such that the restriction map $f|B$ is somewhat continuous for every $B \in \mathcal{B}$". The useful characterization of somewhat continuity is " a map $f : T \to M$ is somewhat continuous if and only if for any dense set $D \subset T$ the set $f(D)$ is dense in $f(T)$." As a simple consequence of this result. There is a similar result for quasicontinuous maps which follow as, a map $f : T \to M$ is quasicontinuous has equivalent to any dense set W in T the set $f(W \cap G)$ is dense in $f(G)$ for any open set G. Also, they introduced a simple extension of somewhat continuity to a multi-valued map as, a multi-valued map $F : T \to M$ is upper(lower) somewhat continuous if for any open $V \subset M$ for which $F^+(V) \neq \phi((F^-(V) \neq \phi)$ we have $(F^+(V))^\circ \neq \phi$ $((F^-(V))^\circ \neq \phi))$. Also, the restriction of multifunction somewhat upper(lower) continuity is related to upper(lower) quasicontinuity of multifunction as we can see," a multi-valued map $F : T \to M$ is upper(lower) quasicontinuous if and only if restriction of $F|B$ is upper(lower) somewhat continuous for any set $B \in \mathcal{B}$ of open set.

Now, we are going to look at how quasicontinuity can be defined in terms of the continuous restriction of a function under certain conditions upon the underlying spaces. First, consider a single-valued function f defined from a first countable Hausdorff space T to a first countable space M then f is quasicontinuous at $x \in T$ if and only if there exists a quasi-open set A containing x such that restriction $f|A$ is continuous at x. Next, define the quasicontinuity of multi-valued function $F : T \to M$ at a point x. It demonstrates that "if T is the first countable Hausdorff space 'and assume that M is first countable at collection $\mathcal{K} = \{F(T) : x \in T\}$, Than F If and only if there is a quasi-open set A containing T such that constraint $F|A$ is upper(lower) continuous at point $x \in T$." Yet a certain sequential characterization of the quasicontinuity may be possible in the case of that characterization through restriction fails. Specifically, it established that if T and M are Hausdorff spaces, a multi-valued map $F : T \to M$ is lower quasicontinuous at $x \in T$ if and only if for any $y \in F(x)$, $\exists$ a quasi-open set A containing x such that, for any sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \to x$, $\exists$ $\{y_n\}$ $y_n \in F(x_n)$ and $n \in \mathbb{N},\ y_n \to y$..

The continuity points of quasicontinuous maps with the values in space with a base of given cardinality. The set of all continuity points (discontinuity points) of a function f is denoted by $C(f)$ (and $D(f)$ respectively), and $D_l(F)$ $(D_u(F))$ represents the set of all points where the multi-valued map F is not lower (upper) continuous. Additionally, let $C_l(F) = T - D_l(F)$ and $C_u(F) = T - D_u(F)$. N.Levine proved a fundamental result related to the continuity points of a quasicontinuous function, that result is " a map $f : T \to M$ is quasicontinuous and M is second countable, then $D(f)$ is a first category set". Extension of this is possible in two ways, the first way is that assume the multi-valued map and the second way is that consider spaces more general than second countable. Also, they proved results for multifunction such as a map $F : T \to M$ is lower quasicontinuous and M has a base of cardinality less than k, than $D_l(F)$ is a first k-category set. Also if F is upper quasicontinuous compact valued and M with base of cardinality less than k. Then $D_u(F)$ is of first k-category. They demonstrated the quasicontinuous nature of $f : T \to M$ for a single-valued map. Let M have cardinality smaller than k and let T be a k-Baire space. Then $C(F)$ is dense.

Various topological properties that are known to be preserved under continuous maps also are preserved by a quasicontinuous map. Preserving separability is one of the important properties of quasicontinuity. Somewhat continuity is sufficient, so it is evident that, given T is a separable space and if $f : T \to M$ as a somewhat continuous map onto M, then, M is separable. Thus, quasicontinuity is also applicable, as quasicontinuity and a single-valued map are somewhat similar. But if f is a bijective quasicontinuous map and M is separable then T need not be separable, for example, the identity map $i : \mathbb{R}_D \to \mathbb{R}_U$ is bijective quasicontinuous, where $\mathbb{R}_D$ and $\mathbb{R}_U$ are discrete and usual topologies on $\mathbb{R}$, respectively. The space $\mathbb{R}_U$ is separable but $\mathbb{R}_D$ is not. Next, considering, a bijective and quasicontinuous map $f : T \to M$ having a condition that $(f(G))^\circ \neq \phi$ for each non-empty open $G \subset T$, then the following outcome occurs, M is separable if and only if T is separable. Somewhat continuous generalized types of quasicontinuous functions preserve Baire space. Since T is k-Baire space and a map $f : T \to M$ is quasicontinuous onto M, it formulated such a conclusion for k-Baire space and the same holds for Baire spaces. If for any non empty open set $G \subset T$ we have $(f(G))^\circ \neq \phi$, then M is k-Baire space. Many topological applications of mappings are closely related to quasicontinuity. For this, the concept of quasi-homeomorphism or semi-homeomorphism is introduced by S.G Crossly and S.K. Hildebrand in [29], [30]. They examined the preservation of different topological properties related to the quasicontinuous map.

Initial study of the sequence of quasicontinuous and its convergence. It found that the sequence of the quasicontinuous function may not converge to the quasicontinuous function. For example, f_n is a sequence of quasicontinuous functions as $f_n(x) = x^n$ shows that may not converge to quasicontinuous function, However, at that time the result "f_n is a sequence of quasicontinuous function converges to a quasicontinuous function f, then $D(f)$ is a first category set" is well known. Some more work on convergence is done in [31], [32], and still is an active research area.

2.2 Topologies on set of quasicontinuous functions

In 2011, L.Hola and D.Holy [33] investigated the idea of pointwise convergence of a real-valued quasicontinuous function sequence. They used the Choquet game to derive the conditions under which the sequence of real-valued quasicontinuous function pointwise converges to a real-valued quasicontinuous function. Further, it was demonstrated that f is quasicontinuous if and only if $\{f_n : n \in \mathbb{N}\}$ is equi-quasicontinuous, given T to be a Baire space and $\{f_n : n \in \mathbb{N}\}$ to be a sequence of real-valued quasicontinuous functions that pointwise converges to real-valued function f. Also, the conclusion for pointwise convergence in the quasi-regular T_1 space with locally countable π-base and metrizable space classes is derived. f is quasicontinuous if and only if $\{f_n : n \in \mathbb{N}\}$ is equi-quasicontinuous. If T is metric space, then T is Baire space, and the $\{f_n : n \in \mathbb{N}\}$ is a sequence of real-valued quasicontinuous function that pointwise converges to real-valued function f. If T is Regular T_1-space with locally countable π-base holds, a similar outcome would occur.

In 2016, L.Hola and D.Holy [34] explored convergence topologies, specifically pointwise convergence τ_p and uniform convergence on compact sets τ_{uc}, within the space $F(T, M)$. Their investigation revealed that these two topologies coincide on a subset $\mathcal{E}$ of densely equi-quasicontinuous functions, where T is any space and (M, d) is an metric space. Furthermore, the authors derived significant results concerning the compactness of subsets in $(QS(T, M), \tau_{uc})$. Specifically, for T being locally compact and (M, d) being a metric space, they demonstrated that " any subset $\mathcal{E}$ of $(QS(T, M), \tau_{uc})$ is compact if and only if it is closed, densely equi-quasicontinuous, and compactly bounded". In the context of compactness, the researchers extended their findings to cover the scenario where T is locally compact, and (M, d) is a complete metric space. In this case, they established that " a subset $\mathcal{E}$ of $(QS(T, M), \tau_{uc})$ is compact if and only if it is closed, densely equi-quasicontinuous, and pointwise bounded". Concluding their study, the authors provided a noteworthy result regarding metric completeness. They proved compactness of $\mathcal{E}$ of $(QS(T, M), \tau_{uc})$ is equivalent to its densely equi-quasicontinuity, closedness, and pointwise boundedness, where T is locally compact and M is complete metric space. They demonstrated that, given any space T and an metric space (M, d), any densely equi-quasicontinuous $\mathcal{E}$ subset of $(QS(T, M)$ then τ_p and τ_{uc} coincide on $\mathcal{E}$. They also obtained strong results indicating that any subset of $(QS(T, M), \tau_{uc})$ is compact. Assuming T to be locally compact and (M, d) to be an metric space, proved $\mathcal{E} \subset (QS(T, M), \tau_{uc})$ is compact only if it is densely equi-quasicontinuous, compactly bounded, and closed. They also demonstrated compactness in the case the T is locally compact and (M, d) is a complete metric space. Any closed, densely equi-quasicontinuous, and pointwise bounded subset $\mathcal{E}$ of $(QS(T, M), \tau_{uc})$ is compact. At the end of the paper they prove that " if T is a locally compact space and (M, d) is a complete metric space, then any subset $\mathcal{E}$ of $(QS(T, M), \tau_{uc})$ is compact if and only if it is pointwise bounded, closed, and densely equi-quasicontinuous".

In 2017, L.Hola and D.Holy [17] investigated the topology of compact convergence on $Q(T, M)$. They discovered that every subset of $Q_C(T, M)$ must meet both necessary and sufficient conditions to be compact. Their first finding demonstrated that the compactness of $\mathcal{E} \subset Q_C(T, M)$ is equivalent to its densely equi-quasicontinuity, closedness, supported at a point where $\mathcal{E}$ is non-locally bounded and pointwise boundedness, where T is a locally compact space and M is complete metric space. In other results the condition densely equi-quasicontinuous and supported at the point of nonlocal boundedness of $\mathcal{E}$ in the above result is replaced by the condition that is, there is a densely open set W in M s.t. $\mathcal{E}$ is densely equi-quasicontinuous at each point $x \in W$ and $\mathcal{E}$ is supported at every point $x \in T \setminus W$. OR there is a dense G_δ set G s.t. $\mathcal{E}$ is equi-continuous at every point $x \in G$ and it has supported at each point $x \in T \setminus G$.

In 2018, L.Hola and D.Holy [35] introduced a novel perspective on the topology of compact convergence by employing uniformity on M^T. In this framework, where T is a Hausdorff space and (M, d) is a metric space, they defined as; "the set $\mathcal{K}(T) = \{K \subset T : K \neq \phi$ and K is compact$\}$. The topology of compact convergence, denoted as τ_{uc} on M^T, is induced by the uniformity $\mathcal{U}_{uc}$, whose base comprises sets of the form $W(K, \epsilon) = \{(f, g) : \forall x \in K, d(f(x), g(x)) < \epsilon\}$ where $\epsilon > 0$ and $K \in \mathcal{K}(T)$". They explored the properties of the subspace of $Q_C(T, M)$, showing that it is first countable, metrizable, and completely metrizable. Subsequently, they established equivalence between several statements:" the uniformity $\mathcal{U}_{uc}$ on $Q(T, M)$ being induced by a metric, $Q(T, M)$ being metrizable, $Q(T, M)$ being first countable, and T being hemicompact". When M is replaced by $\mathbb{R}$, two additional statements were found to be equivalent: $Q(T, M)$ being pointwise countable and $Q(T, M)$ being a q-space. Subsequently, when M is an regular space with countable pseudo-character and M is any metric space, these results were shown to hold when T is a space and $M = \mathbb{R}$. Further, for complete metrizability, the authors demonstrated that if T is an locally compact space and (M, d) is a complete metric space, then the uniformity $\mathcal{U}_{uc}$ on $Q(T, M)$ is induced by a complete metric, and $Q(T, M)$ is completely metrizable, first countable, and T is hemicompact, all of which are equivalent. In the final part of their paper, they presented an application to characterize compact and sequentially compact subsets of $Q(T, M)$. Specifically, "for a locally compact hemicompact space T and boundedly compact metric space (M, d), any subset of $Q^*(T, M) = \{f | f : T \to M$ and quasicontinuous locally boundedly function$\}$, is compact (sequentially compact) if and only if it is closed, pointwise bounded, and densely equi-quasicontinuous".

In 2020, L.Hola and D.Holy [18] introduced the topology of pointwise convergence, denoted as τ_p, on $\mathbb{R}^T$ by utilizing a uniform structure. Specifically, if T is a Hausdorff space and $\Im = \{A \subset T : A \text{ is finite}\}$, the topology τ_p is induced by the uniformity $\mathcal{U}_p$. The basis sets of $\mathcal{U}_p$ are of the form $W(A, \epsilon) = \{(f, g) : \forall x \in A, \text{and}|f(x) - g(x)| < \epsilon\}$, where $\epsilon > 0$ and $A \in \Im$. It is noteworthy that the topology of pointwise convergence coincides with the product topology on $\mathbb{R}^T$. Consequently, the authors demonstrated that $Q_p(T)$ is dense in $\mathbb{R}^T$ equipped with the product topology. Also they studied some cardinal invariant (like weight w, density d, cellularity c, network weight nw, character χ, π-character π_χ and uniform weight u) of $Q_p(T)$ and proved that for a space T, weight, character, π-character and uniform weight of $Q_p(T)$ are equal to cardinality of T and also $d(Q_p(T)) \leq w(T)$ and with the help of results on cardinal invariant, they prove that for any T is space all conditions, T is countable, $Q_p(T)$ is metrizable, first countable, has countable base and has countable π base. At the end of the paper, they compared the cardinal invariant of $Q_p(\mathbb{R})$ and $C_p(\mathbb{R})$ by taking examples.

In 2021, L.Hola and D.Holy [36] studied space of quasicontinuous function endowed with uniform topology. Provided that for a Huasdorff space T the uniform weight, π-character, and character of $Q_C(T)$ coincide with k-cofinality of T and if T is a locally compact space then weight and network weight of $Q_C(T)$ coincides. Also, they studied some cardinal invariants (like density, cellularity, network weight, character, π-character, and spreads) of $Q_C(T)$ under different conditions on T and particularly obtained different cardinal functions for $Q_C(\mathbb{R})$. With the help of results on cardinal invariant, they proved that hemicompactness of T, metrizability of $Q_C(T)$, and $Q_C(T)$ are first countable spaces all are equivalents.

Further, in 2022, M.Kumar and B.K.Tyagi [37] examined the topology of pointwise convergence τ_p on $F(T, M)$ and demonstrated that $Q_p(T, M)$ is dense in $F_p(T, M)$. Several cardinal invariants of $Q_p(T, M)$ are also studied, including weight u, character χ, π-character π_χ, density d, cellularity c, network weight nw, and weight w. They demonstrated that given a space T, the cardinality of T is equal to the character and π-character of $Q_p(T, M)$; additionally, $d(Q_p(T, M)) \leq w(T)$. In addition, they demonstrated that for any T that is a space, then all conditions, T is countable, $Q_p(T, M)$ is metrizable, first countable, has a countable base, and has a countable π base are equivalent. They characterized the cardinal invariant character, density, spread, weight, cellularity, and pseudocharacter of $Q_p(T, M)$. Some properties of restriction maps, evaluation maps, and induced maps on $Q_p(T)$.

2.3 Conclusion and research gap

All through the literature on topological spaces the preservation of different properties of topological spaces under continuous maps is one of the most important areas of research, see in [38]. Similarly, do these properties are also preserved by quasicontinuous maps, is also an interesting topic but in the case of quasicontinuity there are only a few properties whose preservation is studied, see in [39]. Yet, the preservation of connectedness, separation axioms, countability, etc. under quasicontinuous maps is not studied. This leaves us with an opportunity to explore this gap and conduct further study.

In the existing literature on quasicontinuity, the space $Q(T, M)$ has been studied under only two different topologies, pointwise convergence topology [18] and compact convergence topology [17]. In comparison the space $C(T, M)$ has been studied through the lenses of a huge number of different topologies, such as regular topology [40], Cauchy convergence topology [41], graph topology [6] etc. The space $Q(T, M)$ being a larger space than $C(T, M)$ gives us the scope to explore the properties of $Q(T, M)$ under various topologies.

The available literature on the properties of cardinal functions on the space $Q_p(T, M)$ is only focused on weight, density, character, cellularity, and spread [18, 37]. This gives us a chance to explore many other left-out cardinal functions such as tightness, weak covering number, Lindelöf number, etc for the space $Q_p(T, M)$. Also, special maps such as induced, evaluation, and restriction maps are defined on $Q_p(T, M)$ and studied to a little extent in [37]. Hence, we can carry on our study on properties of these above-mentioned maps and also many more special maps for the space $Q_p(T, M)$.

The existing study on the space $Q_C(T, M)$ mostly focused around metrizability, compactness and countability properties, see in [17, 35]. Thus in case of $Q_C(T, M)$ we have vast area of other properties such as, sequential, Baire and covering properties, which can be considered for further study. Also, in case of cardinal functions, there are some cardinal functions like weight, density and cellularity on the space $Q(T)$ which are studied under the light of compact convergence topology, see in [42]. So this leaves us with opportunity to explore various other cardinal functions such as density tightness, fan tightness, strongly fan tightness and density of $Q_C(T)$.

2.4 Research Objectives

After the study of literature on different topologies in the function space, especially topologies in the space of the quasicontinuous functions, these objectives are fixed for further study:

1. Comprehensive study of quasicontinuous functions on Topological space and its applications.
2. Analysis of topological properties of the space of quasicontinuous functions under different topologies.
3. Investigation on the existence of weak or strong topology on the space of quasi-continuous functions.

Chapter 3

Quasicontinuity

3.1 Introduction

Baire discovered the condition of quasicontinuous functions for the first time in 1899 in [43] when examining the continuity points of separately continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. Later, the quasicontinuity introduced in the paper [21] for real functions of several real variables was thoroughly and extensively tested by Kempisty in 1932. When we take a closer look, it turns out that the researchers found this study interesting for a few big reasons. The two most important of these are: the first reason is the fairly strong relationship between continuity and quasi-continuity, despite the broad applicability of the latter concept. The second reason pertains to the significant connection between quasi-continuity and the fields of mathematical analysis and topology. The reader can see survey articles authored by Piotrowski, specifically [44] and [45] that encompass a range of intriguing outcomes in this area. However, these papers do not exclusively focus on this subject. In addition to these papers, readers are encouraged to see the survey paper [39] written by Neubrunn, which is a collection of interesting and important results on quasicontinuous functions. Quasicontinuous functions played an important role in the study of topological groups, the characterization of minimal usco and minimal cusco maps, the CHART group which is the key object for the study of topological dynamics quasicontinuity is a concept in topology that bridges the gap between continuity and discontinuity. In this guide, we will explore the definition of quasicontinuity, discuss its algebraic properties and characteristics, and provide examples of quasicontinuous functions. We will also examine the relation between quasicontinuity and continuity, and explore applications of quasicontinuity in space.

In section (3.2), we study quasicontinuous functions and some other generalizations of continuous functions, which are equivalent to quasicontinuous functions under certain conditions. Further, we study the several different ways to define quasicontinuity of functions and the methods of construction of quasicontinuous functions in space.

In section (3.3), we study the preservation of topological properties under quasicontinuous functions, especially strong forms of connected spaces. Moreover, we study the Intermediate value theorem for the quasicontinuous function.

3.2 Quasicontinuous function

In this section, we delve into the definition and characteristics of quasicontinuous functions. We explore how these functions generalize the traditional notion of continuity and introduce the reader to the construction of quasicontinuous functions. Understanding the fundamental properties and structure of quasicontinuity lays the groundwork for further exploration.

Definition 3.2.1 (Continuous)**.** [38] A mapping $f : T \to M$ is called as continuous if the pre-image of each open set in M is also an open set in T.

Definition 3.2.2 (Quasicontinuous function)**.** [39] A map $f : T \to M$ is quasicontinuous at $t \in T$ if $\exists$ a non-void subset H of A such that $f(H) \subset B$ for any open sets A, B such that $t \in A$ and $f(t) \in B$. If it is quasicontinuous at any t in T, it is said to be quasicontinuous.

Theorem 3.2.1. *[39] Every continuous function is quasicontinuous function, but the converse is not true.*

Example 3.2.1. *Let $\mathbb{R}_s$ and $\mathbb{R}_l$ represent the set of real numbers in standard topology and the lower limit topology, respectively. Let $f : \mathbb{R}_s \to \mathbb{R}_l$ be defined as a quasicontinuous function but not a continuous function.*

N.Levine [26] first proposed the idea of semi-continuity in 1963. He did this by defining a semi-open set as " A subset A of space T is said to be semi-open set if $A \subset Cl(Int(A))$, according to [46]. Semi-closed set are those whose complement a semi-open set." The intersection of any semi-closed set that contains A is the semi-closure (scl) of a set A.

Definition 3.2.3. [26] A function $f : T \to M$ is called as semi-continuous if the pre-image of an open set in M is semi-open set in T.

Theorem 3.2.2. *[27] If a map $\mu : T \to M$ is single-valued, it is semi-continuous if and only if it is quasicontinuous.*

Proof. Let μ be quasicontinuous on T. Choose any non-void open set E containing t. Let $t \in \mu^{-1}(E)$ and let V be any open set containing t. Therefore $\exists$ a non-void U contained in V such that $\mu(U) \subset E$, $U \subset \mu^{-1}(E)$. Hence $U \subset (\mu^{-1}(E))^{\circ}$ and $\phi \neq U = V \cap U \subset V \cap (\mu^{-1}(E))^{\circ}$. This implies that any open set V containing t.

$$V \cap (\mu^{-1}(E))^{\circ} \neq \phi.$$

Then $t \in (\mu^{-1}(E))^{\circ}$. Since t is arbitrary element in $\mu^{-1}(E)$ we have $\mu^{-1}(E) \subset (\mu^{-1}(G))^{\circ}$. Thus $\mu^{-1}(E)$ is semi-open set.
Conversely, Let μ be semi-continuous. Let $t \in T$ and E any open set such that $\mu(t) \in E$. Let V be any open set containing t. Under the assumption $\mu^{-1}(E)$ is a semi-open set. Hence

$$\mu^{-1}(E) \subset \overline{(\mu^{-1}(E))^{\circ}}.$$

Put $U = V \cap (\mu^{-1}(E))^{\circ}$. Since $t \in \mu^{-1}(E) \subset \overline{(\mu^{-1}(E))^{\circ}}$, there is a point in V belonging to $(\mu^{-1}(E))$, hence U is a non-void set. Evidently $U \subset V$ and $\mu(U) = \mu(V \cap (\mu^{-1}(E))^{\circ}) \subset \mu(\mu^{-1}(E)) \subset E$. Hence μ is quasicontinuous at t. Since t was arbitrarily chosen, the function T is quasicontinuous on T. □

In 1971, K.R.Gentry and H.B.Hoyle[28] introduced the concept of a somewhat continuous map.

Definition 3.2.4. [28] A function $f : T \to M$ is somewhat if the inverse image and interior of the inverse image for each open set V in M are $f^{-1}(V) \neq \phi$ and $(f^{-1}(V))^{\circ} \neq \phi$.

Every quasicontinuous map is somewhat continuous but not conversely.

Example 3.2.2. *Let $\mu : \mathbb{R} \to \mathbb{R}$ defined as;*

$$\mu(t) = \begin{cases} 0 & \text{if } t < 0 \text{ and } t \text{ is rational,} \\ 1 & \text{if } t < 0 \text{ and } t \text{ is irrational,} \\ 0 & \text{if } t \in [0,1], \\ 1 & \text{if } t \in (1,\infty). \end{cases}$$

Then μ is somewhat continuous but not quasicontinuous.

W.W. Bledose [24] introduced the concept of neighbourly function.

Definition 3.2.5. [24] Consider the metric space (T,ρ) and $(M,\rho^{'})$. A map $f : T \to M$ is a neighbourly function if an open sphere $S \subset T$ exists for a given $\epsilon > 0$ s.t.

$$\rho(x,y) + \rho^{'}(f(x),f(y)) < \epsilon \,\forall\, y \in S.$$

Theorem 3.2.3. *[25] If T and M are metric space, and $f : T \to M$ is a function, it is neighbourly if and only if it is quasicontinuous.*

In 1982, T. Noiri [46] introduced the concept of α-continuous which is also known as a strongly quasicontinuous map by using the α-open set which are defined as; " A subset A of a space T is said to be α-open set (also known as α-set)[46] if $A \subset Int(Cl(Int(A)))$." α-closed set are those whose complements are α-open set.

Definition 3.2.6. [46] A function $f : T \to M$ is termed as α-continuous(strongly quasicontinuous) if the pre-image of any open set V in M is a α-set in T.

Theorem 3.2.4. *Consider T and M to be space. If $\mu : T \to M$ is used, the following are equivalent.*

1. *μ is quasicontinuous.*
2. *The $\mu^{-1}(B)$ is a semi-closed subset of T for any closed subset B of M.*
3. *[28] There exists a $\mathcal{B}$ basis of T such that the restriction map $\mu|B$ is somewhat continuous for any $B \in \mathcal{B}$.*
4. *[28] $\mu(W \cap G)$ is dense in $\mu(G)$ for any open set G for any dense set $W \subset T$.*

Theorem 3.2.5. *[47] Consider a function$f : T \to M$ that is quasicontinuous (or strongly quasicontinuous). Then*
$Scl(f^{-1}(B)) \subseteq f^{-1}(Scl(B))$ (resp. $\alpha Cl(f^{-1}(B)) \subseteq f^{-1}(\alpha Cl(B))$.

In 2011 L.Hola and D.Holy investigated the notion of convergence of a sequence of real-valued quasicontinuous functions pointwise manner.

Theorem 3.2.6. *[33] If $\{f_n : n \in \mathbb{N}\}$ is a sequence of functions in $Q(T)$ is pointwisely convergent to f, which is a real-valued function defined on T. If $\{f_n : n \in \mathbb{N}\}$ is equi-quasicontinuous, then f is quasicontinuous at x.*

Theorem 3.2.7. *[33] If T be Baire space and $\{f_n : n \in \mathbb{N}\}$ i sequence of functions in $Q(T)$is pointwisely converges to f, which is a real-valued function defined on T. Then f is quasicontinuous if and only if $\{f_n : n \in \mathbb{N}\}$ is equi-quasicontinuous.*

Theorem 3.2.8. *[33] If a metric space (T, d), then following conditions are equivalent,*

1. *T is Baire space.*
2. *The $\{f_n : n \in \mathbb{N}\}$ be a sequence of functions in $Q(T)$is pointwisely converges to f, which is a real-valued function defined on T. Then f is quasicontinuous if and only if $\{f_n : n \in \mathbb{N}\}$ is equi-quasicontinuous.*

Theorem 3.2.9. *[33] Consider a quasi-regular T_1 space (T, d) having countable local π-base, then following conditions are equivalent,*

1. *T is Baire space.*
2. *The $\{f_n : n \in \mathbb{N}\}$ be a sequence of functions in $Q(T)$is pointwisely converges to f, which is a real-valued function defined on T. Then f is quasicontinuous if and only if $\{f_n : n \in \mathbb{N}\}$ is equi-quasicontinuous.*

Construction of quasicontinuous functions

Theorem 3.2.10. *Let T, M and Z be space,*

1. *(**Constant function**) If the function $g : T \to M$ maps the entire set T to a single point y_0 in the codomain M, then g is quasicontinuous.*

   Proof. Constant functions are continuous. Since every continuous function is quasicontinuous. Hence constant functions are quasicontinuous. □

2. *(**Inclusion**) The inclusion function $k : A \to T$ becomes quasicontinuous if A is a subspace of T.*

   Proof. Let V be an open set in T. Then $k^{-1}(V) = A \cap V$ by Definition of subspace topology is open set in A. So by Theorem (3.2.2) inclusion map k is quasicontinuous. □

3. *(**Composition**)[[26], Remark 12] If $f : T \to M$ and $g : M \to Z$ being quasi-continuous, it is not necessary for the composite function $g \circ f : T \to Z$ to be quasicontinuous.*

   Example 3.2.3. *Assume $T = M = [0, 2]$ and $Z = R$ are spaces with standard topology. Let us defines $f : T \to M$ and $g : M \to Z$ as*

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ 1 & \text{if } 1 \leq t < 2, \\ 2 & \text{if } t = 2. \end{cases}$$

it is monotone and left continuous at every point. Hence quasicontinuous.

$$g(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1/2, \\ 1 & \text{if } 1/2 < t \leq 1, \\ 2 & \text{if } 1 < t \leq 2. \end{cases}$$

then the inverse of g is

$$g^{-1}(t) = \begin{cases} [0,1/2] & \textit{if } t = 1, \\ (1/2,1] & \textit{if } t = 2, \\ 2 & \textit{if } 0 < t < 1. \end{cases}$$

clearly inverse image of every open set in Z is semi-open set in M. So by Theorem (3.2.2) g is quasicontinuous. The composition of functions $g \circ f : T \to Z$ is given by

$$(g \circ f)(t) = \begin{cases} 1 & \textit{if } 0 \leq t < 1 \\ 2 & \textit{if } 1 \leq t < 2 \\ 1 & \textit{if } t = 2 \end{cases}$$

Here $V = (0,2)$ is open set in Z but $f^{-1}(V) = [0,1) \cup \{2\}$ is semi-open set [2]. *Therefore by Theorem (3.2.2) $g \circ f$ is not quasicontinuous.*

4. *(**Composition with continuous**) If the function $f : T \to M$ is quasicontinuous, and the function $g : M \to Z$ is continuous, then the composition $g \circ f : T \to Z$ is also quasicontinuous.*

   Proof. Given $g \circ f : T \to Z$ is a map. To prove $g \circ f$ is quasicontinuous, it is sufficient to show that inverse image of every open in Z is semi-open set in T. Let W be an open set of Z. Since g is continuous map M to Z so by Def. of continuity $g^{-1}(W)$ is open set in M. Also given that f is a quasicontinuous function from T to M. By Theorem (3.2.2) $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is semi-open set in T. Therefore for every open set W of Z inverse image $(g \circ f)^{-1}(W)$ is semi-open set in T. Hence proved $g \circ f$ is quasicontinuous. □

5. *(**Restricting Domain**) If $g : T \to M$ is quasicontinuous and B is a subspace of T. The restricted function $f|B : B \to M$ does not have to be quasicontinuous.*

   Example 3.2.4. *Let $T = [0,1] = M$ are space under usual topology and $g : T \to M$ is a function defined as;*

$$g(t) = \begin{cases} 1/2 & \textit{if } 1/2 \leq t < 1, \\ 0 & \textit{if } 0 \leq t < 1/2, \end{cases}$$

   inverse of g is

$$g^{-1}(t) = \begin{cases} [0,1/2) & \textit{if } t = 0 \\ [1/2,1] & \textit{if } t = 1/2 \end{cases}$$

[2] A subset A of space is said to be semi-open set if it is contained in the closure of its interior(i.e. $A \subset cl(int(A))$).

inverse image of every open set in M is semi-open set in T. So by Theorem (3.2.2) g is quasicontinuous.
Let $B = [0, 1/2]$ is subspace of T then restricted function to set B is

$$g|B(t) = \begin{cases} 0 & \text{if } 0 \le t < 1/2, \\ 1/2 & \text{if } t = 1/2. \end{cases}$$

Let $V = (1/3, 3/4)$ is open set in T. But $(g|B)^{-1}(V) = \{2\}$ is not semi-open. By Theorem (3.2.2) restriction $g|B$ is not quasicontinuous.

6. ***(Restricting domain to open subspace)*** *If the function $g : T \to M$ is quasicontinuous, and B is an open subspace of T, then the restricted function $g|B : B \to M$ is also quasicontinuous.*

   Proof. Given $g : T \to M$ is a quasicontinuous function and B is open subspace of T. To prove restriction $f|B$ is quasicontinuous.
   Let V be an open of M. Then

   $$(g|B)^{-1}(V) = g^{-1}(V) \cap B$$

   Since g is quasicontinuous then by Theorem (3.2.2) $g^{-1}(V)$ is semi-open set in T. Given B is open set and $g^{-1}(V)$ is semi-open set then $g^{-1}(V) \cap B$ is semi-open set. Also $g^{-1}(V) \cap B \subset B \subset T$. So $g^{-1}(V) \cap B$ is semi-open set in B. Hence by Theorem (3.2.2) restriction $g|B$ is quasicontinuous. □

7. ***(Restricting or expanding the range)*** *If the function $g : T \to M$ is quasicontinuous and Z is a subspace of M containing the image set $g(T)$, then the function $h : T \to Z$ derived by reducing the range of g is also quasicontinuous. If Z is a space with M as a subspace, then the function $k : T \to Z$ is also quasicontinuous when the range of g is expanded.*

   Proof. **Restricting range** Since $g : T \to M$ is quasicontinuous and $g(T) \subset Z \subset M$
   To prove $h : T \to Z$ is quasicontinuous.
   Let B be any open set of Z and $B = V \cap Z$, where V is open set in M. Then $g^{-1}(V) = h^{-1}(B)$ because Z contain image set $g(T)$. Since g is quasicontinuous so by Theorem (3.2.2) $g^{-1}(V)$ is semi-open set in T. Therefore $h^{-1}(V)$ is semi-open set in T. Hence by Theorem (3.2.2) h is quasicontinuous.
   Expanding domain Given that $g : T \to M$ is quasicontinuous and M is a subset of Z. Then the inclusion map $j : M \to Z$ is continuous. To show that $k : T \to Z$

is quasicontinuous.
Clearly h is composition of g and j (i.e $h = j \circ g$). As we proved in part[4] Composition of quasicontinuous and continuous is quasicontinuous. Hence k is quasicontinuous. □

8. *(**Local formation of quasicontinuity**) If T can be expressed as the union of open sets O_α such that $h|O_\alpha$ is quasicontinuous for each α, the map $h : T \to M$ is quasicontinuous.*

Proof. Since $T = \bigcup_{\alpha \in J} O_\alpha$ such that $g|O_\alpha$ is quasicontinuous for each $\alpha \in J$. To prove g is quasicontinuous.
Let V be an open subset of M. Then

$$g^{-1}(V) \cap O_\alpha = (g|O_\alpha)^{-1}(V) \tag{3.1}$$

for each $\alpha \in J$ function $g|O_\alpha$ is quasicontinuous so by Theorem (3.2.2) $(g|O_\alpha)^{-1}(V)$ is semi-open set in O_α for each $\alpha \in J$. Then $(g|O_\alpha)^{-1}(V)$ is semi-open set in T. So expression (3.1) denotes the set of all points x that lie in O_α for which $g(x) \in V$. Therefore

$$g^{-1}(V) = \bigcup_{\alpha \in J} (g^{-1}(V) \cap O_\alpha)$$

for every $\alpha \in J$, $(g^{-1}(V) \cap O_\alpha)$ is open set in T, So $\bigcup_{\alpha \in J}(g^{-1}(V) \cap O_\alpha$ ie semi-open set [1]. Therefore $g^{-1}(V)$ is semi-open set in T. Hence by Theorem (3.2.2) g is quasicontinuous. □

9. *(**Maps into product**)[[26], Theorem 15] Given $g : B \to T \times M$, which is given by $g(a) = (g_1(a), g_2(a))$ for all $a \in B$, and assuming that g is quasicontinuous, then we have quasicontinuous functions $g_1 : B \to M$ and $g_2 : B \to M$, which are coordinate mappings of g. The converse is not true.*

Proof. Given $g : B \to T \times M$ be given by $g(a) = (g_1(a), g_2(a))$ for all $a \in B$ is quasicontinuous. To prove g_1 and g_2 are quasicontinuous.
Firstly we prove $g_1 : B \to T$ is quasicontinuous. Let V_1 be any open set in T, therefore $V_1 \times M$ is open set in $T \times M$. Since g is quasicontinuous so by Theorem (3.2.2) $g^{-1}(V_1 \times M)$ is semi-open in B, as by Definition of function $g^{-1}(V_1 \times M) = (g_1)^{-1}(V_1)$ therefore $(g_1)^{-1}(V_1)$ is semi-open set in B. Hence g_1 is quasicontinuous. Similarly, we can prove for g_2. □

[1]Arbitrary union of semi-open set is semi-open.

Example 3.2.5. *Let $B = T_1 = [0,1] = T_2$ are space with usual topology. Let $g_i : B \to T_i$ for $i = 1, 2.$ as follows:*

$$g_1(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1/2, \\ 0 & \text{if } 1/2 < t < 1. \end{cases}$$

$$g_2(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1/2, \\ 0 & \text{if } 1/2 \leq t < 1. \end{cases}$$

Both functions g_1 and g_2 are quasi-continuous. but $g : B \to T \times M$ defined as $g(x) = (g_1(x), g_2(x))$ for all $x \in B$ is not quasicontinuous because $V = S_{1/2}(1,0)$ is a open set in $T \times M$. Then $g^{-1}(V) = {g_1}^{-1}(V) \cap {g_1}^{-1}(V)$=$[0,1/2] \cap [1/2,1) = \{2\}$ is not semi-open set in B.

10. *(**Map on product**)[[26], Theorem 14] If quasicontinuous map $g_i : T_i \to M_i$ and $g : T_1 \times T_2 \to M_1 \times M_2$, they are defined as follows: $g(x_1, x_2) = (g_1(x_1), g_2(x_2))$. Then g is quasicontinuous.*

Proof. Given $g_i : T_i \to M_i$ be quasicontinuous map and $g : T_1 \times T_2 \to M_1 \times M_2$ and defined as $g(x_1, x_2) = (g_1(x_1), g_2(x_2))$.Let $p = (p_1, p_2) \in T_1 \times T_2$. To prove g is quasicontinuous. for this we show that for any $p \in W$ open set in $T_1 \times T_2$ and $g(p) \in Z$ open set in $M_1 \times M_2$ $\exists$ non-void open set A such that $A \subset W$ and $g(A) \subset Z$. Let $W = U_1 \times U_2$ for any U_1 and U_2 open set in T_1 and T_2 resp. And $Z = V_1 \times V_2$ for any V_1 and V_2 open set in M_1 and M_2, respectively. Therefore $p_1 \in U_1$ and $g_1(p_1) \in V_1$. Since g_1 is quasicontinuous so by Definition $\exists$ a non-void set $G_1 \subset U_1$ such that $g_1(G_1) \subset V_1$. Similarly $p_2 \in U_2$ and $g_2(p_2) \in V_2$. Since g_2 is quasicontinuous so by Definition $\exists$ a non-void set $G_2 \subset U_2$ such that $g_2(G_2) \subset V_1$. Therefore $\exists$ a non-void open set $G_1 \times G_2 \subset U_1 \times U_2 = W$ such that $g(G_1 \times G_2) = g_1(G_1) \times g_2(G_2) \subset V_1 \times V_2 = Z$. This implies g is quasicontinuous at point p. But p is an arbitrary point of $T_1 \times T_2$. Hence g is quasicontinuous on $T_1 \times T_2$. □

11. *(**Algebraic operation on quasicontinuity**)[[26], Remark 13] Let T be space and if $h, k : T \to \mathbb{R}$ are quasicontinuous function. Then $h + k$, $h - k$, and $h \cdot k$ need not be quasicontinuous. If $k(z) \neq 0$ for all z, then h/k need not be a quasicontinuous function.*

Example 3.2.6. ***For the sum of two quasicontinuous functions*** *Let $T = [0,2]$, $M = \mathbb{R}$ be space with usual topology and $h, k : T \to M$ are two function given* *by*

$$h(t) = \begin{cases} 3 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} < t \leq 1, \\ -t, & \text{if } 1 < t \leq 2. \end{cases}$$

$$k(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} < t < 1, \\ t+1 & \text{if } 1 \leq t \leq 2. \end{cases}$$

Both h and k are quasicontinuous and their sum is defined as

$$(h+k)(t) = \begin{cases} 3 & \text{if } 0 \leq t \leq 1/2, \\ 0 & \text{if } 1/2 < t < 1, \\ 3 & \text{if } t = 1, \\ 1 & \text{if } 1 < t \leq 2. \end{cases}$$

take $V = (2,4)$ be open set of M but $(h+k)^{-1}(V) = [0,1/2] \cup \{1\}$ is not semi-open. Hence by Theorem (3.2.2) $h+k$ is not quasicontinuous.

Example 3.2.7. ***For the product of two quasicontinuous functions*** *Let $T = [0,2], M = \mathbb{R}$ be space with usual topology and $h, k : T \to M$ are to function defined as;*

$$h(t) = \begin{cases} 2 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < t \leq 1, \\ -t & \text{if } 1 < t \leq 2. \end{cases}$$

$$k(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} < t < 1, \\ 2 & \text{if } 1 \leq t \leq 2. \end{cases}$$

both h and k are quasicontinuous and their product is defined as;

$$(h \cdot k)(t) = \begin{cases} 2 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} < t < 1, \\ 2 & \text{if } t = 1, \\ -2t & \text{if } 1 < t \leq 2. \end{cases}$$

Take $V = (1,3)$ is open set in M, but $(h \cdot k)^{-1}(V) = [0,1/2) \cup \{1\}$ is not semi-open. So by Theorem (3.2.2) $h \cdot k$ is not quasicontinuous.

3.3 Topological properties and quasicontinuous function

Preservation of properties under continuous functions on space is a very important tool for the classification of space. However, in some cases, the quaiscontinuous functions are more useful than the continuous functions for classifying space. However, the preservation of a properties under quasicontinuous map implies preservation under a continuous map, but not conversely. As we already have separable space and Baire space are invariant under a quasicontinuous map in [39]. In [30], the properties like compactness, connectedness, and T_0-axiom are not preserved by quasicontinuous map but persevered by continuous map. For example " Let $T = [0, 1)$ and $\tau_1 = \{\{\phi, T\} \cup \{[0, 2^{-n}) : n \in \mathbb{N}\}\}$ and $\tau_2 = \{\{\phi, T\} \cup \{[0, a) : 0 < a \leq 1\}\}$ are two topology on T. Let the identity map $i : (T, \tau_1) \to (T, \tau_2)$ is defined by $i(x) = x$ is quasicontinuous onto map. Since (T, τ_1) is compact but (T, τ_2) is not compact space. Therefore, quasicontinuous map does not preserve compactness, The space $T = \{a, b, c\}$ be set with two topology $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, T\}$ and $\tau_2 = \{\phi, \{a\}, \{b, c\}, T\}$. Then, identity map $f : (T, \tau_1) \to (T, \tau_2)$ is quasicontinuous. As (T, τ_1) is connected space but (T, τ_2) is not a connected space and if $T = [0, 1)$ and $\tau_1 = \{\{\phi, T\} \cup \{[0, a) : 0 < a \leq 1\}\}$ and $\tau_1 = \{\phi, T\} \cup \{[0, 2^{-n}) : n \in \mathbb{N}\}$ are two topology on T. Let identity map $i : (T, \tau_1) \to (T, \tau_2)$ defined by $i(x) = x$ is quasicontinuous onto map. Since (T, τ_1) is T_0-space but (T, τ_2) is not T_0-space. We focus on the preservation of a strong form of connected space under a quasicontinuous map.

3.3.1 Strong forms of connected space

Roughly speaking, a connected space is a single piece, but in proper way it is defined as follows: any space (T, τ) is said to be connected if there does not exist any separation (i.e. it is not possible to find non-void disjoint open sets A and B such that $T = A \cup B$). Connectedness is a topological property and has great importance in the study of space. Some of the generalized form of connectedness like semi-connectedness, α-connectedness, β-connectedness, b-connectedness, half connected, half semi-connectedness, half α-connectedness and half β-connectedness have been introduced and studied in [47–52]. The Cl-Cl connectedness introduced by Modak and Noiri in [52] is a weak form of connectedness and all others are strong forms of connectedness. In this section, we study quasicontinuous functions on the mentioned forms of connected space. In the second section, we mention some results that we will use in the subsequent sections.

Various types of connected space If T, M and Z be our space. Thus, any subset D of T is said to be b-open [53] (resp. β-open set [54]) if $A \subset (Int(Cl(D))) \cup Cl(Int(D))$ (resp. $A \subset Cl(Int(Cl(D))))$. The complement of b-open set (resp. β-open set) set is said to be b-closed (resp. β-closed set). The b-closure(bcl) of D is the intersection of all b-closed set containing D. Similar definitions for β-closure (βCl).

Definition 3.3.1. Two non-void subsets C and D in a space T are said to be

1. semi-separated [48] (resp. α-separated [49], b-separated [51]) if $C \cap Scl(D) = \emptyset = Scl(C) \cap D$, (resp. $C \cap \alpha Cl(D) = \emptyset = \alpha Cl(C) \cap D$, $C \cap bcl(D) = \emptyset = bcl(C) \cap D$).
2. half semi-separated (resp. half α-separated) [47] if $C \cap Scl(D) = \emptyset$ or $Scl(C) \cap D = \emptyset$, (resp. $C \cap \alpha Cl(D) = \emptyset$ or $\emptyset = \alpha Cl(C) \cap D$)
3. $Cl - Cl$ separated sets [52] if closure of two set C and D is disjoint.

Definition 3.3.2. [48] A subset S of a space T is said to be semi-connected (resp., α-connected) if there are no two semi-separated subsets C and D (resp. α-separated) such that $S = C \cup D$.

Theorem 3.3.1. *[48] Semi-connected space is connected, but not conversely.*

Definition 3.3.3. [47] A set $C \subset T$ is said to be half semi-connected (resp. half α-connected) if C is not the union of two non-void half semi-separated (resp. half α-separated) sets in T.

Theorem 3.3.2. *[47] Half semi-connected space is semi-connected, but not conversely.*

Theorem 3.3.3. *[47] Every semi-connected space is α-connected, but not conversely.*

Definition 3.3.4. [52] A set $C \subset T$ is said to be Cl-Cl connected if C f C is not the union of two non-void Cl-Cl separated sets in T.

Theorem 3.3.4. *[52] Connected spaces are always Cl-Cl connected, but not conversely.*

Scl-Scl Connected space We are going to introduce a strong form of semi-connectedness which lies between lies between semi-connectedness and Cl-Cl connectedness. Next, we study its relations with other forms of connectedness.

Definition 3.3.5 (Scl-Scl separated sets)**.** Let C and D be two non-void subsets of the space T are called Scl-Scl separated sets if $Scl(C) \cap Scl(D) = \emptyset$.

Theorem 3.3.5. *The Cl-Cl separated set is always Scl-Scl separated, but not conversely.*

Proof. Let C and D be non-void Cl-Cl separated sets so $Cl(C) \cap Cl(D) = \emptyset$. Since for any subset A of T, then $Scl(A) \subset Cl(A)$. This implies that $Scl(C) \cap Scl(D) = \emptyset$. Hence C and D are Scl-Scl separated.

Converse, let $T = \{1, 2, 3\}$ having topology $\tau = \{\emptyset, \{1\}, \{1, 2\}, T\}$. Semi-open set in T are $\emptyset, \{1\}, \{1, 2\}, \{1, 3\}$ and T. Take $C = \{2\}$ and $D = \{3\}$, then $Scl(C) = \{2\}$ and $Scl(D) = \{3\}$. But $Cl(C) = \{2, 3\}$ and $Cl(D) = \{3\}$, then $Scl(C) \cap Scl(D) = \emptyset$ and $Cl(C) \cap Cl(D) = \{3\}$. Hence C and D are Scl-Scl separated but not Cl-Cl separated. □

Theorem 3.3.6. *The Scl-Scl separated sets are always semi-separated, but not conversely.*

Proof. Given C and D are non-void Scl-Scl separated sets, so $Scl(C) \cap Scl(D) = \emptyset$. Since for any subset B of T, then $B \subset Scl(B)$. This implies that $C \cap Scl(D) = \emptyset$ and $Scl(C) \cap D = \emptyset$. Hence C and D are semi-separated.

Converse, take $\mathbb{R}$ having standard topology. $C = \{(-1)^n \frac{1}{2^n} | n \in \mathbb{N}\}$ and $D = \{(-1)^n \frac{1}{3^n} | n \in \mathbb{N}\}$, then $Scl(C) = \{0\} \cup C = Cl(C)$ and $Scl(D) = \{0\} \cup D = Cl(D)$. Therefore $Scl(C) \cap Scl(D) = \{0\}$ and $Scl(C) \cap D = \emptyset = C \cap Scl(D)$. Hence C and D are semi-separated but not Scl-Scl Separated. □

A Scl-Scl separation has no relation with separation. Let $T = \{1, 2, 3\}$ having topology $\tau = \{\emptyset, \{1\}, \{1, 2\}, T\}$. Semi-open subset of T are $\emptyset, \{1\}, \{1, 2\}, \{1, 3\}$ and T. Then set $C = \{2\}$ and $D = \{3\}$ are Scl-Scl separated but $Cl(C) \cap D = \{3\}$, hence C and D are not separated.

Consider $\mathbb{R}$ having standard topology, $C = \{(-1)^n \frac{1}{2^n} | n \in \mathbb{N}\}$ and $D = \{(-1)^n \frac{1}{3^n} | n \in \mathbb{N}\}$, then $C \cap Cl(D) = \emptyset = Cl(C) \cap D$, but $Scl(C) \cap Scl(D) = \{0\}$. Hence C and D are separated but not Scl-Scl separated.

Definition 3.3.6 (Scl-Scl connectedness)**.** A subset A of T is said to be Scl-Scl connected if A cannot be written as a union of two Scl-Scl separated sets in T.

Theorem 3.3.7. *A space T is Scl-Scl connected if and only if it is not possible to express T as the union of two non-void and disjoint semi-clopen sets.*

Proof. Firstly, assume that T is Scl-Scl connected. If possible, let us assume $T = C \cup D$ such that C and D are non-void disjoint semi clopen sets. Therefore $Scl(C) = C$ and $Scl(D) = D$, then $Scl(C) \cap Scl(D) = \emptyset$, which is contradiction to T is Scl-Scl connected. Hence x is not equal to the disjoint union of two non-void semi clopen sets.

Conversely, let us assume T is not Scl-Scl connected space. Therefore $\exists$ two non empty sets C and D such that $T = C \cup D$ and $Scl(C) \cap Scl(D) = \emptyset$. Then $T = Scl(C) \cup Scl(D)$.

Both sets $U = Scl(C)$ and $V = Scl(D)$ are non-void disjoint semi clopen sets and $T = U \cup V$, which is a contraction. Therefore T is Scl-Scl connected. □

Theorem 3.3.8. *Each Scl-Scl connected space is always $Cl - Cl$ connected, but not conversely.*

Proof. As T Scl-Scl connected if there are no two Scl-Scl separated subsets C and D such that $T = C \cup D$. Thus, by Theorem (3.3.5) there are no two Cl-Cl separated subsets C and D with $T = C \cup D$. Hence T is Cl-Cl connected.

Converse, let $T = \{1, 2, 3\}$ having topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, T\}$ is connected. Take $C = \{1\}$ and $D = \{2, 3\}$, then C and D are disjoint semi clopen sets with $T = C \cup D$. Therefore by Theorem (3.3.7) space T is not Scl-Scl connected space. □

Theorem 3.3.9. *Semi-connected space is always Scl-Scl connected, but not conversely.*

Proof. Let T be a semi-connected space, then there do not exist two semi-separated subsets C and D such that $T = C \cup D$. By Theorem (3.3.6), there are no two Scl-Scl separated subsets C and D with $T = C \cup D$. Hence T is Scl-Scl connected. Conversely it need not be hold, by example in the proof of the Theorem (3.3.6). □

We found that Scl-Scl connected space has no relation with connected space. This can be verified with the following examples. For connected space $\nRightarrow$ Scl-Scl connected, take a set $T = \{a, b, c\}$ having topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, T\}$. Then, space have semi-open set sets are $\emptyset, \{a\}, \{b, c\}, \{b\}, \{a, b\}, \{a, c\}, T$. Let $C = \{a\}$ and $D = \{b, c\}$ are non-void semi clopen sets and $T = C \cup D$, then by Theorem (3.3.7), T is not Scl-Scl connected but it is connected.

For Scl-Scl connected space $\nRightarrow$ connected, take $\mathbb{R}$ having usual topology is connected but not Scl-Scl connected because $\mathbb{R} = (-\infty, 1] \cup (1, \infty)$ both are disjoint and Scl-Scl separated sets.

Thus, from the above discussions, we have the following diagram:

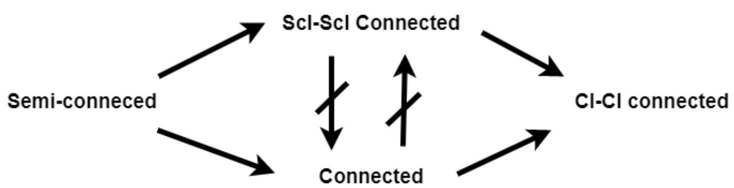

FIGURE 3.1: Relationship between different connected spaces

3.3.2 Preservation under the quasicontinuous maps

In this subsection, we study the preservation of some strong forms of the connectedness of the space under a quasicontinuous map. In the end, we prove the intermediate value theorem for a quasicontinuous map. All the stronger forms of quasicontinuous maps are denoted by QS-map and it is understood that any QS-map is also a quasicontinuous map but its converse need not be true, for example, continuous maps and Strongly quasicontinuous map both are stronger then quasicontinuous map.

Theorem 3.3.10. *If P property is preserved under a quasicontinuous map, then P is preserved under a QS-map.*

Proof. Given that property P is preserved under a quasicontinuous map, that is for any quasicontinuous map $f\colon T \to M$ if T has P property then $f(T)$ also has P property. To prove P is preserved under a QS-map. Take $g\colon T \to M$ to be any QS-map and T has P property. We must prove $g(T)$ has P. Since every QS-map is a quasicontinuous map. So g is quasicontinuous map. Therefore $g(T)$ has P property. Hence QS-map preserves the P property. □

Corollary 3.3.1. *If P property is preserved under a quasicontinuous map, then P is preserved during a continuous map, but not conversely.*

Proof. Since a continuous map is stronger than quasicontinuous map. Hence it preserves property P. Conversely, Take $T = \{1, 2, 3\}$ with topologies $\tau_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, T\}$ and $\tau_2 = \{\emptyset, \{1\}, \{2, 3\}, T\}$. Then, identity map $f : (T, \tau_1) \to (T, \tau_2)$ is a quasicontinuous. As (T, τ_1) and (T, τ_2) are connected and not connected space, respectively. □

Theorem 3.3.11. *The image of semi-connected space under the quasicontinuous map is semi-connected.*

Proof. Given T is semi-connected and $g : T \to M$ is a quasicontinuous map. If possible, assume $g(T)$ is not semi-connected, so by Definition (3.3.2) $\exists$ two non-void sets C and D with $g(T) = C \cup D$ such that $C \cap Scl(D) = \emptyset = Scl(C) \cap D$. Then $T = g^{-1}(C) \cup g^{-1}(D)$. Firstly, $C \cap Scl(D) = \emptyset$ implies $g^{-1}(C \cap Scl(D)) = \emptyset$ and since g is a quasicontinuous map, so by using Theorem (3.2.5), we get

$$g^{-1}(C) \cap Sclg^{-1}(D) \subset g^{-1}(C \cap Scl(D)) = \emptyset.$$

Similarly

$$Sclg^{-1}(C) \cap g^{-1}(D) \subset g^{-1}(Scl(C) \cap (D)) = \emptyset.$$

Which contradicts that T is semi-connected. Therefore $g(T)$ must be semi-connected. ☐

Remark 3.1. The image of α-connected space under the strongly quasicontinuous map is α-connected.

Corollary 3.3.2. *The image of semi-connected space under a quasicontinuous map is connected.*

Proof. By the above Theorem (3.3.11) the image of semi-connected space under the quasicontinuous map is semi-connected, then by Theorem (3.3.1) every semi-connected is connected. ☐

Subsequently we can prove results for a strongly quasicontinuous map.

Remark 3.2. The image of α-connected space under the strongly quasicontinuous map is connected.

Corollary 3.3.3. *[[47]] The image of semi-connected(resp. α-connected) space under a quasicontinuous map(resp. strongly quasicontinuous map) is Cl-Cl connected.*

Proof. By the above Corollary (3.3.2), the quasicontinuous image of a semi-connected space is connected, then by Theorem (3.3.4), a connected space is Cl-Cl connected. ☐

Corollary 3.3.4. *The image of semi-connected space under a quasicontinuous map is β-connected, α-connected, α_β-connected [55].*

Proof. By using Theorem (3.3.11). ☐

Theorem 3.3.12. *The image of half semi-connected space under a quasicontinuous map is half semi-connected.*

Proof. Given that T is half semi-connected and $g : T \to M$ is a quasicontinuous map. Let us assume that $g(T)$ is not half semi-connected so by Definition (3.3.3) $\exists$ two non-void sets C and D with $g(T) = C \cup D$ such that $C \cap Scl(D) = \emptyset$ or $\emptyset = Scl(C) \cap D$. By hypotheses $T = g^{-1}(C) \cup g^{-1}(D)$. If $C \cap Scl(D) = \emptyset$ implies $g^{-1}(C \cap Scl(D)) = \emptyset$ and since g is a quasicontinuous map so by using Theorem (3.2.5) we get

$$g^{-1}(C) \cap Sclg^{-1}(D) \subset g^{-1}(C \cap Scl(D)) = \emptyset.$$

On the other hand, if $Scl(C) \cap D = \emptyset$ implies $Sclg^{-1}(C) \cap g^{-1}(D) \subset g^{-1}(Scl(C) \cap (D)) = \emptyset$. Which is a contradiction to that T is half semi-connected. Thus $g(T)$ must be half semi-connected. ☐

Remark 3.3. The image of half α-connected space under a strongly quasicontinuous map is half α-connected.

Corollary 3.3.5 ([47], Theorem5.4). *The image of half semi-connected space under a quasicontinuous map is Cl-Cl connected.*

Theorem 3.3.13. *The image of semi-connected space under a strongly quasicontinuous map is semi-connected.*

Proof. From Theorem (3.3.11), the quasicontinuous image of semi-connected space is semi-connected. Since the strongly quasicontinuous map is stronger than the quasi-continuous map, so by Theorem (3.3.10), the strongly quasicontinuous image of semi-connected space is semi-connected. □

Remark 3.4. The image of half semi-connected space under a strongly quasicontinuous map is half semi-connected.

Theorem 3.3.14. *The image of Scl-Scl connected space under a quasicontinuous map is Scl-Scl connected.*

Proof. Given $g\colon T \to M$ be a quasicontinuous map, T be Scl-Scl connected and M be any space. If possible, assume that $g(T)$ is not Scl-Scl connected, $\exists$ two non-void disjoint subsets of $g(T)$ with $g(T) = C \cup D$ such that $Scl(C) \cap Scl(D) = \emptyset$. Then $T = g^{-1}(C) \cup g^{-1}(D)$. As $Scl(C) \cap Scl(D) = \emptyset$ implies

$$g^{-1}(Scl(C) \cap Scl(D)) = \emptyset,$$

$$g^{-1}(Scl(C)) \cap g^{-1}(Scl(D)) = \emptyset.$$

by Theorem (3.2.5) we have $Sclg^{-1}(C) \subset g^{-1}(Scl(D))$ so

$$Sclg^{-1}(C) \cap Sclg^{-1}(D) = \emptyset,$$

therefore, the sets $g^{-1}(C)$ and $g^{-1}(D)$ a form Scl-Scl separation which contradicts our assumption. Thus $g(T)$ is Scl-Scl connected. □

Theorem 3.3.15. *The image of Scl-Scl connected space under a strongly quasicontinuous map is Scl-Scl connected.*

Proof. From Theorem (3.3.14), the quasicontinuous image of Scl-Scl connected space is Scl-Scl connected. Since the strongly quasicontinuous map is stronger than quasi-continuous map, so by Theorem (3.3.10) the strongly quasicontinuous image of Scl-Scl connected space is Scl-Scl connected. □

Theorem 3.3.16. *The image of b-connected space under a quasicontinuous map is semi-connected.*

Proof. Given a b-connected space T and $g\colon T \to M$ is a quasicontinuous map. Let us assume that $g(T)$ is not semi-connected, so by Definition (3.3.2) $\exists$ two non-void sets C and D with $g(T) = C \cup D$ such that $C \cap Scl(D) = \emptyset = Scl(C) \cap D$. Then $T = g^{-1}(C) \cup g^{-1}(D)$. Firstly, we take $C \cap Scl(D) = \emptyset$ implies $g^{-1}(C \cap sCl(D)) = \emptyset$ and since g is a quasicontinuous map, so by using Theorem (3.2.5) we get

$$g^{-1}(C) \cap Sclg^{-1}(D) \subset g^{-1}(C \cap Scl(D)) = \emptyset.$$

Since we know $bcl(C) \subset Scl(C)$ then, $g^{-1}(C) \cap bclg^{-1}(D) \subset \emptyset$. In a similar way from $Scl(C) \cap D = \emptyset$, we get $bclg^{-1}(C) \cap g^{-1}(D)$ is empty. This shows that T is not semi-connected, contrary to our assumption. Thus $g(T)$ must be semi-connected. □

Remark 3.5. The image of β-connected space under a strongly quasicontinuous map is semi-connected.

Theorem 3.3.17. *The image of b-connected(resp. β-connected [49]) space under a strongly quasicontinuous map is semi-connected.*

Proof. By Theorem (3.3.16) the image of b-connected space under a quasicontinuous map is semi-connected. Since a strongly quasicontinuous map is stronger than a quasicontinuous map. So by Theorem (3.3.10) the image of b-connected space under a strongly quasicontinuous map is semi-connected. □

Theorem 3.3.18. *The image of half b-connected space under a quasicontinuous map is half semi-connected.*

Proof. Given a half b-connected space T and $g\colon T \to M$ is a quasicontinuous map. Let us assume that $g(T)$ is not half semi-connected so by Definition (3.3.3) $\exists$ two non-void sets A and B with $M = A \cup B$ and $A \cap Scl(B) = \emptyset$ or $\emptyset = Scl(A) \cap B$. By hypotheses $T = g^{-1}(A) \cup f^{-1}(B)$. If $A \cap Scl(B) = \emptyset$ implies $g^{-1}(A \cap Scl(B)) = \emptyset$ and since g is a quasicontinuous map so by using Theorem (3.2.5) we get

$$g^{-1}(A) \cap Sclg^{-1}(B) \subset g^{-1}(A \cap Scl(B)) = \emptyset,$$

by using $bcl(A) \subset Scl(A)$, we get $g^{-1}(A) \cap bclg^{-1}(B) = \emptyset$. On the other hand, if $Scl(A) \cap B = \emptyset$, in similar way we get

$$bclg^{-1}(A) \cap g^{-1}(B) = \emptyset.$$

This shows that T is not half semi-connected which is a contradiction. Thus $g(T)$ must be half semi-connected. □

Remark 3.6. The image of half β-connected space under a quasicontinuous map is half semi-connected.

Theorem 3.3.19. *The image of half b-connected(resp. half β-connected) space under a strongly quasicontinuous map is half semi-connected.*

Proof. From Theorem (3.3.18), the image of half b-connected under a quasicontinuous map is half semi-connected. Since Strongly quasicontinuous -the map is stronger than a quasicontinuous map. By Theorem (3.3.10), the image of half b-connected space under a Strongly quasicontinuous map is half semi-connected. □

The main consequence of the intermediate value theorem of calculus is the study of real-valued continuous functions on closed intervals $[a, b]$ of the real line, where we consider the interval as a subset of $\mathbb{R}$. When we consider this interval as a space, then this theorem does not depend only on the continuity of function but on the properties of the space also. Now the connectedness as a topological property inherited by the space comes into the picture, which gives the general form of this theorem. In the next result, we see how the intermediate value theorem in calculus is generalized using a topological approach as:

Theorem 3.3.20 (Intermediate value theorem)**.** *[38] For a connected space T, M is an ordered space, and $f\colon T \to M$ is a continuous map. If $a, b \in T$ and $r \in M$ such that $r \in (f(a), f(b))$. Then $\exists\ c \in T$ such that $f(c) = r$.*

Now the question arises from the general form of the intermediate value theorem: Is it possible that the theorem also holds for a more general form of continuous map that is a quasicontinuous map? Unfortunately, we can see from the below example that it does not hold good for a quasicontinuous map.

Example 3.3.1. *Let $T = [0, 2], M = \mathbb{R}$ be space having standard topology and $g\colon T \to M$ is a map defined as follows:*

$$g(z) = \begin{cases} 3 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < z \leq 1, \\ -z & \text{if } 1 < z \leq 2. \end{cases}$$

The map g is a quasicontinuous map. Take $r = 2 \in M$, then r lies between $g(1/2) = 3$ and $g(1) = 1$. But there is no $c \in T$ such that $g(c) = r$. Hence the intermediate value theorem does not hold when we replace continuous by a quasicontinuous map.

Now we can find that the general intermediate value theorem is not possible for a quasicontinuous map. But in the case of semi-connected space, it works.

Theorem 3.3.21 (Intermediate value theorem for quasicontinuous map)**.** *For a semi-connected space T, M is an ordered space and $g\colon T \to M$ is a quasicontinuous map. If $a, b \in T$ and $r \in M$ s.t. $r \in (g(a), g(b))$. Then $\exists\ c \in T$ such that $g(c) = r$.*

Proof. Given a semi-connected space T, M is an ordered space and $g\colon T \to M$ is a quasicontinuous map. The sets $C = g(T) \cap (-\infty, r)$ and $D = g(T) \cap (r, \infty)$ are non-void and disjoint because $g(a) \in A$ and $g(b) \in D$. Both sets are open set in $g(T)$ because both rays are open set in M. If there is no $c \in T$ such that $g(c) = r$, then $g(T)$ would be the union of C and D. Then C and D form a separation for $g(T)$, which contradicts that the image of semi-connected space under a quasicontinuous map is connected. Therefore $g(c) = r$, for a $c \in T$. □

Remark 3.7. If T is a α-connected space and g is a strongly quasicontinuous map, then the intermediate value theorem also holds.

3.4 CONCLUSION

In this chapter, we summarizing the key findings and insights gained in the exploration of quasicontinuity. The set of all continuous functions $C(T, M)$ is a subset of the set of all quasicontinuous functions $Q(T, M)$. This chapter serves as a reflection on the significance of the preservations of different forms of connectedness under a quasicontinuous function. Finally, we examined the Intermediate Value Theorem in the context of quasicontinuous function and set the stage for further chapters in this study of functions in space.

Chapter 4

The topology of pointwise convergence on the space $Q(T, M)$

4.1 Introduction

In previous chapter, we already studied that a set of continuous functions is contained in a set of quasicontinuous functions. The introduction of the theory of function spaces of continuous function equipped with the topology of point convergence, now called C_p-theory, is attributed to Alexander Vladimir Arhangel'skii. In 1992, Arhangel'skii studied important results of C_p-theory in [60]. Subsequently, numerous researchers dedicated their efforts to enhancing C_p-theory, bestowing upon it the elegance and magnificence it possesses today. In the latter half of the 20th century, C_p theory underwent significant development, emerging as one of the most dynamic and actively researched areas within set-theoretic topology. Tkachuk made substantial contributions to this field, presenting a wealth of results and posing numerous unresolved questions in his influential works [56, 57, 75, 77]. These works not only advanced the theoretical foundations of C_p theory but also stimulated extensive subsequent research.

In 2018, McCoy et al. [14] consolidated and expanded upon the existing body of knowledge by compiling comprehensive general results concerning topological function spaces. Their work encompassed a diverse range of topologies such as the uniform topology, fine topology, and graph topology, among others, which are crucial in the study of topological structures and their properties [14]. This compilation not only provided a synthesis of existing findings but also highlighted avenues for further exploration and refinement within the broader context of topological function spaces. In recent years Mishra and Bhaumik [90], and Aaliya and Mishra [82] studied properties of topological function spaces under Cauchy convergence topology and regular topology, respectively. Thus, the evolution of C_p theory and related areas underscores its pivotal role in shaping contemporary research in set-theoretic topology, marked by both foundational advancements and ongoing inquiries into unresolved problems and novel theoretical perspectives.

Now, let us see how the study of topological quasicontinuous function space begins where functions were considered quasicontinuous functions. Hola and Holy studied various properties of the space of quasicontinuous functions under different topologies in the literature in [17] and [35]. Recently, in 2020, Hola and Holy studied cardinal invariants of the space $Q_p(T)$. Furthermore, in 2022, Kumar and Tyagi studied cardinal invariants of the space $Q_p(T,M)$ in [37] a more general form. Extending these investigations, we present in this chapter additional findings concerning cardinal invariants of the space $Q_p(T)$ and further study some maps on both $Q_p(T)$ and $Q_p(T,M)$ spaces. Roughly speaking, cardinal functions extend such important topological properties as countable base, separable, and first countable to higher cardinality. Cardinal functions then allow one to formulate, generalize, and prove results of the type just discussed systematically and elegantly. In addition, cardinal functions allow one to make precise quantitative comparisons between certain topological properties. For example, it is well known that a space with a countable base has a countable dense set. A converse of this result from the theory of cardinal functions states that a regular space with a countable dense set has a base of cardinality $\leq 2^{\omega}$.

In the section (4.2), we define the topology of pointwise convergence on function space and recall some important definitions of cardinal invariants (weight, network, tightness, Lindelöff no. and pseudocharacter, etc.) and their interrelation. Moreover, we discuss results that help us to understand our main results.

In section (4.3), we study the coincidence of network weight and weight of $Q_p(T)$. For a regular space T, we characterize the cardinal invariants such as tightness, pseudo character, network weight, and i-weight for $Q_p(T)$ and $Q_p(T,M)$. Further, we prove a regular space is Lindelöf if $Q_p(T)$ has countable tightness and provide a condition of separability of an regular space in terms of the cardinal function of a compact subset of $Q_p(T,M)$.

In section (4.4), we prove the openness of restriction map from space $Q_p(T)$ to $Q(T,M)$, where M is a dense subspace of T, a map q defined on $C_p(T) \times Q_p(T)$ to $Q_p(T)$ by $q(f,g) = f \cdot g$ is a continuous map and the image of the induced map from $Q_p(T,M)$ to $Q_p(T,Z)$ is dense in $Q_p(T,Z)$, where the induced map is $r_*\colon Q_p(T,M) \to Q_p(T,Z)$ defined by $r_*(f) = r \circ f$ whenever r be surjective continuous map from M to Z.

4.2 Preliminaries

We define the topology of pointwise convergence on the function spaces,“ The collection

$$\mathcal{S} = \{S(x,U)\colon x \in T, U \text{ is open set in } M\}, \text{ where}$$
$$S(x,U) = \{f \in F(T,M)\colon f(x) \in U\}.$$

is subbase for the topology on the set $F(T,M)$ called the topology of point-wise convergence?” The topology of point-wise convergence on the $Q(T,M)$,“ The collections

$$\mathcal{S}' = \{[x,U]\colon x \in T, U \text{ is open set in } M\}, \text{ where}$$
$$[x,U] = \{f \in Q(T,M)\colon f(x) \in U\}.$$

and

$$\mathcal{B} = \{[x_1,\ldots,x_n;U_1,\ldots,U_n]\colon x_i \in T, U_i \text{ open set in } M\}, \text{where}$$
$$[x_1,\ldots,x_n;U_1,\ldots,U_n] = \{f \in Q(T,M)\colon f(x_i) \in U_i, 1 \leq i \leq n\}.$$

are respectively subbase and base, for the topology of point-wise convergence $Q(T,M)$.” The space $Q_p(T,M)$ is the subspace of $F_p(T,M)$. For metric space M, the set

$$W(f,A,\epsilon) = \{g \in Q(T,M)\colon d(f(x),g(x)) < \epsilon, \forall x \in A \in \mathcal{F}, \epsilon > 0\}.$$

is an open neighbourhood of f in $Q_p(T,M)$.

Since our main objective is to discuss cardinal invariance for spaces $Q_p(T)$ and $Q_p(T,M)$, we now recall some important definitions of cardinal functions for the space T for better understanding. If T is space, then

1. the weight of the space T is

$$w(T) = \aleph_0 + \min\{|\mathcal{B}|\colon \mathcal{B} \text{ is a basis in } T\}. \tag{4.1}$$

2. the network weight of the space T is

$$nw(T) = \aleph_0 + \min\{|\mathcal{N}|\colon \mathcal{N} \text{ is a network of } T\}. \tag{4.2}$$

   where the network of a space T is a collection $\mathcal{N}$ of subsets of T such that for any $x \in T$ and every open set U containing x, $\exists\ N \in \mathcal{N}$ such that $x \in N \subset U$.

3. the i-weight of the space T is

$$iw(T) = \aleph_0 + \min\{w(M) \colon \exists \text{continuous and bijective}\, f \colon T \to M\}.. \tag{4.3}$$

4. the weak covering number of the space T is

$$wc(T) = \aleph_0 + \min\{|\mathcal{J}| \colon \mathcal{J} \text{ is a weak covering of}\, T\} \tag{4.4}$$

where a weak covering of a space T is a collection $\mathcal{J}$ of open set in T such that $\overline{\bigcup \mathcal{J}} = T$.

5. the tightness of the space T is

$$t(T) = \{t(x, T) \colon x \in T\}, \text{ where} \tag{4.5}$$

$$t(x, T) = \aleph_0 + \sup\{a(x, M) \colon x \in \overline{M} \subset T\} \tag{4.6}$$

is tightness at the point $x \in T$ and $a(x, M) = \min\{|Z| \colon Z \subset M, x \in \overline{Z}\}$.

6. the density of the space T is

$$d(T) = \aleph_0 + \min\{|D| \colon D \text{ is a dense set in}\, T\}. \tag{4.7}$$

7. the character of the space T is

$$\chi(T) = \sup\{\chi(x, T) \colon x \in T\}, \text{ where} \tag{4.8}$$

$$\chi(x, T) = \aleph_0 + \min\{|\mathcal{O}| \colon \mathcal{O} \text{ is local base at}\, x\} \tag{4.9}$$

is character of the point $x \in T$.

8. the pseudo character of the space T is

$$\psi(T) = \sup\{\psi(x, T) \colon x \in T\}, \text{ where} \tag{4.10}$$

$$\psi(x, T) = \aleph_0 + \min\{|\gamma| \colon \gamma \text{ is a family of open set in}\, T \text{ such that } \cap \gamma = \{x\}\} \tag{4.11}$$

is pseudocharacter of the point $x \in T$. The pseudocharacter of the subset A of the space T is

$$\psi(A, T) = \min\{|\mathcal{U}| \colon \mathcal{U} \subset \tau(T), \bigcup \mathcal{U} = A\}. \tag{4.12}$$

We have an interrelation between cardinal functions.

$$d(T) \leq nw(T) \leq w(T), \text{ where } T \text{ is arbitrary space,} \tag{4.13}$$

$$d(T) = nw(T) = w(T), \text{ where } T \text{ is metrizable space.} \tag{4.14}$$

For a more comprehensive understanding of cardinal functions associated with space, refer to [73]. For more details on cardinal invariants for the space of continuous functions see [60, 75]. Let us look at some important lemmas which help us to prove the main results in the next section.

Lemma 4.2.1. *[18] If T and M are spaces and $f\colon T \to M$ is a function such that for every $x \in T$, $\exists$ an open subset U of T containing x such that $f(y) = f(x)$ for any $y \in U$ and x is in the closure of U, then f is said to be quasicontinuous.*

Lemma 4.2.2. *[37] For any $x \in T$, non-empty closed $E \subseteq T$ such that $x \notin E$ and $y_1, y_2 \in M$, where T is a regular space and M is any space, $\exists$ a function f in $Q_p(T, M)$ such that $f(x) = y_1$ and $f(E) = y_2$.*

Lemma 4.2.3. *[37] For any $x \in T$, the evaluation map $e_x\colon Q_p(T, M) \to M$, denoted by $e_x(f) = f(x)$, is continuous.*

4.3 Cardinal invariants of $Q_p(T)$ and $Q_p(T, M)$

As we know for an arbitrary space T, $nw(T) \leq w(T)$ but for metrizable space T network weight and weight coincide. In addition to the metrizable space T, the problem of the coincidence of network weight and weight can be solved in terms of the function space. However, according to Hola and Holy [18, Example 5.1], $nw(C_p(\mathbb{R})) \neq w(C_p(\mathbb{R}))$, since $nw(C_p(\mathbb{R})) = \aleph_0$ and $w(C_p(\mathbb{R})) = \aleph_1$. In continuation, the result [18, Corollary 4.11] tells us the coincidence of network weight and weight for the space $Q_p(T)$ for countable space T but our next result is a coincidence of network weight and weight for the space $Q_p(T)$ in the more general form of the space T.

Theorem 4.3.1. *For a ordered Hausdorff space T, then $nw(Q_p(T)) = w(Q_p(T)) = |T|$.*

Proof. Let T is ordered Hausdorff space. Then, for each $r \in T$, we choose an open set U_r such that $x \in U_r$ for all $x < r$. We define a function $f_r\colon T \to \mathbb{R}$ as,

$$f_r(x) = \begin{cases} 1, & \text{if } x \in \overline{U_r}; \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma (4.2.1) is quasicontinuous. Let $\mathcal{N}$ is network for the space $Q_p(T)$ such that $|\mathcal{N}| = nw(Q_p(T))$. Fix $1 > \epsilon > 0$, then $\{W(f_r, \{r\}, \epsilon) \colon r \in X, \}$ is collection of open neighborhoods of $f_r \in Q_p(T)$. By the definition of network, for all $f_r \in Q_p(T)$ there exists $N_r \in \mathcal{N}$ such that $f_r \in N_r \subset W(f_r, \{r\}, \epsilon)$. The map $g \colon T \to \mathcal{N}$ defined as $g(r) = N_r$ for any $r \in T$. Now we claim $|T| \leq |\mathcal{N}|$. To prove our claim, we show that the map $g \colon T \to \mathcal{N}$ defined as $g(r) = N_r$ for any $r \in T$, is injective. Let us choose two distinct elements r_1 and r_2 of T. By the definition of the function f_r, $f_{r_1} \notin W(f_{r_2}, \{r_2\}, \epsilon)$ i.e. $N_{r_1} \nsubseteq W(f_{r_2}, \{r_2\}, \epsilon)$. Since $N_{r_2} \subseteq W(f_{r_2}, \{r_2\}, \epsilon)$ so $N_{r_1} \neq N_{r_2}$. Since $|\mathcal{N}| = nw(Q_p(T))$ and $|T| \leq |\mathcal{N}|$, therefore $|T| \leq nw(Q_p(T))$. By result [18, Theorem 4.9], $w(Q_p(T)) = |T|$, therefore $w(Q_p(T)) \leq nw(Q_p(T))$. But by the result (2.13), $nw(Q_p(T)) \leq w(Q_p(T))$, therefore $nw(Q_p(T)) = w(Q_p(T))$. □

We are now going to investigate the cardinal function for the space $Q_p(T)$ in terms of the covering property. Let us define some important definitions and results that help in proving the next result. The Lindelöf space is defined by Alexandroff and Urysohn in 1929 as a space T called Lindelöf or has Lindelöf property if every open cover of T is reducible to a countable subcover. The Lindelöf degree of a space T is a cardinal number that provides a measure of the "smallness" of the space in terms of its covering properties. Symbolically the Lindelöf degree of a space T is defined by $L(T) = \aleph_0 + \inf\{\kappa \colon \text{any open cover } \mathcal{V} \text{ of } T \text{ has a subcover } \mathcal{U} \subseteq \mathcal{V} \text{ and } |\mathcal{U}| \leq \kappa\}$. When investigating topological function spaces, it is observed that the tightness of such spaces is intricately linked to the Lindelöf degree of the underlying base space. This relation becomes clear in the ensuing result [59, Theorem 4.7.1] $\alpha L(T) = t(C_\alpha T)$, where α is collection of subsets of T. Motivated by this result, we set out to explore the relation between the tightness of the space $Q_p(T)$ and the Lindelöf degree of the space T in our next result.

Theorem 4.3.2. *For any regular space T, then $L(T) \leq t(Q_p(T))$.*

Proof. Let $t(Q_p(T)) = \eta$ and $\mathcal{O}$ be any open cover of T. Let us choose a subcollection $\mathcal{F}' = \{A \in \mathcal{F} \colon A \subset O_A, O_A \in \mathcal{O}\}$ of $\mathcal{F}$. By the Lemma (4.2.2), for each $A \in \mathcal{F}'$ $\exists$ $f_A \in Q_p(T)$ such that $f_A(O_A) = \{0\}$ and $f_A(T \backslash O_A) = \{1\}$. Let us construct a subset $P = \{f_A \colon A \in \mathcal{F}'\}$ of $Q_p(T)$. By the definition of f_A, it is clear that zero function $f_0 \in \overline{P}$. By the definition of tightness of the space $Q_p(T)$, $\exists$ a subset P' of P such that $|P'| \leq \eta$ and $f_0 \in \overline{P'}$. Let us choose a subcollection $\mathcal{O}' = \{O_A \colon f_A \in P'\}$ of $\mathcal{O}$, where $|\mathcal{O}'| \leq \eta$. Now we claim $\mathcal{O}'$ is cover of T. For this we show that for all $x \in T$, $\exists$ $O_A \in \mathcal{O}'$ such that $x \in O_A$. Let us consider an open neighborhood $[x, (-1, 1)] = \{g \in Q_p(T) \colon g(x) \in (-1, 1)\}$ of the zero function f_0. Since $f_0 \in \overline{P'}$ so $\exists$ some $f_A \in P'$ such that $f_A \in P' \cap [x, (-1, 1)]$. Thus $f_A(x) < 1$, but by definition of f_A, $f_A(x) = 0$. Therefore, $x \in O_A$. Finally by definition of Lindelöf degree, $L(T) \leq t(Q_p(T))$. □

Corollary 4.3.1. *For any regular space T, if $t(Q_p(T)) = \aleph_0$, then T is Lindelöf space.*

According to the Example [18, Example 5.1], $\psi(Q_p(\mathbb{R})) = \mathfrak{c}$. But we know that $wc(\mathbb{R}) = \aleph_0$ i.e. $\psi(Q_p(\mathbb{R})) \neq wc(\mathbb{R})$. An emerging question is how the pseudocharacter and weak covering numbers are generally interconnected. Our next result tells us that under the condition of regularity of space T and for any metric space M, $wc(T) \leq \psi(Q_p(T, M))$.

Theorem 4.3.3. *For a regular space T and M be a metric space, then $wc(T) \leq \psi(Q_p(T, M))$.*

Proof. Let $h_0 \in Q_p(T, M)$ such that $h_0(x) = b$, for all $x \in T$, where b is fixed in M. Let $\psi(h_0, Q_p(T, M)) = |\mathcal{V}|$, where $\mathcal{V} = \{W(h_0, A_i, \epsilon) : i \in J\}$ is collection of open neighborhoods of $h_0 \in Q_p(T, M)$, such that $\bigcap \mathcal{V} = \{h_0\}$, $A_i \in \mathcal{F}$ and J is arbitrary index set. Let us construct a collection $\mathcal{B}_{A_i} = \{U_x : x \in A_i\}$ of open subset of T such that for each $x \in A_i$, we can pick a open set U_x that contain x. Let $\mathcal{J} = \bigcup\{\mathcal{B}_{A_i} : i \in J\}$, where $|\mathcal{J}| \leq |J|$. We claim that $\mathcal{J}$ is the weak covering of T. Let $x \in T$ and $x \notin \overline{\bigcup \mathcal{J}}$. Then by Lemma (4.2.2) $\exists$ a map $h \in Q_p(T, M)$ such that $h(x) = a$ and $h(\overline{\bigcup \mathcal{J}}) = \{b\}$ where $a \neq b$. But each $W(h_0, A_i, \epsilon)$ contains h, which is a contradiction. Thus, $\overline{\bigcup \mathcal{J}} = T$. Since $\mathcal{J}$ is a weak cover of T, therefore, $wc(T) \leq \psi(Q_p(T, M))$. □

By the Problem [76, Problem 175], the separability of the Tychnoff space T is related to the pseudocharacter of some compact subspace G of $C_p(T)$ that is T is separable if $\psi(G, Q_p(T, M)) \leq \aleph_0$. Now, in the next result, we are going to find the condition for the separability of regular space T with the help of the pseudocharacter of some compact subspace of the $Q_p(T, M)$, where M is order space.

Theorem 4.3.4. *Assume that T is an regular space and M is an ordered space with an ordered topology. Given a compact subset $G \subset Q_p(T, M)$ such that $\psi(G, Q_p(T, M)) \leq \aleph_0$, then T is separable.*

Proof. Let $g \in G$ such that $g(x) = b$, for all $x \in T$, where $b \in M$ is fixed. Let open set $U = [x_1, \ldots, x_n; V_1, \ldots, V_n]$ in $Q_p(T, M)$, where $x_i \in T$ and V_i open set in M. Construct a set $K(U) = \{x_1, \ldots, x_n\}$. Since $\psi(G, Q_p(T, M)) \leq \aleph_0$, then by definition of pseudocharacter $\exists$ a collection $\mathcal{V} = \{O_n : n \in \mathbb{N}\}$ of open subsets of $Q_p(T, M)$ such that $\bigcap \mathcal{V} = G$. So, for fixed $n \in \mathbb{N}$ and for each $g \in G$ we have an open set U_g such that $g \in U_g \subset O_n$. Since G is compact so every open cover $\{U_g : g \in G\}$ of G has finite subcover say $\{U_{g_1}, \ldots, U_{g_m}\}$. Construct two sets $P_n = \bigcup_{i=1}^{m} U_{g_i}$ such that $G \subset P_n \subset O_n$ and $D_n = \bigcup_{i=1}^{m} K(U_{g_i})$. It is easy to see that $D = \cup\{D_n : n \in \mathbb{N}\}$ is countable. Now we claim $\overline{D} = T$. Let us assume that $\overline{D} \neq T$, then $x \in T \setminus \overline{D}$. By the Lemma (4.2.3) the map $e_x : Q_p(T, M) \to M$ defined by $e_x(f) = f(x)$ is continuous, therefore the set $e_x(G)$

is bounded in M. So $\exists\, b' > b$ such that $|f(x)| < b'$ for all $f \in G$ and from Lemma (4.2.2) we have a map $h \in Q_p(T, M)$ with $h(x) = b'$ and $h(D) \subset \{b\}$. This implies $h(a) = b'$, so $h \notin G$. However $h|D = g|D$ implies $h \in \bigcap \mathcal{V}$, a contradiction. □

By the result [18, Theorem 4.12] that for a regular space T, network weight of T follows the network weight of $Q_p(T)$ and in general network weight of any space T dominates the pseudocharacter of the space T by Problem [76, Problem 156(iii)]. Now, in the next result, we proved that the pseudocharacter of $Q_p(T, M)$ lies between the network weight of T and the network weight of $Q_p(T, M)$.

Theorem 4.3.5. *For any space M and a regular space T, then $nw(T) \leq \psi(Q_p(T, M))$*

Proof. Let $g(x) = z_0$ for all $x \in T$ and $\psi(g, Q_p(T, M)) = |\mathcal{V}|$, where $\mathcal{V}$ is collection of open subsets of $Q_p(T, M)$ such that $\cap\mathcal{V} = \{g_0\}$ and $\mathcal{V}$ contains element of the form $U = [x_1, \dots, x_n; V_1, \dots, V_n]$. Now construct a set $K(U) = \{x_1, \dots, x_n\}$ and take $\mathcal{N} = \cup\{K(U) \colon U \in \mathcal{V}\}$, clearly $|\mathcal{N}| \leq |\mathcal{V}|$. Since $\psi(g, Q_p(T, M)) \leq \psi(Q_p(T, M))$, so $|\mathcal{V}| \leq \psi(Q_p(T, M))$. Now we claim that $\mathcal{N}$ is a network of T. Let us consider V be any open set in T containing x. By Lemma (4.2.1) define a quasicontinuous function as

$$f(x) = \begin{cases} z_1 & \text{if } x \in V; \\ z_0 & \text{otherwise.} \end{cases}$$

There exists a $U \in \mathcal{V}$ such that $f \notin U$. Then $\exists$ a $N \in \mathcal{N}$ such that $z_0 \notin f(N)$. Therefore for each $y \in N$ the $f(y) \neq z_0$. This implies $y \notin V^C$, thus $y \in N \subset V$. Therefore $\mathcal{N}$ is network for T. Hence $nw(T) \leq \psi(Q_p(T, M))$. □

Corollary 4.3.2. *For any space M and a regular space T, then $nw(T) \leq nw(Q_p(T, M))$.*

Proof. For a space T the $\psi(T) \leq nw(T)$. Then by Theorem (4.3.5), we get $nw(T) \leq nw(Q_p(T, M))$. □

Corollary 4.3.3. *For any metric space M and a regular space T, then $d(T) \leq nw(Q_p(T, M))$. [37, Theorem 4.13]*

Proof. For a space T the $d(T) \leq nw(T)$. Then by Theorem (4.3.5), we get $d(T) \leq nw(T) \leq nw(Q_p(T, M))$. □

Theorem 4.3.6. *For a regular space T, the inequality $wc(T) \cdot \log(nw(T)) \leq iw(Q_p(T))$ holds.*

Proof. For any space Z, $\psi(Z) \cdot log(nw(Z)) \leq iw(Z)$. From Theorem (4.3.3), we have $wc(T) \leq \psi(Q_p(T))$ and by Corollary (4.3.2) $nw(T) \leq nw(Q_p(T))$. Therefore, $wc(T) \cdot log(nw(T)) \leq iw(Q_p(T))$. □

In general $wc(T) \cdot log(nw(T)) \neq iw(Q_p(T))$. For example, from [18, Example 5.1] we have $\mathfrak{c} = \psi(Q_p(\mathbb{R})) \leq iw(Q_p(\mathbb{R}))$, $wc(\mathbb{R}) = \aleph_0$ and $nw(\mathbb{R}) = \aleph_0$, so we have $log(nw(\mathbb{R})) = \aleph_0$. This implies $wc(\mathbb{R}) \cdot log(nw(\mathbb{R})) \neq iw(Q_p(\mathbb{R}))$.

Theorem 4.3.7. *For any Hausdorff space T and any space M, the inequality $d(F_p(T, M)) \leq w(T) \cdot d(M)$ holds.*

Proof. For any space T and dense subset S of T, $d(T) \leq d(S)$. By [37, Theorem 4.15], we have $Q_p(T, M)$ which is dense in $F_p(T, M)$. Therefore $d(F_p(T, M)) \leq d(Q_p(T, M))$. Also from [37, Theorem 4.10], $d(Q_p(T, M)) \leq w(T) \cdot d(M)$. Thus $d(F_p(T, M)) \leq w(T) \cdot d(M)$. □

4.4 Maps on $Q_p(T)$ and $Q_p(T, M)$

Initially, in [59, Chapter-II], McCoy and Ntantu investigated some maps on the space $C_p(T, M)$. Continuing this work, Kumar and Tyagi studied some maps on the space $Q_p(T, M)$ in [37]. Now, in this section, we study properties of maps on $Q_p(T)$ and $Q_p(T, M)$ spaces.

Theorem 4.4.1. *Let M be an open subspace of T and let T be a Hausdorff space. For any $f \in Q_p(T)$, $\pi_M(f) = f|M$ defines a restriction map $\pi_M : Q_p(T) \to Q_p(M)$. Then $\pi_M(Q_p(T)) = Q_p(M)$ and π_M are open continuous.*

Proof. The restriction map $\pi_M : Q_p(T) \to Q_p(M)$ is continuous since it is a projection map. For non-empty open subset Y of T, we prove $\pi_M(Q_p(T)) = Q_p(Y)$. It is oblivious $\pi_M(Q_p(T)) \subset Q_p(M)$. Since M is open set in T and $g \in Q(M)$, now we define a function $h : T \to \mathbb{R}$ as follows

$$h(x) = \begin{cases} g(x), & \text{if } x \in M; \\ 1, & T \setminus M. \end{cases}$$

Clearly $\pi_M(h) = h|M = g$. Let any $x \in T$. Since $h|M = g$ and g is quasicontinuous. Therefore, for $x \in M$ function h is quasicontinuous. If $x \in T \setminus M$, then $h(x) = h(y)$ for all $y \in \overline{T \setminus M} = T \setminus M$ (Since $T \setminus M$ is closed). Hence by Lemma (4.2.1) h is quasicontinuous at T. Finally we have h is quasicontinuous. Now, we prove that

π_M is open. Let $W(f, \{x_1, \ldots, x_k\}, \epsilon)$ is open set in $Q_p(T)$, where $k \in \mathbb{N}$. We claim $\pi_M(W(f, \{x_1, \ldots, x_k\}, \epsilon))$ is open set in $Q_p(M)$. For this we prove

$$\pi_M(W(f, \{x_1, \ldots, x_k\}, \epsilon)) = W(\pi_M(f), \{x_1, \ldots, x_k\}, \epsilon),$$

where $x_i \in M$ for all $1 \leq i \leq k$. By definition of topology of point-wise convergence $W(\pi_M(f), \{x_1, \ldots, x_k\}, \epsilon) \subset Q_p(M)$. By definition of map π_M,

$$\pi_M(W(f, \{x_1, \ldots, x_k\}, \epsilon)) \subset Q_p(M).$$

Let $g \in W(f, \{x_1, \ldots, x_k\}, \epsilon)$ then by definition of open set in $Q_p(T)$, $|g(x_i) - f(x_i)| < \epsilon$ for all either $i = 1, 2, \ldots, k$ or $i = 1, 2, \ldots, l$, where $l \leq k$.

If $x \in M$, implies $\pi_M(g)(x) = g(x)$ and $\pi_M(f)(x) = f(x)$. This show that $\pi_M(g) \in W(\pi_M(f), \{x_1, \ldots, x_l\}, \epsilon)$, therefore

$$\pi_M(W(f, \{x_1, \ldots, x_k\}, \epsilon)) \subset W(\pi_M(f), \{x_1, \ldots, x_l\}, \epsilon)$$

Remain to prove $\pi_M(W(\pi_M(f), \{x_1, \ldots, x_k\}, \epsilon)) \supset W(\pi_M(f), \{x_1, \ldots, x_l\}, \epsilon)$. Take $g \in Q_p(M)$ so $\exists$ a $g_1 \in Q_p(T)$ such that $g = \pi_M(g_1)$. Let

$$m(x) = \begin{cases} g_1(x) & \text{if } x \notin \bigcup_{i=l+1}^{k} \overline{V_i^*}; \\ f(x) & \text{if } x \in \bigcup_{i=l+1}^{k} \overline{V_i^*}. \end{cases}$$

where $V_i^* = V_i \cap (T \backslash M)$ and V_i are disjoint open set containing point x_i, then m is quasicontinuous. Since $\pi_M(m) = \pi_M(g_1) = g$ implies that $m \in W(f, \{x_1, \ldots, x_k\}, \epsilon)$ and $g \in \pi_M(W(f, \{x_1, \ldots, x_k\}, \epsilon))$. So,

$$\pi_M(W(\pi_M(f), \{x_1, \ldots, x_k\}, \epsilon)) \supset W(\pi_M(f), \{x_1, \ldots, x_l\}, \epsilon).$$

□

As we know from the result in [39] semi-continuity and quasi-continuity are equivalence for single-valued function. Now from the result [Remark, 13[26]] and (4.4.2) it is clear that the product of two quasicontinuous functions is not quasicontinuous in general but on the other hand the product of continuous and quasicontinuous function is quasicontinuous respectively. For any $f\colon T \to \mathbb{R}$ and $g\colon T \to \mathbb{R}$ maps. The product of maps f and g is map $f \cdot g\colon T \to \mathbb{R}$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$

Theorem 4.4.2. *The product of continuous and quasicontinuous functions is quasicontinuous.*

Proof. Suppose f and g are continuous and quasicontinuous maps respectively. Now we claim that product map $f \cdot g$ is quasicontinuous. For this assume $h \colon T \to \mathbb{R} \times \mathbb{R}$ defined by $h(x) = (f(x), g(x))$ and $j \colon \mathbb{R} \times \mathbb{R}$ defined by $j(x, y) = x \cdot y$, it is easy to check j is continuous. Now to prove h is quasicontinuous, let any $W = U \times V$ is open set in $\mathbb{R} \times \mathbb{R}$, where U and V are open set in $\mathbb{R}$. Then $h^{-1}(W) = h^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$. Since f is continuous and g is quasicontinuous so by the respective definition, $f^{-1}(U)$ and $g^{-1}(V)$ are open and semi-open set in T, respectively. The intersection of semi-open and open set is semi-open set, thus W is semi-open set. therefore h is quasicontinuous. The map $f \cdot g = j \circ h$ and the composition of the quasicontinuous map and the continuous map is quasicontinuous. Hence $f \cdot g$ is quasicontinuous. □

Theorem 4.4.3. *The map $q \colon C_p(T) \times Q_p(T) \to Q_p(T)$ defined by $q(f, g) = f \cdot g$ is continuous for any space T.*

Proof. Let $h_0 = (f_0, g_0) \in C_p(T) \times Q_p(T)$ and V be any open set in $Q_p(T)$ such that $q(h_0) = f_0 \cdot g_0 \in V$. Then $\exists \{x_1, \ldots, x_n\} \subset T, n \in \mathbb{N}$ and $\epsilon > 0$ such that $q(h_0) \in W(q(h_0), \{x_1, \ldots, x_n\}, \epsilon) \subset V$. Take $M = \Sigma_{i=1}^{n} |f_0(x_i)| + \Sigma_{i=1}^{n} |g_0(x_i)| + 3$ and $\delta = \min\{\frac{\epsilon}{2M}, 1\}$. The set $O = O_1 \times O_2$ is open set in $C_p(T) \times Q_p(T)$ containing h_0, where $O_1 = W(f_0, \{x_1, \ldots, x_n\}, \delta)$ and $O_2 = W(g_0, \{x_1, \ldots, x_n\}, \delta)$ are open neighborhood of f_0 and g_0 in $C_p(T)$ and $Q_p(T)$ respectively. For any $h = (f, g) \in O$ we have, $|g(x_i)| < 1 + |g_0(x_i)| < M$ and $|f(x_i)| < 1 + |f_0(x_i)| < M$, for all $i \leq n$. So,

$$
\begin{aligned}
|q(h)(x_i) - q(h_0)(x_i)| &= |(f \cdot g)(x_i) - (f_0 \cdot g_0)(x_i)| = |f(x_i) \cdot g(x_i) - f_0(x_i) \cdot g_0(x_i)| \\
&= |f(x_i) \cdot g(x_i) - f_0(x_i) \cdot g(x_i) + f_0(x_i) \cdot g(x_i) - f_0(x_i) \cdot g_0(x_i)| \\
&\leq |g(x_i)||(f(x_i) - f_0(x_i))| + |f_0(x_i)||(g(x_i) - g_0(x_i))| \\
&< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} < \epsilon, \text{ for all } i \leq n \\
|q(h)(x_i) - q(h_0)(x_i)| &< \epsilon
\end{aligned}
$$

This show that $q(h) = f \cdot g \in V$ and $q(O) \subset V$, therefore map q is continuous at h_0. Since h_0 is arbitrary element of $C_p(T) \times Q_p(T)$, hence q is continuous map. □

In [37], Kumar and Tyagi studied the continuity of an induced map on $Q_p(T, M)$. In continuation, we are going to prove the denseness of the image of the induced map.

Theorem 4.4.4. *Consider a surjective continuous map $r \colon M \to Z$, and the induced map $r \colon Q_p(T, M) \to Q_p(T, Z)$ defined as $r_*(f) = r \circ f$. Then, the space $r_*(Q_p(T, M))$ is dense in $Q_p(T, Z)$*

Proof. Given $r \colon M \to Z$ surjective continuous map. To prove $r_*(Q_p(T, M))$ is dense in $Q_p(T, Z)$. For this we show that for any $g \in Q_p(T, Z)$ and any open set $[x_1, \ldots, x_n \colon V_1, \ldots, V_n]$

in $Q_p(T,Z)$ containing g, where $x_i \in T$ and V_i open set in Z, then $[x_1,\ldots,x_n;V_1,\ldots,V_n]\cap r_*(Q_p(T,M)) \neq \emptyset$. Now by the definition of continuity and surjectivity of map r implies that, $\forall i \leq n$, $r^{-1}(V_i) = U_i$ for some U_i open set in M. Therefore any element f of $[x_1,\ldots,x_n\colon U_1,\ldots,U_n] \subset Q_p(T,M)$, then $r(f(x_i)) = r_*(f)(x_i) \in V_i$ for all $i \leq n$, which implies $r_*(f) \in [x_1,\ldots,x_n\colon V_1,\ldots,V_n]$. Thus $[x_1,\ldots,x_n;V_1,\ldots,V_n] \cap r_*(Q_p(T,M)) \neq \emptyset$. □

4.5 CONCLUSION

In this chapter, we characterized the cardinal invariants such as pseudocharacter, network weight, weight, and tightness for space $Q_p(T,M)$. We proved that pseudocharacter of $Q_p(T,M)$ dominates the network weight, density and weak covering number of regular space T and we obtained necessary and sufficient conditions on T so that weight and network weight of $Q_p(T)$ coincide. Further, we proved the condition of separability of a regular space in terms of pseudocharacter of a compact subset of $Q_p(T,M)$. Moreover, we studied the openness of the restriction map on $Q_p(T)$ and further proved that the image of $Q_p(T,M)$ under the induced map is dense in $Q_p(T,Z)$.

Chapter 5

The space $Q(T)$ endowed with the compact convergence topology

5.1 Introduction

In previous chapter (4), we studied the space $Q(T, M)$ under pointwise convergence topology, which is coarsest topology on $Q(T, M)$. It helps us to study various properties of $Q(T, M)$. But to its disadvantage, it does not preserves the quasicontinuity under the convergence of sequence of quasicontinuous functions. On the other hand uniform topology gives a metric structure in which most of the results reduce to triviality. This motivate us to dissect the space of $Q(T, M)$ with a topology lies between pointwise and uniform convergence topology, also in which sequence of quasicontinuous functions converge to a quasicontinuous function. Thus, we study the topology of uniform convergence on compacta(or the compact convergence topology) over the space of quasicontinuous functions $Q(T)$, which is finer than the topology of pointwise convergence on $Q(T)$. Also, the function space of continuous function endowed with the compact convergence topology is a subspace of $Q_C(T)$. The study regarding $C_C(T)$ can be found in [14, 56–58]. For better understanding of this chapter, first we take a brief look over the existing study of $Q_C(T, M)$. In 2017, L.Hola and D.Holy [17] studied the Ascoli types theorem for $Q_C(T, M)$. In 2018, L.Hola and D.Holy [35] studied the metrizability and completely metrizability of $Q_C(T, M)$. In 2021, L.Hola and D.Holy [36] studied some cardinal functions of $Q_C(T, M)$. Extending these investigations, we present in this chapter additional findings concerning density and various types of tightness of the space $Q_C(T)$. Moreover, we studied the Frechet-Urysohn property of $Q_C(T)$.

In the section (5.2), we define the compact convergence topology over the function space and recall some important definitions of various types of tightness cardinal functions and results that help to understand our main results.

In section (5.3), we study the coincidence of tightness of $Q_C(T)$ and compact Lindelöf no. of T. Next we characterize the density of $Q_C(T)$ in terms of k-cofinality of T and various types of tightness(density tightness, fan tightness, and strongly fan tightness) of $Q_C(T)$ in terms of covering of space T.

In section (5.4), we study the equivalent condition for the $Q_C(T)$ space to be Fréchet-Urysohn space.

5.2 Preliminaries

The compact convergence topology on $Q(T)$ in [36] is defined as " If Hausdorff space T and $\mathcal{K}(T)$ be the collections of all nonempty compact subsets of T. The compact convergence topology on $F(T,\mathbb{R})$, denoted as τ_{UC}, is induced by the uniformity $\mathcal{U}_{UC}$. The base of this uniformity consists of sets of the form

$$W(C,\epsilon) = \{(f,g) \colon |f(x) - g(x)| < \epsilon, \forall x \in C \in \mathcal{K}(T), \epsilon > 0\},$$

where $F(T,\mathbb{R})$ represents the space of real-valued functions on T.

The general τ_{UC}-basic neighborhood of a function f in $F(T,\mathbb{R})$ is denoted by $W(f,C,\epsilon)$, defined as

$$W(f,C,\epsilon) = \{g \in F(T,\mathbb{R}) \colon |f(x) - g(x)| < \epsilon, \forall x \in C \in \mathcal{K}(T), \epsilon > 0\},"$$

Recall some definitions, any space is σ-compact, if it is equal to the countable union of its compact subsets and A space T is Hemicompact if a countable cofinal subfamily exists in the family of all compact subspaces of T ordered by inclusion. Further they have one-sided implication i.e. Hemicompactness $\implies$ σ-compactness.

Definition 5.2.1 (k-cofinality)**.** [36] The k-cofinality of the space T is defined as:

$$kcof(T) = \aleph_0 + \sup\{|\mathcal{U}| \colon \mathcal{U} \text{ is a cofinal family of } \mathcal{K}(T)\}$$

Definition 5.2.2. [58][κ-dense] A subset M of the space T is called κ-dense if $[M]_\kappa = T$, where $[M]_\kappa$ is κ-closure of set M defined as

$$[M]_\kappa = \bigcup\{|\overline{B}| \colon B \subset M \text{ and} |B| \leq \kappa\}$$

Definition 5.2.3. [58][Density-tightness] The density tightness of the space T is expressed as:

$$dt(T) = \min\{\kappa\colon \text{ Every dense subset of } T \text{ is } \kappa\text{-dense in} \, T\}$$

Definition 5.2.4. [59][Fan-tightness] The Fan-tightness of the space T is defined as:

$$ft(T) = \sup\{ft(x,T)\colon x \in T\}$$

where

$$ft(x,T) = \aleph_0 + \min\{\lambda\colon \text{ for each sequence } \{A_n\} \text{ of subsets } T \text{ and } x \in \bigcap_{n\in\mathbb{N}} \overline{A_n},$$

$$\exists\; B_n \subset A_n \text{ such that } x \in \overline{\bigcup_{n\in\mathbb{N}} A_n} \text{ with } |B_n| \leq \lambda\}.$$

Definition 5.2.5. [59] A space T is said to have countably strongly fan tightness if, for each $x \in T$, $\exists$ a sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of T with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ and then $\exists$ $x_n \in A_n$ such that $x \in \overline{\{x_n\colon n \in \mathbb{N}\}}$.

Definition 5.2.6. [60] A space T is called as Frechet-Urysohn space if, for every point x in T and every subset A of T where x belongs to the closure of A, $\exists$ a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A that converges to x.

Definition 5.2.7. [61] A space T is called as Whyburn if for any subset S of T and any $z \in \overline{S}/S$, there is a set $P \subset S$ such that $\overline{P} = S \cup \{z\}$.

Theorem 5.2.1 ([58], Proposition 2.2)**.** *For any space T, the $dt(T) \leq t(T)$.*

Lemma 5.2.1 ([18], Lemma 4.2)**.** *If T be a Hausdorff spaces and $f : T \to \mathbb{R}$ is a function with property that for any $x \in T$, $\exists$ an open set $U \subset T$ such that $x \in \overline{U}$ and $f(y) = f(x)$ for any $y \in U$. Then $f \in Q(T)$.*

5.3 Various types of tightness of $Q_C(T)$

Definition 5.3.1. Let $\mathcal{V} \subset P(T)$ such that, for each $C \in \mathcal{K}(T)$ $\exists$ an element in $\mathcal{V}$ containing C, then $\mathcal{V}$ is k-cover of T. If the elements of $\mathcal{V}$ are open set, then it is known as an open k-cover.

Definition 5.3.2. The compact Lindelöf number of T is denoted and defined by

$$kL(T) = \aleph_0 + \inf\{\lambda\colon \text{For every open } k\text{-cover } \mathcal{J} \text{ of} \, T \,\exists \text{ sub } k\text{-cover } \mathcal{W} \text{with } |\mathcal{W}| \leq \lambda\}.$$

Theorem 5.3.1. *Every open k-cover $\mathcal{J}$ of a space T, $\exists$ sub k-cover $\mathcal{J}'$ of T with $|\mathcal{J}'| \leq \eta$ if and only the $kL(T) \leq \eta$.*

Proof. Consider that for every open k-cover $\mathcal{J}$ of space T, then we have a sub k-cover $\mathcal{J}'$ of T such that $|\mathcal{J}'| \leq \eta$, then simply by definition of compact Lindelöf number, $kL(T) \leq \eta$.
Conversely, let $kL(T) \leq \eta$ and $\mathcal{J}$ be any open k-cover space T, then by definition of compact Lindelöf number $\exists$ sub k-cover $\mathcal{J}'$ of T such that $|\mathcal{J}'| \leq \eta$. □

Theorem 5.3.2. *A Hausdorff space T has the property $kL(T) = t(Q_C(T))$.*

Proof. Suppose $t(Q_C(T)) = \lambda$. Let $\mathcal{J}$ be any open k-cover of T. Then, for any $C \in \mathcal{K}(T)$, $\exists$ a $U_C \in \mathcal{J}$ such that $C \subset U_C$, according to Definition (5.3.1). Assume that the function $f_C\colon T \to \mathbb{R}$ is defined by $f_C(\overline{U}_C) = \{0\}$ and $f_C(T/\overline{U}_C) \subset \{1\}$, then by Lemma (5.2.1) f_C is quasicontinuous for each $C \in \mathcal{K}(T)$. Let $D = \{f_C\colon C \in \mathcal{K}(T)\}$ be a collection of quasicontinuous function on T. Take f_0 be a zero function on T, then $f_0 \in \overline{D}$. Since $t(Q_C(T)) = \lambda$, then by definition of tightness $\exists$ $D' \subset D$ such that $f_0 \in \overline{D'}$ and $|D'| \leq \lambda$. Let us consider a subfamily $\mathcal{W} = \{U_C : f_C \in D'\}$ of $\mathcal{J}$. Now we claim W is k-cover of T. Take $C \in \mathcal{K}(T)$ and $W(f_0, C, 1)$ is a neighborhood of f_0, therefore $f_{C'} \in D' \cap W(f_0, C, 1)$, for some compact subset C' of T. Thus for $x \in C$, we have

$$f_{C'}(x) < 1 \text{ if } x \in \overline{U_{k'}},\ \ f_{C'}(x) = 1, \text{ otherwise.}$$

This implies that $C \subset U_{C'}$, Therefore $\mathcal{W}$ is a k-cover of T. Hence, $kL(T) \leq t(Q_C(T))$.
Next, to prove $t(Q_C(T)) \leq kL(T)$. Suppose $kL(T) = \eta$. According to the Definition of kL-number, for any open k-cover $\mathcal{J}$ there is a sub k-cover $\mathcal{J}'$ of $\mathcal{J}$ such that $|\mathcal{J}'| \leq \eta$. For any $f \in Q_C(T)$ we can define a real-valued function f_{U_C}on T which satisfies $f_{U_C}(x) = f(x)$ if $x \in U_C$ and $f_{U_C}(x) = 1$ if $x \in T/U_C$, for each $U_C \in \mathcal{J}$. By Lemma (5.2.1) f_{U_C} is quasicontinuous. Let $D = \{f_{U_C}\colon U_C \in \mathcal{J}\}$, from construction of f_{U_C}, easily we have $f(x) \in \overline{D}$ and we have subset $D' = \{f_{U_C}\colon U_C \in \mathcal{J}'\}$ of D whose closure contains f. Then, $t(f, Q_C(T)) = |D'| \leq \eta$. Since f is any function quasicontinuous function, thus $t(Q_C(T)) \leq kL(T)$. □

Corollary 5.3.1. *For a second countable space T, $Q_C(T)$ has countable tightness.*

Theorem 5.3.3. *For any Hausdorff space T, then every open k-cover $\mathcal{J}$ of space T, $\exists$ a sub k-cover $\mathcal{J}'$ of T with $|\mathcal{J}'| \leq \eta$ if and only $t(Q_C(T)) \leq \eta$*

Proof. Simply by using Theorem (5.3.1) and Theorem (5.3.2). □

Theorem 5.3.4. *For any Hausdorff space T, the equality $dt(Q_C(T)) = t(Q_C(T))$ holds.*

Proof. Firstly, we prove that $t(Q_C(T)) \leq dt(Q_C(T))$, for this by Theorem (5.3.3), to prove that $\exists$ a sub k-cover $\mathcal{J}^{'}$ of T with $|\mathcal{J}^{'}| \leq \eta$ for every open k-cover $\mathcal{J}$ of T. Take

$$A = \{f \in Q_C(T)\colon \text{ for some } U \in \mathcal{J} \text{ s.t. } f(T/U) \subset \{0\}\}.$$

Since $\mathcal{J}$ is the open k-cover of T, then for any open set $W(f, C, \epsilon)$ of $Q_C(T)$, $\exists\ U \in \mathcal{J}$ such that $C \subset U$. Take $g \in W(f, C, \epsilon)$, then we have $h \in Q_C(T)$ such that

$$h(x) = g(x) \text{ if } x \in U \text{ and } h(T/U) \subset \{0\}.$$

Then $h \in A \cap W(f, C, \epsilon)$, therefore A is dense in $Q_C(T)$. Let f_0 be a zero function, then by Definition of density tightness, there is a $B \subset A$ satisfying $|B| \leq \eta$ with $f_0 \in \overline{B}$. Take $\mathcal{J}^{'} = \{U_f\colon f \in B\}$, clearly $|\mathcal{J}^{'}| \leq \eta$. Let any $C \in \mathcal{K}(T)$ and $W(f_0, C, 1)$ is a neighborhood of f_0, then $\exists\ g \in B$ such that $g \in W(f_0, C, 1)$, then $C \subset U_g$. Therefore, $\mathcal{J}^{'}$ is k-cover of T.
Next to prove $dt(Q_C(T)) \leq t(Q_C(T))$, it follows from Theorem (5.2.1), we have $dt(Q_C(T)) \leq t(Q_C(T))$. □

For any real-valued continuous and quasicontinuous map f and g on T, respectively. The sum of the maps $f+g$ is quasicontinuous, where it is defined as $(f+g)(x) = f(x)+g(x)$. In [[37]], the mapping $h_f\colon Q_p(T) \to Q_p(T)$ defined as $h_f(g) = f + g$ is continuous, for any $f \in C_p(T)$. Consequently, we have the following results for $Q_C(T)$.

Lemma 5.3.1. *For any real-valued continuous and quasicontinuous map f and g on T, respectively. A mapping $h_f : Q_C(T) \to Q_C(T)$ such that $h_f(g) = f + g$ is a homeomorphism.*

Theorem 5.3.5. *If T be a Hausdorff space, Then $\exists$ a finite subset $\mathcal{J}_n{}^{'}$ of $\mathcal{J}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{J}_n{}^{'}$ is an open k-cover of T for every sequence $\{\mathcal{J}_n\}_{n\in\mathbb{N}}$ of open k-cover of T if and only if $ft(Q_C(T)) = \aleph_0$.*

Proof. Let a sequence $\{\mathcal{J}_n : n \in \mathbb{N}\}$ of open k-cover of T. Take

$$A_n = \{f \in Q_C(T)\colon \text{ For some } U \in \mathcal{J}_n \text{ s.t. } f(T/U) \subset \{0\}\}.$$

for all $n \in \mathbb{N}$. Since $\mathcal{J}n$ is an open k-cover for T, then for any open set $W(f, C, \epsilon)$ of $Q_C(T)$, $\exists$ an element U in $\mathcal{J}_n$ such that $C \subset U$.

Take $g \in W(f, C, \epsilon)$, then we have $h \in Q_C(T)$ such that

$$h(x) = g(x) \text{ if } x \in U \text{ and } h(T/U) \subset \{0\}.$$

Then $h \in A_n \cap W(f, C, \epsilon)$, therefore A_n is dense in $Q_C(T)$.

Now, we take g_1, the constant unit function. Since each A_n is dense, thus $g_1 \in \bigcap_{n=1}^{\infty} \overline{A_n}$ so, $\exists$ set $B_n \subset A_n$ such that $g_1 \in \overline{\bigcup_{n=1}^{\infty} B_n}$, where B_n is finite. Construct $B_n = \{f_{(n,j)} \in A_n : j \leq i(n)\}$, where $i(n)$ is a finite function of n. Thus there is $U_{(n,j)} \in \mathcal{J}_n$ s.t. $f_{(n,j)}(T/U) \subset \{0\}$. Let $\mathcal{J}'_n = \{U_{(n,j)} \in \mathcal{J}_n : j \leq i(n)\}$. Next, to prove $\bigcup_{n\in\mathbb{N}} \mathcal{J}'_n$ is a k-cover of T. For any $C \in \mathcal{K}(T)$, $\exists$ $f_{(n,j)} \in W(g_1, C, 1)$, for some $n \in \mathbb{N}$ and $j \leq i(n)$. Then, $\exists$ a $U_{(n,j)} \in \bigcup_{n\in\mathbb{N}} \mathcal{J}'_n$ with $C \subset U_{(n,j)}$. Therefore, $\bigcup_{n\in\mathbb{N}} \mathcal{J}'_n$ is an open k-cover of T.

Conversely, take f_0 be zero function. By Lemma (5.3.1), $Q_C(T)$ is homogeneous, now it is enough to prove $ft(f_0, Q_C(T)) = \aleph_0$. Let $f_0 \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, where $A_n \subset Q_C(T)$. For each $n \in \mathbb{N}$ and $f \in A_n$, $\exists$ some open subset of T such that image of that set under function f is contained in $(-\frac{1}{n}, \frac{1}{n})$, we denote it as $O_{n,f}$. Construct

$$\mathcal{J}_n = \{O_{n,f} : f \in A_n\}.$$

For each $C \in \mathcal{K}(T)$ $\exists$ a $f \in W(f_0, C, 1/n) \cap A_n$ such that $C \subset O_{n,f}$. Thus $\mathcal{J}_n$ is an open k-cover of T. Take $P = \{n \in \mathbb{N} : T \in \mathcal{J}_n\}$.

Case-1 If P is infinite. For every $W(f_0, C, \epsilon)$ neighborhood of f_0 and $\epsilon > 0$, $\exists$ some $m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$. Then, from the construction of $\mathcal{J}_m$ we have some $g_m \in A_m$ with $g_m(T) = (-\frac{1}{m}, \frac{1}{m})$ and $g_m \in W(f_0, C, \epsilon)$. Thus sequence $\{g_m\}_{m\in\mathbb{N}}$ is convergent to f_0. Here $B_m = \{g_m\}_{m\in\mathbb{N}}$ satisfies that $f_0 \in \overline{\bigcup_{m\in\mathbb{N}} B_m}$.

Case-2 If P is finite. Then $\exists$ $n_0 \in \mathbb{N}$ such that

$$g(T) \neq (-\frac{1}{m}, \frac{1}{m}) \text{ whenever } m \geq n_0,\ g \in A_m.$$

Since $\{\mathcal{J}_m\}_{m\geq n_0}$ is a sequence of open k-cover, thus we have a finite subset $\mathcal{J}'_m$ of $\mathcal{J}_m$ such that $\bigcup_{m\geq n_0} \mathcal{J}'_m$ is an open k-cover of T. Let it be denoted by

$$U'_m = \{U_{(m,j)} \in \mathcal{J}_m : j \leq i(m)\}.$$

Then from construction of $\mathcal{J}_m$, $\exists$ some $f_{(m,j)} \in A_m$ with $f_{(m,j)}(U_{(m,j)}) \subset (-\frac{1}{m}, \frac{1}{m})$. Now we prove $f_0 \in \overline{\{f_{(m,j)} : j \leq i(m)\}}_{m\in\mathbb{N}}$. For any neighborhood $W(f_0, C, \epsilon)$ of f_0, let

$$H = \{\{m, j\} : m \geq n_0,\ j \leq i(m) \text{ and } C \subset U_{(m,j)})\}.$$

Then $H \neq \emptyset$, if H is finite for each $\{m, j\} \in H$, by $U_{(m,j)} \neq T$, taking $x_{(m,j)} \in T/U_{(m,j)}$. Then $\{x_{(m,j)} : \{m, j\} \in H\} \cup C$. But $\exists$ no element $\{x_{(m,j)} : \{m, j\} \in H\} \cup C$ in $\bigcup_{m\geq n_0} \mathcal{J}'_m$, which is contradiction. So H is infinite, then $\exists$ $m \geq n_0$,

$j \leq i(m)$ such that $C \subset U_{(m,j)}$ and $f_{(m,j)}(C) \subset (-\frac{1}{m}, \frac{1}{m})$ with $\frac{1}{m} < \epsilon$. Thus, $f_{(m,j)}(C) \subset (-\epsilon, \epsilon)$ and $f_{(m,j)} \in W(f_0, C, \epsilon)$. Hence $f_0 \in \overline{\{f_{(m,j)} : j \leq i(m)\}}$.

□

Theorem 5.3.6. *For any Hausdorff space T, then $Q_C(T)$ is countably strongly fan tightness if and only if for each sequence $\{\mathcal{J}_n\}_{n\in\mathbb{N}}$ of open k-cover of T, $\exists$ a $U_n \in \mathcal{J}_n$ such that $\{U_n\}_{n\in\mathbb{N}}$ is an open k-cover of T.*

Proof. Let a sequence $\{\mathcal{J}_n : n \in \mathbb{N}\}$ of open k-cover of T. Take

$$A_n = \{f \in Q_C(T) \colon \text{For some}\, U \in \mathcal{J}_n\, \text{s.t.}\, f(T/U) \subset \{0\}\}$$

for all $n \in \mathbb{N}$. Since $\mathcal{J}n$ is an open k-cover for T, then for any open set $W(f, C, \epsilon)$ of $Q_C(T)$, $\exists$ an element U in $\mathcal{J}_n$ such that $C \subset U$. Take $g \in W(f, C, \epsilon)$, then we have $h \in Q_C(T)$ such that

$$h(x) = g(x)\ \text{if}\, x \in U\, \text{and}\, h(T/U) \subset \{0\}.$$

Then $h \in A_n \cap W(f, C, \epsilon)$, therefore A_n is dense in $Q_C(T)$.

Now, we take $g_1 \in Q_C(T)$ such that $g_1 \equiv 1$, since each A_n is dense, thus $g_1 \in \bigcap_{n=1}^{\infty} \overline{A_n}$. Then $\exists$ a $f_n \in A_n$ such that $g_1 \in \overline{\{f_n \colon n \in \mathbb{N}\}}$. For each $n \in \mathbb{N}$ and $f_n \in A_n$, $\exists$ a $U_n \in \mathcal{J}_n$ such that $f_n(T/U) \subset 3$. Now take collection of all such U_n's which denote as $\{U_n : n \in \mathbb{N}\}$. Next to prove $\{U_n : n \in \mathbb{N}\}$ is a k-cover of T. For any $C \in \mathcal{K}(T)$, since $g_1 \in \overline{\{f_n \colon n \in \mathbb{N}\}}$, so $\exists$ $f_n \in W(g_1, C, 1)$, for some $n \in \mathbb{N}$. Then, $\exists$ a U_n with $C \subset U_n$. Therefore, $\{U_n : n \in \mathbb{N}\}$ is an open k-cover of T.

Conversely, take f_0 be zero function and $f_0 \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, where $A_n \subset Q_C(T)$. For each $n \in \mathbb{N}$ and $f \in A_n$, $\exists$ some open subset of T such that image of that set under function f is contained in $(-\frac{1}{n}, \frac{1}{n})$, we denote it as $O_{n,f}$. Construct

$$\mathcal{J}_n = \{O_{n,f} \colon f \in A_n\}$$

For each $C \in \mathcal{K}(T)$ $\exists$ a $f \in W(f_0, C, 1/n) \cap A_n$ such that $C \subset O_{n,f}$. Thus $\mathcal{J}_n$ is an open k-cover of T. Take $M = \{n \in \mathbb{N} \colon T \in \mathcal{J}_n\}$.

Case-1 If P is infinite. For every $W(f_0, C, \epsilon)$ neighborhood of f_0 and $\epsilon > 0$, $\exists$ some $m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$. Then, from the construction of $\mathcal{J}_m$ we have some $g_m \in A_m$ with $g_m(T) = (-\frac{1}{m}, \frac{1}{m})$ and $g_m \in W(f_0, C, \epsilon)$. Thus sequence $\{g_m\}_{m\in\mathbb{N}}$ is convergent to f_0.

Case-2 If P is finite. Then $\exists\, n_0 \in \mathbb{N}$ such that

$$g(T) \neq (-\frac{1}{m}, \frac{1}{m}) \text{ whenever } m \geq n_0,\ g \in A_m.$$

Since $\{\mathcal{J}_m\}_{m\geq n_0}$ is a sequence of open k-cover, thus we have a finite subset $\mathcal{J}_m^{'}$ of $\mathcal{J}_m$ such that $\bigcup_{m\geq n_0} \mathcal{J}^{'}{}_m$ is an open k-cover of T. For $m \geq n_0$, $\exists\, f_m \in A_m$ such that $f_m(U_m) = (-\frac{1}{m}, \frac{1}{m})$. Next we claim that $f_0 \in \overline{\{f_m \colon m \geq n_0\}}$. For any neighborhood $W(f_0, C, \epsilon)$ of f_0, let $\mathcal{J}_C = \{U_m \colon C \subset U_m, m \geq n_0\}$, clearly $U_C \neq \emptyset$. Let us assume $\mathcal{J}_C$ is finite, then $\mathcal{J}_C = \{U_{m_j} \colon j \leq p\}$ where p is some finite natural number. By $U_{m_j} \neq T$ we take $x_{m_j} \in T/U_{m_j}$, then $\{T_{m_j} \colon j \leq p\} \cup C$, so $U_m \cap (\{x_{m_j} \colon j \leq p\} \cup C) = \emptyset$ whenever $m \neq n_0$. Which is a contradiction, therefore, $\mathcal{J}_C$ is infinite. Hence $\exists\, m \neq n_0$ such that $C \subset U_m$ with $f_m(C) \subset f_m(U_m) \subset (-\frac{1}{m}, \frac{1}{m})$, with $\frac{1}{m} < \epsilon$, then $f_m \in W(f_0, C, \epsilon)$. Which implies that $f_0 \in \overline{\{f_m \colon m \geq n_0\}}$

□

Lemma 5.3.2. *Let T be a regular space and M be any space. Then, for a finite number of nonempty disjoint closed subsets $F_1, F_2, \ldots, F_n$ of T and $y_1, y_2, \ldots, y_n \in M$, $\exists$ a quasicontinuous function $f \colon T \to M$ such that $f(F_i) = \{y_i\}$ for all $1 \leq i \leq n$.*

Proof. Since $F_1, F_2, \ldots, F_n$ are compact subsets of T. For each i, $1 \leq i \leq n$, by there exits a $f_i \in Q_C(T)$ such that

$$f_i(x) = \begin{cases} 1 & \text{if } x \in F_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can define a quasicontinuous function $f \colon T \to M$ as

$$f(x) = \begin{cases} y_i f_i(x) & \text{if } x \in F_i, \\ 0 & \text{otherwise.} \end{cases}$$

It satisfies $f(F_i) = \{y_i\}$ for all $1 \leq i \leq n$. □

Theorem 5.3.7. *Let T be a regular space. Then $d(Q_C(T)) \leq kcof(T)$.*

Proof. Let $kcof(T) = |\mathcal{V}|$, where V is cofinal family in $\mathcal{K}(T)$. Now assume $\mathcal{M}$ to be a collection of all finite pairwise disjoint sets of the cofinal family $\mathcal{V}$. Let $C = \{C_1, C_2, \ldots, C_n\} \in \mathcal{M}$ and $r = \{r_1, r_2, \ldots, r_n\}$ be a finite set of rational number. Then by Lemma (5.3.2), there is a quasicontinuous function such that $g_{C,r}(C_i) = \{r_i\}$ for all $1 \leq i \leq n$. Consider a family

$$D = \{g_{C,r} \colon C \in \mathcal{M} \, and \, r \in \mathcal{Q}\}.$$

clearly $|D| \leq |V| \leq \eta$. Now it is sufficient to prove D is dense in $Q_C(T)$. For any $f \in Q_C(T)$, $r \in \mathcal{Q}$ and $C \in \mathcal{K}(T)$. We can define a quasicontinous function

$$g_{k',r}(x) = f(x) \, if x \in C; \; g_{k',r}(x) = r \, if x \in T/C.$$

where $C' = \{C'\}$, $C' \in M$ such that $C' \subset T/C$. Therefore any neighborhood $W(f, C, \epsilon)$ of f, $\exists$ a $g_{C',r}$ quasicontinuous function such that $g_{C',r} \in W(f, C, \epsilon)$. Hence D is dense in $Q_C(T)$. □

5.4 Fréchet-Urysohn Property of $Q_C(T)$

Theorem 5.4.1. *If T is an Hausdorff the and $Q_C(T)$ is a Frechet-Urysohn space, Then for every open k-cover of T, $\exists$ a countable sub k-cover of T.*

Proof. Suppose $Q_C(T)$ is Frechet-Urysohn space. Let $\mathcal{J}$ be any open k-cover of T. So, for each $C \in \mathcal{K}(T)$ $\exists$ $U_C \in \mathcal{J}$ $C \subset U_C$. Then $f_C \in Q_C(T)$ such that $f_C(\overline{U_C}) = \{0\}$ and $f_C(T/\overline{U_C}) = \{1\}$. Take zero function $f_0 \in Q_C(T)$ and $W(f_0, C, \epsilon)$ be any neighborhood of f_0. For C $\exists$ U_C such that $C \subset U_C$. Thus we have a function f_C that lies in $W(f_0, C, \epsilon)$. Therefore $f_0 \in \overline{\{f_C \colon C \in \mathcal{K}(T)\}}$. Then by definition of Frechet-Urysohn space, $\exists$ a sequence $\{f_{C_n} \colon n \in \mathbb{N}\}$ converges to f_0. Construct collection $\{U_{C_n} \colon n \in \mathbb{N}\}$, clearly it is countable. Remain proof that is k-cover of T. Let any $C \in \mathcal{K}(T)$, $\exists$ some $n \in \mathbb{N}$ such that $f_{C_n} \in W(f_0, C, 1)$, then $C \subset U_{C_n}$. Therefore $\{U_{C_n} \colon n \in \mathbb{N}\}$ is k-cover of T. □

In 2007, Ferrando and Moll proved that [[62], Theorem 1], for locally compact Hausdorff space T and $C(T)$ endowed with compact-open topology then following conditions are equivalent: (a) $C(T)$ is Frechet-Urysohn space, (b) $C(T)$ has countable tightness and (c) T is σ-compact. Now we provided the following results for the space $Q_C(T)$.

Theorem 5.4.2. *If T is locally compact Hausdorff space then following are equivalent*

1. *$Q_C(T)$ be Frechet-Urysohn space.*
2. *$Q_C(T)$ has countable tightness.*

3. T is σ-compact space.

Proof. The statement (1) $\Rightarrow$ (2) is easily hold by Theorem (5.4.1) and Theorem (5.3.2). Next, to prove (2) $\Rightarrow$ (3), let $t(Q_C(T)) = \aleph_0$ and $\mathcal{J}$ be any open k-cover of T, then by Theorem (5.3.3), $\exists$ a sub k-cover $\mathcal{J}'$ of T with $|\mathcal{J}'| \leq \aleph_0$. Take $\mathcal{M} = \{\overline{U} : U \in \mathcal{J}'\}$. Since T is locally compact, hence every set in M is compact. Now it is sufficient to prove that $\mathcal{M}$ is a cofinal subfamily in $\mathcal{K}(T)$. For every $C \in \mathcal{K}(T)$, $\exists$ some $\overline{U} \in \mathcal{M}$ such that $C \subset \overline{U}$. Therefore $\mathcal{M}$ is cofinal subfamily in $\mathcal{K}(T)$. Hence T is hemicompact.
To prove (3) $\Rightarrow$ (1), let T is σ-compact, by definition $T = \bigcup_{n\in\mathbb{N}} C_n$, where C_n is compact subset of T for all $n \in \mathbb{N}$ and every $C \in \mathcal{K}(T)$ $\exists$ some $j \in \mathbb{N}$ such that $C \subset C_j$. Construct $C_i' = \bigcup_{n=1}^{i} C_n$ for all $i \in \mathbb{N}$. Since the space $Q_C(T)$ is homogenous. Let f_0 be zero function and F be any subset of $Q_C(T)$ such that $f_0 \in \overline{F}$. Then for every $W(f_0, C_i', \frac{1}{n})$ neighborhood of f_0, $\exists$ some function $f_{C_i',i} \in W(f_0, C_i', \frac{1}{n}) \cap F$ for all $i \in \mathbb{N}$. Now assume that $g_i = f_{C_i',i}$ for all $i \in \mathbb{N}$. Next we claim that sequence $\{g_i : i \in \mathbb{N}\}$ is converges to f_0. Let any $C \in \mathcal{K}(T)$, $\epsilon > 0$, $\exists$ some $p \in \mathbb{N}$ with $\frac{1}{p} < \epsilon$, also $\exists$ some $m \in \mathbb{N}$ such that $C \subset C_m$, implies $C \subset C_i'$ for all $i \geq m$.

Case-1 If $m \geq p$, then $\frac{1}{m} \leq \frac{1}{p}$. Thus

$$|g_i(x) - f_0(x)| < \epsilon \, \forall x \in C \, \forall i \geq m,$$

Case-2 If $p > m$, since $C \subset C_i'$ for all $i \geq m$. Thus $C \subset C_i'$ for all $i \geq p$. Then we have

$$|g_i(x) - f_0(x)| < \epsilon \, \forall x \in C \, \forall i \geq p.$$

Therefore sequence $\{g_i : n \in N\}$ converges to f_0. Hence $Q_C(T)$ is a Frechet-Urysohn space.

□

Corollary 5.4.1. *If T is locally compact metric space then following are equivalent*

1. *$Q_C(T)$ be Frechet-Urysohn space.*
2. *$Q_C(T)$ has countable tightness.*
3. *T is separable.*

Proof. Firstly, we have to show (2) $\Rightarrow$ (3), the space $Q_C(T)$ has countable tightness. Then by above Theorem (5.4.2), space T is σ-compact, since T is metric space therefore it is separable.

Now, we show (3) $\Rightarrow$ (1), Since any space T is separable and locally compact metric space, then it is a σ-compact. Then by the above Theorem (5.4.2), $Q_C(T)$ is Frechet-Urysohn space. □

Definition 5.4.1. A family $\mathcal{V}$ of subsets of a space T is called a k_f-cover if for every finite subfamily $\mathcal{C}'$ of $\mathcal{K}(T)$ $\exists$ $V \in \mathcal{V}$ such that $\bigcup \mathcal{C}' \subset V$. If each element of $\mathcal{V}$ is open, it is called an open k_f-cover.

Suppose T be any space and γ be family of subsets of T, let $\lim \gamma = \{x \in T \colon |\{U \in \gamma \colon x \notin U\} < |\aleph_0\}$.

Theorem 5.4.3. *Let T be any Hausdorff space and $Q_C(T)$ be Frechet-Urysohn space. Then for any open k_f-cover $\mathcal{J}$ of space T, $\exists$ a countable subfamily ξ of $\mathcal{J}$ such that $\lim \xi = T$.*

Proof. Let $Q_C(T)$ be Frechet-Uryshon space. If $T \in \mathcal{J}$ then $\xi = T$, if not then we assume that

$$M = \{f \in Q_C(T) \colon supp(f) \subset U \text{ for some } U \in \mathcal{J}\}.$$

Let g_1 be a constant function such that $g_1(x) = 1$ for all $x \in T$. Then $supp(g_1) = T \notin \mathcal{J}$ this implies $g_1 \notin M$. Now, take $C_1, C_2, \dots C_p \in \mathcal{K}(T)$, where $C_i \cap C_j = \emptyset$, for all $1 \leq i, j \leq p$. By definition of k_f-cover, there is a $U \in \mathcal{J}$ such that $\{C_1, C_2, \dots C_p\} \subset U$. So $\bigcup_{i=1}^{p} C_i$ and T/U are two disjoint closed subsets of T. Therefore, by Lemma (5.3.2) $\exists$ a quasicontinuous function such that

$$f(T/U) = \{0\} \text{ and } f(\bigcup_{i=1}^{p} C_i) = \{1\}.$$

Thus $f \in W(f_1, C_i, \epsilon)$ and $supp(f) = \bigcup_{i=1}^{p} C_i \subset U \in \mathcal{J}$. Then we have $f \in W(f_1, C_i, \epsilon) \cap M$, which implies that $f \in \overline{M}$. Therefore $f \in \overline{M}/M$.
By the definition of Frechet-Urysohn space $Q_C(T)$, $\exists$ a sequence $\{f_n : n \in \mathbb{N}\} \subset M$ such that $f_n \to g_1$. By the selection of sequence, $\exists$ some $U_n \in \mathcal{J}$ such that $supp(f_n) \subset U_n$ for each $n \in \mathbb{N}$. Now, we claim that if $\xi = \{U_n \colon n \in \mathbb{N}\}$ then $lim \xi = T$. For any $x \in T$, then $W(g_1, \{x\}, \epsilon)$ be a neighborhood of f_1, where $0 < \epsilon < 1$. Thus, for $W(g_1, \{x\}, \epsilon)$ $\exists$ some $m \in \mathbb{N}$ such that $f_n \in W(g_1, \{x\}, \epsilon)$ for all $n \geq m$, which implies $x \in supp(f_n) \subset U_n$ for all $n \geq m$. Therefore, $lim \xi = T$. □

Theorem 5.4.4. *If T is a Hausdorff space and $Q_C(T)$ is a Whyburn space. Let $\{\gamma_n\}_{n \in \mathbb{N}}$ be a sequence of open covers of T having the following properties:*

1. $\gamma_n = \{U_m^n \colon m \in \mathbb{N}\}$ *and* $U_m^n \subset U_{m+1}^n$, *for each* $n \in \mathbb{N}$.

2. For any $n \in \mathbb{N}$, we have a closed cover $\mathcal{U}_n = \{F^n_m : m \in \mathbb{N}\}$ of T with $F^n_m \subset U^n_{m+1}$ and $F^n_m \subset F^n_{m+1} \,\forall\, m \in \mathbb{N}$.

Then there exists a k-cover $\{W_n : n \in \mathbb{N}\}$ of T, where $W_n \in \gamma_n$, $\forall\, m \in \mathbb{N}$.

Proof. Let (m, n) be any pair of natural numbers. Then there exists a $f^n_m \in Q_C(T)$ such that $f^n_m | F^n_m \equiv \frac{1}{n}$ and $f^n_m|(T/U^n_m) \equiv 1$. Now, take a sequence $S_n = \{f^n_m : m \in \mathbb{N}\}$, which converges to $h_n \equiv \frac{1}{n}$. Set $S = \bigcup_{n\in\mathbb{N}} S_n$ and a zero function h, then clearly h lies in $\overline{S}$. Since $Q_C(T)$ is a whyburn space, thus we have a subset F of S with $\overline{F} = F \cup \{h\}$. Therefore, for any $n \in \mathbb{N}$, the set $F = F \cap S_n$ will not contain because $\overline{F}/F = \{h\}$ otherwise $h_n \in \overline{F}/F$. Therefore, for every $n \in \mathbb{N}$, we have a number $m(n) \in \mathbb{N}$ such that $F_n \subset \{f^n_m : m \in \mathbb{N}\}$. So for each $n \in \mathbb{N}$ take $W_n = \{U^n_{m(n)}\}$. Next, to prove $\{W_n : n \in \mathbb{N}\}$ is k-cover of T. Since $h \in \overline{F}$, then for each $K \in \mathcal{K}(T)$, there exists a $f^n_m \in F$ such that $f^n_m(x) < 1$ for all $x \in K$. Therefore, $K \cap (T/U^n_m) = \emptyset$, thus $K \subset U^n_m \subset U^n_{m(n)}$. Hence, $\{W_n : n \in \mathbb{N}\}$ is a k-cover of T. □

5.5 CONCLUSION

In this chapter, we studied the density and various types of tightness for space $Q_C(T)$. We proved that tightness of $Q_C(T)$ is equal to compact Lindelöf no. of Hausdorff space T, the density of $Q_C(T)$ is less than k-cofinality of T and we obtained condition on T so that density-tightness and tightness of $Q_C(T)$ coincides. Further we characterized fan tightness and strongly fan-tightness in terms of ccvr of T. Next, we proved that if T locally compact Hausdorff space then $Q_C(T)$ being Frechet-Urysohn, $Q_C(T)$ having countable tightness, and σ-compactness of T are equivalent. Moreover, we proved that if $Q_C(T)$ is Frechet-Urysohn space then every k_f-open covering of T has countable subcover which converges to T.

Chapter 6

Conclusion and future works

6.0.1 Conclusion

In this study, we explored the properties of quasicontinuous functions and their related topological spaces. We established that the set of all continuous functions, $C(T, M)$, is a subset of the set of all quasicontinuous functions, $Q(T, M)$, and examined the Intermediate Value Theorem in this context. We characterized cardinal invariants for the space $Q_p(T, M)$, demonstrating that the pseudocharacter of $Q_p(T, M)$ dominates the network weight, density, and weak covering number of a regular space T. Necessary and sufficient conditions for the weight and network weight of $Q_p(T)$ to coincide were derived. Also, we derived the separability condition of an regular space in terms of the pseudocharacter of a compact subset of $Q_p(T, M)$. We studied the openness of the restriction map on $Q_p(T)$ and the density of the image of $Q_p(T, M)$ under the induced map in $Q_p(T, Z)$. Furthermore, we investigated the density and various types of tightness for the space $Q_C(T)$, proving that the tightness of $Q_C(T)$ equals the compact Lindelöf number of Hausdorff space T and that the density of $Q_C(T)$ is less than the k-cofinality of T. Conditions on T for the density-tightness and tightness of $Q_C(T)$ to coincide were identified. We characterized fan tightness and strongly fan-tightness in terms of the k-cover of T, and showed that if T is locally compact Hausdorff space, then $Q_C(T)$ being Fréchet-Urysohn, $Q_C(T)$ having countable tightness, and the σ-compactness of T are equivalent. Lastly, we demonstrated that if $Q_C(T)$ is a Fréchet-Urysohn space, every k_f-ops covering of T has a countable subcover converging to T. This study provides a foundation for further research on quasicontinuous functions and their impact on topological properties.

6.0.2 Future works

Throughout the thesis, we studied two different topologies (pointwise convergence and compact convergence) on $Q(T, M)$ and some of their properties. Although there is a wide magnitude of area still left to explore. Some of them are:

- In the study of spaces $Q_p(T, M)$ and $Q_C(T, M)$, there is a large gap in the knowledge of various forms of connectivity. This comprises path-connectedness, local connectivity, and connectedness, which have yet to be completely studied. Existing research on $Q_p(T, M)$ and $Q_C(T, M)$ is missing important properties like submetrizability (a space's ability to inherit metrizability from a finer topology), local metrizability (where each point has a metrizable neighborhood), and Baire properties (conditions under which the space is a Baire space). These elements provide fertile territory for future investigation.

- Our current study of $Q(T, M)$ has been limited to two specific topologies, which are pointwise convergence and compact convergence. However, $Q(T, M)$ can be endowed with various other topologies, much like the space $C(T, M)$ which can be topologized in multiple ways, such as regular topology, Cauchy convergence, and fine topology. Exploring $Q(T, M)$ under various topologies allows us to draw comparisons and contrasts with $C(T, M)$, providing a greater grasp of the structure and characteristics of both spaces. This technique offers up new options for investigating and comparing topological properties, giving a comprehensive framework for future research in the topic.

- Till now, the space $Q(T, M)$ has been studied under the consideration that the range is a topological space without any specific algebraic structure, this gives us the scope to explore the space $Q(T, M)$, where the range space M have a specific structure such as group, linear space, uniform space. We can further study the properties of space $Q(T, M)$ under above mentioned conditions and compare it with the existing results.

- Throughout the literature, the space $Q(T, M)$ has been examined under a topological framework. However, several structures can be used to explore function spaces. For example, in functional analysis, function spaces are often studied as normed spaces. We can broaden our understanding of $Q(T, M)$ by examining several structures, including groups, rings, linear spaces, normed spaces, and uniform spaces. Exploring $Q(T, M)$ under these varied structures can provide deeper insights and enrich the overall study of function spaces, leading to a more thorough understanding of their distinctive characteristics and prospective applications.

- An important area of research in function spaces involves understanding the duality between the base space T and the function space $Q(T, M)$, specifically how the properties of T influence $Q(T, M)$ and vice versa. This includes investigating the impact of topological and algebraic structures of T on $Q(T, M)$, such as compactness, connectedness, and uniformity, as well as examining how the characteristics of $Q(T, M)$ can reflect back on T. Similarly, the presence of additional algebraic structures like group operations or vector space properties in T might significantly influence the properties of $Q(T, M)$. Conversely, the study of $Q(T, M)$ can reveal new aspects of the base space T, such as how the completeness or compactness of $Q(T, M)$ provides information about the topology of T. Developing new topological constructs within $Q(T, M)$ based on this duality can lead to novel results and applications. The current lack of extensive literature on these topics underscores a valuable opportunity for future research to explore these aspects, ultimately contributing significantly to the fields of topology and functional analysis.

- Topological function spaces have broad applications across numerous mathematical fields and beyond. By leveraging our findings, we can integrate them into various areas, thereby enriching these disciplines and contributing to their advancement. In functional analysis, for example, incorporating topological insights can lead to a deeper understanding of normed spaces, potentially revealing new connections and results. In game theory, topological function spaces can provide a more robust framework for analyzing strategies and equilibria, leading to more comprehensive models and solutions. Similarly, in graph theory, the integration of function spaces can enhance the study of graph properties and dynamics, offering new perspectives and tools for solving complex.

- The integration of quasicontinuous functions into Topological Data Analysis(TDA) opens up promising directions for handling noisy, incomplete, or irregular data. These functions, which exhibit partial continuity, could enhance persistent homology by capturing finer topological features often smoothed out by traditional methods. They offer potential in functional data analysis, where time-varying or discontinuous data like signals or financial series can be better modeled. quasicontinuous functions can also lead to new filtration techniques and generalized persistence modules, providing deeper insights into machine learning models, such as neural networks with non-smooth mappings. Additionally, they may enhance multi-parameter persistent homology, helping analyze complex systems with multiple variables. Finally, these functions can bridge discrete and continuous spaces in TDA, improving the approximation of topological structures. Overall, quasicontinuous functions bring flexibility and robustness to TDA, extending its applicability across fields like signal processing, machine learning, and beyond.

- Investigating quasicontinuous functions in topologies beyond standard metric and Euclidean spaces offers potential for uncovering new mathematical insights and applications. In non-Hausdorff spaces, where points cannot always be separated by disjoint neighborhoods, quasicontinuous functions may exhibit behaviors distinct from those in Hausdorff spaces, leading to applications in fields like quantum topology and non-classical topology. Exploring quasicontinuity in coarser topologies such as the order topology and Scott topology is also promising. These topologies, often used in domain theory and lattice theory, structure data through inherent orderings, providing a useful framework for studying quasicontinuous functions in contexts where order and hierarchy are critical, such as in computational models. The Scott topology, in particular, allows for understanding quasicontinuity in the context of computational processes that evolve in a structured yet non-continuous way. Additionally, specialized topologies like the lower limit topology (Sorgenfrey line), where intervals are open only to the right, offer another layer of complexity. Quasicontinuous functions in this context could model asymmetrical phenomena, such as systems with one-sided growth or decay. This broader exploration of topological contexts for quasicontinuous functions not only deepens theoretical understanding but also opens up diverse applications in fields like economics, biology, and computational systems.

Bibliography

[1] G.Ascoli. Le curve limite di una varieta data di curve. *Mem. Accad. Lincei.*, 18: 521586,, (1883).

[2] C.Arzella. Funzioni di linee. *Atti della Reale Accademia dei Lincei, Rendiconti*, 5: 342–348, (1889).

[3] J.Hadamard. Sur certaines applications possibles de la theorie des ensembles. *Verhandl. Ersten Intern. Math. Kongresses*, (1898).

[4] M.Frechet. Sur quelques points du calcul functional. *Rend. Circ. Maum.(Palermo, Paris)*, (1906).

[5] R.H.Fox. (X, τ) topologies for function spaces. *Bulletin of American Mathematical Society*, 51:429–432, 1945.

[6] S.A.Naimpally. Graph topology for function spaces. *Trans. Am. Math. Soc*, 123: 267–272, (1966).

[7] D.Gulick. The σ-compact-open topology and its relatives. *Mathematica Scandinavica*, 30(1):159–176, 1972.

[8] K.F.Porter. The open-open topology for function spaces. *Internat. J. Math. and Math. Scl.*, 16(1):111–116, (1993).

[9] S Kundu and AB Raha. The bounded-open topology and its relatives. 1995.

[10] S.Kundu and R.A.McCoy. Weak and support-open topology on $C(X)$. *Roc. Mount. J. of Math.*, 25(2), (1995).

[11] A.V.Osipov. The c-compact open topology on function spaces. *Topol. Appl.*, 159: 3059–3066, (2012).

[12] Gerald Beer. Metric spaces on which continuous functions are uniformly continuous and hausdorff distance. *Proceedings of the American Mathematical Society*, 95(4): 653–658, 1985.

[13] M.Marjanovic. Topologies on collections of closed subsets. *Publ. Inst. Math.(Beograd)*, 20:125–130, 1966.

[14] R.A.McCoy, S.Kundu, and V.Jindal. *Function Spaces with Uniform, Fine and Graph Topologies.* Springer Cham, 2018. doi: https://doi.org/10.1007/978-3-319-77054-3.

[15] J.Nagata. On lattices of functions on topological spaces and of functions on uniform spaces. *Osaka Math. J.*, 1(2):166–181, (1949).

[16] G.W.Whitehead. On products in homotopy groups. *Ann. of Math*, 47(2):460–475, (1946).

[17] L.Hola and D.Holy. Quasicontinuous functions and compactness. *Mediterr.J.Math.*, 14(6):1–11, 2017.

[18] L.Hola and D.Holy. Quasicontinuous functions and the topology of pointwise convergence. *Topol.Appl.*, 282:Article No. 107301, 2020.

[19] R.Arens and J.Dugundji. Topologies for function spaces. *Pac. J. Math.*, 1(1), (1951).

[20] J.R.Jackson. Comparison of topologies on function spaces. *Proc. Amer. Math. Soc.*, 3(1), .

[21] S.Kempisty. Sur les fonctions quasicontinues. *Fundamenta Mathematicae*, 1(19):184–197, 1932.

[22] R.Baire. Sur les fonctions de variables réelles. *Annali di Matematica Pura ed Applicata (1898-1922)*, 3(1):1–123, 1899.

[23] T.Neubrunn. On quasicontinuity of multifunctions. *Mathematica Slovaca*, 32(2):147–154, 1982.

[24] W.W.Bledsoe. Neighborly functions. *Proceedings of the American Mathematical Society*, 3(1):114–115, 1972.

[25] S.Marcus. On quasi-continuous functions in the sense of s. kempisty. 8(1):47–53, 1961.

[26] N.Levine. Semi-open sets and semi-continuity in topological spaces. *The American mathematical monthly*, 70(1):36–41, 1963.

[27] A.Neubrunnov´a. On certain generalizations of the notion of continuity. *Matematick‘y ˇcasopis*, 23(4):374–380, 1973.

[28] K.R.Gentry and H.B.Hoyle III. Somewhat continuous functions. *Czechoslovak Mathematical Journal*, 21(1):5–12, 1971.

[29] S.G. Crossley and S.K. Hildebrand. Semi-closed sets and semi-continuity in topological spaces. *Texas Journal of Science*, 22(2-3):123, 1971.

[30] S.G. Crossley and S.K. Hildebrand. Semi-topological properties. *Fundamenta Mathematicae*, 74(3):233–254, 1972.

[31] J.Ewert. On quasi-continuous and cliquish maps with values in uniform spaces. *Bull. Polish Acad. Sci. Math*, 32(1-2):81–88, 1984.

[32] N.Crivat and T.Banzaru. On the quasicontinuity of the limits for nets of multifunctions, semin. *Math. Fiz. Inst. Politechn. Tymisoara*, pages 37–40, 1983.

[33] L.Hola and D.Holỳ. Pointwise convergence of quasicontinuous mappings and baire spaces. *The Rocky Mountain Journal of Mathematics*, pages 1883–1894, 2011.

[34] L.Hola and D.Holy. Quasicontinuous subcontinuous functions and compactness. *Mediterr.J.Math.*, 13(6):4509–4518, 2016.

[35] L.Hola and D.Holy. Metrizability of the space of quasicontinuous functions. *Topol.Appl.*, 246:137–143, 2018.

[36] L.Holá and D.Holỳ. Quasicontinuous functions and the topology of uniform convergence on compacta. *FILOMAT*, 35:911–917, 2021.

[37] M.Kumar and B.K.Tyagi. Cardinal invariants and special maps of quasicontinuous functions with the topology of pointwise convergence. *Applied General Topology*, 23(2):303–314, 2022.

[38] J.R.Munkres. *Topology.* Indian edition by PHI learning private limited, second edition, 2012. ISBN 978-81-203-2046-8.

[39] T.Neubrunn. Quasi-continuity. *Real Anal. Exchange*, 14(2):259–306, 1988.

[40] Wolf Iberkleid, Ramiro Lafuente-Rodriguez, and Warren Wm McGovern. The regular topology on $c(x)$. *Commentationes Mathematicae Universitatis Carolinae*, 52(3):445–461, 2011.

[41] Zuquan Li. Cauchy convergence topologies on the space of continuous functions. *Topology and its Applications*, 161:321–329, 2014.

[42] L.Hola, D.Holy, and W.Moors. *USCO and Quasicontinuous mappings*, volume 81. De Gruyter, 2021. ISBN 9783110750188.

[43] R.Baire. Sur les functions des varaibles reells. *Ann.Mat.Pura Appl.*, 3:1–122, 1899.

[44] Z.Piotrowski. Separate and joint continuity. *Real Anal. Exchange*, 11:293–322, 1985-86.

[45] Z.Piotrowski. A survey of results concerning generalized continuity in topological spaces. *Acta Math. Univ. Comen.*, 52-53:91–110, 1987.

[46] T.Noiri. A function which preserves connected spaces. *Časopis pro pěstování matematiky*, 107(4):393–396, 1982.

[47] B.K.Tyagi et.al. Some strong forms of connectedness in topological spaces. *Journal of advanced studies in topology*, 10(1):20–27, 2019.

[48] V.Pipitone and G.Russo. Spazi semiconnessi e spazi semiapert. *Rendiconti del Circolo Matematico di Palermo*, 23(4):273–285, 1975.

[49] S.Jafari and T.Noiri. Properties of β-connected spaces. *Acta Mathematica Hungarica*, 101(3):227–236, 2003.

[50] E.Ekici. On separated sets and connected spaces. *Demonstratio Math*, 40(1): 209–217, 2007.

[51] T.Noiri and S.Modak. Half b-connectedness in topological spaces. *Journal of the Chungcheong Mathematical Society*, 29(2):221–221, 2016.

[52] S.Modak and T.Noiri. A weaker form of connectedness. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 65(1): 49–52, 2016.

[53] D.Andrijević. On b-open sets. *Matematički Vesnik*, 48:59–64, 1996.

[54] R.A.Mahmoud M.E.Abd EI-Monsef and S.N.El-Deeb. β-open sets and β-continuous mappings. *Bull. Fac. Sci. Assiut Univ*, 12(1):77–90, 1983.

[55] S.Modak and Md.Islam. More connectedness in topological spaces. *Caspian Journal of Mathematical Sciences (CJMS) peer*, 8(1):74–83, 2019.

[56] V.V.Tkachuk. *A C_p-theory problem book:Compactness in function spaces*. Springer, 2015.

[57] V.V.Tkachuk. *A C_p-theory problem book:Functional equivalencies*. Springer, 2016.

[58] O.G. Okunev and V.V. Tkachuk. Density properties and points of uncountable order for families of open sets in function spaces. *Topology and its Applications*, 122(1-2):397–406, 2002.

[59] R.A.McCoy and I.Ntantu. *Topological Properties of Spaces of Continuous Functions, Lecture Notes in Mathematics*, volume 1315. Springer-Verlag, Berlin, 1988.

[60] A.V.Arkhangel'skii. *Topological function spaces*, volume 78. Kluwer Academic Publishers, 1992. ISBN 0792315316.

[61] A.V.Osipov. Fréchet-urysohn property of quasicontinuous functions. *Rocky Mountain Journal of Mathematics*, 2023.

[62] J.C. Ferrando and S. Moll. $c_c(x)$ spaces with x locally compact. *Acta Mathematica Sinica, English Series*, 23(9):1593–1600, 2007.

[63] S.Lipschutz. *Schaum's outline of theory and problems of general topology.* 1965.

[64] K.D.Joshi. *Introduction to general topology.* New Age International, 1983.

[65] J.Ewert and T.Lipski. Lower and upper quasi-continuous functions. *Demonstratio Mathematica*, 16(1):85–94, 1983.

[66] J.Ewert and T.Lipski. Quasi-continuous multivalued mappings. *Mathematica Slovaca*, 33(1):69–74, 1983.

[67] J.L.Kelley. *General topology.* Courier Dover Publications, 2017.

[68] S.Willard. *General Topology.* Addison-Wesley Publishing Co., (Reading, MA, London, Don Mills, Toronto, ON), (1970).

[69] R.Engelking. *General topology.* Heldermann Berlin, 1989.

[70] E.Kreyszig. *Introductory Functional Analysis With Applications.* John Wiley and Sons. Inc., 1978. ISBN 0-471-50731-8.

[71] M.W.Hirsch. *Differential Topology.* Springer, New York, corrected reprint of the 1976 original edition, (1994).

[72] H.L.Royden and P.M.Fitzpatrick. *Real Analysis.* Prentice Hall, fourth edition, 2010. ISBN 978-0-13-143747-0.

[73] K.Kunen and J.E.Vaughan. *Handbook of set-theoretic topology.* Elsevier Science, 1984. ISBN 0444865802.

[74] J.B.Listing. *Vorstudien zur topologie.* Vandenhoeck und Ruprecht, 1848.

[75] V.V.Tkachuk. *A C_p-theory problem book:Topological and function spaces.* Springer, 2010.

[76] V.V.Tkachuk. *A C_p-theory problem book, Topological and function spaces, Problem Books in Mathematics.* Springer, New York.

[77] V.V.Tkachuk. *A C_p-theory problem book:Special features of function spaces.* Springer, 2014.

[78] L.A.Steen and J.A.Seebach. *Counterexamples in topology*, volume 18. Springer, 1978.

[79] A.V.Osipov. The $\mathcal{C}$-compact-open topology on function spaces. *Topology and its Applications*, 159:3059–3066, 2012.

[80] R.Arens. Topologies for homeomorphism groups. *Am. J. Math*, 68:593–610, (1946).

[81] F.W.Anderson. Approximation in systems of real-valued functions. *Trans. Am. Math. Soc.*, 63:249–271, (1962).

[82] M.Aaliya and S.Mishra. Space of homeomorphisms under regular topology. *Commun.Korean Math.Soc.*, 2023.

[83] C.M.Bishnoi and S.Mishra. Quasicontinuous function on strong forms of connected space. *J.Indones.Math.Soc.*, 29(1):106–115, 2023.

[84] J.Ewert and J.S. Lipiński. On points of continuity, quasicontinuity and cliquishness of real functions. *Real Analysis Exchange*, 8(2):473–478, 1982.

[85] P.Garg and S.Kundu. The compact-$\mathcal{G}_\delta$-open topology on $\mathcal{C}(\mathcal{X})$. *Topol. Appl.*, 159: 2082–2089, 2012.

[86] T.A.Hawary. On generalized preopen sets. *Proyec. J. Math.*, 32(1):47–60, 2013.

[87] E.Hewitt. Rings of real-valued continuous functions i. *Trans. Am. Math. Soc.*, 64: 45–99, (1948).

[88] L.Holá and D.Holý. Spaces of lower semicontinuous set-valued maps. *Math. Slovaca*, 63(4):863–870, (2013).

[89] J.R.Jackson. Comparison of topologies on function spaces. *Proceedings of the American Mathematical Society*, 3(1), .

[90] S.Mishra and A.Bhaumik. Properties of function space under cauchy convergence topology. *Topol.Appl.*, page 108653, 2023. doi: https://doi.org/10.1016/j.topol.2023.108653.

[91] E.H.Moore. On the foundations of the theory of linear integral equations. *Bull. Am. Math. Soc.*, 18:334–362, (1911-1912).

[92] G.Di Maio, L.Holá, D.Holý, and R.A.McCoy. Topologies on the space of continuous functions. *Topol. Appl*, 86:105–122, (1998).

[93] T.Neubrunn. A generalized continuity and product spaces. *Mathematica Slovaca*, 26(2):97–99, 1976.

[94] T.Neubrunn and O.Náther. On a characterization of quasicontinuous multifunctions. *Časopis pro pěstování matematiky*, 107(3):294–300, 1982.

[95] T.Neubrunn. On sequential characterization of quasicontinuous mul-tifunctions. *Acta Math. Univ. Com*, pages 44–45, 1984.

[96] A.V.Osipov. The $\mathcal{C}$-compact-open topology on function spaces. *Topol. Appl.*, 159: 3059–3066, 2012.

[97] P.Garg and S.Kundu. The compact-$\mathcal{G}_\delta$-open topology on $\mathcal{C}(x)$. *Topology and its Applications*, 159:2082–2089, 2012.

[98] B.K.Tyagi, M.Bhardwaj, and S.Singh. α_β-connectedness as a characterization of connectedness. *Journal of advanced studies in topology*, 9(2):119–129, 2018.

[99] R.Arens and J.Dugundji. Topologies for function spaces. *Pacific Journal of Mathematics*, 1:5–31, 1951.

[100] S.Kundu and P.Garg. The pseudocompact-open topology on $\mathcal{C}(x)$. *Topology Proceedings*, 30(1):279–299, 2006.

[101] W.Stephen. Pseudocompact metacompact spaces are compact. *Proceedings of the American Mathematical Society*, 81(1):151–152, 1981.

[102] H.Whitney. Differential manifolds. *Ann. Math.*, 2(37):645–680, (1936).

Publications

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QUASICONTINUOUS FUNCTIONS ON STRONG FORM OF CONNECTED SPACES

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Abstract. Preservation of properties under continuous functions on topological spaces is a very important tool for the classification of topological spaces. However, in some cases the quaiscontinuous functions are more useful than the continuous functions for classifying topological spaces. In this paper, we study preservation of strong forms of connectedness under quasicontinuous function that help to prove the general form of intermediate value theorem.

Key words and Phrases: Quasicontinuous, Half connected, Semi-connected and Half semi-connected.

1. INTRODUCTION

In 1899, Baire [3] used the condition of quasicontinuity to study topological spaces. Later in 1932, Kempisty [6] introduced the concept of quasicontinuous map for several real variables. The conditions for quasicontinuity of function of two variables provided by Volterra [3]. In 1976, Neubrunn [12] reformulated the Kempisty's definition of quasicontinuity for general topological spaces as: "a map $f\colon X \to Y$ is quasicontinuous at $p \in X$ if for any open sets U in X and V in Y such that $p \in U$ and $f(p) \in V$, then there exists a non-empty subset G of U such that $f(G) \subset V$. It is said to be quasicontinuous if it is quasicontinuous at any $p \in X$". All continuous maps are quasicontinuous but its converse not holds. For example $f\colon \mathbb{R} \to \mathbb{R}_l$ defined by $f(x) = x$ is quasicontinuous function but not continuous, where $\mathbb{R}$ and $\mathbb{R}_l$ are set of real numbers with usual and lower limit topology respectively. Quasicontinuity has deep connection with mathematical analysis, topology and many applications in analysis, topology, measure theory, and probability theory.

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- National conference **National Symposium on Mathematics and Applications** at IIT, Madras India, Oral Presentation on "Quasi continuous function on topological space."
- International Conference on Emerging Trends in Applied Mathematics [ICETAM-2022] Oral Presentation on **Some results for quasicontinuous function on strong form of connected spaces** on April2022.

Workshops

- Participated in the “**Workshop on geometry of Banach Spaces and its applications (WGBSA)**”, organized by IIIT, Allahabad, India (December 31, 2021 to January, 2022).
- Participated and presented a talk on topic is “*Topology-Algebra Duality*” in the “**Indian School of Logic and its Applications (ISLA-2022)-Logic and Dualities (9th edition)**”, organized by IIT Kanpur, India (May, 2022).
- Participated in the International workshop on “**Trends in Analysis and Topology** (TIAT-22)” at MNIT Jaipur, on Sep2022.