A STUDY OF FIBONACCI POLYNOMIAL, CHEBYSHEV POLYNOMIAL AND ITS SEQUENCES

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in

Mathematics

By

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2024

DECLARATION

I, hereby declared that the presented work in the thesis entitled "**A STUDY OF FIBONACCI POLYNOMIAL, CHEBYSHEV POLYNOMIAL AND ITS SEQUENCES**" in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of Dr. Vipin Verma, working as Associate Professor, in the Department of Mathematics, School of Chemical Engineering and Physical Sciences of Lovely Professional University, Punjab, India and Dr. Ajay Kumar Sharma, Assistant Professor, Department of Mathematics, Govt. Degree College, Udhampur. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.

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CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled "**A STUDY OF FIBONACCI POLYNOMIAL, CHEBYSHEV POLYNOMIAL AND ITS SEQUENCES**" submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the Department of Mathematics, School of Chemical Engineering and Physical Sciences of Lovely Professional University, Punjab, India is a research work carried out by Mr. Jugal Kishore (**41800284**) is bonafide record of his/her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.

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Abstract

Fibonacci numbers are the amazing numbers discovered by Leonardo of Pisa and are one of God's best-gifted numbers, having a significant impact on our daily lives. These numbers are the outcomes of Leonardo of Pisa's well-known "rabbit problem", which we will cover in more detail later in this thesis. These numbers, in addition to being a part of our everyday lives, have a variety of applications in nature, music, and other fields that cannot be expressed in a few words.

This thesis as a whole concentrate on the notion of these divinely endowed Fibonacci numbers and the associated polynomials that surround them. There are six chapters in this thesis. The first chapter of the thesis provides a brief introduction to the Fibonacci numbers, their history, and their applications in different fields of our lives. In addition, a brief outline of the significant concepts and well-known results pertaining to Fibonacci numbers and the associated polynomials with tabular and graphic illustrations are given, which meets the minimal prerequisite for the establishment of the necessary framework for subsequent chapters. In the section of literature review, a discussion on the existing works done by various researchers in the domain of Fibonacci and related numbers and their associated polynomials is covered, wherein our main focus is on summation representations of finite products of these sequences of numbers and polynomials. A research gap has been identified in this review. This chapter also lays down the objectives and methods that will be employed to bridge these gaps. We extensively employed GeoGebra software to represent various sequences graphically.

The remainder of the thesis is focused on the behaviour and different properties of polynomial sequences that are analogous to sequences of Fibonacci numbers and their inter-linkages. Our work mainly zeros in on the sequences of Lucas, Fibonacci, & Pell numbers & their polynomials, Chebyshev polynomials of the 1st, 2nd, 3rd, & 4th kind, followed by a brief description of Trivariate Lucas and Fibonacci polynomials and their extension to generalized Trivariate Lucas and Fibonacci polynomials, with the development of some results based on their properties and inter-relationships. We employ a variety of methodologies and techniques to accomplish our objectives. By

employing recursive methodology in this thesis, we develop various summation representations for sequences of Lucas and Fibonacci numbers and their polynomials with positive as well as negative indices. After that explicit formulae for the 3rd & 4th kinds of Chebyshev polynomials and their derivatives with odd & even index are obtained, followed by the establishment of their linkages with the Fibonacci polynomial. Furthermore, the sums of the finite products of the 3rd & 4th kinds of Chebyshev polynomials and Pell polynomials are expressed as a linear sum of other orthogonal polynomials using elementary computations. Next, we studied the extensions of Trivariate Lucas and Fibonacci polynomials to *(p, q, r)*-Generalized Trivariate Lucas, and *(p, q, r)*-Generalized Trivariate Fibonacci polynomials and developed their basic properties. Using these properties, we derived the explicit representations of *(p, q, r)*-Generalized Trivariate Fibonacci and *(p, q, r)*-Generalized Trivariate Lucas polynomials and derived several intriguing identities associated with their generating matrices and corresponding determinants.

After introduction to the thesis, we developed various identities on summations of finite products of Lucas & Fibonacci numbers in terms of the 2nd kinds of Chebyshev polynomials and their derivatives. These identities are further extended to the Fibonacci and Lucas numbers with positive as well as negative indices. Next, we derived analogous results for the 3rd & 4th kinds of Chebyshev polynomials followed by some particular cases of these identities. Thereafter, the explicit formulas for the 3rd & 4th kinds of Chebyshev polynomials and their derivatives with odd and even indices were obtained, and their connections with the odd and even indexed Fibonacci polynomials were studied. Further, we obtained some more identities connecting finite product of the 3rd & 4th kinds of Chebyshev polynomials with several other orthogonal polynomials like Pell, Jacobi, Fibonacci, Gegenbauer, Vieta-Fibonacci, and Vieta-Pell polynomials. In terms of these polynomials, analogous results for Lucas & Fibonacci numbers are obtained using the computational method.

Our next step is to establish some new results on representations of finite products of the Lucas & Fibonacci numbers, Fibonacci & Pell polynomials as a linear sum of derivatives of Pell polynomials. Similar results are obtained for the 3rd & 4th kinds of Chebyshev polynomials. Following this pattern, we will introduce similar results for Lucas, Fibonacci, & Complex Fibonacci numbers with negative indices as a linear combination of Pell polynomials. In terms of the 3rd & 4th kinds of Chebyshev polynomials, similar identities were obtained for Pell numbers and Fibonacci polynomials with a negative index. Similar representations for the Chebyshev polynomials of the 3rd & 4th kinds as a linear sum of the Chebyshev polynomials of the 2nd kind are studied.

At the end, we worked on the sequence of Tribonacci numbers and associated polynomials, Trivariate Lucas and Fibonacci polynomials that follows a third-order recursive relation. Following this concept, we will study *(p, q, r)*-Generalized Trivariate Lucas and *(p, q, r)*-Generalized Trivariate Fibonacci polynomials and some of their basic properties and their inter-linkages. These polynomials are characterized recursively as follow:

$$F^*_{\alpha}(\xi,\omega,\zeta) = \begin{cases} 0 & if\ \alpha = 0 \\ 1 & if\ \alpha = 1 \\ p(\xi,\omega,\zeta) & if\ \alpha = 2 \\ p(\xi,\omega,\zeta)\, F^*_{\alpha-1}(\xi,\omega,\zeta) + q(\xi,\omega,\zeta) F^*_{\alpha-2}(\xi,\omega,\zeta) + r(\xi,\omega,\zeta) F^*_{\alpha-3}(\xi,\omega,\zeta), & if\ \alpha > 2 \end{cases}$$

and

$$G^*_{\alpha}(\xi,\omega,\zeta) = \begin{cases} 3 & if\ \alpha = 0 \\ p(\xi,\omega,\zeta) & if\ \alpha = 1 \\ p(\xi,\omega,\zeta)^2 + q(\xi,\omega,\zeta) & if\ \alpha = 2 \\ p(\xi,\omega,\zeta)\, G^*_{\alpha-1}(\xi,\omega,\zeta) + q(\xi,\omega,\zeta) G^*_{\alpha-2}(\xi,\omega,\zeta) + r(\xi,\omega,\zeta) G^*_{\alpha-3}(\xi,\omega,\zeta), & if\ \alpha > 2 \end{cases}$$

where $p(\xi,\omega,\zeta), q(\xi,\omega,\zeta), r(\xi,\omega,\zeta)$ are polynomials of the variables ξ, ω and ζ. Using these recurrence formulas, we will study the sum of the first n-terms of these polynomials, followed by their sum of even and odd number of terms. Some relations involving Jacobian of *(p, q, r)*-Generalized trivariate Lucas and *(p, q, r)*-Generalized trivariate Fibonacci polynomials are also considered.

Using the properties of these polynomials, we will derive the explicit formulae of *(p, q, r)*-Generalized trivariate Fibonacci and *(p, q, r)*-Generalized trivariate Lucas polynomials which are given by

$$F^*{}_\alpha(\xi,\omega,\zeta) = \sum_{t=0}^{\left\lfloor\frac{\alpha-1}{2}\right\rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{\alpha-t-s-1}{t} p^{\alpha-2t-s-1} q^{t-s} r^{s},$$

$$G^*{}_\alpha(\xi,\omega,\zeta) = \sum_{t=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor} \sum_{s=0}^{t} \frac{\alpha}{\alpha-t-s} \binom{t}{s} \binom{\alpha-t-s}{t} p^{\alpha-2t-s} q^{t-s} r^{s},$$

such that $\binom{j}{i} = 0$ for $i > j$ and writing $p = p(\xi,\omega,\zeta), q = q(\xi,\omega,\zeta), r = r(\xi,\omega,\zeta)$.

At the end, we will deduce some identities involving the generating matrices and their determinants. The generating matrix for (*p, q, r)*-Generalized Trivariate Fibonacci and (*p, q, r)*-Generalized Trivariate Lucas polynomials are generated with the help of the following matrix

$$\mathcal{H} = \begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix}$$

and deduced some related determinantal properties.

Finally, we lay out the brief mapping of the future research possibilities based on the content of this thesis.

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$\mathcal{F}_n$	:	n^{th} Fibonacci number
$\mathcal{L}_n$	:	n^{th} Lucas number
$\emptyset$	:	Golden Ratio
$\mathcal{P}_n$	:	n^{th} Pell numbers
$\mathcal{T}_n$	:	n^{th} Chebyshev polynomial of first kind
$\mathcal{U}_n$	:	n^{th} Chebyshev polynomial of second kind
$\mathcal{V}_n$	:	n^{th} Chebyshev polynomial of third kind
$\mathcal{W}_n$	:	n^{th} Chebyshev polynomial of fourth kind
$\mathcal{P}_n$ (x)	:	n^{th} Pell polynomial
$\mathcal{H}_n(x,y,z)$	:	n^{th} Trivariate Fibonacci polynomial
$K_n(x,y,z)$	:	n^{th} Trivariate Fibonacci polynomial
${F^*}_n(x,y,z)$	:	n^{th} $(p.q,r) -$ Generalised trivariate Fibonacci polynomial
${G^*}_n(x,y,z)$	:	n^{th} $(p.q,r) -$ Generalised trivariate Lucas polynomial
$\lfloor . \rfloor$	:	Floor function
$det(A)$	:	determinant of a matrix A
$[.]$	:	Greatest integer function

Chapter 1

Introduction

1.1 Introduction

Leonardo Pisano (1170-1250), an Italian mathematician who is better known by his nick name Fibonacci (an abbreviation of Filius Bonacci), while studying Hindu-Arab numerals, came across what is known as the Fibonacci Sequence, and he compiled his findings in the book *Liber Abaci* which was published in 1202 and later revised in 1228. He visited a number of Mediterranean nations and researched their mathematical practices. Fibonacci's work in Liber Abaci is said to have been influenced by the mathematical work of Egyptian mathematician Abu Kamil. His book opens with the following explanation of the Hindu-Arabic numeral model: The following nine figures have been identified as 1,2,3,4,5,6,7,8,9 allowing any number to be represented using these nine figures and the symbol 0 [3]. First-hand instances of the potential benefits of the new Hindu-Arabic numeral scheme were offered by the problems in this book. Liber Abaci was considered a complete source of mathematical knowledge during the time of Fibonacci. For hundreds of years after its publication, this book served as a crucial source for mathematicians searching for new ideas in algebra and computation.

Now let's focus on Indian mathematicians and their contribution to the Fibonacci numbers. Although Leonardo Fibonacci, who was mentioned in detail above, is the name-bearer of the Fibonacci numbers, the knowledge of these numbers existed long before his time. The Indian mathematician Pingala is credited as being the first to have knowledge of the Fibonacci numbers, according to a number of researchers including Singh [1, 3-4]. The estimated year when he lived is 400 B.C. It is believed that Acarya Virahanka, an Indian mathematician who lived between 600 and 800 A.D., was the first to present the Fibonacci numbers in written form. Gopala is another key figure in the domain of the Fibonacci numbers, born before 1135 A.D. and having significant contributions. Archarya Hemachandra, a renowned Jain writer, presents an estimate of variations in matra-vrttas in Chandonusasana. In Chandonusasana, the translation of his rule, which is referenced from [4], is as follows: *"Sum of the last and the last but one,* *number of variations of the matra-vrttas coming afterward".* (Matras-Vrttas are metres with varying letter counts but consistent amounts of morae). *He continues, "the number 3, which is preserved later and is the number of variants (of meter) having three matras, is the last among the numbers 1, 2, etc., and the last number other than one. The result of adding 3 and 2 is 5, which is kept later, and there are 4 matras in the metre's variations* [5]".

In the classic rabbit problem, the Fibonacci sequence was initially employed to determine how many pairs of rabbits are born out of one pair of rabbits in one year.

This problem is stated as under:

A pair of rabbits was kept in a wall-enclosed region to determine precisely the number of pairs of rabbits that could be bred by a pair of rabbits over an entire year, assuming that each pair of rabbits bears a new pair every month, which becomes productive from the second month onwards, and no rabbit dies during this span of time.

This rabbit problem demonstrated by Fibonacci (1202) is subject to the following ideal conditions:

a) Start with a pair of neonatal rabbits.
b) Maturation period is one month.
c) One month before pregnancy.
d) Imitate a new born couple.
e) Repeating the intimacy, and so on.
f) No rabbit dies.

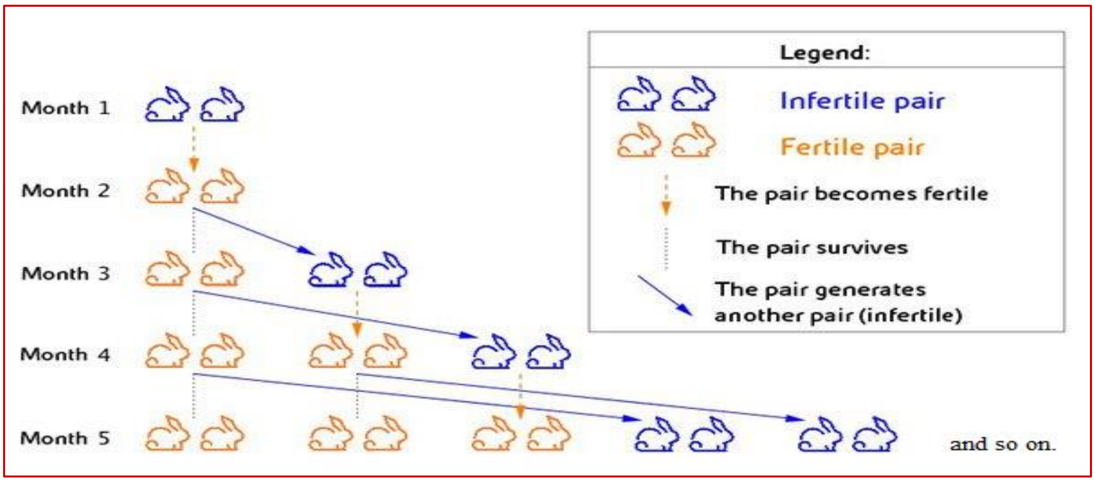

Figure 1.1: Breeding pattern in rabbit Experiment.

Month	*Youth Pairs*	*Matured Pairs*	*Total*
January	1	0	1
February	0	1	1
March	1	1	2
April	1	2	3
May	2	3	5
June	3	5	8
July	5	8	13
August	8	13	21
September	13	21	34
October	21	34	55
November	34	55	89
December	55	89	144
January	89	144	233

Table 1.1: Rabbit problem and Fibonacci numbers.

In the outcome of this experiment, Leonardo found that the rabbit reproduction pattern conforms to a sequence,

1, 2, 3, 5, 8, 13, 21, 34, 55, 89…

This sequence is known as the Fibonacci sequence. In this sequence, every successive term is the sum of the preceding two terms and is generally represented by the recursive relation given by

$$\mathcal{F}_n = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ \mathcal{F}_{n-1} + \mathcal{F}_{n-2}, & n \geq 2, n \in N \end{cases} \tag{1.1}$$

Equivalently, the Fibonacci sequence ($\mathcal{F}_n$) is represented as:

n	0	1	2	3	4	5	6	7	…
$\mathcal{F}_n$	0	1	1	2	5	8	13	21	…

Table 1.2: Fibonacci numbers.

In 1634, A. Gerard arrived at the following recurrence relation for the sequence:

$$\mathcal{U}_{n+2} = \mathcal{U}_{n+1} + \mathcal{U}_n \, , n \geq 1, \tag{1.2}$$

with $\mathcal{U}_1 = 1, \mathcal{U}_2 = 1$.

R. Simpson in 1753, derived a formula implied by Kepler

$$\mathcal{U}_{n+1}\mathcal{U}_{n-1} - \mathcal{U}_n{}^2 = (-1)^{n-1}. \tag{1.3}$$

It was during the period 1878–1891 that Edward Lucas, who dominated the field of recursive series, first attributed Fibonacci's name to the sequence given by (1.1), and since then, it has been called the Fibonacci sequence.

The higher-order Fibonacci numbers are found with the help of Binet's formula. Bernoulli (1724) provided the n^{th} Fibonacci number in Binet's form as:

$$\mathcal{F}_n = \frac{1}{\sqrt{5}}(\, a^n - \, b^n), \tag{1.4}$$

where a and b satisfy the equation

$$x^2 - x - 1 = 0. \tag{1.5}$$

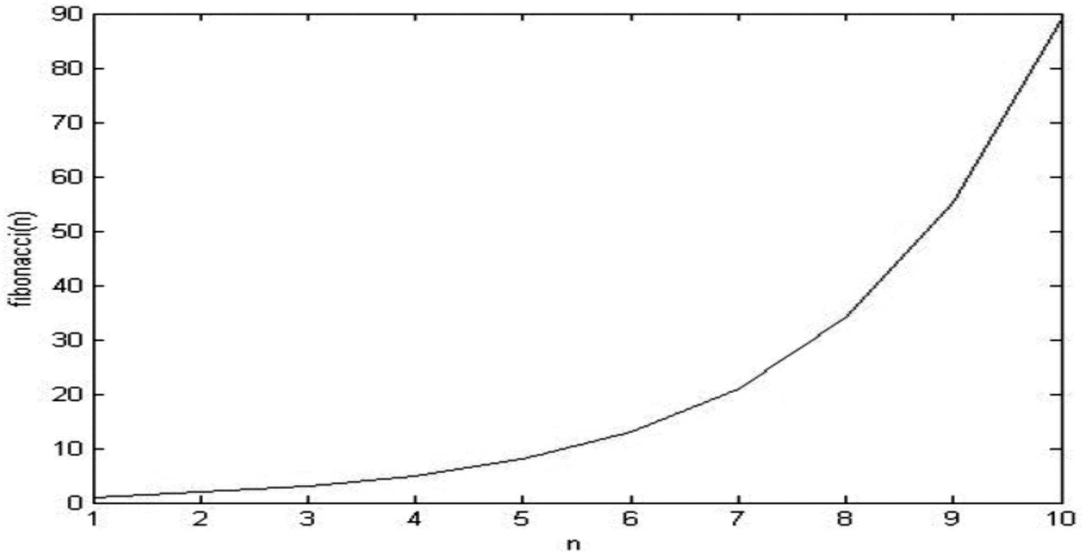

Figure 1.2: Graph of Fibonacci numbers.

Furthermore, the Fibonacci spiral aptly describes the Fibonacci numbers as under:

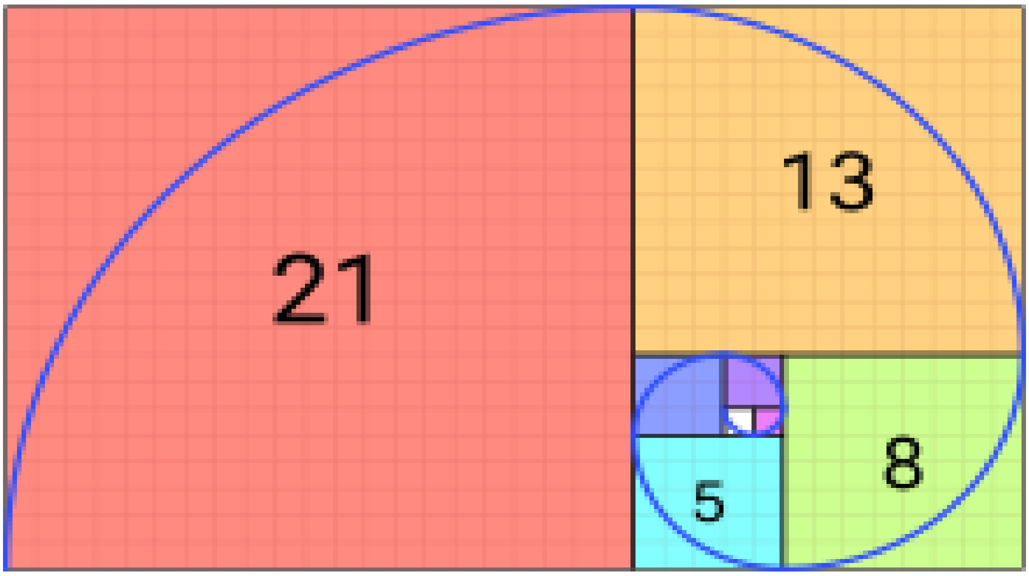

Figure 1.3: Fibonacci Spiral.

Fibonacci numbers have many uses in many different fields of study and are useful in everyday life and the natural world in addition to mathematics. The patella of several blooms generates the Fibonacci number sequence. Lilies, for instance, have three petals; buttercups, five; delphiniums, daisies, and asters, respectively, eight, thirteen, and twenty-one. Additionally, while counting flowers in a clockwise or anticlockwise orientation, some of them display spirals that follow Fibonacci numbers.

Figure 1.4: Flowers with Fibonacci numbers.

Music and Fibonacci numbers are also inextricably linked. Looking at the keyboard of a piano outlines the Fibonacci numbers. [6]. These numbers can also be spotted in pineapples. [7].

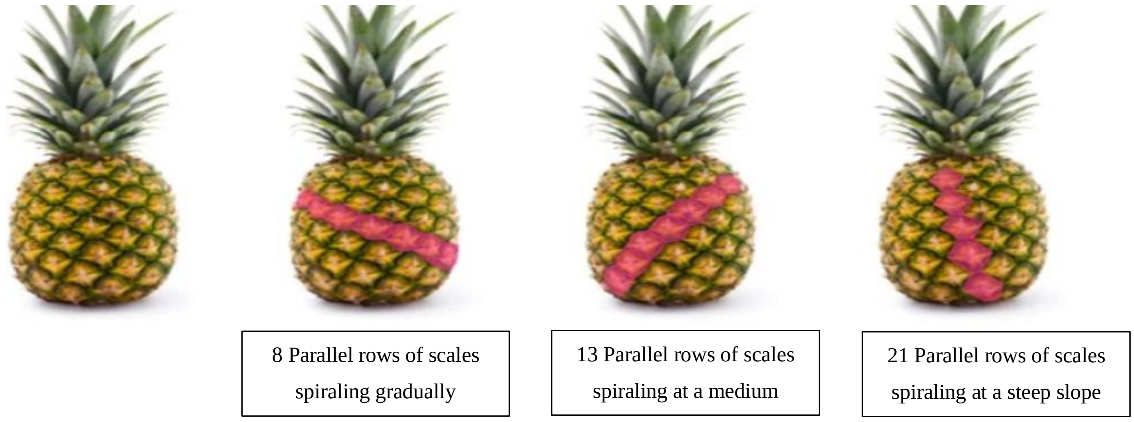

Figure 1.5: Pineapples with Fibonacci numbers.

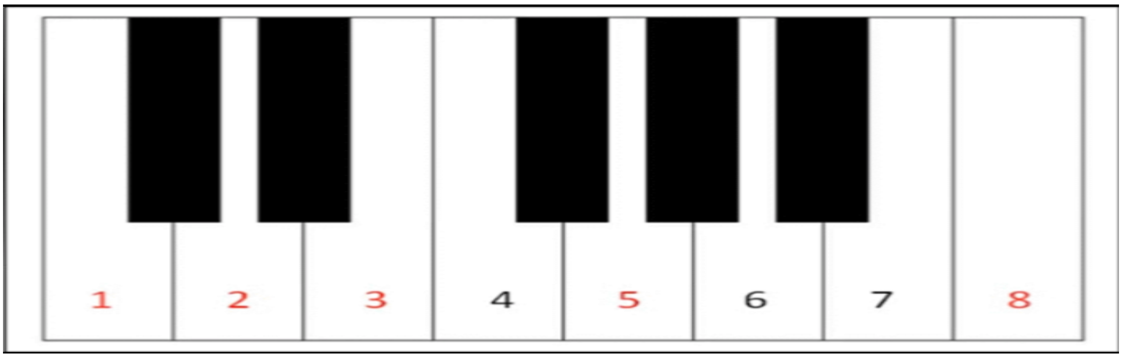

Figure 1.6: Fibonacci numbers on the keyboard of the Piano.

Fibonacci numbers plays a significant role in the life cycle of plants and animals, in bee family trees, in tree growth points, and in various fields that cannot be described in a few words.

1.2 Basic Terminologies and Preliminaries

We employ some fundamental concepts in order to achieve our objective which are discussed as under:

1.2.1 The Golden ratio

The golden ratio is termed as the ratio of the length of the largest portion (L) to the smallest portion (S) being equal to the ratio between the total length and the length of the largest portion of the line segment i.e.

$$\frac{L}{S} = \frac{L+S}{L}. \tag{1.6}$$

For finding the numerical value of the golden ratio, put $\frac{L}{S} = x$ which reduces (1.6) to

$$x^2 - x - 1 = 0.$$

The positive roots of this equation give the "golden ratio," or "golden proportion," or "the golden mean," which is generally denoted by $\emptyset$ and numerically equal to $\emptyset = \frac{1+\sqrt{5}}{2} = 1.616803 \ldots.$

1.2.2 Fibonacci numbers with negative index

The sequence of Fibonacci numbers ($\mathcal{F}_n$) is extended to the negative value of the index n, where n being positive integer, by Abramovich [8] through a relation as follows:

$$\mathcal{F}_{-n} = (-1)^{n-1}\mathcal{F}_n, \tag{1.7}$$

or

$$\mathcal{F}_{n+1} = \mathcal{F}_n + \mathcal{F}_{n-1}, \tag{1.8}$$

with $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = \mathcal{F}_{-1} = 1$.

The extended Fibonacci numbers to negative index are represented by the table as under:

$\boldsymbol{n}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{\mathcal{F}_n}$	0	1	1	2	3	5	8	13	21	34	55
$\boldsymbol{\mathcal{F}_{-n}}$	0	1	-1	2	-3	5	-8	-13	-21	34	-55

Table 1.3. Fibonacci numbers with negative index.

1.2.3 Fibonacci Polynomial

Fibonacci sequence in one of its generalizations extends to polynomials known as Fibonacci polynomials. E. C. Catalan, a Belgian mathematician, and E. Jacobsthal, a German mathematician, studied the Fibonacci polynomials in1883. Catalan defined the Fibonacci polynomials recursively as

$$\mathcal{F}_{\alpha+2}(x) = x\,\mathcal{F}_{\alpha+1}(x) + \mathcal{F}_{\alpha}(x)\,, \tag{1.9}$$

with $\mathcal{F}_1(x) = 1$ and $\mathcal{F}_2(x) = x$ for every integer $\alpha \geq 3$. Also, $\mathcal{F}_{\alpha}(1) = \mathcal{F}_{\alpha}$, α^{th} Fibonacci number.

According to Jacobsthal, Fibonacci polynomials are given by the recursive relation

$$G_{\alpha}(x) = G_{\alpha-1}(x) + x\, G_{\alpha-2}(x), \qquad (1.10)$$

with $G_1(x) = 1 = G_2(x)$, for every integer $\alpha \geq 3$..

Koshy [41], in his book, defines a polynomial sequence called the Fibonacci polynomial, given by

$$\mathcal{F}_{\alpha}(x) = \begin{cases} 0 & \alpha = 0 \\ 1 & \alpha = 1 \\ x\mathcal{F}_{\alpha-1}(x) + \mathcal{F}_{\alpha-2}(x), & \alpha \geq 2, \alpha \in N. \end{cases} \qquad (1.11)$$

The graphical representation is as under:

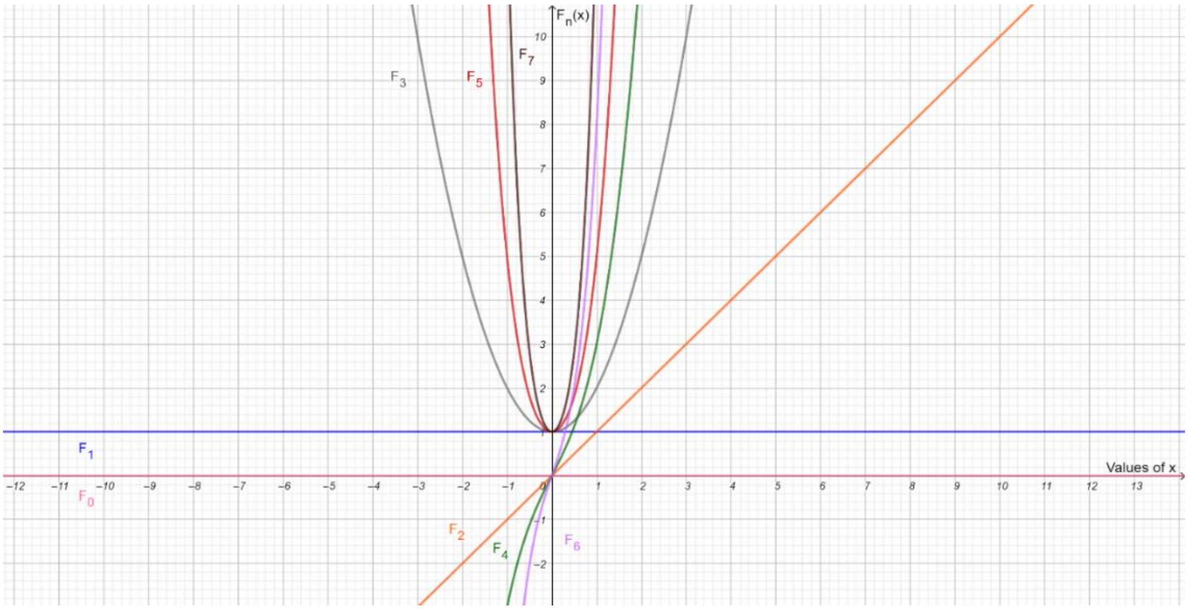

Figure 1.7: Graph of Fibonacci Polynomial.

The sequence of Fibonacci polynomials with negative indices is given by

$$\mathcal{F}_{-\alpha}(x) = (-1)^{\alpha+1}\mathcal{F}_{\alpha}(x), \qquad \alpha \in N, \alpha \geq 1. \qquad (1.12)$$

Some of the useful properties and identities satisfied by the Fibonacci polynomials are:

a) The generating function ($\mathcal{F}(x,t)$) is given by

$$\mathcal{F}(x,t) = \frac{1}{1 - t^2 - tx}. \qquad (1.13)$$

b) The α^{th} Fibonacci polynomials are obtained by the formula

$$\mathcal{F}_{\alpha}(x) = \frac{(c^{\alpha} - d^{\alpha})}{c - d}, \qquad (1.14)$$

where $c = \frac{x+\sqrt{x^2+4}}{2}$, $d = \frac{x-\sqrt{x^2+4}}{2}$ satisfies the equation $t^2 - tx - 1 = 0$.

c) The Fibonacci polynomials are represented by an explicit formula

$$\mathcal{F}_\alpha(x) = \sum_{t=0}^{\left[\frac{\alpha-1}{2}\right]} \binom{\alpha-t-1}{t} (x)^{\alpha-2t-1}. \tag{1.15}$$

d) The Fibonacci polynomials satisfies the relation

$$\mathcal{F}_\alpha(-x) = (-1)^{\alpha+1}\, \mathcal{F}_\alpha(x), \qquad \forall\ \alpha \geq 1. \tag{1.16}$$

1.2.4 Lucas number

The Lucas numbers [13], named after F. E. A. Lucas, a French mathematician, follows a recursive relation similar to that of Fibonacci numbers but differ only in its initial terms. The sequence

$$2, 1, 3, 4, 7, 11, 18, 29 \ldots,$$

represented recursively as

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2}, n \geq 2 \tag{1.17}$$

with $\mathcal{L}_0 = 2$ and $\mathcal{L}_1 = 1$ is called Lucas's sequence, and its terms are called as Lucas numbers.

The higher order Lucas numbers are obtained by using Binet's formula. The Binet's form of n^{th} Lucas number were given by Euler (1726) as:

$$\mathcal{L}_n = a^n + b^n, \tag{1.18}$$

where *a* and *b* satisfy

$$x^2 - x - 1 = 0.$$

The graphical representation is as under:

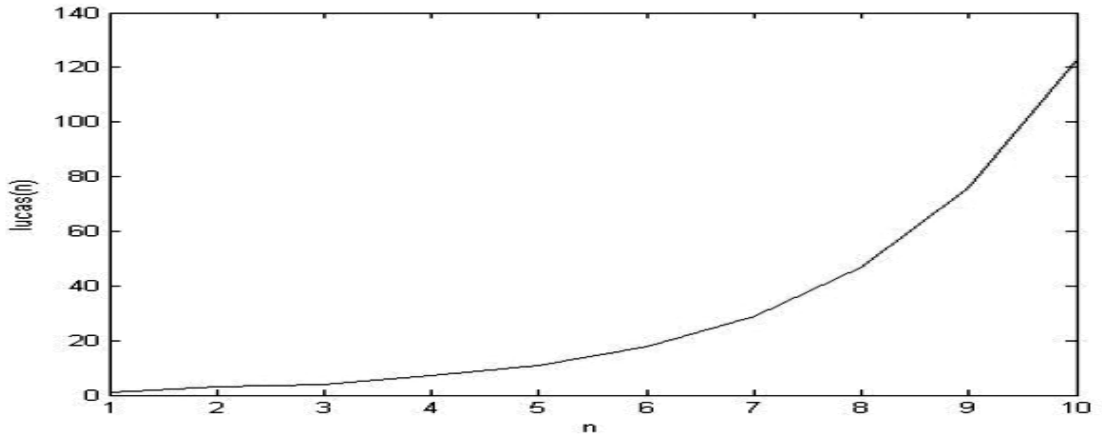

Figure 1.8: Lucas numbers.

Similarly, as the Fibonacci numbers are represented by Fibonacci spiral, Lucas numbers are also well depicted by the Lucas spiral as below

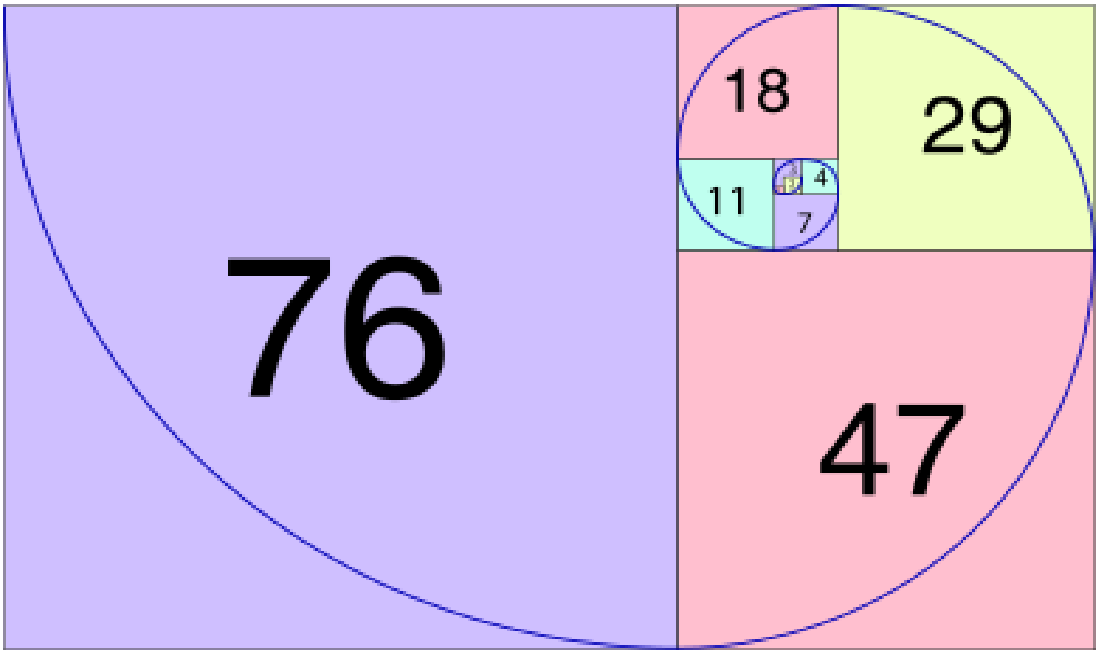

Figure 1.9: Lucas Spiral

1.2.5 Lucas numbers with negative index

Analogous to the Fibonacci sequence, the Lucas sequence with a negative index is given by the following relations:

$$\mathcal{L}_{-n} = (-1)^n \mathcal{L}_n, \tag{1.19}$$

or

$$\mathcal{L}_{n+1} = \mathcal{L}_n + \mathcal{L}_{n-1}, \tag{1.20}$$

where $\mathcal{L}_{-1} = -1, \ \ \mathcal{L}_0 = 2$, & $\ \mathcal{L}_1 = 1$.

A few terms of extended Lucas numbers are as follows:

$\boldsymbol{n}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{\mathcal{L}_n}$	2	1	3	4	7	11	18	29	47	76	123
$\boldsymbol{\mathcal{L}_{-n}}$	2	-1	3	-4	7	-11	18	-29	47	-76	123

Table 1.4: Lucas numbers with negative indices.

1.2.6 Lucas polynomial

Similar to Fibonacci sequence, the Lucas sequence [10] is also extended to polynomials called Lucas polynomials. Lucas polynomials and Fibonacci polynomials are strongly connected because they have the same recursive relation and differs only through their initial conditions. Bicknell (1970) studied the Lucas polynomials, which are defined by

$$\mathcal{L}_\alpha(x) = \begin{cases} 2 & \alpha = 0, \\ x & \alpha = 1, \\ x\mathcal{L}_{\alpha-1}(x) + \mathcal{L}_{\alpha-2}(x) & \alpha \geq 2\,, \alpha \in N\,. \end{cases} \tag{1.21}$$

Furthermore, $\mathcal{L}_\alpha(1) = \mathcal{L}_\alpha$, Lucas number.

The graphical representation is as under;

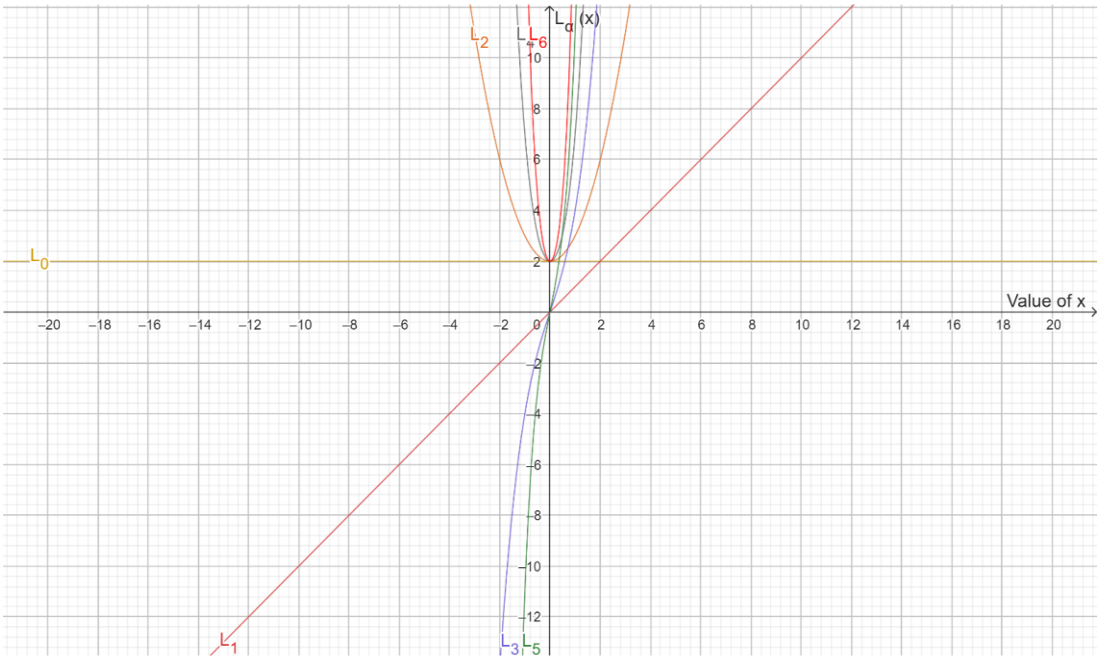

Figure 1.10: Graph of Lucas polynomials.

The sequence of Lucas polynomials can be extended to the set of integers by using the relation

$$\mathcal{L}_{-\alpha}(x) = (-1)^\alpha \mathcal{L}_\alpha(x), \;\; \alpha \in N \;, \alpha \geq 1 \tag{1.22}$$

For any integer $\alpha \geq 1$, some of the useful properties and identities satisfied by the Lucas polynomials are

i). The generating function for Lucas polynomials is

$$\mathcal{L}(x,t) = \frac{2 - xt}{1 - t^2 - tx} \tag{1.23}$$

ii). The α^{th} Lucas polynomials are obtained by the formula

$$\mathcal{L}_{\alpha}(x) = (\,c^{\alpha} + \,d^{\alpha}), \tag{1.24}$$

where $c = \frac{x+\sqrt{x^2+4}}{2}$ and $d = \frac{x-\sqrt{x^2+4}}{2}$ satisfies $t^2 - tx - 1 = 0$.

iii). The Lucas polynomials are represented by an explicit formula

$$\mathcal{L}_{\alpha}(x) \;=\; \sum_{\gamma=0}^{\left[\frac{\alpha}{2}\right]} \frac{\alpha}{\alpha-\gamma}\binom{\alpha-\gamma}{\gamma}(x)^{\alpha-2\gamma}. \tag{1.25}$$

iv). The Lucas polynomials satisfy the identity

$$\mathcal{L}_{\alpha}(-x) = (-1)^{\alpha}\,\mathcal{L}_{\alpha}(x), \quad \alpha \in N, \alpha \geq 1. \tag{1.26}$$

1.2.7 Fibonacci numbers, Lucas numbers, and Golden ratio

The ratio of two consecutive Fibonacci numbers such that the subsequent is divided by the preceding generates a sequence which approaches to ϕ , the golden ratio. A similar, argument holds for Lucas numbers too. Thus,

$$\left.\begin{aligned} \lim_{n\to\infty}\frac{\mathcal{F}_{n+1}}{\mathcal{F}_n} = \phi \\ \lim_{n\to\infty}\frac{\mathcal{L}_{n+1}}{\mathcal{L}_n} = \phi \end{aligned}\right\} \tag{1.27}$$

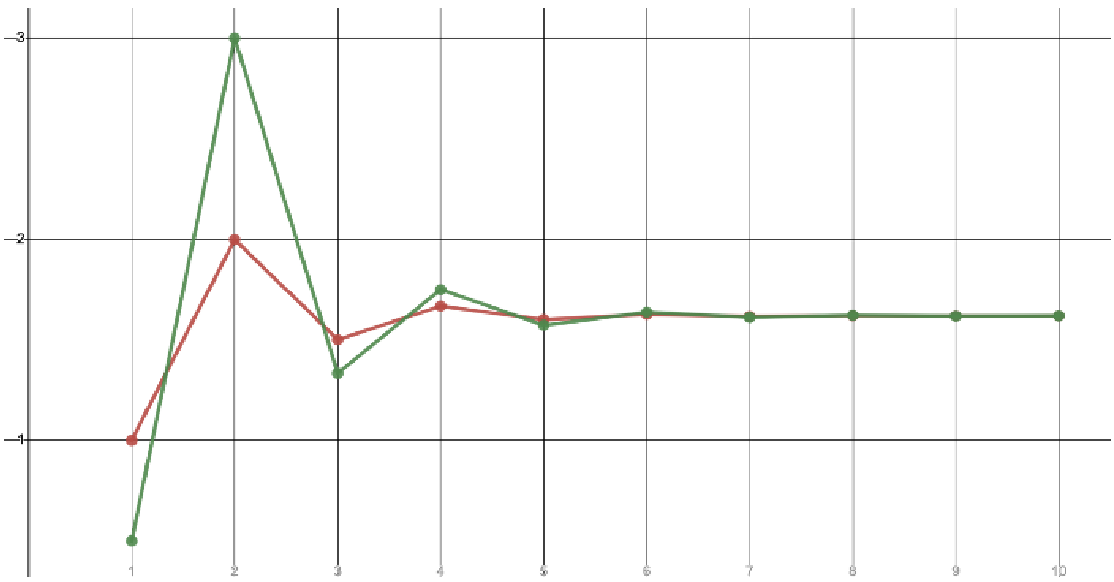

Figure 1.11:(The Fibonacci numbers (Red) and Lucas numbers (Green) has their ratios converge to Golden ratio).

1.2.8 Complex Fibonacci numbers

The Complex Fibonacci numbers [8] are characterized by the relation

$$\mathcal{F}^*{}_{\omega} = \begin{cases} i, & \omega = 0 \\ 1 + i, & \omega = 1 \\ \mathcal{F}^*{}_{\omega-1} + \mathcal{F}^*{}_{\omega-2}\,, & \omega \ \geq \ 2\,, \omega \in Z \end{cases} \tag{1.28}$$

and satisfies the relation

$$\mathcal{F}^*{}_{\omega} = \mathcal{F}_{\omega} + i\,\mathcal{F}_{\omega+1} \tag{1.29}$$

where $i^2 = -1$.

1.2.9 Pell Numbers

Pell numbers [9], derived by John Pell, are given recursively as

$$\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2};\ \forall\, n \geq 2, \tag{1.30}$$

with $\mathcal{P}_0 = 0, \mathcal{P}_1 = 1$.Thus, Pell numbers are the sum of twice of its previous term and the term that precedes it. Pell numbers can be generated by the following formula:

$$\mathcal{P}_n = \frac{\mathscr{f}^n - \mathscr{g}^n}{2\sqrt{2}}, \tag{1.31}$$

where $\mathscr{f}, \mathscr{g}$ satisfies

$$x^2 - 2x - 1 = 0.$$

The graphical representation of Pell numbers is

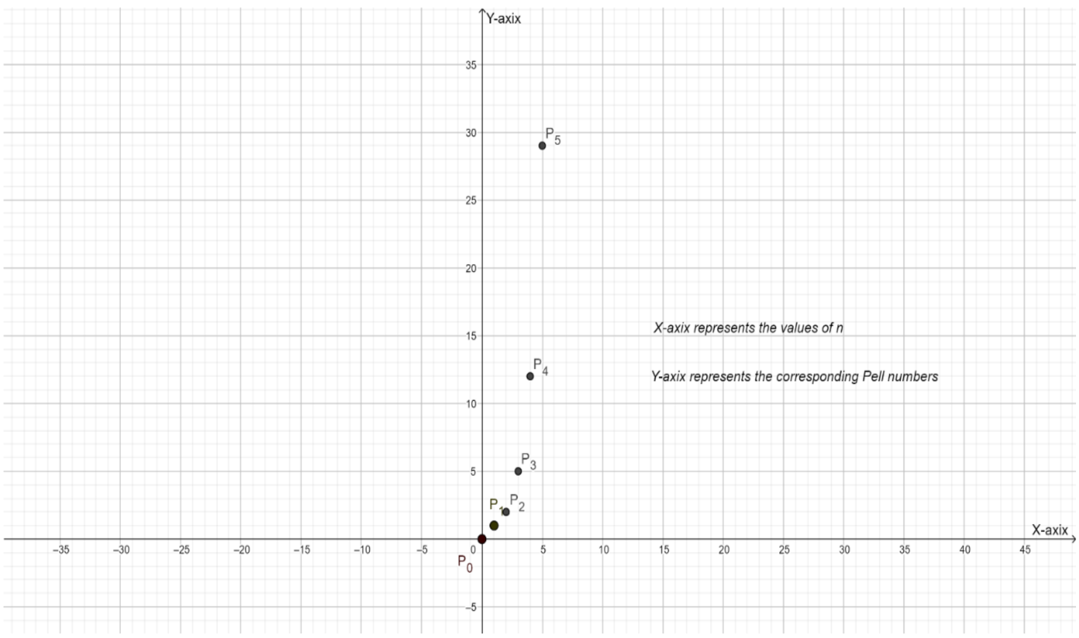

Figure 1.12: Graph of Pell numbers.

1.2.10 Pell Polynomials

Pell polynomials [60], studied by A.F. Horadam (1985), are represented recursively as

$$\mathcal{P}_\nu(x) = \begin{cases} 0 & \nu = 0 \\ 1 & \nu = 1 \\ 2x\mathcal{P}_{\nu-1}(x) + \mathcal{P}_{\nu-2}(x) & \nu \geq 2, \nu \in N. \end{cases} \quad (1.32)$$

Also, $\mathcal{P}_\nu\left(\frac{1}{2}\right) = \mathcal{F}_\nu$ and $\mathcal{P}_\nu(1) = \mathcal{P}_\nu$.

The ν^{th} term of the Pell polynomials is obtained by the formula

$$\mathcal{P}_\nu(x) = \frac{a^\nu - b^\nu}{a - b}, \quad (1.33)$$

where $a = \frac{x+\sqrt{x^2+4}}{2}$, and $b = \frac{x-\sqrt{x^2+4}}{2}$, satisfies $t^2 - xt - 1 = 0$.

The graphical representation of Pell polynomials is

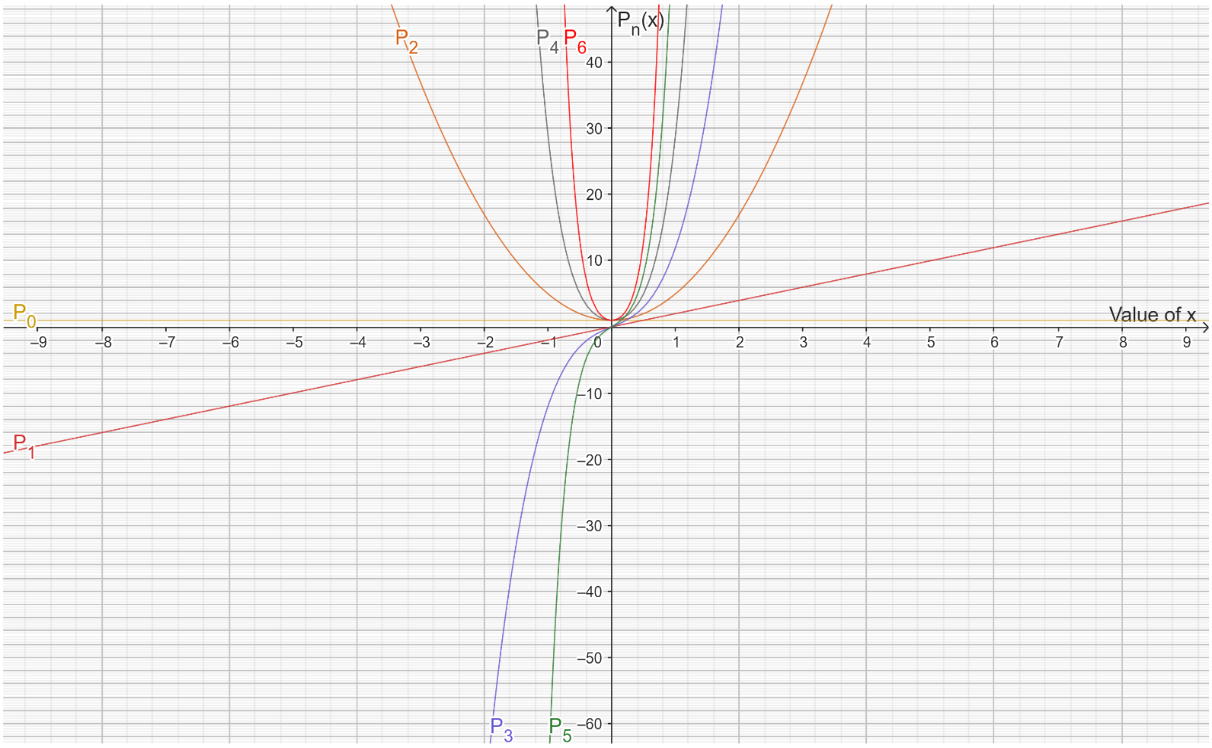

Figure 1.13: Graph of Pell polynomials.

For all integers $\nu \geq 1$, some of the useful properties and identities satisfied by the Pell polynomials are

i) The Pell polynomials are generated by

$$\mathcal{P}(x,t) = \frac{t}{1 - t^2 - 2tx}. \quad (1.34)$$

ii) The Pell polynomials are represented by an explicit formula

$$\mathcal{P}_\nu(x) = \sum_{\gamma=0}^{\left[\frac{\nu-1}{2}\right]} \binom{\nu-\gamma-1}{\gamma} (x)^{\nu-2\gamma-1}. \tag{1.35}$$

iii) The Pell polynomials satisfies

$$\mathcal{P}_\nu(-x) = (-1)^{\nu+1}\,\mathcal{P}_\nu(x). \tag{1.36}$$

1.2.11 Chebyshev polynomials

Chebyshev polynomials were first studied by P. L. Chebyshev (1821-94), a Russian mathematician. In studying the numerical solutions of differential equations, classical orthogonal polynomials are frequently used. Chebyshev polynomials are increasingly used in numerical analysis. Four kinds of Chebyshev polynomials are isolated out of which a wide range of research work is done on the 1st & 2nd kinds of Chebyshev polynomials whereas very little work has been carried out on the 3rd & 4th kinds of Chebyshev polynomials offering a dynamic field for the prospective researchers. These Chebyshev polynomials find application in approximation theory. In this subsection, the existence of Chebyshev polynomials and some of their key characteristics will be discussed [2, 11- 12]. Chebyshev polynomials are solutions of the Chebyshev differential equations [12] which occurs as a special case of the Strum-Liouville problems [52], which we will discuss below:

(i) Chebyshev polynomial of first kind

The solutions of the Chebyshev differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \alpha^2 y = 0,\text{, for } |x| < 1, \text{and } \alpha \in N. \tag{1.37}$$

represented by the polynomials

$$\mathcal{T}_\alpha\ (x) = \cos\alpha\theta\,, \tag{1.38}$$

where $x = \cos\theta$ for all integers $\alpha \geq 0, x \in [-1,1]$ and $\theta \in [0,\pi]$, are called Chebyshev polynomials of first kind.

Furthermore, the application of De Moivre's theorem allows the representation of these polynomials by the recurrence relation as follows:

$$\mathcal{T}_\alpha\ (x) = \begin{cases} 1 & \alpha = 0, \\ x & \alpha = 1, \\ 2x\mathcal{T}_{\alpha-1}\ (x) - \mathcal{T}_{\alpha-2}\ (x) & \alpha \geq 2, \alpha \in N. \end{cases} \tag{1.39}$$

The generating function $G_{\mathcal{T}}(t)$ is

$$\sum_{\alpha=0}^{\infty} \mathcal{T}_{\alpha}\,(x)\,t^{\alpha} = G_{\mathcal{T}}(t) = \frac{1 - xt}{1 - 2xt + t^2}. \tag{1.40}$$

The α^{th} Chebyshev polynomial of first kind is given by

$$\mathcal{T}_{\alpha}\,(x) = \frac{1}{2}[a^{\alpha} + b^{\alpha}] \tag{1.41}$$

where a, b satisfies

$$\lambda^2 - 2x\lambda + 1 = 0\,.$$

It follows the explicit formula

$$\mathcal{T}_{\alpha}\,(x) = \sum_{\ell=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor} \binom{\alpha}{2\ell} x^{\alpha-2\ell}(x^2 - 1)^{\ell} \tag{1.42}$$

Further, for any integer $\alpha, \beta \geq 0$,

$$\int_{-1}^{1} \frac{\mathcal{T}_{\alpha}\,(x)\mathcal{T}_{\beta}\,(x)}{\sqrt{1 - x^2}}\,dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{\pi}{2}, & \alpha = \beta \neq 0 \\ \pi, & \alpha = \beta = 0. \end{cases} \tag{1.43}$$

The graphical representation of these polynomials is

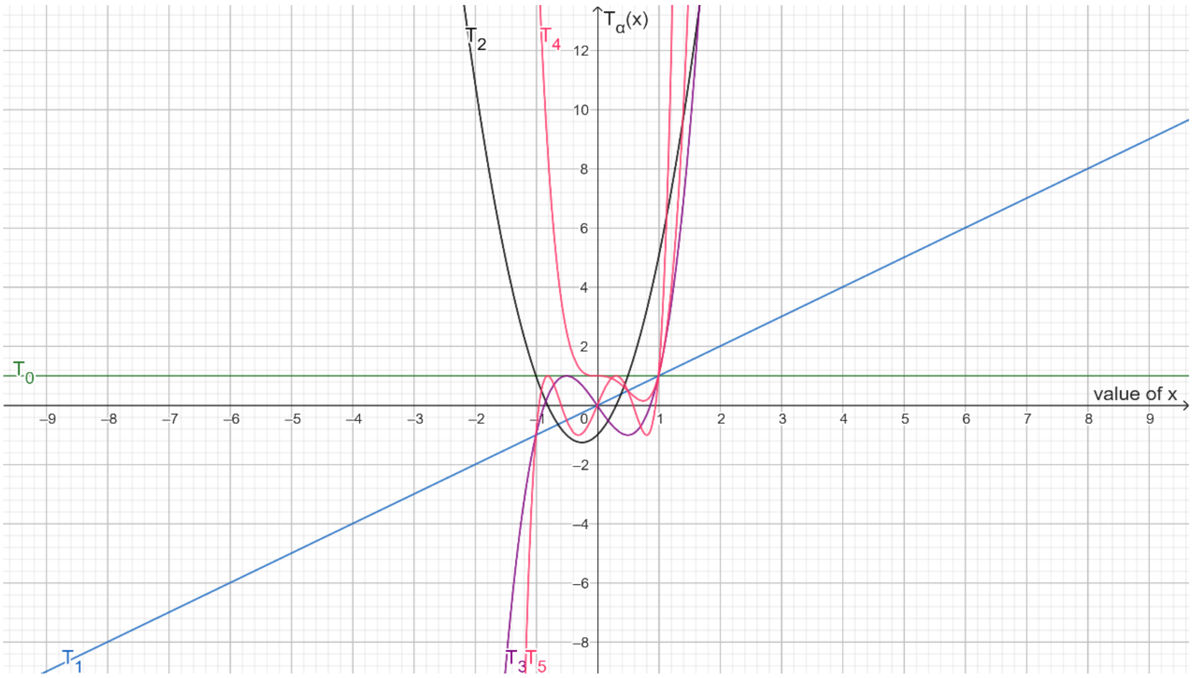

Figure 1.14: Graph of Chebyshev polynomials of first kind ($\alpha = 1$ to $\alpha = 5$)

(ii) Chebyshev polynomials of the second kind

The solutions of the Chebyshev differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + \alpha(\alpha+2)y = 0 \qquad (1.44)$$

represented by the polynomials

$$\mathcal{U}_\alpha\,(x) = \frac{sin(\alpha+1)\theta}{sin\theta} \qquad (1.45)$$

where $x = \cos\theta$, for all integers $\alpha \geq 0, x \in [-1,1]$and $\theta \in [0,\pi]$ are called Chebyshev polynomial of second kind.

Furthermore, the application of De Moivre's theorem allows the representation of these polynomials by the recurrence relation as follows:

$$\mathcal{U}_\alpha\,(x) = \begin{cases} 1 & \alpha = 0, \\ 2x & \alpha = 1, \\ 2x\mathcal{U}_{\alpha-1}\,(x) - \mathcal{U}_{\alpha-2}\,(x) & \alpha \geq 2, \alpha \in N. \end{cases} \qquad (1.46)$$

The generating function $G_{\mathcal{U}}(t)$ is given by

$$\sum_{\alpha=0}^{\infty} \mathcal{U}_\alpha\,(x)\,t^\alpha = G_{\mathcal{U}}(t) = \frac{1}{1-2xt+t^2}. \qquad (1.47)$$

The α^{th} term of this sequence of polynomials $\{\mathcal{U}_\alpha\,(x)\}$ is given by

$$\mathcal{U}_\alpha\,(x) = \frac{\mathfrak{a}^{\alpha+1} + \mathfrak{b}^{\alpha+1}}{\mathfrak{a} - \mathfrak{b}}, \qquad (1.48)$$

where $\mathfrak{a}, \mathfrak{b}$ satisfies

$$\lambda^2 - 2x\lambda + 1 = 0\,.$$

The explicit formula is

$$\mathcal{U}_\alpha\,(x) = \sum_{\ell=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor} \binom{\alpha+1}{2\ell+1} x^{\alpha-2\ell}(x^2-1)^\ell. \qquad (1.49)$$

Further, for any integer $\alpha, \beta \geq 0$,

(Orthogonality Property)

$$\int_{-1}^{1} \mathcal{U}_\alpha\,(x)\mathcal{U}_\beta\,(x)\sqrt{1-x^2}\,dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{\pi}{2}, & \alpha = \beta. \end{cases} \qquad (1.50)$$

The graphical representation is as under:

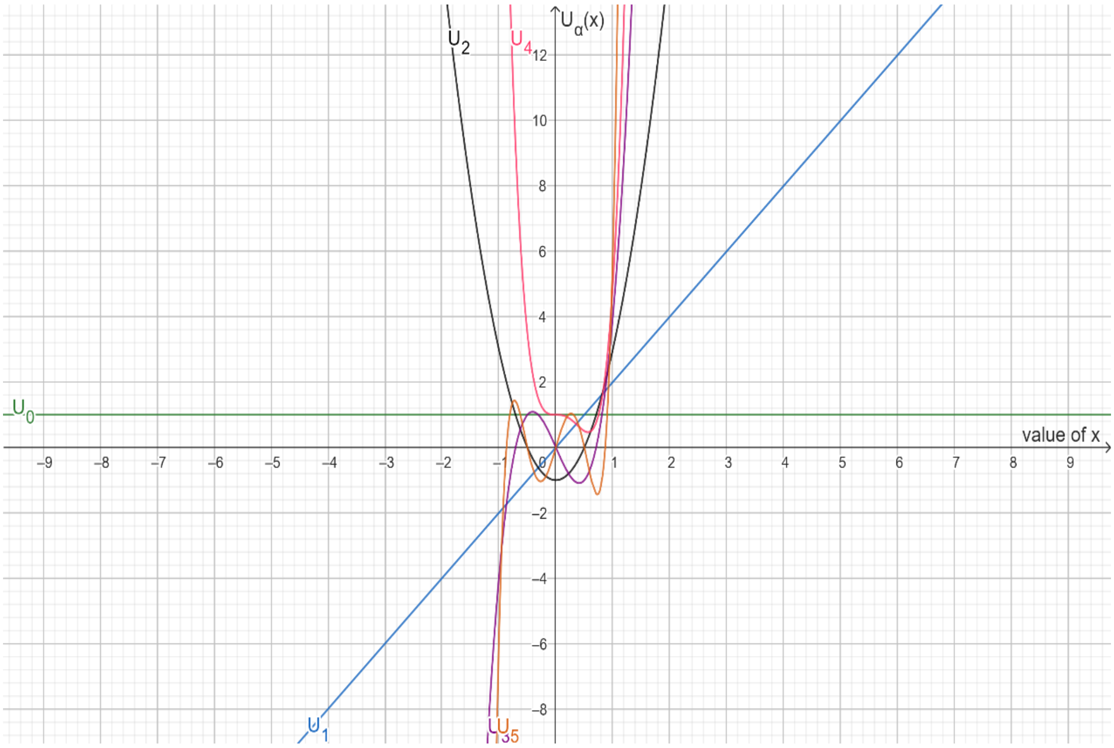

Figure 1.15: Graph of Chebyshev polynomials of second kind ($\alpha = 1$ to $\alpha = 5$)

(iii) Chebyshev polynomials of the third kind

The solutions of the Chebyshev differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} + (1 - 2x)\frac{dy}{dx} + \alpha(\alpha + 1)y = 0,, \text{for } |x| < 1, \text{and } \alpha \in N. \quad (1.51)$$

represented by the polynomials

$$\mathcal{V}_\alpha\,(x) = \frac{cos\left(\alpha + \frac{1}{2}\right)\theta}{cos\,\left(\frac{\theta}{2}\right)}, \quad (1.52)$$

where $x = \cos\theta$, for all integers $\alpha \geq 0, x \in [-1,1]$ and $\theta \in [0, \pi]$ are called Chebyshev polynomial of third kind.

As a consequence of De Moivre's theorem, the above polynomials ($\mathcal{V}_\alpha(x)$) can be represented by

$$\mathcal{V}_\alpha\,(x) = \begin{cases} 1 & \alpha = 0, \\ 2x - 1 & \alpha = 1, \\ 2x\mathcal{V}_{\alpha-1}\,(x) - \mathcal{V}_{\alpha-2}\,(x), & \alpha \geq 2, \alpha \in N. \end{cases} \quad (1.53)$$

The generating function $G_{\mathcal{V}}(t)$ is given by

$$\sum_{\alpha=0}^{\infty} \mathcal{V}_{\alpha}\,(x)\, t^{\alpha} = G_{\mathcal{V}}(t) = \frac{1-t}{1-2xt+t^2}. \tag{1.54}$$

The α^{th} term of the sequence of Chebyshev polynomials of third kind $\{\mathcal{V}_{\alpha}\,(x)\}$ is given by

$$\mathcal{V}_{\alpha}\,(x) = \frac{1}{2^{\alpha}}\left[\frac{f^{2\alpha+1}+g^{2\alpha+1}}{f+g}\right], \tag{1.55}$$

where f, g satisfies

$$\lambda^2 - 2x\lambda + 1 = 0\,.$$

It follows the explicit formula

$$\mathcal{V}_{\alpha}\,(x) = \sum_{\gamma=0}^{\alpha} \frac{(-1)^{\gamma}}{2^{\alpha}}\binom{2\alpha+1}{2\gamma}(1+x)^{\alpha-\gamma}(1-x)^{\gamma}. \tag{1.56}$$

For any integer $\alpha, p \geq 0$,

(Orthogonality Property)

$$\int_{-1}^{1} \mathcal{V}_{\alpha}\,(x)\mathcal{V}_{p}\,(x)\sqrt{\frac{1+x}{1-x}}\,dx = \begin{cases} 0, & \alpha \neq p \\ \pi, & \alpha = p\,. \end{cases} \tag{1.57}$$

The graphical representation is as follows:

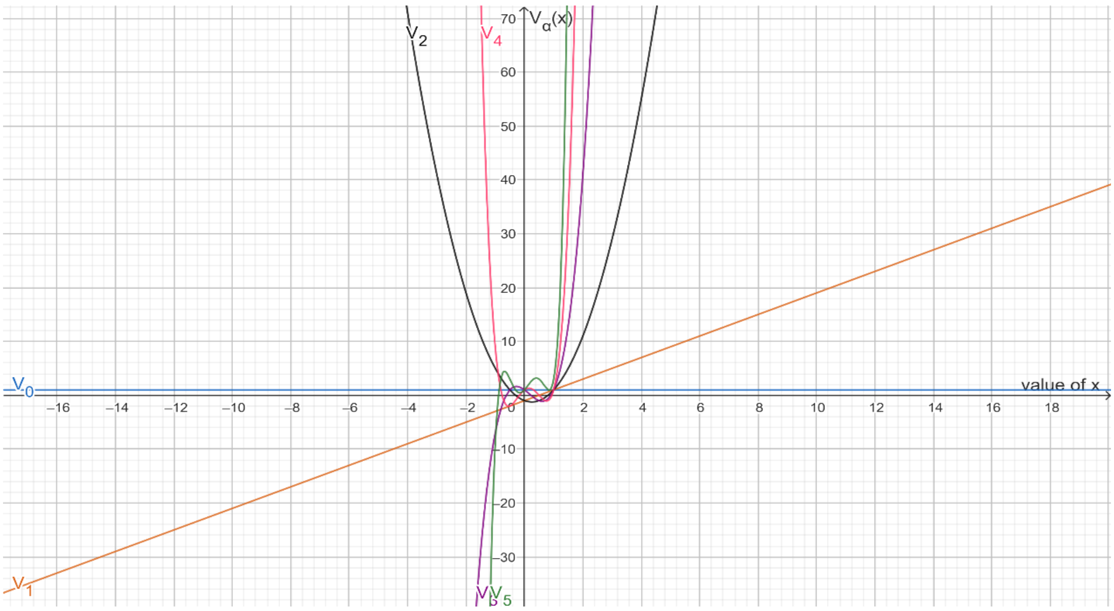

Figure 1.16: Graph of Chebyshev polynomials of third kind ($\alpha = 1$ to $\alpha = 5$)

(iv) Chebyshev polynomials of the fourth kind

The solutions of the Chebyshev differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-(1+2x)\frac{dy}{dx}+\alpha(\alpha+1)y=0, \qquad |x|<1, \alpha \in N, \qquad (1.58)$$

represented by the polynomials

$$\mathcal{W}_\alpha\ (\mathrm{x})=\frac{sin\left(\alpha+\frac{1}{2}\right)\theta}{sin\left(\frac{\theta}{2}\right)}, \qquad (1.59)$$

where $x\ =cos\theta$, for all integers $\alpha\geq 0, x\in[-1,1]$and $\theta\in[0,\pi]$ are called Chebyshev polynomials of fourth kind.

As a consequence of De Moivre's theorem, the above polynomials $(\mathcal{W}_\alpha(x))$ can be represented as

$$\mathcal{W}_\alpha\ (x)=\begin{cases} 1 & \alpha=0, \\ 2x+1 & \alpha=1, \\ 2x\mathcal{W}_{\alpha-1}\ (x)-\mathcal{W}_{\alpha-2}\ (x) & \alpha\geq 2, \alpha\in N. \end{cases} \qquad (1.60)$$

The generating function $G_{\mathcal{W}}(t)$ is

$$\sum_{\alpha=0}^{\infty}\mathcal{W}_\alpha\ (x)\,t^\alpha=G_{\mathcal{W}}(t)=\frac{1+t}{1-2xt+t^2}. \qquad (1.61)$$

The α^{th} term of the sequence of Chebyshev polynomials of third kind $\{\mathcal{W}_\alpha\ (x)\}$ is given by

$$\mathcal{W}_\alpha\ (x)=\frac{1}{2^\alpha}\left[\frac{f^{2\alpha+1}-g^{2\alpha+1}}{f-g}\right], \qquad (1.62)$$

where f, g satisfies

$$\lambda^2-2x\lambda+1=0\,.$$

It follows the explicit formula

$$\mathcal{W}_\alpha\ (x)=\sum_{\gamma=0}^{\alpha}\frac{1}{2^\alpha}\binom{2\alpha+1}{\gamma}(1+x)^{\alpha-\gamma}(x-1)^\gamma. \qquad (1.63)$$

For any integer $\alpha, \beta \geq 0$,

(Orthogonality Property)

$$\int_{-1}^{1} \mathcal{W}_{\alpha}\,(x)\mathcal{W}_{\beta}\,(x)\sqrt{\frac{1-x}{1+x}}\,dx = \begin{cases} 0, & \alpha \neq \beta \\ \pi, & \alpha = \beta. \end{cases} \qquad (1.64)$$

The graphical representation is as under**:**

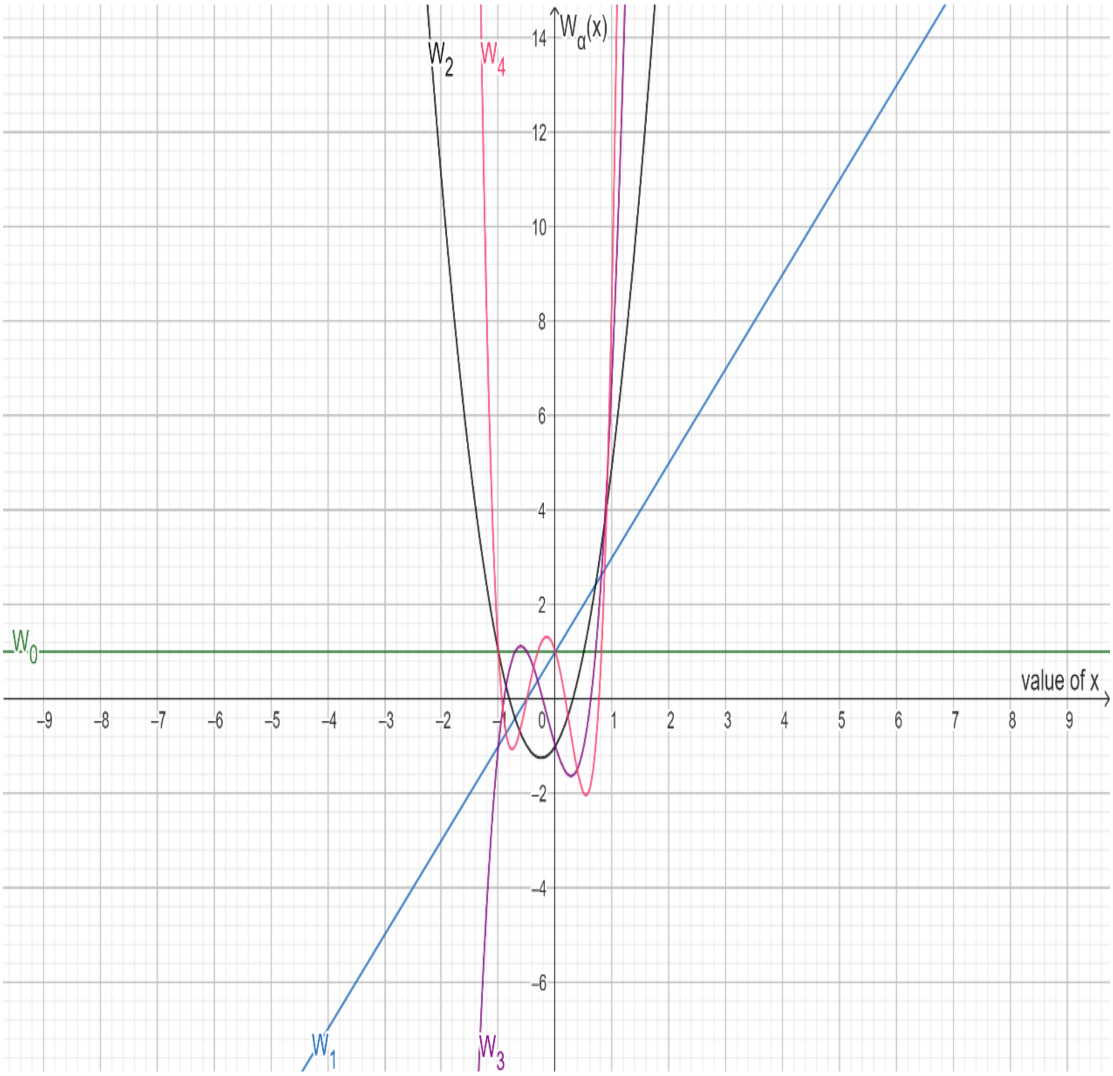

Figure 1.17: Graph of Chebyshev polynomials of fourth kind($\boldsymbol{\alpha = 1}$ **to** $\boldsymbol{\alpha = 4}$)

Some of the important identities connecting these Chebyshev polynomials which are going to be useful in the development of the subsequent results are enumerated as under:

For every integer $\kappa \geq 0$, the Chebyshev polynomials satisfies the following identities:

$$\left.\begin{array}{ll}
i) & 2\,\mathcal{T}_{\kappa}\,(x) = \mathcal{U}_{\kappa}\,(x) - \mathcal{U}_{\kappa-2}\,(x) \\
ii) & \mathcal{V}_{\kappa}\,(x) = \mathcal{U}_{\kappa}\,(x) - \mathcal{U}_{\kappa-1}\,(x) \\
iii) & \mathcal{W}_{\kappa}\,(x) = \mathcal{U}_{\kappa}\,(x) + \mathcal{U}_{\kappa-1}\,(x) \\
iv) & \mathcal{T}_{2\kappa+1}\left(\sqrt{\dfrac{1+x}{2}}\right) = \sqrt{\dfrac{1+x}{2}}\,\mathcal{V}_{\kappa}\,(x) \\
v) & \mathcal{W}_{\kappa}\,(x) = \mathcal{U}_{2\kappa}\left(\sqrt{\dfrac{1+x}{2}}\right) \\
vi) & 2(1-x^2)\,\mathcal{U}_{\kappa}\,(x) = \mathcal{T}_{\kappa}\,(x) - \mathcal{T}_{\kappa+2}\,(x) \\
vii) & (1+x)\,\mathcal{V}_{\kappa}\,(x) = \mathcal{T}_{\kappa}\,(x) + \mathcal{T}_{\kappa+1}\,(x) \\
viii) & \mathcal{V}_{\kappa}\left(\dfrac{3}{2}\right) = \mathcal{F}_{2\kappa+1} \\
ix) & (1-x)\,\mathcal{W}_{\kappa}\,(x) = \mathcal{T}_{\kappa}\,(x) - \mathcal{T}_{\kappa+1}\,(x) \\
x) & \mathcal{W}_{\kappa}\left(\dfrac{3}{2}\right) = \mathcal{L}_{2\kappa+1} \\
xi) & \mathcal{V}_{\kappa}\,(x) + \mathcal{V}_{\kappa-1}\,(x) = 2\,\mathcal{T}_{\kappa}\,(x) \\
xii) & \mathcal{W}_{\kappa}\,(x) = (-1)^{\kappa}\mathcal{V}_{\kappa}\,(-x) \\
xiv) & \mathcal{W}_{\kappa}\,(x) - \mathcal{W}_{\kappa-1}\,(x) = 2\,\mathcal{T}_{\kappa}\,(x) \\
xv) & \mathcal{U}_{\kappa}\,(ix) = i^{\kappa}\mathcal{P}_{\kappa+1}\,(x)
\end{array}\right\} \tag{1.65}$$

These identities can easily be established with the help of basic definitions & fundamental properties of the Chebyshev polynomials

1.2.12 Chebyshev polynomials with negative index

The Chebyshev polynomials can be extended to the negative value of the index [53, 54] by defining the relations as follows:

For any integer $\alpha \geq 0$, and ζ,

$$\mathcal{T}_{-\alpha}\,(\zeta) = \mathcal{T}_{\alpha}\,(\zeta) \tag{1.66}$$

$$\mathcal{U}_{-\alpha}\,(\zeta) = -\mathcal{U}_{\alpha-2}\,(\zeta) \text{ with } \mathcal{U}_{-1}\,(\zeta) = 0 \tag{167}$$

$$\mathcal{V}_{-\alpha}\,(\zeta) = \mathcal{V}_{\alpha-1}\,(\zeta) \tag{1.68}$$

$$\mathcal{W}_{-\alpha}\,(\zeta) = -\mathcal{W}_{\alpha-1}\,(\zeta) \tag{1.69}$$

1.2.13 Vieta-Fibonacci and Vieta-Pell polynomials

A.F. Horadam [36] studied the **Vieta-Fibonacci polynomials** $(S_n(x))$, which are defined recursively by

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x), \tag{1.70}$$

with initial conditions $S_0(x) = 0,\ S_1(x) = 1$ and $n \geq 2$.

Tascı and Yalcın [37] studied **Vieta-Pell polynomials** $(R_n(x))$**,,** which are defined recursively by

$$R_n(x) = 2xR_{n-1}(x) - R_{n-2}(x), \tag{1.71}$$

with initial conditions $R_0(x) = 0$, and $R_1(x) = 1$.

Few of the values of these polynomials are:

N	**Vieta-Fibonacci Polynomials (**$S_n(x)$**)**	**Vieta-Pell Polynomials (**$R_n(x)$**)**
0	0	0
1	1	1
2	x	$2x$
3	$x^2 - 1$	$4x^2 - 1$
4	$x^3 - 2x$	$8x^3 - 4x$
5	$x^4 - 3x^2 + 1$	$16x^4 - 12x^2 + 1$
6	$x^5 - 4x^3 + 3x$	$32x^5 - 32x^3 + 6x$

Table 1.5: **Vieta-Fibonacci and Vieta-Pell polynomials**

The graphical representation of Vieta-Fibonacci and Vieta-Pell Polynomials is as follows:

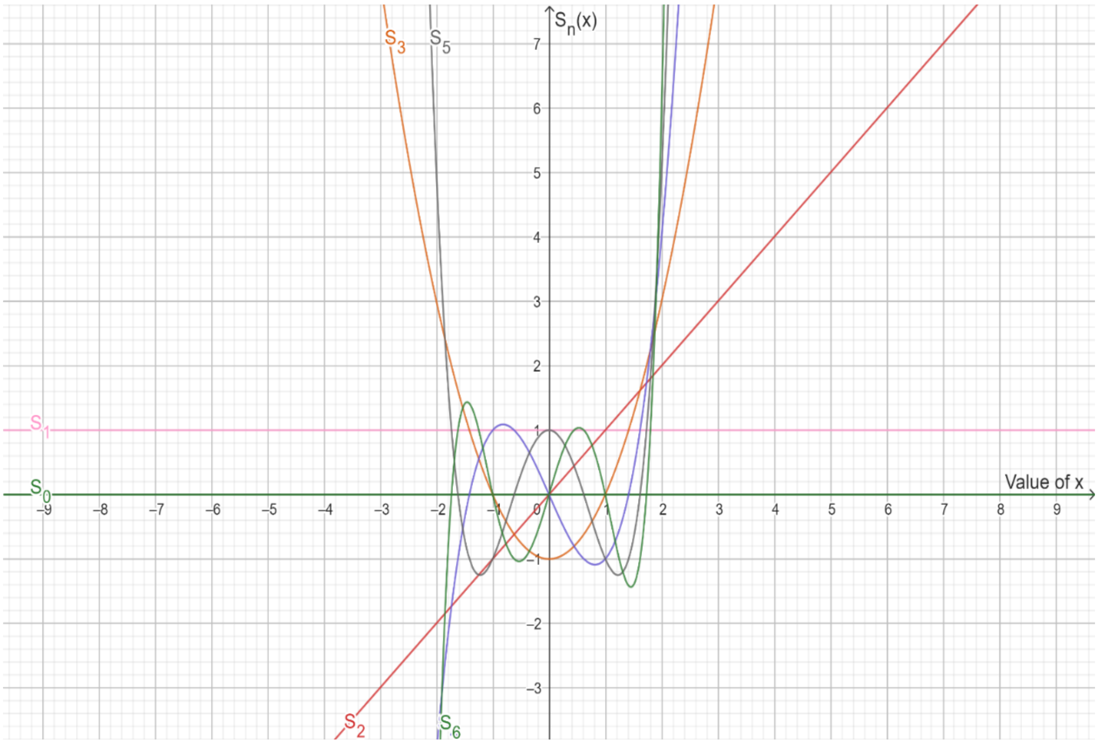

Figure 1.18: Graphical representation of Vieta-Fibonacci polynomials

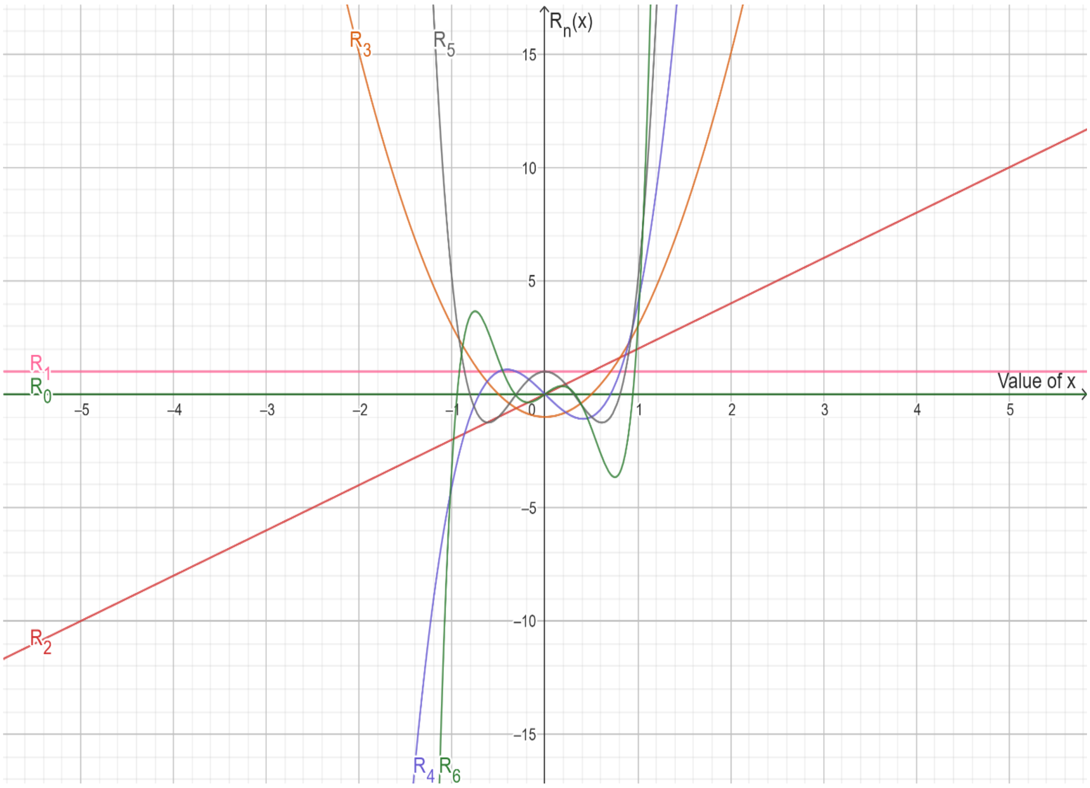

Figure 1.19 Graphical representation of Vieta-Pell polynomials

1.2.14 Jacobi polynomials

The **Jacobi polynomials** ($\mathcal{P}(n:\lambda,\beta)(x)$) [12] are solutions of the Jacobi equation:

$$(1-x^2)\frac{d^2y}{dx^2} - [\beta-\lambda-(\lambda+\beta+2)x]\frac{dy}{dx} + n(n+\lambda+\beta+1)y = 0$$

for $|x| < 1$ and $n \in N$, satisfies the recurrence relation

$$\begin{aligned} &2(n+1)(\lambda+\beta+n+1)(\lambda+\beta+2n)\mathcal{P}(n+1:\lambda,\beta)(x) \\ &= (\lambda+\beta+2n+1)[(\lambda^2-\beta^2)+(\lambda+\beta+2n)(\lambda+\beta+2n+2)x]\mathcal{P}(n:\lambda,\beta)(x) \\ &- 2(\lambda+n)(\beta+n)(\lambda+\beta+2n+2)\mathcal{P}(n \\ &- 1:\lambda,\beta)(x), \end{aligned} \tag{1.72}$$

with initial conditions

$$\mathcal{P}(0:\lambda,\beta)(x) = 1, \qquad \mathcal{P}(1:\lambda,\beta)(x) = \frac{1}{2}[\lambda-\beta+(\lambda+\beta+2)x]$$

1.2.15 Gegenbauer polynomials

The **Gegenbauer polynomials** ($C(\nu:\lambda)(x)$) [12] are given by the Jacobi equation:

$$(1-x^2)\frac{d^2y}{dx^2} - (2\lambda+1)x\frac{dy}{dx} + \nu(\nu+2\lambda)y = 0$$

for $|x| < 1$ and $\nu \in N$, satisfies the recurrence relation

$$C(\nu:\lambda)(x) = \frac{1}{\nu}[2x(\nu+\lambda-1)C(\nu-1:\lambda)(x) - (\nu+2\lambda-2)C(\nu-2:\lambda)(x)], \tag{1.73}$$

with initial conditions

$$C(0:\lambda)(x) = 1, C(1:\lambda)(x) = 2\lambda x.$$

1.2.16 Tribonacci Sequence

Fibonacci sequence in one of its generalisations extends to a sequence called Tribonacci sequence [14]. The sequence

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \ldots$$

where each successive term is a sum of the preceding three terms is called Tribonacci sequence. This sequence is represented by the recursive relation

$$t_\omega = \begin{cases} 0, & \omega = 0, \\ 1, & \omega = 1, \\ 1, & \omega = 2, \\ t_{\omega-1} + t_{\omega-2} + t_{\omega-3}, & \omega \geq 3, \omega \in N \end{cases} \tag{1.74}$$

The graphical representation is:

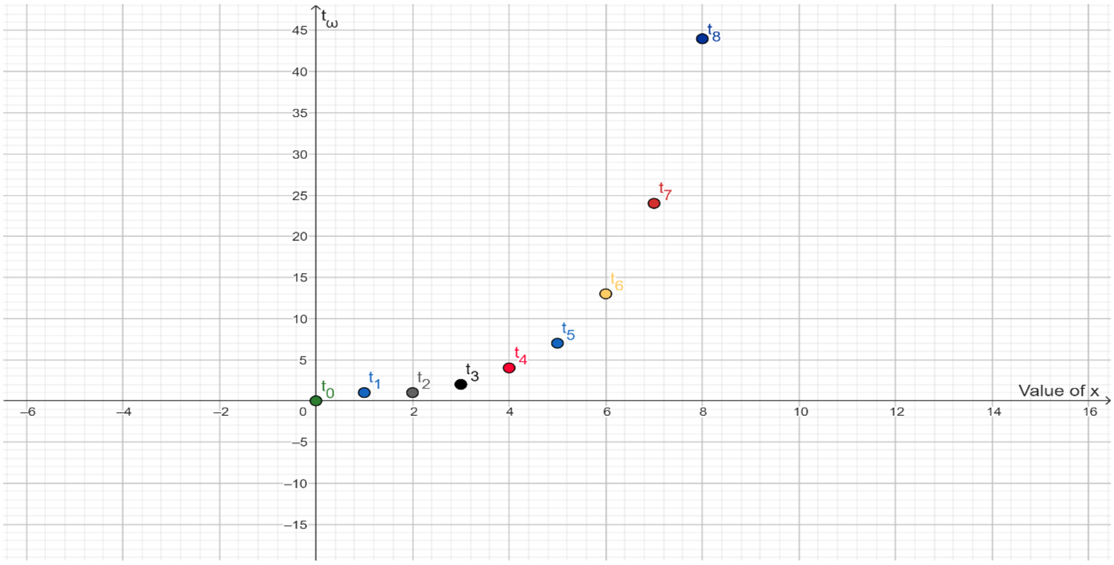

Figure 1.20 Graphical representation of Tribonacci numbers

1.2.17 Tribonacci Polynomials

Hoggatt and Bicknell [15] defined the Tribonacci polynomials in 1973 by the following recursive relation:

$$t_\upsilon(x) = \begin{cases} 1, & \upsilon = 0, \\ 1, & \upsilon = 1, \\ x^2, & \upsilon = 2, \\ x^2 t_{\upsilon-1}(x) - x t_{\upsilon-2}(x) + t_{\upsilon-3}(x), & \upsilon \geq 3, \upsilon \in N. \end{cases} \quad (1.75)$$

Few of the values of these polynomials are:

Value of v	**Tribonacci Polynomials ($t_\upsilon(x)$)**
0	0
1	1
2	x^2
3	$x^4 + x$
4	$x^6 + 2x^3 + 1$
5	$x^8 + 3x^5 + 3x^2$
… and so on	*… and so on*

Table 1.6: Tribonacci polynomials ($t_\upsilon(x)$).

The graphical representation is

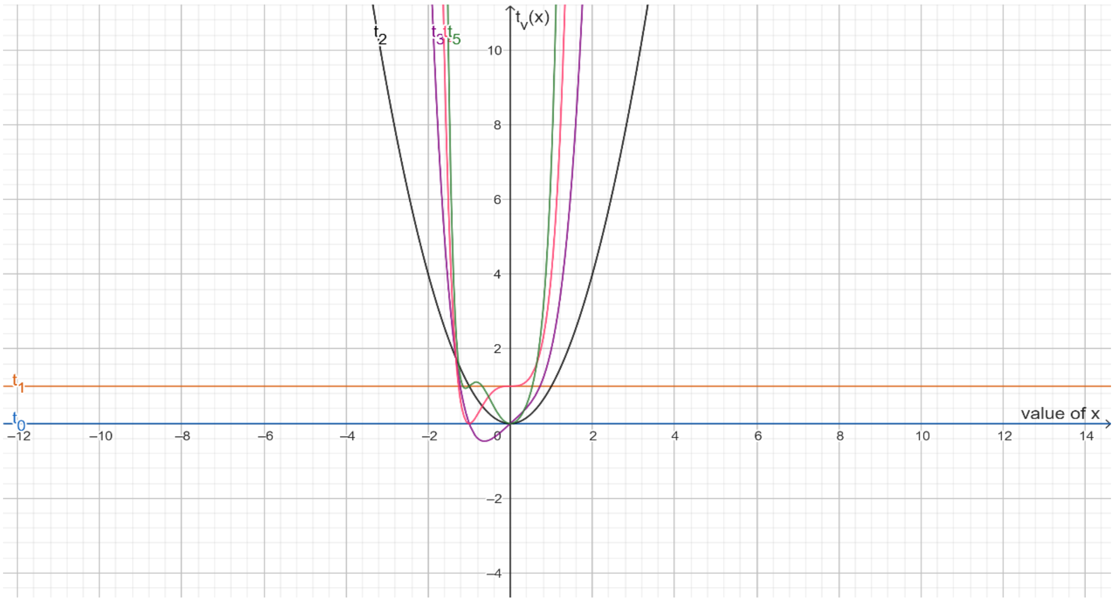

Figure 1.21 Graphical representation of Tribonacci polynomials.

1.2.18 Trivariate Fibonacci polynomials

Let f', ℓ', w' be the given variables. Trivariate Fibonacci [23] polynomials $\mathcal{H}_\alpha(g', \ell', w'), \alpha \in N$ is an extension of Fibonacci polynomials and follows a third-order recursive relation given by

$$\mathcal{H}_\alpha(f', \ell', w') = f'\mathcal{H}_{\alpha-1}(f', \ell', w') + \ell'\mathcal{H}_{\alpha-2}(f', \ell', w') + w'\mathcal{H}_{\alpha-3}(f', \ell', w'), \alpha > 2 \quad (1.76)$$

with $\mathcal{H}_0(f', \ell', w') = 0,\ \mathcal{H}_1(f', \ell', w') = 1,\ \mathcal{H}_2(f', \ell', w') = f'$.

α	**Trivariate Fibonacci Polynomials** $(\mathcal{H}_\alpha(f', \ell', w'))$
0	0
1	1
2	f'
3	$f'^2 + \ell'$
4	$f'^3 + 2f'\ell' + w'$
5	$f'^4 + 3f'^2\ell' + 2f'w' + \ell'^2$
6	$f'^5 + 4f'^3\ell' + 3f'\ell'^2 + 3f'^2w' + 2\ell'w'$

Table 1.7: Trivariate Fibonacci polynomials

The Trivariate Fibonacci Polynomials sequences, by taking different values of f', ℓ', w', take different forms viz. for $f' = 1, \ell' = 1$, and $w' = 0$, $\mathcal{H}_\alpha(f', \ell', w') = t_\alpha$, Tribonacci number, and for $f' = x^2, \ell' = x, \ell' = 1, \mathcal{H}_\alpha(f', \ell', w') = t_\alpha\ (x)$.

1.2.19 Trivariate Lucas polynomials

For any variable quantities f', ℓ', w' and for integer $\alpha \geq 3$, Trivariate Lucas [23] polynomials $\mathcal{L}_\alpha(f', \ell', w')$ is an extension of Lucas polynomials and follows a third-order recursive relation given by

$$\mathcal{L}_\alpha(f', \ell', w') = f'\mathcal{L}_{\alpha-1}(f', \ell', w') + \ell'\mathcal{L}_{\alpha-2}(f', \ell', w') + w'\mathcal{L}_{\alpha-3}(f', \ell', w'), \alpha > 2, \qquad (1.77)$$

with

$$\mathcal{L}_0(f', \ell', w') = 3, \mathcal{L}_1(f', \ell', w') = f', \mathcal{L}_2(f', \ell', w') = f'^2 + 2\ell'.$$

α	**Trivariate Lucas Polynomials**
0	3
1	f'
2	$f'^2 + 2\ell'$
3	$f'^3 + 3f'\ell' + 3w'$
4	$f'^4 + 4f'^2\ell' + 4f'w' + 2\ell'^2$
5	$f'^5 + 5f'^3\ell' + 5f'\ell'^2 + 5f'^2w' + 5\ell'w'$
6	$f'^6 + 6f'^4\ell' + 9f'^2\ell'^2 + 6f'^3w' + 12f'\ell'w' + 2\ell'^3 + 3w'^2$

Table 1.8: Trivariate Lucas polynomials

1.3 Literature Review

The literature on the Fibonacci sequence is vast, as numerous applications of this sequence have been deciphered in different aspects of life, including nature, astronomy, art, and architecture, thereby inspiring many research scholars and mathematicians.

The Vorobyov Brothers, Alfred [17], and Hogatt V.E [18], have given wide spectrum of intriguing properties of the Lucas and Fibonacci numbers. These numbers have been related to almost every kind of number.

Nobel Laureate, the famous physicist Aston [73], has shown the occurrence of the Fibonacci numbers in the atomic world.

Read [19] applied the Fibonacci series to determine how far the moons of Saturn, Uranus, and Jupiter were from their respective axes. He has shown that a particular moon's position is dependent upon the position of previous two moons closer to the primary. Also, the moon seems to reside and, in the case of Jupiter, even congregate at potential levels predicted by the Fibonacci series.

These Fibonacci sequences have been generalized in different way

1. Altering the recurrence relation while keeping the initial terms preserved.
2. Altering the initial term & maintaining the recurrence relations.
3. Modifying the recurrence relation so that each term is the sum of the preceding terms.
4. Others modify recurrence relation so that each term is the sum of four preceding terms.

The sequence

$$\{\mathcal{P}_n\} = 0, 1, 2, 5, 12, \ldots,$$

where $\mathcal{P}_0 = 0$, $\mathcal{P}_1 = 1$ and $\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2}$, $n \geq 2$ is called Pell sequence.

The associated Pell's sequence is defined by

$$J_n = 2J_{n-1} + J_{n-2}, n \geq 2, \text{ with } J_0 = 2 = J_1 .$$

In [20], Horadam replaced the first two Fibonacci numbers by arbitrary integers and defined the sequence $\{g_n\}$

$$g_n = g_{n-1} + g_{n-2}, n \geq 2,$$

where g_0 and g_1 are arbitrary integers.

Waddil and Sacks [21] has considered the sequence $\{K_n\}$ where K_0 , K_1 , and K_2 are arbitrary algebraic integers with

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, n \geq 3$$

In 2007, Falcon and Plaza [22] defined the k-Fibonacci numbers. For every real $k > 0$, the sequence of k-Fibonacci numbers $(\mathcal{F}_{k,\alpha})$ is characterized recursively as

$$\mathcal{F}_{k,\alpha+1} = k\mathcal{F}_{k,\alpha} + \mathcal{F}_{k,\alpha-1},$$

for $\alpha \in N$ with $\mathcal{F}_{k,0} = 0$, $\mathcal{F}_{k,1} = 1$.

If $k = 1$, k-Fibonacci sequence becomes classical Fibonacci sequence and if $k = 2$, it becomes Pell sequence.

Also, in 2017, Elif Tan [23] generalized the Horadam sequence defined by

$$w_n = pw_{n-1} - qw_{n-2}\,, n \geq 2,$$

with w_0, w_1, p, q being arbitrary integers, to a bi-periodic Horadam sequence (w_n) defined by

$$w_n = \begin{cases} bw_{n-1} + w_{n-2} & if\ n\ is\ even, \\ aw_{n-1} + w_{n-2} & if\ n\ is\ odd, \end{cases} \quad , \qquad n \geq 2,$$

with w_0, w_1, a, b are arbitrary non-zero real numbers and obtained various fundamental properties of bi-periodic Horadam Sequence which generalizes the well-established results on bi-periodic Lucas and Fibonacci sequence.

The bi-periodic sequences play an important role in characterizing Fibonacci Octonions and the Lucas Octonions.

In 1963, A.F. Horadam [24] expressed the n^{th} Fibonacci Quaternion and Lucas Quaternion as

$$\mathcal{Q}_n = \mathcal{F}_n + i\mathcal{F}_{n+1} + j\mathcal{F}_{n+2} + k\mathcal{F}_{n+3},$$

$$\mathcal{T}_n = \mathcal{L}_n + i\mathcal{L}_{n+1} + j\mathcal{L}_{n+2} + k\mathcal{L}_{n+3},$$

where $\mathcal{F}_n = n^{th}$ Fibonacci number & $\mathcal{L}_n = n^{th}$ Lucas number & i, j, k obeys the relations

$$jk = i = -kj, \qquad ij = k = -ji, \qquad ki = j = -ik, \quad i^2 = -1 = j^2 = k^2 .$$

In 1969, Muthu Lakshmi R. Iyer [25, 26] derived several relations between Fibonacci Quaternions and Lucas Quaternions and their relation with Fibonacci numbers and Lucas numbers like

$$\mathcal{Q}_n\mathcal{L}_n + \mathcal{T}_n\mathcal{F}_n = 2\mathcal{Q}_{2n}.$$

$$\mathcal{Q}_n\mathcal{L}_n - \mathcal{T}_n\mathcal{F}_n = 2(-1)^n\,\mathcal{Q}_0.$$

$$\mathrm{Q_n} + \mathrm{T_n} = 2\,\mathrm{Q_{n+1}}.$$

$$\mathcal{T}_n - \mathcal{Q}_n = 2\mathcal{Q}_{n-1}.$$

In 2009, Edson and Yayenie [27], for every non-zero reals a and b, defined the sequence of bi-periodic Fibonacci numbers $\{q_n\}$ by the recursive relation

$$q_0 = 0, q_1 = 1, q_n = \begin{cases} aq_{n-1} + q_{n-2} & if\ n\ is\ even \\ bq_{n-1} + q_{n-2} & if\ n\ is\ odd \end{cases}, n \geq 2.$$

In the same line in 2014, Bilgici [28] defined the Bi-periodic Lucas sequence $\{l_n\}$ by the recursive relation

$$l_0 = 0, l_1 = 1, l_n = \begin{cases} bl_{n-1} + l_{n-2} & if\ n\ is\ even \\ al_{n-1} + l_{n-2} & if\ n\ is\ odd \end{cases}, n \geq 2.$$

In 2016, Yilmaz et al [29] using these bi-periodic Fibonacci numbers, they introduced the bi-periodic Fibonacci Octonions as

$$O_n(\mathfrak{a}, \mathfrak{b}) = \sum_{s=0}^{7} q_{n+s}\, e_s,$$

where q_n represents bi-periodic Fibonacci numbers. For negative subscripts, bi-periodic Fibonacci Octonions numbers are

$$O_{-n}(\mathfrak{a}, \mathfrak{b}) = \sum_{s=0}^{7} (-1)^{n-s-1} q_{n-s}\, e_s,$$

where $n \in N$ and derived the generating function for these Octonions as below

$$\sum_{i=0}^{n} O_n(\mathfrak{a}, \mathfrak{b}) x^n = \frac{O_0(\mathfrak{a}, \mathfrak{b}) + x(O_1(\mathfrak{a}, \mathfrak{b}) - \mathfrak{b}\, O_0(\mathfrak{a}, \mathfrak{b})) + R(x)}{1 - \mathfrak{b}x - x^2},$$

where

$$R(x) = \Big(xe_0 + e_1 + \tfrac{1}{x}e_2 + \tfrac{1}{x^2}e_3 + \tfrac{1}{x^3}e_4 + \tfrac{1}{x^4}e_5 + \tfrac{1}{x^5}e_6 + \tfrac{1}{x^6}e_7 f(x) - \Big(xe_1 + e_2 + \Big(\tfrac{1}{x} + (\mathfrak{a}\mathfrak{b} + 1)x\Big)e_3 + \Big(\tfrac{1}{x^2} + \mathfrak{a}\mathfrak{b} + 1\Big)e_4 + \Big(\tfrac{1}{x^3} + (\mathfrak{a}\mathfrak{b} + 1)\tfrac{1}{x} + (\mathfrak{a}^2\mathfrak{b}^2 + 3\mathfrak{a}\mathfrak{b} + 1)x\Big)e_5 + \Big(\tfrac{1}{x^4} + (\mathfrak{a}\mathfrak{b} + 1)\tfrac{1}{x^2} + (\mathfrak{a}^2\mathfrak{b}^2 + 3\mathfrak{a}\mathfrak{b} + 1)\Big)e_6 + \Big(\tfrac{1}{x^5} + (\mathfrak{a}\mathfrak{b} + 1)\tfrac{1}{x^3} + (\mathfrak{a}^2\mathfrak{b}^2 + 3\mathfrak{a}\mathfrak{b} + 1)\tfrac{1}{x} + (\mathfrak{a}^3\mathfrak{b}^3 + 5\mathfrak{a}^2\mathfrak{b}^2 + 6\mathfrak{a}\mathfrak{b} + 1)x\Big)e_7\Big),$$

and

$$f(x) = \frac{x - x^3}{1 - (\mathfrak{a}\mathfrak{b} + 2)x^2 + x^4}.$$

In a similar manner, in 2017, Yilmaz et al [30], using these bi-periodic Lucas numbers, defined the bi-periodic Lucas Octonions and derived their generating functions.

Several mathematicians have investigated the infinite sums of the reciprocals of wide variety of sequences like Fibonacci sequence, Lucas sequence etc. and organized Lucas and Fibonacci numbers as

Fibonacci sequence	**Lucas Sequence**
$\sum_{n=1}^{\infty}\frac{1}{\mathcal{F}_n^2}$	$\sum_{n=1}^{\infty}\frac{1}{\mathcal{L}_n^2}$
$\sum_{n=1}^{\infty}\frac{1}{\mathcal{F}_n^4}$	$\sum_{n=1}^{\infty}\frac{1}{\mathcal{L}_n^4}$
$\sum_{n=1}^{\infty}\frac{1}{\mathcal{F}_n^6}$	$\sum_{n=1}^{\infty}\frac{1}{\mathcal{L}_n^6}$

and expressed each number

$$\sum_{n=1}^{\infty}\frac{1}{\mathcal{F}_n^{2s}} \ and \ \sum_{n=1}^{\infty}\frac{1}{\mathcal{L}_n^{2s}}.$$

$s = 0,1,2,3$ ..., as a rational (respectively algebraic) function over Q. Analogous results in [31-33] were proved for Fibonacci numbers with odd (2008) and even indices (2012).

In 2009, Nakamura and Ohtsuka [34] found the infinite sums for the reciprocal of the Fibonacci numbers and their squares. Taking the floor function of these sums, the authors have obtained very interesting identities for Fibonacci numbers. The main results established by Ohtsuka and Nakamura are as follows:

For all $\mathfrak{u} \geq 1$

$$\left\lfloor\left(\sum_{\ell=\mathfrak{u}}^{\infty}\frac{1}{\mathcal{F}_\ell}\right)^{-1}\right\rfloor = \begin{cases}\mathcal{F}_{\mathfrak{u}-2}, & if\ \mathfrak{u}\ is\ even\\ \mathcal{F}_{\mathfrak{u}-2}-1, & if\ \mathfrak{u}\ is\ odd,\end{cases} \tag{1.78}$$

and

$$\left\lfloor \left(\sum_{\ell=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{F}_{\ell}^{2}} \right)^{-1} \right\rfloor = \begin{cases} -1 + \mathcal{F}_{\mathfrak{u}-1}\mathcal{F}_{\mathfrak{u}}, & if\ \mathfrak{u}\ is\ even \\ \mathcal{F}_{\mathfrak{u}-1}\mathcal{F}_{\mathfrak{u}}, & if\ \mathfrak{u}\ is\ odd, \end{cases} \tag{1.79}$$

where $\lfloor . \rfloor$ stands for the floor function.

For the generalized Fibonacci numbers given by

$$\mathcal{G}_{\mathfrak{u}+2} = a\,\mathcal{G}_{\mathfrak{u}+1} + \mathcal{G}_{\mathfrak{u}}, \mathfrak{u} > 1,$$

with $\mathcal{G}_0 = 0$, $\mathcal{G}_1 = 1$, and a being positive integer, Holliday and Komastsu [35] in 2011, proved the following results:

$$\left\lfloor \left(\sum_{\nu=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{G}_{\nu}} \right)^{-1} \right\rfloor = \begin{cases} \mathcal{G}_{\mathfrak{u}} - \mathcal{G}_{\mathfrak{u}-1}, & if\ \mathfrak{u}\ is\ even \\ \mathcal{G}_{\mathfrak{u}} - \mathcal{G}_{\mathfrak{u}-1} - 1, & if\ \mathfrak{u}\ is\ odd, \end{cases}$$

and

$$\left\lfloor \left(\sum_{\nu=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{G}_{\nu}^{2}} \right)^{-1} \right\rfloor = \begin{cases} a\mathcal{G}_{\mathfrak{u}}\mathcal{G}_{\mathfrak{u}-1} - 1, & if\ \mathfrak{u}\ is\ even \\ a\mathcal{G}_{\mathfrak{u}}\mathcal{G}_{\mathfrak{u}-1}, & if\ \mathfrak{u}\ is\ odd. \end{cases}$$

Wu and Wang [38] in 2011 investigated the similar results for the finite case (i.e., partial finite sums) and observed that

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{\mathcal{F}_{k}} \right)^{-1} \right\rfloor = \mathcal{F}_{n-2}, \quad \forall \quad n \geq 4$$

In 2015, while improving upon the observations of Ohtsuka and Nakamura [24], Wang and Wen [39], examined the case of partial sums for Fibonacci numbers and gave results as follows:

For any integer $\hbar > 2, n > 1$,

$$\left\lfloor \left(\sum_{k=n}^{n\hbar} \frac{1}{\mathcal{F}_{k}} \right)^{-1} \right\rfloor = \begin{cases} \mathcal{F}_{n-2}, & if\ n\ is\ even \\ \mathcal{F}_{n-2} - 1, & if\ n\ is\ odd. \end{cases} \tag{1.80}$$

For any integer $\hbar \geq 0, n \geq 1$,

$$\left\lfloor \left(\sum_{\gamma=n}^{n\hbar} \frac{1}{\mathcal{F}_{\gamma}^{2}} \right)^{-1} \right\rfloor = \begin{cases} \mathcal{F}_{n-1}\mathcal{F}_{n} - 1, & if\ n\ is\ even \\ \mathcal{F}_{n-1}\mathcal{F}_{n}, & if\ n\ is\ odd. \end{cases} \tag{1.81}$$

As $\hbar \to \infty$, (1.80) and (1.81) respectively becomes (1.78) and (1.79).

In 2015, Wang and Zhang [40] obtained the similar results for even and odd indexed Fibonacci numbers which are as under:

For any integers $n \geq 1, \hbar \geq 3$,

$$\left\lfloor \left(\sum_{\gamma=n}^{n\hbar} \frac{1}{\mathcal{F}_{2\gamma}} \right)^{-1} \right\rfloor = \mathcal{F}_{2n-1} - 1,$$

and for any integers $\hbar \geq 2, n \geq 1$,

$$\left\lfloor \left(\sum_{\gamma=n}^{n\hbar} \frac{1}{\mathcal{F}_{2\gamma-1}} \right)^{-1} \right\rfloor = \mathcal{F}_{2n-2} \,.$$

In one of their generalizations, the sequences of Fibonacci and Lucas numbers, extend to polynomials called Fibonacci polynomials and Lucas polynomials, respectively as discussed in section (1.2).

Numerous authors have examined several aspects of the Lucas and Fibonacci polynomials, yielding a host of intriguing results [42].

In 2012, Wu and Zhang [43] extended the results given by Ohtsuka and Nakamura [34] to the Lucas and Fibonacci polynomials and deduced the following significant conclusions:

For all integers $\zeta, \alpha > 0$,

$$\left\lfloor \left(\sum_{\lambda=\alpha}^{\infty} \frac{1}{\mathcal{F}_{\lambda}(\zeta)} \right)^{-1} \right\rfloor = \begin{cases} \mathcal{F}_{\alpha}(\zeta) - \mathcal{F}_{\alpha-1}(\zeta), & if\ \alpha\ is\ even\ with\ \alpha \geq 2 \\ \mathcal{F}_{\alpha}(\zeta) - \mathcal{F}_{\alpha-1}(\zeta) - 1, & if\ \alpha\ is\ odd\ with\ \alpha \geq 1, \end{cases}$$

$$\left\lfloor \left(\sum_{\lambda=\alpha}^{\infty} \frac{1}{{\mathcal{F}_{\lambda}}^{2}(\zeta)} \right)^{-1} \right\rfloor = \begin{cases} x\mathcal{F}_{\alpha}(\zeta) \cdot \mathcal{F}_{\alpha-1}(\zeta) - 1, & if\ \alpha\ is\ even\ with\ \alpha \geq 2 \\ \zeta\mathcal{F}_{\alpha}(\zeta) \cdot \mathcal{F}_{\alpha-1}(\zeta), & if\ \alpha\ is\ odd\ with\ \alpha \geq 1. \end{cases}$$

Similar results are obtained for Lucas polynomials.

In [70], Wu and Zhang (2013) obtained similar results as in [43] by considering the subseries of infinite sums of these polynomials and deducing the results as follows:

For any positive integer $\zeta, \mathfrak{u}$ and even $\mathfrak{a} \geq 2, \mathfrak{b} \geq 1$,

$$\left\lfloor \left(\sum_{\gamma=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{F}_{\mathfrak{a}\gamma}(\zeta)} \right)^{-1} \right\rfloor = \mathcal{F}_{\mathfrak{a}\mathfrak{u}}(\zeta) - \mathcal{F}_{\mathfrak{a}\mathfrak{u}-\mathfrak{a}}(\zeta) - 1.$$

$$\left\lfloor \left(\sum_{\gamma=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{F}_{\mathfrak{a}\gamma}^2(\zeta)} \right)^{-1} \right\rfloor = \mathcal{F}_{\mathfrak{au}}^2(\zeta) - \mathcal{F}_{\mathfrak{au}-\mathfrak{a}}^2(\zeta) - 1.$$

$$\left\lfloor \left(\sum_{\gamma=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{L}_{\mathfrak{a}\gamma}(\zeta)} \right)^{-1} \right\rfloor = \mathcal{L}_{\mathfrak{au}}(\zeta) - \mathcal{L}_{\mathfrak{au}-\mathfrak{a}}(\zeta).$$

$$\left\lfloor \left(\sum_{\gamma=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{L}_{\mathfrak{a}\gamma}^2(\zeta)} \right)^{-1} \right\rfloor = \mathcal{L}_{\mathfrak{au}}^2(\zeta) - \mathcal{L}_{\mathfrak{au}-\mathfrak{a}}^2(\zeta) + 1.$$

and

$$\left\lfloor \left(\sum_{\gamma=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{F}_{\mathfrak{b}\gamma}(\zeta)} \right)^{-1} \right\rfloor = \begin{cases} \mathcal{F}_{\mathfrak{bu}}(\zeta) - \mathcal{F}_{\mathfrak{bu}-\mathfrak{b}}(\zeta), & \textit{if } \mathfrak{u} \textit{ is even} \\ \mathcal{F}_{\mathfrak{bu}}(\zeta) - \mathcal{F}_{\mathfrak{bu}-\mathfrak{b}}(\zeta) - 1, & \textit{if } \mathfrak{u} \textit{ is odd,} \end{cases}$$

$$\left\lfloor \left(\sum_{\gamma=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{F}_{\mathfrak{b}\gamma}^2(\zeta)} \right)^{-1} \right\rfloor = \begin{cases} \mathcal{F}_{\mathfrak{bu}}^2(\zeta) - \mathcal{F}_{\mathfrak{bu}-\mathfrak{b}}^2(\zeta), & \textit{if } \mathfrak{u} \textit{ is even} \\ \mathcal{F}_{\mathfrak{bu}}^2(\zeta) - \mathcal{F}_{\mathfrak{bu}-\mathfrak{b}}^2(\zeta) - 1, & \textit{if } \mathfrak{u} \textit{ is odd,} \end{cases}$$

$$\left\lfloor \left(\sum_{\gamma=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{L}_{\mathfrak{b}\gamma}(\zeta)} \right)^{-1} \right\rfloor = \begin{cases} \mathcal{L}_{\mathfrak{bu}}(\zeta) - \mathcal{L}_{\mathfrak{bu}-\mathfrak{b}}(\zeta) - 1, & \textit{if } \mathfrak{u} \textit{ is even} \\ \mathcal{L}_{\mathfrak{bu}}(\zeta) - \mathcal{L}_{\mathfrak{bu}-\mathfrak{b}}(\zeta), & \textit{if } \mathfrak{u} \textit{ is odd,} \end{cases}$$

$$\left\lfloor \left(\sum_{\gamma=\mathfrak{u}}^{\infty} \frac{1}{\mathcal{L}_{\mathfrak{b}\gamma}^2(\zeta)} \right)^{-1} \right\rfloor = \begin{cases} \mathcal{L}_{\mathfrak{bu}}^2(\zeta) - \mathcal{L}_{\mathfrak{bu}-\mathfrak{b}}^2(\zeta) - 3, & \textit{if } \mathfrak{u} \textit{ is even} \\ \mathcal{L}_{\mathfrak{bu}}^2(\zeta) - \mathcal{L}_{\mathfrak{bu}-\mathfrak{b}}^2(\zeta) + 1, & \textit{if } \mathfrak{u} \textit{ is odd.} \end{cases}$$

where $\lfloor . \rfloor$ is the floor function.

In 2019, Dutta and Ray [44] extended the works of Wang and Wen [39] to the Lucas and Fibonacci polynomials and obtained these results:

For any integer $\zeta, \mathfrak{u} \geq 2$, $m \geq 3$

$$\left\lfloor \left(\sum_{\gamma=\mathfrak{u}}^{m\mathfrak{u}} \frac{1}{\mathcal{F}_{\gamma}(\zeta)} \right)^{-1} \right\rfloor = \mathcal{F}_{\mathfrak{u}}(\zeta) - \mathcal{F}_{\mathfrak{u}-1}(\zeta).$$

For an integer $\zeta < 0$ and integers $\mathfrak{u} \geq 3, m \geq 3$

$$\left[\left(\sum_{\beta=\mathrm{u}}^{m\mathrm{u}}\frac{1}{\mathcal{L}_{\beta}(\zeta)}\right)^{-1}\right]=\mathcal{L}_{\mathrm{u}}(\zeta)-\mathcal{L}_{\mathrm{u}-1}(\zeta).$$

For $\zeta \in Z-0$ and integers $\mathrm{u}>0$ and $sm \geq 2$

$$\left[\left(\sum_{\beta=\mathrm{u}}^{m\mathrm{u}}\frac{1}{\mathcal{F}_{\beta}{}^{2}(\zeta)}\right)^{-1}\right]=\begin{cases}\zeta\mathcal{F}_{\mathrm{u}-1}(\zeta).\mathcal{F}_{\mathrm{u}}(\zeta)-1, & if\ \mathrm{u}\ is\ even\\ \zeta\mathcal{F}_{\mathrm{u}}(\zeta).\mathcal{F}_{\mathrm{u}-1}(\zeta), & if\ \mathrm{u}\ is\ odd.\end{cases}$$

For $\zeta \in Z-0 \pm 1$and integers $\mathrm{u}>0$ and $m \geq 2$,

$$\left[\left(\sum_{\beta=\mathrm{u}}^{m\mathrm{u}}\frac{1}{\mathcal{L}^{2}{}_{\beta}(\zeta)}\right)^{-1}\right]=\begin{cases}\zeta\mathcal{L}_{2\mathrm{u}-1}(\zeta)+1, & if\ \mathrm{u}\ is\ even\ and\ \mathrm{u}\geq 2\\ \zeta\mathcal{L}_{2\mathrm{u}-1}(\zeta)-2, & if\ \mathrm{u}\ is\ odd\ and\ \mathrm{u}\geq 3.\end{cases}$$

Many authors have attempted to draw a relationship between the Chebyshev polynomials, Lucas and Fibonacci polynomials.

Many researchers have analyzed a wide spectrum of properties of the Chebyshev polynomials & deduced a wide spectrum of results. For instance, in 2002, Zhang [55] considered the summations of finite products of Chebyshev polynomials, Lucas and Fibonacci numbers and deduced several intriguing results, particularly

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=n}\mathcal{U}_{\sigma_1}(\zeta)\cdot\mathcal{U}_{\sigma_2}(\zeta)\cdots\mathcal{U}_{\sigma_{r+1}}(\zeta)=\frac{1}{2^r\ r!}\mathcal{U}_{n+r}^{r}(\zeta), \tag{1.82}$$

where $\mathcal{U}_n^r(\zeta) = r^{th}$ derivative of $\mathcal{U}_n(\zeta)$ w.r.t ζ & the sum is taken over $r+1$ dimensional non-negative integral coordinates $(\sigma_1,\sigma_2,\cdots,\sigma_{r+1})$ satisfying $\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=n.$

In 2004, Wenpeng Zhang [45] studied Chebyshev polynomials and their derivatives and deduced their interesting relations with the Lucas and Fibonacci numbers. The main results are:

For integers $k\ ,m\ >0$ and non-negative integer α

$$\sum_{a_1+a_2+\cdots+a_{k+1}=\alpha}\mathcal{F}_{m(a_1+1)}\mathcal{F}_{m(a_2+1)}\ldots\ldots\mathcal{F}_{m(a_{k+1}+1)}$$

$$=(-i)^{m\alpha}\frac{\mathcal{F}_m{}^{k+1}}{2^k\ k!}\mathcal{U}_{\alpha+k}^{k}\left(\frac{i^m}{2}\mathcal{L}_m\right), \tag{1.83}$$

.

$$\sum_{a_1+a_2+\cdots+a_{k+1}=\alpha+k+1} \mathcal{L}_{ma_1}\mathcal{L}_{ma_2}\cdots\cdots\mathcal{L}_{ma_{k+1}}$$

$$= \frac{2\,(-i)^{m(\alpha+1+k)}}{k!}\sum_{\hbar=0}^{k+1}\binom{k+1}{\hbar}\left(\frac{i^{m+2}}{2}\right)^{\hbar}\mathcal{U}_{\alpha+2k+1-\hbar}^{k}\left(\frac{i^m}{2}\quad\mathcal{L}_m\right), \qquad (1.84)$$

Where $\binom{k+1}{\hbar} = \frac{(k+1)!}{\hbar!(k+1-\hbar)!}$ and $\mathcal{U}_\alpha^k$ denotes the *k*th derivative of Chebyshev polynomials of second kind.

In 2009, Falcon and Plaza [46] extended the k-Fibonacci numbers to the k – Fibonacci polynomials by taking k as x, a real variable, then $\mathcal{F}_{k,\alpha} = \mathcal{F}_{x,\alpha}$ and the sequence take the form

$$\mathcal{F}_{\alpha+1}(x) = \begin{cases} 1, & if\ \alpha = 0 \\ x, & if\ \alpha = 1 \\ x\mathcal{F}_\alpha(x) + \mathcal{F}_{\alpha-1}(x), & if\ \alpha > 1. \end{cases}$$

and proved several properties along with the computation of derivatives of these polynomials in the form of convolutions of *k*-Fibonacci polynomials. They obtained the sequence of derivatives of Fibonacci polynomials and generated many integer sequences by giving particular values to the variable *x*, derived the relation between derivatives of Fibonacci polynomials and Fibonacci numbers, and deduced the generating functions for *k*-Fibonacci polynomials and the recurrence relation of the derivative sequence.

In 2014, Yang Li [47] used these ideas of Zhang [45] and Falcon and Plaza [46] and established the relation between the Chebyshev polynomials, Fibonacci polynomial, and the r^{th} derivative of the Chebyshev polynomials. They derived the following relations:

For any integer $\alpha, r > 0$,

$$\mathcal{T}_{2\alpha}^{2r}(x) = \sum_{\mu=1}^{\alpha-r+1}\sum_{\lambda=r}^{\alpha}\frac{(-1)^{\alpha-r-\mu+1}\ 2^{2\lambda+r}(2\mu\alpha-\alpha)(\alpha+\lambda-1)!}{(\alpha-\lambda)!\,(\lambda-r-\mu+1)!\,(\lambda+\mu-r)!}\mathcal{F}_{2\mu-1}(x). \quad (1.85)$$

$$\mathcal{T}_{2\alpha}^{2r-1}(x) = \sum_{\mu=1}^{\alpha-r+1}\sum_{\lambda=0}^{\alpha}\frac{(-1)^{\alpha-r-\mu+1}\ 2^{2\lambda+r}(\alpha+\lambda-1)!\,\mu\alpha}{(\alpha-\lambda)!\,(\lambda-r-\mu+1)!\,(\lambda+\mu-r+1)!}\mathcal{F}_{2\mu}(x). \quad (1.86)$$

Similar results for the Chebyshev and Fibonacci polynomials with odd indices are deduced.

In 2015, Yang Li [48], again derived similar results for the Chebyshev and Fibonacci polynomials:

For any integer $\alpha > 0$,

$$\left.\begin{aligned} \mathcal{T}_{2\alpha}(x) &= \sum_{\mu=1}^{\alpha+1}\sum_{\lambda=0}^{\alpha} \frac{2^{2\lambda}(2\mu\alpha-\alpha)(\alpha+\lambda-1)!}{(-1)^{\mu+\alpha-1}(\alpha-\lambda)!\,(\lambda+\mu)!\,(\lambda-\mu+1)!}\mathcal{F}_{2\mu-1}(x). \\ \mathcal{U}_{2\alpha}(x) &= \sum_{\mu=1}^{\alpha+1}\sum_{\lambda=0}^{\alpha} \frac{2^{2\lambda-1}(1-2\mu)(\alpha+\lambda)!}{(-1)^{\mu+\alpha}(\alpha-\lambda)!\,(\lambda+\mu)!\,(\lambda-\mu+1)!}\mathcal{F}_{2\mu-1}(x). \end{aligned}\right\} \quad (1.87)$$

$$\left.\begin{aligned} \mathcal{F}_{2\alpha}(x) &= \sum_{\lambda=1}^{\alpha-1}\sum_{\gamma=1}^{\alpha-1} \frac{2^{2\gamma-2\alpha+2}\ \lambda(2\alpha-\gamma-1)!}{\gamma!\,(\alpha-\lambda-\gamma)!\,(\alpha+\lambda-\gamma)!}\mathcal{U}_{2\lambda}(x). \\ \mathcal{F}_{2\alpha}(x) &= \sum_{\lambda=1}^{\alpha}\sum_{\gamma=1}^{\alpha-1} \frac{2^{2\gamma+2-2\alpha}(2\alpha-\gamma-1)!}{\gamma!\,(2\alpha+2\lambda-2\gamma-2)!\,(2\alpha-2\lambda-\gamma)!}\mathcal{T}_{2\lambda-1}(x). \end{aligned}\right\} \quad (1.88)$$

Similarly, the relations between odd indexed 1st and 2nd kinds of Chebyshev polynomials and Fibonacci polynomials and vice versa.

In 2015, Xiaoxue Li [49], derived some identities of summation formula for powers of Chebyshev polynomials and discussed few divisibility properties involving these polynomials as follows:

For any integer $\hbar, n > 0$ and variable ζ,

a)

$$\sum_{\lambda=0}^{\hbar} \mathcal{T}_{2\lambda+1}^{2n+1}(\zeta) = \frac{1}{2^{2n+1}} \sum_{\mu=0}^{n} \binom{2n+1}{n-\mu} \frac{\mathcal{U}_{2(2\mu+1)(\hbar+1)-1}(\zeta)}{\mathcal{U}_{2\mu}(\zeta)}. \quad (1.89)$$

b)

$$\sum_{\lambda=1}^{\hbar} \mathcal{T}_{2\lambda}^{2n+1}(\zeta) = \frac{1}{2^{2n+1}} \sum_{\mu=0}^{n} \binom{2n+1}{n-\mu} \frac{\mathcal{U}_{(2\mu+1)(2\hbar+1)-1}(\zeta)-\mathcal{U}_{2\mu}(\zeta)}{\mathcal{U}_{2\mu}(\zeta)}. \quad (1.90)$$

and similar identities for odd and even indexed second kinds of Chebyshev polynomials. In addition to this, they also studied a few divisibility properties of these polynomials as an application of the above-stated results.

In 2015, W. Siyi [57], considered the summations of finite products of second kinds of Chebyshev polynomials and improved upon the results of Zhang [55] and derived the interesting results which includes:

$$\sum_{d_1+d_2+\cdots+d_{r+1}=n} \mathcal{U}_{d_1}(\zeta)\cdot\mathcal{U}_{d_2}(\zeta)\cdots\mathcal{U}_{d_{r+1}}(\zeta)=\frac{1}{2^r r!}\mathcal{U}_{n+r}^{r}(\zeta)$$

$$=\frac{1}{2^r r!}\left[\frac{(2r-1)\zeta}{(1-\zeta^2)}\mathcal{U}_{n+r}^{r-1}(\zeta)+\frac{(r-2)r-(n+r)(n+r+2)}{(1-\zeta^2)}\mathcal{U}_{n+r}^{r-2}(\zeta).\right] \tag{1.91}$$

In 2018, T. Kim et al. [50] considered the summations of finite products of second kinds of Chebyshev polynomials and derived the Fourier expansion of the associated functions, which in turn were used to represent these sums in Bernoulli polynomials. Similar results for Fibonacci polynomials are obtained.

They considered two functions

$$\alpha_{\nu,r}(\zeta)=\sum_{c_1+c_2+\cdots\ldots+c_{r+1}=\nu}\mathcal{U}_{c_1}(\zeta)\cdot\ \mathcal{U}_{c_2}(\zeta)\ \ldots\ldots\mathcal{U}_{c_{r+1}}(\zeta),$$

and

$$\beta_{\nu,r}(\zeta)=\sum_{c_1+c_2+\cdots\ldots+c_{r+1}=\nu}\mathcal{F}_{c_1+1}(\zeta)\cdot\ \mathcal{F}_{c_2+1}(\zeta)\ \ldots\ldots\mathcal{F}_{c_r+1}(\zeta),$$

such that the sum is taken over all non-negative integers $c_1, \ldots\ldots c_{r+1}$ with $c_1+c_2+\cdots\ldots+c_{r+1}=\nu$ and gave the following results:

For any integer $r\geq 1$, and $\nu\geq 1$, we let

$$\Delta_{\nu,r}=\frac{1}{2^r r!}\sum_{\omega=0}^{\left[\frac{\nu-1}{2}\right]}(-1)^k(\nu+r-2\omega)_r\binom{\nu+r-\omega}{\omega}\ 2^{\nu+r-2\omega}.$$

(a) Assume that $\Delta_{\nu,r}=0$ for some positive integer ν,r. Then

(i)

$$\sum_{c_1+c_2+\cdots\ldots+c_{r+1}=\nu}\mathcal{U}_{c_1}(\langle\zeta\rangle)\cdot\ \mathcal{U}_{c_2}(\langle\zeta\rangle)\ \ldots\ldots\mathcal{U}_{c_{r+1}}(\langle\zeta\rangle),$$

has the Fourier series expansion

$$\sum_{c_1+c_2+\cdots\ldots+c_{r+1}=\nu} \mathcal{U}_{c_1}(\langle\zeta\rangle)\cdot\mathcal{U}_{c_2}(\langle\zeta\rangle)\ldots\ldots\mathcal{U}_{c_{r+1}}(\langle\zeta\rangle)$$

$$=\frac{1}{2r}\Delta_{\nu+1,r-1}$$

$$-\sum_{n=-\infty,n\neq 0}^{\infty}\left(\frac{1}{2r}\sum_{\lambda=1}^{\nu}\frac{2^{\lambda}(r+\lambda-1)}{(2\pi in)^{\lambda}}\,\Delta_{\nu-\lambda+1,r+\lambda-1}\right)e^{2\pi in\zeta},$$

for all ζ in R when convergence is uniform.

(ii)

$$\sum_{c_1+c_2+\cdots\ldots+c_{r+1}=\nu} \mathcal{U}_{c_1}(\langle\zeta\rangle)\cdot\mathcal{U}_{c_2}(\langle\zeta\rangle)\ldots\ldots\mathcal{U}_{c_{r+1}}(\langle\zeta\rangle)$$

$$=\frac{1}{2r}\sum_{\lambda=0,\lambda\neq 0}^{\nu}2^{\lambda}\binom{\mathrm{r}+\lambda-1}{\mathrm{r}-1}\Delta_{\nu-\lambda+1,r+\lambda-1}B_{\lambda}(\langle\zeta\rangle),$$

$\forall\,\zeta$ in R. Here $(\zeta)_r=\zeta(\zeta-1)\ldots.(\zeta-r+1)$ for $r\geq 1$ and $(\zeta)_0=1$.

(b) Assume that $\Delta_{\nu,r}\neq 0$ for some positive integer r,ν ,

(i)

$$\frac{1}{2r}\Delta_{\nu+1,r-1}-\sum_{n=-\infty,n\neq 0}^{\infty}\left(\frac{1}{2r}\sum_{\lambda=1}^{\nu}\frac{2^{\lambda}(r+\lambda-1)}{(2\pi in)^{\lambda}}\,\Delta_{\nu-\lambda+1,r+\lambda-1}\right)e^{2\pi in\zeta}$$

$$=\begin{cases}\displaystyle\sum_{c_1+c_2+\cdots\ldots+c_{r+1}=\nu}\mathcal{U}_{c_1}(\langle\zeta\rangle)\cdot\mathcal{U}_{c_2}(\langle\zeta\rangle)\ldots\ldots\mathcal{U}_{c_{r+1}}(\langle\zeta\rangle), & if\ \zeta\in R-Z\\ \dfrac{\Delta_{\nu,r}}{2}, & if\ \zeta\in Z\ and\ \nu\ odd\\ (-1)^{\frac{\nu}{2}}\begin{pmatrix}\frac{\nu}{2}+r\\ \frac{\nu}{2}\end{pmatrix}+\dfrac{\Delta_{\nu,r}}{2}, & if\ \zeta\in Z\ and\ \nu\ even.\end{cases}$$

(ii)

$$\frac{1}{2r}\sum_{\lambda=0,\lambda\neq 0}^{\nu} 2^{\lambda}\binom{r+\lambda-1}{r-1}\Delta_{\nu-\lambda+1,r+\lambda-1}B_{\lambda}(\langle\zeta\rangle)$$

$$=\begin{cases}\sum\limits_{c_1+c_2+\cdots\ldots+c_{r+1}=\nu}\mathcal{U}_{c_1}(\langle\zeta\rangle)\cdot\mathcal{U}_{c_2}(\langle\zeta\rangle)\ldots\ldots\mathcal{U}_{c_{r+1}}(\langle\zeta\rangle), & ,if\ \zeta\in R-Z\\ \dfrac{\Delta_{\nu,r}}{2}, & if\ \zeta\in Z\ and\ \nu\ odd\\ (-1)^{\frac{\nu}{2}}\binom{\frac{\nu}{2}+r}{\frac{\nu}{2}}+\dfrac{\Delta_{\nu,r}}{2}\ , & if\ \zeta\in Z\ and\ \nu\ even.\end{cases}$$

and similarly, for any positive integer ν , r,assuming

$$\Omega_{\nu,r}=\sum\nolimits_{k=0}^{\left[\frac{\nu-1}{2}\right]}\binom{\nu+r-l-1}{l}\binom{\nu+r-2l-1}{r-1}.$$

we have

(i)

$$\frac{1}{r-1}\Omega_{\nu+1,r-1}-\sum_{n=-\infty,n\neq 0}^{\infty}\left(\frac{1}{r-1}\sum_{\lambda=1}^{\nu}\frac{(r-2+\lambda)_{\lambda}}{(2\pi in)^{\lambda}}\,\Omega_{\nu-\lambda+1,r+\lambda-1}\right)e^{2\pi in\zeta}$$

$$=\begin{cases}\sum\limits_{c_1+c_2+\cdots\ldots+c_{r+1}=\nu}\mathcal{F}_{c_1}(\langle\zeta\rangle)\cdot\mathcal{F}(\langle\zeta\rangle)\ldots\ldots\mathcal{F}_{c_{r+1}}(\langle\zeta\rangle), & if\ \zeta\in R-Z\\ \dfrac{\Omega_{\nu,r}}{2}, & if\zeta\in Z\ and\ \nu\ odd\\ \binom{\frac{\nu}{2}+r-1}{\frac{\nu}{2}}+\dfrac{\Omega_{\nu,r}}{2}\ , & if\zeta\in Z\ and\ \nu\ even.\end{cases}$$

(ii)

$$\frac{1}{r-1}\sum_{\lambda=0}^{\nu}\binom{r-2+\lambda}{\lambda}\Omega_{\nu-\lambda+1,r+\lambda-1}B_{\lambda}(\langle\zeta\rangle)$$

$$=\begin{cases}\sum\limits_{c_1+c_2+\cdots\ldots+c_{r+1}=\nu}\mathcal{F}_{c_1}(\langle\zeta\rangle)\cdot\mathcal{F}_{c_2}(\langle\zeta\rangle)\ldots\ldots\mathcal{F}_{c_{r+1}}(\langle\zeta\rangle), & if\ \zeta\in R-Z\\ \dfrac{\Omega_{\nu,r}}{2}\ , & if\zeta\in Z\ and\ \nu\ odd\\ \binom{\frac{\nu}{2}+r-1}{\frac{\nu}{2}}+\dfrac{\Omega_{\nu,r}}{2}\ , & if\zeta\in Z\ and\ \nu\ even.\end{cases}$$

T. Kim et al. [51] considered the summations of finite products of the 3rd and 4th kinds of Chebyshev polynomials and obtained the similar Fourier series of the associated functions, which in turn led to the expression of these sums as a linear sum of Bernoulli polynomials. The associated functions used in this case were

$$\alpha_{\nu,r}(\zeta) = \sum_{\gamma=o}^{\nu} \sum_{c_1+c_2+\cdots\ldots+c_{r+1}=\gamma} \binom{r+\nu-\gamma-1}{r-1} \mathcal{V}_{c_1}(\zeta)\cdot\mathcal{V}_{c_2}(\zeta)\ldots\mathcal{V}_{c_{r+1}}(\zeta),$$

and

$$\beta_{\nu,r}(\zeta) = \sum_{\gamma=0}^{\nu} \sum_{c_1+c_2+\cdots\ldots+c_{r+1}=\gamma} (-1)^{\mathrm{m-l}} \binom{r+\nu-\gamma-1}{r-1} \mathcal{W}_{c_1}(\zeta) \cdot \mathcal{W}_{c_2}(\zeta)\ldots\mathcal{W}_{c_{r+1}}(\zeta),$$

and derived the similar Fourier series results for the 3rd and 4th kinds of Chebyshev polynomials as in the case of first and second kinds of Chebyshev polynomials.

In 2019, T. Kim et al. [56] studied the classical linearization problem, expressing the sums of finite product of Chebyshev polynomials as a linear combination of other orthogonal polynomials like Hermite($\mathcal{H}_n(\xi)$), Legendre($\mathcal{L}_n(\xi)$), extended Laguerre ($\mathcal{P}_n(\xi)$), Gegenbauer ($C_n^{(\lambda)}(\xi)$), and Jacobi Polynomials ($\mathcal{P}_n^{(\alpha,\beta)}(\xi)$). The results obtained includes

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=n} \mathcal{V}_{\sigma_1}(\xi)\cdot\mathcal{V}_{\sigma_2}(\xi)\cdot\ldots\cdot\mathcal{V}_{\sigma_{r+1}}(\xi) = \frac{1}{2^r\, r!}\sum_{\gamma=0}^{n}(-1)^{\gamma}\binom{r+1}{\gamma}\mathcal{U}_{n-i+\gamma}^{r}(\xi) \quad (1.92)$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=n} \mathcal{W}_{\sigma_1}(\xi)\cdot\mathcal{W}_{\sigma_2}(\xi)\cdots\mathcal{W}_{\sigma_{r+1}}(\xi) = \frac{1}{2^r\, r!}\sum_{\gamma=0}^{n}\binom{r+1}{\gamma}\mathcal{U}_{n-i+\gamma}^{r}(\xi) \quad (1.93)$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=n} \mathcal{T}_{\sigma_1}(\xi)\cdot\mathcal{T}_{\sigma_2}(\xi)\cdots\mathcal{T}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{r!}\sum_{s=0}^{\left[\frac{n}{2}\right]}\frac{1}{(n-2s)!}\sum_{\gamma=0}^{s}\frac{(-1)^{\gamma}(n+r-\gamma)!}{\gamma!\,(s-\gamma)!}\, {}_2\mathcal{F}_1\left(2\gamma-n,-r-1;\gamma-m-r;\frac{1}{2}\right)\mathcal{H}_{n-2s}(\xi) \quad (1.94)$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=n} \mathcal{V}_{\sigma_1}(\xi)\cdot\mathcal{V}_{\sigma_2}(\xi)\cdot\ldots\cdot\mathcal{V}_{\sigma_{r+1}}(\xi)$$

$$= (-1)^n(r+1)\sum_{\lambda=0}^{n}\frac{(-1)^\lambda}{\lambda!}\sum_{\beta=0}^{\left[\frac{n-\lambda}{2}\right]}\frac{(\lambda+2\beta+r)!}{(n-\lambda-2\beta)!\,(r+1-n+\lambda+2\beta)!\,\beta!}$$
$$ {}_1\mathcal{F}_1(-\beta,-\lambda-2\beta-r;-1)\,\mathcal{H}_\lambda(\xi) \tag{1.95}$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=n} \mathcal{W}_{\sigma_1}(\xi)\cdot\mathcal{W}_{\sigma_2}(\xi)\cdot\ldots\cdot\mathcal{W}_{\sigma_{r+1}}(\xi)$$

$$= (r+1)\sum_{\lambda=0}^{n}\frac{1}{\lambda!}\sum_{\beta=0}^{\left[\frac{n-\lambda}{2}\right]}\frac{(\lambda+2\beta+r)!}{(n-\lambda-2\beta)!\,(r+1-n+\lambda+2\beta)!\,\beta!}$$
$$ {}_1\mathcal{F}_1(-\beta,-\lambda-2\beta-r;-1)\,\mathcal{H}_\lambda(\xi) \tag{1.96}$$

where all sums run over integers $\sigma_1,\sigma_2,\ldots,\sigma_{r+1}(\geq 0)$ satisfying $\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=n$, with $\binom{r+1}{\gamma}=0$ for $\gamma > r+1$.

Similar results for Legendre ($\mathcal{L}_n(\xi)$), extended Laguerre ($\mathcal{P}_n(\xi)$), Gegenbauer ($C_n^{(\lambda)}(\xi)$), and Jacobi Polynomials ($\mathcal{P}_n^{(\alpha,\beta)}(\xi)$) were obtained. In 2022, A. Patra, and G.K. Panda [59], obtained similar results for Pell polynomials.

In another line of generalisation, several authors have generalised and extended Fibonacci and Lucas polynomials to two or more variables and studied their interesting properties and deduced several results.

One such generalisation was studied by M. Catalani [71] wherein the author studied the bivariate Fibonacci polynomials given by

$$\mathcal{H}_n(\omega,\zeta) = \omega\mathcal{H}_{n-1}(\omega,\zeta) + \zeta\mathcal{H}_{n-2}(\omega,\zeta) \tag{1.97}$$

with $\mathcal{H}_0(\omega,\zeta)=\mathfrak{a}_0$ and $\mathcal{H}_1(\omega,\zeta)=\mathfrak{a}_1$, for every $n>1$ and deduced several results involving their generating matrices.

In 2016, E.G. Kocer and S. Tuncez [72] studied the new generalizations of the Fibonacci and Lucas polynomials to two variables and studied their properties and obtained some results. They introduced the bivariate Fibonacci and Lucas polynomials given by

$$\mathcal{F}_n(\omega,\zeta) = p(\omega,\zeta)\mathcal{F}_{n-1}(\omega,\zeta) + q(\omega,\zeta)\mathcal{F}_{n-2}(\omega,\zeta), \tag{1.98}$$

with $\mathcal{F}_0(\omega,\zeta) = 0$ and $\mathcal{F}_1(\omega,\zeta) = 1$.
and

$$\mathcal{L}_n(\omega,\zeta) = p(\omega,\zeta)\mathcal{L}_{n-1}(\omega,\zeta) + q(\omega,\zeta)\mathcal{L}_{n-2}(\omega,\zeta), \tag{1.99}$$

with $\mathcal{L}_0(\omega,\zeta) = 2$ and $\mathcal{L}_1(\omega,\zeta) = p(\omega,\zeta)$, for every $n > 1$, where $p(\omega,\zeta)$ and $q(\omega,\zeta)$ are polynomials with real coefficients. Similar studies were done by Tan and Yang [68]. Further generalization of Lucas and Fibonacci polynomials to trivariate Lucas and Fibonacci polynomials were studied by Kocer and Gedikce [16, 63] obtaining several interesting properties.

1.4 Research Gap

A generalization is an abstraction wherein common characteristics of particular instances are expressed as general concepts or claims. Generalizations presumes the existence of a domain or set of elements as well as one or more common properties shared by those elements (thus evolving a conceptual method). Thus, they are fundamental to all the valid deductive inferences. In mathematics, the sequence of Fibonacci polynomials can be viewed as a generalization of the sequence of Fibonacci numbers. Lucas polynomials are the polynomials generated from the Lucas numbers in a similar manner.

The thorough review of the cited literature leads to the following inferences regarding the research gap which is proposed to be bridged during tenure of our research work that the Fibonacci polynomials and Lucas polynomials have been generalized mostly for up to two variables and their properties have been established so for, generalizations of Fibonacci and Lucas polynomials for more than three variables is to be explored for this we may extend the recurrence relation, or the recurrence relation is preserved but the coefficients of polynomial are replaced by some new coefficients with more variables or by changing the initial conditions and established their properties.

Many researchers have worked on Chebyshev polynomials of the first and second kind in one or two variables; properties and applications of the 3rd and 4th kinds of Chebyshev polynomials in two or more variables are to be studied, and new relations are to be established. Relations between the 3rd and 4th kinds of Chebyshev polynomials and Pell, Lucas and Fibonacci numbers and polynomials are to be obtained.

The divisibility properties of Chebyshev polynomials can also be explored, and the Fourier series expansion associated with them can be obtained along the same lines as that of Chebyshev Polynomials of 1st and 2nd kind and similar concepts can be extended to Chebyshev-like polynomials also.

1.5 Proposed Objectives of the Research Work

In our research work, we propose to consider the following problems:

- To obtained new generalization of Fibonacci and Lucas polynomials for three or more variables and established their properties.
- New generalization of Chebyshev like polynomials of third and fourth kind are to be find out and to discuss their properties.
- To find out relations between Chebyshev polynomials of third and fourth kind with Fibonacci, Lucas and Pell numbers and polynomials.
- To discuss the application of Chebyshev polynomials and Fibonacci like polynomials

1.6 Proposed Methodology of the Research Work

During our research work, we propose to use the usual method of pure mathematics to achieve our goals.

The Fibonacci and Chebyshev-like polynomials are generalized by extending the recurrence relation; the recurrence relation is preserved, but the coefficients of the polynomial are replaced by some other coefficients with more variables or by changing the initial conditions. We will use these techniques to obtain new generalizations.

Methods of mathematical induction and the techniques of combinatory are used for proving the properties obtained in the form of theorems and lemmas.

1.7 Structure of Thesis

The proposed work, entitled "**A STUDY OF FIBONACCI POLYNOMIAL, CHEBYSHEV POLYNOMIAL, AND ITS SEQUENCES**," is inspired by the study of the sequence of the Fibonacci numbers and their generalizations from Fibonacci polynomials to Chebyshev polynomials and like polynomials. The core of the subject matter of the manuscript grows from a series of our research papers that are cited at the end. The following overview summarizes the thesis:

In the first chapter, an introduction to Fibonacci numbers, their history, their applications in diverse fields, and their polynomial expansions are presented. Additionally, we will give a quick review of a few definitions and well-known results relating to the Fibonacci numbers, Chebyshev polynomials, and Fibonacci numbers, which meet the minimal requirements for the evolution of the emerging chapters. This chapter includes a section of literature review focused on the work done by various researchers in the field of the Fibonacci numbers and their polynomial generalisations through the first, second, third, and fourth kinds of Chebyshev and similar polynomials. This review has identified the research gap. Furthermore, this chapter has also outlined the objectives and methodology to bridge these gaps.

In chapter 2, we will deal with the second kind Chebyshev polynomials. Here we have discussed the identities of the second-kind Chebyshev polynomials and Lucas, Fibonacci, and complex Fibonacci numbers. Several identities connecting sums of finite products of Lucas, Fibonacci, and complex Fibonacci numbers and the second kind Chebyshev polynomials with positive as well as negative odd indices are investigated.

In chapter 3, we will consider the interaction between the 3rd and 4th kinds of Chebyshev polynomials and the Lucas and Fibonacci numbers and the second kind Chebyshev polynomials. In terms of second-kind Chebyshev polynomials and their derivatives, we will develop certain identities involving sums of their finite products. We also discussed some specific cases of these summation identities that result from different values of $r = 1,2,3$.

In Chapter 4, explicit formulae for the 3rd and 4th kinds of Chebyshev polynomials and their derivatives with odd and even index are established. Further, their links with Fibonacci polynomials with negative odd and even indices are also obtained. In the second section, some works on summations of the finite products of the third and fourth-kind Chebyshev polynomials and Pell polynomials as a linear sum of other orthogonal polynomials are considered.

Chapter 5 is composed of two sections focused mainly on the interrelationship between the 3rd and 4th kinds of Chebyshev polynomials and Lucas, Fibonacci, and Pell numbers and their polynomials. In the first section of this chapter, we introduced some more identities expressing summation of finite products of Lucas, Fibonacci, and Pell numbers and Fibonacci polynomials as a linear sum of derived Pell polynomials with even and odd indices, using their basic properties through elementary computations. Similar identities are obtained for the 3rd and 4th kinds of Chebyshev polynomials. We also analyzed these identities by taking particular cases with $r = 1,2,3$.

And in the second section, we will establish few more similar identities for negative indexed Lucas, Fibonacci, and Complex Fibonacci numbers in terms of Pell polynomials with negative even and odd indices, using their basic properties through elementary computations. In terms of the 3rd and 4th kinds of Chebyshev polynomials, similar identities were obtained for Pell numbers and Fibonacci polynomials. Special cases of these identities are also discussed.

At the end in the Chapter 6, we developed the concepts of *(p, q, r)-Generalized trivariate Fibonacci* and *(p, q, r)-Generalized trivariate Lucas* polynomials and their sequences and discussed their properties. Several results involving the relationships of (*p, q, r)-Generalized trivariate Fibonacci* and *(p, q, r)-generalized trivariate Lucas* polynomials are discussed. Using these properties and results, we derived the explicit formula of (*p, q, r)-Generalized trivariate Lucas and Fibonacci* polynomials and deduced several identities involving the generating matrices and their determinants.

Chapter 2

SOME CONNECTIONS BETWEEN FINITE PRODUCTS OF FIBONACCI AND LUCAS NUMBERS AND CHEBYSHEV POLYNOMIALS OF SECOND KIND

2.1 Introduction

This chapter will focus on the development of some results on the representation of the summations of finite products of the Lucas, the Fibonacci numbers, and the Complex Fibonacci numbers as a linear sum of the 2^{nd}-kind Chebyshev polynomials through elementary computations.

2.2 Representations of finite products of Fibonacci and Lucas Numbers in Chebyshev polynomials of the second kind

Here, we will develop some results expressing summations of finite products of the Lucas, Fibonacci, and the complex Fibonacci numbers as a linear sum of derivatives of 2^{nd} kinds of Chebyshev polynomials.

Chebyshev polynomials have drawn the attention of numerous researchers, who have investigated their properties and developed a wide range of results. Zhang [55] for instance, considered the summation formulae for finite products of Chebyshev polynomials, Lucas and Fibonacci numbers and deduced several intriguing results, specifically, given by equation (1.82). Similarly, in [56], the authors have deduced analogous results which include equations (1.92)- (1.94) especially,

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(\xi)\cdot\mathcal{V}_{\sigma_2}(\xi)\ldots\mathcal{V}_{\sigma_{r+1}}(\xi) = \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}\mathcal{U}^{r}_{\alpha-\gamma+r}(\xi)$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(\xi)\cdot\mathcal{W}_{\sigma_2}(\xi)\cdots\mathcal{W}_{\sigma_{r+1}}(\xi) = \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{U}^{r}_{\alpha-\gamma+r}(\xi)$$

where these sums run over all $\sigma_{\hbar}\ (\geq 0)$ *in* $\mathbf{Z}\ (\hbar = 1,2,\ldots,r+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha$ *and* $\binom{r+1}{\gamma}=0$ *for* $\gamma > r+1$.

In the same line of action, we considered a few more identities on summations of finite products of the Lucas and Fibonacci numbers and expressed them as the linear combinations of the derivative of the 2nd kinds of Chebyshev polynomials. The main findings are:

Theorem 2.2.1. *For integers* $\alpha, r \geq 0$,

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{F}_{2\sigma_1+1}\cdot\mathcal{F}_{2\sigma_2+1}\cdots\mathcal{F}_{2\sigma_{r+1}+1} = \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{3}{2}\right),$$

where $\binom{r+1}{\gamma}=0$, for $\gamma > r+1$.

Proof. Taking $\xi = \frac{3}{2}$ in equation (1.92), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}\left(\frac{3}{2}\right)\cdot\mathcal{V}_{\sigma_2}\left(\frac{3}{2}\right)\cdots\mathcal{V}_{\sigma_{r+1}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{3}{2}\right), \tag{2.4}$$

Using $\mathcal{U}_{\alpha}\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+2}$ in equation (1.65) *(ii)* to get $\mathcal{V}_{\alpha}\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+1}$ and using this in turn, in equation (2.4), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{F}_{2\sigma_1+1}\cdot\mathcal{F}_{2\sigma_2+1}\cdots\mathcal{F}_{2\sigma_{r+1}+1} = \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{3}{2}\right).$$

Thus Theorem 2.2.1 is established. ■

Theorem 2.2.2. *For integers* $\alpha, r \geq 0$,

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{L}_{2\sigma_1+1}\cdot\mathcal{L}_{2\sigma_2+1}\cdots\mathcal{L}_{2\sigma_{r+1}+1} = \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{3}{2}\right),$$

where $\binom{r+1}{\gamma}=0$, for $\gamma > r+1$.

Proof. Taking $\xi = \frac{3}{2}$ in equation (1.93), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma}\left(\frac{3}{2}\right)\cdot\mathcal{W}_{\sigma_2}\left(\frac{3}{2}\right)\cdots\mathcal{W}_{\sigma_{r+1}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{3}{2}\right), \tag{2.5}$$

Using $\mathcal{U}_\alpha\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+2}$ in equation (1.65) *(iii)* to get $\mathcal{W}_\alpha\left(\frac{3}{2}\right) = \mathcal{L}_{2\alpha+1}$ and using this in turn in equation (2.5), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{L}_{2\sigma_1+1}\cdot\mathcal{L}_{2\sigma_2+1}\cdots\mathcal{L}_{2\sigma_{r+1}+1} = \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{3}{2}\right).$$

Hence the Theorem 2.2.2. ■

Theorem 2.2.3: *For integers $\alpha, r \geq 0$,*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{F}^*{}_{\sigma_1}\cdot\mathcal{F}^*{}_{\sigma_2}\cdots\mathcal{F}^*{}_{\sigma_{r+1}}$$

$$= \frac{i^{\alpha+r+1}}{2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(-\frac{i}{2}\right)$$

$$= \frac{1}{i^{\alpha-(r+1)}\ 2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{i}{2}\right),$$

where $\binom{r+1}{\gamma} = 0$, for $\gamma > r+1$ and $\mathcal{F}^*{}_\alpha$ is complex Fibonacci number.

Proof. Taking $\xi = -\frac{i}{2}$ in equation (1.92), and $\xi = \frac{i}{2}$ in equation (1.93), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}\left(-\frac{i}{2}\right)\cdot\mathcal{V}_{\sigma_2}\left(-\frac{i}{2}\right)\cdots\mathcal{V}_{\sigma_{r+1}}\left(-\frac{i}{2}\right)$$

$$= \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(-\frac{i}{2}\right) \tag{2.6}$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}\left(\frac{i}{2}\right)\cdot\mathcal{W}_{\sigma_2}\left(\frac{i}{2}\right)\cdots\mathcal{W}_{\sigma_{r+1}}\left(\frac{i}{2}\right)$$

$$= \frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{i}{2}\right) \tag{2.7}$$

Using $\mathcal{U}_\alpha\left(\frac{i}{2}\right) = i^\alpha\mathcal{F}_{\alpha+1}$ in equation (1.65) *(iii)* we get

$$\mathcal{W}_\alpha\left(\frac{i}{2}\right) = i^{\alpha-1}\mathcal{F}_\alpha^* \tag{2.8}$$

Again using equation (2.8) in equation (1.65) *(xii)* we get

$$\mathcal{V}_\alpha\left(-\frac{i}{2}\right)=\frac{\mathcal{F}_\alpha^*}{i^{\alpha+1}} \tag{2.9}$$

Using equation (2.9), in equation (2.6), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha}\mathcal{F}^*{}_{\sigma_1}\cdot\mathcal{F}^*{}_{\sigma_2}\cdots\mathcal{F}^*{}_{\sigma_{r+1}}$$

$$=\frac{i^{\alpha+r+1}}{2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(-\frac{i}{2}\right)$$

Similarly, using equation (2.8), in equation (2.7), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha}\mathcal{F}^*{}_{2\sigma_1+1}\cdot\mathcal{F}^*{}_{2\sigma_2+1}\cdots\mathcal{F}^*{}_{2\sigma_{r+1}+1}$$

$$=\frac{1}{i^{\alpha-(r+1)}\ 2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{i}{2}\right),$$

This establishes the Theorem 2.2.3 ■

Corollary 2.2.1: For integers $\alpha, r \geq 0$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha}\mathcal{F}_{-(2\sigma_1+1)}\cdot\mathcal{F}_{-(2\sigma_2+1)}\cdots\mathcal{F}_{-(2\sigma_{r+1}+1)}$$

$$=\frac{1}{2^r\ r!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{3}{2}\right),$$

where $\binom{r+1}{\gamma}=0$, for $\gamma > r+1$.

Proof. Using $\mathcal{F}_{-\alpha}=(-1)^{\alpha+1}\mathcal{F}_\alpha$ in Theorem 2.2.1 to establish the results. ■

Corollary 2.2.2: For integers $\alpha, r \geq 0$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha}\mathcal{L}_{-(2\sigma_1+1)}\cdot\mathcal{L}_{-(2\sigma_2+1)}\cdots\mathcal{L}_{-(2\sigma_{r+1}+1)}$$

$$=\frac{(-1)^{r+1}}{2^r\ r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{3}{2}\right),$$

where $\binom{r+1}{\gamma}=0$, for $\gamma > r+1$.

Proof. Using $\mathcal{L}_{-\alpha} = (-1)^{\alpha}\, \mathcal{L}_{\alpha}$ in Theorem 2.2.2, to achieve the desired results. ■

Corollary 2.2. 3: For an integers $\alpha, r \geq 0$,

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{F}^*{}_{-\sigma_1} \cdot \mathcal{F}^*{}_{-\sigma_2} \cdots \mathcal{F}^*{}_{-\sigma_{r+1}}$$

$$= \frac{i^{\alpha+r+1}}{2^r\ r!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma} \binom{r+1}{\gamma} \mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{i}{2}\right)$$

$$= \frac{1}{i^{\alpha-(r+1)}\ 2^r\ r!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma} \binom{r+1}{\gamma} \mathcal{U}_{\alpha-\gamma+r}^{r}\left(-\frac{i}{2}\right)$$

where $\binom{r+1}{\gamma} = 0$, for $\gamma > r+1$, and $\mathcal{F}^*{}_{\alpha}$ is a complex Fibonacci number.

Proof. Taking conjugate of $\mathcal{F}^*{}_{\alpha}$ in Theorem 2.2.3 and using $\mathcal{F}^*{}_{-\alpha} = (-1)^{\alpha+1}\overline{\mathcal{F}^*{}_{\alpha}}$, where $\overline{\mathcal{F}^*{}_{\alpha}}$ is the complex conjugate of $\mathcal{F}^*{}_{\alpha}$, we can achieve the desired result. ■

Chapter 3

IDENTITIES ON CHEBYSHEV POLYNOMIALS OF THIRD AND FOURTH KIND AND FIBONACCI AND LUCAS NUMBERS IN TERMS OF SECOND KINDS OF CHEBYSHEV POLYNOMIALS

3.1 Introduction

we will discuss a few identities representing summations of finite products of the 3rd and 4th kinds of Chebyshev polynomials, Lucas, and Fibonacci numbers in the 2nd kinds of Chebyshev polynomials and their derivatives, using the elementary computational method.

3.2 Sums of finite products of third and fourth kinds of Chebyshev polynomials, Lucas and Fibonacci numbers in terms of the second kinds of Chebyshev polynomials.

Several researchers have investigated Chebyshev polynomials and their properties and deduced a broad spectrum of results. One such area is the classical linearization problem considered by Zhang [55], in 2002, wherein the sums of finite products of 2nd-kind Chebyshev polynomials, Lucas and Fibonacci numbers were represented in the linear sums of the derived 2nd-kind Chebyshev polynomials as given by the equation (1.82). Similar results were given by T. Kim et al. [56] in 2019, especially, given by the equations (1.92)- (1.93) in Section 1.2 of Chapter 1. In 2020, D. Han and L. Xinging [74], working on the same idea, introduced some more summation representations of Lucas, Fibonacci and Chebyshev polynomials as a linear sum of Lucas and the1st-kind Chebyshev polynomials.

With the same motivation, we will consider a few more identities connecting summations of finite products of the 3rd and 4th kinds of Chebyshev polynomials, Lucas, and Fibonacci numbers with the Chebyshev polynomials of the 2nd kind. The main results are:

Theorem 3.2.1. *For integers $\alpha, r \geq 0$, we have*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(\xi)\cdot\mathcal{V}_{\sigma_2}(\xi)\cdots\mathcal{V}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{2^r\, r!\,(1-\xi^2)}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{r+1}{\lambda}\Big[(2r-1)\xi\,\mathcal{U}_{\alpha-\lambda+r}^{r-1}(\xi)$$

$$-\left[(\alpha-\lambda+2)(\alpha-\lambda+2r)\right]\mathcal{U}_{\alpha-\lambda+r}^{r-2}(\xi)\Big],$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) in $\boldsymbol{Z}$ $(\hbar = 1,2,\ldots,r+1)$ with $\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha$ and $\binom{r+1}{\lambda}=0$ for $\lambda > r+1$.

Proof. From [57], we first note that for any positive integer $\alpha \geq r > 0$,

$$\mathcal{U}_{\alpha+r}^{r}(\xi) = \frac{(2r-1)\xi}{(1-\xi^2)}\mathcal{U}_{\alpha+r}^{r-1}(\xi) + \frac{(r-2)r-(\alpha+r)(\alpha+r+2)}{(1-\xi^2)}\mathcal{U}_{\alpha+r}^{r-2}(\xi). \quad (3.1)$$

Thus ,

$$\mathcal{U}_{\alpha-\lambda+r}^{r}(\xi) = \frac{(2r-1)\xi}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-1}(\xi)$$

$$+\frac{(r-2)r-(\alpha-\lambda+r)(\alpha-\lambda+r+2)}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-2}(\xi),$$

$$= \frac{(2r-1)\xi}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-1}(\xi)$$

$$-\frac{[\alpha(\alpha-\lambda+r)-\lambda(\alpha-\lambda+r)+r(\alpha-\lambda+2)+2(\alpha-\lambda+r)]}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-2}(\xi),$$

$$= \frac{(2r-1)\xi}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-1}(\xi)$$

$$-\frac{[(\alpha-\lambda+2)(\alpha-\lambda+r)+r(\alpha-\lambda+2)]}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-2}(\xi),$$

$$= \frac{(2r-1)\xi}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-1}(\xi) - \frac{[(\alpha-\lambda+2)(\alpha-\lambda+2r)]}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-2}(\xi).$$

Therefore,

$$\mathcal{U}_{\alpha-\lambda+r}^{r}(\xi) = \frac{(2r-1)\xi}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-1}(\xi) - \frac{[(\alpha-\lambda+2)(\alpha-\lambda+2r)]}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-2}(\xi). \quad (3.2)$$

Using equations (3.1), (3.2) and (1.92), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(\xi)\cdot\mathcal{V}_{\sigma_2}(\xi)\cdots\mathcal{V}_{\sigma_{r+1}}(\xi) = \frac{1}{2^r\ r!}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{r+1}{\lambda}\mathcal{U}^{r}_{\alpha-\lambda+\lambda}(\xi),$$

$$= \frac{1}{2^r\ r!}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{r+1}{\lambda}\left[\frac{(2r-1)\xi}{(1-\xi^2)}\mathcal{U}^{r-1}_{\alpha-\lambda+r}(\xi)\right.$$

$$\left. - \frac{[(\alpha-\lambda+2)(\alpha-\lambda+2r)]}{(1-\xi^2)}\mathcal{U}^{r-2}_{\alpha-\lambda+r}(\xi)\right],$$

$$= \frac{1}{2^r\ r!\,(1-\xi^2)}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{r+1}{\lambda}\left[(2r-1)\xi\mathcal{U}^{r-1}_{\alpha-\lambda+r}(\xi)\right.$$

$$\left. - [(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}^{r-2}_{\alpha-\lambda+r}(\xi)\right]$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(\xi)\cdot\mathcal{V}_{\sigma_2}(\xi)\cdots\mathcal{V}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{2^r\ r!\,(1-\xi^2)}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{r+1}{\lambda}\left[(2r-1)\xi\mathcal{U}^{r-1}_{\alpha-\lambda+r}(\xi)\right.$$

$$\left. - [(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}^{r-2}_{\alpha-\lambda+r}(\xi)\right]$$

Hence, the Theorem is established. ■

Theorem 3.2.2. *For any integer $\alpha, r \geq 0$,*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(\xi)\cdot\mathcal{W}_{\sigma_2}(\xi)\cdots\mathcal{W}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{2^r\ r!\,(1-\xi^2)}\sum_{\lambda=0}^{\alpha}\binom{r+1}{\lambda}\left[(2r-1)\xi\mathcal{U}^{r-1}_{\alpha-\lambda+r}(\xi)\right.$$

$$\left. - [(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}^{r-2}_{\alpha-\lambda+r}(\xi)\right],$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) in $\mathbf{Z}$ $(\hbar = 1,2,\ldots,r+1)$ with $\sigma_1+\sigma_2+\cdots+\sigma_{r+1} = \alpha$ and $\binom{r+1}{\lambda} = 0$ for $\lambda > r+1$..

Proof. We will proceed in a similar manner by using equation (3.1), (3.2) in equation (1.93). So, we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(\xi)\cdot\mathcal{W}_{\sigma_2}(\xi)\cdots\mathcal{W}_{\sigma_{r+1}}(\xi) = \frac{1}{2^r\ r!}\sum_{\lambda=0}^{\alpha}\binom{r+1}{\lambda}\mathcal{U}^{r}_{\alpha-\lambda+r}(\xi),$$

$$= \frac{1}{2^r\ r!}\sum_{\lambda=0}^{\alpha}\binom{r+1}{\lambda}\left[\frac{(2r-1)\xi}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-1}(\xi)\right.$$

$$\left. - \frac{[(\alpha-\lambda+2)(\alpha-\lambda+2r)]}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+r}^{r-2}(\xi)\right],$$

$$= \frac{1}{2^r\ r!\,(1-\xi^2)}\sum_{\lambda=0}^{\alpha}\binom{r+1}{\lambda}\left[(2r-1)\xi\mathcal{U}_{\alpha-\lambda+r}^{r-1}(\xi)\right.$$

$$\left. - [(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}(\xi)\right].$$

Hence, the Theorem is established. ■

Theorem 3.2.3. *For any integers* $\alpha, r \geq 0$,

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha}\mathcal{F}_{2\sigma_1+1}\cdot\mathcal{F}_{2\sigma_2+1}\cdots\mathcal{F}_{2\sigma_{r+1}+1}$$

$$= \frac{1}{2^{r-1}\ r!}\sum_{\lambda=0}^{\alpha}\frac{(-1)^{\lambda}}{5}\binom{r+1}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)\right.$$

$$\left. - 3(2r-1)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)\right],$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\ldots,r+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{r+1} = \alpha$ *and* $\binom{r+1}{\lambda} = 0$ *for* $\lambda > r+1$.

Proof. We use the fact that

$$\mathcal{U}_{\alpha}\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+2}.$$

in equation (1.65) (*ii*) to deduce equation (1.65) (*viii*) and using this in turn in Theorem 3.2.1, with $\xi = \frac{3}{2}$, we get

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha}\mathcal{V}_{\sigma_1}\left(\frac{3}{2}\right)\cdot\mathcal{V}_{\sigma_2}\left(\frac{3}{2}\right)\cdots\mathcal{V}_{\sigma_{r+1}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^r\ r!\left(1-\left(\frac{3}{2}\right)^2\right)}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{r+1}{\lambda}\left[(2r-1)\left(\frac{3}{2}\right)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)\right.$$

$$\left. - [(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)\right].$$

which in turn yields

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{F}_{2\sigma_1+1}\cdot\mathcal{F}_{2\sigma_2+1}\cdots\mathcal{F}_{2\sigma_{r+1}+1}$$

$$= \frac{1}{2^r\ r!}\left(-\frac{4}{5}\right)\sum_{\lambda=0}^{\alpha}(-1)^\lambda\binom{r+1}{\lambda}\Big[(2r-1)\left(\frac{3}{2}\right)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)$$

$$-\left[(\alpha-\lambda+2)(\alpha-\lambda+2r)\right]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)\Big]$$

$$= \frac{1}{2^{r+1}\ r!}\left(\frac{4}{5}\right)\sum_{\lambda=0}^{\alpha}(-1)^\lambda\binom{r+1}{\lambda}\Big[2[(\alpha-\lambda+2)(\alpha-\lambda$$

$$+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)-3(2r-1)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)\Big]$$

$$= \frac{1}{2^{r-1}\ r!}\sum_{\lambda=0}^{\alpha}\frac{(-1)^\lambda}{5}\binom{r+1}{\lambda}\Big[2[(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)$$

$$-3(2r-1)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)\Big].$$

That is,

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{F}_{2\sigma_1+1}\cdot\mathcal{F}_{2\sigma_2+1}\cdots\mathcal{F}_{2\sigma_{r+1}+1}$$

$$= \frac{1}{2^{r-1}\ r!}\sum_{\lambda=0}^{\alpha}\frac{(-1)^\lambda}{5}\binom{r+1}{\lambda}\Big[2[(\alpha-\lambda+2)(\alpha-\lambda$$

$$+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)-3(2r-1)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)\Big].$$

Hence, the Theorem is established. ■

Theorem 3.2.4. *For any integers $\alpha, r \geq 0$,*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{L}_{2\sigma_1+1}\cdot\mathcal{L}_{2\sigma_2+1}\cdots\mathcal{L}_{2\sigma_{r+1}+1}$$

$$= \frac{1}{2^{r-1}\ r!}\sum_{\lambda=0}^{\alpha}\frac{1}{5}\cdot\binom{r+1}{\lambda}\Big[2[(\alpha-\lambda+2r)(\alpha-\lambda+2)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)$$

$$-3(2r-1)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)\Big],$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\dots,r+1)$ *with* $\sigma_1 + \sigma_2 + \cdots + \sigma_{r+1} = \alpha$ *and* $\binom{r+1}{\lambda} = 0$ *for* $\lambda > r+1$.

Proof. To establish this Theorem 3.2.4, we will proceed as in the case of the Theorem 3.2.3 by using the fact

$$\mathcal{U}_{\alpha}\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+2},$$

in equation (1.65) (*iii*) to equation (1.65) (*x*) and then using this in turn in Theorem 3.2.2 with $\xi = \frac{3}{2}$, resulting in

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}\left(\frac{3}{2}\right)\cdot \mathcal{W}_{\sigma_2}\left(\frac{3}{2}\right)\cdots \mathcal{W}_{\sigma_{r+1}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^r\ r!\left(1-\left(\frac{3}{2}\right)^2\right)} \sum_{\lambda=0}^{\alpha} \binom{r+1}{\lambda} \Big[(2r-1)\left(\frac{3}{2}\right)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)$$

$$- [(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)\Big].$$

which in turn yields

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{L}_{2\sigma_1+1}\cdot \mathcal{L}_{2\sigma_2+1}\cdots \mathcal{L}_{2\sigma_{r+1}+1}$$

$$= \frac{1}{2^r\ r!}\left(-\frac{4}{5}\right) \sum_{\lambda=0}^{\alpha} \binom{r+1}{\lambda} \Big[(2r-1)\left(\frac{3}{2}\right)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)$$

$$- [(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)\Big]$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{L}_{2\sigma_1+1}\cdot \mathcal{L}_{2\sigma_2+1}\cdots \mathcal{L}_{2\sigma_{r+1}+1}$$

$$= \frac{1}{2^{r+1}\ r!}\left(\frac{4}{5}\right) \sum_{\lambda=0}^{\alpha} \binom{r+1}{\lambda} \Big[2[(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)$$

$$- 3(2r-1)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)\Big]$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{L}_{2\sigma_1+1}\cdot\mathcal{L}_{2\sigma_2+1}\cdots\mathcal{L}_{2\sigma_{r+1}+1}$$

$$= \frac{1}{2^{r-1}\, r!}\sum_{\lambda=0}^{\alpha}\frac{1}{5}\cdot\binom{r+1}{\lambda}\Big[2[(\alpha-\lambda+2r)(\alpha-\lambda+2)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)$$

$$-3(2r-1)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)\Big].$$

That is,

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha} \mathcal{L}_{2\sigma_1+1}\cdot\mathcal{L}_{2\sigma_2+1}\cdots\mathcal{L}_{2\sigma_{r+1}+1}$$

$$= \frac{1}{2^{r-1}\, r!}\sum_{\lambda=0}^{\alpha}\frac{1}{5}\cdot\binom{r+1}{\lambda}\Big[2[(\alpha-\lambda+2)(\alpha-\lambda+2r)]\mathcal{U}_{\alpha-\lambda+r}^{r-2}\left(\frac{3}{2}\right)$$

$$-3(2r-1)\mathcal{U}_{\alpha-\lambda+r}^{r-1}\left(\frac{3}{2}\right)\Big].$$

Hence, the Theorem is established. ■

Corollary 3.2.1 *For integers* $\alpha \geq 0$, *we have*

$$\sum_{a+b+c=\alpha} \mathcal{V}_{a}(\xi)\cdot\mathcal{V}_{b}(\xi)\cdot\mathcal{V}_{c}(\xi)$$

$$= \sum_{\lambda=0}^{\alpha} (-1)^{\lambda}\binom{3}{\lambda}\,[\mathcal{P}(\alpha,\lambda,\xi)\,\mathcal{U}_{\alpha-\lambda+1}(\xi) - \mathcal{Q}(\alpha,\lambda,\xi)\,\mathcal{U}_{\alpha-\lambda+2}(\xi)],$$

where

$$\mathcal{P}(\alpha,\lambda,\xi) = \left(\frac{3\xi(\alpha-\lambda+3)}{8(1-\xi^2)^2}\right),$$

and

$$\mathcal{Q}(\alpha,\lambda,\xi) = \frac{(\alpha-\lambda+2)}{8\ (1-\xi^2)^2}\left((\alpha-\lambda+4)-(\alpha-\lambda-1)\xi^2\right).$$

Proof. Take $r = 2$ in Theorem 3.2.1 coupled with the identity [57],

$$\mathcal{U}'_{\alpha}(\xi) = \frac{(\alpha+1)}{(1-\xi^2)}\mathcal{U}_{\alpha-1}(\xi) - \frac{\alpha\xi}{(1-\xi^2)}\mathcal{U}_{\alpha}(\xi). \qquad (3.3)$$

So, that we have,

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{V}_{\mathfrak{a}}(\xi)\cdot\mathcal{V}_{\mathfrak{b}}(\xi)\cdot\mathcal{V}_{\mathfrak{c}}(\xi)$$

$$= \frac{1}{2^2\ \ 2!\ (1-\xi^2)} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda}\binom{3}{\lambda}[3\xi\mathcal{U}'_{\alpha+2-\lambda}(\xi)$$

$$-[(\alpha-\lambda+2)(\alpha-\lambda+4)]\mathcal{U}_{\alpha-\lambda+2}(\xi)],$$

$$= \frac{1}{8\ (1-\xi^2)} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda}\binom{3}{\lambda}\left\{\left[3\xi\left(\frac{(\alpha-\lambda+3)}{(1-\xi^2)}\mathcal{U}_{\alpha+1-\lambda}(\xi)\right.\right.\right.$$

$$\left.\left.-\frac{\xi(\alpha-\lambda+2)}{(1-\xi^2)}\mathcal{U}_{\alpha+2-\lambda}(\xi)\right)\right]$$

$$\left.-[(\alpha-\lambda+2)(\alpha-\lambda+4)]\mathcal{U}_{\alpha-\lambda+2}(\xi)\right\},$$

$$= \sum_{\lambda=0}^{\alpha} (-1)^{\lambda}\binom{3}{\lambda}\left[\left(\frac{3\xi(\alpha-\lambda+3)}{8(1-\xi^2)^2}\right)\mathcal{U}_{\alpha-\lambda+1}(\xi)\right.$$

$$\left.-\left(\frac{3\xi^2(\alpha+2-\lambda)}{8(1-\xi^2)^2}+\frac{[(\alpha+2-\lambda)(\alpha+4-\lambda)]}{8\ (1-\xi^2)}\right)\mathcal{U}_{\alpha-\lambda+2}(\xi)\right],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{V}_{\mathfrak{a}}(\xi)\cdot\mathcal{V}_{\mathfrak{b}}(\xi)\cdot\mathcal{V}_{\mathfrak{c}}(\xi)$$

$$= \sum_{\lambda=0}^{\alpha} (-1)^{\lambda}\binom{3}{\lambda}\left[\left(\frac{3\xi(\alpha-\lambda+3)}{8(1-\xi^2)^2}\right)\mathcal{U}_{\alpha-\lambda+1}(\xi)\right.$$

$$\left.-\frac{(\alpha-\lambda+2)}{8\ (1-\xi^2)}\left(\frac{3\xi^2}{(1-\xi^2)}+(\alpha-\lambda+4)\right)\mathcal{U}_{\alpha-\lambda+2}(\xi)\right]$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{V}_{\mathfrak{a}}(\xi)\cdot\mathcal{V}_{\mathfrak{b}}(\xi)\cdot\mathcal{V}_{\mathfrak{c}}(\xi)$$

$$= \sum_{\lambda=0}^{\alpha} (-1)^{\lambda}\binom{3}{\lambda}\left[\left(\frac{3\xi(\alpha-\lambda+3)}{8(1-\xi^2)^2}\right)\mathcal{U}_{\alpha-\lambda+1}(\xi)\right.$$

$$\left.-\frac{(\alpha-\lambda+2)}{8\ (1-\xi^2)^2}\big((\alpha-\lambda+4)-(\alpha-\lambda-1)\xi^2\big)\mathcal{U}_{\alpha-\lambda+2}(\xi)\right].$$

Therefore,

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{V}_{\mathfrak{a}}(\xi)\cdot\mathcal{V}_{\mathfrak{b}}(\xi)\cdot\mathcal{V}_{\mathfrak{c}}(\xi)$$

$$=\sum_{\lambda=0}^{\alpha} (-1)^{\lambda}\binom{3}{\lambda}\left[\mathcal{P}(\alpha,\lambda,\xi)\,\mathcal{U}_{\alpha-\lambda+1}(\xi)-\mathcal{Q}(\alpha,\lambda,\xi)\,\mathcal{U}_{\alpha-\lambda+2}(\xi)\right],$$

where

$$\mathcal{P}(\alpha,\lambda,\xi)=\left(\frac{3\xi(\alpha-\lambda+3)}{8(1-\xi^2)^2}\right),$$

and

$$\mathcal{Q}(\alpha,\lambda,\xi)=\frac{(\alpha-\lambda+2)}{8\ (1-\xi^2)^2}\big((\alpha-\lambda+4)-(\alpha-\lambda-1)\xi^2\big).$$

This establishes the Corollary. ■

Corollary 3.2.2. *For integers $\alpha\geq 0$, we have*

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha} \mathcal{V}_{\mathfrak{a}}(\xi)\cdot\mathcal{V}_{\mathfrak{b}}(\xi)\cdot\mathcal{V}_{\mathfrak{c}}(\xi)\cdot\mathcal{V}_{d}(\xi)$$

$$=\sum_{\lambda=0}^{\alpha}\frac{(-1)^{\lambda}}{48}\binom{4}{\lambda}\left[R(\alpha,\lambda,\xi)\mathcal{U}_{\alpha-\lambda+2}(\xi)-S(\alpha.\lambda,\xi)\mathcal{U}_{\alpha-\lambda+3}(\xi)\right],$$

where

$$R(\alpha,\lambda,\xi)=\left[\frac{15\xi^2-(\alpha-\lambda+6)(\alpha-\lambda+2)(1-\xi^2)}{(1-\xi^2)^3}\right](\alpha-\lambda+4),$$

and

$$S(\alpha.\lambda,\xi)=(\alpha-\lambda+3)\xi\left(\frac{15\xi^2-[(\alpha-\lambda+6)(\alpha-\lambda+2)+5(\alpha-\lambda+5)](1-\xi^2)}{(1-\xi^2)^3}\right).$$

Proof. Take $r\ =\ 3$ in Theorem 3.2.1, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha} \mathcal{V}_{\mathfrak{a}}(\xi)\cdot\mathcal{V}_{\mathfrak{b}}(\xi)\cdot\mathcal{V}_{\mathfrak{c}}(\xi)\cdot\mathcal{V}_{d}(\xi)$$

$$=\frac{1}{2^3\ \ 3!\ (1-\xi^2)}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\Big[5\xi\,\mathcal{U}''_{\alpha-\lambda+3}(\xi)$$

$$-\big((\alpha-\lambda+2)(\alpha-\lambda+6)\big)\mathcal{U}'_{\alpha-\lambda+3}(\xi)\Big],$$

Using the identity [57]

$$(1-\xi^2)\mathcal{U}''_\alpha(\xi) = 3\xi\,\mathcal{U}'_\alpha(\xi) - \alpha(\alpha+2)\,\mathcal{U}_\alpha(\xi)\,,$$

We have

$$\sum_{\mathrm{a}+\mathrm{b}+\mathrm{c}+d=\alpha} \mathcal{V}_\mathrm{a}(\xi)\cdot\mathcal{V}_\mathrm{b}(\xi)\cdot\mathcal{V}_\mathrm{c}(\xi)\cdot\mathcal{V}_d(\xi)$$

$$= \frac{1}{48\,(1-\xi^2)} \sum_{\lambda=0}^{\alpha} (-1)^\lambda \binom{4}{\lambda}\Bigg(5\xi\left(\frac{3\xi}{(1-\xi^2)}\mathcal{U}'_{\alpha-\lambda+3}(\xi) - \frac{(\alpha-\lambda+3)(\alpha-\lambda+5)}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+3}(\xi)\right) - \big((\alpha-\lambda+6)(\alpha-\lambda+2)\big)\mathcal{U}'_{\alpha-\lambda+3}(\xi)\Bigg)$$

$$= \frac{1}{48\,(1-\xi^2)} \sum_{\lambda=0}^{\alpha} (-1)^\lambda \binom{4}{\lambda}\left[\frac{15\xi^2}{(1-\xi^2)} - (\alpha-\lambda+2)(\alpha-\lambda+6)\right]\mathcal{U}'_{\alpha-\lambda+3}(\xi) - \frac{5\xi(\alpha-\lambda+5)(\alpha-\lambda+3)}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+3}(\xi),$$

Now using equation (3.3), we have

$$\sum_{\mathrm{a}+\mathrm{b}+\mathrm{c}+d=\alpha} \mathcal{V}_\mathrm{a}(\xi)\cdot\mathcal{V}_\mathrm{b}(\xi)\cdot\mathcal{V}_\mathrm{c}(\xi)\cdot\mathcal{V}_d(\xi)$$

$$= \frac{1}{48\,(1-\xi^2)} \sum_{\lambda=0}^{\alpha} (-1)^\lambda \binom{4}{\lambda} \times$$

$$\left\{\left[\frac{15\xi^2 - (\alpha-\lambda+2)(\alpha-\lambda+6)(1-\xi^2)}{(1-\xi^2)}\right]\frac{(\alpha-\lambda+4)}{(1-\xi^2)}\mathcal{U}_{\alpha-\lambda+2}(\xi) - (\alpha-\lambda+3)\xi\left(\frac{15\xi^2 - (\alpha-\lambda+2)(\alpha-\lambda+6)(1-\xi^2)}{(1-\xi^2)^2} + \frac{5(\alpha-\lambda+5)}{(1-\xi^2)}\right)\mathcal{U}_{\alpha-\lambda+3}(\xi)\right\},$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha} \mathcal{V}_{\mathfrak{a}}(\xi)\cdot\mathcal{V}_{\mathfrak{b}}(\xi)\cdot\mathcal{V}_{\mathfrak{c}}(\xi)\cdot\mathcal{V}_{d}(\xi)$$

$$= \sum_{\lambda=0}^{\alpha} \frac{(-1)^{\lambda}}{48}\binom{4}{\lambda}\left[\frac{15\xi^2-(\alpha-\lambda+2)(\alpha-\lambda+6)(1-\xi^2)}{(1-\xi^2)^3}\right](\alpha-\lambda+4)\mathcal{U}_{\alpha-\lambda+2}(\xi)$$

$$-(\alpha-\lambda$$

$$+3)\xi\left(\frac{15\xi^2-[(\alpha-\lambda+2)(\alpha-\lambda+6)+5(\alpha-\lambda+5)](1-\xi^2)}{(1-\xi^2)^3}\right)\mathcal{U}_{\alpha-\lambda+3}(\xi),$$

Therefore,

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha} \mathcal{V}_{\mathfrak{a}}(\xi)\cdot\mathcal{V}_{\mathfrak{b}}(\xi)\cdot\mathcal{V}_{\mathfrak{c}}(\xi)\cdot\mathcal{V}_{d}(\xi)$$

$$= \sum_{\lambda=0}^{\alpha} \frac{(-1)^{\lambda}}{48}\binom{4}{\lambda}[R(\alpha,\lambda,\xi)\mathcal{U}_{\alpha-\lambda+2}(\xi)-S(\alpha.\lambda,\xi)\mathcal{U}_{\alpha-\lambda+3}(\xi)],$$

where,

$$R(\alpha,\lambda,\xi)=\left[\frac{15\xi^2-(\alpha-\lambda+2)(\alpha-\lambda+6)(1-\xi^2)}{(1-\xi^2)^3}\right](\alpha-\lambda+4),$$

and

$S(\alpha.\lambda,\xi)=$

$$(\alpha-\lambda+3)\xi\left(\frac{15\xi^2-[(\alpha-\lambda+2)(\alpha-\lambda+6)+5(\alpha-\lambda+5)](1-\xi^2)}{(1-\xi^2)^3}\right).$$

This establishes the Corollary. ■

Corollary 3.2.3. *For integers* $\alpha \geq 0$, *we have*

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{W}_{\mathfrak{a}}(\xi)\cdot\mathcal{W}_{\mathfrak{b}}(\xi)\cdot\mathcal{W}_{\mathfrak{c}}(\xi)$$

$$= \sum_{\lambda=0}^{\alpha} \binom{3}{\lambda}[\mathcal{P}(\alpha,\lambda,\xi)\,\mathcal{U}_{\alpha-\lambda+1}(\xi)-\mathcal{Q}(\alpha,\lambda,\xi)\,\mathcal{U}_{\alpha-\lambda+2}(\xi)],$$

where

$$\mathcal{P}(\alpha,\lambda,\xi)=\left(\frac{3\xi(\alpha-\lambda+3)}{8(1-\xi^2)^2}\right),$$

and

$$\mathcal{Q}(\alpha,\lambda,\xi)=\frac{(\alpha-\lambda+2)}{8\ (1-\xi^2)^2}\big((\alpha-\lambda+4)-(\alpha-\lambda-1)\xi^2\big).$$

Proof. For the proof of the Corollary, we will take $r\ =\ 2$ in Theorem 3.2.2 and proceed similarly as in the case of the Corollary 3.2.1 to achieve the desired results. ■

Corollary 3.2.4. *For integer $\alpha\ \geq 0$, we have*

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha}\mathcal{W}_{\mathfrak{a}}(\xi)\cdot\mathcal{W}_{\mathfrak{b}}(\xi)\cdot\mathcal{W}_{\mathfrak{c}}(\xi)\cdot\mathcal{W}_{d}(\xi)$$

$$=\sum_{\lambda=0}^{\alpha}\frac{1}{48}\binom{4}{\lambda}\left[R(\alpha,\lambda,\xi)\mathcal{U}_{\alpha-\lambda+2}(\xi)-S(\alpha.\lambda,\xi)\mathcal{U}_{\alpha-\lambda+3}(\xi)\right],$$

where

$$R(\alpha,\lambda,\xi)=\left[\frac{15\xi^2-(\alpha-\lambda+2)(\alpha-\lambda+6)(1-\xi^2)}{(1-\xi^2)^3}\right](\alpha-\lambda+4),$$

and

$$S(\alpha.\lambda,\xi)=(\alpha-\lambda+3)\xi\left(\frac{15\xi^2-[(\alpha-\lambda+2)(\alpha-\lambda+6)+5(\alpha-\lambda+5)](1-\xi^2)}{(1-\xi^2)^3}\right).$$

Proof. For the proof of the Corollary 3.2.4, we will take $r\ =\ 3$ in Theorem 3.2.2 and proceed similarly as in case of the Corollary 3.2.2 to achieve the desired result. ■

Corollary 3.2.5. *For integer $\alpha\ \geq 0$,*

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha}\mathcal{F}_{2\mathfrak{a}+1}\cdot\mathcal{F}_{2\mathfrak{b}+1}\cdot\mathcal{F}_{2\mathfrak{c}+1}=\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[A_{\alpha,\lambda}\,\mathcal{F}_{2\alpha-2\lambda+4}+B_{\alpha,\lambda}\mathcal{F}_{2\alpha-2\lambda+6}\right],$$

where

$A_{\alpha,\lambda}=\frac{9}{25}(\alpha-\lambda+3)$ and $B_{\alpha,\lambda}=\frac{1}{50}(\alpha-\lambda+2)(5\alpha-5\lambda-7).$

Proof. Using equation (1.65) *(viii)* together with $\mathcal{U}_\alpha\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+2}$ in Theorem 3.2.3 for $\xi = \frac{3}{2}$ with *r =2*, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{F}_{2\mathfrak{a}+1}\cdot\mathcal{F}_{2\mathfrak{b}+1}\cdot\mathcal{F}_{2\mathfrak{c}+1}$$

$$= \frac{1}{2\cdot 2!}\sum_{\lambda=0}^{\alpha}\frac{(-1)^\lambda}{5}\binom{3}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+4)]\mathcal{U}_{\alpha-\lambda+2}\left(\frac{3}{2}\right)\right.$$

$$\left. -3(3)\mathcal{U}'_{\alpha-\lambda+2}\left(\frac{3}{2}\right)\right]$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{F}_{2\mathfrak{a}+1}\cdot\mathcal{F}_{2\mathfrak{b}+1}\cdot\mathcal{F}_{2\mathfrak{c}+1}$$

$$= \frac{1}{4}\sum_{\lambda=0}^{\alpha}\frac{(-1)^\lambda}{5}\binom{3}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+4)]\mathcal{U}_{\alpha-\lambda+2}\left(\frac{3}{2}\right)\right.$$

$$\left. -9\left(\frac{(\alpha-\lambda+3)}{\left(1-\left(\frac{3}{2}\right)^2\right)}\ \mathcal{U}_{\alpha-\lambda+1}\left(\frac{3}{2}\right) - \frac{(\alpha-\lambda+2)\left(\frac{3}{2}\right)}{\left(1-\left(\frac{3}{2}\right)^2\right)}\mathcal{U}_{\alpha-\lambda+2}\left(\frac{3}{2}\right)\right)\right].$$

$$= \frac{1}{20}\sum_{\lambda=0}^{\alpha}(-1)^\lambda\binom{3}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+4)]\mathcal{F}_{2\alpha-2\lambda+6}\right.$$

$$\left. -9\left(\frac{(\alpha-\lambda+3)}{\left(-\frac{5}{4}\right)}\ \mathcal{F}_{2\alpha-2\lambda+4} - \frac{(\alpha-\lambda+2)\left(\frac{3}{2}\right)}{\left(-\frac{5}{4}\right)}\mathcal{F}_{2\alpha-2\lambda+6}\right)\right],$$

$$= \frac{1}{20}\sum_{\lambda=0}^{\alpha}(-1)^\lambda\binom{3}{\lambda}\left[2[(\alpha-\lambda+2)(\alpha-\lambda+4)]\mathcal{F}_{2\alpha-2\lambda+6}\right.$$

$$-9\left(\left(-\frac{4}{5}\right)(\alpha-\lambda+3)\,\mathcal{F}_{2\alpha-2\lambda+4}\right.$$

$$\left.\left. -\left(-\frac{4}{5}\right)(\alpha-\lambda+2)\left(\frac{3}{2}\right)\mathcal{F}_{2\alpha-2\lambda+6}\right)\right],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{F}_{2\mathfrak{a}+1}\cdot\mathcal{F}_{2\mathfrak{b}+1}\cdot\mathcal{F}_{2\mathfrak{c}+1}$$

$$=\frac{1}{10}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\Bigg[[(\alpha-\lambda+2)(\alpha-\lambda+4)]\mathcal{F}_{2\alpha-2\lambda+6}$$

$$-9\left(\left(-\frac{2}{5}\right)(\alpha-\lambda+3)\,\mathcal{F}_{2\alpha-2\lambda+4}-\left(-\frac{3}{5}\right)(\alpha-\lambda+2)\mathcal{F}_{2\alpha-2\lambda+6}\right)\Bigg],$$

$$=\frac{1}{10}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\Bigg[[(\alpha-\lambda+2)(\alpha-\lambda+4)]\mathcal{F}_{2\alpha-2\lambda+6}$$

$$+9\left(\left(\frac{2}{5}\right)(\alpha-\lambda+3)\,\mathcal{F}_{2\alpha-2\lambda+4}\left(\frac{3}{2}\right)-\left(\frac{3}{5}\right)(\alpha-\lambda+2)\mathcal{F}_{2\alpha-2\lambda+6}\right)\Bigg],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{F}_{2\mathfrak{a}+1}\cdot\mathcal{F}_{2\mathfrak{b}+1}\cdot\mathcal{F}_{2\mathfrak{c}+1}$$

$$=\frac{1}{10}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\Bigg[(\alpha-\lambda+2)\left((\alpha-\lambda+4)-\frac{27}{5}\right)\mathcal{F}_{2\alpha-2\lambda+6}$$

$$+\frac{18}{5}(\alpha-\lambda+3)\,\mathcal{F}_{2\alpha-2\lambda+4}\Bigg],$$

$$=\frac{1}{10}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\Bigg[\frac{1}{5}(\alpha-\lambda+2)(5\alpha-5\lambda-7)\mathcal{F}_{2\alpha-2\lambda+6}$$

$$+\frac{18}{5}(\alpha-\lambda+3)\,\mathcal{F}_{2\alpha-2\lambda+4}\Bigg],$$

$$=\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\Bigg[\frac{9}{25}(\alpha-\lambda+3)\,\mathcal{F}_{2\alpha-2\lambda+4}$$

$$+\frac{1}{50}(\alpha-\lambda+2)(5\alpha-5\lambda-7)\mathcal{F}_{2\alpha-2\lambda+6}\Bigg].$$

Therefore,

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{F}_{2\mathfrak{a}+1}\cdot\mathcal{F}_{2\mathfrak{b}+1}\cdot\mathcal{F}_{2\mathfrak{c}+1}=\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{3}{\lambda}\left[A_{\alpha,\lambda}\,\mathcal{F}_{2\alpha-2\lambda+4}+B_{\alpha,\lambda}\mathcal{F}_{2\alpha-2\lambda+6}\right],$$

where

$$A_{\alpha,\lambda}=\tfrac{9}{25}(\alpha-\lambda+3)\text{ and }B_{\alpha,\lambda}=\tfrac{1}{50}(\alpha-\lambda+2)(5\alpha-5\lambda-7).$$

This proves the Corollary 3.2.5. ■

Corollary 3.2.6. *For integer* $\alpha \geq 0$, *we have*

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha} \mathcal{F}_{2\mathfrak{a}+1} \cdot \mathcal{F}_{2\mathfrak{b}+1} \cdot \mathcal{F}_{2\mathfrak{c}+1} \cdot \mathcal{F}_{2d+1}$$

$$= \frac{1}{150} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{4}{\lambda} \left[C_{\alpha,\lambda} \mathcal{F}_{2\alpha-2\lambda+8} - D_{\alpha,\lambda} \mathcal{F}_{2\alpha-2\lambda+6}\right],$$

where

$$C_{\alpha,\lambda} = 3(\alpha - \lambda + 3)\Big(\big((\alpha - \lambda + 2)(\alpha - \lambda + 6) + 27\big) - 5(\alpha - \lambda + 5)\Big),$$

and

$$D_{\alpha,\lambda} = 2[(\alpha - \lambda + 2)(\alpha - \lambda + 6) + 27](\alpha - \lambda + 4).$$

Proof. To prove the Corollary, we will proceed as in the case of Corollary 3.2.5 and use equation (1.65) (*viii)* in Theorem 3.2.3 for $\xi = \frac{3}{2}$ with $r = 3$, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha} \mathcal{F}_{2\mathfrak{a}+1} \cdot \mathcal{F}_{2\mathfrak{b}+1} \cdot \mathcal{F}_{2\mathfrak{c}+1} \cdot \mathcal{F}_{2d+1}$$

$$= \frac{1}{5} \cdot \frac{1}{2^2 3!} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{4}{\lambda} \left[2[(\alpha - \lambda + 2)(\alpha - \lambda + 6)]\mathcal{U}'_{\alpha-\lambda+3}\left(\frac{3}{2}\right) - 3 \right.$$

$$\left. \cdot 5\, \mathcal{U}''_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\right],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha} \mathcal{F}_{2\mathfrak{a}+1} \cdot \mathcal{F}_{2\mathfrak{b}+1} \cdot \mathcal{F}_{2\mathfrak{c}+1} \cdot \mathcal{F}_{2d+1}$$

$$= \frac{1}{120} \sum_{\lambda=0}^{\alpha} (-1)^{\lambda} \binom{4}{\lambda} \left[2[(\alpha - \lambda + 2)(\alpha - \lambda + 6)]\mathcal{U}'_{\alpha-\lambda+3}\left(\frac{3}{2}\right) \right.$$

$$- 15 \left(\frac{3\left(\frac{3}{2}\right)}{\left(1 - \left(\frac{3}{2}\right)^2\right)} \mathcal{U}'_{\alpha-\lambda+3}\left(\frac{3}{2}\right) \right.$$

$$\left.\left. - \frac{(\alpha - \lambda + 3)(\alpha - \lambda + 5)}{\left(1 - \left(\frac{3}{2}\right)^2\right)} \mathcal{U}_{\alpha-\lambda+3}\left(\frac{3}{2}\right) \right) \right],$$

$$= \frac{1}{120}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\Bigg[[(\alpha-\lambda+2)(\alpha-\lambda+6)+\ 54]\mathcal{U}'_{\alpha-\lambda+3}\left(\frac{3}{2}\right)$$

$$-12\left((\alpha-\lambda+3)(\alpha-\lambda+5)\mathcal{U}_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\right)\Bigg],$$

$$\sum_{\mathrm{a+b+c}+d=\alpha}\mathcal{F}_{2\mathrm{a}+1}\cdot\mathcal{F}_{2\mathrm{b}+1}\cdot\mathcal{F}_{2\mathrm{c}+1}\cdot\mathcal{F}_{2d+1}$$

$$= \frac{1}{120}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\Bigg[[(\alpha-\lambda+2)(\alpha-\lambda+6)$$

$$+\ 54]\left(\frac{(\alpha-\lambda+4)}{\left(1-\left(\frac{3}{2}\right)^{2}\right)}\ \mathcal{U}_{\alpha-\lambda+2}\left(\frac{3}{2}\right)-\frac{(\alpha-\lambda+3)\left(\frac{3}{2}\right)}{\left(1-\left(\frac{3}{2}\right)^{2}\right)}\mathcal{U}_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\right)$$

$$-12(\alpha-\lambda+3)(\alpha-\lambda+5)\mathcal{U}_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\Bigg],$$

$$= \frac{1}{120}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\Bigg[[(\alpha-\lambda+2)(\alpha-\lambda+6)$$

$$+\ 54]\left(\frac{(\alpha-\lambda+4)}{\left(-\frac{5}{4}\right)}\ \mathcal{U}_{\alpha-\lambda+2}\left(\frac{3}{2}\right)-\frac{(\alpha-\lambda+3)\left(\frac{3}{2}\right)}{\left(-\frac{5}{4}\right)}\mathcal{U}_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\right)$$

$$-12(\alpha-\lambda+3)(\alpha-\lambda+5)\mathcal{U}_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\Bigg],$$

$$= \frac{1}{60}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\Bigg[\left(-\frac{4}{5}\right)[(\alpha-\lambda+2)(\alpha-\lambda+6)+\ 27](\alpha-\lambda+4)\mathcal{U}_{\alpha-\lambda+2}\left(\frac{3}{2}\right)$$

$$+\left(\frac{6}{5}\right)\big((\alpha-\lambda+2)(\alpha-\lambda+6)+\ 27\big)(\alpha-\lambda+3)\mathcal{U}_{\alpha-\lambda+3}\left(\frac{3}{2}\right)$$

$$-6(\alpha-\lambda+3)(\alpha-\lambda+5)\mathcal{U}_{\alpha-\lambda+3}\left(\frac{3}{2}\right)\Bigg],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha}\mathcal{F}_{2\mathfrak{a}+1}\cdot\mathcal{F}_{2\mathfrak{b}+1}\cdot\mathcal{F}_{2\mathfrak{c}+1}\cdot\mathcal{F}_{2d+1}$$

$$=\frac{1}{60}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\Bigg[\left(-\frac{4}{5}\right)[(\alpha-\lambda+2)(\alpha-\lambda+6)+\ 27](\alpha-\lambda$$

$$+4)\mathcal{F}_{2\alpha-2\lambda+6}$$

$$+(\alpha-\lambda+3)\left(\left(\frac{6}{5}\right)\big((\alpha-\lambda+2)(\alpha-\lambda+6)+\ 27\big)\right.$$

$$\left.-6(\alpha-\lambda+5)\right)\mathcal{F}_{2\alpha-2\lambda+8}\Bigg],$$

$$=\frac{1}{30}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\Bigg[\left(-\frac{2}{5}\right)[(\alpha-\lambda+2)(\alpha-\lambda+6)+\ 27](\alpha-\lambda$$

$$+4)\mathcal{F}_{2\alpha-2\lambda+6}$$

$$+\left(\frac{3}{5}\right)(\alpha-\lambda+3)\Big(\big((\alpha-\lambda+2)(\alpha-\lambda+6)+\ 27\big)$$

$$-5(\alpha-\lambda+5)\Big)\mathcal{F}_{2\alpha-2\lambda+8}\Bigg],$$

$$=\frac{1}{150}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}\Big[3(\alpha-\lambda+3)\Big(\big((\alpha-\lambda+2)(\alpha-\lambda+6)+\ 27\big)$$

$$-5(\alpha-\lambda+5)\Big)\mathcal{F}_{2\alpha-2\lambda+8}\Big]$$

$$-2[(\alpha-\lambda+2)(\alpha-\lambda+6)+\ 27](\alpha-\lambda+4)\mathcal{F}_{2\alpha-2\lambda+6}.$$

Therefore,

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+d=\alpha}\mathcal{F}_{2\mathfrak{a}+1}\cdot\mathcal{F}_{2\mathfrak{b}+1}\cdot\mathcal{F}_{2\mathfrak{c}+1}\cdot\mathcal{F}_{2d+1}$$

$$=\frac{1}{150}\sum_{\lambda=0}^{\alpha}(-1)^{\lambda}\binom{4}{\lambda}C_{\alpha,\lambda}\mathcal{F}_{2\alpha-2\lambda+8}-D_{\alpha,\lambda}\mathcal{F}_{2\alpha-2\lambda+6},$$

where

$$C_{\alpha,\lambda} = 3(\alpha - \lambda + 3)\Big(\big((\alpha - \lambda + 2)(\alpha - \lambda + 6) + 27\big) - 5(\alpha - \lambda + 5)\Big),$$

and

$$D_{\alpha,\lambda} = 2[(\alpha - \lambda + 2)(\alpha - \lambda + 6) + 27](\alpha - \lambda + 4).$$

This establishes the Corollary. ■

Corollary 3.2.7. *For integers $n, r \geq 0$, we have*

$$\sum_{a+b+c=\alpha} \mathcal{L}_{2a+1} \cdot \mathcal{L}_{2b+1} \cdot \mathcal{L}_{2c+1} = \sum_{\lambda=0}^{\alpha} \binom{3}{\lambda} \left[A_{\alpha,\lambda}\, \mathcal{F}_{2\alpha-2\lambda+4} + B_{\alpha,\lambda} \mathcal{F}_{2\alpha-2\lambda+6}\right]$$

where

$$A_{\alpha,\lambda} = \frac{9}{25}(\alpha - \lambda + 3) \; and \;\; B_{\alpha,\lambda} = \frac{1}{50}(\alpha - \lambda + 2)(5\alpha - 5\lambda - 7).$$

Proof. For the proof of the Corollary 3.2.7, we will proceed similarly as in the case of Corollary 3.2.5 and use equation (1.65) (*x*) in Theorem 3.2.4 for $\xi = \frac{3}{2}$ with $r = 2$ to achieve the desired results. ■

Corollary 3.2.8. *For integer $\alpha \geq 0$, we have*

$$\sum_{a+b+c+d=\alpha} \mathcal{L}_{2a+1} \cdot \mathcal{L}_{2b+1} \cdot \mathcal{L}_{2c+1} \cdot l_{2d+1}$$

$$= \frac{1}{150} \sum_{\lambda=0}^{\alpha} \binom{4}{\lambda} \left[C_{\alpha,\lambda} \mathcal{F}_{2\alpha-2\lambda+8} - D_{\alpha,\lambda} \mathcal{F}_{2\alpha-2\lambda+6}\right],$$

where

$$C_{\alpha,\lambda} = 3(\alpha - \lambda + 3)\Big(\big((\alpha - \lambda + 2)(\alpha - \lambda + 6) + 27\big) - 5(\alpha - \lambda + 5)\Big),$$

and

$$D_{\alpha,\lambda} = 2[(\alpha - \lambda + 2)(\alpha - \lambda + 6) + 27](\alpha - \lambda + 4).$$

Proof. For the proof of the Corollary 3.2.8, we will proceed similarly as in case of Corollary 3.2.6 and use equation (1.65) (*x*) in Theorem 3.2.4 for $\xi = \frac{3}{2}$ with $r = 3$ to achieve the desired results. ■

CHAPTER 4

IDENTITIES ON CHEBYSHEV POLYNOMIALS OF THIRD AND FOURTH KINDS AND THEIR DERIVATIVES

4.1 Introduction

In the first section of this chapter, we shall derive the explicit formulae for the 3rd and 4th kinds of Chebyshev polynomials and investigate their connections with the negative indexed Fibonacci polynomials. Similar results for their derivatives are obtained.

In the second section, we will express sums of finite products of the 3rd and 4th kinds of Chebyshev polynomials as a linear combination of Jacobi, Fibonacci, Gegenbauer, Pell, Vieta-Fibonacci, and Vieta-Pell polynomials. Similar identities for Lucas and Fibonacci numbers are obtained.

4.2 Explicit formulae on Chebyshev polynomial

This section focuses on the development of explicit formulae for the of Chebyshev polynomials of 3rd and 4th kinds and their derivatives and express their connections with the negative indexed Fibonacci polynomials.

Many authors have investigated the Chebyshev polynomials and obtained several explicit formulations [45-49, 58]. For instance, Yang Li in [47,48] has derived the explicit formulae for the 1st and 2nd kinds of Chebyshev polynomials. Similarly, in this section, explicit formulae for the 3rd and 4th kinds of Chebyshev polynomials with odd and even indices as well as their derivatives will be derived, followed by an investigation of their relation with the negative indexed Fibonacci polynomials. The main findings are:

Theorem 4.2.1. *For any positive integer α and $\zeta \in R$,*

$$\mathcal{V}_{2\alpha}(\zeta) = (2\zeta)^{2\alpha} + \sum_{\nu=0}^{\alpha-1} (-1)^{\alpha-\nu}\, 2^{2\nu+1} \binom{\alpha+\nu}{2\nu} \left[\frac{(\zeta)^{2\nu}}{2} + \left(\frac{\alpha-\nu}{2\nu+1}\right) (\zeta)^{2\nu+1} \right],$$

$$\mathcal{V}_{2\alpha+1}(\zeta) = \sum_{\nu=0}^{\alpha}(-1)^{\alpha-\nu}\, 2^{2\nu+1}\binom{\alpha+\nu}{2\nu}\left[\left(\frac{\alpha+\nu+1}{2\nu+1}\right)(\zeta)^{2\nu+1} - \frac{(\zeta)^{2\nu}}{2}\right].$$

Proof. From [48], for any positive integer α, we have

$$\mathcal{T}_{2\alpha}(\zeta) = \sum_{\nu=0}^{\alpha}\frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, \alpha}{\alpha+\nu}\binom{\alpha+\nu}{2\nu}(\zeta)^{2\nu}. \tag{4.1}$$

$$\mathcal{T}_{2\alpha+1}(\zeta) = \sum_{\nu=0}^{\alpha}\frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\,(2\alpha+1)}{\alpha+\nu+1}\binom{\alpha+\nu+1}{2\nu+1}(\zeta)^{2\nu+1}. \tag{4.2}$$

Using the fact**,**

$$\mathcal{T}'_{\alpha}(\zeta) = \alpha\,\mathcal{U}_{\alpha-1}(\zeta). \tag{4.3}$$

$$\mathcal{U}_{2\alpha}(\zeta) = \frac{1}{(2\alpha+1)}\mathcal{T}'_{2\alpha+1}(\zeta).$$

and

$$\mathcal{T}'_{2\alpha+1}(\zeta) = \sum_{\nu=0}^{\alpha}\frac{(-1)^{\alpha-\nu}2^{2\nu}(2\alpha+1)}{\alpha+\nu+1}\binom{\alpha+\nu+1}{2\nu+1}(2\nu+1)(\zeta)^{2\nu}.$$

which implies

$$\mathcal{U}_{2\alpha}(\zeta) = \sum_{\nu=0}^{\alpha}\frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\,(2\nu+1)}{\alpha+\nu+1}\binom{\alpha+\nu+1}{2\nu+1}(\zeta)^{2\nu}. \tag{4.4}$$

Similarly,

$$\mathcal{U}_{2\alpha+1}(\zeta) = \frac{1}{2(\alpha+1)}\mathcal{T}'_{2(\alpha+1)}(\zeta),$$

implies

$$\mathcal{U}_{2\alpha+1}(\zeta) = \sum_{\nu=0}^{\alpha}\frac{(-1)^{\alpha-\nu}\, 2^{2\nu+2}\,(\nu+1)}{\alpha+\nu+2}\binom{\alpha+\nu+2}{2\nu+2}(\zeta)^{2\nu+1}. \tag{4.5}$$

Consequently,

$$\mathcal{U}_{2\alpha-1}(\zeta) = \sum_{\nu=0}^{\alpha-1}\frac{(-1)^{\alpha-\nu-1}\, 2^{2\nu+2}\,(\nu+1)}{\alpha+\nu+1}\binom{\alpha+\nu+1}{2\nu+2}(\zeta)^{2\nu+1}. \tag{4.6}$$

Thus, using Theorem 1.65 *(ii)* with equation (4.4) and (4.6), we proceed as,

$$\begin{aligned}
\mathcal{V}_{2\alpha}(\zeta) = \mathcal{U}_{2\alpha}(\zeta) - \mathcal{U}_{2\alpha-1}(\zeta)& \\
&= \sum_{\nu=0}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (2\nu+1)}{\alpha+\nu+1} \binom{\alpha+\nu+1}{2\nu+1} (\zeta)^{2\nu} \\
&- \sum_{\nu=0}^{\alpha} \frac{(-1)^{\alpha-\nu-1}\, 2^{2\nu+2}\, (\nu+1)}{\alpha+\nu+1} \binom{\alpha+\nu+2}{2\nu+2} (\zeta)^{2\nu+1}, \\
&= \sum_{\nu=0}^{\alpha} (-1)^{\alpha-\nu}\, 2^{2\nu} \binom{\alpha+\nu}{2\nu} (\zeta)^{2\nu} \\
&- \sum_{\nu=0}^{\alpha-1} \frac{(-1)^{\alpha-\nu-1}\, 2^{2\nu+1}\, (\alpha-\nu)}{(2\nu+1)} \binom{\alpha+\nu}{2\nu} (\zeta)^{2\nu+1}, \\
&= (2\zeta)^{2\alpha} + \sum_{\nu=0}^{\alpha-1} (-1)^{\alpha-\nu}\, 2^{2\nu} \binom{\alpha+\nu}{2\nu} (\zeta)^{2\nu} \\
&+ \sum_{\nu=0}^{\alpha-1} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+1}\, (\alpha-\nu)}{(2\nu+1)} \binom{\alpha+\nu}{2\nu} (\zeta)^{2\nu+1}.
\end{aligned}$$

$$\mathcal{V}_{2\alpha}(\zeta) = (2\zeta)^{2\alpha} + \sum_{\nu=0}^{\alpha-1} (-1)^{\alpha-\nu}\, 2^{2\nu+1} \binom{\alpha+\nu}{2\nu} \left[\frac{(\zeta)^{2\nu}}{2} + \frac{(\alpha-\nu)}{(2\nu+1)} (\zeta)^{2\nu+1} \right]. \quad (4.7)$$

Similarly, using Theorem 1.65 *(ii)*) and (4.4) and (4.5)

$$\begin{aligned}
\mathcal{V}_{2\alpha+1}(\zeta) = \mathcal{U}_{2\alpha+1}(\zeta) - \mathcal{U}_{2\alpha}(\zeta)& \\
&= \sum_{\nu=0}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+2}\, (\nu+1)}{\alpha+\nu+2} \binom{\alpha+\nu+2}{2\nu+2} (\zeta)^{2\nu+1} \\
&- \sum_{\nu=0}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (2\nu+1)}{\alpha+\nu+1} \binom{\alpha+\nu+1}{2\nu+1} (\zeta)^{2\nu}, \\
&= \sum_{\nu=0}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+1}\, (\alpha+\nu+1)}{(2\nu+1)} \binom{\alpha+\nu}{2\nu} (\zeta)^{2\nu+1} \\
&- \sum_{\nu=0}^{\alpha} (-1)^{\alpha-\nu}\, 2^{2\nu} \binom{\alpha+\nu}{2\nu} (\zeta)^{2\nu},
\end{aligned}$$

$$\mathcal{V}_{2\alpha+1}(\zeta) = \sum_{\nu=0}^{\alpha}(-1)^{\alpha-\nu}\, 2^{2\nu}\binom{\alpha+\nu}{2\nu}\left[\frac{2(\alpha+\nu+1)}{(2\nu+1)}\,(\zeta)^{2\nu+1} - (\zeta)^{2\nu}\right],$$

$$= \sum_{\nu=0}^{\alpha}(-1)^{\alpha-\nu}\, 2^{2\nu+1}\binom{\alpha+\nu}{2\nu}\left[\frac{(\alpha+\nu+1)}{(2\nu+1)}\,(\zeta)^{2\nu+1} - \frac{(\zeta)^{2\nu}}{2}\right],$$

Therefore,

$$\mathcal{V}_{2\alpha+1}(\zeta) = \sum_{\nu=0}^{\alpha}(-1)^{\alpha-\nu}\, 2^{2\nu+1}\binom{\alpha+\nu}{2\nu}\left[\frac{(\alpha+\nu+1)}{(2\nu+1)}\,(\zeta)^{2\nu+1} - \frac{(\zeta)^{2\nu}}{2}\right]. \qquad (4.8)$$

This proves the Theorem 4.2.1. ■

Theorem 4.2.2. *For any positive integer α and $\zeta \in R$,*

$$\mathcal{W}_{2\alpha}(\zeta) = (2\zeta)^{2\alpha} + \sum_{\nu=0}^{\alpha-1}(-1)^{\alpha-\nu}\, 2^{2\nu+1}\binom{\alpha+\nu}{2\nu}\left[\frac{(\zeta)^{2\nu}}{2} - \left(\frac{\alpha-\nu}{2\nu+1}\right)(\zeta)^{2\nu+1}\right],$$

$$\mathcal{W}_{2\alpha+1}(\zeta) = \sum_{\nu=0}^{\alpha}(-1)^{\alpha-\nu}\, 2^{2\nu+1}\binom{\alpha+\nu}{2\nu}\left[\frac{(\zeta)^{2\nu}}{2} + \left(\frac{\alpha+\nu+1}{2\nu+1}\right)(\zeta)^{2\nu+1}\right].$$

Proof. Using Theorem 1.65 *(xii)* and equation (4.7), we have

$$\mathcal{W}_{2\alpha}(\zeta) = (-1)^{2\alpha}\mathcal{V}_{2\alpha}(-\zeta)$$

$$= \mathcal{V}_{2\alpha}(-\zeta) = (-2\zeta)^{2\alpha}$$

$$+ \sum_{\nu=0}^{\alpha-1}(-1)^{\alpha-\nu}\, 2^{2\nu+1}\binom{\alpha+\nu}{2\nu}\left[\frac{(-\zeta)^{2\nu}}{2} + \frac{(\alpha-\nu)}{(2\nu+1)}\,(-\zeta)^{2\nu+1}\right],$$

$$= (2\zeta)^{2\alpha} + \sum_{\nu=0}^{\alpha-1}(-1)^{\alpha-\nu}\, 2^{2\nu+1}\binom{\alpha+\nu}{2\nu}\left[\frac{(\zeta)^{2\nu}}{2} - \frac{(\alpha-\nu)}{(2\nu+1)}\,(\zeta)^{2\nu+1}\right].$$

Therefore

$$\mathcal{W}_{2\alpha}(\zeta) = (2\zeta)^{2\alpha} + \sum_{\nu=0}^{\alpha-1}(-1)^{\alpha-\nu}\, 2^{2\nu+1}\binom{\alpha+\nu}{2\nu}\left[\frac{(\zeta)^{2\nu}}{2} - \frac{(\alpha-\nu)}{(2\nu+1)}\,(\zeta)^{2\nu+1}\right].$$

Similarly, using Theorem 1.65 *(xii)* and equation (4.8),

$$\mathcal{W}_{2\alpha+1}(\zeta) = (-1)^{2\alpha+1}\mathcal{V}_{2\alpha+1}(-\zeta) = -\mathcal{V}_{2\alpha+1}(-\zeta),$$

$$= -\sum_{\nu=0}^{\alpha}(-1)^{\alpha-\nu}\, 2^{2\nu+1}\binom{\alpha+\nu}{2\nu}\left[\frac{(\alpha+\nu+1)}{(2\nu+1)}\,(-\zeta)^{2\nu+1} - \frac{(-\zeta)^{2\nu}}{2}\right]$$

$$= \sum_{\nu=0}^{\alpha} (-1)^{\alpha-\nu}\, 2^{2\nu+1} \binom{\alpha+\nu}{2\nu} \left[\frac{(\alpha+\nu+1)}{(2\nu+1)}\, (\zeta)^{2\nu+1} + \frac{(\zeta)^{2\nu}}{2} \right],$$

$$= \sum_{\nu=0}^{\alpha} (-1)^{\alpha-\nu}\, 2^{2\nu+1} \binom{\alpha+\nu}{2\nu} \left[\frac{(\zeta)^{2\nu}}{2} + \frac{(\alpha+\nu+1)}{(2\nu+1)}\, (\zeta)^{2\nu+1} \right],$$

Therefore,

$$\mathcal{W}_{2\alpha+1}(\zeta) = \sum_{\nu=0}^{\alpha} (-1)^{\alpha-\nu}\, 2^{2\nu+1} \binom{\alpha+\nu}{2\nu} \left[\frac{(\zeta)^{2\nu}}{2} + \frac{(\alpha+\nu+1)}{(2\nu+1)}\, (\zeta)^{2\nu+1} \right].$$

This proves the Theorem 4.2.2. ■

Theorem 4.2.3. *For integer n, r* (> 0) *and* $\zeta \in R$,

$$\mathcal{V}_{2\alpha}{}^{r}(\zeta) = \sum_{\nu=\left\lceil \frac{r}{2} \right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (\alpha+\nu)!}{(\alpha-\nu)!\, (2\nu-r)!} \zeta^{2\nu-r}$$
$$- \sum_{\nu=\left\lceil \frac{r-1}{2} \right\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\nu-1}\, 2^{2\nu+1}\, (\alpha+\nu)!}{(\alpha-\nu-1)!\, (2\nu+1-r)!} \zeta^{(2\nu+1)-r}$$

$$\mathcal{V}_{2\alpha+1}{}^{r}(\zeta) = \sum_{\nu=\left\lceil \frac{r-1}{2} \right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+1}\, (\alpha+\nu+1)!}{(\alpha-\nu)!\, (2\nu+1-r)!} \zeta^{(2\nu+1)-r}$$
$$- \sum_{\nu=\left\lceil \frac{r}{2} \right\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (\alpha+\nu)!}{(\alpha-\nu)!\, (2\nu-r)!} \zeta^{(2\nu-r)}$$

where $\lceil \zeta \rceil$ *denotes ceiling function.*

Proof. Differentiating equations (4.4), (4.5) and (4.6) r times, we have

$$\mathcal{U}^{r}{}_{2\alpha}(\zeta) = \sum_{\nu=\left\lceil \frac{r}{2} \right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (2\nu+1)}{\alpha+\nu+1} \binom{\alpha+\nu+1}{2\nu+1} (2\nu)\, (2\nu-1)\, (2\nu - 2) \ldots \ldots (2\nu-r+1)\, (\zeta)^{2\nu-r},$$

$$\mathcal{U}^r{}_{2\alpha}(\zeta) = \sum_{\nu=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (2\nu+1)(2\nu)!}{(\alpha+\nu+1)(2\nu-r)!} \binom{\alpha+\nu+1}{2\nu+1} (\zeta)^{2\nu-r},$$

$$= \sum_{\nu=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (2\nu+1)(2\nu)!\,(\alpha+\nu+1)!}{(\alpha+\nu+1)(2\nu-r)!\,(2k+1)!\,(\alpha-k)!} (\zeta)^{2\nu-r}.$$

$$\therefore \qquad \mathcal{U}^r{}_{2\alpha}(\zeta) = \sum_{\nu=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu-r)!} (\zeta)^{2\nu-r}. \tag{4.9}$$

Similarly differentiating equations (4.5) and (4.6) r times, we have

$$\mathcal{U}^r{}_{2\alpha+1}(\zeta) = \sum_{\nu=\lceil\frac{r-1}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+1}(\alpha+\nu+1)!}{(\alpha-\nu)!\,(2\nu+1-r)!} (\zeta)^{2\nu+1-r}. \tag{4.10}$$

$$\mathcal{U}^r{}_{2\alpha-1}(\zeta) = \sum_{\nu=\lceil\frac{r-1}{2}\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\nu-1}\, 2^{2\nu+1}\, (\alpha+\nu)!}{(2\nu+1-r)!\,(\alpha-\nu-1)!} (\zeta)^{2\nu+1-r}. \tag{4.11}$$

Now, differentiating Theorem 1.65 *(ii)*,

$$\mathcal{V}^r{}_{\alpha}(\zeta) = \mathcal{U}^r{}_{\alpha}(\zeta) - \mathcal{U}^r{}_{\alpha-1}(\zeta).$$

which implies

$$\mathcal{V}^r{}_{2\alpha}(\zeta) = \mathcal{U}^r{}_{2\alpha}(\zeta) - \mathcal{U}^r{}_{2\alpha-1}(\zeta)$$

$$= \sum_{\nu=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu-r)!} (\zeta)^{2\nu-r}$$

$$- \sum_{\nu=\lceil\frac{r-1}{2}\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\nu-1}\, 2^{2\nu+1}\, (\alpha+\nu)!}{(2\nu+1-r)!\,(\alpha-\nu-1)!} (\zeta)^{2\nu+1-r},$$

$$\mathcal{V}^r{}_{2\alpha}(\zeta) = \sum_{\nu=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\, (\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu-r)!} (\zeta)^{2\nu-r}$$

$$+ \sum_{\nu=\lceil\frac{r-1}{2}\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+1}\, (\alpha+\nu)!}{(2\nu+1-r)!\,(\alpha-\nu-1)!} (\zeta)^{2\nu+1-r}. \tag{4.12}$$

Also,

$$\mathcal{V}^r{}_{2\alpha+1}(\zeta) = \mathcal{U}^r{}_{2\alpha+1}(\zeta) - \mathcal{U}^r{}_{2\alpha}(\zeta)$$

$$= \sum_{\nu=\left\lceil\frac{r-1}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+1}(\alpha+\nu+1)!}{(\alpha-\nu)!\,(2\nu+1-r)!}\,(\zeta)^{2\nu+1-r}$$

$$- \sum_{\nu=\left\lceil\frac{r}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\,(\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu-r)!}\,(\zeta)^{2\nu-r},$$

Therefore,

$$\mathcal{V}^r{}_{2\alpha+1}(\zeta) = \sum_{\nu=\left\lceil\frac{r-1}{2}\right\rceil}^{\alpha} (-1)^{\alpha-\nu}\, 2^{2\nu+1} \frac{(\alpha+\nu+1)!}{(\alpha-\nu)!\,(2\nu+1-r)!}\,(\zeta)^{2\nu+1-r}$$

$$- \sum_{\nu=\left\lceil\frac{r}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\,(\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu-r)!}\,(\zeta)^{2\nu-r} \tag{4.13}$$

This proves the Theorem 4.2.3. ■

Theorem 4.2.4. *For any integer $r > 0$ and $\zeta \in R$,*

$$\mathcal{W}_{2\alpha}{}^r(\zeta) = \sum_{\nu=\left\lceil\frac{r}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\,(\alpha+\nu)!}{(\alpha-\nu)!\ (2\nu-r)!}\,\zeta^{2\nu-r}$$

$$- \sum_{\nu=\left\lceil\frac{r-1}{2}\right\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+1}\,(\alpha+\nu)!}{(\alpha-\nu-1)!\ (2\nu+1-r)!}\,\zeta^{(2\nu+1)-r},$$

$$\mathcal{W}_{2\alpha+1}{}^r(\zeta) = \sum_{\nu=\left\lceil\frac{r-1}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+1}\,(\alpha+\nu+1)!}{(\alpha-\nu)!\ (2\nu+1-r)!}\,\zeta^{2\nu-r}$$

$$+ \sum_{\nu=\left\lceil\frac{r}{2}\right\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\,(\alpha+\nu)!}{(\alpha-\nu)!\ (2\nu-r)!}\,\zeta^{(2\nu-r)},$$

where $\lceil\zeta\rceil$ represents ceiling function.

Proof. Differentiating 1.65 *(xii)* r times we have

$$\mathcal{W}^r{}_\alpha(\zeta) = (-1)^{\alpha+1}\mathcal{V}^r{}_\alpha(-\zeta).$$

On replacing α by 2α and using equations 4.12 and 4.13, we have

$$\mathcal{W}^r{}_{2\alpha}(\zeta) = (-1)^{2\alpha+1}\mathcal{V}^r{}_{2\alpha}(-\zeta) = -\mathcal{V}^r{}_{2\alpha}(-\zeta)$$

$$= \sum_{\nu=\left\lceil\frac{r}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu+1}\, 2^{2\nu}\,(\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu-r)!}(-\zeta)^{2\nu-r}$$

$$- \sum_{\nu=\left\lceil\frac{r-1}{2}\right\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\nu-2}\, 2^{2\nu+1}\,(\alpha+\nu)!}{(2\nu+1-r)!\,(\alpha-\nu-1)!}(-\zeta)^{2\nu+1-r}$$

$$\mathcal{W}^r{}_{2\alpha}(\zeta) = \sum_{\nu=\left\lceil\frac{r}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu-r+1}\, 2^{2\nu}\,(\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu-r)!}(\zeta)^{2\nu-r}$$

$$+ \sum_{\nu=\left\lceil\frac{r-1}{2}\right\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\nu-r}\, 2^{2\nu+1}\,(\alpha+\nu)!}{(2\nu+1-r)!\,(\alpha-\nu-1)!}(\zeta)^{2\nu+1-r}$$

Similarly,

$$\mathcal{W}^r{}_{2\alpha+1}(\zeta) = (-1)^{2\alpha+2}\,\mathcal{V}^r{}_{2\alpha+1}(-\zeta) = \mathcal{V}^r{}_{2\alpha+1}(-\zeta)$$

$$= \left[\sum_{\nu=\left\lceil\frac{r-1}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu+1}(\alpha+\nu+1)!}{(\alpha-\nu)!\,(2\nu+1-r)!}\,(-\zeta)^{2\nu+1-r}\right.$$

$$\left. - \sum_{\nu=\left\lceil\frac{r}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu}\, 2^{2\nu}\,(\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu-r)!}(-\zeta)^{2\nu-r}\right]$$

$$\mathcal{W}^r{}_{2\alpha+1}(\zeta) = \sum_{\nu=\left\lceil\frac{r-1}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu-r+1}\, 2^{2\nu+1}(\alpha+\nu+1)!}{(\alpha-\nu)!\,(2\nu+1-r)!}\,(\zeta)^{2\nu+1-r}$$

$$- \sum_{\nu=\left\lceil\frac{r}{2}\right\rceil}^{\alpha} \frac{(-1)^{\alpha-\nu-r}\, 2^{2\nu}\,(\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu-r)!}(\zeta)^{2\nu-r}$$

This proves the Theorem 4.2.4.■

Theorem 4.2.5. *For any positive integer α and $\zeta \in R$,*

$$\mathcal{V}_{2\alpha}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha}\frac{(-1)^{\delta+\alpha}\ 2^{2\nu-1}\ (1-2\delta)\ (\alpha+\nu)!}{(\alpha-\nu)!\ (\nu+\delta)!\,(\nu-\delta+1)!}\times \mathcal{F}_{-(2\delta-1)}(\zeta)$$
$$+\sum_{\delta=1}^{\alpha}\sum_{\nu=0}^{\alpha-1}\frac{(-1)^{\delta+\alpha}\, 2^{2\nu+2}\,\delta\,(\alpha+\nu-1)!}{(\alpha-\nu+1)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!}\times \mathcal{F}_{-(2\delta)}(\zeta),$$

$$\mathcal{V}_{2\alpha+1}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha}\frac{(-1)^{\delta+\alpha+2}(\alpha+\nu)!}{(\nu-\delta+1)!}\left[\frac{2^{2\nu+2}\,\delta}{(\alpha-\nu+2)!\,(\nu+\delta+1)!}\mathcal{F}_{-(2\delta)}(\zeta) - \frac{2^{2\nu-1}\,(1-2\delta)}{(\alpha-\nu)!\,(\nu+\delta)!}\mathcal{F}_{-(2\delta-1)}(\zeta)\right].$$

Proof. For integer $\alpha > 0$, one can see that [48],

$$\mathcal{U}_{2\alpha}(\zeta) = \sum_{\delta=1}^{+\infty} c_{2\alpha,\delta}\,\mathcal{F}_{\delta}(\zeta),$$

and

$$\mathcal{U}_{2\alpha-1}(\zeta) = \sum_{\delta=1}^{+\infty} c_{2\alpha-1,\delta}\,\mathcal{F}_{\delta}(\zeta),$$

where,

$$c_{2\alpha,\delta} = \begin{cases} \displaystyle\sum_{\nu=0}^{\alpha}\frac{2^{4\nu+1}\, i^{3\delta+2\alpha+1}\delta(\alpha+\nu)!}{(\alpha-\nu)!\,(2\nu+\delta+1)!!\,(2\nu-\delta+1)!!}, & \delta \text{ is odd} \\ 0, & \text{otherwise} \end{cases},$$

$$c_{2\alpha-1,\delta} = \begin{cases} \displaystyle\sum_{\nu=0}^{\alpha}\frac{2^{4\nu+3}\, i^{3\delta+2\alpha}\,\delta(\alpha+\nu-1)!}{(\alpha-\nu-1)!\,(2\nu+\delta+2)!!\,(2\nu-\delta+2)!!} & \delta \text{ is odd} \\ 0, & \text{otherwise} \end{cases}.$$

Using this, from [48], for any positive integer α, we have

$$\mathcal{U}_{2\alpha}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha}\frac{(-1)^{\delta+\alpha}\ 2^{2\nu-1}\ (1-2\delta)(\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!}\times \mathcal{F}_{2\delta-1}(\zeta), \tag{4.14}$$

$$\mathcal{U}_{2\alpha-1}(\zeta) = \sum_{\delta=1}^{\alpha}\sum_{\nu=0}^{\alpha-1}\frac{(-1)^{\delta+\alpha}\ 2^{2\nu+2}\ \delta\ (\alpha+\nu-1)!}{(\alpha-\nu+1)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!}\times \mathcal{F}_{2\delta}(\zeta), \tag{4.15}$$

Again,

$$\mathcal{U}_{2\alpha+1}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha+1}\, 2^{2\nu+2}\, \delta\, (\alpha+\nu)!}{(\alpha-\nu+2)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!} \times \mathcal{F}_{2\delta}(\zeta). \qquad (4.16)$$

Now using Theorem 1.65 (*ii*), equations (4.14), (4.15) and (4.16), we have

$$\mathcal{V}_{2\alpha}(\zeta) = \mathcal{U}_{2\alpha}(\zeta) - \mathcal{U}_{2\alpha-1}(\zeta)$$

$$= \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu-1}\, (1-2\delta)(\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!} \times \mathcal{F}_{2\delta-1}(\zeta)$$

$$- \sum_{\delta=1}^{\alpha}\sum_{\nu=0}^{\alpha-1} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu+2}\, \delta\, (\alpha+\nu-1)!}{(\alpha-\nu+1)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!} \times \mathcal{F}_{2\delta}(\zeta),$$

Using equation 1.12 (section 1.2, Chapter 1), $\mathcal{F}_{2\delta-1}(\zeta) = \mathcal{F}_{-(2\delta-1)}(\zeta)$ and $\mathcal{F}_{2\delta}(\zeta) = -\mathcal{F}_{-(2\delta)}(\zeta)$

$$\mathcal{V}_{2\alpha}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu-1}\, (1-2\delta)(\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta-1)}(\zeta)$$

$$- \sum_{\delta=1}^{\alpha}\sum_{\nu=0}^{\alpha-1} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu+2}\, \delta\, (\alpha+\nu-1)!}{(\alpha-\nu+1)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!} \times (-1)\, \mathcal{F}_{-(2\delta)}(\zeta),$$

$$\mathcal{V}_{2\alpha}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu-1}\, (1-2\delta)\, (\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta-1)}(\zeta)$$

$$+ \sum_{\delta=1}^{\alpha}\sum_{\nu=0}^{\alpha-1} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu+2}\, \delta\, (\alpha+\nu-1)!}{(\alpha-\nu+1)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!}$$

$$\times \mathcal{F}_{-(2\delta)}(\zeta) \qquad (4.17)$$

Similarly,

$$\mathcal{V}_{2\alpha+1}(\zeta) = \mathcal{U}_{2\alpha+1}(\zeta) - \mathcal{U}_{2\alpha}(\zeta)$$

$$= \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha+1}\, 2^{2\nu+2}\, \delta\, (\alpha+\nu)!}{(\alpha-\nu+2)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!} \times \mathcal{F}_{2\delta}(\zeta)$$

$$- \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu-1}\, (1-2\delta)(\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!} \times \mathcal{F}_{2\delta-1}(\zeta)$$

$$\mathcal{V}_{2\alpha+1}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha+1}\, 2^{2\nu+2}\, \delta\, (\alpha+\nu)!}{(\alpha-\nu+2)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!} \times (-1)\mathcal{F}_{-2\delta}(\zeta)$$
$$+ \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha+1}\, 2^{2\nu-1}\, (1-2\delta)(\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta-1)}(\zeta)$$

$$\mathcal{V}_{2\alpha+1}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}(\alpha+\nu)!}{(\nu-\delta+1)!}\left[\frac{2^{2\nu+2}\,\delta}{(\alpha-\nu+2)!\,(\nu+\delta+1)!}\mathcal{F}_{-(2\delta)}(\zeta) - \frac{2^{2\nu-1}\,(1-2\delta)}{(\alpha-\nu)!\,(\nu+\delta)!}\mathcal{F}_{-(2\delta-1)}(\zeta)\right] \tag{4.18}$$

This proves the Theorem 4.2.5.■

Theorem 4.2.6. *For any integer $\alpha > 0$ and $\zeta \in R$,*

$$\mathcal{W}_{2\alpha}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu-1}\, (1-2\delta)(\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta-1)}(\zeta)$$
$$- \sum_{\delta=1}^{\alpha}\sum_{\nu=0}^{\alpha-1} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu+2}\, \delta\, (\alpha+\nu-1)!}{(\alpha-\nu+1)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta)}(\zeta),$$

$$\mathcal{W}_{2\alpha+1}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}(\alpha+\nu)!}{(\nu-\delta+1)!}\left[\begin{array}{c}\dfrac{2^{2\nu+2}\,\delta}{(\alpha-\nu+2)!\,(\nu+\delta+1)!}\mathcal{F}_{-(2\delta)}(\zeta) \\ +\dfrac{2^{2\nu-1}\,(1-2\delta)}{(\alpha-\nu)!\,(\nu+\delta)!}\mathcal{F}_{-(2\delta-1)}(\zeta)\end{array}\right].$$

Proof. Using equation 1.12 & 1.16 (section 1.2, Chapter 1), we have

$$\mathcal{F}_{-(2\delta-1)}(-\zeta) = \mathcal{F}_{-(2\delta-1)}(\zeta), \tag{4.19}$$

and

$$\mathcal{F}_{-(2\delta)}(-\zeta) = -\mathcal{F}_{-(2\delta)}(\zeta). \tag{4.20}$$

Using equations (4.19) and (4.20) in equation (4.17), we have

$$\mathcal{W}_{2\alpha}(\zeta) = (-1)^{2\alpha}\mathcal{V}_{2\alpha}(-\zeta) = \mathcal{V}_{2\alpha}(-\zeta)$$

$$= \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu-1}\,(1-2\delta)(\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta-1)}(-\zeta)$$

$$+ \sum_{\delta=1}^{\alpha}\sum_{\nu=0}^{\alpha-1} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu+2}\,\delta\,(\alpha+\nu-1)!}{(\alpha-\nu+1)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta)}(-\zeta)$$

$$\Rightarrow \mathcal{W}_{2\alpha}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu-1}\,(1-2\delta)(\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta-1)}(\zeta)$$

$$+ \sum_{\delta=1}^{\alpha}\sum_{\nu=0}^{\alpha-1} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu+2}\,\delta\,(\alpha+\nu-1)!}{(\alpha-\nu+1)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!}$$

$$\times (-1)\, \mathcal{F}_{-(2\delta)}(-\zeta)$$

$$\therefore \mathcal{W}_{2\alpha}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu-1}\,(1-2\delta)(\alpha+\nu)!}{(\alpha-\nu)!\,(\nu+\delta)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta-1)}(\zeta)$$

$$- \sum_{\delta=1}^{\alpha}\sum_{\nu=0}^{\alpha-1} \frac{(-1)^{\delta+\alpha}\, 2^{2\nu+2}\,\delta\,(\alpha+\nu-1)!}{(\alpha-\nu+1)!\,(\nu+\delta+1)!\,(\nu-\delta+1)!} \times \mathcal{F}_{-(2\delta)}(\zeta)$$

Similarly, using equations (4.19) and (4.20) in equation (4.18), we have

$$\mathcal{W}_{2\alpha+1}(\zeta) = (-1)^{2\alpha+1}\mathcal{V}_{2\alpha+1}(-\zeta) = -\mathcal{V}_{2\alpha+1}(-\zeta)$$

$$= (-1)\sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}(\alpha+\nu)!}{(\nu-\delta+1)!}\Bigg[\frac{2^{2\nu+2}\,\delta}{(\alpha-\nu+2)!\,(\nu+\delta+1)!}\mathcal{F}_{-(2\delta)}(-\zeta)$$

$$- \frac{2^{2\nu-1}\,(1-2\delta)}{(\alpha-\nu)!\,(\nu+\delta)!}\mathcal{F}_{-(2\delta-1)}(-\zeta)\Bigg]$$

$$\Rightarrow \mathcal{W}_{2\alpha+1}(\zeta)$$

$$= (-1)\sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha} \frac{(-1)^{\delta+\alpha}(\alpha+\nu)!}{(\nu-\delta+1)!}\Bigg[\frac{2^{2\nu+2}\,\delta}{(\alpha-\nu+2)!\,(\nu+\delta+1)!}(-1)\mathcal{F}_{-(2\delta)}(\zeta)$$

$$- \frac{2^{2\nu-1}\,(1-2\delta)}{(\alpha-\nu)!\,(\nu+\delta)!}\mathcal{F}_{-(2\delta-1)}(\zeta)\Bigg]$$

$$\therefore \mathcal{W}_{2\alpha+1}(\zeta) = \sum_{\delta=1}^{\alpha+1}\sum_{\nu=0}^{\alpha}\frac{(-1)^{\delta+\alpha}(\alpha+\nu)!}{(\nu-\delta+1)!}\left[\frac{2^{2\nu+2}\,\delta}{(\alpha-\nu+2)!\,(\nu+\delta+1)!}\mathcal{F}_{-(2\delta)}(\zeta) + \frac{2^{2\nu-1}\,(1-2\delta)}{(\alpha-\nu)!\,(\nu+\delta)!}\mathcal{F}_{-(2\delta-1)}(\zeta)\right]$$

This proves the Theorem 4.2.5. ■

4.3 Sums of finite products of Chebyshev polynomials of the third and fourth kinds in other orthogonal polynomials.

Before coming to the main results, it is important to revisit the basic definitions and concepts already discussed in section 1.2 of Chapter 1 regarding Jacobi, Pell, Gegenbauer, Fibonacci, Vieta-Pell, and Vieta-Fibonacci polynomials [11, 12, 37, 55, and 60] which are instrumental in the development of the essence of the content of this section. Here the summation representations of finite products of the 3rd and 4th kinds of Chebyshev polynomials in other orthogonal polynomials are studied.

Many authors have analyzed and investigated Chebyshev polynomials and one such area is the classical linearization problem. For instance, Zhang [55], in 2002, studied summation problems of finite products of 2nd-kind Chebyshev polynomials, Lucas and Fibonacci numbers as given by the equation (1.82). Similar study was conducted by T. Kim et al. [56] in 2019 and obtained interesting results, especially, given by the equations (1.92)- (1.93). Following the pattern, D. Han and L. Xinging [73], similar summation representations for Lucas, Fibonacci and Chebyshev polynomials in terms of 1st-kind Chebyshev and Lucas polynomials are deduced.

Similarly, following this pattern, we will write sums of the finite products of the 3rd and 4th kinds of Chebyshev polynomials as a linear sum of Jacobi, Pell, Gegenbauer, Fibonacci, Vieta-Pell, and Vieta-Fibonacci polynomials. Analogous results for the Lucas and Fibonacci numbers are considered. The main results are:

Lemma 4.3.1. *For all positive integers α and $\xi \in R$,*

$$\mathcal{P}_{\alpha}(\xi) = \mathcal{F}_{\alpha}(2\xi). \tag{4.21}$$

$$\mathcal{P}_{\alpha+1}(\xi) = \frac{1}{\sqrt{(-1)^{\alpha}}}\mathcal{U}_{\alpha}\left(\sqrt{-1}\xi\right) = \frac{1}{i^{\alpha}}\mathcal{U}_{\alpha}(i\xi). \tag{4.22}$$

$$S_{\alpha}(\xi) = \mathcal{U}_{\alpha}\left(\frac{1}{2}\xi\right). \tag{4.23}$$

$$\mathcal{U}_{\alpha}(\xi) = R_{\alpha+1}(\xi). \tag{4.24}$$

$$\mathcal{U}_{\alpha}(\xi) = C(\alpha{:}\,1)(\xi). \tag{4.25}$$

$$\mathcal{U}_{\alpha}(\xi) = \frac{(\alpha+1)!\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\mathcal{P}\left(\alpha{:}\frac{1}{2},\frac{1}{2}\right)(\xi). \tag{4.26}$$

Proof. This Lemma can easily be developed by utilizing the basic definitions and recurrence relations for Pell polynomials $\mathcal{P}_{\alpha}(\xi)$ (sub-section 1.2.10), Chebyshev polynomials of second kind $\mathcal{U}_{\alpha}(\xi)$ (sub-section 1.2.11(*i*)) Vieta-Fibonacci polynomials $S_{\alpha}(\xi)$ and Vieta-Pell polynomials $C(\alpha{:}\,\lambda)(\xi)$ (sub-section 1.2.13), Jacobi Polynomials(sub-section 1.2.14), Gegenbauer polynomials $\mathcal{P}(\alpha{:}\,\lambda,\beta)(\xi)$ (sub-section 1.2.15).■

Theorem 4.3.1 *For any integer* $\alpha, r \geq 0$, *we have*

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(i\xi)\,.\,\mathcal{V}_{\sigma_2}(i\xi).\,\mathcal{V}_{\sigma_3}(i\xi).\,\ldots\,\mathcal{V}_{\sigma_{r+1}}(i\xi)$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}\,i^{\alpha-r}\mathcal{P}^{r}_{\alpha-r+\gamma+1}(\xi),$$

$$= \frac{1}{r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}\,i^{\alpha-r}\mathcal{F}^{r}_{\alpha-r+\gamma+1}(2\xi),$$

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(i\xi).\mathcal{W}_{\sigma_2}(i\xi).\mathcal{W}_{\sigma_3}(i\xi).\ldots\mathcal{W}_{\sigma_{r+1}}(i\xi)$$

$$=\frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{P}^r_{\alpha-r+\gamma+1}(\xi),$$

$$=\frac{1}{r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{F}^r_{\alpha-r+\gamma+1}(2\xi),$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ *(*$\hbar = 1,2,\ldots,r+1$*) with* $\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha$ *&* $\binom{r+1}{\gamma}=0$ *for* $\gamma > r+1$, $i=\sqrt{-1}$ *and* $\mathcal{P}^r_\alpha(\xi)$ *&* $\mathcal{F}^r_\alpha(\xi)$ *is* r^{th} *derivative of Pell polynomial and Fibonacci polynomial respectively.*

Proof. Replacing ξ by $i\xi$ in equations (1.92) and (1.93), we have

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(i\xi).\mathcal{V}_{\sigma_2}(i\xi).\mathcal{V}_{\sigma_3}(i\xi).\ldots\mathcal{V}_{\sigma_{r+1}}(i\xi)$$

$$=\frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{r+1}{\gamma}\mathcal{U}^r_{\alpha-\gamma+r}(i\xi), \tag{4.27}$$

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(i\xi).\mathcal{W}_{\sigma_2}(i\xi).\mathcal{W}_{\sigma_3}(i\xi).\ldots\mathcal{W}_{\sigma_{r+1}}(i\xi)$$

$$=\frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{U}^r_{\alpha-\gamma+r}(i\xi), \tag{4.28}$$

Differentiating equations (4.21) and (4.22), *r*- times w.r.t ξ, we get

$$\mathcal{P}^r_\alpha(\xi) = 2^r\mathcal{F}^r_\alpha(2\xi), \tag{4.29}$$

$$\mathcal{U}^r_\alpha(i\xi) = i^{\alpha-r}\mathcal{P}^r_{\alpha+1}(\xi), \tag{4.30}$$

Using equations (4.29) and (4.30) in equations (4.27) and (4.28), we have

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(i\xi).\mathcal{V}_{\sigma_2}(i\xi).\mathcal{V}_{\sigma_3}(i\xi).\ldots\mathcal{V}_{\sigma_{r+1}}(i\xi)$$

$$=\frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{P}^r_{\alpha-\gamma+r+1}(\xi),$$

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(i\xi)\,.\,\mathcal{V}_{\sigma_2}(i\xi).\,\mathcal{V}_{\sigma_3}(i\xi).\ldots\mathcal{V}_{\sigma_{r+1}}(i\xi)$$

$$= \frac{1}{r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{F}^{r}_{\alpha-\gamma+r+1}(2\xi),$$

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(i\xi).\,\mathcal{W}_{\sigma_2}(i\xi).\,\mathcal{W}_{\sigma_3}(i\xi).\ldots\mathcal{W}_{\sigma_{r+1}}(i\xi)$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{P}^{r}_{\alpha-\gamma+r+1}(\xi)$$

$$= \frac{1}{r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{F}^{r}_{\alpha-\gamma+r+1}(2\xi)\,.$$

Hence the Theorem 4.3.1 is established. ■

Theorem 4.3.2 *For any integer* $\alpha, r \geq 0$ *and* $\xi \in R$, *we have*

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(\xi)\,.\,\mathcal{V}_{\sigma_2}(\xi).\,\mathcal{V}_{\sigma_3}(\xi).\ldots\mathcal{V}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{2^r r!}\frac{(\alpha+1)!\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}\mathcal{P}^{r}\left(\alpha-\gamma+r{:}\,\frac{1}{2},\frac{1}{2}\right)(\xi)$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma} C^{r}(\alpha-\gamma+r{:}\,1)(\xi),$$

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(\xi).\,\mathcal{W}_{\sigma_2}(\xi).\,\mathcal{W}_{\sigma_3}(\xi).\ldots\mathcal{W}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{2^r r!}\frac{(\alpha+1)!\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{P}^{r}\left(\alpha-\gamma+r{:}\,\frac{1}{2},\frac{1}{2}\right)(\xi)$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} C^{r}(\alpha-\gamma+r{:}\,1)(\xi),$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\ldots,r+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha$ *and* $\binom{r+1}{\gamma}=0$ *for* $\gamma > r+1, i=\sqrt{-1}$ *and* $\mathcal{P}^{r}(\alpha{:}\,\beta,\gamma)(\xi)$ *and*

$C^r(\alpha:\beta)(\xi)$ *is the r^th derivative of Jacobi's polynomial and Gegenbauer polynomials respectively.*

Proof. Differentiating equations (4.25) and (4.26) r times, we have

$$\mathcal{U}_\alpha^r(\xi) = C^r(\alpha:1)(\xi) \tag{4.31}$$

$$\mathcal{U}_\alpha^r(\xi) = \frac{(\alpha+1)!\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)} \mathcal{P}^r\left(\alpha:\frac{1}{2},\frac{1}{2}\right)(\xi) \tag{4.32}$$

Using equations (4.31) and (4.32) in equations (1.92) and (1.93), we have

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(\xi)\,.\,\mathcal{V}_{\sigma_2}(\xi).\mathcal{V}_{\sigma_3}(\xi).\ldots\mathcal{V}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{2^r r!} \frac{(\alpha+1)!\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)} \sum_{\gamma=0}^{\alpha} (-1)^\gamma \binom{r+1}{\gamma} \mathcal{P}^r\left(\alpha-\gamma+r:\frac{1}{2},\frac{1}{2}\right)(\xi)$$

$$\therefore \sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(\xi)\,.\,\mathcal{V}_{\sigma_2}(\xi).\mathcal{V}_{\sigma_3}(\xi).\ldots\mathcal{V}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{2^r r!} \sum_{\gamma=0}^{\alpha} (-1)^\gamma \binom{r+1}{\gamma} C^r(\alpha-\gamma+r:1)(\xi),$$

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(\xi)\,.\,\mathcal{W}_{\sigma_2}(\xi).\mathcal{W}_{\sigma_3}(\xi).\ldots\mathcal{W}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{2^r r!} \frac{(\alpha+1)!\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)} \sum_{\gamma=0}^{\alpha} \binom{r+1}{\gamma} \mathcal{P}^r\left(\alpha-\gamma+r:\frac{1}{2},\frac{1}{2}\right)(\xi),$$

$$= \frac{1}{2^r r!} \sum_{\gamma=0}^{\alpha} \binom{r+1}{\gamma} C^r(\alpha-\gamma+r:1)(\xi).$$

Hence the Theorem 4.3.2 is proved. ■

Theorem 4.3.3. *For any integer* $\alpha, r \geq 0$ *and* $\xi \in R$,

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}\left(\frac{\xi}{2}\right).\mathcal{V}_{\sigma_2}\left(\frac{\xi}{2}\right).\mathcal{V}_{\sigma_3}\left(\frac{\xi}{2}\right).\dots\mathcal{V}_{\sigma_{r+1}}\left(\frac{\xi}{2}\right)$$
$$=\frac{1}{r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}S_{\alpha-\gamma+r}^{r}(\xi),$$
$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}\left(\frac{\xi}{2}\right).\mathcal{W}_{\sigma_2}\left(\frac{\xi}{2}\right).\mathcal{W}_{\sigma_3}\left(\frac{\xi}{2}\right).\dots\mathcal{W}_{\sigma_{r+1}}\left(\frac{\xi}{2}\right)$$
$$=\frac{1}{r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}S_{\alpha-\gamma+r}^{r}(\xi),$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\dots,r+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha$ *and* $\binom{r+1}{\gamma}=0$ *for* $\gamma > r+1, i=\sqrt{-1}$ *and* $S_{\alpha}^{r}(\xi)$ *represents the* r^{th} *derivative of Vieta- Fibonacci polynomials.*

Proof. Replacing ξ by $\frac{\xi}{2}$ in equations (1.92) and (1.93), we get

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}\left(\frac{\xi}{2}\right).\mathcal{V}_{\sigma_2}\left(\frac{\xi}{2}\right).\mathcal{V}_{\sigma_3}\left(\frac{\xi}{2}\right).\dots\mathcal{V}_{\sigma_{r+1}}\left(\frac{\xi}{2}\right)$$
$$=\frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{\xi}{2}\right). \qquad (4.33)$$

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}\left(\frac{\xi}{2}\right).\mathcal{W}_{\sigma_2}\left(\frac{\xi}{2}\right).\mathcal{W}_{\sigma_3}\left(\frac{\xi}{2}\right).\dots\mathcal{W}_{\sigma_{r+1}}\left(\frac{\xi}{2}\right)$$
$$=\frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{U}_{\alpha-\gamma+r}^{r}\left(\frac{\xi}{2}\right). \qquad (4.34)$$

Differentiating (4.23) r-times, we have

$$S_{\alpha}^{r}(\xi)=\frac{1}{2^r}\mathcal{U}_{\alpha}^{r}\left(\frac{1}{2}\xi\right). \qquad (4.35)$$

Using equation (4.35) in equations (4.33) and (4.34), we get

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}\left(\frac{\xi}{2}\right).\mathcal{V}_{\sigma_2}\left(\frac{\xi}{2}\right).\mathcal{V}_{\sigma_3}\left(\frac{\xi}{2}\right).\dots\mathcal{V}_{\sigma_{r+1}}\left(\frac{\xi}{2}\right)$$
$$=\frac{1}{r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}S_{\alpha-\gamma+r}^{r}(\xi),$$
$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}\left(\frac{\xi}{2}\right).\mathcal{W}_{\sigma_2}\left(\frac{\xi}{2}\right).\mathcal{W}_{\sigma_3}\left(\frac{\xi}{2}\right).\dots\mathcal{W}_{\sigma_{r+1}}\left(\frac{\xi}{2}\right)$$
$$=\frac{1}{r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}S_{\alpha-\gamma+r}^{r}(\xi).$$

This establishes the Theorem 4.3.3. ■

Theorem 4.3.4. *For integers $\alpha, r \geq 0$ and $\xi \in R$, we have*

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(\xi).\mathcal{V}_{\sigma_2}(\xi).\mathcal{V}_{\sigma_3}(\xi).\dots\mathcal{V}_{\sigma_{r+1}}(\xi)$$
$$=\frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}R_{\alpha-\gamma+r+1}^{r}(\xi),$$
$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(\xi).\mathcal{W}_{\sigma_2}(\xi).\mathcal{W}_{\sigma_3}(\xi).\dots\mathcal{W}_{\sigma_{r+1}}(\xi)$$
$$=\frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}R_{\alpha-\gamma+r+1}^{r}(\xi)$$

*where sum runs over all $\sigma_{\hbar}$ (≥ 0) in **Z** $(\hbar = 1,2,\dots,r+1)$ with $\sigma_1+\sigma_2+\cdots+\sigma_{r+1}=\alpha$ and $\binom{r+1}{\gamma}=0$ for $\gamma > r+1, i=\sqrt{-1}$* and *$R_{\alpha}^{r}(\xi)$ is r^{th} derivative of Vieta- Pell polynomial.*

Proof. Differentiating (4.24) r -times, we have

$$\mathcal{U}_{\alpha}^{r}(\xi) = R_{\alpha+1}^{r}(\xi). \tag{4.38}$$

Using equation (4.38) in equations (1.92) and (1.93), we see that

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{V}_{\sigma_1}(\xi).\mathcal{V}_{\sigma_2}(\xi).\mathcal{V}_{\sigma_3}(\xi).\dots\mathcal{V}_{\sigma_{r+1}}(\xi)$$
$$=\frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}R_{\alpha-\gamma+r+1}^{r}(\xi).$$

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{W}_{\sigma_1}(\xi)\,.\,\mathcal{W}_{\sigma_2}(\xi).\,\mathcal{W}_{\sigma_3}(\xi).\,\ldots\,\mathcal{W}_{\sigma_{r+1}}(\xi)$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} R^r_{\alpha-\gamma+r+1}(\xi).$$

This establishes the Theorem 4.3.4. ■

Corollary 4.3.1. *For integer* $\alpha, r \geq 0$ *and* $\xi \in R$, *we have*

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{F}_{\sigma_1}\,.\,\mathcal{F}_{\sigma_2}.\,\mathcal{F}_{\sigma_3}.\,\ldots\,\mathcal{F}_{\sigma_{r+1}}$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{P}^r_{\alpha-\gamma+r+1}\left(-\frac{3}{2}i\right)$$

$$= \frac{1}{r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{F}^r_{\alpha-\gamma+r+1}(-3i)$$

$$= \frac{1}{2^r r!}\frac{(\alpha+1)}{\mathcal{P}_{\alpha}\left(\alpha{:}\,\frac{1}{2},\frac{1}{2}\right)(1)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma}\mathcal{P}^r\left(\alpha-\gamma+r{:}\,\frac{1}{2},\frac{1}{2}\right)\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma} C^r(\alpha-\gamma+r{:}\,1)\left(\frac{3}{2}\right)$$

$$= \frac{1}{r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma} S^r_{\alpha-\gamma+r}(3)$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{r+1}{\gamma} R^r_{\alpha-\gamma+r+1}\left(\frac{3}{2}\right)$$

Proof. By taking $\xi = \frac{3}{2i}$ in Theorem 4.3.1, $\xi = \frac{3}{2}$ in Theorem 4.3.2, $\xi = 3$ in Theorem 4.3.3, $\xi = \frac{3}{2}$ in Theorem 4.3.4, and using equation in Theorem 1.65(*viii*) establishes the Corollary 4.3.1. ■

Corollary 4.3.2. *For any integers* $\alpha, r \geq 0$ *and* $\xi \in R$,

$$\sum_{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{r+1}=\alpha} \mathcal{L}_{\sigma_1}.\mathcal{L}_{\sigma_2}.\mathcal{L}_{\sigma_3}.\ldots\mathcal{L}_{\sigma_{r+1}}$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{P}_{\alpha-\gamma+r+1}^{r}\left(-\frac{3}{2}i\right)$$

$$= \frac{1}{r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} i^{\alpha-r}\mathcal{F}_{\alpha-\gamma+r+1}^{r}(-3i)$$

$$= \frac{1}{2^r r!}\frac{(\alpha+1)}{\mathcal{P}_{\alpha}\left(\alpha:\frac{1}{2},\frac{1}{2}\right)(1)}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma}\mathcal{P}^{r}\left(\alpha-\gamma+r:\frac{1}{2},\frac{1}{2}\right)\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} C^{r}(\alpha-\gamma+r:1)\left(\frac{3}{2}\right)$$

$$= \frac{1}{r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} S_{\alpha-\gamma+r}^{r}(3)$$

$$= \frac{1}{2^r r!}\sum_{\gamma=0}^{\alpha}\binom{r+1}{\gamma} R_{\alpha-\gamma+r+1}^{r}\left(\frac{3}{2}\right)$$

Proof. Similarly, by taking $\xi = \frac{3}{2i}$ in Theorem 4.3.1, $\xi = \frac{3}{2}$ in Theorem 4.3.2, $\xi = 3$ in Theorem 4.3.3, $\xi = \frac{3}{2}$ in Theorem 4.3.4, and using Theorem 1.65(x) establishes the Corollary 4.3.2. ■

Chapter 5

SOME REPRESENTATIONS OF SUMS OF FINITE PRODUCTS OF PELL, FIBONACCI AND CHEBYSHEV POLYNOMIALS

5.1 Introduction

The first section will focus on establishment of some more identities on representations of summations of finite products of Lucas and Fibonacci numbers and Fibonacci and Pell polynomials as a linear sum of derivatives of Pell polynomials, using their basic properties through elementary computations. Similar identities are obtained for the 3rd and 4th kinds of Chebyshev polynomials.

In the second section, we will prove some more similar identities on finite products of negative indexed Lucas, Fibonacci, and Complex Fibonacci numbers. In terms of the 3rd and 4th kinds of Chebyshev polynomials, analogous results are obtained for Pell numbers and Fibonacci polynomials.

5.2 Sums of finite products of Pell, Fibonacci, and Chebyshev polynomials of third and fourth kinds in Pell polynomials

Here we will develop some results expressing finite products of Lucas and Fibonacci numbers, Pell and Fibonacci polynomials as a linear sum of derived Pell polynomials, through elementary computations. Analogous identities are obtained for the 3rd and 4th kinds of Chebyshev polynomials.

Zhang [55] investigated the linear sum problem on 2nd kinds of Chebyshev polynomials and derived many identities, particularly, given by the equation (1.82). Similar results were observed by T. Kim et al. [51] for 1st kinds of Chebyshev polynomials and Lucas polynomials. In [56], T. Kim et al. have observed the sums of finite products of the 3rd and 4th kinds of Chebyshev polynomials which among others includes which are represented by equations (1.92)- (1.93). Analogous results were developed by W. Siyi [57] and D. Han and L. Xinging [74]. A. Patra and G.K. Panda

[59] also developed similar identities expressing finite products of Pell polynomials in other orthogonal polynomials.

According to the preceding literature, previous works have developed identities representing finite products of Lucas and Fibonacci numbers, Fibonacci, Pell and Lucas polynomials, and Chebyshev polynomials of 3rd and 4th kind as a linear sum of derivatives of Lucas Polynomials, Fibonacci polynomials, or Chebyshev polynomials, but the similar results in terms of Pell polynomials have not been studied. So, this section is dedicated to the development of some more similar identities representing finite products of the Lucas and Fibonacci numbers and Pell, Fibonacci, and Chebyshev polynomials of 3rd and 4th kinds, primarily in terms of derivatives of the Pell polynomials, are obtained. The main findings of this section are:

Lemma 5.2.1. *For any non-negative integers α, the following identities holds*

i). $\mathcal{P}_{\alpha+1}\left(-\frac{3}{2}\,i\right) = i^{-\alpha}\,\mathcal{F}_{2(\alpha+1)}$.

ii). $\mathcal{P}_{\alpha+1}\left(\frac{3}{2}\,i\right) = i^{\alpha}\,\mathcal{F}_{2(\alpha+1)}$.

iii). $\mathcal{P}_{\alpha+1}(-2) = \frac{i^{\alpha}}{2}\mathcal{F}_{3(\alpha+1)}$.

iv). $\mathcal{V}_{\alpha}\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+1}$.

v). $\mathcal{W}_{\alpha}\left(\frac{3}{2}\right) = \mathcal{L}_{2\alpha+1}$.

Proof. *(i)* Take $\xi = -\frac{3i}{2}$ in equation 1.65 *(xv)*, we have

$$\mathcal{U}_{\alpha}\left(\frac{3}{2}\right) = i^{\alpha}\mathcal{P}_{\alpha+1}\left(-\frac{3}{2}\,i\right).$$

$$\mathcal{P}_{\alpha+1}\left(-\frac{3}{2}\,i\right) = i^{-\alpha}\mathcal{U}_{\alpha}\left(\frac{3}{2}\right). \tag{5.1}$$

Using $\mathcal{U}_{\alpha}\left(\frac{3}{2}\right) = \mathcal{F}_{2(\alpha+1)}$ in equation (5.1) we have

$$\mathcal{P}_{\alpha+1}\left(-\frac{3}{2}\,i\right) = i^{-\alpha}\mathcal{F}_{2(\alpha+1)}.$$

ii) To establish this identity, we will proceed as above in case of (*i)* and using $\mathcal{U}_{\alpha}\left(-\frac{3}{2}\right) = (-1)^{\alpha}\,\mathcal{F}_{2(\alpha+1)}$.

iii) Taking $\xi = -2$ in equation 1.65 (*xv), we* have

$$\mathcal{U}_{\alpha}(-2i) = i^{\alpha}\mathcal{P}_{\alpha+1}(-2).$$

$$\mathcal{P}_{\alpha+1}\,(-2) = i^{-\alpha}\mathcal{U}_{\alpha}\,(\,-2i)\,. \tag{5.2}$$

Using $\mathcal{U}_{\alpha}\,(-2i) = \frac{(-1)^{\alpha}}{2}\mathcal{F}_{3(\alpha+1)}$ in equation (5.2), we have

$$\mathcal{P}_{\alpha+1}\,(-2) = \frac{i^{\alpha}}{2}\mathcal{F}_{3(\alpha+1)}.$$

iv) From equation 1.65 (*ii*), we have

$$\mathcal{V}_{\alpha}\,(\,\xi) = \mathcal{U}_{\alpha}\,(\,\xi) - \mathcal{U}_{\alpha-1}\,(\,\xi) \tag{5.3}$$

Taking $\xi = \frac{3}{2}$ and using $\mathcal{U}_{\alpha}\left(\frac{3}{2}\right) = \mathcal{F}_{2(\alpha+1)}$ in equation (5.3), we have

$$\mathcal{V}_{\alpha}\left(\frac{3}{2}\right) = \mathcal{U}_{\alpha}\left(\frac{3}{2}\right) - \mathcal{U}_{\alpha-1}\left(\frac{3}{2}\right) = \mathcal{F}_{2\alpha+2} - \mathcal{F}_{2\alpha} = \mathcal{F}_{2\alpha+1}$$

v) Similarly, using equation 1.65 (*iii*) and proceeding as above in (*iv*), we can establish the result ■

***Lemma* 5.2.2**. *For any integer $\alpha \geq 0$, and $\xi \in R$, we have the identity*

$$\mathcal{P}'_{\alpha+1}(\xi) = \frac{(\alpha+1)}{(1+\xi^2)}\,\mathcal{P}_{\alpha}(\xi) + \frac{\alpha\xi}{(1+\xi^2)}\,\mathcal{P}_{\alpha+1}(\xi).$$

where $\mathcal{P}_{\alpha}(\xi)$ is a Pell polynomial.

Proof. From [57], we have

$$(1-\xi^2)\mathcal{U}'_{\alpha}(\xi) = (\alpha+1)\mathcal{U}_{\alpha-1}(\xi) - \alpha\xi\mathcal{U}_{\alpha}(\xi). \tag{5.4}$$

Replacing ξ by $i\xi$ in equation (5.4), we have

$$(1+\xi^2)\mathcal{U}'_{\alpha}(i\xi) = (\alpha+1)\mathcal{U}_{\alpha-1}(i\xi) - \alpha i\xi\mathcal{U}_{\alpha}(i\xi). \tag{5.5}$$

Differentiating equation 1.65 (*xv*), we have

$$\mathcal{U}'_{\alpha}(i\xi) = i^{\alpha-1}\,\mathcal{P}'_{\alpha+1}\,(\xi). \tag{5.6}$$

Using equation (5.6) in equation (5.5), we have

$$(1+\xi^2)i^{\alpha-1}\mathcal{P}'_{\alpha}(\xi) = (\alpha+1)\,i^{\alpha-1}\mathcal{P}_{\alpha}(\xi) - \alpha\,i\,\xi\,i^{\alpha}\,\mathcal{P}_{\alpha}(\xi),$$

$$(1+\xi^2)\mathcal{P}'_{\alpha+1}(\xi) = (\alpha+1)\,\mathcal{P}_{\alpha}(\xi) + \alpha\xi\,\mathcal{P}_{\alpha+1}(\xi),$$

$$\mathcal{P}'_{\alpha+1}(\xi) = \frac{(\alpha+1)}{(1+\xi^2)}\,\mathcal{P}_{\alpha}(\xi) + \frac{\alpha\xi}{(1+\xi^2)}\,\mathcal{P}_{\alpha+1}(\xi).$$

This proves the Lemma 5.2.2. ■

***Lemma* 5.2.3**. *For any integer $\alpha \geq 0$, and $\xi \in R$, we have the identity*

$$\mathcal{P}''_{\alpha}(\xi) = \frac{\alpha(\alpha+2)}{(1+\xi^2)}\mathcal{P}_{\alpha+1}(\xi) - \frac{3\xi}{(1+\xi^2)}\mathcal{P}'_{\alpha+1}(\xi).$$

where $\mathcal{P}_{\alpha}(\xi)$ is a Pell polynomial.

Proof. From [57], we have

$$(1-\xi^2)\mathcal{U}''_\alpha(\xi) = 3\xi\,\mathcal{U}'_\alpha(\xi) - \alpha(\alpha+2)\,\mathcal{U}_\alpha(\xi), \tag{5.7}$$

Replacing ξ by $i\xi$ in equation (5.7), we have

$$(1+\xi^2)\,\mathcal{U}''_\alpha(i\xi) = 3\xi i\,\mathcal{U}'_\alpha(i\xi) - \alpha(\alpha+2)\,\mathcal{U}_\alpha(i\xi), \tag{5.8}$$

Differentiating equation 1.65 (*xv*), we have

$$\mathcal{U}'_\alpha(i\xi) = i^{\alpha-1}\,\mathcal{P}'_{\alpha+1}(\xi), \tag{5.9}$$

$$\mathcal{U}''_\alpha(i\xi) = -i^{\alpha}\,\mathcal{P}''_{\alpha+1}(\xi). \tag{5.10}$$

Using equation (5.9) and equation (5.10) in equation (5.8) and proceeding as above in Lemma 5.2.2, we have

$$(1+\xi^2)\mathcal{P}''_\alpha(\xi) = \alpha(\alpha+2)\mathcal{P}_{\alpha+1}(\xi) - 3\xi\mathcal{P}'_{\alpha+1}(\xi),$$

$$\mathcal{P}''_\alpha(\xi) = \frac{\alpha(\alpha+2)}{(1+\xi^2)}\mathcal{P}_{\alpha+1}(\xi) - \frac{3\xi}{(1+\xi^2)}\mathcal{P}'_{\alpha+1}(\xi).$$

This proves the Lemma 5.2.3. ■

Lemma 5.2.4. *For any integer $\alpha \geq \lambda > 0$, and $\xi \in R$, we have the identity*

$$\mathcal{P}^{\lambda}_{\alpha+1}(\xi) = -\frac{1}{(1+\xi^2)}\left[(2\lambda-1)\xi\;\mathcal{P}^{\lambda-1}_{\alpha+1}(\xi) + \big((\lambda-2)\lambda - \alpha(\alpha+2)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+1}(\xi)\right].$$

where $\mathcal{P}_\alpha(\xi)$ is a Pell polynomial.

Proof. From Lemma 5.2.2 and Lemma 5.2.3,

$$(1+\xi^2)\mathcal{P}'_{\alpha+1}(\xi) \;=\; (\alpha+1)\,\mathcal{P}_\alpha(\xi) + \alpha\xi\;\mathcal{P}_{\alpha+1}(\xi), \tag{5.11}$$

$$(1+\xi^2)\mathcal{P}''_\alpha(\xi) = \alpha(\alpha+2)\mathcal{P}_{\alpha+1}(\xi) - 3\xi\mathcal{P}'_{\alpha+1}(\xi), \tag{5.12}$$

Differentiating equation (5.12) $(\lambda-2)$ times, and using equation (5.11) we obtain

$$(1+\xi^2)\,\mathcal{P}^{\lambda}_{\alpha+1}(\xi) = -(2\lambda-1)\xi\;\mathcal{P}^{\lambda-1}_{\alpha+1}(\xi) - \big((\lambda-2)\lambda - \alpha(\alpha+2)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+1}(\xi),$$

$$\mathcal{P}^{\lambda}_{\alpha+1}(\xi) = -\frac{1}{(1+\xi^2)}\left[(2\lambda-1)\xi\;\mathcal{P}^{\lambda-1}_{\alpha+1}(\xi) + \big((\lambda-2)\lambda - \alpha(\alpha+2)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+1}(\xi)\right],$$

This proves the Lemma 5.2.4. ■

Lemma 5.2.5. *For any non-negative integers α,k, and $\xi \in R$,*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}(\xi)\cdot\mathcal{P}_{\sigma_2+1}(\xi)\cdots\mathcal{P}_{\sigma_{\lambda+1}+1}(\xi) = \frac{1}{2^{\lambda}\,\lambda!}\mathcal{P}^{\lambda}_{\alpha+\lambda+1}(\xi),$$

*where sum runs over all $\sigma_{\hbar}$ (≥ 0) in **Z** $(\hbar = 1,2,\ldots,\lambda+1)$ with $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1} = \alpha$ and where $\mathcal{P}_\alpha(\xi)$ is a Pell polynomial.*

Proof. The Lemma 5.2.5 can be easily established by using the equation 1.65 *(xv)* in equation (1.82) ■.

Theorem 5.2.1. *For any non-negative integers* $\alpha \geq \lambda > 0$, *and* $\xi \in R$ *then*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}(\xi)\cdot\mathcal{P}_{\sigma_2+1}(\xi)\cdots\mathcal{P}_{\sigma_{\lambda+1}+1}(\xi)$$

$$= -\frac{1}{2^{\lambda}\ \lambda!\,(1+\xi^2)}\left[(2\lambda-1)\xi\ \mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}(\xi)\right.$$

$$\left.+\ \big(\lambda(\lambda-2)-(\alpha+\lambda+2)(\alpha+\lambda)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}(\xi)\right]$$

where sum runs over all $\sigma_\hbar\ (\geq 0)$ *in* $\boldsymbol{Z}$ *(*$\hbar = 1,2,\ldots,\lambda+1$*) with* $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha$.

Proof. Using Lemma 5.2.4 and Lemma 5.2.5, we get the desired result. ■

Theorem 5.2.2. *For any non-negative integers* $\alpha \geq \lambda > 0$, *then the following identities hold:*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{\sigma_1+1}(\xi)\cdot\mathcal{F}_{\sigma_2+1}(\xi)\cdots\mathcal{F}_{\sigma_{\lambda+1}+1}(\xi)$$

$$= \frac{(-1)^{\alpha}}{2^{\lambda-1}\ \lambda!\,(\xi^2+4)}\left[(2\lambda-1)\xi\ \mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}\left(-\frac{\xi}{2}\right)\right.$$

$$\left.-\ 2\big((\lambda-2)\lambda-(\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}\left(-\frac{\xi}{2}\right)\right]$$

where sum runs over all $\sigma_\hbar\ (\geq 0)$ *in* $\boldsymbol{Z}$ *(*$\hbar = 1,2,\ldots,\lambda+1$*) with* $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha$.

Proof: Replacing ξ by $-\frac{\xi}{2}$ in Theorem 5.2.1, we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}\left(-\frac{\xi}{2}\right)\cdot\mathcal{P}_{\sigma_2+1}\left(-\frac{\xi}{2}\right)\cdots\mathcal{P}_{\sigma_{\lambda+1}+1}\left(-\frac{\xi}{2}\right)$$

$$= -\frac{1}{2^{\lambda}\ \lambda!\left(1+\left(-\frac{\xi}{2}\right)^2\right)}\left[(2\lambda-1)\left(-\frac{\xi}{2}\right)\mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}\left(-\frac{\xi}{2}\right)\right.$$

$$+\big((\lambda-2)\lambda$$

$$\left.-(\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}\left(-\frac{\xi}{2}\right)\right], \qquad (5.13)$$

Replacing ξ by $\frac{\xi}{2}$ in equation 1.65 *(xv)* and using $\mathcal{F}_{\alpha}(\xi) = i^{\alpha-1}\ \mathcal{U}_{\alpha-1}\left(-\frac{\xi i}{2}\right)$, we get

$$\mathcal{P}_{\alpha+1}\left(-\frac{\xi}{2}\right)=(-1)^{\alpha}\,\mathcal{F}_{\alpha+1}(\xi), \tag{5.14}$$

Using equation (5.14) in equation (5.13), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha}\mathcal{F}_{\sigma_1+1}(\xi)\cdot\mathcal{F}_{\sigma_2+1}(\xi)\cdots\mathcal{F}_{\sigma_{\lambda+1}+1}(\xi)$$

$$=\frac{(-1)^{\alpha+1}}{2^{\lambda}\ \lambda!\left(1+\left(-\frac{\xi}{2}\right)^{2}\right)}\left[(2\lambda-1)\left(-\frac{\xi}{2}\right)\ \mathcal{P}_{\alpha+\lambda+1}^{\lambda-1}\left(-\frac{\xi}{2}\right)+\big((\lambda-2)\lambda\right.$$

$$\left.-(\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}_{\alpha+\lambda+1}^{\lambda-2}\left(-\frac{\xi}{2}\right)\right],$$

$$=\frac{(-1)^{\alpha+2}\,2^{2}}{2^{\lambda+1}\ \lambda!\,(4+\xi^{2})}\left[(2\lambda-1)\xi\ \mathcal{P}_{\alpha+\lambda+1}^{\lambda-1}\left(-\frac{\xi}{2}\right)-2\big((\lambda-2)\lambda\right.$$

$$\left.-(\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}_{\alpha+\lambda+1}^{\lambda-2}\left(-\frac{\xi}{2}\right)\right],$$

$$\therefore\ \sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha}\mathcal{F}_{\sigma_1+1}(\xi)\cdot\mathcal{F}_{\sigma_2+1}(\xi)\cdots\mathcal{F}_{\sigma_{\lambda+1}+1}(\xi)$$

$$=\frac{(-1)^{\alpha}}{2^{\lambda-1}\ \lambda!\,(\xi^{2}+4)}\left[(2\lambda-1)\xi\ \mathcal{P}_{\alpha+\lambda+1}^{\lambda-1}\left(-\frac{\xi}{2}\right)-2\big((\lambda-2)\right.$$

$$\left.-(\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}_{\alpha+\lambda+1}^{\lambda-2}\left(-\frac{\xi}{2}\right)\right].$$

This establishes the Theorem 5.2.2. ■

Theorem 5.2.3. *For any non-negative integers* $\alpha\geq\lambda>0$, *we have the following identities:*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha}\mathcal{F}_{2(\sigma_1+1)}\cdot\mathcal{F}_{2(\sigma_2+1)}\cdots\mathcal{F}_{2(\sigma_{\lambda+1}+1)}$$

$$=-\frac{i^{\alpha}}{2^{\lambda-1}.5.\lambda!}\left[3i\,(2\lambda-1)\,\mathcal{P}_{\alpha+\lambda+1}^{\lambda-1}\left(-\frac{3i}{2}\right)\right.$$

$$\left.+\ 2\big((\alpha+\lambda)(\alpha+\lambda+2)-(\lambda-2)\lambda\big)\,\mathcal{P}_{\alpha+\lambda+1}^{\lambda-2}\left(-\frac{3i}{2}\right)\right]$$

$$=\frac{1}{2^{\lambda-1}\,.5.\,i^{\alpha}.\lambda!}\left[3i\,(2\lambda-1)\,\mathcal{P}_{\alpha+\lambda+1}^{\lambda-1}\left(\frac{3i}{2}\right)\right.$$

$$\left.+\ 2\big((\lambda-2)\lambda-(\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}_{\alpha+\lambda+1}^{\lambda-2}\left(\frac{3i}{2}\right)\right],$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\dots,\lambda+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1} = \alpha$ *with* $\binom{\lambda+1}{\gamma} = 0$, *for* $\gamma > \lambda+1$.

Proof. Taking $\xi = -\frac{3i}{2}$ in Theorem 5.2.1, we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}\left(-\frac{3i}{2}\right)\cdot\mathcal{P}_{\sigma_2+1}\left(-\frac{3i}{2}\right)\cdots\mathcal{P}_{\sigma_{\lambda+1}+1}\left(-\frac{3i}{2}\right)$$

$$= -\frac{1}{2^{\lambda}\ \lambda!\left(1+\left(-\frac{3i}{2}\right)^2\right)}\Bigg[(2\lambda-1)\left(-\frac{3i}{2}\right)\ \mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}\left(-\frac{3i}{2}\right)+\big((\lambda-2)\lambda$$

$$-(\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}\left(-\frac{3i}{2}\right)\Bigg],$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}\left(-\frac{3i}{2}\right)\cdot\mathcal{P}_{\sigma_2+1}\left(-\frac{3i}{2}\right)\cdots\mathcal{P}_{\sigma_{\lambda+1}+1}\left(-\frac{3i}{2}\right)$$

$$= -\frac{1}{2^{\lambda-1}\ \lambda!\ 5}\Bigg[3i\ (2\lambda-1)\ \mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}\left(-\frac{3i}{2}\right)-2\,\big((\lambda-2)\lambda$$

$$-(\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}\left(-\frac{3i}{2}\right)\Bigg].$$

Now using Lemma 5.2.1*(i)*, we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{2(\sigma_1+1)}\cdot\mathcal{F}_{2(\sigma_2+1)}\cdots\mathcal{F}_{2(\sigma_{\lambda+1}+1)}$$

$$= -\frac{i^{\alpha}}{2^{\lambda-1}.5.\lambda!}\Bigg[3i\ (2\lambda-1)\,\mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}\left(-\frac{3i}{2}\right)$$

$$+\ 2\big((\alpha+\lambda)(\alpha+\lambda+2)-(\lambda-2)\lambda\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}\left(-\frac{3i}{2}\right)\Bigg].$$

Again, taking $\xi = \frac{3i}{2}$ in Theorem 5.2. 1 and using Lemma 5.2.1*(ii)*, and proceeding as above, we get

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{2(\sigma_1+1)}\cdot\mathcal{F}_{2(\sigma_2+1)}\cdots\mathcal{F}_{2(\sigma_{\lambda+1}+1)}$$

$$= \frac{1}{2^{\lambda-1}\ .5.i^{\alpha}.\lambda!}\Bigg[3i\ (2\lambda-1)\,\mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}\left(\frac{3i}{2}\right)$$

$$+\ 2\big((\lambda-2)\lambda-(\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}\left(\frac{3i}{2}\right)\Bigg].$$

Thus, the Theorem 5.2.3 is established. ■

Theorem 5.2.4. *For any non-negative integers* $\alpha \geq \lambda > 0$, *we have the following identities:*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{3(\sigma_1+1)} \cdot \mathcal{F}_{3(\sigma_2+1)} \cdots \mathcal{F}_{3(\sigma_{\lambda+1}+1)}$$

$$= \frac{(-1)^\alpha}{5.\lambda!}\Big[2(2\lambda-1)\,\mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}(-2)$$

$$+ \big((\alpha+\lambda)(\alpha+\lambda+2)-(\lambda-2)\lambda\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}(-2)\Big],$$

where sum runs over all $\sigma_\hbar\ (\geq 0)$ *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\ldots,\lambda+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1} = \alpha$ *with* $\binom{\lambda+1}{\gamma} = 0$, *for* $\gamma > \lambda+1$.

Proof. Taking $\xi = -2$ in Theorem 5.2.1, we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}(-2) \cdot \mathcal{P}_{\sigma_2+1}(-2) \cdots \mathcal{P}_{\sigma_{\lambda+1}+1}(-2)$$

$$= -\frac{1}{2^\lambda\,\lambda!\,(1+(-2)^2)}\Big[(2\lambda-1)(-2)\ \mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}(-2) + \big((\lambda-2)\lambda$$

$$- (\alpha+\lambda)(\alpha+\lambda+2)\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}(-2)\Big],$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}(-2) \cdot \mathcal{P}_{\sigma_2+1}(-2) \cdots \mathcal{P}_{\sigma_{\lambda+1}+1}(-2)$$

$$= \frac{1}{2^\lambda\,\lambda!\,5}\Big[2(2\lambda-1)\ \mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}(-2)$$

$$+ \big((\alpha+\lambda)(\alpha+\lambda+2)-(\lambda-2)\lambda\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}(-2)\Big],$$

Now using Lemma 5.2.1(iii), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{3(\sigma_1+1)} \cdot \mathcal{F}_{3(\sigma_2+1)} \cdots \mathcal{F}_{3(\sigma_{\lambda+1}+1)}$$

$$= \frac{(-1)^\alpha}{5.\lambda!}\Big[2(2\lambda-1)\,\mathcal{P}^{\lambda-1}_{\alpha+\lambda+1}(-2)$$

$$+ \big((\alpha+\lambda)(\alpha+\lambda+2)-(\lambda-2)\lambda\big)\,\mathcal{P}^{\lambda-2}_{\alpha+\lambda+1}(-2)\Big].$$

Hence the Theorem 5.2.4 is established. ■

Theorem 5.2.5. *For any non-negative integers* $\alpha \geq \lambda > 0$, *we have the following identities:*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{\sigma_1}(i\xi)\cdot\mathcal{V}_{\sigma_2}(i\xi)\cdots\mathcal{V}_{\sigma_{\lambda+1}}(i\xi)$$
$$= \frac{1}{2^\lambda\,\lambda!\,(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\big[(2\lambda-1)\xi\ \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-1}(\xi)$$
$$+(\alpha-\gamma+2)(\alpha-\gamma+2\lambda)\,\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-2}(\xi)\big],$$

where sum runs over all $\sigma_\hbar\ (\geq 0)$ *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\dots,\lambda+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha$ *with* $\binom{\lambda+1}{\gamma}=0$, *for* $\gamma > \lambda+1$ *and* $i=\sqrt{-1}$.

Proof. Replacing ξ by $i\xi$ in equation (1.93), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{\sigma_1}(i\xi)\cdot\mathcal{V}_{\sigma_2}(i\xi)\cdots\mathcal{V}_{\sigma_{\lambda+1}}(i\xi)$$
$$= \frac{1}{2^\lambda\,\lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{\lambda+1}{\gamma}\mathcal{U}_{\alpha-\gamma+\lambda}^{\lambda}(i\xi). \qquad (5.15)$$

Differentiating equation 1.65 (*xv*) w.r.t ξ, we get

$$\mathcal{U}_{\alpha}^{\lambda}(i\xi) = i^{\alpha-\lambda}\ \mathcal{P}_{\alpha+1}(\xi), \qquad (5.16)$$

Using equation (5.16) in equation (5.15), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{\sigma_1}(i\xi)\cdot\mathcal{V}_{\sigma_2}(i\xi)\cdots\mathcal{V}_{\sigma_{\lambda+1}}(i\xi)$$
$$= \frac{1}{2^\lambda\,\lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma} i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda}(\xi) \qquad (5.17)$$

Using Lemma 5.2.4 in equation (5.17), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{\sigma_1}(i\xi)\cdot\mathcal{V}_{\sigma_2}(i\xi)\cdots\mathcal{V}_{\sigma_{\lambda+1}}(i\xi)$$
$$= \frac{1}{2^\lambda\,\lambda!\,(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\big[(2\lambda-1)\xi\ \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-1}(\xi)$$
$$+\ \big((\lambda-2)\lambda-\alpha(\alpha+2)\big)\,\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-2}(\xi)\big]$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{\sigma_1}(i\xi)\cdot\mathcal{V}_{\sigma_2}(i\xi)\cdots\mathcal{V}_{\sigma_{\lambda+1}}(i\xi)$$
$$= \frac{1}{2^\lambda\,\lambda!\,(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\big[(2\lambda-1)\xi\ \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-1}(\xi)$$
$$+(\alpha-\gamma+2)(\alpha-\gamma+2\lambda)\,\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-2}(\xi)\big]$$

which establishes the Theorem 5.2.5. ■

Theorem 5.2.6. *For any non-negative integers* $\alpha \geq \lambda > 0$, *the following identities holds:*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{W}_{\sigma_1}(i\xi)\cdot\mathcal{W}_{\sigma_2}(i\xi)\cdots\mathcal{W}_{\sigma_{\lambda+1}}(i\xi)$$

$$= -\frac{1}{2^{\lambda}\ \lambda!\,(1+\xi^2)}\sum_{\gamma=0}^{\alpha} i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\Big[(2\lambda-1)\xi\ \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-1}(\xi)+(\alpha$$

$$-\gamma+2)(\alpha-\gamma+2\lambda)\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-2}(\xi)\Big],$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\dots,\lambda+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha$ *with* $\binom{\lambda+1}{\gamma}=0$ *for* $\gamma > \lambda+1$ *and* $i=\sqrt{-1}$.

Proof. Using equation (1.93) and proceeding as in Theorem 5.2.5, we can easily establish Theorem 5.2.6. ■

Theorem 5.2.7. *For any non-negative integers* $\alpha \geq \lambda > 0$, *we have the following identities:*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{2\sigma_1+1}\cdot\mathcal{F}_{2\sigma_2+1}\cdots\mathcal{F}_{2\sigma_{\lambda+1}+1}$$

$$= \frac{1}{2^{\lambda-1}.5.\lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1}i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\Big[3i(2\lambda$$

$$-1)\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-1}\left(-\frac{3}{2}i\right)$$

$$-2(\alpha-\gamma+2)(\alpha-\gamma+2\lambda)\,\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-2}\left(-\frac{3}{2}i\right)\Big],$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\dots,\lambda+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha$ *with* $\binom{\lambda+1}{\gamma}=0$ *for* $\gamma > \lambda+1$ *and* $i=\sqrt{-1}$.

Proof. Replacing z by $\xi = -\frac{3}{2}i$ in Theorem 5.2.5, we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{\sigma_1}\left(\frac{3}{2}\right)\cdot \mathcal{V}_{\sigma_2}\left(\frac{3}{2}\right)\cdots \mathcal{V}_{\sigma_{\lambda+1}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^{\lambda}\lambda!\left(1+\left(-\frac{3}{2}i\right)^2\right)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\Big[(2\lambda$$

$$-1)\left(-\frac{3}{2}i\right)\ \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-1}\left(-\frac{3}{2}i\right)+(\alpha-\gamma+2)(\alpha-\gamma$$

$$+2\lambda)\ \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-2}\left(-\frac{3}{2}i\right)\Big],$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{\sigma_1}\left(\frac{3}{2}\right)\cdot \mathcal{V}_{\sigma_2}\left(\frac{3}{2}\right)\cdots \mathcal{V}_{\sigma_{\lambda+1}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^{\lambda-1}.5.\lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\Big[3i(2\lambda$$

$$-1)\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-1}\left(-\frac{3}{2}i\right)-2(\alpha-\gamma+2)(\alpha-\gamma$$

$$+2\lambda)\ \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-2}\left(-\frac{3}{2}i\right)\Big],$$

Using Lemma 5.2.1*(iv)*, we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{2\sigma_1+1}\cdot \mathcal{F}_{2\sigma_2+1}\cdots \mathcal{F}_{2\sigma_{\lambda+1}+1}$$

$$= \frac{1}{2^{\lambda-1}.5.\lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\Big[3i(2\lambda$$

$$-1)\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-1}\left(-\frac{3}{2}i\right)$$

$$-2(\alpha-\gamma+2)(\alpha-\gamma+2\lambda)\ \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-2}\left(-\frac{3}{2}i\right)\Big].\ \blacksquare$$

Theorem 5.2.8. *For any non-negative integers* $\alpha \geq \lambda > 0$*, we have the following identities:*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{L}_{2\sigma_1+1}\cdot\mathcal{L}_{2\sigma_2+1}\cdots\mathcal{L}_{2\sigma_{\lambda+1}+1}$$

$$= -\frac{1}{2^{\lambda-1}.5.\lambda!}\sum_{\gamma=0}^{\alpha} i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\Big[3i(2\lambda-1)\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-1}\left(-\frac{3}{2}i\right)$$

$$-2(\alpha-\gamma+2)(\alpha-\gamma+2\lambda)\,\mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda-2}\left(-\frac{3}{2}i\right)\Big]$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\ldots,\lambda+1)$ *with* $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha$ *with* $\binom{\lambda+1}{\gamma}=0$ *for* $\gamma > \lambda+1$ *and* $i=\sqrt{-1}$.

Proof. Replacing z by $\xi = -\frac{3}{2}i$ in Theorem 5.2.6 and proceeding as in Theorem 5.2.7, we get the desired result. ■

Corollary 5.2.1. *For any non-negative integer* α, *and* $\xi \in R$, *the following identities holds:*

$$\sum_{a+b+c=\alpha}\mathcal{P}_{a+1}(\xi)\cdot\mathcal{P}_{b+1}(\xi).\mathcal{P}_{c+1}(\xi) = A_\alpha(\xi)\,\mathcal{P}_{\alpha+3}(\xi) - B_\alpha(\xi)\mathcal{P}_{\alpha+2}(\xi),$$

$$\sum_{a+b+c+d=\alpha}\mathcal{P}_{a+1}(\xi)\cdot\mathcal{P}_{b+1}(\xi).\mathcal{P}_{c+1}(\xi).\mathcal{P}_{d+1}(\xi) = C_\alpha(\xi)\,\mathcal{P}_{\alpha+3}(\xi) + D_\alpha(\xi)\,\mathcal{P}_{\alpha+4}(\xi),$$

where

$$A_\alpha(\xi) = \frac{(\alpha+2)}{8(1+\xi^2)^2}[(\alpha+1)\xi^2+(\alpha+4)],$$

$$B_\alpha(\xi) = \frac{3\xi(\alpha+3)}{8(1+\xi^2)^2},$$

$$C_\alpha(\xi) = \frac{(\alpha+4)}{48(1+\xi^2)^3}[(\alpha^2+8\alpha+27)\xi^2+(\alpha^2+8\alpha+12)],$$

$$D_\alpha(\xi) = \frac{(\alpha+3)\xi}{48(1+\xi^2)^3}\,[(\alpha^2+3\alpha+2)\xi^2+(\alpha^2+3\alpha-13)].$$

Proof. Taking $\lambda = 2$ in Theorem 5.2.1, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{P}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{P}_{\mathfrak{b}+1}(\xi).\mathcal{P}_{c+1}(\xi)$$

$$= -\frac{1}{8(1+\xi^2)}[3\xi\mathcal{P}'_{\alpha+3}(\xi) - (\alpha+2)(\alpha+4)\mathcal{P}_{\alpha+3}(\xi)],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{P}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{P}_{\mathfrak{b}+1}(\xi).\mathcal{P}_{c+1}(\xi)$$

$$= \frac{1}{8(1+\xi^2)}[(\alpha+2)(\alpha+4)\mathcal{P}_{\alpha+3}(\xi) - 3\xi\mathcal{P}'_{\alpha+3}(\xi)],$$

$$= \frac{1}{8(1+\xi^2)}(\alpha+2)(\alpha+4)\mathcal{P}_{\alpha+3}(\xi)$$

$$- \frac{3\xi}{8(1+\xi^2)}\left[\frac{(\alpha+3)}{(1+\xi^2)}\mathcal{P}_{\alpha+2}(\xi) + \frac{(\alpha+2)}{(1+\xi^2)}\,\xi\,\mathcal{P}_{\alpha+3}(\xi)\right],$$

$$= \frac{(\alpha+2)}{8(1+\xi^2)}\left[(\alpha+4) - \frac{3\xi^2}{(1+\xi^2)}\right]\mathcal{P}_{\alpha+3}(\xi) - \frac{3\xi(\alpha+3)}{8(1+\xi^2)^2}\mathcal{P}_{\alpha+2}(\xi)$$

$$= \frac{(\alpha+2)}{8(1+\xi^2)^2}[(\alpha+1)\xi^2 + (\alpha+4)]\,\mathcal{P}_{\alpha+3}(\xi) - \frac{3\xi(\alpha+3)}{8(1+\xi^2)^2}\mathcal{P}_{\alpha+2}(\xi),$$

Therefore,

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{P}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{P}_{\mathfrak{b}+1}(\xi).\mathcal{P}_{c+1}(\xi) = A_\alpha(\xi)\,\mathcal{P}_{\alpha+3}(\xi) - B_\alpha(\xi)\mathcal{P}_{\alpha+2}(\xi),$$

where,

$$A_\alpha(\xi) = \frac{(\alpha+2)}{8(1+\xi^2)^2}[(\alpha+1)\xi^2 + (\alpha+4)], \quad B_\alpha(\xi) = \frac{3\xi(\alpha+3)}{8(1+\xi^2)^2}.$$

Taking $\lambda = 3$ in Theorem 5.2.1, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c+d=\alpha} \mathcal{P}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{P}_{\mathfrak{b}+1}(\xi).\mathcal{P}_{c+1}(\xi).\mathcal{P}_{d+1}(\xi)$$

$$= -\frac{1}{48(1+\xi^2)}\left[5\xi\mathcal{P}''_{\alpha+4}(\xi) + \big(3 - (\alpha+3)(\alpha+5)\big)\mathcal{P}'_{\alpha+4}(\xi)\right],$$

$$= \frac{1}{48(1+\xi^2)}[(\alpha+3)(\alpha+5) - 3]\mathcal{P}'_{\alpha+4}(\xi) - 5\xi\mathcal{P}''_{\alpha+4}(\xi),$$

$$= \frac{1}{48(1+\xi^2)}[(\alpha+3)(\alpha+5) - 3]\mathcal{P}'_{\alpha+4}(\xi)$$

$$- \frac{5\xi}{48(1+\xi^2)}\left[\frac{(\alpha+3)(\alpha+5)}{(1+\xi^2)}\mathcal{P}_{\alpha+4}(\xi) - \frac{3\xi}{(1+\xi^2)}\;\mathcal{P}'_{\alpha+4}(\xi)\right],$$

$$= \frac{1}{48(1+\xi^2)}\left[\left((\alpha+3)(\alpha+5)-3\right)+\frac{15\xi^2}{(1+\xi^2)}\right]\mathcal{P}'_{\alpha+4}(\xi)$$
$$-\frac{5\xi\,(\alpha+3)(\alpha+5)}{48\,(1+\xi^2)^2}\mathcal{P}_{\alpha+4}(\xi),$$

$$\sum_{a+b+c+d=\alpha} \mathcal{P}_{a+1}(\xi)\cdot\mathcal{P}_{b+1}(\xi).\mathcal{P}_{c+1}(\xi).\mathcal{P}_{d+1}(\xi)$$
$$= \frac{1}{48(1+\xi^2)}\left[\left((\alpha+3)(\alpha+5)-3\right)+\frac{15\xi^2}{(1+\xi^2)}\right]\left[\frac{(\alpha+4)}{(1+\xi^2)}\mathcal{P}_{\alpha+3}(\xi)\right.$$
$$\left.+\frac{(\alpha+3)\xi}{(1+\xi^2)}\,\mathcal{P}_{\alpha+4}(\xi)\right]-\frac{5\xi\,(\alpha+3)(\alpha+5)}{48\,(1+\xi^2)^2}\mathcal{P}_{\alpha+4}(\xi),$$
$$= \frac{(\alpha+4)}{48(1+\xi^2)^3}\left[\left((\alpha+3)(\alpha+5)-3\right)(1+\xi^2)\right.$$
$$\left.+15\xi^2\right]\mathcal{P}_{\alpha+3}(\xi)$$
$$+\frac{(\alpha+3)\xi}{48(1+\xi^2)^3}\,\left[\left((\alpha+3)(\alpha+5)-3\right)(1+\xi^2)+15\xi^2\right.$$
$$\left.-5\,(\alpha+5)(1+\xi^2)\right]\mathcal{P}_{\alpha+4}(\xi),$$

$$= \frac{(\alpha+4)}{48(1+\xi^2)^3}[(\alpha^2+8\alpha+27)\xi^2+(\alpha^2+8\alpha+12)]\,\mathcal{P}_{\alpha+3}(\xi)$$
$$+\frac{(\alpha+3)\xi}{48(1+\xi^2)^3}\,[(\alpha^2+3\alpha+2)\xi^2+(\alpha^2+3\alpha-13)]\mathcal{P}_{\alpha+4}(\xi),$$

Therefore,

$$\sum_{a+b+c+d=\alpha} \mathcal{P}_{a+1}(\xi)\cdot\mathcal{P}_{b+1}(\xi).\mathcal{P}_{c+1}(\xi).\mathcal{P}_{d+1}(\xi) = C_\alpha(\xi)\,\mathcal{P}_{\alpha+3}(\xi)+D_\alpha(\xi)\,\mathcal{P}_{\alpha+4}(\xi),$$

where

$$C_\alpha(\xi) = \frac{(\alpha+4)}{48(1+\xi^2)^3}[(\alpha^2+8\alpha+27)\xi^2+(\alpha^2+8\alpha+12)],$$

$$D_\alpha(\xi) = \frac{(\alpha+3)\xi}{48(1+\xi^2)^3}\,[(\alpha^2+3\alpha+2)\xi^2+(\alpha^2+3\alpha-13)].$$

Thus, the Corollary 5.2.1 is established. ■

Corollary 5.2.2. *For any non-negative integer α, and $\xi \in R$, the following identities holds:*

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{F}_{\mathfrak{b}+1}(\xi).\mathcal{F}_{c+1}(\xi)$$

$$= \frac{(\alpha+2)}{2(\xi^2+4)^2}[(\alpha+1)\xi^2+4(\alpha+4)]\mathcal{F}_{\alpha+3}(\xi) - \frac{3\xi(\alpha+3)}{(\xi^2+4)^2}\mathcal{F}_{\alpha+2}(\xi),$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{F}_{\mathfrak{b}+1}(\xi).\mathcal{F}_{c+1}(\xi)\,\mathcal{F}_{d+1}(\xi)$$

$$= \frac{(\alpha+4)}{3(\xi^2+4)^3}[(\alpha^2+8\alpha+27)\xi^2+4(\alpha^2+8\alpha+12)]\mathcal{F}_{\alpha+3}(\xi)$$

$$+ \frac{(\alpha+3)\,\xi}{6(\xi^2+4)^3}[(\alpha^2+3\alpha+2)\xi^2+4(\alpha^2+3\alpha-13)]\,\mathcal{F}_{\alpha+4}(\xi),$$

Proof. Taking $\lambda = 2$ in Theorem 5.2.2, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{F}_{\mathfrak{b}+1}(\xi).\mathcal{F}_{c+1}(\xi)$$

$$= \frac{(-1)^{\alpha}}{4\ (\xi^2+4)}\left[3\xi\,\mathcal{P'}_{\alpha+3}\left(-\frac{\xi}{2}\right) + 2(\alpha+2)(\alpha + 4)\ \mathcal{P}_{\alpha+3}\left(-\frac{\xi}{2}\right)\right], \tag{5.18}$$

Using Lemma 5.2.1, we have

$$\mathcal{P}'_{\alpha+3}\left(-\frac{\xi}{2}\right) = \frac{4\,(\alpha+3)}{(\xi^2+4)}\,\mathcal{P}_{\alpha+2}\left(-\frac{\xi}{2}\right) - \frac{2(\alpha+2)\xi}{(\xi^2+4)}\,\mathcal{P}_{\alpha+3}\left(-\frac{\xi}{2}\right), \tag{5.19}$$

By using equation (5.18) in equation (5.19), we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{F}_{\mathfrak{b}+1}(\xi).\mathcal{F}_{c+1}(\xi)$$

$$= \frac{(-1)^{\alpha}}{4\ (\xi^2+4)}\left[3\xi\left(\frac{4\,(\alpha+3)}{(\xi^2+4)}\,\mathcal{P}_{\alpha+2}\left(-\frac{\xi}{2}\right) - \frac{2(\alpha+2)\xi}{(\xi^2+4)}\,\mathcal{P}_{\alpha+3}\left(-\frac{\xi}{2}\right)\right) + 2(\alpha+2)(\alpha+4)\ \mathcal{P}_{\alpha+3}\left(-\frac{\xi}{2}\right)\right],$$

$$= \frac{(-1)^{\alpha}}{4\ (\xi^2+4)}\left[\frac{12\xi\ (\alpha+3)}{(\xi^2+4)}\ \mathcal{P}_{\alpha+2}\left(-\frac{\xi}{2}\right)\right.$$

$$\left. - 2(\alpha+2)\left(\frac{3\xi^2}{(\xi^2+4)} - (\alpha+4)\ \right)\mathcal{P}_{\alpha+3}\left(-\frac{\xi}{2}\right)\right],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{F}_{\mathfrak{b}+1}(\xi).\mathcal{F}_{c+1}(\xi)$$

$$= \frac{(-1)^{\alpha}}{4\ (\xi^2+4)}\left[\frac{12\xi\ (\alpha+3)}{(\xi^2+4)}\ \mathcal{P}_{\alpha+2}\left(-\frac{\xi}{2}\right)\right.$$

$$\left. - 2(\alpha+2)\left(\frac{3\xi^2}{(\xi^2+4)} - (\alpha+4)\ \right)\mathcal{P}_{\alpha+3}\left(-\frac{\xi}{2}\right)\right],$$

Now using equation (5.14), we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{F}_{\mathfrak{b}+1}(\xi).\mathcal{F}_{c+1}(\xi)$$

$$= -\frac{1}{2\ (\xi^2+4)^2}\{6\xi\ (\alpha+3)\ \mathcal{F}_{\alpha+2}(\xi)$$

$$- (\alpha+2)[(\alpha+1)\xi^2+4(\alpha+4)]\mathcal{F}_{\alpha+3}(\xi)\},$$

$$= \frac{(\alpha+2)}{2(\xi^2+4)^2}[(\alpha+1)\xi^2+4(\alpha+4)]\mathcal{F}_{\alpha+3}(\xi) - \frac{3\xi(\alpha+3)}{(\xi^2+4)^2}\ \mathcal{F}_{\alpha+2}(\xi).$$

Again Taking $\lambda = 3$ in Theorem 5.2.2, and using Lemma 5.2.2 and Lemma 5.2.3 and using equation (5.14) and proceeding as above, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{\mathfrak{a}+1}(\xi)\cdot\mathcal{F}_{\mathfrak{b}+1}(\xi).\mathcal{F}_{c+1}(\xi)\,\mathcal{F}_{d+1}(\xi)$$

$$= \frac{(\alpha+4)}{3(\xi^2+4)^3}[(\alpha^2+8\alpha+27)\xi^2+4(\alpha^2+8\alpha+12)]\mathcal{F}_{\alpha+3}(\xi)$$

$$+ \frac{(\alpha+3)\ \xi}{6(\xi^2+4)^3}[(\alpha^2+3\alpha+2)\xi^2+4(\alpha^2+3\alpha-13)]\ \mathcal{F}_{\alpha+4}(\xi).$$

This establishes the Corollary 5.2.2. ■

Corollary 5.2. 3. *For any non-negative integers α, we have the following identities*

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{2(\mathfrak{a}+1)}\cdot\mathcal{F}_{2(\mathfrak{b}+1)}.\mathcal{F}_{2(c+1)}$$

$$= \frac{1}{50}[18\ (\alpha+3)\mathcal{F}_{2\alpha+4} + (\alpha+2)(5\alpha-7)\ \mathcal{F}_{2\alpha+6}].$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+c+d=\alpha} \mathcal{F}_{2(\mathfrak{a}+1)} \cdot \mathcal{F}_{2(\mathfrak{b}+1)} . \mathcal{F}_{2(c+1)} . \mathcal{F}_{2(d+1)}$$

$$= \frac{1}{150} \left[3(\alpha+3)(\alpha^2+3\alpha+14)\,\mathcal{F}_{2\alpha+8} - 2(\alpha+4)(\alpha^2+8\alpha+39)\mathcal{F}_{2\alpha+6}\right].$$

Proof. Taking $\lambda = 2$ in Theorem 5.2.3, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{2(\mathfrak{a}+1)} \cdot \mathcal{F}_{2(\mathfrak{b}+1)} . \mathcal{F}_{2(c+1)}$$

$$= -\frac{i^\alpha}{20}\left[9i\,\mathcal{P}'_{\alpha+3}\left(-\frac{3}{2}\,i\right) + 2(\alpha+2)(\alpha+4)\mathcal{P}_{\alpha+3}\left(-\frac{3}{2}\,i\right)\right], \quad (5.20)$$

Using Lemma 5.2.2

$$\mathcal{P}'_{\alpha+3}\left(-\frac{3}{2}\,i\right) = -\frac{4(\alpha+3)}{5}\,\mathcal{P}_{\alpha+2}\left(-\frac{3}{2}\,i\right) + \frac{6i}{5}\,(\alpha+2)\mathcal{P}_{\alpha+3}\left(-\frac{3}{2}\,i\right), \quad (5.21)$$

From equation (5.21) and equation (5.20), we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{2(\mathfrak{a}+1)} \cdot \mathcal{F}_{2(\mathfrak{b}+1)} . \mathcal{F}_{2(c+1)}$$

$$= -\frac{i^\alpha}{20}\left[9i\left(-\frac{4(\alpha+3)}{5}\,\mathcal{P}_{\alpha+2}\left(-\frac{3}{2}\,i\right) + \frac{6i}{5}\,(\alpha+2)\mathcal{P}_{\alpha+3}\left(-\frac{3}{2}\,i\right)\right) + 2(\alpha+2)(\alpha+4)\mathcal{P}_{\alpha+3}\left(-\frac{3}{2}\,i\right)\right],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{2(\mathfrak{a}+1)} \cdot \mathcal{F}_{2(\mathfrak{b}+1)} . \mathcal{F}_{2(c+1)}$$

$$= -\frac{i^\alpha}{50}\left[-18i(\alpha+3)\,\mathcal{P}_{\alpha+2}\left(-\frac{3}{2}\,i\right) + (\alpha+2)(5\alpha-7)\mathcal{P}_{\alpha+3}\left(-\frac{3}{2}\,i\right)\right],$$

Using Lemma 5.2.1*(i)*, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{2(\mathfrak{a}+1)} \cdot \mathcal{F}_{2(\mathfrak{b}+1)} . \mathcal{F}_{2(c+1)}$$

$$= -\frac{i^\alpha}{50}\left[-18i(\alpha+3)\,\frac{\mathcal{F}_{2\alpha+4}}{i^{\alpha+1}} + (\alpha+2)(5\alpha-7)\,\frac{\mathcal{F}_{2\alpha+6}}{i^{\alpha+2}}\right],$$

$$\sum_{a+b+c=\alpha} \mathcal{F}_{2(a+1)} \cdot \mathcal{F}_{2(b+1)} . \mathcal{F}_{2(c+1)}$$

$$= \frac{1}{50}[18\,(\alpha+3)\mathcal{F}_{2\alpha+4} + (\alpha+2)(5\alpha-7)\,\mathcal{F}_{2\alpha+6}].$$

Again, taking $\lambda = 3$ in Theorem 5.2.3 and using Lemma 5.2.2 and Lemma 5.2.3, with $\xi = -\frac{3}{2}\,i$ and proceeding as above, we have

$$\sum_{a+b+c+d=\alpha} \mathcal{F}_{2(a+1)} \cdot \mathcal{F}_{2(b+1)} . \mathcal{F}_{2(c+1)} . \mathcal{F}_{2(d+1)}$$

$$= \frac{1}{150}\,[3(\alpha+3)(\alpha^2+3\alpha+14)\,\mathcal{F}_{2\alpha+8}$$

$$- 2(\alpha+4)(\alpha^2+8\alpha+39)\mathcal{F}_{2\alpha+6}].$$

This establishes the Corollary 5.2.3.■

Corollary 5.2.4. *For any non-negative integer α , the following identities holds:*

(i) $$\sum_{a+b+c=\alpha} \mathcal{F}_{3(a+1)} \cdot \mathcal{F}_{3(b+1)} . \mathcal{F}_{3(c+1)}$$

$$= \frac{1}{100}[(\alpha+2)(5\alpha-18)\mathcal{F}_{3\alpha+9} - 2(\alpha+3)\,\mathcal{F}_{3\alpha+6}],$$

(ii) $$\sum_{a+b+c+d=\alpha} \mathcal{F}_{3(a+1)} \cdot \mathcal{F}_{3(b+1)} . \mathcal{F}_{3(c+1)} . \mathcal{F}_{3(d+1)}$$

$$= \frac{1}{150}\,[(\alpha+4)(\alpha^2+8\alpha+24)\,\mathcal{F}_{3\alpha+9}$$

$$+ 2(\alpha+3)(\alpha^2+3\alpha-1)\mathcal{F}_{3\alpha+12}].$$

Proof. Taking $\lambda = 2$ in Theorem 5.2.4, we have

$$\sum_{a+b+c=\alpha} \mathcal{F}_{3(a+1)} \cdot \mathcal{F}_{3(b+1)} . \mathcal{F}_{3(c+1)}$$

$$= \frac{(-1)^\alpha}{10}[6\,\mathcal{P}'_{\alpha+3}(-2)$$

$$+ (\alpha+2)(\alpha+4)\mathcal{P}_{\alpha+3}(-2)] \qquad (5.22)$$

Using Lemma 5.2.2

$$\mathcal{P}'_{\alpha+3}(-2) = \frac{(\alpha+3)}{5}\,\mathcal{P}_{\alpha+2}(-2) - \frac{2}{5}\,(\alpha+2)\mathcal{P}_{\alpha+3}(-2) \qquad (5.23)$$

From equation (5.23) and equation (5.22) with Lemma 5.2.1(*iii*)

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{3(\mathfrak{a}+1)} \cdot \mathcal{F}_{3(\mathfrak{b}+1)} . \mathcal{F}_{3(c+1)}$$
$$= \frac{(-1)^{\alpha}}{10}\left[6\left(\frac{(\alpha+3)}{5}\,\mathcal{P}_{\alpha+2}(-2) - \frac{2}{5}\,(\alpha+2)\mathcal{P}_{\alpha+3}(-2)\right) + (\alpha+2)(\alpha+4)\mathcal{P}_{\alpha+3}(-2)\right]$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{3(\mathfrak{a}+1)} \cdot \mathcal{F}_{3(\mathfrak{b}+1)} . \mathcal{F}_{3(c+1)}$$
$$= \frac{(-1)^{\alpha}}{50}\left[2(\alpha+3)\,\mathcal{P}_{\alpha+2}(-2) + (\alpha+2)(5\alpha-18)\mathcal{P}_{\alpha+3}(-2)\right]$$

Using Lemma 5.2.1 *(iii)*, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{F}_{3(\mathfrak{a}+1)} \cdot \mathcal{F}_{3(\mathfrak{b}+1)} . \mathcal{F}_{3(c+1)}$$
$$= \frac{1}{100}\left[\,(\alpha+2)\,(5\alpha-18)\,\mathcal{F}_{3\alpha+9} - 2\,(\alpha+3)\,\mathcal{F}_{3\alpha+6}\right]$$

Again, taking $\lambda = 3$ in Theorem 5.2.4, and using Lemma 5.2.2 and Lemma 5.2.3, with $\xi = -2$ and proceeding as above, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c+d=\alpha} \mathcal{F}_{3(\mathfrak{a}+1)} \cdot \mathcal{F}_{3(\mathfrak{b}+1)} . \mathcal{F}_{3(c+1)} . \mathcal{F}_{3(d+1)}$$
$$= \frac{1}{150}\left[(\alpha+4)(\alpha^2+8\alpha+24)\,\mathcal{F}_{3\alpha+9} + 2(\alpha+3)(\alpha^2+3\alpha-1)\mathcal{F}_{3\alpha+12}\right]$$

This establishes the Corollary 5.2.4. ■

Corollary 5.2.5. *For any non-negative integers α,and $\xi \in R$, we have the following identities:*

$$\sum_{a+b+c=\alpha} \mathcal{V}_a(i\xi)\cdot\mathcal{V}_b(i\xi)\,\mathcal{V}_c(i\xi)$$

$$= \frac{1}{8(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1}i^{\alpha-\gamma}\binom{3}{\gamma}\Big\{3\xi\,(\alpha-\gamma+3)\;\mathcal{P}_{\alpha-\gamma+2}(\xi)$$

$$-(\alpha-\gamma+2)[(\alpha-\gamma+1)\xi^2+(\alpha-\gamma+4)]\,\mathcal{P}_{\alpha-\gamma+3}(\xi)\Big\},$$

$$\sum_{a+b+c+d=\alpha} \mathcal{V}_a(i\xi)\cdot\mathcal{V}_b(i\xi)\,\mathcal{V}_c(i\xi)\mathcal{V}_d(i\xi)$$

$$= \frac{1}{48(1+\xi^2)^3}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1}i^{\alpha-\gamma}\binom{4}{\gamma}\Big[(\alpha-\gamma$$

$$+3)\xi\Big[\big(5(\alpha-\gamma+5)-(\alpha-\gamma+2)(\alpha-\gamma+6)\big)(1+\xi^2)$$

$$-15\xi^2\Big]\,\mathcal{P}_{\alpha-\gamma+4}(\xi)\;-(\alpha-\gamma+4)[15\,\xi^2$$

$$+(\alpha-\gamma+2)(\alpha-\gamma+6)(1+\xi^2)\;]\mathcal{P}_{\alpha-\gamma+3}(\xi)\Big]$$

Proof. Taking in Theorem 5.2.5, we have

$$\sum_{a+b+c=\alpha} \mathcal{V}_a(i\xi)\cdot\mathcal{V}_b(i\xi)\,\mathcal{V}_c(i\xi) = \frac{1}{8(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1}i^{\alpha-\gamma}\binom{3}{\gamma}3\xi\,\{\mathcal{P}'_{\alpha-\gamma+3}(\xi)$$
$$-(\alpha-\gamma+2)(\alpha-\gamma+4)\,\mathcal{P}_{\alpha-\gamma+3}(\xi)\,\},\quad(5.24)$$

From Lemma 5.2.2

$$\mathcal{P}'_{\alpha-\gamma+3}(\xi)\;=\frac{(\alpha-\gamma+3)}{(1+\xi^2)}\,\mathcal{P}_{\alpha-\gamma+2}(\xi)+\frac{(\alpha-\gamma+2)\,\xi}{(1+\xi^2)}\;\mathcal{P}_{\alpha-\gamma+3}(\xi),\qquad(5.25)$$

Using equation (5.25) in equation (5.24), we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{V}_{\mathfrak{a}}(i\xi)\cdot\mathcal{V}_{\mathfrak{b}}(i\xi)\,\mathcal{V}_c(i\xi)$$

$$= \frac{1}{8(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{3}{\gamma}\left\{3\xi\left[\frac{(\alpha-\gamma+3)}{(1+\xi^2)}\,\mathcal{P}_{\alpha-\gamma+2}(\xi)\right.\right.$$

$$\left.\left.+\frac{(\alpha-\gamma+2)\,\xi}{(1+\xi^2)}\;\mathcal{P}_{\alpha-\gamma+3}(\xi)\right] - (\alpha-\gamma+2)(\alpha-\gamma+4)\,\mathcal{P}_{\alpha-\gamma+3}(\xi)\right\},$$

$$= \frac{1}{8(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{3}{\gamma}\left\{\frac{3\xi\,(\alpha-\gamma+3)}{(1+\xi^2)}\,\mathcal{P}_{\alpha-\gamma+2}(\xi)\right.$$

$$\left.+\frac{(\alpha-\gamma+2)}{(1+\xi^2)}\,[3\xi^2-(\alpha-\gamma+4)(1+\xi^2)]\,\mathcal{P}_{\alpha-\gamma+3}(\xi)\right\},$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{V}_{\mathfrak{a}}(i\xi)\cdot\mathcal{V}_{\mathfrak{b}}(i\xi)\,\mathcal{V}_c(i\xi)$$

$$= \frac{1}{8(1+\xi^2)^2}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{3}{\gamma}\{3\xi\,(\alpha-\gamma+3)\,\mathcal{P}_{\alpha-\gamma+2}(\xi)$$

$$-(\alpha-\gamma+2)\,[(\alpha-\gamma+1)\xi^2+(\alpha-\gamma+4)]\,\mathcal{P}_{\alpha-\gamma+3}(\xi)\,\},$$

Now, taking $\lambda = 3$ in Theorem 5.2.5, we have

$$\sum_{\mathfrak{a}+\mathfrak{b}+c+d=\alpha} \mathcal{V}_{\mathfrak{a}}(i\xi)\cdot\mathcal{V}_{\mathfrak{b}}(i\xi)\,\mathcal{V}_c(i\xi)\mathcal{V}_d(i\xi)$$

$$= \frac{1}{48(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{4}{\gamma}\{5\xi\,\mathcal{P}''_{\alpha-\gamma+4}(\xi)$$

$$-(\alpha-\gamma+2)(\alpha-\gamma$$

$$+6)\,\mathcal{P}'_{\alpha-\gamma+3}(\xi)\}, \tag{5.26}$$

From Lemma 5.2.3, we have

$$\mathcal{P}''_{\alpha-\gamma+4}(\xi) = \frac{(\alpha-\gamma+3)(\alpha-\gamma+5)}{(1+\xi^2)}\,\mathcal{P}_{\alpha-\gamma+4}(\xi) - \frac{3\,\xi}{(1+\xi^2)}\;\mathcal{P}'_{\alpha-\gamma+4}(\xi), \tag{5.27}$$

Using equation (5.27) in equation (5.26), we have

$$\sum_{a+b+c+d=\alpha} \mathcal{V}_a(i\xi)\cdot\mathcal{V}_b(i\xi)\,\mathcal{V}_c(i\xi)\mathcal{V}_d(i\xi)$$

$$= \frac{1}{48(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{4}{\gamma}\left\{5\xi\left[\frac{(\alpha-\gamma+3)(\alpha-\gamma+5)}{(1+\xi^2)}\,\mathcal{P}_{\alpha-\gamma+4}(\xi)\right.\right.$$

$$\left.\left.-\frac{3\,\xi}{(1+\xi^2)}\,\mathcal{P}'_{\alpha-\gamma+4}(\xi)\right] - (\alpha-\gamma+2)(\alpha-\gamma+6)\,\mathcal{P}'_{\alpha-\gamma+3}(\xi)\right\},$$

$$= \frac{1}{48(1+\xi^2)}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{4}{\gamma}\left\{\frac{5\xi(\alpha-\gamma+3)(\alpha-\gamma+5)}{(1+\xi^2)}\mathcal{P}_{\alpha-\gamma+4}(\xi)\right.$$

$$\left.-\left[\frac{15\,\xi^2}{(1+\xi^2)} + (\alpha-\gamma+2)(\alpha-\gamma+6)\right]\mathcal{P}'_{\alpha-\gamma+4}(\xi)\right\},$$

$$\sum_{a+b+c+d=\alpha} \mathcal{V}_a(i\xi)\cdot\mathcal{V}_b(i\xi)\,\mathcal{V}_c(i\xi)\mathcal{V}_d(i\xi)$$

$$= \frac{1}{48(1+\xi^2)^2}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{4}{\gamma}\{5\xi(\alpha-\gamma+3)(\alpha-\gamma$$

$$+5)\mathcal{P}_{\alpha-\gamma+4}(\xi)$$

$$-[15\,\xi^2 + (\alpha-\gamma+2)(\alpha-\gamma+6)(1+\xi^2)\,]\mathcal{P}'_{\alpha-\gamma+4}(\xi)\},$$

Again, from Lemma 5.2.2,

$$\mathcal{P}'_{\alpha-\gamma+4}(\xi) = \frac{(\alpha-\gamma+4)}{(1+\xi^2)}\,\mathcal{P}_{\alpha-\gamma+3}(\xi) + \frac{(\alpha-\gamma+3)\,\xi}{(1+\xi^2)}\,\mathcal{P}_{\alpha-\gamma+4}(\xi), \qquad (5.28)$$

Using equation (5.28), we have

$$\sum_{a+b+c+d=\alpha} \mathcal{V}_a(i\xi)\cdot\mathcal{V}_b(i\xi)\,\mathcal{V}_c(i\xi)\mathcal{V}_d(i\xi)$$

$$= \frac{1}{48(1+\xi^2)^3}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1} i^{\alpha-\gamma}\binom{4}{\gamma}\{(\alpha-\gamma$$

$$+3)\xi\big[\big(5(\alpha-\gamma+5)-(\alpha-\gamma+2)(\alpha-\gamma+6)\big)(1+\xi^2)$$

$$-15\xi^2\big]\,\mathcal{P}_{\alpha-\gamma+4}(\xi)$$

$$-(\alpha-\gamma+4)[15\,\xi^2$$

$$+(\alpha-\gamma+2)(\alpha-\gamma+6)(1+\xi^2)\,]\}\mathcal{P}_{\alpha-\gamma+3}(\xi).$$

This establishes the Corollary 5.2.5. ■

Corollary 5.2.6. *For any non-negative integers* α, *and* $\xi \in R$, *we have the following identities*

$$\sum_{\mathfrak{a}+\mathfrak{b}+c=\alpha} \mathcal{W}_{\mathfrak{a}}(i\xi)\cdot\mathcal{W}_{\mathfrak{b}}(i\xi)\,\mathcal{W}_{c}(i\xi)$$

$$= -\frac{1}{8(1+\xi^2)}\sum_{\gamma=0}^{\alpha} i^{\alpha-\gamma}\binom{3}{\gamma}\{3\xi\,(\alpha-\gamma+3)\;\mathcal{P}_{\alpha-\gamma+2}(\xi)$$

$$-(\alpha-\gamma+2)[(\alpha-\gamma+1)\xi^2+(\alpha-\gamma+4)]\,\mathcal{P}_{\alpha-\gamma+3}(\xi)\},$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+c+d=\alpha} \mathcal{W}_{\mathfrak{a}}(i\xi)\cdot\mathcal{W}_{\mathfrak{b}}(i\xi)\,\mathcal{W}_{c}(i\xi)\mathcal{W}_{d}(i\xi)$$

$$= -\frac{1}{48(1+\xi^2)^3}\sum_{\gamma=0}^{\alpha} i^{\alpha-\gamma}\binom{4}{\gamma}\{(\alpha-\gamma$$

$$+3)\xi\big[\big(5(\alpha-\gamma+5)-(\alpha-\gamma+2)(\alpha-\gamma+6)\big)(1+\xi^2)$$

$$-15\xi^2\big]\,\mathcal{P}_{\alpha-\gamma+4}(\xi)$$

$$-(\alpha-\gamma+4)[15\,\xi^2$$

$$+(\alpha-\gamma+2)(\alpha-\gamma+6)(1+\xi^2)\;]\mathcal{P}_{\alpha-\gamma+3}(\xi)\},$$

Proof. Taking $\lambda = 2,3$ in Theorem 5.2.6, and proceeding as in Corollary 5.2.6, we can establish this Corollary. ■

5.3 Representations of sums of finite products of Pell, Fibonacci, and Chebyshev polynomials with negative indices

Here, we develop some results representing summations of finite products of negative indexed Lucas, Fibonacci, and Complex Fibonacci numbers as a linear sum of Pell polynomials. In terms of the 3rd and 4th kinds of Chebyshev polynomials, similar identities are obtained for Pell numbers and Fibonacci polynomials with the same line of action as in Section 5.2. The main findings are:

Theorem 5.3.1. *For integers* $\alpha, \lambda \geq 0$,

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{-(2\sigma_1+1)}\cdot\mathcal{F}_{-(2\sigma_2+1)}\cdots\mathcal{F}_{-(2\sigma_{\lambda+1}+1)}$$

$$= \frac{1}{2^{\lambda}\,\lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\,i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\,\mathcal{P}^{\lambda}_{\alpha-\gamma+\lambda+1}\left(-\frac{3}{2}\,i\right)$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\mathbf{Z}$ $(\hbar = 1,2,\ldots,\lambda+1)$ *with* $\sigma_1 + \sigma_2 + \cdots + \sigma_{\lambda+1} = \alpha$ *with* $\binom{\lambda+1}{\gamma} = 0$ *for* $\gamma > \lambda + 1$ *and* $i = \sqrt{-1}$.

Proof. Taking $\xi = \frac{3}{2}$ in equation (1.92), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{\sigma_1}\left(\frac{3}{2}\right)\cdot\mathcal{V}_{\sigma_2}\left(\frac{3}{2}\right)\cdots\mathcal{V}_{\sigma_{\lambda+1}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^{\lambda}\ \lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{\lambda+1}{\gamma}\mathcal{U}_{\alpha-i+\gamma}^{\lambda}\left(\frac{3}{2}\right) \tag{5.26}$$

Using Lemma 5.2.1 *(iv)* in equation (5.29), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{2\sigma_1+1}\cdot\mathcal{F}_{2\sigma_2+1}\cdots\mathcal{F}_{2\sigma_{\lambda+1}+1}$$

$$= \frac{1}{2^{\lambda}\ \lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{\lambda+1}{\gamma}\mathcal{U}_{\alpha-i+\gamma}^{\lambda}\left(\frac{3}{2}\right), \tag{5.30}$$

Using equation 1.12 (section 1.2) in equation (5.30), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{-(2\sigma_1+1)}\cdot\mathcal{F}_{-(2\sigma_2+1)}\cdots\mathcal{F}_{-(2\sigma_{\lambda+1}+1)}$$

$$= \frac{1}{2^{\lambda}\ \lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{\lambda+1}{\gamma}\mathcal{U}_{\alpha-i+\gamma}^{\lambda}\left(\frac{3}{2}\right), \tag{5.31}$$

Differentiating equation 1.65 *(xv) w.r.t x* , we have

$$\mathcal{U}_{\alpha}^{\lambda}(i\xi) = i^{\alpha-\lambda}\ \mathcal{P}_{\alpha+1}(\xi), \tag{5.32}$$

Taking $\xi = -\frac{3}{2}i$ in equation (5.32), we have

$$\mathcal{U}_{\alpha}^{\lambda}\left(\frac{3}{2}\right) = i^{\alpha-\lambda}\ \mathcal{P}_{\alpha+1}\left(-\frac{3}{2}i\right), \tag{5.33}$$

From equations (5.31) and (5.33), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{-(2\sigma_1+1)}\cdot\mathcal{F}_{-(2\sigma_2+1)}\cdots\mathcal{F}_{-(2\sigma_{\lambda+1}+1)}$$

$$= \frac{1}{2^{\lambda}\ \lambda!}\ \sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\ i^{\alpha-\gamma}\binom{\lambda+1}{\gamma}\ \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda}\left(-\frac{3}{2}\ i\right). \tag{5.34}$$

This establishes the Theorem 5.3.1. ■

Theorem 5.3.2. *For integers $\alpha, \lambda \geq 0$,*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{L}_{-(2\sigma_1+1)} \cdot \mathcal{L}_{-(2\sigma_2+1)} \cdots \mathcal{L}_{-(2\sigma_{\lambda+1}+1)}$$

$$= \frac{(-1)^{2\alpha+\lambda+1}}{2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} i^{\alpha-\gamma} \binom{\lambda+1}{\gamma} \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda}\left(-\frac{3}{2}\ i\right),$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) in $\mathbf{Z}$ $(\hbar = 1,2,\ldots,\lambda+1)$ with $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1} = \alpha$ with $\binom{\lambda+1}{\gamma} = 0$ for $\gamma > \lambda+1$ and $i = \sqrt{-1}$.

Proof. Taking $\xi = \frac{3}{2}$ in equation (1.93), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{W}_{\sigma}\left(\frac{3}{2}\right) \cdot \mathcal{W}_{\sigma_2}\left(\frac{3}{2}\right) \cdots \mathcal{W}_{\sigma_{\lambda+1}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} \binom{\lambda+1}{\gamma} \mathcal{U}_{\alpha-\gamma+\lambda}^{\lambda}\left(\frac{3}{2}\right), \tag{5.35}$$

Using Lemma 5.2.1 *(v)* in equation (5.35), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{L}_{2\sigma_1+1} \cdot \mathcal{L}_{2\sigma_2+1} \cdots \mathcal{L}_{2\sigma_{\lambda+1}+1}$$

$$= \frac{1}{2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} \binom{\lambda+1}{\gamma} \mathcal{U}_{\alpha-\gamma+\lambda}^{\lambda}\left(\frac{3}{2}\right), \tag{5.36}$$

Using $\mathcal{L}_{-\alpha} = (-1)^\alpha\ \mathcal{L}_\alpha$ in (5.36), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{L}_{-(2\sigma_1+1)} \cdot \mathcal{L}_{-(2\sigma_2+1)} \cdots \mathcal{L}_{-(2\sigma_{\lambda+1}+1)}$$

$$= \frac{(-1)^{2\alpha+\lambda+1}}{2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} \binom{\lambda+1}{\gamma} \mathcal{U}_{\alpha-\gamma+\lambda}^{\lambda}\left(\frac{3}{2}\right), \tag{5.37}$$

Differentiating equation 1.65 (*xv*) w.r.t *z*, and taking $\xi = -\frac{3}{2}i$ and using this in (5.37), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{L}_{-(2\sigma_1+1)} \cdot \mathcal{L}_{-(2\sigma_2+1)} \cdots \mathcal{L}_{-(2\sigma_{\lambda+1}+1)}$$

$$= \frac{(-1)^{2\alpha+\lambda+1}}{2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} i^{\alpha-\gamma} \binom{\lambda+1}{\gamma} \mathcal{P}_{\alpha-\gamma+\lambda+1}^{\lambda}\left(-\frac{3}{2}\ i\right). \tag{5.38}$$

This establishes the Theorem 5.3.2. ■

Theorem 5.3. 3. *For integers $\alpha, \lambda \geq 0$,*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}^*{}_{-(2\sigma_1+1)} \cdot \mathcal{F}^*{}_{-(2\sigma_2+1)} \dots \mathcal{F}^*{}_{-(2\sigma_{\lambda+1}+1)}$$

$$= \frac{\left((i)^{2\alpha+2\lambda+2}\right)}{2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} (-1)^\gamma\, i^{\alpha-\gamma} \binom{\lambda+1}{\gamma} \mathcal{P}^\lambda_{\alpha-\gamma+\lambda+1}\left(\frac{1}{2}\right),$$

$$= \frac{1}{(i)^{2\alpha}\ 2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} (-1)^\gamma\, i^{\alpha-\gamma} \binom{\lambda+1}{\gamma} \mathcal{P}^\lambda_{\alpha-\gamma+\lambda+1}\left(-\frac{1}{2}\right),$$

*where sum runs over all $\sigma_\hbar$ (≥ 0) in **Z** $(\hbar = 1,2,\dots,\lambda+1)$ with $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1} = \alpha$ with $\binom{\lambda+1}{\gamma} = 0$ for $\gamma > \lambda+1$ and $i = \sqrt{-1}$* and *$\mathcal{F}^*{}_\alpha$ is a Complex Fibonacci number.*

Proof. Taking $\xi = -\frac{i}{2}$ in equation (1.92), and $\xi = \frac{i}{2}$ in equation (1.93), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{\sigma_1}\left(-\frac{i}{2}\right) \cdot \mathcal{V}_{\sigma_2}\left(-\frac{i}{2}\right) \cdots \mathcal{V}_{\sigma_{\lambda+1}}\left(-\frac{i}{2}\right)$$

$$= \frac{1}{2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} (-1)^\gamma \binom{\lambda+1}{\gamma} \mathcal{U}^\lambda_{\alpha-\gamma+\lambda}\left(-\frac{i}{2}\right), \tag{5.39}$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{W}_{\sigma_1}\left(\frac{i}{2}\right) \cdot \mathcal{W}_{\sigma_2}\left(\frac{i}{2}\right) \cdots \mathcal{W}_{\sigma_{\lambda+1}}\left(\frac{i}{2}\right)$$

$$= \frac{1}{2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} \binom{\lambda+1}{\gamma} \mathcal{U}^\lambda_{\alpha-\gamma+\lambda}\left(\frac{i}{2}\right), \tag{5.40}$$

Using, $\mathcal{U}_\alpha\left(\frac{i}{2}\right) = i^\alpha \mathcal{F}^*_{\alpha+1}$ in equation 1.65 (*iii*) to get $\mathcal{W}_\alpha\left(\frac{i}{2}\right) = i^{\alpha-1}\mathcal{F}^*_\alpha$ and using this in turn in equation 1.65 (*xii*), we get $\mathcal{V}_\alpha\left(-\frac{i}{2}\right) = \frac{\mathcal{F}^*_\alpha}{i^{\alpha+1}}$.

Using this, therefore, reduces (5.39) and (5.40) to

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}^*{}_{2\sigma_1+1} \cdot \mathcal{F}^*{}_{2\sigma_2+1} \cdots \mathcal{F}^*{}_{2\sigma_{\lambda+1}+1}$$

$$= \frac{\left(i^{2\alpha+2\lambda+2}\right)}{2^\lambda\ \lambda!} \sum_{\gamma=0}^{\alpha} (-1)^\gamma \binom{\lambda+1}{\gamma} \mathcal{U}^\lambda_{\alpha-\gamma+\lambda}\left(-\frac{i}{2}\right), \tag{5.41}$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}^*{}_{2\sigma_1+1} \cdot \mathcal{F}^*{}_{2\sigma_2+1} \cdots \mathcal{F}^*{}_{2\sigma_{\lambda+1}+1}$$

$$= \frac{1}{i^{2\alpha}\, 2^{\lambda}\, \lambda!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma} \binom{\lambda+1}{\gamma} \mathcal{U}^{\lambda}_{\alpha-\gamma+\lambda}\left(\frac{i}{2}\right). \tag{5.42}$$

Taking conjugate of $\mathcal{F}^*{}_{\alpha}$ in (5.41) and (5.42), using $\mathcal{F}^*{}_{-\alpha} = (-1)^{\alpha+1}\overline{\mathcal{F}^*{}_{\alpha}}$, where $\overline{\mathcal{F}^*{}_{\alpha}}$ represents complex conjugate of $\mathcal{F}^*{}_{\alpha}$, we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}^*{}_{-(2\sigma_1+1)} \cdot \mathcal{F}^*{}_{-(2\sigma_2+1)} \cdots \mathcal{F}^*{}_{-(2\sigma_{\lambda+1}+1)}$$

$$= \frac{\left((i)^{2\alpha+2\lambda+2}\right)}{2^{\lambda}\, \lambda!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma} \binom{\lambda+1}{\gamma} \mathcal{U}^{\lambda}_{\alpha-\gamma+\lambda}\left(\frac{i}{2}\right),$$

$$= \frac{1}{(i)^{2\alpha}\, 2^{\lambda}\, \lambda!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma} \binom{\lambda+1}{\gamma} \mathcal{U}^{\lambda}_{\alpha-\gamma+\lambda}\left(-\frac{i}{2}\right). \tag{5.43}$$

Differentiating equation 1.65 *(xv)* *r*- times w. r. t ξ and putting $\xi = \frac{1}{2}$ and $\xi = -\frac{1}{2}$, we get $\mathcal{U}^{\lambda}{}_{\alpha}\left(\frac{i}{2}\right) = i^{\alpha-\lambda}\ \mathcal{P}_{\alpha+1}\left(\frac{1}{2}\right)$ and $\mathcal{U}^{\lambda}{}_{\alpha}\left(-\frac{i}{2}\right) = i^{\alpha-\lambda}\ \mathcal{P}_{\alpha+1}\left(-\frac{1}{2}\right)$. Using this in (5.43) gives

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}^*{}_{-(2\sigma_1+1)} \cdot \mathcal{F}^*{}_{-(2\sigma_2+1)} \cdots \mathcal{F}^*{}_{-(2\sigma_{\lambda+1}+1)}$$

$$= \frac{\left((i)^{2\alpha+2\lambda+2}\right)}{2^{\lambda}\, \lambda!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma}\, i^{\alpha-\gamma} \binom{\lambda+1}{\gamma} \mathcal{P}^{\lambda}_{\alpha-\gamma+\lambda+1}\left(\frac{1}{2}\right)$$

$$= \frac{1}{(i)^{2\alpha}\, 2^{\lambda}\, \lambda!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma}\, i^{\alpha-\gamma} \binom{\lambda+1}{\gamma} \mathcal{P}^{\lambda}_{\alpha-\gamma+\lambda+1}\left(-\frac{1}{2}\right). \tag{5.44}$$

This establishes the desired result. ■

Theorem 5.3.4. *For integer $\alpha, \lambda \geq 0$, and $\xi \in R$*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{-(\sigma_1+1)}(\xi) \cdot\ \mathcal{P}_{-(\sigma_2+1)}(\xi) \cdots\ \mathcal{P}_{-(\sigma_{\lambda+1}+1)}(\xi)$$

$$= \sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{-(\sigma_1+1)}\ \cdot\ \mathcal{P}_{-(\sigma_2+1)}\ \cdots\ \mathcal{P}_{-(\sigma_{\lambda+1}+1)}$$

$$= \frac{i^{\alpha}}{\lambda!} \sum_{\gamma=0}^{\alpha} \binom{\lambda + \left[\frac{\gamma}{2}\right]}{\lambda} \left(\alpha + \lambda - \left[\frac{\gamma}{2}\right]\right)_{\lambda} \mathcal{V}_{\alpha-\gamma}(i)$$

$$= \frac{i^{\alpha}}{\lambda!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma} \binom{\lambda + \left[\frac{\gamma}{2}\right]}{\lambda} \left(\alpha + \lambda - \left[\frac{\gamma}{2}\right]\right)_{\lambda} \mathcal{W}_{\alpha-\gamma}(i)$$

where sum runs over all $\sigma_{\hbar}$ (≥ 0) *in* $\boldsymbol{Z}$ $(\hbar = 1,2,\dots,\lambda+1)$ *with* $\sigma_1 + \sigma_2 + \cdots + \sigma_{\lambda+1} = \alpha$ *with* $\binom{\lambda+1}{\gamma} = 0$ *for* $\gamma > \lambda + 1$ *and* $i = \sqrt{-1}$ and $(s)_{\alpha} = s(s-1)(s-2)\dots(s-\alpha+1)$ *is falling factorial polynomial.*

Proof. From [59],

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}(\xi) \cdot \mathcal{P}_{\sigma_2+1}(\xi) \cdots \mathcal{P}_{\sigma_{\lambda+1}+1}(\xi) = \frac{1}{i^{\alpha}\,\lambda!} \sum_{\gamma=0}^{\alpha} \binom{\lambda + \left[\frac{\gamma}{2}\right]}{\lambda} \left(\alpha + \lambda - \left[\frac{\gamma}{2}\right]\right)_{\lambda} \mathcal{V}_{\alpha-\gamma}(i\xi),$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}(\xi) \cdot \mathcal{P}_{\sigma_2+1}(\xi) \cdots \mathcal{P}_{\sigma_{\lambda+1}+1}(\xi) = \frac{1}{i^{\alpha}\,\lambda!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma} \binom{\lambda + \left[\frac{\gamma}{2}\right]}{\lambda} \left(\alpha + \lambda - \left[\frac{\gamma}{2}\right]\right)_{\lambda} \mathcal{W}_{\alpha-\gamma}(i\xi), \tag{5.45}$$

Using $\mathcal{P}_{-\alpha}(\xi) = (-1)^{\alpha+1}\mathcal{P}_{\alpha}(\xi)$ in (5.45), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{-(\sigma_1+1)}(\xi) \cdot \mathcal{P}_{-(\sigma_2+1)}(\xi) \cdots \mathcal{P}_{-(\sigma_{\lambda+1}+1)}(\xi) = \frac{i^{\alpha}}{\lambda!} \sum_{\gamma=0}^{\alpha} \binom{\lambda + \left[\frac{\gamma}{2}\right]}{\lambda} \left(\alpha + \lambda - \left[\frac{\gamma}{2}\right]\right)_{\lambda} \mathcal{V}_{\alpha-\gamma}(i\xi),$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{-(\sigma_1+1)}(\xi) \cdot \mathcal{P}_{-(\sigma_2+1)}(\xi) \cdots \mathcal{P}_{-(\sigma_{\lambda+1}+1)}(\xi) = \frac{i^{\alpha}}{\lambda!} \sum_{\gamma=0}^{\alpha} (-1)^{\gamma} \binom{\lambda + \left[\frac{\gamma}{2}\right]}{\lambda} \left(\alpha + \lambda - \left[\frac{\gamma}{2}\right]\right)_{\lambda} \mathcal{W}_{\alpha-\gamma}(i\xi), \tag{5.46}$$

Using $\mathcal{P}_{-\alpha}(1) = \mathcal{P}_{-\alpha}$ in (5.46), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{-(\sigma_1+1)} \cdot \mathcal{P}_{-(\sigma_2+1)} \cdots \mathcal{P}_{-(\sigma_{\lambda+1}+1)} = \frac{i^\alpha}{\lambda!}\sum_{\gamma=0}^{\alpha}\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\left(\alpha+\lambda-\left[\frac{\gamma}{2}\right]\right)_\lambda \mathcal{V}_{\alpha-\gamma}(i),$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{-(\sigma_1+1)} \cdot \mathcal{P}_{-(\sigma_2+1)} \cdots \mathcal{P}_{-(\sigma_{\lambda+1}+1)} = \frac{i^\alpha}{\lambda!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\left(\alpha+\lambda-\left[\frac{\gamma}{2}\right]\right)_\lambda \mathcal{W}_{\alpha-\gamma}(i). \tag{5.47}$$

Hence the Theorem is established. ■

Theorem 5.3.5. *For integers $\alpha,\lambda \geq 0$ and $\xi \in R$*

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{-(\sigma_1+1)}(\xi) \cdot \mathcal{F}_{-(\sigma_2+1)}(\xi) \cdots \mathcal{F}_{-(\sigma_{\lambda+1}+1)}(\xi) = \frac{(-1)^{\alpha+\lambda+1}}{i^\alpha\ \lambda!}\sum_{\gamma=0}^{\alpha}\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\left(\alpha+\lambda-\left[\frac{\gamma}{2}\right]\right)_\lambda \mathcal{V}_{\alpha-\gamma}\left(\frac{\xi}{2}i\right),$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{-(\sigma_1+1)}(\xi) \cdot \mathcal{F}_{-(\sigma_2+1)}(\xi) \cdots \mathcal{F}_{-(\sigma_{\lambda+1}+1)}(\xi) = \frac{(-1)^{\alpha+\lambda+1}}{i^\alpha\ \lambda!}\sum_{\gamma=0}^{\alpha}(-1)^\gamma\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\left(\alpha+\lambda-\left[\frac{\gamma}{2}\right]\right)_\lambda \mathcal{W}_{\alpha-\gamma}\left(\frac{\xi}{2}i\right),$$

*where sum runs over all $\sigma_\hbar$ (≥ 0) in **Z** $(\hbar = 1,2,\dots,\lambda+1)$ with $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1} = \alpha$ with $\binom{\lambda+1}{\gamma} = 0$ for $\gamma > \lambda+1$ and $i = \sqrt{-1}$* and $(s)_\alpha = s(s-1)(s-2)\dots(s-\alpha+1)$ *is falling factorial polynomial*.

Proof. Replacing ξ by $\frac{\xi}{2}$ in equation (5.45), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}\left(\frac{\xi}{2}\right) \cdot \mathcal{P}_{\sigma_2+1}\left(\frac{\xi}{2}\right) \cdots \mathcal{P}_{\sigma_{\lambda+1}+1}\left(\frac{\xi}{2}\right) = \frac{1}{i^\alpha\ \lambda!}\sum_{\gamma=0}^{\alpha}\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\left(\alpha+\lambda-\left[\frac{\gamma}{2}\right]\right)_\lambda \mathcal{V}_{\alpha-\gamma}\left(\frac{\xi}{2}i\right),$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{P}_{\sigma_1+1}\left(\frac{\xi}{2}\right)\cdot\ \mathcal{P}_{\sigma_2+1}\left(\frac{\xi}{2}\right)\cdots\ \mathcal{P}_{\sigma_{\lambda+1}+1}\left(\frac{\xi}{2}\right)$$

$$= \frac{1}{i^{\alpha}\ \lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\Big(\alpha+\lambda$$

$$-\left[\frac{\gamma}{2}\right]\Big)_{\lambda}\mathcal{W}_{\alpha-\gamma}\left(\frac{\xi}{2}i\right), \tag{5.48}$$

Using $\mathcal{F}_{\alpha}(\xi)=\mathcal{P}_{\alpha}\left(\frac{\xi}{2}\right)$ in equation (5.47), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{(\sigma_1+1)}(\xi)\cdot\ \mathcal{F}_{(\sigma_2+1)}(\xi)\cdots\ \mathcal{F}_{(\sigma_{\lambda+1}+1)}(\xi)$$

$$= \frac{1}{i^{\alpha}\ \lambda!}\sum_{\gamma=0}^{\alpha}\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\left(\alpha+\lambda-\left[\frac{\gamma}{2}\right]\right)_{\lambda}\mathcal{V}_{\alpha-\gamma}\left(\frac{\xi}{2}i\right),$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{(\sigma_1+1)}(\xi)\cdot\ \mathcal{F}_{(\sigma_2+1)}(\xi)\cdots\ \mathcal{F}_{(\sigma_{\lambda+1}+1)}(\xi)$$

$$= \frac{1}{i^{\alpha}\ \lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\left(\alpha+\lambda-\left[\frac{\gamma}{2}\right]\right)_{\lambda}\mathcal{W}_{\alpha-\gamma}\left(\frac{\xi}{2}i\right), \tag{5.49}$$

Again, using $\mathcal{F}_{-\alpha}(\xi)=(-1)^{\alpha}\mathcal{F}_{\alpha}(\xi)$in equation (5.49), we get the desired result.

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{-(\sigma_1+1)}(\xi)\cdot\ \mathcal{F}_{-(\sigma_2+1)}(\xi)\cdots\ \mathcal{F}_{-(\sigma_{\lambda+1}+1)}(\xi)$$

$$= \frac{(-1)^{\alpha+\lambda+1}}{i^{\alpha}\ \lambda!}\sum_{\gamma=0}^{\alpha}\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\left(\alpha+\lambda-\left[\frac{\gamma}{2}\right]\right)_{\lambda}\mathcal{V}_{\alpha-\gamma}\left(\frac{\xi}{2}i\right),$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{F}_{-(\sigma_1+1)}(\xi)\cdot\ \mathcal{F}_{-(\sigma_2+1)}(\xi)\cdots\ \mathcal{F}_{-(\sigma_{\lambda+1}+1)}(\xi)$$

$$= \frac{(-1)^{\alpha+\lambda+1}}{i^{\alpha}\ \lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{\lambda+\left[\frac{\gamma}{2}\right]}{\lambda}\Big(\alpha+\lambda$$

$$-\left[\frac{\gamma}{2}\right]\Big)_{\lambda}\mathcal{W}_{\alpha-\gamma}\left(\frac{\xi}{2}i\right). \tag{5.50}$$

Hence the Theorem is established. ■

Theorem 5.3.6. *For any integer* $\alpha \geq 0$, *and* $\xi \in R$,

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{-(\sigma_1+1)}(\xi)\cdot\mathcal{V}_{-(\sigma_2+1)}(\xi)\cdot\ldots\cdot\mathcal{V}_{-(\sigma_{\lambda+1}+1)}(\xi)$$

$$=\frac{1}{2^\lambda\,\lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1}\binom{\lambda+1}{\gamma}\mathcal{U}^{\lambda}_{-(\alpha-\gamma+\lambda+2)}(\xi)$$

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{W}_{-(\sigma_1+1)}(\xi)\cdot\mathcal{W}_{-(\sigma_2+1)}(\xi)\cdot\ldots\cdot\mathcal{W}_{-(\sigma_{\lambda+1}+1)}(\xi)$$

$$=\frac{(-1)^\lambda}{2^\lambda\,\lambda!}\sum_{\gamma=0}^{\alpha}\binom{\lambda+1}{\gamma}\mathcal{U}^{\lambda}_{-(\alpha-\gamma+\lambda+2)}(\xi)$$

where all sums run over all non-negative integers $(\sigma_1, \sigma_2, \ldots, \sigma_{\lambda+1})$ *such that* $\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha$ *with* $\binom{\lambda+1}{\gamma}=0$ *for* $\gamma > r+1$.

Proof. From [53],

$$\mathcal{U}_{-\alpha}(\xi) = -\mathcal{U}_{\alpha-2}(\xi) \text{ with } \mathcal{U}_{-1}(\xi) = 0 \tag{5.51}$$

Using equation (5.51) in equations 1.65 (*ii*) and 1.65 (*iii*) we have

$$\mathcal{V}_{-\alpha}(\xi) = \mathcal{V}_{\alpha-1}(\xi) \tag{5.52}$$

and

$$\mathcal{W}_{-\alpha}(\xi) = -\mathcal{W}_{\alpha-1}(\xi) \tag{5.53}$$

Using equation (5.52) in equation (1.92), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{V}_{-(\sigma_1+1)}(\xi)\cdot\mathcal{V}_{-(\sigma_2+1)}(\xi)\cdot\ldots\cdot\mathcal{V}_{-(\sigma_{\lambda+1}+1)}(\xi)$$

$$=\frac{1}{2^\lambda\,\lambda!}\sum_{\gamma=0}^{\alpha}(-1)^{\gamma+1}\binom{\lambda+1}{\gamma}\mathcal{U}^{\lambda}_{-(\alpha-\gamma+\lambda+2)}(\xi) \tag{5.54}$$

Similarly, using equation (5.53) in equation (1.93), we have

$$\sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} (-1)^{\lambda+1}\,\mathcal{W}_{-(\sigma_1+1)}(\xi)\cdot\mathcal{W}_{-(\sigma_2+1)}(\xi)\cdot\ldots\cdot\mathcal{W}_{-(\sigma_{\lambda+1}+1)}(\xi)$$

$$=\frac{1}{2^\lambda\,\lambda!}\sum_{\gamma=0}^{\alpha}\binom{\lambda+1}{\gamma}\mathcal{U}^{\lambda}_{-(\alpha-\gamma+\lambda+2)}(\xi)\,.$$

$$\therefore \sum_{\sigma_1+\sigma_2+\cdots+\sigma_{\lambda+1}=\alpha} \mathcal{W}_{-(\sigma_1+1)}(\xi)\cdot\mathcal{W}_{-(\sigma_2+1)}(\xi)\cdot\ldots\cdot\mathcal{W}_{-(\sigma_{\lambda+1}+1)}(\xi)$$

$$= \frac{(-1)^{\lambda}}{2^{\lambda}\,\lambda!}\sum_{\gamma=0}^{\alpha}\binom{\lambda+1}{\gamma}\mathcal{U}^{\lambda}_{-(\alpha-\gamma+\lambda+2)}(\xi) \tag{5.55}$$

Thus, the equations (5.54) and (5.55) establishes the Theorem. ■

Corollary 5.3.1*. For integer $\alpha \geq 0$,*

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{F}_{-(2\mathfrak{a}+1)}\cdot\mathcal{F}_{-(2\mathfrak{b}+1)}\,\mathcal{F}_{-(2\mathfrak{c}+1)}$$

$$= \sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\binom{3}{\gamma}\left[\frac{9}{25}\,A_{\alpha,\gamma}\,\mathcal{F}_{(2\alpha-2\gamma+4)} - \frac{1}{50}B_{\alpha,\gamma}\,\mathcal{F}_{(2\alpha-2\gamma+6)}\right],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{F}_{-(2\mathfrak{a}+1)}\cdot\mathcal{F}_{-(2\mathfrak{b}+1)}\,\mathcal{F}_{-(2\mathfrak{c}+1)}$$

$$= \sum_{\gamma=0}^{\alpha}(-1)^{\gamma}\,(i)^{\alpha-\gamma}\binom{3}{\gamma}\left[\frac{1}{50}B_{\alpha,\gamma}\mathcal{P}_{(\alpha-\gamma+3)}\left(-\frac{3i}{2}\right)\right.$$

$$\left. - \frac{9i}{25}\,A_{\alpha,\gamma}\mathcal{P}_{(\alpha-\gamma+2)}\left(-\frac{3i}{2}\right)\right],$$

where $A_{\alpha,\gamma} = (\alpha-\gamma+3), B_{\alpha,\gamma} = (\alpha-\gamma+2)(7-5\alpha-5\gamma),\ \binom{3}{\gamma} = 0,$ for $\gamma > 3$ and $i = \sqrt{-1}$.

Proof. Taking $\lambda = 2$ in Theorem 5.3.1 and equation (5.31) using the identities [57, 59]

$$(1-\xi^2)\mathcal{U}'_{\alpha}(\xi) = (\alpha+1)\mathcal{U}_{\alpha-1}(\xi) - \alpha\xi\mathcal{U}_{\alpha}(\xi). \tag{5.56}$$

$$(1-\xi^2)\mathcal{U}''_{\alpha}(\xi) = 3\xi\mathcal{U}'_{\alpha}(\xi) - \alpha(\alpha+2)\mathcal{U}_{\alpha}(\xi). \tag{5.57}$$

$$(1+\xi^2)\mathcal{P}'_{\alpha+1}(\xi) = (\alpha+1)\mathcal{P}_{\alpha}(\xi) + \alpha\xi\mathcal{P}_{\alpha+1}(\xi). \tag{5.58}$$

$$(1+\xi^2)\mathcal{P}''_{\alpha}(\xi) = \alpha(\alpha+2)\mathcal{P}_{\alpha+1}(\xi) - 3\xi\mathcal{P}'_{\alpha+1}(\xi). \tag{5.59}$$

with $\xi = \frac{3}{2}$ and $\xi = -\frac{3}{2}i$, we get the desired result. ■

Corollary 5.3.2*. For integer $\alpha \geq 0$,*

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{L}_{-(2\mathfrak{a}+1)}\,\mathcal{L}_{-(2\mathfrak{b}+1)}\,\mathcal{L}_{-(2\mathfrak{c}+1)}$$

$$= \sum_{\gamma=0}^{\alpha}\binom{3}{\gamma}\left[\frac{1}{50}B_{\alpha,\gamma}\,\mathcal{F}_{(2\alpha-2\gamma+6)} - \frac{9}{25}\,A_{\alpha,\gamma}\,\mathcal{F}_{(2\alpha-2\gamma+4)}\right],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{L}_{-(2\mathfrak{a}+1)}\, \mathcal{L}_{-(2\mathfrak{b}+1)}\, \mathcal{L}_{-(2\mathfrak{c}+1)}$$

$$= \sum_{\gamma=0}^{\alpha} i^{\alpha-\gamma} \binom{3}{\gamma} \left[\frac{9i}{25}\, A_{\alpha,\gamma} \mathcal{P}_{(\alpha-\gamma+2)}\left(-\frac{3i}{2}\right) - \frac{1}{50} B_{\alpha,\gamma} \mathcal{P}_{(\alpha-\gamma+3)}\left(-\frac{3i}{2}\right)\right],$$

where $A_{\alpha,\gamma} = (\alpha-\gamma+3), B_{\alpha,\gamma} = (\alpha-\gamma+2)(7-5\alpha-5\gamma)$, $\binom{3}{\gamma} = 0$, *for* $\gamma > 3$ *and* $i = \sqrt{-1}$.

Proof. Taking $\lambda = 2$ in Theorem 5.3.2 and equation (5.36) and using the identities (5.56) - (5.59) with $\xi = \frac{3}{2}, -\frac{3}{2}i$, we get the desired result. ■

Corollary 5.3.3. *For integer* $\alpha \geq 0$,

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{F}^*_{-(2\mathfrak{a}+1)} \cdot \mathcal{F}^*_{-(2\mathfrak{b}+1)} . \mathcal{F}^*_{-(2\mathfrak{c}+1)}$$

$$= \sum_{\gamma=0}^{\alpha} (-1)^{\alpha+\gamma+3} \binom{3}{\gamma} \left[\frac{3i}{25} C_{\alpha,\gamma} \mathcal{U}_{(\alpha-\gamma+1)}\left(\frac{i}{2}\right) - \frac{1}{50} D_{\alpha,\gamma} \mathcal{U}_{(\alpha-\gamma+2)}\left(\frac{i}{2}\right)\right]$$

$$= \sum_{\gamma=0}^{\alpha} (-1)^{\alpha+\gamma+1} \binom{3}{\gamma} \left[\frac{3i}{25} C_{\alpha,\gamma} \mathcal{U}_{(\alpha-\gamma+1)}\left(-\frac{i}{2}\right)\right.$$

$$\left. + \frac{1}{50} D_{\alpha,\gamma} \mathcal{U}_{(\alpha-\gamma+2)}\left(-\frac{i}{2}\right)\right],$$

$$\sum_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}=\alpha} \mathcal{F}^*_{-(2\mathfrak{a}+1)} \cdot \mathcal{F}^*_{-(2\mathfrak{b}+1)} . \mathcal{F}^*_{-(2\mathfrak{c}+1)}$$

$$= \sum_{\gamma=0}^{\alpha} (-1)^{\alpha+\gamma+3}\, i^{\alpha-\gamma} \binom{3}{\gamma} \left[\frac{1}{50}\, D_{\alpha,\gamma}\, \mathcal{F}_{(\alpha-\gamma+3)} - \frac{3}{25} C_{\alpha,\gamma}\, \mathcal{F}_{(\alpha-\gamma+2)}\right],$$

$$= \sum_{\gamma=0}^{\alpha} (-1)^{\alpha+\gamma}\, i^{\alpha-\gamma} \binom{3}{\gamma} \left[\frac{1}{50}\, D_{\alpha,\gamma}\, \mathcal{P}_{(\alpha-\gamma+3)}\left(-\frac{1}{2}\right)\right.$$

$$\left. + \frac{3}{25} C_{\alpha,\gamma}\, \mathcal{P}_{(\alpha-\gamma+2)}\left(-\frac{1}{2}\right)\right],$$

where $C_{\alpha,\gamma} = (\alpha-\gamma+3)$, $D_{\alpha,\gamma} = (\alpha-\gamma+2)(5\alpha-5\gamma+17)$, $\binom{3}{\gamma} = 0$ for $\gamma > 3$ *and* $\mathcal{F}^*_{\alpha}$ *is a complex Fibonacci number.*

Proof. Taking $\lambda = 2$ in Theorem 5.3.3 and equation (5.43) and using the identities (5.56)- (5.59) with $\xi = \frac{i}{2}, -\frac{i}{2}, \frac{1}{2}, -\frac{1}{2}$, we get the desired result. ■

CHAPTER 6

GENERALIZED TRIVARIATE FIBONACCI AND LUCAS POLYNOMIALS

6.1 Introduction

This chapter will focus on the study of *(p, q, r)-Generalized Trivariate Fibonacci* and *(p, q, r)-Generalized Trivariate Lucas* polynomials and their basic properties. Using these properties, we will derive the explicit formula of (*p, q, r)-Generalized Trivariate Lucas* and *Fibonacci* polynomials and deduce some intriguing identities involving the generating matrices and their determinants.

6.2 Generalized Trivariate Fibonacci and Lucas polynomials

The Fibonacci and Lucas numbers and their generalizations have been widely studied, and many interesting properties have been established. For any positive $\alpha \geq 2$, the Fibonacci and Lucas numbers are recursively defined as in chapter 1,

$$\mathcal{F}_\alpha = \mathcal{F}_{\alpha-1} + \mathcal{F}_{\alpha-2}, \qquad \mathcal{F}_0 = 0, \qquad \mathcal{F}_1 = 1,$$

and

$$\mathcal{L}_\alpha = \mathcal{L}_{\alpha-1} + \mathcal{L}_{\alpha-2}, \qquad \mathcal{L}_0 = 2, \qquad \mathcal{L}_1 = 1.$$

As an extension of the Fibonacci numbers, the Tribonacci numbers [14, 41] were first studied by M. Feinberg [75] in 1963 by defining the recursive relation as

$$\mathcal{T}_\alpha = \mathcal{T}_{\alpha-1} + \mathcal{T}_{\alpha-2} + \mathcal{T}_{\alpha-3} \quad , \qquad \alpha > 2,$$

with initial conditions

$$\mathcal{T}_0 = 0, \qquad \mathcal{T}_1 = 1, \qquad \mathcal{T}_2 = 1.$$

In [14, 62, 64-67], different authors have studied the Tribonacci numbers and deduced various properties and generalizations and obtained several identities thereof. Alladi and Hoggatt [61] studied the Tribonacci numbers by defining the Tribonacci triangle as below

$\alpha \backslash \beta$	**0**	**1**	**2**	**3**	**4**	**5**	**.**	**.**	**.**
0	**1**								
1	**1**	**1**							
2	**1**	**3**	**1**						
3	**1**	**5**	**5**	**1**					
4	**1**	**7**	**13**	**7**	**1**				
.	**.**	**.**	**.**	**.**	**.**	**.**			
.	**.**	**.**	**.**	**.**	**.**	**.**			
.	**.**	**.**	**.**	**.**	**.**	**.**			

Table 6.1: Tribonacci number triangle

If $A(\alpha, \beta)$ represents the element in the α^{th} row & $\beta^{\,th}$ columns of the Tribonacci Triangle, the we can see that

$$A(\alpha + 1, \beta) = A(\alpha, \beta) + A(\alpha, \beta - 1) + A(\alpha - 1, \beta - 1).$$

and

$$\mathcal{T}_{\alpha} = \sum_{\alpha=0}^{\left\lfloor \frac{\alpha}{2} \right\rfloor} A(\alpha - 1, \beta).$$

which represents the aggregate of the elements that constitute the rising diagonals which generates Tribonacci numbers.

In one of the branches of extension of Fibonacci numbers, E.C. Catalan in 1883 studied the Fibonacci polynomials characterized by the recursive relation:

$$\mathcal{F}_{\alpha}(\xi) = \xi \mathcal{F}_{\alpha-1}(\xi) + \mathcal{F}_{\alpha-2}(\xi), \text{for all } \alpha > 2, \text{with } \mathcal{F}_{1}(\xi) = 1, \mathcal{F}_{2}(\xi) = \xi.$$

Similarly, in 1970, Bicknel originally studied the Lucas polynomials by defining the recursive relation as

$$\mathcal{L}_{\alpha}(\xi) = \xi\mathcal{L}_{\alpha-1}(\xi) + \mathcal{L}_{\alpha-2}(\xi), \text{forall } \alpha \geq 2, \text{with } \mathcal{L}_0(\xi) = 2, \mathcal{L}_1(\xi) = \xi.$$

In 1973, Hoggatt and Bicknell [15] gave a new generalization in the form of Tribonacci polynomials defined recursively as

$$t_{\alpha}(\xi) = \xi^2 t_{\alpha-1}(\xi) + \xi t_{\alpha-2}(\xi) + t_{\alpha-3}(\xi), \text{for all } \alpha > 2$$

with

$$t_0(\xi) = 0, t_1(\xi) = 1, t_2(\xi) = \xi^2.$$

Further generalization of Lucas and Fibonacci polynomials to Bivariate Lucas and Fibonacci polynomials were studied by Tan and Yang [68] by obtaining some of their interesting properties. Kocer and Gedikce [16, 63] studied the Trivariate Fibonacci and Lucas polynomials with recurrence relations defined as follows:

$$\mathcal{H}_{\alpha}(\xi,\omega,\zeta) = \xi\mathcal{H}_{\alpha-1}(\xi,\omega,\zeta) + \omega\mathcal{H}_{\alpha-2}(\xi,\omega,\zeta) + \zeta\mathcal{H}_{\alpha-3}(\xi,\omega,\zeta), \qquad \alpha > 2,$$

with

$$\mathcal{H}_0(\xi,\omega,\zeta) = 0, \qquad \mathcal{H}_1(\xi,\omega,\zeta) = 1, \qquad \mathcal{H}_2(\xi,\omega,\zeta) = \xi.$$

and

$$K_{\alpha}(\xi,\omega,\zeta) = \xi K_{\alpha-1}(\xi,\omega,\zeta) + \omega K_{\alpha-2}(\xi,\omega,\zeta) + \zeta K_{\alpha-3}(\xi,\omega,\zeta), \qquad \alpha > 2,$$

with

$$K_0(\xi,\omega,\zeta) = 3, \qquad K_1(\xi,\omega,\zeta) = \xi, \qquad K_2(\xi,\omega,\zeta) = \xi^2 + 2\omega,$$

respectively and derived several properties thereof.

Continuing in the same line of action, in this study, we will study new generalizations of the Trivariate Fibonacci and Lucas polynomials.

Definition 6.2.1. For integer $\alpha > 2$, the recurrence relation of the *(p, q, r)* -Generalized Trivariate Fibonacci polynomials is defined as:

$$\begin{aligned} F^*{}_{\alpha}(\xi,\omega,\zeta) = p(\xi,\omega,\zeta)\, F^*{}_{\alpha-1}(\xi,\omega,\zeta) + q(\xi,\omega,\zeta)F^*{}_{\alpha-2}(\xi,\omega,\zeta) \\ +r(\xi,\omega,\zeta)F^*{}_{\alpha-3}(\xi,\omega,\zeta), \end{aligned} \tag{6.1}$$

with

$$F^*{}_0(\xi,\omega,\zeta) = 0, \qquad F^*{}_1(\xi,\omega,\zeta) = 1, \qquad F^*{}_2(\xi,\omega,\zeta) = p(\xi,\omega,\zeta),$$

where $p(\xi,\omega,\zeta), q(\xi,\omega,\zeta), r(\xi,\omega,\zeta)$are polynomials of ξ, ω and ζ respectively.

Definition 6.2.2. For integer $\alpha > 2$, the recurrence relation of the *(p, q, r)*-Generalized Trivariate Lucas polynomials is defined as follows:

$$G^*_\alpha(\xi,\omega,\zeta) = p(\xi,\omega,\zeta)G^*_{\alpha-1}(\xi,\omega,\zeta) + q(\xi,\omega,\zeta)G^*_{\alpha-2}(\xi,\omega,\zeta) + r(\xi,\omega,\zeta)G^*_{\alpha-3}(\xi,\omega,\zeta) \tag{6.2}$$

with

$G^*_0(\xi,\omega,\zeta) = 3\,, G^*_1(\xi,\omega,\zeta) = p(\xi),\qquad G^*_2(\xi,\omega,\zeta) = p(\xi,\omega,\zeta)^2 + 2q(\xi,\omega,\zeta).$

For different values of $p(\xi,\omega,\zeta), q(\xi,\omega,\zeta), r(\xi,\omega,\zeta)$ these recursive relations give rise to different polynomials. As for $p(\xi,\omega,\zeta) = \xi, q(\xi,\omega,\zeta) = \omega,\ r(\xi,\omega,\zeta) = \zeta$, we have $F^*_\alpha(\xi,\omega,\zeta) = \mathcal{H}_\alpha(\xi,\omega,\zeta)$, Trivariate Fibonacci polynomials and $G^*_\alpha(\xi,\omega,\zeta) = K_\alpha(\xi,\omega,\zeta)$, Trivariate Lucas polynomials and for $p(\xi,\omega,\zeta) = 1, q(\xi,\omega,\zeta) = 1, r(\xi,\omega,\zeta) = 1$ gives $F^*_\alpha(1,1,1) = \mathcal{T}_\alpha$, Tribonacci numbers and $p(\xi,\omega,\zeta) = \xi^2, q(\xi,\omega,\zeta) = \xi, r(\xi,\omega,\zeta) = 1$ $F^*_\alpha(\xi,\omega,\zeta) = t_\alpha(\xi)$, Tribonacci polynomials. Some of the values of the *(p, q, r)-Generalized Trivariate Lucas* and *Fibonacci* polynomials are written as below (writing $p(\xi,\omega,\zeta) = p, q(\xi,\omega,\zeta) = q, r(\xi,\omega,\zeta) = r$).

$\boldsymbol{\alpha}$	$\boldsymbol{F^*_\alpha(\xi,\omega,\zeta)}$	$\boldsymbol{G^*_\alpha(\xi,\omega,\zeta)}$
0	0	3
1	1	p
2	p	$p^2 + 2q$
3	$p^2 + q$	$p^3 + 3pq + 3r$
4	$p^3 + 2pq + q$	$p^4 + 4p^2q + 4pr + 2q^2$
5	$p^4 + 3p^2q + 2pr + q^2$	$p^5 + 5p^3q + 5pq^2 + 5p^2r + 5qr$
…	…	…

Table 6.2: *(p, q, r)*-Generalized Trivariate Fibonacci and Lucas polynomials

Further, the characteristic equation corresponding to the recursive relations (6.1) and (6.2) is

$$\mu^3 - p(\xi,\omega,\zeta)\mu^2 - q(\xi,\omega,\zeta)\mu - r(\xi,\omega,\zeta) = 0. \tag{6.3}$$

and the corresponding Binet's formula are

$$F^*_{\alpha}(\xi,\omega,\zeta) = \frac{a^{\alpha+1}}{(a-b)(a-c)} + \frac{b^{\alpha+1}}{(b-a)(b-c)} + \frac{c^{\alpha+1}}{(c-a)(c-b)}. \tag{6.4}$$

and

$$G^*_{\alpha}(\xi,\omega,\zeta) = a^{\alpha} + b^{\alpha} + c^{\alpha}. \tag{6.5}$$

where a, b, c satisfies the characteristic equation

$$\mu^3 - p(\xi,\omega,\zeta)\mu^2 - q(\xi,\omega,\zeta)\mu - r(\xi,\omega,\zeta) = 0.$$

Again, the generating functions of *(p, q, r)*-Generalized Trivariate Fibonacci and Lucas polynomials respectively are:

$$F^*(t) = \sum_{\alpha=0}^{\infty} F^*_{\alpha}(\xi,\omega,\zeta) = \frac{t}{1-pt-qt^2-rt^3}. \tag{6.6}$$

and

$$G^*(t) = \sum_{\alpha=0}^{\infty} G^*_{\alpha}(\xi,\omega,\zeta) = \frac{3-2pt-qt^2}{1-pt-qt^2-rt^3}. \tag{6.7}$$

Again taking $p(\xi,\omega,\zeta) = 1$, $q(\xi,\omega,\zeta) = 1$, $r(\xi,\omega,\zeta) = 1$ equation (6.6) gives generating function for Tribonacci numbers ($\mathcal{T}_{\alpha}$) and taking $p(\xi,\omega,\zeta) = \xi$, $(\xi,\omega,\zeta) = \omega$, $(\xi,\omega,\zeta) = \zeta$ and then replacing ξ by ξ^2, ω by ξ, ζ by 1 , we get generating function for Tribonacci polynomials ($t_{\alpha}(\xi)$). In the further discussions, we shall write $p = p(\xi,\omega,\zeta), q = (\xi,\omega,\zeta), r = r(\xi,\omega,\zeta)$.

Theorem 6.2.1. *For any integer $\alpha \geq 0$,*

$$G^*_{\alpha}(\xi,\omega,\zeta) = pF^*_{\alpha}(\xi,\omega,\zeta) + 2qF^*_{\alpha-1}(\xi,\omega,\zeta) + 3rF^*_{\alpha-2}(\xi,\omega,\zeta). \tag{6.8}$$

Proof. Using the generating functions for *(p, q, r)*-Generalized Lucas polynomials given by equation (6.7), the Theorem 6.2.1 can easily be established. ■

Theorem 6.2.2. *For any integer $\alpha \geq 0$,*

$$\sum_{s=0}^{\alpha} F^*_{s}(\xi,\omega,\zeta) = \frac{F^*_{\alpha+2}(\xi,\omega,\zeta) + (1-p)F^*_{\alpha+1}(\xi,\omega,\zeta) + rF^*_{\alpha}(\xi,\omega,\zeta) - 1}{p+q+r-1}, \tag{6.9}$$

and

$$\sum_{s=0}^{\alpha} G^*{}_s(\xi,\omega,\zeta)$$
$$= \frac{G^*{}_{\alpha+2}(\xi,\omega,\zeta) + (p-1)G^*{}_{\alpha+1}(\xi,\omega,\zeta) + rG^*{}_{\alpha}(\xi,\omega,\zeta) - (3-2p-q)}{p+q+r-1}, \quad (6.10)$$

provided $p+q+r \neq 1$

Proof. We shall prove equation (6.9) and equation (6.10) by using method of mathematical induction. For equation (6.9), we proceed as follows

For $\alpha = 1$, we have to show

$$\sum_{s=0}^{1} F^*{}_s(\xi,\omega,\zeta) = \frac{F^*{}_3(\xi,\omega,\zeta) + (1-p)F^*{}_2(\xi,\omega,\zeta) + rF^*{}_1(\xi,\omega,\zeta) - 1}{p+q+r-1},$$

Equivalently,

$$F^*{}_0(\xi,\omega,\zeta) + F^*{}_1(\xi,\omega,\zeta)$$
$$= \frac{F^*{}_3(\xi,\omega,\zeta) + (1-p)F^*{}_2(\xi,\omega,\zeta) + rF^*{}_1(\xi,\omega,\zeta) - 1}{p+q+r-1}.$$

$$R.\mathcal{H}.S = \frac{F^*{}_3(\xi,\omega,\zeta) + (1-p)F^*{}_2(\xi,\omega,\zeta) + rF^*{}_1(\xi,\omega,\zeta) - 1}{p+q+r-1}$$
$$= \frac{p^2 + q + (1-p)p + r - 1}{p+q+r-1} = 1 + 0 = F^*{}_0(\xi,\omega,\zeta) + F^*{}_1(\xi,\omega,\zeta)$$
$$= R.\mathcal{H}.S$$

Hence for $\alpha = 1$, the result is true.

Suppose for $\alpha = \eta$, the result is true i.e.

$$\sum_{s=0}^{\eta} F^*{}_s(\xi,\omega,\zeta) = \frac{F^*{}_{\eta+2}(\xi,\omega,\zeta) + (1-p)F^*{}_{\eta+1}(\xi,\omega,\zeta) + rF^*{}_{\eta}(\xi,\omega,\zeta) - 1}{p+q+r-1}$$

Next, we shall prove the result for $\alpha = \eta + 1$, that is,

$$\sum_{s=0}^{\eta+1} F^*{}_s(\xi,\omega,\zeta) = \frac{F^*{}_{\eta+3}(\xi,\omega,\zeta) + (1-p)F^*{}_{\eta+2}(\xi,\omega,\zeta) + rF^*{}_{\eta+1}(\xi,\omega,\zeta) - 1}{p+q+r-1}$$

Now

$$R.\mathcal{H}.S. = \sum_{s=0}^{\eta+1} F^*{}_s(\xi,\omega,\zeta) = \sum_{s=0}^{\eta} F^*{}_s(\xi,\omega,\zeta) + F^*{}_{\eta+1}(\xi,\omega,\zeta)$$
$$= \frac{F^*{}_{\eta+2}(\xi,\omega,\zeta) + (1-p)F^*{}_{\eta+1}(\xi,\omega,\zeta) + rF^*{}_{\eta}(\xi,\omega,\zeta) - 1}{p+q+r-1}$$
$$+ F^*{}_{\eta+1}(\xi,\omega,\zeta)$$

$$= \frac{F^*{}_{\eta+2}(\xi,\omega,\zeta) + (1-p)F^*{}_{\eta+1}(\xi,\omega,\zeta) + rF^*{}_{\eta}(\xi,\omega,\zeta) - 1 + (p+q+r-1)F^*{}_{\eta+1}(\xi,\omega,\zeta)}{p+q+r-1}$$

$$= \frac{F^*{}_{\eta+2}(\xi,\omega,\zeta) + F^*{}_{\eta+1}(\xi,\omega,\zeta) + F^*{}_{\eta+3}(\xi,\omega,\zeta) - pF^*{}_{\eta+2}(\xi,\omega,\zeta) + rF^*{}_{\eta+2}(\xi,\omega,\zeta) - 1}{p+q+r-1}$$

$$= \frac{F^*{}_{\eta+3}(\xi,\omega,\zeta) + (1-p)F^*{}_{\eta+2}(\xi,\omega,\zeta) + rF^*{}_{\eta+1}(\xi,\omega,\zeta) - 1}{p+q+r-1}$$

$$= R.\mathcal{H}.S$$

Hence equation (6.9) holds for all positive α.

Similarly, we can see that equation (6.10) also holds true. That is,

$$\sum_{s=0}^{\alpha} G^*{}_s(\xi,\omega,\zeta)$$
$$= \frac{G^*{}_{\alpha+2}(\xi,\omega,\zeta) + (p-1)G^*{}_{\alpha+1}(\xi,\omega,\zeta) + rG^*{}_{\alpha}(\xi,\omega,\zeta) - (3-2p-q)}{p+q+r-1}.$$

This proves the theorem. ■

Taking $p(\xi,\omega,\zeta) = \xi, q(\xi,\omega,\zeta) = \omega, r(\xi,\omega,\zeta) = \zeta$ at $\xi = \omega = \zeta = 1$, we get the sum for α- Tribonacci numbers and at $\xi = \xi^2, \omega = \xi, \zeta = 1$, we have sum of α-Tribonacci Polynomials respectively.

Theorem 6.2.3. *For any integer $\alpha \geq 0$,*

$$\sum_{\eta=1}^{\alpha} F^*{}_{2\eta}(\xi,\omega,\zeta)$$
$$= \frac{F^*{}_{2\alpha+2}(\xi,\omega,\zeta) + r^2\, F^*_{2\alpha-2}(\xi,\omega,\zeta) + (r^2 - q^2 + 2rp)F^*{}_{2\alpha}(\xi,\omega,\zeta) - (p+r)}{[(p+q)^2 - (1-q)^2]},$$

and

$$\sum_{\eta=1}^{\alpha} F^*{}_{2\eta-1}(\xi,\omega,\zeta)$$
$$= \frac{F^*{}_{2\alpha+3}(\xi,\omega,\zeta) + (1-2q-p^2)F^*{}_{2\alpha+1}(\xi,\omega,\zeta) + r^2 F^*_{2\alpha-1}(\xi,\omega,\zeta) - (1-q)}{[(p+q)^2 - (1-q)^2]},$$

provided $(p+q)^2 - (1-q)^2 \neq 0$.

Proof. From the recurrence relation (6.1), we have

$$pF^*{}_{\alpha}(\xi,\omega,\zeta) + rF^*{}_{\alpha-2}(\xi,\omega,\zeta) = F^*{}_{\alpha+1}(\xi,\omega,\zeta) - qF^*{}_{\alpha-1}(\xi,\omega,\zeta) \qquad (6.11)$$

Writing the equation (6.11) for different values of α , we have

$$pF^*{}_0(\xi,\omega,\zeta) + rF^*{}_{-2}(\xi,\omega,\zeta) = F^*{}_1(\xi,\omega,\zeta) - qF^*{}_{-1}(\xi,\omega,\zeta)$$
$$pF^*{}_2(\xi,\omega,\zeta) + rF^*{}_0(\xi,\omega,\zeta) = F^*{}_3(\xi,\omega,\zeta) - qF^*{}_1(\xi,\omega,\zeta)$$
$$pF^*{}_4(\xi,\omega,\zeta) + rF^*{}_2(\xi,\omega,\zeta) = F^*{}_5(\xi,\omega,\zeta) - qF^*{}_3(\xi,\omega,\zeta)$$

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$$pF^*{}_{2\alpha}(\xi,\omega,\zeta) + rF^*{}_{2\alpha-2}(\xi,\omega,\zeta) = F^*{}_{2\alpha+1}(\xi,\omega,\zeta) - qF^*{}_{2\alpha-1}(\xi,\omega,\zeta)$$

Adding these equations, we have

$$1 + (p+r)\sum_{\eta=1}^{\alpha} F^*_{2\eta-2}(\xi,\omega,\zeta) + pF^*{}_{2\alpha}(\xi,\omega,\zeta)$$
$$= F^*{}_{2\alpha+1}(\xi,\omega,\zeta) + (1-q)\sum_{\eta=1}^{\alpha} F^*_{2\eta-1}(\xi,\omega,\zeta)$$

After simplification, we have

$$(p+r)\sum_{\eta=1}^{\alpha} F^*_{2\eta}(\xi,\omega,\zeta)$$
$$= F^*{}_{2\alpha+1}(\xi,\omega,\zeta) + rF^*{}_{2\alpha}(\xi,\omega,\zeta) - 1 + (1-q)\sum_{\eta=1}^{\alpha} F^*_{2\eta-1}(\xi,\omega,\zeta) \qquad (6.12)$$

Again, using the (6.11) and proceeding as above, we can write

$$(p+r)\sum_{\eta=1}^{\alpha} F^*_{2\eta-1}(\xi,\omega,\zeta)$$
$$= F^*{}_{2\alpha}(\xi,\omega,\zeta) + rF^*{}_{2\alpha-1}(\xi,\omega,\zeta) + (1-q)\sum_{\eta=1}^{\alpha} F^*_{2\eta-2}(\xi,\omega,\zeta)$$

After simplification, we can write

$$(p+r)\sum_{\eta=1}^{\alpha} F^*_{2\eta-1}(\xi,\omega,\zeta)$$
$$= qF^*{}_{2\alpha}(\xi,\omega,\zeta) + rF^*{}_{2\alpha-1}(\xi,\omega,\zeta) + (1-q)\sum_{\eta=1}^{\alpha} F^*_{2\eta}(\xi,\omega,\zeta) \qquad (6.13)$$

Using (6.12) in (6.13), we get

$$\sum_{\eta=1}^{\alpha} F^*{}_{2k}(\xi,\omega,\zeta)$$

$$= \frac{F^*{}_{2\alpha+2}(\xi,\omega,\zeta) + r^2 F^*_{2\alpha-2}(\xi,\omega,\zeta) + (r^2 - q^2 + 2rp)F^*{}_{2\alpha}(\xi,\omega,\zeta) - (p+r)}{[(p+q)^2 - (1-q)^2]}$$

Similarly, using (6.13) in (6.12), we have

$$\sum_{\eta=1}^{\alpha} F^*{}_{2\eta-1}(\xi,\omega,\zeta)$$

$$= \frac{F^*{}_{2\alpha+3}(\xi,\omega,\zeta) + (1-2q-p^2)F^*{}_{2\alpha+1}(\xi,\omega,\zeta) + r^2 F^*_{2\alpha-1}(\xi,\omega,\zeta) - (1-q)}{[(p+q)^2 - (1-q)^2]}$$

This establishes the Theorem. ■

Theorem 6.2.4 *For any integer* $\alpha \geq 0$,

$$\sum_{\eta=1}^{\alpha} G^*{}_{2\eta}(\xi,\omega,\zeta)$$

$$= \frac{G^*{}_{2\alpha+2}(\xi,\omega,\zeta) + r^2 G^*_{2\alpha-2}(\xi,\omega,\zeta) + (r^2 - q^2 + 2rp)G^*{}_{2\alpha}(\xi,\omega,\zeta) - [(3r+p)(p+r) + 2q(1-q)]}{[(p+q)^2 - (1-q)^2]},$$

and

$$\sum_{\eta=1}^{\alpha} G^*{}_{2\eta-1}(\xi,\omega,\zeta)$$

$$= \frac{G^*{}_{2\alpha+3}(\xi,\omega,\zeta) + (1-2q-p^2)G^*{}_{2\alpha+1}(\xi,\omega,\zeta) + r^2 G^*_{2\alpha-1}(\xi,\omega,\zeta) - [(q+1)p + (3-q)r]}{[(p+q)^2 - (1-q)^2]}$$

provided $(p+q)^2 - (1-q)^2 \neq 0$.

Proof: Proceeding as above in Theorem 6.2.3, the desired results can be established. ■

Now we shall discuss explicit formulas for (p,q,r) -*Generalized Trivariate Fibonacci* and *Lucas* polynomials. Firstly, we will write the (p,q,r)-*Generalised Trivariate Fibonacci* polynomials triangle and (p,q,r) −*Generalized Trivariate Lucus* polynomials triangle as under:

α \ t	**0**	**1**	**2**	**3**	**4**	**. . .**
0	1					
1	p	q				
2	p^2	$2pq+r$	q^2			
3	p^3	$3p^2q + 2pr$	$3pq^2 + 2qr$	q^3		
4	p^4	$4p^3q + 3p^2r$	$6p^2q^2 + 6pqr + r^2$	$4pq^3 + 3q^2r$	q^4	
...	…	...	…	...	..	

Table 6.3: ***(p, q, r)*-Generalized Trivariate Fibonacci polynomials triangle**

α \ t	**0**	**1**	**2**	**3**	**4**	**. . .**
0	3					
1	p	$2q$				
2	p^2	$3pq+3r$	$2q^2$			
3	p^3	$4p^2q + 4pr$	$5pq^2 + 5qr$	$2q^3$		
4	p^4	$5p^3q + 5p^2r$	$9p^2q^2 + 11pqr + 3r^2$	$7pq^3 + 6q^2r$	$2q^4$	
...	…	...	…	...	..	

Table 6.4: ***(p, q, r)*-Generalized Trivariate Lucas polynomials triangle**

If $\mathcal{B}_{F^*}(\alpha,t)$ and $\mathcal{B}_{G^*}(\alpha,t)$represents the element in the $\alpha^{th}-row$ and $t^{th}-column$ of the (p,q,r)-*Generalised Trivariate Fibonacci* polynomial triangle and (p,q,r)-*Generalised Trivariate Lucas* polynomial triangle respectively, then we can write

$$\mathcal{B}_{F^*}(\alpha,t)=\sum_{s=0}^{t}\binom{t}{s}\binom{\alpha-s}{t}p^{\alpha-t-s}q^{t-s}r^{s},$$

and

$$\mathcal{B}_{G^*}(\alpha,t)=\sum_{s=0}^{t}\frac{\alpha+t}{\alpha-s}\binom{t}{s}\binom{\alpha-s}{t}p^{\alpha-t-s}q^{t-s}r^{s},$$

Consequently, it can be easily seen that,

$$\mathcal{B}_{F^*}(\alpha+1,t)=p\mathcal{B}_{F^*}(\alpha,t)+q\mathcal{B}_{F^*}(\alpha,t-1)+r\mathcal{B}_{F^*}(\alpha-1,t-1),$$

with

$$\mathcal{B}_{F^*}(\alpha,0)=p^{\alpha}\ ,\qquad \mathcal{B}_{F^*}(\alpha,\alpha)=q^{\alpha}.$$

$$\mathcal{B}_{G^*}(\alpha+1,t)=p\mathcal{B}_{G^*}(\alpha,t)+q\mathcal{B}_{G^*}(\alpha,t-1)+r\mathcal{B}_{G^*}(\alpha-1,t-1),$$

with

$$\mathcal{B}_{G^*}(\alpha,0)=p^{\alpha},\qquad \mathcal{B}_{G^*}(\alpha,\alpha)=2q^{\alpha}.$$

Further, we can easily write that,

$$F^*{}_{\alpha}(\xi,\omega,\zeta)=\sum_{t=0}^{\left\lfloor\frac{\alpha-1}{2}\right\rfloor}\mathcal{B}_{F^*}(\alpha-t-1,t),$$

and

$$G^*{}_{\alpha}(\xi,\omega,\zeta)=\sum_{t=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor}\mathcal{B}_{G^*}(\alpha-t,t).$$

Now, we are in a position to write the explicit formulae for (p,q,r) -*Generalized Trivariate Fibonacci* and *Lucas* polynomials respectively as under:

Theorem 6.2.5. *The explicit representation of (p,q,r) −Generalized Trivariate Fibonacci and Lucas polynomials is as follows:*

$$F^*{}_{\alpha}(\xi,\omega,\zeta)=\sum_{t=0}^{\left\lfloor\frac{\alpha-1}{2}\right\rfloor}\sum_{s=0}^{t}\binom{t}{s}\binom{\alpha-t-s-1}{t}p^{\alpha-2t-s-1}q^{t-s}r^{s}, \qquad (6.14)$$

$$G^*{}_{\alpha}(\xi,\omega,\zeta)=\sum_{t=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor}\sum_{s=0}^{t}\frac{\alpha}{\alpha-t-s}\binom{t}{s}\binom{\alpha-t-s}{t}p^{\alpha-2t-s}q^{t-s}r^{s}, \qquad (6.15)$$

such that $\binom{j}{i}=0$ whenever $i>j$.

Proof. We will prove (6.14) by using mathematical induction**.**

For $=1,2,3,4$, the result (6.11) is true.

Suppose the result is true for $\alpha=\eta$, that is,

$$F^*{}_{\eta}(\xi,\omega,\zeta)=\sum_{t=0}^{\left\lfloor\frac{\eta-1}{2}\right\rfloor}\sum_{s=0}^{t}\binom{t}{s}\binom{\eta-t-s-1}{t}p^{\eta-2t-s-1}q^{t-s}r^{s}.$$

Next, we will show that the result is true for $\alpha=\eta+1$, that is,

$$F^*{}_{\eta+1}(\xi,\omega,\zeta)=\sum_{t=0}^{\left\lfloor\frac{\eta}{2}\right\rfloor}\sum_{s=0}^{t}\binom{t}{s}\binom{\eta-t-s}{t}p^{\eta-2t-s}q^{t-s}r^{s}.$$

Consider

$$F^*{}_{\eta+1}(\xi,\omega,\zeta)=pF^*{}_{\eta}(\xi,\omega,\zeta)+qF^*{}_{\eta-1}(\xi,\omega,\zeta)+rF^*{}_{\eta-2}(\xi,\omega,\zeta)$$

$$=p\left[\sum_{t=0}^{\left\lfloor\frac{\eta-1}{2}\right\rfloor}\sum_{s=0}^{t}\mathcal{B}_{F^*}(\eta-t-1,t)\right]+q\left[\sum_{t=0}^{\left\lfloor\frac{\eta-2}{2}\right\rfloor}\sum_{s=0}^{t}\mathcal{B}_{F^*}(\eta-t-2,t)\right]$$

$$+r\left[\sum_{t=0}^{\left\lfloor\frac{\eta-3}{2}\right\rfloor}\sum_{s=0}^{t}\mathcal{B}_{F^*}(\eta-t-3,t)\right],$$

$$= p\left[\mathcal{B}_{F^*}(\eta-1,0)+\mathcal{B}_{F^*}(\eta-2,1)+\mathcal{B}_{F^*}(\eta-3,2)+\cdots+\mathcal{B}_{F^*}\left(\frac{\eta-1}{2},\frac{\eta-1}{2}\right)\right]$$

$$+ q\left[\mathcal{B}_{F^*}(\eta-2,0)+\mathcal{B}_{F^*}(\eta-3,1)+ +\mathcal{B}_{F^*}(\eta-4,2)\right.$$

$$\left.+\cdots\mathcal{B}_{F^*}\left(\frac{\eta-2}{2},\frac{\eta-2}{2}\right)\right]$$

$$+ r\left[\mathcal{B}_{F^*}(\eta-3,0)+\mathcal{B}_{F^*}(\eta-4,1)+\mathcal{B}_{F^*}(\eta-5,2)+\cdots\right.$$

$$\left.+\mathcal{B}_{F^*}\left(\frac{\eta-3}{2},\frac{\eta-3}{2}\right)\right],$$

$$= \mathcal{B}_{F^*}(\eta,0)+\mathcal{B}_{F^*}(\eta-1,1)+\mathcal{B}_{F^*}(\eta-2,2)+\mathcal{B}_{F^*}(\eta-3,3)+\cdots+\mathcal{B}_{F^*}\left(\frac{\eta+1}{2},\frac{\eta-1}{2}\right)$$

$$+\mathcal{B}_{F^*}\left(\frac{\eta}{2},\frac{\eta}{2}\right),$$

$$\therefore F^*{}_{\eta+1}(\xi,\omega,\zeta)=\sum_{t=0}^{\left\lfloor\frac{\eta}{2}\right\rfloor}\mathcal{B}_{F^*}(\eta-t,t)=\sum_{t=0}^{\left\lfloor\frac{\eta}{2}\right\rfloor}\sum_{s=0}^{t}\binom{t}{s}\binom{\eta-t-s}{t}p^{\eta-2t-s}q^{t-s}r^{s}.$$

Thus, by induction, the result holds for all positive integer α.

Similarly, we can obtain (6.15) for (p,q,r)-*Generalized Trivariate Lucas* polynomials. ■

Theorem 6.2.6. *Let $F^*{}_{\alpha}(\xi,\omega,\zeta)$ and $G^*{}_{\alpha}(\xi,\omega,\zeta)$ be (p,q,r) −Generalized Trivariate Fibonacci and Lucas Polynomials respectively. Then*

$$\frac{\partial(p,G^*{}_{\alpha}(\xi,\omega,\zeta),r)}{\partial(\xi,\omega,\zeta)}=\alpha F^*{}_{\alpha-1}(\xi,\omega,\zeta)\frac{\partial(p,q,r)}{\partial(\xi,\omega,\zeta)}.$$

Proof. From Theorem 6.2.5, we have

$$G^*{}_{\alpha}(\xi,\omega,\zeta)=\sum_{t=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor}\sum_{s=0}^{t}\frac{\alpha}{\alpha-t-s}\binom{t}{s}\binom{\alpha-t-s}{t}p^{\alpha-2t-s}q^{t-s}r^{s}. \quad (6.16)$$

Differentiating equation (6.16) w.r.t ξ, partially, we have

$$\frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \xi} = \sum_{t=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor}\sum_{s=0}^{t}\frac{\alpha}{\alpha-t-s}\binom{t}{s}\binom{\alpha-t-s}{t}(\alpha-2t-s)p^{\alpha-2t-s-1}\, p_\xi\, q^{t-s}r^s$$

$$+\sum_{t=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor}\sum_{s=0}^{t}\frac{\alpha}{\alpha-t-s}\binom{t}{s}\binom{\alpha-t-s}{t}p^{\alpha-2t-s}(t-s)\;\; q_\xi\, q^{t-s-1}r^s$$

$$+\sum_{t=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor}\sum_{s=0}^{t}\frac{\alpha}{\alpha-t-s}\binom{t}{s}\binom{\alpha-t-s}{t}p^{\alpha-2t-s}q^{t-s}\, r_\xi\, s\, r^s$$

$$= \alpha p_\xi \sum_{t=0}^{\left\lfloor\frac{\alpha-1}{2}\right\rfloor}\sum_{s=0}^{t}\binom{t}{s}\binom{\alpha-t-s-1}{t}p^{\alpha-2t-s-1}q^{t-s}r^s$$

$$+\alpha q_\xi \sum_{t=0}^{\left\lfloor\frac{\alpha-2}{2}\right\rfloor}\sum_{s=0}^{t}\binom{t}{s}\binom{\alpha-t-s-2}{t}p^{\alpha-2t-s-2}q^{t-s}r^s$$

$$+\alpha r_\xi \sum_{t=0}^{\left\lfloor\frac{\alpha-3}{2}\right\rfloor}\sum_{s=0}^{t}\binom{t}{s}\binom{\alpha-t-s-3}{t}p^{\alpha-2t-s-3}q^{t-s}r^s$$

$$= \alpha p_\xi F^*{}_\alpha(\xi,\omega,\zeta) + \alpha q_\xi F^*{}_{\alpha-1}(\xi,\omega,\zeta) + \alpha r_\xi F^*{}_{\alpha-2}(\xi,\omega,\zeta).$$

Therefore,

$$\frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \xi} = \alpha p_\xi F^*{}_\alpha(\xi,\omega,\zeta) + \alpha q_\xi F^*{}_{\alpha-1}(\xi,\omega,\zeta) + \alpha r_\xi F^*{}_{\alpha-2}(\xi,\omega,\zeta). \tag{6.17}$$

Similarly,

$$\frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \omega} = \alpha p_\omega F^*_\alpha(\xi,\omega,\zeta) + \alpha q_\omega F^*_{\alpha-1}(\xi,\omega,\zeta) + \alpha r_\omega F^*_{\alpha-2}(\xi,\omega,\zeta). \tag{6.18}$$

$$\frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \zeta} = \alpha p_\zeta F^*_\alpha(\xi,\omega,\zeta) + \alpha q_\omega F^*_{\alpha-1}(\xi,\omega,\zeta) + \alpha r_\zeta F^*_{\alpha-2}(\xi,\omega,\zeta). \tag{6.19}$$

Multiplying (6.18) by $r_\zeta$and (6.19) by r_ω and subtracting we have,

$$r_\zeta \frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \omega} - r_\omega \frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \zeta}$$
$$= \alpha\left[p_\omega r_\zeta - p_\zeta r_\omega\right] F_\alpha(\xi,\omega,\zeta) + \alpha\left[q_\omega r_\zeta - q_\zeta r_\omega\right] F_{\alpha-1}(\xi,\omega,\zeta) \qquad (6.20)$$

Multiplying (6.17) by r_ζ and (6.19) by r_ξ and subtracting we have,

$$r_\zeta \frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \xi} - r_\xi \frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \zeta}$$
$$= \alpha\left[p_\xi r_\zeta - p_\zeta r_\xi\right] F_\alpha(\xi,\omega,\zeta) + \alpha\left[q_\xi r_\zeta - q_\zeta r_\xi\right] F_{\alpha-1}(\xi,\omega,\zeta) \qquad (6.21)$$

Multiplying (6.17) by r_ω and (6.18) by r_ξ and subtracting we have,

$$r_\omega \frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \xi} - r_\xi \frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial \omega}$$
$$= \alpha\left[p_\xi r_\omega - p_\omega r_\xi\right] F_\alpha(\xi,\omega,\zeta) + \alpha\left[q_\xi r_\omega - q_\omega r_\xi\right] F_{\alpha-1}(\xi,\omega,\zeta) \qquad (6.22)$$

Now, using (6.20), (6.21) and (6.22), we have

$$\frac{\partial(p, G^*{}_\alpha(\xi,\omega,\zeta), r)}{\partial(\xi,\omega,\zeta)} = \alpha F^*{}_{\alpha-1}(\xi,\omega,\zeta) \frac{\partial(p,q,r)}{\partial(\xi,\omega,\zeta)}$$

This completes the proof. ■

Theorem 6.2.7. *Let $F^*{}_\alpha(\xi,\omega,\zeta)$ and $G^*{}_\alpha(\xi,\omega,\zeta)$ be (p,q,r) −Generalized Trivariate Fibonacci and Lucas Polynomials respectively. Then*

$$p \frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial p} + q \frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial q} + r \frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial r} = \alpha F^*{}_\alpha(\xi,\omega,\zeta). \qquad (6.23)$$

Proof. From Theorem 6.2.3, we have,

$$G^*{}_\alpha(\xi,\omega,\zeta) = \sum_{t=0}^{\left\lfloor \frac{\alpha}{2} \right\rfloor} \sum_{s=0}^{t} \frac{\alpha}{\alpha - t - s} \binom{t}{s} \binom{\alpha - t - s}{t} p^{\alpha-2t-s} q^{t-s} r^s. \qquad (6.24)$$

Differentiating equation (6.24) w.r.t p partially, we have

$$\frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial p} = \sum_{t=0}^{\left\lfloor \frac{\alpha}{2} \right\rfloor} \sum_{s=0}^{t} \frac{\alpha}{\alpha - t - s} \binom{t}{s} \binom{\alpha - t - s}{t} (\alpha - 2t - s) p^{\alpha-2t-s-1} q^{t-s} r^s$$
$$= \alpha \sum_{t=0}^{\left\lfloor \frac{\alpha-1}{2} \right\rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{\alpha - t - s - 1}{t} p^{\alpha-2t-s-1} q^{t-s} r^s = \alpha F^*{}_\alpha(\xi,\omega,\zeta),$$

Therefore,

$$\frac{\partial G^*{}_\alpha(\xi,\omega,\zeta)}{\partial p} = \alpha \; F^*{}_\alpha(\xi,\omega,\zeta).$$

Again, differentiating equation (6.24) w.r.t q partially, we have

$$\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial q}=\sum_{t=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor}\sum_{s=0}^{t}\frac{\alpha}{\alpha-t-s}\binom{t}{s}\binom{\alpha-t-s}{t}p^{\alpha-2t-s}(t-s)\,q^{t-s-1}r^{s}$$

$$=\alpha\sum_{t=0}^{\left\lfloor\frac{\alpha-2}{2}\right\rfloor}\sum_{s=0}^{t}\binom{t}{s}\binom{\alpha-t-s-2}{t}p^{\alpha-2t-s-2}q^{t-s}r^{s}$$

$$=\alpha F^*{}_{\alpha-1}(\xi,\omega,\zeta),$$

Therefore,

$$\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial q}=\alpha F^*{}_{\alpha-1}(\xi,\omega,\zeta).$$

Again, differentiating equation (6.14) w.r.t r partially, we have

$$\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial r}=\sum_{t=0}^{\left\lfloor\frac{\alpha}{2}\right\rfloor}\sum_{s=0}^{t}\frac{\alpha}{\alpha-t-s}\binom{t}{s}\binom{\alpha-t-s}{t}p^{\alpha-2t-s}q^{t-s}sr^{s}$$

$$=\alpha\sum_{t=0}^{\left\lfloor\frac{\alpha-3}{2}\right\rfloor}\sum_{s=0}^{t}\binom{t}{s}\binom{\alpha-t-s-3}{t}p^{\alpha-2t-s-3}q^{t-s}r^{s}$$

$$=\alpha\,F^*{}_{\alpha-2}(\xi,\omega,\zeta),$$

Therefore,

$$\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial r}=\alpha F^*{}_{\alpha-2}(\xi,\omega,\zeta).$$

Now, we have

$$G.\mathcal{H}.S=p\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial p}+q\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial q}+r\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial r}$$

$$=\alpha pF^*{}_{\alpha}(\xi,\omega,\zeta)+\alpha qF^*{}_{\alpha-1}(\xi,\omega,\zeta)+\alpha rF^*{}_{\alpha-2}(\xi,\omega,\zeta)$$

$$=\alpha F^*{}_{\alpha+1}(\xi,\omega,\zeta)=R.\mathcal{H}.S$$

Therefore,

$$p\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial p}+q\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial q}+r\frac{\partial G^*{}_{\alpha}(\xi,\omega,\zeta)}{\partial r}=\alpha\,F^*{}_{\alpha}(\xi,\omega,\zeta)\,.\quad\blacksquare$$

6.2.1. Generating matrix for $(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})-$Generalized Trivariate Fibonacci polynomials

As in [62, 64] the generating matrix for (p,q,r) −*Generalized Trivariate Fibonacci* polynomials is

$$\mathcal{H} = \begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix}.$$

Using mathematical Induction, we can easily deduce

$$\mathcal{H}^{\alpha} = \begin{bmatrix} {F^*}_{\alpha+1} & {F^*}_{\alpha} & {F^*}_{\alpha-1} \\ q{F^*}_{\alpha} + r{F^*}_{\alpha-1} & q{F^*}_{\alpha-1} + r{F^*}_{\alpha-2} & q{F^*}_{\alpha-2} + r{F^*}_{\alpha-3} \\ r{F^*}_{\alpha} & r{F^*}_{\alpha-1} & r{F^*}_{\alpha-2} \end{bmatrix},$$

where

$${F^*}_{\alpha} = {F^*}_{\alpha}(\xi,\omega,\zeta).$$

Theorem 6.2.8. *For any positive integers* α, β

$$\begin{aligned} {F^*}_{\alpha+\beta}(\xi,\omega,\zeta) = {} & {F^*}_{\beta+1}(\xi,\omega,\zeta){F^*}_{\alpha}(\xi,\omega,\zeta) + {F^*}_{\beta}(\xi,\omega,\zeta){F^*}_{\alpha+1}(\xi,\omega,\zeta) \\ & + \zeta {F^*}_{\beta-1}(\xi,\omega,\zeta){F^*}_{\alpha-1}(\xi,\omega,\zeta) \\ & - \xi {F^*}_{\beta}(\xi,\omega,\zeta)\,{F^*}_{\alpha}(\xi,\omega,\zeta). \end{aligned} \tag{6.25}$$

Proof. With the help of the identity $\mathcal{H}^{\alpha+\beta} = \mathcal{H}^{\alpha}\mathcal{H}^{\beta}$ and equality of matrices, the desired result can be established. ■

Corollary 6.2.1. *For any positive integers* α, β

$$\begin{aligned} {F^*}_{2\alpha}(\xi,\omega,\zeta) = {} & r{F^*}^2_{\beta+1}(\xi,\omega,\zeta) - p{F^*}^2_{\beta}(\xi,\omega,\zeta) \\ & + 2{F^*}_{\alpha+1}(\xi,\omega,\zeta){F^*}_{\alpha}(\xi,\omega,\zeta). \end{aligned} \tag{6.26}$$

Proof. By using $\alpha = \beta$ in equation (6.26), the desired result can be established. ■

Corollary 6.2.2. For any positive integers α, β

$${F^*}_{2\alpha+1} = {F^*}^2_{\alpha+1}(\xi,\omega,\zeta) + q{F^*}^2_{\alpha}(\xi,\omega,\zeta) + 2r{F^*}_{\alpha}(\xi,\omega,\zeta){F^*}_{\alpha-1}(\xi,\omega,\zeta)$$

Proof. By using $\beta = \alpha + 1$ equation (6.26), the desired result can be established. ■

Theorem 6.2.9. *For any positive integer n,*

$$\begin{vmatrix} {F^*}_{\alpha+2} & {F^*}_{\alpha+1} & {F^*}_{\alpha} \\ {F^*}_{\alpha+1} & {F^*}_{\alpha} & {F^*}_{\alpha-1} \\ {F^*}_{\alpha} & {F^*}_{\alpha-1} & {F^*}_{\alpha-2} \end{vmatrix} = -r^{\alpha-1}, \tag{6.27}$$

where ${F^*}_{\alpha} = {F^*}_{\alpha}(\xi,\omega,\zeta)$.

Proof. Evidently $\det(\mathcal{H}) = r$ and hence $\det(\mathcal{H}^{\alpha}) = r^{\alpha}$, implies

$$\begin{vmatrix} F^*{}_{\alpha+1} & F^*{}_{\alpha} & F^*{}_{\alpha-1} \\ qF^*{}_{\alpha}+rF^*{}_{\alpha-1} & qF^*{}_{\alpha-1}+rF^*{}_{\alpha-2} & qF^*{}_{\alpha-2}+rF^*{}_{\alpha-3} \\ rF^*{}_{\alpha} & rF^*{}_{\alpha-1} & rF^*{}_{\alpha-2} \end{vmatrix} = r^{\alpha},$$

Operating $\mathrm{R}_2 + p\mathrm{R}_1$ and interchanging R_1 and R_2, we have

$$\begin{vmatrix} F^*{}_{\alpha+2} & F^*{}_{\alpha+1} & F^*{}_{\alpha} \\ F^*{}_{\alpha+1} & F^*{}_{\alpha} & F^*{}_{\alpha-1} \\ rF^*{}_{\alpha} & rF^*{}_{\alpha-1} & rF^*{}_{\alpha-2} \end{vmatrix} = -r^{\alpha},$$

Which further implies,

$$\begin{vmatrix} F^*{}_{\alpha+2} & F^*{}_{\alpha+1} & F^*{}_{\alpha} \\ F^*{}_{\alpha+1} & F^*{}_{\alpha} & F^*{}_{\alpha-1} \\ F^*{}_{\alpha} & F^*{}_{\alpha-1} & F^*{}_{\alpha-2} \end{vmatrix} = -r^{\alpha-1},$$

This establishes the determinant properties of (p,q,r)-*Generalized Trivariate Fibonacci* polynomials. Taking $p = q = r = 1$, we obtain the determinant property of Tribonacci numbers and by taking $p = \xi^2, q = \xi, r = 1$, determinant property of Tribonacci polynomials is obtained. Next, we will attempt to establish the determinant properties of (p,q,r)-*Generalized Trivariate Lucas* polynomials. The (p,q,r)-*Generalized Trivariate Lucas* polynomials are generated by a matrix M_1 with the help of the following matrices:

$$\mathcal{H} = \begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix},$$

and

$$M_0 = \begin{bmatrix} G^*{}_2 & G^*{}_1 & G^*{}_0 \\ G^*{}_1 & G^*{}_0 & G^*{}_{-1} \\ G^*{}_0 & G^*{}_{-1} & G^*{}_{-2} \end{bmatrix} = \begin{bmatrix} p^2+2q & p & 3 \\ p & 3 & -\frac{q}{r} \\ 3 & -\frac{q}{r} & \frac{q^2-2pr}{r^2} \end{bmatrix},$$

such that

$$M_1 = M_0\mathcal{H} = \begin{bmatrix} G^*{}_2 & G^*{}_1 & G^*{}_0 \\ G^*{}_1 & G^*{}_0 & G^*{}_{-1} \\ G^*{}_0 & G^*{}_{-1} & G^*{}_{-2} \end{bmatrix} \begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix} = \begin{bmatrix} p^3+3pq+3r & p^2+2q & p \\ p^2+2q & p & 3 \\ p & 3 & -\frac{q}{r} \end{bmatrix}$$

$$= \begin{bmatrix} G^*{}_3 & G^*{}_2 & G^*{}_1 \\ G^*{}_2 & G^*{}_1 & G^*{}_0 \\ G^*{}_1 & G^*{}_0 & G^*{}_{-1} \end{bmatrix}.$$

Proceeding inductively, we can easily see that

$$M_\alpha = M_{\alpha-1}\mathcal{H} = \begin{bmatrix} G^*{}_{\alpha+2} & G^*{}_{\alpha+1} & G^*{}_{\alpha} \\ G^*{}_{\alpha+1} & G^*{}_{\alpha} & G^*{}_{\alpha-1} \\ G^*{}_{\alpha} & G^*{}_{\alpha-1} & G^*{}_{\alpha-2} \end{bmatrix}.$$

Theorem 6.2.9. For any positive integer α,

$$M_\alpha = M_0\mathcal{H}^\alpha, \tag{6.28}$$

where $\mathcal{H}^1 = \mathcal{H}$.

Proof. The result can be easily established using induction hypothesis.

For $\alpha = 1$, clearly $M_1 = M_0\mathcal{H}^1 = M_0\mathcal{H}$

As

$$M_0\mathcal{H} = \begin{bmatrix} p^2+2q & p & 3 \\ p & 3 & -\frac{q}{r} \\ 3 & -\frac{q}{r} & \frac{q^2-2pr}{r^2} \end{bmatrix}\begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix},$$

$$= \begin{bmatrix} p^3+3pq+3r & p^2+2q & p \\ p^2+2q & p & 3 \\ p & 3 & -\frac{q}{r} \end{bmatrix}$$

$$= \begin{bmatrix} G^*{}_3 & G^*{}_2 & G^*{}_1 \\ G^*{}_2, & G^*{}_1 & G^*{}_0 \\ G^*{}_1 & G^*{}_0 & G^*{}_{-1} \end{bmatrix} = M_1.$$

Suppose the result is true for $\alpha = \eta$, that is,

$$M_\eta = M_0\mathcal{H}^\eta.$$

Next, we shall prove that the result is true for $\alpha = \eta + 1$, that is,

$$M_{\eta+1} = M_0\mathcal{H}^{\eta+1}.$$

$$M_0\mathcal{H}^{\eta+1} = M_0\mathcal{H}^\eta\mathcal{H} = M_\eta\mathcal{H} = \begin{bmatrix} G^*{}_{\eta+2} & G^*{}_{\eta+1} & G^*{}_{\eta} \\ G^*{}_{\eta+1} & G^*{}_{\eta} & G^*{}_{\eta-1} \\ G^*{}_{\eta} & G^*{}_{\eta-1} & G^*{}_{\eta-2} \end{bmatrix}\begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} pG^*{}_{\eta+2} + qG^*{}_{\eta+1} + rG^*{}_{\eta} & G^*{}_{\eta+1} & G^*{}_{\eta} \\ pG^*{}_{\eta+2} + qG^*{}_{\eta+1} + rG^*{}_{\eta} & G^*{}_{\eta}, & G^*{}_{\eta-1} \\ pG^*{}_{\eta} + qG^*{}_{\eta-1} + rG^*{}_{\eta-2} & G^*{}_{\eta-1} & G^*{}_{\eta-2} \end{bmatrix},$$

$$M_0\mathcal{H}^{\eta+1} = \begin{bmatrix} G^*{}_{\eta+3} & G^*{}_{\eta+2} & G^*{}_{\eta+1} \\ G^*{}_{\eta+2} & G^*{}_{\eta+1} & G^*{}_{\eta} \\ G^*{}_{\eta+1} & G^*{}_{\eta} & G^*{}_{\eta-1} \end{bmatrix} = M_{\eta+1}.$$

Hence, the result holds for all positive integers α. ■

Summary and Conclusions.

In chapter 2, we derived identities expressing sums of finite product of the Lucas numbers ($\mathcal{L}_n$), the Fibonacci ($\mathcal{F}_n$), & the Complex Fibonacci numbers($\mathcal{F}^*{}_n$) as linear sum of derivatives of the 2nd kinds of Chebyshev polynomials ($\mathcal{U}_n\ (z)$) through elementary computations.

In chapter 3, we introduced a few more results on sums of finite product of the 3rd and 4th kinds of Chebyshev polynomials, Lucas and Fibonacci numbers in terms of the 2nd kind Chebyshev polynomials and their derivatives. Also, we discussed some particular cases of the results obtained in this chapter in the form of corollaries by taking different values of $r = 1,2,3$.

In chapter 4, using elementary methods, we deduced the explicit formulae for the 3rd and 4th kinds of Chebyshev Polynomials and their derivatives with odd and even indices and obtained a relationship connecting the 3rd and 4th kinds of Chebyshev Polynomials and negative indexed Fibonacci polynomials.

In first section of chapter 5, we introduced a few more results expressing summations of finite products of Lucas & Fibonacci numbers, Fibonacci and Pell polynomials as a linear sum of the derivatives of Pell polynomials, using their basic properties through elementary computations. Similar identities are obtained for the 3rd and 4th kinds of Chebyshev polynomials. In the next section, we established similar identities for the negative indexed Lucas, Fibonacci, and Complex Fibonacci numbers. In terms of the 3rd and 4th kinds of Chebyshev polynomials, similar identities were obtained for Pell numbers and Fibonacci polynomials

At the end in the chapter 6, we developed the concept of *(p, q, r)-Generalized Trivariate Fibonacci* and *(p, q, r)-Generalized Trivariate Lucas* polynomials and discussed their properties. Using these properties, we derived the explicit formula of *(p,q,r)-Generalized Trivariate Fibonacci* and *Lucas* polynomials and deduce some results on the generating matrices and their determinants.

Future and Scope

1. Identities on the sums of the finite product of the Pell numbers, the Jacobsthal numbers, and polynomials in terms of the derivatives of the 1^{st}, 2^{nd}, 3^{rd}, and 4^{th} kinds of Chebyshev polynomials can be obtained using elementary computational method.
2. Identities on sums of finite products of Lucas and Fibonacci numbers, Pell and Fibonacci polynomials as a linear sum of derivatives of Jacobsthal polynomials, using their basic properties through elementary computations can be obtained.
3. Identities on sums of finite products of negative indexed Lucas, Fibonacci,and Complex Fibonacci numbers in terms of Jacobsthal polynomials and Jacobsthal Lucas polynomials can be obtained using their basic properties through elementary computations.

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List of Publications and Communications

S.No	Name of the Journal	Title of the Paper	Status
1	Notes on Number Theory and Discrete Mathematics (Web of Science)	Some identities involving Chebyshev Polynomials, Fibonacci Polynomials and their derivatives	Published . https://nntdm.net/volume-29-2023/number-2/204-215/
2	Presented in this International Conference on Mathematical and statistical Computation (ICMSC-2022) and accepted for publication to Journal of Rajasthan academy of Physical Sciences-(Web of Science).	Sums of Finite Product of Chebyshev Polynomials of Third and Fourth Kind and Fibonacci and Lucas Numbers	Accepted for publication
3	Journal of algebraic statistics (Web of Science)	Some identities on Finite sums of product of Fibonacci and Lucas Numbers in Chebyshev Polynomials of second Kind	Published https://www.publishoa.com/index.php/journal/article/view/1136
4	Communication in Mathematics and applications (Web of Science)	Some identities on Sums of Finite Product ofChebyshev Polynomials of third and fourth kind.	Published **DOI:** https://doi.org/10.26713/cma.v14i1.2079
5	Indian Journal of Science & Technology (Web of Science)	Some more Identities on sums of Finite Product of the Pell, Fibonacci and Chebyshev Polynomials	Published https://indjst.org/articles/some-identities-on-sums-of-finite-products-of-the-pell-fibonacci-and-chebyshev-polynomials
6	Notes on Number Theory and Discrete Mathematics (Web of Science)	Generalised Trivariate Fibonacci and Lucas polynomials and their identities	Communicated

List of Conferences

S.No	Name of the Journal	Title of the Paper	Date
1.	National conference on Integrated Approach in science and Technology for sustainable Future (IAST-F) organised by MAM, college Jammu (UT of J&K)	Some identities involving Chebyshev Polynomials, Fibonacci Polynomials and their derivatives	27-28 Feb 2022
2.	International Conference on Mathematical and statistical Computation (ICMSC-2022) organised by department of Mathematics, SKITMG, Jaipur, Rajasthan.	Sums of Finite Product of Chebyshev Polynomials of Third and Fourth Kind and Fibonacci and Lucas Numbers	March 3-5, 2022
3.	International Conference on Fractional Calculus: Theory, Applications and Numeric (IFCTAN-2023) Organised by NIT, Puduchchery, Karaikal	Some Identities on Finite Sums of Product of Fibonacci type Numbers and Polynomials	27-29, Jan 2023

Certificates of Presentation

National Conference on Integrated Approach in Science and Technology for Sustainable Future (IAST-SF)
27-28 February 2022

CERTIFICATE

This is to Certify that Mr./Ms./Dr./Prof. Jugal Kishore
from Lovely Professional University, Punjab *has delivered an oral presentation on*
some identities on Chebshev Polynomial, fibonacci polynomial and their derivatives

in the two day National Conference on "Integrated Approach in Science and Technology for Sustainable Future" held at Govt. MAM college, Jammu on 27- 28 Feb., 2022.

Prof. G.S Rakwal Principal MAM College, Jammu	Prof. Ranvijay Singh Principal GDC Ukhral, Ramban	Dr. Vishal Sharma Convener IAST-SF 2022 MAM College, Jammu	Prof. J.S Tara Convener ISCA Jammu Chapter

Organized by
Maulana Azad Memorial College Jammu, &
Govt. Degree College Ukhral, Ramban, J&K

In Collaboration with
Indian Science Congress Association
Jammu Chapter

International Conference
on
Mathematical and Statistical Computation
(ICMSC-2022)
3-5 March, 2022

Organized by
Department of Mathematics
Swami Keshvanand Institute of Technology,Management & Gramothan
Jaipur(Rajasthan), India
In association with Rajasthan Academy of Physical Sciences

This Certificate is presented to
Jugal Kishore
Lovely Professional University, Phagwara, Punjab

for presenting a paper entitled **"Sums of Finite Product of Chebyshev Polynomials of Third and Fourth Kind and Fibonacci and Lucas Numbers"** in ICMSC-2022 held during 3-5 March, 2022.

Dr. S. L. Surana Director (Academics)	Dr. Ramesh Kumar Pachar Principal	Dr. Sangeeta Choudhary Convenor	Dr. Shalini Shekhawat Convenor

IFCTAN 2023

Springer

UNIVERSAL WISER PUBLISHER

International Conference on Fractional Calculus: Theory, Applications and Numerics

27−29 January, 2023

Organized by

Department of Mathematics
National Institute of Technology Puducherry, Karaikal

Certificate of Appreciation

This is to certify that **Mr. JUGAL KISHORE**, Lovely Professional University has participated in the International Conference on Fractional Calculus: Theory, Applications & Numerics (ICFCTAN) held during January 27-29, 2023 conducted by National Institute of Technology Puducherry, Karaikal and presented a paper entitled **New identities on finite sums of product of fibonacci type numbers and polynomials** through virtual mode.

Dr. V. Govindaraj
(Organizing Chairman)

Prof. K. Sankaranarayanasamy
(Director, NIT Puducherry)

List of Conferences/workshops attended

S.No	Name of the Conference/Workshop	Title of the Paper	Date
1.	International Conference on Recent Advances in Fundamental and applied Sciences Organiesd by LPU, Jalandhar	Participated only	25-26, June 2021
2.	Attended One Week Workshop on mathematical analysis organised by Loyola College, Chennai.	Attended only	13-18, Sept 2021
3.	Attended international webinar on Mathematics of Computer Vision organised by M M (Deemed to be University) Mullana Amabala India	Attended only	28 April 2022
4.	Attended International Webinar on Mathematical modelling of Biology and Medicine, Vidhyasagar Metropolitan College	Attended only	13 May 2022
5.	Attended one Week Workshop on Discrete Mathemtics, Mathematical Modelling and probability Probability and	Attended only	14-20, June 2022

	Statistics organised by Calcutta Mathematical Society, Kolkatta.		
6.	Attended 15 days international FDP on Frontiers of Mathematics organised by SRM Institute of Science and Technology, Chennai.	Attended only	19.05.2022 - 02.06.2022
7.	Attended One day workshop on “Scholarly Publishing: Do’s and Don’t’s” organised by Tumkur University	Attended only	16 July 2022

Certificate No. 227587

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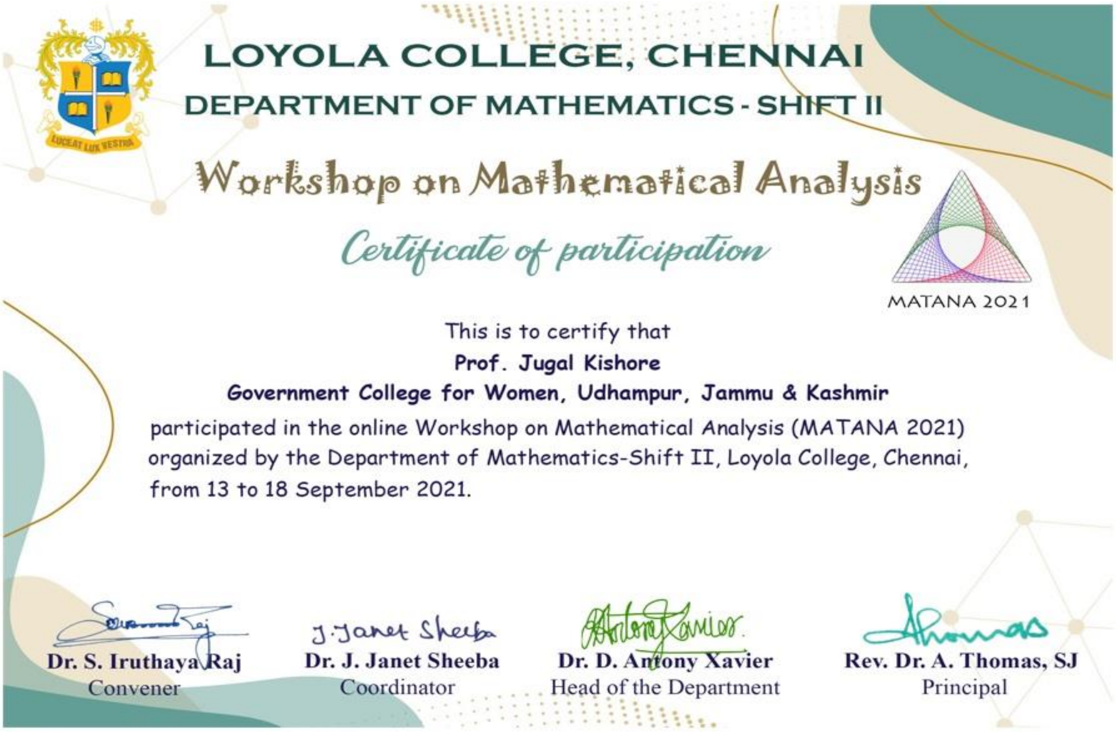
RAFAS

Certificate of Participation

This is to certify that Mr. Jugal Kishore of Lovely Professional University has participated in the International Conference on "Recent Advances in Fundamental and Applied Sciences" (RAFAS 2021) held on June 25-26, 2021, organized by School of Chemical Engineering and Physical Sciences, Lovely Faculty of Technology and Sciences, Lovely Professional University, Punjab.

Date of Issue : 09-08-2021
Place of Issue: Phagwara (India)

Prepared by (Administrative Officer-Records) | Organizing Secretary (RAFAS 2021) | Convener (RAFAS 2021)



LOYOLA COLLEGE, CHENNAI

DEPARTMENT OF MATHEMATICS - SHIFT II

Workshop on Mathematical Analysis

Certificate of participation

MATANA 2021

This is to certify that

Prof. Jugal Kishore

Government College for Women, Udhampur, Jammu & Kashmir

participated in the online Workshop on Mathematical Analysis (MATANA 2021) organized by the Department of Mathematics-Shift II, Loyola College, Chennai, from 13 to 18 September 2021.

Dr. S. Iruthaya Raj	Dr. J. Janet Sheeba	Dr. D. Antony Xavier	Rev. Dr. A. Thomas, SJ
Convener	Coordinator	Head of the Department	Principal

MAHARISHI MARKANDESHWAR

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(Established under Section 3 of the UGC Act, 1956)

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www.mmumullana.org

Certificate

OF PARTICIPATION

This is to certify that Jugal Kishore, Government College for Women,Udhampur has attended an international webinar on **"Mathematics of Computer Vision"** held on April 28, 2022 organized by **Department of Mathematics and Humanities, M. M. (Deemed to be University), Mullana-Ambala, India.**

Dr. Aliya Naaz Siddiqui	Dr. Kamran Ahmad	Dr. Deepak Gupta
Co-convener	Convener	Head of Department

INTERNATIONAL WEBINAR
ON
MATHEMATICAL MODELLING OF BIOLOGY AND MEDICINE

CERTIFICATE OF PARTICIPATION

This is to certify that Mr Jugal Kishore, Assistant Professor of Government College for Women, Udhampur has participated in the International Webinar on "MATHEMATICAL MODELLING OF BIOLOGY AND MEDICINE" organized by Department of Mathematics, Vidyasagar Metropolitan College on the 13th May, 2022 from 7:00 pm to 8:00 pm.

Dr. Mohsin Islam
Convenor
Vidyasagar Metropolitan College

Dr. Ram Swarup Gangopadhyay
Principal
Vidyasagar Metropolitan College

Calcutta Mathematical Society

AE-374, Sector I, Salt Lake City, Kolkata - 700064, WB, India

Workshop on Discrete Mathematics, Mathematical Modelling and Probability & Statistics (WDMMMPS-2022)
(One Week Faculty Development Programme)

14-20 June, 2022

Certificate of Participation

This is to certify that **Mr. Jugal Kishore** of **Government College for Women, Udhampur** participated in the **Workshop on Discrete Mathematics, Mathematical Modelling and Probability & Statistics (WDMMMPS-2022), One Week Faculty Development Programme,** of total **thirty six hours duration** *organised by* **Calcutta Mathematical Society**, Kolkata, India, during **14-20 June 2022.**

Dr. Kalyan Halder
Jt. Convenor

Dr. Gokul Saha
Jt. Convenor

Prof. Arindam Bhattacharyya
Secretary, Calcutta Mathematical Society

Prof. Rasajit Kumar Bera
President, Calcutta Mathematical Society

RAMAPURAM SRM

SRM Institute of Science and Technology

Ramapuram Campus, Chennai, INDIA

DEPARTMENT OF MATHEMATICS

in association with

"1 DECEMBRIE 1918 UNIVERSITY" OF ALBA IULIA

ALBA IULIA, ROMANIA

UAB

15 days Virtual International FDP

FRONTIERS OF MATHEMATICS

E - Certificate of Participation

This is to certify that **Jugal Kishore, Assistant Professor in Mathematics** from **Government College for Women,Udhampur** has attended Fifteen Days International Virtual FDP "Frontiers of Mathematics" organised by the Department of Mathematics, SRMIST, Ramapuram, Chennai, India in association with "1 Decembrie 1918 University" of Alba Iulia, Romania from 19.05.2022 to 02.06.2022.

Dr. Shakeela Sathish
Prof & Head, Dept of Mathematics
SRMIST- Ramapuram,Chennai

CERTIFICATE ID
SRMIST - FDP -000158

Prof. Daniel Breaz
Rector, "1 Decembrie 1918 University"
Alba Iulia, Romania

TUMKUR UNIVERSITY

Department of Studies & Research in Business Administration

In association with

Management Research Forum

CERTIFICATE OF PARTICIPATION

This is to Certify that

Jugal Kishore

Asst.Prof, Tumkur Government College for Women ,Udhampur

participated in the Workshop on

"Scholarly Publishing: Dos and Don'ts" held on Saturday 16th July 2022

Dr. K. shivachithappa Registrar Tumkur University, Tumkur	Dr.Noor Afza Professor & Chairperson DOS&R in Business	Dr. S. Sathyeshwar Secretary- MRF

LOYOLA COLLEGE (AUTONOMOUS), CHENNAI-34
DEPARTMENT OF MATHEMATICS

LUCEAT LUX VESTRA

International Conference on Recent Trends in Applied Mathematics (ICRTAM 2022)

March 3 & 4, 2022

Certificate

This is to certify that Mr. / ~~Ms.~~ / ~~Dr.~~ / ~~Prof.~~ .... **Jugal Kishore, Government College for Women, Udhampur, Jammu & Kashmir** .... presented a paper in the International Conference on Recent Trends in Applied Mathematics (ICRTAM 2022) organized by the Department of Mathematics, Loyola College, Chennai on March 3 & 4, 2022, titled ... **Some Identities on Sums of Finite Product of Fibanacci and Lucas Numbers in Chebyshev Polynomials of Second Kind**

Dr. D. Antony Xavier
Convener

Rev. Dr. A. Thomas SJ
Principal

National Multi Disciplinary Conference
on
"Recent Trends in Agriculture, Bio Sciences, Computer Applications, Environment & Humanities

Certificate ID: GDCB/Conf/22/33

ORGANIZED BY
GOVERNMENT DEGREE COLLEGE BILLAWAR

CERTIFIED

Certificate

This is certify that Mr/Miss/Mrs/Dr Jugal Kishore Designation ________
Institute/College/University LPU Phagwara has participated in a National Conference on "Recent Trends in Agriculture, Bio Sciences, Computer Applications, Environment & Humanities" (RTABCEH-2022) on 24th of March 2022 as Chairman/Co Chairman Rapporteur/Delegate/Invited Speaker and /Presented Paper/Poster entitled Some Identities on Sums of Finite Product of Fibonacci And Lucas Numbers in Chebysheb Polynomials of Second Kind
We Wish him/her all success in life.

Dr. Shamim Ahmed Banday
Organising Secretary

Prof. Lekh Raj
Convener

Prof. Anita Jamwal
Principal

BABA GHULAM SHAH BADSHAH UNIVERSITY RAJOURI
Jammu & Kashmir, INDIA

INTERNATIONAL CONFERENCE ON MATHEMATICAL ANALYSIS & APPLICATIONS
March 30 & 31, 2022

CERTIFICATE

Session Coordinators

Dr. Javid Iqbal

Dr. Naveen Sharma

This is to certify that **Jugal Kishore of Government College for Women, Udhampur** has attended the "International Conference on Mathematical Analysis & Applications" organised by the Department of Mathematical Sciences, Baba Ghulam Shah Badshah University Rajouri, w. e. f. March 30, 2022 to March 31, 2022, and presented a paper entitled **Some identities on Sums of Finite Product of Fibonacci and Lucas Numbers in Chebyshev Polynomials of second Kind**.

Convener & Head of the Department

Dr. Zaheer Abbas

ICMAA

GOVT. DEGREE COLLEGE BILLAWAR
TWO DAY INTERNATIONAL CONFERENCE
NOVEMBER 25-26, 2022

CERTIFICATE

No: GDCB/2022/1954
Date: 31-12-2022

This certificate is awarded to Prof./Dr./Mr./Ms. Jugal Kishore, Assistant Prof./ Research Scholar of GCW, Udhampur/ LPU, Phagwara for his/her Invited Lecture/Chairperson/Co-Chairperson/Adjudicator/Participation/Presentation of Research Paper (Oral/Poster) entitled "Some Identities on Chebyshev Polynomial of Third and Fourth Kind" in TWO DAY INTERNATIONAL CONFERENCE on "ETHNOBIOLOGY: PRESENT AND FUTURE PERSPECTIVES "held at Govt. Degree College Billawar on November 25-26, 2022.

PROF. YASH PAUL
Organizing Secretary

PROF. LEKH RAJ
Co-Convener

PROF. ANITA JAMWAL
Principal and Convener

SAINT PATRICK
PATRON

PATRICIAN COLLEGE OF ARTS AND SCIENCE

Affiliated to the University of Madras & Re-accredited 'A+' Grade by NAAC in 2021
Ranked 1st in TN and 18th among the Non-Autonomous Colleges in India (EW)
Awarded 4 Star with Mentor Status by Innovation Cell, MoE, Govt. of India

BISHOP DANIEL DELANY
FOUNDER

INTERNATIONAL CONFERENCE ON RESEARCH TRENDS IN CONTEMPORARY MATHEMATICS (ICRTCM 2023)

Certificate of Participation

This is to certify that JUGAL KISHORE of LOVELY PROFESSIONAL UNIVERSITY has ~~Participated~~ / Presented a Paper on the Title NEW IDENTITIES OF FINITE SUMS OF PRODUCT OF FIBONACCI TYPE NUMBERS AND POLYNOMIALS during the International Conference on Research Trends in Contemporary Mathematics(ICRTCM 2023) organized by P.G Department of Mathematics, Patrician College of Arts and Science, Chennai in colloboration with PG & Research Department of Mathematics, Sacred Heart College, Tirupattur on 3rd & 4th February 2023.

Dr. S.Sriram
Convener

Dr. Usha George
Principal