SEMI-ANALYTICAL METHODS FOR SOLUTION OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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By

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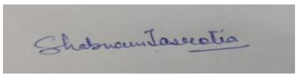
DECLARATION OF AUTHORSHIP

I, hereby declared that the presented work in the thesis entitled **“Semi-Analytical Methods for solution of Nonlinear Partial Differential Equations”** in fulfilment of degree of **Doctor of Philosophy (Ph.D.)** is outcome of research work carried out by me under the supervision Dr. Prince, working as Associate Professor, in the Department of Mathematics, School of Chemical Engineering and Physical Sciences of Lovely Professional University, Punjab, India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.



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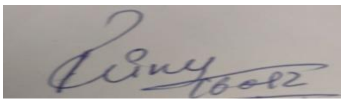
<u>CERTIFICATE</u>

This is to certify that the work reported in the Ph.D. thesis entitled **" Semi-Analytical Methods for solution of Nonlinear Partial Differential Equations**" submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the Department of Mathematics, School of Chemical Engineering and Physical Sciences, is a research work carried out by **Shabnam Jasrotia, 41900659**, is bonafide record of her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.



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ABSTRACT

The thesis is concerned with enhancing the area of analytical techniques through the creation of a hybrid approach that blends integral transformation methods such as Laplace and other pertinent transformations. This enhancement in analytical methodologies is envisioned to facilitate more effective problem-solving within the domains of mathematical physics and engineering. Furthermore, our study aims to apply these newly discovered methodologies to complex physical issues, including PDEs and FDEs, as well as to a various boundary conditions and restrictions. This goal has wide-ranging practical implications, including fluid dynamics, heat transport, and materials science. Moreover, our research includes a thorough examination of error and convergence in numerical simulations of the investigated models. Furthermore, we strive to corroborate our findings by thorough comparisons with well-documented outcomes in the scientific literature. This essential validation stage establishes the dependability and precision of our suggested approaches, establishing them as invaluable instruments for future scientific investigations and academic activities.

Chapter 1 presents an in-depth analysis of differential equations. We begin by explaining the basic concepts related to these equations. In this context, we dive into the complexities of partial differential equations, categorizing them to emphasize their diversity. In addition, we discuss a different form of integral transformation that we have used in our research work and also define some properties of integral transformation. Further, we go through the historical evolution of the Homotopy Perturbation Method (HPM) in the later half of chapter 1, tracking its growth alongside numerous analytical approaches used to solve partial differential equations and fractional differential equations. This historical framework not only contextualizes our study, but also emphasizes the long-term importance of these techniques.

In chapter 2, the analysis delves into studies on PDEs, FDEs, and the utilization of semi-analytical methodologies such as the Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), and Homotopy Perturbation Sumudu Transformation Method (HPSTM), among others, which have been discussed by some authors and are relevant to our research. Each method's strengths, adaptability, and limitations in dealing with nonlinear and fractional equations are meticulously examined.

In **chapter 3**, the focus lies on solving nonlinear partial differential equations using a hybrid technique. An Accelerated Homotopy Perturbation Elzaki transformation Method (AHPETM)

is introduced for this purpose. The methodology aims to provide approximate series solutions. To evaluate their effectiveness, a rigorous comparison is conducted between the outcomes of this method, the exact solution, and approximate solutions found in available literature. The results are depicted using surface graphs and line graphs.

In Chapter 4, two hybrid techniques are employed for solving nonlinear PDEs. These techniques are the accelerated Homotopy Perturbation transformation method and the Accelerated homotopy perturbation Sumudu transformation Method. Various equations are examined, including one-dimensional (1D) and two-dimensional (2D) Burgers equations, as well as the 1D BBMB equation. These methods offer approximate series solutions. To gauge their effectiveness, a rigorous comparison is made between the outcomes of these techniques and both the exact and approximate solutions already aviliable in literature. The results are presented using surface graphs and line graphs.

Chapter 5 thoroughly explains the technique employed to address the research objectives of this study. The nonlinear fractional PDE in the Caputo sense is used to analyse a variety of equations, such as the Swift-Hohenberg equation, Inviscid Burger's equation, the F-W equation, and Fisher's equation. The technique is validated, and approximate solutions are obtained using rigorous computational procedures in Mathematica. The results are then presented into graphical representations, including surface and line graphs.

In **Chapter 6**, the nonlinear fractional PDE is utilized in the Caputo-Fabrizio sense to analyze various equations, such as the Burgers' equation, KdV equation, and K-G equation. In this chapter, AHPTM is employed to achieve the research goals in this study. Through rigorous computational procedures in Mathematica, approximate solutions are validated. The findings are then visually presented through graphical representations, including surface and line graphs.

In **Chapter 7**, the conclusions drawn from all chapters are presented.

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List of Abbreviations

ADM – Adomain Decomposition Method

C-F – Caputo-Fabrizio fractional derivative

DTM – Differential Transform Method

ET - Elzaki Transform

FC – Fractional Calculus

FEM – Finite Element Method

F-W - Forenberg-Whitham equation

HAM- Homotopy Analysis Method

HPETM - Homotopy Perturbation Elzaki Transformation Method

HPM - Homotopy Perturbation Method

HPSTM - Homotopy Perturbation Sumudu Transformation Method

HPTM - Homotopy Perturbation Transformation Method

KdV – Korteweg-de Vries

K-G - Klein-Gordon equation

LDM – Laplace Decomposition Method

LT – Laplace Transform

M-L – Mittag-Leffler

ODE – Ordinary Differential Equation

PDE – Partial Differential Equation

R-L – Riemann-Liouville

S-H - Swift-Hohenberg Equation

S-T – Sumudu Transform

VIM – Variation Iteration Metho

Chapter 1

1. INTRODUCTION

1.1 Preliminary

Differential Equation:

Differential equations have been a significant part of mathematical research, contributing to the growth of mathematics and its implications, particularly in the domains of physics and engineering. A differential equation is an expression in mathematics that includes independent and dependent variables, along with their derivatives. It specifically explains the relationship between the rate of change of a dependent variable 'y' and one or more independent variables 'x'.

$$\frac{dy}{dx} = f(x)$$

For example: $\frac{dy}{dx} = 3x + 2$

Differential equations are fundamental in modeling and understanding various phenomena in science, engineering, economics, and more. The study of the differential equations is divided into two categories.

1. Ordinary differential equation (ODE)
2. Partial differential equation (PDE)

1.1.1 Ordinary Differential Equation

If the unknown function of a differential equation depends only on one independent variable, the equation is defined as an ordinary differential equation. The generic form of the nth order ODE is presented as

$$\dot{F}(w, v, v' \dots \dots v^{n}) = 0; \qquad w \in I \tag{1.1.1}$$

Where v represent an unknown function with regard to the single independent variable w over an interval I and $\dot{F}$ is a specified function. The equation (1.1) is called the nth ordinary differential equation.

1.1.2 Partial Differential Equation

A PDE involves more than one independent variable, an unknown function (the dependent variable), and partial derivatives of the unknown function with regard to the independent variable. Such as

$$F\left(x_1, x_2, x_3 \dots x_n, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2} \dots \frac{\partial y}{\partial x_n}\right) = 0, \tag{1.1.2}$$

Here, $x_1, x_2, x_3 \dots x_n$ represents the independent variables, whereas y represents the dependent variable. Solving PDEs is a challenging yet rewarding task, often requiring a combination of analytical, numerical, and computational methods. But these equations are really important for describing things that happen in nature, like in science and engineering. As a result, it's important to understand the several conventional and contemporary approaches of solving these mathematical problems and using them practically. Fluid dynamics, electromagnetic, thermodynamics, quantum mechanics, and many other engineering and science fields make use of PDEs. PDEs are used in fluid dynamics to explain the movement of fluids and the effect of forces acting on them. They are used in electromagnetism to describe the electromagnetic field that occurs in nature. PDEs are also used to represent chemical processes and to describe the mobility of heat and sound waves. PDEs are employed in quantum mechanics to describe particle behaviour and interactions with other particles. PDEs may also be used to tackle engineering challenges such as optimum structure and system design.

The focus of our research being PDEs, we will provide a brief overview of them, including any conditions such as initial and boundary conditions.

1.2 Basic Definitions:

Definition 1.2.1 Order of PDE

The order of a PDE is determined by the highest order of the partial derivative present in the equation.

e.g. (1) $\varphi_x - \varphi_y = 0$, is the PDE of order one.

(2) $\varphi_{xx} - \varphi_{yy} = 0$, is the PDE of order two.

Definition 1.2.2 Linear PDE:

A linear PDE is one in which both the dependent variable and its partial derivative is linear.

e.g. (1) $\varphi_{xx} - \varphi_{yy} = 0$ (Laplace equation)

(2) $\varphi_t + V\varphi_x = 0$ (Transport equation)

Definition 1.2.3 Quasi-Linear PDE:

If a PDE is linear in the highest-order derivative of an unknown function, it is said to be quasi-linear.

e.g. (1) $\varphi_t = k\varphi_{xx}$ (Heat equation)

(2) $\varphi_{xx} = c^2\varphi_{yy}$ (Wave equation)

Definition 1.2.4 Semi-Linear PDE:

The equation is said to be semi-linear PDE if the coefficient of the highest order derivative of a PDE does not depend on the dependant variable or its derivatives.

e.g. (1) $\frac{\partial \varphi}{\partial t} + \varphi \frac{\partial \varphi}{\partial x} + 6\frac{\partial^3 \varphi}{\partial x^3} = 0$ (Korteweg-de Vries equation)

(2) $\frac{\partial \vartheta}{\partial t} + \frac{\partial \vartheta}{\partial x} + \vartheta^2 = 0$ (Transport equation)

Definition 1.2.5 Non-Linear PDE:

A PDE is considered to be non-linear if the derivative of the highest order term in its equation is dependent on the dependent variable.

e.g. (1) $\varphi_t - \varphi\varphi_x = \alpha\varphi_{xx}$ (Burgers equation)

(2) $\varphi_{tt} - \varphi_{xx} + 3(\varphi^2)_{xx} - \varphi_{xxx} = 0$ (Boussinesq equation)

Definition 1.2.6 Initial condition:

The initial condition is the state of the functions and their derivatives at the beginning of time $(t = 0)$. For e.g. in a wave equation, the initial condition might be the displacement of the wave and its speed at the same time$(t = 0)$.

Definition 1.2.7 Boundary condition:

A PDE is used to express the mathematical behaviour of physical processes within a given space. The dependent variable, *v*, is typically given at the edges of the domain, *D*. this boundary data is known as boundary conditions. Boundary circumstances are classified into three categories:

Definition 1.2.8 Dirichlet boundary condition:

A Dirichlet boundary condition is defined as a set of conditions that establish the determination of the dependent variable's value at the boundary (∂C) of a given domain. Imagine the boundary value problem as an example.

$$\frac{\partial^2 \vartheta}{\partial t^2} - c^2 \nabla^2 \vartheta = 0, x, y \in R, t \geq 0, \tag{1.2.1}$$

where $$B(\vartheta) = 0 \text{ on } \partial C \tag{1.2.2}$$

If $B(\vartheta) = 0$ denote for following boundary condition

$$\vartheta = 0 \text{ on } \partial C \qquad \text{(Dirichlet condition)}$$

Definition 1.1.9 Neumann boundary condition:

A Neumann boundary condition is described as a set of conditions that fulfil the value of the derivative of the dependent variable at the boundary (∂C) of a given domain. For instance, if the boundary conditions in the problem (1.3) are $B(\vartheta) = 0$ is of the form

$$\frac{\partial \vartheta}{\partial x} = 0 \text{ on } \partial C \qquad \text{(Neumann condition)}$$

Definition 1.1.10 Robin boundary condition:

A Robin boundary condition, also referred to as mixed boundary conditions, is defined as a set of conditions that involve both the dependent variable and its derivative at the boundary (∂C) of a given domain.

For instance, if the boundary conditions in problem (1.3) are $B(\vartheta) = 0$ is of the form

$$\frac{\partial \vartheta}{\partial x} + \vartheta = 0 \text{ on } \partial C \qquad \text{(Robin condition)}$$

1.3 Fractional Calculus [FC]

The expansion of classical calculus to FC is a significant advancement that has opened up an endless number of new study options, with implications for both mathematics in general and particular topics or subjects. FC may be regarded of as an extension of classical calculus. FC is an area of mathematics that focuses on the study of differentiating and integrating functions at fractional orders, expanding beyond the constraints of traditional integer-order calculus. It provides tools for solving differential and integral problems involving arbitrary orders, enabling a more nuanced analysis of systems and phenomena with fractional dynamics.

This branch of mathematics has been around since the beginning of calculus but has become increasingly important over the last few decades due to its wide application in fields such as fluid flow, viscoelasticity, probability, statistics, solid mechanics and signal processing.

The calculus of fractional order is an expansion of the traditional calculus developed by Newton and Leinbitz to include positive and negative fractions. This concept is as old as conventional calculus, which was developed by Newton in 1665 and Leibnitz in 1674. Subsequent to this, Euler and others extended the calculus to include higher-order derivatives. L'Hôpital addressed a letter to Leibniz in 1695 describing derivatives of non-integer orders, including half derivatives. On September 30th of the same year, Leibniz responded, noting that this may lead to a paradox from which important conclusions could be formed. Since then, many prominent mathematicians, including Liouville, Riemann, Weyl, Fourier, Grunwald, Letnikov, Abel, Lacroix, and Leibniz, have made a contribution to the study of FC.

In 1823, Niels Abel presented the first ever applications of fractional calculus by solving the tautochrone problem. This was followed by Joseph Liouville's papers from 1832-1837, in which he developed a similar theory of fractional operators. Riemann also worked independently on the same, coming up with the RL fractional derivative, which is achieved by an integral approach.

Grunwald and Letnikov (1867-68) proposed the concept of “differ-integral” as a generalization of the definition of an integer order derivative, which has been referred to as the limit of the difference quotient. This definition is an important milestone in fractional calculus, as it is algorithmic and thus useful in computations involving fractional derivatives. In 1967, Michele Caputo proposed a different formulation of fractional derivatives which is more useful for practical applications as it requires initial conditions in terms of ordinary derivatives rather than fractional derivatives, which have an unclear physical interpretation. Therefore, Caputo derivatives have gained much attention in recent years.

Fractional calculus has gained popularity in recent years mainly because of its application in a variety of scientific and technical domains, including electronics and signal processing, neural networks, bioengineering, cryptography, image processing, fluid mechanics, and electrodynamics. In contrast to their integer-order counterparts, fractional-order models have been seen to deliver improved results due to their inherently non-local nature, which allows them to take into account memory effects more accurately. As such, FC is now widely used in many domains of science and technology. Since then, important mathematicians have suggested various types of definitions of fractional-order derivatives, some of which are well-known, including the Grunwald-Letnikov, RL, and Caputo derivatives.

1.3.1 Definition: Grunwald-Letnikov: The Grunwald-Letnikov fractional derivative is a concept in mathematics that extends the theory of differentiation to non-integer orders. The mathematicians Anton Karl Grunwald and Aleksey Vasilievich Letnikov separately developed it around the beginning of the 20th century.

$$D^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} f(t - nh)$$

Where $n \in \mathbb{N}$, and the Gamma function is used to determine the binomial coefficient.

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}$$

1.3.2 Riemann-Liouville [R-L]: The basic ideas of differentiation and integration are expanded to non-integer orders using a mathematical framework called RL fractional calculus. Since its development in the 19th century by the mathematicians Bernhard Riemann and Joseph Liouville, it has demonstrated its utility across a multitude of academic disciplines, including but not limited to engineering, physics, and signal analysis. The process of determining the rate at which a function changes in relation to its independent variable is known as differentiation in classical calculus. The area under a curve is sought after during integration, on the other hand. The definition of these operations is restricted to integer orders, such as first-order (derivative) and second-order (integral). The differentiation and integration of non-integer orders is a feature of Riemann-Liouville fractional calculus, which generalizes these operations. Fractional derivatives and integrals can be defined for any real or complex number, rather than just taking into account integer values for the order.

An operator of fractional-order derivative is used on a function to obtain its fractional derivative. This operator generalizes the idea of differentiation to non-integer orders. For example, the fractional derivative operator D^(1/2) represents a derivative of order 1/2, indicating a fractional number of derivative operations applied to the function. The function having a fractional order is iteratively integrated to produce the fractional integral. There are numerous intriguing characteristics of the RL in fractional calculus. The fractional chain rule, according to which the derivative of a composite function may be represented in the context of the fractional derivatives of its constituent components, is one of the important features. Due to this fact, many mathematical identities and formulas may be used in fractional calculus. Additionally, fractional calculus offers a potent tool for modelling memory- or long-range-dependent systems. It is employed in the study of fractional differential equations, anomalous diffusion, and viscoelasticity.

RL can be defined as

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\eta)}{(t-\eta)^{\alpha-n+1}} d\eta, (n-1) \leq \alpha < 1.$$

The concept of R-L definition was developed by Leibniz in 1690, which was further extended by Letnikov in four papers from 1868 to 1872. Riemann formulated his theory of fractional integration as a student, but he didn't publish it during his lifetime. Instead, it was posthumously included in his Gesammelte Werke, published in 1892. The generalisation of Leibniz's *n*th derivative of product when *n* is not a positive integer was further discussed by C.J. Hargreave and Liouville.

1.3.3 Caputo fractional derivative: The Caputo fractional derivative of a function f of order α is defined as

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\eta)}{(t-\eta)^{\alpha-n+1}} d\eta, (n-1) \leq \alpha < 1.$$

1.3.4 Caputo-Fabrizio Fractional derivative[C-F]: A novel fractional methodology was presented in 2015 by Caputo and Fabrizio. This concept attracted attention because it was necessary to explain a class of non-local systems that could not be adequately characterized by fractional models with a solitary kernel.

$$D_t^{\alpha} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f(\eta) \exp\left[-\frac{\alpha(t-\eta)}{1-\alpha}\right] d\eta, (n-1) \leq \alpha < 1.$$

$M(\alpha)$ denote the normalization function such that $M(0) = M(1) = 1$.

Definition 1.3.5 Mittag-Leffler function [M-L function]: The M-L function serves as a generalization of the exponential function, playing a crucial role in both fractional calculus (FC) and fractional modeling. The M-L function [54] expressed as

$$E_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(\alpha m+1)}, \alpha > 0.$$

Definition 1.3.6 Gamma function: In the 18th century, Bernoulli and Goldbach, renowned mathematicians, endeavoured to extend the factorial function $m!$, where m is a natural number. Nevertheless, it was Euler (1707–1783), a Swiss mathematician, who ultimately achieved success in this pursuit. [55].

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt.$$

1.4 Integral Transform:

Integral transform have a long history that began in the 18th century with the invention of mathematical methods for analyzing functions in many fields and solving differential equations. As mathematical understanding progressed, the concept of converting functions via integral operations developed, leading to the development of a variety of integral transforms and their uses in diverse areas of mathematics. One of the oldest examples of integral transforms may be found in the early 19th-century work of Jean-Baptiste Joseph Fourier. Fourier developed the Fourier series, which condensed trigonometric function to express periodic functions. Functions in the frequency domain can be studied using this form to determine their harmonic components. Another notable advancement in integral transforms occurred in the middle of the 19th century because to the work of Joseph Liouville and the pupil Michel Chasles. The French mathematician Augustin Louis Cauchy's pupil Emile Mellin furthered their research and created what is known as the Millin transform. Different mathematicians and scientists developed the theory and uses of integral transforms during

19th and 20th centuries. Application for these transformations may be found in tomography, wave propogation analysis, and picture reconstruction, among other fields. Since then, the study of integral transforms has developed, and it has been used in several areas of mathematics, physics, engineering, and other disciplines. Integral transforms are now regarded as important tools for mathematical analysis, offering strong methods for problem-solving, function analysis, and mathematical structure exploration.

1.4.1 Laplace Transform [LT]:

The Laplace transform, which Pierre-Simon Laplace created in the late 18th century, made a substantial addition to integral transforms. Laplace used this technique to analyze systems with time-dependent variables and to solve differential equations. By converting differential equations into algebraic equations, the Laplace transform offers a potent tool for streamlining the solution of these problems.

The Laplace transform is defined as

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\ dt.$$

Properties

1. $\mathcal{L}\{1\} = \frac{1}{s}$
2. $\mathcal{L}\{t^p\} = \frac{p!}{s^{p+1}}, p\ a\ positive\ integer.$
3. $\mathcal{L}\{\sin pt\} = \frac{p}{p^2+s^2}$
4. $\mathcal{L}\{\cos pt\} = \frac{s}{p^2+s^2}$

1.4.2. Sumudu Transform [ST]:

G. K. Watugala [176] presented the Sumudu transformation 1993. It is defined as a transformation where differentiation and integration operations in the time domain (t-domain) correspond to division and multiplication by a variable u in the transform domain (u-domain). The Sumudu transform is defined as follows:

$$S[f(t)] = \frac{1}{u}\int_0^\infty f(t)\, e^{-\frac{t}{u}}\, dt, t > 0.$$

Some basic properties are

1. $\text{S}\{1\} = 1$
2. $\text{S}\{t\} = u$
3. $\text{S}\{t^n\} = n!\, u^n$
4. $\text{S}\{e^{at}\} = \frac{1}{1-au}$
5. $\text{S}\{\sin(at)\} = \frac{au}{1+a^2u^2}$
6. $\text{S}\{\cos(at)\} = \frac{u}{1+a^2u^2}$

1.4.3. Elzaki Transform [ET]:

Tarig Elzaki invented the Elzaki Transform in 2011 as a tool for solving ODEs and PDEs in the time domain. The Elzaki transform is represented by the integral equation's operator E (.).

$$E[f(t)] = v \int_0^\infty f(t)\, e^{-\frac{t}{v}}\, dt, t > 0.$$

Some basic properties are

1. $E\{1\} = v^2$
2. $E\{t\} = v^3$
3. $\text{E}\{t^n\} = n!\, v^{n+2}$
4. $\text{E}\{e^{at}\} = \frac{v^2}{1-av}$
5. $\text{E}\{\sin(at)\} = \frac{av^3}{1+a^2v^2}$
6. $\text{E}\{\cos(at)\} = \frac{v^2}{1+a^2v^2}$
7. $\text{E}\{f'(t)\} = \frac{T(v)}{v} - vf(0), if \;\; \text{E}\{f(t)\} = T(v).$

1.5 Semi-Analytical and Numerical Techniques for Non-Linear PDEs

The solution of non-linear PDEs poses substantial challenges due to their complex nature, often lacking analytical solutions. In order to properly address these issues, researchers are now focusing on numerical and semi-analytical methodologies. These techniques provide useful insight on how to resolve challenging issues in science and engineering with increased precision and effectiveness. The solutions to non-linear, coupled, partial equation and fractional differential equations have been found using a number of semi- analytical methods, including

1. Homotopy Perturbation Method [HPM]
2. Homotopy Analysis Method [HAM]
3. Adomain Decomposition method [ADM]
4. Variation Iteration Method [VIM]
5. Laplace Decomposition Method [LDM]
6. Finite Element Method [FEM]
7. Differential Transform Method [DTM]
8. Homotopy Perturbation Transformation Method [HPTM]
9. Homotopy Perturbation Sumudu Transformation Method [HPSTM]
10. Homotopy Perturbation Elzaki Transformation Method [HPETM]

In this research endeavor, our primary emphasis lies on investigating methodologies employing the HPM. The subsequent section will provide an in-depth elucidation of the HPM and a thorough description of HPM variations, namely HPTM, HPETM and HPSTM.

1.5.1 Homotopy

Homotopy is a basic idea in topology that deals with the continuous transformation of one topological space into another. It offers a method for comprehending the "shape" of spaces without paying attention to more minute geometrical features. A homotopy, by definition, is a smooth transformation of one map into another that keeps the endpoint at each moment. Alternatively, the maps f and homotopy capture the continuous deformation of g into each other that can occur without any point being torn or adhered. Homotopy offers a method for categorizing topological spaces according to their "homotopy type." Homotopy equivalent spaces have the same homotopy type, which denotes that they may continuously deform into one another. When separating spaces that have various topological aspects but nevertheless have some qualitative traits, this idea is especially helpful.

A continuous transformation that smoothly connects f and g is described as a homotopy between two continuous functions s and t defined on a topological space R and taking values in another topological space S. Formally, s and t consider homotopic if $\exists$ a continuous function $H: R \times [0,1] \to S$ such that $\forall\ x \in R, H(x,0) = s(x)$ and $H(x,1) = t(x)$. The function H is commonly referred to as a homotopy between s and t, and it represents a continuous deformation or path between the two functions over the interval [0,1].

1.5.2 Perturbation Theory

The use of perturbation techniques requires the presence of small or large parameters or variables, often known as perturbation quantities. The conversion of nonlinear problems into linear sub-problems using an infinite sequence of terms, approximated by solving the initial sub-problems, is a significant application of extensive perturbation techniques.

However, perturbation techniques have several restrictions. The technique is severely constrained since not all nonlinear problems can be described by a perturbation quantity. Additionally, specialized analytic approximations for nonlinear issues frequently fail as nonlinearity grows stronger, making perturbation approximations only useful for nonlinear problems with nonlinearity discontinuities.

1.6 Homotopy Perturbation Method

The HPM, presented by Ji Huan He in 1999 and further refined in 2000, combines the principles of Homotopy theory and perturbation theory to solve nonlinear problems. Homotopy theory, derived from algebraic topology, involves continuous deformation of topological space. In HPM, a homotopy boundary is defined by an embedding parameter "$p \in [0,1]$", where "p" represents a small parameter. When "p" is set to 0, the issue is simplified, resulting in a comparatively simple solution. As "p" approaches 1, the system experiences a series of deformations, each of which closely resembles the preceding stage. At $p = 0$, the original problem is restored, and the desired solution is obtained.

Notably, the HPM does not rely on a minor parameter in the equations, thereby eliminating the drawbacks of additional perturbation methods. This feature distinguishes the HPM from other techniques proposed in the late 1990s that aimed to solve nonlinear equations without a small parameter.

The HPM has proven advantageous in various scientific and engineering computations. Its ability to bypass the requirement for a tiny parameter in the equations eliminates any restrictions of conventional perturbation approaches. Additionally, the computational procedures involved in the HPM are straightforward and direct.

Ji-Huan He and other authors have successfully employed the Homotopy Perturbation Method (HPM) to analyze the nonlinear vibration behavior of N/MEMS (Nano/Micro-Electro-Mechanical Systems) oscillators, including the Duffing oscillator, the Fangzhu oscillator, and nonlinear oscillators with coordinate-dependent mass. Ordinary differential equations (ODEs) [1, 2, 3, 4] can be used to characterise these systems. The use of HPM with Partial Differential Equations (PDEs) has also been thoroughly investigated by researchers [5-8, 11-12]. For instance, Kaur G., et al. [9] used HPM to solve population balance equations including fragmentation and aggregation, while Gupta S., et al. [10] used it to resolve convection-diffusion equations. Utilising HPM, Jassim H. K. [13] was able to solve the Newell-Whitehead Segel problem.

Additionally, using the HPM, fractional equations, including those with fractional derivatives, is successfully solved [14-29]. Yasir Khan et al. broadened the utilisation of HPM to obtain analytical solutions for the Klein-Gordon fractional PDE [15]. The fractional Black-Scholes equation was investigated by Asma Ali Elbeleze et al. [16], and Chun-Fu WEI employed HPM to solve non-linear and singular fractional Lane-Emden type equations [19]. Additionally, with amazing effectiveness, the HPM has been expanded to address additional mathematical models including integral equations and delay differential equations [30-44]. To explore linear Volterra integral equations with discontinuous second-kind kernels, Samad Noeiaghdam et al. used HPM [45].

Furthermore, Fatemeh Shakeri and Mehdi Dehghan provided numerical examples to illustrate their solutions to delay differential equations using HPM [31].

To demonstrate the core concept behind this approach, we will use a general nonlinear differential equation.

$$K(\vartheta) - f(s) = 0, s \in \Omega \qquad (1.6.1)$$

with boundary condition

$$B\left(\vartheta, \frac{\partial \vartheta}{\partial n}\right) = 0,\ s \in \Gamma \qquad (1.6.2)$$

Here, K represent a differential operator of a general nature, B is an operator associated with the boundary, $f(s)$ is an analytic function and Γ denotes the boundary of the domain Ω. The operator K usually has two parts: R and P. R is for the linear part, and P is for the nonlinear part.Equation (1.6.1) may so be expressed as below.

$$R(\vartheta) + P(\vartheta) - f(s) = 0 \quad (1.6.3)$$

By the Homotopy technique, we generate a homotopy $\varphi(s,p) : \Omega \times [0,1] \to \mathbb{R}$ [84], which satisfies.

$$H(\varphi,p) = (1-p)[R(\varphi) - R(\vartheta_0)] + p[K(\varphi) - f(s)] = 0, \ \ p \in [0,1], s\epsilon\Omega$$

or $$H(\varphi,p) = R(\varphi) - R(\vartheta_0) + pR(\vartheta_0) + p[P(\varphi) - f(s)] = 0 \quad (1.6.4)$$

where p is parameter such that $0 \le p \le 1$, u_0 is an initial approximation of equation (1.6.1), which satisfies the boundary condition, we have

$$H(\varphi, 0) = R(\varphi) - R(\vartheta_0) = 0$$

$$H(\varphi, 1) = K(\varphi) - f(s) = 0,$$

The procedure of changing of p from zero to one is like transforming $\varphi(s,p)$ from $\vartheta_0(s)$ to $\vartheta(s)$. The main idea behind this method is that we can represent the solution of equation (1.6.1) as

$$\varphi = \varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots \quad (1.6.5)$$

The approximate solution of the equation is found by setting equal p to 1.

$$\vartheta = \lim_{p\to 1} \varphi = \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \cdots$$

The Homotopy Perturbation Method combines the Perturbation method and the Homotopy method. It helps to avoid the limitations of traditional perturbation methods while still retaining their advantages.

In general, the HPM proves to be a versatile and efficient approach for solving various types of mathematical problems, including fractional equations, integral equations, and differential equations, delay differential equations, and both ordinary and PDE. It is a useful tool for practitioners and researchers across a range of academic and scientific disciplines due to its efficacy and broad applications.

1.7 Existing Methods in the literature

1.7.1 Homotopy Analysis Method [HAM]:

The HAM is a semi-analytical approach employed to solve linear and nonlinear differential equations. In order to generate precise or convergent series-based solutions, it combines the traditional perturbation approach with topological principles. Both ODE and PDE have been used to illustrate the effectiveness of HAM. Liao originally introduced an approximation scheme within his doctoral dissertation, which served as a means to ascertain series solutions for nonlinear problems. Subsequently, he refined and expounded upon this methodology, culminating in the publication of the book titled "Beyond Perturbation: Introduction to the Homotopy Analysis Method" in 2003[46]. Abbasasbandy and Shivanian [47] successfully solved a difficult linear vibrational boundary value problem with singularity by using the HAM. The results of Ganjiani's [48] application of HAM to a nonlinear fractional differential equation showed excellent congruence with precise solutions. This technique has attracted attention because of its favourable potential in calculation pertaining to quantum field theory and quantum statistical mechanics [51]. Furthermore, HAM [52] was successfully used to obtain analytical solutions controlling the radial oscillations of a multielectron bubble encased in liquid helium. The generalized Benjamin-Bona-Mahony equation and other problems requiring series based solutions are addressed by HAM in a novel way [53]. Notably, the practitioner can regulate the convergence zone for the series solution by using an auxiliary parameter. This achievement demonstrated the method's effectiveness in tackling complex, severly nonlinear dynamical systems. Along with these successes, the research of Hassami et al. [49] and Nadeem and Lee [50] demonstrated the value of HAM in tackling the difficulties associated with boundary layer nanofluid flow across stretched surfaces. The predictor homotopy analysis method (PHAM), which expands the capabilities of the HAM to capture numerous solutions of nonlinear differential equations, also helped to expand the frontiers of the technique.

1.7.2 Homotopy Perturbation Sumudu Transformation Method [HPSTM]:

HPSTM is a technique for solving nonlinear equations that integrates the ST in combination with the HPM. Round-off errors are avoided by this method since it does not need discretization or restrictive assumptions. The HPSTM offers a clear and effective solution for a variety of nonlinear conditions by utilizing He's polynomials to control nonlinear terms. Singh et al. concentrated on studying solutions for both linear and nonlinear partial differential equations using the HPSTM. Equations containing fractional derivatives or integrals are particularly well-suited for its use. With regard to both linear and nonlinear K-G Equations, HPSTM has proven to be effective in solving initial value boundary problems.

1.7.3 Homotopy Perturbation Elzaki Transformation Method [HPETM]:

Tarig M. Elzaki and J. Biazar developed this approach in 2013. The suggested approach was developed by merging the ET with the HPM. This approach has been recognised for its high efficiency and simplicity in solving linear and nonlinear differential equations. Furthermore, it presents certain advantages over previous semi-analytical methodologies such as the HAM and the ADM, in that it avoids the requirement to construct Adomain Ploynomials and convergence parameters.

1.7.4 Homotopy Perturbation Transformation Method [HPTM]:

The HPTM is an incredibly powerful mathematical approach that has revolutionized the field of nonlinear differential equations. Combining the HPM and the LT resulted in the HPTM. The strength of the HPTM lies in its versatility. It has found applications in a wide array of fields, particularly in physics and engineering. One area where it has truly made its mark is in solving FDEs. Academician and researchers from all over the world have embraced the HPTM to solve a wide range of problems, both linear and non-linear. For example, Yasir Khan and Wu Q. used HPTM to solve nonlinear advection equations, obtaining exact closed-form solutions for both homogeneous and non-homogeneous instances. Gupta and Gupta demonstrated how the HPTM can be used to tackle initial boundary value problems with variable coefficients. Furthermore, among other prominent uses in the literature, Devendra Kumar et al. utilized the HPTM to tackle linear and nonlinear Schrödinger equations. Using this effective strategy, they were able to find solutions that have significant implications in the field of quantum mechanics.

1.8 Nonlinear Partial Differential Equation in Mathematical Physics:

1.8.1 Burgers' Equation:

The development of dependable computational methodologies capable of addressing nonlinear PDEs frequently used in fluid mechanics and heat transfer applications has received significant attention in recent years. The Burgers' equation stands out as a widely celebrated example of an equation that incorporates both nonlinear propagation effects and diffusive effects. Burgers' equation is applicable to a variety of challenging physical issues faced in engineering fields, but its non-linear character makes solving it difficult by nature [177].

Harry Bateman (1882-1946), an English mathematician, introduced the Burgers' equation and its accompanying initial and boundary conditions in a 1915 article, which is where the equation first appeared, and given as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}, 0 < x < L,\ \ 0 < t < \tau, \tag{1.8.1}$$

$$u(x,0) = \varphi(x), 0 < x < L,$$
$$u(0,t) = \vartheta_1(t)\ , u(L,t) = \vartheta_2(t)\ , 0 < t < \tau,$$

Here variables *u, v, t,* and *x* represent the velocity, spatial coordinate, and time, respectively. Depending on the specific conditions in which the problem must be solved, the variables $\varphi, \vartheta_1, \vartheta_2$ are provided as functions of variables. The equation (1.8.1) is known as inviscid Burgers' equation when the viscosity v is equal to zero. Gas dynamics are modelled using this equation.

The mathematical modelling of turbulence was subsequently explained by Dutch scientist Johannes Martinus Burgers' (1895-1981) using eq. (1.8.1), and as a result of his substantial contributions to the field of fluid mechanics, he gained renown. Burgers' is recognised for his substantial contribution, hence this equation is often referred to as the Burgers' equation.

1.8.2 Korteweg-de Vries [KdV] Equation:

The KdV equation is a dispersive, nonlinear mathematical model that is utilised to explain how shallow water surface wave's respond. The KdV equation can be interpreted as

$$\varphi_t - 6\varphi\varphi_x + \varphi_{xxx} = 0,$$

Boussinesq first proposed it in 1877, and Korteweg and de Vries later developed it in 1895 to simulate soliton events. Solitons are nonlinear waves that resemble pulses and are present in a variety of physical systems. They have special qualities including keeping their speed and form while travelling without distortion. Solitons are an important phenomenon in a variety of nonlinear media, such as plasma, optics, and fluid flow, and they stay stable even when they clash with one another.

Subsequent research by Shi et al. [57] demonstrated that the modified KdV equation exhibits a fractal set of fragile amplitude, quasio-periodic solutions. Dutykh et al. [58] investigated the energy transfer process in the modified KdV equation during the nonlinear stage of modulation instability, with an emphasis on Fourier space analysis. Triki H., & Wazwaz, A. M. [59] investigated two families of fifth-order KdV equations featuring time-varying coefficient and linear damping term, these equations are suitable for describing envelope wave dynamics in inhomogeneous systems governed by KdV-type equations.

In order to handle the KdV problem, Gardner, C. S., [60] created a variational and Hamiltonian formulation, and Gardner et al. [61] provided a number of techniques. These advancements have helped to improve our understanding of the KdV equation and its application to the studyof shallow water wave dynamics and associated phenomena.

1.8.3 Klein –Gordon equation [K-G equation]:

The K-G equation is a fundamental PDE in physics and mathematics. It was developed independently in 1926 by scientists Oskar Klein and Walter Gordon, and is named after them. The Klein-Gordon equation is a relativistic wave equation used to describe the behavior of spinless particles, such as mesons, within the framework of quantum field theory.

The K-G equation is written as described below: [62]

$$\varphi_{tt}(x,t) + a\varphi(x,t) + h\big(\varphi(x,t)\big) = g(x,t).$$

With initial condition

$$\varphi(x,0) = f(x), \ \ \varphi_t = k(x).$$

Where 'a' represents the constant, the source term is denoted by $g(x,t)$, and $h\big(\varphi(x,t)\big)$ is the nonlinear function of $\varphi(x,t)$. The K-G equation has been used to simulate several problems in science and engineering, including those related to solitons, condensed physics, classical and quantum mechanics, and solitons. Numerous researchers have successfully solved the K-G equation using a variety of techniques, like according to Birrell and Davies [63], it is likely to investigate how quantum fields, such as scalar fields governed by the K-G equation, interact with space time and how this affects particle behaviour and the evolution of the universe. Maireche A. [64] explains K-G equation in a noncommutative three-dimensional space with a modified Coulomb plus inverse-square potential. Detweiler S. [65] investigated K-G equation for a scalar field of mass in revolving black hole’s geometry.

1.8.4 Fornberg-Whitham Equation [F-W Equation]:

The F-W equation was designed for the purpose to investigate nonlinear shattering dispersive waves from the ocean. The F-W equation is defined as follows [66]:

$$\theta_t - \theta_{xxt} + \theta_x = \theta\theta_{xxx} - \theta\theta_x + 3\theta_x\theta_{xx}.$$

It is a highly significant mathematical model in the field of mathematical physics. The study of these waves is essential for a variety of practical applications since they frequently display complicated behaviours. The F-W equation can provide peak on solutions, which depict waves with restricted heights and wave breaking events. Many academicians, like Gupta and Singh [67] and Alderremy et al. [68], have studied this equation's fractional version, focusing on the fractional Caputo derivative. Additionally, this field of study has benefited from the work of other academicians including Haroon et al. [69], Alsidrani et al. [70], Sunthrayuth et al. [71], and Singh et al. [72]. The presence of a single kernel causes difficulties for the implementation of the fractional Caputo derivative, which restricts the scope of its practical use.

1.8.5 Fisher's Equation:

The Fisher equation,

$$\theta_t = \theta_{xx} + \theta(1 - \theta).$$

It is also known as the Fisher-KPP equation or Kolmogorove Petrovsky-Piscounov equation and is a widely known mathematical expression applied in several scientific fields, including biology, chemistry, heat and mass transport, and ecology. Fishers' equation describes the process of interaction between diffusion and reaction. Numerous studies of the Fisher equation have examined different facts in the literature. In their study, Bin Jebreen et al. [73] devised an effective algorithm that successfully solves the Fisher equation by combining the wavelet Galerkin technique with the finite difference approach. While this was going on, Lou et al. [74] investigated all of the solutions to the Fishers-KPP equation with the Dirichlet boundary condition, paying special attention to the half line. For the one-dimensional nonlinear Fisher equation, on the other hand, Chandraker et al. [75] created a semi-implicit finite difference method. Rose et al. [76] also investigated a generalised version of the Fisher problem and developed reductions to ordinary differential equations using the notion of symmetry reduction.

1.8.6 Swift-Hohenberg Equation [S-H Equation]:

The Swift-Hohenberg [S-H] equation,

$$\partial_t\varphi = r\varphi - (1 + \nabla^2)^2\varphi + N(\varphi).$$

Which was developed in 1977 by Jack Swift and Pierre Hohenberg, is a basic model that describes fluid velocity in thermal convection and the dynamics of temperature [77]. It has several purposes in science and engineering, such as fluid dynamics, hydrodynamics, laser research, and other domains such as biology and physics. Moreover, due to its nonlinear parabolic structure and its ability to elucidate pattern formation in fluid layers confined between well-conducting horizontal boundaries, the S-H equation holds significant importance. Numerous physical systems have been studied and described in terms of pattern generation using the S-H equation. It has been used in Rayleigh-Benard convection to mimic the formation and development of patterns like rolls and hexagons [78]. The equation hasnalso been used to represent wave-vector selection principles in the context of optical parametric oscillators, taking into account both degenerate and nondegenerate cases [79]. Extending the scalar version, the vector complex S-H equation has proved useful in analyzing pattern creation and structures in nonlinear optical systems, such as lasers. The study of travelling waves, spiral waves, flaws, and the rivalry between stable solutions has all been conducted using this method [80]. The cubic-quintic S-H equation has also been useful in explaining convective systems with reflection symmetry. This equation has provided insight into the behaviour of moving structures and possible collisions by analysing the impact of symmetry breakdown on spatially localised structures [81].

In addition, studies of the S-H equation on manifolds with conical singularities have revealed important insights into the existence, singularity, and regularity of solutions. These investigations have also emphasised the connection between the manifolds' local geometry and the solutions [82]. The wide application and significance of the S-H equation in comprehending complex pattern generation and dynamics in many physical systems are shown by these diverse experiments across numerous areas.

1.9 Research Objectives:

1. To develop semi-analytical techniques which is hybrid of the existing techniques using integral transformation methods like Laplace and other integral transformations.

2. Implementation of developed techniques to find solutions of complex physical problems like partial differential equations and fractional differential equation under various conditions in view of applications.

3. To perform the error analysis and convergence analysis for the numerical simulation of the considered model and validation of findings with available results in literature.

1.9.1 Organisation of the Thesis

This thesis is primarily concerned with creating a hybrid method for approximating series solutions to FDEs and PDEs. Throughout this study, we investigate the convergence of the solutions, perform error analysis, and compare our findings to the exact solution as well as the results that have previously been published in the literature.

In the second chapter, we will discuss papers that are related to our research work. It also provides us with a wonderful starting point for planning and carrying out our study.

In the third chapter, the novel technique employed, namely AHPETM, will be explored for addressing the PDEs.

In the fourth chapter, two methods will be delved into: AHPTM and AHPSTM. These methods will be used to solve various nonlinear partial differential equations.

In the fifth chapter, a demonstration of how the suggested technique solves FDEs in the Caputo sense will be presented. Convergence analysis will also be conducted, along with the presentation of surface and line graphs.

In the sixth chapter, an illustration of how the suggested technique solves FDEs in the Caputo-Fabrizio sense will be provided. A comparison with the exact solution will be made, and surface and line graphs will be presented.

Chapter 2

Literature Review

Weerakoon, S. (1994) [83]**.** The research article gives a thorough analysis of the partial derivative derivation of the ST and discusses its possible use in solving PDEs. Firstly, the ST illustrates that it is a valuable technique for solving PDEs. The second proof of the research article is that ST of partial derivatives may be produced by integrating by parts. Last but not least, the Sumudu transform's effectiveness in resolving three separate PDEs serve as an illustration of how broadly applicable it is. Overall, the study presents a fresh way for handling PDEs, providing a useful technique that may be used in a variety of scientific and technical fields.

He, J.H. (1999) [84]**.** The author demonstrates a novel perturbation approach combined with a homotopy technique. This approach is appropriate not just for small parameters, but also for big values, and the constraints of the classic perturbation method are readily overcome.

He, J.H. (2000) [85]. The author discussed the nonlinear problem, which is solved by proposing the merging approach of a homotopy technique with a perturbation methodology. The coupling approach fully utilises the conventional perturbation method. Some examples in this paper demonstrate the ease and success of the new technique. In this case, the starting approximation can be arbitrarily chosen with an unknown constant.

He, J.H. (2003) [86]. In his study, the author makes the case that the HPM is a new perturbation approach that does not necessitate the inclusion of a minor parameter in an equation. It can fully utilise homotopy techniques and conventional perturbation methods. Compared to the conventional answer at the second order of approximation, the findings provided by this method at the first degree of approximation are significantly more accurate. Application of the approach to a duffing equation with high-order nonlinearity demonstrates the method's efficacy and accuracy. The first order of approximation derived using the suggested method is more effective in comparison to the perturbation solution, and it is true uniformly even for large parameters.

El-Sayed, S. M. (2003) [87]**.** In this study, the solution of linear and nonlinear K-G equations and Sine-Gordon equations employing ADM is demonstrated. Analytical and numerical researches are both provided, and the outcomes demonstrate advancements above the current method. As shown through analytical and numerical studies, the paper comes to the conclusion that Adomain's decomposition methodology is effective for solving linear and nonlinear K-G and Sin-Gordon equations.

Liao, S. (2004) [88]**.** They explain in this paper that the nonlinear problem is generally solved with an easy-to-use tool, namely the HAM, and that this method further improves and systematically describe a typical nonlinear problem, namely the algebraically decaying viscous boundary layer flow caused by moving sheets. Two rules are proposed in this paper: the first rule of solution expression and the second rule of coefficient ergodicity. Both play an important part in identifying the HAM and providing an analytical solution.

He, J. H. (2004) [89]. In this research paper, the HPM and the HAM are compared, and it is found that HPM is more effective than HAM. A novel perturbation approach called HPM seeks an asymptotic solution with a minimal number of terms, whereas HAM is a generalised Taylor series method that seeks an infinite series solution. There is no requirement for convergence theory. The paper also describes how recent perturbation techniques are used to advance HPM.

Liao, S. (2005) [90]**.** The author compared the HAM to the HPM. The homotopy perturbation method is just an extension of the HAM. The Taylor series with regard to an embedding parameter serves as the foundation for these two procedures. Both approaches are good in various ways, such as HPM having a sufficient initial estimate; however, this is not required for the homotopy analysis approach since HAM includes the auxiliary parameter 'h'. And at the end of the result, it shows that HAM is more general as compared to HPM.

Momani, S., and Odibat, Z. (2007) [14**].** In this study, they present an enhanced version of the HPM for nonlinear PDEs with fractional time derivatives. An approximate solution is produced by the proposed strategy in the format of a simple-to-calculate convergent series. The results show that this strategy is resilient, successful, and simple, and they are consistent with previous findings in the literature. The technique performs more accurately than previous approaches like variational iteration and adomain decomposition. Furthermore, it has been proven that, given the right initial approximation, the modified HPM produces results that are equal to those of the variational iteration approach.

Gorji, M. et al. (2007) [91]**.** The research shows that the coupled Schrödinger-KdV and shallow water equations may be efficiently and conveniently utilised by the HPM. Because it does not necessitate either large or tiny parameters, the technique is ideal for discovering equations that are uniformly valid for both small and large parameters in nonlinear problems. The method has several uses in research and engineering.

Rana, M. A., et al. (2007) [92]**.** In this article, the authors of the study conclude that He's HPM was successfully employed to compute the ST. Instead of requiring integration, as is the case with conventional procedures, this method is a straightforward and effective mathematical tool.

He, J.H. (2008) [93]**.** The paper titled 'Recent Development of the Homotopy Perturbation Method' offers a simple introduction to the homotopy perturbation method's fundamental solution process. It places emphasis on creating an appropriate homotopy equation and understanding how to break down a difficult problem into a series of similar tasks.

Mohyud-Din, S. T., and Noor, M. A. (2009) [5**]**. In their study, they employed the HPM to solve both linear and nonlinear PDEs. This approach finds the solution without the need of linearization, discretization, or constrictive assumptions. They include Helmholtz, Fisher's, Boussinesq, unique fourth-order PDEs, systems of PDEs, and higher-dimensional initial boundary value issues. In this study, a method's precision, dependability, and effectiveness are examined. One ingenious benefit of this approach is that it eliminates the requirement to locate Adomian's polynomials.

Kilicman, A., and Gadan, H. E. (2010) [94]**.** The ST and its link to the LT are discussed in this study. It also presents an example of the double ST to solve the wave equation in one dimension with singularity in the beginning circumstances. According to the article, the ST can be a useful tool for solving differential equations, especially those with singularities. Additionally, it draws attention to the connection between the PDE's pre- and post-convolution solutions.

Dalir, M., & Bashour, M. (2010) [95]. The paper discusses several interpretations of fractional derivatives and integrals and then uses them to develop precise formulas and graphs for a few exceptional functions. Additionally, various applications of the theory of fractional calculus are reviewed.

Abbasbandy, S., and Shirzadi, A. (2010) [96]**.** The first integral method, a method used to deal with nonlinear equations and resolve various nonlinear issues, is put into practice in this study. New exact solutions to the modified Benjamin-Bona-Mahony (mBBM) problem were found when the approach was explicitly applied to it. Using hyperbolic and exponential functions, two of these solutions are complex, and two are real. These solutions’ traits are consistent with those found in earlier research, supporting their validity. Importantly, these recently discovered solutions could have a big impact on how we understand real-world physical issues.

Khan, Y., and Wu, Q. (2011) [97]. To solve nonlinear equations, the authors use the LT method with the HPM in this study. The HPTM is the name given to the technique. He's polynomials have made working with nonlinear terms easy to understand. The suggested approach finds the solution without requiring any assumptions about discretization and prevents round-off mistakes. The suggested strategy solves nonlinear problems without the need for Adomain’s polynomials, which is a significant benefit over the decomposition method.

Gupta, P. K., and Singh, M. (2011) [67]. The HPM is used in this study to identify approximations of solutions to the nonlinear fractional F-W equation. In comparison to other perturbation approaches, the numerical results demonstrated that this method has good accuracy and decreases the number of calculations. The HPM is a strong, simple, and efficient method for a broad spectrum of nonlinear issues in science and engineering without a lot of presumptions or limitations. In real-world situations where intricate boundary conditions and nonlinear differential equations dominate the process, it can also be used.

Singh, J. et al. (2011) [175]**.** In this manuscript, the authors put forward a novel approach referred to as the HPSTM to address nonlinear equations. This method amalgamates the ST with the HPM, thereby yielding a solution devoid of discretization or restrictive assumptions and circumventing round-off errors. The implementation of He's polynomials in the nonlinear term is presented, which is perceived as an advantageous departure from the application of Adomian's polynomials in the decomposition method. The proposed algorithm offers a prompt convergent series solution, potentially leading to a solution. The efficacy of the HPSTM is exhibited in resolving nonlinear advection equations, encompassing both homogeneous and non-homogeneous cases.

Kilicman, A., and Eltayab, H. (2012) [98]. The authors compare and contrast the LT with the ST in this work. They solve the steady-state temperature distribution function for this, and after solving it with both the LT and ST, they come to the conclusion that while a solution to a differential equation found by the ST may exist, a solution obtained by the LT does not always imply that it does.

Vishal, K. et al. (2012) [99]. The authors employ the HAM to derive approximate analytical solutions for the nonlinear S-H equation with a fractional time derivative in the Caputo sense in this study. The study uses the residual error formula to analyse the effect of physical relevant elements on the probability density function and the convergence of the approximation series solution. The authors exhibit these effects in graphs and tables for numerous specific examples, including fractional Brownian movements and standard motion.

Elzaki, T. M., & Hilal, E. M. A. (2012) [100]**.** In their research, they outline a reliable merging of the ET and the HPM to examine a few nonlinear PDEs. The suggested HPM is employed to rewrite the first- and second-order initial value problems, resulting in a solution in terms of changed variables. The inverse transformation is then used to find the series solution. According to the conclusion, the findings validated the methodology's efficacy.

El-Sayed, A. M. et al. (2012) [101]**.** In this paper, the authors offer a HPM for solving initial-boundary condition problems related to fractional order PDE in finite domains. To demonstrate the efficacy of the suggested strategy, examples are given. The method can effectively address a category of initial-boundary value problems for partial differential equations with fractional orders within finite domains. The exact solution reported in prior works is compatible with the approximate solution derived by the HPM.

Singh, J. et al. (2012) [102]**.** According to the study paper's findings, the linear and nonlinear Klein-Gordon equations may be accomplished simply and effectively using the HPTM. This method avoids round-off error by combining the LT and the HPM to generate the solutions without using discretization or constraining assumptions. Another benefit of this approach is that, unlike the decomposition technique, it does not require Adomain's polynomials to solve nonlinear problems. The study outcomes demonstrate how the proposed strategy may be utilised to handle a range of other nonlinear problems.

Nazari-Golshan, A. et al. (2013) [103]**.** They use the modified HPM to solve nonlinear and singular Lane-Emden equations, including He's polynomial with HPM and combining it with the Fourier transform. Using the approach, the suggested solutions to the three unique linear and nonlinear differential equations of Lone-Emden validated the physical property of the Lane-Emden problem's equilibrium, as $x \to \infty$ the solution approaches monotonically constant.

Kashuri, A. et al. (2013) [104**].** In this study, they describe a new approach for solving nonlinear PDEs that combines the HPM and a new integral transform. In this paper, they solve a nonlinear PDEs with initial conditions and He's polynomial is also used in this method for calculation. Compare to previous approaches, this approach reduces computing work.

Atangana, A., & Kilicman, A. (2013) [105]**.** In this research, they employed the Sumudu transform characteristics to solve nonlinear FDEs expressing heat-like equations with variable constants in this research. This approach combines the ST as well as the HPM, utilising He's polynomial. This technique is easy to use and does not require anything like Adomain polynomial. This approach is employed to solve a variety of linear and nonlinear FDEs encountered in many disciplines. The calculation used in this method is simple and easy to calculate.

Bizara, J., & Eslami, M. (2013) [106**].** The New Homotopy Perturbation Method was presented as a modified version of HPM. The nonlinear two-dimensional wave equation has been solved using NHPM in this research. In this study, they used the first approximation solution to arrive at the exact solution. The new technique that is used gives a better result than the HPM.

Yousif, E. A., & Hamed, S. H. (2014) [107**].** The authors have successfully derived precise analytical solutions for nonlinear FDEs by employing an integrated approach involving the HPM and the ST. These solutions are expressed in closed forms, utilising ML functions. The consideration of fractional derivatives is based on the Caputo sense. This method is simple, easy to apply, and effective for solving the nonlinear differential equation.

Daga, A., and Pradhan, V. (2014) [108]**.** In their research articles, they used variational PDEs to solve the nonlinear generalised long wave (GRLW) problem. The combined technique was utilised, which is a mix of the variational iterative method and the HPM. To demonstrate the accuracy of this technology, an example of the propagation of a single soliton is shown. They create an analytical solution for the regularised wave equation. The study uses graphics to demonstrate solitons and waves. The method's speedy convergence demonstrates its reliability and brings a considerable improvement in solving PDEs over prior approaches.

Elzaki, T. M., and Kim, H. (2014) [109**].** In this study, the authors propose a highly efficient method for solving Burger's equation by incorporating both the ET and the HPM. Burger's equation analytical conclusion has been expressed in terms of a convergent series with a calculable component, and PDE can be used to treat the equations nonlinear term. After solving Burger's equation, they conclude that the proposed approach is more effective, and this approach yields results that are nearly exact.

Ziane, D., and Cherif, M. H. (2015) [110]**.** The author's primary objective in this article is to broaden the applicability of the ET decomposition technique by solving PDEs of the first, second, and third orders. Upon conducting this comparison, it is determined that the ET decomposition method produces highly precise approximate solutions with minimal iterations, especially in the equations of the third order, and the emergence of the decomposition series solution after calculating the two first terms only.

Hamza, A. E., & Elzaki, T. M. (2015) [111]**.** In this study, the exact solution to Burger's equation is obtained. For obtaining, the solution, the author's combines two techniques: homotopy perturbation and the Sumudu transform. They demonstrate that the suggested technique is a refinement of an existing numerical technique.

Gupta, S. et al. (2015) [10]. In this paper, the convection-diffusion problem was solved by using the HPTM to find the analytical solution. They use He's polynomial, and show that He's polynomial, is more powerful than the Adomain polynomial. By using He's polynomial the solution converges very quickly, and negligible errors have been observed even with just a few terms of HPTM. They conclude that HPTM solves nonlinear problems without using Adomain polynomials.

Touchent, K. A., & Belgacem, F. B. M. (2015) [112**].** The authors use the HPSTM to solve a nonlinear FDE in this study. The HPSTM is turn out a significant technique over the Adomain decomposition method; it solves the nonlinear fractional PDEs without using the Adomain polynomial. Finally, based on the results obtained, the HPSTM is considered to clarify the existing numerical techniques, and it is also an effective way of solving the FDEs.

Caputo, M., and Fabrizio, M. (2015) [113]**.** The presented paper introduces a novel definition of fractional derivative that possesses a smooth kernel. The temporal and spatial variables are represented in two distinct ways in this definition. Consequently, the utilisation of the Laplace transform becomes feasible for the temporal variables, while the spatial variables can be subjected to the Fourier transform. The novel non-local fractional derivative, proposed within the paper, offers the capability to describe material heterogeneities and structures of varying scales. These characteristics are not adequately captured by conventional local theories. Thus, the spatial fractional derivative emerges as a potentially significant tool for investigating the macroscopic behaviours of materials featuring nonlocal interactions. Additionally, the paper encompasses applications and simulations of the novel derivatives, which are applied to classical functions, such as trigonometric functions.

Ortigueira, M. D., and Machado, J. A. T. (2015) [114]**.** The idea of a fractional derivative is covered in this paper, along with the rules that an operator following it must abide by. The wide sense criterion is one of two sets of criteria for identifying a certain operator as an FD that were put forward. Several recent formulations of functional derivatives were examined based on those requirements. It was established that some topics are failed by the classical fractional derivatives. The derivatives of Grünwald-Letnikov, RL, and Caputo, on the other hand, were responded to and examined in the context of the suggested criteria. As a result, the paper's conclusions are that the Grünwald-Letnikov, RL, and Caputo fractional derivatives satisfy the suggested requirements for being categorised as fractional derivatives, but the classical fractional derivative do not satisfy all of the proposed criteria.

Baleanu, D. et al. (2016) [115].To solve an advection partial differential equation with time-fractional Caputo and C-F derivatives, the authors of this article used the q-HAM and the variational homotopic perturbation method (VHPM). Both approaches' results showed that both derivatives' exhibit similarities.

Neamaty, A. et al. (2016) [116**].** They demonstrated the use of two methods: HPM and ET. They investigate certain nonlinear PDEs of fractional order in this study. This paper shows that ADM is simple for finding the solutions, but determining Adomian's polynomial is difficult and complex. When there is a comparison between the decomposition method and HPET, the major advantage is that nonlinear problems can be solved without the need for Adomian's polynomial.

Tarasov, E. V. (2016) [117]**.** The paper put out a unique geometric interpretation of non-integer orders' R-L and Caputo derivatives, based on jet bundle geometry and modern differential geometry. The idea of infinite jets of functions serves as the foundation for the suggested interpretation of fractional derivatives. The fractional-order derivatives are represented as infinite series with integer-order derivatives in the paper in order to establish a geometric interpretation of them. The study shows how certain types of infinite jets are related to non-integer-order derivatives. In terms of order, the proposed infinite jets are seen as a rebuild of standard jets. The paper offers a viewpoint on the geometrical analysis of fractional order derivatives as an outcome.

Sharma, D. et al. (2016) [118]. In order to solve fractional PDEs, the HPSTM is applied in this study. The method is seen to be very straightforward, very effective, and error–free. The paper gives instances of applying the HPSTM to solve several equations. The findings show that while solving nonlinear systems of PDEs, the HPSTM takes much less computing work and demonstrates quick convergence. The paper's overall conclusion is that the HPSTM overcomes the drawbacks of existing approaches, such as the HPM, and is a powerful and simple method for solving fractional nonlinear PDEs.

Zarebnia, M., and Parvaz, R. (2016) [119**].** The Benjamin-Bona-Mahony-Burgers (BBMB) problem is solved using a cubic B-spline collection approach in this study. The method's stability study demonstrates that it is infallibly stable. The order of convergence is $o(h^2 + \Delta t)$. The numerical results produced using the suggested approach show high accuracy and stability. As a result, the research draws the conclusion that the suggested approach works well for resolving the BBMB equation.

Morales-Delgado, V. F., et al. (2016) [120**].** By integrating the LT with homotopy approaches, the study provides a unique method for solving FPDEs. This technique works well for getting approximations for FPDEs, making it a useful tool for physicists and engineers working in a variety of scientific domains. Due to the Liouville-Caputo representation's inaccurate depiction of memory effects, the CF formulation of the fractional operator is preferred. By contrasting the derived solutions with the exact solutions of the fractional equations, the research proposes a generic framework for locating approximations to FPDEs and illustrates its effectiveness.

Gomez-Aguilar, J. F. et al. (2017) [121]**.** The authors provide the HPTM for solving nonlinear FPDEs with the C-F fractional operator in this paper. To get the infinite series solution, perturbative expansion polynomials are taken into account. To establish the efficacy of this method, the authors successfully tackle the fractional equations specified for the singular scenario, wherein the boundary of the integral degree of the temporal derivative is taken into account. The authors also provide a general scheme for approximating solutions of fractional equations, which quickly converges and is presented to a series form.

Singh, P., and Sharma, D. (2017) [122]. The authors of this article discuss the problem of convergence in the HPTM and provide a resolution. In addition, they calculate the series solution's largest absolute truncation error. A non-linear fractional partial differential equation is also subjected to the HPTM, with an approximation of the solution produced. A surface graph of the fifth-order approximation solution is used by the authors to further analyze the results. In conclusion, the study contributes by providing a solution to the convergence issue in HPTM, calculating truncation error, using the approach on a non-linear equation, and interpreting the outcomes using a surface graph.

Patel, M.A., & Desai, N.B. (2017) [123**].** The author describes the HAM that was employed to solve a nonlinear PDE resulting from countercurrent imbibitions in a homogeneous porous medium. The HAM provides an approximate analytical solution to a nonlinear PDE.

Khadar, M. M. (2017) [124]**.** The explained HPSTM is implemented to obtain the approximate solution of the multidimensional fractional heat equation. The major benefit of the HPSTM is that it finds solutions to the non-linear differential equation without the use of Adomain polynomials. By using the HPSTM, the approximate solution that is obtained is in coincides accurately with the exact solution and also illustrates the high potential and validity of the technique.

Turkyilmazoglu, M. (2017) [125]**.** In this paper, the author presents a parametrized variation of the ADM, referred to as the optimum ADM. The objective of this method is to enhance the convergence and rate of convergence of the classical ADM. It exhibits that the optimum ADM achieves convergence to the precise solution in cases where the classical ADM fails to converge. The optimal ADM greatly broadens the conventional ADM's constrained domain of convergent physical solutions to a more refined interval. Consequently, the newly proposed algorithm, the optimum Adomian decomposition method, exhibits superior accuracy in comparison to the recently popular HAM.

Li, W. & Pang, Y. (2018) [126]. The authors offer an iterative approach for obtaining both estimated and accurate solutions to time-FDEs in this paper. To demonstrate the potency of the technique, the authors apply it to linear and nonlinear S-H equations and come up with approximate analytic solutions backed up by numerical figures. This demonstrates that the iterative approach is both easy to apply and successful when dealing with Cauchy problems with time-fractional differential equations.

Elbadri, M. (2018) [127]. The author compares the HPTM and HPM. For comparison, they apply the methods to the inhomogeneous equation, the non-linear K-G equation, and the non-homogeneous equation. After finding the series solution, it was concluded that the HPTM has a more rapid pace of convergence than the HPM.

Elzaki, T. M., & Chamekh, M. (2018) [128]**.** The authors discussed the New Decomposition Method (NDM) introduced for solving nonlinear fractional initial value problems by applying a method that is proposed by both the Elzaki and Adomain decomposition method. The combined technique is a useful strategy for solving nonlinear fractional differential equations.

Gad-Allah, M. R & Elzaki, T.M (2018) [129]**.** Discussed new novel technique, i.e., the new HPM, and how it is employed to solve of linear, nonlinear, and integral problems. In this article, they used two important steps: first, an appropriate homotopy equation, and second, choosing suitable initial conditions. To check the ability of the approach, some instances are provided. The NHPM is more reliable and efficient than the HPM, as well as easier to use as well as more precise in solving linear and nonlinear differential equations.

Noeiaghdam, S. et al. (2018) [130]. In this paper, they investigate the susceptible-infected–recovered model of computer viruses as a nonlinear system of ODE by applying the HAM. They have certain additional parameters and functions in this work, and one of them is convergent to the control parameter curves and identifies the convergence. They show the advantages of this method. The remaining errors were utilised to demonstrate the method's accuracy and effectiveness.

Al-Nemrat, A., & Zainuddin, Z. (2018) [131]**.** In this paper, the author provides and examines the results. Examining different frequencies across approximate, HPSTM, HPLTM, and precise solutions reveals that the HPSTM approach is exceedingly efficient, straightforward, and applicable to different forms of nonlinear boundary value problems.

Kharrat, B. N., and Toma, G. (2018) [132**].** The authors offer a novel approximation approach in this study that is produced by merging Sumudu's transform with the HPM. This approach may be utilised to solve a variety of nonlinear PDE problems. The findings of this approach were then compared to those of ETHPM, and they determined that MHPSTM gives more successful results than ETHPM.

Singh, P., and Sharma, D. (2018) [133**].** The article proposes a hybrid approach for solving nonlinear PDEs, which includes ST and HPM. He's polynomial, expressing the nonlinear terms, and provides the series solution. The study defines the convergence and uniqueness conditions, offers information on convergence, and analyses the solution's error. The resolution of Newell-Whitehead Segel, and Fisher's equations provides evidence for the established fact. Examples are provided to support HPSTM's Numerical and Error analysis.

Matlob, M. A., and Jamali, Y. (2019) [134]. The study investigates using fractional-order differential calculus to model viscoelastic systems, addressing its fundamentals and offering an overview of its use. In modelling viscoelastic systems, it emphasises the benefits of fractional order calculus, such as its ability to accurately capture system behaviour, and its advantages over traditional approaches.

Singh, P., & Sharma, D. (2019) [135**]**. In this paper, the author presents the HPTM as an amalgamation of the HPM, the LT, and the HPETM, which are used to obtain a series of equations from a nonlinear fractional PDE. After applying both strategies, namely HPTM and HPETM, to solve nonlinear homogeneous and non-homogeneous fractional PDEs, the outcomes demonstrate a high level of accuracy, simplicity, and efficiency in tackling challenging equations.

Jena, R. M., & Chakraverty, S. (2019) [29]. In this study, two strategies are explained: one is the HPM, and the other is the ET, which is combined to make the HPETM. The HPETM is used to solve the Navier-Stokes equation of fractional order, and the results obtained are good. In lieu of Adomain polynomials, this is a powerful and efficient approach for getting analytical and approximation solutions for fractional order nonlinear PDEs.

Kaya, F., & Yilmaz, Y. (2019) [136]**.** In this paper, they explaine that the integral transform is the solution method for ODEs and PDEs. This paper explains how newly introduced Sumudu transform properties and their application provide an effective tool for the study of some ODEs and PDEs.

Turkyilmazoglu, M. (2019) [137]. The present paper offers a rigorous mathematical framework that validates the classical Adomain method. Specifically, it prevents divergence and accelerates convergence in the case of a least change in the interval of the approximate series and the optimal value of the added parameter, resulting in faster convergence. The important outcome of the result generated after applying the Adomain decomposition method is that the ADM signals are no longer validated via other numerical means.

Aggarwal, S., et al. (2019) [138] -The authors' goal in this paper is to present duality relations between the Elzaki transform and some useful integral transforms, such as the LT and the ST, and to show that the Elzaki transform and other transforms are strongly related and used in many advanced problems**.**

Olubanwo, O. O., and Odetunde, O. S., (2019) [139] **.** The Laplace Homotopy Perturbation Method (LHPM), a new method for resolving nonlinear PDE, is presented in this study. The LHPM provides a powerful approach to solving problems by blending the LT method and the HPM. Since the results of LHPM are represented in terms of transformed variables, it is possible to drive a series solution by applying the inverse properties of the LT. The LHPM produces equivalent results, as shown by comparison with other approaches like HPETM. As a result, the research indicates that the LHPM is an appropriate and useful approach for dealing with such issues. In conclusion, the LHPM demonstrates its value in solving nonlinear PDEs.

Singh, P., and Sharma, D. (2020) [56]. In the present article, the author employs the HPTM and the HPETM for the purpose of resolving nonlinear fractional PDEs.The HPTM is an amalgamation of the HPM and LT, whereas the HPETM entails the ET. The aforementioned techniques are employed in tackling the Fractional Fisher's equation, time-fractional F-W equation, and time-fractional Inviscid Burgers' equation. The outcomes attained through these methods manifest in the form of power series, which exhibit rapid convergence and furnish highly accurate solutions with a mere handful of iterations. By enabling the facile computation of additional terms, these methodologies effectively diminish the computational expense entailed by the resolution of intricate problems.

Loyinmi, A. C., and Akinfe, T. K. (2020) [140**].** In this research paper, the ETHPM and two novel hybrid algorithms are suggested to search for a precise solution. Fisher's equations are classified into three categories. The equations are well-known in mathematical biology and have many applications, including genetic transmission, population dynamics, stochastic processes, the combustion theorem, and a model for a spreading flame. Convergence and error analysis were used to provide the validity and usefulness of this approach, and EHPTM findings are an effective and dependable technique for offering an accurate solution to a broader range of nonlinear PDE in an easy-to-understand way without any discretization, linearization, Adomain polynomial calculation, or erroneous. These results were compared to the precise results, homotopy results, and other existing literature results.

Kharrat, B. N., & Toma, G. (2020) [141**].** In this paper, they present the combination of two methods, i.e. the Sumudu transformation and the HPM. The main objective of this approach is to acquire an approximate solution. After applying the ST-HPM, the outcome demonstrates that it is a capable and efficient approach to finding precise and approximate equations.

Li, W., & Pang, Y., (2020) [142**].** This work illustrates a few nonlinear issues pertaining to the ADM, including its efficient and straightforward convergence analysis and iterative procedure. In this paper ADM is applying to find the approximate solution to the algebraic equation fractional differential equation (time-fractional Riccati equation), integero differential equation, differential equation, fractional PDE (time-fractional Kawahara equation, modified time-fractional Kawahara equation), and so on.

Anjum, N. et al. (2020) [143**].** In this paper, for the first time, the Elzaki transform of the variational iteration algorithm is used to identify the Lagrange multiplier. The iteration method that is used in this paper is converging rapidly and only one iteration results in a highly accurate solution. This approach to solve the nonlinear PDEs is incredibly effective.

Dawood, L., et al. (2020) [144**].** The article presents a novel approach, the Laplace Discrete Adomian Decomposition Method, developed based on discretization principles. This method is designed to address nonlinear Volterra-Fredholm integro-differential equations efficiently. By leveraging the inherent properties of discretization, the proposed technique offers a modified framework for solving such equations numerically. This method provides the approximate solution iteratively with less computation as compared to the ADM.

Kumar , K. H., & Jiwari, R. (2020) [145] . In this paper, the authors present their work on the solution of time-dependent 1D, 2D, and 3D Benjamin-Bona-Mahony-Burgers (BBMB) and Sobolev equations. Legendre wavelets are employed in the proposed approach to resolve these equations. In this paper, they extend wavelet-based approaches for tying dependent three-dimensional (3D) issues in the temporal domain without using a finite difference approach. The Chebyshev wavelet is used to test all of the examples in this study. The sequence of errors is identical to that in the Legendre wavelet. In terms of grid size, it was discovered that the suggested technique is more efficient than the techniques existing in the literature.

Deepak, G. et al. (2020) [146]. In this paper, a novel semi-analytical approach called accelerated HPSTM is suggested to get approximate analytical solutions for non-linear PDE with proportionate delay. The suggested method merges the ST with the accelerated HPM. The approach provides a quickly convergent series and is observed to converge more quickly than HPM, VIM, and DTM, which are some additional semi-analytical approaches. In order to validate the proposed technique's effectiveness and dependability, the condition of convergence and the approximations come quite close to the precise solutions of the models under consideration under the specified initial conditions. Therefore, the suggested approach is a reasonable and efficient way to resolve proportional nonlinear partial differential equations.

Matwal, A. A. H., and Alkaeeli, S. (2020) [147], The Shehu transform homotopy method (STHM), which the authors propose, is an entirely novel method for proportionally resolving time-fractional PDEs. The process combines the HPM with the Shehu transform method. The fractional derivative is defined in the sense of Caputo. The series' suggested solution quickly arrived at the exact solution. Three test problems are provided in the study to validate and demonstrate the effectiveness of the strategy. Under perturbation-restrictive conditions, the approximative solutions produced without any discretization converge fairly rapidly. The STHM is an effective and simple approach for solving TFPDEs with proportional delay.

Ahmed, S., et al. (2021) [148]. In this research article, the authors present the Yang transform homotopy perturbation method (YTHPM), a new approach. They construct Yang transform equations for Caputo-Fabrizio fractional-order derivatives. The author next presents a technique for solving CF fractional-order PDEs, and evidence of its correctness is provided by demonstrating convergence to the precise solution when renowned in well-known nonlinear PDEs with names like the KdV equation and Burger's problem.

Nonlaopon, K., et al. (2021) [149]. The authors of this paper created a new technique, the Elzaki Transform Decomposition Method (ETDM), by combining the ET and the ADM. This method is used to resolve time-fractional S-H equations, which can be used to explain how liquid surfaces grow in the absence of horizontally well-conducting barriers. The Caputo sense is used for computing the fractional derivative. To show and validate the suggested technique's correctness, multiple figures are used to analyse linear and nonlinear S-H equations with varying beginning circumstances.

Alrabaiah, H. et al. (2021) [150]. The authors of this paper use the Laplace Adomain Decomposition Method (LADM) to drive approximate analytical solutions to fractional order 'S-H'. They explore the effects of different types of dispersive terms and Caputo fractional-order derivatives. They conclude that the LADM is effective in dealing with both linear and nonlinear fractional-order PDEs and outperforms other analytical methods in terms of its convergence. However, it is slower when it comes to stability rate. Furthermore, they compare the results obtained via LADM with those obtained via the Homotopy Analysis Method (HAM) and determine that, when the Homotopy parameter h is chosen correctly, the solutions provided by the two methods are quite close.

Kehaili, A., Benali, S., and Hakem, A. (2021) [151]. The homotopy perturbation transformation approach is used by the authors of this article to discover an approximate solution to a system of nonlinear fractional partial differential equations. A comprehensive examination is conducted to demonstrate the efficacy of the method by exploring individual, likewise interconnected equations that originate from wave phenomena in diverse disciplines, including but not limited to physics, geochemistry, chemical kinetics, and mathematical biology.

Chu, Y. M., et al. (2021) [152]. To examine the fractional order of the Cauchy reaction-diffusion equation, two analytical methodologies were employed: the HPTM and the innovative iterative transform method. They improved and steamlined the implementation process by utilising the ET. The accuracy of the current strategy is demonstrated by the results, which closely match the precise solution of the model. For solving fractional-order PDEs, the recommended approaches showed adequate convergence rates. Furthermore, the commutations required by these procedures were simple and easy to understand.

He, W. et al. (2021) [153]. A hybrid approach called the Iteration Transform approach is proposed in the paper "Fractional System KdV Equation via ET" to deal with the fractional-order linked KdV problem. This technique combines the ET with the new iteration approach to provide solutions in series form for assessing the analytical findings. A few numerical issues are also presented in this study to aid in comprehension of the iteration transform method's analytical process. The numerical solutions demonstrate that just a small number of terms are required for efficiency, accuracy, and dependably arriving at an approximation. The analysis demonstrates that the results achieved with the present approach are extremely near to the actual ones.

Alrabaiah, H. (2021) [154]. This paper discusses the fractional order F-W equation under the non-singular kernel type derivative, as proposed by Atangana and Baleanu, along with a modified homotopy perturbation method (MHPM) to discover approximate solutions. No discretization is necessary for the straightforward procedure. Surface plot interpretations provide graphical representations that highlight the analytical dynamics of the problem. The study's findings demonstrate the viability of the suggested method for estimating analytical solutions to nonlinear problems.

Jani, H. P., and Singh, T. R. (2022) [155]. The fractional model of the Swift-Hohenberg equation was successfully solved using the Aboodh transform homotopy perturbation method (ATHPM), as detailed in the research. The Swift-Hohenberg equation was used to explain the formation and progression of patterns in a variety of systems, such as fluid dynamics, temperature, and thermal convection. To present the approximate analytical solutions to nonlinear differential equations, this approach was contrasted with the current approaches such as the q-HATM iterative method and ETDM, and was determined to be in excellent agreement. The problem's convergence analysis was also presented. Additionally, convergence analysis is carried out, and accuracy is evaluated by comparing the ATHPM solution to the exact solution and LADM. Overall, the research indicates that ATHPM is an effective method for resolving the fractional model of the S-H problem, offering highly accurate solutions to nonlinear systems, and facilitating the exploration of various nonlinear scenarios.

Mohamed, M. et al. (2022) [156]. In this study, the authors combine the Elzaki transform with a novel homotopy perturbation approach. By using this method, they solve a nonlinear fractional partial differential equation, and after solving the nonlinear problem, they represent their solution with the help of a surface and line graph. They compare their results with the exact solution.

Obi, C. N. (2022) [157]. The author extends the utilisation of the HPM to the advection equation in one or two dimensions in the present research. It employs the HPM to provide a semi-analytic solution to nonlinear advection issues. Many writers have used the approach to tackle linear and nonlinear problems, and this study contributes to its utilisation in the detection equation. The work focuses on solving nonlinear PDEs with beginning and boundary conditions, which can have applications in a variety of domains. The HPM yields analytic and approximate solutions.

Salman, A. T. et al. (2022) [158]. In this article, the analytical solutions to the coupled Burger's equation and the time-fractional Burger's equation are obtained using the EHPM. Fractional derivatives are expressed using the Caputo sense. This approach is useful for solving fractional partial differential equations as the result is almost the precise solution. It is a very beneficial tool for fractional PDE solutions.

Huseen, S., and Okposo, N. I. (2022) [159]. The fractional natural transform decomposition method (FNTDM), a modified integral transform technique, is presented in this paper as a means of obtaining approximate analytical solutions for certain time-fractional versions of the nonlinear Swift-Hohenberg (S-H) equation with fractional derivatives in the sense of Caputo. The Adomain decomposition method and natural transform are used in the FNTDM to provide series solutions with a high degree of accuracy and few calculations. There is no discretization, linearization, or perturbation involved in this method's direct equation solution. The results of the simulations performed support the applicability of the proposed technique to even more difficult issues that arise in a variety of applications of applied mathematics and physics by demonstrating similarities with those in the related literature that has already been published. The technique of solution under consideration is acknowledged to have a significant capability to generate an optimal convergence area for the solution, which contributes to its dependability, effectiveness, and simplicity.

Iqbal, S., et al. (2022) [160]. The conformable Elzaki Transform Homotopy Perturbation Method (C_DETHPM), a new method for resolving nonlinear time-fractional partial differential equations (N-TFPDEs), is presented by the authors. In comparison to conventional techniques, the suggested method greatly decreases the amount of computational work necessary while maintaining excellent numerical precision. It is successful in locating precise and approximate solutions for N-TFPDEs. The solution's uniqueness and convergence have also been determined by the results. Through the use of four distinct problems, the effectiveness and approximation of the suggested approach have been confirmed.

Alshehry, A. S. et al. (2022) [161]. To solve real-world issues, they concentrate on developing numerical analytical solutions and building fractional-order mathematical models. It presents the Laplace residual-power-series method (LRPSM), an entirely novel and dependent approach for resolving fractional partial differential equations. The need for an effective approach to solving such equations is the fundamental problem this research attempts to address. The results of the study indicate that the use of LRPSM is a successful strategy for resolving fractional PDEs, with a series solution that quickly converges to the exact solution. The method successfully resolves fractional PDE and manages the fractional derivative in the context of Caputo sense. The results obtained from LRPSM align with those of the natural homotopy perturbation approach, demonstrating their compatibility.

Rahman, M. M., et al. (2022) [162]. The KdV equation was solved using both HAM and HPM in the study. For three separate examples, both approaches l_2-error were determined using an appropriate exact solution. According to the results, HPM performed better than HAM when the time value (t) was small. In all three situations, there was an agreement between the two techniques on the exact solution. It was found that the HAM and HPM solutions coincided when $\mathcal{H}$ and $\hbar$ values met the requirement $\mathcal{H}\hbar = -1$ and the answer was computed using MATLAB algorithms.

Saifullah, S., et al. (2022) [163]. The nonlinear K-G equation with the Caputo fractional derivative is investigated in this study. By combining the double LT with the decomposition technique, the authors were able to establish the general series solution of the system. They observed that the resulting solution precisely matches the mathematical model's solution. It is examined if the model holds up in the presence of the Caputo fractional derivative. With proper auxiliary circumstances and particular instances, the applicability and accuracy of the suggested approach are proven. The authors deduced that the suggested system permits soliton solution to the numerical solutions. The amplitude of the wave solution is observed to rise with time derivative, which leads to the conclusion that the factor significantly increases the amplitude and breaks the dispersion and nonlinearity features, potentially allowing the dynamical system to be excited. Additionally, they have shown the physical activity that demonstrates the development of localised mode excitation inside the system.

He, Y., and Zhang, W. (2023) [164]. They investigated an iterative transformation approach for analysing PDEs that incorporates ET and iterative methods in their research article**.** The technique works well to produce series-based numerical solutions. The answers to the K-G problem obtained using the HPM and the method described in this paper are identical. Additionally, this method's steps and outcomes for resolving the novel generalised fractional Hirota-Satusuma linked KdV equation are provided in the study. As a result, utilising the ET with an iterative technique for obtaining fractional PDEs is efficient.

Chauhan, A., & Arora, R (2022) [165]. In this study, the fifth-order KdV equations of various forms were approximation-analytically solved using the HAM. These equations are often used in the domains of fluid dynamics and mathematical physics. The derived solutions were carefully compared with exact solutions, and their correctness was shown by displaying the absolute errors graphically. The effectiveness of HAM as a reliable numerical method for resolving non-linear PDEs is highlighted in the paper. The method's great precision in finding approximate solutions is due in part to its ability to adjust the rate of approximation series and the convergence zone. With a solid method to handle non-linear KdV, the reported findings offer an invaluable resource for researchers working in the field.

Kapoor, M., & Joshi, V. (2023) [166]. Sumudu HPM and Elzaki HPM, two hybrid methods for solving coupled Burger's equations, are compared by the authors. Applying these techniques to three separate instances of the problem shows their efficiency and accuracy. The study's error and convergence assessments for the suggested approaches demonstrate their applicability and demonstrate. In comparison to complicated numerical systems, the analytical regimes suggested in the paper are more effective at handling partial differential equations. It is emphasised that precise and approximation solutions can coexist, and it is claimed that the suggested regimes have numerical convergence. Ultimately, the work offers a useful comparison of the two hybrid techniques for resolving coupled Burgers equations.

Yasmin, H. (2023) [167]. In this paper, the author presents an efficient method for solving fractional nonlinear convection-diffusion equations: the Aboodh homotopy perturbation transformation method. Using the Aboodh transformation to make the presented issues simpler, this approach utilises the capabilities of the CD and AB operators. Fractional-order problem solutions are accurate representations of the issues' actual dynamics. The paper also demonstrates the practical applicability of these techniques in a variety of fields, including combustion and detonation theory, heat transfer in dranning film, mathematical biology and population dynamics, fluid flow, finance, and transport chemistry in the atmosphere. Fractional nonlinear convection, reaction, and diffusion equations, including fractional Atangana-Baleanu and Caputo derivatives, are analysed using the Aboodh homotopy perturbation transform method. The research develops a modified method to approximate these derivatives, analyses the suggested model using graphical and tabular simulations, and emphasises the model's usefulness in real-world applications in several domains.

Naeem, M., et al. (2023) [168]. The study of fractional PDEs is made easier by these techniques, which combine the Yang transform, HPTM, and YTDM with the Caputo fractional derivative. It is shown that the HPTM and YTDM are used for solving both linear and nonlinear FDEs. The key features of the YTDM and HPTM are highlighted in the study. The study possibly offers an alternative to current approaches and has extensive scientific and engineering implications. Future goals include putting the new method to the test on more fractional differential equation problems and testing quickly convergent realistic series.

Albogami, D. et al. (2023) [169]. Time-fractional linear and nonlinear Klein-Gordon equations were solved analytically using the Adomain Decomposition method. A well-known and convergent approach was used to deal with linear and nonlinear time-fractional K-G equations. The efficiency of the process was confirmed by comparing the numerical solutions produced using the decomposition method with the exact solutions and finding that they were quite close to each other. These incredibly accurate results demonstrate the decomposition method’s capacity to quickly and effectively produce numerical results that match the exact solution.

Elbadri, M., (2023) [170]. The linear and nonlinear time-fractional K-G equations are investigated in this study using the Natural Transform Decomposition Method (NTDM). The process combines two approaches: ADM and the natural transform method. In series form, the equation’s approximate solutions are obtained. Three examples are given in order to show the method’s effectiveness. The solutions have been developed for a variety of time power values.

Naeem, M. et al. (2023) [171]. The HPTM and the Yang transform decomposition method (YTDM), are two different techniques for solving various forms of nonlinear PDEs, including fractional PDEs. To solve fractional PDEs, the Yang transform, HPTM, and YTDM are combined with the Caputo fractional derivative. In-depth analysis, illustrations, and tabular numerical data are provided for large strategies in the study. Using HPTM and YTDM, the authors have successfully solved an extensive number of linear and nonlinear FDEs. The research comes to the conclusion that these techniques may be used in place of present approaches and have an extensive variety of applications in research and engineering. The authors want to demonstrate rapid convergence in realistic series and use these methods for more fractional differential equation applications in the future.

Conclusion:

In conclusion, this literature review provided a comprehensive overview of semi-analytical techniques to perform analysis of FDEs and nonlinear PDEs. These ideas have grown in the realm of mathematical modelling into strong tools with a broad variety of applications in science and engineering. In this study, we investigated a several types of semi-analytical methods, including HPM, HPSTM, HPTM, and many more, and the different equations on which these methods are applied. The advantages and disadvantages of each of these approaches were reviewed, as well as how well they handled nonlinear and fractional problems.

Chapter 3

Solution of Nonlinear Partial Differential Equations using Accelerated Homotopy perturbation Elzaki Transformation Method

3.1 Introduction

This chapter provides a full explanation of the technique used to accomplish the research objectives set out in this study. The main aim of this study is to create a semi-analytical technique that combines traditional approaches with integral transformation theories, such as Laplace and other approaches of a similar kind. Accelerated algorithms are computational techniques designed to speed up the convergence or minimize the number of iterations needed to solve a specific problem. These algorithms often use advanced optimisation techniques or parallel processing to achieve faster results compared to traditional algorithms. Kalla, I. L. has introduced an approach that utilizes a formula based on He's polynomial to construct accelerated He's polynomials, thereby enhancing the convergence rate. Numerical examples are presented to demonstrate the approach's effectiveness.

In our research work, we apply different types of semi-analytical techniques utilizing the accelerated He's polynomial. After using this technique, we compare the findings with the precise answer and with results that have already been published in the literature. The research is primarily centred on addressing nonlinear partial differential equations. In this study, a variety of equations, including advection problems and non-homogeneous PDEs are utilised and subsequently verified using Mathematica.

3.2 Convergence Analysis:

Theorem: If ∃ η varying from $0 < \eta < 1$ for the ω and $\omega_n(x,t)$ stated in Banach space, [56] thus the given series solution converges to the solution
if $\| \omega_{n+1} \| \leq \eta \| \omega_n \|$.

Proof: We determine that S_n is a Cauchy sequence in $(C[0,K], \|.\|)$, to ascertain the convergence of sequence $\{S_n\}$ of the partial sums of the series.

$$\|s_{n+1} - s_n\| = \| \omega_{n+1} \| \leq \eta \| \omega_n \|,$$

$$\leq \eta^2 \|\omega_{n-1}\| \leq \cdots \leq \eta^{n+1} \|\omega_0\|,$$

$$\text{Here } \|s_n - s_m\| = \|\textstyle\sum_{i=m+1}^{n} \omega_i\| \le \sum_{i=m+1}^{n} \| \omega_i \|,$$

$$\le \eta^{m+1}\left(\textstyle\sum_{i=0}^{n-m} \eta^i\right) \|\omega_0\|,$$

$$= \eta^{m+1} \frac{(1-\eta^{n-m})}{(1-\eta)} \|\omega_0\|, \qquad n, m \in \mathbb{N}$$

As of $0 < \eta < 1$, we have

$$\|s_n - s_m\| \le \frac{\eta^{n+1}}{1-\eta} \|\omega_0\| ,$$

Moreover, ω_0 is bounded, so $\|s_n - s_m\| \to 0 \; as \; m, n \to \infty$. so $\{S_n\}$ is a Cauchy sequence in $C[0, K]$. As a result $\sum_{n=0}^{\infty} \omega_n(\varphi, t)$ is convergent.

3.3 Accelerated Homotopy Perturbation Elzaki Transformation Method

Assume the nonlinear PDE [173]

$$\frac{\partial^n \varphi}{\partial t^n} + L\varphi(x,t) + N\varphi(x,t) = G(x,t) \tag{3.3.1}$$

With condition $\varphi^i(x, 0) = k_i(x), i = 0,1,2 \dots \dots n-1$.

Using ET in equation (3.3.1) we obtain

$$E\left[\frac{\partial^n \varphi}{\partial t^n} + L\varphi(x,t) + N\varphi(x,t)\right] = E[G(x,t)], \tag{3.3.2}$$

When the Elzaki transformation characteristics are applied to equation (3.3.2), the result is

$$E[\varphi(x,t)] = \textstyle\sum_{k=0}^{n-1} v^{k+2} \varphi^k(x,0) + v^n \, E[G(x,t) - \{L\varphi(x,t) + N\varphi(x,t)\}], \tag{3.3.3}$$

using the inverse ET to equation (3.3.3)

$$\varphi(x,t) = \textstyle\sum_{k=0}^{n-1} \frac{t^k}{k!} \varphi^k(x,0) + E^{-1}\{v^n \, E[G(x,t) - \{L\varphi(x,t) + N\varphi(x,t)\}]\}, \tag{3.3.4}$$

Using the HPM on equation (3.3.4), we obtain

$$0 = (1-p)(\varphi(x,t) - \varphi(x,0) + p(\varphi(x,t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \varphi^k(x,0)$$
$$+ p\left\{E^{-1}\{v^n \, E[G(x,t) - \{L\varphi(x,t) + N\varphi(x,t)\}]\}\right\},$$

Here, $p \in [0,1]$ defined the parameter. Let

$$\varphi(x,t) = \textstyle\sum_{n=0}^{\infty} \varphi_n p^n, \tag{3.3.5}$$

$$N\varphi(x,t) = \textstyle\sum_{n=0}^{\infty} \widetilde{H}_n p^n, \tag{3.3.6}$$

Where $\widetilde{H}_n$ represent accelerated He's polynomial with

$$\widetilde{H}_n(\varphi_0, \varphi_1, \varphi_2, \dots \dots . \varphi_n) = N(S_n) - \textstyle\sum_{i=0}^{n-1} \widetilde{H}_i, \tag{3.3.7}$$

$\widetilde{H}_n = N(\varphi(x_0))$, and $S_k = (\varphi_1 + \varphi_2 + \varphi_3 \dots \dots \varphi_k)$ using equation (3.3.5), (3.3.6) and (3.3.7) in equation (3.3.4) gives

$$\sum_{n=0}^{\infty} \varphi_n p^n = \varphi(x,0) + p\left\{\sum_{k=0}^{n-1} \frac{t^k}{k!}\varphi^k(x,0) + E^{-1}\left\{v^n\, E\left[G(x,t) - \left\{L\sum_{n=0}^{\infty}\varphi_n p^n + \sum_{n=0}^{\infty}\widetilde{H}_n p^n\right\}\right]\right\}\right\}, \tag{3.3.8}$$

When we compare the coefficients of the comparable powers of p, we obtain

$$p^0 = \varphi_0 = \varphi(x,0),$$

$$p^1 : \varphi_1 = \sum_{k=0}^{n-1} \frac{t^k}{k!}\varphi^k(x,0) + E^{-1}\left\{v^n\, E\left\{G(x,t) - \{L\varphi_0 + \widetilde{H}_0\}\right\}\right\},$$

$$p^2 : \varphi_2 = -E^{-1}\left\{v^n\, E\{L\varphi_0 + \widetilde{H}_0\}\right\},$$

$$\vdots$$

As a result, when $p \to 1$ the solution of equation is obtained as

$$\varphi(x,t) = \sum_{n=0}^{\infty} \varphi_n. \tag{3.3.9}$$

Example 3.3.1: Let us assume the homogenous advection problem [173]

$$u_t + uu_x = 0. \tag{3.3.10}$$

with initial condition$u(x,0) = -x$.

We have the Elzaki transform of equation (3.3.10)

$$E[u(x,t)] = -xv^2 - vE[uu_x], \tag{3.3.11}$$

Using the inverse Elzaki transform means that

$$u(x,t) = -x - E^{-1}\{vE[uu_x]\},$$

Applying AHPETM to equation (3.3.11) yields

$$\sum_{n=0}^{\infty} u_n(x,t)\, p^n = -x - p\,\{E^{-1}\{vE[\sum_{n=0}^{\infty}\widetilde{H}_n P^n]\,\}\}, \tag{3.3.12}$$

First terms of $\widetilde{H}_n$ are given as

$$\widetilde{H}_0 = x;$$

$$\widetilde{H}_1 = t(2+t)x;$$

$$\widetilde{H}_2 = \frac{1}{9}t^2(18+24t+15t^2+6t^3+t^4)x;$$

$$\widetilde{H}_3 = \frac{1}{3969}t^3(5292+10584t+13230t^2+12348t^3+8946t^4+5418t^5+2772t^6+1155t^7+378t^8+91t^9+14t^{10}+t^{11})x;$$

$$\vdots$$

When we examine the like powers of p in equation (3.3.12), we obtain

$$p^0: u_0 = \text{-}x,$$

$$p^1: u_1 = -E^{-1}\{vE[\widetilde{H}_0 u]\} = -xt,$$

$$p^2: u_2 = -E^{-1}\{vE[\widetilde{H}_1 u]\} = -xt - \frac{xt^3}{3},$$

$$p^3: u_3 = -E^{-1}\{vE[\widetilde{H}_2 u]\} = \frac{1}{63}(-42t^3x - 42t^4x - 21t^5x - 7t^6x - t^7x),$$

$$\begin{aligned} p^4: u_4 &= -E^{-1}\{vE[\widetilde{H}_3 u]\} \\ &= \frac{1}{238140}(-79380t^4x - 127008t^5x - 132300t^6x - 105840t^7x \\ &- 67095t^8x - 36120t^9x - 16632t^{10}x - 6300t^{11}x - 1890t^{12}x \\ &- 420t^{13}x - 60t^{14}x - 4t^{15}x), \end{aligned}$$

$$\vdots$$

Taking $p \rightarrow 1$ yields the solution of the problem.

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_o + u_1 + u_2 + \cdots$$

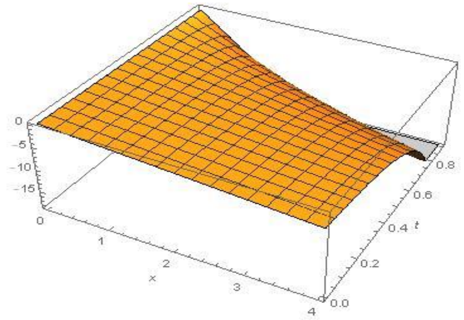
$$\begin{aligned} u(x,t) - x - tx - t^2x - \frac{t^3x}{3} + \frac{1}{63}(-42t^3x - 42t^4x - 21t^5x - 7t^6x - t^7x) \\ + \frac{1}{238140}(-79380t^4x - 127008t^5x - 132300t^6x - 105840t^7x \\ - 67095t^8x - 36120t^9x - 16632t^{10}x - 6300t^{11}x - 1890t^{12}x \\ - 420t^{13}x - 60t^{14}x - 4t^{15}x) + \cdots \end{aligned}$$

The precise answer to equation (3.3.10) is

$$u(x,t) = \frac{x}{t-1}.$$

the approximate solution of eq.(3.3.10) is given by

$$u(x,t) = -x - tx - t^2x - t^3x - t^4x - \cdots$$



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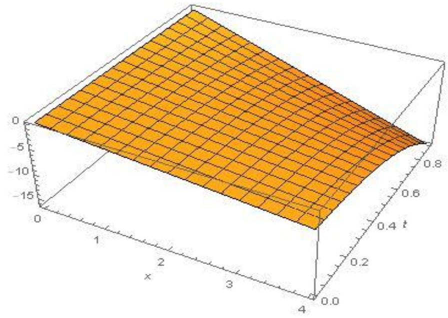

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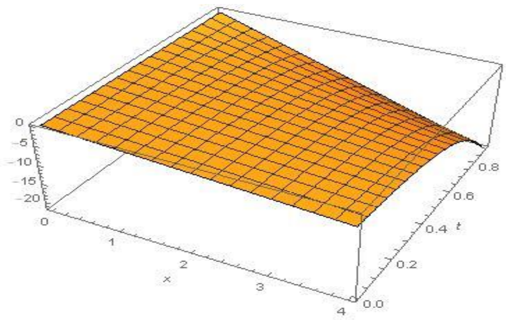
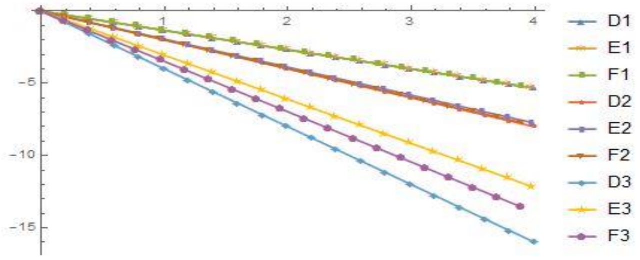


(c)

Fig 3.3.1 (i): (a) surface graph of exact solution at $0 \le x \le 4, 0 \le t \le 0.9$, (b) surface graph HPTM at $0 \le x \le 4, 0 \le t \le 0.9$, (c) surface graph of AHPETM at $0 \le x \le 4, 0 \le t \le 0.9$.



(d)

Fig.3.2.1 (ii) :(d) Line graph where D1= u_{Exact} , E1=u_{HPTM}, F1=u_{AHETM} at $t = 0.25$, $0 \le x \le 4$, ;

D2= u_{Exact} , E2=u_{HPTM}, F2=u_{AHETM} at $t = 0.5$, $0 \le x \le 4$, ; D3= u_{Exact} , E3=u_{HPTM}, F3=u_{AHETM} at $t = 0.75$, $0 \le x \le 4$.

Example 3.3.2: Assuming the system of PDE [21,173]

$$\frac{\partial u}{\partial t} - v\left(\frac{\partial u}{\partial x}\right)^2 = 0, \qquad (3.3.13)$$

$$\frac{\partial v}{\partial t}+u(\frac{\partial v}{\partial x})^2=0, \tag{3.3.14}$$

with initial conditions

$$u(x,0)=e^x, v(x,0)=e^{-x}$$

Taking Elzaki transform of equations (3.3.13) and (3.3.14), we have

$$E[u(x,t)]=e^x v^2+vE[v(u_x)^2],$$

$$E[v(x,t)]=e^{-x}v^2-vE[u(u_x)^2],$$

Inverse Elzaki transform given as:

$$u(x,t)=e^x+E^{-1}\{vE[v(u_x)^2]\},$$

$$v(x,t)=e^{-x}-E^{-1}\{vE[u(u_x)^2]\},$$

Now apply AHPETM

$$\sum_{n=0}^{\infty} u_n(x,t)\,p^n=e^x+p\,\{E^{-1}\{vE[\sum_{n=0}^{\infty}\widetilde{H}_n P^n]\}\}, \tag{3.3.15}$$

$$\sum_{n=0}^{\infty} v_n(x,t)\,p^n=e^{-x}-p\,\{E^{-1}\{vE[\sum_{n=0}^{\infty}\widetilde{H}_n P^n]\}\}, \tag{3.3.16}$$

And the first few terms of $\widetilde{H}_n$ are given as

$$\widetilde{H}_0(u)=e^x,$$

$$\widetilde{H}_0(v)=e^{-x},$$

$$\widetilde{H}_1(u)=-e^x t(-1+t+t^2),$$

$$\widetilde{H}_1(v)=e^{-x} t(-1-t+t^2),$$

$$\widetilde{H}_2(u)=-\frac{1}{1728}e^x t^2(-2592-1152t-288t^2-720t^3+264t^4+864t^5+384t^6 -316t^7-210t^8+36t^9+27t^{10}),$$

$$\widetilde{H}_2(v)=-\frac{1}{1728}e^{-x} t^2(-2592+1152t-288t^2+720t^3+264t^4-864t^5+384t^6 +316t^7-210t^8-36t^9+27t^{10}$$

$$\vdots$$

When we examine the like powers of p in equations (3.3.15) and (3.3.16), we obtain

$$p^0: u_0(x,t)=e^x,$$

$$p^0: v_0(x,t)=e^{-x},$$

$$p^1: u_1=E^{-1}\{vE[\widetilde{H}_0 u]\}=te^x,$$

$$p^1: v_1 = -E^{-1}\{vE[\widetilde{H}_0 v]\} = -te^{-x},$$

$$p^2: u_2 = E^{-1}\{vE[\widetilde{H}_1 u]\} = -\frac{1}{12}e^x t^2(-6+4t+3t^2),$$

$$p^2: v_2 = -E^{-1}\{vE[\widetilde{H}_1 v]\} = -\frac{1}{12}e^{-x} t^2(-6-4t+3t^2),$$

$$\begin{aligned} p^3: u_3 &= E^{-1}\{vE[\widetilde{H}_2 u]\} \\ &= -\frac{1}{25945920}e^x t^3(-12972960 - 4324320t - 864864t^2 - 1801800t^3 \\ &+ 566280t^4 + 1621620t^5 + 640640t^6 - 474474t^7 - 286650t^8 \\ &+ 45045t^9 + 31185t^{10}) \end{aligned}$$

$$\begin{aligned} p^3: v_3 &= -E^{-1}\{vE[\widetilde{H}_2 v]\} \\ &= \frac{1}{25945920}e^{-x} t^3(-12972960 + 4324320t - 864864t^2 + 1801800t^3 \\ &+ 566280t^4 - 1621620t^5 + 640640t^6 + 474474t^7 - 286650t^8 \\ &- 45045t^9 + 31185t^{10})e^{-x} \end{aligned}$$

$$\vdots$$

Taking p→1 yields the solution to the problem.

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_o + u_1 + u_2 + \cdots$$

$$v(x,t) = \sum_{n=0}^{\infty} v_n = v_0 + v_1 + v_3 + \cdots$$

$$\begin{aligned} u(x,t) = e^x \Bigg(&1 + t + \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{12} + \frac{t^5}{30} + \frac{5t^6}{72} - \frac{11t^7}{504} - \frac{t^8}{16} - \frac{2t^9}{81} + \frac{79t^{10}}{4320} + \frac{35t^{11}}{3168} \\ &- \frac{t^{12}}{576} - \frac{t^{13}}{832}\Bigg) + \cdots \end{aligned}$$

$$\begin{aligned} v(x,t) = \frac{1}{25945920}e^{-x}(&25945920 - 25945920t + 12972960t^2 - 4324320t^3 \\ &- 2162160t^4 - 864864t^5 + 1801800t^6 + 566280t^7 - 1621620t^8 \\ &+ 640640t^9 + 474474t^{10} - 286650t^{11} - 45045t^{12} + 31185t^{13}) + \cdots \end{aligned}$$

The exact solution of equation (3.3.13) and (3.3.14) is

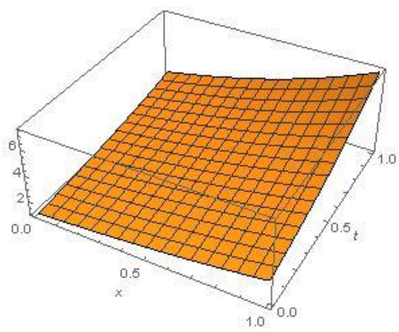
$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = e^{x+t}$$

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) = e^{x-t}$$

Approximate solution of (3.3.13) and (3.3.14) using HPTM [21] is

$$u(x,t) = e^{x} + e^{x}t + \frac{e^{x}t^{2}}{2} + \frac{e^{x}t^{3}}{6} + \cdots$$

$$v(x,t) = e^{-x} - e^{-x}t + \frac{1}{2}e^{-x}t^{2} - \frac{1}{6}e^{-x}t^{3} + \cdots$$



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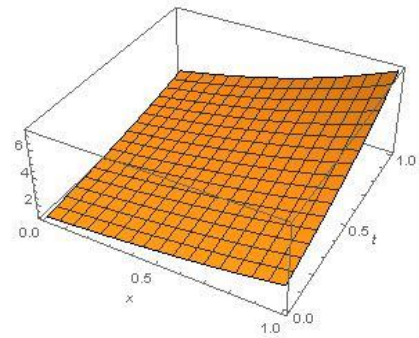

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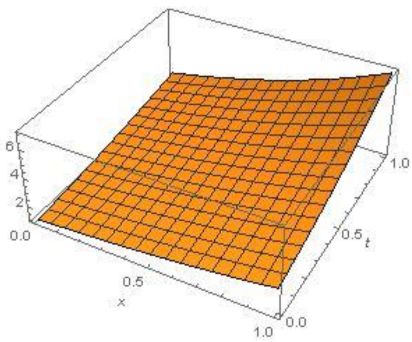
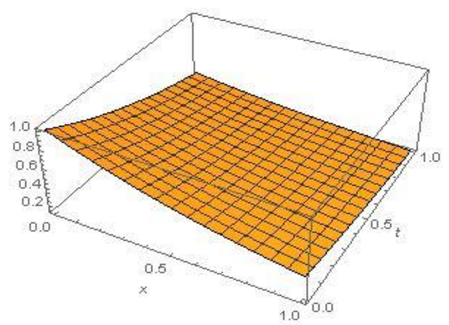


(c)

Fig. 3.3.2(i): (a) surface graph of u_{Exact} of eq. (3.2.13) at $0 \le x \le 1, 0 \le t \le 1$, (b) surface graph of u_{HPTM} of eq. (3.2.13) at $0 \le x \le 1, 0 \le t \le 1$, (c) surface graph of u_{AHETM} of eq. (3.2.13) at $0 \le x \le 1, 0 \le t \le 1$.



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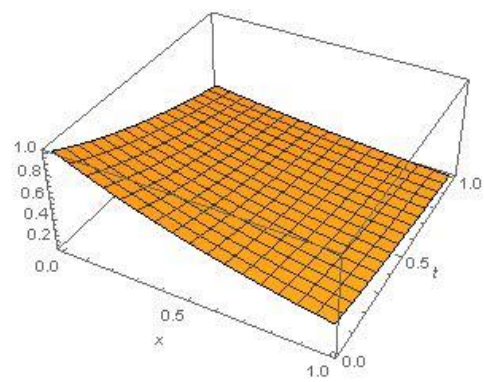

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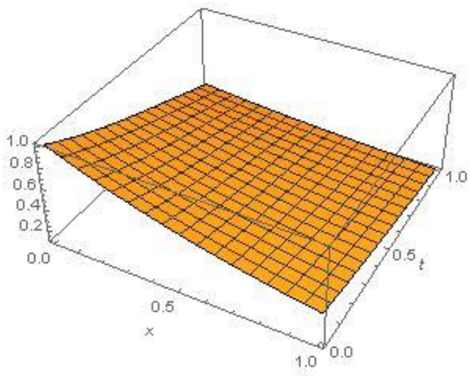
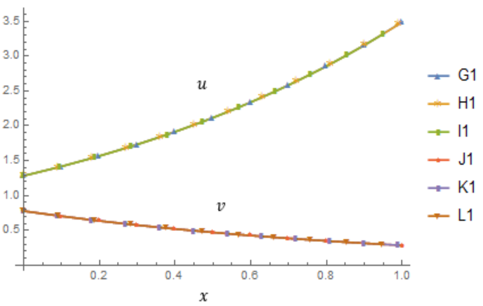


(c)

Fig. 3.3.2(ii): (a) surface graph of v_{Exact} of eq. (3.2.14) at $0 \le x \le 1, 0 \le t \le 1$, (b) surface graph of v_{HPTM} of eq. (3.2.14) at $0 \le x \le 1, 0 \le t \le 1$, (c) surface graph of v_{AHETM} of eq. (3.2.14) at $0 \le x \le 1, 0 \le t \le 1$.



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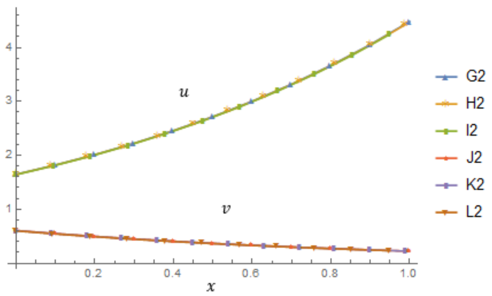

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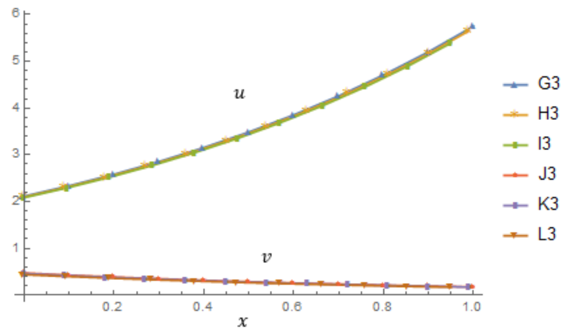


(c)

Fig 3.3.2 (iii): **(a)**line graph of G1= u_{Exact} , H1=u_{HPTM}, I1=u_{AHETM} at $t = 0.25$, $0 \le x \le 1$, J1= v_{Exact} , K1=v_{HPTM}, L1=v_{AHETM} at $t = 0.25$, $0 \le x \le 1$, **(b)** G2= u_{Exact} , H2=u_{HPTM}, I2=u_{AHETM} at $t = 0.5$, $0 \le x \le 1$, J2= v_{Exact} , K2=v_{HPTM}, L2=v_{AHETM} at $t = 0.5$, $0 \le x \le 1$, **(c)** : G3= u_{Exact} , H3=u_{HPTM}, I3=u_{AHETM} at $t = 0.75$, $0 \le x \le 1$,J3= v_{Exact} , K3=v_{HPTM}, L3=v_{AHETM} at $t = 0.75$, $0 \le x \le 1$.

Example 3.3.3:.Examine the problem of non-homogeneous advection. [97]

$$u_t + uu_x = 2t + x + t^3 + xt^2 \tag{3.3.17}$$

With initial condition $u(x,0) = 0$

Taking ET of eq. (3.3.17), we have

$$E[u(x,t)] = vE[2t + x + t^3 + xt^2 - uu_x] \tag{3.3.18}$$

Take the inverse of ET of equation (3.3.18) we get

$$[u(x,t)] = E^{-1}\{vE[2t + x + t^3 + xt^2 - uu_x]\} \tag{3.3.19}$$

Now, when we apply AHPETM to equation (3.3.19), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t)\, p^n = p\,\{E^{-1}\{vE[\sum_{n=0}^{\infty} u_n p^n - \sum_{n=0}^{\infty} \widetilde{H}_n P^n]\}\} \tag{3.3.20}$$

And the first few of terms of $\widetilde{H}_n$ are represented as

$$\widetilde{H}_0(u) = 0$$

$$\widetilde{H}_1(u) = t^3 + \frac{7t^5}{12} + \frac{t^7}{12} + t^2x + \frac{2t^4x}{3} + \frac{t^6x}{9}$$

$$\widetilde{H}_2(u) = -\frac{7t^5}{12} - \frac{113t^7}{360} - \frac{53t^9}{2016} + \frac{7t^{11}}{540} + \frac{19t^{13}}{6480} + \frac{t^{15}}{6048} - \frac{2t^4x}{3} - \frac{17t^6x}{45} - \frac{2t^8x}{63} + \frac{4t^{10}x}{225} + \frac{4t^{12}x}{945} + \frac{t^{14}x}{3969}$$

$$\vdots$$

By contrasting the equivalent powers of p in equation (3.3.20), we obtain

$$p^0: u_0(x,t) = 0$$

$$p^1: u_1(x,t) = t^2 + \frac{t^4}{4} + tx + \frac{t^3x}{3}$$

$$p^2: u_2(x,t) = -\frac{1}{10080}t^3(2520t + 980t^3 + 105t^5 + 3360x + 1344t^2x + 160t^4x)$$

$$p^3: u_3(x,t) = \frac{1}{21794572800}(2118916800t^6 + 855134280t^8 + 57297240t^{10} - 23543520t^{12} - 4564560t^{14} - 225225t^{16} + 2905943040t^5x + 1176215040t^7x + 76876800t^9x - 35223552t^{11}x - 7096320t^{13}x - 366080t^{15}x)$$

$$\vdots$$

the solution of equation is obtained by taking $p \to 1$

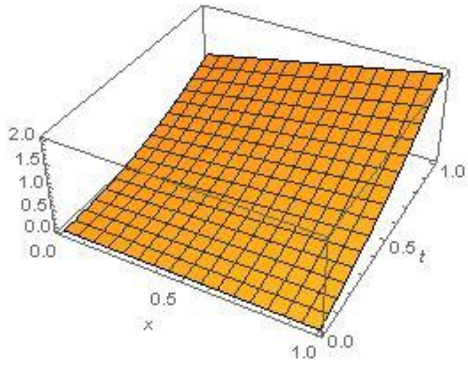
$$u(x,t)=\sum_{n=0}^{\infty}u_n=u_o+u_1+u_2+\cdots$$

$$u(x,t)=t^2+\frac{83t^8}{2880}+\frac{53t^{10}}{20160}-\frac{7t^{12}}{6480}-\frac{19t^{14}}{90720}-\frac{t^{16}}{96768}+tx+\frac{4t^7x}{105}+\frac{2t^9x}{567}-\frac{4t^{11}x}{2475}-\frac{4t^{13}x}{12285}-\frac{t^{15}x}{59535}+\cdots$$

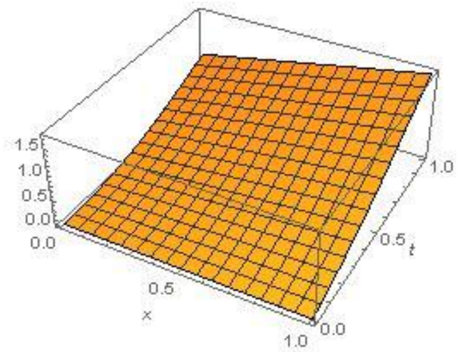
The exact solution is $u(x,t)=t^2+xt$

Also, the approximate HPTM [21] solution is given by

$$u(x,t)=t^2-\frac{7t^6}{72}-\frac{t^8}{96}+tx-\frac{2t^5x}{15}-\frac{t^7x}{63}\ldots$$



(a)



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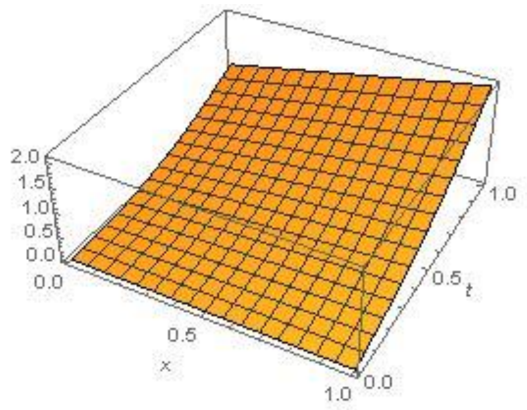
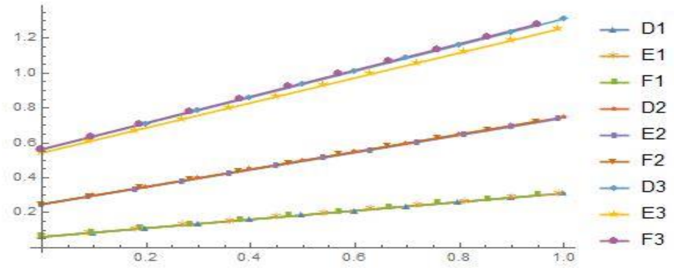


(c)

Fig 3.3.3 (i): (a) surface graph of Exact solution at $0\le x\le 1, 0\le t\le 1$**, (b)** surface graph HPTM at $0\le x\le 1, 0\le t\le 1$**, (c)** Surface graph of AHPETM at $0\le x\le 1$,$0\le t\le 1$.



(d)

Fig. 3.3.3(ii): (d) Line graph where D1= u_{Exact} , E1=u_{HPTM}, F1=u_{AHETM} at $t = 0.25, 0 \leq x \leq 1,$; D2= u_{Exact} , E2=u_{HPTM}, F2=u_{AHETM} at $t = 0.5, 0 \leq x \leq 4,$; D3= u_{Exact} , E3=u_{HPTM}, F3=u_{AHETM} at $t = 0.75, 0 \leq x \leq 1.$

Conclusion:

In summary, this chapter provides a comprehensive outline of the methodology employed to fulfill the objectives of the study. We utilized nonlinear Partial Differential Equations (PDEs), encompassing the homogeneous advection problem and the nonhomogeneous advection problem within the study's framework. Our approach involved extensive computations in Mathematica to verify the efficacy of our methods, yielding approximate solutions. Subsequently, we visually represented these findings through surface and line graphs.

Chapter 4

Solution of Nonlinear Partial Differential Equations using Accelerated Homotopy Perturbation Transformation Method and Accelerated Homotopy Perturbation Sumudu Transformation Method

4.1 Introduction:

In this chapter, the investigation of nonlinear partial differential equations (PDEs) is further advanced using sophisticated semi-analytical methods. Building upon the hybrid techniques introduced in Chapter 3, this chapter presents a more extensive application of these methods to a diverse range of nonlinear equations, showcasing their versatility and adaptability. By broadening the scope of these techniques, a more comprehensive evaluation of their efficacy and utility is provided, demonstrating their potential and effectiveness in solving intricate mathematical problems. In this chapter, we delve into two distinct methodologies for tackling nonlinear PDEs. The first method is the Accelerated Homotopy Perturbation Transformation Method (AHPTM), a hybrid technique formed by merging the HPM with the LT. This method is applied to solve both one-dimensional (1D) and two-dimensional (2D) Burgers' equations. The secondary method utilised is the Homotopy Perturbation Sumudu Transform Method (HPSTM), formed by combining the HPM with the ST. With this method, we address the 1D Benjamin-Bona-Mahoney-Burgers equation. Additionally, we conduct convergence analysis, presenting the results in a tabular format. Furthermore, we compare the obtained solutions with exact solutions and visualize the outcomes through surface and line graphs generated using Mathematica.

4.2 Accelerated Homotopy Perturbation Transformation Method

To understand the basic idea behind this method, examine the general problem of nonlinear, non-homogeneous partial differential equations with an initial condition.

$$\frac{\partial\theta}{\partial t} + R\theta(x,t) + N\theta(x,t) = g(x,t), \qquad (4.2.1)$$

$$\theta(x,0) = h(x,t),$$

When we apply LT to both sides of the eq. (4.2.1), we obtain

$$\mathcal{L}\left\{\frac{\partial\theta}{\partial t}+R\theta(x,t)+N\theta(x,t)\right\}=\mathcal{L}[g(x,t)]\,, \tag{4.2.2}$$

Then, applying properties of LT to equation (4.2.2), we have

$$\theta(x,s)=\frac{1}{s}\theta(x,0)+\frac{1}{s}\mathcal{L}[g(x,t)\ -\{R\theta(x,t)+N\theta(x,t)\}], \tag{4.2.3}$$

Further, by applying the inverse LT to equation (4.2.3), we get

$$\theta(x,t)=\theta(x,0)-\mathcal{L}^{-1}\{\frac{1}{s}\ \mathcal{L}[g(x,t)-\{R\theta(x,t)+N\theta(x,t)\}]\}\,, \tag{4.2.4}$$

Now, use the HPM. We have

$$0=(1-p)(\theta(x,t)-\theta(x,0)+p\ (\theta(x,t)-\mathcal{L}^{-1}\{\frac{1}{s}\ \mathcal{L}[g(x,t)-\{R\theta(x,t)+N\theta(\mathrm{x,t})\}]\},$$

where $p\in[0,1]$ is a parameter. Let

$$\theta(x,t)=\sum_{n=0}^{\infty}p^n\theta_n, \tag{4.2.5}$$

and nonlinear can be decompose as

$$N\theta(x,t)=\sum_{n=0}^{\infty}p^n\widetilde{H}_n\,(V), \tag{4.2.6}$$

Where $\widetilde{H}_n$ denotes accelerated He's polynomial with

$$\widetilde{H}_{\mathrm{n}}(\theta_0,\theta_1,\theta_2\,....\,\theta_n)\ =N(S_k)-\sum_{j=0}^{n-1}\widetilde{H}_j\,,n\geq 1 \tag{4.2.7}$$

$\widetilde{H}_n=N(\theta(x_0))$, and $S_k=(\theta_0+\theta_1+\theta_2\ldots\ldots..+\theta_k)$ using equation (4.2.5), (4.2.6) and (4.2.7) in equation (4.2.4) gives

$$\sum_{n=0}^{\infty}p^n\theta_n=\theta(x,0)-p\left(\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\big[G(x,t)-[\,R\sum_{n=0}^{\infty}p^n\theta_n+\sum_{n=0}^{\infty}p^n\widetilde{H}_n]\big]\right]\right),$$

By comparing the coefficients of corresponding powers of p, we obtain

$$p^0\colon\theta_0(x,t)\ =\theta(x,0),$$

$$p^1\colon\theta_1=\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\big[G(x,t)-\{R\theta_0+\widetilde{H}_0\}\big]\right\},$$

$$p^2\colon\theta_2=-\mathcal{L}^{-1}\left\{\frac{1}{s}\big[R\theta_1+\widetilde{H}_1\big]\right\},$$

$$\vdots$$

Therefore, the Acc. HPTM series solution of equation (4.2.1) is obtained as $p\to 1$

$$\theta(x,t) = \sum_{n=0}^{\infty} \theta_n.$$

Example 4.2.1: Consider the Burgers' equation [111]

$$V_t + VV_x = V_{xx} \tag{4.2.8}$$

With initial condition

$$V(x,0) = x,$$

and $$V(x,t) = \frac{x}{1+t},$$

Apply the LT to equation (4.2.8), we get

$$\mathcal{L}[V(x,t)] = \frac{x}{s} + \mathcal{L}[V_{xx} - VV_x]$$

Using the inverse of the LT, we get

$$V(x,t) = x + \ \mathcal{L}^{-1}\left[\frac{1}{s}(\mathcal{L}[V_{xx} - VV_x])\right]$$

Now, use AHPTM

$$\sum_{n=0}^{\infty} V_n\,(x,t) = x + p\mathcal{L}^{-1}\left[\frac{1}{s}\left(\mathcal{L}\left[\sum_{n=0}^{\infty} p^n V(x,t) - \sum_{n=0}^{\infty} p^n \widetilde{H}_n(V)\right]\right)\right]$$

Where $\widetilde{H}_n(V)$ denotes the polynomials that represent the nonlinear term.

$$\widetilde{H}_0(V) = V_0 V_{0x} = x$$

$$\widetilde{H}_1(V) = V_0 V_{1x} + V_1 V_{0x} + V_1 V_{1x}$$

$$\widetilde{H}_2(V) = V_0 V_{2x} + V_1 V_{2x} + V_2 V_{0x} + V_2 V_{1x} + V_2 V_{2x}$$

$$\vdots$$

Contrasting the coefficient of p we have

$$p^0 : V_0(x,t) = x$$

$$p^1: V_1(x,t) = -xt$$

$$p^2: V_2(x,t) = xt^2 - \frac{xt^3}{3}$$

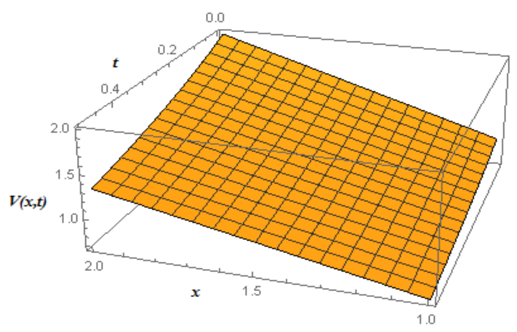
$$p^3: V_3(x,t) = \frac{1}{63}(-42xt^3 + 42xt^4 - 21xt^5 + 7xt^6 - xt^7)$$

$$\vdots$$

Hence the solution of eq. (4.2.8) is

$$V(x,t) = x(1 - t + t^2 - t^3 + t^4 - t^5 + \cdots)$$

$$V(x,t) = \frac{x}{1+t}$$



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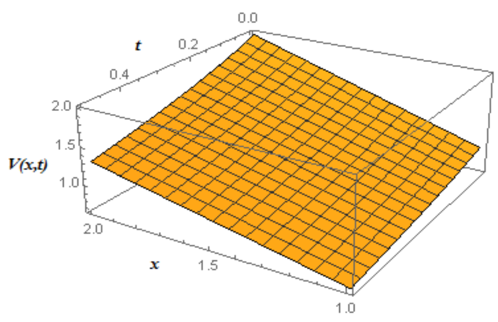

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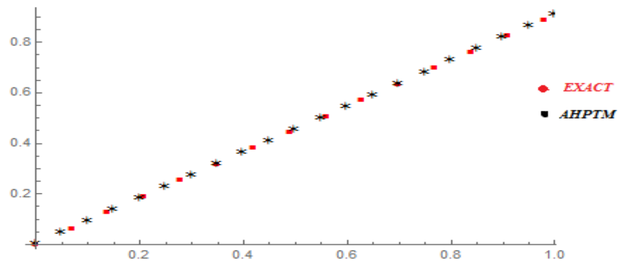


(c)

Fig.4.2.1: (a) surface graph of V_{EXACT} when $t = .1$ (b) surface graph of V_{AHPTM} when $t = .1$ (c) line graph of V_{Exact}, V_{HPTM} $at\ t = .1,\ 0 \leq x \leq 1$.

Table 4.2.1: Error analysis equation (4.2.8) when $t = 0.1, 0.3, 0.5$.

t	x	Exact Solution	AHPTM	Abs. Error	$\|\omega_1\|$	$\|\omega_2\|$	$\|\omega_3\|$
0.1	0.2	0.1818182	0.1818127	5.49324E-06	0.02	0.0019333	0.0001206
	0.4	0.36363636	0.3636254	1.09865E-05	0.04	0.0038667	0.0002413
	0.6	0.54545455	0.5454381	1.64797E-05	0.06	0.0058	0.0003619
	0.8	0.72727273	0.7272508	2.1973E-05	0.08	0.0077333	0.0004826
	1	0.90909091	0.9090634	2.74662E-05	0.1	0.0096667	0.0006032
0.3	0.2	0.15384615	0.1535335	0.000312648	0.06	0.0162	0.0026665
	0.4	0.30769231	0.307067	0.000625296	0.12	0.0324	0.005333
	0.6	0.46153846	0.4606005	0.000937944	0.18	0.0486	0.0079995
	0.8	0.61538462	0.614134	0.001250593	0.24	0.0648	0.010666
	1	0.76923077	0.7676675	0.001563241	0.3	0.081	0.0133325
0.5	0.2	0.13333333	0.1315724	0.001760913	0.1	0.0416667	0.0100942
	0.4	0.26666667	0.2631448	0.003521825	0.2	0.0833333	0.0201885
	0.6	0.4	0.3947173	0.005282738	0.3	0.125	0.0302827
	0.8	0.53333333	0.5262897	0.007043651	0.4	1.67E-01	0.040377
	1	0.66666667	0.6578621	0.008804563	0.5	0.2083333	0.0504712

Example .4.2.2: Consider the Burgers' equation [111]

$$V_t + VV_x = V_{xx} \tag{4.2.9}$$

With initial condition

$$V(x,0) = 1 - \frac{2}{x}\,, x > 0$$

and $$V(x,t) = 1 - \frac{2}{x-t}$$

Apply the LT to equation (4.2.9), we obtain

$$\mathcal{L}[V(x,t)] = \frac{1}{s}\left(1 - \frac{2}{x}\right) + \frac{1}{s}(\mathcal{L}[V_{xx} - VV_x])$$

After applying the inverse of the LT, we get

$$V(x,t) = \left(1 - \frac{2}{x}\right) + \mathcal{L}^{-1}\left[\frac{1}{s}(\mathcal{L}[V_{xx} - VV_x])\right]$$

Now apply AHPTM,

$$\sum_{n=0}^{\infty} V_n\,(x,t) = \left(1-\frac{2}{x}\right) + p\mathcal{L}^{-1}\left[\frac{1}{s}\left(\mathcal{L}\left[\sum_{n=0}^{\infty} p^n V(x,t) - \sum_{n=0}^{\infty} p^n \widetilde{H}_n(V)\right]\right)\right]$$

Where $\widetilde{H}_n(V)$ represent the polynomials that represent the nonlinear term.

$$\widetilde{H}_0(V) = V_0 V_{0x} = \frac{2(-2+x)}{x^3}$$

$$\widetilde{H}_1(V) = V_0 V_{1x} + V_1 U V_{0x} + V_1 V_{1x}$$

$$\widetilde{H}_2(V) = V_0 V_{2x} + V_1 V_{2x} + V_2 V_{0x} + V_2 V_{1x} + V_2 V_{2x}$$

$$\vdots$$

By comparison the coefficient of p, we get

$$p^0 : V_0(x,t) = 1 - \frac{2}{x}$$

$$p^1 : V_1(x,t) = -\frac{2t}{x^2}$$

$$p^2 : V_2(x,t) = -\frac{2t^2}{x^3} + \frac{8t^3}{3x^5}$$

$$p^3 : V_3(x,t) = -\frac{2t^3(4+3x)}{3x^5} + \frac{t^4(36+25x)}{3x^7} + \frac{4t^5(-28+9x)}{15x^8} - \frac{64t^6}{9x^9} + \frac{320t^7}{63x^{11}}$$

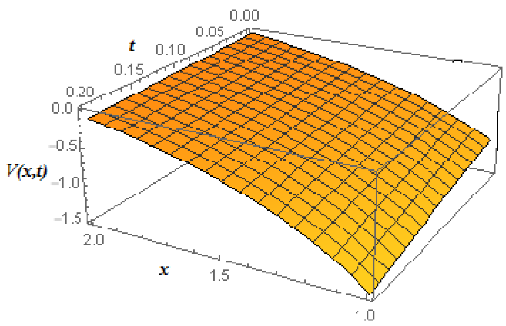
$$\vdots$$

Hence, the approximate series solution of eq. (4.2.9) is

$$V(x,t) = 1 - \frac{2}{x} - \frac{2t}{x^2} - \frac{2t^2}{x^3} - \frac{2t^3}{x^4} - \cdots$$

$$V(x,t) = 1 - \frac{2}{x}\left(1 + \frac{t}{x} + \frac{t^2}{x^2} + \frac{t^3}{x^3} + \cdots\right)$$

$$V(x,t) = 1 - \frac{2}{x-t}$$



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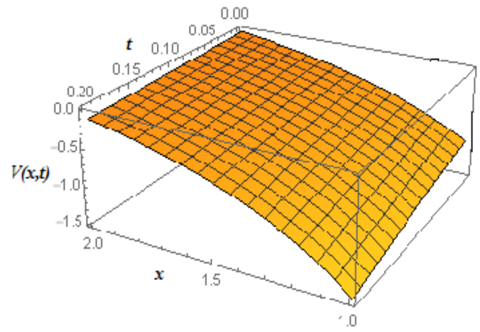

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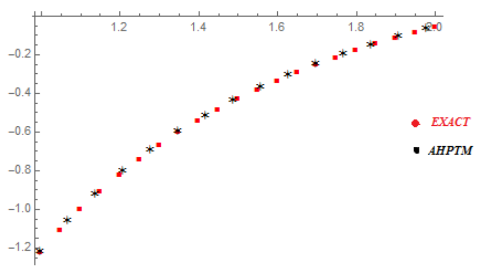


(c)

Fig 4.2.2: (a) surface graph of V_{EXACT} when $t = .1$ **(b)** surface graph of V_{AHPTM} when $t = .1$ **(c)** line graph of V_{Exact}, V_{HPTM} $at\ t = 0.1,\ 0 \le x \le 1.$

Table 4.2.2 –Error analysis of equation (4.2.9) at *t*=0.1,0.2,0.3.

t	*x*	Exact sol.	AHPTM	Absol. Error	$\|\omega_1\|$	$\|\omega_2\|$	$\|\omega_3\|$
0.1	1	1.2222222	1.2200239	0.002198286	0.2	0.01733333	0.0026906
	1.2	0.81818182	0.8174921	0.000689685	0.13888889	0.0105024	0.0014342
	1.4	0.53846154	0.5382001	0.000261446	0.10204082	0.0067928	0.000795
	1.6	0.33333333	0.3332196	0.000113774	0.078125	0.0046285	0.0004661
	1.8	0.17647059	0.1764156	5.49882E-05	0.05	0.00328823	0.0002879
0.2	1	1.5	1.4654781	0.034521905	0.4	0.05866667	0.0068114
	1.2	1	0.9890539	0.010946083	0.27777777	0.03772291	0.0068866
	1.4	0.66666667	0.6624898	0.00417689	0.20408163	0.02518792	0.0046488
	1.6	0.42857143	0.4267459	0.001825483	0.15625	0.01749674	0.0029992
	1.8	0.25	0.2491152	0.000884818	0.12345679	0.01258842	0.0019589
0.3	1	1.85714286	1.6856851	0.171457714	0.6	0.108	0.0223149
	1.2	1.22222222	1.1672566	0.054965609	0.41666667	0.07523148	0.0086918
	1.4	0.81818182	0.7970623	0.021119486	0.30612245	0.05221039	0.0101581
	1.6	0.53846154	0.5291901	0.009271464	0.234375	3.71E-02	0.0077362
	1.8	0.33333333	0.3288261	0.004507259	0.18518519	0.0270538	0.005476

Example 4.2.3: Given the (2+1)-dimensional Burgers' equation

$$V_t + VV_x + VV_y = V_{xx} + V_{yy} \tag{4.2.10}$$

$$V(x, y, 0) = x + y$$

And $$V(x, y, t) = \frac{x+y}{1+2t}$$

Apply the LT to equation (4.2.10), we get

$$\mathcal{L}[V(x, y, t)] = \frac{1}{s}\left(1 - \frac{2}{x}\right) + \frac{1}{s}\left(\mathcal{L}\left[(V_{xx} + V_{yy}) - (VV_x + VV_y)\right]\right),$$

After applying the inverse of the LT, we obtain

$$V(x, y, t) = (x + y) + \mathcal{L}^{-1}\left[\frac{1}{s}\left(\mathcal{L}\left[(V_{xx} + V_{yy}) - (VV_x + VV_y)\right]\right)\right],$$

Now apply AHPTM,

$$\sum_{n=0}^{\infty} V_n\,(x, y, t) = (x + y) + p\mathcal{L}^{-1}\left[\frac{1}{s}\left(\mathcal{L}\left[\sum_{n=0}^{\infty} p^n V(x, t) - \sum_{n=0}^{\infty} p^n \widetilde{H}_n(V)\right]\right)\right],$$

$\widetilde{H}_n(V)$ denote the polynomials that represent the nonlinear terms.

$$\widetilde{H}_0(V) = V_0V_{0x} + V_0V_{0y} = 2(x + y),$$

$$\widetilde{H}_1(V) = V_0V_{1x} + V_1V_{0x} + V_1V_{1x} + V_0V_{1y} + V_1V_{0y} + V_1V_{1y},$$

$$\widetilde{H}_2(V) = V_0V_{2x} + V_1V_{2x} + V_2V_{0x} + V_2V_{1x} + V_2V_{2x} + V_0V_{2y} + V_1V_{2y} + V_2V_{0y} + V_2V_{1y} + V_2V_{2y},$$

$$\vdots$$

Comparing the coefficient of *p*, we obtain

$$p^0\colon V_0(x, y, t) = x + y,$$

$$p^1\colon V_1(x, y, t) = -2t(x + y),$$

$$p^2\colon V_2(x, y, t) = 4t^2(x + y) - \frac{8}{3}t^3(x + y),$$

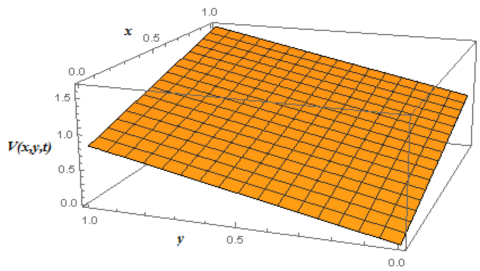
$$p^3: V_3(x,y,t) = -\frac{16}{3}t^3(x+y) + \frac{32}{3}t^4(x+y) - \frac{32}{3}\mathrm{t}^5(x+y) + \frac{64}{9}\mathrm{t}^6(x+y) - \frac{128}{63}\mathrm{t}^7(x+y),$$

$$\vdots$$

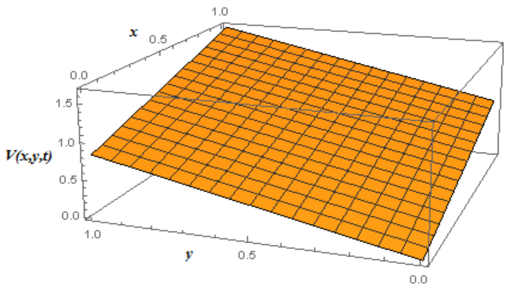
Hence the solution of eq. (4.2.10) is

$$V(x,y,t) = -\frac{2}{63}\mathrm{t}(63 - 126\mathrm{t} + 252\mathrm{t}^2 - 336\mathrm{t}^3 + \cdots)(x+y),$$

$$V(x,y,t) = \frac{x+y}{1+2\mathrm{t}}$$



(a)



(b)

Fig .4.2.3(a) surface graph of V_{EXACT}, when $t = .1$ (b) surface graph of V_{AHPTM}, when $t = .1$

4.3 Accelerated Homotopy Perturbation Sumudu Transformation Method

Examine the nonlinear equations below to better understand the proposed technique. [56]

$$\frac{\partial^n \psi}{\partial t^n} + L\psi(x,t) + N\psi(x,t) = K(x,t) \qquad (4.3.1)$$

With condition $\quad \psi^i(\mathrm{x},0) = \mathrm{k_i}(x), \mathrm{i} = 0,1,2 \ldots \ldots \mathrm{n}-1$

Using the ST in eq. (4.3.1) we get,

$$S\left[\frac{\partial^n \psi}{\partial t^n} + L\psi(x,t) + N\psi(x,t)\right] = S[K(x,t)], \qquad (4.3.2)$$

Using properties of ST to eq. (4.3.2), gives

$$S\{\psi(x,t)\} = u^n \sum_{k=0}^{k=n-1} \frac{1}{u^{n-k}} f^k(x,0) + u^n S\left(K(x,t) - \left(L\psi(x,t) + N\psi(x,t)\right)\right), \qquad (4.3.3)$$

Applying the inverse ST to eq. (4.3.3), we have

$$\{\psi(x,t)\} = u^n \sum_{k=0}^{k=n-1} \frac{t^k}{k!} f^k(x,0) + S^{-1}(u^n S(K(x,t) - (L\psi(x,t) + N\psi(x,t)))), \quad (4.3.4)$$

Utilising the HPM on equation (4.3.4), we get

$$0 = (1-p)(\psi(x,t) - \psi(x,0) + p\left(\psi(x,t) - \sum_{k=0}^{n-1} \frac{t^k}{k!}\psi^k(x,0)\right) - p\left\{S^{-1}\{u^n\, S[K(x,t) - \{L\psi(x,t) + N\psi(x,t)\}]\}\right\},$$

Where $p \in [0,1]$ is a parameter. Let

$$\psi(x,t) = \sum_{n=0}^{\infty} \psi_n p^n, \quad (4.3.5)$$

and

$$N\psi(x,t) = \sum_{n=0}^{\infty} \widetilde{H}_n p^n, \quad (4.3.6)$$

Where $\widetilde{H}_n$ represents the accelerated He's polynomial with

$$\widetilde{H}_{\mathrm{n}}\,(\psi_0, \psi_1, \psi_2, \dots \dots . \psi_n) = N(S_k) - \sum_{i=0}^{n-1} \widetilde{H}_i, \quad (4.3.7)$$

$\widetilde{H}_n = N(\psi_0)$, and $S_k = (\psi_0 + \psi_1 + \psi_2 + \psi_3 \dots \dots \psi_k)$. Using equation (4.3.5), (4.3.6) and (4.3.7) in equation (4.3.4) gives

$$\sum_{n=0}^{\infty} \psi_n p^n = \psi(x,0) + p\left\{\sum_{k=0}^{n-1} \frac{t^k}{k!}\psi^k(x,0) + S^{-1}\left\{u^n\, S\left[K(x,t) - \left\{L\sum_{n=0}^{\infty}\psi_n p^n + \sum_{n=0}^{\infty}\widetilde{H}_n p^n\right\}\right]\right\}\right\},$$

on contrasting the coefficient of the similar powers of p, we get

$$p^0 = \psi_0 = \varphi(x,0),$$

$$p^1 : \varphi_1 = \sum_{k=0}^{n-1} \frac{t^k}{k!}\varphi^k(x,0) + S^{-1}\left\{u^n\, S\left\{G(x,t) - \{L\varphi_0 + \widetilde{H}_n\}\right\}\right\},$$

$$p^2 : \psi_2 = -S^{-1}\left\{u^n\, S\{L\varphi_1 + \widetilde{H}_1\}\right\},$$

$$\vdots$$

The equation's solution is achieved by choosing $p \to 1$ i.e.-

$$\psi(x,t) = \sum_{n=0}^{\infty} \psi_n.$$

Example 4.3.1: Examine the one-dimensional (1D) time-dependent BBMB equation

$$\frac{\partial\varphi}{\partial t}=\frac{\partial^3\varphi}{\partial x^2\partial t}+\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial\varphi}{\partial x}+f(x,t)$$

with initial boundary conditions [172]

$$\varphi(x,0)=\sin x\,,\quad 0\le x\le\pi \tag{4.3.8}$$

$$\varphi(x,t)=e^{-t}\sin x\,,\ x\in 0,\pi,\ \ t\in(0,T]$$

and $$f(x,t)=e^{-t}\left(\cos x-\sin x+\frac{1}{2}\,e^{-t}\sin 2x\right).$$

By applying Sumudu transformation equation (4.3.8) subjected to the initial condition, we have

$$S\left(\frac{\partial\varphi}{\partial t}\right)=S\left(\frac{\partial^3\varphi}{\partial x^2\partial t}+\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial\varphi}{\partial x}\right)+S\big(f(x,t)\big),$$

or

$$S\left(\frac{\partial\varphi}{\partial t}\right)=u\left(S\left(\frac{\partial^3\varphi}{\partial x^2\,\partial t}+\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial\varphi}{\partial x}+\varphi\frac{\partial\varphi}{\partial x}\right)\right)+\frac{u}{(1+u)}\left(e^{-t}\left(\cos x-\sin x+\frac{1}{2}e^{-t}\sin 2x\right)\right), \tag{4.3.9}$$

Apply inverse Sumudu transformation in equation (4.3.9) we get,

$$\frac{\partial\varphi}{\partial t}=S^{-1}\left(u\left(\frac{\partial^3\varphi}{\partial x^2\,\partial t}+\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial\varphi}{\partial x}+\varphi\frac{\partial\varphi}{\partial x}\right)\right)+S^{-1}\left(\frac{u}{(1+u)}\left(e^{-t}\left(\cos x-\sin x+\frac{1}{2}e^{-t}\sin 2x\right)\right)\right), \tag{4.3.10}$$

Now apply AHPSTM on equation (4.3.10) we get,

$$\sum_{n=0}^{\infty}\varphi_n(x,t)\,p^n=\varphi(x,o)+p\left\{S^{-1}\left\{u\left[\sum_{n=0}^{\infty}\widetilde{H}_nP^n\right]\right\}\right\},$$

And the first few terms of $\widetilde{H}_n$ are given as

$$\widetilde{H}_0=\cos x\sin x,$$

$$\begin{aligned}\widetilde{H}_1=&-\cos x\sin x\\&+e^{-2t}\big(1+e^t(-1+t)\cos x\,(-1+\sin x)\\&+\big((1-e^tt)\sin x\big)\big((1-e^tt)\cos x\big)+\big(1+e^t(-1+t)\big)\cos^2 x\\&-\big(1+e^t(-1+t)\big)(-1+\sin x)\sin x\big),\end{aligned}$$

$$\widetilde{H}_2 = -e^{-2t}((1+e^t(-1+t))\cos x(-1+\sin x)+(1-e^t t)\sin x)((1-e^t t)\cos x + (1+e^t(-1+t))\cos^2 x - (1+e^t(-1+t))(-1+\sin x)\sin x) + \frac{1}{576}e^{-4t}((3+42e^t+e^{2t}(-21+48t-3t^2+2t^3))\cos x \ldots$$

$$\vdots$$

Comparing the similar powers of p , we get

$$p^0: \varphi_0 = \sin x,$$

$$\mathrm{p}^1: \varphi_1 = -\frac{1}{2}e^{-2t}\cos x\sin x + e^{-t}(-\cos x+\sin x) + t(-\cos x - \sin x - \cos x\sin x) + \frac{1}{2}(2\cos x - 2\sin x + \cos x\sin x),$$

$$p^2: \varphi_2 = \frac{1}{8}e^{-2t}(-6\cos^2 x - \cos^3 x + 8\cos x\sin x + 2\cos^2 x\sin x + 6\sin^2 x + 2\cos x\sin^2 x - \sin^3 x) + \frac{1}{6}e^{-3t}(\cos^3 x - 2\cos^2 x\sin x - 2\cos x\sin^2 x + \sin^3 x) + e^{-t}t\cos^3 x - 4\cos x\sin x - 2\cos^2 x\sin x - 2\cos x\sin^2 x + \sin^3 x + \frac{1}{2}e^{-t}2\cos x + 2\cos^2 x + \cos^3 x + 2\sin x - 4\cos x\sin x - 2\cos^2 x\sin x - 2\sin^2 x - 2\cos x\sin^2 x + \sin^3 x + \frac{1}{4}t^2(4\cos x + 4\cos^2 x + 3\cos^3 x + 4\cos x\sin x + 2\cos^2 x\sin x + 2\cos^3 x\sin x - 4\sin^2 x - 6\cos x\sin^2 x - \sin^3 x - 2\cos x\sin^3 x) + \frac{1}{4}e^{-2t}t(\cos^3 x + 2\cos^2 x\sin x\, 2\cos^3 x\sin x - 2\cos x\sin^2 x - \sin^3 x - 2\cos x\sin^3 x) + \frac{1}{16}e^{-4t}(\cos^3 x\sin x - \cos x\sin^3 x) - \frac{1}{3}t^3(\cos^2 x + \cos^3 x + 2\cos^2 x\sin x + \cos^3 x\sin x - \sin^2 x - 2\cos x\sin^2 x - \sin^3 x - \cos x) + \frac{1}{4}t(4\cos x - 2\cos^2 x - 2\cos^3 x + 12\sin x + 16\cos x\sin x - \cos^3 x\sin x + 2\sin^2 x + 4\cos x\sin^2 x + \cos x\sin^3 x) + \frac{1}{48}(-48\cos x - 12\cos^2 x - 26\cos^3 x - 48\sin x + 48\cos x\sin x + 52\cos^2 x\sin x - 3\cos^3 x\sin x + 12\sin^2 x + 52\cos x\sin^2 x - 26\sin^3 x + 3\cos x\sin^3 x),$$

$$p^3: \varphi_3 = \frac{1}{87091200}e^{-8t}(87091200e^{7t}\cos x - 87091200e^{8t}\cos x + 87091200e^{8t}t\cos x - 261273600e^{8t}t^2\cos x - 29030400e^{8t}t^3\cos x + \cdots$$

⋮

the solution of equation is obtained by taking $p \to 1$

$$\varphi(x,t) = \sum_{n=0}^{\infty} \varphi_n = \varphi_o + \varphi_1 + \varphi_2 + \cdots$$

$$\varphi(x,t) = \sin x - \frac{1}{2}e^{-2t}\cos x \sin x + e^{-t}(-\cos x + \sin x) + t(-\cos x - \sin x - \cos x \sin x) + \cdots$$

The exact solution is $\varphi(x,t) = \mathrm{e}^{-\mathrm{t}} \sin x.$

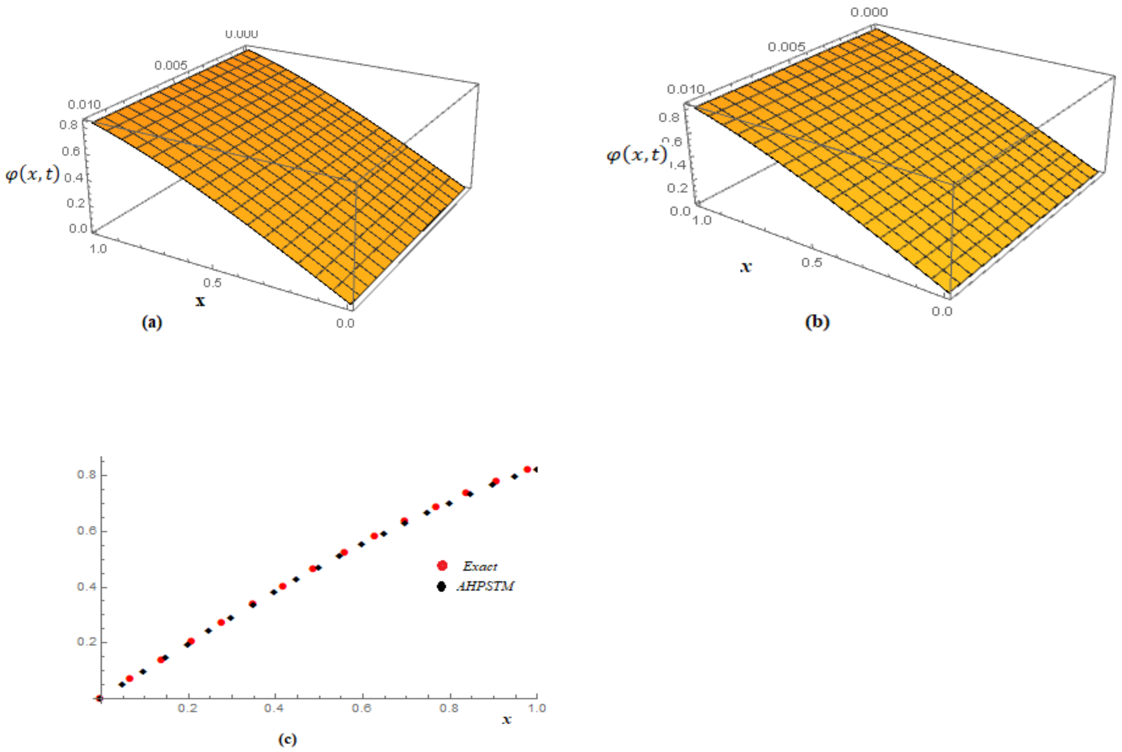


Fig.4.3.1: (a)surface graph of exact solution of example 3.1 at $0 \leq x \leq 1, at\ t = 0.01$, (b) surface graph of AHPSTM solution of example 3.1 at $0 \leq x \leq 1, at\ t = 0.01$, (c) line graph of : φ_{Exact} , φ_{AHSTM} at $t = 0.01$, $0 \leq x \leq 1$.

Conclusion:

In conclusion, this chapter gives a thorough overview of the methods used to achieve the study's objectives. To solve a number of problems, we employed nonlinear PDEs, including the Burgers' equation, the homogeneous advection problem, the nonhomogeneous advection problem, and the BBMB equation, in the context of this study. We performed computations in Mathematica to validate our methods, producing approximate results. These findings were then visualised using surface and line graphs.

Chapter 5

Solution of Nonlinear Fractional Partial Differential Equations using Accelerated Homotopy Perturbation Transformation Method in Caputo Sense

5.1 Introduction:

In this chapter, the focus shifts from solving regular nonlinear partial differential equations (PDEs) using hybrid techniques, such as the Accelerated Homotopy Perturbation Transformation Method (Acc. HPTM) and the Accelerated Homotopy Perturbation Sumudu Transformation Method (Acc. HPSTM), explored in the previous chapter, to addressing nonlinear fractional PDEs in the Caputo sense. The techniques previously employed for standard nonlinear PDEs are now adapted and extended in this chapter to solve fractional PDEs, demonstrating the continuity and versatility of the methodology while broadening its application to fractional calculus. This chapter provides a full explanation of the technique used to accomplish the research objectives set out in this study. We use nonlinear fractional PDEs in the Caputo sense to analyze different equations. This chapter provides comprehensive details about the methods proposed for achieving the research objectives in this study. In this study, a variety of equations, including but not limited to the Burgers' equation, Fisher's equation, and S-H equation, F-W equation are utilized and subsequently verified using Mathematica and the approximate series solution is obtained by utillising the approaches, which are then represented in the forms of surface and line graphs.

5 .2 Fractional Partial Differential Equations in Caputo sense:

To illustrate the core concept of this approach, consider a general nonlinear, non-homogeneous fractional PDE.

$$D_t^\alpha \omega(\varphi,t) + R(\varphi,t) + N\omega(\varphi,t) = g(\varphi,t), \text{ with condition } \omega(\varphi,0) = k(\varphi). \qquad (5.2.1)$$

Where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ denotes fractional Liouville-Caputo derivative of the function $\omega(\varphi,t)$, R is linear differential operator and N is nonlinear differential operator and $g(\varphi,t)$ represents the source term.

Apply LT on both side of eq. (5.2.1), we get

$$\mathcal{L}[D_t^{\alpha}\omega(\varphi,t)] + \mathcal{L}\omega[R(\varphi,t)] + \mathcal{L}[N\omega(\varphi,t)] = \mathcal{L}[g(\varphi,t)]\,, \qquad (5.2.2)$$

Apply basic properties LT to eq. (5.2.2), we get

$$\mathcal{L}[\omega(\varphi,t)] = \frac{1}{s^{\alpha}}\omega(\varphi,0) + \frac{1}{s^{\alpha}}\mathcal{L}\big(g(\varphi,t) - R\omega(\varphi,t) - N\omega(\varphi,t)\big), \qquad (5.2.3)$$

Implementing the inverse of the LT to the equation (5.2.3), we obtain

$$\omega(\varphi,t) = (\varphi,0) + \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\big(g(\varphi,t) - R\omega(\varphi,t) - N\omega(\varphi,t)\big)\right\}, \qquad (5.2.4)$$

using HPM, we get

$$0 = (1-p)[\omega(\varphi,t) - \omega(\varphi,0)] + p\left[\omega(\varphi,t) - \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\big(g(\varphi,t) - R\omega(\varphi,t) - N\omega(\varphi,t)\big)\right\}\right], \qquad (5.2.5)$$

where $p \in [0,1]$ is a parameter. Let

$$\omega(\varphi,t) = \sum_{n=0}^{\infty} p^n \omega_n(\varphi,t), \qquad (5.2.6)$$

and nonlinear term decompose as

$$N\omega(\varphi,t) = \sum_{n=0}^{\infty} p^n \widetilde{H}_n, \qquad (5.2.7)$$

Where $\widetilde{H}_n$ indicate accelerated He’s polynomial with

$$\widetilde{H}_{\mathrm{n}}(\omega_0,\omega_1,\omega_2 \ldots \omega_n) = N(S_k) - \sum_{i=0}^{n-1}\widetilde{H}_n, \qquad (5.2.8)$$

where $\widetilde{H}_0 = N(\omega(\varphi_0))$, and $S_k = (\omega_0 + \omega_1 + \cdots + \omega_k)$,

Substituting the equation (5.2.6), (5.2.7) in equation (5.2.5) we get,

$$\sum_{n=0}^{\infty} p^n \omega_n(\varphi,t) = \omega(\varphi,0) + p\left[\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left\{\sum_{n=0}^{\infty} p^n \omega_n(\varphi,t) - \sum_{n=0}^{\infty} p^n \widetilde{H}_n\big(\omega(\varphi,t)\big)\right\}\right\}\right], \qquad (5.2.9)$$

compare same powers of *p*, we obtain

$$p^0: \omega_0 = \omega(\varphi,0),$$

$$p^1: \omega_1 = \omega_1(\varphi,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\big[g(\varphi,t) - R\omega_0 - \widetilde{H}_0\omega\big]\right\},$$

$$p^2 : \omega_2 = \omega_2(\varphi, t) = -\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left[R\omega_1 + \widetilde{H}_1\omega\right]\right\},$$

$$p^3 : \omega_3 = \omega_3(\varphi, t) = -\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left[R\omega_2 + \widetilde{H}_2\omega\right]\right\},$$

$$\vdots$$

Hence, when $p \to 1$the series solution of equation (5.2.1) is obtained as

$$\omega(\varphi, t) = \omega_0 + \omega_1 + \omega_3 \ldots \tag{5.2.10}$$

Example 5.2.1: In the Liouville-Caputo sense, consider the non-linear time fractional S-H equation. [149,174].

$$\frac{\partial^\alpha \omega(\varphi,t)}{\partial t^\alpha} + \frac{\partial^4 \omega(\varphi,t)}{\partial \varphi^4} + (1-\beta)\omega(\varphi,t) + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2} - \omega^2(\varphi,t) + \left(\frac{\partial \omega(\varphi,t)}{\partial \varphi}\right)^2 = 0\,, 0 < \alpha \leq 1, \tag{5.2.11}$$

with initial condition $\omega(\varphi, 0) = e^\varphi$.

Using AHPTM, we get

$$\sum_{n=0}^{\infty} p^n \omega_n = e^\varphi - p\left[\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left[\left\{\frac{\partial^4 \omega}{\partial \varphi^4} + (1-\beta)\omega(\varphi,t) + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2}\right\} - \left\{\sum_{n=0}^{\infty} p^n \widetilde{H}_n(\omega)\right\}\right]\right\}\right] \tag{5.2.12}$$

the first few components of $\widetilde{H}_n$ are given as

$$\widetilde{H}_0 = (\omega_0)^2 - (\omega_{0\varphi})^2,$$

$$\widetilde{H}_1 = 2\omega_0\omega_1 + {\omega_1}^2 - \omega_{1\varphi}\left(2\omega_{0\varphi} + \omega_{1\varphi}\right),$$

$$\widetilde{H}_2 = 2\omega_0\omega_2 + 2\omega_1\omega_2 + (\omega_2)^2 - 2\omega_{0\varphi}\omega_{2\varphi} - 2\omega_{1\varphi}\omega_{2\varphi} - (\omega_{2\varphi})^2,$$

$$\vdots$$

When compared to the like power of *p* on each side of equation (5.2.12) yields the following result:

$$\omega_0 = \mathrm{e}^\varphi,$$

$$\omega_1 = \frac{\mathrm{e}^\varphi \mathrm{t}^\alpha(\beta - 4)}{\Gamma(\alpha + 1)},$$

$$\omega_2 = \frac{\mathrm{e}^\varphi \mathrm{t}^{2\alpha}(\beta - 4)^2}{\Gamma(2\alpha + 1)},$$

$$\omega_3 = \frac{e^{\varphi}t^{3\alpha}(\beta-4)^3}{\Gamma(3\alpha+1)},$$

$$\vdots$$

the solution of equation (5.2.11) obtained by AHPTM.

$$\omega(\varphi,t) = \sum_{m=0}^{\infty} \omega_m(\varphi,t) = \omega_0(\varphi,t) + \omega_1(\varphi,t) + \omega_2(\varphi,t) + \cdots$$

$$\omega(\varphi,t) = e^{\varphi} + \frac{e^{\varphi}t^{\alpha}(\beta-4)}{\Gamma(\alpha+1)} + \frac{e^{\varphi}t^{2\alpha}(\beta-4)^2}{\Gamma(2\alpha+1)} + \frac{e^{\varphi}t^{3\alpha}(\beta-4)^3}{\Gamma(3\alpha+1)} + \cdots \tag{5.2.13}$$

and exact solution of eq. (5.2.11) is

$$\omega(\varphi,t) = e^{\varphi}E_{\alpha}\big((\beta-4)t^{\alpha}\big).$$

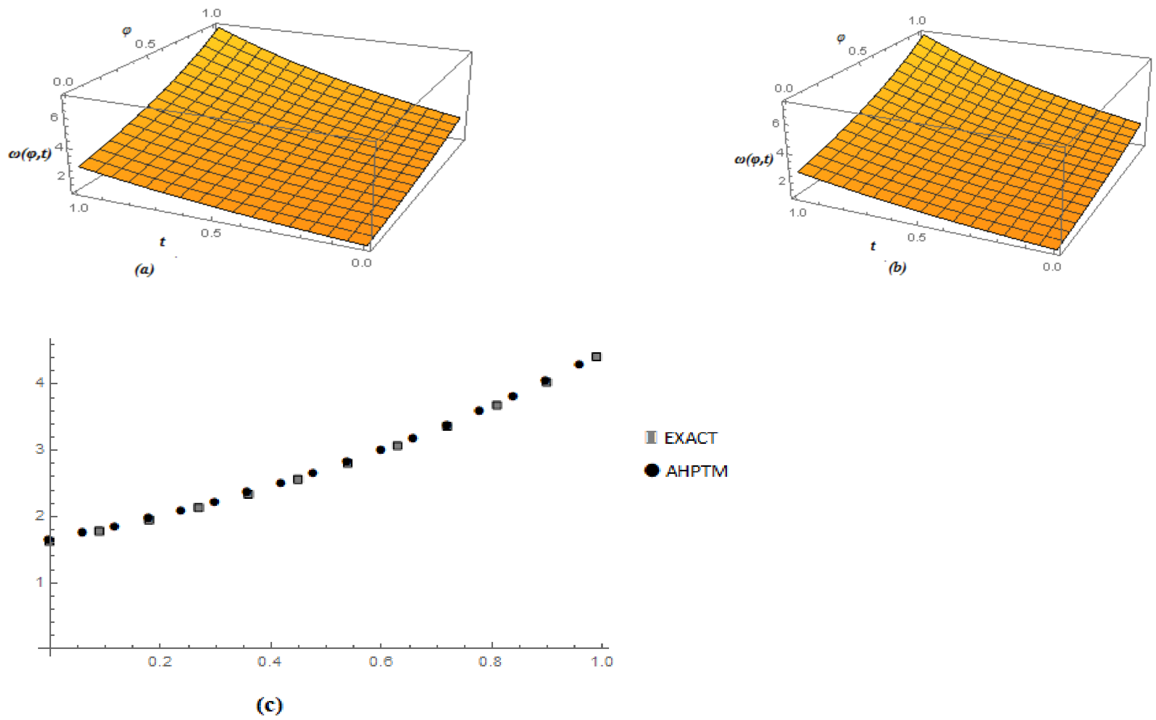
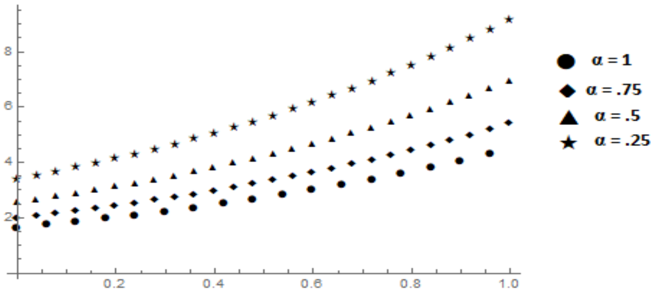


Fig. 5.2.1 (i):(a) surface graph of exact solution at $\alpha = 1$, **(b)** surface graph AHPTM at $\alpha = 1$, **(c)** line graph of exact solution and AHPTM at $\alpha = 1, \beta = 5, 0 < \varphi \le 1$ and $t = 0.5$.



(d)

Fig.5.2.1(ii)(d). the solution of AHPTM at various fractional order $\alpha = 1{,}0.75{,}0.5{,}0{,}25, \beta = 5, 0 < \varphi \leq 1$ and $t = 0.5$.

Table 5.2.1: Error analysis of Example (5.2.1) at $\alpha = 1$(upto the fourth order).

t	x	Exact	AHPTM	Abs. Error	$\|\|\omega_1\|\|$	$\|\|\omega_2\|\|$	$\|\|\omega_3\|\|$
0.1	0.1	1.2214027	1.2213981	4.6985E-06	0.11051709	0.0055259	0.00018419
	0.3	1.4918247	1.491819	5.7388E-06	0.13498588	0.0067493	0.00022497
	0.5	1.8221188	1.8221118	7.0093E-06	0.16487213	0.0082436	0.00024787
	0.7	2.2255409	2.2255324	8.56129E-06	0.20137527	0.0100688	0.00033563
	0.9	2.7182818	2.7182714	1.04568E-05	0.24596031	0.012298	0.00040993
0.3	0.1	1.4918247	1.4914282	0.00039654	0.33155128	0.0497327	0.00497327
	0.3	1.8221188	1.8216345	0.00048434	0.40495764	0.0607436	0.00607436
	0.5	2.2255409	2.2249494	0.00059157	0.49461638	0.0741925	0.00741925
	0.7	2.7182818	2.7175593	0.00072255	0.60412581	0.0906189	0.00906189
	0.9	3.3201169	3.3192344	0.00088252	0.73788093	0.1106821	0.01106821
0.5	0.1	1.8221188	1.8189271	0.00319166	0.55258546	0.1381464	0.0230244
	0.3	2.2255409	2.2216426	0.00389830	0.6749294	0.1687324	0.0281221
	0.5	2.7182818	2.7135204	0.00476140	0.82436064	0.2060902	0.0343484
	0.7	3.3201169	3.3143013	0.00581559	1.00687635	0.2517191	0.0419532
	0.9	4.0551999	4.0480968	0.00710318	1.22980156	0.3074504	0.0512417

Example 5.2.2: Considering the time fractional nonlinear S-H equation. [149,174].

$$\frac{\partial^\alpha \omega(\varphi,t)}{\partial t^\alpha} + \frac{\partial^4 \omega(\varphi,t)}{\partial \varphi^4} + (1-\beta)\omega(\varphi,t) + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2} - \rho\frac{\partial^3 \omega(\varphi,t)}{\partial \varphi^3} - \omega^2(\varphi,t) + \left(\frac{\partial \omega(\varphi,t)}{\partial \varphi}\right)^2 = 0\,,$$

$$0 < \alpha \leq 1, \qquad (5.2.13)$$

With initial condition $\omega(\varphi,0) = e^{\varphi}$.

Apply AHPTM , we get,

$$\sum_{n=0}^{\infty} p^n \omega_n = e^{\varphi} - p\left[\mathcal{L}^{-1}\ \left\{\frac{1}{s^\alpha}\mathcal{L}\left[\left\{\frac{\partial^4 \omega}{\partial \varphi^4} + (1-\beta)\omega(\varphi,t) + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2} - \rho\frac{\partial^3 \omega(\varphi,t)}{\partial \varphi^3}\right\} - \left\{\sum_{n=0}^{\infty} p^n \widetilde{H}_n\,(\omega)\right\}\right]\right\}\right],$$

(5.2.14)

the first components of $\widetilde{H}_n$ are given as

$$\widetilde{H}_0 = (\omega_0)^2 - (\omega_{0\varphi})^2,$$

$$\widetilde{H}_1 = 2\omega_0\omega_1 + {\omega_1}^2 - \omega_{1\varphi}\big(2\omega_{0\varphi} + \omega_{1\varphi}\big),$$

$$\widetilde{H}_2 = 2\omega_0\omega_2 + 2\omega_1\omega_2 + {\omega_2}^2 - 2\omega_{0\varphi}\omega_{2\varphi} - 2\omega_{1\varphi}\omega_{2\varphi} - {\omega_{2\varphi}}^2,$$

$$\vdots$$

Comparing the similar power of *p* on each sides of the equation (5.2.14) we obtain,

$$\omega_0 = \mathrm{e}^{\varphi},$$

$$\omega_1 = \frac{e^x t^\alpha(\rho+\beta-4)}{\Gamma(\alpha+1)},$$

$$\omega_2 = \frac{\mathrm{e}^{\varphi}\mathrm{t}^{2\alpha}(\rho+\beta-4)^2}{\Gamma(2\alpha+1)},$$

$$\omega_3 = \frac{\mathrm{e}^{\varphi}\mathrm{t}^{3\alpha}(\rho+\beta-4)^3}{\Gamma(3\alpha+1)},$$

$$\vdots$$

the approximate series solution of equation (5.2.13) obtained by AHPTM.

$$\omega(\varphi,t) = \sum_{m=0}^{\infty} \omega_m(\varphi,t) = \omega_0(\varphi,t) + \omega_1(\varphi,t) + \omega_2(\varphi,t) + \cdots$$

$$\omega(\varphi,t) = \mathrm{e}^{\varphi} + \frac{e^x t^\alpha(\rho+\beta-4)}{\Gamma(\alpha+1)} + \frac{\mathrm{e}^{\varphi}\mathrm{t}^{2\alpha}(\rho+\beta-4)^2}{\Gamma(2\alpha+1)} + \frac{\mathrm{e}^{\varphi}\mathrm{t}^{3\alpha}(\rho+\beta-4)^3}{\Gamma(3\alpha+1)} + \cdots \qquad (5.2.15)$$

and exact sol. of eq. (5.2.13) is

$$\omega(\varphi,t) = \mathrm{e}^{\varphi}\mathrm{E}_\alpha\big((\rho+\beta-4)\mathrm{t}^\alpha\big).$$

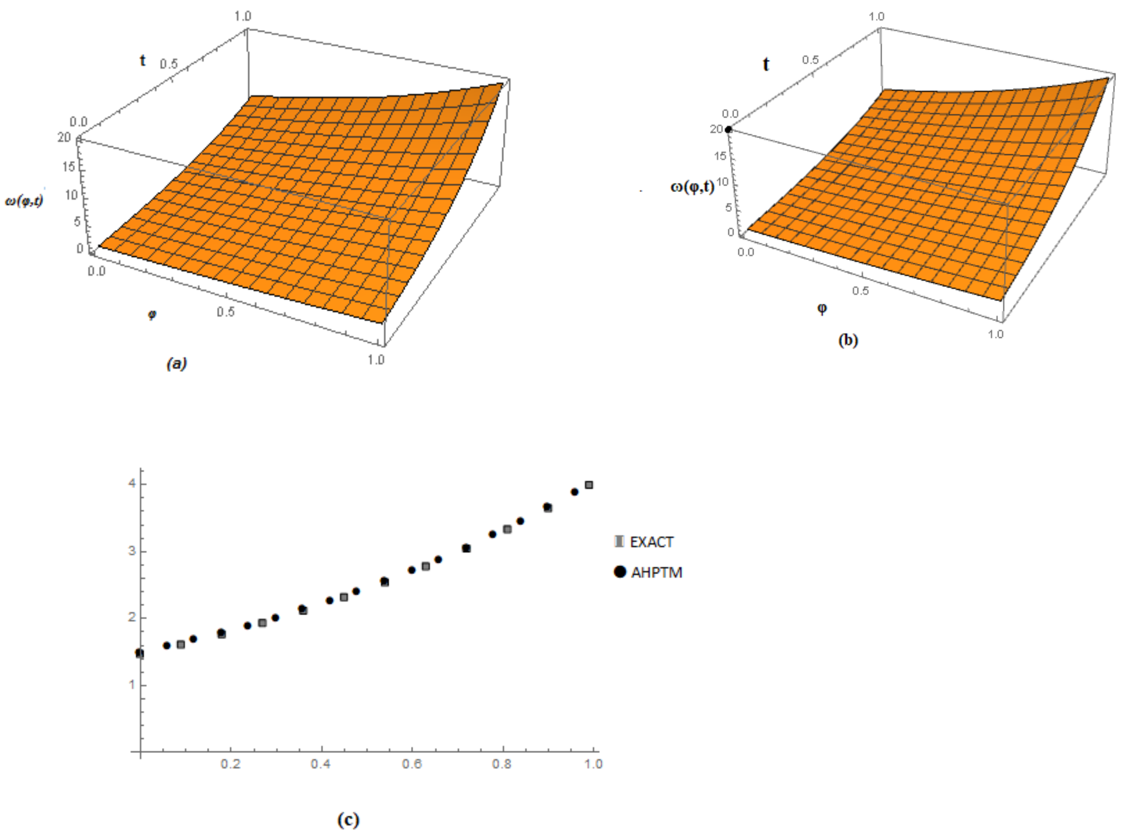
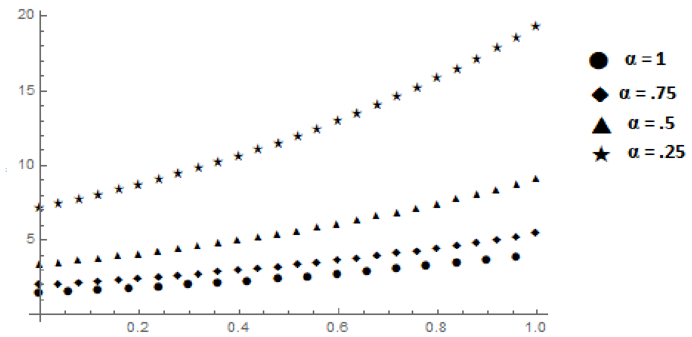


Fig. 5.2.2(i). (a)the surface graph of exact solution at $\alpha = 1$, **(b)** surface graph AHPTM at $\alpha = 1$, **(c)** line graph of exact solution and AHPTM at $\alpha = 1, \beta = 5, \rho = 1, 0 < \varphi \leq 1 \; and \; t = 0.2$.



(d)

Fig. 5.2.2(ii) (d). solution of AHPTM at various fractional order $\alpha = 1, 0.75, 0.5, 0.25, \beta = 5, \; \rho = 1, 0 < \varphi \leq 1 \; and \; t = 0.2$.

Table 5.2.2: Error analysis equation (5.2.13) at $\alpha = 1$(upto the fourth order)

t	x	Exact	AHPTM	Abs. error	$\|\|\omega_1\|\|$	$\|\|\omega_2\|\|$	$\|\|\omega_3\|\|$
0.1	0.1	1.349858808	1.3497821	0.000076726	0.2210342	0.022103418	0.001473561
	0.3	1.648721271	1.6486276	9.37137E-05	0.269972	0.026997176	0.001799812
	0.5	2.013752707	2.0136382	0.000114462	0.3297443	0.032974425	0.002198295
	0.7	2.459603111	2.4594633	0.000139804	0.4027505	0.040275054	0.002685
	0.9	3.004166024	3.0039953	0.000170758	0.4919206	0.049192062	0.003279471
0.3	0.1	2.013752707	2.0069904	0.00676232	0.6631026	0.198930765	0.039786153
	0.3	2.459603111	2.4513436	0.008259517	0.8099153	0.242974585	0.048594917
	0.5	3.004166024	2.9940778	0.010088196	0.9892328	0.296769829	0.059353966
	0.7	3.669296668	3.6569749	0.012321751	1.2082516	0.362475487	0.072495097
	0.9	4.48168907	4.4666392	0.01504982	1.4757619	0.44272856	0.088545712
0.5	0.1	3.004166023	2.9471224	0.057043576	1.1051709	0.552585459	0.184195153
	0.3	3.669296668	3.5996235	0.069673181	1.3498588	0.674929404	0.224976468
	0.5	4.48168907	4.3965901	0.085099015	1.6487213	0.824360635	0.274786878
	0.7	5.473947392	5.3700072	0.103940172	2.0137527	1.006876353	0.335625451
	0.9	6.685894442	6.5589416	0.126952813	2.4596031	1.229801556	0.409933852

Example 5.2.3: Examine the non-linear time fractional S-H equation [149,174].

$$\frac{\partial^\alpha \omega(\varphi,t)}{\partial t^\alpha} + \frac{\partial^4 \omega(\varphi,t)}{\partial \varphi^4} + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2} - \rho\frac{\partial^3 \omega(\varphi,t)}{\partial \varphi^3} + \beta\omega(\varphi,t) - 2\omega^2(\varphi,t) + \omega^3(\varphi,t) = 0, 0 < \alpha \le 1, \quad (5.2.16)$$

With initial condition $\omega(\varphi, 0) = \cos x$.

using AHPTM , we get,

$$\sum_{n=0}^{\infty} p^n \omega_n = \cos x - p\left[\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left[\left\{\frac{\partial^4 \omega}{\partial \varphi^4} + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2} - \rho\frac{\partial^3 \omega(\varphi,t)}{\partial \varphi^3} + \beta\omega(\varphi,t)\right\} - \left\{\sum_{n=0}^{\infty} p^n \widetilde{H}_n(\omega)\right\}\right]\right\}\right], \quad (5.2.17)$$

the first few components of $\widetilde{H}_n$ are given as

$$\widetilde{H}_0 = 2(\omega_0)^2 - (\omega_0)^3,$$

$$\widetilde{H}_1 = 2\omega_1{}^2 + 4\omega_0\omega_1 - 3\omega_0{}^2\omega_1 - 3\omega_0\omega_1{}^3,$$

$$\widetilde{H}_2 = 4\omega_0\omega_2 + 4\omega_1\omega_2 + 2\omega_2{}^2 - 3\omega_0{}^2\omega_2 - 6\omega_0\omega_1\omega_2 - 3\omega_2{}^2\omega_0 - 3\omega_1{}^2\omega_2 - 3\omega_2{}^2 - \omega_2{}^3,$$

$$\vdots$$

similar powers of p comparing each sides of equation (5.2.17), we obtain

$$\omega_0 = \cos x,$$

$$\omega_1 = \left((1+\beta)\cos x + 2\cos^2 x - \cos^3 x + \rho \sin x\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$\begin{aligned}\omega_2 = \frac{1}{64}[-(4t^{\alpha}(-2+3\cos x)\,\Gamma(2\alpha \\ &+ 1)\,(4+\cos x) + 4\beta\cos x + 4\cos 2x - \cos 3x \\ &+ 4\rho\sin x)^2)]\,\frac{t^{2\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} \\ &- \Gamma(3\alpha \\ &+ 1)(4+\cos x + 4\beta\cos x + 4\cos 2x - \cos 3x \\ &+ 4\rho\sin x)^3\,\frac{t^{2\alpha}}{\Gamma(\alpha+1)^3\Gamma(4\alpha+1)} \\ &+ \frac{1}{\Gamma(2\alpha+1)}64\big(-4(-5+\beta)\cos x^3 - 10\cos x)^4 + 3\cos x^5 \\ &+ \cos^2 x(-4+6\beta-24\rho\sin x) \\ &+ \cos x\,(-\rho^2+(1+\beta)^2+20k\sin x - 48\sin x^2) \\ &+ 2\sin x\,(\rho+\rho\beta+4\sin x+3\rho)\big)],\end{aligned}$$

$$\vdots$$

the approximate series solution of equation (5.2.16) obtained by AHPTM.

$$\omega(\varphi,t) = \sum_{m=0}^{\infty}\omega_m(\varphi,t) = \omega_0(\varphi,t)+\omega_1(\varphi,t)+\omega_2(\varphi,t)+\cdots$$

$$\omega(\varphi,t) = \cos x + \left((1+\beta)\cos x + 2\cos^2 x - \cos^3 x + \rho\sin x\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \cdots \qquad (5.2.18)$$

In Fig. 5.2.3, we use ρ= 0.3 and β= 0.5 to approximate the solution of the fractional S-H equation up to three terms.

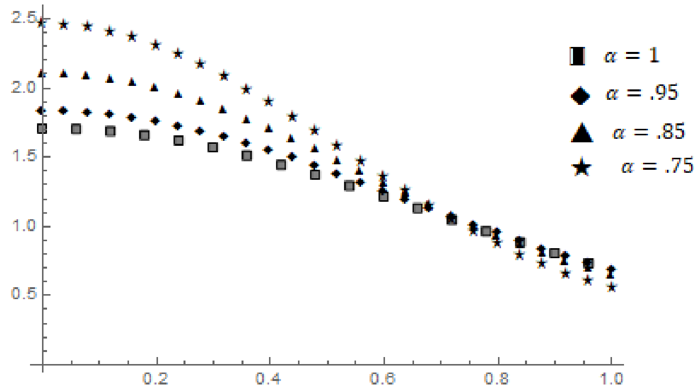


Fig. 5.2.3. Solution of AHPTM at various fractional order $\alpha = 1,0.95,0.85,0.75, \beta = 0.5, \rho = 0.3\,, 0 < \varphi \leq 1$ and $t = 0.2$.

Example 5.2.4: Assume the time fractional nonlinear Fisher's equation [56].

$$\frac{\partial^\alpha \omega}{\partial t^\alpha} = \frac{\partial^2 \omega}{\partial x^2} + 6\omega(1-\omega), t > 0, x \in \mathbb{R}\,, 0 < \alpha \leq 1. \quad (5.2.19)$$

Initial condition $\omega(x,0) = \frac{1}{(1+e^x)^2}$

And $\omega(x,t) = \frac{1}{(1+e^{x-5t})^2}$

Applying AHPTM on (5.2.19), we have

$$\sum_{n=0}^{\infty} p^n \omega_n = \frac{1}{(1+e^x)^2} - p\left[\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\{\sum_{n=0}^{\infty}(p^n\omega_n)_{xx} + 6\{\sum_{n=0}^{\infty} p^n\omega_n - \sum_{n=0}^{\infty} p^n \widetilde{H}_n\,(\omega)\}\}\right\}\right], \quad (5.2.20)$$

The first few terms of $\widetilde{H}_n$ are given as

$$\widetilde{H}_0 = \left(\frac{1}{(1+e^x)^2}\right)^2$$

$$\widetilde{H}_1 = \frac{20e^x t^\alpha(5e^x t^\alpha + (1+e^x)\Gamma[1+\alpha])}{(1+e^x)^6\Gamma[1+\alpha]^2}$$

$$\tilde{H}_2 = \frac{-\frac{(1+e^x)^6(10e^x t^\alpha + (1+e^x)\Gamma[1+\alpha])^2}{\Gamma[1+\alpha]^2}}{(1+e^x)^{12}} + \frac{\left((1+e^x)^4 + \frac{10e^x(1+e^x)^3 t^\alpha}{\Gamma[1+\alpha]} + \frac{50e^x t^{2\alpha}\left((1+e^x)^2(-1+2e^x) - \frac{12e^x t^\alpha \Gamma[1+2\alpha]^2}{\Gamma[1+\alpha]^2\Gamma[1+3\alpha]}\right)}{\Gamma[1+2\alpha]}\right)^2}{(1+e^x)^{12}}$$

⋮

When the similar powers of p on both sides of (5.2.20) are compared, we get,

$$p^0 : \omega_0 = \frac{1}{(1+e^x)^2};$$

$$p^1 : \omega_1 = \frac{10e^x t^\alpha}{(1+e^x)^3\Gamma[1+\alpha]};$$

$$p^2 : \omega_2 = \frac{50e^x t^{2\alpha}\left((1+e^x)^2(-1+2e^x) - \frac{12e^x t^\alpha \Gamma[1+2\alpha]^2}{\Gamma[1+\alpha]^2\Gamma[1+3\alpha]}\right)}{(1+e^x)^6\Gamma[1+2\alpha]}$$

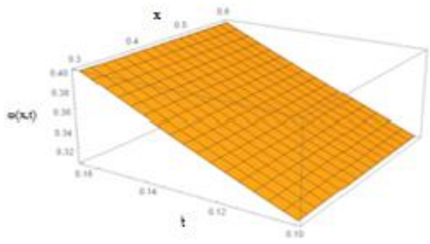
$$p^3 : \omega_3 = \frac{1}{(1+e^x)^{12}} 50e^x t^{3\alpha}\left(-\frac{120e^x(1+e^x)^5(-1+2e^x)t^\alpha\Gamma[1+3\alpha]}{\Gamma[1+\alpha]\Gamma[1+2\alpha]\Gamma[1+4\alpha]} - \frac{300e^x(1-2e^x)^2(1+e^x)^4 t^{2\alpha}\Gamma[1+4\alpha]}{\Gamma[1+2\alpha]^2\Gamma[1+5\alpha]} - \frac{24e^x(1+e^x)^3 t^\alpha \Gamma[1+2\alpha](-60e^x t^\alpha \Gamma[1+4\alpha]^2 + (-1-6e^x+6e^{2x}+11e^{3x})\Gamma[1+\alpha]\Gamma[1+3\alpha]\Gamma[1+5\alpha])}{\Gamma[1+\alpha]^3\Gamma[1+3\alpha]\Gamma[1+4\alpha]\Gamma[1+5\alpha]} + \frac{1}{\Gamma[1+3\alpha]}(1+e^x)^2\left((1+e^x)^4(5-6e^x-15e^{2x}+20e^{3x}) + \frac{7200e^{2x}(-1+2e^x)t^{3\alpha}\Gamma[1+5\alpha]}{\Gamma[1+\alpha]^2\Gamma[1+6\alpha]}\right) - \frac{43200e^{3x}t^{4\alpha}\Gamma[1+2\alpha]^2\Gamma[1+6\alpha]}{\Gamma[1+\alpha]^4\Gamma[1+3\alpha]^2\Gamma[1+7\alpha]}\right)$$

⋮

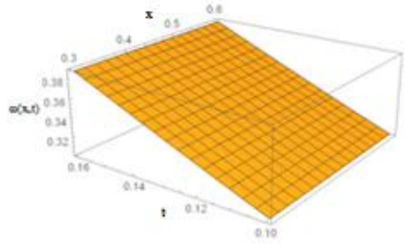
The solution to the problem is achieved by taking $p \to 1$

$$\omega(x,t) = \omega_0 + \omega_1 + \omega_3 \ldots$$

$$\omega(x,t) = \frac{1}{(1+e^x)^2} + \frac{10e^x t^\alpha}{(1+e^x)^3\Gamma[1+\alpha]} + \frac{50e^x(2e^x-1)t^{2\alpha}}{(1+e^x)^4\Gamma[1+2\alpha]} + \left(600e^{2x}\frac{\Gamma[1+2\alpha]}{(1+e^x)^6\Gamma[1+\alpha]^2\Gamma[1+3\alpha]}\right)\frac{t^{3\alpha}}{\Gamma[1+3\alpha]}\ldots$$



(a)



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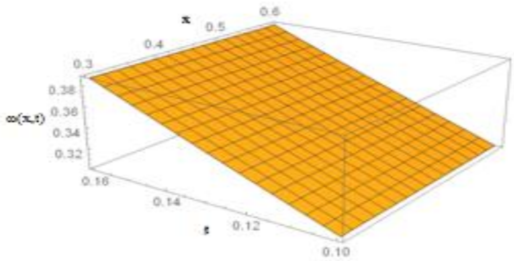
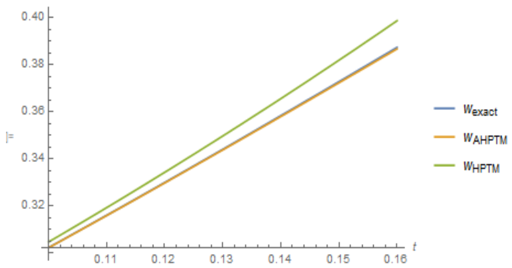


(c)

Fig. 5.2.4 (i): (a) surface graph of ω_{HPTM} of eq. (5.2.19) when $\alpha = 1$, (b) surface graph of ω_{AHPTM} of eq. (5.2.19) when $\alpha = 1$, **(c)** surface graph of Exact sol. of eq. (5.2.19) when $\alpha = 1$.



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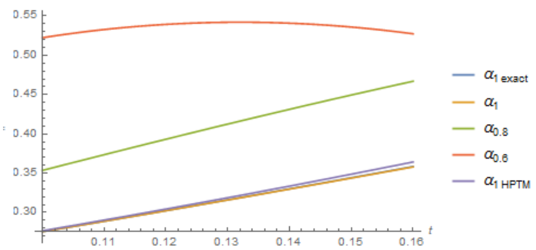


(e)

Fig. 5.2.4 (ii): (d) $\omega_{exact}, \omega_{AHPTM}, \omega_{HPTM}$ $at\ x = 0.3, 0 \le t \le 0.16$, **(e)** plot of $\omega(x,t)$ of eq. (5.2.19) when x =2 and $\alpha = 0.6.,0.8\ and\ 1$**.**

Table 5.2.4 Approximate solution of Fisher's equation upto fourth order ($\alpha = 1$)

x	t	ω(exact sol.)	ω_{AHPTM}	Abs. Error	ω_{HPTM}	Abs. error	$\lVert\omega_1\rVert$	$\lVert\omega_2\rVert$	$\lVert\omega_3\rVert$
0.3	0.1	0.3023174	0.30222259	0.00009482	0.30469113	0.00237371	0.104031	0.016648	0.000445
	0.11	0.3160424	0.31590336	0.00013905	0.31929263	0.00325021	0.114434	0.019882	0.000488
	0.12	0.3299842	0.32978627	0.00019792	0.33431978	0.00433558	0.124837	0.023349	0.000501
	0.13	0.3441202	0.34384527	0.00027491	0.34977709	0.00565691	0.13524	0.027037	0.000469

	0.14	0.3584269	0.35805283	0.00037408	0.36566905	0.00724213	0.145643	0.030932	0.000378
0.4	0.1	0.2756031	0.27552116	0.00008198	0.27661106	0.00100792	0.096419	0.01733	0.00072
	0.11	0.2888308	0.28871262	0.00011822	0.29026645	0.00143561	0.106062	0.020744	0.000856
	0.12	0.3023174	0.30215183	0.00016559	0.30430237	0.00198495	0.115704	0.024419	0.000977
	0.13	0.3160424	0.31581587	0.00022654	0.31871854	0.00267612	0.125345	0.028344	0.001074
	0.14	0.3299842	0.32968020	0.00030400	0.33351465	0.00353044	0.134987	0.032508	0.001133
0.5	0.1	0.25	0.24992567	0.00007432	0.24976552	0.00023449	0.088723	0.017665	0.001
	0.11	0.2626536	0.26254875	0.00010482	0.26243511	0.00021848	0.097596	0.021184	0.001232
	0.12	0.2756031	0.27545964	0.00014350	0.27544103	0.00016212	0.106468	0.024984	0.001471
	0.13	0.2888308	0.28863901	0.00019182	0.28877889	0.00005191	0.11534	0.029055	0.001706
	0.14	0.3023174	0.30206589	0.00025153	0.30244426	0.00012684	0.124213	0.033389	0.009273

Example 5.2.5: Consider the time-fractional F-W equation [56]

$$\frac{\partial^\alpha}{\partial t^\alpha}\omega(x,t) = \frac{\partial^3\omega}{\partial x^2\partial t} - \frac{\partial\omega}{\partial x} + \omega\frac{\partial^3\omega}{\partial x^3} - \omega\frac{\partial\omega}{\partial x} + 3\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2},$$

$$t > 0, x \in \mathbb{R}, 0 < \alpha \le 1 \qquad (5.2.21)$$

With initial condition $\omega(x,0) = e^{\frac{x}{2}}$

And $\qquad \omega(x,t) = e^{\frac{1}{2}(x-\frac{4t}{3})}$

by applying AHPTM on (5.2.21), we have,

$\sum_{n=0}^{\infty} p^n\omega_n = e^{\frac{x}{2}} - p\left[\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\left(\mathcal{L}\left\{\sum_{n=0}^{\infty}(p^n\omega_n)_{xxt} + \sum_{n=0}^{\infty}p^n(-\omega_n)_x -\right.\right.\right.\right.$

$\left.\left.\left.\left.\left\{\sum_{n=0}^{\infty}p^n\widetilde{H}_n(\omega)\right\}\right\}\right)\right\}\right]$ (5.2.22) and the first few terms of $\widetilde{H}_n$ are represented as

$$\widetilde{H}_0 = \omega_0\omega_{0xxx} - \omega_0\omega_{0x} + 3\omega_{0x}\omega_{0xx} = 0$$

$$\widetilde{H}_1 = \omega_0\omega_{1xxx} + \omega_1\omega_{0xxx} + \omega_1\omega_{1xxx} - \omega_0\omega_{1xx} - \omega_1\omega_{0x} - \omega_1\omega_{1x} + 3\omega_{0x}\omega_{1xx} + 3\omega_{1x}\omega_{0xx} + 3\omega_{1x}\omega_{1xx}$$

$$\widetilde{H}_2 = \omega_0\omega_{2xxx} + \omega_1\omega_{2xxx} + \omega_2\omega_{0xxx} + \omega_2\omega_{1xxx} + \omega_2\omega_{2xxx} - \omega_0\omega_{2x} - \omega_2\omega_{0x} - \omega_2\omega_{1x} - \omega_2\omega_{2x} + 3\omega_{0x}\omega_{2xx} + 3\omega_{1x}\omega_{2xx} + 3\omega_{2x}\omega_{0xx} + 3\omega_{2x}\omega_{1xx} + 3\omega_{2x}\omega_{2xx}$$

$$\vdots$$

When similar powers of p on each side of (5.2.22) are compared, we get

$$p^0 : \omega_0 = e^{\frac{x}{2}}$$

$$p^1 : \omega_1 = -\frac{e^{\frac{x}{2}}}{2}\frac{t^\alpha}{\Gamma(1+\alpha)};$$

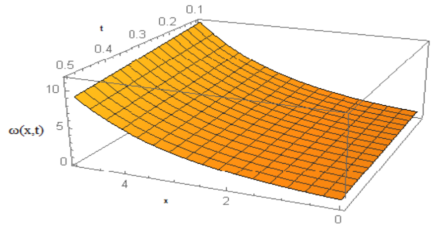
$$p^2 : \omega_2 = \frac{-e^{\frac{x}{2}}}{8}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{e^{\frac{x}{2}}t^{2\alpha}}{4\Gamma(2\alpha+1)};$$

$$p^3 : \omega_3 = \mathrm{e}^{\frac{x}{2}} \left(\frac{3}{32} \frac{\mathrm{t}^{3\alpha-2}}{\Gamma(3\alpha-1)} - \frac{3}{16} \frac{\mathrm{t}^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{\mathrm{t}^{3\alpha}}{8\Gamma(3\alpha+1)} \right);$$

$$\vdots$$

Hence, the obatined solution is

$$\omega(x,t) = \mathrm{e}^{\frac{x}{2}} - \frac{\mathrm{e}^{\frac{x}{2}}}{2} \frac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)} - \frac{\mathrm{e}^{\frac{x}{2}}}{8} \frac{\mathrm{t}^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{\mathrm{e}^{\frac{x}{2}} \mathrm{t}^{2\alpha}}{4\Gamma(2\alpha+1)} + \mathrm{e}^{\frac{x}{2}} \left(\frac{3}{32} \frac{\mathrm{t}^{3\alpha-2}}{\Gamma(3\alpha-1)} - \frac{3}{16} \frac{\mathrm{t}^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{\mathrm{t}^{3\alpha}}{8\Gamma(3\alpha+1)} \right) + \cdots \tag{5.2.23}$$



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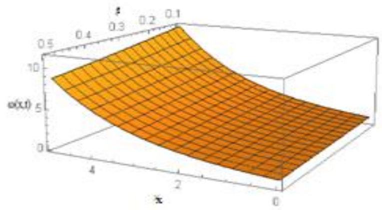

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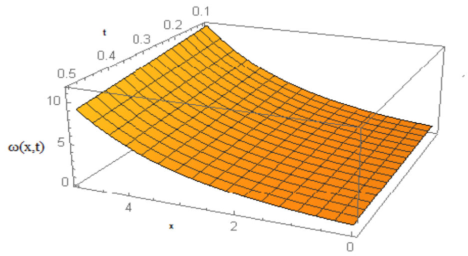
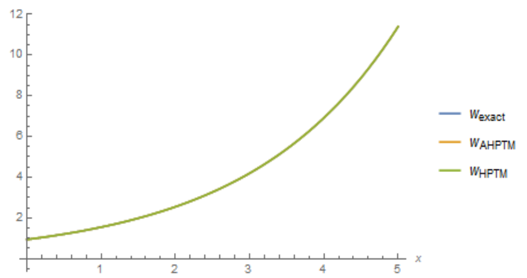


(c)

Fig. 5.2.5 (i): (a) surface graph of ω_{AHPTM} eq. (5.2.21), when $= 1$, **(b)**: surface graph of ω_{HPTM} of eq. (5.2.21),when $\alpha = 1$, **(c)** surface graph of $\omega(x,t)$ eq.(5.2.21), when $\alpha = 1$(exact sol.)



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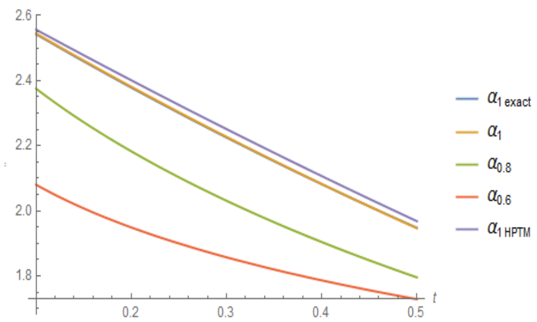


(e)

Fig. 5.2.5 (ii): (d) $\omega(\text{exact sol.}), \omega_{AHPTM}, \omega_{HPTM}$ $at\ t = 0.1, 0 \leq x \leq 5$.when $\alpha = 1$, **(e)** plot of $\omega(x,t)$ of eq. (5.2.21) when x=2 and $\alpha = 0.6.,0.8\ and\ 1.$

Table 5.2.5. Approximate solution of time-fractional F-W equation upto fourth order ($\alpha = 1$)

t	x	Exact Sol.	ω_{AHPTM}	Abs. Error	ω_{HPTM}	Abs. Error	$\lVert\omega_1\rVert$	$\lVert\omega_2\rVert$	$\lVert\omega_3\rVert$
0.1	1	1.542390265	1.543580941	1.19E-03	1.543580941	1.19E-03	0.082436064	0.018548114	0.004156152
	2	2.542971638	2.544934731	1.96E-03	2.544934731	1.96E-03	0.135914091	0.030580671	0.006852335
	3	4.19265143	4.195888023	3.23E-03	4.195888024	3.24E-03	0.224084453	0.050419002	0.011297591
	4	6.912513593	6.917849834	5.33E-03	6.917849834	5.34E-03	0.369452804	0.083125881	0.018626578
	5	11.39680819	11.40560618	8.79E03	11.40560617	8.80E-03	0.609124698	0.137053057	0.030710037
0.3	1	1.349858807	1.351024036	1.17E-03	1.351024036	1.17E-03	0.247308191	0.043278933	0.007110110
	2	2.225540928	2.227462065	1.92E-03	2.227462066	1.92E-03	0.407742274	0.07135549	0.011722590
	3	3.669296667	3.672464087	3.17E-03	3.672464088	3.17E-03	0.672253361	0.117644338	0.019327284
	4	6.049647464	6.054869657	5.22E-03	6.054869657	5.22E-03	1.108358415	0.193962723	0.031865304
	5	9.974182455	9.982792394	8.61E-03	9.982792395	8.61E-03	1.827374094	0.319790466	0.052537005
0.5	1	1.181360413	1.180724868	6.36E-04	1.180724868	6.36E-04	0.412180318	0.05152254	0.004293544
	2	1.947734041	1.946686205	1.05E-03	1.946686205	1.05E-03	0.679570457	0.084946307	0.007078858
	3	3.211270543	3.209542954	1.73E-03	3.209542954	1.73E-03	1.120422268	0.140052783	0.011671065
	4	5.294400504	5.291641738	2.85E-03	5.291641738	2.85E-03	1.847264025	0.230908003	0.019242333
	5	8.729138364	8.72444229	4.70E-03	8.72444229	4.70E-03	3.04562349	0.380702936	0.031725244

Example 5.2.6: Examine the nonlinear homogeneous time fractional Inviscid Burgers' equation [56].

$$D_t^{\alpha}\omega + \omega\omega_x = 1 + x + t, \tag{5.2.24}$$

With initial condition $\omega(x,0) = x, 0 < \alpha \leq 1$

and $\omega(x,t) = x + t$

By applying AHPTM on equation (5.2.24) we get,

$$\sum_{n=0}^{\infty} p^n \omega_n = x - p\left(\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left(\{1 + x + t\} - \left\{\sum_{n=0}^{\infty} p^n \widetilde{H}_n(\omega)\right\}\right)\right\}\right) \tag{5.2.25}$$

And the first few of terms of $\widetilde{H}_n$ are represented as

$$\widetilde{H}_0 = x$$

$$\widetilde{H}_1 = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)};$$

$$\widetilde{H}_2 = \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right);$$

$$\vdots$$

When we compare comparable powers of p on both sides of (5.2.25), we get

$$p^0: \omega_0 = x$$

$$p^1: \omega_1 = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)};$$

$$p^2 : \omega_2 = -\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right);$$

$$p^3 : \omega_3 = \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right),$$

$$\vdots$$

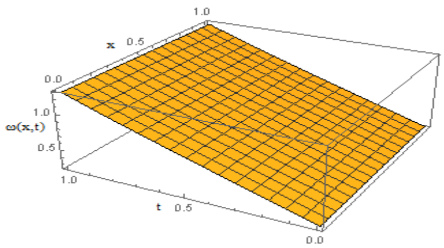
Hence the solution of eq. (5.2.24) is

$$\omega(x,t) = x + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) + \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) \ldots$$

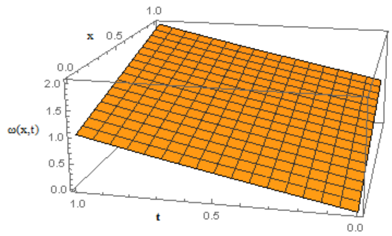
or $\quad \omega(x,t) = x + \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right) + \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots\right)$

or $\quad \omega(x,t) = x - \sum_{n=1}^{\infty} \frac{(-1)^n t^{n\propto}}{\Gamma(n\alpha+1)} - t\sum_{n=1}^{\infty} \frac{(-1)^n t^{n\propto}}{\Gamma(n\alpha+2)}$

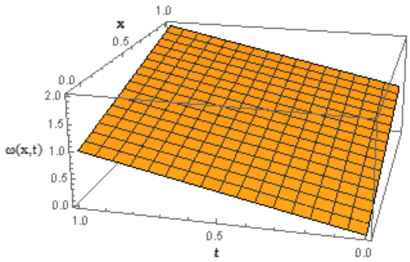
or $\quad \omega(x,t) = x + 1 + t - E_{\propto,1}(-t^{\propto}) - tE_{\propto,2}(-t^{\propto})$



(a)

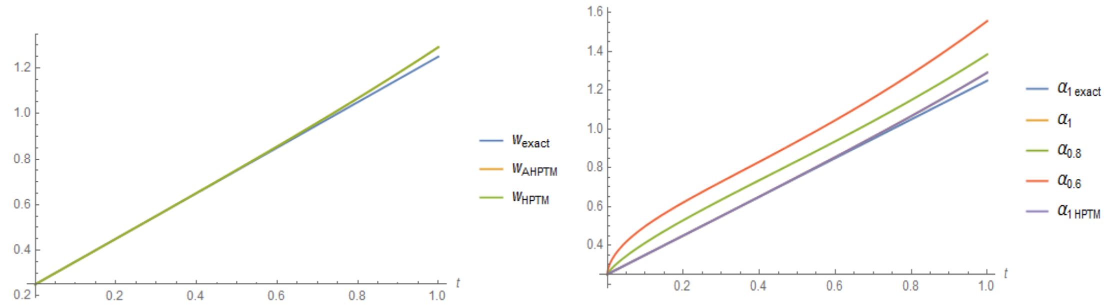


(b)



(c)

Fig. 5.2.6 (i): (a) surface graph of ω_{AHPTM} of eq. (5.2.24) ,when $\alpha = 1$,**(b)**: surface graph of ω_{HPTM} of eq. (5.2.24) ,when $\alpha = 1$,**(c)** surface graph of $\omega(x,t)$ of eq.(5.2.24) ,when $\alpha = 1$(exact sol.).



(d) (e)

Fig. 5.2.6 (ii): (d): $\omega_{exact}, \omega_{AHPTM}, \omega_{HPTM}$ $at\ x = 0.25, 0 \leq t \leq 1.$ when $= 1$, **(e)** plot of $\omega(x,t)$ of eq. (4.2.21) ,when x=.25 and $\alpha = 0.6.$,0.8 and $1.$

Table 5.2.6. Approximate solution of Inviscid Burgers' equation upto fourth order ($\alpha = 1$)

x	t	Exact solution	ω_{AHPTM}	Abs. Error	ω_{HPTM}	Abs. Error	$\|\omega_1\|$	$\|\omega_2\|$	$\|\omega_3\|$
0.25	0.25	0.5	0.5001628	0.0001628	0.5001628	0.0001628	0.28125	0.0338542	0.002766927
	0.5	0.75	0.7526042	0.0026042	0.7526042	0.0026042	0.625	0.1458333	0.0234375
	0.75	1	1.0131836	0.0131836	1.0131836	0.0131836	1.03125	0.3515625	0.083496094
	1	1.25	1.2916667	0.0416667	1.2916667	0.0416667	1.5	0.6666667	0.208333333
0.5	0.25	0.75	0.7501628	0.0001628	0.7501628	0.0001628	0.28125	0.0338542	0.002766927
	0.5	1	1.0026042	0.0026042	1.0026042	0.0026042	0.625	0.1458333	0.0234375
	0.75	1.25	1.2631836	0.0131836	1.2631836	0.0131836	1.03125	0.3515625	0.083496094
	1	1.5	1.5416667	0.0416667	1.5416667	0.0416667	1.5	0.6666667	0.20833333
0.75	0.25	1	1.0001628	0.0001628	1.0001628	0.0001628	0.28125	0.0338542	0.002766927
	0.5	1.25	1.2526042	0.0026042	1.2526042	0.0026042	0.625	0.1458333	0.0234375
	0.75	1.5	1.5131836	0.0131836	1.5131836	0.0131836	1.03125	0.3515625	0.083496094
	1	1.75	1.7916667	0.0416667	1.7916667	0.0416667	1.5	0.6666667	0.20833333

Conclusion: In conclusion, this chapter has offered a thorough explanation of the strategies used to achieve the study objectives. Within the scope of this investigation, we used the nonlinear fractional PDEs, as defined by the Caputo formulations, to address a variety of equations, including the Swift-Hohenberg equation, Inviscid Burgers' equation, F-W equation, and Fisher's equation. To validate our technique, rigorous computations were performed in Methematica, obtaining approximate results. These results were then visually rendered in surface and line graph forms. Furthermore, a thorough investigation of this method's convergence qualities was carried out, adding to a better understanding of its effectiveness in solving complicated mathematical problems.

Chapter 6

Nonlinear Fractional Partial Differential Equations using Accelerated Homotopy Perturbation Transformation Method in Caputo-Fabrizio Sense

6.1 Introduction:

In this chapter, the focus shifts from solving nonlinear fractional partial differential equations (PDEs) in the Caputo sense, explored in the previous chapter, to addressing nonlinear fractional PDEs in the Caputo-Fabrizio sense. While Chapter 5 applied the Accelerated Homotopy Perturbation Transformation Method (Acc. HPTM) to fractional PDEs using Caputo derivatives, this chapter expands the methodology to equations that incorporate the Caputo-Fabrizio fractional derivative, characterized by a non-singular kernel, thereby broadening the scope of application. This chapter provides a full explanation of the technique used to achieve the research aims specified in this study. Nonlinear fractional PDEs in the C-F sense are used to solve a variety of problems, including the Burgers' equation, KdV equation, and K-G equation. The recommended techniques for achieving the study objectives are extensively detailed. The equations are used and validated in Mathematica, and the approximate solution is found using specific procedures. To provide a more precise depiction, the results are presented in the form of surface and line graphs.

6 .2 Fractional Partial Differential Equations in Caputo-Fabrizio sense

We can describe the key concept of this technique by looking at a general nonlinear, non-homogeneous fractional PDE

$$ {}^{\mathbb{CF}}_{0}\mathrm{D}_t^{\alpha}\varphi(x,t) + R\varphi(x,t) + N\varphi(x,t) = g(x,t), t > 0, x \in \mathrm{R}, \mathrm{n} - 1 < \alpha \leq n, \quad (6.2.1) $$

with differential operator initial condition $\varphi(x,0) = k(x)$

Where ${}^{\mathbb{CF}}_{0}\mathrm{D}_t^{\alpha}$ is represents the fractional C-F derivative with respect to t, R and N denotes the linear differential operator and non-linear differential operator and the source term is represented by $g(x,t)$.

now, taking LT on both side of equation (6.2.1),

$$ \mathcal{L}\left[{}^{\mathbb{CF}}_{0}\mathrm{D}_t^{\alpha}\varphi(x,t)\right] + \mathcal{L}\varphi[R(x,t)] + \mathcal{L}[N\varphi(x,t)] = \mathcal{L}[g(x,t)] $$

Employing the differential property

$$\mathcal{L}[\varphi(x,t)] = \frac{1}{s}\left(\varphi(x,0)\right) + \left(\frac{s+\mu(1-s)}{s}\right)\mathcal{L}\left(g(x,t) - R\varphi(x,t) - N\varphi(x,t)\right) \tag{6.2.2}$$

Now, applying both sides of eq. (6.2.1) inverse LT, we get

$$\varphi(x,t) = \varphi(x,0) + \mathcal{L}^{-1}\left\{\left(\frac{s+\mu(1-s)}{s}\right)\left[\mathcal{L}\left(g(x,t) - R\varphi(x,t) - N\varphi(x,t)\right)\right]\right\},$$

employing HPM, we obtain

$$0 = (1-p)[\varphi(x,t) - \varphi(x,0)] + p\left[\varphi(x,t) + \mathcal{L}^{-1}\left\{\left(\frac{s+\mu(1-s)}{s}\right)\left[\mathcal{L}\left(g(x,t) - R\varphi(x,t) - N\varphi(x,t)\right)\right]\right\}\right] \tag{6.2.3}$$

Let

$$\varphi(\mathrm{x},t) = \sum_{n=0}^{\infty} p^n \varphi_n(x,t), \tag{6.2.4}$$

and nonlinear term can be decompose as

$$N\varphi(x,t) = \sum_{n=0}^{\infty} p^n \widetilde{H}_n(\varphi(x,t)) \tag{6.2.5}$$

Where $\widetilde{H}_n$ represent accelerated He's polynomial with

$$\widetilde{H}_{\mathrm{n}}(\varphi_0, \varphi_1, \varphi_2 \ldots \varphi_n) = N(S_k) - \sum_{i=0}^{n-1} \widetilde{H}_n \tag{6.2.6}$$

$\widetilde{H}_n = N\left(\varphi(x_0)\right)$, and $S_k = (\varphi_0 + \varphi_1 + \cdots + \varphi_k)$

Substituting the equation (6.2.5), (6.2.6) in equation (6.2.3) we get,

$$\sum_{n=0}^{\infty} p^n \varphi_n(x,t) = \varphi(x,0) - p\left[\mathcal{L}^{-1}\left\{\left(\frac{s+\mu(1-s)}{s}\right)\left[\mathcal{L}\{g(x,t) - (\sum_{n=0}^{\infty} p^n \varphi_n(x,t) + \sum_{n=0}^{\infty} p^n \widetilde{H}_n(\varphi(x,t)))\}\right]\right\}\right],$$

On comparing like powers of p we get

$$p^0: \varphi_0 = \varphi(x,0)$$

$$p^1: \varphi_1(x,t) = \mathcal{L}^{-1}\left\{\left(\frac{s+\mu(1-s)}{s}\right)\left[\mathcal{L}\left[g(x,t) - (R\varphi_0 + \widetilde{H}_0\varphi)\right]\right]\right\},$$

$$p^2: \varphi_2(x,t) = -\mathcal{L}^{-1}\left\{\left(\frac{s+\mu(1-s)}{s}\right)\left[\mathcal{L}\left[R\varphi_1 + \widetilde{H}_1\varphi\right]\right]\right\},$$

$$p^3: \varphi_3(x,t) = -\mathcal{L}^{-1}\left\{\left(\frac{s+\mu(1-s)}{s}\right)\left[\mathcal{L}\left[R\varphi_2 + \widetilde{H}_2\varphi\right]\right]\right\},$$

$$\vdots$$

Hence, when $p \to 1$ the solution of equation is obtained as

$$\varphi(x,t) = \varphi_0 + \varphi_1 + \varphi_3 \dots$$

Example 6.2.**1**: In the Caputo-Fabrizio sense, consider the following nonlinear KdV equation [121].

$$^{\mathbb{C}\mathrm{F}}_{0}\mathrm{D}_t^{\alpha}\varphi(x,t) = -\varphi\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial^3\varphi}{\partial x^3},\ 0 < \alpha \leq 1, \tag{6.2.7}$$

With initial condition

$$\varphi(x,0) = x,$$

by operating the LT, we get

$$\mathcal{L}[\varphi(x,t)] = \frac{1}{s}\varphi(x,0) - \left(\frac{s+\mu(1-s)}{s}\right)\mathcal{L}\left[\varphi\frac{\partial\varphi}{\partial x} + \varphi\frac{\partial^3\varphi}{\partial x^3}\right]$$

using the inverse of the LT to

$$\varphi(x,t) = \varphi(x,0) - \mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[\varphi\frac{\partial\varphi}{\partial x} + \varphi\frac{\partial^3\varphi}{\partial x^3}\right]\right]$$

Now apply AHPTM

$$\sum_{n=0}^{\infty} p^n\varphi_n(x,t) = x - p\mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[\sum_{n=0}^{\infty} p^n\varphi_n(x,t) + \sum_{n=0}^{\infty} p^n\widetilde{H}_n(\varphi)\right]\right]$$

Where $\widetilde{H}_n(\varphi)$ are the polynomials that expressing the nonlinear terms

$$\widetilde{H}_0(\varphi) = \varphi_0\varphi_{0x} - \varphi_0\varphi_{0xxx}$$

$$\widetilde{H}_1(\varphi) = \varphi_0\varphi_{1x} + \varphi_1\varphi_{0x} + \varphi_1\varphi_{1x} - \varphi_0\varphi_{0xxx} - \varphi_1\varphi_{0xxx} - \varphi_1\varphi_{1xxx}$$

$$\widetilde{H}_2(\varphi) = \varphi_0\varphi_{2x} + \varphi_1\varphi_{2x} + \varphi_2\varphi_{0x} + \varphi_2\varphi_{1x} + \varphi_2\varphi_{2x} - \varphi_0\varphi_{2xxx} - \varphi_1\varphi_{2xxx} - \varphi_2\varphi_{0xxx} - \varphi_2\varphi_{1xxx} - \varphi_2\varphi_{2xxx}$$

$$\vdots$$

Comparing the similar power of *p*, we get

$$\varphi_0(x,t) = x,$$

$$\varphi_1(x,t) = -x + x\alpha - xt\alpha,$$

$$\varphi_2(x,t) = x - x\alpha - x\alpha^2 + x\alpha^3 + t(x\alpha + 2x\alpha^2 - 3x\alpha^3) + t^2(-x\alpha^2 + 2x\alpha^3) - \frac{1}{3}t^3x\alpha^3,$$

$$\begin{aligned}\varphi_3(x,t) = {} & -x + x\alpha + 3x\alpha^2 - 5x\alpha^3 + x\alpha^4 + 3x\alpha^5 - 3x\alpha^6 + x\alpha^7 \\ & + t(-x\alpha - 6x\alpha^2 + 15x\alpha^3 - 4x\alpha^4 - 15x\alpha^5 + 18x\alpha^6 - 7x\alpha^7) \\ & + t^2(3x\alpha^2 - 10x\alpha^3 + x\alpha^4 + 26x\alpha^5 - 36x\alpha^6 + 16x\alpha^7) \\ & - \frac{1}{3}t^3(-5x\alpha^3 - 6x\alpha^4 + 62x\alpha^5 - 100x\alpha^6 + 51x\alpha^7) \\ & + \frac{1}{6}t^4(-5x\alpha^4 + 44x\alpha^5 - 90x\alpha^6 + 55x\alpha^7) \\ & - \frac{1}{15}t^5(13x\alpha^5 - 46x\alpha^6 + 38x\alpha^7) + \frac{1}{9}t^6(-2x\alpha^6 + 3x\alpha^7) - \frac{1}{63}t^7x\alpha^7,\end{aligned}$$

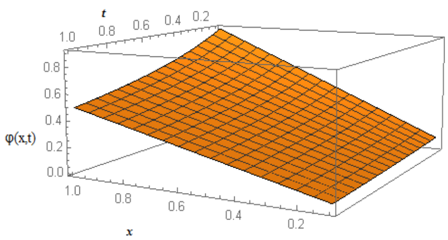
$$\vdots$$

So, the approximate solution is given by

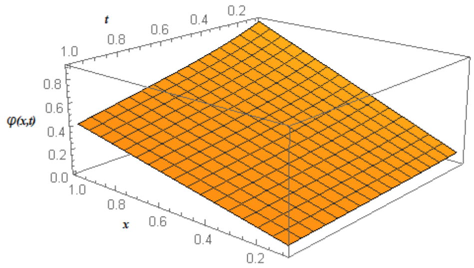
$$\varphi(x,t) = \sum_{n=0}^{\infty}\varphi_n(x,t) = \varphi_0 + \varphi_1 + \varphi_2 + \cdots$$

$$\varphi(x,t) = x(1 + 2\alpha^2 - 4\alpha^3 - \tfrac{1}{3}t^3\alpha^3 + \alpha^4 + 3\alpha^5 - 3\alpha^6 + \alpha^7 + \cdots)$$

Hence, the resulting solution was obtained when $\alpha \to 1$ i.e. $\varphi(x,t) = \frac{x}{1+t}$.



(a)



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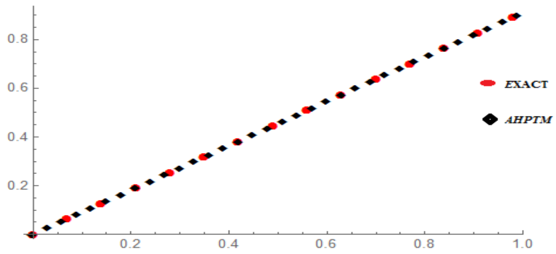


(c)

Fig. 6.2.1(i): (a) surface graph of exact solution at $\alpha = 1$, (b) surface graph AHPTM at $\alpha = 1$, (c) line graph of exact solution and AHPTM at $\alpha = 1, 0 < \varphi \leq 1$ and $t = 0.1$.

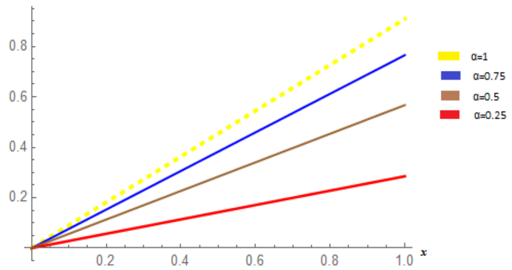


Fig. 6.2.1 (ii): the solution of eq. (4.3.7) at different fractional order $\alpha = 1, 0.75, 0.5, 0,25, 0 < \varphi \leq 1$ and $0 < t \leq 1$.

Table: 6.2.1- Error analysis of example (4.3.7) at $t = 0.1, 0.3, 0.5$ and $\alpha = 1$(upto fourth order)

t	x	φ_{Exact}	φ_{AHPTM}	Abs. error	$\|\varphi_1\|$	$\|\varphi_2\|$	$\|\varphi_3\|$
0.1	0.1	0.0909091	0.0909063	2.74E-06	0.01	0.000967	6.03224E-05
	0.3	0.2727273	0.272719	8.24E-06	0.03	0.0029	0.000180967
	0.5	0.4545455	0.4545317	1.373E-05	0.05	0.004833	0.000301612
	0.7	0.6363636	0.6363444	1.923E-05	0.07	0.006767	0.000422257
	0.9	0.8181818	0.8181571	2.472E-05	0.09	0.0087	0.000542901
0.3	0.1	0.0769231	0.0767668	0.0001563	0.03	0.020967	6.03224E-05
	0.3	0.2307692	0.2303003	0.000469	0.09	0.0629	0.000180967
	0.5	0.3846154	0.3838338	0.0007816	0.15	0.104833	0.000301612
	0.7	0.5384615	0.5373673	0.0010943	0.21	0.146767	0.000422257
	0.9	0.6923077	0.6909008	0.0014069	0.27	0.146767	0.000542901
0.5	0.1	0.0666667	0.0657862	0.0008805	0.05	0.040967	6.03224E-05
	0.3	0.2	0.1973586	0.0026414	0.15	0.1229	0.000180967
	0.5	0.3333333	0.3289311	0.0044023	0.25	0.204833	0.000301612
	0.7	0.4666667	0.4605035	0.0061632	0.35	0.286767	0.000422257
	0.9	0.6	0.5920759	0.0079241	0.45	0.3687	0.000542901

Example 6.2.2: Consider the nonlinear KdV Equation in the C-F sense [121].

$$ {}^{\mathbb{C}\mathrm{F}}_{0}\mathrm{D}_t^{\alpha}\varphi(x,t) = -\varphi\frac{\partial\varphi}{\partial x} - \frac{\partial^3\varphi}{\partial x^3} + x^2 + 2x^3t^2,\ 0 < \alpha \le 1, \tag{6.2.8} $$

with initial condition $\varphi(x,0) = 0,$

apply the L.T to equation (6.2.8), we have

$$ \mathcal{L}[\varphi(x,t)] = \left(\frac{s+\alpha(1-s)}{s}\right)\left(\frac{x^2}{s} + \frac{4x^3}{s^3}\right) - \left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[\varphi\frac{\partial\varphi}{\partial x} + \frac{\partial^3\varphi}{\partial x^3}\right] $$

using the inverse of the LT to

$$ \varphi(x,t) = x^2(1-\alpha+t\alpha) + x^3\left(\frac{2}{3}t^3\alpha + 2t^2 - 2t^2\alpha\right) - \mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[\varphi\frac{\partial\varphi}{\partial x} + \frac{\partial^3\varphi}{\partial x^3}\right]\right] $$

Now apply AHPTM

$$\sum_{n=0}^{\infty} p^n \varphi_n (x,t)$$

$$= x^2(1-\alpha+t\alpha) + x^3\left(\frac{2}{3}t^3\alpha + 2t^2 - 2t^2\alpha\right)$$
$$- p\mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[\sum_{n=0}^{\infty} p^n\varphi_n(x,t) + \sum_{n=0}^{\infty} p^n \widetilde{H}_n(\varphi)\right]\right]$$

where $\widetilde{H}_n(\varphi)$ component given as,

$$\widetilde{H}_0(\varphi) = \varphi_0\varphi_{0x}$$

$$\widetilde{H}_1(\varphi) = \varphi_1\varphi_{0x} + \varphi_0\varphi_{1x} + \varphi_1\varphi_{1x}$$

$$\widetilde{H}_2(\varphi) = \varphi_0\varphi_{2x} + \varphi_1\varphi_{2x} + \varphi_2\varphi_{0x} + \varphi_2\varphi_{1x} + \varphi_2\varphi_{2x}$$

$$\vdots$$

Comparing the similar power of p we have

$$\varphi_0(x,t) = 0$$

$$\varphi_1(x,t) = \frac{1}{3}x^2(3 - 6t^2x(-1+\alpha) - 3\alpha + 3t\alpha + 2t^3x\alpha)$$

$$\varphi_2(x,t) = -\frac{4}{21}t^7x^5\alpha^3 + 2(-x^3 + 3x^3\alpha - 3x^3\alpha^2 + x^3\alpha^3) - 6t(x^3\alpha - 2x^3\alpha^2 + x^3\alpha^3)$$
$$+ 2t^2(-6 - 5x^4 + 12\alpha + 15x^4\alpha - 6\alpha^2 - 2x^3\alpha^2 - 15x^4\alpha^2 + 2x^3\alpha^3$$
$$+ 5x^4\alpha^3) - \frac{2}{3}t^3(12\alpha + 25x^4\alpha - 12\alpha^2 - 50x^4\alpha^2 + x^3\alpha^3 + 25x^4\alpha^3)$$
$$+ \frac{8}{3}t^6(-x^5\alpha^2 + x^5\alpha^3) + \frac{1}{3}t^4(-36x^5 + 108x^5\alpha - 3\alpha^2 - 20x^4\alpha^2$$
$$- 108x^5\alpha^2 + 20x^4\alpha^3 + 36x^5\alpha^3) - \frac{2}{15}t^5(78x^5\alpha - 156x^5\alpha^2 + 5x^4\alpha^3$$
$$+ 78x^5\alpha^3)$$

$$\varphi_3(x,t) = 2(6 - 5x^4(-1+\alpha) + 6x^5(-1+\alpha)^3)(-1+\alpha)^4$$
$$- 2t(24 - 25x^4(-1+\alpha) + 42x^5(-1+\alpha)^3)(-1+\alpha)^3\alpha$$
$$+ 2t^2(-1+\alpha)^2(132x(-1+\alpha)^2 - 36x^2(-1+\alpha)^4 + 70x^6(-1+\alpha)^5 + 21\alpha^2$$
$$- 35x^4(-1+\alpha)\alpha^2 + 6x^5(-1+\alpha)^3(-7+16\alpha^2)) - 4t^3(-1$$
$$+ \alpha)\alpha(132x(-1+\alpha)^2 - 72x^2(-1+\alpha)^4 + 175x^6(-1+\alpha)^5 + 3\alpha^2 - 10x^4(-1$$
$$+ \alpha)\alpha^2 + x^5(-1+\alpha)^3(-67+51\alpha^2)) + \frac{1}{6}t^4(-2880x^3(-1+\alpha)^6$$
$$+ 3552x^7(-1+\alpha)^7 + 1728x(-1+\alpha)^2\alpha^2 + 6\alpha^4 - 55x^4(-1+\alpha)\alpha^4$$
$$- 432x^2(-1+\alpha)^4(-11+5\alpha^2) + 12x^5(-1+\alpha)^3\alpha^2(-136+55\alpha^2)$$
$$+ 28x^6(-1+\alpha)^5(-48+265\alpha^2)) - \frac{2}{15}t^5x\alpha(-9120x^2(-1+\alpha)^5$$
$$+ 16456x^6(-1+\alpha)^6 + 402(-1+\alpha)\alpha^2 - 5x^3\alpha^4$$
$$+ 12x^4(-1+\alpha)^2\alpha^2(-69+19\alpha^2) - 36x(-1+\alpha)^3(-178+41\alpha^2)$$
$$+ 14x^5(-1+\alpha)^4(-258+545\alpha^2))$$

$$\frac{1}{45}t^6x\big(-32400x^3(-1+\alpha)^6 + 48600x^7(-1+\alpha)^7 - 48600x^2(-1+\alpha)^4\alpha^2$$
$$+ 135\alpha^4 - 45x(-1+\alpha)^2\alpha^2(-298+51\alpha^2)$$
$$+ 28x^5(-1+\alpha)^3\alpha^2(-539+685\alpha^2) + 96x^6(-1+\alpha)^5(-90+1367\alpha^2)$$
$$+ 90x^4\alpha^4(9 - 9\alpha - 2\alpha^2 + 2\alpha^3)\big)$$
$$- \frac{2}{315}t^7x^2\alpha\big(-190080x^2(-1+\alpha)^5 + 455220x^6(-1+\alpha)^6$$
$$- 69240x(-1+\alpha)^3\alpha^2 + 30x^3\alpha^4(-5+\alpha^2)$$
$$+ 28x^4(-1+\alpha)^2\alpha^2(-551+515\alpha^2) + 72x^5(-1+\alpha)^4(-564+4019\alpha^2)$$
$$- 135\alpha^2(46 - 46\alpha - 7\alpha^2 + 7\alpha^3)\big) + \frac{t^8x^2}{1260}(907200x^7(-1+\alpha)^7$$
$$- 975240x^2(-1+\alpha)^4\alpha^2 + 3606120x^6(-1+\alpha)^5\alpha^2$$
$$- 109620x(-1+\alpha)^2\alpha^4 + 45\alpha^4(47 - 7\alpha^2) + 16x^5(-1+\alpha)^3\alpha^2(-9996$$
$$+ 47279\alpha^2) + 1680x^4\alpha^4(9 - 9\alpha - 7\alpha^2 + 7\alpha^3))$$
$$+ \frac{2}{105}t^{10}x^3\alpha^2(51744x^6(-1+\alpha)^5 - 1993x(-1+\alpha)^2\alpha^2$$
$$+ 19020x^5(-1+\alpha)^3\alpha^2 - 14\alpha^4 + 8x^4\alpha^2(18 - 18\alpha - 55\alpha^2 + 55\alpha^3))$$

$$-\frac{4t^{11}}{3465}\big(327828x^9(-1+\alpha)^4\alpha^3 - 2475x^4(-1+\alpha)\alpha^5 + 43524x^8(-1+$$
$$\alpha)^2\alpha^5 + 20x^7\alpha^5(-4+11\alpha^2)\big) + \frac{1}{63}t^{12}(5064x^9(-1+\alpha)^3\alpha^4 - 5x^4\alpha^6 +$$

$$216x^8(-1+\alpha)\alpha^6) - \frac{8}{819}t^{13}x^8\alpha^5(956x(-1+\alpha)^2 + 9\alpha^2) + \frac{80}{147}t^{14}x^9(-1+\alpha)\alpha^6 - \frac{16t^{15}x^9\alpha^7}{1323}$$

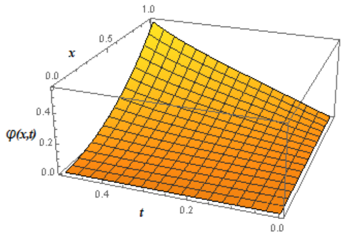
$$\vdots$$

so, the approximate solution are given as

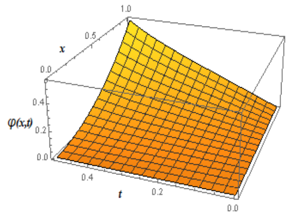
$$\varphi(x,t) = \sum_{n=0}^{\infty} \varphi_n(x,t) = \varphi_0 + \varphi_1 + \varphi_2 + \cdots$$

$$\varphi(x,t) = x^2(1-\alpha+t\alpha) + x^3\left(\frac{2}{3}t^3\alpha + 2t^2 - 2t^2\alpha\right) + 2(-x^3 + 3x^3\alpha - 3x^3\alpha^2 + x^3\alpha^3) + \cdots$$

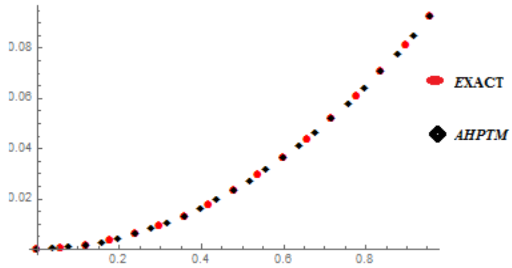
The preceding solution demonstrates that the approximate solution generated from the prior technique is quite close to the exact solution. i.e. $\varphi(x,t) = x^2\text{t}$ when $\alpha \to 1$.



(a)

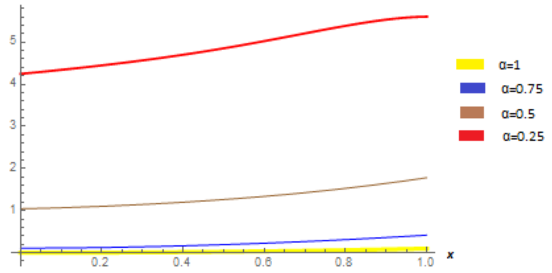


(b)



(c)

Fig 6.2.2(i): (a) surface graph of exact solution at $\alpha = 1$, (b) surface graph AHPTM at $\alpha = 1$, (c) line graph of exact solution and AHPTM at $\alpha = 1, 0 < \varphi \leq 1$ *and* $t = .1$.



(d)

Fig 6.2.2 (ii):(d) the solution of eq. (4.3.8) at different fractional order $\alpha = 1, 0.75, 0.5, 0.25,\ 0 < \varphi \leq 1$ and $t = 0.1$

Table: 6.2.2- Error analysis of example (6.2.8) at $t = 0.1,0.2,0.3$ and $\alpha = 1$(upto fourth order).

t	x	φ_{Exact}	φ_{AHPTM}	Abs. Error	$\|\varphi_1\|$	$\|\varphi_2\|$	$\|\varphi_3\|$
0.1	0.1	0.001	0.0010003	3.00143E-07	0.001001	0.000101	0.0001
	0.3	0.009	0.009000901	9.01423E-07	0.009018	0.000118	0.000101
	0.5	0.025	0.025001505	1.50536E-06	0.025083	0.000184	0.000102
	0.7	0.049	0.049002117	2.11661E-06	0.049229	0.00033	0.000104
	0.9	0.081	0.081002745	2.75E-06	0.081486	0.00059	0.000104
0.2	0.1	0.002	0.002019237	1.92366E-05	0.002005	0.001605	0.001619
	0.3	0.018	0.018057946	5.79462E-05	0.018144	0.001746	0.00166
	0.5	0.05	0.050097141	9.71406E-05	0.050667	0.00228	0.001711
	0.7	0.098	0.098137421	0.000137421	0.099829	0.003481	0.001789
	0.9	0.162	0.1621801	0.0001801	0.165888	0.005629	0.001922
0.3	0.1	0.003	0.003219637	0.000219637	0.003018	0.008118	0.00832
	0.3	0.027	0.027664798	0.000664798	0.027486	0.008599	0.008778
	0.5	0.075	0.076120681	0.001120681	0.07725	0.010453	0.009323
	0.7	0.147	0.148597599	0.001597599	0.153174	0.01467	0.010094
	0.9	0.243	0.245118246	0.002118246	0.256122	0.022309	0.011306

Example 6.2.3: Consider the nonlinear Klein-Gordon equation [121].

$$ {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\alpha+1}\varphi(x,t) = \frac{\partial^2\varphi}{\partial x^2} - \varphi^2 + x^2t^2,\ 0 < \alpha \leq 1, \tag{6.2.9}$$

with initial conditions

$$\varphi(x,0) = 0\ , \qquad \varphi_t(x,0) = x$$

operating the LT to equation (6.2.9), we have

$$\mathcal{L}[\varphi(x,t)] = \frac{1}{s^2}\varphi_t(x,0) + 2\left(\frac{s+\alpha(1-s)}{s^5}\right)x^2 + \left(\frac{s+\alpha(1-s)}{s^2}\right)\mathcal{L}\left[\frac{\partial^2\varphi}{\partial x^2} - \varphi^2\right] \qquad (6.2.10)$$

using the inverse LT to eq.(6.2.10)

$$\varphi(x,t) = xt + \frac{\alpha t^4 x^2}{12} + \frac{(1-\alpha)t^3x^2}{3} + \mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right)\mathcal{L}\left[\frac{\partial^2\varphi}{\partial x^2} - \varphi^2\right]\right]$$

Now apply the AHPTM,

$$\sum_{n=0}^{\infty} p^n\varphi_n(x,t) = xt + \frac{\alpha t^4x^2}{12} + \frac{(1-\alpha)t^3x^2}{3} - p\mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right)\mathcal{L}\left[\sum_{n=0}^{\infty} p^n\varphi_n(x,t) + \sum_{n=0}^{\infty} p^n\widetilde{H}_n(\varphi)\right]\right],$$

Where $\widetilde{H}_n(\varphi)$ express the nonlinear terms,

$$\widetilde{H}_0(\varphi) = (\varphi_0)^2,$$

$$\widetilde{H}_1(\varphi) = (\varphi_0)^2 + 2\varphi_0\varphi_1,$$

$$\widetilde{H}_2(\varphi) = (\varphi_2)^2 + 2\varphi_0\varphi_2 + 2\varphi_1\varphi_2,$$

$$\vdots$$

on comparing the similar power of p we have

$$\varphi_0(x,t) = xt + \frac{\alpha t^4x^2}{12} + \frac{(1-\alpha)t^3x^2}{3},$$

$$\begin{aligned}\varphi_1(x,t) = &-\frac{t^{10}x^4\alpha^3}{12960} + \frac{1}{3}t^3(-x^2 + x^2\alpha) - \frac{1}{12}t^4(-2 + 4\alpha + x^2\alpha - 2\alpha^2) \\ &- \frac{1}{15}t^5(2x^3 - \alpha - 4x^3\alpha + \alpha^2 + 2x^3\alpha^2) + \frac{1}{180}t^6(-9x^3\alpha + \alpha^2 + 9x^3\alpha^2) \\ &- \frac{1}{112}t^8(x^4\alpha - 2x^4\alpha^2 + x^4\alpha^3) + \frac{1}{648}t^9(-x^4\alpha^2 + x^4\alpha^3) \\ &+ \frac{1}{252}t^7(-4x^4 + 12x^4\alpha - x^3\alpha^2 - 12x^4\alpha^2 + 4x^4\alpha^3),\end{aligned}$$

$$\varphi_2(x,t) = -\frac{1}{6}t^4(-1+\alpha)^2 + \frac{1}{15}t^5(-1+\alpha)(2x^3(-1+\alpha)+\alpha)$$
$$-\frac{1}{180}t^6(-34x(-1+\alpha)^3 + 9x^3(-1+\alpha)\alpha + \alpha^2)$$
$$-\frac{t^7x(68x^3(-1+\alpha)^3 + 112(-1+\alpha)^2\alpha - 5x^2\alpha^2)}{1260}$$
$$+\frac{t^8x(-1+\alpha)(-120x(-1+\alpha)^3 + 132x^3(-1+\alpha)\alpha + 61\alpha^2)}{5040}$$
$$+\frac{t^9}{45360}(160x^5(-1+\alpha)^4 + 140(-1+\alpha)^5 + 660x^2(-1+\alpha)^3\alpha - 173x^4(-1+\alpha)\alpha^2$$
$$-22x\alpha^3)$$
$$-\frac{1}{453600}t^{10}(2016x^3(-1+\alpha)^5 + 970x^5(-1+\alpha)^3\alpha + 1148(-1+\alpha)^4\alpha$$
$$+1380x^2(-1+\alpha)^2\alpha^2 - 75x^4\alpha^3) + \frac{1}{2494800}t^{11}(-1+\alpha)(4032x^6(-1+\alpha)^4$$
$$+8820x^3(-1+\alpha)^3\alpha + 1105x^5(-1+\alpha)\alpha^2 + 1932(-1+\alpha)^2\alpha^2 + 630x^2\alpha^3)$$

$$-\frac{1}{29937600}t^{12}(-13200x^4(-1+\alpha)^6 + 37296x^6(-1+\alpha)^4\alpha + 31440x^3(-1+\alpha)^3\alpha^2$$
$$+1085x^5(-1+\alpha)\alpha^3 + 3276(-1+\alpha)^2\alpha^3$$
$$+210x^2\alpha^4) - \frac{1}{35380800}t^{13}(11520x^7(-1+\alpha)^6 + 15060x^4(-1+\alpha)^5\alpha$$
$$-12708x^6(-1+\alpha)^3\alpha^2 - 5100x^3(-1+\alpha)^2\alpha^3 - 35x^5\alpha^4 - 252(-1+\alpha)\alpha^4)$$
$$+\frac{1}{123832800}t^{14}\alpha(37980x^7(-1+\alpha)^5 + 20105x^4(-1+\alpha)^4\alpha - 5931x^6(-1+\alpha)^2\alpha^2$$
$$-1128x^3(-1+\alpha)\alpha^3 - 21\alpha^4)$$

$$+\frac{1}{28576800}t^{15}x^3(480x^5(-1+\alpha)^7 - 3265x^4(-1+\alpha)^4\alpha^2 - 886x(-1+\alpha)^3\alpha^3$$
$$+84x^3(-1+\alpha)\alpha^4 + 6\alpha^5)$$
$$-\frac{1}{228614400}t^{16}x^4\alpha(4290x^4(-1+\alpha)^6 - 4874x^3(-1+\alpha)^3\alpha^2 - 707(-1+\alpha)^2\alpha^3$$
$$+15x^2\alpha^4)$$
$$+\frac{1}{7772889600}t^{17}x^4(-1+\alpha)\alpha^2(66950x^4(-1+\alpha)^4 - 16151x^3(-1+\alpha)\alpha^2 - 1176\alpha^3)$$
$$-\frac{1}{69956006400}t^{18}(146045x^8(-1+\alpha)^4\alpha^3 - 6944x^7(-1+\alpha)\alpha^5 - 196x^4\alpha^6)$$
$$+\frac{t^{19}x^7\alpha^4(319x(-1+\alpha)^3 - 2\alpha^2)}{1116944640} - \frac{487t^{20}x^8(-1+\alpha)^2\alpha^5}{22338892800} + \frac{t^{21}x^8(-1+\alpha)\alpha^6}{1175731200}$$
$$-\frac{t^{22}x^8\alpha^7}{77598259200},$$

$$\vdots$$

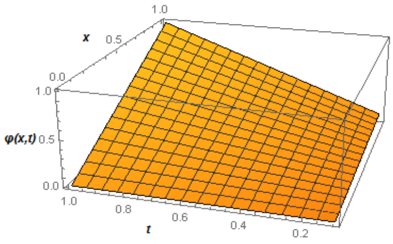
So, the approximate solution of given by the equation

$$\varphi(x,t) = \sum_{n=0}^{\infty} \varphi_n(x,t) = \varphi_0 + \varphi_1 + \varphi_2 + \cdots,$$

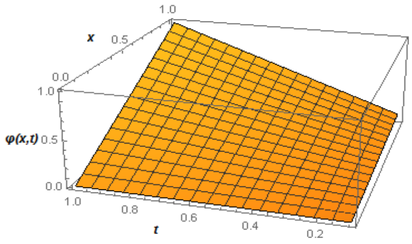
$$\varphi(x,t) = tx + \frac{1}{3}t^3x^2(1-\alpha) + \cdots.$$

The result is the same as the closed form solution, when $\alpha \to 1$, i.e.

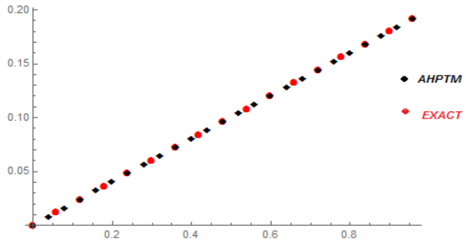
$$\varphi(x,t) = xt.$$



(a)

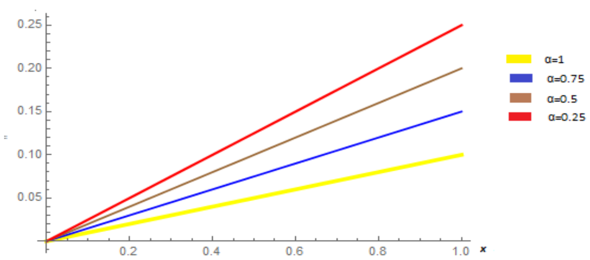


(b)



(c)

Fig.6.2.3(i): (a) surface graph of exact solution at $\alpha = 1$, (b) surface graph AHPTM at $\alpha = 1$, (c) line graph of exact solution and AHPTM at $\alpha = 1, 0 < \varphi \leq 1$ and $t = .1$.



(d)

Fig. 6.2.3 (ii): (d). the solution of eq. (6.2.9) at different fractional order $\alpha = 1, 0.75, 0.5, 0,25, 0 < \varphi \leq 1$ and $t = .1$.

Table: 6.2.3 - Error analysis of example (6.2.9) at $t = 0.2, 0.4, 0.6$ and $\alpha = 1$(upto fourth order)

t	x	φ_{Exact}	φ_{AHPTM}	Abs. Error	$\|\varphi_1\|$	$\|\varphi_2\|$	$\|\varphi_3\|$
0.2	0.1	0.02	0.02	6.14E-16	9.77829E-07	3.5553E-07	2.48324E-11
	0.3	0.06	0.06	5.65E-15	1.16458E-05	3.54258E-07	7.44325E-11
	0.5	0.1	0.01	1.57E-14	3.29841E-05	3.49329E-07	1.23621E-10
	0.7	0.14	0.14	3.07E-14	6.49952E-05	3.38303E-07	1.71704E-10
	0.9	0.18	0.18	5.04E-14	0.000107681	3.18739E-07	2.17641E-10
0.4	0.1	0.04	0.04	2.37E-12	1.41572E-06	2.27618E-05	1.27173E-08
	0.3	0.12	0.12	2.30E-11	0.00016942	2.2618E-05	3.81018E-08
	0.5	0.2	0.2	6.40E-11	0.000511391	2.20054E-05	6.30868E-08
	0.7	0.28	0.28	1.25E-10	0.00102481	2.06104E-05	8.69612E-08
	0.9	0.36	0.36	2.02E-10	0.001709989	1.8119E-05	1.08657E-07
0.6	0.1	0.06	0.06	2.74E-10	0.000151089	0.000259578	4.89284E-07
	0.3	0.18	0.18	2.95E-09	0.000715803	0.00025766	1.46646E-06
	0.5	0.3	0.3	8.27E-09	0.002454715	0.0002477	2.42272E-06
	0.7	0.42	0.42	1.61E-08	0.005071014	0.000224286	3.31689E-06
	0.9	0.54	0.54	2.62E-08	0.008570088	0.000181973	4.08704E-06

Example 6.2.4: Consider the nonlinear Burgers' equation in the C-F sense [121].

$$ {}^{\mathbb{C}\mathrm{F}}_{0}\mathrm{D}_t^{\alpha}\varphi(x,t) + \varphi\frac{\partial\varphi}{\partial x} = \eta\frac{\partial^2\varphi}{\partial x^2},\ 0 < \alpha \le 1, \qquad (6.2.12)$$

With initial condition

$$\varphi(x,0) = nx,$$

using LT to eq. (6.2.12), we have

$$\mathcal{L}[\varphi(x,t)] = \frac{1}{s}\varphi(x,0) - \left(\frac{s+\mu(1-s)}{s}\right)\mathcal{L}\left[\varphi\frac{\partial\varphi}{\partial x} - \eta\frac{\partial^2\varphi}{\partial x^2}\right] \qquad (6.2.13)$$

on operating the inverse of the LT to eq. (6.2.13)

$$\varphi(x,t) = \varphi(x,0) - \mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[\varphi\frac{\partial\varphi}{\partial x} - \eta\frac{\partial^2\varphi}{\partial x^2}\right]\right]$$

Now, apply AHPTM

$$\sum_{n=0}^{\infty} p^n \varphi_n\,(x,t) = nx + p\mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[\eta\frac{\partial^2\varphi}{\partial x^2}\sum_{n=0}^{\infty} p^n\varphi_n(x,t) - \sum_{n=0}^{\infty} p^n\widetilde{H}_n(\varphi)\right]\right]$$

Where first component of $\widetilde{H}_n(\varphi)$ given as,

$$\widetilde{H}_0(\varphi) = \varphi_0\varphi_{0x}$$

$$\widetilde{H}_1(\varphi) = \varphi_0\varphi_{1x} + \varphi_1\varphi_{0x} + \varphi_1\varphi_{1x}$$

$$\widetilde{H}_2(\varphi) = \varphi_0\varphi_{2x} + \varphi_1\varphi_{2x} + \varphi_2\varphi_{0x} + \varphi_2\varphi_{1x} + \varphi_2\varphi_{2x}$$

$$\vdots$$

On comparing the similar power of *p*, we have

$$\varphi_0(x,t) = nx,$$

$$\varphi_1(x,t) = -n^2x(1-\alpha+t\alpha),$$

$$\begin{aligned}\varphi_2(x,t) &= 2n^2x(1-2\alpha+\alpha^2) - n^4x(1-3\alpha+3\alpha^2-\alpha^3) \\ &+ t(4n^3x\alpha - 3n^4x\alpha - 4n^3x\alpha^2 + 6n^4x\alpha^2 - 3n^4x\alpha^3) \\ &+ t^2(n^3x\alpha^2 - 2n^4x\alpha^2 + 2n^4x\alpha^3) - \frac{1}{3}n^4t^3x\alpha^3,\end{aligned}$$

$$\varphi_3(x,t) = -4n^4x + 6n^5x - 6n^6x + 4n^7x - n^8x + 12n^4x\alpha - 24n^5x\alpha + 30n^6x\alpha$$
$$- 24n^7x\alpha + 7n^8x$$
$$- n^4tx(12 + 24n(-1+\alpha) + 30n^2(-1+\alpha)^2 + 24n^3(-1+\alpha)^3$$
$$+ 7n^4(-1+\alpha)^4)(-1+\alpha)^2\alpha - 12n^4x\alpha^2 + 36n^5x\alpha^2 - 60n^6x\alpha^2$$
$$+ 60n^7x\alpha^2 - 21n^8x\alpha^2$$
$$+ n^4t^2x(6 + 23n(-1+\alpha) + 42n^2(-1+\alpha)^2 + 44n^3(-1+\alpha)^3$$
$$+ 16n^4(-1+\alpha)^4)(-1+\alpha)\alpha^2 + 4n^4x\alpha^3 - 24n^5x\alpha^3 + 60n^6x\alpha^3$$
$$- 80n^7x\alpha^3 + 35n^8x\alpha^3$$
$$-\frac{1}{3}n^4t^3x(2 + 22n(-1+\alpha) + 68n^2(-1+\alpha)^2 + 104n^3(-1+\alpha)^3$$
$$+ 51n^4(-1+\alpha)^4)\alpha^3 + 6n^5x\alpha^4 - 30n^6x\alpha^4 + 60n^7x\alpha^4 — 35n^8x\alpha^4$$
$$+\frac{1}{6}n^5t^4x(4 + 29n(-1+\alpha) + 75n^2(-1+\alpha)^2 + 55n^3(-1+\alpha)^3)\alpha^4$$
$$+ 6n^6x\alpha^5 - 24n^7x\alpha^5 + 21n^8x\alpha^5$$
$$-\frac{1}{15}n^6t^5x(5 + 30n(-1+\alpha) + 38n^2(-1+\alpha)^2)\alpha^5 + 4n^7x\alpha^6 - 7n^8x\alpha^6$$
$$+\frac{1}{9}n^7t^6x\big(1 + 3n(-1+\alpha)\big)\alpha^6 + n^8x\alpha^7 - \frac{1}{63}n^8t^7x\alpha^7,$$

$$\vdots$$

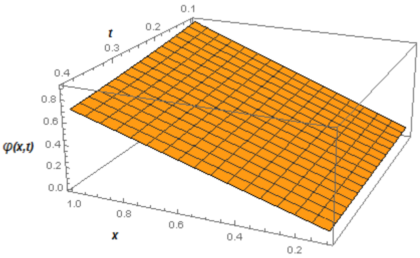
So the approximate solution of given by the equation

$$\varphi(x,t) = \sum_{n=0}^{\infty}\varphi_n(x,t) = \varphi_0 + \varphi_1 + \varphi_2 + \cdots$$

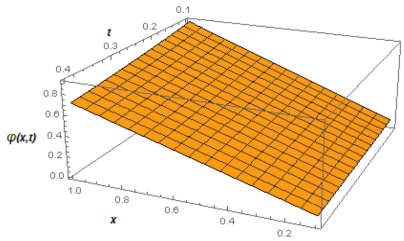
$$\varphi(x,t) = nx + 2n^3x - 5n^4x + 6n^5x - 6n^6x + 4n^7x - n^8x - 4n^3x\alpha \ldots$$

solution obtain when $\alpha \to 1$, i.e.

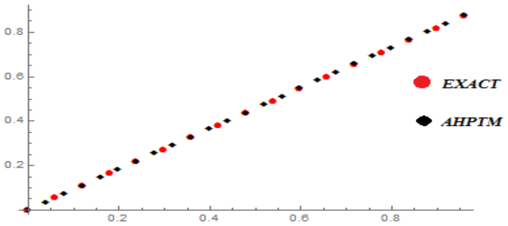
$$\varphi(x,t) = \frac{nx}{1+nt}$$



(a)

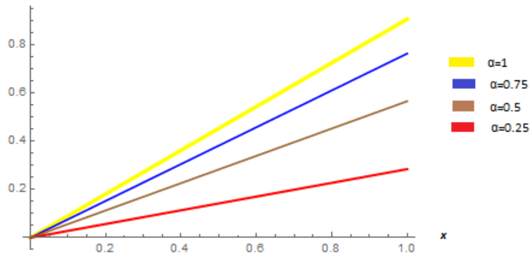


(b)



(c)

Fig 6.2.4 (i): (a) surface graph of exact solution at $\alpha = 1$, (b) surface graph AHPTM at $\alpha = 1$, (c) line graph of exact solution and AHPTM at $\alpha = 1, 0 < \varphi \leq 1 \; and \; t = 0.1$.



(d)

Fig 6.2.4(ii): (d) the solution of eq. (5.2.10) at different fractional order $\alpha = 1, 0.75, 0.5, 0.25, 0 < \varphi \leq 1 \; and \; t = 0.1$.

Table: 6.2.4- Error analysis of example (6.2.10) at $t = 0.2, 0.4, 0.6$, $n = 1$ and $\alpha = 1$(upto fourth order)

t	x	φ_{Exact}	φ_{AHPTM}	Abs. error	$\|\varphi_1\|$	$\|\varphi_2\|$	$\|\varphi_3\|$
0.1	0.1	0.090909091	0.090906344	2.74662E-06	0.01	0.006266667	0.000436643
	0.3	0.27272727	0.272719033	8.23987E-06	0.03	0.0188	0.001309928
	0.5	0.45454545	0.454531721	1.37E-05	0.05	0.031333333	0.002183213
	0.7	0.636363636	0.63634441	1.92E-05	0.07	0.043866667	0.003056498
	0.9	0.818181818	0.818157099	2.47E-05	0.09	0.0564	0.003929783
0.3	0.1	0.076923077	0.076766753	1.56E-04	0.03	0.0081	0.000436643
	0.3	0.230769231	0.230300259	4.69E-04	0.09	0.0243	0.001309928
	0.5	0.384615385	0.383833764	7.82E-04	0.15	0.0405	0.002183213
	0.7	0.538461538	0.53736727	1.09E-03	0.21	0.0567	0.003056498
	0.9	0.692307692	0.690900776	1.41E-03	0.27	0.0729	0.003929783
0.5	0.1	0.066666667	0.06578621	8.80E-04	0.05	0.020833333	0.005047123
	0.3	0.2	0.197358631	2.64E-03	0.15	0.0625	0.015141369
	0.5	0.333333333	0.328931052	4.40E-03	0.25	0.104166667	0.025235615
	0.7	0.466666667	0.460503472	6.16E-03	0.35	0.145833333	0.035329861
	0.9	0.6	0.592075893	7.92E-03	0.45	0.1875	0.045424107

Conclusion: In conclusion, this chapter has provided a comprehensive explanation of the methodologies employed to fulfill the study objectives. Within the scope of this investigation, nonlinear fractional partial differential equations (PDEs), characterized by both Caputo and C-F formulations, were utilized to address various equations, including the Burgers' Equation, KdV equation, and the K-G equation. To validate our approach, meticulous computations were conducted using Mathematica, yielding approximate results. These findings were then visually represented through surface and line graphs. Additionally, a thorough examination of the convergence properties of this method was undertaken, contributing to a deeper understanding of its efficacy in solving complex mathematical problems.

7. Conclusion

The thesis titled "Semi- analytical Methods for Solution of Nonlinear Partial Differential Equations" focuses on extending analytical approaches by presenting a hybrid approach. The fundamental research goal is to create novel hybrid semi-analytical methods by combining integral transformation techniques, particularly Laplace and other integral transformations. These advanced methods are designed to solve complicated physical issues, notably PDEs and fractional differential equations. The ultimate objective is to enable the resolution of these complex problems under various circumstances in order to cater to real-life applications in diverse domain Finally, we validated the study's outcomes by comparing them to established results in the existing literature, confirming the accuracy and reliability of our findings. Our research focuses on the use of accelerated algorithms, specifically He's polynomial, to improve convergence and reduce the number of iterations. We used a variety of semi-analytical methods, including the Accelerated Homotopy Perturbation Elzaki Transform Method (AHPETM), the Accelerated Homotopy Perturbation Sumudu Transform Method (AHPSTM), the Accelerated Homotopy Perturbation Transformation Method (AHPTM). In our research, we apply proposed method to the Burgers' equation, the Advection problem, the Benjamin-Bona-Mahoney-Burgers equations (BBMBEs), Fisher's equation, the KdV equation, and the K-G equation. The approximate series solution was then represented and visually shown using surface and line graphs, providing insights into our approaches. In addition, we performed error analysis and convergence analysis for the numerical simulation of the given model, as well as confirmation of findings with published date.

Finally, this thesis presents a comprehensive review of our study, divided into six distinct chapters. Chapter 1 lays the groundwork by providing a fundamental understanding of differential equations and approaches for solving nonlinear problems. Chapter 2 delves into a study of pertinent literature, providing a thorough framework for our work. Chapter 3 outlines suggested methods, such as Acc. HPETM, for tackling PDEs, which are rigorously evaluated for efficacy. Chapter 4 applies two distinct methods to various equations, yielding approximate solutions. Chapter 5 implements a novel method for nonlinear fractional PDEs in the Caputo sense, introducing fractional calculus. Chapter 6 employs Acc. HPTM for nonlinear fractional PDEs in the C-F sense, demonstrating versatility. Each chapter builds upon the previous one, showcasing a logical progression in applying semi-analytical methods to increasingly complex nonlinear PDEs. Chapter 4 expands hybrid methods to solve broader nonlinear equations, while Chapter 5 extends applications to nonlinear fractional PDEs in the Caputo sense. Chapter 6 further develops this by transitioning to the Caputo-Fabrizio sense, demonstrating the adaptability of these techniques.

Future Scope:

Although our research accomplished its objectives, there are still unresolved issues that we want to address in future endeavors.

1. We develop some other semi-analytical techniques that are novel to the existing techniques using different integral transformation methods like the Sehu transform, the Aboodh transform, etc., and find better and more accurate results than the existing techniques.
2. In future work, we will obtain the results by using other fractional derivatives, like the Atangana-Baleanu fractional derivative.

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Paper Publications

Papers Published from the thesis

1. Shabnam Jasrotia and Prince Singh, Accelerated Homotopy Perturbation Elzaki Transformation Method for Solving Nonlinear Partial Differential Equations. In *Journal of Physics: Conference Series* Vol. 2267(2021), No. 1, p. 012106. IOP Publishing.
2. Shabnam Jasrotia and Prince Singh, A Semi-Analytical Method for Solving Nonlinear Fractional-Order Swift- Hohenberg Equations. In *Contemporary Mathematics*, Vol.1062-1075.

Paper Communicated from the thesis

1. Shabnam Jasrotia and Prince Singh, A Comparison of the Accelerated Homotopy Perturbation Transformation with the Homotopy Perturbation Transformation Technique for solving Nonlinear Fractional PDE
2. Shabnam Jasrotia and Prince Singh, The Solution of Burgers' Equation by Accelerated Homotopy Perturbation Transformation Method.
3. Shabnam Jasrotia and Prince Singh, Semi-Analytical solution of Nonlinear Fractional Partial Differential Equations Using Accelerated HPTM.

Conferences and Workshop

Attended "International conference on Emerging trends in Pure and applied Mathematics"**,** Department of applied science, School of Engineering in association with department of department of Mathematical sciences, School of Tezpur University. March 12-13, 2022.

Presented in 6th International Conference on Recent Advances in Science (ICRAS-2022), Invertis University, Bareilly, April 1-2, 2022.'Accelerated Homotopy Perturbation Sumudu Transformation method for solving 1D and 2D Benjamin-Bona-Mahony-Burgers Equations."

Presented in National Conference on Ancient Mathematics and its Emerging Areas (AMEA-2022), Teerthanker Mahaveer University (TMU), Moradabad, November 18, 2022. 'The Solution of Burgers' Equation by Accelerated Homotopy Perturbation Transformation Method.'

Presented in International Conference on Fractional Calculus: Theory, Application and Numerics (ICFCTAN) (Springer, Universal Wiser Publisher)**.** National Institute of Technology Puducherry, Karaikal, January 27-29, 2023. 'Semi-Analytical solution of Nonlinear Fractional Partial Differential Equations Using Accelerated HPTM'.

Attended "Three days live online Workshop on Mathematica for Beginners (A Technical Computing System)", Department of Applied Sciences Chitkara University, Punjab, August 17-19, 2020.

Attended two week faculty Development programme on "Managing Online Classes and Co-Creating Moocs:2.0" organised by Teaching learning centre Ramanujan college, university of Delhi and sponsored by Ministry Of Human Resource Development Pandit Madan Mohan Malaviya National Mission on Teachers and Teaching from may 18 to june 03, 2020

Attended one week faculty Development programme on "Open Source Tool For Research" organised by Teaching learning centre Ramanujan college, university of Delhi and sponsored by Ministry Of Human Resource Development Pandit Madan Mohan Malaviya National Mission on Teachers and Teaching from june 08 to june 14, 2020

Attend “Two-day National Webinar on Applied Mathematics and its Recent Applications” organised by Department of mathematics Ghatal Rabindra Satabarsiki Mahavidyala , West Bengal on 16-17th June 2022

Participate in the "Academic Writing" course for four credits from Swayam in September 2020.