A STUDY ON ROUGH CONVERGENCE OF SOME SEQUENCES IN CERTAIN SPACES

Thesis Submitted for the Award of the Degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

By

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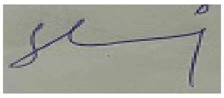
2024

DECLARATION

I, hereby declare that the presented work in the thesis entitled “A Study on Rough Convergence of Some Sequences in Certain Spaces” in fulfilment of degree of **Doctor of Philosophy (Ph. D.)** is outcome of research work carried out by me under the supervision of Dr. Pankaj Pandey, working as Associate Professor, in the Department of Mathematics of Lovely Professional University, Punjab, India and Dr. Sanjay Mishra, working as Professor, in the Department of Mathematics of Amity University, Lucknow Campus, U.P., India. In keeping with general practice of reporting scientific observations, due acknowledgements have been made whenever work described here has been based on findings of other investigator. This work has not been submitted in part or full to any other University or Institute for the award of any degree.



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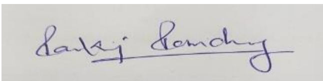
CERTIFICATE

This is to certify that the work reported in the Ph. D. thesis entitled “A Study on Rough Convergence of Some Sequences in Certain Spaces” submitted in fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Ph.D.)** in the Department of Mathematics, is a research work carried out by Shivani Sharma, 42000333, is bonafide record of her original work carried out under my supervision and that no part of thesis has been submitted for any other degree, diploma or equivalent course.



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“Pure Mathematics is, in its way, the poetry of logical ideas.”

Albert Einstein

Abstract

The convergence of sequences plays an indispensable role in mathematics. As the sequence progresses it is said to be convergent if its terms eventually (all but finitely many) become arbitrarily close to a certain unique finite value and is said to be divergent if no such finite value exists. Convergent sequences are extensively used to approximate real numbers, which has practical applications in numerical methods for solving differential equations and optimization problems. Since convergent sequences play a pivotal role in analysis, it is always desirable to have convergent sequences. Thus, various efforts were made to define new types of convergence under which a sequence that diverges or oscillates in the usual sense can be considered convergent. In 1935, a new type of convergence was given, which is founded on the concept of the "natural density" of the indexing set. In this type of convergence, the focus shifts from the behaviour of the majority of the terms in any sequence, to the terms that do not come close to a certain finite value under consideration. This convergence is called statistical convergence, is in some way similar to the concept of 'almost everywhere' in measure theory and has been referred to as almost convergence by Zygmund.
In 2000, Kostyrko extended statistical convergence and developed ideal convergence. He shifted from so called, "natural density" (of positive $\mathbb{Z}$), to a specific collection of subsets of any arbitrary set, which he named an ideal. He established that there are sequences that are convergent with respect to ideal convergence but fail to converge statistically. In 2001, Phu introduced rough convergence, which differs from all the above mentioned convergences as it allows a sequence to converge to any point in any neighbourhood. Recently, rough convergence has been invoking interest from researchers and has magnetized a lot of focus.

In this thesis, "A Study on Rough Convergence of Some Sequences in Certain Spaces," we have examined rough convergence, rough ideal convergence, generalized it for sequence spaces and investigated the topology defined by rough convergent sequences. The work was done under the able guidance of Dr. Pankaj Pandey, Associate Professor at the Department of Mathematics, LPU, Phagwara, Punjab and co-guided by Dr. Sanjay Mishra, Professor at the Department of Mathematics, Amity University, Lucknow, Uttar Pradesh and is being submitted at LPU,

Phagwara, Punjab, India for the award of the degree of Doctor of Philosophy in Mathematics.

This thesis has been divided into four chapters. The first chapter is the Introduction. It consists of the literature and background on convergence and rough convergence. This chapter has seven sections. The first section is devoted to the history and background on rough convergence. In the following section, we have considered rough statistically convergent sequences and gave various examples and important theorems that suggest rough statistical convergence extends the concept of rough convergence. The third section of this chapter introduces rough ideal convergence. Several examples of different types of ideals have been given in this section and some results based on this concept have been given, which will be useful in the subsequent chapters of this thesis. In the fourth section of this chapter, we have recalled some common sequence spaces. This fifth section focuses on the research gap and the motivation for taking up this topic for the research work. In the sixth section we have given the objectives of the work done and the last section gives a brief outline of the work presented in this thesis.
The significance of bounded linear operators in the analysis is undeniable. This pushed us to consider the sequences of bounded linear operators. The second chapter introduces rough ideal convergence for sequences with elements as bounded linear operators between normed spaces (n.l.s.). We have also analysed the algebraic properties of the rough ideal convergent sequence spaces(S-spaces) so obtained. We have also looked into the inclusion relations between these rough ideal convergent S-spaces. The results of the second chapter have been published in South East Asian Journal of Mathematics and Mathematical Sciences Vol. 19, No. 2 (2023), pages 297-310 DOI: 10.56827/SEAJMMS.2023.1902.22 and the results of the third chapter has been accepted to be published in Palestine Journal of Mathematics.

After investigating the rough ideal convergent sequence spaces for the sequences of bounded linear operators, we were intrigued to view the sequence spaces of real numbers under rough ideal convergence and generalise the sequence spaces so obtained with the help of the Orlicz function. Throughout the third chapter, we have considered the sequences of real numbers. Works of [84],[64],[39], laid the foundation for this chapter.

Much of what we need to know about the real line and the functions $g\colon \mathbb{R} \to \mathbb{R}$, can be explained using convergent sequences. We were therefore drawn to study the possibility of generating a topology using rough convergent sequences. In this quest, we have used Kurtatowski's closure operator and defined Kuratowski topology. Continuity and compactness can also be viewed via convergent sequences

A.K. Banerjee and A. Paul [7] attempted to replicate the same with the help of rough convergent sequences. They gave various illustrations to support their idea and showed that rough continuous maps fail preserve compactness and connectedness. In the last chapter of this thesis, we have tried to address this issue and have introduced and explored rough sequential continuity, rough sequential compactness and rough sequential connectedness.

We have presented new theorems and proofs based on the approach of contradiction and generalization. We have also included conclusion and future scope of this work at the end of this thesis.

Acknowledgements

I consider myself extremely fortunate to not have journeyed alone in this endeavour. This work progressed smoothly to its conclusion with the unwavering, support, guidance, blessings and encouragement of various individuals. At this moment of accomplishment, I would like to express my heartfelt and sincere gratitude towards everyone who played a part in their diverse capacities.

First and foremost, I am profoundly thankful to God, the one to whom I owe my very being, for choosing me for this work and bestowing upon me the wisdom and guidance to accomplish this work.

No words can quantify my deep gratitude for my guide Dr. Pankaj Panday, Associate Professor, Department of Mathematics, Lovely Professional University and my co-guide, Dr. Sanjay Mishra, Professor, Department of Mathematics, Amity University Lucknow Campus, U.P., for picking me as a student and their strong and able shoulders never let me feel the pressure of this work. It was a blessing to work under the guidance of such dedicated, experienced and energetic individuals. I am reminded of the countless moments of inspiration, constructive criticism and stimulating suggestions that have propelled me forward on this scholarly journey and have been instrumental in shaping this thesis.

I extend my sincere and warm thanks to my dearest friend, Dr. V. Renuka Devi, Associate Professor, Department of Mathematics, Central University of Tamil Naidu, for rendering selfless and unflinching support, which helped me take the first step in this journey. A pure, altruistic and nobel soul who spared her valuable time, addressed my concerns, busted my stress and gave insightful ideas for this work.

I take this opportunity with much pleasure to acknowledge the encouragement and support of my seniors, research mates, comrades and members of the Department of Mathematics who extended help in any way I asked for.

This work would not have seen the light of day, if my family had not been supportive. With deep respect and admiration, I extend my appreciation and gratitude to my parents, my siblings, my better half and my daughter for the sacrifices they had to make so that I could focus on this work.

Contents

Abbreviations

Abbreviations	**List of Abbreviations**
a.i.	**a**dmissible **i**deal of Natural numbers, unless and otherwise stated
b.l.o.	**b**ounded **l**inear **o**perator
n.l.s.	**n**ormed **l**inear **s**pace
rc	**r**ough **c**losed
r-sc	**r**ough **s**equentially **c**ompact
r-scontinuous	**r**ough **s**equentially continuous
r-sconnected	**r**ough **s**equentially connected
S-space	**S**equence **space**
WLOG	**W**ithout **L**oss **O**f **G**enerality

Symbols

Symbol	Name
$\mathbb{N}$	Set of Natural numbers
$\mathbb{R}$	Set of Real numbers
$\mathbb{Z}$	Set of Integers
$\mathbb{C}$	Set of Complex numbers
$\mathfrak{I}$	Ideal on Natural numbers, unless and otherwise stated
$\mathfrak{F}$	Filter on Natural numbers, unless and otherwise stated
$\mathfrak{F}(\mathfrak{I})$	Filter associated with ideal
$\exists$	There exists
$\nexists$	There does not exist
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\neq$	Not equal to
$\cap$	Intersection
$\cup$	Union
$\subset$	Inclusion
$\lim_n$	Limit tending to infinity
$\sup_k$	Supremum over all values of k
$\inf_k$	Infimum over all values of k
$\sum_k$	Summation from 1 to infinity
ω	The set of real or complex sequences
c^0	The set of all sequences converging to zero.

ℓ^∞	The set of bounded sequences
ℓ^p	The set of pth summable series

Dedicated to my beloved family.

Chapter 1

Introduction

Our objective in this chapter is to offer a fast overview of a few key concepts and results from real analysis. Our goal is to set the stage for the subsequent chapters and gather in one place some of the notations that are already more or less familiar.

1.1 Convergence of Sequences

Sequences and their convergence are fundamental tools in mathematics, providing insights into the behavior of mathematical objects, facilitating approximation and analysis, and serving as the basis for many important mathematical theories and applications. Sequences play a pivotal role in the study of limits, continuity, differentiation, integration, and other concepts in mathematical analysis. Intuitively, a sequence can be considered an ordered list of objects or events. Let us now introduce sequences formally and give some definitions and examples that will be frequently used in the subsequent chapters of this thesis.

Definition 1.1.1. (Sequence) A function $s\colon \mathbb{N} \to A$ defined as

$$s(k) = a_k,$$

is a sequence $\{a_k\}$ in A, where A is any non-empty set. Here, a_k denotes the k^{th} element of the sequence. Clearly, sequence can be considered an ordered list $a_1, a_2, a_3..., a_k, ...$ indexed by natural numbers.

Example 1.1.1. *Let* $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$. *Here, the* k^{th} *term is* $\frac{1}{k}$. *So, we write in short as* $\{\frac{1}{k}\}_{k\in\mathbb{N}}$ *or simply* $\{\frac{1}{k}\}, (\frac{1}{k})$.

For large k, a sequence may converge, diverge, or oscillate.

Definition 1.1.2. (Convergent sequence) Consider $\{a_n\}$ with real entries and $a^* \in \mathbb{R}$. Then we say that a_n converges to a^* if, for any pre assigned number $\epsilon > 0$, $\exists$ some p $\in \mathbb{N}$ and for any q $\geq$ p,

$$|a_q - a^*| < \epsilon.$$

The following notations are commonly used for the convergence of a sequence:

1. $a_n \to a^*$ as n $\to \infty$.
2. or $\lim_{n\to\infty} a_n = a^*$,

and a^* is referred to as the limit of $\{a_n\}$.

If $\nexists$ any finite number, $a^* \in \mathbb{R}$, for which the majority of terms cluster near a^*, that is $a_n \nrightarrow a^*$ as n $\to \infty$ then the sequence $\{a_n\}$ is not convergent or is divergent. We use the following notations for divergence of a sequence:

1. $a_n \to +\infty$ *or* $-\infty$ as $n \to \infty$.
2. or $\lim_{n\to\infty} a_n = +\infty$ *or* $-\infty$.

Definition 1.1.3. (Bounded Sequence) If we can find some finite real number κ satisfying,

$$|a_n| \leq \kappa, \forall\, n,$$

then we consider the sequence $\{a_n\}$ to be bounded.

A sequence that is neither convergent nor divergent is said to be oscillatory.

Definition 1.1.4. (Subsequence) A subsequence $\{b_n\}$ of a sequence $\{a_n\}$ is defined to be a sequence given by $b_k = a_{n_k}$, where $\{n_k\}$ is a sequence of positive integers with $n_1 < n_2 < n_3 \cdots$.

Definition 1.1.5. (Cauchy Sequence) A sequence $\{a_n\}$ is Cauchy if, after a certain point sequence's elements approach each other. Mathematically, if for given $\epsilon > 0$ however small, $\exists\ p\ (0 < p\ \in \mathbb{N})$ satisfying,

$$|a_q - a_r| < \epsilon, \forall q,\ r \geq p.$$

We now give some important theorems on convergent sequences.

Theorem 1.1.1. *Convergent sequence has exactly one limit.*

Theorem 1.1.2. *A convergent sequence necessarily have bounds.*

However, the above condition fails to be sufficient for the convergence of a sequence.

Theorem 1.1.3. *Let $\{p_n\}$ and $\{q_n\}$ be any two convergent sequences converging to some p and q respectively. Then the following holds good:*

1. $\lim_{n\to\infty} cp_n = cp$, *where c is any constant.*
2. $\lim_{n\to\infty}(p_n \pm q_n) = p \pm q$.
3. $\lim_{n\to\infty} p_n q_n = ab$.
4. $\lim_{n\to\infty} \frac{p_n}{q_n} = \frac{p}{q}$, $q_n \neq 0$, $q \neq 0$.

Theorem 1.1.4. *For a real sequence $\{a_n\}$, the statements given under are interchangeable:*

1. $\{a_n\}$ *is Cauchy.*
2. $\{a_n\}$ *is convergent.*

Theorem 1.1.5. *(Bolzano-Weierstrass Theorem) "Every bounded sequence has a convergent subsequence."*

1.2 Rough Convergence

H.X. Phu [66] conceived the notion of rough convergence for finite dimensional n.l.s. and defined rough convergence as follows: For any sequence (a_k) in some n.l.s. $(N, \|\cdot\|)$ and $r > 0$ is any non-negative, (a_k) is r-convergent to a_*, symbolized as $a_k \xrightarrow{r} a_*$, provided there is some $k_\epsilon \in \mathbb{N}$ with

$$k \geq k_\epsilon \Rightarrow \|a_k - a_*\| < r + \epsilon, \ \forall \epsilon > 0,$$

where r and a_* are called the "roughness degree" and the r-limit point of (a_k) respectively.
Under this definition, a sequence is allowed to converge to different limits for different values of r. This means for a r-convergent sequence, r-limit is not unique. Thus, we consider a set of r-limit points denoted by $LIM^r a_k$ and defined as,

$$LIM^r a_k = \{a_* \colon a_k \xrightarrow{r} a_*\}.$$

Introduction of rough convergence can be supported by various reasons. It is widely recognized that a convergent sequence frequently cannot be precisely determined, measured or calculated. As a result, one often deals with an approximated sequence. To support this concept numerically and to explain the need for the concept of rough convergence Phu gave the following illustration:

Example 1.2.1. *[66] Let*

$$a_k = 0.5 + \frac{2(-1)^k}{k} \ , \quad k = 1, 2, ...$$

As k tends to infinity, calculating the limit for (a_k) gets difficult. So, to calculate, the limit of a_k, we choose $b_k = rd\,(a_k) = c, c \in \mathbb{Z},\ c - 0.5 \ \leq a_k \ < c + 0.5$.
Then
$b_1 = -1,\ b_2 = 2, b_{2n-1} = 0, b_{2n} = 1 \ \ for \ \ n = 2, 3, ...$
Clearly, (b_k) does not converge to any real number under usual convergence. But $b_k \xrightarrow{r} 0.5$ for r=0.5 and for $r > 0.5$ $LIM^r_{b_k} = [1 - r, r]$. However, it fails to r-converge to any limit for $r < 0.5$.

Phu focused his work on study of the properties (algebraic and geometric) of the r-limit set so obtained and established the convexity and boundedness of this set.

He viewed Cauchy sequences through the lens of rough convergence and came up with the concept of ρ-Cauchy sequences and established that ρ =2r is the least Cauchy degree. He also explored the correlation between Cauchy and ρ-Cauchy sequences and showed that a for some ρ >0, a bounded sequence is equivalent to a ρ-Cauchy sequence. Additionally, he also proved that for r > $\sqrt{\frac{n}{2(n+1)}}\rho$, every ρ-Cauchy sequence is r-convergent.
It was a natural quest to examine if similar results hold for an infinite dimensional n.l.s. Phu [68] introduced rough convergence for infinite n.l.s. and examined various results analogous to the usual convergent sequences.
Using this idea of "convergence in a relaxed neighbourhood", Phu [67] attempted to give a novel concept of continuity named rough continuity.

Definition 1.2.1. (Rough Continuity) A function $\mu\colon U \to V$ between any two n.l.s. $(U, \|.\|)$ and $(V, \|.\|)$ is defined to be roughly continuous at $m \in$ U, (in short, r-continuous at u) if for some pre assigned $\epsilon > 0$, there is some positive $\delta > 0$ satisfying,

$$\left\|m' - m\right\|_U < r + \delta \quad \Longrightarrow \quad dist\left(\mu(m'),\ \mu(d(m,r))\right) < \epsilon.$$

where, r is some positive integer,

$$d(m,r) = \{w \in U : \|w - m\|_U \le r\} \text{ and}$$

$$dist\left(\mu(m'), \mu(d(m,r))\right) = inf_{v\in\mu(d(m,r))}\ \left\|\mu(m') - v\right\|.$$

We give an illustration to understand rough continuity.

Example 1.2.2. *The Dirichlet function, $d\colon \mathbb{R} \to \mathbb{R}$ which takes rational numbers to 1 and irrational numbers to 0, is discontinuous at each $x \in \mathbb{R}$, but for r=1, d is r-continuous at each $x \in \mathbb{R}$.*

The introduction of this concept sparked the interest of researchers worldwide in exploring different structures like sequences such as double sequences, difference sequences, or spaces, normed spaces like 2-normed space etc., for studying rough convergence within that framework.
In 2017, Debnath and Rakshit [17] introduced rough convergence for metric spaces.

Definition 1.2.2. (Rough Convergence in a Metric Space) For any sequence $\{a_n\}$, r-convergence of $\{a_n\}$ to a point $a_* \in$ in a metric space (M,d), is defined as follows: Given $\epsilon > 0$, then for some $p_0 \in \mathbb{N}$, $d(a_p, a_*) < r+\epsilon$, holds for every p $\geq\ p_0,\ r > 0$.

Examination of rough convergence for various other related areas ("cone metric spaces, S-metric spaces, generalized metric spaces"), was then taken up by several scholars.

1.3 Rough Statistical Convergence

Problems with series summation, prompted the introduction of statistical convergence. Statistical convergence (in short, s-convergence) focuses on those points of a sequence that do not go arbitrarily close to the limit of the sequence, thus generalizing the regular concept of convergence. Statistical convergence first appeared in the much talked about *monograph of Zygmund* [88] published in Warsaw (first edition) in 1935, where it was called "almost convergence". Fast [24] and Steinhaus [79] are credited for formally introducing the concept of statistical convergence for sequences in $\mathbb{R}$. However, Schoenberg [76] reintroduced statistical convergence in the year 1959. Buck [11] introduced a concept "convergence in density" which in fact is statistical convergence but under a different name. Consider a set A in $\mathbb{N}$ of natural numbers. Set

$$A_n = \{a \in A : a \leq n\}.$$

Definition 1.3.1. (Natural Density) For any set A in $\mathbb{N}$, its natural density is usually symbolized as $\delta(A)$ and is given as:

$$\delta(A) = \lim_{n\to\infty} \tfrac{1}{n} |A_n|,$$

whenever, the right hand side of the above equation is a finite number; $|A_n|$, stands for the cardinality of $A_n = \{a \in A : a \leq n\}$.

Example 1.3.1. *Let us find out the density of the set of positive integers divisible by 2 (even natural numbers),* $A = \{2, 4, 6, ...\}$ *. Then*

$$|\{a \in A : a \leq 1\}| = 0,$$

$$|\{a \in A : a \leq 2\}| = 1,$$

$$|\{a \in A : a \leq 3\}| = 1,$$

$$|\{a \in A : a \leq 4\}| = 2,$$

$$|\{a \in A : a \leq 5\}| = 2,$$

$$|\{a \in A : a \leq 6\}| = 3 \cdots$$

Continuing in this manner, we get

$$|\{a \in A : a \leq n\}| = \begin{cases} \frac{n}{2}; & \text{if n is divisible by 2,} \\ \frac{n-1}{2}; & \text{otherwise.} \end{cases}$$

$$\text{and} \qquad \lim_{n\to\infty} \frac{1}{n} |\{a \in A : a \leq n\}| = \begin{cases} \frac{1}{2}; & \text{if n is divisible by 2,} \\ \frac{1}{2}; & \text{otherwise.} \end{cases}$$

Therefore, we get $\delta(A) = \frac{1}{2}$.
Also, for $A^c = \{1, 3, 5, ...\}$, that is the set of odd natural numbers proceedings as above we have,

$$|\{a \in A^c : a \leq 1\}| = 1,$$

$$|\{a \in A^c : a \leq 2\}| = 1,$$

$$|\{a \in A^c : a \leq 3\}| = 2,$$

$$|\{a \in A^c : a \leq 4\}| = 2,$$

$$|\{a \in A^c : a \leq 5\}| = 3,$$

$$|\{a \in A^c : a \leq 6\}| = 3 \cdots$$

We have,

$$|\{a \in A^c : a \leq n\}| = \begin{cases} \frac{n}{2}; & \text{if n is divisible by 2,} \\ \frac{n+1}{2}; & \text{otherwise.} \end{cases}$$

We conclude that, $\delta\left(A^{c}\right)=\frac{1}{2}$.
Hence,

$$\delta\left(A^{c}\right)=1-\delta\left(A\right).$$

Remark 1.1. Consider a subset, A = $\{m_1, m_2, m_3, ..., m_t\}$ of $\mathbb{N}$ having t elements. Then,

$$\left|\{m_k \in A : m_k \leq n\}\right| \leq |A| = t.$$

$$\text{Therefore,} \qquad \delta\left(A\right)=\lim_{n\to\infty}\frac{1}{n}\left|\{m_k \in A : m_k \leq n\}\right| \leq \lim_{n\to\infty}\frac{t}{n}=0.$$

Thus, every finite set in $\mathbb{N}$ has natural density 0.

Let us now see if the converse of the above remark holds or not.

Example 1.3.2. *For,* $A=\{1^2, 2^2, 3^2, 4^2, ...\}$,

we observe,

$$\begin{aligned}
|\{a \in A : a \leq 1\}| &= 1 \leq \sqrt{1},\\
|\{a \in A : a \leq 2\}| &= 1 \leq \sqrt{2},\\
|\{a \in A : a \leq 3\}| &= 1 \leq \sqrt{3},\\
|\{a \in A : a \leq 4\}| &= 2 \leq \sqrt{4}\cdots
\end{aligned}$$

implying,

$$|\{a \in A : a \leq n\}| \leq \sqrt{n}, \forall\ n \in \mathbb{N}.$$

This further suggests that,

$$0=\lim_{n\to\infty}\frac{\sqrt{n}}{n} \geq \lim_{n\to\infty}\frac{1}{n}\left|\{a \in A : a \leq n\}\right|.$$

Here $\delta(A)=0$, but A is not finite.

We are now going to give definition for statistical convergence for sequences in $\mathbb{R}$.

Definition 1.3.2. (Statistical Convergence) A sequence $\{a_n\}$ is statistically convergent to a, provided that any pre assigned value $\epsilon > 0$, we have

$$\delta\left(\{n \in \mathbb{N} : |a_n - a| \geq \epsilon\}\right)=0.$$

Clearly, usual convergence implies statistically convergence but not conversely.

Example 1.3.3. *Define a sequence $a = \{a_n\}$ by*

$$x_k = \begin{cases} 1\ ; & \text{if } k \text{ is a pefect square,} \\ 0\ ; & \text{otherwise.} \end{cases}$$

That is,

$$a = \{1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0, ...\}.$$

The sequence defined above is statistically convergent but not convergent.
We observe that,

$$|\{k \leq n : a_k \neq 0\}| \leq \sqrt{n}.$$

$$(since\ a_k\ is\ non-zero\ only\ when\ k\ is\ a\ square).$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : a_k \neq 0\}| \\ \leq \lim_{n\to\infty} \frac{\sqrt{n}}{n}, \\ = \lim_{n\to\infty} \frac{1}{\sqrt{n}}, \\ = 0. \end{aligned}$$

Thus, $s - \lim_k a_k = 0$.
Also, it is evident that $\{a_k\}$ is not convergent, for it is finitely oscillating, however the sequence a is statistically convergent.
Statistical convergence thus generalizes the usual concept of convergence of sequences.
We now give some examples to draw comparison between statistical convergence and regular convergence.
In case of usual convergence, convergent sequence guarantees the convergence of it's subsequences, but it is no longer carried forward by statistical convergence.

Example 1.3.4. *Consider*

$$a = \{a_n\} = \begin{cases} n\ ; & \text{if } k \text{ is a perfect square,} \\ 0\ ; & \text{otherwise.} \end{cases}$$

That is,

$$a = \{1,\ 0,\ 0,\ 4,\ 0,\ 0,\ 0,\ 0,\ 9,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 16,\ 0,\ 0, ...\}.$$

Then a is statistically convergent. But the subsequence

$$a_{n^2} = \{1,\ 4,\ 9,\ 16, ...\}$$

of a is not statistically convergent.

Another difference between regular convergence and statistical convergence is evident from the following instance.

Example 1.3.5. *Consider*

$$a = \{a_n\} \quad = \begin{cases} k\ ; & \textit{if } n \textit{ is a perfect square,} \\ 0\ ; & \textit{otherwise.} \end{cases}$$

Then $\{a_k\}$ is unbounded but statistically convergent to 0 as

$$\delta\left(A\right) = 0, \quad where \ \ A = \left\{n^2 : n = 1,\ 2,\ 3,\ 4, ...\right\}.$$

Under statistical convergence, we consider two types of points statistical limit point and statistical cluster points [26]. Our familiarity with limit points (w.r.t. usual convergence) could make us think that statistical limit point and statistical cluster points are identical but that is not the case here.

Definition 1.3.3. (Thin Subsequence) Any subsequence $\{a_K\}$ of a sequence $\{a_n\}$ is a thin subsequence if natural density of its indexing set, K is zero.
Mathematically, $\delta\left(K\right) = 0$.
and if the indexing set, K has a positive natural density or does not have any natural density, then $\{a_K\}$ is called a non-thin subsequence.

Definition 1.3.4. (Ordinary Limit Point) Consider a sequence $\{a_n\}$. Then a number b $\in \mathbb{R}$ is the limit point of $\{a_n\}$ if there is some subsequence $\{b_n\}$ of $\{a_n\}$ such that $b_n \to b$ w.r.t usual convergence. We symbolize the collection of all such b for $\{a_n\}$, by $\mathbf{OL_{a_n}}$.

Definition 1.3.5. (Statistical Limit Point) Consider a sequence $\{a_n\}$ and $\alpha \in \mathbb{R}$. If a non-thin subsequence of $\{a_n\}$ converges to α, then we say that α is the statistical limit point of $\{a_n\}$. We symbolize the family of all these points α for $\{a_n\}$, by $\mathbf{SL_{a_n}}$.

Definition 1.3.6. (Statistical Cluster Point) A number $\beta \in \mathbb{R}$ is the statistical cluster point of $\{a_n\}$, whenever the following holds:

$$\{k \in \mathbb{N} \colon \ |a_k - \beta| < \epsilon, \ \epsilon > 0\},$$

has positive natural density. We denote the collection of all such β for $\{a_n\}$, by $\mathbf{SC_{a_n}}$.

Let us now see the relation among the limit point sets so defined.

Theorem 1.3.1. *For any sequence a,*

$$SL_a \subseteq SC_a \subseteq OL_a.$$

In 2008, Aytar [3] gave the statistical version for the rough convergence and investigated various results analogous to the results for infinite dimensional n.l.s. In this paper, Aytar defined "rough statistical limit points and rough statistical cluster points" and established relation between them. In another paper related to this topic, [4] Aytar considered rough statistical convergence for the sequences with elements as functions and defined statistical condensation point. Several attempts were made by various researchers to view existing literature on statistical convergence in the light of rough convergence.

1.4 Rough Ideal Convergence

Since the natural extension of statistical convergence is ideal convergence, attempts were made to combine rough convergence with ideal convergence. S.K. Pal et al. developed rough ideal convergence in 2013. Later in 2014, E. Dundar et al. separately presented this concept in a n.l.s.
Kostyrko is credited for giving the notion of ideal convergence in [62], which extends and generalizes statistical convergence and depends on a collection of sets called an ideal.

Definition 1.4.1. (Ideal) If any non-void collection $\mathfrak{I}$ of sets in a non-void set X meets the following criteria:

1. $\mathfrak{I}$ is stable under finite union, $H, K \in \mathfrak{I} \implies H \cup K \in \mathfrak{I}$ and
2. $\mathfrak{I}$ is stable under subsets, $H \in \mathfrak{I}$, and $K \subseteq H \implies K \in \mathfrak{I}$.

Then such a collection is called "ideal of X". Additionally, $\mathfrak{I}$ is admissible (a.i.) if, all the subsets of X with cardinality one (singletons) belong to $\mathfrak{I}$ and it is non-trivial, whenever "$\mathfrak{I} \neq \{\emptyset\}$ and $X \notin \mathfrak{I}$".

Let X be any set and $\mathfrak{I}$ be any ideal on X. Then

Definition 1.4.2. (Maximal Ideal) $\mathfrak{I}$ is called maximal whenever, $\nexists$ any proper ideal K of X, with $K \neq \mathfrak{I}$ and $\mathfrak{I} \subset K$.

Definition 1.4.3. (Filter) If any non-void collection $\mathfrak{F}$, of sets in a non-void set X meets the following criteria:

1. $\emptyset \notin \mathfrak{F}$,
2. $\mathfrak{F}$ is stable under finite intersection, $H, K \in \mathfrak{F} \implies H \cap K \in \mathfrak{F}$ and
3. $\mathfrak{F}$ is stable under super-sets, $H \in \mathfrak{F}$, and $H \subseteq K \implies K \in \mathfrak{F}$.

Then such a collection is called filter of X.

Definition 1.4.4. (Filter associated with Ideal) A family of complements of members of $\mathfrak{I}$, denoted by $\mathfrak{F}(\mathfrak{I})$ and defined as the set "$\{P \subset X : \exists Q \in \mathfrak{I}, P = X \backslash Q\}$, is called filter associated with the ideal $\mathfrak{I}$".

We now give the definition of convergence defined by these ideals.

Definition 1.4.5. (Ideal Convergence) Ideal convergence of a sequence $\{a_n\}$ in a n.l.s. $(X, \|\cdot\|)$ to a, is denoted by $\mathfrak{I} - \lim a_n = a$ and is defined as: For any positive pre assigned number ϵ,

$$\{n \in \mathbb{N} \colon \|a_n - a\| \geq \epsilon\} \in \mathfrak{I}.$$

Several examples of ideals have been discussed in [62]. Let us consider some important examples of ideals to have a better understanding of ideal convergence.

Example 1.4.1. *1. The collection $\mathfrak{I}_{fn}$ of all sets in $\mathbb{N}$ with finite cardinality, is a.i.. Furthermore, $\mathfrak{I}_{fn}$ convergence matches the regular convergence of real numbers.*

2. For $\mathfrak{I} = \mathfrak{I}_\phi = \{\{\phi\}\}$, a sequence is $\mathfrak{I}_\phi$ convergent iff it is constant.

3. Let $\mathfrak{I}_{\delta_0}$ be family of sets in $\mathbb{N}$ having "natural density" 0. Then $\mathfrak{I}_{\delta_0}$ convergence gives statistical convergence.

One may be interested to know how much is ideal convergence similar to the usual convergence. Keeping this in mind, we now give comparison of ideal convergence with regular convergence.

Theorem 1.4.1. *Consider a set M with $|M| \geq 2$. For any a.i. $\mathfrak{I}$ of M the following hold good:*

1 Each constant sequence $\{a,a,a,a,\ldots,a,..\}$ in M is $\mathfrak{I}$- convergent to a.

2 $\mathfrak{I}$- convergent sequence cannot converge to more than one point.

3 If a is the $\mathfrak{I}$- limit of every subsequence of any sequence (a_n). Then a is the $\mathfrak{I}$-limit of (a_n) as well.

However, there exist sequences whose subsequences may $\mathfrak{I}$ converge to an $\mathfrak{I}$ limit other than the $\mathfrak{I}$ limit of the sequence.

Example 1.4.2. *In the example, (1.3.3) if we consider $\mathfrak{I} = \mathfrak{I}_{\delta_0}$, then this serves as a counter example where the sequence is $\mathfrak{I}$ convergent to 0, but it contains a constant subsequences which has $\mathfrak{I}$ -limit 1.*

Every bounded sequence may not be convergent but $\mathfrak{I}$ -convergence allows bounded sequences to converge.

Theorem 1.4.2. *For a maximal a.i. $\mathfrak{I}$ of $\mathbb{N}$, every bounded real sequence is $\mathfrak{I}$-convergent.*

Ideal convergence also satisfies algebra of limits similar to the algebra of limits for usual convergent sequences.

Theorem 1.4.3. *For any proper ideal $\mathfrak{I}$ of $\mathbb{N}$, and any two sequences $\{s_n\}$ and $\{t_n\}$ we have the following:*

1. $\mathfrak{I} - \lim(s_n + t_n) = s + t$, *whenever* $\mathfrak{I} - \lim s_n = s$ *and* $\mathfrak{I} - \lim t_n = t$.
2. $\mathfrak{I} - \lim(s_n t_n) = st$, *whenever* $\mathfrak{I} - \lim s_n = s$ *and* $\mathfrak{I} - \lim t_n = t$.

Theorem 1.4.4. *Ideal convergence is preserved under continuous functions.*

Significant contributions in developing the theory of ideal convergence were given by Hazarika, Gurdal, Kolk, V.A. Khan, E. Savas, S.A. Mohiuddine, B.C. Tripathy. Recently with the help of ideal convergence, a new type of integration has been introduced and investigated in [16]. Explicit detail about the development of ideal convergence and its application can be found in [29], [57], [20], [27], [28], and [76]. These papers contribute to the understanding of ideal convergence and highlight the importance of generalized notion of convergence and suggest open questions for further research.

The concept of ideals was applied to rough convergence, as a result rough ideal convergence was introduced, which in turn led to the expansion of already known convergences, particularly rough convergence and rough statistical convergence [63].

Definition 1.4.6. (Rough Ideal Convergence) For some $\mathfrak{I}$ non-trivial a.i. on $\mathbb{N}$ and $r > 0$ in $\mathbb{R}$, a sequence $\{a_n\}$ in a n.l.s. $(X, \|\cdot\|)$ is considered to be $r\mathfrak{I}$-convergent to a, denoted by $a_n \xrightarrow{r\mathfrak{I}} a$, if

$$\{n \in \mathbb{N} \colon \|a_n - a\| \geq r + \epsilon\} \in \mathfrak{I}, \ \forall \ \epsilon > 0.$$

The limit of the above convergence may vary for different choice of r. Therefore the following set is considered for the rough limit points,

$$LIM^{r\mathfrak{I}}a_n = \{a\colon a_n \overset{r\mathfrak{I}}{\to} a\}.$$

Additionally, it has been observed that,

"$LIM^{r\mathfrak{I}}a_n = [\mathfrak{I} - lim\ sup\ a_n - r, \mathfrak{I} - lim\ inf\ a_n - r]$."

Various properties of this limit so obtained were investigated in the lines of the rough limit point set and it was established that for any sequence $\{a_n\}$, $LIM^{r\mathfrak{I}}a_n$ is closed, bounded convex and diameter of this set cannot exceed 2r, for any given r.

Let us now dive deeper into this concept with the help of an example. We have seen that there are unbounded sequences which may not be rough convergent for any r, [66]. This sequence may be rough ideal convergent for some ideal.

Example 1.4.3. *For a proper a.i.* $\mathfrak{I}$ *of* $\mathbb{N}$ *containing an infinite set M. Define a sequence* $\{a_n\}$ *as:*

$$a_k = \begin{cases} (-1)^k\ ; & \text{if } k \text{ is not in } M, \\ k\ ; & \text{otherwise.} \end{cases}$$

Then this sequence is $r\mathfrak{I}$*-convergent but not r-convergent for any* $r \geq 0$*.*

Theorem 1.4.5. *For any sequence* $a = (a_i)$ *in a n.l.s. X, the following are interchangeable:*

1. *a is* $r\mathfrak{I}$*-convergent to* a_*, $r > 0$.
2. $\exists$ *a sequence* $b = (b_i)$, $\|a_i - b_i\| \leq r$, *for* $i \in \mathbb{N}$ *and* $\mathfrak{I} - \lim b = a_*$.

Rough ideal convergence has been examined for sequences (like double, triple, n-sequences) in spaces like, fuzzy normed spaces, generalized metric spaces[41] and [53]. Rough convergence and its numerous extensions have been a focal point for scholars for last one decade. Continuous attempts are being made by scholars to add to its literature by extending it to abstract spaces like topological spaces [46] and explore its applications. This thesis is a small step in this big stride.

1.5 Sequence Space

From algebra of sequences, we know that sequences are stable under addition and scalar multiplication and therefore form a vector space. A sequence space is any subspace of linear space of sequences. We now give some well known sequence spaces which can be found easily in [33] and [45].

"The space of real or complex sequences," $\omega = \{a = (a_n)\colon a_n \in \mathbb{R}\ or\ \mathbb{C}\}$.
"The space of bounded sequences," $\ell^\infty = \{a = (a_n) \in \omega\colon \sup_n |a_n| < \infty\}$.
"The space of null sequences," $c^0 = \{a = (a_n) \in \omega\colon \lim_n |a_n| = 0\}$.
"The space of finite sequences," $c^{00} = \{a = (a_n) \in \omega\colon a_n = 0$ for all but finitely many n$\}$.
"The space of convergent sequences," $c = \{a = (a_n) \in \omega\colon \lim_n a_n = l, l \in \mathbb{R}\ (or\ \mathbb{C})\}$.
"The space of absolutely convergent series," $\ell^1 = \{a = (a_n) \in \omega\colon \sum_n |a_n| < \infty\}$.
"The space of strongly Cesaro-bounded sequences," $w^\infty = \{a = (a_k) \in \omega\colon \sup_n \frac{1}{n} \sum_k |a_k| < \infty\}$.
"The space of pth summable series," $\ell^p = \{a = (a_n) \in \omega\colon \sum_n |a_n|^p < \infty, \quad 0 < p < \infty\}$.
"The space of all pth Cesaro summable sequences," $w^p = \{a = (a_k) \in \omega\colon \lim_n \frac{1}{n} \sum_k |a_k - l|^p = 0, l \in \mathbb{R}\ (or\ \mathbb{C})\}$.
For a non negative sequence of real numbers, $t = (t_k)$,
$\ell^\infty(t) = \{a = (a_k) \in \omega : \sup_k |a_k|^{t_k} < \infty\}$.
$c(t) = \{a = (a_k) \in \omega : \lim_k |a_k - l|^{t_k} = 0,$ for some $l \in \mathbb{R}\ (or\ \mathbb{C})\}$.
$c^0(t) = \{a = (a_k) \in \omega : \lim_k |a_k|^{t_k} = 0\}$.
$w^\infty(t) = \{a \in \omega : \sup_k(\frac{1}{n} \sum_{k=1}^n |a_k|^{t_k} < \infty\}$.
$w(t) = \{a = (a_k \in \omega : \lim_n(\frac{1}{n} \sum_{k=1}^n |a_k - l|^{t_k}) = 0,$ for some $l \in \mathbb{R}\ (or\ \mathbb{C})\}$.
$w^0(t) = \{a = (a_k) \in \omega : \lim_n(\frac{1}{n} \sum_{k=1}^n |a_k|^{t_k}) = 0\}$.
$\ell(t) = \{a = (a_k) \in \omega : \lim_n \frac{1}{n} \sum_{k=1} |a_k|^{t_k} = 0\}$.

1.6 Motivation for research work

After giving significant readings to the literature related to rough convergence, it was observed that the researchers were focusing on studying rough convergence for either different types of sequences like, sequences of fuzzy numbers, double

sequences, triple sequences, sequences of sets, difference sequences etc., or for sequences in different spaces where distance between any two points can be given. However, the investigations in these attempts were concentrated on the rough limits set so obtained and its geometric and topological properties.This motivated us to generalize this type of convergence and obtain new spaces (sequence spaces) and study their properties using this type of convergence. We also got interested in exploring the possibility of generalizing it for the topological spaces and study the topology generated by rough convergent sequences.
Rough convergence is concerned with the numerical approximation of any real-world situation. It aids in verifying the accuracy of answers acquired from computer programs and drawing conclusions from scientific investigations. In the last decade numerous academicians have been working on Neutrosophic set theory to address imprecise and uncertain thinking [38] and [40]. Recently, R. Antal and et al. used the idea of rough convergence to put forward the notion of rough statistical convergence in neutrosophic normed spaces and offered some useful functional tools for real-world inconsistency and indeterminacy scenarios by looking at some features associated with rough convergence in these spaces [1]. In 2023, F. Samarandache [78] described nine new topologies on neutrosophic sets for the first time. This is a new and unexplored area of research which, together with rough convergence can pave the way for large applications in the real world problems.

1.7 Objectives

We resolve to achieve the following objectives:

1. To investigate algebraic properties of rough convergence for different types of sequences.

2. To analyze the algebraic properties, rough continuity, relation of rough convergence with other types of convergence in various spaces.

3. To investigate the rough statistical convergence and rough ideal convergence of different types of sequences and study their properties.

1.8 Outline of the Thesis

The thesis “A Study on Rough Convergence of Some Sequences in Certain Spaces,” has been divided into four chapters, the first of which is Introduction. It consists of the literature review and background of Rough Ideal Convergence.

In the second chapter, we have introduced rough ideal convergence for sequences with elements as linear operators between two normed spaces (n.l.s.). This chapter consists of six sections and deals with the importance of bounded linear operators in the analysis and motivation behind choosing sequences of bounded linear operators for studying rough ideal convergence. After defining rough ideal convergence for these sequences, we have considered some rough ideal convergent sequence spaces of bounded linear operators and investigated some inclusion relations of these spaces, decomposition theorem and algebraic properties.

The third chapter deals with sequence spaces, where we have considered the rough ideal convergence of the sequences of real numbers and we have used the Orlicz function for generalizing these sequence spaces. The results in this chapter have been motivated by the works of [84],[64], and [39]. This chapter has been divided into five sections, which are mainly devoted to the study of Orlicz functions and rough ideal convergent sequences defined by the Orlicz function. In this chapter, we have established that the sequence spaces so obtained are solid and monotone and given an example to show that they fail to be convergence free.

In the fourth chapter, we have investigated the possibilities of obtaining a topology via rough convergent sequences. This chapter has been divided into four sections. In these sections, we have introduced, rough closure, proved that it satisfies the conditions for Kurtatowski’s closure operator and then defined a Kuratowski topology. We have also given rough sequential definitions for continuity, compactness and connectedness and given results parallel to the results for connected and compact sets under the usual topology on $\mathbb{R}$. We have also shown that rough sequentially continuous maps between any two spaces preserve rough sequential compactness and rough sequential connectedness.

We have given summary at the end of each chapter. We have considered the space (normed linear space or metric space) of real numbers and sequences of real numbers. However, similar results can also be obtained for the sequences of complex numbers.

Chapter 2

Rough Ideal Convergence for Sequences of Bounded Linear Operators

The idea of sequence spaces can be traced back to the work of mathematicians like Hilbert and Fischer in the late 19th and early 20th centuries. Hilbert's introduction of the space of square-summable sequences, now known as, l^2 laid the groundwork for further study. In this chapter, we have extended the notion of rough ideal convergence to the sequences of bounded linear operators and studied the sequence spaces so obtained. Throughout this chapter, $\mathfrak{I}$ refers to any non-trivial a.i. on $\mathbb{N}$ and r will always be a non-negative real number, unless stated otherwise. All the operators have been considered between the n.l.s. of real numbers over the field of real numbers, O is the zero operator.

2.1 Introduction

Bounded linear operators and their sequences are a common occurrence, often associated with problems such as Fourier series convergence, interpolation polynomial sequences, and numerical integration techniques, as seen in [2], [45], and [32]. In these scenarios, the focus typically revolves around the convergence of operator sequences, the boundedness of associated norm sequences, or similar characteristics. Bounded linear operators exhibit certain properties which make them desirable

to study. For instance, there are various interchangeable conditions to see to the continuity of a map between two metric spaces. Whereas, for a linear map between two normed spaces boundedness of the map guarantees continuity and this property of bounded operators is of great importance and has been exploited for *closed graph theorem, uniform boundedness principle and open map theorem.*
Over the period of time, various types of sequence convergence has been defined and studies for sequences of operators. However, the sequence convergence for sequences of operators introduced by Nuemann namely, weak, strong and uniform convergence has its own prominence. Introduction of statistical and $\mathfrak{I}$ convergence for sequences with elements as functions, motivated researchers to investigate statistical and ideal versions for the convergence introduced in [61]. Kolk, Bhardwaj, Balcerzak, Pehlivan [9], [42], [43], [65], and [5] independently gave insights to the application of statistical and ideal convergence to the sequences of operators.
After the introduction of rough ideal convergence, it was a natural instinct to study the convergence of sequences with elements as linear operators in the light of this new type of convergence. This chapter is devoted to extension of notion of rough ideal convergence to bounded linear operators using the results in [63]. We have defined some rough ideal convergent sequence spaces of operators. We have also investigated some properties of these spaces when topologized through a norm and investigated inclusion relations, equivalent conditions, decomposition theorem and algebraic properties of such spaces.

2.2 Bounded Linear Operators

Before giving main results of this chapter, we first break down some important definitions and results to have a clear understanding of normed linear spaces and bounded linear operators.

Definition 2.2.1. (Normed Linear Space) A positive real valued function on a vector space $W(F)$, often written as,

$$\|\cdot\|: W \to \mathbb{R}^+,$$

is a norm on W if, it meets the following criteria:

1. $\|\cdot\|$ takes only zero vector to 0.

2. $0 < \|w\|$, whenever $0 < w, \ w \in W$ (Definiteness)

3. For any scalar $c \in F$ and $w \ \in W, \ \|cw\| = |c|\|w\|$. (Absolute Homogeneity)

4. Any two vectors $v, w \in W$ satisfy, $\|v\| + \|w\| \geq \|v + w\|$. (Triangular inequality)

$(W\| \cdot \|)$ is then called a normed linear space (shortly, n.l.s.).
The presence of a norm on a vector space facilitates the study of infinite dimensional vector spaces as it induces a metric ($\rho(a, b) = \|a - b\|$) which is in agreement with the linearity of the vector space. In a metric space, mainly three properties are of great interest namely, connectedness, compactness and completeness. But for a n.l.s. only completeness property is crucial, for every n.l.s. is connected and is never compact (except zero space). A complete n.l.s. is called a Banach space.

Example 2.2.1. *It is well known that linear combination of bounded(continuous) functions is again bounded(continuous). Therefore, the collection on a metric space* M*,*

$$C_\infty[M;\mathbb{R}] = \{h\colon M \to \mathbb{R}\colon \textit{ h is continuous and } |h(x)| < \infty\},$$

is a n.l.s. under the norm given as:

$$\|h\|_\infty = \sup_{m \in M} |h(m)|.$$

Example 2.2.2. *Let* $l^\infty = \{b = (b_n)\colon$ *a is bounded real sequence*$\}$. *Then* l^∞ *is a n.l.s. where,*

$$\|b\| = \sup_n |b_n|.$$

Definition 2.2.2. (Linear Operator) For any two linear spaces $P(\mathbb{F})$ and $Q(\mathbb{F})$, a mapping between them which takes linear combinations to linear combinations is called a linear operator. Equivalently,

$$f\colon P \to Q,$$

is a linear map if $f(ap + bq) = af(p) + bf(q), \ a, b \in \mathbb{F}, \ p \in P$and $q \in Q$.

Definition 2.2.3. (Bounded Linear Operator) A linear operator f between two n.l.s. $(P, \| \cdot \|_1)$ and $(Q, \| \cdot \|_2)$ is said to be bounded (in short, b.l.o.), if for some constant h such that for all p in P, the norm of fp in Q is bounded by h times the norm of p in P. Mathematically,

$$\|fp\|_2 \leq h\|p\|_1.$$

One may wonder if there is any smallest constant h such that the above inequality holds for all non-zero $p \in P$. In other words, we are interested in the smallest h such that,

$$\frac{\|fp\|_2}{\|p\|_1} \leq h.$$

The quantity on the right side of the above inequality must be at least as large as the quantity on the left. The desirable smallest possible quantity is the "supremum" of the values on the left side and is called the "norm of the linear operator" f. Mathematically,

$$\|f\| = \sup_{0 \neq x \in P} \frac{\|fp\|}{\|p\|}.$$

Let P and Q be any two n.l.s. Consider a collection of linear operators between P to Q given below:

$$\mathbb{L}(U) = \{U = (U_k) \colon U_k \colon P \to Q, \text{ is linear } \forall\ k \in \mathbb{N}\}.$$

Let $\mathbb{B}^\infty(U)$ be the collection of sequences with elements as b.l.o. from a n.l.s. P to Q. Then by the same argument as above, $\mathbb{B}^\infty(U)$ is a linear space. Additionally, $\mathbb{B}^\infty(U)$ is a n.l.s. with norm given by

$$\|U\| = \sup_k \|U_k(x)\|.$$

Whenever Q is a Banach space, $\mathbb{B}^\infty(U)$ is also Banach.

2.3 Sequences of Operators

As introduced by [61], there are mainly three types of convergence defined for the sequences of operators which are as given under :

Definition 2.3.1. A sequence (U_k) of operators between any two n.l.s. P and Q, $U_k \in \mathbb{B}^\infty(U)$ is defined to be,

1. Uniformly convergent, whenever (U_k) converges in the norm on $\mathbb{B}^{\infty}(U)$ i.e. $\|U_k - T\| \to 0$.

2. Strongly convergent, whenever $(U_k p)$ converges strongly in Q for every $p \in P$ i.e. $\|U_k(p) - T(p)\| \to 0$, for every $p \in P$.

3. Weakly convergent, whenever $(U_k p)$ converges weakly in Q for every $p \in P$ i.e. $|h(U_k p) - h(Tp)| \to 0$, for every $p \in P$ and $h \in Q^{'}$.

Kamthan and Gupta in their celebrated monograph "Sequence spaces and series" [33], collected and complied several applications of sequence spaces. They primarily focused on the topological aspects of sequence spaces (new as well as classical) and its application in topological vector spaces. The following definitions have been cited from their monograph.

Definition 2.3.2. (Sequence Space) Let Λ be a vector space of sequences. Then any vector subspace κ of Λ is known as sequence space (in short, S-space).

Definition 2.3.3. (Sequence Algebra) A S-space κ is a sequence algebra if it is stable under multiplication. That is,

$$(a_n.b_n) \in \kappa, \text{ whenever } (a_n), (b_n) \in \kappa.$$

Definition 2.3.4. (Sectional Subspace) Let $L = \{l_1 < l_2 < l_3 \ldots\}$ be a subsequence in $\mathbb{N}$ and a S-space, κ Then

$$\kappa_L = \{(x_{l_n})\colon x_l \in \kappa\},$$

is said to be the L-step space or sectional subspace.

Definition 2.3.5. (Canonical Pre-image) For any sequence in (x_l) in L-step space, the sequence a_l defined as

$$a_l = \begin{cases} x_l; & \text{if } l \text{ is in } L, \\ 0; & \text{otherwise.} \end{cases}$$

is the canonical pre-image of a sequence (x_l). The collection of all canonical pre-images of each sequence in a step sequence is called the canonical pre-image of a S-space.

Definition 2.3.6. (Monotone Space) Whenever a S-space κ contains pre-images of each of its step spaces it is known as monotone space.

Definition 2.3.7. (Solid Space) A S-space κ where $b_n \in \kappa$, whenever there is some $a_n \in \kappa$ with $|b_n| \leq |a_n|$, $n \in \mathbb{N}$ is called solid.

We now give these definitions for sequence space of operators.

Definition 2.3.8. (Solid Sequence Space of Operators) A S-space P of operators, where $V_k \in P$ whenever there is some $U_k \in P$ and $\|V_k\| \leq \|U_k\|$, all $k \in \mathbb{N}$ is called solid (or normal).

Definition 2.3.9. If the canonical pre-images of each step space of S-space P of operators is included in P, then we call P a monotone S-space.

Remark 2.1. Solid S-space is always monotone.

Definition 2.3.10. (Lipschitz Function) Let X be any non-empty space. A function $l\colon X \to \mathbb{R}$ is Lipschitz if it meets the following condition (Lipschitz condition),

$$|l(x) - l(y)| \leq H|x - y|,$$

where H is some constant known as the Lipschitz constant.

Lipschitz condition ensures that the function's values do not change too rapidly as its inputs vary. Lipschitz continuity plays a crucial role in *metric spaces and fixed-point theorems.* These mappings are functions that adhere to a Lipschitz condition, with Lipschitz constant H<1.

Example 2.3.1. *The absolute value function m(x) =$|x|$, $x \in \mathbb{R}$ is Lipschitz. Here Lipschitz constant, H is 1.*
But the real valued function l on $\mathbb{R}$, given as $l(x) = x^2$, serves as a non-example for Lipschitz function.

2.4 Rough Ideal Convergence of Sequence of Operators

Kostyrko [62] extended the notion of ideal convergence to sequence of functions. [5] and [44] showed that ideal convergence has an advantage over usual convergence.

Ideal convergence of sequence of functions has been applied to family of continuous and equicontinuous functions which has been exploited to present Arzela-Ascoli [80].

Definition 2.4.1. (Ideal Convergence of Sequence of Functions) Consider a sequence of functions $\{h_n\}_{n\in\mathbb{N}}$ from a non void set H into a metric space M and $\mathfrak{I}$ be an a.i. of $\mathbb{N}$. Then $\{h_n\}_{n\in\mathbb{N}}$ is said to $\mathfrak{I}$-converge to a function $h\colon H\to M$ if, $\forall\ x\in H$, $\mathfrak{I}-\lim_{n\to\infty} h_n(x)=h(x)$. We refer to h as the $\mathfrak{I}$-limit function of $\{h_n\}_{n\in\mathbb{N}}$ and written as,

$$\mathfrak{I}-lim h_n=h.$$

In 2014, Khan and Shafiq introduced $\mathfrak{I}$ convergence for sequence of operators and used modulus function for generating some new $\mathfrak{I}$ convergent sequence spaces [39].

Motivated by the $\mathfrak{I}$ convergence of sequence of operators, we try to apply rough convergence and expand the notion of ideal convergence of sequence of operators.

Definition 2.4.2. (Rough Ideal Convergence of Sequence of Operators) For a sequence $U=(U_k)\in\mathbb{B}^\infty(U)$, $r\mathfrak{I}$-convergence is defined as follows: for any $r>0$ and for any chosen $\epsilon>0$, the set

$$\{k\in\mathbb{N}\colon \|U_k(x)-V(x)\|\geq r+\epsilon\}\in\mathfrak{I}.$$

where $V\in\mathbb{L}(U)$. Mathematically, $r\mathfrak{I}-\lim U_k=V$.

We give the some rough ideal convergent classes of sequences of operators.

$$\mathcal{C}^{R\mathfrak{I}}(T)=\{U=(U_k)\in\mathbb{B}^\infty(U)\colon \{k\in\mathbb{N}\colon \|U_k(x)-V(x)\|\geq r+\epsilon\}\in\mathfrak{I},\ \text{where}\, V\in\mathbb{L}(U), r>0\},$$

$$\mathcal{C}_0^{R\mathfrak{I}}(T)=\{U=(U_k)\in\mathbb{B}^\infty(U)\colon \{k\in\mathbb{N}\colon \|U_k(x)\|\geq r+\epsilon\}\in\mathfrak{I}, r>0\},\ \text{and}$$

$$\mathbb{B}(T)=\{U=(U_k)\in\mathbb{B}^\infty(U)\colon \sup_k \|U_k(x)\|<\infty\}.$$

Also,

$$\mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)=\mathbb{B}(T)\cap\mathcal{C}^{R\mathfrak{I}}(T),\ \mathcal{G}_{\mathcal{C}_0}^{R\mathfrak{I}}(T)=\mathbb{B}(T)\cap\mathcal{C}_0^{R\mathfrak{I}}(T),\ \text{and}$$

$$\mathcal{C}^{\mathfrak{I}}(T)=\{U=(U_k)\in\mathbb{B}^\infty(U)\colon \{k\in\mathbb{N}\colon \|U_k(x)-V(x)\|\geq \epsilon\}\in\mathfrak{I},\ \text{where}\, V\in\mathbb{L}(U)\}.$$

2.5 Main Results

In this section, we undertake the investigation and study of some algebraic characteristics exhibited by the rough ideal convergent classes of sequences of b.l.o., which we introduced in the previous section. We have also explored Cauchy like criteria for rough ideal convergent sequences of b.l.o. and proved some equivalent conditions.

We first show that the classes of rough ideal convergent sequences of b.l.o. are vector spaces over the field of $\mathbb{R}$.

Theorem 2.5.1. *The classes $\mathcal{C}^{R\mathfrak{I}}(T), \mathcal{C}_0^{R\mathfrak{I}}(T), \mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ and $\mathcal{G}_{\mathcal{C}_0}^{R\mathfrak{I}}(T)$ form vector spaces over $\mathbb{R}$.*

Proof. Let $U = (U_k),\ V = (V_k) \in \mathcal{C}^{R\mathfrak{I}}(T)$ and $\alpha, \beta, \in \mathbb{R}$ then for some $r_1, r_2 > 0$ and for given $\epsilon > 0$, there exist some $T_1, T_2 \in \mathbb{L}(U)$ such that

$$\{k \in \mathbb{N} \colon \|U_k(x) - T_1(x)\| \geq r_1 + \tfrac{\epsilon}{2}\}, \{k \in \mathbb{N} \colon \|V_k(x) - T_2(x)\| \geq r_2 + \tfrac{\epsilon}{2}\} \in \mathfrak{I}.$$

Let $r = \max\{r_1, r_2\}$, then

$$\{k \in \mathbb{N} \colon \|U_k(x) - T_1(x)\| \geq r + \tfrac{\epsilon}{2}\}, \{k \in \mathbb{N} \colon \|V_k(x) - T_2(x)\| \geq r + \tfrac{\epsilon}{2}\} \in \mathfrak{I}.$$

Let

$$P_1 = \{k \in \mathbb{N} \colon \|U_k(x) - T_1(x)\| < r + \tfrac{\epsilon}{2}\}, P_2 = \{k \in \mathbb{N} \colon \|V_k(x) - T_2(x)\| < r + \tfrac{\epsilon}{2}\} \in \mathfrak{F}(\mathfrak{I}),$$

are such that ${P_1}^c, {P_2}^c \in \mathfrak{I}$. Then

$$P_3 = \{k \in \mathbb{N} \colon \|(\alpha U_k)(x) + (\beta V_k)(x) - (\alpha T_1 - \beta T_2)\| \leq 2r + \epsilon\} \supseteq (P_1 \cap P_2) \in \mathfrak{F}(\mathfrak{I}).$$

Thus, $\alpha U_k + \beta V_k$ is rough ideal convergent, for all scalars α, β and $(U_k), (V_k) \in \mathcal{C}^{R\mathfrak{I}}(T)$. Therefore, $\mathcal{C}^{R\mathfrak{I}}(T)$ is linear. The proof for other classes, $\mathcal{C}_0^{R\mathfrak{I}}(T), \mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ and $\mathcal{G}_{\mathcal{C}_0}^{R\mathfrak{I}}(T)$ can be done similarly. □

Let us now equip the linear spaces $\mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ and $\mathcal{G}_{\mathcal{C}_0}^{R\mathfrak{I}}(T)$ with a norm.

Theorem 2.5.2. *A positive real valued function $\|\cdot\|$ on X, defined as*

$$\|T\|_* = \sup_k \|T_k(x)\|.$$

where, $X = \mathcal{G}^{R\mathfrak{I}}_{\mathcal{C}}(T)$*,* $\mathcal{G}^{R\mathfrak{I}}_{\mathcal{C}_0}(T)$ *is a norm on* X*.*

Proof. Proof is similar to the proof given in [64]. □

We now establish a relationship between rough ideal convergent sequences of b.l.o. and rough Cauchy criteria.

Theorem 2.5.3. *The necessary and sufficient condition for a sequence* $U = (U_k) \in \mathbb{B}^{\infty}(U)$ *to be* $r\mathfrak{I}$*-convergent is that for* $r > 0$ *and* $\epsilon > 0$*,* $\exists$ *a set* $N_{r\epsilon} \in \mathbb{N}$ *so that*

$$\{k \in \mathbb{N}\colon \|U_k(x) - U_{N_{r\epsilon}}(x)\| < 2r + \epsilon\} \in \mathfrak{F}(\mathfrak{I}).$$

Proof. Let $U = (U_k) \in \mathbb{B}^{\infty}(U)$ be $r\mathfrak{I}$- convergent for some $r > 0$ and let T be the $r\mathfrak{I}$-limit of (U_k). Then,

$$B_{r\epsilon} = \left\{k \in \mathbb{N}\colon \|U_k(x) - T(x)\| < r + \tfrac{\epsilon}{2}\right\} \in \mathfrak{F}(\mathfrak{I}),\ \forall \epsilon > 0.$$

Fix an $N_{r\epsilon} \in B_{r\epsilon}$. Then we observe that,

$$\|U_k(x) - U_{N_{r\epsilon}}(x)\| \leq \|U_k(x) - T(x)\| + \|U_{N_{r\epsilon}}(x) - T(x)\| < r + \tfrac{\epsilon}{2} + r + \tfrac{\epsilon}{2} = 2r + \epsilon$$

$\forall k \in B_{r\epsilon}$. Thus, $\{k \in \mathbb{N}\colon \|U_k(x) - U_{N_{r\epsilon}}(x)\| < 2r + \epsilon\} \in \mathfrak{F}(\mathfrak{I})$. For the converse part, assume

$$\{k \in \mathbb{N} : \|U_k(x) - U_{N_{r\epsilon}}(x)\| < 2r + \epsilon\} \in \mathfrak{F}(\mathfrak{I}),\ \forall \epsilon > 0.$$

Clearly,

$$K_{r\epsilon} = \{k \in \mathbb{N}\colon U_k(x) \in [U_{N_{r\epsilon}}(x) - (2r - \epsilon), U_{N_{r\epsilon}}(x) - (2r + \epsilon)]\} \in \mathfrak{F}(\mathfrak{I}),\ \forall \epsilon > 0.$$

Let $I_{r\epsilon} = [U_{N_{r\epsilon}}(x) - (2r - \epsilon), U_{N_{r\epsilon}}(x) - (2r + \epsilon)]$. If we can fix $r > 0$ and $\epsilon > 0$ then we get $K_{r\epsilon}, K_{\frac{r\epsilon}{2}} \in \mathfrak{F}(\mathfrak{I})$, implies $K_{r\epsilon} \cap K_{\frac{r\epsilon}{2}} \in \mathfrak{F}(\mathfrak{I})$. Consequently, $I_{\epsilon} = I_{r\epsilon} \cap I_{\frac{r\epsilon}{2}} \neq \emptyset$. That is $\{k \in \mathbb{N}\colon U_k(x) \in I_{\epsilon}\} \in \mathfrak{F}(\mathfrak{I})$. Let $diam\, I_{\epsilon}$ be the length of the interval I_{ϵ}. Clearly, $diam\, I_{\epsilon} \leq diam\, I_{r\epsilon}$. Proceeding inductively, a sequence of closed intervals can be obtained which satisfy the following inclusion relation.

$$I_{r\epsilon} = A_0 \supset A_1 \supset \ldots \supset A_k \supset \ldots$$

with the property that $diam\, A_k \leq \frac{1}{2}\, diam A_{k-1}, 1 < k \in \mathbb{N}$ and $\{k \in \mathbb{N} \colon U_k(x) \in A_k\} \in \mathfrak{F}(\mathfrak{I})$ for $k \in \mathbb{N}$. By nested interval property, there must be some $\zeta \in \cap A_k,\ k \in \mathbb{N}$ with $\zeta = r\mathfrak{I} - \lim U_k(x)$. Thus, $U = (U_k) \in \mathbb{B}^{\infty}(U)$ is $r\mathfrak{I}$-convergent. □

In the following theorem, we explore the relationship between ideal convergence and rough ideal convergence and give some equivalent conditions.

Theorem 2.5.4. *The following are interchangeable:*

1. $(U_k) \in \mathcal{C}^{R\mathfrak{I}}(T)$,
2. *For every* $k \in \mathfrak{I}$ *we have* $(V_k) \in \mathcal{C}^{\mathfrak{I}}(T)$ *so that* $\|U_k(x) - V_k(x)\| \leq r,\ r > 0$,
3. *For all* $k \in \mathfrak{I}$ *there exists* $(V_k) \in \mathcal{C}^{\mathfrak{I}}(T)$ *and* $(W_k) \in {\mathcal{C}_0}^{R\mathfrak{I}}(T)$ *such that* $U_k = V_k + W_k$,
4. *There is a subsequence* U_{k_n} *of* U_k *for which* $\lim_{n\to\infty} \|U_{k_n}(x) - T(x)\| < r$, *where* $K = \{k_1, k_2, \ldots\} \in \mathfrak{F}(\mathfrak{I})$ *and* $U_k \xrightarrow{r} T$.

Proof. **(1) ⇒ (2)** Let $(U_k) \in \mathcal{C}^{R\mathfrak{I}}(T)$. Then there must be some $T \in \mathbb{L}(U)$ such that for some $r > 0$ and chosen $\epsilon > 0$,

$$\{k \in \mathbb{N} \colon \|U_k(x) - T(x)\| \geq r + \epsilon\} \in \mathfrak{I}.$$

Then by Theorem 1.4.5, there exists a sequence (V_k) as

$$V_k = \begin{cases} T, & \|U_k(x) - T(x)\| < r, \\ U_k + r\frac{T - U_k}{\|U_k - T\|}, & \text{otherwise.} \end{cases}$$

Clearly, $(V_k) \in \mathcal{C}^{\mathfrak{I}}(T)$ and $\|U_k(x) - V_k(x)\| \leq r$, for all $k \in \mathbb{N}$.

(2) ⇒ (3) We are given that for $(U_k) \in \mathcal{C}^{R\mathfrak{I}}(T)$, then there exists $(V_k) \in \mathcal{C}^{\mathfrak{I}}(T)$ so that for every $k \in \mathfrak{I}$, $\|U_k(x) - V_k(x)\| \leq r$, where $r > 0$. Now, $K \in \mathfrak{I}$ where $K = \{k \in \mathbb{N} \colon \|U_k - V_k\| > r\}$. Consider,

$$W_k = \begin{cases} U_k - V_k, & k \in K, \\ O, & \text{otherwise.} \end{cases}$$

Then, $(W_k) \in \mathcal{C}_0^{R\mathfrak{I}}(T)$.

(3) $\Rightarrow$ (4) Let $A = \{k \in \mathbb{N} \colon \|W_k(x)\| > r + \frac{\epsilon}{2}\}$. Then $A^c \in \mathfrak{F}(\mathfrak{I})$. Let $A^c = K = \{k_1, k_2, k_3 \ldots\}$. Then,

$$\|W_{k_n}(x)\| < r + \tfrac{\epsilon}{2}.$$

This implies that

$$\|U_{k_n}(x) - V_{k_n}(x)\| < r + \tfrac{\epsilon}{2}.$$

Now,

$$\|U_{k_n}(x) - T(x)\| \leq \|U_{k_n}(x) - V_{k_n}(x)\| + \|V_{k_n}(x) - T(x)\| < r + \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = r + \epsilon$$

Therefore, $\lim_{n\to\infty} \|U_{k_n}(x) - T(x)\| < r$.

(4) $\Rightarrow$ (1) For $\epsilon > 0$,

$$\{k \in \mathbb{N} \colon \|U_k(x) - T(x)\| \geq r + \epsilon\} \subseteq K^c \cup \{k \in K \colon \|U_k(x) - T(x)\| \geq r + \epsilon\}.$$

Hence, $(U_k) \in \mathcal{C}^{R\mathfrak{I}}(T)$.

□

Theorem 2.5.5. *The spaces ${\mathcal{C}_0}^{R\mathfrak{I}}(T), \mathcal{C}^{R\mathfrak{I}}(T)$ and $\mathbb{B}^{\infty}(U)$ satisfy the following inclusions: $\mathbb{B}^{\infty}(U) \supset \mathcal{C}^{R\mathfrak{I}}(T) \supset {\mathcal{C}_0}^{R\mathfrak{I}}(T)$.*

Proof. Since every sequence that r$\mathfrak{I}$ converges to zero operator is obviously r$\mathfrak{I}$ convergent, we clearly have ${\mathcal{C}_0}^{R\mathfrak{I}}(T) \subset \mathcal{C}^{R\mathfrak{I}}(T)$. To show $\mathcal{C}^{R\mathfrak{I}}(T) \subset \mathbb{B}^{\infty}(U)$, let $(U_k) \in \mathcal{C}^{R\mathfrak{I}}(T)$ implies there must be some, $T \in \mathbb{B}^{\infty}(U)$ with

$$\{k \in \mathbb{N} \colon \|U_k(x) - T(x)\| \geq r + \epsilon\} \in \mathfrak{I}.$$

Now,

$$\|U_k(x)\| = \|U_k(x) - T(x) + T(x)\| \leq \|U_k(x) - T(x)\| + \|T(x)\|.$$

Taking supremum over k from both sides (in the above inequality) we conclude, $(U_k) \in \mathbb{B}^{\infty}(U)$. □

We now construct a Lipschitz function with the help of rough ideal convergent sequence space of b.l.o.

Theorem 2.5.6. *Consider a function* $\mathfrak{L}\colon \mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T) \to \mathbb{R}$ *given as* $\mathfrak{L}(U) = \|r\mathfrak{I} - \lim U\|$, *for a fixed* $r > 0$. *Then* $\mathfrak{L}$ *is Lipschitz and hence continuous (uniformly continuous).*

Proof. For a fixed $r > 0$, let $U, V \in \mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ be such that

$$U = V \Rightarrow r\mathfrak{I} - \lim U = r\mathfrak{I} - \lim V \Rightarrow \|r\mathfrak{I} - \lim U\| = \|r\mathfrak{I} - \lim V\|.$$

Thus, $\mathfrak{L}$ is well defined. Then the sets

$$K_U = \{k \in \mathbb{N}\colon \|U_k(x) - \mathfrak{L}(U)\| \geq r + \|U - V\|\} \in \mathfrak{I}, \text{ and}$$
$$K_V = \{k \in \mathbb{N}\colon \|V_k(x) - \mathfrak{L}(V)\| \geq r + \|U - V\|\} \in \mathfrak{I},$$

where $U = (U_k), V = (V_k)$, and $\|U - V\| = \sup_k \|(U_k - V_k)(x)\|$. Clearly,

$${K_U}^c = \{k \in \mathbb{N}\colon \|U_k(x) - \mathfrak{L}(U)\| < r + \|U - V\|\} \in \mathfrak{F}(\mathfrak{I}), \text{ and}$$
$${K_V}^c = \{k \in \mathbb{N}\colon \|V_k(x) - \mathfrak{L}(V)\| < r + \|U - V\|\} \in \mathfrak{F}(\mathfrak{I}).$$

Hence, $K = {K_U}^c \cap {K_V}^c \in \mathfrak{F}(\mathfrak{I})$ is nonempty. For, $k \in K$

$$\begin{aligned}\|\mathfrak{L}(U) - \mathfrak{L}(V)\| &\leq \|\mathfrak{L}(U) - U_k(x)\| + \|U_k(x) - V_k(x)\| + \|V_k(x) - \mathfrak{L}(V)\| \\ &< (2r + 1)\|U - V\|.\end{aligned}$$

□

In the consequent theorems, we investigate some algebraic properties of the rough ideal convergent sequence spaces of bounded linear operators.

Theorem 2.5.7. *If* $U = (U_k), V = (V_k) \in \mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ *with* $U_kV_k(x) = U_k(x) \cdot V_k(x)$, *then* $(U \cdot V) \in \mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ *but* $\mathfrak{L}(U \cdot V) \neq \mathfrak{L}(U)\mathfrak{L}(V)$, *where* $\mathfrak{L}(U) = \|r\mathfrak{I} - \lim U\|$, *for a fixed* $r > 0$.

Proof. For $\epsilon = \|U - V\| = \sup_k \|(U_k - V_k)(x)\|$, we have

$$K_1 = \{k \in \mathbb{N}\colon \|U_k(x) - \mathfrak{L}(U)\| < r + \epsilon\} \in \mathfrak{F}(\mathfrak{I}), \text{ and}$$
$$K_2 = \{k \in \mathbb{N}\colon \|V_k(x) - \mathfrak{L}(V)\| < r + \epsilon\} \in \mathfrak{F}(\mathfrak{I}).$$

Now,

$$\begin{aligned}\|U_kV_k(x) - \mathfrak{L}(U)\mathfrak{L}(V)\| &= \|U_k(x)\cdot V_k(x) - U_k(x)\mathfrak{L}(V) + U_k(x)\mathfrak{L}(V) - \mathfrak{L}(U)\mathfrak{L}(V)\| \\ &\leq \|U_k(x)\|\|V_k(x) - \mathfrak{L}(V)\| + \|\mathfrak{L}(V)\|\|U_k(x) - \mathfrak{L}(U)\|.\end{aligned}$$

Since $\mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T) \subset \mathbb{B}^{\infty}(U)$, implies that we have some $M \in \mathbb{R}$ satisfying $\|U_k(x)\| \leq M$. Therefore, we have

$$\begin{aligned}\|U_kV_k(x) - \mathfrak{L}(U)\mathfrak{L}(V)\| &\leq M(r+\epsilon) + \|\mathfrak{L}(V)\|(r+\epsilon), \\ &= r(M + \|\mathfrak{L}(V)\|) + \epsilon(M + \|\mathfrak{L}(V)\|), \\ &= r_* + \epsilon_*, \, \forall k \in K_1 \cap K_2.\end{aligned}$$

□

Theorem 2.5.8. *The S-spaces, $\mathcal{C}_0^{R\mathfrak{I}}(T)$ and $\mathcal{G}_{\mathcal{C}_0}^{R\mathfrak{I}}(T)$ are solid and therefore monotone S-spaces.*

Proof. Let $(U_k) \in \mathcal{C}_0^{R\mathfrak{I}}(T)$. Consequently,

$$\{k \in \mathbb{N} \colon \|U_k(x)\| \geq r + \epsilon\} \in \mathfrak{I}.$$

For any sequence (a_k) of scalars with $\|a_k\| \leq 1$ for all $k \in \mathbb{N}$,

$$\|a_kU_k(x)\| \leq \|U_k(x)\|, \, \forall k \in \mathbb{N}.$$

Therefore,

$$\{k \in \mathbb{N} \colon \|\alpha_kU_k(x)\| \geq r + \epsilon\} \in \mathfrak{I}.$$

Thus, $\mathcal{C}_0^{R\mathfrak{I}}(T)$ is solid and since every solid space is monotone, it follows that $\mathcal{C}_0^{R\mathfrak{I}}(T)$ is solid and monotone. Proceeding in similar way, we get $\mathcal{G}_{\mathcal{C}_0}^{R\mathfrak{I}}(T)$ is solid and monotone. □

Theorem 2.5.9. *The space $\mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ is closed in $\mathbb{B}^{\infty}(U)$.*

Proof. We shall show that ever Cauchy sequence in $\mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ converges to some operator in $\mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$. For this, consider a Cauchy sequence, $U^n = ({U_k}^{(n)})$ in $\mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ converging to U. Since ${U_k}^{(n)} \in \mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$, there exists F_n such that for $r > 0$

$$\{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - F_n\| \geq r + \epsilon\} \in \mathfrak{I}.$$

We proceed in the following manner. We shall prove that

1. (F_n) converges to F.

2. $V^c \in \mathfrak{I}$, where $V = \{k \in \mathbb{N} \colon \|U_k(x) - F\| < r + \epsilon\}$.

1 For $({U_k}^{(n)})$ is a Cauchy in $\mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$, so by definition for any choice of $\epsilon > 0$, we obtain $n_0 \in \mathbb{N}$ for which,

$$\sup_k \|{U_k}^{(n)}(x) - {U_k}^{(m)}(x)\| < \tfrac{\epsilon}{3}, \forall\, n, m \geq n_0.$$

For a given $\epsilon > 0$, and some $r > 0$ consider

$$\begin{aligned} P_{nm} &= \{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - {U_k}^{(m)}(x)\| < \tfrac{r+\epsilon}{3}\}, \\ P_m &= \{k \in \mathbb{N} \colon \|{U_k}^{(m)}(x) - F_m\| < \tfrac{r+\epsilon}{3}\}, \\ P_n &= \{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - F_n\| < \tfrac{r+\epsilon}{3}\}. \end{aligned}$$

Then, ${P_{nm}}^c, {P_n}^c, {P_m}^c \in \mathfrak{I}$. Let $P^c = {P_{nm}}^c \cup {P_m}^c \cup {P_n}^c$, where

$$P = \{k \in \mathbb{N} \colon \|F_m - F_n\| < r + \epsilon\}.$$

Thus, $P^c \in \mathfrak{I}$. For $n_0 \in P^c$ and $n, m \geq n_0$, we get

$$\begin{aligned} \{k \in \mathbb{N} \colon \|F_m - F_n\| < r + \epsilon\} \supseteq \{&\{k \in \mathbb{N} \colon \|F_m - {U_k}^{(m)}(x)\| < \tfrac{r+\epsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - {U_k}^{(m)}(x)\| < \tfrac{r+\epsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon \|{U_k}^{(n)}(x) - F_n\| < \tfrac{r+\epsilon}{3}\}\}. \end{aligned}$$

Then (F_n) is a ρ-Cauchy sequence in $\mathbb{R}$ and since $\mathbb{R}$ is r-complete, suggests the existence of some F in $\mathbb{R}$, F_n is r-convergent to F for some $r > 2^{-1}J(\mathbb{R})\rho$, where J is Jung's constant [68].

2 Given that ${U_k}^{(n)} \to U$ implies there is $n_0 \in \mathbb{N}$ such that for any pre assigned positive number ϵ, we have,

$$B = \{k \in \mathbb{N} \colon \|{U_k}^{(n_0)}(x) - U_k(x)\| < \tfrac{r+\epsilon}{3}, \} \qquad (*)$$

implies $B^c \in \mathfrak{I}$. Particularly, n_0 be chosen in a way which along with $(*)$, gives

$$C = \{k \in \mathbb{N} \colon \|F_{n_0} - F\| < \tfrac{r+\epsilon}{3}\},$$

implying $C^c \in \mathfrak{I}$. Let $D^c = \{k \in \mathbb{N} \colon \|{U_k}^{(n_0)}(x) - F_{n_0}\| \geq \frac{r+\epsilon}{3}\}$ then $D^c \in \mathfrak{I}$. Let $V^c = B^c \cup C^c \cup D^c$, where $V = \{k \in \mathbb{N} \colon \|U_k(x) - F\| < r + \epsilon\}$. So, $\forall k \in V^c$, we get

$$\begin{aligned}\{k \in \mathbb{N} \colon \|U_k(x) - F\| < r + \epsilon\} \supseteq \{&\{k \in \mathbb{N} \colon \|U_k(x) - {U_k}^{(n_0)}(x)\| < \tfrac{r+\epsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon \|{U_k}^{(n_0)}(x) - F_{n_0}\| < \tfrac{r+\epsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon \|F_{n_0} - F\| < \tfrac{r+\epsilon}{3}\}\}.\end{aligned}$$

Thus $V^c \in \mathfrak{I}$.

Therefore, $\mathcal{G}_{\mathcal{C}}^{R\mathfrak{I}}(T)$ is closed in $\mathbb{B}^{\infty}(U)$.

□

2.6 Summary

This work is the extension of idea of rough ideal convergence to the sequences of bounded linear operators. We have defined some new sequence spaces which boil down to the already known ideal convergent sequence spaces of bounded linear operators for r=0. Theorem (2.5.4) establishes a relationship between rough ideal convergent sequences and ideal convergent sequences. In theorem (2.5.6), we have constructed a Lipschitz function with the help of rough ideal convergent sequence space of bounded linear operators. We have given a rough Cauchy criteria for sequences of bounded linear operators. We have also studied the topological aspects and algebraic properties of these new S-spaces. Delving into sequences of unbounded linear operators in future research promises to be intriguing. Furthermore, extending the concept of rough ideal convergence to sequences of linear operators in spaces of analytic functions could lead to the discovery of new and intriguing results.

Chapter 3

Sequence Spaces of Rough Ideal Convergent Sequences

After considering rough ideal convergent sequence spaces of bounded linear operators, we try to replicate the same with sequences of real or complex number and generalise the already known sequence spaces with the help of Orlicz function. In this chapter, we will explore various types of sequence spaces under the prism of rough convergence and examine their properties and interrelationships.

3.1 Introduction

Orlicz function was first introduced by the Polish mathematician Orlicz in 1931. Lindberg initiated the use of the Orlicz functions to solve an open problem to find a Banach space having subspaces isomorphic to c^0, the space of null sequences or ℓ^p spaces. Their work evoked the interest of Lindenstrauss and Tzafriri [47] and they were successful in constructing a sequence space, ℓ^S with the help of the Orlicz function S, which furthermore solved a long pending open problem of finding a complete n.l.s. which has a subspace isomorphic to some, $\ell^p = \{a = (a_n) \in \omega\colon \sum_n |a_n|^p < \infty\}, (1 \leq p < \infty)$. And

$$\ell^S := \{x \in \omega : \sum_{k=1}^{\infty} S(\frac{|x_k|}{\psi}) < \infty, \text{for some } \psi > 0\}, \text{where}$$

ℓ^S is a complete normed linear space under the norm,

$$\|x\| = \inf\{\psi > 0 : \sum_{k=1}^{\infty} S(\frac{|x_k|}{\psi}) \leq 1\},$$

and is referred to as "Orlicz" S-space.

In [64] Parashar and Choudhary defined certain paranorms for Orlicz S-space, laying the foundation for topologization of various generalized Orlicz S-spaces. Orlicz S-spaces has always been a centre of interest for researchers as it generalizes and unifies several known S-spaces for, the space ℓ^S becomes $\ell^p, (\ 1 \leq p < \infty)$ if we choose $S(x) = x^p$. After the introduction of statistical and ideal convergence, several researchers introduced statistical and ideal convergent S-spaces determined by the Orlicz functions and investigated their algebraic and topological properties (see [86], [75], [23], [69], [84], and [39]).

This chapter is aimed at introducing and generalising rough ideal convergence for S-space using an Orlicz function S, which is the generalization of ℓ^S, the Orlicz S-space and "$[C, 1, p], [C, 1, p]_0, [C, 1, p]_\infty$", (the S-spaces of strongly summable sequences [51]). We have also given some properties of these spaces when topologized through a paranorm and investigated inclusion relations, equivalent conditions, decomposition theorem and algebraic properties of such spaces.

3.2 Orlicz function

We first give some important theorems and results that will be used in the main results of this chapter.

Definition 3.2.1. (Convex Function) A map $g\colon [a, b] \to \mathbb{R}$ is convex if,

$$g(t_1 c + t_2 d) \leq t_1 g(c) + t_2 g(d), \text{where } c, d \in [a, b] \text{ and } t_1 + t_2 = 1.$$

Example 3.2.1. *Any real valued linear map g on any interval of $\mathbb{R}$ defined as $g(x) = \varrho x + \varsigma$, where ϱ, ς are constant, is referred to as a convex function.*

Definition 3.2.2. (Orlicz Function) Consider a map S between non negative real numbers, $S\colon [0, \infty) \to [0, \infty)$. Then S is an "Orlicz" function if,

1. S is continuous,
2. S is convex,
3. S is non decreasing,
4. S takes zero to zero, $S(0) = 0$,
5. S takes positive values to positive values, $S(u) > 0$ for $u > 0$,
6. S takes large values to large values, $S(u) \to \infty$ *as* $u \to \infty$.

Definition 3.2.3. (Δ_2-condition) Let S be an Orlicz function. If for every positive real number k, there exist constant $M > 0$ such that $S(2k) \leq MS(k)$, then we say S satisfies Δ_2-condition.

The Δ_2-condition, can also be considered as the inequality, $S(pk) \leq M\ pS(k)$, $\forall k$ and for $p > 1$.

Corollary 3.2.1. *For* $0 < p < 1$, $S(pk) \leq p\ S(k)$, *where* S *is an Orlicz function.*

Throughout this chapter, $\mathfrak{I}$ refers to a non-trivial a.i. on $\mathbb{N}$ and r is some positive integer.

It is well established that the spaces,

1. $\omega := \{a = (a_n) \colon a_n \in \mathbb{R} \text{ or } \mathbb{C}\}$,
2. $\ell^\infty := \{a = (a_n) \in \omega \colon \sup_n \|a_n\| < \infty\}$,
3. $c^0 := \{a = (a_n) \in \omega \colon \lim_n \|a_n\| = 0\}$,

are Banach spaces with norm $\|a\| = \sup_n |a_n|$.

For an Orlicz function S and $t = (t_k)$, where $t_k > 0$ and some real number $r > 0$. We give the following definitions:

$$c^{R\mathfrak{I}}(S,t) = \{x = (x_k) \in \omega \colon \{k \in \mathbb{N} \colon S(\tfrac{|x_k - L|}{\rho})^{t_k} \geq r + \epsilon\} \in \mathfrak{I}, L \in \mathbb{R}, \rho > 0\},$$
$${c_0}^{R\mathfrak{I}}(S,t) = \{x = (x_k) \in \omega \colon \{k \in \mathbb{N} \colon S(\tfrac{|x_k|}{\rho})^{t_k} \geq r + \epsilon\} \in \mathfrak{I}, \rho > 0\},$$
$$\ell_\infty(S,t) = \{x = (x_k) \in \omega \colon \sup_k S(\tfrac{|x_k|}{\rho})^{t_k} < \infty, \rho > 0\}.$$

We also denote

$$\mathcal{G}^{R\mathfrak{I}}{}_{c}(S,t) = \ell_\infty(S,t) \cap c^{R\mathfrak{I}}(S,t),$$
$$\mathcal{G}^{R\mathfrak{I}}{}_{c_0}(S,t) = \ell_\infty(S,t) \cap c_0{}^{R\mathfrak{I}}(S,t).$$

3.3 Paranormed Space

Norm is a generalized notion of distance and paranorm is the generalized absolute value function. We first give the definition of a paranorm on a linear space.

Definition 3.3.1. (Paranormed Space) A mapping $\upsilon\colon X \to \mathbb{R}$ on a linear space X is a paranorm, if it meets the following conditions:

1. υ takes non-negative vectors to non-negative values, $0 \leq \upsilon(m), \ \forall\ 0 \leq m,\ m \in X$.
2. υ is an even function, $\upsilon(-m) = m, \forall\ u \in X$.
3. υ complies with triangular inequality, $\upsilon(m) + \upsilon(n) \geq \upsilon(m+n), \forall m, n \in X$,
4. υ is continuous under multiplication by scalars, for any sequence of scalars, $\{\alpha_n\}$ with $\alpha_n \to \alpha$ and a vector sequence (a_n) such that $\upsilon(a_n - a) \to 0$ as $n \to \infty$, we have $\upsilon(\alpha_n a_n - \alpha a) \to 0$, as $n \to \infty$.

Then (X, υ), is a paranormed space. Additionally, (X, υ), is called a total paranormed space, whenever $\upsilon(\varsigma) = 0$, implies ς is the zero vector in X. If we define a map $\varrho\colon X \to \mathbb{R}$ as $\varrho(m,n) = \upsilon(m-n)$ then ϱ meets the conditions to be a metric on X and X is known as a linear metric space.

We now consider some illustrations of paranormed spaces.

Example 3.3.1. *For a bounded sequence $t = t_k$,*

1. *The space $c^0(t) = \{x = (x_k) \in \omega : \lim_k |x_k|^{t_k} = 0\}$ is a paranormed by:*

$$\theta(x) = \sup_k |x_k|^{\frac{t_k}{H}},$$

where $H = max(1, \sup_k t_k)$.

2. $w(t)$ and $\ell(t)$ are paramomed spaces, paranormed by:

$$\theta(x) = (\sum_k |x_k|^{t_k})^{\frac{1}{H}}.$$

3.4 Main Results

Theorem 3.4.1. *For $t = (t_k) \in \ell^\infty$ and an Orlicz function S, the classes of sequence*

$$c^{R\mathfrak{I}}(S,t), {c_0}^{R\mathfrak{I}}(S,t), \mathcal{G}^{R\mathfrak{I}}{}_c(S,t), \mathcal{G}^{R\mathfrak{I}}{}_{c_0}(S,t),$$

are vector spaces over $\mathbb{R}$.

Proof. Consider $a = (a_n),\ b = (b_n) \in c^{R\mathfrak{I}}(S,t)$ be and let $a^{'}, b^{'}$ be any two scalars. Since $a = (a_n),\ b = (b_n) \in c^{R\mathfrak{I}}(S,t)$ by definition of $c^{R\mathfrak{I}}(S,t)$, for any $\epsilon > 0$ there are $\psi_1, \psi_2 > 0$ such that the sets,

$$A^1 = \{k \in \mathbb{N}\colon S(\tfrac{|a_k - a_*|}{\psi_1})^{t_k} \geq r_1 + \tfrac{\epsilon}{2}\} \in \mathfrak{I}\}, \text{for some } a_* \in \mathbb{R} \text{ and } r_1 > 0. \quad (3.4.1)$$

$$A^2 = \{k \in \mathbb{N}\colon S(\tfrac{|b_k - b_*|}{\psi_2})^{t_k} \geq r_2 + \tfrac{\epsilon}{2}\} \in \mathfrak{I}\}, \text{for some } b_* \in \mathbb{R} \text{ and } r_2 > 0. \quad (3.4.2)$$

Let $r = \max\{r_1, r_2\}$ and $\psi_3 = \max\{2|a^{'}|\psi_1, 2|b^{'}|\psi_2\}$. Furthermore, S being an Orlicz function is convex and non decreasing, we have the following inequality

$$\begin{aligned} S(\tfrac{|(a^{'}a_k + b^{'}b_k) - (a^{'}a_* + b^{'}b_*)|}{\psi_3})^{t_k} &\leq S(\tfrac{|a^{'}||a_k - a_*|}{\psi_3})^{t_k} + S(\tfrac{|b^{'}||b_k - b_*|}{\psi_3})^{t_k}, \\ &\leq S(\tfrac{|a^{'}||a_k - a_*|}{\psi_1})^{t_k} + S(\tfrac{|b^{'}||b_k - b_*|}{\psi_2})^{t_k}. \end{aligned}$$

Then from above inequality along with (3.4.1) and (3.4.2) we have,

$$\{k \in \mathbb{N}\colon S(\tfrac{|(a^{'}a_k + b^{'}b_k) - (a^{'}a_* + b^{'}b_*)|}{\psi_3})^{t_k} \geq 2r + \epsilon\} \subseteq A^1 \cup A^2 \in \mathfrak{I},$$

implies that

$$\{k \in \mathbb{N}\colon S(\tfrac{|(a^{'}a_k + b^{'}b_k) - (a^{'}a_* + \beta b_*)|}{\psi_3})^{t_k} \geq 2r + \epsilon\} \in \mathfrak{I}.$$

Thus, $a^{'}a_k + b^{'}b_k \in c^{R\mathfrak{I}}(S,t)$. Hence, $c^{R\mathfrak{I}}(S,t)$ is a vector space. The proof for ${c_0}^{R\mathfrak{I}}(S,t), \mathcal{G}^{R\mathfrak{I}}{}_c(S,t), \mathcal{G}^{R\mathfrak{I}}{}_{c_0}(S,t)$ can be obtained similarly. □

Theorem 3.4.2. *Let S be an Orlicz function and $t = (t_k) \in \ell^\infty$, then the map $\upsilon(x)$ defined as,*

$$\upsilon(x) = \inf_{k\geq 1}\{\psi^{\frac{t_k}{M}}: \sup_k S(\tfrac{|x_k|}{\psi})^{t_k} \leq 1, \text{ where } \psi > 0\}, \text{ where}$$

$M = \max\{1, \sup_k t_k\}$ is a paranorm and the spaces $\mathcal{G}^{R\mathfrak{I}}{}_c(S,t), \mathcal{G}^{R\mathfrak{I}}{}_{c_0}(S,t)$ are paranormed spaces, paranormed by $\upsilon(x)$.

Proof. Proof omitted as it is simple and similar to the proof given in [64]. □

Theorem 3.4.3. *For any two Orlicz functions S_1 and S_2 which satisfy Δ_2-condition, the following inclusions hold:*

1. *$\zeta(S_1S_2, t)$ contains $\zeta(S_2, t)$,*
2. *$\zeta(S_1, t)\cap\zeta(S_2, t)$ is included in $\zeta(S_1+S_2, t)$, where $\zeta = c^{R\mathfrak{I}}, {c_0}^{R\mathfrak{I}}, \mathcal{G}^{R\mathfrak{I}}{}_c, \mathcal{G}^{R\mathfrak{I}}{}_{c_0}$.*

Proof. 1. Let $x = (x_k) \in {c_0}^{R\mathfrak{I}}(S_2, t)$ be any arbitrary element. Then by definition of ${c_0}^{R\mathfrak{I}}(S_2, t)$, for any pre assigned $\epsilon > 0$, we find some $\psi > 0$ with,

$$\{k \in \mathbb{N}\colon S_2(\tfrac{|x_k|}{\psi})^{t_k} \geq r + \epsilon \in \mathfrak{I}\}, \text{where } r > 0. \tag{3.4.3}$$

For a suitable choice of $\eta \in (0, 1)$, we obtain $S_1(t) < r + \epsilon$, for $t \in [0, \eta]$. Put $s_k = S_2(\frac{|x_k|}{\psi})^{t_k}$, then

$$\lim_k S_1(s_k) = \lim_{s_k\leq\eta} S_1(s_k) + \lim_{s_k>\eta, k\in\mathbb{N}} S_1(s_k).$$

Case 1 If $s_k > \eta$. Since $\eta < 1$, we get $s_k < \frac{s_k}{\eta} < 1 + \frac{s_k}{\eta}$, S_1 is an Orlicz function, by property 2 and 3 in (3.2.2) and (3.2.1), we have,

$$S_1(s_k) < S_1(1 + \tfrac{s_k}{\eta}) < \tfrac{1}{2}S_1(2) + \tfrac{1}{2}S_1(\tfrac{2s_k}{\eta}).$$

Also, by (3.2.3), $S_1(s_k) < \frac{1}{2}M\frac{s_k}{\eta}S_1(2) + \frac{1}{2}M\frac{s_k}{\eta}S_1(2)$, it follows that $S_1(s_k) < M\frac{s_k}{\eta}S_1(2)$. This further implies,

$$\lim_{s_k>\eta, k\in\mathbb{N}} S_1(s_k) \leq \max\{r, M\tfrac{1}{\eta}S_1(2) \lim_{s_k>\eta, k\in\mathbb{N}}(s_k)\}. \tag{3.4.4}$$

Case 2 If $s_k \leq \eta$, Then

$$\lim_{s_k\leq\eta, k\in\mathbb{N}} S_1(s_k) \leq r. \tag{3.4.5}$$

From (3.4.3), (3.4.4) and (3.4.5), we conclude that

$$\{k \in \mathbb{N} : S_1S_2(\tfrac{|x_k|}{\psi})^{t_k} \geq r+\epsilon \in \mathfrak{I}, \text{for some } r>0\}.$$

Hence, $x=(x_k) \in {c_0}^{R\mathfrak{I}}(S_1S_2,t)$. This proves that $\zeta(S_2,t) \subseteq \zeta(S_1S_2,t)$.

2. Consider $x=(x_k) \in {c_0}^{R\mathfrak{I}}(S_1,t) \cap {c_0}^{R\mathfrak{I}}(S_2,t)$. Then by definition we have,

$$\{k \in \mathbb{N} : S_1(\tfrac{|x_k|}{\psi})^{t_k} \geq r+\epsilon\} \in \mathfrak{I},$$

and

$$\{k \in \mathbb{N} : S_2(\tfrac{|x_k|}{\psi})^{t_k} \geq r+\epsilon \in \mathfrak{I} \text{ for some } r>0\}, \text{ where}$$

$\epsilon > 0$ and $\psi > 0$. Also,

$$\{k \in \mathbb{N} : (S_1+S_2)(\tfrac{|x_k|}{\psi})^{t_k} \geq r+\epsilon\} \subseteq [\{k \in \mathbb{N} : S_1(\tfrac{|x_k|}{\psi})^{t_k} \geq r+\epsilon\} \cup \{k \in \mathbb{N} : S_2(\tfrac{|x_k|}{\psi})^{t_k} \geq r+\epsilon\}],$$

suggests that,

$$\{k \in \mathbb{N}\colon (S_1+S_2)(\tfrac{|x_k|}{\psi})^{t_k} \geq r+\epsilon\} \in \mathfrak{I}.$$

Thus, $x=(x_k) \in {c_0}^{R\mathfrak{I}}(S_1+S_2,t)$.

We can prove the other inclusions by proceeding in the same fashion. □

Theorem 3.4.4. *The spaces ${c_0}^{R\mathfrak{I}}(S,t)$ and ${\mathcal{G}^{R\mathfrak{I}}}_{c_0}(S,t)$ are solid S-spaces.*

Proof. Let $x=(x_k) \in {c_0}^{R\mathfrak{I}}(S,t)$. Then

$$\{k \in \mathbb{N} : S(\tfrac{|x_k|}{\psi})^{t_k} \geq r+\epsilon \in \mathfrak{I}\}, \text{ for some } r>0, \epsilon>0, \psi>0.$$

For S is an Orlicz function, by (3.2.1) we obtain,

$$S(\tfrac{|\alpha_k x_k|}{\psi})^{t_k} \leq |\alpha_k|^{t_k} S(\tfrac{|x_k|}{\psi})^{t_k} \leq S(\tfrac{|x_k|}{\psi})^{t_k},$$

for some sequence of scalars α_k with $|\alpha_k| \leq 1, \forall k \in \mathbb{N}$. Clearly, $\alpha_k x_k \in {c_0}^{R\mathfrak{I}}(S,t)$. The proof for ${\mathcal{G}^{R\mathfrak{I}}}_{c_0}(S,t)$ can be obtained similarly. □

In the light of (2.1), we conclude that the spaces ${c_0}^{R\mathfrak{I}}(S,t)$ and ${\mathcal{G}^{R\mathfrak{I}}}_{c_0}(S,t)$ are monotone S-spaces.

Our experience with $c^I(M)$ and $m^I(M)$ [84], may lead us to believe that $c^{R\mathfrak{I}}(S,t)$ and $\mathcal{G}^{R\mathfrak{I}}{}_c(S,t)$ are not monotone and therefore not solid. However rough convergence gives sequences liberty to converge to any limit in a relaxed neighbourhood, (for any suitable r>0). Let us understand this with the help of an example.

Example 3.4.1. *For $S(x) = x^2, t_k = 1, \mathfrak{I} = \mathfrak{I}_\delta$, consider the constant sequence $\{a_n\}$, where $a_n = 1, \forall n$. Then $a_n \in c^{\mathfrak{I}}(S,t)$, but there is a canonical pre-image $\{b_n\}$ of $\{a_n\}$, defined as:*

$$b_k = \begin{cases} a_k \ ; & \text{if } k \text{ is even,} \\ 0 \ ; & \text{otherwise.} \end{cases}$$

which does not belong to $c^{\mathfrak{I}}(S,t)$ [84]. However, for $r = 1$, b_k is $r - \mathfrak{I}$ convergent to 0. Thus, canonical pre-images of all the step spaces, for a suitable choice of 'r' are in $c^{R\mathfrak{I}}(S,t)$ and $\mathcal{G}^{R\mathfrak{I}}{}_c(S,t)$.

Thus, $c^{R\mathfrak{I}}(S,t)$ and $\mathcal{G}^{R\mathfrak{I}}{}_c(S,t)$ are solid and monotone for some $r > 0$.

Theorem 3.4.5. *$c^{R\mathfrak{I}}(S,t)$ and $\mathcal{G}^{R\mathfrak{I}}{}_c(S,t)$ fail to be symmetric.*

Proof. We give the following illustration to establish the conclusion of the theorem. □

Example 3.4.2. *For $S(x) = x, t_k = 1 = \psi$, let $\{a_n\}$ be a sequence in $\mathbb{R}$. Let $\mathfrak{I}$ be a.i. of $\mathbb{N}$ such that it contains an infinite set P. Define*

$$a_k = \begin{cases} k \ ; & \text{for } k \in P, \\ (-1)^k; & \text{otherwise.} \end{cases}$$

Then $a_n \in c^{R\mathfrak{I}}(S,t)$. However, the permutation sequence, defined as:

$$a_k = \begin{cases} (-1)^k; & \text{for } k \in P, \\ k; & \text{otherwise.} \end{cases}$$

is not $r - \mathfrak{I}$ convergent. Thus, $c^{R\mathfrak{I}}(S,t)$ is not symmetric in general. Also, it fails to be convergence free ([84]example 1).

Theorem 3.4.6. *For an admissible ideal $\mathfrak{I}$, the following are interchangeable:*

1. $(a_k) \in c^{R\mathfrak{I}}(S,t)$,
2. *For all* $k \in \mathfrak{I}$, *there exists* $(b_k) \in c^{\mathfrak{I}}(S,t)$ *such that* $\|a_k - b_k\| \leq r, \quad r > 0$,
3. *For all* $k \in \mathfrak{I}$, *we have* $(b_k) \in c^{\mathfrak{I}}(S,t)$ *and* $(c_k) \in {c_0}^{R\mathfrak{I}}(S,t)$ *with* $a_k = b_k + c_k$,
4. $\lim_{n\to\infty} S(\frac{|a_{j_n}-l|}{\psi})^{t_{j_n}} = r$, *where* l *is the* r-$\mathfrak{I}$ *limit of* $S(\frac{|a_k|}{\psi})^{t_k}$ *and* $J = \{j_1, j_2, \ldots\}$ *of* $\mathbb{N}$ *such that* $M \in \mathfrak{F}(\mathfrak{I})$.

Proof. **(1) ⇒ (2)** Let $(a_k) \in c^{R\mathfrak{I}}(S,t)$. Then for some $r > 0$ and $\epsilon > 0$ and l ,

$$\{k \in \mathbb{N} \colon S(\frac{|a_k - l|}{\psi})^{t_k} \geq r + \epsilon\} \in \mathfrak{I}\}.$$

Again by (1.4.5), we have a sequence (b_k) as

$$b_k = \begin{cases} a_k; & S(\frac{|a_k-l|}{\psi})^{t_k} < r, \\ l; & \text{otherwise.} \end{cases}$$

Clearly, $(b_k) \in c^{\mathfrak{I}}(S,t)$ and for all $k \in \mathfrak{I}$, $\|a_k - b_k\| \leq r$.

(2) ⇒ (3) We are given that $(a_k) \in c^{R\mathfrak{I}}(S,t)$, then there exists $(b_k) \in c^{\mathfrak{I}}(S,t)$ with $|a_k - b_k| \leq r$, where $r > 0$, $k \in \mathfrak{I}$. Let $J = \{k \in \mathbb{N} \colon \|a_k - b_k\| \geq r\}$, then $J \in \mathfrak{I}$. Define a sequence

$$c_k = \begin{cases} a_k - b_k; & k \in J, \\ 0; & \text{otherwise.} \end{cases}$$

Then, $(c_k) \in {c_0}^{R\mathfrak{I}}(S,t)$.

(3) ⇒ (4) Let $A = \{k \in \mathbb{N} \colon S(\frac{|c_k|}{\psi})^{t_k} \geq r + \frac{\epsilon}{2}\}$. Then $A^c \in \mathfrak{F}(\mathfrak{I})$. Let $A^c = J = \{j_1, j_2, \ldots\}$. Then we have, $\lim_{n\to\infty} S(\frac{|a_{j_n}-l|}{\psi})^{t_{j_n}} = r$.

(4) ⇒ (1) Let $\epsilon > 0$, then we have,

$$\{k \in \mathbb{N} \colon S(\frac{|a_k-l|}{\psi})^{t_k} \geq r + \epsilon\} \subseteq J^c \cup \{j_n \in J \colon S(\frac{|a_{j_n}-l|}{\psi})^{p_{j_n}} = r \geq r + \epsilon\} \in \mathfrak{I}.$$

Hence, $(a_k) \in c^{R\mathfrak{I}}(S,t)$.

□

Theorem 3.4.7. *The set $\mathcal{G}^{R\mathfrak{I}}{}_c(S,t)$ is closed in $\ell_\infty(S,t)$.*

Proof. Consider a Cauchy sequence $({a_k}^{(n)})$ in $\mathcal{G}^{R\mathfrak{I}}{}_c(S,t)$ with ${a_k}^{(n)} \to a$. Since ${a_k}^{(n)} \in \mathcal{G}^{R\mathfrak{I}}{}_c(S,t)$, there exists b_n such that for some $r > 0$,

$$\{k \in \mathbb{N} \colon S(\tfrac{|{a_k}^{(n)} - b_n|}{\psi})^{t_k} \geq r + \epsilon\} \in \mathfrak{I}.$$

We shall establish that

(a) (b_n) converges to b,

(b) If $V = \{k \in \mathbb{N} \colon S(\frac{|{a_k}^{(n)} - b|}{\psi})^{t_k} < r + \epsilon\}$, then $V^c \in \mathfrak{I}$.

(a) Now, $({a_k}^{(n)}) \in \mathcal{G}^{R\mathfrak{I}}{}_c(S,t)$ is Cauchy $\implies \exists\ n_0 \in \mathbb{N}$ such that

$$sup_k S(\tfrac{|{a_k}^{(n)} - {a_k}^{(m)}|}{\psi})^{t_k} < \frac{\epsilon}{3}, \forall n, m \geq n_0, \epsilon > 0.$$

For any pre assigned $\epsilon > 0$, and some $r > 0$ consider

$$P_{nm} = \{k \in \mathbb{N} \colon S(\tfrac{|{a_k}^{(n)} - {a_k}^{(m)}|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\},$$
$$P_m = \{k \in \mathbb{N} \colon S(\tfrac{|{a_k}^{(m)} - b_m|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\},$$
$$P_n = \{k \in \mathbb{N} \colon S(\tfrac{|{a_k}^{(n)} - b_n|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\}.$$

Then, ${P_{nm}}^c, {P_n}^c, {P_m}^c \in \mathfrak{I}$. Also if

$$P = \{k \in \mathbb{N} \colon S(\tfrac{|b_m - b_n|}{)} \psi^{t_k} < r + \epsilon\},$$

then, $P^c = {P_{nm}}^c \cup {P_m}^c \cup {P_n}^c$ lies in $\mathfrak{I}$. Let $n, m \geq n_0$, where $n_0 \in P^c$, we get

$$\begin{aligned}\{k \in \mathbb{N} \colon S(\tfrac{|b_m - b_n|}{\psi})^{t_k} < r + \epsilon\} \supseteq \{&\{k \in \mathbb{N} \colon S(\tfrac{|b_m - {a_k}^{(m)}|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon S(\tfrac{|{a_k}^{(m)} - {a_k}^{(n)}|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\} \\ &\cap \{k \in \mathbb{N} \colon S(\tfrac{|{a_k}^{(n)} - b_n|}{\psi})^{t_k} < \frac{r+\epsilon}{3}\}\}.\end{aligned}$$

This establishes that (b_n) is a ρ-Cauchy sequence in $\mathbb{R}$ and $\mathbb{R}$ being r-complete, we get some b in $\mathbb{R}$ with b_n is r-convergent to b for some $r > 2^{-1}\mathbb{J}(\mathbb{R})\rho$, where $\mathbb{J}$ is "Jung's constant" [68].

(b) For $a_k^{(n)} \to x$, there exists $n_0 \in \mathbb{N}$ with $B = \{k \in \mathbb{N}\colon S(\frac{|a_k^{(n_0)} - a_k|}{\psi})^{t_k} < (\frac{r+\epsilon}{3N})^M\}$,

where $r > 0,\ \epsilon > 0$,

$M = \max\{1, \sup_k t_k\},\ N = \max\{1, 2^{P-1}\},\ P = \sup_k t_k \qquad *$ implies $B^c \in \mathfrak{I}$.

For a suitably chosen, n_0 together with $*$, we have

$$C = \{k \in \mathbb{N}\colon S(\tfrac{|b_{n_0} - b|}{\psi})^{t_k} < (\tfrac{r+\epsilon}{3N})^M\},$$

such that $C^c \in \mathfrak{I}$. Let $D^c = \{k \in \mathbb{N}\colon S(\frac{|a_k^{(n_0)} - b_{n_0}|}{\psi})^{t_k} \geq (\frac{r+\epsilon}{3N})^M\}$ then $D^c \in \mathfrak{I}$. Let $V^c = B^c \cup C^c \cup D^c$, where $V = \{k \in \mathbb{N}\colon S(\frac{|a_k - b\|}{\psi}^{t_k}) < r+\epsilon\}$. Consequently, for any $k \in V^c$,

$$\begin{aligned}\{k \in \mathbb{N}\colon S(\tfrac{|a_k - b|}{\psi})^{t_k} < r + \epsilon\} \supseteq \{\{k \in \mathbb{N}\colon S(\tfrac{|a_k - a_k^{(n_0)}|}{\psi}^{t_k} < (\frac{r+\epsilon}{3N})^M\} \\ \cap \{k \in \mathbb{N}\colon S(\tfrac{|a_k^{(n_0)} - b_{n_0}|}{\psi})^{t_k} < (\frac{r+\epsilon}{3N})^M\} \\ \cap \{k \in \mathbb{N}\colon S(\tfrac{|b_{n_0} - b|}{\psi})^{t_k} < (\frac{r+\epsilon}{3N})^M\}\}.\end{aligned}$$

Thus, $V^c \in \mathfrak{I}$.

□

3.5 Summary

In this chapter, we have given some novel classes of S-spaces using Orlicz function. Theorem 3.5 establishes a some new relationship between rough ideal convergent sequences and ideal convergent sequences. The S-spaces are solid and monotone which was not the case with ideal convergent sequence spaces in general. However, it still fails to be convergence free and symmetric. It will be interesting to see that under what conditions or function the spaces so formed will become convergence free. Investigation for rough ideal convergent S-spaces defined by other functions for instance (compact operator and modulus functions) will be intriguing to obtain some new and interesting results.

Chapter 4

On Rough Continuity and Rough Compactness

Rough convergence was initially introduced for n.l.s and then latter extended to metric spaces. Since the limit of a rough convergent sequence is not unique and in particular we allow a sequence to convergence in any neighbourhood it becomes interesting to study the topology it generates or to extend this notion to sequences in a topological space. This chapter is an attempt in the same direction. We have introduced closed sets with the help rough convergent sequences in $\mathbb{R}$. We will also consider rough sequential approach for continuity, compactness and connectedness and give results involving all these three.

4.1 Introduction

Rough convergence has been attracting a lot of attention from researchers ever since its introduction. Rough convergence allows a sequence to converge in a relaxed manner and therefore bounded sequences in $\mathbb{R}$ are rough convergent even if they oscillate finitely. H.X. Phu [66], conceived the idea of this relaxed convergence, which he called rough convergence and initially introduced it for finite dimensional n.l.s. as: for any sequence $\{a_n\}$ is a sequence in a n.l.s. $(X, \|\cdot\|)$ and for some $0 \leq r \in \mathbb{R}$ if, if there is $k_\epsilon \in \mathbb{N}$ with

$$k \geq k_\epsilon \Rightarrow \|a_k - a_*\| < r + \epsilon, \text{ for all } \epsilon > 0.$$

Then $\{a_n\}$ is called r-convergent to a_*, written as, $a_n \xrightarrow{r} a_*$, where r and a_* are called the roughness degree and the r-limit point of (x_i) respectively. The immediate consequence of this definition is that every bounded sequence is convergent and the limit is not unique. The collection of all the r-limit points of r convergent sequence $\{a_n\}$ is symbolised as $LIM^r a_n$ which equals to $\{a_* \colon a_n \xrightarrow{r} a_*\}$.
In his paper, he investigated the properties of the limit set so obtained and also introduced rough Cauchy sequences. He also went on to expand this notion of relaxing the ϵ neighbourhood for continuous functions and in [67], he introduced rough continuity for linear operators between two n.l.s.
For two n.l.s. $(P, \|\cdot\|)$ and $(Q, \|\cdot\|)$ and r>0, the map $\varrho \colon P \to Q$ is defined to be roughly continuous at $p \in P$ (or r-continuous at p) if, for any pre assigned $\epsilon > 0$ there is a $\delta > 0$ with,

$$\|p - q\|_P < r + \delta \implies dist(f(q), f(B(p, r))) < \epsilon,$$

where,

$$B(p, r) = \{s \colon \|s - p\|_P \leq r\} \; and \; dist(f(q), f(B(p, r))) = \inf_{s \in B(p,r)} \|f(q) - s\|.$$

Later on, he also investigated similar results for infinite dimensional n.l.s. [68]. For broadening its applicability, statistical and ideal versions of rough convergence were introduced and studied (see [3, 4, 21, 54, 55, 63]. Subsequently, investigations were made for various several spaces like metric spaces [17], cone metric spaces [6], S-metric spaces [58] etc., where the investigations mainly revolved around the geometric and algebraic charactistics of the rough limit set. Some of the recent work in this direction can be found in [7], [18], [19], [77]. In [7], the authors studied the idea of rough continuity and its ideal version for real valued functions and gave examples to show that compactness and connectedness is not preserved by a rough continuous function.
This prompted us to take a rough sequential approach for compactness. Any subset H in a metric space (X,ρ), is referred to as sequentially compact, whenever any sequence of elements in H has a convergent subsequence with limit in H [12]. In this chapter, we have defined r-closure of a set with the help of rough convergent sequences. Using this r-closure we have defined r-closed sets and investigated Kuratowski topology so induced. We have given definitions for r-sequential continuity, r-sequential compact sets investigated their properties and proved that the image of a r-sequentially compact set under r-sequential continuous map is

r-sequentially compact.

4.2 Basic Definitions and Results

We now give some definitions and results that will be used in the subsequent section of this chapter.

Definition 4.2.1. (Rough convergence in a Metric Space) For any sequence $\{a_n\}$ in a metric space (X,ρ) with,

$$\rho(a_m, a) < r + \epsilon, \text{ for every } m \geq m_0, \ r \geq 0,$$

where $\epsilon > 0$ is any pre assigned value and $m_0 \in \mathbb{N}$, a $\in$ X. Then we say a_n is rough convergent, (written shortly as, r-convergent) to the point a $\in$ X. This is rough convergence with "degree of roughness" r. It is evident that for r =0, rough convergence equals usual convergence in (X,ρ).

Remark 4.1. For a r-convergent sequence $\{a_n\}$ can converge to different limits for different choice of r, so we consider a set of limit points denoted by $LIM^r a_n$ and is defined as,

$$LIM^r a_n = \{a \in X : a_n \xrightarrow{r} a\}.$$

For A$\subseteq$ $\mathbb{R}$, the set $LIM^r a_n$ lying in A is denoted by $LIM^{A,r} a_n$ and is defined as

$$LIM^{A,r} a_n = \{a \in A : a_n \xrightarrow{r} a\}.$$

Theorem 4.2.1. *A sequence $\{a_n\}$ is bounded iff for some $r \geq 0$, such that $LIM^r a_n \neq \phi$.*

Theorem 4.2.2. *There is always an r-convergent subsequence ($r \geq 0$), for every bounded sequence.*

Theorem 4.2.3. *For any subsequence $\{a_{n_i}\}$ of a sequence $\{a_n\}$, $LIM^r a_n \subseteq LIM^r a_{n_i}$.*

Definition 4.2.2. (Topological Space) Let $\mathcal{P}$ be a non-void collection of sets in a non-void set M. Then $\mathcal{P}$ is said to be a topology on M, if the following holds:

1. ϕ, $M \in \mathcal{P}$,

2. $\mathcal{P}$ is stable under arbitrary union,

3. $\mathcal{P}$ is stable under finite intersection.

Then (M, $\mathcal{P}$) is said to be a topological space. The open sets in (M, $\mathcal{P}$), are the elements in $\mathcal{P}$ and the closed sets in (M, $\mathcal{P}$) are the complements of open sets.

Clearly, a topological space is the generalization of a metric space.

Remark 4.2. Every metric induces a topology on a set, but a topology may not define a metric or in other words all metric spaces are topological spaces but the converse need not be true.

Definition 4.2.3. (Kuratowski Closure Operator) For any topological space X, closure of a set A in X , symbolized as cl(A) or $\overline{A}$ can be considered as a function from powerset $\mathbb{P}(X)$ to X given by

$$cl(A) = \overline{A} = \{x \in X : \text{there is a sequence } \{a_n\} \in A \text{ such that } a_n \to x.\}$$

Theorem 4.2.4. *For a topological space X, the operation $K \to cl(K)$ meets the following:*

1. $\overline{\phi} = \phi$.

2. $K \subset \overline{K}$.

3. $\overline{\overline{K}} = \overline{K}$.

4. $\overline{K \cup L} = \overline{K} \cup \overline{L}$.

5. *K is a closed set iff $\overline{K} = K$.*

Moreover, for any set X with a mapping $\mathbb{P}(X) \to \mathbb{P}(X)$ which takes sets to its closure, K $\to$ cl(K) satisfying 1 to 4 is called a Kuratowski closure operator and if we define closed sets using 5, in X, the resulting topology on X has the same closure operation K $\to$ cl(K), with which we began.

Definition 4.2.4. (Kuratowski Topology) Let M be any non-void set. Then a collect $\mathcal{K}$ comprising of subsets of M satisfying :

1. ϕ, M $\in$ $\mathcal{K}$.

2. $\mathcal{K}$ is stable under finite union.

3. $\mathcal{K}$ is stable under arbitrary intersection.

forms Kuratowski's topology and (M, $\mathcal{K}$) is referred to as Kuratowski topological space. The closed sets in (M, $\mathcal{K}$), are the elements in $\mathcal{P}$ and the open sets in (M, $\mathcal{K}$) are the complements of the closed sets.

Definition 4.2.5. (Rough Continuity) Let A $\subset \mathbb{R}$. We first give rough continuity at a point a $\in$ A. Let $\{a_n\}$ be any sequence converging to a, if for some real number $r_a \geq 0$, the sequence of the image of $\{a_n\}$ under g , that is $\{g(a_n)\}$ is r-convergent to g(a), with roughness degree r_a then we say g is r-continuous at a. If g is r-continuous at each a $\in$ A and $\zeta = sup\{r_a : a \in A\}$ is finite, then g is called r-continuous on the set A with roughness degree ζ.

Remark 4.3. Every continuous function is r-continuous. However, the converse may not necessarily hold good. It is well known that the *characteristic function* of the Cantor set is a discontinuous map. But for r$\geq \frac{1}{3}$, it is r-continuous.

Remark 4.4. In the light of the above definition, we observe that the Intermediate Value Theorem does not hold good for r-continuity. For r=1, the Dirichlet function is r-continuous but it does not assume any value other than 1 and 0.

Consider a metric space, (M,ρ). Then

Definition 4.2.6. (Separated Sets) For any G,H $\subseteq$ M, G and H are said to be separated if neither has a point in common with the closure of each other.

$$H \cap \overline{G} \;=\; \phi, \; and \; \overline{H} \cap G \;=\; \phi.$$

Clearly, If G and H are separated then they are disjoint.

Definition 4.2.7. (Connected Sets) For any H$\subset$ M, is connected if it cannot be expressed as the union of two non empty separated sets in M. A set which is not connected is said to be a disconnected set.

Theorem 4.2.5. *Let $H \subset M$, then the statements given below are interchangeable*:

1. *H is connected.*

2. *There does not exist any non void closed sets A and B, $A \cap B = \phi$ in H with H= A $\cup$ B.*

3. *ϕ and H are the only clopen sets in H.*

4.3 Rough Closed Sets

All the results in this section, are given for $\mathbb{R}$ with usual metric. In this section, we first define rough closure of subsets of $\mathbb{R}$ and then study the topology induced by the r-closed sets so obtained.

Definition 4.3.1. (r-Closure) Let A$\subseteq$ $\mathbb{R}$. Then we say that $a \in \mathbb{R}$ belongs to rough closure (in short r-closure) of A, if there exists a sequence $\{a_n\}$ in A such that $a_n \xrightarrow{r} a$, for some r. We denote r-closure of A by $\overline{A}^r$.
If K$\subseteq$ A, we symbolize r-closure of K in A by $\overline{K}^r{}_A$ and is defined as $\overline{K}^r{}_A = \overline{K}^r \cap A$.

Clearly, if B $\subseteq$ A, then $\overline{B}^r \subseteq \overline{A}^r$.

Theorem 4.3.1. *For any space X and for any subsets A, B of X, we have the following:*

1. $\overline{\phi}^r = \phi$.
2. $\overline{A}^r \supset A$.
3. $B \supset A \implies \overline{B}^r \supset \overline{A}^r$.
4. $\overline{\overline{A}^r}^r = \overline{A}^r$.
5. $\overline{A \cup B}^r = \overline{A}^r \cup \overline{B}^r$.

Proof.

1 Clearly, 1 holds for there is no sequence in ϕ.

2 Let $a \in A$. Consider a sequence $\{a_n\}, a_n = a, \forall\ n$ Then $a_n \xrightarrow{r} a$ implies a $\in$ $\overline{A}^r$.

3 Let $a \in \overline{A}^r$. Then there is a sequence $\{a_n\}$ in A with $a_n \xrightarrow{r} a$. Since A $\subseteq$ B, $\{a_n\}$ in B with $a_n \xrightarrow{r} a$ implies $a \in \overline{B}^r$.

4 Let a $\in \overline{\overline{A}^r}^r$. Then there is a some r-convergent sequence $\{a_n\}$ in $\overline{A}^r$ such that $a_n \xrightarrow{r} a$. Again, $a_n \in \overline{A}^r$ implies that for each a_n, we have a sequence say, $\{b_{kn}\}$ in A with $b_{kn} \xrightarrow{r} a_n, \forall k, n \in \mathbb{N}$. If possible, let us assume that a $\notin \overline{A}^r$. This means there does not exist any sequence in A which is r-convergent to a, for any r>0. However, if we consider the diagonal sequence b_{kk} in A then $b_{kk} \xrightarrow{r} a$. Thus, a $\in \overline{A}^r$. Hence, $\overline{\overline{A}^r}^r \subseteq \overline{A}^r$. Also by 2, we have $\overline{A}^r \subseteq \overline{\overline{A}^r}^r$.

5 Let a $\in \overline{A \cup B}^r$. Consequently, we have a sequence $\{a_n\}$ in $A \cup B$ with $a_n \xrightarrow{r} a$. WLOG, let us assume that A contains infinitely many terms of $\{a_n\}$. Then there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ and $a_{n_k} \xrightarrow{r} a$. Thus, $a \in \overline{A}^r$. In a similar manner, it can be shown that $a \in \overline{B}^r$. Thus, $a \in \overline{A}^r \cup \overline{B}^r$. Also, by 3, we have $\overline{A}^r \cup \overline{B}^r \subseteq \overline{A \cup B}^r$.

□

From the above theorem, it is clear that r-closure is a Kuratowski's closure operator on any space X. We are now in a position to define r-closed sets with the help of the definition of r-closure.

Definition 4.3.2. (r-Closed Sets) Let A$\subseteq \mathbb{R}$. Then we say that A is rough closed set or r-closed,(in short rc) if all of the points in its r-closure are in A. That is,

$$\overline{A}^r \subset A.$$

We define r-open sets to be the complements of r-closed sets.

We first show that rc sets is stable under arbitrary intersection.

Theorem 4.3.2. *For a fixed r >0, let $\{A_\alpha : \alpha \in \Omega\}$ be a family of rc subsets of $\mathbb{R}$. Then $\cap_{\alpha \in \Omega} A_\alpha$ is rc.*

Proof: Consider a sequence $\{a_n\}$ in $\cap_{\alpha \in \Omega} A_\alpha$. This implies $a_n \in A_\alpha, \forall \alpha$. Since each A_α is rc, $\{a \in \mathbb{R} : a_n \xrightarrow{r} a\} \subset A_\alpha, \forall \alpha$. Thus, $\{a \in \mathbb{R} : a_n \xrightarrow{r} a\} \subset \cap_{\alpha \in \Omega} A_\alpha$. This proves that $\cap_{\alpha \in \Omega} A_\alpha$ is rc.

Theorem 4.3.3. *For a fixed r >0, let H and K be any two rc sets in $\mathbb{R}$. Then $H \cup K$ is also rc.*

Proof: Consider a r-convergent sequence, $\{a_n\}$ in $H \cup K$. Consequently, $\{a_n\}$ is bounded. Then $\exists$ convergent subsequence $b_n = a_{n_k}$ of $\{a_n\}$ in $H \cup K$. WLOG, let us assume that H contains all except a finite number of entries of the sequence b_n. Since every convergent sequence is r -convergent, b_n is r-convergent. Also by (4.2.3), we have $LIM^r a_n \subseteq LIM^r a_{n_k}$. Now H is rc, so we have $LIM^r a_{n_k} \subset H$. Therefore, $LIM^r a_n \subseteq H \cup K$. Hence $H \cup K$ is rc.

Corollary 4.3.1. *From (4.3.2) and (4.3.3), we deduce that a family of rc sets in* $\mathbb{R}$ *induce a Kuratowski Topology on* $\mathbb{R}$.

In the next result we study, the link between the rc sets and closed sets on $\mathbb{R}$.

Theorem 4.3.4. *Every closed set is rc in* $\mathbb{R}$.

Proof: Consider a closed set A in $\mathbb{R}$. We shall show that A is rc in $\mathbb{R}$. For this, let a $\in \overline{A}^r$. Then there must be a sequence $\{a_n\}$ in A, such that a_n r-converges to a. Consequently, there is a subsequence of $\{a_n\}$ which converges to a. Since every convergent sequence is r-convergent and A is a closed set, implies that a $\in$ A. This proves that A is a rc set.

Remark 4.5. The converse of theorem holds for r =0. For r > 0, the converse may not hold.

Example 4.3.1. *Consider* $\mathbb{R}$ *with usual topology. Then the set S= (0,1] is not closed w.r.t. the usual topology on* $\mathbb{R}$, *for there is a sequence* $\{\frac{1}{n}\}$*in S which does not have a limit in S. However for r>0,* $\{\frac{1}{n}\}$ *is r-convergent to points in S. Thus, S is r-closed but not closed.*

Thus, Kuratowski topology so obtained on $\mathbb{R}$, is coarser than the usual topology on $\mathbb{R}$.

Theorem 4.3.5. *For* $B \subseteq A \subseteq \mathbb{R}$, *B is rc in* $\mathbb{R}$ *whenever A is rc in* $\mathbb{R}$ *and B is rc in A.*

Proof: Let $b \in \overline{B}^r$, by definition of r -closure $\exists$ a sequence $\{b_n\}$, with $b_n \xrightarrow{r}$ b. Now, B $\subseteq$ A, implies $\overline{B}^r \subseteq \overline{A}^r$. Therefore, b $\in \overline{A}^r$. Again, A is rc in $\mathbb{R}$, suggests b $\in$ A. Thus b $\in \overline{B}^r{}_A$ and B is rc in A, implies b $\in$ B. Consequently, B is rc in $\mathbb{R}$.

Let us now examine the behaviour of rc sets under a continuous map.

Theorem 4.3.6. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous map and M be a rc set. Then* $g^{-1}(M)$ *is also rc.*

Proof: Let $N = g^{-1}(M)$ and let v $\in \overline{N}^r$. Then $\exists$ is a sequence $\{a_n\}$ in N such that $a_n \xrightarrow{r} v$. Since g is continuous, $g(a_n) \xrightarrow{r} g(v)$. Now, M is a rc set, so $g(v) \in$ U. This implies that $v \in N$. Thus, $\overline{N}^r \subseteq N$. This proves that N is a rc set.

4.4 Rough Sequential Compactness and Connectedness

The next best thing to finiteness is compactness and therefore one is always drawn to check for the compactness of a given set. It is a well known fact that compactness can be viewed through the lens of convergent sequences. This motivated us to present rough compactness as under:

Definition 4.4.1. (r-Compact Sets) A set U in $\mathbb{R}$ is called rough sequentially compact (in short r-sc) if, for any sequence of points in U, there is a subsequence whose r-limit set is in U.

We now give some results for r-compact sets analogous to the results for compact sets in $\mathbb{R}$.

Theorem 4.4.1. *Any rc subset of r-sc set is r-sc.*

Proof: Let A be any r-sc set and B $\subseteq$ A be a rc set. Take any sequence $\{a_n\}$ in B, then $\{a_n\}$ is a sequence in A. Additionally, A is given to be a r-sc set, so we get a subsequence, $\{b_n\}$ of $\{a_n\}$ with $\text{LIM}^r b_n \subseteq A$. Now, B is rc, this implies that $\text{LIM}^r b_n \subseteq B$. Thus, B is r-sc set.

Theorem 4.4.2. *Any r-sc subset of $\mathbb{R}$ is rc.*

Proof: Assume $A \subset \mathbb{R}$ is r-sc set and a $\in \overline{A}^r$. Then there is a r-convergent sequence $\{a_n\}$ in A such that $a_n \xrightarrow{r} a$. Subsequently, $\{a_n\}$ is bounded, we have a subsequence $\{b_n\}$ of $\{a_n\}$ with $b_n \to a$. Again, A is r-sc and $b_n \to a$, by definition of r-sc set, there is a r-convergent subsequence $\{c_n\}$ of $\{b_n\}$. By (4.2.3), it follows that $a \in LIM^r c_n$. Thus, A is a rc set.

We have seen that r-continuity does not preserve compactness and connectedness and therefore we are motivated to define r-sequential continuity and investigate if, it preserves r-sequential compact sets.

Definition 4.4.2. (r-Sequential Continuous Map) Let A $\subset \mathbb{R}$. A function $g\colon A \to \mathbb{R}$ is defined to be rough sequential continuous (in short, r-scontinuous) at a point a $\in$ A of roughness degree r_a if, for any sequence $\{a_n\}$ r-convergent to a, the sequence $\{g(a_n)\}$ is r-convergent to g(a) for r $=r_a$. g is said to be rough sequential continuous on A, if it is r-scontinuous at each point of A, where r $=\rho$ provided, $\rho = sup\{r_a : x \in A\}$ exists finitely.

Theorem 4.4.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a r-scontinuous map. Then f takes r-sc set to a r-sc set.*

Proof: Let A be any r-sc set in $\mathbb{R}$ and $\{a_n\}$ be a sequence in A. Also, let $u_n = f(a_n)$ is a sequence in $f(A)$. Since A is r-sc set, $\{a_n\}$ contains a r-convergent subsequence, $\{b_n\}$. Clearly, $f(b_n)$ is a subsequence of u_n in $f(A)$. Since f is r-scontinuous, implies $f(b_n)$ is r-convergent in $f(A)$. Thus, $f(A)$ is r-sc.

Next, we give rough sequential definition for connectedness.

Definition 4.4.3. (r-Connectedness) A non-void set G in $\mathbb{R}$ is referred to as rough sequentially connected (in short, r-sconnected) if, there are no non-empty disjoint rc sets H and K in $\mathbb{R}$ with G $\subseteq H \cup K$,and $G \cap H \neq \phi$, $G \cap K \neq \phi$.

Theorem 4.4.4. *For any subset $G \subset \mathbb{R}$, the following are interchangeable :*

1. *G is r-sc.*
2. *There do not exist disjoint rc sets H,K in G with, $G = H \cup K$.*
3. *There do not exist disjoint r-open sets H,K in G with, $G = H \cup K$.*
4. *G does not contain any proper r-clopen (rc as well as r-open) subset.*

The proof of this theorem is obvious from the definition (4.4.3) and (4.3.2). Let now examine if r-sconnectedness is preserved under r-scontinuous maps.

Theorem 4.4.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a r-scontinuous map. Then image of a r-sconnected set under f is r-sconnected.*

Proof: If possible, let us assume that for a r-sconnected set A, $f(A)$ is not r-sconnected. Then there exists two rc sets $\phi \neq H, K$ and $H \cap K = \phi$ in $\mathbb{R}$ with $H \cup K \supset f(A)$, $f(A) \cap\ H \neq \phi$, $f(A) \cap\ K \neq \phi$. By theorem (4.3.6), $f^{-1}(H)$ and

$f^{-1}(K)$ are disjoint rc sets in $\mathbb{R}$, $A \subseteq f^{-1}(H) \cup f^{-1}(K)$, implying that A is not r-sconnected, a contradiction. Thus, $f(A)$ is r-sconnected.
We now give some results analogous to the results for connected sets on $\mathbb{R}$.

Theorem 4.4.6. *Let G be a r-sconnected set in $\mathbb{R}$ and I,J be any two non void disjoint rc sets in $\mathbb{R}$ with $G \subseteq I \cup J$, then either $G \subseteq I$ or $G \subseteq J$.*

Proof: Let us assume that neither $G \subseteq I$ nor $G \subseteq J$. Since $G \nsubseteq I$, there must be an element a $\in G$ with a $\notin I$. Now, $G \subseteq I \cup J$ and a $\in G$, implies a $\in J$. Thus, a $\in G \cap J$, which is not true. Using the same argument for $G \nsubseteq J$, we get that $G \cap J \neq \phi$. Hence, either $G \subseteq I$ or $G \subseteq J$.

Theorem 4.4.7. *Let P be a proper r-sequentially clopen set in $\mathbb{R}$. If G is r-sconnected set in $\mathbb{R}$ then either $G \subseteq P$ or $G \subseteq P^c$.*

Proof: Clearly, P and P^c are two disjoint rc sets in $\mathbb{R}$ such that G $\subseteq P \cup P^c$. Since A is r-sconnected, by (4.4.6) we have, either G $\subseteq$ P or G $\subseteq P^c$.

Theorem 4.4.8. *Let $I,\ J \subseteq \mathbb{R}$ with $I \subseteq J \subseteq \overline{I}^r$. If I is r-sconnected then J is also r-sconnected.*

Proof: Let I be r-sconnected and $I \subseteq J \subseteq \overline{I}^r$. Now, $J \subseteq \overline{I}^r$ implies $J \subseteq \overline{I}^r \cap J = \overline{I}^r{}_J$ (by (4.3.1)). This means, $J \subseteq \overline{I}^r{}_J$. Also, $I \subseteq J \implies \overline{I}^r{}_J \subseteq J$. Thus we have, J=$\overline{I}^r{}_J$.
If possible, let us assume that J is not r-sconnected. Then $\exists$ rc sets $U, V \neq \phi$ and $U \cap V = \phi$, $J \subseteq U \cup V$ and J meets both U and V. Since I is connected and $I \subseteq J$ by (4.4.6) ,either $I \subseteq U$ or $I \subseteq V$. WLOG, Let $I \subseteq U$. Then $\overline{I}^r \subseteq \overline{U}^r$ and $\overline{I}^r{}_J \subseteq \overline{U}^r \cap J$. For U is rc, we have $\overline{U}^r = U$. Consequently, $\overline{I}^r{}_J \subseteq U \cap J$. Again, $J = \overline{I}^r{}_J$, implies $J \subseteq U \cap J$. Thus $J = J \cap U$, a contradiction. Hence the proof.

Remark 4.6. In the light of the above theorem, if H is r-sconnected then $\overline{H}^r$ is also r-sconnected.

Theorem 4.4.9. *Let $\{U_\alpha : \alpha \in \Delta\}$ be a collection of r-sconnected subsets of $\mathbb{R}$ with nonempty intersection. Then $\cup_{\alpha\in\Delta} U_\alpha$ is r-sconnected.*

Proof: Let us assume that $\cup_{\alpha\in\Delta} U_\alpha$ is not r-sconnected. Then there exists non-empty disjoint rc sets H, K in $\mathbb{R}$ which cover $\cup_{\alpha\in\Delta} U_\alpha$. Since each U_α is r-sconnected, by (4.4.6), either $U_m \subseteq H$ or $U_m \subseteq K$, $\forall\ m \in \Delta$. If for some

$m \neq n \in \Delta$, $U_m \subseteq H$ *and* $U_n \subseteq K$, implies $U_m \cap U_n = \phi$, which is a contradiction. Thus, either $\{U_\alpha \colon \alpha \in \Delta\} \subseteq H$ or $\{U_\alpha \colon \alpha \in \Delta\} \subseteq K$. This implies that either $\{U_\alpha \colon \alpha \in \Delta\} = H \cap \{U_\alpha \colon \alpha \in \Delta\}$ or $\{U_\alpha \colon \alpha \in \Delta\} = K \cap \{U_\alpha \colon \alpha \in \Delta\}$, again a contradiction. Thus, our supposition is wrong. Thus, $\{U_\alpha \colon \alpha \in \Delta\}$ is r-sconnected.

Corollary 4.4.1. *Let* $\{U_\alpha \colon \alpha \in \Delta\}$ *be a collection of r-sconnected subsets of* $\mathbb{R}$ *and* $V \subseteq \mathbb{R}$ *be a r-sconnected set with* $V \cap U_\alpha \neq \phi$, $\forall \alpha \in \Delta$. *Then* $V \cup (\cup_{\alpha \in \Delta} U_\alpha)$ *is also r-sconnected.*

4.5 Summary

In this chapter, we have given rough sequential definitions of continuity, compactness, connectedness and proved that under r-sequential continuous map image of a r-sequentially compact(connected) set is r- sequentially compact(connected). Using these compact(connected) sets a topology can be generated and interesting results can be obtained. Since under rough convergence the limit is not unique, the topology generated by the rough convergent sequences is not Hausdroff. Recently, Leonatti [46] extended the idea of rough ideal convergence to sequences in topological spaces and provided some properties of the set of limit points through the ideal cluster points. However, defining rough convergence for nets and filters in a topological space is still unexplored.

Chapter 5

Conclusion and Future Scope

This thesis revolves around the study of rough ideal convergence, which generalises rough convergence. Firstly, we applied the idea of rough ideal convergence to the sequence of bounded linear operators. These sequences are frequently encountered in numerical integration methods, Fourier series convergence and interpolation polynomial sequences. With the help of these rough ideal convergent sequences, we have introduced a novel sequence space of rough ideal convergent sequence spaces of bounded linear operators. We have given equivalent conditions that suggest a relation between ideal convergence and rough ideal convergence.

In pure mathematics, generalizations play an important role. This encouraged us to generalize the rough ideal convergent sequence spaces. For its generalization, we have used the Orlicz function and established that rough ideal convergent sequence spaces so obtained are solid and monotone, which isn't the case with ideal convergent sequence spaces. This highlights the importance of rough convergence. We have also given some inclusion relations, which indicate that the space of rough ideal convergent sequences includes the space convergent sequences. All this is done for normed linear spaces of real numbers over the field of reals.

Rough convergence has been explored so far for normed linear spaces and metric spaces. In 2023, when the concept of rough continuity was examined by A.K. Banerjee et al. for real valued functions, rough continuity failed to preserve the connected and compact sets. We worked on this problem and came up with rough sequential definitions for continuity, compactness and connectedness and established that the rough sequential continuous map preserves rough sequentially compact and rough sequentially connected sets.

Rough convergence is a relatively new area of research. It has been applied to

various kinds of sequences in spaces where there is a concept of distance between two points. There is a need to address the primary challenge we encountered when applying this concept to an abstract space like a topological space where we move from distance to neighborhoods. Since rough convergence allows a sequence to converge in any neighbourhood, for a sequence $\{x_n\}$ to converge to a point x in a topological space X, $x_n \in U_r \cup U, \forall n \geq m$, for each neighbourbhood U of x in X, where $U_r \subset X$. The main challenge is to choose the set U_r. Thus, it will be intriguing to extend the idea of rough convergence to topological spaces. This in turn will open the doors for the study of rough convergence in topological vector space, rough equicontinuity, rough versions of the Arzela-Ascoli theorem etc.
Although we have successfully achieved the objectives of this research project, there is still a lot to be explored. Rough uniform continuity, rough convergence for series and various other concepts that involve convergence at its core can therefore be studied.

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Conferences and Workshops attended

Participated in the 3rd International Conference on Recent Advances in Fundamental and Applied Sciences (RAFAS 2021), organized by the School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab, held on 25 -26 June, 2021.

Attended a Short Term Course on Scientific and Technical Drawing with LaTeX, organized by the School of Chemical Engineering and Physical Sciences and Human Resource Development Center (HRDC) Lovely Professional University, Phagwara, Punjab, held from 25-30 July, 2022.

Attended a Refresher Course in Applicable Mathematics, from 26 July to 08 August, 2022 organized by the Teaching Learning Centre, Ramanujan College University of Delhi in collaboration with Department Of Mathematics Sri Jayachamarajendra College of Engineering (SJCE) JSS Science and Technology University, Mysuru.

Attended An International Online Workshop on Trends in Analysis and Topology, from 08 -12 September, 2022 organized by the Department of Mathematics, Malaviya National Institute of Technology Jaipur.

Participated in a National Conference on Recent Advances in Mathematics, organized by the Department of Mathematics Central University of Jammu held on 20-21 October, 2022.

Participated in the 30th Annual Conference of Jammu Mathematical Society (JMS), organized at the Department of Mathematics, University of Jammu, held on 04-06 May, 2023.

Attended a Refresher Course in Mathematics (Theme Python and Vedic Mathematics), from 13 - 26 July, 2023 organized by Teaching Learning Centre, Ramanujan College University of Delhi in collaboration with Department Of Mathematics Sri Jayachamarajendra College of Engineering (SJCE) JSS Science and Technology University, Mysuru.

Participated in the 5th International Conference on Recent Advances in Fundamental and Applied Sciences (RAFAS-2024), organized by the School of Chemical Engineering and Physical Science at Lovely Professional University, Phagwara, Punjab, during 19-20 April, 2024.

Paper Presentations and Publications

Paper Presentations

Shivani Sharma and Sanjay Mishra, On Generalized Rough Ideal Convergent Sequence Spaces, presented at the National Conference on Recent Advances in Mathematics, organized by the Department of Mathematics Central University of Jammu held on 20-21 October,2022.

Shivani Sharma and Sanjay Mishra, On Some Rough Ideal Convergent Sequence Spaces of Bounded Linear Operators, presented at the 30th Annual Conference of Jammu Mathematical Society (JMS), organized at the Department of Mathematics, University of Jammu, held on 04-06 May 2023.

Shivani Sharma, Sanjay Mishra and Pankaj Pandey, On Some Rough Ideal Convergent Sequence Spaces, presented at the 5th International Conference on Recent Advances in Fundamental and Applied Sciences (RAFAS-2024), organized by the School of Chemical Engineering and Physical Science at Lovely Professional University, Phagwara, Punjab, during 19-20 April,2024.

Publications

Shivani Sharma and Sanjay Mishra, Rough Ideal Convergent Sequence Spaces of Bounded Linear Operators, have been published in the South East Asian Journal of Mathematics and Mathematical Sciences Vol. 19, No. 2 (2023), pages 297-310 DOI: 10.56827/SEAJMMS.2023.1902.22

Submitted for Journal Publications

Shivani Sharma, Sanjay Mishra and Pankaj Pandey, On Some Rough Ideal Convergent Sequence Spaces. (Accepted to be published in Palestine Journal of Mathematics)

Shivani Sharma, Sanjay Mishra and Pankaj Pandey, On Rough Continuity and Rough Compactness. (Communicated)